

The affine curve-lengthening flow

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Abstract. The motion of any smooth closed convex curve in the plane in the direction of steepest increase of its affine arc length can be continued smoothly for all time. The evolving curve remains strictly convex while expanding to infinite size and approaching a homothetically expanding ellipse.

1. Introduction

In this paper we study an affine-geometric, fourth-order parabolic evolution equation for closed convex curves in the plane. This is defined by following the direction of steepest ascent of the affine arc length functional L , with respect to a natural affine-invariant inner product. The definitions of these are given in Section 2, along with a discussion of other aspects of the affine differential geometry of curves.

The evolution equation can be written as follows:

$$(1) \quad \frac{\partial x}{\partial t} = -\mathcal{K}\mathcal{N}$$

where \mathcal{N} is the affine normal vector, and \mathcal{K} is the affine curvature. In terms of Euclidean-geometric invariants, this can be written as follows:

$$\frac{\partial x}{\partial t} = \left(\frac{1}{3} \frac{\partial}{\partial u} \left(\kappa^{-5/3} \frac{\partial \kappa}{\partial u} \right) + \kappa^{4/3} \right) \left(\kappa^{1/3} \mathbf{n} + \frac{1}{3} \kappa^{-5/3} \frac{\partial \kappa}{\partial u} \mathbf{t} \right)$$

where κ is the Euclidean curvature of the curve, u is an anticlockwise Euclidean arc-length parameter, \mathbf{n} is the outward unit normal, and \mathbf{t} is the Euclidean unit tangent vector. This is a fifth-order system of partial differential equations which is equivalent to a scalar fourth-order parabolic equation (see Section 3). Equation (1) is the parabolic analogue of the affine-geometric maximal surface equation, which has been an important problem in affine differential geometry, and with regard to which many problems remain unsolved (see [NS] and [LSZ]).

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The analogous Euclidean-geometric evolution equation, the curve-shortening flow, has been studied extensively by Gage ([Ga1], [Ga2]), Gage and Hamilton [GH], Grayson [Gr] and many others. For that evolution equation it has been shown that any initial closed, smooth embedded curve produces a solution which exists for a finite time, contracting at the end of that time to a single point, in such a way that the curves can be rescaled about the final point to converge to a circle. The curve-shortening flow is a second-order partial differential equation, and many details of its analysis are based on the parabolic maximum principle, which is not available to us in the present problem. However, the result we prove in this paper is somewhat analogous to that for the curve-shortening flow:

Theorem 1. *Let $x_0 : \mathcal{C} \rightarrow \mathbb{R}^2$ be a smooth, strictly locally convex embedding of a curve \mathcal{C} into the plane. Then there exists a unique, smooth solution $\{x_t\}$ of the evolution equation (1), which exists for all positive times. There exists a smooth embedding $x_\infty : \mathcal{C} \rightarrow \mathbb{R}^2$, a point $p \in \mathbb{R}^2$, and a real number $t_0 \geq -\frac{3}{8} \left(\frac{A[\mathcal{C}_0]}{\pi} \right)^{4/3}$, such that the image of x_∞ is an ellipse of enclosed area π with centre at the origin, and*

$$\left\| x_t - \left(\frac{8}{3} (t - t_0) \right)^{3/8} x_\infty - p \right\|_{C^k} \leq C(k, \varepsilon) t^{-(4-\varepsilon)} \quad \text{for any } \varepsilon > 0.$$

Any ellipse evolves by expanding homothetically about its centre.

Some other evolution equations of higher order for curves in the plane have been considered before – Polden [Po1] has considered the L^2 gradient flow of the integral of the square of the curvature, which is a fourth-order parabolic equation (note that Langer and Singer [LS] earlier introduced a different gradient flow, in which the integral of the squared curvature is treated as a functional on the space of $W^{2,1}$ indicatrices of curves of fixed length – this produces a non-local flow of zeroth order; Wen [W1–2] has also considered the gradient flow of the same functional on the space of L^2 indicatrices of curves of fixed length, which is a second order parabolic equation). Higher order equations have also been considered for the evolution of Riemannian metrics, by Rendall [Re], Singleton [Si], Chruściel [Ch], and Polden [Po2].

The main result used in characterising the long-term behaviour of solutions (presented in Section 4) is closely related to integral estimates which the author developed in applications to second-order evolution equations for curves and for hypersurfaces ([A1]–[A3]). The regularity estimates for the flow, given here in Section 7, are inspired by the work of Alexander Polden [Po1–2] on higher order evolution equations for curves and for metrics on surfaces. The equation under investigation here has some remarkable special features not shared by most other higher-order equations, however: First, we have the advantage of a large group of symmetries (namely, the special affine group); second, the equation has a certain degeneracy (see Eq. (19)) which has important consequences for the analysis – although in many situations such degeneracy may be considered a disadvantage, it is vital to the argument of the present paper, because non-degenerate fourth-order equations do not preserve pointwise inequalities, and in particular could not keep arbitrary convex curves convex.

The organisation of the paper is as follows: In Section 2 we summarise various aspects of the affine differential geometry of curves, support functions and their use in describing convex sets, and other results required in the later analysis. In Section 3 we introduce the evolution equation, establish the short-time existence of solutions, and discuss the normalisation of solution curves using scalings, translations, and area-preserving affine boosts. Section 4 accomplishes the key step in the proof, using the Brunn-Minkowski theorem about concavity of the area functional to deduce an integral estimate with two important consequences: First, it implies that solutions which exist for all time must evolve purely by scaling in the large-time limit; and secondly it implies that the evolving curve remains strictly convex as long as it exists, as we demonstrate in Section 5. Section 6 completes our characterisation of the long-term behaviour by showing that the only solutions which evolve purely by scaling are ellipses. This is proved by partially integrating the fourth-order ordinary differential equation characterising such solutions, and deriving a contradiction to the affine isoperimetric inequality in all cases except ellipses. In particular we make use of an affine-geometric version of the four-vertex theorem (Proposition 18). Section 7 provides regularity estimates which imply that solutions exist for all time, and Section 8 combines all these elements to complete the proof of Theorem 1.

We note that the classification of homothetic solutions for this evolution equation has also been carried out by Lima and Montenegro [LM], and their analysis also gives a classification of non-embedded homothetic solutions. A different affine-geometric evolution equation, the flow by the affine normal vector, has been studied by Tannenbaum and Sapiro [TS1–2] and the author [A4]. That evolution equation is of second order, and the methods associated with it are more closely related to those applied to the curve-shortening flow.

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2. Background and notation

2.1. Support functions. Let \mathcal{C} be a closed, smooth, convex embedded curve in the plane, and Ω the region enclosed by \mathcal{C} . The support function $h : S^1 \rightarrow \mathbb{R}$ of \mathcal{C} (or Ω) is defined by

$$(2) \quad h(z) = \sup_{y \in \mathcal{C}} \langle y, z \rangle.$$

The support function contains all geometric information about the curve. In particular, there is a natural embedding $x : S^1 \rightarrow \mathbb{R}^2$ with image equal to \mathcal{C} , defined by

$$(3) \quad \bar{x}(z) = h(z)z + \frac{\partial h(z)}{\partial \theta} \frac{\partial z}{\partial \theta}$$

for all $z \in S^1$, where we think of S^1 as naturally included in \mathbb{R}^2 , and θ is the usual angle coordinate on S^1 . This embedding associates to a point z in S^1 the point of \mathcal{C} with normal direction z . The radius of curvature of \mathcal{C} at $\bar{x}(z)$ is given by

$$(4) \quad r = h_{\theta\theta} + h,$$

where the subscripts denote derivatives. More generally, for a C^2 function $f: S^1 \rightarrow \mathbb{R}$ we define $r[f] = f_{\theta\theta} + f$, and $x[f] = f(z)z + f_{\theta}z_{\theta}$. These are related by the fact that $\frac{\partial}{\partial\theta} x[f] = r[f] \frac{\partial z}{\partial\theta}$. Any function f with $r[f] > 0$ everywhere is the support function of a convex curve, given by the embedding $\bar{x}[f]$.

2.2. The Brunn-Minkowski Theorem. Let Ω_i , $i = 0, 1$ be bounded open convex sets in \mathbb{R}^2 , with support functions h_i . Then their Minkowski sum $\Omega_0 + \Omega_1$ is the convex set $\Omega_0 + \Omega_1 = \{a + b : a \in \Omega_0, b \in \Omega_1\}$. Let h be the support function of $\Omega_0 + \Omega_1$. Then

$$\begin{aligned} h(z) &= \sup_{y \in \Omega_0 + \Omega_1} \langle y, z \rangle \\ &= \sup_{a \in \Omega_0, b \in \Omega_1} \langle a + b, z \rangle \\ &= \sup_{a \in \Omega_0} \langle a, z \rangle + \sup_{b \in \Omega_1} \langle b, z \rangle \\ &= h_0(z) + h_1(z), \end{aligned}$$

so support functions are additive under Minkowski addition.

The Brunn-Minkowski Theorem ([Br], [M], [B13], sections 21–22) concerns the concavity properties of the area functional under Minkowski addition:

Theorem 2 (The Brunn-Minkowski Theorem). *For any bounded convex sets Ω_0 and Ω_1 in \mathbb{R}^2 , $A[(1-t)\Omega_0 + t\Omega_1]^{1/2}$ is a concave function of $t \in [0, 1]$, and is strictly concave unless $c_0\Omega_0 = c_1\Omega_1 + p$ for some $c_i \geq 0$ and some $p \in \mathbb{R}^2$.*

The Brunn-Minkowski Theorem plays a crucial role in the proof of our main estimate in Section 4.

2.3. Affine differential geometry of curves. Affine differential geometry concerns those properties of \mathcal{C} which are invariant under the action of area-preserving affine transformations of \mathbb{R}^2 . An affine-geometric arc-length element ds can be defined by

$$(5) \quad ds = \left| \frac{\partial x}{\partial u} \times \frac{\partial^2 x}{\partial u^2} \right|^{1/3} du$$

for any non-degenerate parameter u . Since the quantity in the bracket is just the area of the parallelogram formed by $\partial x/\partial u$ and $\partial^2 x/\partial u^2$, and the special affine transformations preserve area and equivariantly transform the first and second derivatives, this definition is affine-invariant; the invariance under reparametrisation is clear. The affine arc-length parameter s is obtained by integrating ds . Then $\mathcal{T} = \frac{\partial x}{\partial s}$ is the affine unit tangent vector, and $\mathcal{N} = \frac{\partial^2 x}{\partial s^2}$ is the affine normal vector.

In order to write these in terms of Euclidean invariants, we take u to be Euclidean arc-length, \mathbf{t} the Euclidean unit tangent vector x_u , and \mathbf{n} the Euclidean unit normal. Then $ds = \kappa^{1/3} du = r^{2/3} d\theta$ where κ is the Euclidean curvature of \mathcal{C} , so s is a non-degenerate parameter if \mathcal{C} is strictly convex. In that case $\mathcal{T} = \kappa^{-1/3} \mathbf{t}$ and

$$\mathcal{N} = -\left(\kappa^{1/3} \mathbf{n} + \frac{1}{\kappa} \frac{\partial}{\partial u} (\kappa^{1/3}) \mathbf{t}\right) = -x[r^{-1/3}].$$

The definition of s , with the convention that the curve be parametrised anticlockwise, gives that $\mathcal{T} \times \mathcal{N} = 1$ everywhere, so \mathcal{N} and \mathcal{T} are linearly independent at each point, and we can define an affine-invariant inner product $\langle\langle \dots, \dots \rangle\rangle$ on the bundle $x_* \mathbb{R}^2$ over \mathcal{C} by taking \mathcal{N} and \mathcal{T} to be an orthonormal basis.

The affine curvature \mathcal{K} is defined by $\mathcal{K} = -\langle\langle \frac{\partial \mathcal{N}}{\partial s}, \mathcal{T} \rangle\rangle$. Since $\mathcal{T} \times \mathcal{N} = 1$, we have $0 = \frac{\partial \mathcal{T}}{\partial s} \times \mathcal{N} + \mathcal{T} \times \frac{\partial \mathcal{N}}{\partial s} = \mathcal{N} \times \mathcal{N} + \mathcal{T} \times \frac{\partial \mathcal{N}}{\partial s} = \mathcal{T} \times \frac{\partial \mathcal{N}}{\partial s}$. Therefore $\frac{\partial \mathcal{N}}{\partial s} = -\mathcal{K} \mathcal{T}$. In terms of the support function, $\mathcal{K} = r^{-1} r[r^{-1/3}]$.

If V is an \mathbb{R}^2 -valued vector field on \mathcal{C} , then $V = \langle\langle V, \mathcal{T} \rangle\rangle \mathcal{T} + \langle\langle V, \mathcal{N} \rangle\rangle \mathcal{N}$. Differentiating, we obtain expressions for the derivatives of $\langle\langle V, \mathcal{T} \rangle\rangle$ and $\langle\langle V, \mathcal{N} \rangle\rangle$ along \mathcal{C} :

$$(6) \quad \langle\langle V, \mathcal{N} \rangle\rangle_s = \langle\langle V_s, \mathcal{N} \rangle\rangle - \langle\langle V, \mathcal{T} \rangle\rangle$$

and

$$(7) \quad \langle\langle V, \mathcal{T} \rangle\rangle_s = \langle\langle V_s, \mathcal{T} \rangle\rangle + \mathcal{K} \langle\langle V, \mathcal{N} \rangle\rangle.$$

We will also find it useful to consider the function $\sigma = hr^{1/3}$, which we call the *affine support function*. σ is invariant under special linear transformations, but not translations.

Useful affine integral invariants are the enclosed area $A = \frac{1}{2} \int_{S^1} hr[h] d\theta$ of the curve, the affine arc length $L = \int_{\mathcal{C}} ds$, and the total affine curvature $\int_{\mathcal{C}} \mathcal{K} ds$.

We note some useful identities: First, the enclosed area can be written as follows:

$$(8) \quad A = \frac{1}{2} \int_{S^1} hr d\theta = \frac{1}{2} \int_{\mathcal{C}} hr^{1/3} r^{2/3} d\theta = \frac{1}{2} \int_{\mathcal{C}} \sigma ds.$$

The affine arc length can be written in several ways:

$$(9) \quad L = \int_{\mathcal{C}} ds = \int_{S^1} r^{2/3} d\theta = \int_{S^1} r^{-1/3} r d\theta = \int_{S^1} hr[r^{-1/3}] d\theta = \int_{S^1} \sigma \mathcal{K} ds$$

where we integrated by parts twice.

Given $f \in C^\infty(S^1)$, write $F = fr^{1/3}$. From the definition of $x[f]$ we have

$$\begin{aligned}
 (10) \quad x[f] &= fz + f_\theta z_\theta \\
 &= f(-r^{1/3}\mathcal{N} - (r^{-1/3})_\theta \mathcal{T}) + f_\theta r^{-1/3} \mathcal{T} \\
 &= -F\mathcal{N} + (f(r^{1/3})_s + f_s r^{1/3}) \mathcal{T} \\
 &= -F\mathcal{N} + F_s \mathcal{T}.
 \end{aligned}$$

Differentiating, we find

$$x[f]_s = (F_{ss} + \mathcal{K}F) \mathcal{T},$$

and the left hand side can be rewritten as

$$\begin{aligned}
 x[f]_s &= x[f]_\theta \theta_s \\
 &= r[f] z_\theta r^{-2/3} \\
 &= \frac{r[f]}{r} \mathcal{T}.
 \end{aligned}$$

Combining these, we obtain the identity

$$(11) \quad \frac{r[f]}{r} = F_{ss} + \mathcal{K}F.$$

In particular, the cases $F = \sigma$ and $F = \langle\langle \mathcal{N}, e \rangle\rangle$ for $e \in \mathbb{R}^2$ yield $x[f] = x$ and $x[f] = e$ respectively, and lead to the identities

$$(12) \quad 1 = \sigma_{ss} + \mathcal{K}\sigma$$

and

$$(13) \quad 0 = \langle\langle \mathcal{N}, e \rangle\rangle_{ss} + \mathcal{K}\langle\langle \mathcal{N}, e \rangle\rangle.$$

A famous result in affine geometry is the following, due to Blaschke (see [Bl1] or [Bl4], p. 61):

Theorem 3 (The affine isoperimetric inequality). *For any smooth, strictly convex curve \mathcal{C} , $L \leq 2\pi^{2/3} A^{1/3}$, with equality if and only if \mathcal{C} is an ellipse.*

The following result is also due to Blaschke in the planar case [Bl2]:

Theorem 4 (The Blaschke-Santaló inequality). *For any smooth, strictly convex \mathcal{C} with support function h there exists a point $p \in \mathbb{R}^2$ enclosed by \mathcal{C} such that*

$$A[\mathcal{C}] \int_{S^1} \frac{1}{(h - \langle z, p \rangle)^2} d\theta \leq 2\pi^2.$$

The equality can be made strict except when \mathcal{C} is an ellipse.

2.4. Affine-geometric variation formulae. In this paper we consider the flow of steepest ascent of L with respect to the inner product $\langle \dots, \dots \rangle$: That is, we wish to find a smooth family of embeddings x_t starting from a prescribed embedding x_0 , such that

$$\frac{dL}{dt} / \left(\int_{\mathcal{C}} \left\| \frac{\partial x}{\partial t} \right\|^2 ds \right)^{1/2}$$

is as large as possible at each time $t \geq 0$. In order to derive an expression for this evolution equation we consider the initial rate of change of L under an arbitrary smooth variation $\{x_t\}$. For convenience we write $\dot{x}|_{t=0} = \alpha \mathcal{N} + \beta \mathcal{T}$, and suppose without loss of generality that x_0 is an affine-arc-length parametrisation of its image.

The variation of the arc length element ds can be calculated from (5):

$$\begin{aligned} \left. \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial u} \times \frac{\partial^2 x}{\partial u^2} \right) \right|_{t=0} &= \frac{\partial}{\partial u} (\alpha \mathcal{N} + \beta \mathcal{T}) \times \mathcal{N} + \mathcal{T} \times \frac{\partial^2}{\partial u^2} (\alpha \mathcal{N} + \beta \mathcal{T}) \\ &= (\alpha \mathcal{N} + \beta \mathcal{T})_s \times \mathcal{N} + \mathcal{T} \times (\alpha \mathcal{N} + \beta \mathcal{T})_{ss} \end{aligned}$$

since at $t = 0$ the curve is parametrised by affine arc length. The above definitions give $\mathcal{T}_s = \mathcal{N}$ and $\mathcal{N}_s = -\mathcal{K}\mathcal{T}$, and we have $\mathcal{T} \times \mathcal{N} = 1$ everywhere. This gives

$$\left. \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial u} \times \frac{\partial^2 x}{\partial u^2} \right) \right|_{t=0} = \alpha_{ss} - 2\mathcal{K}\alpha + 3\beta_s,$$

and

$$\begin{aligned} (14) \quad \left. \frac{\partial}{\partial t} ds \right|_{t=0} &= \left. \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial u} \times \frac{\partial^2 x}{\partial u^2} \right)^{1/3} du \right|_{t=0} \\ &= \frac{1}{3} (\alpha_{ss} - 2\mathcal{K}\alpha + 3\beta_s) ds \end{aligned}$$

since $du = ds$ at $t = 0$. Integration over the curve gives the variation in L :

$$\begin{aligned} (15) \quad \frac{d}{dt} L|_{t=0} &= \frac{1}{3} \int_{\mathcal{C}} (\alpha_{ss} - 2\mathcal{K}\alpha + 3\beta_s) ds \\ &= -\frac{2}{3} \int_{\mathcal{C}} \mathcal{K}\alpha ds \\ &= -\frac{2}{3} \int_{\mathcal{C}} \langle \mathcal{K}\mathcal{N}, \dot{x} \rangle ds. \end{aligned}$$

The variation in enclosed area can also be written in affine-geometric language:

$$(16) \quad \frac{d}{dt} A|_{t=0} = - \int_{\mathcal{C}} \langle V, \mathcal{N} \rangle ds.$$

The total affine curvature changes as follows:

$$\begin{aligned}
 (17) \quad \frac{d}{dt} \int_{\mathcal{C}} \mathcal{K} ds|_{t=0} &= \frac{d}{dt} \int_{S^1} r^{-1/3} r [r^{-1/3}] d\theta|_{t=0} \\
 &= -\frac{2}{3} \int_{S^1} r^{-4/3} r [\dot{h}] r [r^{-1/3}] d\theta \\
 &= -\frac{2}{3} \int_{\mathcal{C}} \mathcal{K} ((\dot{h} r^{1/3})_{ss} + \mathcal{K} (\dot{h} r^{1/3})) ds \\
 &= \frac{2}{3} \int_{\mathcal{C}} \mathcal{K} (\alpha_{ss} + \mathcal{K} \alpha) ds
 \end{aligned}$$

where we used identity (11) in the third line.

3. The evolution equation

It follows from Eq. (15) that the steepest ascent flow for the affine arc length with respect to the affine L^2 metric is (up to a factor) the evolution equation

$$\frac{\partial x}{\partial t} = -\mathcal{K} \mathcal{N}.$$

This evolution equation is equivalent to a scalar, fourth-order parabolic equation, since in order to describe the evolving curve we need only specify the support function at each time. The support function evolves according to the equation

$$(18) \quad \frac{\partial h}{\partial t} = r^{-4/3} r [r^{-1/3}],$$

or more fully

$$(19) \quad \frac{\partial h}{\partial t} = -\frac{h_{\theta\theta\theta\theta} + h_{\theta\theta}}{3(h_{\theta\theta} + h)^{8/3}} + \frac{4(h_{\theta\theta\theta} + h_{\theta})^2}{9(h_{\theta\theta} + h)^{11/3}} + (h_{\theta\theta} + h)^{-5/3}.$$

It can be seen that there are special solutions given by $h(z, t) = \left(C + \frac{8}{3}t\right)^{3/8}$ for any C , which correspond to expanding circles centred at the origin. Eq. (19) is a nonlinear fourth-order scalar equation which is parabolic provided $h_{\theta\theta} + h > 0$ everywhere. Proposition 2.3 of [Lu] applies to show that if the initial support function is smooth and has $r > 0$, then there exists a unique, smooth solution of equation (19) for a short time. This solution can then be used to reconstruct a unique solution of the original equation (1) for a short time by the same argument as in [A1], Lemma I1.2.

The analysis of solutions of equation (1) is simplified by normalising the solution curves to have fixed enclosed area. Suppose $x : \mathcal{C} \times [0, T) \rightarrow \mathbb{R}^2$ is a solution of (1), and

define $\tilde{x}(p, \tau) = \sqrt{\frac{\pi}{A[\mathcal{C}_t]}} x_t$, where $\tau(t) = \int_0^t \left(\frac{\pi}{A[\mathcal{C}_t]}\right)^{4/3} dt'$. Then \tilde{x} satisfies a modified evolution equation which we derive as follows, where we denote with a tilde those invariants associated with the rescaled solution \tilde{x} :

$$\begin{aligned} (20) \quad \frac{\partial}{\partial \tau} \tilde{x} &= \frac{\partial t}{\partial \tau} \frac{\partial}{\partial t} \left(\sqrt{\frac{\pi}{A[\mathcal{C}_t]}} x \right) \\ &= \left(\frac{A[\mathcal{C}_t]}{\pi}\right)^{4/3} \left(-\sqrt{\frac{\pi}{A[\mathcal{C}_t]}} \mathcal{K}\mathcal{N} - \frac{1}{2\pi} \left(\frac{\pi}{A[\mathcal{C}_t]}\right)^{3/2} \int_{\mathcal{C}_t} \mathcal{K} ds \right) \\ &= -\tilde{\mathcal{K}}\tilde{\mathcal{N}} - \frac{\tilde{x}}{2\pi} \int_{\tilde{\mathcal{C}}} \tilde{\mathcal{K}} ds, \end{aligned}$$

since $\tilde{\mathcal{K}} = \left(\frac{A[\mathcal{C}_t]}{\pi}\right)^{2/3} \mathcal{K}$, $\tilde{\mathcal{N}} = \left(\frac{A[\mathcal{C}_t]}{\pi}\right)^{1/6} \mathcal{N}$, and $d\tilde{s} = \left(\frac{A[\mathcal{C}_t]}{\pi}\right)^{-1/3} ds$.

Since the equation (1) is equivariant under special affine transformations, it is also useful to normalise the solutions by performing translations to centre them as much as possible at the origin (in view of Theorem 4, it is useful to translate to minimise $\int_{S^1} h^{-2} d\theta$), and also to apply optimal special linear transformations (we choose to minimise the Euclidean length $\ell = \int_{\mathcal{C}} du$ of the curve). These requirements are equivalent to the following:

The enclosed area condition

$$(21a) \quad A[\mathcal{C}] = \pi$$

the translation normalisation conditions

$$(21b) \quad 0 = \int_{\mathcal{C}} h^{-3} \langle z, e \rangle = \int_{\mathcal{C}} \sigma^{-3} \langle \mathcal{N}, e \rangle ds$$

for all $e \in \mathbb{R}^2$, and the length-minimisation condition

$$(21c) \quad 0 = \int_{\mathcal{C}} D_{L_i} h d\theta$$

for $i = 1, 2$, where $L_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In the remainder of the paper, *normalised curves* are smooth, strictly convex embedded closed curves satisfying the normalisation conditions (21a–c).

The resultant normalised solution, which we denote by \hat{x} , satisfies the following evolution equation:

$$(22) \quad \frac{\partial}{\partial \tau} \hat{x} = -\hat{\mathcal{X}}\hat{\mathcal{N}} - \frac{\hat{x}}{2\pi} \int_{\hat{\mathcal{C}}} \hat{\mathcal{X}} d\hat{s} - \sum_{i,j=1}^2 (M^{-1})^{ij} \int_{\hat{\mathcal{C}}} \frac{\hat{\mathcal{X}}}{\hat{\sigma}^4} \langle\langle e_i, \mathcal{N} \rangle\rangle d\hat{s} e_j \\ - \sum_{i,j=1}^2 (N^{-1})^{ij} \int_{\hat{\mathcal{C}}} \hat{\mathcal{X}} G_i d\hat{s} L_j \hat{x}$$

where $G_1 = \hat{r}^{-5/3} \cos 2\theta$ and $G_2 = \hat{r}^{-5/3} \sin 2\theta$; M is the matrix with elements

$$M_{ij} = \int_{\hat{\mathcal{C}}} \hat{\sigma}^{-4} \langle\langle e_i, \mathcal{N} \rangle\rangle \langle\langle e_j, \mathcal{N} \rangle\rangle d\hat{s};$$

N is the matrix

$$\frac{\ell}{2} - \frac{1}{6} \begin{bmatrix} \int_{\hat{\mathcal{C}}} \hat{r} \cos 4\theta d\theta & \int_{\hat{\mathcal{C}}} \hat{r} \sin 4\theta d\theta \\ \int_{\hat{\mathcal{C}}} \hat{r} \sin 4\theta d\theta & -\int_{\hat{\mathcal{C}}} \hat{r} \cos 4\theta d\theta \end{bmatrix};$$

and finally, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note that both M and N are positive definite, since $\text{tr } M = \int_{\hat{\mathcal{C}}} h^{-4} d\theta > 0$ and $\det M = \int_{\hat{\mathcal{C}}} h^{-4} \cos^2 \theta d\theta \int_{\hat{\mathcal{C}}} h^{-4} \sin^2 \theta d\theta - (\int_{\hat{\mathcal{C}}} h^{-4} \cos \theta \sin \theta d\theta)^2 > 0$, while $\text{tr } N = \ell > 0$ and

$$\det N = \frac{1}{4} \ell^2 - \frac{1}{36} \left(\left(\int_{\hat{\mathcal{C}}} \hat{r} \cos 4\theta d\theta \right)^2 + \left(\int_{\hat{\mathcal{C}}} \hat{r} \sin 4\theta d\theta \right)^2 \right) \\ \geq \frac{1}{4} \ell^2 - \frac{1}{36} \left(\int_{\hat{\mathcal{C}}} \hat{r} d\theta \int_{\hat{\mathcal{C}}} \hat{r} \cos^2 4\theta d\theta + \int_{\hat{\mathcal{C}}} \hat{r} d\theta \int_{\hat{\mathcal{C}}} \hat{r} \sin^2 4\theta d\theta \right) \\ = \frac{2}{9} \ell^2 \\ > 0.$$

The support function \hat{h} corresponding to the fully normalised curves $\hat{\mathcal{C}}$ evolves according to the equation

$$(23) \quad \frac{\partial}{\partial \tau} \hat{h} = \hat{r}^{-4/3} r[\hat{r}^{-1/3}] - \frac{\hat{h}}{2\pi} \int_{S^1} \hat{r}^{-1/3} r[\hat{r}^{-1/3}] d\theta \\ - \left[\int_{\hat{\mathcal{C}}} \frac{r[\hat{r}^{-1/3}]}{\hat{r}^2} \cos 2\theta d\theta \int_{\hat{\mathcal{C}}} \frac{r[\hat{r}^{-1/3}]}{\hat{r}^2} \sin 2\theta d\theta \right] N^{-1} \begin{bmatrix} \hat{h} \cos 2\theta - \hat{h}_\theta \sin 2\theta \\ \hat{h} \sin 2\theta + \hat{h}_\theta \cos 2\theta \end{bmatrix} \\ - \left[\int_{\hat{\mathcal{C}}} \frac{r[\hat{r}^{-1/3}]}{\hat{h}^4 \hat{r}^{4/3}} \cos \theta d\theta \int_{\hat{\mathcal{C}}} \frac{r[\hat{r}^{-1/3}]}{\hat{h}^4 \hat{r}^{4/3}} \sin \theta d\theta \right] M^{-1} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}.$$

4. Convexity of the area functional

In this section we prove a crucial estimate which will control the (Euclidean) curvature of the curve, and also imply that the curve is asymptotically homothetically expanding. The result is the following:

Theorem 5. *For any smooth, strictly convex solution $x : \mathcal{C} \times [0, T) \rightarrow \mathbb{R}^2$ of Eq. (1), $A[\mathcal{C}_t]^{4/3}$ is a convex function of t , strictly convex unless there exists some $c > 0$ and some point $p \in \mathbb{R}^2$ such that $\mathcal{K} = c\tau^{1/3} \langle x - p, z \rangle$, in which case the curve is homothetically expanding about p :*

$$\mathcal{C}_t = p + \left(1 + \frac{8ct}{3}\right)^{3/8} (\mathcal{C}_0 - p).$$

Proof. The statement of this theorem has an obvious similarity to the statement of the Brunn-Minkowski Theorem (Theorem 2), and indeed the main step in the proof is an application of the Brunn-Minkowski Theorem.

In order to make use of Theorem 2, we first derive from it a family of inequalities: Let \mathcal{C} be a fixed smooth, strictly convex curve enclosing a region Ω with support function h , and let f be a fixed smooth function on \mathcal{C} . For ε sufficiently small, the function $h + \varepsilon f$ is the support function of a smooth, strictly convex region Ω_1 . Consider the Brunn-Minkowski Theorem applied in this case, with $\Omega_0 = \Omega$. Then $\Omega_t = (1 - t)\Omega_0 + t\Omega_1$ has support function $h_t = (1 - t)h + t(h + \varepsilon f) = h + t\varepsilon f$. Let $F = f\tau[h]^{1/3}$. We compute:

$$\frac{d}{dt} A[\Omega_t] = \varepsilon \int_{S^1} f\tau[h] d\theta = \varepsilon \int_{\mathcal{C}} F ds$$

and

$$\frac{d^2}{dt^2} A[\Omega_t] = \varepsilon^2 \int_{S^1} f\tau[f] d\theta = \varepsilon^2 \int_{\mathcal{C}} F(F_{ss} + \mathcal{K}F) ds,$$

where we used the identity (11). The concavity of $A[\Omega_t]^{1/2}$ is equivalent to the inequality

$$\begin{aligned} 0 &\geq \frac{d^2}{dt^2} A^{1/2} = \frac{1}{2} A^{-1/2} \left(\frac{d^2 A}{dt^2} - \frac{1}{2A} \left(\frac{dA}{dt} \right)^2 \right) \\ &= \frac{1}{2} A^{-1/2} \left(\int_{\mathcal{C}} F(F_{ss} + \mathcal{K}F) ds - \frac{1}{2A} \left(\int_{\mathcal{C}} F ds \right)^2 \right). \end{aligned}$$

Now integrate by parts and note that F is arbitrary to prove the following:

Lemma 6 (Affine-geometric Wirtinger inequality). *For any smooth, strictly convex embedded curve \mathcal{C} and any smooth $F : \mathcal{C} \rightarrow \mathbb{R}$,*

$$\int_{\mathcal{C}} F_s^2 ds \geq \int_{\mathcal{C}} \mathcal{K}F^2 ds - \frac{1}{2A} \left(\int_{\mathcal{C}} F ds \right)^2,$$

with equality if and only if $F = c\sigma + \langle q, \mathcal{N} \rangle$ for some $c \in \mathbb{R}$ and some $q \in \mathbb{R}^2$.

Now we continue the proof of Theorem 5 by computing the second time derivative of $A[C_t]^{4/3}$, using the variation equations (16)–(17):

$$\begin{aligned}
 (24) \quad \frac{d^2}{dt^2} A^{4/3} &= \frac{4}{3} \frac{d}{dt} (A^{1/3} \int_{\mathcal{C}} \mathcal{K} ds) \\
 &= -\frac{8}{9} A^{1/3} \int_{\mathcal{C}} \mathcal{K} (\mathcal{K}_{ss} + \mathcal{K}^2) ds + \frac{4}{9} A^{-2/3} (\int_{\mathcal{C}} \mathcal{K} ds)^2 \\
 &= \frac{8}{9} A^{1/3} (\int_{\mathcal{C}} \mathcal{K}_s^2 ds - \int_{\mathcal{C}} \mathcal{K}^3 ds + \frac{1}{2A} (\int_{\mathcal{C}} \mathcal{K} ds)^2) \\
 &\geq 0
 \end{aligned}$$

by Lemma 6 with $F = \mathcal{K}$. Furthermore, by Lemma 6 equality occurs only if

$$\mathcal{K} = c\sigma + \langle\langle q, \mathcal{N} \rangle\rangle$$

for some constant c and some point $q \in \mathbb{R}^2$. Note that c cannot equal zero, since then we would have $\mathcal{K} = \langle\langle q, \mathcal{N} \rangle\rangle$. Multiplying by $\langle\langle q, \mathcal{N} \rangle\rangle$ and integrating over \mathcal{C} , we obtain using (7) with $V = q$

$$\int_{\mathcal{C}} \langle\langle \mathcal{N}, q \rangle\rangle^2 ds = \int_{\mathcal{C}} \mathcal{K} \langle\langle \mathcal{N}, q \rangle\rangle ds = \int_{\mathcal{C}} \langle\langle \mathcal{T}, q \rangle\rangle_s ds = 0,$$

and so $q = 0$ and $\mathcal{K} \equiv 0$, which is impossible in view of identity (9). Therefore, taking $p = -q/c$ we find $\mathcal{K} = c(\sigma - \langle\langle p, \mathcal{N} \rangle\rangle)$. Hence the constant c must be positive, since otherwise we have $0 < L = \int_{S^1} \mathcal{K} (\sigma - \langle\langle p, \mathcal{N} \rangle\rangle) ds = c \int_{S^1} (\sigma - \langle\langle p, \mathcal{N} \rangle\rangle)^2 ds \leq 0$. Since

$$\mathcal{K} = c(\sigma - \langle\langle p, \mathcal{N} \rangle\rangle) = -c \langle\langle x - p, \mathcal{N} \rangle\rangle,$$

we have

$$\begin{aligned}
 -\mathcal{K}\mathcal{N} &= c \langle\langle x - p, \mathcal{N} \rangle\rangle \mathcal{N} \\
 &= c(x - p) - c \langle\langle x - p, \mathcal{T} \rangle\rangle (x - p)_s.
 \end{aligned}$$

Thus if x_0 is an affine-arc-length parametrisation of \mathcal{C} , Eq. (1) has the solution

$$x(\xi, t) = p + \varrho(t)(x_0(\varphi(\xi, t)) - p)$$

where $\varrho(t) = \left(1 + \frac{8ct}{3}\right)^{3/8}$, and $\varphi(x, t)$ is obtained by solving for each ξ the ordinary differential equation

$$(25) \quad \begin{aligned} \frac{\partial \varphi(\xi, t)}{\partial t} &= -cV(\varphi(\xi, t)), \\ \varphi(\xi, 0) &= \xi \end{aligned}$$

where $V(\xi) = \langle\langle x_0(\xi) - p, \mathcal{T} \rangle\rangle$. \square

The convexity of the enclosed area has the following important consequence:

Corollary 7. *For any smooth, strictly convex solution of Eq.(1), the quantity*

$$\mathcal{L} = 1 - \frac{1}{2\pi^{4/3}} A^{1/3} \int_{\mathcal{C}_t} \mathcal{K} ds$$

is non-negative and decreasing in time, strictly unless the solution curves are homothetically expanding about some centre.

The non-negativity of \mathcal{L} follows from Lemma 6 (with $F = 1$) together with Theorem 3. \mathcal{L} decreases since $\frac{d^2}{dt^2} A^{4/3} = \frac{4}{3} \frac{d}{dt} (A^{1/3} \int \mathcal{K} ds)$ is non-negative by the proof of Theorem 5, and strictly positive unless the solution is homothetic.

5. Preserving convexity and bounding the curvature

We will use the result of Corollary 7 to deduce upper and lower bounds for the Euclidean curvature of the solution \hat{x} of the rescaled equation (22). To prepare for this, we note the following automatic bounds which hold for any convex curve which is normalised by special affine transformations to satisfy (21 a–c), and so in particular for the solutions of Eq. (22):

Proposition 8. *There exist absolute positive constants c_1 and c_2 such that whenever \mathcal{C} satisfies (21 a–c),*

$$c_1 \leq h(z) \leq c_2$$

for all $z \in S^1$, where h is the support function of \mathcal{C} .

Proof. Write $h = h_0 + \sum_{k=1}^{\infty} h_k \cos k(\theta - \theta_k)$. By (21 c) we have for $i = 1, 2$

$$\begin{aligned} 0 &= D_{L_i} \ell \\ &= \int_{S^1} D_{L_i} h d\theta. \end{aligned}$$

We compute

$$\begin{aligned}
 D_{L_1} h &= D_{T_1} \langle x, z \rangle \\
 &= \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x, z \right\rangle \\
 &= \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} h \cos \theta - h_\theta \sin \theta \\ h \sin \theta + h_\theta \cos \theta \end{bmatrix}, \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \right\rangle \\
 &= h(\cos^2 \theta - \sin^2 \theta) - 2h_\theta \cos \theta \sin \theta \\
 &= h \cos 2\theta = h_\theta \sin 2\theta
 \end{aligned}$$

and similarly

$$D_{L_2} h = h \sin 2\theta + h_\theta \cos 2\theta.$$

Therefore

$$(26a) \quad 0 = \int_{S^1} h \cos 2\theta - h_\theta \sin 2\theta d\theta = 3 \int_{S^1} h \cos 2\theta d\theta,$$

and

$$(26b) \quad 0 = \int_{S^1} h \sin 2\theta + h_\theta \cos 2\theta d\theta = 3 \int_{S^1} h \sin 2\theta d\theta.$$

It follows that $h_2 = 0$.

Then $r = h_0 + \sum_{k=3}^{\infty} h_k (1 - k^2) \cos k(\theta - \theta_k) > 0$ and, integrating against the positive function $1 \pm \cos k(\theta - \theta_k)$ for any $k \geq 3$, we obtain

$$0 \leq \int_{S^1} r(1 \pm \cos k(\theta - \theta_k)) = 2\pi h_0 \pm \pi h_k (1 - k^2),$$

so that $|h_k| \leq \frac{2}{k^2 - 1} h_0$ for every $k \geq 3$.

Now we use the fact that the enclosed area is normalised to be π :

$$\begin{aligned}
 \pi &= \frac{1}{2} \int_{S^1} h r d\theta \\
 &= \pi h_0^2 + \frac{\pi}{2} \sum_{k=3}^{\infty} (1 - k^2) h_k^2 \\
 &\geq \pi h_0^2 \left(1 - \frac{1}{2} \sum_{k=3}^{\infty} \frac{4}{k^2 - 1} \right) \\
 &= \frac{1}{6} \pi h_0^2.
 \end{aligned}$$

Therefore $h_0 \leq \sqrt{6}$. For all θ_0 , we have since $h \geq 0$

$$\begin{aligned} h(\theta_0) &\leq |h(\theta_0) + h(\theta_0 + \pi)| \\ &= \frac{1}{2} \left| \int_{S^1} r |\sin(\theta - \theta_0)| d\theta \right| \\ &\leq \frac{1}{2} \int_{S^1} r d\theta \\ &= \pi h_0 \\ &\leq \pi \sqrt{6}. \end{aligned}$$

This proves the upper bound.

Fix θ_0 . Then \mathcal{C} is contained in $\{x \in \mathbb{R}^2 : |x| \leq \sqrt{6}\pi, \langle x, e^{i\theta_0} \rangle \leq h(\theta_0)\}$, so h satisfies

$$\frac{h(\theta)}{\sqrt{6}\pi} \leq \begin{cases} \frac{h(\theta_0)}{\sqrt{6}\pi} \cos(\theta - \theta_0) + \sqrt{1 - \frac{h(\theta_0)^2}{6\pi^2}} |\sin(\theta - \theta_0)|, & |\cos(\theta - \theta_0)| \geq \frac{h(\theta_0)}{\sqrt{6}\pi}, \\ 1, & \text{otherwise.} \end{cases}$$

Hence by Theorem 4,

$$\begin{aligned} 2\pi &\geq \int_{\mathcal{C}} \frac{1}{\sigma^2} ds = \int_{S^1} \frac{1}{h^2} d\theta \\ &\geq \frac{2}{6\pi^2} \int_0^{\arccos(h(\theta_0)/\sqrt{6}\pi)} \frac{d\theta}{\left(\frac{h(\theta_0)}{\sqrt{6}\pi} \cos \theta + \sqrt{1 - \frac{h(\theta_0)^2}{6\pi^2}} \sin \theta\right)^2} \\ &\quad + \frac{2}{6\pi^2} \left(\pi - \arccos\left(\frac{h(\theta_0)}{\sqrt{6}\pi}\right)\right) \\ &\geq \frac{1}{3\pi^2 \sqrt{6\pi^2 - h(\theta_0)^2}} (h(\theta_0)^{-1} - h(\theta_0)). \end{aligned}$$

Since this approaches infinity as $h(\theta_0)$ approaches zero, h is bounded below. \square

Now we proceed to our main estimate:

Theorem 9. *For any normalised curve C ,*

$$\frac{1}{1 + \pi\sqrt{\mathcal{L}}} \leq \sigma \leq \left(1 + \frac{\pi}{2} \sqrt{\mathcal{L}}\right)^2.$$

Proof. Since $A[\mathcal{C}] = \pi$, we have

$$(27) \quad 2\pi(1 - \mathcal{L}[\mathcal{C}]) = \int_{\mathcal{C}} \mathcal{K} ds = \int_{\mathcal{C}} \frac{1}{\sigma} ds - \int_{\mathcal{C}} \frac{\sigma_s^2}{\sigma^2} ds.$$

By the Hölder inequality, the Blaschke-Santaló inequality and the affine isoperimetric inequality we have

$$(28) \quad \int_{\mathcal{C}} \frac{1}{\sigma} ds \leq L^{1/2} \left(\int_{\mathcal{C}} \frac{1}{\sigma^2} ds \right)^{1/2} \leq (2\pi)^{1/2} (2\pi)^{1/2} = 2\pi.$$

This gives the estimate

$$\int_{\mathcal{C}} \frac{\sigma_s^2}{\sigma^2} ds \leq 2\pi \mathcal{L}.$$

Define a length element $d\xi$ by $d\xi = \sigma^p ds$ for p to be chosen later. Then we have

$$\int_{\mathcal{C}} \frac{\sigma_s^2}{\sigma^2} ds = \int_{\mathcal{C}} (\sigma^{p/2})_{\xi}^2 d\xi.$$

Therefore

$$(29) \quad \sup(\sigma^{p/2}) - \inf(\sigma^{p/2}) \leq \frac{p^2}{16} \mathcal{L} \int_{\mathcal{C}} d\xi.$$

The following result allows us to deduce that $\inf(\sigma^{p/2}) \leq 1$ for any p :

Lemma 10. *If \mathcal{C} is a smooth embedded convex curve with enclosed area π , then there exists $x_0 \in \mathcal{C}$ with $\sigma(x_0) = 1$.*

Proof. We have by Theorem 3, $\int_{\mathcal{C}} \sigma ds = 2\pi \geq L$, and so $\sup(\sigma) \geq \frac{1}{L} \int_{\mathcal{C}} \sigma ds \geq 1$. Thus the only possibility for the lemma to fail is if $\sigma > 1$ everywhere. Suppose this is the case.

Let $\mu = \langle \mathcal{N}, e \rangle$ for any $e \neq 0$ in \mathbb{R}^2 . Then $\mu = 0$ precisely at the two points of \mathcal{C} where \mathcal{T} is parallel to e . Let $f = \mu/\sigma$, and $d\ell = \sigma ds$. Then we have $\int_{\mathcal{C}} d\ell = 2\pi$, $\mu_{ss} + \mathcal{K}\mu = 0$, and $\sigma_{ss} + \mathcal{K}\sigma = 1$ by (8), (13), and (12). Therefore

$$\begin{aligned} f_{ss} &= \frac{1}{\sigma} (\mu_{ss} - f\sigma_{ss}) - \frac{2\sigma_s}{\sigma} f_s \\ &= -\frac{f}{\sigma} - \frac{2\sigma_s}{\sigma} f_s. \end{aligned}$$

Also $f_{ss} = \sigma^2 f_{\ell\ell} + \sigma\sigma_{\ell} f_{\ell}$, $\sigma_s = \sigma\sigma_{\ell}$, and $f_s = \sigma f_{\ell}$. Combining these, we find

$$(30) \quad \frac{\partial}{\partial \ell} \left(\sigma^3 \frac{\partial f}{\partial \ell} \right) + f = 0.$$

In the case where $\sigma > 1$ everywhere, the Sturm comparison theorem ([Ha], Theorem 3.1 of Chapter XI) implies that the interval of ℓ between zeroes of f is larger than π . Since there are two zeroes of f , this implies that $\int_{\mathcal{C}} d\ell > 2\pi$, which is a contradiction. \square

Now we choose two values of p in (29) to get the two inequalities in Theorem 9: First choose $p = 1$, which gives $\int_{\mathcal{C}} d\xi = \int_{\mathcal{C}} \sigma ds = 2\pi$ and hence

$$\sup \sigma^{1/2} - 1 \leq \frac{\pi}{2} \sqrt{\mathcal{L}},$$

which is the upper bound.

Next we take $p = -2$, so that by Theorem 4, $\int_{\mathcal{C}} d\xi = \int_{\mathcal{C}} \sigma^{-2} ds \leq 2\pi$, and so

$$\sup(\sigma^{-1}) - 1 \leq \pi \sqrt{\mathcal{L}},$$

which is the lower bound in Theorem 9. \square

Corollary 11. *For any normalised curve \mathcal{C} , we have*

$$\frac{c_1^3}{c_2^3} \frac{1}{(1 + \pi \sqrt{\mathcal{L}})^3} \leq r \leq \frac{c_2^3}{c_1^3} \left(1 + \frac{\pi}{2} \sqrt{\mathcal{L}}\right)^6.$$

Proof. $\sigma = hr^{1/3}$ and $c_1 \leq h \leq c_2$ by Proposition 8, so the result follows directly from Theorem 9. \square

This result shows that the normalised curves evolving under Eq. (22) remain uniformly convex and have uniformly bounded curvature. As another consequence of this estimate we have control above and below on the affine length of the normalised curves:

Corollary 12. *Any convex curve \mathcal{C} of enclosed area π satisfies*

$$\frac{2\pi}{\left(1 + \frac{\pi}{2} \sqrt{\mathcal{L}}\right)^2} \leq L \leq 2\pi.$$

Proof. We have $2\pi = 2A = \int_{\mathcal{C}} \sigma ds \leq \left(1 + \frac{\pi}{2} \sqrt{\mathcal{L}}\right)^2 \int_{\mathcal{C}} ds$ by Theorem 9. \square

6. Classification of limits

In this section we show that the only smooth embedded convex curves which evolve by homothetic expansion are ellipses. Recall from the statement and proof of Theorem 5 that homothetically expanding solutions are those which satisfy the identity $\mathcal{K} = c\sigma$ for some choice of origin and some $c > 0$. If \mathcal{C} satisfies this identity, then a suitably scaled copy of \mathcal{C} satisfies $\mathcal{K} = \sigma$.

Theorem 13. *If \mathcal{C} is a smooth, strictly convex embedded closed curve in \mathbb{R}^2 , satisfying the identity $\mathcal{K} = \sigma$ everywhere, then \mathcal{C} is an ellipse of enclosed area π centred at the origin.*

Proof. The method of proof is to deduce a contradiction to the affine isoperimetric inequality if \mathcal{C} is not an ellipse. First, we characterise ellipses:

Lemma 14. *\mathcal{C} is an ellipse of enclosed area π centred at the origin if and only if $\sigma \equiv 1$.*

Proof. If \mathcal{C} is an ellipse centred at the origin with area π , then \mathcal{C} is a special linear image of the unit circle, and hence $\sigma \equiv 1$ by the invariance of σ .

If $\sigma \equiv 1$, then $r = h^{-3}$. Therefore $h_{\theta\theta} + h = h^{-3}$, and integration gives

$$h_{\theta}^2 = 2C - h^{-2} - h^2$$

for some constant C . Also

$$(h^2)_{\theta\theta} + 4h^2 = 2h(h_{\theta\theta} + h) + 2h_{\theta}^2 + 2h^2 = 2h^{-2} + 2h^2 + 4C - 2h^{-2} - 2h^2 = 4C.$$

Therefore $h^2 = C + \sqrt{C^2 - 1} \cos 2(\theta - \theta_0)$ for some $C \geq 1$ and some θ_0 .

The support function of an ellipse may be calculated as follows: An ellipse of area π centred at 0 is the set $\{x \in \mathbb{R}^2 : x^T M x = 1\}$ for some symmetric matrix M with $\det M = 1$. Therefore $h(z) = \sup \langle z, x \rangle$ is attained at a point with $z = \lambda M x$ for some constant λ , and taking the inner product with x gives $h(z) = \lambda$. Thus $x = h(z)^{-1} M^{-1} z$, and the inner product with z gives $h(z)^2 = z^T M^{-1} z$ for all $z \in S^1$. Writing

$$M^{-1} = \begin{bmatrix} a & d \\ d & b \end{bmatrix}$$

with $ab - d^2 = 1$, we have

$$(31) \quad \begin{aligned} h^2 &= \frac{a+b}{2} + \frac{a-b}{2} \cos 2\theta + d \sin 2\theta \\ &= C + \sqrt{C^2 - 1} \cos 2(\theta - \theta_0), \end{aligned}$$

where $C = (a+b)/2$, $\cos 2\theta_0 = (a-b)/\sqrt{C^2 - 1}$, and $\sin 2\theta_0 = d/\sqrt{C^2 - 1}$. \square

Next we obtain a bound above on the enclosed area of a homothetic solution:

Lemma 15. *If $\mathcal{K} = \sigma$ then $A[\mathcal{C}] \leq \pi$, with equality if and only if $\sigma \equiv 1$.*

Proof. Since $\mathcal{K} = \sigma$, we have

$$A[\mathcal{C}]^{4/3} = A[\mathcal{C}]^{1/3} \frac{1}{2} \int_{\mathcal{C}} \sigma ds = \frac{1}{2} A[\mathcal{C}]^{1/3} \frac{1}{2} \int_{\mathcal{C}} \mathcal{K} ds = \pi^{4/3} (1 - \mathcal{L}[\mathcal{C}]) \leq \pi^{4/3}$$

by Lemma 6 (with $F = 1$) together with Theorem 3. Hence $A[\mathcal{C}] \leq \pi$, and equality holds only if $\sigma = c + \langle\langle \mathcal{N}, p \rangle\rangle$ for some constant c and some $p \in \mathbb{R}^2$. But then

$$\mathcal{X} = (1 - \sigma_{ss})/\sigma = (1 - \langle\langle p, \mathcal{N} \rangle\rangle_{ss})/\sigma = (1 + \mathcal{X} \langle\langle p, \mathcal{N} \rangle\rangle)/(c + \langle\langle p, \mathcal{N} \rangle\rangle)$$

by (12) and (13). Therefore $c^{-1} = \mathcal{X} = \sigma = c + \langle\langle \mathcal{N}, p \rangle\rangle$, so that $c = 1$ and $p = 0$. \square

Finally we consider the affine length L :

Lemma 16. *If $\mathcal{X} = \sigma$, then either $\sigma \equiv 1$ or σ is periodic in s with period greater than $\sqrt{2}\pi$.*

Proof. The affine support function σ can be computed explicitly in terms of s as the solution of an ordinary differential equation: $\sigma = \mathcal{X} = \frac{1 - \sigma_{ss}}{\sigma}$, so we have

$$\sigma_{ss} + \sigma^2 - 1 = 0.$$

Hence $\frac{1}{2} \sigma_s^2 + \frac{1}{3} \sigma^3 - \sigma = \frac{1}{2} E$ for some constant E . Since σ is bounded we must have $-4/3 \leq E \leq 4/3$, and we can write

$$\sigma_s^2 = E + 2\sigma - \frac{2}{3} \sigma^3 = \frac{2}{3} (\sigma_+ - \sigma)(\sigma - \sigma_-)(\sigma - \sigma_*),$$

where $\sigma_+ = 2 \cos \beta$, $\sigma_- = -\cos \beta + \sqrt{3} \sin \beta$, and $\sigma_* = -\cos \beta - \sqrt{3} \sin \beta$, and

$$\beta = \frac{1}{3} \arccos(3E/4),$$

which is a decreasing function of E . If we choose $s = 0$ to be at a point where σ attains its maximum, then ([AS], section 16)

$$\sigma(s) = \frac{m - 2}{\sqrt{1 - m + m^2}} + \frac{3m}{\sqrt{1 - m + m^2}} \operatorname{sn}^2\left(\frac{s}{\sqrt{2}(1 - m + m^2)^{1/4}} \mid m\right),$$

where $m = (3 \cos \beta - \sqrt{3} \sin \beta)/(3 \cos \beta + \sqrt{3} \sin \beta)$, which is decreasing in β and hence increasing in E , and $\operatorname{sn}(u \mid m)$ is the Jacobian elliptic function defined by $\operatorname{sn}(u \mid m) = \sin \varphi$, $u = \int_0^\varphi (1 - m \sin^2 \theta)^{-1/2} d\theta$. Note that as E increases from $-4/3$ to $4/3$, m increases from 0 to 1.

It follows that the half-period S of σ is given in terms of the complete elliptic integral

$$K(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{-1/2} d\theta$$

$$S(m) = \sqrt{2}(1 - m + m^2)^{1/4} K(m).$$

Lemma 17. $\lim_{m \rightarrow 0} S(m) = \pi/\sqrt{2}$, and $S(m)$ is strictly increasing in m .

Proof. $K(m)$ is given for $|m| < 1$ by the convergent power series ([AS], Equation 17.3.11)

$$K(m) = \frac{\pi}{2} \left(\sum_{n=0}^{\infty} \frac{((2n)!)^2}{2^{4n}(n!)^4} m^n \right).$$

Hence we also have

$$\begin{aligned} \frac{d}{dm} K(m) &= \frac{\pi}{2} \left(\sum_{n=0}^{\infty} \frac{(n+1)((2n+2)!)^2}{2^{4(n+1)}((n+1)!)^4} m^n \right) \\ &= \frac{\pi}{2} \left(\sum_{n=0}^{\infty} \left(n + \frac{1}{4(n+1)} \right) \frac{((2n)!)^2}{2^{4n}(n!)^4} m^n \right) \\ &\geq \frac{1}{4} K(m). \end{aligned}$$

Equality holds only for $m = 0$.

If we write $\mu(m) = 1 - m + m^2$, then we also have

$$\frac{1}{\mu} \frac{d\mu}{dm} = \frac{2 \left(m - \frac{1}{2} \right)}{\left(m - \frac{1}{2} \right)^2 + \frac{3}{4}} \geq -1$$

for $0 \leq m \leq 1$, with equality if and only if $m = 0$. Combining these, we have

$$\frac{\frac{d}{dm} K(m)}{K(m)} - \frac{1}{4} \frac{\frac{d}{dm} \mu(m)}{\mu(m)} \geq 0,$$

with equality only when $m = 0$. Hence $S(m) = \sqrt{2} \mu(m)^{1/4} K(m)$ is strictly increasing for $m > 0$, as required.

This completes the proof of Lemma 16. \square

It follows that $L[\mathcal{C}] = 2kS(m) \geq \sqrt{2}k\pi$ for some integer $k \geq 1$. Theorem 3 and Lemma 14 imply that $L < 2\pi(A/\pi)^{1/3} < 2\pi$ if \mathcal{C} is not an ellipse, which is a contradiction except in the case $k = 1$. The following rules out that possibility:

Proposition 18 (An affine four-vertex theorem). *If \mathcal{C} is a closed, smooth, strictly convex embedded curve in \mathbb{R}^2 , then the affine curvature \mathcal{K} of \mathcal{C} has at least 4 critical points.*

Proof. For any $x \in \mathcal{C}$, there is a unique point $x^* \neq x$ for which $\mathcal{F}(x) \times \mathcal{F}(x^*) = 0$. Define a function $\Psi: \mathcal{C} \rightarrow \mathbb{R}$ by $\Psi(x) = \mathcal{K}(x) - \mathcal{K}(x^*)$. Then $\Psi(x^*) = -\Psi(x)$ for all

$x \in \mathcal{C}$, so there exists some x_0 such that $\Psi(x) = 0$. Then $\mathcal{K}(x_0) = \mathcal{K}(x_0^*)$, and we denote this value by \mathcal{K}_0 .

If there are less than 4 critical points of \mathcal{K} , then there is a unique maximum and a unique minimum. Therefore $\mathcal{C} \setminus \{x_0, x_0^*\}$ consists of two components \mathcal{C}_+ and \mathcal{C}_- such that $\mathcal{K}(x) > \mathcal{K}_0$ for $x \in \mathcal{C}_+$, and $\mathcal{K}(x) < \mathcal{K}_0$ for $x \in \mathcal{C}_-$.

Define $\varrho(x) = \mathcal{T}(x) \times \mathcal{T}(x_0)$. Then $\varrho = 0$ only when $x = x_0$ or $x = x_0^*$. Without loss of generality $\varrho(x) > 0$ for $x \in \mathcal{C}_+$ and $\varrho(x) < 0$ for $x \in \mathcal{C}_-$.

Note that $\mathcal{K}(x)\varrho(x) = -(\mathcal{N}(x) \times \mathcal{T}(x_0))_s$, and also $\varrho(x) = (x \times \mathcal{T}(x_0))_s$. Therefore $\int_{\mathcal{C}} \mathcal{K}\varrho ds = \int_{\mathcal{C}} \varrho ds = 0$, and

$$0 = \int_{\mathcal{C}} \mathcal{K}\varrho ds - \mathcal{K}_0 \int_{\mathcal{C}} \varrho ds = \int_{\mathcal{C}_+} (\mathcal{K} - \mathcal{K}_0)\varrho ds + \int_{\mathcal{C}_-} (\mathcal{K} - \mathcal{K}_0)\varrho ds.$$

But the construction above shows that the integrand is strictly positive on both \mathcal{C}_+ and \mathcal{C}_- . \square

This completes the proof of Theorem 13, since if $k = 1$ then \mathcal{K} has only two critical points, contradicting Proposition 18, and if $k \geq 2$ then $L[\mathcal{C}] \geq 2\sqrt{2}\pi$, contradicting Theorem 3 and Lemma 14. \square

We note that Lima and Montenegro [LM] have also recently given a classification of the solutions of $\mathcal{K} = \sigma$, including non-embedded cases. Also, in the case where the curves are symmetric it can be shown that the affine isoperimetric ratio $LA^{-1/3}$ is strictly increasing under Eq. (1) unless \mathcal{C} is an ellipse, by a method similar to that employed by Gage [Ga1] for the curve shortening flow. However the symmetrisation argument used by Gage to deduce this also for non-symmetric curves does not generalise easily to this situation because $\int_{\mathcal{C}} \mathcal{K} ds$ depends on third derivatives of the curve, while the symmetrised curves are only $C^{1,1}$.

7. Regularity estimates

In this section we give the estimates required to prove that the evolving curves remain smooth, and that the solution exists for all time. These estimates depend strongly on the bound on $\int_{\mathcal{C}} \mathcal{K} ds$ obtained in Section 4.

7.1. Gagliardo-Nirenberg estimates. First we prove a slightly generalised version of the Gagliardo-Nirenberg estimates, which will be the main tool in the proof of the estimates of this section:

Theorem 19. *Given integers $m \geq 1$ and $0 \leq i \leq m - 1$, and real numbers p, q , and r with $1 \leq p, q, r \leq \infty$ and $1/p \leq 1/r + (i/m)(1/q - 1/r)$, there exists a constant C depending on i, m, p, q , and r such that for any curve \mathcal{C} and any smooth function u on \mathcal{C} which attains a value of zero somewhere,*

$$\|D^i u\|_{L^p} \leq C \|D^m u\|_{L^q}^a \|u\|_{L^r}^{1-a},$$

where

$$a = \frac{i + \frac{1}{r} - \frac{1}{p}}{m + \frac{1}{r} - \frac{1}{q}}.$$

Proof. See [Au], page 93, for all cases except for $i = 0$, $m = 1$. Consider the latter case, with $p = \infty$: If $q = 1$ we have $\|Du\| \geq 2\|u\|_\infty$ trivially. If $q > 1$, define

$$A(K) = \{x \in \mathcal{C} : |u(x)| \geq K \sup |u|\}$$

for each $K \in [0, 1]$. u is Hölder continuous, with

$$\sup_{x \neq y \in \mathcal{C}} \frac{|u(x) - u(y)|}{|x - y|^{1-1/q}} \leq \|Du\|_q.$$

If $x \in \mathcal{C}$ with $|u(x)| = \sup |u|$, then $|u(y)| \geq \|u\|_\infty - \|Du\|_q |x - y|^{1-1/q}$ for all $y \in \mathcal{C}$. It follows that

$$|A(K)| \geq 2(1 - K)^{q/(q-1)} \left(\frac{\|u\|_\infty}{\|Du\|_q} \right)^{q/(q-1)}$$

for all $K \in [0, 1]$, since u must attain zero somewhere. Now

$$\begin{aligned} \|u\|_r^r &= r \|u\|_\infty^r \int_0^1 K^{r-1} |A(K)| dK \\ &\geq 2r \|u\|_\infty^{r+q/(q-1)} \|Du\|_q^{-q/(q-1)} \int_0^1 K^{r-1} (1 - K)^{q/(q-1)} dK. \end{aligned}$$

After some re-arrangement, this becomes

$$\|u\|_\infty \leq C \|Du\|_q^a \|u\|_r^{1-a},$$

where

$$a = \frac{1/r}{1 + 1/r - 1/q}$$

and

$$C = (2r \int_0^1 K^{r-1} (1 - K)^{q/(q-1)} dK)^{-(q-1)/(qr+q-r)}.$$

If $i = 0$, $m = 1$ with any $p > r$, we have by the Hölder inequality

$$\|u\|_p \leq \|u\|_r^{r/p} \|u\|_\infty^{1-r/p},$$

and the result follows by applying the previous case to $\|u\|_\infty$. \square

7.2. Preliminary estimates. We derive some preliminary estimates which will simplify the calculations later:

Theorem 20. For any number $\mathcal{L}_0 \geq 0$, there exist constants $C_0(\mathcal{L}_0)$ and $C_1(\mathcal{L}_0)$ such that every smooth embedded convex closed curve \mathcal{C} with enclosed area π and $\mathcal{L}[\mathcal{C}] \leq \mathcal{L}_0$ satisfies

$$\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \leq \max \{C_0(\mathcal{L}_0), C_1(\mathcal{L}_0)(\int_{\mathcal{C}} \mathcal{K}_s^2 ds)^{1/2}\}.$$

Proof. Since both sides of the above equation are invariant under special affine transformations, we can translate to satisfy the condition (21b), and so the estimates of Propositions 8–12 hold. Also equations (27) and (28) show that

$$\mathcal{L}[\mathcal{C}] = \frac{1}{2\pi} \int_{\mathcal{C}} \frac{\sigma_s^2}{\sigma^2} ds + \left(1 - \frac{1}{2\pi} \int_{\mathcal{C}} \frac{1}{\sigma} ds\right),$$

in which both terms are non-negative. For convenience we let $u = -\log \sigma$. Then we have

$$\int_{\mathcal{C}} u_s^2 ds \leq 2\pi \mathcal{L}[\mathcal{C}],$$

and $\mathcal{K} = e^u + u_{ss} - u_s^2$ by Eq. (12). Hence

$$(32) \quad \int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds = \int_{\mathcal{C}} (e^u - 1)^2 + u_{ss}^2 + u_s^4 + 2u_s^2 - 4e^u u_s^2 ds,$$

and

$$(33) \quad \int_{\mathcal{C}} \mathcal{K}_s^2 ds = \int_{\mathcal{C}} u_{sss}^2 + 4u_{ss}^2 u_s^2 + e^{2u} u_s^2 ds \\ + \int_{\mathcal{C}} 2u_{ss}^3 + 2e^u u_s^4 - 2e^u u_{ss}^2 ds.$$

Now

$$\int_{\mathcal{C}} (e^u - 1)^2 ds \leq L(\sup |e^u - 1|)^2 \\ \leq L\left(\int_{\mathcal{C}} e^u |u_s| ds\right)^2 \\ \leq L \int_{\mathcal{C}} e^{2u} ds \int_{\mathcal{C}} u_s^2 ds \\ \leq 4\pi^2 \int_{\mathcal{C}} u_{ss}^2 ds,$$

where we used the fact that $u = 0$ somewhere by Lemma 10, the bound on $\int e^{2u} ds = \int \sigma^{-2} ds$ from Theorem 4, and the Poincaré inequality. Also $\int u_s^2 ds \leq \int u_{ss}^2 ds$ (since $L \leq 2\pi$), and by Theorem 19,

$$\int_{\mathcal{C}} u_s^4 ds \leq C\left(\int_{\mathcal{C}} u_{ss}^2 ds\right)^{1/2} \left(\int_{\mathcal{C}} u_s^2 ds\right)^{3/2} \leq C\mathcal{L}^{1/2} \int_{\mathcal{C}} u_{ss}^2 ds.$$

Therefore $\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \leq C(\mathcal{L}) \int_{\mathcal{C}} u_{ss}^2 ds$.

A further application of Theorem 19 gives

$$\begin{aligned} \int_{\mathcal{C}} u_{ss}^3 ds &\leq C \left(\int_{\mathcal{C}} u_s^2 \right)^{5/8} \left(\int_{\mathcal{C}} u_{sss}^2 \right)^{7/8} \\ &\leq C \mathcal{L}^{5/8} \left(\int_{\mathcal{C}} u_{sss}^2 \right)^{7/8}, \end{aligned}$$

and, by Theorem 9 and the Poincaré inequality,

$$\int_{\mathcal{C}} e^u u_{ss}^2 \leq C \mathcal{L}^{1/2} \int_{\mathcal{C}} u_{ss}^2 ds \leq C \mathcal{L} \left(\int_{\mathcal{C}} u_{sss}^2 ds \right)^2.$$

Combining these gives

$$\int_{\mathcal{C}} \mathcal{K}_s^2 ds \geq \int_{\mathcal{C}} u_{sss}^2 ds \left(1 - C \mathcal{L} \left(\int_{\mathcal{C}} u_{sss}^2 ds \right)^{-1/2} - C \mathcal{L}^{5/8} \left(\int_{\mathcal{C}} u_{sss}^2 ds \right)^{-1/8} \right).$$

In particular, for any C_0 , either $\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \leq C_0$ or

$$\int_{\mathcal{C}} u_{sss}^2 ds \geq \int_{\mathcal{C}} u_{ss}^2 ds \geq C \int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \geq CC_0.$$

Taking C_0 sufficiently large (depending only on \mathcal{L}), this implies if $\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds > C_0$ that

$$\int_{\mathcal{C}} \mathcal{K}_s^2 ds \geq C \int_{\mathcal{C}} u_{sss}^2 ds.$$

This implies by Theorem 19 that either $\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds < C_0$, or

$$\begin{aligned} \int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds &\leq C \int_{\mathcal{C}} u_{ss}^2 ds \\ &\leq C \left(\int_{\mathcal{C}} u_s^2 ds \right)^{1/2} \left(\int_{\mathcal{C}} u_{sss}^2 ds \right)^{1/2} \\ &\leq C \mathcal{L}^{1/2} \left(\int_{\mathcal{C}} \mathcal{K}_s^2 ds \right)^{1/2}, \end{aligned}$$

proving the result. The application of Theorem 19 to $\mathcal{K} - 1$ requires the following lemma:

Lemma 21. *For any strictly convex normalised curve \mathcal{C} there exists a point $x_0 \in \mathcal{C}$ such that $\mathcal{K}(x_0) = 1$.*

Proof. First note that

$$1 \geq \frac{L}{2\pi} = \frac{\int_{\mathcal{C}} \mathcal{K} \sigma ds}{\int_{\mathcal{C}} \sigma ds} \geq \inf_{\mathcal{C}} \mathcal{K}.$$

Hence the result holds unless $\mathcal{K} < 1$ everywhere. Let $\mu = \mathcal{T} \times e$ for any $e \neq 0$ in \mathbb{R}^2 . $\mu = 0$ at the two points of \mathcal{C} with \mathcal{T} parallel to e , and μ satisfies $\mu_{ss} + \mathcal{K}\mu = 0$ by (13). If $\mathcal{K} < 1$

everywhere, the interval of s between zeroes is greater than π by the Sturm comparison theorem ([Ha], Chapter XI), so $L > 2\pi$, contradicting Theorem 3. \square

Corollary 22. For any $\mathcal{L}_0 \geq 0$ and any $k \geq 1$, every normalised curve \mathcal{C} with $\mathcal{L}[\mathcal{C}] \leq \mathcal{L}_0$ satisfies

$$\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \leq \max \left\{ C_0(\mathcal{L}_0), C_1(\mathcal{L}_0)^{2k/(k+1)} \left(\int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds \right)^{1/(k+1)} \right\}.$$

Proof. First we prove that for any k ,

$$(34) \quad \int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds \leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/(k+1)} \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+1} \mathcal{K}}{\partial s^{k+1}} \right)^2 ds \right)^{k/(k+1)}.$$

This follows for $k = 1$ by applying the Cauchy-Schwarz inequality to the right-hand side of the identity

$$(35) \quad \int_{\mathcal{C}} \mathcal{K}_s^2 ds = - \int_{\mathcal{C}} (\mathcal{K} - 1) \mathcal{K}_{ss} ds.$$

Proceeding by induction, suppose (34) holds for $k = 1, \dots, \ell - 1$, so in particular

$$(36) \quad \int_{\mathcal{C}} \left(\frac{\partial^{\ell-1} \mathcal{K}}{\partial s^{\ell-1}} \right)^2 ds \leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/\ell} \left(\int_{\mathcal{C}} \left(\frac{\partial^{\ell} \mathcal{K}}{\partial s^{\ell}} \right)^2 ds \right)^{(\ell-1)/\ell}.$$

Applying the Cauchy-Schwarz inequality to

$$\int_{\mathcal{C}} \left(\frac{\partial^{\ell} \mathcal{K}}{\partial s^{\ell}} \right)^2 ds = - \int_{\mathcal{C}} \left(\frac{\partial^{\ell-1} \mathcal{K}}{\partial s^{\ell-1}} \right) \left(\frac{\partial^{\ell+1} \mathcal{K}}{\partial s^{\ell+1}} \right) ds,$$

we obtain

$$(37) \quad \int_{\mathcal{C}} \left(\frac{\partial^{\ell} \mathcal{K}}{\partial s^{\ell}} \right)^2 ds \leq \left(\int_{\mathcal{C}} \left(\frac{\partial^{\ell-1} \mathcal{K}}{\partial s^{\ell-1}} \right)^2 ds \right)^{1/2} \left(\int_{\mathcal{C}} \left(\frac{\partial^{\ell+1} \mathcal{K}}{\partial s^{\ell+1}} \right)^2 ds \right)^{1/2}.$$

Estimating the first factor on the left using (36), we find

$$\left(\int_{\mathcal{C}} \left(\frac{\partial^{\ell} \mathcal{K}}{\partial s^{\ell}} \right)^2 ds \right)^{(\ell+1)/(2\ell)} \leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/(2\ell)} \left(\int_{\mathcal{C}} \left(\frac{\partial^{\ell+1} \mathcal{K}}{\partial s^{\ell+1}} \right)^2 ds \right)^{1/2},$$

which gives (34) for $k = \ell$, completing the induction.

Now we prove the corollary by induction on k : Theorem 20 gives the case $k = 1$. Suppose the claim holds for $k = \ell - 1$:

$$\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \leq \max \left\{ C_0, C_1^{2(\ell-1)/\ell} \left(\int_{\mathcal{C}} \left(\frac{\partial^{\ell-1} \mathcal{K}}{\partial s^{\ell-1}} \right)^2 ds \right)^{1/\ell} \right\}.$$

Then if $\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds > C_0$, an application of (34) to the integral on the right of the previous estimate gives

$$\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \leq C_1^{2(\ell-1)/\ell} \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/\ell^2} \left(\int_{\mathcal{C}} \left(\frac{\partial^\ell \mathcal{K}}{\partial s^\ell} \right)^2 ds \right)^{(\ell-1)/\ell^2},$$

which completes the induction after re-arrangement. \square

Corollary 23. For any k and \mathcal{L}_0 there exists a constant C such that every normalised curve \mathcal{C} with $\mathcal{L}[\mathcal{C}] \leq \mathcal{L}_0$ satisfies

$$\int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds \leq C \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{(k+1)/(k+3)}.$$

Proof. By the identity (34), we have

$$\int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds \leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/(k+1)} \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+1} \mathcal{K}}{\partial s^{k+1}} \right)^2 ds \right)^{k/(k+1)}.$$

Applying (34) again to the second factor on the right, we find

$$\int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds \leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{2/(k+2)} \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{k/(k+2)}.$$

Corollary 22 (with k replaced by $k + 2$) applies to the first factor to give the desired result. \square

7.3. A bound on $\int \mathcal{K}^2 ds$.

Theorem 24. Let $x : \mathcal{C} \times [0, T] \rightarrow \mathbb{R}^2$ be a solution of Eq. (1). Then there exist constants C_1, C_2 and C_3 depending only on $\mathcal{L}[\mathcal{C}_0]$ such that

$$\int_{\mathcal{C}_t} \mathcal{K}^2 ds \leq \max \{ C_1 A[\mathcal{C}_0]^{-1}, ((\int_{\mathcal{C}_0} \mathcal{K}^2 ds)^{-2} + C_2 t A[\mathcal{C}_0]^{2/3})^{-1/2} \}$$

for $0 \leq t \leq C_3 A[\mathcal{C}_0]^{4/3}$.

Proof. First we compute the evolution equation for $\int_{\mathcal{C}} \mathcal{K}^2 ds$ under Eq. (1):

Lemma 25. Under Eq. (1),

$$\frac{d}{dt} \int_{\mathcal{C}} \mathcal{K}^2 ds = -\frac{2}{3} \int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds + \frac{16}{3} \int_{\mathcal{C}} \mathcal{K} \mathcal{K}_s^2 ds - 2 \int_{\mathcal{C}} \mathcal{K}^4 ds.$$

Proof. Recall that $\mathcal{K} = \frac{r[r^{-1/3}]}{r}$, so $\int_{\mathcal{C}} \mathcal{K}^2 ds = \int_{S^1} r^{-4/3} r [r^{-1/3}]^2 d\theta$. Therefore

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{C}} \mathcal{K}^2 ds &= -\frac{2}{3} \int_{S^1} r^{-4/3} r [r^{-1/3}] r [r^{-4/3} r [r^{-4/3} r [r^{-1/3}]]] d\theta \\ &\quad - \frac{4}{3} \int_{S^1} r^{-7/3} r [r^{-1/3}]^2 r [r^{-4/3} r [r^{-1/3}]] d\theta. \end{aligned}$$

Using the identity (11), we re-write this as

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{C}} \mathcal{K}^2 ds &= -\frac{2}{3} \int_{\mathcal{C}} \mathcal{K} ((\mathcal{K}_{ss} + \mathcal{K}^2)_{ss} + \mathcal{K} (\mathcal{K}_{ss} + \mathcal{K}^2)) ds \\ &\quad - \frac{4}{3} \int_{\mathcal{C}} \mathcal{K}^2 (\mathcal{K}_{ss} + \mathcal{K}^2) ds \\ &= -\frac{2}{3} \int_{\mathcal{C}} \mathcal{K}_{ss} (\mathcal{K}_{ss} + \mathcal{K}^2) ds - 2 \int_{\mathcal{C}} \mathcal{K}^2 (\mathcal{K}_{ss} + \mathcal{K}^2) ds \\ &= -\frac{2}{3} \int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds + \frac{16}{3} \int_{\mathcal{C}} \mathcal{K} \mathcal{K}_s^2 ds - 2 \int_{\mathcal{C}} \mathcal{K}^4 ds. \quad \square \end{aligned}$$

Hence the only term that must be controlled is the second.

Lemma 26. For any $\varepsilon > 0$ and any $\mathcal{L}_0 \geq 0$ there exists a constant $C(\varepsilon, \mathcal{L}_0)$ such that every smooth, strictly convex embedded curve \mathcal{C} with $\mathcal{L}[\mathcal{C}] \leq \mathcal{L}_0$ satisfies

$$\int_{\mathcal{C}} \mathcal{K} \mathcal{K}_s^2 ds \leq \varepsilon \int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds + CA[\mathcal{C}]^{-7/3}.$$

Proof. Note that his estimate is scaling-invariant: If $\tilde{\mathcal{C}} = \sqrt{\frac{\pi}{A[\mathcal{C}]}} \mathcal{C}$, then

$$\int_{\mathcal{C}} \mathcal{K} \mathcal{K}_s^2 ds = \left(\frac{\pi}{A[\mathcal{C}]} \right)^{7/3} \int_{\tilde{\mathcal{C}}} \tilde{\mathcal{K}} \tilde{\mathcal{K}}_s^2 d\tilde{s},$$

and

$$\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds = \left(\frac{\pi}{A[\mathcal{C}]} \right)^{7/3} \int_{\tilde{\mathcal{C}}} \tilde{\mathcal{K}}_{ss}^2 d\tilde{s}.$$

Hence it suffices to consider the case where $A[\mathcal{C}] = \pi$.

Now we estimate:

$$\begin{aligned} \int_{\mathcal{C}} \mathcal{K} \mathcal{K}_s^2 ds &= \int_{\mathcal{C}} \mathcal{K}_s^2 ds + \int_{\mathcal{C}} (\mathcal{K} - 1) \mathcal{K}_s^2 ds \\ &\leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/2} \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{1/2} \\ &\quad + \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/2} \left(\int_{\mathcal{C}} \mathcal{K}_s^4 ds \right)^{1/2}, \end{aligned}$$

where we used the estimate (34) with $k = 1$, and the Cauchy-Schwartz inequality. The last term can be estimated by an application of Theorem 19, noting again that this can be applied since $\mathcal{K} - 1$ is zero somewhere by Lemma 21:

$$\int_{\mathcal{C}} \mathcal{K}_s^4 ds \leq C \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{5/16} \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{3/16}.$$

Also, we estimate $\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds$ using Corollary 22. This gives

$$\begin{aligned} \int_{\mathcal{C}} \mathcal{K} \mathcal{K}_s^2 ds &\leq \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{1/2} \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{1/2} \\ &\quad + C \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{7/8} \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{5/8} \\ &\leq C \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{1/2} + C \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{2/3} \\ &\quad + C \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{5/8} + C \left(\int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds \right)^{11/12} \\ &\leq \varepsilon \int_{\mathcal{C}} \mathcal{K}_{ss}^2 ds + C(\varepsilon), \end{aligned}$$

where the last step follows by Young's inequality. \square

To simplify the proof of Theorem 24, we note that there is some time period on which the enclosed area remains comparable to its initial value:

Lemma 27. *For any solution of Eq. (1),*

$$A[\mathcal{C}_0]^{4/3} + \frac{8}{3} \pi^{4/3} (1 - \mathcal{L}[\mathcal{C}_0])t \leq A[\mathcal{C}_t]^{4/3} \leq A[\mathcal{C}_0]^{4/3} + \frac{8}{3} \pi^{4/3} t.$$

In particular, $\frac{1}{2} A[\mathcal{C}_0] \leq A[\mathcal{C}_t] \leq 2A[\mathcal{C}_0]$ for $0 \leq t \leq \delta(A[\mathcal{C}_0]/\pi)^{4/3}$, where $\delta = 3/8$ if $\mathcal{L}[\mathcal{C}_0] \leq 3/2$, and $\delta = (3/16)(\mathcal{L}[\mathcal{C}_0] - 1)^{-1}$ if $\mathcal{L}[\mathcal{C}_0] \geq 3/2$.

Proof. This follows immediately from the identity

$$\frac{d}{dt} A[\mathcal{C}_t]^{4/3} = \frac{8}{3} \pi^{4/3} (1 - \mathcal{L}[\mathcal{C}_t]),$$

together with the monotonicity of $\mathcal{L}[\mathcal{C}_t]$ from Corollary 7. \square

It follows from Lemmas 25 and 26 that we have for $0 \leq t \leq \delta(A[\mathcal{C}_0]/\pi)^{4/3}$

$$\frac{d}{dt} \int_{\mathcal{C}} \mathcal{X}^2 ds \leq -\frac{1}{3} \int_{\mathcal{C}} \mathcal{X}_{ss}^2 + C(\mathcal{L}[\mathcal{C}_0])A[\mathcal{C}_0]^{-7/3}.$$

Corollary 22, together with the area bound, applies to the first term to give

$$\int_{\mathcal{C}} \mathcal{X}_{ss}^2 ds \geq CA[\mathcal{C}_0]^{2/3} \left(\int_{\mathcal{C}} \mathcal{X}^2 ds \right)^3 - CA[\mathcal{C}_0]^{-7/3},$$

so that

$$\frac{d}{dt} \int_{\mathcal{C}} \mathcal{X}^2 ds \leq -CA[\mathcal{C}_0]^{2/3} \left(\int_{\mathcal{C}} \mathcal{X}^2 ds \right)^3 + CA[\mathcal{C}_0]^{-7/3}.$$

Hence, there exists $C(\mathcal{L}_0)$ such that either $\int_{\mathcal{C}} \mathcal{X}^2 ds \leq C(\mathcal{L}_0)$ or

$$\frac{d}{dt} \int_{\mathcal{C}} \mathcal{X}^2 ds \leq -CA[\mathcal{C}_0]^{2/3} \left(\int_{\mathcal{C}} \mathcal{X}^2 ds \right)^3.$$

The result follows by integrating this inequality. \square

Since Theorem 24 can be applied over arbitrary time intervals within the interval of existence of a solution, we immediately obtain the following:

Corollary 28. *If $\{\mathcal{C}_t\}_{0 \leq t \leq T}$ is a smooth solution of Eq. (1), there exists a constant C depending only on $A[\mathcal{C}_0]$, $\int_{\mathcal{C}_0} \mathcal{X}^2 ds$ and on $\mathcal{L}[\mathcal{C}_0]$ such that*

$$A[\mathcal{C}_t] \int_{\mathcal{C}_t} \mathcal{X}^2 ds \leq C$$

for all $t \in [0, T]$.

7.4. Higher regularity. Next we seek bounds on integrals of derivatives of \mathcal{X} . First we compute the required evolution equations:

Proposition 29. *Under Eq. (19), we can write*

$$\frac{\partial}{\partial t} \left(\frac{\partial^k \mathcal{X}}{\partial s^k} \right) = -\frac{1}{3} \left(\frac{\partial^{k+4} \mathcal{X}}{\partial s^{k+4}} \right) + P_k,$$

where P_k is a polynomial of degree $3 + 2k$ in \mathcal{X} and its derivatives up to order $k + 2$:

$$P_k = \sum_i c_i \prod_{j=0}^{k+2} \left(\frac{\partial^j \mathcal{K}}{\partial s^j} \right)^{\alpha(i,j)},$$

where $\alpha(i, j) \in \{0, 1, \dots, 3 + 2k\}$, and for each i

$$\sum_{j=0}^{k+2} \alpha(i, j)(j+2) = k+6.$$

Proof. This follows for $k=1$ from the proof of Lemma 25, and for higher k by induction using the identity

$$(38) \quad \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial F}{\partial t} \right) - \frac{2}{3} (\mathcal{K}_{ss} + \mathcal{K}^2) \left(\frac{\partial F}{\partial s} \right). \quad \square$$

The result of Proposition 29, together with a computation of the time derivative of the length element ds under Eq. (1), gives an expression for the time derivative of the integral of the square of any derivative of \mathcal{K} . After integrating by parts all the terms which involve a single factor of $(\partial/\partial s)^{k+2} \mathcal{K}$, we obtain the following:

Corollary 30.

$$\frac{d}{dt} \int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds = -\frac{2}{3} \int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds + \sum_i c_i \int_{\mathcal{C}} \prod_{j=0}^{k+1} \left(\frac{\partial^j \mathcal{K}}{\partial s^j} \right)^{\beta(i,j)} ds$$

where $\beta(i, j)$ is a non-negative integer, and for each i

$$\sum_{j=0}^{k+1} \beta(i, j)(j+2) = 2k+8.$$

As in the proof of the estimate on $\int_{\mathcal{C}} \mathcal{K}^2 ds$, we will use the first term in this evolution equation to control all of the remaining terms. This is accomplished using the following result:

Proposition 31. *Let k be a positive integer and $\beta_0, \dots, \beta_{k+1}$ non-negative integers with $\sum_{j=0}^{k+1} \beta_j(j+2) = 2k+8$. Then for any $\varepsilon > 0$ and $\mathcal{L}_0 \geq 0$, there exists a constant C depending only on $k, \beta_0, \dots, \beta_{k+1}, \mathcal{L}_0$ and ε , such that every smooth, strictly convex curve C with $\mathcal{L}[C] \leq \mathcal{L}_0$ satisfies*

$$\left| \int_{\mathcal{C}} \prod_{j=0}^{k+1} \left(\frac{\partial^j \mathcal{K}}{\partial s^j} \right)^{\beta_j} ds \right| \leq \varepsilon \int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds + CA[\mathcal{C}]^{-(7+2k)/3}.$$

Proof. As in Lemma 26, we can use the scaling invariance to reduce the problem to the case $A[\mathcal{C}] = \pi$. We first use the Hölder inequality, with an arbitrary non-trivial collection of exponents p_0, \dots, p_{k+1} with $\sum p_j^{-1} = 1$:

$$\left| \int_{\mathcal{C}} \prod_{j=0}^{k+1} \left(\frac{\partial^j \mathcal{K}}{\partial s^j} \right)^{\beta_j} ds \right| \leq \prod_{j=0}^{k+1} \left(\int_{\mathcal{C}} \left| \frac{\partial^j \mathcal{K}}{\partial s^j} \right|^{\beta_j p_j} ds \right)^{1/p_j}.$$

For each of the integrals on the right-hand side, we use Theorem 19. First, in the case $j = 0$, we have

$$\begin{aligned} \left(\int_{\mathcal{C}} |\mathcal{K}|^{\beta_0 p_0} ds \right)^{1/p_0} &= \|\mathcal{K}\|_{p_0 \beta_0}^{\beta_0} \\ &\leq (\|\mathcal{K} - 1\|_{p_0 \beta_0} + L^{1/(p_0 \beta_0)})^{\beta_0} \\ &\leq 2^{\beta_0 - 1} \left(\int_{\mathcal{C}} |\mathcal{K} - 1|^{\beta_0 p_0} ds \right)^{1/p_0} + 2^{\beta_0 - 1} L^{1/p_0} \\ &\leq C \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{\beta_0/2 - \gamma_0} \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{\gamma_0} \\ &\quad + 2^{\beta_0 - 1} (2\pi)^{1/p_0}, \end{aligned}$$

where $2(k+2)\gamma_0 = \beta_0/2 - 1/p_0$. For $j > 0$, Theorem 19 gives

$$\left(\int_{\mathcal{C}} \left| \frac{\partial^j \mathcal{K}}{\partial s^j} \right|^{\beta_j p_j} ds \right)^{1/p_j} \leq C \left(\int_{\mathcal{C}} (\mathcal{K} - 1)^2 ds \right)^{\beta_j/2 - \gamma_j} \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{\gamma_j},$$

where $2(k+2)\gamma_j = (j+1/2)\beta_j - 1/p_j$. In each of these cases we can apply Corollary 22 to bound the first factor, yielding:

$$\left(\int_{\mathcal{C}} \left| \frac{\partial^j \mathcal{K}}{\partial s^j} \right|^{\beta_j p_j} ds \right)^{1/p_j} \leq C \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{\delta_j} + C,$$

where

$$\delta_j = \frac{\beta_j(j+3/2) - 1/p_j}{2(k+3)}.$$

Note that

$$\delta_j - \frac{\beta_j(j+2) - 1/p_j}{2k+7} = -\frac{\beta_j(k-j+3/2) + 1/p_j}{2(k+3)(2k+7)} < 0.$$

Therefore we have

$$\begin{aligned} \left| \int_{\mathcal{C}} \prod_{j=0}^{k+1} \left(\frac{\partial^j \mathcal{K}}{\partial s^j} \right)^{\beta_j} ds \right| &\leq \prod_{j=0}^{k+1} \left(C + \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{\delta_j} \right) \\ &\leq C + \left(\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{K}}{\partial s^{k+2}} \right)^2 ds \right)^{\sum_{j=0}^{k+1} \delta_j}. \end{aligned}$$

The result then follows by Young's inequality, since

$$\sum_{j=0}^{k+1} \delta_j < \frac{1}{2k+7} \sum_{j=0}^{k+1} (\beta_j(j+2) - 1/p_j) = \frac{1}{2k+7} (2k+8-1) = 1$$

because $\sum p_j^{-1} = 1$ and $\sum (j+2)\beta_j = 2k+8$. \square

We are now in a position to derive bounds on the integrals of squares of derivatives of \mathcal{X} of arbitrarily high order:

Theorem 32. *Let $\{C_t\}_{0 \leq t \leq T}$ be a solution of Eq. (1). Then for any integer $k \geq 0$ there exist constants C_1, C_2 , and C_3 depending only on $\mathcal{X}[\mathcal{C}_0]$ and k such that*

$$\int_{\mathcal{C}_t} \left(\frac{\partial^k \mathcal{X}}{\partial s^k} \right)^2 ds \leq \max \left\{ C_1 A[\mathcal{C}_0]^{-\frac{2k+3}{3}}, q(1 + C_2 A[\mathcal{C}_0]^{-4/3} q^{\frac{2}{k+1}} t)^{-\frac{k+1}{2}} \right\}$$

for $0 \leq t \leq C_3 A[\mathcal{C}_0]^{4/3}$, where

$$q = \int_{\mathcal{C}_0} \left(\frac{\partial^k \mathcal{X}}{\partial s^k} \right)^2 ds.$$

Proof. By Corollary 30 and Proposition 31, we have the estimate

$$(39) \quad \frac{d}{dt} \left(\int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{X}}{\partial s^k} \right)^2 ds \right) \leq -\frac{1}{3} \int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{X}}{\partial s^{k+2}} \right)^2 ds + CA[\mathcal{C}]^{-(7+2k)/3}.$$

Corollary 23 and scaling give

$$\int_{\mathcal{C}} \left(\frac{\partial^{k+2} \mathcal{X}}{\partial s^{k+2}} \right)^2 ds \geq CA[\mathcal{C}]^{2/(3(k+2))} \left(\int_{\mathcal{C}} \left(\frac{\partial^k \mathcal{X}}{\partial s^k} \right)^2 ds \right)^{(k+3)/(k+1)},$$

which we apply to the first term in (39). Lemma 27 implies that $A[\mathcal{C}_t]$ is comparable to $A[\mathcal{C}_0]$ for suitably small t . Writing

$$q(t) = A[\mathcal{C}_0]^{(2k+3)/3} \int_{\mathcal{C}_t} \left(\frac{\partial^k \mathcal{X}}{\partial s^k} \right)^2 ds,$$

we find that if q is sufficiently large, then

$$\frac{d}{dt} q(t) \leq -CA[\mathcal{C}_0]^{-4/3} q(t)^{(k+3)/(k+1)},$$

which implies the result by comparison with the solution of the ordinary differential equation $\dot{q} = -Cq^{(k+3)/(k+1)}$. \square

As in the case $k = 0$, this implies a scaling-invariant bound as long as the solution exists:

Corollary 33. Let $\{\mathcal{C}_t\}_{0 \leq t \leq T}$ be a smooth solution of Eq.(1), and write for each positive integer k

$$q_k(t) = A[\mathcal{C}_t]^{(2k+3)/3} \int_{\mathcal{C}_t} \left(\frac{\partial^k \mathcal{K}}{\partial s^k} \right)^2 ds .$$

Then there exists a constant C depending only on k , $\mathcal{L}[\mathcal{C}_0]$ and $q_k(0)$, such that $q_k(t) \leq C$ for all $t \in [0, T]$.

8. The convergence theorem

In this section we apply the results of the previous sections to prove Theorem 1. First we show that solutions exist for all time:

Theorem 34. Let x_0 be a smooth, strictly convex embedded closed curve in \mathbb{R}^2 . Then the solution of Eq. (1) with initial conditions given by x_0 exists for all positive times.

Proof. Suppose this is not the case – that is, the solution exists only up to a finite time T . Note that the enclosed area $A[\mathcal{C}_t]$ is bounded above and below on this time interval: Lemma 27 gives a bound above for all time, and Theorem 3 gives a bound below since the affine length is increasing:

$$A[\mathcal{C}_t] \geq \pi \left(\frac{L[\mathcal{C}_t]}{2\pi} \right)^3 \geq \pi \left(\frac{L[\mathcal{C}_0]}{2\pi} \right)^3 .$$

Therefore by Corollaries 28 and 33, \mathcal{K} and all of its derivatives are bounded on $\mathcal{C} \times [0, T)$. Under Eq. (19), we also have

$$\frac{\partial}{\partial t} r = r(\mathcal{K}_{ss} + \mathcal{K}^2) ,$$

so that the total change in the logarithm of r is bounded, and r remains bounded above and below. The evolution of the support function is given by

$$\frac{\partial h}{\partial t} = -\mathcal{K} r^{-1/3} ,$$

which is bounded, and so h remains bounded. It follows that the curves \mathcal{C}_t are uniformly bounded in C^k for every k . Furthermore, the parametrisation x_t remain smooth and non-degenerate, since they are given by $x_t = \bar{x}_t \circ \varphi_t$,

$$(40) \quad \dot{\varphi} = - (r^{-4/3} \mathcal{K}_\theta) \circ \varphi .$$

Hence the total change in φ is bounded, and by differentiating (40) we see that φ remains non-degenerate and the derivatives of φ remain bounded.

It follows by the Arzela-Ascoli theorem that the maps x_t converge to a limit x_T for some subsequence of times approaching T , and x_T is a smooth, non-degenerate parametrisation of a strictly convex embedded closed curve. Since the time-derivative of x and all

of its derivatives remain bounded, x_t converges in C^∞ to x_T as t approaches T . The short-time existence theorem then shows that the solution can be extended beyond T , contradicting the maximality of T . \square

Next we wish to show that the rescaled solutions converge to a limit as t approaches ∞ :

Proposition 35. *There exists a sequence of times $\{t_k\}$ approaching infinity, such that the curves \hat{C}_{t_k} given by the fully normalised equation (22) converge in C^∞ to a unit circle centred at the origin.*

Proof. We note that the rescaled time parameter τ also approaches infinity as t does, since

$$\tau = \int_0^t \left(\frac{\pi}{A[\mathcal{C}_{t'}]} \right)^{4/3} dt'$$

and

$$\left(\frac{A[\mathcal{C}_t]}{\pi} \right)^{4/3} \leq \left(\frac{A[\mathcal{C}_0]}{\pi} \right)^{4/3} + \frac{8t}{3},$$

so

$$\tau \geq \frac{3}{8} \log \left(1 + \frac{8t}{3} \left(\frac{\pi}{A[\mathcal{C}_0]} \right)^{4/3} \right)$$

which approaches infinity as t does.

Let $\hat{h}(\theta, \tau)$ be the support function of the solution of the fully normalised equation (22). Then for each $T > 0$ define $\hat{h}_T(\theta, \tau) = \hat{h}(\theta, T + \tau)$. Then \hat{h}_T is a solution of the fully normalised support function evolution equation (23).

Corollaries 28 and 33 give uniform bounds on the affine curvature \mathcal{K} and its derivatives for the fully normalised solutions, for all time. By Proposition 8, the support function \hat{h} is uniformly bounded above and below, and by Corollary 11, $\hat{\nu}$ is uniformly bounded above and below. It follows that the support function \hat{h} is uniformly bounded in C^∞ , with $\hat{\nu}$ strictly positive.

Therefore there exists a sequence of times T_k such that $\hat{h}_{T_k} : S^1 \times [0, \infty) \rightarrow \mathbb{R}$ converges in C^∞ as $k \rightarrow \infty$ to a limit $\hat{h}_\infty : S^1 \times [0, \infty) \rightarrow \mathbb{R}$, which is therefore also a solution of (23).

We claim that $\hat{h}_\infty \equiv 1$. By Theorem 13, the equation $\mathcal{K} = c\sigma$ is satisfied if and only if \hat{h} is the support function of an ellipse of enclosed area π centred at the origin, and since \hat{h} is fully normalised, this occurs if and only if $\hat{h} \equiv 1$. By Corollary 7, the time derivative of \mathcal{L} is strictly negative if this is not the case. Choose a time interval $[\tau_0, \tau_1]$ such that

$$\frac{d}{d\tau} \mathcal{L}[\hat{h}_\infty(\tau)] \leq -\varepsilon < 0$$

for all $\tau_0 \leq \tau \leq \tau_1$. Then choose k_0 sufficiently large to ensure that

$$\left| \frac{d}{d\tau} \mathcal{L}[\hat{h}_{T_k}(\tau)] - \frac{d}{d\tau} \mathcal{L}[\hat{h}_\infty(\tau)] \right| \leq \varepsilon/2$$

for all $\tau_0 \leq \tau \leq \tau_1$ and all $k \geq k_0$. Choose a subsequence of $\{T_k\}$, also denoted $\{T_k\}$, such that for each $k > k_0$, $T_{k+1} + \tau_0 > T_k + \tau_1$. That is, all of the intervals $[T_k + \tau_0, T_k + \tau_1]$ are disjoint for $k \geq k_0$.

Then

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \mathcal{L}[\hat{C}_\tau] &= \mathcal{L}[\hat{C}_{T_{k_0}}] + \int_{T_{k_0}}^{\infty} \frac{d}{d\tau} \mathcal{L} d\tau \\ &\leq \mathcal{L}[\hat{C}_{T_{k_0}}] + \sum_{k=k_0}^{\infty} \int_{T_k + \tau_0}^{T_k + \tau_1} \frac{d}{d\tau} \mathcal{L} d\tau \\ &\leq \mathcal{L}[\hat{C}_{T_{k_0}}] - \sum_{k=k_0}^{\infty} \frac{1}{2} \varepsilon (\tau_1 - \tau_0) \\ &= -\infty. \end{aligned}$$

But this is impossible since $\mathcal{L} \geq 0$ always. \square

Stronger convergence to the limit will follow from a consideration of behaviour for solutions close to ellipses: First we show stability for the fully normalised solutions.

Proposition 36. *If $\hat{h}: S^1 \times [0, \infty)$ satisfies (23), then for any k and any $\varepsilon > 0$ there exists C such that*

$$\|\hat{h} - 1\|_{W^{k,2}} \leq C e^{-(3.5/3 - \varepsilon)\tau}.$$

Proof. For h close to 1, Eq. (23) can be written as follows:

$$\begin{aligned} (41) \quad \frac{\partial}{\partial t} (h - 1) &= -\frac{1}{3} r[r[h - 1]] - \frac{4}{3} r[h - 1] - (h - 1) + \frac{1}{3\pi} \int_{S^1} (h - 1) d\theta \\ &\quad - \frac{3}{\pi} \int_{S^1} (h - 1) \cos 2\theta d\theta \cos 2\theta - \frac{3}{\pi} \int_{S^1} (h - 1) \sin 2\theta d\theta \sin 2\theta \\ &\quad + \frac{4}{\pi} \int_{S^1} (h - 1) \cos \theta d\theta \cos \theta + \frac{4}{\pi} \int_{S^1} (h - 1) \sin \theta d\theta \sin \theta + R, \end{aligned}$$

where R is quadratic in $h - 1$ and its derivatives up to fourth order. In order to control this, we note that many of these terms are negligible:

Lemma 37. For any normalised curve with support function h ,

$$\begin{aligned} \int_{S^1} (h-1) \cos 2(\theta - \theta_0) d\theta &= 0; \\ \left| \int_{S^1} (h-1) \cos(\theta - \theta_0) d\theta \right| &\leq C \int_{S^1} (h-1)^2 d\theta; \\ \int_{S^1} (h-1) d\theta &= -\frac{1}{2} \int_{S^1} (h-1) r[h-1] d\theta. \end{aligned}$$

Proof. The first identity is a restatement of (21c). The last is a consequence of (21a), since

$$\int_{S^1} (h-1) r[h-1] d\theta = \int_{S^1} h r[h] d\theta + 2\pi - 2 \int_{S^1} h d\theta = 2 \int_{S^1} 1 - h d\theta$$

since $\int_{S^1} h r[h] d\theta = 2A = 2\pi$.

The second identity comes from condition (21b), since (in view of the bounds above and below on h from Proposition 8), $|h^{-3} - 1 + 3(h-1)| \leq C(h-1)^2$. Integration against $\cos \theta$ gives by (21b)

$$3 \left| \int_{S^1} (h-1) \cos \theta d\theta \right| \leq C \int_{S^1} (h-1)^2 d\theta,$$

and similarly with $\cos \theta$ replaced by $\sin \theta$. \square

This gives the estimate

$$(42) \quad \frac{\partial}{\partial \tau} (h-1) = -\frac{1}{3} r[r[h-1]] - \frac{4}{3} r[h-1] - (h-1) + \tilde{R}$$

where \tilde{R} is again quadratic in $h-1$ and its derivatives.

Lemma 38. For any solution of Eq. (23) and any $\varepsilon > 0$, there exists $C(\varepsilon)$ such that

$$\left| \int_{S^1} (h-1) \tilde{R} d\theta \right| \leq C(\varepsilon) \left(\int_{S^1} (h-1)^2 d\theta \right)^{3/2 - \varepsilon}.$$

Proof. This follows from the form of \tilde{R} , the uniform C^k bounds on the solution supplied by Corollary 33, and Theorem 19 (applied with $r = 2$, and m sufficiently large). \square

Calculating the time derivative of $\int_{S^1} (h-1)^2 d\theta$ using Eq. (42) and applying Lemma 38 with $\varepsilon = 1/4$, we find

$$\begin{aligned} \frac{d}{dt} \int_{S^1} (h-1)^2 d\theta &\leq -\frac{2}{3} \int_{S^1} r[h-1]^2 d\theta - \frac{8}{3} \int_{S^1} (h-1) r[h-1]^2 d\theta \\ &\quad - 2 \int_{S^1} (h-1)^2 d\theta + C \left(\int_{S^1} (h-1)^2 d\theta \right)^{5/4}. \end{aligned}$$

Lemma 39. *If $\eta \in \mathcal{C}^\infty(S^1)$ satisfies*

$$\int_{S^1} \eta d\theta = \int_{S^1} \eta \cos(\theta - \theta_0) d\theta = \int_{S^1} \eta \cos 2(\theta - \theta_0) d\theta,$$

then

$$\frac{2}{3} \int_{S^1} \mathfrak{r}[\eta]^2 d\theta + \frac{8}{3} \int_{S^1} \eta \mathfrak{r}[\eta] d\theta + 2 \int_{S^1} \eta^2 d\theta \geq \frac{70}{3} \int_{S^1} (h-1)^2 d\theta.$$

Proof. Expanding η in a trigonometric series $\sum_{j \geq 3} \eta_j \cos j(\theta - \theta_j)$, we find

$$\begin{aligned} & \frac{2}{3} \int_{S^1} \mathfrak{r}[\eta]^2 d\theta + \frac{8}{3} \int_{S^1} \eta \mathfrak{r}[\eta] d\theta + 2 \int_{S^1} \eta^2 d\theta \\ &= \frac{2}{3} \sum_{j \geq 3} \eta_j^2 (1-j^2)^2 \pi + \frac{8}{3} \sum_{j \geq 3} \eta_j^2 (1-j^2) \pi + 2 \sum_{j \geq 3} \eta_j^2 \pi \\ &\geq \frac{70}{3} \pi \sum_{j \geq 3} \eta_j^2, \end{aligned}$$

since $\frac{2}{3} (1-j^2)^2 + \frac{8}{3} (1-j^2) + 2$ is increasing for $j \geq 3$. \square

Hence the part of $h-1$ orthogonal in L^2 to 1 , $\cos(\theta - \theta_0)$, and $\cos 2(\theta - \theta_0)$ satisfies this estimate, and by Lemma 37 the remaining components can be bounded by a $\int (h-1)^2 d\theta$ and $\int (h-1) \mathfrak{r}[h-1] d\theta$. It follows that for any $\varepsilon > 0$ there exists a radius $r(\varepsilon) > 0$ such that whenever $\int_{S^1} (h-1)^2 d\theta \leq r(\varepsilon)^2$,

$$\frac{d}{dt} \int_{S^1} (h-1)^2 d\theta \leq -\left(\frac{70}{3} - \varepsilon\right) \int_{S^1} (h-1)^2 d\theta.$$

The result of Proposition 36 follows, because the regularity estimates of Corollary 33 allow any $W^{k,2}$ norm of $h-1$ to be bounded by a power arbitrarily close to 1 of the L^2 norm (by taking m sufficiently large in Theorem 19). \square

Next we use these stability estimates to control the scalings, special linear transformations, and translations which relate the fully normalised solutions to the un-normalised solutions of (19):

Proposition 40. *Let $h : S^1 \times [0, \infty)$ satisfy (19), and let \hat{h} be the corresponding solution of (23). Write $\mathcal{C}_t = R_t T_t \hat{\mathcal{C}}_{t(t)} + p_t$, where $R_t \in \mathbb{R}$, $T_t \in \text{SL}(2, \mathbb{R})$ and $p_t \in \mathbb{R}^2$. There exists $T_\infty \in \text{SL}(2, \mathbb{R})$, $p_\infty \in \mathbb{R}^2$ and t_0 and $\tau_0 \in \mathbb{R}$ such that*

$$d_{\text{SL}(2)}(T_t, T_\infty) \leq C(\varepsilon) t^{-(35/4 - \varepsilon)};$$

$$|p_t - p_\infty| \leq C(\varepsilon) t^{-(75/8 - \varepsilon)};$$

$$\left| R_t - \left(\frac{8}{3} (t - t_0) \right)^{3/8} \right| \leq C(\varepsilon) t^{-(67/8 - \varepsilon)};$$

and

$$|\tau - \tau_0 - \frac{3}{8} \log(t - t_0)| \leq C(\varepsilon) t^{-(31/4 - \varepsilon)}$$

for any $\varepsilon > 0$.

Proof. We can recover the special linear transformations T_τ using the formula (compare Eq. (22))

$$T_\tau^{-1} \dot{T}_\tau = \sum_{i,j=1}^2 (N^{-1})^{ij} \int_{\mathcal{C}} \hat{\mathcal{X}} G_i d\hat{s} L_j$$

where $\dot{T}_\tau = dT_\tau/d\tau$, and L_1 and L_2 are the Lie algebra elements given after Eq. (21c). Since the eigenvalues of N are bounded above and below, we have

$$\|\dot{T}_\tau\|^2 \leq C \sum_{i=1}^2 \left(\int_{\mathcal{C}} \hat{\mathcal{X}} G_i d\hat{s} \right)^2.$$

Now expanding about $\hat{h} = 1$, we find

$$\int_{\mathcal{C}} \hat{\mathcal{X}} G_1 d\hat{s} = 3 \int_{S^1} (\hat{h} - 1) \cos 2\theta d\theta + O(h - 1)_{\mathcal{C}^4}^2,$$

and by Lemma 37, the first term on the right is zero. Therefore by Proposition 36,

$$\|\dot{T}_\tau\| \leq C(\varepsilon) e^{-(70/3 - \varepsilon)\tau}.$$

Integrating, we deduce that $d_{\text{SL}(2)}(T_\tau, T_\infty) \leq C(\varepsilon) e^{-(70/3 - \varepsilon)\tau}$ for any $\varepsilon > 0$.

Next we consider the scalings R_τ : We have $R(\tau) = \sqrt{\frac{A[\mathcal{C}_\tau]}{\pi}}$, and so

$$\frac{d}{d\tau} R(t) = \frac{d}{dt} \sqrt{\frac{A[\mathcal{C}_t]}{\pi}} \frac{dt}{d\tau} = \sqrt{\frac{\pi}{A[\mathcal{C}_t]}} \frac{1}{2\pi} \int_{\mathcal{C}_t} \mathcal{X} ds = (1 - \mathcal{Z}[\mathcal{C}_t]) R(t).$$

Note that $\mathcal{Z}[\mathcal{C}_t] = \mathcal{Z}[\hat{\mathcal{C}}_\tau] = \frac{1}{3\pi} \int_{S^1} (h - 1) d\theta + O(h - 1)_{\mathcal{C}^3}^2$, so by Proposition 36 and Lemma 37 this is bounded by $C(\varepsilon) e^{-(70/3 - \varepsilon)\tau}$. It follows that

$$R(t) = e^{\tau - \tau_0} (1 + O(e^{-(70/3 - \varepsilon)\tau})).$$

From the definition of τ in terms of t , this implies

$$\tau - \tau_0 = \frac{3}{8} \log(t - t_0) + O(e^{-(35/4 - \epsilon)\tau}).$$

Finally, we consider the translations p_t : We have from Eq. (23)

$$\frac{d}{dt} (x - R(t)L(t)\hat{x}) = \sum_{i,j=1}^2 (\bar{M}^{-1})^{ij} \int_{S^1} \frac{z_j r[r^{-1/3}]}{h^4 r^{4/3}} d\theta e_i,$$

where $z_1 = \cos \theta$ and $z_2 = \sin \theta$, and

$$\bar{M}_{ij} = \int_{S^1} h^{-4} z_i z_j d\theta.$$

$R^4 M$ has eigenvalues bounded above and below for large times, and so the square of the speed of translation is bounded by a constant times

$$R^8 \sum_{i=1}^2 \left(\int_{S^1} \frac{z_i r[r^{-1/3}]}{h^4 r^{4/3}} d\theta \right)^2.$$

Expanding about $h = R(t)$ and using the bound from Lemma 37 on $\int_{S^1} \hat{h} z_j d\theta$, we find

$$\left| \frac{dp}{dt} \right|^2 \leq CR^{-5/3} \int_{S^1} (\hat{h} - 1)^2 d\theta \leq Ct^{-(75/8 - \epsilon)}.$$

The result follows by integrating this inequality. \square

The final step in the proof of Theorem 1 is to show that the embedding x_t given by the solution of (1) also remain non-degenerate and converge, completing the proof of Theorem 1:

Proposition 41. *If $x : \mathcal{C} \times [0, \infty) \rightarrow \mathbb{R}^2$ satisfies Eq. (1), and $\bar{x} : S^1 \times [0, \infty) \rightarrow \mathbb{R}^2$ is the family of embeddings given by formula (3) applied to the support function h of the corresponding solution of Eq. (19), then $x_t = \bar{x}_t \circ \varphi_t$ for some smooth family of embeddings $\varphi_t : \mathcal{C} \times [0, \infty) \rightarrow S^1$. There exists a smooth embedding $\varphi_\infty : \mathcal{C} \rightarrow S^1$ such that φ_t converges in C^∞ to φ_∞ as t approaches ∞ .*

Proof. By computing the rate of change of the Euclidean unit normal under Eq. (1), we find:

$$\frac{\partial}{\partial t} \varphi = -(\mathbf{r}^{-2/3} \mathcal{K}_s) \circ \varphi.$$

By scaling and expanding about $\hat{r} = 1$, we find:

$$\left| \frac{\partial \varphi}{\partial t} \right| \leq R(t)^{-2} (\mathbf{r}[\hat{h} - 1] + \mathbf{r}[\hat{h} - 1]_\theta + \frac{1}{3} \mathbf{r}[\mathbf{r}[\hat{h} - 1]]_\theta + O(\|\hat{h} - 1\|_{C^2}^2)).$$

The bounds on $R(t)$ from Proposition 40 and on $\hat{h} - 1$ from Proposition 36 therefore give

$$\left| \frac{\partial \varphi}{\partial t} \right| \leq C(\varepsilon) t^{-(41/8 - \varepsilon)}.$$

Hence the total change in φ is bounded, and φ converges to a limiting map φ_∞ .

We similarly bound the change in φ_θ , using the identity

$$\frac{\partial}{\partial t} \varphi_\theta = -(\mathcal{K}_{ss} - \frac{2}{3} r^{-5/3} r_\theta \mathcal{K}_s) \varphi_\theta.$$

Hence, estimating as above,

$$\left| \frac{\partial}{\partial t} \log |\varphi_\theta| \right| \leq C(\varepsilon) t^{-(41/8 - \varepsilon)}.$$

Therefore the limiting map φ_∞ is nondegenerate, hence an embedding. Similarly, the total change in any higher derivative of φ is bounded, and so φ_∞ is a C^∞ embedding and the convergence of φ_t to φ_∞ is in C^∞ . \square

This completes the proof of Theorem 1, since the embeddings \bar{x} are smooth and non-degenerate whenever h is smooth and $r > 0$.

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