A STUDY OF THE SIMPLE BRANCHING PROCESS WITH INFINITE MEAN.

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A dissertation submitted for the degree of Master's of Science at the Australian National University, Canberra, A.C.T..

Signature I. L. Hudson

Date 16th March 1977.

*Work done partly at Cambridge University, England.

Preface:

I hereby declare that my dissertation entitled "A Study of the Simple Branching Process with Infinite Mean " is not substantially the same as any that I have submitted for a degree, diploma or other qualification at any other university.

Signed T. L. Hudson.

Date 16/3/77.

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INTRODUCTION AND SUMMARY

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We consider the Bienaymé-Galton Watson process with offspring distribution having infinite mean; and foremost we generalize theorem 1 and 2 [17] and theorem 3 [18] for the process with and without immigration. We further strengthen the convergence assertions of these theorems to that of almost sure convergence.

We assume the reader will have some familiarity with the theory of branching processes [1], where we denote the process $\{Z_n\}_{n\geq 1}$ $(Z_o = 1)$ and in particular we suppose $F'(1-) = \infty$, where $F(s)$ $s \in [0, 1]$, is the probability generating function (p.g.f) of its offspring distribution. As usual q denotes the extinction probability of the process and we assume $q < 1$.

When immigration is permitted in each generation after the initial, it is according to a p.g.f. $B(s) = \sum_{j=o}^{\infty} b_j s^j$, $B(o) \neq 1$ and we denote this process by $\{X_n\}$ and suppose for simplicity that $X_o = 0$.

Defining

$$\emptyset(t) = U\ \{(1 - q)/(1 - F\ (1 - [{}^{(1-q)}/W(t)]\ ))\}\ ;\quad t > 0$$

we assume that $\emptyset(t)$ is either convex or concave on $[0, \infty)$ and $c = \lim\ (t \to \infty)\ \emptyset(t)/t$ satisfies $0 < c < 1$. These conditions ensure that there exists a sequence of positive constants $\{\beta_n\}$ such that $\{\beta_n\ U(Z_n + 1)\}$ converges almost surely to a proper nondegenerate random variable, where U is a function slowly varying at infinity, defined on $[1, \infty)$, continuous and strictly increasing with $U(1) = 0$, $U(\infty) = \infty$.

Denote by $W(t)$ its inverse function defined on $[0, \infty)$ with $W(\infty) = \infty$.

These results reckon among earlier ones in [17], [18] with $U(t) = \log t$ and can be considered as a partial study of the infinite mean Galton Watson process. More specifically for $c_{(o)} = {}^1/_{F'}(1-)$, $(= 0)$, say, then for $U(t) = \log t$ and writing for c as in [17], $c = c_{(1)}$ we find that the above results may be considered to be in part, a study of the case when $c_{(o)} = 0$. Analogously, a similar study of the case $c_{(1)} = 0$ can be attempted by taking $U(t) = \log \{1 + \log t\}$ and so in effect a hierarchy of results emerges, where $U(t)$ takes the form of an iterated logarithm.

In addition we show that there is a natural relationship between our general approach which appeals to functional equations and the theory of slowly varying functions. Indeed we can take $\beta_n = c^n/L(c^{-n})$ where $L(t)$ is a monotone function, slowly varying at infinity. When we pose questions, however, as in [18] related to the limit distribution function our general theory presents some difficulties; but it bears mentioning that from the viewpoint of generality our theory in terms of a general U is to be preferred.

Also we make comment and modify a method of D.A. Darling[4] to interpret results pertaining to the specific case when $U(t) = \log t$. This approach which uses the continuity theorem of Laplace transforms however proved laborious and often unfruitful for $U(t) = \log \{1 + \log t\}$ or logarithmic iterates of this.

We can again prove the convergence in law of $\beta_n U(Z_n + 1)$

to a proper distribution for the particular case when $U(t) = \log t$ by working along similar lines to a paper on the Böttcher functional equation by M Kuczma [11]. But when we extend these ideas to further logarithmic iterated versions of log t (in the sense described earlier) further difficulties emerge. Our findings then lead us to a generalization of Kuczma's theorem to a more general functional equation in terms of a general function.

As stated earlier our assumptions ensure that a sequence $\{\beta_n\}$ as before exists such that for U as before, $\{\beta_n U(Z_n + 1)\}$ converges almost surely to a proper non-degenerate random variable. In fact the almost sure convergence follows by a direct application of a theorem by H. Cohn, once convergence in distribution is established.[2] We shall demonstrate how Cohn's theorem applies in our development.

For the super-critical process the criterion of almost sure convergence was proved by C.C. Heyde [7] via a martingale argument. Hence it is logical to question, as we do, whether a bounded positive martingale of the form $(1 - {}^{(1-q)}/W(1/\beta_n))^{Z_n}$ provides the required convergence after the application of the martingale convergence theorem. (We were motivated to consider such a martingale by a remark in [12]). We shall demonstrate that the martingale does not provide the required almost sure convergence to a proper limit.

Then turning to the Bienaymé-Galton Watson process allowing immigration, $\{X_n\}$ as before, we find that together with the same conditions which ensure the existence of $\{\beta_n\}$ such that $\{\beta_n U(Z_n + 1)\}$ converges in law to a proper distribution, for $\{Z_n\}$ the process without immigration, we require a further condition that $\sum_{j=2}^{\infty} b_j \log U(j) < \infty$ to ensure that $\{\beta_n\ U(X_n + 1\ )\}$ is convergent as above. We denote the limit distribution function by p(t), say, which is positive, finite, continuous and strictly increasing on $[0, \infty)$ if the latter condition holds, otherwise $p(t) \equiv 0$ and $\{\beta_n\ U(X_n + 1)\}$ approaches infinity in distribution and hence in probability.

NOTE: A paper entitled "A Note on Simple Branching Processes with Infinite Mean", Irene L. Hudson and E. Seneta is to appear which summarizes the main scope and conclusions of this dissertation. (J. Appl. Prob. (Dec). 1977).

CHAPTER I

A GENERALIZED APPROACH

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Section 1, Introduction and Preliminaries

The topic under consideration is the Bienaymé-Galton Watson process $\{Z_n\}_{n=0}^{\infty}$ $(Z_0 = 1)$ with an infinite mean [1]. More explicitly the process is such that $F'(1-) = \infty$ where $F(s)$, $s \in [0, 1]$ is the probability generating function (p.g.f) of its offspring distribution. By convention q denotes the extinction probability of the process and we assume $q < 1$.

In Seneta [17] it was shown that under certain conditions there exists a sequence $\{\rho_n\}$ of positive constants such that $\{\rho_n \log (Z_n + 1)\}$ (*) converges in law to a proper limit distribution, whose distribution function is continuous and strictly increasing on $(0, \infty)$ with a jump of magnitude q at the origin.

The theory employed a function (and its iterates) defined by

$$\emptyset (t) = - \log \left[\frac{1 - F(1 - (1 - q)e^{-t})}{(1 - q)} \right] , \quad t \geq 0$$

and

$$\emptyset_n(t) = - \log \left[\frac{1 - F_n(1 - (1 - q)e^{-t})}{(1 - q)} \right] \quad \text{for each } n \geq 1, \ (t \geq 0).$$

Under the assumption of convexity or concavity of $\emptyset(t)$ on $[0, \infty)$ it follows that the limit as $t \to \infty$ of $\emptyset(t)/t$ exists, say $c_{(1)}$ such that $0 \leq c_{(1)} \leq 1$. By restricting $c_{(1)}$ such that $0 < c_{(1)} < 1$, the above sequence of constants $\{\rho_n\}$ was shown to exist such that the earlier convergence result (*) holds true. In addition a necessary and sufficient condition for $\rho_n \sim$ constant . c^n $(n \to \infty)$ was produced.

We note that all generating functions such that $F'(1-) < \infty$ give $\emptyset(t)/t \to 1$ as $t \to \infty$ but also, as pointed out in [17] there exist "pathological" F(s) with infinite mean and c = 1. The latter case is one to which the functional-iterate development of [17] is not applicable.

In this chapter we formulate results which essentially generalize the theorems of [17] and are written in terms of slowly varying functions. That is, generalized conditions analogous to those of [17] are given which ensure that there exists a sequence $\{\beta_n\}$, say of positive constants such that $\{\beta_n U(Z_n + 1)\}$ converges in distribution to a proper non-degenerate limit distribution; whose distribution function has properties as in [17]. Our function U is slowly varying at infinity (in the sense described in Seneta [19]), defined on $[1, \infty)$, continuous and strictly increasing from $U(1) = 0$ to $U(\infty) = \infty$. Further under our generalized conditions a necessary and sufficient condition for $\beta_n \sim$ constant . c^n as $n \to \infty$ is given and it is a generalization of the analogous condition of Theorem 2 [17].

We will define functions of which our previous $\emptyset(t)$ [17] is a special case and such that when they satisfy certain conditions a probabilistic interpretation analogous to (2.4) of [17] is derivable.

We repeat that our $U(t)$ be defined, continuous and strictly increasing on $[1, \infty)$ such that $U(1) = 0$, $U(\infty) = \infty$

and suppose U is slowly varying at infinity [19]. Denote by W(t) its inverse function which is defined on $[0, \infty)$ with $W(o) = 1$, $W(\infty) = \infty$.

We now define

$$\phi(t) = U\left\{\frac{(1-q)}{1 - F\,(1 - (1-q)/W(t))}\right\} \qquad t \geqslant 0,$$

and for each $n \geqslant 1$

$$\phi_n(t) = U\left\{\frac{(1-q)}{1 - F_n\,(1 - (1-q)/W(t))}\right\} \qquad t \geqslant 0,$$

where $F_n(s)$ denotes the n^{th} functional iterate of F(s). It can be easily checked that ϕ_n is the n^{th} functional iterate of ϕ and that $\phi(0) = \phi_n(0) = 0$ for each n.

Our primary assumption is

(A) $\phi(t)$ is convex or concave on $[0, \infty)$.

Define the limit $c = \lim_{t\to\infty} \phi(t)/_t$.

It follows from the nature of F, U and (A) that the limit c exists and always satisfies $0 \leqslant c \leqslant 1$. Further note that if $F'(1-) < \infty$ then $c = 1$. We are now in a position to make our next assumption that is,

(B) $0 < c < 1$.

We then have the following two theorems.

<u>THEOREM 1.1</u>

Under conditions (A) and (B) there exists a sequence of positive constants $\{\beta_n\}$, $\beta_n \to 0$ as $n \to \infty$, such that

$\{\beta_n U(Z_n + 1)\}$ converges in law to a proper limit distribution, with distribution function defined by

$$u(t) = 1 - (1 - q)/ W(\Delta(t)), \quad t > 0,$$

where $\Delta(t)$ is continuous and strictly increasing from $\Delta(0+) = 0$ to ∞ and is convex or concave on $(0, \infty)$ as is $\emptyset(t)$.

Remark:

(1) These constants are essentially unique by Khintchine's theorem for convergence of (positive) types. The proof of theorem 1.1 gives an explicit form for them. Further properties of $u(t)$ will be implicit from the functional equation approach in the proof and these will be mentioned later.

(2) In the particular case when $U(t) = \log t$, Theorem 1.1 has been proved in [17], [18].

THEOREM 1.2

Under conditions (A) and (B) the norming sequence $\{\beta_n\}$ of theorem 1.1 satisfies $\beta_n \sim$ constant $. c^n$ as $n \to \infty$ if and only if,

$$\int_A^\infty t^{-1} \left| c - \frac{\emptyset(t)}{t} \right| dt < \infty ,$$

where A is any fixed number > 0.

Section 2, Proofs of Theorems 1.1 and 1.2

(a) Functional-iterate approach

Suppose (A) and (B) hold. The definition of the probability of extinction q, such that $F(q) = q$ gives us that

$F_n(q) = q$ for all $n \geqslant 1$.

Define the inverse function of $\phi(t)$, $t \geqslant 0$, by $d(.)$ where

$$d(t) = U\left[\frac{1-q}{G\left(\frac{1-q}{W(t)}\right)}\right] \quad t \geqslant 0.$$

and G is the inverse function to $1 - F(1 - x)$; $0 \leqslant x \leqslant 1$. Similarly for all $n \geqslant 1$

$$d_n(t) = U\left[\frac{1-q}{G_n\left(\frac{1-q}{W(t)}\right)}\right] \quad t \geqslant 0$$

denotes the inverse function to $\phi_n(t)$ $t \geqslant 0$ and clearly is the n^{th} functional iterate of d where $G_n(x)$ is the inverse function to $1 - F_n(1 - x)$, $0 \leqslant x \leqslant 1$. The function ϕ satisfies $\phi(t) < t$, $0 < t < \infty$ because $F(x) < x$ for $q < x < 1$ and hence by inverses we have that

(1.1) $\qquad t < d(t)\ , \qquad t > 0.$

We note that $\phi(o) = \phi_n(o) = 0$ and we follow precisely as in [17] and define

$$\tilde{\phi}(x) = \frac{1}{\phi(1/x)}\ ; \qquad 0 \leqslant x < \infty$$

$$\tilde{d}(x) = \frac{1}{d(1/x)}\ ; \qquad 0 \leqslant x < \infty$$

and similarly we have $\tilde{\phi}(0) = \tilde{d}(o) = 0$. The corresponding n^{th} functional iterates are indicated by a subscript n, that is

$$\tilde{\phi}_n(x) = \frac{1}{\phi_n(1/x)}\ ; \quad 0 \leqslant x < \infty$$

$$\tilde{d}_n(x) = \frac{1}{d_n(1/x)}; \qquad 0 \leqslant x < \infty$$

and it is easily checked that they are an inverse pair for each n and $\tilde{\phi}_n(o) = \tilde{d}_n(o) = 0$.

We can now prove a lemma analogous to lemma 2.1 [17].

<u>LEMMA 2.1</u>

For $x \in [0,\infty)$ the sequence $\dfrac{\tilde{d}_n(x)}{\tilde{d}_n(x_o)}$, $x_o \in (0,\infty)$ arbitrary but fixed, approaches the function $\tilde{D}(x)$ as $n \to \infty$, where the limit function $\tilde{D}(x)$ is positive on $(0, \infty)$ and is a solution to the Schröder functional equation

$$(1.2) \qquad \tilde{D}(\tilde{d}(x)) = c\,\tilde{D}(x),$$

where $\tilde{D}(x)$ is the unique solution (up to constant factors) of (1.2) such that $\tilde{D}(x)/x$ is monotonic in $(0, \infty)$.

<u>Proof</u>: As in the proof of lemma 2.1 [17] it is only necessary to check that the conditions of Kuczma [10] are satisfied. By definition $\phi(t)$ is continuous and strictly increasing to infinity on $[0, \infty)$, this follows from the nature of U. We then conclude by the properties of inverse functions that $d(t)$ is continuous and strictly increasing to infinity on $(0, \infty)$. Hence our definition of $\tilde{d}(x)$ and (1.1) yield that $\tilde{d}(x)$ is such that $\tilde{d}(o) = 0$ and $0 < \tilde{d}(x) < x$ for $0 < x < \infty$. We write

$$(1.3) \qquad \lim_{x\to 0+} \frac{\tilde{d}(x)}{x} = \lim_{t\to\infty} \frac{\phi(d(t))}{d(t)}, \quad \text{putting } t = {}^1/_x$$

and noting that $d(t) \uparrow \infty$ as $t \to \infty$ and that $\phi(t)/t \to c$ as $t \to \infty$ we have

$$\lim_{x \to 0+} \frac{\tilde{d}(x)}{x} \text{ exists and } = c \quad \text{and then by}$$

(B) we have $0 < c < 1$.

The remaining condition to check is whether $\frac{\tilde{d}(x)}{x}$ is monotonic on $(0, \infty)$. However this monotonicity holds since $\tilde{d}(x)$ is convex or concave on $(0, \infty)$ as $\emptyset(t)$, and $\tilde{d}(0) = 0$. Thus all the conditions of Kuczma's theorem [10] are satisfied and hence the lemma is proved.

We denote

$$(1.4) \qquad \frac{\tilde{d}_n(x)}{\tilde{d}_n(x_o)} \equiv \tilde{D}_n(x), \text{ say}$$

where $\tilde{D}_n(x) \to \tilde{D}(x)$ as $n \to \infty$; $x > 0$ and $x_o \in (0, \infty)$ is arbitrary but fixed and $\tilde{D}(x)$ satisfies the Schröder functional equation as stated in the above lemma.

Exactly as in [17] we require to prove the next lemma 2.2 and then we note the application of lemma 4.4 [16] on inversion and limits on (1.4) gives us that as $n \to \infty$

$$(1.5) \qquad E\left[\left(1 - \frac{(1-q)}{W(t/\beta_n)}\right)^{Z_n}\right] \to u(t), \quad t > 0$$

with $\{\beta_n\}$ and $u(t)$ as specified in theorem 1.1.

<u>LEMMA 2.2</u>

$\tilde{D}(x)$ is continuous and strictly monotone increasing on $[0, \infty)$ such that $\tilde{D}(x) \to \infty$ as $x \to \infty$.

<u>Proof</u>: Rewrite (1.4) as

$$\frac{\tilde{d}_n(x)}{\tilde{d}_n(x_o)} = \frac{d_n(t_o)}{d_n(t)} \underset{n \to \infty}{\to} \frac{1}{D(t)} = \tilde{D}(1/t),$$

$0 < t < \infty$ where we put $t_o = 1/x_o$ and $t = 1/x$ and denote $\tilde{D}(x) = 1/D(1/x)$, $0 \leqslant x < \infty$.

By definition $d_n(t)$ is convex or concave on $(0, \infty)$ as $d(t)$. Then $D(t)$, being the limit of convex or concave functions is either convex or concave respectively on $(0, \infty)$, and hence continuous thereon. Thus $\tilde{D}(x)$ is concave or convex on $(0, \infty)$ and continuous on the same interval.

As $\tilde{d}_n(x)$ is decreasing for each n we have that $\lim_{x \to 0+} \tilde{D}(x) = \tilde{D}(0+)$ exists. Letting $x \to 0 +$ in (1.2) and noting that $\tilde{d}(0) = 0$ yields $\tilde{D}(0 +) = c\,\tilde{D}(0 +)$. Thus $\tilde{D}(0 +) = \tilde{D}(0) = 0$ holds by the nondecreasing nature and positivity of $\tilde{D}(x)$ on $(0, \infty)$.

We have now demonstrated that $\tilde{D}(x)$ is continuous on $[0, \infty)$ and will prove by contradiction that $\tilde{D}(x)$ must be strictly monotone increasing on this interval.

Assume to the contrary that $\tilde{D}(x) = k$ = constant for $x \in [\alpha, \beta] \subset (0, \infty)$ where $\beta > \alpha$. From (1.2) $\tilde{D}(x) = k/c$ for $x \in [\tilde{d}(\alpha), \tilde{d}(\beta)]$ and we now have the impossible situation where a convex or concave function takes constant values on two line segments, these values being unequal. Hence we conclude that $\tilde{D}(x)$ is strictly monotone increasing on $[0, \infty)$.

Recall that $\tilde{d}_n(0) = 0$ for each $n \geqslant 1$ and that (1.4) gives $\tilde{D}_n(0) = 0$ for each n. Similarly since $d_n(0) = 0$ for each $n \geqslant 1$ we have that for every $n \geqslant 1$ $\tilde{D}_n(x) \to \infty$ as $x \to \infty$.

Letting $x \to \infty$ in (1.3) and noting that $d(t) \to 0$ as $t \to 0 +$ we have that $\tilde{D}(\infty) = c\ \tilde{D}(\infty)$ and $\tilde{D}(x) \to \infty$ as $x \to \infty$; the existence of the latter limit is a consequence of the positivity and non-decreasing nature of $\tilde{D}$. This concludes the proof of the Lemma.

Proof of Theorem 1.1:

We shall show precisely as in [17] that the convergence result (1.5) holds with $\{\beta_n\}$ and $u(t)$ having the structure specified in the statement of the theorem.

Summarizing the results of lemma 2.2, we have the sequence $\{\tilde{D}_n(x)\}_{n\geqslant 0}$ such that for each $n \geqslant 1$, $\tilde{D}_n(x)$ is defined, continuous and strictly monotone increasing on $[0, \infty)$ with $\tilde{D}_n(0) = 0$ and $\tilde{D}_n(\infty) = \infty$.

By the convergence $\tilde{D}_n(x) \underset{n\to\infty}{\to} \tilde{D}(x)$ $0 < x < \infty$ and the behaviour of $\tilde{D}$ (x) as stated in lemma 1.1 and 1.2 we have the conditions for the dual version of lemma 4.4 [16] satisfied; with $a_n \equiv a = \infty$. The inverse functions $\tilde{\phi}_n(\tilde{d}_n(x_o)x)$, of $\tilde{d}_n(x)/\tilde{d}_n(x_o)$ are well defined for all $0 \leqslant x < \infty$ and hence from (1.4) and the application of the dual version of lemma 4.4 it follows that

$$\tilde{\phi}_n(\tilde{d}_n(x_o).x) \underset{n\to\infty}{\to} \tilde{D}^{-1}(x) \;; \qquad 0 \leqslant x < \infty.$$

Here $\tilde{D}^{-1}(x)$ is continuous, strictly increasing from zero to infinity on $[0, \infty)$. Using the definitions of $\tilde{\phi}$, $\tilde{d}$, and $\tilde{D}(x) = 1/D(1/x)$, (substitute $t = 1/x$, $t_o = 1/x_o$) the latter convergence result can be rewritten in terms of ϕ and D^{-1}, viz.,

$$\phi_n(t.\ d_n(t_o)) \underset{n\to\infty}{\to} D^{-1}(t) = \Delta(t), \text{ say,}$$

$0 < t < \infty$, where $\Delta(t)$ is defined, continuous and strictly increasing on $(0, \infty)$ with $\Delta(0+) = 0$, $\Delta(\infty) = \infty$. Write $\beta_n = \tilde{d}_n(x_o) = 1/d_n(t_o)$ then by the definition of $\phi(t)$,

$$(1.6) \qquad F_n(1 - [(1-q)/W(t/\beta_n)]) \underset{n\to\infty}{\to} 1 - [(1-q)/W(\Delta(t))]$$

$$= u(t), \text{ say} \quad 0 < t < \infty .$$

Rewriting (1.6) we have clearly (1.5) where $\beta_n \downarrow 0$ as $n \to \infty$ and $u(t)$ is defined, strictly increasing and continuous on $(0, \infty)$ with $u(0+) = q$ and $u(\infty) = 1$.

We will show that (1.5) is tantamount to

$$(1.7) \qquad P(\beta_n U(Z_n + 1) \leq t) \underset{n\to\infty}{\to} u(t), \qquad t > 0.$$

Hence convergence in law in theorem 1.1 is then established and it bears mentioning that a direct application of theorem 2 of H. Cohn [2] yields almost sure convergence of $\{\beta_n U(Z_n + 1)\}$ to a nondegenerate limit random variable. This will be discussed later in Chapter IV.

Section 2 (b) Proof of Weak Convergence

With the assistance of the subsequent lemma we will show that (1.7) can be assigned to be the probabilistic interpretation of (1.5). We are indebted to G.E.H. Reuter for the special case of the lemma when $k = 1$, $U(t) = \log t$.*

LEMMA 1.3

Let $\{Z_n\}_{n\geq 0}$ be any sequence of non negative random variables and $\{\beta_n\}$ any sequence of positive constants such that

* The approach suggested by G.E.H. Reuter provided us with an alternative method by which to derive (2.4) from (2.3) [17] to that of D.A. Darling [4]. The development of Darling utilizes Laplace transforms and a gamma function approximation. A parallel approach to Darling [4] proved unsuccessful for the case when U is general and has properties as stated in Theorem 1.1.

$\beta_n \to 0$ as $n \to \infty$ and

$$(1.8) \qquad E\left\{\left(1-\frac{k}{W(t/\beta_n)}\right)^{Z_n}\right\} \underset{n\to\infty}{\to} u(t); \quad t > 0$$

where k is a positive constant, $0 < k \leq 1$, W is as previously specified and u(t) is continuous and nondecreasing on $(0, \infty)$. Then

$$(1.9) \qquad P(\beta_n \, U(Z_n + 1) \leq t) \underset{n\to\infty}{\to} u(t); \qquad t > 0 .$$

<u>Proof:</u> Let us denote by T_n; $T_n = Z_n + 1$. Then (1.8) remains true when we substitute T_n for Z_n by the nature of W and β_n. Let us initially prove the lemma for k = 1. Allow $t > 0$ be fixed, but arbitrary and define

$$A_n = \{\beta_n \, U(T_n) \leq t\} = \{T_n \leq W(t/\beta_n)\}$$

for each $n \geq 1$, the last equality is ensured by the existence of W the strictly increasing and continuous inverse function of U. We choose t', t" such that $0 < t' < t < t"$ and write

$$Y_n{}' = [1 - \alpha_n{}']^{T_n} ; \qquad \alpha_n = 1/W(t'/\beta_n)$$

and $\alpha_n{}''$, $Y_n{}''$ are defined similarly in terms of t". We note that $\alpha_n{}'$, $\alpha_n{}''$ are small for n large enough since $\beta_n \to 0$ as $n \to \infty$ and note it is true that for $0 < \alpha < 3/5$

$$(1.10) \qquad -\alpha(1+\alpha) < \log(1-\alpha) < -\alpha .$$

We now derive estimates for $P(A_n)$. Let P denote the appropriate probability measure.

(i) $$E(Y_n') = \int_{A_n} Y_n' \, dP + \int_{A_n^c} Y_n' \, dP$$

$$\leqslant P(A_n) + \sup (Y_n' \text{ on } A_n^c)$$

On A_n^c, $T_n > W(t/\beta_n)$, so by the right hand side of inequality (1.10)

$$Y_n' = (1 - \alpha_n')^{T_n} = \exp (T_n \log (1 - \alpha_n')) < \exp (-\alpha_n' T_n) .$$

In addition

$$\alpha_n' T_n > W(t/\beta_n)/W(t'/\beta_n) = \lambda_n, \text{ say}$$

where $\lambda_n \to \infty$ as $n \to \infty$ since $t > t'$, $\beta_n \downarrow 0$ and $W(t\mu)/W(t) \to \infty$ as $t \to \infty$ for each $\mu > 1$. [For the latter result see [19], here this result is incorrectly labelled as theorem 1.9 instead of theorem 1.11.] Thus on A_n^c we have $Y_n' < e^{-\lambda_n} = \xi_n' \to 0$ as $n \to \infty$. Hence taking the limit as $n \to \infty$ in $P(A_n) \geqslant E\, Y_n' - \xi_n'$ gives

$$\liminf P(A_n) \geqslant \lim E(Y_n') = u(t') .$$

(ii) In a similar manner,

$$E(Y_n'') \geqslant \int_{A_n} Y_n'' \, dP \geqslant P(A_n) \,.\, \inf \{Y_n'' \text{ on } A_n\} .$$

By (1.10) it follows that for n sufficiently large,

$$Y_n'' = (1 - \alpha_n'')^{T_n} = \exp (T_n \log (1 - \alpha_n'')) \geqslant \exp(-\alpha_n''(1+\alpha_n'')T_n).$$

However, on A_n, $T_n \leqslant W(t/\beta_n)$ so

$$(1 + \alpha_n'') \alpha_n'' T_n \leqslant (1 + \alpha_n'') \left[\frac{W(t/\beta_n)}{W(t''/\beta_n)} \right] = \xi_n'', \text{ say}$$

where $\xi_n'' \to 0$ as $n \to \infty$ since $t'' > t$, $\beta_n \downarrow 0$ and from [19] p. 41, $W(t\gamma)/W(t) \to 0$ as $t \to \infty$ for each γ, $0 < \gamma < 1$.

Thus on A_n, $\inf Y_n''$ is $\geqslant e^{-\xi_n''}$. Subsequently, if we let $n \to \infty$ in $P(A_n) \leqslant e^{\xi_n''} E(Y_n'')$ we obtain

$$\limsup P(A_n) \leqslant \lim_{n\to\infty} E(Y_n'') = u(t'') \text{ .}$$

Combining the results of (i) and (ii) and allowing $t' \uparrow t$ and $t'' \downarrow t$ it then follows by the continuity of $u(.)$ that $\lim P(A_n)$ exists and equals $u(t)$.

Now assume (1.8) for the particular case when k is some positive constant such that $0 < k < 1$. Then it is straightforward that

$$u(t) \geqslant \limsup \; E\left[\left\{1 - \frac{1}{W(t/\beta_n)}\right\}^{Z_n}\right].$$

If we take $0 < t' < t$, then for n large enough,

$$k \geqslant W(t'/\beta_n)/W(t/\beta_n);$$

in view of the property of W stated earlier and that $\beta_n \downarrow 0$. So

$$(1 - k/W(t'/\beta_n))^{Z_n} \leqslant (1 - 1/W(t/\beta_n))^{Z_n}$$

and taking the limit of expectations yields

$$u(t') \leqslant \liminf \; E\left[(1 - \frac{1}{W(t/\beta_n)})^{Z_n}\right] .$$

Then using the continuity of $u(.)$ we have the convergence result (1.8) satisfied when $k \leq 1$; wherefore (1.9) follows.

This completes the proof of theorem 1.1.

<u>Remark:</u>

Later we will demonstrate that the above argument is reversible, that is ultimately (1.7) implies (1.5) and we shall discuss its significance in our development.

Proof of theorem 1.2:

Khintchine's convergence of types theorem establishes the essential uniqueness of normalizing constants for convergence to a nondegenerate limit distribution. Then it follows that $\beta_n \sim$ constant . $\tilde{d}_n(x_o)$ $(n \to \infty)$, so to derive a necessary and sufficient condition for $\beta_n \sim$ constant. c^n we need only consider the asymptotic behaviour of $\tilde{d}_n(x_o)$. Such iterated functions and their corresponding behaviour as $n \to \infty$ is dealt with in [15]. The results of [15] give us that a necessary and sufficient condition for $\tilde{d}_n(x_o) \sim$ constant . c^n as $n \to \infty$ is that

$$(1.11) \qquad \int_o^\delta \frac{|\tilde{d}(x) - c\,x|}{x^2}\, dx < \infty,$$

where $\delta \in (0, \infty)$ is an arbitrary but fixed number.

We can transform the above condition into one in terms of $\emptyset(t)$. Let $x = \tilde{\emptyset}(y)$ and recall that if $y = 1/t$ then $\tilde{\emptyset}(y) \sim y/c$ as $y \to 0+$. The method is outlined below.

Outline of method By definition $\tilde{\emptyset}(.) = \tilde{d}^{-1}(.)$, so $\tilde{d}(x) = y$. Rewriting (1.11) in terms of y yields

$$\int_o^\delta \frac{|y - c\,\tilde{\emptyset}(y)|}{(\tilde{\emptyset}(y))^2}\, d\,\tilde{\emptyset}(y) < \infty.$$

The integrand is

$$\left| \frac{y}{\tilde{\phi}(y)^2} - c\,\tilde{\phi}(y) \right| \quad \text{with respect to } d\,\tilde{\phi}(y).$$

By definition $\tilde{\phi}(y) = 1/\phi(t)$, $t \in (A, \infty)$, $A = 1/_\delta$, and

$$\left| \frac{d\,\tilde{\phi}(y)}{dy} \right| \sim \left| c/\phi(t)^2 \right|$$

since $\tilde{\phi}(y) \sim y/c$ $(y \to 0+)$ implies $\phi(t) \sim t\,c$ $(t \to \infty)$.
Hence the integrand is $\sim |y - c/\phi(t)|\, c\, dt$
and can be rewritten as $|(c - \phi(t)/t)/t|\, dt$. We can now
write (1.11) equivalently as $\int_A^\infty \frac{|c - \phi(t)/t|}{t}\, dt < \infty$,
hence completing the proof.

Section 3; The Nature of our Generalization

- relevance of the constant c.

In the special case that $U(t) = \log t$ theorem 1.1 has been proved in [17], [18]. For this specific case we write $c = c_{(1)}$ and note that if $c_{(1)} = 1$ then all p.g.f. such that $F'(1-) < \infty$ give $c_{(1)} = 1$. It was noted in [17] that there exist "pathological" p.g.f. such that $F'(1-) = \infty$ yet $c_{(1)} = 1$. It bears mentioning that the functional iterate development in [17] and our generalized theory do not apply to this specific situation.

We can now demonstrate the nature of our present generalization. If we write $c_{(o)} = 1/F'(1-)$ then theorem 1.1 with $U(t) = \log t$, $c = c_{(1)}$ may be regarded as a (partial) study of the case $c_{(o)} = 0$. Then it is reasonable to consider whether an analogous study of the case $c_{(1)} = 0$ can be achieved. This is (partly) accomplished by considering the specific case when $U(t) = \log(1 + \log t)$ and writing here $c = c_{(2)}$. We note that the assumption $0 < c_{(2)} < 1$

indeed implies that $c_{(1)} = 0$. Thus we can visualize a hierarchy or "spectrum" of results developing if we continue in this fashion. In this hierarchical classification, $U(t)$ always takes the form of an iterated logarithm. We will show that such iterates do provide us with a tool to study the infinite mean branching process (in part). We introduce the notion of orders of the infinite mean Galton Watson process, where the first order is regarded as the case when $c_{(o)} = 0$, then in view of our argument the next logical order is when $c_{(1)} = 0$ and so we can continue in this manner to build up a whole succession of results.

To emphasize the significance of our hierarchy we shall generalize further and define the limit $c_{(i)}$ for some $i = 1, 2, \ldots,$ where $c_{(i)}$ exists and is given by $c_{(i)} = \lim (t \to \infty)\ \phi_{(i)}(t)/t$ where

$$\phi_{(i)}(t) = U_{(i)}\left[\frac{1 - q}{1 - F(1 - \frac{(1 - q)}{W_{(i)}(t)})}\right], \quad 0 \leqslant t < \infty .$$

Note also that for $n \geqslant 1$,

$$\phi_{(i),n}(t) = U_{(i)}\left[\frac{1 - q}{1 - F_n(1 - \frac{(1 - q)}{W_{(i)}(t)})}\right]$$

is the n^{th} functional iterate of $\phi_{(i)}(t)$. Then under an assumption analogous to (A), we suppose that for this specific i, $0 \lesssim c_{(i)} \leqslant 1$ and $\phi_{(i)}(t)$ is either convex or concave on $[0, \infty)$. As before, $U_{(i)}$, $W_{(i)}$ is an inverse pair, where $U_{(i)}$ is slowly varying at infinity, defined on $[1, \infty)$; continuous and strictly increasing with $U_{(i)}(1) = 0$, $U_{(i)}(\infty) = \infty$.

Next we define

$$\phi_{(i+1)}(t) = U_{(i+1)} \left[\frac{(1-q)}{1 - F\left(1 - \frac{(1-q)}{W_{(i+1)}(t)}\right)} \right] \quad 0 \leqslant t < \infty,$$

where

$$U_{(i+1)}(t) = \log \left(U_{(i)}(t) + 1\right), \quad 1 \leqslant t < \infty$$

and hence

$$W_{(i+1)}(t) = \exp \left(W_{(i)}(t) - 1\right), \quad 0 \leqslant t < \infty.$$

Clearly the above is an inverse pair where $U_{(i+1)}$ is slowly varying at infinity, continuous, strictly increasing on $[1, \infty)$ such that $U_{(i+1)}(1) = 0$, $U_{(i+1)}(\infty) = \infty$.

Now we make the primary assumption that $\phi_{(i+1)}(t)$ is either convex or concave on $[0, \infty)$ and for $c_{(i)}$ defined above, it follows by the latter assumption, the nature of F and the log. function that the limit $c_{(i+1)}$ (= lim $(t \to \infty)$ $\phi_{(i+1)}(t)/t$) exists and must always satisfy $0 \leqslant c_{(i+1)} \leqslant 1$.

Logically, we now assume that $0 < c_{(i+1)} < 1$ and will show that under this assumption $c_{(i)} = 0$ and indeed if $c_{(i)}$ is such that $0 < c_{(i)} \leqslant 1$ then $c_{(i+1)} = 1$. Again we can envisage a "spectrum" of results developing as we vary i over the positive integers.

(i) Let $0 < c_{(i+1)} < 1$ then by definition, $c_{(i)}$ can be rewritten equivalently as

$$c_{(i)} = \lim_{t\to\infty} \frac{\phi_{(i)}\left(U_{(i-1)}(t)\right) + 1}{U_{(i-1)}(t) + 1}$$

where $U_{(i-1)}(t)$ satisfies the properties (for general U) specified in theorem 1.1; in particular it is continuous and strictly increasing to ∞ on $[1, \infty)$.

Taking the log of the reciprocal of both sides we then have by the definition of $U_{(i+1)}(t)$ as an iterated logarithm that

$$\log 1/c_{(i)} = \lim_{t\to\infty} U_{(i)}(t) . \left[1 - \frac{\log[\emptyset_{(i)}(U_{(i-1)}(t)) + 1]}{U_{(i)}(t)} \right]$$

Given the definition of $\emptyset_{(i+1)}(t)$ above then

$$\log 1/c_{(i)} = \lim_{t\to\infty} U_{(i)}(t) \left[1 - U_{(i+1)} \left\{ \frac{1-q}{1 - F\left(1 - \frac{(1-q)}{W_{(i)}(U_{(i-1)}(t))}\right)} \right\} \right]$$

and noting that $W_{(i+1)}(U_{(i)}(t)) = W_{(i)}(U_{(i-1)}(t))$ for each $t \in [1, \infty)$ we obtain

$$(1.12) \qquad \log \frac{1}{c_{(i)}} = \lim_{t\to\infty} U_{(i)}(t) \left[1 - \frac{\emptyset_{(i+1)}(U_{(i)}(t))}{U_{(i)}(t)} \right]$$

Now $\emptyset_{(i+1)}(U_{(i)}(t))/U_{(i)}(t) \to c_{(i+1)}$ as $t \to \infty$, since $U_{(i)}(t) \to \infty$ as $t \to \infty$ and is continuous on $[1, \infty)$. By our previous assumption $0 < c_{(i+1)} < 1$ and thus we have from (1.12) that $c_{(i)} = 0$.

(ii) Let $0 < c_{(i)} \leq 1$.

Now by definition,

$$c_{(i+1)} = \lim_{t\to\infty} \phi_{(i+1)}(t)/t$$

$$= \lim_{t\to\infty} \frac{\log\left[U_{(i)}\left\{\dfrac{1-q}{1-F(1-\dfrac{(1-q)}{W_{(i+1)}(t)})}\right\} + 1\right]}{t}$$

and noting that $W_{(i+1)}(t) = W_{(i)}(e^t - 1)$ we have that

$$c_{(i+1)} = \lim_{t\to\infty} \frac{\left[\log \phi_{(i)}(e^t - 1) + 1\right]}{t}$$

$$= \lim_{t\to\infty} \left\{\log\left[\frac{\phi_{(i)}(e^t - 1) + 1}{e^t}\right] + t\right\} \bigg/ t \ .$$

Clearly $c_{(i)} = \lim \quad (t \to \infty)\ \phi_{(i)}\ (e^t - 1)/(e^t - 1)$

$$= \lim(t\to\infty) \ \frac{\phi_{(i)}(e^t - 1) + 1}{e^t}$$

where $0 < c_{(i)} \leq 1$. Thus by the continuity and increasing nature of the log function we have $c_{(i+1)} = 1$, the required result.

For the specific case of $c = c_{(1)}$, $U(t) = \log t$ theorem 1.1 provides a sequence $\{\beta_n\}$, say, $\beta_n \to 0$ such that $\{\beta_n \log (Z_n + 1)\}$ converges in law to a proper distribution. Our β_n correspond to the ρ_n of Seneta [17]. In view of our hierarchy it is natural to question if for the case when $c_{(1)} = 0$ $(0 < c_{(2)} < 1)$ a sequence $\{\beta_n'\}$ of positive constants exists, such that for the same logarithmic function of Z_n, $\{\beta_n' \log (Z_n + 1)\}$ converges as specified in theorem 1.1. We shall show that no such sequence $\{\beta_n'\}$ exists and will generalize this in the following theorem.

THEOREM 1.3:

For $\{Z_n\}_{n\geq 0}$ such that $Z_o = 1$ and $c_{(o)} = 0$ then given some i, (i = 1, 2 ...) we define

$$c_{(i)} = \lim_{t\to\infty} \phi_{(i)}(t)/t$$

where $$\phi_{(i)}(t) = U_{(i)}\left[\frac{1-q}{1-F\left(1-\frac{(1-q)}{W_{(i)}(t)}\right)}\right], \quad 0 \leq t < \infty;$$

and $\phi_{(i)}(t)$ is assumed to be either convex or concave on $[0, \infty)$ and the constant $c_{(i)}$ is such that $0 < c_{(i)} < 1$. Also for each $i \geq 1$, $U_{(i)}(t) = \log(U_{(i-1)}(t) + 1)$, $1 \leq t < \infty$,

$W_{(i)}(t) = \exp(W_{(i-1)}(t) - 1)$, $0 \leq t < \infty$,

for $U_{(i-1)}$, $W_{(i-1)}$ and inverse pair, where $U_{(i-1)}$ is (as in theorem 1.1 for general U) slowly varying at infinity defined, continuous and stricitly increasing on $[1, \infty)$ with $U_{(i-1)}(1) = 0$, $U_{(i-1)}(\infty) = \infty$. Also $U_{(o)}(t) = (t-1)$, $1 \leq t < \infty$. Then for the given i, there exists no sequence of positive constants $\{\beta_n'\}$, say ($\beta_n' \to 0$ as $n \to \infty$) such that $\{\beta_n' U_{(i-1)}(Z_n + 1)\}$ converges in law to a proper nondegenerate random variable.

REMARKS (1) Clearly from our generalized argument we have that for a specific i (i = 1, 2 ...), there exists a sequence of positive constants $\{\beta_n\}$ such that $\{\beta_n U_{(i)}(Z_n + 1)\}$ converges in law to a proper distribution.

(2) It was previously noted that after a direct application of a theorem by H. Cohn [2], the convergence assertion of theorem 1.1 can be strengthened to that of almost sure convergence. It is straightforward from our theorem that

indeed no sequence $\{\beta_n'\}$ as above exists such that $\{\beta_n' U_{(i-1)} (Z_n + 1)\}$ converges almost surely to a proper nondegenerate random variable.

(3) The specific case of theorem 1.3 when $0 < c_{(1)} < 1$ was essentially proven in Seneta [16] where it was demonstrated that for the infinite mean Galton Watson process no sequence of positive constants $\{c_n\}$, say exists such that $\{Z_n/c_n\}$ converges in law to a proper nondegenerate random variable.

<u>Proof of theorem 1.3</u>: Given i, some positive integer ≥ 1, and $c_{(i)}$ such that $0 < c_{(i)} < 1$ we have shown that $c_{(i-1)} = 0$ where $c_{(i-1)} = \lim_{t\to\infty} \phi_{(i-1)}(t)/t$, (see (ii) of this section). We assume to the contrary that a sequence of positive constants $\{\beta_n'\}$ exists such that $\beta_n' \to 0$ as $n \to \infty$ and $\{\beta_n' U_{(i-1)}(Z_n + 1)\}$ converges as stated in the theorem. Hence we have convergence in distribution such that

$$(1.13) \qquad P(\beta_n' U_{(i-1)} (Z_n + 1) \leq t) \underset{n\to\infty}{\to} u(t) \qquad 0 < t < \infty,$$

where $u(t)$ is the proper limit distribution function of the form specified in theorem 1.1, that is in terms of $W_{(i-1)}(t)$ and $\Delta_{(i-1)}(t) = D^{-1}_{(i-1)}(t)$ where

$$\phi_{(i-1)}(t/\beta_n') \to \Delta_{(i-1)}(t) \quad \text{as } n \to \infty \quad , \quad 0 < t < \infty.$$

For K some positive constant, $0 < K \leq 1$, Lemma 1.3 gives us that (1.13) follows if

$$E\left[\left(1 - \frac{K}{W_{(i-1)}\left(\frac{t}{\beta_n'}\right)}\right)^{Z_n}\right] \to u(t)$$

as $n \to \infty$.

We shall show that the argument of Lemma 1.3 is reversible and that this ultimately leads to a contradiction of the assumption which yielded (1.13).

Let $T_n = Z_n + 1$ and $t > 0$ be arbitrary but fixed. Define $A_n = \{\beta_n' \ U_{(i-1)} \ (T_n) \leq t\} = \{T_n \leq W_{(i-1)} \ (t/\beta_n')\}$ for each $n \geq 1$. The last equality is ensured by the existence of $W_{(i-1)}(.) = U^{-1}_{(i-1)} \ (.)$, defined, strictly increasing and continuous on $[0, \infty)$.

Choose t', t'' such that $0 < t' < t < t''$ and write $A_n' = \{\beta_n' \ U_{(i-1)}(T_n) \leq t'\} \equiv \{ T_n \leq W_{(i-1)}(t'/\beta_n')\}$ and A_n'' is defined similarly. Analogous to lemma 1.3, put

$$Y_{(i-1),n} = [1 - \alpha_{(i-1),n}]^{T_n} \quad \text{where} \quad \alpha_{(i-1),n} = \frac{K}{W_{(i-1)}(t/\beta_n')},$$

for K a positive constant, $0 < K \leq 1$. Note that since $\beta_n' \to 0$ as $n \to \infty$, $\alpha_{(i-1),n}$ is small for n sufficiently large. Again employ the following inequality $-\alpha(1 + \alpha) < \log(1 - \alpha) < -\alpha$, true for $0 < \alpha < 3/5$. Estimates for $E[Y_{(i-1),n}]$ now follow. For the appropriate probability measure P,

$$\text{(i)} \quad E[Y_{(i-1),n}] = \int_{A_n''} Y_{(i-1),n} \, dP + \int_{A_n''^{c}} Y_{(i-1),n} \, dP$$

$$\leq P(A_n'') + \sup \{Y_{(i-1),n} \text{ on } A_n''^{c}\}$$

On $A_n''^{c}$, $T_n > W_{(i-1)}(t''/\beta_n')$ so

$$Y_{(i-1),n} = (1 - \alpha_{(i-1),n})^{T_n} = \exp(T_n \log(1 - \alpha_{(i-1),n}))$$

$$< \exp\{-\alpha_{(i-1),n} T_n\} < e^{-\lambda_{(i-1),n}} \to 0 \text{ as } n \to \infty$$

where $t'' > t$ and $\beta_n' \to 0$ and then $\lambda_{(i-1),n} = \dfrac{K\, W_{(i-1)}(t''/\beta_n')}{W_{(i-1)}(t/\beta_n')} \to \infty$

as $n \to \infty$ by a property of rapidly varying functions [19]. Taking the limit as $n \to \infty$ in $E(Y_{(i-1),n}) \leqslant P(A_n'') + e^{-\lambda_{(i-1),n}}$. yields $\limsup E\,[Y_{(i-1),n}] \leqslant \lim_{n\to\infty} P(A_n'') = u(t'')$.

(ii) Similarly

$$E\,[Y_{(i-1),n}] \geqslant P(A_n').\ \inf\ \{\, Y_{(i-1),n} \text{ on } A_n'\}$$

For n sufficiently large

$$\begin{aligned} Y_{(i-1),n} &= \exp\,\{T_n \log\,(1-\alpha_{(i-1),n})\} \\ &\geqslant \exp\,\{-\alpha_{(i-1),n}\ (1+\alpha_{(i-1),n})\ T_n\,\} \end{aligned}$$

And on A_n' $\quad T_n \leqslant W_{(i-1)}(t'/\beta_n')$ so

$$\begin{aligned} (1+\alpha_{(i-1),n})\ \alpha_{(i-1),n}\ T_n &\leqslant (1+\alpha_{(i-1),n}) \left\{ \frac{K\, W_{(i-1)}(t'/\beta_n')}{W_{i-1}\ (t/\beta_n')} \right\} \\ &= \mu_{(i-1),n}\text{, say.} \end{aligned}$$

Since $t' < t$ and $\beta_n' \to 0$ as $n \to \infty$ it follows by the rapidly varying nature of $W_{(i-1)}$ at infinity that (from [19]) $W_{(i-1)}(t'/\beta_n')/W_{(i-1)}(t/\beta_n') \to 0$ as $n \to \infty$. Thus on A_n'

$\inf Y_{(i-1),n}$ is $\geqslant e^{-\mu_{(i-1),n}} \to 1$ as $n \to \infty$.

Subsequently letting $n \to \infty$ in $E(Y_{(i-1),n}) \geqslant P(A_n')\, e^{-\mu_{(i-1),n}}$ yields $\liminf E(Y_{(i-1),n}) \geqslant \lim P(A_n') = u(t')$. Unifying the results of (i) and (ii) and letting $t' \uparrow t$ and $t'' \downarrow t$ we have by the continuity of $u(.)$ that $\lim E\,[Y_{(i-1),n}]$ exists and equals $u(t)$. For $K = (1 - q)$ we have that

$$F_n \left[\left(1 - \frac{(1-q)}{W_{(i-1)}(t/\beta_n')}\right) \right] \underset{n\to\infty}{\to} u(t) \;, \quad 0 < t < \infty,$$

so then

$$(1.14) \quad U_{(i-1)} \left[\frac{1-q}{1 - F_n \left[1 - \frac{(1-q)}{W_{(i-1)}(t/\beta_n')} \right]} \right] \to U_{(i-1)} \left[\frac{1-q}{1-u(t)} \right]$$
$$= \beta_{(i-1)}(t), \text{ say}$$

$0 < t < \infty$ the latter convergence is a consequence of the continuity properties of $U_{(i-1)}$, F and u(t). Recall the definition of $\emptyset_{(i)}(t)$ in theorem 1.3 then clearly the left hand side of (1.14) is $\emptyset_{(i-1),n}(t)$ the n^{th} functional iterate of $\emptyset_{(i-1)}(t)$ thus

$$\emptyset_{(i-1),n}(t/\beta_n') \underset{n\to\infty}{\to} \beta_{(i-1)}(t) \qquad 0 < t < \infty,$$

where the behaviour of u(.) ensures that $\beta_{(i-1)}(t)$ is defined, strictly increasing and continuous on $(0, \infty)$ such that $\beta_{(i-1)}(0+) = 0$, $\beta_{(i-1)}(\infty) = \infty$. Also the conditions of lemma 4.4. [16] on inversion and limits are satisfied, hence

$$(1.15) \qquad \beta_n' \; d_{(i-1),n}(t) \underset{n\to\infty}{\to} \gamma_{(i-1)}(t), \qquad 0 < t < \infty,$$

where $d_{(i-1),n}(.) = \emptyset^{-1}_{(i-1),n}(.)$; $\gamma_{(i-1)}(.) = \beta^{-1}_{(i-1)}(.)$.

Note the iterative nature of $d_{(i-1),n}$ that is $d_{(i-1),n+1}(t) = d_{(i-1),n}(d_{(i-1)}(t))$ for each n, so by the continuity of $d_{(i-1)}(.)$ and $d_{(i-1),n}$ for each n,

$$(1.16) \qquad \beta_n' \; d_{(i-1),n+1}(t) \underset{n\to\infty}{\to} \gamma_{(i-1)}(d(t)), \quad 0 < t < \infty .$$

Write $\beta_n' \, d_{(i-1),n}(d_{(i-1)}(t)) = \beta'_{n+1} \; d_{(i-1),n+1}(t) \left[\dfrac{\beta'_n}{\beta'_{n+1}}\right]$;

Then (1.15) and (1.16) yield

$$\frac{\beta'_n \, d_{(i-1),n}(d_{(i-1)}(t))}{\beta'_{n+1} \, d_{(i-1),n+1}(t)} \underset{n\to\infty}{\to} \frac{\gamma_{(i-1)}(d(t))}{\gamma_{(i-1)}(t)}, \qquad 0 < t < \infty.$$

Thus

$$(1.17) \qquad \frac{\beta'_n}{\beta'_{n+1}} \to \frac{\gamma_{(i-1)}(d(t))}{\gamma_{(i-1)}(t)} = \alpha^*_{(i-1)}, \quad \text{say}$$

where $\alpha_{(i-1)}{}^*$ exists and $0 < \alpha^*_{(i-1)} < \infty$.

It is true by asymptotic equivalence that

$$\beta'_{n+1} \; d_{(i-1),n+1}(t) \sim \gamma_{(i-1)}(t) \sim \beta'_n \, d_{(i-1),n}(t) \qquad (n \to \infty).$$

hence $\displaystyle \lim_{n\to\infty} \frac{\beta'_n}{\beta'_{n+1}} = \lim_{n\to\infty} \frac{d_{(i-1),n+1}(t)}{d_{(i-1),n}(t)}$.

Recall that $d_{(i-1),n}(t) \to \infty$ as $n \to \infty$ and by definition $d_{(i-1)}(t)/t \to 1/c_{(i-1)}$ as $t \to \infty$. However, if $0 < c_{(i)} < 1$ it is true that $c_{(i-1)} = 0$, hence $1/c_{(i-1)}$ is infinite, contrary to the finiteness of $\alpha^*_{(i-1)}$ in (1.17). Hence a contradiction, which completes the proof.

CHAPTER II

LIMIT STRUCTURE OF THE PROCESS

(WITH INFINITE MEAN)

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LIMIT STRUCTURE OF THE PROCESS(WITH INFINITE MEAN)

Section 1: Introduction and Preliminaries

It was demonstrated in [18, Section 3] that there is with respect to the functional equation approach of [17] "an intimate and natural relationship" between the latter and regularly varying functions. The existence of such a connection is corroborated by our generalized theory, as it appears in Chapter I. In particular we will show that, analogous to theorem 3 [18], the constants $\{\beta_n\}$ as in theorem 1.1 [Chapter I] are asymptotically related to a function slowly varying at infinity.

Following the development of Seneta [18] it is natural to question whether we can gain further information regarding the asymptotic behaviour of the tail of the limit distribution u(t); also whether an almost everywhere expression of the corresponding density function can be derived. We shall show that when we attempt to answer questions pertaining to the limit distribution u(t) in the framework of our generalized argument (where U(t) is as in theorem 1.1 of the form of an iterated logarithm) considerable difficulties occur.

Nevertheless, despite such problems a (general) theory in terms of slowly varying functions seems preferable in view of generality bearing in mind that the early concept of a slowly varying function partly evolved from the notion of an iterated logarithm.

Recall from theorem 1.1 that under assumptions (A) and (B), for $\emptyset(t)$, $0 \leq t < \infty$, U and W as specified there, a sequence

$\{\beta_n\}$ of positive constants exists such that $\{\beta_n U(Z_n + 1)\}$ converges in law to a nondegenerate limit distribution. The limit random variable has distribution function u(t) given by

$$u(t) = 1 - (1 - q)/ W(\Delta(t)), \qquad 0 < t < \infty.$$

We note that $\Delta(t)$ is continuous, strictly increasing on $(0, \infty)$ with $\Delta(0 +) = 0$, $\Delta(\infty) = \infty$ and $\Delta(t)$ convex or concave on $(0, \infty)$ as is $\emptyset(t)$, (assumption (A)). Further $\Delta(t) = D^{-1}(t)$, $0 < t < \infty$, where D(t) positive on $(0, \infty)$ is a solution to Schröder's functional equation

$$(2.1) \qquad D(d(t)) = \frac{1}{c} D(t) \qquad 0 < t < \infty,$$

and is up to constant factors, the unique solution such that D(t)/t is monotonic on $(0, \infty)$. We can now derive an explicit form for the constants β_n in terms of a function L, say slowly varying at infinity and also rewrite (1.11) of theorem 1.2 in terms of the latter function.

Section 2: Theorem and Proof

Theorem 2.1

The function $\Delta(t)$ is given by

$$\Delta(t) = t\, L(t) \qquad 0 < t < \infty,$$

where L(t) is a monotone function, slowly varying at infinity such that as $t \to \infty$, $0 < \lim L(t) \leq \infty$ with $\emptyset(t)$ convex on $[0, \infty)$ and $0 \leq \lim L(t) < \infty$ with concavity of $\emptyset(t)$ thereon. Also $0 < \lim L(t) < \infty$ if and only if

$$(2.2) \qquad \int_A^\infty t^{-1} \left| c - \frac{\emptyset(t)}{t} \right| dt < \infty ,$$

where A is any fixed number > 0. In addition we may take

$$\beta_n = c^n/L(c^{-n}) \quad .$$

Proof:

Taking inverses in (2.1) gives it the following Poincaré form $D^{-1}(c\,t) = d^{-1}(D^{-1}(t))$, $0 < t < \infty$, which by the definition of Δ and $\emptyset$ is equivalent to

$$(2.3) \qquad \Delta(c\,t) = \emptyset(\Delta(t)), \qquad 0 < t < \infty .$$

Recall that $\emptyset(t)/t \xrightarrow{t\to\infty} c$ and note that $\Delta(t)$ is strictly increasing and continuous on $(0, \infty)$ with $\Delta(\infty) = \infty$. Then from (2.3) it follows that

$$(2.4) \qquad \frac{\Delta\ (c\ t)}{\Delta\ (t)} = \frac{\emptyset\ (\Delta(t))}{\Delta(t)} \to c \qquad \text{as } t \to \infty.$$

If we assume that $\Delta(t)$ is convex on $(0, \infty)$, since $\Delta(0+) = 0$, then $\Delta(t)/t$ increases with t, so for λ, where $c \leqslant \lambda \leqslant 1$,

$$\frac{\Delta\ (c\ t)/ct}{\Delta\ (t)/t} \leqslant \frac{\Delta\ (\lambda t)/\lambda t}{\Delta\ (t)/t} \leqslant 1.$$

However, if $\Delta(t)$ is concave on $(0, \infty)$ the above inequalities are reversed. But in both cases it follows from (2.4) and the Sandwich theorem that $\Delta(\lambda t)/\Delta(t) \to \lambda$ as $t \to \infty$. Thus $\Delta(t)$ is a function regularly varying at infinity and of index 1. Then $\Delta(t)$ may be written as $\Delta(t) = t\,L(t)$, for $L(t)$ slowly varying at infinity. Since $\Delta(t)$ is convex or concave as is $\emptyset(t)$ on $[0, \infty)$ it is clear that the further properties of L stated in the theorem are satisfied.

Now iterating (2.1) we have

$$D(d_n(t_o)) = \frac{1}{c^n} D(t_o) = \frac{1}{c^n}, \quad \text{as } D(t_o) \equiv 1.$$

Hence write

$$\beta_n = 1/d_n(t_o) = [D^{-1}(D(d_n(t_o)))]^{-1}$$

$$= [\Delta(c^{-n})]^{-1}$$

$$= c^n/L\ (c^{-n}).$$

We can write $\beta_n \sim c^n/L(c^{-n})$ as $n \to \infty$ where we note $c^{-n} \to \infty$ as $n \to \infty$. When $0 < \lim L(t) < \infty$ $(t \to \infty)$ the latter asymptotic relation is equivalent to $\beta_n \sim$ constant. $c^{n(*)}$ for some finite, non zero constant. By theorem 1.2 a necessary and sufficient condition for (*) is (2.2), which completes the proof.

<u>Section 3: The limit distribution function</u>

In the particular case when $U(t) = \log t$ it was shown in [18] that the limit distribution $u(t)$ is "closely akin to the exponential and has an exponentially decreasing tail" if and only if (2.2) is true. According to theorem 1.1 and 2.1 for general U, W the tail of the limit distribution is described by

$$1 - u(t) = (1 - q)/\ W(t\ L(t)) \qquad 0 < t < \infty.$$

Recall that by the property of slowly varying functions $t^{\delta}L(t) \to \infty$ as $t \to \infty$ for any $\delta > 0$. Arguing as in [18] for $U(t) = \log t$ it is clear that when $0 < \lim L(t) < \infty$ $(t \to \infty)$ (or equivalently (2.2) by theorem 2.1) the stated relation between the behaviour of the tail of the limit distribution and the exponential $(W(t) = e^t)$ is true.

However, for our general W, we cannot generalize the argument of [18] to make statements about the behaviour of the tail of the distribution based on whether $0 < \lim_{t\to\infty} L(t) < \infty$ (or (2.2)) holds. This is due to the generality of our functions W and the fact that they are rapidly varying at infinity and so simple analysis of the limiting behaviour of $W(t\,L(t))$ as $t \to \infty$ does not provide the answers.

Also for the case when $U(t) = \log t$ [18] an almost everywhere expression for the density of u(t) was derived, viz. (3.7). In terms of our general W however, an almost everywhere derivative of u(t) may not exist if W is not absolutely continuous on each finite subinterval of $[0, \infty)$,[14]. Thus we cannot obtain results analogous to theorem 4 [18] in terms of our general U and W.

It bears mentioning however, that for functions defined in Chapter I, section 3 by,

$$W_{(i)}(t) = \exp\,(W_{(i-1)}(t) - 1), \quad 0 \leqslant t < \infty\,, \quad i = 1,2\ldots$$

and $W_{(o)}(t) = t+1$, we can under certain assumptions derive an almost everywhere expression for the density corresponding to $u_{(i)}(t)$, via an analogous argument to that of [18]. Note that for each $i = 1, 2, \ldots,$ $W_{(i)}$ is differenciable on $(0, \infty)$ and $W_{(i)}(0) = 1$. Indeed for each integral i we can show that $1/W_{(i)}(t)$ is absolutely continuous on finite subintervals of $[0, \infty)$.

Note that

$$\int_0^t \left\{- \frac{1}{W_{(1)}(u)}\right\} du + 1 = \frac{1}{W_{(1)}(t)} \quad .$$

Then after some manipulation, it follows by the above definition of $W_{(i)}(t)$ for each i that

$$(*)\quad \int_0^t \left[- \frac{1}{W_{(i)}(u)} \right] W_{(i-1)}(u) \cdot W_{(i-2)}(u) \cdot \quad \ldots\ldots W_{(1)}(u)\,du + 1$$

$$= 1/W_{(i)}(t).$$

As before for i = 1, 2 ... we define

$$\phi_{(i)}(t) = U_{(i)}(t) \left[\frac{(1 - q)}{1 - F\left\{1 - \frac{(1 - q)}{W_{(i)}(t)}\right\}} \right], \qquad 0 \leq t < \infty,$$

and the limit distribution is given by

$$u_{(i)}(t) = 1 - \frac{(1 - q)}{W_{(i)}(\Delta_{(i)}(t))}, \qquad 0 < t < \infty.$$

Here $\Delta_{(i)}(0+) = \Delta_{(i)}(0) = 0$ and $\Delta_{(i)}(t)$ is continuous and strictly increasing on $(0, \infty)$. Appealing to lemma 1.3 and theorem 1.1 [9], for $\Delta_{(i)}(t)$ convex on $[0, \infty)$ we have $\Delta_{(i)}(t)$ absolutely continuous on each finite subinterval of $[0, \infty)$; hence it can be written as the integral of its almost everywhere derivative, $p_{(i)}$ say, viz.,

$$(2.5)\qquad \Delta_{(i)}(t) = \int_0^t p_{(i)}(u)\,du,$$

where $p_{(i)}(t) = \Delta_{(i)}'(t)$ almost everywhere, and $p_{(i)}$ is strictly positive, nondecreasing and right continuous on $(0, \infty)$ as is $\Delta_{(i)}(t)$.

As $1/W_{(i)}(t)$ is absolutely continuous on finite subintervals of $[0, \infty)$, its a.e. derivative exists on such intervals and so a density expression for $u_{(i)}(t)$ analogous to

(3.7) [18] is possible. Note also that the limit distribution $u_{(i)}(t)$ is absolutely continuous on finite subintervals of $(0, \infty)$.

If $\Delta_{(i)}(t)$ is concave define $\bar{\Delta}_{(i)}(t) = -\Delta_{(i)}(t)$, which is convex on $[0, \infty)$ so by analogy we have,

$$\bar{\Delta}_{(i)}(t) = \int_0^t \bar{p}_{(i)}(u)\, du,$$

where $\bar{p}_i(t)$ is right continuous, strictly negative and increasing for $t > 0$. Substituting $p_{(i)}(t) = -\bar{p}_{(i)}(t)$, gives results as stated above.

REMARK ON ANOTHER FUNCTIONAL EQUATION

We can develop a characterization of $u(t)$ in terms of $\Delta(t)$ via (2.3). Indeed the distribution function will be uniquely determined by it since $D(t)$ is the unique (up to constant factors) solution to the Schröder functional equation (2.1). Analogous to the development of Darling [4] we can also show that $u(t)$ satisfies the Poincaré functional equation, (2.6) $F(u(t/c)) = u(t)$, $0 < t < \infty$. Appealing to the results of chapter I and the iterative nature of the p.g.f. of $\{Z_n\}$, we have

$$F_n(1 - (1 - q)/W(t/\beta_n)) \underset{n\to\infty}{\to} u(t),$$

$$F\left[F_n(1 - (1 - q)/W(t/\beta_{n+1}))\right] \underset{n\to\infty}{\to} u(t) \qquad , \qquad 0 < t < \infty.$$

Note that by the definitions of c, ϕ_n and β_n, $\{\beta_n/\beta_{n+1}\} \to c^{-1}$ as $n \to \infty$. Write $W(t/\beta_{n+1}) = W(t/\beta_n \cdot \beta_n/\beta_{n+1})$ where W is as before rapidly varying at infinity. Clearly $F_n(1 - (1 - q)/W(t/c\,\beta_n)) \underset{n\to\infty}{\to} u(t/c)$. After some manipulation (2.6) follows from these relations by dominated convergence, the continuity of F, F_n and W. However, the distribution function $u(t)$ is not in general uniquely determined by (2.6) and so we cannot obtain further properties of $u(t)$ from it. Thus from the viewpoint of uniqueness and utility the characterization in terms of $\Delta(t)$ is to be preferred.

CHAPTER III

A NOTE AND A THEOREM

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Section 1: A note on D.A. Darling's approach

In the particular case when $U(t) = \log t$ Seneta [17] developed a theory which produced the following convergence result,

$$(3.1) \qquad E\left[(1 - (1 - q)e^{-t/\rho_n})^{Z_n}\right] \underset{n\to\infty}{\to} 1 - (1 - q)\, e^{-\Delta(t)}$$

$$= u(t), \text{ say, } 0 < t < \infty,$$

where ρ_n, $\Delta(t)$ are as in the sequel. The author states that by "imitating the development of Darling" [4] "but replacing his b^{-n} by ρ_n" (3.1) is tantamount to

$$(3.2) \qquad P(\rho_n \log (Z_n + 1) \leqslant t) \underset{n\to\infty}{\to} u(t),\ t > 0 .$$

The author goes on to remark that (3.2) can be obtained using Stirling's formula (with remainder) [3] at the point in Darling's proof where he makes a key statement involving Stirling's formula for $\Gamma(x)$.

We shall elucidate this statement and show how the suggested method is feasible and provides the required result.

First we multiply both sides of (3.1) with $\exp(-\xi t)$, where $\xi > 0$ is arbitrary. Then we integrate over $(0, \infty)$. It follows by taking $n \to \infty$ that the integrand on the left hand side is positive and bounded for each $n \geqslant 1$ by $\exp(-\xi t)$. So interchange of the limit and integration signs is permissible and gives us a positive integrand on the left hand side. Then via Fubini's theorem we can further interchange the expectation and integration signs to give

$$\lim_{n\to\infty} E\left[\int_o^\infty (1 - (1-q)\, e^{-t/\rho_n})^{Z_n}\right] = \int_o^\infty e^{-\xi t}\; u(t)\, dt, \quad 0 < t < \infty.$$

If we make the substituition $x = (1 - q)\, e^{-t/\rho_n}$, the integral on the left hand side becomes the recognizable beta function and thus we have

$$\frac{-\rho_n}{(1 - q)^{\xi\rho_n}}\; B(\xi\rho_n, Z_n + 1) = \frac{-\Gamma(1 + \xi\rho_n)}{\xi(1 - q)^{\xi\rho_n}} \; \frac{\Gamma(Z_n + 1)}{\Gamma(Z_n + 1 + \xi\rho_n)} \;.$$

We now seek an asymptotic expression for $\Gamma(Z_n + 1)/\,\Gamma(Z_n + 1 + \xi\rho_n)$ and note that $(1 - q)^{\xi\rho_n} \to 1$ as $n \to \infty$. It follows from [3] by Cramér's expression for $\log\, \Gamma(p)$, $p > o$ that

$$\text{(3.3)} \quad \log\left[\frac{\Gamma(Z_n + 1)}{\Gamma(Z_n + 1 + \xi\rho_n)}\right] = (Z_n + \tfrac{1}{2})\Big[\log\,(Z_n + 1) - \log\,(Z_n + 1 + \xi\rho_n)$$

$$- \xi\rho_n \log\,(Z_n + 1 + \xi\rho_n) + \xi\rho_n$$

$$+ \int_o^\infty \frac{P_1(x)}{Z_n + 1 + x}\, dx - \int_o^\infty \frac{P_1(x)}{Z_n + 1 + \xi\rho_n + x}\, dx$$

where $P_1(x) = [x] - x + \tfrac{1}{2}$; $0 \leqslant x < \infty$, so $|P_1(x)| \leqslant \tfrac{1}{2}$ for each x in this interval. Employing the mean value theorem yields

$$\log(Z_n + 1 + \xi\rho_n) = \log\,(Z_n + 1) \quad + \frac{\xi\rho_n}{Z_n + 1 + \mu_n}$$

where $0 < \mu_n < \xi\rho_n$ for each $n \geq 1$. Noting that $\rho_n \downarrow 0$ as $n \to \infty$, we then write

$$-\xi\rho_n \log\,(Z_n + 1 + \xi\rho_n) = -\xi\rho_n \log\,(Z_n + 1) + o(1) \quad (n \to \infty)$$

where $o(1)$ denotes bounded random variables where these bounds tend to zero as $n \to \infty$.

Similarly, as $n \to \infty$

$$(Z_n + \tfrac{1}{2}) \quad \left[\log\,(Z_n + 1) - \log\,(Z_n + 1 + \xi\rho_n)\right] \to 0.$$

The following inequality is true for the remaining terms of (3.3)

$$\int_0^\infty \frac{P_1(x)}{Z_n + 1 + x}\,dx - \int_0^\infty \frac{P_1(x)}{Z_n + 1 + \xi\rho_n + x}\,dx \leq \frac{\xi\rho_n}{2} \int_0^\infty \frac{1}{1 + x^2}\,dx$$

where by the properties of the Cauchy distribution we have that the latter is bounded above by $\frac{\xi\,\pi\,\rho_n}{4}$ which tends to zero as $n \to \infty$. So despite the asymptotic behaviour of Z_n i.e. whether $Z_n \to \infty$ or eventually becomes zero it holds that

$$\log\left[\frac{\Gamma(Z_n + 1)}{\Gamma(Z_n + 1 + \xi\rho_n)}\right] = -\,\xi\rho_n \log\,(Z_n + 1) + o(1) \quad (n \to \infty)$$

for $o(1)$ as specified previously.

Equivalently we can write

$$\frac{\Gamma(Z_n + 1)}{\Gamma(Z_n + 1 + \xi\rho_n)} = (Z_n + 1)^{-\xi\rho_n} [1 + o(1)] \qquad (n \to \infty),$$

whence,

$$\xi^{-1} \; E\left[\exp(-\xi\rho_n \log(Z_n + 1))\right] \underset{n\to\infty}{\to} \int_0^\infty e^{-\xi t} u(t)\, dt, \; 0 < t < \infty.$$

The continuity theorem of Laplace transforms gives (3.2).

At this point we recall that for β_n, $U_{(i)}$, $W_{(i)}$ as defined in section 3 of chapter I for some specific $i(i = 1,2 \ldots.)$ we showed that

$$\text{(i)} \quad E\left[\left(1 - \frac{(1 - q)}{W_{(i)}(t/\beta_n)}\right)^{Z_n}\right] \underset{n\to\infty}{\to} u_{(i)}(t), \qquad 0 < t < \infty$$

held and can be interpreted probabilistically as

$$\text{(ii)} \quad P(\beta_n U_{(i)} (Z_n + 1) \leqslant t) \overset{n\to\infty}{\to} u_{(i)}(t), \quad t > 0.$$

It bears mentioning that when a proof of the latter (i.e. (ii) follows from (i)) was attempted along similar lines to the development above, substantial difficulties emerged. This Laplace transform approach produced integrals which were extremely difficult (if at all possible) to evaluate. Hence the approach developed from Lemma 1.3 is to be preferred from the viewpoint of simplicity and managability.

<u>Section 2: Generalization of a theorem by M. Kuczma</u>

Some initial motivation for examining the special case of $U(t) = \log t$ [17] came from a paper on the Böttcher functional equation by M. Kuczma [11].

We shall demonstrate that if we follow the line of approach of his theorem and demonstration thereof, a function different to the $\emptyset(.)$ of Chapter I can be defined which after some manipulation still provides us with the main convergence result of theorem 1.1 in the particular case when $U(t) = \log t$. More specifically we have a sequence of positive constants $\{\beta_n\}$ such that $\{\beta_n \log (Z_n + 1)\}$ converges in law to a proper distribution. However, we notice that when we attempt a similar study for the next case when $c_{(1)} = 0$ some difficulties emerge (bearing in mind the hierarchy defined in section 3 of Chapter I, where we take $U(t) = \log (1 + \log t)$ and $c_{(2)} \in (o, 1)$).

We shall further show that Kuczma's result [11] can be generalized to a less specific functional equation by considering a general $U(t)$.

Recall from Chapter I, section 3, that for $U(t) = \log t$, $c = c_{(1)} \in (O, 1)$ we defined

$$\emptyset_{(1)}(t) = - \log \left[\frac{1 - F\,(1 - (1 - q)e^{-t})}{1 - q} \right], \qquad t \geqslant 0 .$$

We now define

$$\emptyset^* (t) = \log \emptyset_{(1)}(e^t), \qquad - \infty < t < \infty,$$

and noting that $0 < \emptyset_{(1)}(t) \leqslant t$ for $t \geqslant 0$ (with strict inequality except at $t = 0$), it follows that

$$(3.4) \qquad - \infty < \emptyset^* (t) < \infty \quad , \qquad - \infty < t < \infty,$$

since $q < F(x) < x$ for $q < x < 1$.

We have $\emptyset_{(1)}(t)$ strictly increasing and continuous on $[0, \infty)$ with $\emptyset_{(1)}(0) = 0$, $\emptyset_{(1)}(\infty) = \infty$, hence as $t \to \infty$ $\emptyset^*(t) \to \infty$ and as $t \to -\infty$, $\emptyset^*(t) \to -\infty$.

Clearly $\emptyset^*(t) > 0$ if and only if $\emptyset_{(1)}(e^t) > 1$; note that $\emptyset_{(1)}(t) > 1$ iff $1 - (1 - q)\, e^{-1} \leqslant F(1 - (1 - q)\, e^{-t})$ and so $\emptyset^*(t)$ is positive for t sufficiently large. It is at this point that we make an assumption analogous to (A) of Chapter I; that is we suppose $\emptyset^*(t)$ is convex or concave on its whole domain of definition, namely $(-\infty, \infty)$. (A*)

Thus for some finite B, $\dfrac{\emptyset^*(t) - \emptyset^*(B)}{t - B}$ is either nondecreasing with convexity of $\emptyset^*$ in $[B, \infty)$ or nonincreasing with concavity of $\emptyset^*$ on the same interval. So as $t \to \infty$ the limit of $\emptyset^*(t) - \emptyset^*(B)/(t - B)$ exists (though possibly infinite) and so c^* exists where $c^* = \lim\limits_{(t\to\infty)} \dfrac{\emptyset^*(t)}{t}$ and must always satisfy $0 \leqslant c^* \leqslant 1$, since $\emptyset^*(t)$ satisfies (3.4) and is positive for t large enough. Recall from Chapter I that

$$c_{(2)} = \lim_{t\to\infty} \emptyset_{(2)}(t)/t \qquad \text{where}$$

$$\emptyset_2(t) = U_{(2)} \left[\frac{1 - q}{1 - F\left(1 - \dfrac{(1 - q)}{W_{(2)}(t)}\right.} \right]$$

for $U_{(2)}$, $W_{(2)}$ as before. We now show that the limit $c_{(2)}$ is equal to c^*. By definition we can write

$$\frac{\emptyset^*(t)}{t} = \frac{\log \emptyset_{(1)}(e^t)}{t}$$

$$= \frac{\log\left[\dfrac{\emptyset_{(1)}(e^t - 1) + 1}{e^t}\right] + t}{t} \quad .$$

Earlier we showed that for each integral $i \geqslant 1$, we could define

$$\phi_{(i)}(t) = U_{(i)} \left[\frac{1 - q}{1 - F\left(1 - \frac{(1-q)}{W_{(i)}(t)}\right)} \right] ; \quad t \geqslant 0,$$

where $W_{(i)}(t) = W_{(i-1)}(e^t - 1)$, $\quad 0 \leqslant t < \infty,$

and $U_{(i)}(t) = \log(U_{(i-1)}(t) + 1), \quad 1 \leqslant t < \infty,$

where $U_o(t) = t - 1$.

Then for $i = 2$ we have

$$\phi_{(2)}(t) = \log \left[\frac{U_{(1)}\left[\frac{1 - q}{1 - F\left(1 - \frac{(1-q)}{W_{(1)}(e^t - 1)}\right)}\right] + 1}{e^t} \right] + t$$

$$= \log \left[\frac{\phi_{(1)} (e^t - 1) + 1}{e^t} \right] + t \quad .$$

Thus $c_{(2)} = c^*$ and under assumption (B) of chapter I we then have $0 < c^* < 1$ and indeed then $c_{(1)} = o$.

We define for each $n \geqslant 1$,

$$\phi_n^* (t) = \log \phi_{(1),n}(e^t)$$

$$= - \log \left[\frac{1 - F_n(1 - (1 - q)\, e^{-e^t})}{(1 - q)} \right] ; \quad t \geqslant 0$$

which clearly is the n^{th} functional iterate of $\emptyset^*(t)$.

We shall show that to study the case when $c_{(1)} = o$ a theory analogous to that of Chapter I can be formulated in terms of $\emptyset_n^*(t)$ and produce the equivalent probabilistic interpretation (3.2).

Define the inverse function of $\emptyset^*(.)$ by $d^*(.)$ where $d^*(t) = \log d_{(1)}(e^t)$, $-\infty < t < \infty$ and for $n \geq 1$, $d_n^*(t) = \log d_{(1),n}(e^t)$ where $d_{(1)}(.) = \emptyset_{(1)}^{-1}(.)$; $d_{(1),n}(.) = \emptyset_{(1),n}^{-1}(.)$ and $\emptyset_n^*$, d_n^* are the n^{th} functional iterates of $\emptyset^*$ and d^* respectively. By (3.4) we have

$$(3.5) \qquad d^*(t) > t$$

and also as $t \to \infty$, $d^*(t) \to \infty$ whereas $d^*(t) \to -\infty$ as $t \to -\infty$. By the definition of c^* we have $d^*(t)/t \to 1/c^*$ as $t \to \infty$.

Write

$$F^*(t) = -d^*(-t) \quad ,$$

$$F_n^*(t) = -d_n^*(-t) \quad , -\infty < t < \infty \quad , \quad n \geq 1.$$

We note from (3.5) that for each t, $-\infty < t < \infty$ we have $F^*(t) < t$.

Now we are in a position to apply the results of the theorem and demonstration of [11] directly to our problem. In particular, we replace his a with $a = \infty$ (hence $A = \infty$) and then obtain that for t_o arbitrary but fixed, $-\infty < t_o < \infty$,

$$\lim_{n\to\infty} \frac{F_n^*(t)}{F_n^*(t_o)} = \frac{G^*(t)}{G^*(t_o)}$$

where $G^*(t)$ satisfies (9) [11] with $p = 1/c^*$, and the uniqueness of solutions follows similarly. Rewriting in terms of d^*, d_n^*, we have

$$(3.6) \qquad \frac{d_n^*(t)}{d_n^*(t_o)} \to D^*(t) \qquad -\infty < t < \infty$$

where $D^*(t)$ satisfies the following functional equation,

$$D^*(d^*(t)) = \frac{1}{c^*} D^*(t), \qquad -\infty < t < \infty \text{ ; and its}$$

uniqueness up to constant factors etc. as before. Also $D^*(t)$ is strictly monotone increasing and continuous on $(-\infty, \infty)$ such that as $t \to \infty$ $D^*(t) \to \infty$ and by the positivity of D^* we have that as $t \to -\infty$ $D^*(t) \to 0$. An application of results of Lemma 4.4 [16] on inversion and limits yields, as $n \to \infty$

$$\emptyset_n^* (d_n^*(t_o) . t) \to D^{*-1}(t) = \Delta^*(t), \text{ say}, \quad 0 < t < \infty .$$

Let $\beta_n^* = 1/d_n^* (t_o)$ for each $n \geqslant 1$, then

$$\log \left[- \log \left\{ \frac{1 - F_n(1 - (1 - q) \exp \{- e^{t/\beta_n^*}\})}{(1 - q)} \right\} \right] \underset{n\to\infty}{\to} \Delta^*(t),$$

so that

$$(3.7) \quad F_n(1 - (1 - q) \exp \{- e^{t/\beta_n^*}\}) \to 1 - (1 - q) \exp \{- e^{\Delta^*(t)}\}$$

$$= u^*(t), \text{ say } 0 < t < \infty ,$$

where $u^*(t)$ is continuous, strictly increasing on $(0, \infty)$ and satisfies $u^*(0 +) = q$.

By the application of lemma 1.3 with $k = (1 - q)$ and after some manipulation we have that (3.7) is equivalent to

$$P(\beta_n^* \log (\log (Z_n + 1) + 1) \leqslant t) \underset{n\to\infty}{\to} u^*(t), \quad 0 < t < \infty .$$

The next logical step would be, as we do, to attempt a similar study of the case when $c_{(2)} = 0$. This is partly achieved by defining and working with the following functions,

$$\emptyset^{**}(t) = \log \emptyset^{*}(e^{t}) \; ; \quad t_o^{*} < t < \infty$$

and for each $n \geqslant 1$,

$$\emptyset_n^{**}(t) = \log \emptyset_n^{*}(e^{t}); \quad t_o^{*} < t < \infty \quad \text{where } t_o^{*} \text{ is}$$

such that for each $n \geqslant 1$

$$\emptyset^{*}(e^{t_o^{*}}) = \emptyset_n^{*}(e^{t_o^{*}}) = 0.$$

Analogous to earlier work $\emptyset^{**}(t)$ is assumed either convex or concave on its whole domain of definition and that $0 < c^{**} < 1$, where c^{**} is such that $\emptyset^{**}(t)/t \to c^{**}$ as $t \to \infty$.

Analogous to our earlier discussion it is straightforward that c^{**} is equivalent to $c_{(3)}$ where $c_{(3)} = \lim\,(t \to \infty)\; \emptyset_{(3)}(t)/t$.

Then manipulation of inverse functions and the application of Kuczma's approach provide results analogous to (3.6) in terms of D^{**}, say. Again we can arrive at the analogous probabilistic result from theorem 1.1 for the specific case when $U(t) = \log\,[\log\,\{1 + \log t\} + 1\,]$. Ultimately we obtain for β_n^{**}, $\Delta^{**}(t)$ analogous to our previous β_n^{*}, $\Delta^{*}(t)$, that

$$P(\beta_n^{**} \log\,[\log\,\{1 + \log t\} + 1] \leqslant t) \underset{n\to\infty}{\to} u^{**}(t), \quad 0 < t < \infty$$

where $u^{**}(t) = 1 - (1 - q)\exp\{-\exp\,[e^{\Delta^{**}(t)}]\}$, $\quad 0 < t < \infty$

We note that whereas the theory of Chapter I produced (in this specific case of U) a limit distribution with a jump of magnitude q at the origin, here our distribution function is such that $u^{**}(0+) = 1-(1-q)\,e^{-1}$; here lies the divergence between our method above motivated by [11] and our previous generalized approach. Hence the latter method is preferrable from the viewpoint of generality. However, our findings motivate and lead us to the following extension of Kuczma's theorem, in terms of a general function $U(t)$.

<u>Theorem 3.1</u>:

Let $h(x)$ be defined, positive and strictly increasing on $(0, a)$, $0 < a \leq \infty$ and satisfying $h(x) < x$ on this interval. Let $U(x)$ be defined on $(0, \infty)$, continuous and strictly increasing with $U(0+) = -\infty$, $U(\infty) = \infty$; denote by $W(.)$ the inverse function of $U(.)$.

Suppose further that,

(i) $U(h(W(t)))$ is convex or concave on $(-\infty, A)$, where $A = U(a)$, and

(ii) $$\lim_{x\to 0+} \frac{U(h(x))}{U(x)} = \alpha > 1.$$

Then all solutions ρ of the functional equation

$$(3.8) \qquad \rho(h(x)) = W\,[\alpha U\,(\rho\,(x))] \qquad 0 < x < a,$$

satisfying the condition that $U(\rho(W(t))$ is convex or concave on $(-\infty, A)$ form the one parameter family

$$\rho(x) = \lim_{n\to\infty} W\left[\gamma\,\frac{U(h_n(x))}{U(h_n(x_0))}\right]$$

x_0 arbitrary, but fixed, $0 < x_0 < a$, $\gamma \in (-\infty, \infty)$.

<u>Proof of theorem 3.1</u>:

Let us define

$$d(t) = U(h(W(t))), \quad -\infty < t < A$$

and for each $n \geqslant 1$ its corresponding n^{th} functional iterate

$$d_n(t) = U(h_n(W(t)), \qquad -\infty < t < A, \quad \text{for}$$

h_n the n^{th} functional iterate of h.

The assumption (ii) is equivalent to $\lim_{t\to\infty} \frac{d(t)}{t} = \alpha$, by the clearly continuous and strictly increasing nature of W on $(-\infty, \infty)$.

Also write

$$\psi(t) = U[\rho(W(t))], \qquad -\infty < t < A,$$

then (3.8) yields

$$\text{(3.9)} \qquad \psi(d(t)) = \alpha\,\psi(t), \qquad -\infty < t < A.$$

We note that $d(t)$ satisfies those properties of "$F(t)$" in [11] and the required monotonicity of $d(t)$ on $(-\infty, 0) \cup (0, A)$ is satisfied by the concavity/convexity assumption of $d(t)$ on $(-\infty, A)$ (i.e. (i)). Hence the results of Kuczma [11] follow directly and so the solutions ψ of (3.9) may be written as

$$\psi(t) = \gamma \lim_{n\to\infty} \frac{d_n(t)}{d_n(t_o)}$$

where for x_o, $0 < x_o < a$ arbitrary but fixed, $t_o = U(x_o)$. Thus

$$\rho(x) = W\left[\gamma \lim_{n\to\infty} \frac{d_n(U(x))}{d_n(t_o)}\right] \quad \text{so by}$$

the definition of d_n

$$\rho(x) = W\left[\gamma \lim_{n\to\infty} \frac{U(h_n(x))}{U(h_n(x_o))}\right]$$

$$= \lim_{n\to\infty} W\left[\gamma \frac{U\ (h_n(x))}{U\ (h_n(x_o))}\right] ,$$

The last line follows by the continuity of W, this completes the proof. ▫

CHAPTER IV

ALMOST SURE CONVERGENCE AND IMMIGRATION

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SECTION 1:

Introduction and Preliminaries

In Chapter I we noted that strong convergence in theorem 1.1 follows from a direct application of a theorem by H. Cohn [2] We shall verify this and show further that our distribution function $u(t)$, as before, satisfies the required properties of this latter theorem.

Heyde [7] derived a martingale which lead to almost sure convergence for the supercritical case; thus it is natural to ask whether for the infinite mean Galton Watson process almost sure convergence can be obtained via a martingale argument. In particular we investigate whether a bounded positive martingale based on $\{\beta_n U(Z_n + 1)\}$, $U(.)$, β_n as in theorem 1.1; a generalization of a martingale suggested in [12], provides the required almost sure convergence after the application of the martingale convergence theorem.

To this point we have restricted our attention to the nonimmigration Galton Watson process and we now consider whether the results of Chapter I can be generalized to the immigration case [1], [6]. Again we suppose that the offspring distribution has infinite mean and when independent immigration is permitted we denote this process by $\{X_n\}$ and assume for simplicity that $X_0 = 0$, as before $F'(1-) = \infty$ for $F(s)$, $s \in [0, 1]$ the p.g.f. of the offspring distribution and q denotes the extinction probability where we assume $q < 1$. The distribution of the number of immigrants entering a given generation is given by $B(s) = \sum_{j=0}^{\infty} b_j s^j$, $B(o) \neq 1$. We refer here also to assumptions (A) and (B) of Chapter I.

We then prove the following theorem which generalizes theorem 1.1 to the immigration case.

Theorem 4.1

Under conditions (A) and (B), with the sequence of positive constants $\{\beta_n\}$ as in theorem 1.1, $\{\beta_n U(X_n + 1)\}$ converges almost surely to a proper random variable on $(0, \infty)$, with continuous distribution function given by

$$p(t) = \prod_{m=1}^{\infty} B(u(t c^{-m})), \qquad t > 0$$

where u(.) is as in theorem 1.1, if

$$\sum_{j=2}^{\infty} b_j \log U(j) < \infty \quad ;$$

and diverges to infinity in probability otherwise.

SECTION 2:

Almost Sure Convergence

(a) A theorem by H. Cohn

The following limit theorem of Cohn [2] for nonhomogenous Markov chains, based intrinsically on martingales shows that the convergence in distribution of Theorem 1.1 implies almost sure convergence. We restate his Theorem 2 replacing U by V, say, c_n by $1/\beta_n$ and his limiting distribution F by u.

Suppose that $\{Z_n;\ n \geq 0\}$ is a Galton Waston process with $m(= F'(1-)) = \infty$, V strictly increasing, continuous and slowly varying function defined on $[0, \infty)$ with $\lim_{x\to\infty} V(x) = \infty$ and $\{1/\beta_n\}$ an increasing sequence of constants with $\lim_{n\to\infty} 1/\beta_n = \infty$. Suppose further that $\{\beta_n V(Z_n);\ n \geq 0\}$ converges in distribution to a nondegenerate limiting distribution u, which is continuous on $(0, \infty)$. The $\{\beta_n V(Z_n):\ n \geq 0\}$ converges almost surely, $\lim_{n\to\infty} \frac{\beta_{n-1}}{\beta_n} = \alpha < 1$ exists and is finite, and u satisfies the conditions

$$(4.1)\qquad \lim_{n\to\infty} u(\alpha^n x)^{[U^{-1}((x-\xi)/\beta_n)]} = 1$$

and

$$(4.2)\qquad \lim_{n\to\infty} u(\alpha^n x)^{[U^{-1}((x+\xi)/\beta_n)]} = 0$$

for any $x > 0$ and $\xi > 0$.

We include a further specification that $V(0) = 0$ and writing

$V(t) = U(t + 1) \quad 0 \leqslant t < \infty$ where U is as in theorem 1.1 where we proved the convergence of $\{\beta_n U(Z_n + 1)\}$ in law to a proper nondegenerate limit distribution, $u(t)$.

Recall that U is defined on $[1, \infty)$, continuous and strictly increasing from $U(1) = 0$ to $U(\infty) = \infty$. In particular U is slowly varying at infinity so clearly $V(t)$ $(V(0) = 0)$ satisfies the properties required by Cohn's theorem. We know from theorem 1.1 that $u(x)$ is strictly increasing and continuous on $(0, \infty)$ such that $u(0+) = q$. Also the sequence of positive constants $\{\beta_n\}$ is such that $\beta_n \downarrow 0$ as $n \to \infty$ and $\lim\limits_{n\to\infty} \dfrac{\beta_{n-1}}{\beta_n} = \dfrac{1}{c}$, where c is as in theorem 1.1 and in particular $0 < c < 1$. So writing $\alpha = 1/c$ it is clear that the almost sure convergence of $\{\beta_n U(Z_n + 1)\}$ to a proper nondegenerate random variable is implied by Cohn's theorem. We need only check that $u(t)$ satisfies properties (4.1) and (4.2).

Then following the development of Cohn, for any $x > 0$, $a > 0$ we can write

$$(4.3) \qquad \lim_{n\to\infty} P\left\{\frac{Z_n + 1}{U^{-1}\left(\frac{x}{\beta_n}\right)} \leqslant a\right\} = u(x)$$

since we have

$$\lim_{n\to\infty} \beta_n U(a\, U^{-1}(x/\beta_n)) = \lim_{n\to\infty} \frac{U(a\, U^{-1}(\frac{x}{\beta_n}))}{\frac{x}{\beta_n}} \cdot \frac{x}{\beta_n} \cdot \beta_n$$

and by the properties of slowly varying functions [19] the right hand side $= 1 \cdot x = x$, and so by the continuity of $u(.)$ (4.3) follows.

Writing $T_n = \beta_n U(Z_n + 1)$ for $n = 1, 2...$ we then obtain from (4.3) that

$$P(T_n \in [0, x) \mid Z_m = i) = P\left\{ \frac{Z_n + 1}{U^{-1}(x/\beta_n)} \in [0, 1) \mid Z_m = i \right\}$$

$$(4.4) \qquad = P\left\{ \frac{Z_1^{(n-m)} + \dots + Z_i^{(n-m)} + 1}{U^{-1}(x/\beta_n)} \in [0, 1) \right\}$$

where $Z_1^{(n-m)}$, $Z_2^{(n-m)}$ .... are independent, identically distributed random variables with the same distribution function as Z_{n-m}, (as defined in Heyde [8]).

By our construction we note that

$$\frac{\beta_{n-m}}{\beta_n} = \frac{\beta_{n-m}}{\beta_{n-m+1}} \cdot \frac{\beta_{n-m+1}}{\beta_{n-m+2}} \cdots\cdots \frac{\beta_{n-1}}{\beta_n}$$

$$\underset{n\to\infty}{\to} (1/c)(1/c) \dots\dots (1/c) = (1/c)^m$$

so appealing to (4.3), (4.4) and the continuity of u we get

$$\lim_{n\to\infty} P(T_n \in [0, x) \mid Z_m = i) = \lim_{n\to\infty} P^{(i)} \left\{ \frac{Z_1^{(n-m)} + 1}{U^{-1}\left(\frac{1}{\beta_{n-m}} \left(\frac{x}{c^m}\right)\right)} < 1 \right\}$$

$$= u^{(i)} \left(\frac{x}{c^m}\right)$$

where $u^{(i)}$ is analogous to $F^{(i)}$ in [2].

It is then straightforward (by analogy) that

$$u^{(i)}\left(\frac{x}{c^m}\right) \geqslant u\left(\frac{x}{c^m}\right)^{[U^{-1}(\frac{x-\xi}{\beta_m})]} \qquad \text{for } i < U^{-1}\left(\frac{x-\xi}{\beta_m}\right)$$

and

$$u^{(i)}\left(\frac{x}{c^m}\right) \leqslant u\left(\frac{x}{c^m}\right)^{[U^{-1}(\frac{x+\xi}{\beta_m})]} \qquad \text{for } i > U^{-1}\left(\frac{x+\xi}{\beta_m}\right)$$

$U^{-1} \equiv W$ from before and in particular it is rapidly varying at infinity.

We require to prove (4.1) and (4.2) for any $\xi > 0$, $x > 0$, and so we consider a subsequence $\{m_\ell\}$ say such that for a certain x_o,

$$\lim_{\ell\to\infty} u\left(\frac{x}{c^{m_\ell}}\right)^{[U^{-1}(\frac{x_o}{\beta_{m_\ell}})]} = \delta > 0.$$

Then analogous to the development of Cohn we can show that

$\lim_{\ell\to\infty} u\left(\frac{x}{c^{m_\ell}}\right)^{[U^{-1}(\frac{x'}{\beta_{m_\ell}})]} = 1$ if $x' < x_o$, whereas if $\{m_\ell\}$ is such that

$\lim_{\ell\to\infty} u\left(\frac{x}{c^{m_\ell}}\right)^{[U^{-1}(\frac{x_o}{\beta_{m_\ell}})]} = 0$, then

$$\lim_{\ell\to\infty} u^{[U^{-1}(\frac{x'}{\beta_{m_\ell}})]} \left(\frac{x}{c^{m_\ell}}\right) = 0 \qquad \text{for any } x' > x_o$$

We note that the exponent can be written as

$$U^{-1}\left(\frac{x'}{\beta_{m_\ell}}\right) = W\left(\frac{x_o}{\beta_{m_\ell}}\right) \frac{W(x'/\beta_{m_\ell})}{W(x_o/\beta_{m_\ell})}$$

so as in [2] from the property of rapidly varying functions [19] we know that $W(x'/\beta_{m_\ell})/W(x/\beta_{m_\ell}) \to 0$ as $\ell \to \infty$ $(x' < x_o)$. Hence as in [2] it is true that there exists a certain x_o such that

$$\lim_{\ell\to\infty} u^{[W(\frac{x'}{\beta_{m_\ell}})]} \left(\frac{x}{c^{m_\ell}}\right) = \begin{cases} 1 & \text{if } x' < x_o \\ 0 & \text{if } x' > x_o \end{cases}$$

and similarly we can see that the above limit does not depend on the particular choice of $\{m_\ell\}$ and hence the required result.

We remark that for $U(t) \equiv U_{(i)}(t)$ $i = 1,2 \ldots$ defined as in section 3 Chapter I by

$$U_{(i+1)}(t) = \log (U_{(i)}(t) + 1)$$

$$U_{(o)}(t) = t - 1,$$

then for any i (a positive non zero integer) $U_{(i)}(t)$ satisfies the properties of Cohn's theorem and so for $\{\beta_n\}$ as before, the convergence of $\{\beta_n U_{(i)}(Z_n + 1)\}$ to a continuous law now implies almost sure convergence to a proper non degenerate limit random variable.

Section 2 (b):

Martingales

As indicated earlier it is natural to question whether the almost sure convergence of $\{\beta_n U(Z_n + 1)\}$; U as in theorem 1.1, can be obtained by applying to the martingale convergence theorem; in effect can we find a bounded positive martingale bearing some relationship with U which provides the required result?

Again we limit ourselves to the explosive Galton Watson process without immigration. In the particular case when $U(t) = \log t$, $c = c_{(1)}$, Pakes [12] puts forward a possible martingale; we shall generalize this (in terms of a general U and hence W) but demonstrate that these martingales do not provide the required almost sure convergence to a nondegenerate limit.

With U, W, u(t), $\{\beta_n\}$ as in the proof of Theorem 1.1 we have from (1.16) that as $n \to \infty$

$$E\left[\left(1 - \frac{(1-q)}{W(t/\beta_n)}\right)\right]^{Z_n} \to u(t), \qquad 0 < t < \infty.$$

Define

$$T_n^* = \left(1 - \frac{(1-q)}{W(1/\beta_n)}\right)^{Z_n} \qquad \text{for } n = 1, 2 \ldots.$$

We denote by $\mathcal{F}_n$ the σ-field generated by $Z_1, \ldots, Z_n$ then for each $n \geq 1$ we have

$$E(T_{n+1}^{*}|\mathcal{F}_n) = \left\{ E\left[1 - \frac{(1-q)}{W(\frac{1}{\beta_{n+1}})}\right]^{Z_1} \right\}^{Z_n}$$

$$= \left\{ F(1 - \frac{(1-q)}{W(\frac{1}{\beta_{n+1}})}) \right\}^{Z_n} .$$

Then appealing to the definition of $\emptyset(t)$ as in Chapter I yields

$$E(T_{n+1}^{*}|\mathcal{F}_n) = \left\{ 1 - \frac{(1-q)}{W(\emptyset(\frac{1}{\beta_{n+1}}))} \right\}^{Z_n}$$

Now $\beta_n \equiv 1/d_n(t_o)$ for each $n \geq 1$, $0 < t_o < \infty$ arbitrary but fixed and $\emptyset$ and d are an inverse pair; it follows by the iterative nature of d_n that

$$E(T_{n+1}^{*}|\mathcal{F}_n) = \left\{ 1 - \frac{(1-q)}{W(\frac{1}{\beta_n})} \right\}^{Z_n} = T_n^{*},$$

thus $\left\{ (1 - \frac{(1-q)}{W(\frac{1}{\beta_n})}) \right\}^{Z_n}_{n\geq 1}$ is a martingale.

Since $|T_n^{*}| \leq 1$ for each $n \geq 1$ using the martingale convergence theorem we get

$$T_n^{*} \underset{n\to\infty}{\to} T^{*},$$

almost surely with $0 \leq T^* \leq 1$ and $E(T^{*}) = u(1) < \infty$,

We shall show that this implies that

$$(4.5) \qquad \frac{Z_n + 1}{W\left(\frac{1}{\beta_n}\right)} \underset{\rightarrow}{n\to\infty} \frac{(-\log T^*)}{(1-q)} \qquad \text{for} \quad 0 < T^* \leqslant 1,$$

$$\frac{Z_n + 1}{W\left(\frac{1}{\beta_n}\right)} \underset{n \to \infty}{\rightarrow} \infty \qquad \text{for} \quad T^* = 0.$$

Appealing to the definition of T^*_n and noting that $W(\frac{1}{\beta_n}) \uparrow \infty$ as $n \to \infty$ we have

$$\left\{1 - \frac{(1-q)}{W\left(\frac{1}{\beta_n}\right)}\right\}^{Z_n + 1} \overset{a.s}{\underset{n\to\infty}{\rightarrow}} T^*.$$

Taking logarithms of both sides preserves a.s. convergence so

$$(Z_n + 1) \log \left(1 - \frac{(1-q)}{W\left(\frac{1}{\beta_n}\right)}\right) \overset{a.s}{\rightarrow} \log T^*.$$

Indeed we can write the above as

$$-(Z_n + 1) \frac{(1-q)}{W\left(\frac{1}{\beta_n}\right)} \left[\frac{\log\left(1 - \frac{(1-q)}{W(1/\beta_n)}\right)}{(1-q)/\,W\left(\frac{1}{\beta_n}\right)}\right] \overset{a.s}{\underset{n\to\infty}{\rightarrow}} -\log T^*$$

and thus the required result follows since for $|x| \leqslant 1$, $-\log(1-x) \sim 1$ as $x \to 0+$; and we note that $W(t) \geqslant 1$ for all $t \geqslant 0$ where W is continuous on $[0, \infty)$ and strictly increasing to infinity.

Seneta [16] showed that for the infinite mean case Z_n

cannot be normed to give a proper non degenerate limit distribution; hence (4.5) for $0 < T^* \leqslant 1$ is invalid. If we then assume that T^* is zero with probability one this contradicts our assertion that $E(T^*) = u(1) > u(0+) = q$. So our martingale does not provide us with a means of validating the almost sure convergence of $\{\beta_n U(Z_n + 1)\}$ to a non degenerate finite limit.

REMARK: We can consider some $i = 1,2 \ldots$ and in view of the hierarchy discussed in Chapter I may question whether the above martingale can give almost sure convergence of $\beta_n U_{(i)}(Z_n + 1)$ to a finite, nondegenerate limit random variable. ($U(t) \equiv U_{(i)}(t)$, β_n as before). We now assume that the limit corresponding to T^*, $T^*_{(i)}$ say, has some mass on $(0, \infty)$. Then by taking the log of $\dfrac{Z_n + 1}{W_{(i)}(\frac{1}{\beta_n})}$ and multiplying by $1/W_{(i-1)}(\frac{1}{\beta_n})$, then taking the log again and multiplying the latter by $1/W_{(i-2)}(\frac{1}{\beta_n})$ and continuing in this fashion for $(i-3)$, $(i-4)$.... we ultimately obtain that

$$(*) \quad \beta_n U_{(i)}(Z_n + 1) \underset{n\to\infty}{\overset{a.s}{\to}} 1.$$

The latter result contradicts our general result that $\{\beta_n U_{(i)}(Z_n + 1)\}$ is convergent in law to a nondegenerate limit distribution, with distribution function continuous and strictly increasing on $(0,\infty)$. Since the probability concentration at a point is zero the trajectories such that (*) holds have zero probability, which contradicts our earlier assumption regarding $T^*_{(i)}$. So, again a direct martingale approach is unsuccessful.

SECTION 3:

The Case with Immigration

When independent immigration is permitted we denote this process by $\{X_n\}$ and under the conditions stated earlier ($F'(1-) = \infty$, $F(s)$ the p.g.f. of the offspring distribution; $B(o) \neq 1$, $B(s)$ generates the distribution of the number of immigrants entering at a given generation) we shall generalize theorem 1.1 to the immigration case; these results are summarised in theorem 4.1 which we now prove.

PROOF OF THEOREM 4.1:

Assume conditions (A) and (B) of theorem 1.1 hold. Let $P_n(s)$ denote the p.g.f. of X_n, where it is well known [1] that

$$P_n(s) = \prod_{k=1}^{n} B(F_k(s))$$

and we assume $P_o(O) = 1$.

Now we consider the behaviour of the following function $p_n(t)$ for $t > O$, where

$$p_n(t) = E\left[\left(1 - \frac{(1-q)}{W\left(\frac{t}{\beta_n}\right)}\right)^{X_n}\right]$$

$$= P_n\left(1 - \frac{(1-q)}{W(t/\beta_n)}\right)$$

$$= \prod_{m=1}^{n} B\left(1 - \frac{(1-q)}{W\left(\emptyset_{n-m}\left(\frac{t}{\beta_n}\right)\right)}\right),$$

the last equality follows from the recurrence relation for $P_n(s)$ stated above.

Recall that

$$\phi_n\left(\frac{t}{\beta_n}\right) \underset{n\to\infty}{\to} \Delta(t) \qquad\qquad 0 < t < \infty$$

where $\Delta(t)$ satisfies the following equation of the Poincaré form

$$\Delta(c\,t) = \phi(\Delta(t)) \qquad\qquad 0 < t < \infty.$$

By iteration and consideration of inverses we have

$$\phi_n(t/\beta_n) = \Delta\left(c^n\,\Delta^{-1}\left(\frac{t}{\beta_n}\right)\right),$$

thus the continuity of functions considered yields

$$(4.6) \qquad c^n\,\Delta^{-1}\,(t/\beta_n) \to t \qquad \text{as} \qquad n \to \infty.$$

We note the relations

$$\phi_{n-m}\,(t/\beta_n) = \phi_{n-m}\,\left(\frac{t}{\beta_{n-m}} \cdot \frac{\beta_{n-m}}{\beta_n}\right)$$

where as $n \to \infty \left\{\frac{\beta_{n-m}}{\beta_n}\right\} \to c^{-m}$, also

$$\phi_{n-m}\,\left(\frac{t/c^m}{\beta_{n-m}}\right) = \Delta\left(c^{n-m}\,\Delta^{-1}\,\left(\frac{t}{c^m\,\beta_{n-m}}\right)\right).$$

$$\underset{n\to\infty}{\to} \Delta(t/c^m) \quad \text{by} \quad (4.6).$$

Thus we have as $n \to \infty$

$$(4.7) \qquad P_n(t) \to \prod_{m=1}^{\infty} B(u(t/c^m)) = p(t), \text{ say} \qquad 0 < t < \infty,$$

where as in Chapter I

$$u(t) = 1 - \frac{(1-q)}{W\,(\Delta(t))} \qquad\qquad 0 < t < \infty.$$

Clearly $p(c\,t) = p(t)\,B(u\,(t))$, and letting $t \to 0+$ implies $p(0+) = 0$.

Lemma 4.2:

The function $p(t)$ is positive for all $t > 0$ if

$$\sum_{j=2}^{\infty} b_j \log U(j) < \infty \tag{4.8}$$

and zero for all $t > 0$ otherwise.

PROOF:

We note that $\Delta(t)$ is strictly increasing to infinity on $(0, \infty)$ and continuous thereon. For $t > 0$ fixed $K_1\, a^n \leq \Delta(t/c^n) \leq K_2\, b^n$ for $n = 1,2...$ and some constants $0 < K_1, K_2 < \infty$ and $1 < a < b < \infty$. These inequalities follow since $\Delta(t) = t\,L(t)$ and for any slowly varying function at infinity L, we have for any fixed $\delta > 0$, $t^{\delta}\,L(t) \to \infty$ and $t^{-\delta}\,L(t) \to 0$ as $t \to \infty$.

We argue as in [12] and appeal to the inequality, $e^{-x/2} \leq 1 - x < e^{-x}$, true for x positive and sufficiently small. It follows from (4.7) and the criterion for convergence of infinite products that the proposition of the lemma is true if (4.8) is tantamount to the finiteness of

$$\sum_n [1 - B\,(\exp\,\{-G/W(K\delta^n)\})]$$

for any positive constants G, K and $\delta > 1$. So by Cauchy's integral test and Fubini's theorem we require to show that (4.8) is equivalent to the finiteness of

$$\sum_j b_j\, I_j$$

where

$$I_j = \int_A^\infty [1 - \exp \{\frac{-j^G}{W(K\delta^x)}\}] \, dx.$$

and A > O is a fixed positive number. By change of varaible to

$$z = \frac{W(K\delta^x)}{j}$$

we obtain

$$I_j = \frac{1}{\log \delta} \int_{z=\frac{B}{j}}^{\infty} \{1 - \exp(\frac{-G}{z})\} \frac{d\,U(zj)}{U(zj)}$$

where B is a positive constant independent of j. We shall estimate the behaviour of I_j for large j. For j sufficiently large we rewrite the integral as

$$\frac{1}{\log\delta} \int_{j=\frac{B}{j}}^{1} \{1 - \exp(\frac{-G}{z})\} \frac{dU(zj)}{U(zj)} + \frac{1}{\log\delta} \int_{1}^{\infty} \{1 - \exp(\frac{-G}{z})\} \frac{dU(zj)}{U(zj)} .$$

The first integral is bounded above and below by positive numbers independent of j, each multiplied by the factor

$$\int_{j=\frac{B}{j}}^{1} \frac{d\,U(zj)}{U(zj)} = \log U(j) - \log U(B).$$

Likewise, the second integral is bounded by positive multiples of

$$\int_1^\infty \frac{1}{z} \frac{d\,U(zj)}{U(zj)} = -\log U(j) + j \int_j^\infty \frac{\log U(y)}{y^2} dy ,$$

where the right-hand side follows by integration by parts. We rewrite this as

$$= \log U(j) \left\{ -1 + \frac{\int_j^\infty y^{-2} \log U(y)\, dy}{j^{-1} \log U(j)} \right\}$$

$$= \log U(j) \,.\, o(1) \qquad \text{as } j \to \infty,$$

the latter is obtained by a known property of regularly varying functions [p. 87, 19], since $\log U(y)$ is slowly varying. ∎

We continue with the proof of theorem 4.1. Now if (4.8) is not true we may apply Lemma 1.3 with $Z_n = X_n$, $u(t) \equiv 0$ to give us that $\beta_n U(X_n + 1)$ approaches infinity in distribution and hence in probability.

If (4.8) holds, $p(t)$ is a proper and continuous distribution function on $[0,\infty)$. In particular it is the distribution function of

$$I' = \sum_{m=1}^{\infty} \frac{Y_m}{c^m}$$

where Y_m are identically and independently distributed with distribution function $B(u(t))$ which is clearly nondegenerate. The conclusion then follows as in [20] via a result of Smith [21]. We note that according to [21], I' is finite with probability one if and only if $E[\log^+ Y_m] < \infty$, and after some manipulation we can see that this is tantamount to (4.8).

Hence if (4.8) holds we can employ lemma 1.3 with $Z_n = X_n$, $u(t) = p(t)$ to give the required conclusion of theorem 4.1 in terms of convergence in distribution. In this case almost sure convergence can be obtained by a relatively simple modification of theorem 2 of Cohn [2]; this we shall omit.

We note also that $\{T_n^*\}$ of section 2(b), of this chapter is a submartingale in the immigration case. Analogous to the non immigration case it does not provide us with a means of validating almost sure convergence via the submartingale convergence theorem; however, as mentioned earlier when (4.8) holds almost sure convergence may be obtained by an adaption of Cohn's theorem.

This completes the proof of the theorem. ∎

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