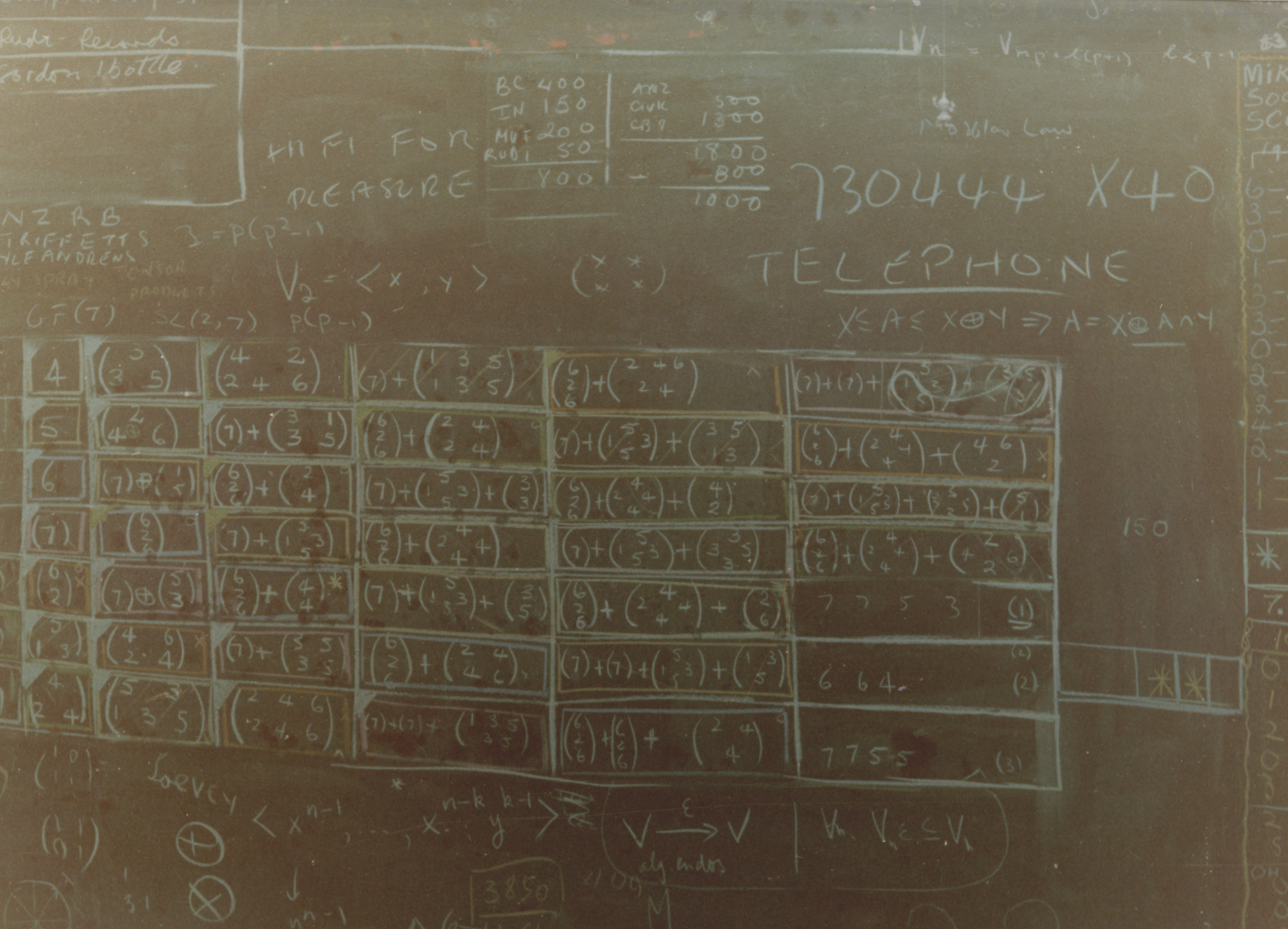
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A STUDY OF CERTAIN MODULAR REPRESENTATIONS

by

D.J. Glover

A thesis submitted for the degree of

Doctor of Philosophy

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STATEMENT

The work contained in this thesis is my own except where otherwise stated.

D.J. Glover

D.J. Glover

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ABSTRACT

Let p be a prime number, F the field of p elements, M the semigroup of all 2×2 matrices over F, G the group $GL(2, p)$ of invertible elements of M, and S the normal subgroup $SL(2, p)$ of G consisting of the matrices of determinant one. The aim of this thesis is to study the representations of M, G and S over F.

A certain construction is of great help. Let V be the commutative polynomial algebra in two indeterminants, x and y say, over F. For each positive integer m, the homogeneous polynomials of degree $m - 1$ form a subspace V_m, of dimension m, in V, and $V = \bigoplus_{m=1}^{\infty} V_m$. Each V_m may be regarded as an FM-module by considering the elements of M as homogeneous linear substitutions, so an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of M applied to a monomial $x^i y^j$ yields $(ax+by)^i(cx+dy)^j$.

The one dimensional FM-modules, other than V_1, are the tensor powers D^n $(n = 1, 2, \ldots, p-1)$ of the module D which affords the determinant representation (so on this each element of M acts as the scalar which is its determinant); we put $D^0 = V_1$. We show that M has precisely p^2 (isomorphism classes of) irreducible modules over F; namely the $V_m \otimes D^n$ with $1 \leq m \leq p$, $0 \leq n \leq p-1$. These modules, after restriction, yield all the irreducibles of G and S as well, but to keep them pairwise ionisomorphic the upper end of the range of n has to be brought down to $p - 2$ and 0 respectively. For each of M, G and S the principal indecomposable modules are described in sufficient detail to reveal their complete submodule structure. For G and S, all principal indecomposables occur as direct summands in the restrictions of the V_m,

while the same will hold for some, but definitely not all, principal indecomposables in the case of M. For G and S, direct decompositions of restrictions of the V_m have at most one nonprojective summand each; this is also false for M. For G and S the nonprojective indecomposable direct summands of the V_m form periodic sequences with period $p(p-1)$. In the (repeated) initial segment of length $p(p-1)$ of this sequence, every pth term is 0 while the others are nonzero and pairwise nonisomorphic. In the case of S every nonprojective indecomposable is isomorphic to one and only one of these, while in the case of G every nonprojective indecomposable is isomorphic to one and only one of these tensored with a D^n. The nonprojective indecomposables for G and S can therefore be described in sufficient detail to reveal a great deal of their submodule structure. Each G-module can be made into an M-module by making all elements of M outside G annihilate it: call such M-modules singular. Now M has infinitely many isomorphism types of nonsingular indecomposables, but only finitely many of them occur as direct summands of the V_m, and we do not attempt to classify even those which do.

To complete this summary of the results in this thesis, it remains to indicate what we know about the structure of the V_m. As far as the action of G or S is concerned, each V_m is a direct summand of $V_{m+p(p-1)}$ with a complement which depends only on the residue class of $m \bmod p^2-1$. This makes it possible to describe all V_m by dealing, as we do, with the first few. By contrast, with respect to the action of M the V_m do not fit such an arithmetic pattern: for instance, V_2 is not a direct summand of any other V_m, and is not even embeddable in V_m unless $m-1$ is a power of p.

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CHAPTER 1

INTRODUCTION

Let p be a prime number F the field of p elements, M the semigroup of all 2×2 matrices over F, G the group $GL(2, p)$ of invertible elements of M, and S the normal subgroup $SL(2, p)$ of G consisting of the matrices of determinant one. The aim of this thesis is to study the representations of M, G and S over F. A certain construction is of great help towards this aim as well as being of considerable interest in its own right. We describe this next.

Let V be the commutative polynomial algebra in two indeterminants, x and y say, over F. For each positive integer m, the homogeneous polynomials of degree $m - 1$ form a subspace V_m, of dimension m, in V, and $V = \bigoplus_{m=1}^{\infty} V_m$. Each V_m may be regarded as an FM-module by considering the elements of M as homogeneous linear substitutions, so an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of M applied to a monomial $x^i y^j$ yields $(ax+by)^i(cx+dy)^j$. (As V_1 consists of the "constant" polynomials, its elements are left unchanged by *all* elements of M, including the zero matrix.) The maps so representing the elements of M on V are not only linear transformations but are also F-algebra endomorphisms of V.

This construction was used by Brauer and Nesbitt [1] to determine the irreducibles of S. Various other results given below for S or G occur in the literature or folklore of the subject, even to the explicit description of the submodule structure of half the indecomposables for S by Janusz in [7]. Instead of quoting these, or applying the general theories from which they are derived, we give virtually bare-handed proofs, many of which seem to have distinct attractions. Our need is for an approach that will enable us to handle the case of M. The representation theory of (finite) semigroups as expounded for instance in [2, 3] by Clifford and Preston, or in [8, 9] by McAllister, seems to succeed only with modules that can be induced from maximal subgroups. One of the results of this thesis is that M has infinitely many isomorphism types of indecomposables over F, a result one would hardly have expected along the lines of this theory, for the maximal subgroups of M are G and cyclic groups of order 1 and $p - 1$. Groups of this type have only finitely many isomorphic types of indecomposables over F. This illustrates the reasons for proceeding virtually from first principles.

Next we summarize our results. We aim at finding the irreducibles, the indecomposables and the submodule structure of at least the principal indecomposable S, G and M modules. We would also like to be able to describe all the V_m and, although we have not been able to give a complete description of these, at least we can say as much as we would like to say about the $V_m|_G$.

The one dimensional FM-modules, other than V_1, are the tensor powers D^n $(n = 1, 2, \ldots, p-1)$ of the module D which affords the determinant representation (so on this each element of M acts as the scalar which is its determinant); we put $D^0 = V_1$. We show that M has precisely p^2 (isomorphism classes of) irreducible modules over F; namely the $V_m \otimes D^n$ with $1 \leq m \leq p$, $0 \leq n \leq p-1$. These modules, after restruction, yield all the irreducibles of G and S as well, but to keep them pairwise nonisomorphic the upper end of the range of n has to be brought down to $p - 2$ and 0 respectively. For each of M, G and S the principal indecomposable modules are described in sufficient detail to reveal their complete submodule structure. For G and S, all principal indecomposables occur as direct summands in the restrictions of the V_m, while the same will hold for some, but definitely not all, principal indecomposables in the case of M. For G and S, direct decompositions of restructions of the V_m have at most one nonprojective summand each; this is also false for M. For G and S the nonprojective indecomposable direct summands of the V_m form periodic sequences with period $p(p-1)$. In the (repeated) initial segment of length $p(p-1)$ of this sequence, every pth term is 0 while the others are nonzero and pairwise nonisomorphic. In the case of S every nonprojective indecomposable is isomorphic to one and only one of these, while in the case of G every nonprojective indecomposable is isomorphic to one and only one of these tensored with a D^n. The nonprojective indecomposables for G and S can therefore be described in sufficient detail to reveal a great deal of their submodule structure. Each G-module can be made into an M-module by making all elements of M outside G annihilate it: call such M-modules singular. Now M has infinitely many isomorphism types of nonsingular indecomposables, but only finitely many of them occur as direct summands of the V_m, and we do not attempt to classify even those which do.

To complete this summary of the results in this thesis, it remains to indicate what we know about the structure of the V_m. As far as the action of G or S is concerned, each V_m is a direct summand of $V_{m+p(p-1)}$ with a complement which depends only on the residue class of $m \bmod p^2-1$. This makes it possible to describe all V_m by dealing, as we do, with the first few. By contrast, with respect to the action of M the V_m do not fit such an arithmetic pattern: for instance, V_2 is not a direct summand of any other V_m , and is not even embeddable in V_m unless $m - 1$ is a power of p . For this reason we cannot complete the discussion of the V_m as M-modules. This is all the more surprising as there is "very little" M-action (other than G-action) on each V_m in the following sense: for $m > p$ the unique largest singular submodule [namely $(x^p y - xy^p)V_{m-p-1}$] has codimension $p + 1$ in V_m and the unique smallest nonsingular submodule has dimension at most $p + 1$. In any composition series of V_m all but one factor is singular, so in any direct decomposition of a V_m all but one summand is singular and so on.

As we have said, we work essentially from first principles. We do rely on some general representation theory, say from the book [4] by Curtis and Reiner. Our only other sources are Dickson and Kelly [5], which we invoke to show that M is of infinite representation type, Huppert [6] for a general result about modules and an unpublished manuscript by Professor G.E. Wall to whom we are greatly indebted for making it available. A crucially important tool in the investigation of the V_m is a family of short exact sequences

$$0 \to V_m \otimes V_n \otimes D \to V_{m+1} \otimes V_{n+1} \to V_{m+n+1} \to 0$$

which comes from that manuscript, where it is used for analysing the first $2p$ of the V_m as S-modules, the tensor products of irreducible S-modules and the principal indecomposable S-modules. While we reach our conclusions by somewhat different arguments, we greatfully acknowledge the influence of that manuscript through much of this thesis.

The thesis is arranged in seven chapters the first of which is this introduction. In the second chapter we present some repliminary results, mainly from general representation theory. They are all well known and fit into two classes, the first being results that do not seem to be presented in the literature in a form suitable for use here. Proofs are provided for these. The second class consists of results that are frequently used throughout the thesis and it is convenient to collect them together in one place. More importantly however, we can use these results to exhibit the differences relating to group algebras and the more general class of algebra that we are interested in.

In Chapter 3 we look at the matrices representing, on a V_m , certain (invertible) elements of M . In this way we are able to determine some useful results relating to the restriction of the V_m to subgroups of G . These results are of interest in themselves and some may, in later chapters, be used to obtain some of the structure of the V_m themselves. Important results in this chapter include the fact that, for any m , a decomposition of $V_m|_G$ into indecomposables yields at most one that is nonprojective. Also we show that these nonprojective summands form a periodic sequence. Other results include the proof of the irreducibility of the first p of the V_m and the fact that any nonprojective indecomposable FG (FS) module is isomorphic to a direct summand of $\left(V_m \otimes D^n\right)\Big|_G \left(V_m|_S\right)$ for a unique $m \in \{1, 2, \ldots, p(p-1)\}$.

In Chapter 4 we switch back to considering the V_m , not just their restructions to subgroups. It is here that we show that for $m \geq p+1$, each V_m has a unique largest singular submodule $\left(x^p y - xy^p\right) V_{m-p-1}$ and a unique smallest nonsingular submodule $V_m FN$. We see that the quotients $V_m / \left(x^p y - xy^p\right) V_{m-p-1}$ are indecomposable and form a periodic sequence, of period $p - 1$. This fact enables us to show that the $V_m|_G$ fit an arithmetic pattern in the sense of (4.3) and already referred to in this introduction. The submodule X_m , of V_m , spanned by the $(m-1)$st powers of elements of V_2 turns out to be in fact V_mFN , giving alternative descriptions for that module. We see that X_m is indecomposable and (for $m \neq 1$) $X_m/X_m \cap (x^py-xy^p)V_m$ is isomorphic to V_n with $n \in \{2, 3, \ldots, p\}$, $n \equiv m \bmod p-1$. These two submodules, X_m and $(x^py-xy^p)V_m$, will play a large role in this thesis.

Chapter 5 shows another change in direction. In fact Chapters 3, 4 and 5 are almost independent of each other, results from Chapters 3 and 4 creeping in right at the end of chapter 5. It is here that the "exact sequences" enter the picture and we can begin to describe the structure of the V_m for $m > p$. The chapter concludes with a description of V_{pk} and V_{p+k} for $k \in \{0, 1, \ldots, p-1\}$ with these two series of modules providing us with some useful side results for the final two chapters. Other results in Chapter 5 are more technical in nature and we do not list them here.

Chapter 6 opens with a description of all irreducible and principal indecomposable FG-modules. The work for this has been done in Chapter 5 and in Chapter 6 we do little more than summarize that. The first $p(p-1)$ of the $V_m|_G$ are described and we produce some reduction formulas that would enable us to describe a V_m in terms of direct sums of some $V_{n_i}|_G$ with $n_i \leq p(p-1)$.

The last chapter exhibits the difference between the V_m and $V_m|_G$. First we look at arbitrary FM-modules, by considering all principal indecomposables, and we note they have a more complex structure than that of the principal indecomposable FG-modules. These principal indecomposables show us that the algebra FM is of infinite representation type. Finally we show that only a finite number of different indecomposable FM-modules occur as direct summands of the V_m but a simple arithmetic pattern as in (4.3) is not evident for the whole semigroup algebra.

The case $p = 2$ is one of degenerate simplicity in some of the more technical statements. In order not to clutter up an already rather complicated situation by this, from Chapter 5 on we deal only with the case $p > 2$. As a result, some of the claims made in this introduction are not explicitly established here for $p = 2$: however, they are valid (and provable by the same methods) also for $p = 2$.

CHAPTER 2

REPRESENTATION THEORY

Most of the notation and quoted results in this thesis can be taken straight from Curtis and Reiner [4]. A useful table of notation is included in the preface to that book, the two main points of difference being our use of right, not left, modules and "exponential" notation. We use this chapter partly to provide a sample of some results from Curtis and Reiner that illustrate theorems about group algebras which may, or may not, be extended to corresponding theorems concerning a more general class of algebra. We find that the results that do not generalize in the manner we would like fail, even in the "relatively nice" algebra FM. Most of the results in this chapter will be used frequently throughout the thesis, often without reference.

Everything mentioned in this thesis (with the exception of V itself) is finite. This fact will be assumed in all results in this chapter, even though most are ture in much greater generality. Also we will assume that all algebras mentioned have unity element 1 and we do not consider any other fields than F, $F = \mathrm{GF}(p)$.

Within any module U there are two important chains of submodules. Defining $\varphi^0 U = U$ the first of these chains has two equivalent definitions. For $n \geq 1$ we define $\varphi^n U$ to be the smallest submodule such that $\varphi^{n-1}U/\varphi^n U$ is semisimple. Equivalently $\varphi^n U$ is the intersection of the maximal submodules of $\varphi^{n-1}U$. The other chain is defined by $\sigma^0 U = 0$ and for $n \geq 1$, $\varphi^n U$ is to be the largest submodule of U such that $\sigma^n U/\sigma^{n-1}U$ is semi-simple. We begin by exploring a few elementary facts about these submodules.

(2.1). *For any indecomposable but reducible module* U , $\sigma U \leq \varphi U$.

Proof. Let Y be an irreducible submodule of U and choose W such that

$$\frac{U}{\varphi U} = \frac{Y+\varphi U}{\varphi U} \oplus \frac{W}{\varphi U} .$$

If $W \cap Y = 0$ then $U = W \oplus Y$ contradicting the indecomposability of U and if $W \cap Y \neq 0$ then $Y = W \cap Y \leq W \cap (Y+\varphi U) \leq \varphi U$ and hence (2.1) holds. □

(2.2). *If* U *is a module and* α *is an endomorphism of* U , *then* $(\varphi^n U)\alpha \leq \varphi^n(U\alpha)$.

Proof. Induction on n gives this easily, provided one knows that $(\varphi U)\alpha \leq \varphi(U\alpha)$. As $\varphi(U\alpha)$ is the intersection of all the maximal submodules Y of $U\alpha$, to this end, it is sufficient to show that $(\varphi U)\alpha \leq Y$ for every such Y . But the complete inverse image W of Y under α is a maximal submodule of U and thus $\varphi U \leq W$ which gives $(\varphi U)\alpha \leq W\alpha = Y$. □

An algebra A can itself be considered as a right A-module (sometimes denoted A_A). If we write $A = \oplus A_i$, A_i indecomposable, then the A_i are the so called principal indecomposable A-modules. We can say a surprising amount about these modules.

(2.3) ([4], 54.11, p. 372). *If* A_i *is a principal indecomposable* A*-module, then* $A_i/\varphi A_i$ *is irreducible. Two principal indecomposable modules* A_i *and* A_j *are isomorphic if and only if* $A_i/\varphi A_i \cong A_j/\varphi A_j$.

([4], 54.13, p. 374). *Every irreducible* A*-module is isomorphic to* $A_i/\varphi A_i$ *for some principal indecomposable* A*-module* A_i . □

In view of (2.3), for any irreducible A-module U , it makes sense to talk of $P(U)$, the principal indecomposable A-module having U as its unique irreducible factor module. The following result is well known but as restrictions are usually placed on the field F , A being an F-algebra, I cannot find it in an easily quotable form. We therefore go through the standard proof.

(2.4). *If* U *is an absolutely irreducible* A*-module, then the number of summands isomorphic to* $P(U)$ *in an unrefinable direct decomposition of* A *is equal to the dimension of* U .

Proof. If $A = \oplus A_i$, A_i indecomposable, then $\varphi A = \oplus \varphi A_i$ so $A/\varphi A \cong \oplus A_i/\varphi A_i$. By (2.3) the A_i are irreducible. Let the A_i be numbered so that $A_i/\varphi A_i \cong U$ if and only if $1 \le i \le m$. Thus

$$\bigoplus_{i=1}^{m} A_i/\varphi A_i \cong U^{\oplus m}$$

(the direct sum of m copies of U,).

Therefore we have $U^{\oplus m}$, a homomorphic image of A , which is generated by one element. Thus $U^{\oplus m}$ can itself be generated by a single element and we will use this fact to show $m \le \dim U$. On the other hand we will show that $U^{\oplus \dim U}$ is generated by one element and must therefore be a homomorphic image of A , the free A-module of rank one. This will show $\dim U \le m$ and thus equality.

A direct sum $U^{\oplus m}$ of m copies of U is given by m projections $\pi_i : U^{\oplus m} \twoheadrightarrow U$ and m embeddings $\eta_i : U \rightarrowtail U^{\oplus m}$ such that $\eta_i \pi_j$ is 1 or 0 in End U as $i = j$ or $i \ne j$, and $\sum \pi_i \eta_i = 1$, 1 the identity element in End $U^{\oplus m}$.

If $U^{\oplus m}$ is generated by an element v and a linear combination $\sum \alpha_i (v\pi_i)$ of the projections $v\pi_i$ vanishes then the kernel of the homomorphism $\sum \alpha_i \pi_i : U^{\oplus m} \to U$ contains v and therefore all of $U^{\oplus m}$. Therefore $\sum \alpha_i \pi_i = 0$ and so, for all j, $0 = \eta_j \sum \alpha_i \pi_i = \alpha_j$. This shows that the $v\pi_i$ are linearly independent in U and thus $m \le \dim U$.

Conversely if $u_1, u_2, \ldots, u_m$ are linearly independent elements of U and $v = \sum u_i \eta_i$, then v generates $U^{\oplus m}$. To see this, assume not so that v lies in a proper submodule of $U^{\oplus m}$ and thus in the kernel of some nonzero homomorphism $\varphi : U^{\oplus m} \to U$. Now $\varphi = \sum \pi_i \eta_i \varphi$ so $\eta_j \varphi \neq 0$ for some j. As $\eta_i \varphi \in \operatorname{End} U$, by ([4], 29.13, p. 202) each $\eta_i \varphi$ acts as multiplication by a scalar, α_i say, where of course $\alpha_j \neq 0$. On the other hand $0 = v\varphi = v \sum \pi_i \eta_i \varphi = \sum \alpha_i u_i$ as $v\pi_i = u_i$ by the definition of v and now $\alpha_j \neq 0$ contradicts the linear independence of the u_i. □

Several characterizations of projective modules will be used. We call an A-module P projective if and only if:

(a) it is isomorphic to a direct sum of principal indecomposable modules;

(b) whenever X and Y are A-modules such that $0 \to X \to Y \to P \to 0$ is an exact sequence, then this sequence splits;

(c) if X and Y are A-modules and $\varphi : P \to Y$, $\psi : X \twoheadrightarrow Y$ are homomorphisms, the latter being onto Y, then there exists a ξ in $\operatorname{Hom}(P, X)$ such that $\xi\psi = \varphi$.

An immediate corollary of this last definition is that if W is any module with $W/\varphi W$ irreducible then W is a homomorphic image of $P(W/\varphi W)$. This fact will be used often without reference.

In a similar way to (b) above we can define an A-module I to be injective if and only if all exact sequences $0 \to I \to K \to Y \to 0$ split. A definition similar to (c) also holds but in general the indecomposables which occur in the appropriate version of (a) are not the principal indecomposables. We do have however, for group algebras

(2.5) ([4], 58.14, p. 402). *A module for a group algebra is injective if and only if it is projective.* □

For group algebras we can say even more about the principal indecomposables. In view of the comments at the end of ([4], 58.12, p. 401) we have

(2.6). *If A_i is a principal indecomposable module for a group algebra then σA_i is irreducible and $\sigma A_i \cong A_i/\varphi A_i$.* □

We will see later that neither (2.5) nor (2.6) hold for FM-modules.

The following result is well known and the proof quite standard. As I can not find a quotable reference for it we go through the proof.

(2.7). *Let A be an algebra whose projective indecomposable modules and injective indecomposable modules are all uniserial (modules with only one composition series). Then every indecomposable A-module is uniserial.*

Proof. Let U be an indecomposable A-module and V/W a uniserial section of U of maximal composition length. Consider the submodules X of U which satisfy $V \cap X = W$. The set of such submodules contains W and so is not empty. Let X be a maximal element of this set. If $0 < Y/X \leq U/X$ then, by the choice of X , $W < V \cap Y$. Therefore, as V/W is uniserial, $\sigma(V/W) \leq Y/W$. This shows that U/W is an essential extension of $\sigma(V/W)$ so U/W is isomorphic to a submodule of the injective hull of $\sigma(V/W)$, which is, by assumption, uniserial. Thus U/W is uniserial and the maximality of V/W guarantees $U = V$.

Next let X' be minimal among the submodules of U satisfying $X' + W = U$. Put $\varphi(U/W) = V'/W$. As $U = X' + W$ we have

$V' = (X' \cap V') + W$. Now $Y' \leq X' \cap V' < X'$ so the maximality of Y' in X' gives that $Y' = X' \cap V'$. Thus X' has only one maximal submodule, namely $X' \cap Y'$, so it is a homomorphic image of the projective cover of the irreducible module $X'/X' \cap Y'$. By assumption that principal indecomposable is uniserial so X' is uniserial. Now $U/W = X'+W/W \cong X'/X' \cap W$ and U/W was chosen to have maximal composition length among the uniserial sections, so $X' \cap W = 0$. Thus $U = X' \oplus W$ and as U is assumed indecomposable this means $W = 0$ and so U is uniserial.

For group algebras we also have the Tensor Product Theorem which is just the essential special case of ([4], 44.3, p. 325).

(2.8). *Let* L *be a subgroup of a group* H , U *an* FL*-module,* W *an* FH*-module. Then* $U^H \otimes W \cong (U \otimes W_L)^H$. □

Before we derive some corollaries of (2.8) we note that for a group H , the module FH (group algebra considered as a module) is induced from the one dimensional module for the trivial subgroup E . That is $FH \cong (FE)^H$, or in the tensor product language of Curtis and Reiner $FH \cong FE \otimes_{FE} FH$. Thus we can characterize projective FH-modules as modules that are direct summands of modules induced from E . A similar sort of thing holds for a more general class of algebras but we do not need that in this thesis.

(2.9). *If* U *and* W *are* FH*-modules for a group* H *and* W *is projective, then so is* $U \otimes W$.

Proof. As W is projective, it is a direct summand of X^H where X is a module for the trivial subgroup E . That is $W \oplus \overline{W} = X^H$ for some FH-module $\overline{W}$ so $U \otimes (W \oplus \overline{W}) \cong (U \otimes W) \oplus (U \otimes \overline{W})$. On the other hand

$$U \otimes (W \oplus \overline{W}) \cong U \otimes X^H$$
$$\cong \left(U_E \otimes X\right)^H$$

by (2.8) where $\left(U_E \otimes X\right)^H$ is also a module induced from E to H and, by the Krull Schmidt Theorem, $U \otimes W$ is isomorphic to a direct summand of it. Therefore $U \otimes W$ is projective. □

The thing that makes (2.9) work is the Tensor Product Theorem which as stated is not applicable to arbitrary algebras. We will see that (2.9) does not hold for FM-modules, even if U happens to be one dimensional. In that situation, that is U one dimensional, if W is a principal indecomposable module for a group algebra then so is $U \otimes W$. For FM modules this tensor product need not even be indecomposable. The reason for this difference is that for group algebras, to each one dimensional module U there exists another one dimensional module $\overline{U}$ such that on $\overline{U} \otimes U$ the group acts trivially and so $\overline{U} \otimes U \otimes W \cong W$. It follows that the submodule lattices of W and $U \otimes W$ are isomorphic. There is no reason however that one dimensional modules for arbitrary algebras need have "inverses".

We also need to know something about projectivity of induced and restructed modules. We sum this up in:

(2.10). *If L is a subgroup of a group H, P the Sylow p-subgroup of H, U an FL-module and W an FH-module then*

(a) U projective implies U^H projective,

(b) W projective implies W_L projective, and

(c) W_P projective implies W projective.

Proof. Using the approach of (2.9) the proof of (a) and (b) is straightforward. In (c), as W_P is projective, there exists a $\overline{W}$ such that $W_P \oplus \overline{W} \cong X^P$ for some E module X. Therefore, on the one hand

$(W_P \oplus \overline{W})^H \cong (W_P)^H \oplus (\overline{W})^H$ and on the other hand $(W_P \oplus \overline{W})^H \cong (X^P)^H \cong X^H$.

By ([4], 63.2, p. 427 and 63.7, p. 430) W is a component of $(W_P)^H$ so by the Krull-Schmidt Theorem (c) holds. □

One other related fact that will be used often mostly without reference is

(2.11) ([4], 65.17, p. 439). *If P is a Sylow p-subgroup of a group H and U is an FH-module, then U is projective only if the order of P divides the dimension of U .* □

We conclude the representation part of this chapter with a result relating to arbitrary modules.

(2.12). *If I and U are disjoint submodules of a module X , with I injective, then there exists $Y \leq X$ such that $U \leq Y$ and $X = I \oplus Y$.*

Proof. Choose Y maximal with respect to $U \leq Y$ and $Y \cap I = 0$. Then $(Y \oplus I)/Y$ is isomorphic to I and is thus injective. Therefore there exists a submodule W such that $X/Y \cong (Y \oplus I)/Y \oplus W/Y$. If $Y < Z \leq X$ then $W \cap I > 0$ by the maximality of Y . Therefore $W \cap (I \oplus Y) > Y$ providing a contradiction. Thus $Y = W$ and $X/Y \cong (Y \oplus I)/Y$. □

Finally two arithmetic results that we shall need later.

(2.13).

$$\sum_{m=1}^{p-1} m^t \text{ is congruent to } \begin{cases} 0 \bmod p & \text{if } t \not\equiv 0 \bmod p-1 \\ -1 \bmod p & \text{if } t \equiv 0 \bmod p-1 . \end{cases}$$

Proof. Writing $m^t = m^{n+j(p-1)}$, $0 \leq n < p-1$ we have

$$\sum_{m=1}^{p-1} m^t \equiv \left(\sum_{m=1}^{p-1} m^n\right) \bmod p .$$

If $n = 0$ then

$$\sum_{m=1}^{p-1} m^0 = p - 1$$

proving the second part. With

$$\binom{m}{i} = \frac{1}{i\,!}\ (m-1)(m-2)\ \ldots\ (m-i+1)$$

we can write

$$m^t = \sum_{i=1}^{t} \beta_i \binom{m}{i}\ ,$$

with β_i integers, so that for $1 \le n < p-1$,

$$\sum_{m=1}^{p-1} m^n = \sum_{m=1}^{p-1} \sum_{i=1}^{n} \beta_i \binom{m}{i}$$

$$= \sum_{i=1}^{n} \beta_i \sum_{m=1}^{p-1} \binom{m}{i}$$

$$= \sum_{i=1}^{n} \beta_i \binom{p}{i+1}$$

by a well known formular relating binomial coefficients. This sum is always congruent to 0 mod p as $1 \le n \le p-2$, proving the first part of (2.13).

(2.14). *For every* r *and* i *such that* $0 \le i \le p^r-1$,

$$\binom{p^r-1}{i} \equiv (-1)^i \bmod p\ .$$

Proof. This is proved by induction on i noting that

$$\binom{p^2-1}{0} = 1 = (-1)^0\ .$$

As is well known $\binom{p^r}{i}$ is divisible by p whenever $0 < i < p^r$ so for $0 < i \le p^r-1$ we have

$$\binom{p^r-1}{i} = \binom{p^r}{i} - \binom{p^r-1}{i-1} \equiv 0 - (-1)^{i-1} \bmod p\ ,$$

the last step by the inductive hypothesis. □

CHAPTER 3

RESTRICTIONS TO SUBGROUPS

In the introduction we defined the polynomial algebra V and the FM-modules V_m consisting of the homogeneous polynomials of degree $m - 1$ in V. Recall that M acts on V by F-algebra endomorphisms. The V_m can also be considered as FH-modules for subgroups H of M. We, as usual, denote these restrictions to H by $V_m|_H$ or $(V_m)_H$. One of the aims of this thesis is to describe the $V_m|_G$ but besides this there are two other reasons for studying the restrictions to subgroups.

Firstly, results about restrictions to certain small subgroups of G can be used to determine results about restrictions to larger subgroups or hopefully to results about the V_m themselves. An example of this is (3.1) where we see that for $m \in \{1, 2, \ldots, p\}$, the $V_m|_S$ are irreducible and consequently so must be the V_m. Another example is (3.3) which states that for any m, a decomposition of $V_m|_G$ into indecomposable summands yields at most one that is nonprojective. This crucial result is a simple consequence of (3.2), a result concerning restriction to a p-cycle.

The other reason for examining these restrictions is that the V_m provide a very convenient medium for studying representations of these subgroups. For example, in (3.9) we show that if W is any nonprojective indecomposable FS-module then there exists an m such that W is isomorphic to a direct summand of $V_m|_S$. It is also true that projective indecomposable FS-modules arise in the same way. For indecomposable FG-modules the situation is nearly as good. We see in (3.8) that for any nonprojective indecomposable FG-module W there exists m, n such that W

is isomorphic to a direct summand of $\left(V_m \otimes D^n\right)_G$.

The other important result in this chapter is (3.7) which says that the nonprojective summands of the $V_m|_G$ form a periodic sequence. This is used often throughout the thesis and in particular in Chapter 4 where we show that the $V_m|_G$ themselves fit into a conveniently describable pattern (see (4.3)).

The two subgroups not already defined that play a big part in this chapter are the subgroup T consisting of the lower triangular matrices, that is, the matrices of the form $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$ $(a, b, c \in F, ab \neq 0)$, and P , the subgroup consisting of the matrices of the form $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in F$. The subgroup P is a Sylow p-subgroup of T, S, $T \cap S$ and G . It is the cyclicity of P that makes a lot of this thesis possible.

A convenient basis to pick for each V_m is the set of monomials $\{x^i y^{m-i-1};\ 0 \leq i \leq m-1\}$; clearly, V_m has dimension m . For two u, v $\in V$ with u $\in V_m$ and v $\in V_n$, uv $\in V_{m+n-1}$. Note that V_1 is the one dimensional module on which $v^t = v$ for all $v \in V_1$ and *all* $t \in M$ (including the case where t is the zero element of M , however strange that may look). If 1 is the identity element in V , we may as well let this be "the" basis vector for V_1 . We use the convention that $V_0 = 0$.

We have already introduced the one-dimensional modules D^j where for $t \in M$ and $d \in D^j$, $d^t = (\det t)^j d$. Then D^j is the j-fold tensor power of D ; and in general for every module A we take A^j to mean the j-fold tensor power, with the convention that $A^0 = V_1$.

The first result in this chapter was proved in ([1], 30, p. 588).

However the statement of the result is not in the form we require. The proof given here is shorter and sets the spirit for the chapter as we exploit this approach repeatedly. The notation $\langle u_1, u_2, \ldots, u_n \rangle$ denotes the module spanned by $u_1, \ldots, u_n$.

(3.1). *For* $1 \le m \le p$ *, the modules* $V_m|_S$ *are absolutely irreducible and thus the* $V_m \otimes D^n$ *and the* $\left(V_m \otimes D^n\right)_G$ *are absolutely irreducible for all values of* n .

Proof. Take $h = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ in P . On the chosen basis of V_m this element h is represented by the matrix Y_m whose i, j entry is the binomial coefficient $\binom{i-1}{j-1}$ reduced modulo p to yield an element of F . Note that $\binom{i-1}{i-2} = i - 1$ and thus Y_m has the form

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & 0 & & \\ & 2 & 1 & & & \\ & & 3 & 1 & & \\ & & & \ddots & \ddots & \\ & * & & & \ddots & \ddots \\ & & & & m-1 & 1 \end{pmatrix} .$$

From this we see that $Y_m - I_m$ (the $m \times m$ identity matrix) has rank $m - 1$ (it is here we use $m \le p$), and thus the fixed points of Y_m form a subspace of dimension one. As all irreducible FP-modules are one dimensional and P acts trivially on them, this means that $V_m|_P$ has a unique minimal submodule.

But clearly, by the same argument, any factor module has a unique minimal submodule. Therefore $V_m|_P$ is uniserial with the unique composition series

$$V_1|_P \cong \langle x^{m-1}\rangle < \langle x^{m-1}, x^{m-2}y\rangle < \ldots < \langle x^{m-1}, x^{m-2}y, \ldots, y^{m-1}\rangle \cong V_m|_P .$$

Thus every nonzero submodule of $V_m|_P$ contains x^{m-1} but no proper submodule can contain y^{m-1}. Since $(x^{m-1})^s = y^{m-1}$ for $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in S$, no submodule of $V_m|_S$ can contain x^{m-1} without containing y^{m-1} and hence $V_m|_S$ can have no nonzero proper submodules.

Even if we were to extend the field of scalars, the same argument would apply, so $V_m|_S$ is absolutely irreducible. As $\left(V_m \otimes D^n\right)_S \cong V_m|_S$, the other assertions of (3.1) are clear. $\square$

This result does more than describe *some* irreducible S, M and G modules. In later chapters we show that this in fact describes *all* of them.

We also note that as the $V_m|_P$ with $1 \le m \le p$ are uniserial, and so indecomposable, and as P has just p isomorphism types of indecomposables (see Curtis and Reiner ([4], 64.2, p. 431)), every indecomposable FP-module is isomorphic to one of *these* $V_m|_P$. We exploit this in the proof of the next result which will have as an almost immediate corollary one of the (two) most useful tools in this thesis.

(3.2). *For $m \ge 1$ write $m = kp + r$ with $0 \le r \le p-1$. Then $V_m|_P$ is the direct sum of $V_r|_P$ and k copies of the regular FP-module.*

Proof. We proceed as in the proof of (3.1); the first point of departure being that $Y_m - I_m$ has rank $(kp+r) - (k+1)$ if $r > 0$ and $(kp+r) - k$ if $r = 0$. The space of fixed points of Y_m therefore has dimension k if $r = 0$ and $k + 1$ if $r > 0$. We also notice that the span of the first kp basis vectors is a submodule, U say, on which h is represented by Y_{kp}. This kp-dimensional submodule has fixed point space of dimension k and is therefore the direct sum of at most k

indecomposables, each of dimension at most p. Thus U must be the direct sum of precisely k indecomposables, each of dimension p and hence regular. Consequently U is a direct summand with its direct complement having dimension r and (if $r > 0$) fixed point space of dimension one. Thus this complement is indecomposable and in fact isomorphic to $V_r|_P$. □

(3.3). *If* $p|m$ *then* $V_m|_G$ *is projective and if* $p\nmid m$ *then* $V_m|_G$ *is the direct sum of a projective module and a nonprojective indecomposable module.*

Proof. From (3.2), if $p|m$, then $V_m|_P$ is projective so by (2.10) so is $V_m|_G$. On the other hand if $p\nmid m$ then by (2.11) $V_m|_G$ is not projective. Write $V_m|_G$ as the direct sum of indecomposable modules. Should more than one of these summands be nonprojective then by (2.10), so would their restrictions to P. This would contradict (3.2) so we conclude that there is exactly one nonprojective summand. □

The subgroup T has been defined to consist of the matrices h of the form $\begin{pmatrix} a(h) & 0 \\ b(h) & c(h) \end{pmatrix}$ with $a(h)$, $b(h)$, $c(h) \in F$ and $a(h)b(h) \neq 0$. Two one dimensional FT-modules are already implicit in this description: A with $\mathrm{a}^h = a(h)\mathrm{a}$ for all a in A and h in T, and B with $\mathrm{b}^h = b(h)\mathrm{b}$ for all b in B. It is clear that $A^i \otimes B^j \cong A^k \otimes B^l$ if and only if $r \equiv k \bmod p-1$ and $j \equiv l \bmod p-1$.

Note next that T is generated by P and two more elements $\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}$ where f is a generator of the multiplicative group of F. Consideration of the restrictions $V_m|_T$ is greatly facilitated by the fact that the standard basis elements are all eigenvectors for the two extra generators of T. In particular $\langle \mathrm{x}^{m-1} \rangle$ is a submodule of $V_m|_T$, and it is easily seen to be isomorphic to A^{m-1}. Thus $\langle x^{p-1} \rangle$ is a submodule of $V_p|_T$ isomorphic to $V_1|_T$, and each $V_p|_T \otimes A^i \otimes B^j$ has a submodule isomorphic to $A^i \otimes B^j$. As $\left(V_p|_T \otimes A^i \otimes B^j\right)_P = V_p|_P$, these modules are uniserial and injective so $V_p|_T \otimes A^i \otimes B^j$ is the injective hull of $A^i \otimes B^j$. It follows that the $V_p|_T \otimes A^i \otimes B^j$ with $i, j \in \{0, 1, \ldots, p-2\}$ are pairwise nonisomorphic and as the dimension of their direct sum is the order of T, that direct sum must be precisely the regular FT-module. This shows that every irreducible FT-module is isomorphic to $A^i \otimes B^j$ and every principal indecomposable, being isomorphic to some $V_p|_T \otimes A^i \otimes B^j$, is uniserial. We can now invoke (2.7) to conclude that every indecomposable FT-module is uniserial and hence isomorphic to a submodule of the injective hull of its unique minimal submodule. This shows that every indecomposable is isomorphic to a submodule of some $V_p|_T \otimes A^i \otimes B^j$, and so there are at most $p(p-1)^2$ isomorphism classes of nonzero indecomposables.

Similarly we observe that the $V_r|_T \otimes A^i \otimes B^j$ with $1 \le r \le p$ and $i, j \in \{0, 1, \ldots, p-2\}$ form a list of $p(p-1)^2$ modules. The restriction to P of each of these is uniserial, so they are all indecomposables. The unique minimal submodule of $V_r|_T \otimes A^i \otimes B^j$ is isomorphic to $A^{r-1+i} \otimes B^j$, and comparison of these enables us to show that no two modules in the above list which have the same dimension can be isomorphic. Thus we have proved the following result.

(3.4). *Every indecomposable FT-module is isomorphic to $V_r|_T \otimes A^i \otimes B^j$ for some i, j, r with $1 \le r \le p$ and $i, j \in \{0, 1, \ldots, p-2\}$.* $\square$

NOTE. A convenient side result of the argument above is that any two indecomposable FT-modules which have the same dimension and isomorphic socles are themselves isomorphic. This will be used in the last step of the proof of the next result.

(3.5). *For* $m \geq 1$ *write* $m = kp + r$ *with* $0 \leq r \leq p-1$. *Then*

$$V_m|_T \cong V_r|_T \otimes B^k \oplus \bigoplus_{l=0}^{k-1} \left(V_p|_T \otimes A^{k+r-1-l} \otimes B^l\right) .$$

Proof. For each $l \in \{0, 1, \ldots, k-1\}$ let U_{l+1} denote the span of the first $p(p-1)$ elements of the standard basis of V_m and put $U_0 = 0$, $U_{k+1} = V_m|_T$. As in the proof of (3.2) we note that $U_1, U_2, \ldots, U_k$ admit P and indeed as P-modules they are isomorphic to $V_p|_P, \ldots, V_{kp}|_P$. Since $U_1, U_2, \ldots, U_k$ have bases consisting of eigenvectors of the additional generators of T , they admit T as well. Also by (3.2) their restrictions to P are injective so by (2.10) they are injective T-modules. It follows that

$$V_m|_T \cong \bigoplus_{l=0}^{k} U_{l+1}/U_l .$$

Each summand U_{l+1}/U_l has an irreducible submodule spanned by $x^{m-1-lp}y^{lp} + U_l$ and hence isomorphic to $A^{m-1-lp} \otimes B^{lp}$, that is, to $A^{k+r-1-l} \otimes B^l$ as $m - 1 - lp \equiv k + r - 1 - l \bmod p-1$. We know from (3.2) that $V_m|_P$ does not have a direct decomposition with more summands than

$$V_m|_T \cong \oplus\, U_{l+1}/U_l ,$$

so this must be an unrefinable decomposition. As each indecomposable is identifiable by its dimension and isomorphism type of an irreducible submodule in it, (see the note after (3.4)),

$$U_{l+1}/U_l \cong V_p|_T \otimes A^{k+r-1-l} \otimes B^l$$

for $l < k$ while $U_{k+1}/U_k \cong V_r|_T \otimes B^k$. $\square$

The following result provides a way of determining some of the structure of a KG-module by examining the restriction of it to T .

(3.6). *For any* FG*-module* U , $(U_T)^G \cong U \oplus (U \otimes V_p|_G)$.

NOTE. A similar result is true if we substitute S for G and $T \cap S$ for T . The same proof applies.

Proof of (3.6). By the Tensor Product Theorem (2.8),

$$(U_T)^G \cong (U_T \otimes V_1|_T)^G \cong U \otimes (V_1|_T)^G .$$

To prove (3.6) therefore, we have to show $(V_1|_T)^G$, which is the permutation representation of G on the cosets of T , is isomorphic to $V_1|_G \oplus V_p|_G$.

Take $\mathsf{u} = 1 + \mathsf{x}^{p-1} \in V_1 \oplus V_p$ and look at $\{\mathsf{u}^g \mid g \in G\}$. The elements of this set are permuted transitively by G and together span a submodule of $(V_1 \oplus V_p)_G$ which is neither $V_1|_G$ nor $V_p|_G$. Therefore $\langle \mathsf{u}^g \mid g \in G \rangle = (V_1 \oplus V_p)_G$ and $\{\mathsf{u}^g \mid g \in G\}$ has at least $p + 1$ elements.

Next we observe that if $h = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \in T$ (so $a \neq 0$) then $\mathsf{u}^h = 1 + (a\mathsf{x})^{p-1} = 1 + \mathsf{x}^{p-1} = \mathsf{u}$. Therefore if g and g' are in the same coset of T (that is $g' = hg$ for $h \in T$) then $\mathsf{u}^{g'}$ $(= \mathsf{u}^{hg})$ $= \mathsf{u}^g$ and we conclude that the number of distinct u^g is at most the number of cosets of T in G , that is, at most $p + 1$. By the conclusion of the last paragraph therefore the number of distinct u^g is exactly $p + 1$ and so they are in one to one correspondence with the cosets. As G acts on the u^g as it does on the cosets and the u^g form a basis for $(V_1 \oplus V_p)_G$ it follows that $(V_1|_T)^G \cong V_1|_G \oplus V_p|_G$ and we remarked in the opening paragraph that this is sufficient to prove (3.6). $\square$

Before studying some of the consequences of this result we need to introduce some notation. By (3.3), $V_m|_G = \overline{V}_m \oplus \overline{\overline{V}}_m$ where $\overline{\overline{V}}_m$ is a suitable projective and $\overline{V}_m$ is zero or a nonprojective indecomposable. While $V_m|_G$ could have several such decompositions, the Krull-Schmidt Theorem guarantees that $\overline{V}_m$ and $\overline{\overline{V}}_m$ are unique up to isomorphism. We should note that while V_m is an M-module, $\overline{V}_m$ and $\overline{\overline{V}}_m$ are defined as G-modules. The next result establishes that the $\overline{V}_m$ form a periodic sequence.

(3.7). *The nonprojective parts of the* $V_m|_G$ *form a periodic sequence:* $\overline{V}_{m+p(p-1)} \cong \overline{V}_m$ *for all* m .

Proof. Write $m = kp + r$, $0 \le r \le p-1$ so that $m + p(p-1) = \big(k+(p-1)\big)p + r$. Now use (3.5) to show

$$V_{(k+(p-1))p+r}|_T \cong V_r|_T \otimes B^{k+p-1} \oplus \bigoplus_{l=0}^{k+p-2} \left(V_p|_T \otimes A^{k+p-2-l} \otimes B^l\right)$$

$$\cong V_r|_T \otimes B^k \oplus \bigoplus_{l=0}^{k+p-2} \left(V_p|_T \otimes A^{k-1-l} \otimes B^l\right)$$

$$\cong V_m|_T \oplus \bigoplus_{l=k}^{k+p-2} \left(V_p|_T \otimes A^{k-1-l} \otimes B^l\right)$$

$$= V_m|_T \oplus X$$

where X is projective by (2.10). Therefore, by (3.6),

$$(V_{m+p(p-1)}|_T)^G \cong (V_m|_T \oplus X)^G$$

$$\cong V_m|_G \oplus (V_m|_G \otimes V_p|_G) \oplus X^G$$

$$= \overline{V}_m \oplus \overline{\overline{V}}_m \oplus (V_m \otimes V_p)_G \oplus X^G .$$

As X^G is projective by (2.10) the only nonprojective summand of

$(V_{m+p(p-1)}|_T)^G$ is $\overline{V}_m$. On the other hand a direct application of (3.6) shows

$$(V_{m+p(p-1)}|_T)^G \cong \overline{V}_{m+p(p-1)} \oplus \overline{\overline{V}}_{m+p(p-1)} \oplus (V_{m+p(p-1)}|_G \otimes V_p|_G)$$

so the only nonprojective indecomposable summand in another decomposition is $\overline{V}_{m+p(p-1)}$. The Krull-Schmidt Theorem now yields our claim.

We conclude this chapter by showing that a detailed knowledge of the V_m would provide much to the general representation theory of G and S .

(3.8). *Let W be a nonprojective indecomposable FG-module. Then there exist unique integers m, n such that $1 \leq m \leq p(p-1)$, $0 \leq n \leq p-2$ and $W \cong \overline{V}_m \otimes D_G^n$.*

Proof. First we establish the analogous claim for T in place of G . Let U be a nonprojective indecomposable FT-module. From (3.4) we see that $U \cong V_r|_T \otimes A^i \otimes B^j$ for some i, j, r with $1 \leq r < p$. Noting that $(A \otimes B)^i \cong D_T^i$ rewrite this so that $U \cong V_r|_T \otimes B^{j-i} \otimes (A \otimes B)^i$ where, by reducing modulo $p - 1$, we assume $j - i$ to be nonnegative. By (3.5), U is isomorphic to the unique nonprojective indecomposable summand of $\left(V_{(j-i)p+r} \otimes D^i\right)_T$. It follows that there exist m, n in the given range, such that U is isomorphic to the unique nonprojective indecomposable direct summand of $\left(V_m \otimes D^n\right)_T$. Note that there are $(p-1)^3$ choices of U , and the same number of choices for m, n in the given range subject to the obvious restraint that $p \nmid m$, and so the uniqueness is assured. We are now ready to proceed with the case of G-modules.

Write $W_T = \bigoplus_i U_i$. Since $(W_T)^G \cong W \oplus X$, where X is projective by (3.6), and also $(W_T)^G \cong \bigoplus_i U_i^G$, it follows from (2.10) that exactly one

U_i , say U_1 , is nonprojective. Therefore, by the Krull Schmidt Theorem W is isomorphic to a direct summand of U_i^G .

We have noted that there exist m, n , unique in the given range such that $\left(V_m \otimes D^n\right)_T$ contains a direct summand isomorphic to U_1 . Therefore $\left(\left(V_m \otimes D^n\right)_T\right)^G$ contains a summand isomorphic to U_1^G , which in turn contains a summand isomorphic to W . Use (3.6) to argue that

$$\left(\left(V_m \otimes D^n\right)_T\right)^G \cong \left(\left(\overline{V}_m \otimes D^n\right)_T\right)^G \oplus \left(\left(\overline{\overline{V}}_m \otimes D^n\right)_T\right)^G$$

$$\cong \left(\overline{V}_m \otimes D^n\right)_G \oplus \left(\overline{V}_m \otimes D^n \otimes V_p\right)_G \oplus \left(\left(\overline{\overline{V}}_m \otimes D^n\right)_T\right)^G .$$

Now the Krull Schmidt Theorem gives that every nonprojective indecomposable direct summand of $\left(\left(V_m \otimes D^n\right)_T\right)^G$ is isomorphic to $\overline{V}_m \otimes D_G^n$ and this applies to W . If some other $\overline{V}_m \otimes D_G^n$ also had a direct summand isomorphic to W , upon restriction to T we would contradict the uniqueness of m, n relative to U_1 , and so the proof is complete. □

For nonprojective indecomposable FS-modules the situation is better still as we do not have to tensor with a power of D .

(3.9). *Let W be a nonprojective indecomposable FS-module. Then there exists a unique positive integer m such that $1 \leq m \leq p(p-1)$ and $W \cong \overline{V}_m|_S$.* □

We do not prove this result as the proof follows very closely the proof of (3.8).

CHAPTER 4

TWO IMPORTANT SUBMODULES

Let N be the set of singular matrices in M and in each V_m denote the annihilator of N by V_m^* : this is clearly a submodule. In the introduction we called a module singular if it is annihilated by N, and nonsingular otherwise; V_m^* is the largest singular submodule of V_m. Another submodule of importance is V_mFN. As x^{m-1} is fixed by $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$, we have that V_mFN is nonsingular. In fact V_mFN will turn out to be minimal with respect to being nonsingular.

The observation that $x^{m-1} \in V_m$ and (3.1) show that $V_m^* = 0$ for $m \le p$. The first thing we prove in this chapter is $V_m^* = \left(x^py-xy^p\right)V_{m-p-1}$ when $m \ge p+1$. The obvious question to ask then is what the $(p+1)$-dimensional quotients look like. We have not got enought guns to describe them fully in this chapter but in (4.2) we note that for $m \ge p+1$ the V_m/V_m^* form a periodic sequence with period $p-1$. In (4.6) we note that they are indecomposable with unique minimal submodules.

Although we do not know the structure of the V_m/V_m^* we can make good use of the isomorphisms between them to show in (4.3) that the $V_m|_G$, with m in any residue class modulo $p\left(p^2-1\right)$, form an arithmetic sequence.

We complete the chapter by showing that V_mFN is in fact X_m - the submodule of V_m generated by the $(m-1)$st powers v^{m-1} $\left(v \in V_2\right)$ and thus has dimension at most $p+1$. In (4.6) we see that X_m $\left(= V_mFN\right)$ is indecomposable with unique maximal submodule.

(4.1). *For* $m \geq p+1$ *the maximal singular submodule* V_m^* *of* V_m *is* $(x^p y - xy^p)V_{m-p-1}$ *and is isomorphic to* $D \otimes V_{m-p-1}$.

Proof. Take $t = \binom{a\ b}{c\ d} \in M$. Using the fact that pth powering is an endomorphism of the F-algebra V , we get:

$$\begin{aligned}(x^p y - xy^p)^t &= (ax+by)^p(cx+dy) - (ax+by)(cx+dy)^p \\ &= (ax^p+by^p)(cx+dy) - (ax+by)(cx^p+cy^p) \\ &= (\det t)(x^p y - xy^p) .\end{aligned}$$

Thus $D \cong \langle x^p y - xy^p \rangle \leq V_{p+2}$. It follows that $(x^p y - xy^p)V_{m-p-1}$ is a submodule of V_m^* and is isomorphic to $D \otimes V_{m-p-1}$.

Next we prove that $\{x^{m-1}, x^{m-2}y, \ldots, x^{m-p-1}y^p\} \cup \{y^{m-1}\}$ is linearly independent even modulo V_m^* . This will complete the proof for it shows that V_m^* cannot be larger than its submodule $(x^p y - xy^p)V_{m-p-1}$, which already has codimension $p+1$ in V_m .

Let us suppose that there exist scalars a_i ,nnot all zero such that the element w , defined by

$$w = \sum_{i=0}^{p-1} a_i x^{m-1-i} y^i + a_p y^{m-1} ,$$

belongs to V_m^* . That is, w is annihilated by all of N , and in particular by the elements $t = \binom{0\ 0}{0\ 1}$ and $t(b) = \binom{1\ 0}{b\ 0}$, $b \in F$. As $w^t = a_p y^{m-1}$ and $w^{t(0)} = a_0 x^{m-1}$, $a_0 = a_p = 0$. Suppose a_n is the first nonzero coefficient in the defining expression of w and take $b \in F$, $b \neq 0$. Since

$$0 = w^{t(b)} = x^{m-1} \sum_{i=n}^{p-1} a_i b^i$$

we have

$$a_n = -\sum_{i=n+1}^{p-1} a_i b^{i-n} .$$

As this holds for each choice of b , we have

$$(p-1)a_n = -\sum_{b=1}^{p-1}\sum_{i=n+1}^{p-1} a_i b^{i-n} = \sum_{i=n+1}^{p-1}\left(a_i \sum_{b=1}^{p-1} b^{i-n}\right) = 0$$

by (2.13), and this contradiction completes the proof. □

The previous result identifies a submodule V_m^* of V_m , of co-dimension $p+1$, on which only the action of G is relevant. Hopefully, if we can identify the quotient V_m/V_m^* , then all we have to do is find a way of "sticking them together". At this stage we cannot describe these quotients fully but the following result shows that there are not too many nonisomorphic ones.

(4.2). *For* $m \geq 2p$ *, write* $n = k + j(p-1)$ *with* $p+1 \leq k \leq 2p$ *. Then* V_m/V_m^* *is isomorphic to* V_k/V_k^* .

Proof. The standard basis for V_k is $\{x^{k-1}, x^{k-2}y, \ldots, y^{k-1}\}$. Define $\psi : V_k \to V_m$ on these basis vectors by

$$x^i y^{k-i-1}\psi = \begin{cases} x^{m-k}\, x^i x^{k-i-1} & \text{if } i \geq 1 \\ \\ y^{m-1} & \text{if } i = 0 \end{cases}$$

and extend to V_k by linearity. We note that for elements u in the subset xV_{k-1} of V_k we have $u\psi = x^{m-k}u$ and for any $v \in \{ay^{k-1} \mid a \in F\}$ we have $v\psi = y^{m-k}v$. Therefore, since

$$V_k^* = \left(x^p y - xy^p\right)V_{k-p-1} \leq xV_{k-1} ,$$

we have

$$V_k^*\psi = x^{m-k}V_k^* = \left(x^py-xy^p\right)x^{m-k}V_{k-p-1} \le V_m^* .$$

We will establish (4.2) by proving that $\left(u^t\right)\psi - (u\psi)^t \in V_m^*$ whenever $u \in V_k$. This will show that ψ followed by the canonical map η of V_m onto V_m/V_m^* is an *FM*-homomorphism and, as we noted above that V_k^* is in the kernel of $\psi\eta$, this map induces a homomorphism of V_k/V_k^* into V_m/V_m^*. This homomorphism is onto V_m/V_m^* because, as we saw in (4.1), V_m/V_m^* has basis

$$\left\{x^{m-1}+V_m^*,\ x^{m-2}y+V_m^*,\ \dots,\ x^{m-p}y^{p-1}+V_m^*,\ y^{m-1}+V_m^*\right\}$$

and this basis is the image under $\psi\eta$ of

$$\{x^{k-1},\ x^{k-2}y,\ \dots,\ x^{k-p}y^{p-1},\ x^{k-1}\} .$$

A dimension count then shows that we have an isomorphism.

We prove $\left(u^t\right)\psi - (u\psi)^t \in V_m^*$ using the definition of V_m^* as the annihilator of N, that is, by showing that $\left(\left(u^t\right)\psi-(u\psi)^t\right)^{t'} = 0$ for all t' in N. Since ψ is linear it is sufficient to do this separately for the cases $u \in xV_{k-1}$ and $u = y^{k-1}$.

In the first case $u = xu_1$ with $u_1 \in V_{k-1}$. Write $u^t = xu_2 + ay^{k-1}$ where $u_2 \in V_{k-1}$ and $a \in F$. If t' is a nonzero element of N then $\dim V_2^{t'} = 1$ so $V_2^{t'} = \langle v \rangle$ say, and for any $r \ge 2$, $\left(V_r\right)^{t'} = \langle v^{r-1}\rangle$ as

$$\left(x^iy^{r-i-1}\right)^{t'} = \left(x^{t'}\right)^i\left(y^{t'}\right)^{r-i-1} \in \langle v^{r-1}\rangle$$

for each i. Therefore let

$$x^{t'} = bv,\ y^{t'} = cv,\ x^{tt'} = dv^{k-2},\ u^{tt'} = ev^{k-2},\ u_2^{t'} = fv^{k-2} .$$

We note

$$u^{tt'} = x^{tt'}u_1^{tt'} = (de)v^{k-1}$$

and on the other hand

$$u^{tt'} = \left(xu_2+ay^{k-1}\right)^{t'} = \left(bf+ac^{k-1}\right)v^{k-1} .$$

Thus $de = bf + ac^{k-1}$ and since $m \equiv k \bmod p-1$ we have $bf + ac^{m-1} = de$.

Now let us see how t' acts on $\left(u^t\right)\psi$ and on $(u\psi)^t$:

$$\left(\left(u^t\right)\psi\right)^{t'} = \left(\left(xu_2+ay^{k-1}\right)\psi\right)^{t'} = \left(x^{m-k}xu_2+ay^{m-1}\right)^{t'} = \left(b^{m-k+1}f+ac^{m-1}\right)v^{m-1}$$

while

$$\left((u\psi)^t\right)^{t'} = \left(x^{m-k}xu_1\right)^{tt'} = d^{m-k+1}ev^{m-1} = dev^{m-1} .$$

The last equation of the previous paragraph now shows that $\left(\left(u^t\right)\psi\right)^{t'} = \left((u\psi)^t\right)^{t'}$ as claimed.

In the second case $u = y^{k-1}$ and again we write $u^t = xu_2 + ay^k$. In addition to the notation above we need $y^{tt'} = gv$ say. This time

$$u^{tt'} = \left(y^{k-1}\right)^{tt'} = g^{k-1}v^{k-1}$$

and

$$u^{tt'} = \left(xu_2+ay^k\right)^{t'} = \left(bf+ac^{k-1}\right)v^{k-1}$$

yield that $bf + ac^{m-1} = g^{m-1}$. Now apply t' to $\left(u^t\right)\psi$ and $(u\psi)^t$.

$$\begin{aligned} \left(\left(u^t\right)\psi\right)^{t'} &= \left(\left(xu_2+ay^k\right)\psi\right)^{t'} = \left(x^{m-k}xu_2+ay^{m-1}\right)^{t'} = \left(b^{m-k+1}f+ac^{m-1}\right)v^{m-1} \\ &= \left(bf+ac^{m-1}\right)v^{m-1} , \\ \left((u\psi)^t\right)^{t'} &= \left(y^m\right)^{tt'} = g^{m-1}v^{m-1} , \end{aligned}$$

and so again $\left(\left(u^t\right)\psi\right)^{t'} = \left((u\psi)^t\right)^{t'}$ as claimed. This completes the proof of (4.2). □

The following result tells us that a description of the first $p\left(p^2-1\right)$ of the $V_m|_G$ is sufficient to describe them all. An interesting observation is that this result can be derived while knowing the structure of the first p of the $V_m|_G$ only. A tighter result is obtained in Chapter 6 but there we know, and use, much about the structure of the $V_m|_G$.

(4.3). *For each positive integer* m *there is an* FG*-module* P_m, *depending only on the residue class of* m *modulo* $p-1$, *such that* $V_{m+p(p^2-1)}\big|_G \cong V_m|_G \oplus P_m$

Proof. By $p(p-1)$ applications of (4.1), or directly, we observe that $(x^py-xy^p)^{p(p-1)}\overline{\overline{V}}_m$ is an injective submodule of $V_{m+p(p^2-1)}\big|_G$ isomorphic to $\overline{\overline{V}}_m$. By the Krull-Schmidt Theorem it follows that $\overline{\overline{V}}_{m+p(p^2-1)} \cong \overline{\overline{V}}_m \oplus P_m$ for some FG-module P_m. As we know from (3.7) that $\overline{V}_m \cong \overline{V}_{m+p(p^2-1)}$, this means that $V_{m+p(p^2-1)}\big|_G \cong V_m|_G \oplus P_m$. It remains to show that $P_{m+p-1} \cong P_m$ for all m.

First we establish that at least restrictions to T behave in the required manner, that is, $P_{m+p-1}|_T \cong P_m|_T$. Using (3.5) and putting $m = kp + r$ we have

$$V_{m+p(p^2-1)}\big|_T = V_{(p^2-1+k)p+r}$$

$$\cong V_r|_T \otimes B^{p^2-1+k} \oplus \bigoplus_{l=0}^{p^2+k-2} \left(V_p|_T \otimes A^{p^2-2+k+r-l} \otimes B^l\right)$$

$$\cong V_r|_T \otimes B^k \oplus \bigoplus_{l=0}^{p^2+k-2} \left(V_p|_T \otimes A^{k+r-l-1} \otimes B^l\right)$$

after reducing exponents modulo $p-1$. Also we have

$$V_m|_T \cong V_r|_T \otimes B^k \oplus \bigoplus_{l=0}^{k-1} \left(V_p|_T \otimes A^{k+r-l-1} \otimes B^l\right)$$

so

$$V_{m+(p^2-1)p}\big|_T \cong V_m\big|_T \oplus \bigoplus_{l=k}^{p^2-2+k} \left(V_p\big|_T \otimes A^{k+r-l-1} \otimes B^l\right)$$

$$\cong V_m\big|_T \oplus \left[\bigoplus_{l=0}^{p-2} \left(V_p\big|_T \otimes A^{k+r-l-1} \otimes B^l\right)\right]^{\oplus(p+1)} .$$

Now, writing $m + p - 1 = (k+1)p + (r-1)$ or $kp + p - 1$ depending on whether $r \geq 1$ or $r = 0$ respectively we see that

$$V_{m+p-1+(p^2-1)p}\Big|_T \cong V_{m+p-1}\big|_T \oplus \left[\bigoplus_{l=0}^{p-2} \left(V_p\big|_T \otimes A^{k+r-l-1} \otimes B^l\right)\right]^{\oplus(p+1)}$$

and thus $P_{m+p-1}\big|_T \cong P_m\big|_T$ as claimed.

It will now be sufficient to establish that $P_{m+p-1} \otimes V_p\big|_G \cong P_m \otimes V_p\big|_G$ because

$$\left((P_{m+p-1})_T\right)^G \cong \left(P_m\big|_T\right)^G \cong P_m \oplus P_m \otimes V_p\big|_G$$

by the previous paragraph and (3.6) while (3.6) gives directly

$$\left((P_{m+p-1})_T\right)^G \cong P_{m+p-1} \oplus P_{m+p-1} \otimes V_p\big|_G$$

and so the Krull-Schmidt Theorem will finish the proof.

By repeated use of (4.1), or directly, we observe that the submodules U_i of $V_{m+p(p^2-1)}$ defined by

$$U_i = (x^py-xy^p)^{p(p-1)-i}V_{m+i(p+1)} , \quad i = 0, 1, \ldots, p(p-1)$$

form a chain such that $U_0|G \cong V_m|_G$ and

$$U_i/U_{i-1} \cong D^{p(p-1)-i} \otimes \left(V_{m+i(p+1)}/V^*_{m+i(p+1)}\right) .$$

Here the isomorphism type of U_i/U_{i-1} depends only on i and the residue class of m modulo $p - 1$. It follows that the $(U_i \otimes V_p)_G$ form a chain of injective submodules of $(V_{m+p(p^2-1)} \otimes V_p)_G$ and thus

$$(V_{m+p(p^2-1)} \otimes V_p)_G \cong (U_0 \otimes V_p)_G \oplus \bigoplus_{i=1}^{p(p-1)} \left((U_i/U_{i-1}) \otimes V_p\right)_G .$$

On the other hand, $(U_0 \otimes V_p)_G \cong (V_m \otimes V_p)_G$ and

$$(V_{m+p(p^2-1)} \otimes V_p)_G \cong (V_m|_G \oplus P_m) \otimes V_p|_G$$

$$\cong U_0 \otimes V_p|_G \oplus P_m \otimes V_p|_G .$$

The Krull Schmidt Theorem yields that

$$P_m \otimes V_p|_G \cong \bigoplus_{i=1}^{p(p-1)} \left((U_i/U_{i-1}) \otimes V_p\right)_G .$$

As observed above, the right hand side depends only on the residue class of m modulo $p - 1$ and so $P_m \otimes V_p|_G \cong P_{m+p-1} \otimes V_p|_G$ follows as required. □

We turn our attention to the V_mFN. Define X_m as the span of the $(m-1)$st powers of the elements of V_2. As each element of V_2 is a scalar multiple of one of

$$x,\ y,\ x+y,\ 2x+y,\ \ldots,\ (p-1)x+y\ ,$$

we see that X_m is spanned by

$$x^{m-1},\ y^{m-1},\ (x+y)^{m-1},\ (2x+y)^{m-1},\ \ldots,\ \left((p-1)x+y\right)^{m-1} ,$$

so X_m has dimension at most $p + 1$. We also note that, on account of (3.1), $X_m = V_m$ when $m \leq p$.

(4.4). $V_mFN = X_m$.

Proof. For each $v \in V_2$ there exists a $t \in N$ such that $x^t = v$. Therefore

$$v^{m-1} = (x^t)^{m-1} = (x^{m-1})^t \in V_mFN ,$$

that is, $X_m \leq V_mFN$.

On the other hand, if t is a nonzero element of N then V_2^t is one dimensional, say $V_2^t = \langle v \rangle$. Then $x^t = av$, $y^t = bv$ for some $a, b \in F$ and

$$(x^i y^{m-1-i})t = a^i b^{m-1-i} v^{m-1} \in X_{m-1} .$$

Thus $V_m FN \leq X_m$. □

Let X_m^* be the largest singular submodule of X_m , so $X_m^* = X_m \cap V_m^*$. As $X_m = V_m FN$, the quotient V_m/X_m is singular. Our next result will show that X_m/X_m^* is irreducible; thus a composition series of V_m through X_m^* and X_m has only one nonsingular composition factor (namely X_m/X_m^* itself). By the Jordan-Hölder Theorem, it follows that each composition series of V_m has precisely one nonsingular composition factor.

(4.5). *For* $m \geq p+1$ *write* $m = r(p-1) + k$ *with* $2 \leq k \leq p$. *Then* X_m/X_m^* *is isomorphic to* V_k *(which is irreducible by* (3.1)). *In particular, each composition series of* V_m *has precisely one nonsingular composition factor.*

Proof. Let X be an F-space of dimension $p + 1$ with basis $x_0, x_1, \ldots, x_p$. Define for $n = 2, k, m$ the F linear map ρ_n of X onto X_n by

$$x_0\rho_n = x^{n-1} ,$$

$$x_i\rho_n = (ix+y)^{n-1} \text{ for } i = 1, 2, \ldots, p .$$

Observe that

$$x_i\rho_n = (x_i\rho_2)^{n-1} \text{ for } i = 0, 1, 2, \ldots, p$$

and that every nonzero element of V_2 $(= X_2)$ is of the form $a(x_j\rho_2)$ with with unique a from F and j from $\{0, 1, \ldots, p\}$. Using this latter fact, we may define an M-action on X by putting, for each t in M ,

$$x_i^t = \begin{cases} 0 & \text{if } (x_i\rho_2)^t = 0 \\ a^{k-1}x_j\rho_2 & \text{if } (x_i\rho_2)^t = ax_j\rho_2 \,. \end{cases}$$

This makes X into an FM-module.

Now

$$(x_i\rho_k)^t = \left[(x_i\rho_2)^{k-1}\right]^t = \left[(x_i\rho_2)^t\right]^{k-1} = [a(x_j\rho_2)]^{k-1} = a^{k-1}(x_j\rho_2)^{k-1}$$

and

$$(x_i^t)\rho_k = \left(a^{k-1}x_j\right)\rho_k = a^{k-1}(x_j\rho_k) = a^{k-1}(x_j\rho_2)^{k-1}$$

show that ρ_k is an FM-homomorphism. The same calculation also works with ρ_m in place of ρ_k and as $m \equiv k \bmod p-1$, $m > 1$, $k > 1$ ensure $a^{k-1} = a^{m-1}$ we conclude that ρ_m is also an FM-homomorphism.

Next we establish that $(\ker \rho_k)\rho_m \leq X_m^*$. To this end we need to show that

$$\left(\sum_{i=0}^{p} b_i x_i\right)\rho_k = 0$$

implies

$$\left[\left(\sum_{i=0}^{p} b_i x_i\right)\rho_m\right]^t = 0$$

for all nonzero t in N . Given such a t , as V_2^t is one dimensional, $(x_i\rho_2)^t = a_i(x_j\rho_2)$ with the same j for all i [with $a_i = 0$ this expression can also cover the case $(x_i\rho_2)^t = 0$]. Thus

$$\left(\sum b_i x_i \rho_m\right)^t = \sum b_i \left[(x_i \rho_2)^{m-1}\right]^t$$

$$= \sum b_i \left[(x_i \rho_2)^t\right]^{m-1}$$

$$= \sum b_i a_i^{m-1} (x_j \rho_2)^{m-1} .$$

On the other hand, by a similar calculation we have

$$0 = 0^t = \left(\sum b_i x_i \rho_k\right)^t = \left(\sum b_i a_i^{k-1}\right)(x_j \rho_2)^{k-1}$$

so $\sum b_i a_i^{k-1} = 0$. Again $m \equiv k \bmod p-1$, $m > 1$, $k > 1$ guarantee that $a_i^{m-1} = a_i^{k-1}$ for all a_i , so $\sum b_i a_i^{m-1} = 0$ and $\left(\sum b_i x_i \rho_m\right)^t = 0$ as required.

Now define an isomorphism of V_k onto X_m/X_m^* by choosing to each $u \in V_k$ an inverse image x in X under ρ_k and then mapping u to $x\rho_m + X_m^*$. Since $(\ker \rho_k)\rho_m \leq X_m^*$ this map of V_k is well defined, and as it is defined in terms of FM-homomorphisms it is an FM-homomorphism. As ρ_m was onto, this map is onto, and as V_k is irreducible this map must be an isomorphism unless $X_m/X_m^* = 0$. Of course $x^{m-1} \in X_m$ and $\left(x^{m-1}\right)^t = x^{m-1}$ for $t = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in N$ shows $x^{m-1} \notin X_m^*$ so the last proviso does not apply and the proof is complete. $\square$

As we mentioned after the proof of (4.4), each V_m has a unique nonsingular composition factor. We use this fact in proving the final result in this chapter.

(4.6). *For any integer* $m \geq 1$, X_m *and* V_m/V_m^* *are indecomposable; in fact* X_m *has a unique maximal submodule namely* X_m^* *and* V_m/V_m^* *has a unique minimal submodule namely* $\left(X_m+V_m^*\right)/V_m^*$.

Proof. To show by a contradiction argument that X_m^* is the unique maximal submodule of X_m suppose that there exists a maximal submodule U in X such that $U \nleq X_m^*$. Then $U + X_m^* = X_m$ so

$$X_m^*/X_m^* \cap U \cong \left(X_m^*+U\right)/X_m^* \cong X_m/X_m^*$$

so that X_m^* has a nonsingular factor module. This is clearly impossible as X_m^* is singular by definition.

To establish the second claim, suppose W/V_m^* is an irreducible submodule of V_m/V_m^* other than $X_m+V_m^*/V_m^*$. Then $WN \leq V_m FN = X_m$ by (4.4) so $WN \leq W \cap X_m \leq V_m^*$ and therefore $(WN)N \leq V_m^* N = 0$. It is easy to see (see note below) that $N^2 = N$ and therefore $W \leq V_m^*$ contrary to assumption and hence we conclude that $\left(X_m+V_m^*\right)/V_m^*$ is the unique minimal submodule of V_m/V_m^*. □

NOTE. There exist idempotents $e_i \in N$ such that $N = \bigcup_i e_i N$ and so

$$N^2 = \bigcup_i e_i N^2 \geq \bigcup_i e_i e_i N = \bigcup_i e_i N = N .$$

CHAPTER 5

THE EXACT SEQUENCES AND SOME IMMEDIATE COROLLARIES

As mentioned in the introduction, this thesis depends heavily on the use of some exact sequences involving tensor products of the V_m . We begin this chapter by describing those exact sequences and then go on to derive properties of the V_m that will be useful for the last two chapters. Many of these results are technical in nature and we won't provide a summary of them here. At the end of the chapter we describe V_{p+k} and V_{kp} for $1 \le k \le p-1$. These modules provide us with enough information to describe the principal indecomposable FG-modules but we postpone that to the following chapter where we consider the FG-structure of the V_m .

It is in this chapter, from (5.7) onwards, that the restriction $p > 2$ becomes necessary: it will be in force throughout, without further notice.

(5.1). *For all* $m, n \ge 1$ *there is an exact sequence*

$$0 \to V_m \otimes V_n \otimes D \xrightarrow{\theta_{m,n}} V_{m+1} \otimes V_{n+1} \xrightarrow{\varphi_{m,n}} V_{m+n+1} \to 0 .$$

NOTE. Often, because we know the domain or codomain of $\theta_{m,n}$, $\varphi_{m,n}$, we can drop the suffixes from these maps and denoting them by θ, φ respectively will cause no confusion.

Proof of (5.1). Define $\varphi_{m,n}$ on a basis for $V_{m+1} \otimes V_{n+1}$ by $(u \otimes v)\varphi = uv$ as u and v run through bases of V_{m+1} and V_{n+1} respectively, and extend to $V_{m+1} \otimes V_{n+1}$ by linearity. Note that $(u \otimes v)\varphi = uv$ for all $u \in V_{m+1}$, $v \in V_{n+1}$, not only for the basis elements used in the definition and that:

$$[(u \otimes v)\varphi]^t = (uv)^t$$
$$= u^t v^t$$
$$= (u^t \otimes v^t)\varphi$$
$$= [(u \otimes v)^t]\varphi$$

so by linearity φ is an FM-homomorphism. Noting that

$$(x^i y^{m-i} \otimes x^j y^{n-j})\varphi = x^{i+j} y^{m+n-i-j}$$

and

$$\{(i+j,\ m+n-i-j) \mid 0 \le i \le m,\ 0 \le j \le n\} = \{(k,\ m+n-k) \mid 0 \le k \le m+n\}$$

we see that $\varphi_{m,n}$ maps onto V_{n+m+1} .

Next define $\theta_{m,n}$ on a basis for $V_m \otimes V_n \otimes D$ by

$$(u \otimes v \otimes w)\theta_{m,n} = (ux \otimes vy) - (uy \otimes vx)$$

where u and v run through bases of V_m and V_n respectively and w is a basis element of D . Extend to $V_m \otimes V_n \otimes D$ by linearity. As in the case of φ we note that

$$(u \otimes v \otimes w)\theta = (ux \otimes vy) - (uy \otimes vx)$$

holds not only for the basis elements but also for every $u \in V_m$, $v \in V_n$, $w \in D$, and that for $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$,

$$[(u \otimes v \otimes w)\theta]^t = [(ux \otimes vy) - (uy \otimes vx)]^t$$
$$= [u^t(ax+by) \otimes v^t(cx+dy)] - [u^t(cx+dy) \otimes v^t(ax+by)]$$
$$= (u^t x \otimes v^t y)(ad-bc) + (u^t y \otimes v^t x)(bc-da)$$
$$= (\det t).[(u^t x \otimes v^t y) - (u^t y \otimes v^t x)]$$
$$= (\det t).(u^t \otimes v^t \otimes w)\theta$$
$$= [(u \otimes v \otimes w)^t]\theta .$$

Thus θ is an FM-homomorphism which we now check to be an isomorphism into

$V_{m+1} \otimes V_{n+1}$ by showing $\ker \theta = 0$.

Consider the usual basis $\{x^{m-1}, x^{m-2}y, \ldots, y^{m-1}\}$ of V_m and any basis $\{w\}$ of D . Every element of $V_m \otimes V_n \otimes D$ can be expressed in "normal form" as

$$\sum_{i=1}^{m} x^{m-i}y^{i-1} \otimes v_i \otimes w$$

with uniquely determined elements $v_1, \ldots, v_m$ of V_m . Take an arbitrary element of $\ker \theta$ in this form so

$$\sum_{i=1}^{m} \left[\left(x^{m-i}y^{i-1}x \otimes v_i y\right) - \left(x^{m-i}y^{i-1}y \otimes v_i x\right)\right] = 0 .$$

Collect terms of the left hand side to get that

$$\left(x^m \otimes v_1 y\right) - \left(y^m \otimes v_m x\right) + \sum_{i=1}^{m-1} x^{m-i}y^i \otimes \left(v_{i+1}y - v_i x\right) = 0 .$$

The left hand side is now in a similar "normal form" for elements $V_{m+1} \otimes V_{n+1}$ and we must have $v_1 y = 0$, $v_m x = 0$ and $v_{i+1}y - v_i x = 0$ for $i = 1, \ldots, m-1$. As the algebra V has no zero divisors, the first of these relations yields $v_1 = 0$, hence the third gives $v_2 = 0$; therefore the next gives $v_3 = 0$ and so on until $v_m = 0$. This shows that the arbitrary element of $\ker \theta$ that we started with was in fact 0 .

A check on dimensions, together with the obvious fact that $\theta\varphi = 0$, completes the proof of (5.1). □

It can be shown that for $m + n \leq p + 1$ that the above sequences split. We do not need that result as such but instead the following.

(5.2). *If* $p \nmid m$ *then* $V_m \otimes V_2 \cong \left(V_{m-1} \otimes D\right) \oplus V_{m+1}$.

Proof. The case $m = 1$ is covered by the convention $V_0 = 0$ so take $m > 1$. From (5.1) we have the exact sequence

$$0 \to V_{m-1} \otimes D \to V_m \otimes V_2 \xrightarrow{\varphi} V_{m+1} \to 0$$

where φ simply forgets the tensor sign. If we can find a $\bar{\varphi} : V_{m+1} \to V_m \otimes V_2$ such that $\bar{\varphi}\varphi$ is the identity automorphism ι_{m+1} of V_{m+1} then the sequence will split as claimed. We define $\delta_m : V_{m+1} \to V_m \otimes V_2$ by setting

$$x^l y^{m-l} \delta_m = l\left(x^{l-1} y^{m-l} \otimes x\right) + (m-l)\left(x^l y^{m-l-1} \otimes y\right)$$

(for $m = 1$ let $u\delta_1 = 1 \otimes u$) and extend to V_{m+1} by linearity. We note that $\delta_m \varphi_{m-1,1} = m\iota_{m+1}$. Therefore, if we can show that δ_m is an FM-homomorphism then we can set $\bar{\varphi} = \frac{1}{m}\delta_m$ whenever $p \nmid m$ to get the desired result.

We can see that δ_m defined above is closely related to differentiation in that for a polynomial $f \in V_{m+1}$, $f\delta_m = \left(f_x \otimes x\right) + \left(f_y \otimes y\right)$ where f_x, f_y denote the partial derivatives. It is likely in fact that this differentiation approach has been used to prove results similar to (5.2) in the classical setting over the field of real numbers. While acknowledging its inspiration, we do not appeal to the authority of calculus but prefer to adopt a barefisted approach in proving (5.2).

As δ is linear, it is sufficient to establish $\left(w^t\right)\delta = (w\delta)^t$ for all t in M and all monomials w in V_{m+1}. We shall do this by induction on m, exploiting a version of the "product rule" which we deal with first. To this end let $u \in V_{i+1}$, $v \in V_{m-i+1}$. We shall show that

$$(u \otimes v)\varphi_{i,m-i}\delta_m = \left(u \otimes v\delta_{m-i}\right)\left(\varphi_{i,m-i} \otimes \iota_2\right) + \left(v \otimes u\delta_i\right)\left(\varphi_{m-i,i-1} \otimes \iota_2\right)$$

always holds. The suffixes in the maps have been retained in this formula to warn against a tempting misinterpretation: $\left(u \otimes v\delta_{m-i}\right)\left(\varphi_{i,m-i-1} \otimes \iota_2\right)$

does not stand for the meaningless $u\varphi_{i,m-i-1} \otimes v\delta_{m-i}$ but for $(uv_x \otimes x) + (uv_y \otimes y)$ with v_x, v_y in V_{m-i} defined by $v\delta_{m-i} = v_x \otimes x + v_y \otimes y$, and the second summand of the right hand side to be read similarly. It is clear that the maps $\varphi_{i,m-i} \otimes \iota_2$ and $\varphi_{m-i,i-1} \otimes \iota_2$ are *FM*-homomorphisms. As all maps involved are linear, we need only prove this "rule" for the case u and v are monomials, say for $u = x^j y^{i-j}$, $v = x^k y^{m-i-k}$. The left hand side is $(x^{k+j}y^{m-k-j})\delta_m$ or

$$(k+j)(x^{k+j-1}y^{m-k-j} \otimes x) + (m-k-j)(x^{k+j}y^{m-k-j-1} \otimes y)$$

and the right hand side is

$$[(x^j y^{i-j} k x^{k-1} y^{m-i-k} \otimes x) + (x^j y^{i-j} (m-i-k) x^k y^{m-i-k-1} \otimes y)]$$
$$+ [(x^k y^{m-i-k} j x^{j-1} y^{i-j} \otimes x) + (x^k y^{m-i-k} (i-j) x^j y^{i-j-1} \otimes y)]$$

from which we conclude the two sides are equal.

Now to the proof, by induction on m, that $(w^t)\delta_m = (w\delta_m)^t$ for all w in V_{m+1} and all t in M. The initial case of $m = 1$, with $\delta_1 : w \mapsto 1 \otimes w$, is trivial. For $m > 1$ we have already noted that we only need deal with monomials w. In fact we only assume that w is in the image of some φ, say $w = (u \otimes v)\varphi_{i,m-i}$. The inductive hypothesis ensures that δ_i and δ_{m-i} are *FM*-homomorphisms so all maps on the right hand side of the relevant case of the "product rule" are *FM*-homomorphisms:

$$(w^t)\delta_m = [(u \otimes v)\varphi_{i,m-i}]^t\delta_m$$

$$= (u^t \otimes v^t)\varphi_{i,m-i}\delta_m$$

$$= \left[u^t \otimes v^t\delta_{m-i}\right](\varphi_{i,m-i} \otimes \iota_2) + \left[v^t \otimes u^t\delta_i\right](\varphi_{m-i,i} \otimes \iota_2)$$

$$= \left[u^t \otimes (v\delta_{m-i})^t\right](\varphi_{i,m-i} \otimes \iota_2) + \left[v^t \otimes (u\delta_i)^t\right](\varphi_{m-i,i} \otimes \iota_2)$$

$$= [(u \otimes v\delta_{m-i})(\varphi_{i,m-i} \otimes \iota_2)+(v \otimes u\delta_i)(\varphi_{m-i,i} \otimes \iota_2)]^t$$

$$= (w\delta_m)^t \ . \qquad \square$$

The proof of the next result introduces yet another set of homomorphisms and, as we will soon see, there is an interesting link between these and the δ_m introduced in the last result.

(5.3). $V_m \otimes V_p \cong V_{mp}$ *for all* m .

Proof. First note that $u\beta_m = u^p$ defines an FM-isomorphism of V_{m+1} into V_{mp+1} [as $(au+v)^p = au^p + v^p$ and $(u^t)^p = (u^p)^t$ for all $a \in F$, $u, v \in V$, $t \in M$]. The suffix of β is usually omitted. We note that $\beta_m\delta_{mp} = 0$ holds trivially; this will be used in the proof of (5.7). In fact it is not hard to see that $0 \to V_{m+1} \xrightarrow{\beta_m} V_{mp+1} \xrightarrow{\delta_{mp}} V_{mp} \otimes V_2$ is exact but we shall never use that.

The initial case of $m = 1$ is trivial. For $m > 1$ it is convenient to replace m by $m + 1$. We have the sequence of homomorphisms

$$V_{m+1} \otimes V_p \xrightarrow{\beta \otimes \iota_p} V_{mp+1} \otimes V_p \xrightarrow{\varphi_{mp,m-1}} V_{(m+1)p}$$

and for $x^l y^{m-l} \in V_{m+1}$, $x^i y^{p-1-i} \in V_p$,

$$(x^l y^{m-l} \otimes x^i y^{p-1-i})(\beta_m \otimes \iota_p)\varphi_{mp,m-1} = (x^{lp} y^{(m-l)p} \otimes x^i y^{p-1-i})\varphi_{mp,m-1}$$

$$= x^{lp+i} y^{(m-l)p+p-i-1} \ .$$

Since

$$\{pl+i \mid 0 \le l \le m,\ 0 \le i \le p-1\} = \{n \mid 0 \le n \le (m+1)p-1\} ,$$

$(\beta_m \otimes \iota_p)\varphi_{mp,m-1}$ is clearly onto $V_{(m+1)p}$, so that, by a dimension count, $V_{(m+1)} \otimes V_p \cong V_{(m+1)p}$. □

(5.4) (Corollary). *If* $p \nmid m$ *and* p^k *is any power of* p *, we have*

$$V_2 \otimes V_{mp^k} \cong \left(V_{(m-1)p^k} \otimes D\right) \oplus V_{(m+1)p^k} .$$

Proof. The proof is immediate using (5.2) and (5.3). □

Provided $p \nmid m$, (5.2) gives us a good method of exploring V_{m+1} once V_m and V_{m-1} are known. The proof of the following result shows that (5.2) is indeed a very powerful tool.

(5.5). *Let* $1 \le m \le n \le p$. *If* $m + n \le p + 1$ *then*

(a) $$V_m \otimes V_n \cong V_{m+n-1} \oplus \left(V_{m-1} \otimes V_{n-1} \otimes D\right)$$

$$\cong \bigoplus_{i=0}^{m-1} \left(V_{m+n-1-2i} \otimes D^i\right)$$

while if $p \le m+n \le 2p$ *then*

(b) $$V_m \otimes V_n \cong V_{p(m+n-p)} \oplus \left(V_{p-n} \otimes V_{p-m} \otimes D^{m+n-p}\right)$$

$$\cong V_{p(m+n-p)} \oplus \bigoplus_{i=0}^{p-n-1} \left(V_{(p-n)+(p-m)-1-2i} \otimes D^{m+n-p+i}\right) .$$

[In interpreting these formulas at certain extremes we use the usual conventions that both V_0 and the sum of an empty set of summands are 0 .]

Proof. For a proof of (a) by induction on m , note that (a) is a tautology when $m = 1$ and it is valid by (5.2) when $m = 2$, for all relevant values of n . Let $2 \le m' < p$ and suppose (a) holds for all relevant values of n whenever $m \le m'$. For the inductive step of the proof we need to deduce that (a) is valud when $m = m' + 1$ and n is in the range $m' + 1 \le n \le (p+1) - (m'+1)$. To this end note that

$$V_2 \otimes \left(V_{m'} \otimes V_n\right) \cong V_2 \otimes \left(V_{m'+n-1} \oplus \left(V_{m'-1} \otimes V_{n'-1} \otimes D\right)\right)$$

by the inductive hypothesis and so, by two applications of (5.2) [the first of which is possible because $(m'+1)+n-1 < p$], we get

$$V_2 \otimes V_{m'} \otimes V_n \cong V_{m'+n} \oplus \left(V_{m'+n-2} \otimes D\right) \oplus \left(V_{m'} \otimes V_{n-1} \otimes D\right)$$
$$\oplus V_{m'-2} \otimes V_{n-1} \otimes D^2 .$$

On the other hand by (5.2),

$$\begin{aligned} V_2 \otimes V_{m'} \otimes V_n &\cong \left(V_{m'+1} \oplus \left(V_{m'-1} \otimes D\right)\right) \otimes V_n \\ &\cong \left(V_{m'+1} \otimes V_n\right) \oplus \left(V_{m'-1} \otimes V_n \otimes D\right) \\ &\cong \left(V_{m'+1} \otimes V_n\right) \oplus \left(V_{m'+n-2} \otimes D\right) \oplus \left(V_{m'-2} \otimes V_{n-1} \otimes D^2\right) \end{aligned}$$

by the inductive hypothesis. Comparing these two expressions of $V_2 \otimes V_{m'} \otimes V_n$ and using the Krull Schmidt Theorem we conclude that the first half of (a) holds at $m = m' + 1$. As the second half is an obvious consequence, this completes the inductive step and with it the proof of (a).

As the second half of (b) follows immediately from (a) and the first half, we need only prove the first part. This will also be done by induction on m . For the initial step note that when $m = 1$ (and so $n \geq p-1$) we have a tautology, and when $m = 2$ so n is $p - 2$ or $p - 1$ or p , we have a tautology, a direct application of (5.2), or a direct application of (5.3). For the inductive step let $2 \leq m' < p$ and suppose (b) holds for all relevant n provided $m \leq m'$. We have to deduce that (b) holds at $m = m' + 1$ for all n satisfying $m' + 1 \leq n \leq p$ and $p \leq (m'+1) + n \leq 2p$. When $(m'+1) + n = p$ and $m = m' + 1$, the first half of (b) is a tautology. In the case $(m'+1) + n = p + 1$, part (a) is applicable and the expression of $V_{m'+1} \otimes V_n$ obtained there is, in this special case, just what is claimed in (b). Therefore we may assume $p + 1 < (m'+1) + n$. We exploit this immediately, for now $m' + n \geq p + 1$ with n in the relevant range so we will be able to apply the inductive hypothesis. That is:

$$V_2 \otimes V_{m'} \otimes V_n \cong V_2 \otimes \left(V_{p(m'+n-p)} \oplus \left(V_{p-n} \otimes V_{p-m'} \otimes D^{m'+n-p}\right)\right)$$

$$\cong \left(V_{p(m'+1+n-p)} \oplus V_{p(m'-1+n-p)} \otimes D\right)$$

$$\oplus \left(V_{p-n} \otimes V_{p-m'+1} \otimes D^{m'+n-p}\right) \oplus \left(V_{p-n} \otimes V_{p-(m'+1)} \otimes D^{m'+1+n-p}\right)$$

by (5.2) and (5.4). On the other hand, n is in the relevant range for (b) even with $m = m' - 1$, so by (5.2),

$$V_2 \otimes V_{m'} \otimes V_n$$

$$\cong \left(V_{m'+1} \oplus \left(V_{m'-1} \otimes D\right)\right) \otimes V_n$$

$$\cong \left(V_{m'+1} \otimes V_n\right) \oplus \left(V_{m'-1} \otimes V_n \otimes D\right)$$

$$\cong \left(V_{m'+1} \otimes V_n\right) \oplus \left(V_{p(m'-1+n-p)} \otimes D\right) \oplus \left(V_{p-n} \otimes V_{p-m'+1} \otimes D^{m'+n-p}\right)$$

by the inductive hypothesis. Comparing these decompositions of $V_2 \otimes V_{m'} \otimes V_n$, the Krull Schmidt Theorem yields (b) with $m = m' + 1$. This completes the inductive step. □

In Chapter 6, the following paraphrase of (5.5) will be more convenient.

(5.5'). *Let* $m, n \in \{1, 2, \ldots, p\}$. *The largest projective direct summand of* $V_m \otimes V_n$ *is* 0 *when* $m + n \leq p$ *and* $V_{p(m+n-p)}$ *otherwise, while the direct complement is always completely reducible and has the explicit form*

$$\bigoplus_{i=0}^{\min\{m,n\}} V_{m+n-1-2i} \otimes D^i$$

or

$$\bigoplus_{i=0}^{p-1-\max\{m,n\}} V_{2p-m-n-1-2i} \otimes D^{m+n-p+i}$$

according to whether $m + n \leq p$ *or* $m + n > p$.

An easy corollary of (5.5) that we will find useful in Chapter 6 is:

(5.6). *If* $1 \le i \le j \le p$ *, then*

$$V_i \otimes V_{j-1} \cong \left(V_{i-1} \otimes V_j\right) \oplus \left(V_{j-i} \otimes D^{i-1}\right) .$$

Proof. This is tautology if $i = j$ or $i = 1$ so suppose $1 < i < j$. If $i + j - 1 \le p + 1$ we have from (5.5) (a) that

$$V_i \otimes V_{j-1} \cong \bigoplus_{l=0}^{i-1} V_{i+j-2-2l} \otimes D^l$$

and

$$V_{i-1} \otimes V_j \cong \bigoplus_{l=0}^{i-2} V_{i+j-2-2l} \otimes D^l$$

so our claim follows. If $i + j - 1 \ge p + 1$ then from (5.5) (b),

$$V_i \otimes V_{j-1} \cong V_{p(i+j-1-p)} \oplus \bigoplus_{l=0}^{p-j} V_{p(p-j+1)+(p-i)-1-2l} \otimes D^{l+i+j-p-1}$$

and

$$V_{i-1} \otimes V_j \cong V_{p(i+j-1-p)} \oplus \bigoplus_{l=0}^{p-j-1} V_{(p-j)+(p-i-1)-1-2l} \otimes D^{l+i+j-p-1}$$

and again our result follows. □

In (3.1) we saw that the first p of the V_m are irreducible. We are now in a position to be able to describe the next $p - 1$ of them. These particular modules will prove to be very useful in later results. Also in V_{2p-1} we see, for the first time, a V_m which is indecomposable whereas $V_m|_G$ is not.

From now on, $p > 2$.

(5.7). *For* $r \in \{1, 2, \ldots, p-1\}$ *,* V_{p+r} *is indecomposable (and in fact for* $r \in \{1, 2, \ldots, p-2\}$ *,* $V_{p+r}|_G$ *is indecomposable) with*

$$\varphi V_{p+r} = \sigma V_{p+r} \cong \left(V_{r-1} \otimes D\right) \oplus V_{r+1}$$

and

$$V_{p+r}/\sigma V_{p+r} \cong V_{p-r} \otimes D^r .$$

Proof. In view of (2.1) the claim that $\varphi V_{p+r} = \sigma V_{p+r}$ follows from the rest of the assertion; we shall not refer to φV_{p+r} again in this proof. By (5.3), $V_2 \otimes V_p \cong V_{2p}$ and by (5.1) we have the exact sequence

$$0 \to V_{p-1} \otimes D \to V_2 \otimes V_p \to V_{p+1} \to 0 .$$

Thus V_{2p} contains a submodule U_1 with $U_1 \cong V_{p-1} \otimes D$ and $V_{2p}/U_1 \cong V_{p+1}$. By (3.3), $V_{2p}|_G$ is projective and thus either a principal indecomposable or the direct sum of two principal indecomposables each of dimension p. The latter possibility is ruled out by (2.6) and the presence of the irreducible U_1 of dimension $p - 1$. (It is in this last sentence that $p > 2$ is exploited for the first time.)

Recalling the definition of β_1 from the proof of (5.3) we next note that $\langle x^p, y^p \rangle$ is a submodule of V_{p+1} and $\langle x^p, y^p \rangle = V_2\beta_1 \cong V_2$. Thus V_{2p} has a submodule U with $U/U_1 \cong V_2$. Therefore, by (2.6), $V_{2p}|_G$, and thus V_{2p}, must be uniserial with U_1 and U being the only proper nonzero submodules.* Also by (2.6) we know that $(V_{2p}/U)_G \cong (V_{p-1} \otimes D)_G$ and so V_{p+1} is uniserial with $\sigma V_{p+1} \cong V_2$ and $(V_{p+1}/\sigma V_{p+1})_G \cong (V_{p-1} \otimes D)_G$. In the proof of (5.2) we showed that the map $\delta_p : V_{p+1} \to V_p \otimes V_2$ defined by

$$x^i y^{p-i} \delta_p = \left(i x^{i-1} y^{p-i} \otimes x\right) + \left((p-i) x^i y^{p-i-1} \otimes y\right)$$

is a homomorphism and we noted in the proof of (5.3) that $\beta_1\delta_p = 0$. As δ_p is clearly nonzero, it follows that $\ker \delta_p = V_2\beta_1$ and so $V_{p+1}\delta_p$ is irreducible. Thus $V_{p+1}/\sigma V_{p+1} \cong V_{p-1} \otimes D$ and (5.7) holds for $r = 1$.

---

* It follows that $X_{2p} = U$: a fact we do not need here but will need in the proof of (5.9).

For $r > 1$ we first identify two submodules of V_{p+r} , the first being X_{p+r} as defined in (4.4). In (5.1) we defined $\varphi_{m,n} : V_{m+1} \otimes V_{n+1} \to V_{m+n+1}$ to be the map which "forgets the tensor signs". We note that as X_{p+r+1} is generated by the u^{p+r} with $u \in V_2$ and as $u^{p+r} = \left(u^{p+r-1} \otimes u\right)\varphi_{p+r-1,1}$ with $u^{p+r-1} \in X_{p+r}$, clearly $X_{p+r+1} \leq \left(X_{p+r} \otimes V_2\right)\varphi_{p+r-1,1}$. Above we established that $X_{p+1} \cong V_2$. Let us use this as the initial step in a proof, by induction on r , for the claim that $X_{p+r} \cong V_{r+1}$ whenever $r \in \{1, 2, \ldots, p-1\}$. Indeed, if $X_{p+r} \cong V_{r+1}$ and $r < p-1$ then X_{p+r+1} is a submodule of a homomorphic image of $V_{r+1} \otimes V_2$ which, by (5.2) is just $\left(V_r \otimes D\right) \oplus V_{r+2}$. But by (4.4) we know that X_{p+r+1} is nonsingular and is indecomposable by (4.6), so the only possibility is that $X_{p+r+1} \cong V_{r+2}$. Thus for $r \in \{1, 2, \ldots, p-1\}$, V_{p+r} has a submodule isomorphic to V_{r+1} . By (4.1), V_{p+r} has also a submodule isomorphic to $V_{r-1} \otimes D$, so to complete the proof of (5.7) all we need show is that V_{p+r} is indecomposable (with $V_{p+r}|_G$ indecomposable if $r \leq p-2$) and has composition factors $V_{r-1} \otimes D$, V_{r+1} , $V_{p-r} \otimes D^r$.

Use the description of the composition factors of V_{p+1} and (5.5) to show that $V_{p+1} \otimes V_r$ has a submodule isomorphic to $\left(V_{r-1} \otimes D\right) \oplus V_{r+1}$ with factor module isomorphic to $\left(V_{p(r-1)} \otimes D\right) \oplus \left(V_{p-r} \otimes D^r\right)$. Therefore, by (5.3) and the exact sequence

$$0 \to \left(V_p \otimes V_{r-1}\right) \otimes D \to V_{p+1} \otimes V_r \to V_{p+r} \to 0 ,$$

and the Jordan-Hölder Theorem ([4], 13.7, p. 79), V_{p+r} has the desired composition factors.

From the knowledge of the dimensions of the composition factors of V_{p+r} we can see for $r < p-1$ that $V_{p+r}|_G$ cannot have any submodules of dimension p. Therefore, by (3.3), $V_{p+r}|_G$ is indecomposable. For $r = p - 1$ we cannot use this argument as $V_{r+1}|_G$ has dimension p and does in fact split off as a direct summand of $V_{2p-1}|_G$; the other summand, by (3.3), being uniserial of dimension $p - 1$. In particular, any irreducible direct summand of V_{2p-1} would have to be isomorphic to V_p. Suppose that V_{2p-1} is decomposable. Then $V_{2p-1} = A \oplus B$ with $A \cong V_p$, $\sigma B \cong V_{p-2} \otimes D$ and $B/\sigma B \cong V_1 \otimes D^{p-1}$. We have observed (at the end of Chapter 4) that $N^2 = N$ so $B(FN) = B(FN)^2 < (\sigma B)FN = 0$ contradicting the fact that $(x^p y - xy^p)V_{p-2}$ is the largest singular submodule of V_{2p-1} (see (4.1)). Therefore V_{2p-1} is indecomposable. $\square$

NOTE (1). In the proof of (6.1) we will use the fact that $V_{2p-1}|_G = A \oplus B$ where $A \cong V_p|_G$ and B is indecomposable with a submodule isomorphic to $(V_{p-2} \otimes D)_G$ and factor module isomorphic to $\left(V_1 \otimes D^{p-1}\right)_G$.

NOTE (2). The modules V_{p+r}, $1 \le r \le p-1$ exhibit the $p - 1$ pairwise nonisomorphic factor modules V_n/V_n^* as described in (4.1) and (4.2). From the proof of (5.7) we see that

$$\sigma\left(V_{p+r}/V_{p+r}^*\right) \cong V_{r+1}$$

and

$$\left(V_{p+r}/V_{p+r}^*\right)/\left(\sigma\left(V_{p+r}/V_{p+r}^*\right)\right) \cong V_{p-r} \otimes D^r .$$

Now since for $k \ge 1$, we can write $kp + 1$ as $(k-1)(p-1) + p + k$, we have by (4.2):

(5.8). *For* $1 \le k \le p-1$, V_{kp+1}/V_{kp+1}^* *has socle isomorphic to* V_{k+1}

and factor module isomorphic to $V_{p-k} \otimes D^k$. □

The next result gives us the structure of $V_{2p}, V_{3p}, \ldots, V_{p^2}$, by determining their indecomposable direct summands. Upon restriction to G , these summands give principal indecomposable FG-modules: the information obtained here will be exploited in the next chapter.

(5.9). *For* $1 \le k \le p$,

(a) $\varphi X_{(k+1)p} \cong V_{p-k} \otimes D^k$ *and* $X_{(k+1)p}/\varphi X_{(k+1)p} \cong V_{k+1}$, *while of course* $X_p = V_p$;

(b) *there is a submodule* U_k *in* $V_{(k+1)p}$, *containing* $X_{(k+1)p}$, *for which* $V_{(k+1)p} \cong U_k \oplus V_{(k-1)p} \otimes D$, *so with the convention that* $U_0 = V_p$ *we have*

$$V_{(k+1)p} \cong \bigoplus_{m=0}^{[k/2]} \left(U_{k-2m} \otimes D^m\right) ;$$

(c) $\varphi^2 U_k = \sigma U_k \cong U_k/\varphi U_k \cong V_{p-k} \otimes D^k$ *while*

$$\varphi U_k/\sigma U_k \cong \left(V_{k-1} \otimes D\right) \oplus V_{k+1} .$$

Proof. Note first that $\varphi^2 U_k = \sigma U_k$ follows from the rest of the claims; we shall not refer again to $\varphi^2 U_k$ here. It can be seen from the first two paragraphs of the proof of (5.7) that all the claims are valid at $k = 1$. We shall proceed by induction on k . We let $k' \in \{2, 3, \ldots, p-1\}$, suppose (5.9) is true whenever $k < k'$, and aim to deduce that it holds also when $k = k'$. For a start use (5.5) and (5.3) to amend the exact sequence

$$0 \to V_{p-1} \otimes V_{k'} \otimes D \xrightarrow{\theta_{p-1,k'}} V_p \otimes V_{k'+1} \xrightarrow{\varphi_{p-1,k'}} V_{p+k'} \to 0$$

to

$$0 \to \left(V_{p-k'} \otimes D^{k'}\right) \oplus \left(V_{(k'-1)p} \otimes D\right) \xrightarrow{\theta_{p-1,k'}} V_{(k'+1)p} \xrightarrow{\varphi_{p-1,k'}} V_{p+k'} \to 0 .$$

Thus $V_{(k'+1)p}$ has a submodule W [namely $\left(V_{(k'-1)p} \otimes D\right)\theta_{p-1,k'}$] which is isomorphic to $V_{(k'-1)p} \otimes D$. By the inductive hypothesis if $k' > 2$, and obviously at $k' = 2$, it follows that

$$\sigma W = \bigoplus_{m=0}^{\left[\frac{k'-2}{2}\right]} V_{p-(k'-2)+2m} \otimes D^{k'-m-1}$$

and so U has no submodules of dimension less than $p - k' + 2$.

Next we use (4.5) to deduce that $X_{(k'+1)p}/X^*_{(k'+1)p} \cong V_{k'+1}$. As $WN = 0$ we have that $X^*_{(k'+1)p} \geq X_{(k'+1)p} \cap W$. If $X_{(k'+1)p} \cap W$ were not zero, the observation at the end of the previous paragraph would therefore yield $\dim X^*_{(k'+1)p} \geq p - k' + 2$ and so

$$\dim X_{(k'+1)p} \geq (k'+1) + (p-k'+2) = p + 3 ,$$

which is impossible as $\dim X_m \leq p + 1$ for all m . Thus $X_{(k'+1)p} \cap W = 0$. As $W \cong V_{(k'-1)p} \otimes D$, by (3.3) we have that W_G is injective and hence (2.12) guarantees that $X_{(k'+1)p}$ is contained in some G-admissible complement, U say, of W in $V_{(k'+1)p}$. By (4.4), $UN \leq V_{(k'+1)p}FN = X_{(k'+1)p} \leq U$ so this U is in fact an FM-submodule. This proves (b) for $k = k'$. It is easy to see that (a) follows from (c); all that remains to be shown is that U has the properties claimed in (c) for U_k .

Recall that W was chosen so that the image of $\theta_{p-1,k'}$ is $W \oplus W'$ with $W' = \left(V_{p-k'} \otimes D^{k'}\right)\theta_{p-1,k'} \cong V_{p-k'} \otimes D^{k'}$. Put $U' = U \cap (W \oplus W')$. As $W \oplus W' \leq W \oplus U$ $\left(= V_{(k'+1)p}\right)$, Dedekinds law gives $W \oplus W' = W \oplus U'$ and so

$$U' \cong W \oplus W'/W \cong W' \cong V_{p-k'} \otimes D^{k'} .$$

By an isomorphism theorem

$$U/U' \cong (W \oplus W' + U)/(W \oplus W') \ \big(= V_{(k'+1)p}/(W \oplus W)\big)$$

and as $W \oplus W'$ (being in the image of $\theta_{p-1,k'}$) is the kernel of $\varphi_{p-1,k'}$, this means that $U/U' \cong V_{p+k'}$. In view of (5.7) it suffices to establish $\sigma U = U'$.

As $U \oplus W = V_{(k'+1)p}$, (3.3) tells us that U_G is projective and in light of the information we already have about the composition factors of U , it is easy to deduce that (in the notation of (2.4))

$$U_G \cong \begin{cases} P\Big[\big(V_{p-k'} \otimes D^{k'}\big)_G\Big] & \text{if } k' < p-1 \\ \\ P\big(V_1|_G\big) \oplus V_p|_G & \text{if } k' = p-1 . \end{cases}$$

When $k' < p-1$, this implies that every nonzero submodule of U_G , and hence every nonzero submodule of U , must contain U' , so we are done. When $k' = p - 1$, this consideration shows that U_G has a unique nonzero submodule U'' which fails to contain U' . Here $U'' \cong V_p|_G$ with $U_G/U'' \cong P\big(V_1|_G\big)$. This shows that U_G/U'' has no p-dimensional composition factor. As we have seen above $X_{p^2}/X^*_{p^2} \cong V_p$ [and as $X_{p^2} = \varphi X_{p^2}$ by (4.6)], either U'' does not admit M or $X_{p^2} = U''$. To complete the proof, we exclude the second possibility by showing that X_{p^2} has a one-dimensional submodule. This is done by considering the element $(\mathrm{x}^p\mathrm{y}-\mathrm{x}\mathrm{y}^p)^{p-1}$, which obviously spans a one-dimensional submodule in V_{p^2} , and proving that it is equal to the element

$$-x^{p^2-1} - y^{p^2-1} - \sum_{i=1}^{p-1} (ix+y)^{p^2-1}$$

of X_{p^2}. Indeed, in terms of the standard basis of V_{p^2},

$$\begin{aligned}
(x^p y - xy^p)^{p-1} &= (xy)^{p-1} \sum_{k=0}^{p-1} \binom{p-1}{k} x^{(p-1)k} \left(-y^{p-1}\right)^{p-1-k} \\
&= (xy)^{p-1} \sum_{k=0}^{p-1} x^{(p-1)k} y^{(p-1)(p-1-k)} \quad \text{by (2.14)} \\
&= \sum_{k=0}^{p-1} x^{(p-1)(k+1)} y^{(p-1)(p-k)}
\end{aligned}$$

while

$$\begin{aligned}
&-x^{p^2-1} - y^{p^2-1} - \sum_{i=1}^{p-1} (ix+y)^{p^2-1} \\
&= -x^{p^2-1} - y^{p^2-1} - \sum_{i=1}^{p-1} \sum_{j=0}^{p^2-1} \binom{p^2-1}{j} i^j x^j y^{p^2-1-j} \\
&= -x^{p^2-1} - y^{p^2-1} + \sum_{n=0}^{p+1} \binom{p^2-1}{n(p-1)} x^{n(p-1)} y^{p^2-1-n(p-1)} \quad \text{by (2.13)} \\
&= \sum_{n=1}^{p} x^{n(p-1)} y^{(p+1-n)p-1} \quad \text{by (2.14)} \\
&= \sum_{k=0}^{p-1} x^{(p-1)(k+1)} y^{(p-1)(p-k)} \quad \text{with } n = k+1 .
\end{aligned}$$

Comparing these two expressions we see that

$$(x^p y - xy^p)^{p-1} = -x^{p^2-1} - y^{p^2-1} - \sum_{i=1}^{p-1} (ix+y)^{p^2-1} \in X_{p^2}$$

completing the proof of (5.9).

CHAPTER 6

RESTRICTION TO THE GROUP

In this chapter we will be concerned purely with the restriction of modules to G. To simplify notation therefore we drop the suffix G from any module U_G and simply denote it by U. We revert to the old notation in Chapter 7.

Towards the end of this chapter, in (6.7) to be precise, we show that for certain projective modules $Q_0, Q_1, \ldots, Q_p$ (whose detailed structure is given in (6.8)), and for every nonnegative m,

$$V_{m+p(p-1)} = V_m \oplus \left(Q_i \otimes D^n\right)$$

with k and n defined by $m = n(p+1) + k$, $0 \leq k \leq p$. This theorem is a refinement of (4.3), and implies that it is only necessary to describe the first $p(p-1)$ of the V_m to describe all of them. To that end, write $V_m = \overline{V}_m \oplus \overline{\overline{V}}_m$ as in (3.7). For $m \leq p(p-1)$, the $\overline{V}_m$ are described in (6.4) and this result also describes the $\overline{\overline{V}}_m$ in terms of the V_{kp} with $k \leq p-1$. These V_{kp} are expressed in terms of principal indecomposables in (6.3). The principal indecomposable modules are fully described using (6.1) and (6.2), the latter result stating that

$$\left\{V_m \otimes D^n \mid 1 \leq m \leq p,\ 0 \leq n \leq p-2\right\}$$

is the full set of irreducible KG-modules.

For $1 \leq m \leq p$ and $0 \leq n \leq p-2$ let $P[m, n]$ denote the principal indecomposable FG-module for which

$$\sigma P[m, n] \cong P[m, n]/\varphi P[m, n] \cong V_m \otimes D^n .$$

Our first result describes the structure of these.

(6.1).

$$\varphi P[1,\ n]/\sigma P[1,\ n] \cong V_{p-2} \otimes D^{n+1} ,$$

$$\varphi P[m,\ n]/\sigma P[m,\ n] \cong \left(V_{p-1-n} \otimes D^{p+1-m+n}\right) \oplus \left(V_{p+1-m} \otimes D^{p-m+n}\right)$$

when $1 < n < p$ *and*

$$P[p,\ n] = V_p \otimes D^n .$$

Proof. The last claim is a direct consequence of (3.3) and (2.9). For $1 \le m < p$ we set the modules $P[m,\ p-m]$ in the proof of (5.9) and our present claim is just a restatement of what we had there. It remains to note that $P[m,\ p-m] \otimes D^{n+m-1}$ is projective by (2.9), has a homomorphism onto $V_m \otimes D^n$ and its submodules are in one-to-one correspondence with those of $P[m,\ p-m]$ because of the fact that

$$\left(P[m,\ p-m] \otimes D^{n+m-1}\right) \otimes D^{2p-m-n-1} \cong P[m,\ n] .$$

From this correspondence we deduce first that $P[m,\ p-m] \otimes D^{n+m-1}$ is indecomposable so it must be $P[m,\ n]$ and then our claims concerning submodules of $P[m,\ n]$ will follow. □

We can use the previous result to deduce:

(6.2). *Any irreducible* FG*-module is isomorphic to* $V_m \otimes D^n$ *for some* $m \in \{1, 2, \ldots, p\}$ *and* $n \in \{0, 1, \ldots, p-2\}$. *The* $p(p-1)$ *irreducibles so described are pairwise nonisomorphic.*

Proof. First we show that the irreducibles described above are in fact distinct. Write U for the trivial one dimensional FS-module $V_1|S$ and observe that the induced module U^G is just the regular G/S-module regarded as a G-module so that in fact

$$U^G = \bigoplus_{j=0}^{p-2} D^j .$$

Take an arbitrary $V_m \otimes D^n$ and for the moment denote it by W. Then

$$\begin{aligned}
(W_S)^G &\cong (W_S \otimes U)^G \\
&\cong W \otimes U^G \quad \text{by } (2.8) \\
&\cong W \otimes \left(\bigoplus_{j=0}^{p-2} D^j\right) \\
&\cong \bigoplus_{j=0}^{p-2} \left(V_m \otimes D^j\right) .
\end{aligned}$$

From (3.1) we know that W_S is absolutely irreducible so, by ([4], 29.13, p. 302), $\mathrm{Hom}(W_S, W_S) = F$. On the other hand, by ([6], 16.6, p. 556), we have $\mathrm{Hom}\left(W, (W_S)^G\right) \cong \mathrm{Hom}(W_S, W_S)$, and so we conclude that $W \left(= V_m \otimes D^n\right)$ has only one copy in $W_S^G \left(= \bigoplus_{j=0}^{p-2} V_m \otimes D^j\right)$. Thus $V_m \otimes D^n$ is not isomorphic to any of the other $V_m \otimes D^j$ with $0 \le j \le p-2$ and of course dimension distinguishes it from all the $V_i \otimes D^j$ with $i \ne m$.

To show that the list of modules in (6.2) is in fact a full set of irreducibles we use (2.4) and the size of the principal indecomposables in (6.1). That is

$$\begin{aligned}
\dim FG &\le \sum_{n=0}^{p-2} \sum_{m=1}^{p} m(\dim P[m, n]) \\
&= (p-1)\left[1.p+2p \,.\, \sum_{m=2}^{p-1} m+p.p\right] \\
&= p(p-1)\left(p^2-1\right) \\
&= \dim FG .
\end{aligned}$$

Thus equality holds and we conclude that there is no room for further irreducibles in FG. □

We now turn our attention to calculating the first $p(p-1)$ of the V_m.

When $p|m$ all we have to do is rewrite (5.9).

(6.3). *For* $0 \le k \le p-2$,

$$V_{(k+1)p} \cong \left(V_{(k-1)p} \otimes D\right) \oplus P[p-k,\ k]$$

$$\cong \bigoplus_{m=0}^{[k/2]} P[p-k+2m,\ k-m]$$

while

$$V_{p^2} \cong \left(V_{(p-2)p} \otimes D\right) \oplus P[1,\ 0] \oplus V_p$$

$$\cong \left(\bigoplus_{m=0}^{(p-1)/2} P[p-k+2m,\ k-m]\right) \oplus V_p \ . \qquad \square$$

This adds the first $(p-1)$ of the $V_{(k+1)p}$ to our stock of building blocks. It will be convenient to use these, rather than the individual principal indecomposables, in describing the projective parts $\overline{\overline{V}}_m$ of the V_m . For the $\overline{V}_m$ we have $\overline{V}_m = \sigma^2\overline{V}_m$, and we shall explicitly determine $\sigma\overline{V}_m$ and $\sigma\overline{V}_m/\ \overline{V}_m$.

(6.4). *For* $0 \le k \le p-2$ *and* $0 \le r \le p-1$,

$$\sigma\overline{V}_{kp+r} \cong \begin{cases} \displaystyle\bigoplus_{i=0}^{\min\{k+1,r\}-1} \left(V_{k+r-2i} \otimes D^i\right) & \text{if } k+r < p \\ \displaystyle\bigoplus_{i=1}^{p-\max\{k+1,r\}} \left(V_{2p-k-r-2i} \otimes D^{k+r+i}\right) & \text{if } k+r \ge p\ , \end{cases}$$

$$\overline{V}_{kp+r}/\sigma\overline{V}_{kp+r} \cong \begin{cases} \displaystyle\bigoplus_{i=0}^{\min\{p-k,r\}-1} V_{p-k+r-1-2i} \otimes D^{k+i} & \text{if } k \ge r \\ \displaystyle\bigoplus_{i=0}^{p-1-\max\{p-k,r\}} V_{p+k-r-1-2i} \otimes D^{r+i} & \text{if } k < r \end{cases}$$

and

$$\overline{\overline{V}}_{kp+r} \cong \begin{cases} P \oplus V_{(k-r)p} \otimes D^r & \text{if} \quad k \le r \\ P & \text{if} \quad k < r \end{cases}$$

where

$$P \cong \begin{cases} 0 & \text{if} \quad k+r < p \\ V_{p(k+1+r-p)} & \text{if} \quad k+r \ge p \ . \end{cases}$$

Proof. When $k = 0$ or $r = 0$ the claims are virtually tautologies, so suppose $k \ge 1$ and $r \ge 1$. Let Y_k denote V_{kp+1}/V^*_{kp+1} and note that V^*_{kp+1} is isomorphic to $V_{(k-1)p} \otimes D$ which is injective by (3.3). Thus

$$V_{kp+1} \cong Y_k \oplus \left(V_{(k-1)p} \otimes D\right)$$

and hence, using (5.3),

$$V_{kp+1} \otimes V_r \cong \left(Y_k \otimes V_r\right) \oplus \left(V_{k-1} \otimes V_r \otimes V_p \otimes D\right) \ .$$

On the other hand we have the exact sequence

$$0 \to V_{kp} \otimes V_{r-1} \otimes D \to V_{kp+1} \otimes V_r \to V_{kp+r} \to 0$$

which, because of the injectivity of V_{kp}, splits and so using (5.3) once more we have

$$V_{kp+1} \otimes V_r \cong V_{kp+r} \oplus \left(V_k \otimes V_{r-1} \otimes V_p \otimes D\right) \ .$$

In applying the Krull-Schmidt Theorem to these decompositions of $V_{kp+1} \otimes V_r$ we distinguish two cases:

(i) when $r \le k$, (5.6) enables us to substitute

$$\left(V_{r-1} \otimes V_k\right) \oplus \left[V_{k-r} \otimes D^{r-1}\right]$$

for $V_{k-1} \otimes V_r$ in the first decomposition, so after cancellations and use of (5.3) we get

$$\left(Y_k \otimes V_r\right) \oplus V_{p(k-r)} \otimes D^r \cong V_{kp+r} \ ;$$

(ii) when $r > k$ the substitution justified by (5.6) is that of

$$\left(V_{k-1} \otimes V_r\right) \oplus \left[V_{r-k} \otimes D^{k-1}\right]$$

for $V_{r-1} \otimes V_k$ in the second decomposition, yielding

$$Y_k \otimes V_r \cong V_{kp+r} \oplus \left(V_{p(r-k)} \otimes D^k\right) .$$

We are going to derive the structure of V_{kp+r} from these expressions. In either case the crux of the matter is to calculate $Y_k \otimes V_r$. By (5.8), $\sigma Y_k \cong V_{k+1}$ and $Y_k/\sigma Y_k \cong V_{p-k} \otimes D^k$. Thus $Y_k \otimes V_r$ has a submodule isomorphic to $V_{k+1} \otimes V_r$ with factor module isomorphic to $V_{p-k} \otimes V_r \otimes D^k$. We get the structure of this submodule and this factor module from (5.5'). That explicit description splits into several cases, but before we go into those details we note the qualitative aspects which enable us to proceed. From (5.5') we see that the tensor product of any two irreducibles is a direct sum of a projective module and a completely reducible module. Thus we may write the submodule $(\sigma Y_k) \otimes V_r$ of $Y_k \otimes V_r$ as $P \oplus R_1$ with P projective and R_1 completely reducible without any projective summand. From a similar decomposition of the factor module we get an R_2 such that $R_2/(\sigma Y_k \otimes V_r)$ is completely reducible without projective summands and such that $Y_k \otimes V_r/R_2$ is projective. By (2.12), $R_2 = P \oplus R$ with $R \geq R_1$ and $R/R_1 \cong R_2/(\sigma Y_k \otimes V_r)$ while of course $Y_k \otimes V_r = Q \oplus R_2$ with a projective Q . Thus $V_k \otimes V_r = P \oplus Q \oplus R$. As $\varphi^2 R = 0$ and R has no p-dimensional composition factor, the description of the principal indecomposables shows that R has no projective direct summand. The Krull-Schmidt Theorem now shows, from (i) or (ii) as the case may be that $R \cong \overline{V}_{kp+r}$, so R is indecomposable. It follows from (2.1) that $\sigma R = \varphi R = R_1$ unless R is 0 or irreducible. Thus $\sigma\overline{V}_{kp+r}$ is the nonprojective summand in the direct decomposition of $V_{k+1} \otimes V_r$ given by (5.5'), while $\overline{V}_{kp+r}/\sigma\overline{V}_{kp+r}$ is the nonprojective summand in the direct decomposition of $V_{p-k} \otimes V_r \otimes D^k$

obtainable from (5.5'). This enables the corresponding claims of (6.4).

Turning to the projective parts, first note that P is 0 when $k + r < p$ and $V_{p(k+1+r-p)}$ otherwise. In case (i) [that is, when $r \leq k$], the Krull-Schmidt Theorem gives that

$$\overline{\overline{V}}_{kp+r} \cong P \oplus Q \oplus \left(V_{p(k-r)} \otimes D^r\right)$$

and (5.5') yields $Q = 0$. Similarly in case (ii) $[r > k]$ we get

$$\overline{\overline{V}}_{kp+r} \oplus \left(V_{p(r-k)} \otimes D^k\right) \cong P \oplus Q$$

while (5.5') gives that $Q \cong V_{p(r-k)} \otimes D^k$. Thus one more application of the Krull-Schmidt Theorem yields $\overline{\overline{V}}_{kp+r} \cong P$. In either case we have the result as claimed in (6.4). □

The rest of the chapter is devoted to producing reduction formulas which have as an end result the consequences set out in the introduction to this chapter. Much of the hard work in that direction is carried out in the proof of the following result.

(6.5). *For any* $l \geq 0$ *and any* $r \geq 0$,

$$V_{r+p+1+lp(p-1)} \oplus \left(V_r \otimes D\right) \cong \left(V_{r+lp(p-1)} \otimes D\right) \oplus V_{r+p+1} .$$

Proof. This is a tautology for $l = 0$ so assume $l \geq 1$. We first to establish (6.5) in the case $p|r$ where we can write $r + p + 1 + lp(p-1)$ as $sp + 1 + lp(p-1)$ for some $s \geq 1$. As at the beginning of the proof of (6.4) define Y_s as V_{sp+1}/V^*_{sp+1} (for all $s \geq 1$) and then use (4.1), (4.2) and the injectivity of $V_{(s-1)p+lp(p-1)} \otimes D$ to split this off as a direct summand. That is

$$\begin{aligned} V_{r+p+1+lp(p-1)} \oplus \left(V_r \otimes D\right) &= V_{(s-1)p+p+1+lp(p-1)} \oplus \left(V_{(s-1)p} \otimes D\right) \\ &\cong \left(\left(V_{(s-1)p+lp(p-1)} \otimes D\right) \oplus Y_s\right) \oplus \left(V_{(s-1)p} \otimes D\right) \\ &\cong \left(V_{(s-1)p+lp(p-1)} \otimes D\right) \oplus \left(V_{(s-1)p} \otimes D \oplus Y_s\right) \\ &\cong \left(V_{r+lp(p-1)} \otimes D\right) \oplus V_{r+p+1} \end{aligned}$$

Thus (6.5) is proved in the case $p|r$.

Consider next the case $p|r-1$; say $r = sp + 1$ with $s \geq 0$. We need to prove

$$V_{p[s+1+l(p-1)]+2} \oplus \left(V_{sp+1} \otimes D\right) \cong \left(V_{p[s+lp(p-1)]+1} \otimes D\right) \oplus V_{(s+1)p+2} \text{ .}$$

Using two applications of (4.1) we note that there exists an A such that

$$V_{p\left(s+1+lp(p-1)\right)+2} = \left(x^py-xy^p\right)^2 V_{p\left(s-1+lp(p-1)\right)} \oplus A \text{ .}$$

Put $A \cap V^*_{p\left(s+1+lp(p-1)\right)+2} = B$ so using (4.1) we have

$$\begin{aligned}\left(x^py-xy^p\right)^2 V_{p\left(s-1+lp(p-1)\right)} &\leq V^*_{p\left(s+1+lp(p-1)\right)+2}\\ &\leq \left(x^py-xy^p\right)^2 V_{p\left(s-1+lp(p-1)\right)} \oplus A\end{aligned}$$

so by Dedekind's law we have

$$V^*_{p\left(s+1+lp(p-1)\right)+2} \cong \left(x^py-xy^p\right)^2 V_{p\left(s-1+lp(p-1)\right)} \oplus B \text{ .}$$

That is

$$\begin{aligned}V_{p\left(s+lp(p-1)\right)+1} \otimes D &\cong \left[\left(x^py-xy^p\right) V_{p\left(s-1+lp(p-1)\right)} \otimes D\right] \oplus B\\ &= \left(V^*_{p\left(s+lp(p-1)\right)+1} \otimes D\right) \oplus B\end{aligned}$$

so by (4.2), $B \cong Y_{s+p-1} \otimes D$. Also from (4.2) and one of the isomorphism theorems we have

$$\begin{aligned}\frac{A}{B} &\cong \frac{\left(V_{p\left(s+1+lp(p-1)\right)+2}\right) / \left[\left(x^py-xy^p\right)^2 V_{p\left(s-1+lp(p-1)\right)}\right]}{\left[\left(x^py-xy^p\right) V_{p\left(s-1+lp(p-1)\right)}\right] / V^*_{p\left(s+1+lp(p-1)\right)+2}}\\ &\cong V_{p(s+1)+2} / V^*_{p(s+1)+2}\\ &\cong Y_{s+2} \text{ .}\end{aligned}$$

By (4.1) and (4.2) we also know that

$$\begin{aligned}V_{p\left(s+lp(p-1)\right)+1} \otimes D &\cong \left[V_{p\left(s-1+lp(p-1)\right)} \otimes D^2\right] \oplus \left(Y_{s+p-1} \otimes D\right)\\ &\cong \left(x^py-xy^p\right)^2 V_{p\left(s-1+lp(p-1)\right)} \oplus B\end{aligned}$$

so to show that

$$V_{p(s+1+lp(p-1))+2} \oplus (V_{sp+1} \otimes D) \cong (V_{p(s+lp(p-1))+1} \otimes D) \oplus V_{(s+1)p+2}$$

it suffices to show that

$$A \oplus V_{sp+1} \otimes D \cong B \oplus V_{(s+1)p+2} .$$

We need one more fact about A : namely that it follows from (3.7) and the Krull-Schmidt Theorem that $A \cong \overline{V}_{(s+1)p+2} \oplus P$ for some projective P . Now let us deal with the case of $s = 0$. We have then from above that $B \cong Y_{p-1} \otimes D$ so by (5.8), $B \cong V_p \otimes D \oplus V_1 \otimes D$. If $p \neq 3$ then $\overline{V}_{p+2} = V_{p+2}$ so $A \cong V_{p+2}$ as required. If $p = 3$ then $V_{p+2} = \overline{V}_{p+2} \oplus V_p$ by (6.4) but also $A/B \cong Y_2 \cong V_p \oplus V_1$ so

$$A = \overline{V}_{p+2} \oplus V_p \oplus (V_p \otimes D) = V_{p+2} \oplus (V_p \otimes D)$$

and again we are done. Suppose next that $s > 0$ so we may write

$$V_{sp+1} \cong V_{(s-1)p} \otimes D \oplus Y_s$$

and amy also define A', B', P' similarly to A, B, P above so that

$$V_{(s+1)p+2} = V_{(s-1)p} \otimes D^2 \oplus A' ,$$

$$A' = \overline{V}_{(s+1)p+2} \oplus P'$$

and $B' \cong B$, $A'/B' \cong A/B$ by (4.2). It follows that P and P' are projectives of dimension at most $2p$ with the same composition factors so from the knowledge of the principal indecomposables from (6.1) we conclude that $P \cong P'$, $A = A'$. Thus

$$\begin{aligned} A \oplus (V_{sp+1} \otimes D) &\cong A \oplus \left(V_{(s-1)p} \otimes D^2\right) \oplus (Y_s \otimes D) \\ &\cong B \oplus V_{(s+1)p+2} \end{aligned}$$

follows as required from the observation that $B \cong Y_s \otimes D \cong Y_{s+p-1} \otimes D$ by (4.2). This completes the proof for the case $p | r-1$.

In particular we have established (6.5) for $r = 0, 1$. We proceed to prove (6.5) by induction on r . Suppose then that $r \geq 2$ and

$$V_{r-1+p+1+lp(p-1)} \oplus \left(V_{r-1} \otimes D\right) \cong \left(V_{r-1+lp(p-1)} \otimes D\right) \oplus V_{r-1+p+1}$$

and also that

$$V_{r-2+p+1+lp(p-1)} \oplus \left(V_{r-2} \otimes D\right) \cong \left(V_{r-2+lp(p-1)} \otimes D\right) \oplus V_{r-2+p+1} .$$

If p divides r or $r - 1$ there is nothing to do, while if p divides neither it is easy to calculate from (5.2) the tensor product of V_2 with each side of the first isomorphism in the inductive hypothesis above so we get that

$$\left(V_{r-2+p+1+lp(p-1)} \otimes D\right) \oplus V_{r+p+1+lp(p-1)} \oplus \left(V_{r-2} \otimes D^2\right) \oplus \left(V_r \otimes D\right)$$
$$\cong \left(V_{r-2+lp(p-1)} \otimes D^2\right) \oplus \left(V_{r+lp(p-1)} \otimes D\right) \oplus \left(V_{r-2+p+1} \otimes D\right) \oplus V_{r+p+1} .$$

From this, the second isomorphism of the inductive hypothesis above and the Krull Schmidt Theorem give (6.5). $\square$

Using (6.5) as the initial step of a simple induction argument on we get the general lemma we have been aiming at

(6.6). *For* $r, l \geq 0$, $n \geq 1$,

$$V_{r+n(p+1)+lp(p-1)} \oplus \left(V_r \otimes D^n\right) \cong \left(V_{r+lp(p-1)} \otimes D^n\right) \oplus V_{r+n(p+1)} .$$

Proof. We take the inductive step as the implication from n to $n + 1$. First we use (6.5) with $r + (n-1)(p+1)$ in place of r :

$$V_{r+n(p+1)+lp(p-1)} \oplus \left(V_{r+(n-1)(p+1)} \otimes D\right) \oplus \left(V_r \otimes D^n\right)$$
$$\cong V_{r+(n-1)(p+1)+lp(p-1)} \otimes D \oplus V_{r+n(p+1)} \oplus V_r \otimes D^n$$
$$= \left(V_{r+(n-1)(p+1)+lp(p-1)} \oplus \left(V_r \otimes D^{n-1}\right)\right) \otimes D \oplus V_{r+n(p+1)}$$
$$\cong \left(\left(V_{r+lp(p-1)} \otimes D^{n-1}\right) \oplus V_{r+(n-1)(p+1)}\right) \otimes D \oplus V_{r+n(p+1)}$$

by the inductive hypothesis. Therefore by the Krull-Schmidt Theorem the conclusion of the inductive step follows.

We are now ready to establish the main reduction theorem. For $0 \leq k \leq p$, define Q_k so that

$$V_{k+p(p-1)} \cong Q_k \oplus V_k \ ,$$

namely by

$$Q_k = \begin{cases} \overline{\overline{V}}_{k+p(p-1)} & \text{if} \quad k < p \\ \\ V_{p(p-2)} \otimes D \oplus P[1,\ 0] & \text{if} \quad k = p\ : \end{cases}$$

when $k < p$ this works by (3.7) and when $k = p$ by (6.3). The theorem is:

(6.7). *If* $m = n(p+1) + k$ *with* $0 \leq k \leq p$, *then*

$$V_{m+p(p-1)} \cong V_m \oplus \left(Q_k \otimes D^n\right) .$$

Repeated application of this enables one to express all V_m in terms of the first $p(p-1)$ which have already been completely described in (6.4), and in terms of the Q_k which will be described in (6.8) below. The theorem is a refinement of (4.3). For a qualitative comparison one may paraphase (6.7) as follows: to each m , $V_{m+p(p-1)}$ is the direct sum of V_m and a projective module which depends only on the residue class of m modulo p^2-1 . Quantitatively, it yields the structure of the P_m defined in (4.3), but we do not need the explicit formula.

Proof of (6.7). For $n = 0$, this is a repeat of the defining property of Q_k . For $n > 0$, we use (6.6):

$$\begin{aligned} V_{m+p(p-1)} \oplus \left(V_k \otimes D^n\right) &\cong V_{k+p(p-1)} \otimes D^n \oplus V_m \\ &\cong \left(Q_k \otimes D^n\right) \oplus \left(V_k \otimes D^n\right) \oplus V_m \end{aligned}$$

and so the Krull-Schmidt Theorem yields the result.

(6.8). *For* $0 \leq k \leq p-1$ *we have* $Q_k \cong \left(V_{p(p-k-1)} \otimes D^k\right) \oplus V_{pk}$.

Proof. Note that at $k = 0$ this is the definition of Q_0 , while at $k = 1$ we have, from (4.1) and (5.8) as so many times before, that

$$\begin{aligned} V_{1+p(p-1)} &\cong V_{p(p-2)} \otimes D \oplus Y_{p-1} \\ &\cong V_{p(p-2)} \otimes D \oplus V_p \oplus V_1 \,. \end{aligned}$$

Hence Q_1 is as claimed. Suppose then that $2 \leq k \leq p-1$. The exact sequence

$$0 \to V_{k-1} \otimes V_{p(p-1)} \otimes D \to V_k \otimes V_{1+p(p-1)} \to V_{k+p(p-1)} \to 0$$

from (5.1) splits as $V_{k-1} \otimes V_{p(p-1)} \otimes D$ is injective, so we have

$$V_k \otimes V_{1+p(p-1)} \cong \left(V_{k-1} \otimes V_{p(p-1)} \otimes D\right) \oplus V_{k+p(p-1)}$$

and by (5.3), (5.5) and (5.3) again

$$\begin{aligned} V_{k-1} \otimes V_{p(p-1)} &\cong V_{k-1} \otimes V_{p-1} \otimes V_p \\ &\cong \left(V_{p(k-2)} \oplus \left[V_{p-k+1} \otimes D^{k-2}\right]\right) \otimes V_p \\ &\cong V_{p^2(k-2)} \oplus V_{p(p-k+1)} \otimes D^{k-2} \,. \end{aligned}$$

Therefore we have

$$V_k \otimes V_{1+p(p-1)} \cong \left(V_{p^2(k-2)} \otimes D\right) \oplus \left(V_{p(p-k+1)} \otimes D^{k-1}\right) \oplus V_{k+p(p-1)} \,,$$

the first of two direct decompositions to be exploited in an application of the Krull-Schmidt Theorem. The other arises from the expression obtained above for $V_{1+p(p-1)}$, again using (5.3) and (5.5):

$$\begin{aligned} &V_k \otimes V_{1+p(p-1)} \\ &\quad\cong V_k \otimes \left[\left(V_{p(p-2)} \otimes D\right) \oplus V_p \oplus V_1\right] \\ &\quad\cong \left(V_k \otimes V_{p-2}\right) \otimes \left(V_p \otimes D\right) \oplus V_{pk} \oplus V_k \\ &\quad\cong \left[V_{p(k-2)} \oplus \left(V_{p-k+1} \otimes D^{k-2}\right) \oplus \left(V_{p-k-1} \otimes D^{k-1}\right)\right] \otimes \left(V_p \otimes D\right) \oplus V_{pk} \oplus V_k \\ &\quad\cong \left(V_{p^2(k-2)} \otimes D\right) \oplus \left(V_{p(p-k+1)} \otimes D^{k-1}\right) \oplus V_{p(p-k-1)} \otimes D^k \oplus V_{pk} \oplus V_k \,. \end{aligned}$$

The Krull-Schmidt Theorem now yields the result. □

This completes the story on the V_m , a complete description now being possible along the lines outlined in the introduction to this chapter.

The last result in this chapter is one we shall need in Chapter 7.

(6.9). *For every* m ,

$$1 - p \leq \dim \overline{V}_{m+p+1} - \dim \overline{V}_m \leq 1 + p \ .$$

Proof. The second of these inequalities follows from the fact that $(x^p y - xy^p)\overline{\overline{V}}_m$ is an injective submodule and hence a direct summand of V_{m+p+1} and therefore also of $\overline{\overline{V}}_{m+p+1}$. To derive the first inequality, we need to apply (3.7) and (6.4); we omit the laborious but straightforward arithmetic. □

CHAPTER 7

ON FM-MODULES

This chapter starts with a complete description of the irreducible and of the principal indecomposable *FM*-modules, in close analogy with the first part of Chapter 6. We then deduce from the details of this information that *FM* is of infinite representation type; that is, there exist indecomposable *FM*-modules of arbitrary large (finite) dimension. This threatens that the conclusive results we obtained for the $V_m|_G$ have no analogues for the V_m themselves. The next result shows that only finitely many isomorphism types of indecomposables occur as direct summands of the V_m, so hope returns again. However, in the last result we see that in spite of this the V_m cannot fit any "arithmetic" pattern similar to that obtained for their restrictions.

The $V_m \otimes D^n$ with $1 \leq m \leq p$ and $0 \leq n \leq p-1$ are absolutely irreducible and pairwise nonisomorphic, as the V_m themselves $\left(\text{the } V_m \otimes D^0\right)$ are all nonsingular while the singular $V_m \otimes D^n$ (with $n > 0$) are pairwise nonisomorphic as *FG*-modules, by (6.2). Let P[m, n] stand for the principal indecomposable *FM*-module which has $V_m \otimes D^n$ as a homomorphic image. There is no reason to assume that the restriction of P[m, n] to G is the $P[m, n]$ of Chapter 6. We shall see that this does happen for some, but certainly not all, relevant values of the parameters m, n. We shall therefore find it useful to define *FM*-modules $P[m, n]$ whose restrictions to G are *always* the $P[m, n]$ of Chapter 6: namely let $P[m, n]$ be the *FM*-module which is annihilated by N and which, as an *FG*-module, is isomorphic to the principal indecomposable *FG*-module with

$\left(V_m \otimes D^n\right)_G$ as homomorphic image. The structure of $P[m, n]$ is then exactly as given in (6.1) except that every occurence of D^0 in (6.1) is replaced by D^{p-1} and that n is no longer restricted to the range $\{0, 1, \ldots, p-2\}$. (The latter change is of convention of convenience for of course $P[m, n] \cong P[m, n+(p-1)]$.) It is immediate that $P[m, n]$ is a homomorphic image of $\mathrm{P}[\mathrm{m}, \mathrm{n}]$ whenever $n > 0$. This and the original (5.9) itself is all we need for establishing our main result.

(7.1). *(a) Each irreducible FM-module is isomorphic to some* $V_m \otimes D^n$ *with* $1 \leq m \leq p$ *and* $0 \leq n \leq p-1$. *These* p^2 *irreducibles are pairwise nonisomorphic.*

(b) If $n > 0$ *and* $m \neq p - n$ *then* $\mathrm{P}[\mathrm{m}, \mathrm{n}]$ *is just* $P[m, n]$, *so its composition structure can be read off* (6.1).

(c) When $n = 0$ *we have* $\mathrm{P}[1, 0] \cong V_1$ *and for* $2 \leq m \leq p$ *we have* $\mathrm{P}[\mathrm{m}, \mathrm{n}] \cong X_{mp}$ *so* $\varphi\mathrm{P}[\mathrm{m}, \mathrm{n}] \cong V_{p-m+1} \otimes D^{m-1}$.

(d) When $n > 0$ *and* $m = p - n$, *we have*

$$\varphi^2\mathrm{P}[\mathrm{p-n}, \mathrm{n}] = \sigma\mathrm{P}[\mathrm{p-n}, \mathrm{n}] \cong \left(V_{p-n} \otimes D^n\right)^{\oplus 2}$$

and

$$\varphi\mathrm{P}[\mathrm{p-n}, \mathrm{n}]/\sigma\mathrm{P}[\mathrm{p-n}, \mathrm{n}] \cong \begin{cases} \left(V_{n-1} \otimes D\right) \oplus V_{n+1} \otimes D^{p-1} \oplus V_{n+1} & \text{if } n \neq p-1 \\ V_{p-2} \otimes D \oplus V_p & \text{if } n = p-1 \,. \end{cases}$$

[As usual $U^{\oplus k}$ denotes the direct sum of k copies of U.]

Proof. The second sentence of (a) has been established already. For (b), so far we only know that in this case $P[m, n]$ is a homomorphic image of $\mathrm{P}[\mathrm{m}, \mathrm{n}]$, so in particular

$$\dim \mathsf{P}[\mathsf{m},\ \mathsf{n}] \geq \dim P[m,\ n] = \begin{cases} p & \text{if } m = 1 \text{ or } p \\ 2p & \text{if } 1 < m < p\ . \end{cases}$$

Similarly in case (c) we have of course that V_1 is a homomorphic image of $\mathsf{P}[1,\ 0]$ so $\dim \mathsf{P}[1,\ 0] \geq 1$, while for $2 \leq m \leq p$ we have by (5.9) that X_{mp} has dimension $p + 1$ and is a homomorphic image of $\mathsf{P}[\mathsf{m},\ 0]$ so $\dim \mathsf{P}[\mathsf{m},\ 0] \geq p + 1$. An application of (2.4) below will show these dimension estimates to be accurate and so establish (b) and (c) but first we must get similar estimates for case (d).

Let $n > 0$, $m = p - n$ and $\mathsf{P} = \mathsf{P}[\mathsf{p-n},\ \mathsf{n}]$. As

$$U_n/\varphi U_n \cong P[p-n,\ n]/\varphi P[p-n,\ n] \cong V_{p-n} \otimes D^n$$

for the module U_n defined in (5.9), one may choose submodules U and W in P so that $\mathsf{P}/U \cong U_n$ and $\mathsf{P}/W \cong P[p-n,\ n]$. Recall from (5.9) that U_n contains $X_{(n+1)p}$ and $X_{(n+1)p}/\varphi X_{(n+1)p} \cong V_{n+1}$ while by definition $P[p-n,\ n]$ is singular. Thus if X/U is the copy of $X_{(n+1)p}$ in P/U given by the isomorphism $\mathsf{P}/U \cong U_n$, we must have $X/U \leq U+W/U$ otherwise the nonsingular $(X/U)/\varphi(X/U)$ would be a factor module of

$$\frac{X/U}{(X/U)\cap(U+W/U)}$$

which is isomorphic to the submodule

$$\frac{(X/U)+(U+W/U)}{U+W/U}$$

of

$$\frac{\mathsf{P}/U}{U+W/U}\ ,$$

while the latter is a homomorphic image of the singular P/W . Now $U < X \leq U + W$ and Dedekind's law give $X = U + (X \cap W)$ and so by one of the isomorphism theorems

$$X/U \cong X \cap W / U \cap W \ .$$

Before we proceed, we note that when $n < p - 1$ a similar argument can be used with

$$X_{(n+1)p} \otimes D^{p-1} < P[p-n,\ n]$$

in place of $X_{(n+1)p} < U_n$. Let Y/W be the copy of $X_{(n+1)p} \otimes D^{p-1}$ in P/W given by the isomorphism $\mathrm{P}/W \cong P[p-n,\ n]$. As

$$(Y/W)/\varphi(Y/W) \cong V_{n+1} \otimes D^{p-1}$$

and as U_n has no section isomorphic to $V_{n+1} \otimes D^{p-1}$, we get that $Y \leq U + W$ and so

$$(U \cap Y)/(U \cap W) \cong Y/W \cong X_{(n+1)p} \otimes D^{p-1} \ .$$

We shall need this later, but we first proceed with the main argument without the restriction $n < p - 1$.

At first we only use X to give an estimate:

$$X_{(n+1)p} \cong X/U \cong (X \cap W)/(U \cap W)$$

gives $\dim W \geq p + 1$ and hence, as

$$\dim \mathrm{P}/W = \dim P[p-n,\ n] = \begin{cases} 2p & \text{when } 1 \leq n < p-1 \\ p & \text{when } n = p-1 \end{cases}$$

we have

$$\dim \mathrm{P}[\mathrm{m},\ \mathrm{n}] \geq \begin{cases} 3p+1 & \text{when } 1 \leq n < p-1 \\ 2p+1 & \text{when } n = p-1 \ . \end{cases}$$

We are now ready to apply (2.4):

$$p^4 = \dim FM \geq \sum_{m=1}^{p} \sum_{n=0}^{p-1} m \dim \mathrm{P}[m, n]$$

$$= \dim \mathrm{P}[1, 0] + \sum_{m=2}^{p} m \dim \mathrm{P}[m, 0] + \sum_{n=1}^{p-2} \dim \mathrm{P}[1, n] + \dim \mathrm{P}[1, p-1]$$

$$+ \sum_{m=2}^{p-1} \sum_{\substack{n=1 \\ n \neq p-m}}^{p-1} m \dim \mathrm{P}[m, n] + \sum_{m=2}^{p-1} m \dim \mathrm{P}[m, p-m] + \sum_{n=1}^{p-1} p \dim \mathrm{P}[p, n]$$

$$\geq 1 + \sum_{m=2}^{p} m(p+1) + \sum_{n=1}^{p-2} p + (2p+1)$$

$$+ \sum_{m=2}^{p-1} \sum_{\substack{n=1 \\ n \neq p-m}}^{p-1} m.2p + \sum_{n=2}^{p-1} m(3p+1) + \sum_{n=1}^{p-1} p^2$$

$$= p^4 .$$

This proves that equality must hold throughout and therefore also in all our estimates. In particular, (a), (b) and (c) are proved and towards the proof of (d) we have $W = X \cong X_{(n+1)p}$ and $U \cap W = 0$. This shows that the sum of U and W in P is direct and also that when $n < p - 1$ we have $Y \cap U \cong X_{(n+1)p} \otimes D^{p-1}$. Dimension count now shows that when $n < p - 1$ we must in fact have $Y \cap U = U$. Also, if $n = p - 1$ we just use the Jordan-Hölder Theorem to get $U \cong V_1 \otimes D^{p-1}$. In either case

$$\sigma\mathrm{P} \geq \sigma(U \oplus W) = \sigma U \oplus \sigma W \cong \left(V_{p-n} \otimes D^n\right)^{\oplus 2} .$$

As

$$\sigma\mathrm{P}/\sigma U = \sigma\mathrm{P}/\sigma\mathrm{P}\cap U \cong \sigma\mathrm{P}+U/U \leq \sigma(\mathrm{P}/U) \cong \sigma U_n$$

and σU_n is irreducible, we must have

$$\sigma\mathrm{P} = \sigma U \oplus \sigma W \cong \left(V_{p-n} \otimes D^n\right)^{\oplus 2}$$

as claimed. When $n = p - 1$ we are done, for then

$$\mathrm{P}/\sigma P \cong (\mathrm{P}/U)/(\mathrm{P}/\sigma U) \cong U_{p-1}/\sigma U_{p-1}$$

together with (5.9) gives the rest of our claim. Also from (5.9) when $n < p - 1$ we have two completely reducible modules

$$\varphi\mathrm{P}/(U+\sigma\mathrm{P}) \cong \varphi U_n/\sigma U_n$$

and

$$\varphi\mathrm{P}/W+\sigma\mathrm{P} \cong \varphi P[p-n, n]/\sigma P[p-n, n] ,$$

while of course

$$(U+\sigma\mathrm{P}) \cap (W+\sigma\mathrm{P}) = \sigma\mathrm{P} ,$$

so $\varphi P/\sigma P$ is completely reducible and the Jordan-Hölder Theorem tells us that its summands are as claimed.

It remains to show that $\varphi^2\mathrm{P} = \sigma\mathrm{P}$. We have just seen in effect that $\varphi^2\mathrm{P} \leq \sigma\mathrm{P}$. Now $\varphi^2\mathrm{P} < \sigma\mathrm{P}$ would imply that $\varphi\mathrm{P}/\varphi^2\mathrm{P}$ has a direct summand isomorphic to $V_{p-n} \otimes D^n$. But the kernel of a homomorphism of φP onto $V_{p-n} \otimes D^n$ would have to contain W (as the only singular homomorphic image of W is 0), and $\varphi\mathrm{P}/W$ has no homomorphism onto $V_{p-n} \otimes D^n$ (as by (6.1) the only irreducible homomorphic images of $\varphi P[p-n, n]$ are $V_{n-1} \otimes D$ and $V_{n+1} \otimes D^{p-1}$, the latter only when $n < p - 1$). Thus $\varphi^2 P < \sigma P$ is impossible and the proof is complete. $\square$

NOTE. Using (4.6) and the structure of the above principal indecomposables we note that when $X_m/\varphi X_m \cong V_{n+1}$ then φX_m is either 0 or isomorphic to $V_{p-n} \otimes D^n$. In particular σX_m is always irreducible.

In the group setting we noted that the nonprojective indecomposable direct summands of the V_m repeated in a sequence of period $p(p-1)$. This was perhaps to be expected as G has only a finite number of nonisomorphic indecomposables over F (see [4], 64.1, p. 431). We now use (7.1) to show that by contrast FM is of infinite representation type.

(7.2). *There are indecomposable FM-modules of arbitrary high (finite) dimension.*

Proof. In view of ([5], Theorem 3.1, p. 137) it suffices to produce an indecomposable FM-module P with the following properties:

(i) there exists $W_1 \oplus W_2 \leq \mathrm{P}$ with $W_1 \cong W_2$ and

$W_1\alpha \cap W_2\alpha = 0$ for all endomorphisms α of U ;

(ii) every extension of W_1 by P splits;

(iii) W_1 and W_2 are annihilated by the radical R of the endomorphism ring of U .

We let P be P[p-n, n] for some $n \in \{1, 2, \ldots, p-1\}$. Because P is projective (ii) holds (for any W_1) and from (7.1) we know that there exists $W_1 \oplus W_2 \leq \mathrm{P}$ with $W_1 \cong W_2 \cong V_{p-n} \otimes D^n$. Let α be an endomorphism of P . As $\mathrm{P}/\varphi\mathrm{P}$ is absolutely irreducible, α acts like a scalar, a say, on this quotient (see [4], 29.13, p. 202). Consider the endomorphism $\alpha - a$ which annihilates this top composition factor. We want to see how $\alpha - a$ acts on $\varphi^2\mathrm{P}$ where, by (7.1), $\varphi^2\mathrm{P} = W_1 \oplus W_2$. Using (2.2) we have

$$\begin{aligned}\varphi^2\mathrm{P}(\alpha-a) &\leq \varphi^2\big(\mathrm{P}(\alpha-a)\big)\\ &\leq \varphi^2(\alpha\mathrm{P}) = \varphi^3\mathrm{P} = 0 .\end{aligned}$$

Thus α acts like the scalar a on $W_1 \oplus W_2$ and hence (i) holds. That is, $W_1\alpha \cap W_2\alpha = W_1 \cap W_2 = 0$.

We note also that the radical of an algebra always consists of nilpotent elements. Thus, in particular if α is in the radical of the endomorphism algebra of P then $\mathrm{P}\alpha \leq \mathrm{P}$ and so the scalar a above is 0 . This shows that α annihilates $W_1 \oplus W_2$ which proves (iii) and thus (7.2).□

Having established good reason to be pessimistic we can now go the other way by showing that although FM is of infinite representation type, only finitely many isomorphism types of indecomposables occur as direct summands of the V_m .

(7.3). *In an unrefinable direct decomposition of a* V_m *, precisely one summand is nonsingular. The singular summands regarded as* G*-modules are of course indecomposable. If the nonsingular summand becomes decomposable upon restriction to* G *, then this restriction is the direct sum of at most one nonprojective indecomposable and a projective of dimension at most* $4p$.

Proof. Using (4.1) we define P by

$$V_m|_G = \left[(x^py-xy^p)\overline{\overline{V}}_{m-p-1} \oplus P\right] \oplus \overline{V}_m .$$

By (6.9) we have

$$\begin{aligned}\dim P &= \dim \overline{\overline{V}}_m - \dim \overline{\overline{V}}_{m-p-1} \\ &= m - \dim \overline{V}_m - \left[(m-p-1) - \dim \overline{V}_{m-p-1}\right] \\ &= p + 1 + \left[\dim \overline{V}_{m-p-1} - \dim \overline{V}_m\right] \\ &\leq 2p .\end{aligned}$$

As $(x^py-xy^p)\overline{\overline{V}}_{m-p-1} \leq V_m^*$ we can consider it as an M-module A where $A = \bigoplus_i A_i$ with each $A_i|_G$ a principal indecomposable G-module. Set

$B_j = \bigoplus_{i\neq j} A_i$ and note that there exists a B_j such that $X_m \cap B_j = 0$. If this were not the case, as we have already observed (just after (7.1)) that σX_m is irreducible, then for all j , $\sigma X_m \leq B_j$. This would contradict the fact that $\bigcap_j B_j = 0$. We now use (2.12) to show that there exists an X such that $X_n \leq X$ and $V_m|_G = B_j|_G \oplus X$. By (4.4), X admits M and so we can write $V_m = B_j \oplus X$. But writing

$$V_m|_G = B_j|_G \oplus A_j|_G \oplus P \oplus \overline{V}_m$$

it is clear that $X_G \cong \overline{V}_m \oplus R$ with $\dim R \leq 4p$ and $B_j = \bigoplus_{i \neq j} A_i$ with the $A_i|_G$ principal indecomposables. $\square$

We noted in (5.7) that for $m = 2p - 1$, V_m is indecomposable while $V_m|_G$ is not. Therefore, even if the V_m were to fit into some sort of arithmetic pattern as shown in (6.7), for their restrictions to G, this pattern would differ from that of the $V_m|_G$. We complete this chapter by showing that no such arithmetic pattern exists.

One indication of this can be found by observing that V_2 is not a direct summand of any other V_m since, by (4.1), no other V_m has a singular submodule of codimension 2 and, by (4.5), a direct complement of V_2 would have to be singular. Moreover, we prove:

(7.4). V_m *has a submodule isomorphic to* V_2 *if and only if* $m = p^l + 1$ *for some* l.

Proof. If $m = p^l + 1$ then

$$(ax+by)^{p^l} = (ax)^{p^l} + (by)^{p^l} \in \langle x^{p^l}, y^{p^l} \rangle$$

and using (4.4) we see that $\langle x^{p^l}, y^{p^l} \rangle$ is a submodule of V_m. This submodule is clearly isomorphic to V_2.

On the other hand suppose V_m has a submodule U isomorphic to V_2. We have noted (just after the proof of (4.4)) that V_m has a unique non-singular composition factor so $X_m \cong V_2$. It remains to show that X_m has dimension at least 3 unless $m = p^l + 1$ for some l.

Suppose $m - 1 = p^l s$ where $s > 1$ and $p \nmid s$, and note

$$(x+y)^{p^l s} = \left((x+y)^{p^l}\right)^s = \left(x^{p^l}+y^{p^l}\right)^s$$

$$= x^{p^l s} + y^{p^l s} + s x^{p^l(s-1)} y^{p^l} + \ldots \notin \langle x^{p^l s}, y^{p^l s} \rangle .$$

Thus in this case the three elements x^{m-1}, y^{m-1}, $(x+y)^{m-1}$ of X_m are linearly independent. $\square$

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