

SPEEDING THE CONVERGENCE OF SERIES

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INTRODUCTION

In mathematics, many quantities can be calculated in closed form, i.e. by using a finite number of simple operations. Failing this we may have to use a rapidly convergent series to calculate the quantity. In a sense this is also a finite operation because we estimate the error due to truncating the series and then add up sufficient terms of the series to give whatever accuracy we desire.

If all else fails, our only method of calculating some quantity may be to use a slowly convergent series in which a very large number of terms is required to give the necessary accuracy. This situation arises from time to time and a considerable number of methods exist in the research literature of mathematics for dealing with these series. The methods range from highly specialised methods which make use of some special property of the series, through varying degrees of generality, to very general methods which assume very little about the properties of the series.

The work in this submission deals with fairly general series in which practically nothing is assumed. The main assumptions are that the ratio of successive terms varies slowly along the series and that the series is either summable or transformable to an equivalent summable series. The simplest case is the purely geometric or exponential series in which the ratio of successive terms is constant so the series can be summed in one step.

This broad class of series includes the rapidly convergent series, like the series for e^x , which do not need their convergence speeded. Next, we have the almost geometric both convergent and divergent series, like the binomial, logarithmic and hypergeometric series, and finally the highly divergent asymptotic series. A

class of series not amenable to this general treatment are the Fourier or Sturm-Liouville series which are not simply alternating or monotonic and need some special treatment, such as Shanks's e_2 transformation or Wynn's change to complex variables, before summing.

PREVIOUS WORK

The best known linear transformation of a series is Euler's transformation which uses finite differences and works best on a slowly convergent alternating series. Also a weighted form of Euler's transformation can be used on power series provided only that the ratio of successive terms is not close to +1.

This transformation was vastly improved by A.C. Aitken in 1926. In a small section of a paper about other matters he used a non-linear transformation in order to sum his series. Aitken's transformation was re-discovered in 1955 by D. Shanks who extended it and also used it to manipulate divergent series.

The significance of this non linear transformation is two-fold: (i) It considerably improves the convergence of series so we do not need to devise rapidly convergent or extended series to calculate functions or evaluate physical quantities.

(ii) The method which turns slowly convergent series into rapidly convergent series also turns divergent series into convergent series. If we are careful to use unbiased transformations and justify our actions on grounds such as analytic continuation then we may use divergent series in practical calculations. /

Several other contributions to general methods for summing series are also cited in my papers.

PRESENT WORK

I first became interested in summing series when I wished to calculate the velocity of propagation of an oscillating sheet in a viscous liquid. In section 2(b) of my first paper, "Propulsion by Oscillating Sheets and Tubes in a Viscous Fluid", I summed a series for which four terms were known and which appeared to be convergent for $bk < 1$. I then claimed to know the result to within a few percent for $b^2k^2 < 2$. My reason for doing so was that I was using Aitken's transformation. (At the time I attributed it to Shanks, not realising that Aitken had a prior claim.) If this transformation transforms all the later terms of the series into a rapidly convergent series and the transformation is unbiased and the two series agree for $bk < 1$, then we obtain an analytic continuation of the series beyond its original radius of convergence.

Since then I have published two papers and a short article on accelerating the convergence of general series and have had two further papers accepted for publication.

The basic ideas I have developed for summing series are:

- (i) The series may be rearranged by term splitting and recombining.
- (ii) The series can be written like a G.P. as a power series in the ratio of successive terms.
- (iii) A general n -term formula must fit the first n terms of a series exactly.
- (iv) A weighting method may be used to construct more complicated formulae.

Term splitting is an extremely simple but powerful general method of transforming series. It consists in splitting each term of a series into two parts of which the first part approximates to the sum of the remainder of the series starting at the term being split. The series is recombined by joining the second part of each term to the first part of the following term. This creates a new series whose first term is approximately equal to the sum of the series and whose remaining terms are all much smaller than the corresponding original terms. The method is simple enough for high school students to understand yet it forms a very effective general basis for most of the known methods of transforming series. It is described in my second paper, "Summing Series by Term Splitting".

To briefly sketch the second principle let us take a series ,

$$S = u_0 + u_1 + u_2 + u_3 + \dots ,$$

and define the ratio of successive terms as R_n , where

$$R_n = u_{n+1}/u_n ,$$

then we can write

$$S = u_0 (1 + R_0 + R_0 R_1 + R_0 R_1 R_2 + \dots)$$

which looks like a power series in the R's.

As an example of the third principle, Aitken's approximation to the sum using two terms is

$$\frac{u_0^2}{u_0 - u_1} = \frac{u_0}{1 - R_0} = u_0 (1 + R_0 + R_0^2 + R_0^3 + \dots) .$$

If we compare these two series, assuming that they are rapidly convergent, we see that the first two terms agree *exactly*. For this reason I wish to describe Aitken's formula as a *general* formula. Furthermore, the third terms will almost agree if $R_0 \doteq R_1$, so Aitken's formula closely approximates to the original series if R_n varies slowly along the series. We also require that $(1 - R_0)$ shall not be zero. We then qualify our statement to say that Aitken's formula is a general formula for series in which R_n varies slowly along the series and R_n is not close to +1.

As an example of the fourth principle we can write Aitken's formula using u_0 and u_1 as

$$S \doteq \frac{u_0^2}{u_0 - u_1} = u_0 + \frac{u_0 u_1}{u_0 - u_1} = u_0 + \frac{u_1}{1 - R_0} .$$

Similarly, using u_1 and u_2 ,

$$S \doteq u_0 + \frac{u_1^2}{u_1 - u_2} = u_0 + \frac{u_1}{1 - R_1}.$$

For a rapidly convergent series the errors in these two Aitken formulae are roughly in the ratio $u_0:u_1$ or $u_1:u_2$. Hence a weighted combination of these two formulae will have a much smaller error if we choose weights $u_1:-u_0$ or $u_2:-u_1$ in order to cancel the errors. Thus, two better estimators of the sum are

$$u_0 + u_1 \frac{1 + w}{(1 - R_1) + w(1 - R_0)} \quad \text{where } w \text{ is } -R_0 \text{ or } -R_1$$

namely

$$u_0 + u_1 \frac{1 - R_0}{(1 - R_1) - R_0(1 - R_0)} \quad \text{or} \quad u_0 + u_1 \frac{1 - R_1}{(1 - R_1) - R_1(1 - R_0)}.$$

It should be noted that this particular method of weighting was chosen to satisfy the second and third principles.

These principles are developed in my third paper entitled "Summing Slowly Convergent Series". Section 2 may be omitted on first reading except for the definitions in the first four lines. This is because section 2 is almost identical with my fourth paper on "Summing a Common Type of Slowly Convergent Series of Positive Terms".

In my third paper I derive two three-term formulae, one of which is Lubkin's, six four-term formulae and seven five-term formulae for summing series. The only restriction on the series is that the ratio of successive terms varies slowly along the series. Furthermore, this ratio can be much closer to 1 than is the case with Euler's and Aitken's formulae. These formulae are more power-

ful than Aitken's and Lubkin's and may be used repeatedly on a longer series and can be programmed into a computer. However, I do not expect that they will be much used unless some algorithm can be found to reduce their complexity. At present the complexity of these formulae outweighs their advantage in being slightly more powerful than Aitken's and Lubkin's formulae. Finally, although this work is not directly useful it has indirectly resulted in three useful formulae.

The first of the three useful formulae, which was discovered as an offshoot of this work, is described in section 2 of my third paper or in my fourth paper on "Summing a Common Type of Slowly Convergent Series of Positive Terms". This is a more specialised formula as it requires the further information that the terms of the series behave like n^{-k} . This reduction in generality is justified on the grounds that this is an important class of series which otherwise is exceedingly difficult to sum.

The second formula is a weighted Euler sum using the reciprocals of the terms of the series as weights and it is described in my fifth paper "A Formula for Accelerating the Convergence of a General Series". It was discovered while looking for a pattern in the 2, 3, 4 and 5-term formulae. It is not an exceedingly powerful formula and it is not very useful for a slowly convergent series of positive terms. However, if we have available a reasonably large number of terms of an alternating series this transformation will derive a lot of information with very little effort in a single operation. For this reason it should become a popular transformation.

The third formula was also discovered while looking for a pattern in the 2, 3, 4 and 5-term formulae. It is described in the sixth paper "A Formula for Summing Divergent Series". It is a more complicated formula than the previous one and is most effective in transforming an asymptotic series into another asymptotic series with a smaller, smallest term. If the asymptotic series is truncated at its smallest term the "sum" so obtained appears to be consistent with the values associated with the series by other processes, such as the value of a related integral. The easiest way found to use this formula is to calculate terms successively using an algorithm similar to that of P. Wynn for Shanks's determinants and simpler than the algorithm for Padé approximants.

The seventh, and final paper, is a short article appealing to mathematicians to take some interest in divergent series. It is a general article and not a research paper and it is an attempt to inform a wider audience about some of the simpler aspects of my own and related work on summing series.

Propulsion by oscillating sheets and tubes in a viscous fluid

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Taylor's formula for the velocity of an oscillating sheet in a viscous fluid is extended to larger amplitudes of motion and his formula for the velocity of a rotating or oscillating filament is proved to be valid for amplitudes of motion small but greater than the filament radius. A simple formula is derived for the velocity of a sheet or filament in any general oscillatory motion of small amplitude. The forces and energy involved in oscillating tubes are calculated.

1. Introduction

Taylor (1951, 1952) has treated the problem of motion of an infinite sheet in sinusoidal motion in a viscous liquid at low Reynolds number and has shown that there is a non-zero velocity at which the thrust developed equals the drag. A similar treatment of an infinitely long cylinder oscillating transversely or in a rotating spiral leads him to a similar result, but in this case his choice of mathematical model makes his solution valid only for amplitudes of motion less than the radius of the cylinder.

Taylor's (1951) result for the sheet is extended here to larger amplitudes of motion and a different choice of mathematical model for the filament shows that Taylor's (1952) result for the filament is also valid for amplitudes of motion still small but greater than the filament radius and consistent with Hancock's (1953) results for very thin filaments for all amplitudes of motion.

2. The motion of an oscillating sheet

(a) Taylor (1951) showed that an infinite sheet in steady transverse sinusoidal motion in a plane progressive wave of small amplitude propels a viscous liquid forward in the direction of the wave at a non-zero velocity.

If we apply the same analysis to a sheet in a general transverse motion described by the lateral displacement,

$$y = \sum_n b_n \cos(k_n x - \sigma_n t + a_n) \quad (1)$$

we must relax Taylor's condition that the sheet is inextensible except when all the components of the wave have the same velocity.

However, if we write down the first-order terms for the fluid stream function and equate the normal component of fluid and sheet velocity to first and second order at the surface of the sheet, and, in place of the inextensibility conditions,

we say that the velocity component parallel to the mean position of the sheet is periodic in both x and t to first order, then we obtain the fluid velocity at infinity

$$v = \sum_n \frac{1}{2} k_n \sigma_n b_n^2 + O(\sigma k^3 b^4), \quad (2)$$

provided no pairs of k and σ are identically equal.

The higher-order terms in the expression for v may be affected by the tangential motion (stretching or sliding) of the sheet, but not the first-order terms.

We may interpret this result physically as follows: the normal component of the sheet's motion will push liquid in the direction of the normal and the tangential component of the sheet's motion will drag liquid with it. If the sheet is inextensible and contains a wave of fixed shape of velocity V and wavelength L , while the arc length for the wavelength L is S , then at any point where the sheet is inclined at θ to its mean surface the normal component of velocity is $-V \sin \theta$, and the tangential component of velocity is $V(\cos \theta - S/L)$.

Resolving along the mean line of the sheet, the normal thrust tends to induce a velocity in the fluid which is the mean value of

$$V \sin^2 \theta, \quad (3)$$

while the tangential drag tends to induce a velocity in the opposite direction which may for large amplitudes tend to infinity. We may then conclude that the fluid velocity has an upper bound V and no lower bound can be given for an inextensible sheet whose amplitude of motion is large.

Furthermore, for small values of θ ,

$$V \sin^2 \theta \div \sum_n \frac{1}{2} k_n \sigma_n b_n^2,$$

if σ_n/k_n is constant. Hence, for small amplitudes of motion, the velocity induced in the fluid is approximately that induced by the normal motion and the tangential effect is negligible.

(b) Taylor's (1951) rigorous treatment of viscous motion due to an inextensible sheet in sinusoidal motion for $bk < 1$ gives the fluid velocity at infinity

$$v = \frac{1}{2} \sigma b^2 k \left(1 - \frac{19}{16} b^2 k^2 + \frac{41}{32} b^4 k^4 - \frac{169}{128} b^6 k^6 + \dots \right), \quad (4)$$

on adding two more terms to Taylor's result.

This series is convergent for $bk < 1$ and may be expected on physical grounds to have an analytic continuation for $bk > 1$.

On using Shanks's (1955) e_1 operation repeatedly to improve the convergence of the series, we obtain the following values:

$b^2 k^2$	$\frac{1}{4}$	$\frac{1}{2}$	1	2
vk/σ	0.0958	0.153	0.215	0.26

while for the range $0 < bk < 1$ these values may be fitted reasonably well by the formula

$$v = \frac{\sigma}{k} \frac{0.518 b^2 k^2}{1 + 1.406 b^2 k^2},$$

or

$$v = \frac{\sigma}{k} \frac{0.508b^2k^2 - 0.030b^4k^4}{1 + 1.224b^2k^2}, \quad (5)$$

both formulae agreeing within a few per cent for $b^2k^2 < 2$, but being quite unreliable for larger amplitudes. These formulae may be compared with Hancock's (1953) formulae for plane and spiral waves in a thin filament.

(c) Since any periodic motion with finite oscillation may be analysed into a Fourier series of discrete terms, the displacement given by equation (1) represents any periodic motion, and furthermore if y is as given in equation (1) then

$$Av \left(-\frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \right) = \sum_n \frac{1}{2} k_n \sigma_n b_n^2,$$

where the average is taken over both x and t . This is the liquid velocity given in equation (2). Hence we may conclude that, for any general oscillatory motion of a sheet in which the slope of the sheet is small, the first approximation to the velocity of the liquid relative to the sheet is

$$v = Av \left(-\frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \right). \quad (6)$$

This formula may also be obtained for an oscillating filament from Hancock's (1953) vector formula. The corrected statement of the condition for finite energy in his formula (49) is

$$\lim_{s \rightarrow \infty} \frac{1}{2s} \int_{-s}^s \{f(x)\mathbf{t} + g(x)\mathbf{n} + h(x)\mathbf{b}\} ds = 0, \quad (7)$$

where s is the arc length replacing the linear dimension x and similarly in his formula (51), while his formula (47) gives the surface velocity of the filament, which need not be in the direction of the filament but any direction, so his formula (50) may be generalized to

$$-2f(x)\mathbf{t} - g(x)\mathbf{n} - h(x)\mathbf{b} + \mathbf{V} = \mathbf{U}, \quad (8)$$

where $\mathbf{U}$ is the velocity of the filament and $\mathbf{V}$ is the fluid velocity at infinity. Hence

$$f(x) = \frac{1}{2}\mathbf{t} \cdot (\mathbf{V} - \mathbf{U}), \quad g(x) = \mathbf{n} \cdot (\mathbf{V} - \mathbf{U}),$$

and

$$h(x) = \mathbf{b} \cdot (\mathbf{V} - \mathbf{U}).$$

Substituting in the energy integral,

$$\lim_{s \rightarrow \infty} \frac{1}{2s} \int_{-s}^s [\mathbf{V} - \mathbf{U} - \frac{1}{2}\mathbf{t} \cdot (\mathbf{V} - \mathbf{U})\mathbf{t}] ds = 0. \quad (9)$$

Hancock (1953) also concludes in §3 that, apart from small end effects, finite filaments travel at the same speed as infinite filaments, so the above formula will also hold for a finite filament with the limits of integration changed to the ends of the filament.

For small amplitudes of motion, suppose the filament is close to the x -axis and oscillating transversely in both the y - and z -directions, then

$$\mathbf{U} = \left(0, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \right).$$

Also $\mathbf{V}$ and $\mathbf{t}$ will be close to the x -direction and $ds \doteq dx$ so

$$\mathbf{t} \cdot (\mathbf{V} - \mathbf{U}) \doteq V - \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} - \frac{\partial z}{\partial x} \frac{\partial z}{\partial t}.$$

Substituting in the x -component of equation (9) we find

$$V = -\frac{1}{l} \int_0^l \left(\frac{\partial y}{\partial x} \frac{\partial y}{\partial t} + \frac{\partial z}{\partial x} \frac{\partial z}{\partial t} \right) dx \quad (10)$$

for a filament oscillating with small amplitude of motion.

3. The motion of a viscous fluid round a rotating helix

Let the helix consist of a cylinder of radius a twisted into a helix of radius b and wavelength $2\pi/k$, rotated with frequency $\omega/2\pi$, so the equation of the arc of the helix is

$$x = b \cos(\omega t - kz), \quad y = b \sin(\omega t - kz)$$

for a helix rotating about the z -axis.

Taylor (1952) gives the velocity of propagation of the helix through a viscous liquid and an incorrect expression for the couple required to maintain this motion when the surface of the cylinder does not rotate, and points out that his analysis is valid only for small values of kb and for $b < a$, while it is exact for all values of ka . The reason for this is that, in the mathematical model used to represent the motion of the fluid, his functions would have singularities in the fluid for $b > a$.

To avoid this difficulty we use a skew co-ordinate system with the centre of the cylinder as the origin of cylindrical (r, θ) co-ordinates and z measured along the axis of the helix. The co-ordinate transformation from rectangular (x, y, z) co-ordinates is

$$x = b \cos(\theta - \theta') + r \cos \theta, \quad y = b \sin(\theta - \theta') + r \sin \theta, \quad z = \zeta,$$

where $\theta' = \theta + k\zeta - \omega t$.

The differential equations for the fluid velocity are then put in tensor notation, and then written out in terms of the skew co-ordinates, and a solution for the pressure p and velocity components u_r, u_θ, u_z is sought in the form

$$\begin{aligned} p &= \mu k (A_1 \sin \theta' + b k A_2 \sin 2\theta' + \dots), \\ u_r &= B_1 \sin \theta' + b k B_2 \sin 2\theta' + \dots, \\ u_\theta &= C_0 + C_1 \cos \theta' + b k C_2 \cos 2\theta' + \dots, \\ u_z &= D_0 + D_1 \cos \theta' + b k D_2 \cos 2\theta' + \dots, \end{aligned}$$

where the $ABCD$'s are functions of r only and are bounded for $a \leq r < \infty$ with the boundary condition on the surface of the cylinder $r = a$ that the fluid velocity

$$u_r = b\omega \sin \theta', \quad u_\theta = a\Omega + b\omega \cos \theta', \quad u_z = 0,$$

which is the velocity of the surface of the helix, which rotates with angular velocity ω while its surface skin rotates with angular velocity Ω ($\Omega = \omega$ for a rigid helix and $\Omega = 0$ for the case considered by Taylor (1952)).

On substituting in the differential equations we find that the lowest-order terms in bk are

$$\begin{aligned} A_1 &= a_1 K_1(kr)/e_1, \\ B_1 &= [b_1 K_0(kr) + c_1 K_2(kr) + a_1 kr K_1(kr)]/2e_1, \\ C_1 &= [b_1 K_0(kr) - c_1 K_2(kr)]/2e_1, \\ D_1 &= [d_1 K_1(kr) - a_1 kr K_0(kr)]/2e_1, \end{aligned}$$

where the K 's are modified Bessel functions while a_1, b_1, c_1, d_1 and e_1 are constant.

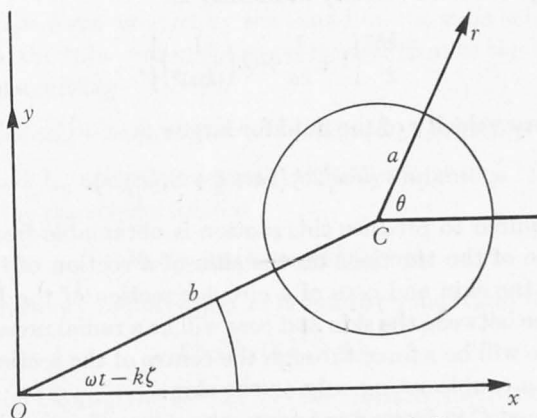


FIGURE 1. Cross-section of the helix.

From the boundary and divergence conditions

$$C_0 = a^2 \Omega / r \quad \text{and} \quad D_0 = 0$$

to first order in kb and

$$\begin{aligned} a_1 &= 2b\omega K_1(2K_1 + uK_0)/u, \\ b_1 &= b\omega[u(K_1^2 - 2K_0^2) - 2K_1K_0 + 4K_1^2/u], \\ c_1 &= -b\omega K_1^2 u, \\ d_1 &= b\omega K_0(2K_1 + uK_0), \\ e_1 &= (K_1 + uK_0)(K_1^2 - K_0^2) + 2K_0K_1^2/u, \\ u &= ka, \quad K_0 = K_0(ka), \quad K_1 = K_1(ka), \end{aligned}$$

where

while to second order in kb

$$C_0 = \frac{1}{r} \left\{ a^2 \Omega + \frac{b^2 \omega}{2e_1} K_1[u^2(K_1^2 - K_0^2) + 4K_1^2] \right\} - \frac{bk}{4e_1} (a_1 + b_1 - c_1) K_1(kr), \quad (11)$$

and

$$D_0 = \frac{\omega kb^2}{2e_1} (2K_1 + uK_0)(K_0^2 - K_1^2 + 2K_0K_1/u) + \frac{bk}{4e_1} \{a_1 kr K_1(kr) - 2(a_1 + d_1) K_0(kr)\}. \quad (12)$$

The velocity of the fluid at infinity produced by rotating the helix is the constant term in D_0 . This is the same result as derived by Taylor (1952), valid for all ka

and a first-order approximation for small kb , but this analysis using spiral co-ordinates shows that the restriction $b < a$ is not necessary.

We now proceed to list some further results of the theory. If ka is small the fluid velocity at infinity is

$$\frac{\omega kb^2}{2K_0 + 1} \left(2K_0 - 1 + \frac{k^2 a^2}{2} \frac{8K_0^2 - 2K_0 + 3}{2K_0 + 1} + O(ka)^3 \right), \quad (13)$$

and the circulatory velocity of the fluid for large r is

$$a^2 \Omega / r + 2b^2 \omega / (2K_0 + 1) r + O(ka), \quad (14)$$

while if ka is large the fluid velocity at infinity is

$$\frac{\omega kb^2}{2} \left\{ 1 + \frac{1}{ka} + O\left(\frac{1}{(ka)^2}\right) \right\}, \quad (15)$$

and the circulatory velocity of the fluid for large r is

$$a^2 \Omega / r + (b^2 \omega / 2r) (ka + \frac{3}{2} + O(ka)^{-1}). \quad (16)$$

The couple required to produce this motion is obtainable from the resultant thrust and couple of the tractions on the skin of a section of the helix. If the contact between the skin and core of a circular section of the helix is smooth, then the only force between the skin and core will be a radial pressure, so the only action on the core will be a force through the centre of the section. The traction will also produce a couple acting only on the skin.

The only action at C in figure 1 is a force whose line of action is perpendicular to OC and opposes the rotation. Its magnitude is

$$\pi \mu b \omega u K_1 (u K_1^2 - u K_0^2 + 2K_0 K_1 + 8K_1^2 / u) / e_1. \quad (17)$$

The couple on the skin of the helix is

$$-\pi \mu \{ 4a^2 \Omega - b^2 \omega u^2 K_1 (K_0^2 - K_1^2 + 2K_0 K_1 / u) / e_1 \}, \quad (18)$$

and the resultant couple about the axis, O , of the helix is

$$-4\pi \mu \{ a^2 \Omega + b^2 \omega K_1 (u^2 K_1^2 - u^2 K_0^2 + 4K_1^2) / 2e_1 \} = -4\pi \mu (r C_0), \quad (19)$$

and the energy output per unit length is

$$\begin{aligned} & \pi \mu \{ 4a^2 \Omega^2 - b^2 \Omega \omega u^2 K_1 (K_0^2 - K_1^2 + 2K_0 K_1 / u) e_1 \\ & + b^2 \omega^2 u K_1 (u K_1^2 - u K_0^2 + 2K_0 K_1 + 8K_1^2 / u) / e_1 \} \div 4\pi \mu b^2 \omega^2 / (K_0 + \frac{1}{2}) \end{aligned} \quad (20)$$

for small ka . The case for $\Omega = 0$ was considered by Taylor (1952). He incorrectly used w'_1 instead of v'_1 in his equations (3.10) and (3.12), and the expression for the couple in his equation (3.13) should be

$$4\pi \mu b^2 \kappa U \{ \{ K_0(ka) + \frac{1}{2} \} \text{ instead of } -4\pi \mu a b^2 \kappa^2 U \left\{ \frac{K_0(ka) - \frac{1}{2}}{K_0(ka) + \frac{1}{2}} \right\}.$$

The former expression agrees with (19) above. The remainder of Taylor's paper is not affected by this error.

4. The motion of an oscillating tube in a viscous fluid

In the case of a tube oscillating in a plane through its axis a set of skew co-ordinates may be used

$$x = b \cos(k\zeta - \omega t) + r \cos \theta, \quad y = r \sin \theta, \quad z = \zeta$$

and the surface of the tube is $r = a$.

The hydrodynamic equations can be expressed in terms of these skew co-ordinates and solved for small kb while a may be as large or as small as we please. Apart from the energy output per unit length the results agree with Taylor's (1952).

If kb is small, the force exerted by the liquid on the tube is in the x -direction at right angles to the tube and in the opposite direction to the tube's transverse velocity and of magnitude

$$\pi \mu b \omega (8K_1^2 + 2uK_0K_1 + u^2(K_1^2 - K_0^2)) (K_1/e_1) \sin(kz - \omega t), \quad (21)$$

where $u = ka$ and K_0 and K_1 are modified Bessel functions of argument ka .

If ka is small the transverse force is

$$4\pi \mu b \omega \sin(kz - \omega t) / (K_0 + \frac{1}{2})$$

and the average energy required per unit length to maintain the motion is

$$2\pi \mu b^2 \omega^2 / (K_0 + \frac{1}{2}). \quad (22)$$

The corresponding value given by Taylor (1952) is half this, owing to the omission of a factor $\frac{1}{2}$ before the quantities B and C in his equation (2.36). The consistency of the present result may be easily confirmed by comparison with the result for the helix for small ka . If the transverse velocity of the tube is

$$v = b\omega \sin(kz - \omega t),$$

then the force on the tube is

$$-4\pi \mu v / (K_0 + \frac{1}{2}).$$

If such a force acts about the axis at a distance b from the axis, it produces a couple

$$-4\pi \mu b v / (K_0 + \frac{1}{2}),$$

which is the value of the couple on the helix for a velocity $b\omega$ with $\Omega = 0$.

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SUMMING SERIES BY TERM SPLITTING J.E. DRUMMOND

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If you have ever tried to find π by summing the series $4 - 4/3 + 4/5 - \dots$ or $\ln 2$ by summing $1 - 1/2 + 1/3 - \dots$ you will soon give up. However, there are a number of methods invented by Euler, Aitken, Shanks and others for making these series more rapidly convergent and hence practically more useful.

The following method is a general method which provides a framework for discussing the effectiveness of the specialist methods. Also, when the terms of the series consist of rational numbers it is both a useful and an amusing way of manipulating series.

Term splitting

Term splitting is a name for a general method of rearranging a series by splitting each term into a first and a second part then recombining the second part of each term with the first part of the next term to form a term of the transformed series.

As an example, if $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

we split each term in half so the series becomes

$$\frac{1+1}{2} - \frac{1+1}{4} + \frac{1+1}{6} - \dots$$

On reassembling the adjacent half terms we obtain

$$\ln 2 = \frac{1}{2} + \frac{1}{2} \left\{ \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \dots \right\}.$$

In this case our series transforms to a first part of the first term and a new series which is more rapidly convergent than the original series. This is actually the first step in constructing the *Euler Sum* for the series. The Euler Sum was invented by the eighteenth century Swiss-German mathematician Leonhard Euler and was expressed in finite differences of ascending order, thus

$$\begin{aligned} u_0 - u_1 + u_2 - \dots &= \frac{1}{2}u_0 + \frac{1}{4}(u_0 - u_1) + \frac{1}{8}(u_0 - 2u_1 + u_2) + \\ &\quad + \frac{1}{16}(u_0 - 3u_1 + 3u_2 - u_3) + \dots \end{aligned}$$

Repeated splittings in half of the terms of each new series give a series of first terms which is the Euler Sum. After three splittings

$$\ln 2 = \frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1.2.3}{2^3} \left\{ \frac{1}{1.2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} - \dots \right\}.$$

Here we have three terms of the Euler Sum.

We may equally well use different splitting ratios and, depending on our skill, do better or worse than the Euler Sum. Two examples of splitting are

$$\begin{aligned}\ln 2 &= \frac{2+1}{3} - \frac{3+2}{2.5} + \frac{4+3}{3.7} - \frac{5+4}{4.9} \\ &= \frac{2}{3} + \left\{ \frac{1}{2.3.5} - \frac{1}{3.5.7} + \frac{1}{4.7.9} - \dots \right\}\end{aligned}$$

or

$$\begin{aligned}\ln 2 &= \frac{3+1}{4} - \frac{5+3}{2.8} + \frac{7+5}{3.12} - \frac{9+7}{4.16} + \dots \\ &= \frac{3}{4} - \frac{1}{4} \left\{ \frac{1}{1.2^2} - \frac{1}{2^2.3^2} + \frac{1}{3^2.4^2} - \dots \right\}.\end{aligned}$$

Both of these series are more rapidly convergent than the first splitting with equal ratios and may be further split.

If we are fortunate or happen to know the sum of a series we may obtain complete cancellation of the recombined terms. For the geometric series the splitting ratio is obviously "minus the common ratio", so

$$\begin{aligned}S &= 1 + a + a^2 + a^3 + a^4 + \dots \\ &= \frac{1-a}{(1-a)} + a \frac{1-a}{(1-a)} + a^2 \frac{1-a}{(1-a)} + \dots \\ &= (1-a)^{-1} \{1 + (-a+a) + (-a^2+a^2) + \dots\} \\ &= (1-a)^{-1}.\end{aligned}$$

Another well known series is

$$\begin{aligned}S &= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots \\ &= \frac{2-1}{1.2} + \frac{3-2}{2.3} + \frac{4-3}{3.4} + \dots \\ &= \frac{2}{1.2} + \left[\frac{-1}{1.2} + \frac{3}{2.3} \right] + \left[\frac{-2}{2.3} + \frac{4}{3.4} \right] + \dots \\ &= 1 + 0 + 0 + \dots\end{aligned}$$

Rules for term splitting

The few examples above should indicate that term splitting is an amusing game in which we can transform series almost at will into a large variety of other series. It also has its practical uses in that, by careful choice of the splitting ratios, we can speed up the summation of a new and unknown series to whatever accuracy we need. It also has theoretical interest in that a number of methods for summing series may be described as term splitting, hence their powers (a powerful method has a small residual) may be compared by comparing the sizes of the residual series after regrouping.

However, before we proceed, we need a theorem that the transformation does not alter the value of the series. A convenient theorem is *If a series is convergent and either part of a split term tends to zero as we move along the series, then the sum of the first term plus the transformed series is equal to the original series.*

Also, if our aim is to make the terms of the new series smaller than the corresponding terms of the original series, or increase its rate of convergence, we observe that the perfect choice of splitting is as follows: if u_n is a typical term of a series $S = u_1 + u_2 + u_3 + \dots$ and the residual sum $R_n = u_{n+1} + u_{n+2} + u_{n+3} + \dots$, then $u_n = R_{n-1} - R_n$. If we use this as the splitting formula, $S = (R_0 - R_1) + (R_1 - R_2) + \dots$, so, on regrouping, $S = R_0 + (-R_1 + R_1) + (-R_2 + R_2) + \dots = R_0$, provided only that S is convergent or $R_n \rightarrow 0$ as $n \rightarrow \infty$.

The practical importance of this result is that even when we do not know the sum or cannot achieve perfect cancellation, we can make the transformed series small by using good approximations to R_{n-1} and $(-R_n)$ in splitting u_n .

Transformations of convergent series

We may split the terms of a convergent series in any way we please provided the split parts tend to zero as we move along the series. This enables us to construct a great variety of series. If we choose our splitting ratios so that they are close to ideal and at the same time some of the factors in the recombined terms cancel, we can construct more rapidly convergent and at the same time relatively simple series.

One way of transforming the series for $\ln 2$ is

$$\begin{aligned} \ln 2 &= 1 - \frac{3+1}{4} + \frac{1}{2} - \frac{4+2}{6} + \frac{1}{3} - \frac{5+3}{8} + \frac{1}{4} - \frac{6+4}{10} + \dots \\ &= \frac{3}{4} - \frac{1}{2} \left(\frac{1}{2 \cdot 3} \cdot \frac{6+2}{8} - \frac{1}{3 \cdot 4} \cdot \frac{7+3}{10} + \frac{1}{4 \cdot 5} \cdot \frac{8+4}{12} - \dots \right) \\ &= \frac{3}{4} \left(1 - \frac{1}{4 \cdot 3} \right) + \frac{1}{2} \left(\frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{9+3}{12} - \frac{1}{4 \cdot 5 \cdot 6} \cdot \frac{10+4}{14} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{11+5}{16} - \dots \right) \\ &= \frac{3}{4} \left(1 - \frac{1}{4 \cdot 3} + \frac{1 \cdot 2}{4^2 \cdot 3 \cdot 5} \right) - \frac{3}{4} \left(\frac{1}{4 \cdot 5 \cdot 6 \cdot 7} \cdot \frac{12+4}{16} - \dots \right) \\ \text{etc.} &= \frac{3}{4} \left(1 - \frac{1}{4 \cdot 3} + \frac{1 \cdot 2}{4^2 \cdot 3 \cdot 5} - \frac{1 \cdot 2 \cdot 3}{4^3 \cdot 3 \cdot 5 \cdot 7} + \dots \right), \end{aligned}$$

which converges faster than the Euler Sum.

A well known series for π which is not of much use because it is very slowly convergent is

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots$$

By repeated splitting in half we obtain the Euler Sum,

$$\pi = 2 + \frac{2}{3} + \frac{2 \cdot 2}{3 \cdot 5} + \frac{2 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9} + \dots$$

If we divide each term into rational fractions choosing a denominator which fits in with the other factors in the denominator and numerators which help to simplify the resulting expression we may be able to produce a rapidly

convergent but not too complicated series. One such attempt is to proceed as follows:

$$\begin{aligned}\pi &= 4\left(1 \frac{5+1}{6} - \frac{1}{3} \frac{7+3}{10} + \frac{1}{5} \frac{9+5}{14} - \dots\right) \\ &= \frac{2.5}{3} - 4\left(\frac{1}{3.5} \frac{11+3}{14} - \frac{1}{5.7} \frac{13+5}{18} + \frac{1}{7.9} \frac{15+7}{22} - \dots\right) \\ &= \frac{2.5}{3} - \frac{2.11}{3.5.7} + 4.2\left(\frac{1}{5.7.9} \frac{17+5}{22} - \frac{1}{7.9.11} \frac{19+7}{26} + \dots\right)\end{aligned}$$

finally reducing to

$$\pi = \frac{2.5.0!}{1.3} - \frac{2.11.1!}{3.5.7} + \frac{2.17.2!}{5.7.9.11} - \dots$$

This series is more rapidly convergent than the Euler Sum but the work of calculating the terms is roughly comparable for the same accuracy.

Transformations of divergent series

G.H. Hardy [1] wrote a textbook about divergent series and showed that meaningful and self-consistent rules can be used in their summation. The author, who was originally taught to avoid divergent series as hopeless, believes that they should be a useful adjunct to every mathematician's repertoire. Also, in recent years two authors, F. Rhodes [2] and R.E. Scraton [3], have discussed the summation of some divergent series in *The Mathematical Gazette*.

Term splitting will also provide a meaningful sum for a divergent series, but we require a more restrictive theorem than with convergent series. Theorems in this field are either limited in usefulness or difficult to prove but a useful practical rule stated somewhat vaguely is as follows:

for a divergent alternating series, treat the positive and negative terms in the same way.

The significance of this restriction is illustrated by the following example.

$$\begin{aligned}\text{Let } S &= 1 - 1 + 1 - 1 + 1 - 1 + \dots \\ &= \frac{(2+1)}{3} - \frac{(1+2)}{3} + \frac{(2+1)}{3} - \frac{(1+2)}{3} + \dots\end{aligned}$$

Using different splittings for the positive and negative terms, we get

$$= \frac{2}{3} + \left(\frac{1}{3} - \frac{1}{3}\right) - \left(\frac{2}{3} - \frac{2}{3}\right) + \dots = \frac{2}{3};$$

whereas with the same splittings we obtain the usually accepted

$$\begin{aligned}S &= \frac{(1+1)}{2} - \frac{(1+1)}{2} + \frac{(1+1)}{2} - \dots \\ &= \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}\right) + \dots = \frac{1}{2}.\end{aligned}$$

Next let $S = 1 - 2 + 3 - 4 + 5 - 6 + \dots$.

Suppose we split $n = [an+b] + [(1-a)n-b]$; then, for perfect cancellation with the next term, we require

$$(1-a)n-b = a(n+1)+b, \quad \text{i.e., } a = 1/2, \quad b = -1/4.$$

$$\begin{aligned}\text{Therefore } S &= (1+3)/4 - (3+5)/4 + (5+7)/4 - \dots \\ &= 1/4, \text{ on regrouping.}\end{aligned}$$

Furthermore in the splitting formula we increase n by 1 in going from a positive to a negative term and similarly in going from a negative to a positive term so the formula is unbiased.

Similarly for an alternating sum of squares we get complete cancellation with the next term if we write $n^2 = n(n-1)/2 + n(n+1)/2$.

$$\begin{aligned}\text{Therefore } S &= 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \dots \\ &= (0+1) - (1+3) + (3+6) - (6+10) + \dots \\ &= 0, \text{ on regrouping.}\end{aligned}$$

This may be verified by splitting each term in half and regrouping three times which yields the Euler sum.

$$\begin{aligned}\text{Finally } \int_1^\infty e^{-t} dt/t &= 0! - 1! + 2! - 3! + \dots \\ &= 0! \cdot \frac{(1+1)}{2} - 1! \cdot \frac{(1+2)}{3} + 2! \cdot \frac{(1+3)}{4} - \dots \\ &= \frac{0!}{2} + \left\{ \frac{1!}{2 \cdot 3} - \frac{2!}{3 \cdot 4} + \frac{3!}{4 \cdot 5} - \dots \right\}\end{aligned}$$

which is a good start for the summation of this highly divergent series.

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SUMMING SLOWLY CONVERGENT SERIES

J.E. Drummond

Abstract Several formulae are given for summing slowly convergent series of positive terms. (i) If the terms behave like n^{-k} where k is an exactly known¹ constant, a modification of Aitken's formula using two terms transforms the series into a series of terms like n^{-k-2} provided $k \neq 1$. (ii) For more general slowly convergent series a four term formula similar to Lubkin's three term formula is derived. This is slightly more powerful but more complicated than Lubkin's formula. The two formulae may be used to make full use of any truncated series except 1, 2 and 5 terms. Several other 4 and 5 term formulae are also derived.

1. INTRODUCTION

Two common classes of slowly convergent monotonic sequences or series of positive terms are the almost geometric series in which the ratio of successive terms is slightly less than 1 and the series whose terms behave like n^{-k} summed over the integers, n , with $k > 1$.

If we assume very little knowledge of any special properties of the series other than that the series is convergent and that any

modification we make to the series does not alter its sum then there are several general methods for transforming the series. The purpose of these transformations is to produce a more rapidly convergent series with the same sum while being economical in computer time and/or number of terms of the series.

The simplest of these general or semi-general transformations is the weighted Euler sum. If the series is known to approximate closely to geometric and the limiting ratio of successive terms is known to be R where

$$(1.1) \quad R = \lim_{n \rightarrow \infty} (u_{n+1}/u_n)$$

then the series,

$$(1.2) \quad \sum_{n=0}^{\infty} u_n,$$

may be transformed using a weighted Euler transformation into

$$(1.3) \quad \frac{u_0}{1-R} + \frac{u_1 - Ru_0}{(1-R)^2} + \frac{u_2 - 2Ru_1 + R^2u_0}{(1-R)^3} + \dots$$

This transformation may however fail to increase the rate of convergence if R is close to 1.

Two generalisations of Euler's transformation have been proposed by T.M. Cherry [2], one being

$$(1.4) \quad \sum_{r=0}^{\infty} C_{n+r} t^{n+r} f(n+r) = t^n \sum_{r=0}^{p-1} t^r D_t^r \phi_n(t) \cdot \Delta^r f(n)/r! + S_{n,p}.$$

This combines a comparison series, $\phi_n(t)$, which can be differentiated and summed in closed form and a set of finite differences as in the Euler sum. In their actual application, Cherry's formulae become less general in that we require some knowledge of the properties of the series in order to choose a comparison series, $\phi_n(t)$, which closely fits the larger terms of the original series.

If less is known about the series then the series, $\sum_{n=0}^{\infty} u_n$,

may be transformed using Aitken's [1] transformation into a series

$\sum u_n^*$ where

$$(1.5) \quad u_n^* = u_n - \frac{u_n^2}{u_n - u_{n+1}} + \frac{u_{n+1}^2}{u_{n+1} - u_{n+2}},$$

taking $u_{-1} = 0$. This transformation is most effective when the ratio of successive terms (u_{n+1}/u_n) varies slowly along the series and is again

not close to 1. The most difficult of the smooth monotonic series to

sum are the series whose terms behave like n^{-k} and hence u_{n+1}/u_n is close to 1 for large n . Two methods have been proposed for dealing

with such series. The first is P. Wynn's [4] ρ -algorithm which is a

set of recurrence relations for calculating successive approximations to the sum of the series assuming that the partial sums may be approximated by rational functions of n . It is discussed in section 2 and it appears

to work best when $k = 2$. If k is known but not necessarily an integer

then a modification of Aitken's formula is given in section 2 for summing the series and compared with Wynn's ρ -algorithm.

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The second method for dealing with series whose terms behave like n^{-k} where k is not known has been given by S. Lubkin [3]. This is discussed in section 3 before going on to other similar formulae.

If k is not known then k may be eliminated between two of the modifications of Aitken's formula to give Lubkin's formula. This formula assumes that the tail of a series, $u_{n+1} + u_{n+2} \dots$, is approximately

$$\frac{u_n u_{n+1} (u_{n+1} - u_{n+2})}{u_n u_{n+1} - 2u_n u_{n+2} + u_{n+1} u_{n+2}}.$$

It may then be used to transform a series $\sum u_n$ into a series $\sum u_n^{**}$ where

$$\begin{aligned} u_{n+1}^{**} = u_{n+1} &- \frac{u_n u_{n+1} (u_{n+1} - u_{n+2})}{u_n u_{n+1} - 2u_n u_{n+2} + u_{n+1} u_{n+2}} \\ &+ \frac{u_{n+1} u_{n+2} (u_{n+2} - u_{n+3})}{u_{n+1} u_{n+2} - 2u_{n+1} u_{n+3} + u_{n+2} u_{n+3}}. \end{aligned}$$

Lubkin's formula is exact if $u_n = A(n+a)!/(n+b)!$ where A , a and b are constants and appears to be effective on a number of different slowly convergent series.

Since Lubkin's estimator of the tail uses three terms of the series the new series derived from a truncated series has three less terms than the original series. The transformation may then be repeated on the new series producing a third series, reducing the number of terms by three each time until (i) less than three terms survive or (ii) oscillations at the start of the series make the transformation

unreliable or (iii) rounding off errors make the later terms of the series unreliable.

A minor disadvantage with Lubkin's transformation is that if the number of known terms of the series is not a multiple of 3, then we do not make full use of the first one or two terms of the series in improving the convergence. If we had a four term formula as good or better than Lubkin's formula then we could use Lubkin's formula repeatedly until the transformed series is reduced to 0, 4 or 8 terms then finish off with 0, 1 or 2 applications of the four term formula. This accounts for all but a series of 1, 2 or 5 known terms. Nothing can be done with one term while a two term series may be slightly improved using Aitken's formula and a five term formula might be occasionally used to make full use of a series truncated at 5 terms. On the other hand a four term formula which is derived below in section 4 can supplement and corroborate Lubkin's formula.

2. SUMMING A SERIES LIKE $\sum n^{-k}$ WHEN k IS KNOWN

Let u_n be a term of a series.

Let Z_n be the sum to infinity of the series starting with u_n , then

$$Z_n = u_n + u_{n+1} + u_{n+2} + \dots$$

Let R_n be the ratio of successive terms, u_{n+1}/u_n .

We first take a special series for which

$$(2.1) \quad Z_n = \frac{A}{(n-k)(n-k+1)\dots(n-2)}$$

then

$$(2.2) \quad u_n = Z_n - Z_{n+1} = \frac{A(k-1)}{(n-k)(n-k+1)\dots(n-1)}$$

and

$$(2.3) \quad R_n = u_{n+1}/u_n = 1 - k/n.$$

Hence

$$\frac{Z_n}{u_n} = \frac{n-1}{k-1} = 1 + \frac{k}{k-1} \frac{R_n}{1-R_n}$$

$$(2.4) \quad \therefore Z_n = u_n + \frac{k}{k-1} \frac{u_n u_{n+1}}{u_n - u_{n+1}}.$$

Now let

$$(2.5) \quad T_{n+1} = \frac{u_n u_{n+1}}{u_n - u_{n+1}}$$

then

$$(2.6) \quad Z_{n+1} = \frac{k}{k-1} T_{n+1}.$$

Aitken's [1] approximant to the ^{tail} of a series, Z_n , is

$$(2.7) \quad A_n = \frac{u_n^2}{u_n - u_{n+1}} = u_n + \frac{u_n u_{n+1}}{u_n - u_{n+1}} = u_n + T_{n+1}.$$

Now Aitken's approximant is exact for a G.P. but when the ratio of successive terms, R_n , is $(1 - k/n)$ then Aitken's approximant underestimates the tail, Z_{n+1} , by a factor $(k-1)/k$. If k is known we may correct for this by multiplying the second part of A_n by $k/(k-1)$.

For the particular series given above this gives the sum exactly, but if we have some other series for which n^{-k} is the dominant part of a term we may still transform the series in the same way.

To compare this modification of Aitken's transformation with Aitken's original transformation we first write out the two transformations:

$$\begin{aligned}
 Z_0 &= u_0 + u_1 + u_2 + \dots \\
 &= (A_0 + u_0 - A_0) + (A_1 + u_1 - A_1) + \dots \\
 (2.8) \quad &= A_0 + (u_0 - A_0 + A_1) + (u_1 - A_1 + A_2) + \dots
 \end{aligned}$$

and

$$\begin{aligned}
 &u_0 + \left(\frac{k}{k-1} T_1 + u_1 - \frac{k}{k-1} T_1 \right) + \left(\frac{k}{k-1} T_2 + u_2 - \frac{k}{k-1} T_2 \right) + \dots \\
 (2.9) \quad &= \left(u_0 + \frac{k}{k-1} T_1 \right) + \left(u_1 - \frac{k}{k-1} T_1 + \frac{k}{k-1} T_2 \right) + \left(u_2 - \frac{k}{k-1} T_2 + \frac{k}{k-1} T_3 \right) \\
 &\quad + \dots
 \end{aligned}$$

We thus transform the series into a large first term A_0 or $(u_0 + kT_1/(k-1))$ and a transformed series of smaller terms.

The size of a typical term of the transformed series may now be calculated using the formulae 2.7 and 2.5 for A_n and T_{n+1} .

$$\text{If } u_n = \frac{1}{n^k} \text{ then } (u_n - A_n + A_{n+1}) \sim \frac{1}{kn^k}$$

$$\text{and } \left(u_{n+1} - \frac{k}{k-1} T_{n+1} + \frac{k}{k-1} T_{n+2} \right) \sim -\frac{(k+1)}{12n^{k+2}}.$$

The significant difference between these transformations is the division of the original terms by n^2 which makes the terms of the second transformed series much smaller. The transformed series may be transformed a second time, but now the series behaves like $\sum n^{-k-2}$ so we take $(k+2)/(k+1)$ times T_n thus increasing k by 2 for each successive transformation.

We demonstrate this on a slowly convergent series for π in table I using eight terms of the series

$$\begin{aligned}
 (2.10) \quad \pi &= 2 \int_0^1 (1-x^2)^{-1/2} dx = 2 \int_0^1 \left(1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \dots\right) dx \\
 &= 2 + \frac{1}{3} + \frac{3}{4 \cdot 5} + \frac{3 \cdot 5}{4 \cdot 6 \cdot 7} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 9} + \dots,
 \end{aligned}$$

and multiplying by 10^9 to avoid tabulating decimals. This series would require about 10^{16} terms to give an error of 10 in the tabulated values or 10^{-8} in π .

TAKE IN TABLE I

If it is known that the first, third, fifth and seventh columns are terms of series which behave like $n^{-3/2}$, $n^{-7/2}$, $n^{-11/2}$, $n^{-15/2}$ then we multiply Aitken's approximant to Z_{n+1} , namely T_{n+1} , by $3/1$, $7/5$, $11/9$, $15/13$ using the formulae at the bottom of the table to construct the table. The numbers marked with an asterisk at the top of each column are summed to give 3141592641 which differs from $10^9 \pi$ by 12. This method may be compared with Wynn's [4] ρ -algorithm,

$$(2.11) \quad \rho_0(S_{m+r}) = S_{m+r}, \quad \rho_1(S_{m+r}) = \frac{1}{u_{m+r+1}},$$

TABLE 1

Summing a series for $10^9 \pi$ knowing that the terms behave like $n^{-3/2}$.

u_n	$3T_n$	u_n^*	$\frac{7T_n^*}{5}$	u_n^{**}	$\frac{11T_n^{**}}{9}$	u_n^{***}	$\frac{15T_n^{***}}{13}$
2000,000000*							
333,333333	1200,000000*						
150,000000	818,181818	-48484848*					
89,285714	661,764706	-6417112	-10354391*				
60,763889	570,652174	-1826818	-3575381	361898*			
44,744318	509,159483	-728802	-1697558	51006	72568*		
34,705528	464,062500	-352665	-956649	12106	19401	-2161*	
27,929686	429,163491	-193480	-600104	3881	6981	-314	-424*

$$T_n = u_{n-1} u_n / (u_{n-1} - u_n)$$

$$u_n^* = u_n^{-k} (T_n + T_{n+1}) / (k-1)$$

$$\rho_s(S_{m+r}) = \rho_{s-2}(S_{m+r+1}) + \frac{S}{\rho_{s-1}(S_{m+r+1}) - \rho_{s-1}(S_{m+r})} \quad (s = 2, 3, \dots)$$

where S_{m+r} is a partial sum of a series up to u_{m+r} and ρ_s is an estimate of the sum of the infinite series. In particular

$$(2.12) \quad \rho_2(S_{n-1}) = S_n + \frac{2u_n u_{n+1}}{u_n - u_{n+1}}.$$

On comparing this with equation 2.4 we see that the ρ -algorithm works best for a series for which $k = 2$ and is less efficient when $k = 3/2$.

The amount of computing for each term, u_n^* , in the modified Aitken formula is a subtraction, division and two multiplications for equation 2.6 followed by an addition and subtraction for each term in equation 2.9. A similar calculation using the ρ -algorithm requires using formula 2.11 twice hence uses two subtractions two divisions and two additions.

To show the ρ -algorithm at its best we transform 12 terms of the series $\sum n^{-2}$. Wynn [4] tabulated the calculation using his ρ -algorithm in his table 3. If we use the modified Aitken formula knowing that $k = 2$ then the first transformation using $2T_n$ is the same as Wynn's set of values for ρ_2 . The second

transformation using $4T_n^*/3$ gives a column of 8 entries which may be compared directly with ρ_4 . The corresponding entries at the bottom of the column are

Wynn's ρ_4	1.64493389
Modified Aitken	1.64493392
$\pi^2/6$	1.64493407 .

In this example the modified Aitken formula has a slightly smaller error than ρ_4 and a further small improvement occurs on ρ_6 . ρ_8 is not comparable because of rounding-off errors in Wynn's calculation.

3. GENERAL THREE TERM FORMULAE

Before considering the four and five term formulae it seems appropriate to make some comments about general formulae and in particular Lubkin's three term formula.

If $T_n(u_n, u_{n+1}, u_{n+2})$ is an estimator of the tail of a series, Z_n , using the first three terms of Z_n and we wish T_n to be a good estimator for a wide variety of series, and Z_n is a linear function of its terms, then we shall restrict T_n also to be a linear function of its terms.

If $R_n = u_{n+1}/u_n$ then the series may be written as an accumulating series in the R 's

$$\begin{aligned}
 (3.1) \quad Z_n &= u_n + u_{n+1} + u_{n+2} + \dots \\
 &= u_n (1 + R_n + R_n R_{n+1} + R_n R_{n+1} R_{n+2} + \dots)
 \end{aligned}$$

and

$$(3.2) \quad T_n = u_n f(R_n, R_{n+1}) .$$

If the series is rapidly convergent and we expand T_n as an accumulating series in the R 's then the expansion of a three term approximant must agree exactly with the first three terms of the expansion of the original series, i.e.

$$(3.3) \quad T_n = u_n (1 + R_n + R_n R_{n+1} + O(R^3)) .$$

Furthermore the approximant will be a good approximant if the later terms, and, in particular the larger of the later terms, closely approximate to the corresponding terms of the original series for a wide range of types of series.

As an example, Aitken's two term approximant may be expanded

$$(3.4) \quad A_n = u_n^2 / (u_n - u_{n+1}) = u_n / (1 - R_n) = u_n (1 + R_n + R_n^2 + \dots) .$$

This agrees with the first two terms exactly and is in good agreement with the third term if R_{n+1}/R_n is small.

Now we construct a formula for summing series like $\sum n^{-k}$, where k is unknown, by eliminating k between two consecutive approximants derived in equation 2.4 .

If

$$(3.5) \quad Z_n = A / (n-k)(n-k+1) \dots (n-2) = u_n + u_{n+1} + u_{n+2} + \dots$$

then

$$(3.6) \quad Z_{n+1} = \frac{k}{k-1} \frac{u_n u_{n+1}}{u_n - u_{n+1}} \quad \text{and} \quad Z_{n+2} = \frac{k}{k-1} \frac{u_{n+1} u_{n+2}}{u_{n+1} - u_{n+2}} .$$

Eliminating k ,

$$(3.7) \quad \frac{Z_{n+2}}{Z_{n+1}} = \frac{u_{n+2} (u_n - u_{n+1})}{u_n (u_{n+1} - u_{n+2})} .$$

On using R_n this may be rearranged to give

$$(3.8) \quad Z_n = L_n \equiv u_n + u_{n+1} + u_{n+2} \frac{1 - R_n}{1 - 2R_{n+1} + R_n R_{n+1}}.$$

This transformation is basically the same as the formula derived by

S. Lubkin [3].

New
Paragraph

(a)

$$\begin{aligned} (3.9) \quad L_n &= u_n + u_{n+1} + u_{n+2} \frac{(1-R_n)}{(1-2R_{n+1} + R_n R_{n+1})} \\ &= u_n + u_{n+1} \frac{(1-R_{n+1})}{(1-2R_{n+1} + R_n R_{n+1})} \\ &= u_n \frac{(1-2R_{n+1} + R_n)}{(1-2R_{n+1} + R_n R_{n+1})} \\ &= u_n + u_{n+1} \frac{T_{n+1}}{(T_{n+1} - T_{n+2})}. \end{aligned}$$

(b) The sum is exact if R_n is constant and if $R_n = 1 + k/n$.

(c) L_n uses the first three terms of the series Z_n .

(d) If L_n is expanded out as an accumulating series in the R 's it agrees exactly with the first three terms of any series. To see this we compare the two expressions

$$(3.10) \quad Z_n = u_n (1 + R_n + R_n R_{n+1} + \dots)$$

$$(3.11) \quad L_n = u_n \frac{(1 - 2R_{n+1} + R_n)}{(1 - 2R_{n+1} + R_n R_{n+1})}.$$

Multiply by the denominator of L_n and pick out terms with the same power of R , then the first three terms agree, so L_n gives the first three terms of any convergent series. Furthermore, the fourth term is

$u_n R_n R_{n+1} (R_{n+2} - 2R_{n+1} + R_n)$ and the general term is

$u_n R_n R_{n+1} \dots R_{n+p-1} (R_{n+p} R_{n+p+1} - 2R_{n+p} R_{n+1} + R_n R_{n+1})$. Hence L_n is a good fit provided $u_{n+p} (R_{n+p} R_{n+p+1} - 2R_{n+p} R_{n+1} + R_n R_{n+1})$ is small for all p .

(e) Lubkin's formula may be written as a weighted extrapolation of Aitken's formula. This may be shown as follows: Aitken's approximation to Z_{n+1} using u_n and u_{n+1} is $u_{n+1}/(1-R_n)$ and using u_{n+1} and u_{n+2} it is $u_{n+1}/(1-R_{n+1})$. The ratio of the errors in these two expressions is approximately $u_n : u_{n+1}$ or $u_{n+1} : u_{n+2}$. To eliminate the errors we combine these two expressions in the negative inverse ratio of the errors. If we want a three term formula we restrict ourselves to using only u_n , u_{n+1} and u_{n+2} so we may combine these two formulae in the ratio $-R_n$ or $-R_{n+1} : 1$. If we want a formula which fits the first three terms of any series and sums any approximately geometric series (if R_n varies slowly with n) we apply these weights to both the numerator and denominator. This gives two three-term formulae

$$(3.12) \quad u_{n+1} \frac{-R_n + 1}{-R_n(1-R_n) + 1 \cdot (1-R_{n+1})} \quad \text{and} \quad u_{n+1} \frac{-R_{n+1} + 1}{-R_{n+1}(1-R_n) + 1 \cdot (1-R_{n+1})}.$$

The first of these formulae appears to be very efficient in transforming divergent alternating series and transforms series like n^{-k} to something like $k^{-1} \sum n^{-k-1}$. The second is Lubkin's formula which is more efficient again in transforming slowly convergent series.

(f) Lubkin's formula transforms a series like $\sum n^{-k}$ into a series like $[(k+1)/6(k-1)] \sum n^{-k-2}$. To show this, let $u_n = n^{-k}$ and $L_n = u_n(1-2R_{n+1}+R_n)/(1-2R_{n+1}+R_n R_{n+1})$ then the n th term of the transformed series

$$\begin{aligned}
 3.13) \quad u_n^* &= u_n - L_n + L_{n+1} = \frac{u_{n+2} [R_n - 2R_{n+1} + R_{n+2} + R_n R_{n+1} - 2R_n R_{n+2} + R_{n+1} R_{n+2}]}{(1 - 2R_{n+1} + R_n R_{n+1})(1 - 2R_{n+2} + R_{n+1} R_{n+2})} \\
 &= \frac{(k+1)u_{n+2}}{6(k-1)n^2} .
 \end{aligned}$$

Hence, provided $k \neq 1$ which gives a divergent series any way, the terms of the series are divided by n^2 in the transformation.

(g) Rounding-off errors are amplified in Lubkin's transformation of a slowly convergent series of positive terms. The reason for this is that the denominator of L_n is $(1 - 2R_{n+1} + R_n R_{n+1})$ so when R_n is close to 1 the denominator is small and so amplifies errors. In a numerical calculation with a truncated series this amplification of errors is first observed where R_n is closest to 1 at the end of the truncated series.

4. FOUR-TERM FORMULAE FOR SLOWLY CONVERGENT SERIES

If we start with two Lubkin transformations first using u_n, u_{n+1}, u_{n+2} and secondly using $u_{n+1}, u_{n+2}, u_{n+3}$ and writing R_a for R_{n+a} for brevity when there is no likelihood of confusion, then

$$(4.1) \quad L_n = u_n + u_{n+1} + u_{n+2} \frac{1 - R_o}{1 - 2R_1 + R_o R_1}$$

and

$$(4.2) \quad u_n + L_{n+1} = u_n + u_{n+1} + u_{n+2} \frac{1 - R_2}{1 - 2R_2 + R_1 R_2} .$$

The errors in using these two formulae will be in the ratio 1:R if R is the ratio of adjacent terms and if R is slowly varying. When we try to improve on these formulae by combining the numerators and denominators in

the ratio $-R:1$ we find that there is little improvement and the formula fails for series like $\sum n^{-2}$.

An improvement can however be achieved if we weight the two Lubkin formulae in the ratio $(R-R^2):(-1+R)$ instead of $R:-1$ where R and R^2 are carefully chosen linear and quadratic functions of R_0 , R_1 and R_2 .

Let

$$(4.3) \quad U_n = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-\sum a_i R_i) - (1-R_0)(\sum b_i R_i - \sum c_{ij} R_i R_j)}{(1-2R_2+R_1 R_2)(1-\sum a_i R_i) - (1-2R_1+R_0 R_1)(\sum b_i R_i - \sum c_{ij} R_i R_j)}.$$

This is a four-term formula provided only that the subscripts i, j belong to the set $0, 1, 2$. If the ratios of successive terms, R or R_i or R_j are small and vary slowly along the series and U_n is expanded as an accumulating series in the R 's then the early terms behave like a G.P. while the first four terms agree exactly with the original series for any a_i , b_i and c_{ij} .

If we equate $U_n = Z_n$ and multiply through by the denominator of U_n then the $(p+6)$ th terms of U_n and Z_n are in good agreement, provided

$$(4.4) \quad \{R_p R_{p+1} [R_{p+2} R_{p+3} - 2R_2 R_{p+2} + R_1 R_2] - \sum a_i R_i R_p [R_{p+1} R_{p+2} - 2R_2 R_{p+1} + R_1 R_2] \\ - \sum b_i R_i R_p [R_{p+1} R_{p+2} - 2R_1 R_{p+1} + R_0 R_1] + \sum c_{ij} R_i R_j [R_p R_{p+1} - 2R_1 R_p + R_0 R_1]\}$$

is small.

For a geometric type of series where, say, $R_i = a+b/(n+i)$ this expression is smallest if $\sum b_i = 1$ and $\sum c_{ij} = \sum a_i$. Also, for a rapidly divergent or rapidly convergent series where, say, $R_i = A(n+i)^\lambda$ the above expression is smallest for all p , λ and A if $\sum b_i = 1$ and $\sum c_{ij} = \sum a_i$.

For a slowly convergent series of positive terms where, say,
 $u_n = an^{-k}$ we may calculate a typical term of the transformed series

$$u_n^* = u_n - U_n + U_{n+1}$$

and we find that u_n^*/u_n is of order n^{-2} if $\sum a_i = 1$ and $\sum a_i + \sum b_i = 2\sum c_{ij}$.
 Finally, u_n^*/u_n is of order n^{-3} if

$$(4.5) \quad \sum a_i = \sum b_i = \sum c_{ij} = 1$$

$$\sum (i+j)c_{ij} = \sum ia_i + \sum ib_i + 3.$$

In order to minimise rounding-off errors in a numerical calculation we further wish to minimise the number of terms in the denominator of U_n . Since U_n is of the form

$$(4.6) \quad u_n + u_{n+1} + u_{n+2} (1-3R+3R^2-R^3)/(1-4R+6R^2-4R^3+R^4)$$

we require that the denominator should contain 4 terms linear in R_0, R_1, R_2 all negative, similarly, 6 quadratic, 4 cubic and one quartic term. There are six integral solutions satisfying the above conditions. The resulting six four-term formulae are:

(4.7) Insert additional page here. 16 (a)

These are all what one might call general formulae in the sense that they fit the first four terms of any series exactly and closely match any series in which R_n varies slowly with n , such as the asymptotic or factorial, nearly geometric and inverse factorial series. They also transform a series like $\sum an^{-k}$ into a series like $\sum bn^{-k-3}$.

To compare the efficiencies of these four-term formulae with Lubkin's three-term formula when we have a truncated series and wish to extract as much information as possible from the available terms, suppose we start with a series of 12 terms. In four transformations with Lubkin's formula using up three terms per operation we divide such terms by n^8

$$(4.7) \quad U_{1n} = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-R_1) - (1-R_0)(2R_0-R_2-R_0R_2)}{(1-2R_2+R_1R_2)(1-R_1) - (1-2R_1+R_0R_1)(2R_0-R_2-R_0R_2)}$$

$$U_{2n} = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-2R_0+R_1) - (1-R_0)(R_1-R_1R_2)}{(1-2R_2+R_1R_2)(1-2R_0+R_1) - (1-2R_1+R_0R_1)(R_1-R_1R_2)}$$

$$U_{3n} = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-R_0) - (1-R_0)(2R_1-R_2-R_1R_2)}{(1-2R_2+R_1R_2)(1-R_0) - (1-2R_1+R_0R_1)(2R_1-R_2-R_1R_2)}$$

$$U_{4n} = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-R_0) - (1-R_0)(R_0-R_1R_2)}{(1-2R_2+R_1R_2)(1-R_0) - (1-2R_1+R_0R_1)(R_0-R_1R_2)}$$

$$U_{5n} = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-R_0) - (1-R_0)(R_1-R_2^2)}{(1-2R_2+R_1R_2)(1-R_0) - (1-2R_1+R_0R_1)(R_1-R_2^2)}$$

$$U_{6n} = u_n + u_{n+1} + u_{n+2} \frac{(1-R_2)(1-R_1) - (1-R_0)(R_0-R_2^2)}{(1-2R_2+R_1R_2)(1-R_1) - (1-2R_1+R_0R_1)(R_0-R_2^2)}$$

apart from a numerical constant. In three operations with a four-term formula we also use up 12 terms and divide each term by n^9 apart from a numerical constant. Hence if n is large, the four-term formula transforms the original series into a new series of smaller terms than Lubkin's formula does at the expense of the same number of terms.

In the transformation from $\sum a_n^{-k}$ to $\sum b_n^{-k-3}$, the coefficient b contains a factor (k^2-1) in the denominator so the sum becomes unreliable when k is near 1 or near -1. Both of these series are already divergent so this situation does not represent a defect in the formula.

If n is small the size of the terms in the transformed series will depend on other details of u_n or initial stability considerations, so we should look at some actual numerical examples. Also, since most, if not all of the well-known series become smoother as we move away from the start of the series we must prefer a weighting of the two Lubkin formulae which uses later rather than earlier terms. For this reason the formula above for U_{5n} which I shall call formula U_5 is regarded as the most stable of the six formulae.

Depending on the calculating facilities available U_{5n} may be represented in several different ways:

$$\begin{aligned}
 (4.8) \quad u_{5n} &= u_n + u_{n+1} + u_{n+2} \frac{(1-R_0)(1-R_1-R_2+R_2^2)}{1-R_0-R_1-2R_2+2R_0R_2+2R_1^2+R_1R_2+R_2^2-R_0R_1^2-R_0R_1R_2-2R_1R_2^2+R_0R_1R_2^2} \\
 &= u_n + u_{n+1} \frac{1-R_0-2R_2-R_0R_1+2R_0R_2+R_1^2+R_2^2-R_1R_2}{1-R_0-R_1-2R_2+2R_0R_2+2R_1^2+R_1R_2+R_2^2-R_0R_1^2-R_0R_1R_2-2R_1R_2^2+R_0R_1R_2^2}
 \end{aligned}$$

or as a weighted sum of Aitken's formula

$$= u_n + u_{n+1} \frac{(1+R_1-R_2)(1-R_0) - 1 \cdot (R_1+R_2-R_0R_2-R_2^2) + 1 \cdot (R_1^2-R_1R_2^2)}{(1-R_2)(1-R_0) - (1-R_1)(R_1+R_2-R_0R_2-R_2^2) + (1-R_0)(R_1^2-R_1R_2^2)}$$

or as Lubkin's formula with a correction term

$$= u_n + u_{n+1} + u_{n+2} \frac{1-R_0}{1-2R_1+R_0R_1 - \frac{R_0-2R_1+R_2+R_0R_1-2R_0R_2+R_1R_2}{1-R_1-R_2+R_2^2}}$$

$$= u_n + u_{n+1} + u_{n+2} \frac{Q_1}{1+Q_0-Q_1 - \frac{Q_1Q_2(Q_0-2Q_1+Q_2)}{Q_1+Q_2^2 - Q_1Q_2}}$$

$$Q_i = \frac{1}{1-R_i}$$

5. TWO EXAMPLES OF REPEATED TRANSFORMATIONS USING LUBKIN'S FORMULA
AND FORMULA U_5

In table 2 we take the first 17 terms of a formula for π and transform it four times using Lubkin's formula. To avoid rounding-off errors the first two transformations were kept as rational fractions and the last two transformations were done to 16 figures in a programmable calculator before abbreviating.

In table 3 a similar calculation was done using the formula U_5 avoiding rounding-off errors in the same way. Both methods use up 12 terms and the residual series of 5 terms is slightly smaller for formula U_5 .

Take in tables 2 and 3

In tables 4 and 5 similar transformations are made to a series for $\pi^2/6$. The transformations using formula U_5 exhibit some instability at the start of the series but where the series is smoother and n is larger, i.e. after the first three terms, the terms of the residual series produced by three repetitions of formula U_5 are smaller than the terms produced by four repetitions of Lubkin's formula.

Take in tables 4 and 5.

TABLE 2

Lubkin's transformations of 17 terms of the series $\pi = 2 + 1/3 + 3/20 + \dots$

u_n	L'_n	$u_n^{**} \times 10^3$	$L'_n \times 10^3$	$u_n^{***} \times 10^6$	$L'_n \times 10^6$	$u_n^{****} \times 10^9$	$L'_n \times 10^9$	$u_n^{*****} \times 10^{12}$
2.0000								
.3333	1.0476							
.1500	.7846	70.3297						
.0893	.6485	13.8809	23.4490					
.0608	.5639	4.6784	9.7128	144.652				
.0447	.5052	2.0528	5.0669	32.452	50.071			
.0347	.4615	1.0563	3.0243	10.128	17.740	120.495		
.0279	.4274	.6050	1.9719	3.891	7.643	30.478	46.062	
.0231	.3998	.3743	1.3687	1.726	3.762	9.500	15.677	93.28
.0195	.3770	.2455	.9952	.851	2.040	3.458	6.200	23.21
.0168	.3576	.1685	.7502	.455	1.190	1.418	2.748	6.98
.0146	.3410	.1200	.5820	.260	.736	.639	1.332	2.41
.0129	.3264	.0880	.4622	.156	.476	.311	.694	.93
.0115	.3136	.0662	.3743	.098	.320	.161		
.0103	.3022	.0509	.3081	.064				
.0093	.2919	.0399						
.0085								

$$\begin{aligned} \pi = & 2.000000,000000,00 + 1.047619,047619,05 + .070329,670329,67 \\ & + .023449,045676,79 + .000144,651830,94 + .000050,071449,18 \\ & + .000000,120494,77 + .000000,046061,84 + .000000,000093,28 + \dots \end{aligned}$$

Formulae used: $L'_n = L_{n-1} - u_{n-1}$ using the second line of equation 3.9 and $u_n^* = u_n - L'_n - L'_{n+1}$.

TABLE 3

Transformations with formulae U_5 of 17 terms of the series $\pi = 2 + 1/3 + 3/20 + \dots$

u_n	U'_{5n}	$u_n^* \times 10^4$	$U'_{5n}^* \times 10^4$	$u_n^{**} \times 10^8$	$U'_{5n}^{***} \times 10^8$	$u_n^{***} \times 10^{12}$
2.0000						
.3333						
.1500	.8093					
.0893	.6587	-6.3403				
.0608	.5691	-2.2179				
.0447	.5083	-.8480	-1.8018			
.0347	.4635	-.3782	.8695	843.658		
.0279	.4288	-.1899	.4866	46.880		
.0231	.4008	-.1043	.2959	7.742	10.786	
.0195	.3778	-.0615	.1914	1.955	3.041	-26.80
.0168	.3582	-.0383	.1298	.632	1.085	-10.05
.0146	.3414	-.0249	.0916	.241	.452	-2.91
.0129	.3268	-.0168	.0667	.104	.211	-.84
.0115	.3139	-.0117	.0498	.049	.107	-.26
.0103	.3024	-.0084	.0381	.025		
.0093	.2921	-.0062				
.0085						

$$\begin{aligned} \pi = & 2.000000,000000,00 + .333333,333333,33 + .809286,308039,43 - .000634,031303,72 \\ & - .000221,787959,71 - .000180,181719,59 + .000008,436578,34 + .000000,468802,81 \\ & + .000000,107859,88 - .000000,000026,80 - \dots \end{aligned}$$

Formulae used: $U'_{5n} = U_{5(n-2)} - u_{n-2} - u_{n-1}$ using the last line of equation 4.8

$$\text{and } u_n^* = u_n - U'_{5n} + U'_{5(n+1)}.$$

TABLE 4

Lubkin's transformations of 17 terms of the series $\pi^2/6 = 1 + 1/4 + 1/9 + \dots$

u_n	L'_n	$u_n^* \times 10^3$	$L_n'^* \times 10^3$	$u_n^{**} \times 10^6$	$L_n'^{***} \times 10^6$	$u_n^{****} \times 10^9$	$L_n'^{****} \times 10^9$	$u_n^{*****} \times 10^{12}$
1.00000								
.25000	.62500							
.11111	.38889	13.88889						
.06250	.28125	3.47222	6.00601					
.04000	.22000	1.25000	2.56148	27.692				
.02778	.18056	.55556	1.31868	7.206	11.443			
.02041	.15306	.28345	.76553	2.404	4.263	26.612		
.01562	.13281	.15944	.48303	.951	1.867	7.224	11.122	
.01235	.11728	.09645	.32402	.426	.918	2.346	3.914	17.16
.01000	.10500	.06173	.22778	.210	.493	.874	1.573	4.77
.00826	.09504	.04132	.16616	.111	.284	.362	.701	1.53
.00694	.08681	.02870	.12490	.063	.172	.164	.339	.55
.00592	.07988	.02054	.09624	.037	.110	.079	.175	.22
.00510	.07398	.01509	.07572	.023	.072	.041		
.00444	.06889	.01134	.06064	.015				
.00391	.06445	.00868						
.00346								

Formulae used: $L_n' = L_{n-1} - u_{n-1}$ using the second line of equation 3.9 and $u_n^* = u_n - L_n' + L_{n+1}'$.

TABLE 5

Transformations with formula U_5 of 17 terms of the series $\pi^2/6 = 1 + 1/4 + 1/9 + \dots$

u_n	U'_{5n}	$u_n^* \times 10^4$	$U'_{5n} \times 10^4$	$u_n^{**} \times 10^8$	$U'_{5n} \times 10^8$	$u_n^{***} \times 10^{12}$
1.00000						
.25000						
.11111	.39494					
.06250	.28387	.39925				
.04000	.22135	-.19827				
.02778	.18134	-.11911	-.04738			
.02041	.15355	-.06235	-.30284	-3745.7417		
.01562	.13314	-.03370	-.09323	1472.5717		
.01235	.11752	-.01918	-.05522	43.1109	52.4227	
.01000	.10517	-.01148	-.03537	6.6983	9.1899	-1218.02
.00826	.09517	-.00717	-.02373	1.6358	2.4971	54.20
.00694	.08690	-.00466	-.01650	.5136	.8617	4.28
.00592	.07996	-.00313	-.01182	.1905	.3482	.42
.00510	.07404	-.00216	-.00869	.0798	.1576	.03
.00444	.06894	-.00153	-.00652	.0367		
.00391	.06449	-.00111				
.00346						

Formulae used:

$$U'_{5n} = U_{5(n-2)} - u_{n-2} - u_{n-1} \quad \text{using the last line of equation 4.8}$$

$$\text{and } u_n^* = u_n - U'_{5n} + U'_{5(n+1)}$$

6. FIVE-TERM FORMULAE

Five-term formulae may be derived as a weighted extrapolation of Lubkin's or the four-term formulae.

If we take three adjacent forms of Lubkin's formula, namely

$$(6.1) \quad L_n = u_n + u_{n+1} + u_{n+2} \frac{1-R_0}{1-2R_1+R_0R_1}$$

$$u_n + L_{n+1} = u_n + u_{n+1} + u_{n+2} \frac{1-R_2}{1-2R_2+R_1R_2}$$

$$u_n + u_{n+1} + L_{n+2} = u_n + u_{n+1} + u_{n+2} \frac{1-2R_3+R_2}{1-2R_3+R_2R_3}$$

which have errors in the ratio $1:R:R^2$ and combine the numerators and denominators of the last term in the ratio $R^2:-2R:1$ where $2R$ and R^2 are linear and quadratic functions of R_0, R_1, R_2 and R_3 we do not greatly reduce the error in estimating the sum of a slowly convergent series of positive terms. In order to improve on the four-term formula we have to use more complicated weights like $(1-2R+R^2):(-2R+4R^2-2R^3):(R^2-2R^3+R^4)$.

A simpler and more limited way of proceeding is to take two forms of one of the four-term formulae and combine these in the ratio $(1-R):(-R+R^2)$. Thus if we take formula U_1 using terms u_{n+1} to u_{n+4} and another formula U_1 using terms u_n to u_{n+3} and combine the numerators and denominators of the formulae in the ratio $(1-\sum a_i R_i):(-\sum b_i R_i + \sum c_{ij} R_i R_j)$, then the various conditions as before imposed on the new formula require that

$$(6.2) \quad \sum a_i = \sum b_i = \sum c_{ij} = 1$$

$$\sum (i+j)c_{ij} = \sum ia_i + \sum ib_i + 6.$$

The only solution to these equations with $i, j = 0, 1, 2$ or 3 and a_i, b_i, c_{ij} positive integers is $a_0 = b_0 = c_{33} = 1$ with the other a 's, b 's and c 's all zero. The resulting weight is $1-R_0:R_0-R_3^2$ and the six four-term formulae give six five-term formulae

(6.3) Insert page 20(a) here

Finally if we look for more general combinations of three of Lubkin's formulae using quadratic weights, one other such formula has been found, namely

(6.4) Insert page 20(b) here

7. USE OF THREE AND FOUR-TERM FORMULAE FOR SPEEDING THE CONVERGENCE OF

A SHORT SERIES

If a short monotonic sequence or series is known for some quantity and it may be assumed that these are the first terms of a convergent series with some degree of smoothness, then the approximation may be improved by using three or four-term formulae or a combination of both.

In order that we may observe the efficiency of the transformations these are demonstrated on a known series. In table 6 are listed the results of using Lubkin's three-term and the four-term formula U_5 on the monotonic series for π . Where not all the terms are used the transformation is made to the later terms of the series to obtain the best result and the earlier unused terms are added on.

$$(6.3) \quad V_{1n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-R_2)(1-R_0) - (1-R_2)[(2R_1-R_3-R_1R_3)(1-R_0)+(1-R_1)(R_0-R_3^2)] + (1-R_0)(2R_0-R_2-R_0R_2)(R_0-R_3^2)}{(1-2R_3+R_2R_3)(1-R_2)(1-R_0) - (1-2R_2+R_1R_2)[(2R_1-R_3-R_1R_3)(1-R_0)+(1-R_1)(R_0-R_3^2)] + (1-2R_1+R_0R_1)(2R_0-R_2-R_0R_2)(R_0-R_3^2)}$$

$$V_{2n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-2R_1+R_2)(1-R_0) - (1-R_2)[R_2(1-R_3)(1-R_0)+(1-2R_0+R_1)(R_0-R_3^2)] + (1-R_0)R_1(1-R_2)(R_0-R_3^2)}{(1-2R_3+R_2R_3)(1-2R_1+R_2)(1-R_0) - (1-2R_2+R_1R_2)[R_2(1-R_3)(1-R_0)+(1-2R_0+R_1)(R_0-R_3^2)] + (1-2R_1+R_0R_1)R_1(1-R_2)(R_0-R_3^2)}$$

$$V_{3n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-R_0)(1-R_1) - (1-R_2)[(2R_2-R_3-R_2R_3)(1-R_0)+(1-R_0)(R_0-R_3^2)] + (1-R_0)(2R_1-R_2-R_1R_2)(R_0-R_3^2)}{(1-2R_3+R_2R_3)(1-R_0)(1-R_1) - (1-2R_2+R_1R_2)[(2R_2-R_3-R_2R_3)(1-R_0)+(1-R_0)(R_0-R_3^2)] + (1-2R_1+R_0R_1)(2R_1-R_2-R_1R_2)(R_0-R_3^2)}$$

$$V_{4n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-R_0)(1-R_1) - (1-R_2)[(R_1-R_2R_3)(1-R_0)+(1-R_0)(R_0-R_3^2)] + (1-R_0)(R_0-R_1R_2)(R_0-R_3^2)}{(1-2R_3+R_2R_3)(1-R_0)(1-R_1) - (1-2R_2+R_1R_2)[(R_1-R_2R_3)(1-R_0)+(1-R_0)(R_0-R_3^2)] + (1-2R_1+R_0R_1)(R_0-R_1R_2)(R_0-R_3^2)}$$

$$V_{5n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-R_0)(1-R_1) - (1-R_2)[(R_2-R_3^2)(1-R_0)+(1-R_0)(R_0-R_3^2)] + (1-R_0)(R_1-R_2^2)(R_0-R_3^2)}{(1-2R_3+R_2R_3)(1-R_0)(1-R_1) - (1-2R_2+R_1R_2)[(R_2-R_3^2)(1-R_0)+(1-R_0)(R_0-R_3^2)] + (1-2R_1+R_0R_1)(R_1-R_2^2)(R_0-R_3^2)}$$

$$V_{6n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-R_0)(1-R_2) - (1-R_2)[(R_1-R_3^2)(1-R_0)+(1-R_1)(R_0-R_3^2)] + (1-R_0)(R_0-R_2^2)(R_0-R_3^2)}{(1-2R_3+R_2R_3)(1-R_0)(1-R_2) - (1-2R_2+R_1R_2)[(R_1-R_3^2)(1-R_0)+(1-R_1)(R_0-R_3^2)] + (1-2R_1+R_0R_1)(R_0-R_2^2)(R_0-R_3^2)}$$

(6.4)

$$v_{7n} = u_n + u_{n+1} + u_{n+2} \frac{(1+R_2-2R_3)(1-R_O)(1-R_1) - 2(1-R_2)(1-R_O)(R_1-R_3^2) + (1-R_O)(R_O-R_2^2)(R_1-R_3^2)}{(1-2R_3+R_2R_3)(1-R_O)(1-R_1) - 2(1-2R_2+R_1R_2)(1-R_O)(R_1-R_3^2) + (1-2R_1+R_OR_1)(R_O-R_2^2)(R_1-R_3^2)}$$

TAKE IN TABLE 6

TABLE 6

Truncated series for π

$$2 + 1/3 + 3/20 + \dots$$

Number of terms used	Type of transformation	Transformed sum
1	None	2.0
2	Aitken	2.4
3	3-term	3.05
4	3-term	3.12
	4-term	3.143
5	3-term	3.13
	4-term	3.1420
6	4-term	3.1418
	3-term twice	3.1414
7	4 and 3-term	3.14161
	3 and 4-term	3.141594

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SUMMING A COMMON TYPE OF SLOWLY CONVERGENT SERIES
OF POSITIVE TERMS

(Short title: SUMMING SLOWLY CONVERGENT SERIES)

J.E. Drummond

Abstract

If the terms of a series behave like n^{-k} where k is an exactly known constant, a formula using two terms transforms the series into a series of terms like n^{-k-2} provided $k \neq 1$. The multiple use of this transformation is demonstrated in summing three series.

1. Introduction

When the n th term of a series is a quotient of powers or polynomials or γ -functions in n and the ratio of successive terms is close to 1 the series is difficult to sum. The three main reasons for this are that a lot of terms of the series are needed, the individual terms are much smaller than the sum and rounding off errors in a numerical calculation can be troublesome.

[1]

Two methods for transforming such series are Lubkin's transformation and Wynn's ρ -algorithm which may be used to repeatedly transform the series or its corresponding sequence into more rapidly convergent forms.

[2]

The following method bears a close resemblance to an improvement on Wynn's ρ -algorithm and is easier to use.

2. Derivation of the formula

Let u_n be the nth term of a series.

Let Z_n be the sum to infinity of the series starting with u_n , thus $Z_n = u_n + u_{n+1} + u_{n+2} + \dots$

We first take a special series for which

$$Z_n = \frac{A}{(n-k)(n-k+1) \dots (n-2)}$$

Hence $u_n = \frac{A(k-1)}{(n-k)(n-k+1) \dots (n-1)}$

and $Z_n = \frac{k}{k-1} \frac{u_{n-1} u_n}{u_{n-1} - u_n} = u_n + \frac{k}{k-1} \frac{u_n u_{n+1}}{u_n - u_{n+1}} \quad 1$

The particular series given above is summed exactly from a knowledge of k and the first two terms of the series. However if we have some other series for which An^{-k} is the dominant part of the nth term we may still transform the series as follows:-

Let $Z_1 = u_1 + u_2 + u_3 + \dots$ and take $u_0 = 0$

and define $T_n = \frac{k}{k-1} \frac{u_{n-1} u_n}{u_{n-1} - u_n}$ so $T_1 = 0 \quad 2$

then by simple rearrangements we get

$$Z_1 = u_1^* + u_2^* + u_3^* + \dots$$

where $u_n^* = u_n - T_n + T_{n+1}$

3

provided that $k \neq 1$ and $\lim_{n \rightarrow \infty} T_n = 0$.

Also if $u_n \sim A n^{-k}$ we substitute in equations 2 and 3 to get

$$u_n^* \sim -[A(k+1)/12] (n-1)^{-k-2} \quad \text{for } n > 1 \text{ and } k \neq 1.$$

The transformed series may be transformed a second time, but now the series behaves like $\sum n^{-k-2}$ so we change k to $(k+2)$ in equation 2 and continue to increase k by 2 for each successive transformation.

Several comments on the transformation formula are:-

(i) Equations (2) and (3) can be written into a calculator programme which stores k and calculates u_n^* when u_{n+1} is inserted.

(ii) Rounding off errors are magnified because of the factor $(u_{n-1} - u_n)$ in the denominator of T_n so where possible it is advisable to do at least the first transformation of the series exactly and the early transformations as accurately as possible.

(iii) k may be estimated using three terms of the series. Thus when the equations (1) are approximately true we eliminate Z_n from them to obtain an approximator

$$k \approx \frac{(u_{n-1} - u_n)(u_n - u_{n+1})}{u_{n-1} u_{n+1} - u_n^2}.$$

(iv) If k is unknown it may be eliminated from equations (1) to give Lubkin's [1] formula 17 namely

$$Z_n = \frac{u_n (1 - R_{n+1})}{1 - 2R_{n+1} + R_n R_{n+1}} \quad \text{where} \quad R_n = \frac{u_n}{u_{n-1}} .$$

3. Demonstration of the formula

The use of equations 2 and 3 is demonstrated in table 1 on the first eight terms of a slowly convergent series for π namely

$$\pi = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n-1} (n!)^2 (2n+1)}$$

multiplied by 10^9 to avoid tabulating decimals. This series would require about 10^{16} terms to give an error of 10 in the tabulated values or 10^{-8} in π .

TAKE IN TABLE I IN SECTION 3

If it is known that the first, third, fifth and seventh columns are terms of series which behave like $n^{-3/2}$, $n^{-7/2}$, $n^{-11/2}$, $n^{-15/2}$ then we take $k = 3/2, 7/2, 11/2, 15/2$ in successive uses of equation 2 or use the equivalent formulae at the bottom of table 1 in order to construct the table. The numbers marked with an asterisk at the top of each column are summed to give 3,141,592,641 which differs from $10^9 \pi$ by 12.

TABLE I

Summing a series for $10^9 \pi$ knowing that the terms behave like $n^{-3/2}$

u_n	$3A_n$	u_n^*	$7A_n^*/5$	u_n^{**}	$11A_n^{**}/9$	u_n^{***}	$15A_n^{***}/13$
2000,000,000*							
333,333,333	1200,000,000*						
150,000,000	818,181,818	-48,484,848*					
89,285,714	661,764,706	-6,417,112	-10,354,391*				
60,763,889	570,652,174	-1,826,818	-3,575,381	361,898*			
44,744,318	509,159,483	-728,802	-1,697,558	51,006	72,568*		
34,705,528	464,062,500	-352,665	-956,649	12,106	19,401	-2,161*	
27,929,686	429,163,491	-193,480	-600,104	3,881	6,981	-314	-424*

$$A_n = u_{n-1} u_n / (u_{n-1} - u_n) \quad T_n = k A_n / (k-1)$$

$$u_n^* = u_n - T_n + T_{n+1}$$

4. Comparison with other formulae

The two transformations most closely related to this are those of Lubkin and Wynn.

If we do not know k we may use Lubkin's transformation, the difference being that we use three terms of the series to reduce the size of the terms by a factor $(k+1)/(6(k-1)n^2)$ instead of using two terms and a knowledge of k to reduce the size of the terms by a factor $(k+1)/12n^2$. The results of summing six terms of three series for which k is 1.5, 2, and 4 are given in table 2. A comparison of columns A and C demonstrates the improvement resulting from a knowledge of k .

TAKE IN TABLE 2 IN SECTION 4

Wynn's [2] ρ -algorithm for transforming a sequence at the start of his section 2 is

$$\rho_s(S_n) = \rho_{s-2}(S_{n+1}) + \frac{s}{\rho_{s-1}(S_{n+1}) - \rho_{s-1}(S_n)} \quad 4$$

where S_n is the sum of a series from u_0 to u_n inclusive and ρ_s for even s is an estimate of the sum of the infinite series. His initial conditions are ideal if $k = 2$ and his three examples are of this type. However if k is known and is not 2 we can take as initial conditions $\rho_{k-3}(S_n) = 0$,

$\rho_{k-2}(S_n) = S_n$ hence $\rho_{k-1}(S_n) = (k-1)/u_{n+1}$ and

$$\rho_k(S_n) = S_{n+1} + \frac{k}{k-1} \frac{u_{n+1} u_{n+2}}{u_{n+1} - u_{n+2}} .$$

TABLE 2

Comparison of formulae

I Six terms of $\sum 2(2n)!/2^{2n}(n!)^2(2n+1)$

	A	B	C
Sum	2.6781273	2.6781273	2.6781273
1st transformation	3.1425424	3.1425424	3.1365079
2nd transformation	3.1415737	3.1415715	3.1413978
3rd transformation	3.1415952	3.1415967	-
Infinite sum		$\pi = 3.1415927$	

II Six terms of $\sum n^{-2}$

	A	B	C
Sum	1.4913889	1.4913889	1.4913889
1st transformation	1.6454293	1.6454293	1.6436111
2nd transformation	1.6449244	1.6449226	1.6448949
3rd transformation	1.6449350	1.6449357	-
Infinite sum		$\pi^2/6 = 1.6449341$	

III Six terms of $\sum n^{-4}$

	A	B	C
Sum	1.08112353	1.08112353	1.08112353
1st transformation	1.08233901	1.08233901	1.08230690
2nd transformation	1.08232242	1.08232208	1.08232213
3rd transformation	1.08232340	1.08232368	-
Infinite sum		$\pi^4/90 = 1.08232323$	

A Present formula, B Modified Wynn's ρ -algorithm, C Lubkin's formula.

The first two steps in this modification of Wynn's formula are now identical with the transformation derived in section 2 but thereafter the two transformations give slightly different results. In the examples in table 2 a comparison of columns A and B shows that the present formula gives results slightly closer to the infinite sum than does the ρ -algorithm. Also the amount of calculation done using equation 4 twice or equations 2 and 3 is comparable but in using a programmable desk calculator where most of the time taken is in inserting numbers the ρ -algorithm is longer.

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(Continued inside back cover)

A formula for accelerating the convergence of a general series

J.E. Drummond

A weighted average of the partial sums of a series provides a quick and moderately powerful sum for any series in which the ratio of successive terms varies slowly along the series and this ratio is not close to $+1$. Some properties of the sum are examined.

Let u_n be the n -th term of a series and S_n given by

$$S_n = u_1 + u_2 + \dots + u_n$$

be a partial sum of n terms of an infinite series, S . Aitken's [1] two term formula for the sum may be written

$$T_{r,r+1} = (S_{r+1}/u_{r+1} - S_r/u_r) / (1/u_{r+1} - 1/u_r)$$

and Lubkin's [2] three term formula may be written

$$T_{r,r+2} = (S_{r+2}/u_{r+2} - 2S_{r+1}/u_{r+1} + S_r/u_r) / (1/u_{r+2} - 2/u_{r+1} + 1/u_r).$$

We now consider a generalization of these formulae,

$$(1) \quad T_{rs} = \frac{\sum_{i=0}^{s-r} W_i S_{r+i}}{\sum_{i=0}^{s-r} W_i} \quad \text{where} \quad W_i = (-1)^i C_i^{s-r} / u_{r+i}$$

and C_i^k is the binomial coefficient $k!/i!(k-i)!$.

T_{rs} is a weighted average of the partial sums of the series and may

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be compared with the Euler sum of a slowly convergent alternating series,

$$S_r + \frac{1}{2} u_{r+1} + \frac{1}{4} (u_{r+1} - u_{r+2}) + \dots + (\text{to } k \text{ terms}) = \sum_{i=0}^k C_i^k S_{r+i} / 2^k.$$

This is a good approximation when the terms alternate and vary slowly in size.

An approximation to the residual $(S - T_{rs})$: Let $T_{rs} = S - E_{rs}$. If S_i is a convergent sequence, then the truncation error $(S - S_i)$ tends to zero as $i \rightarrow \infty$ and T_{rs} is a weighted mean of the S_i , so E_{rs} also tends to zero as r and $s \rightarrow \infty$ provided that the sum of the weights has a non-zero limit.

Let $u_{n+1}/u_n = R_n$; then

$$(2) \quad T_{rs} = S_{r-1} + u_r \frac{\sum_{i=0}^{s-r} (-)^i C_i^{s-r} (R_{s-i} R_{s-i+1} \dots R_{s-1}) \sum_{j=0}^{s-i} (R_r R_{r+1} \dots R_{s-1+j})}{\sum_{i=0}^{s-r} (-)^i C_i^{s-r} (R_{s-i} R_{s-i+1} \dots R_{s-1})}.$$

For a rapidly convergent series (that is all the $|R_n| < 1$)

$$(3) \quad T_{r+1,s+1} - T_{rs} = E_{rs} - E_{r+1,s+1} \doteq u_{s+1} \sum_{i=0}^{s-r} (-)^i C_i^{s-r} R_{s-i}^{1/R_s},$$

on picking out the largest term in the difference. We may conclude from this formula that if the ratio of terms, R_n , varies slowly with n then the $(s-r)$ -th order finite difference of R_n will be small compared with R_n . Hence the first finite difference of E_{rs} will be small compared with u_{s+1} . If, furthermore, u_s decreases rapidly as s increases, so also will E_{rs} decrease as s increases with $(s-r)$ constant, so we can neglect $E_{r+1,s+1}$ in our approximation and equation (2) gives an estimate of E_{rs} . Similarly for a rapidly divergent series (that is all the $|R_n| > 1$) on picking out the largest term in the difference,

$$(4) \quad E_{rs} - E_{r+1,s+1} \doteq u_r R_r \sum_{i=0}^{s-r} (-)^i C_i^{s-r} / R_{r+i}.$$

In both cases we compare the residual in the formula with the smallest term of the original series used in the weighting. If the divergent series has a meaningful sum, then the residual in the formula will be small if the $(s-r)$ -th finite difference of the reciprocal of the R_n is small compared with the reciprocal of R_n . If the transformation makes the series less divergent, $E_{r+1,s+1}$ will be larger than E_{rs} while E_{rs} will be the larger if the divergent series is converted to a convergent series, but in either case E_{rs} will be comparable with the expression (4).

In operating on a slowly convergent or divergent series the expression for the residual is more complicated than (3) or (4) but for a slowly convergent alternating series, formula (1) approximates to the Euler sum which is known to be a good approximation for such a series. However, if u_r is the reciprocal of a polynomial in n , then in equation (1), $\sum W_i$ is the finite difference of a polynomial, so will be zero when $(s-r)$ is sufficiently large. In this case T_{rs} may be unbounded. Lubkin's formula which is $T_{r,r+2}$ fails when u_n is the reciprocal of a linear function of n , but this is no loss because $\sum n^{-1}$ is itself a divergent series. For a slowly convergent series of positive terms formula (1) is found to be unsatisfactory because the sum of the weights is small, but it works well on many other series.

As an example we sum the series $\sum (-)^n n!$ to show the simplicity and power of the sum. We tabulate $(10)!/u_n$ and $(10)!S_n/u_n$; then we may introduce the binomial coefficients and calculate any T_{rs} as desired. The set T_{0n} is listed in the third column of the table. Any one of these may be computed as desired and provides a relatively simple and moderately powerful method for summing a series. If greater accuracy is desired, it may be observed that the T_{0n} column appears to be a convergent sequence, so formula (1) may be applied again to this sequence. In this case the

labour involved becomes more comparable with that of the other more powerful methods such as repeated use of Aitken's transform [1], Shanks' e_n processes [4], Wynn's E and ρ algorithms [5, 6] and Rutishauser's Q - D algorithm [3]. Euler's transform is useless on this series.

TABLE. ESTIMATES OF $\sum (-)^n n!$

$(10)!/n!$	$(10)!S_n/n!$	T_{0n}
3628800	3628800	
3628800	0	.50000
1814400	3628800	.57143
604800	-2419200	.58824
151200	3024000	.59330
30240	-3024000	.59508
5040	3124800	.59578
720	-3182400	.59608
90	3231000	.59621
10	-3269800	.59628
1	331820	.59631

As a second example we sum the series

$$\pi = 4 - 4/3 + 4/5 + \dots$$

In this example $R_n = (2n-1)/(2n+1)$. R_n changes rapidly with n when n is small, so any finite difference of R_n or R_n^{-1} which includes terms with n near zero is not small. In this case we can do better by taking a lower order finite difference further along the series. If we use 16 terms it is best to calculate $T_{6,16}$; then we get an error of 1 in the 10-th digit. This series is also particularly good for Euler's formula since R_n is close to 1. However we lose strength if we include the first terms where R_n is not close to 1. On applying Euler's formula to the latter 10 or 11 of 16 terms, we get an error of 2 in the 9-th digit. Aitken's formula develops the same weakness when R_n is changing

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rapidly. With somewhat more labour and six repetitions of Aitken's transform we use up 12 terms of the series before Aitken's transform becomes unstable at the start of the series. At this stage the error is in the 14-th digit. Aitken's transform may be repeated but each repetition after the sixth uses up three terms because of the initial instability so the convergence is less rapid. Aitken's method however remains the best of these three methods per number of terms used if the amount of effort used is not an important restriction.

If we tabulate T_{1s} as s runs from 2 to 16 we obtain a convergent sequence of estimates of π using equation 1 and if desired we may repeat equation 1 on T_{1s} to get a better estimate of the sum. Alternatively, we may tabulate $T_{r,16}$ as r runs from 1 to 15 and this time we obtain an asymptotic sequence with a minimum at $r = 6$ mentioned above. Equation 1 may be repeated on this sequence to get a better estimate.

In conclusion, equation (1) is a weighted Euler sum which may be used when a suitable Euler weight is unknown.

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A FORMULA FOR SUMMING DIVERGENT SERIES

J.E. DRUMMOND

Abstract: A formula is given for assigning sums to divergent series in which the ratio of adjacent terms varies slowly along the series. This formula consists of a weighted average of partial sums and is shown to be a general formula which can be easily calculated using a simple recurrence relation. It appears to be more powerful than a repeated Aitken or Shanks e_1 process as long as the transformed series remains divergent and it is also compared with the Padé approximants. It is demonstrated on a factorial series, on a nearly geometric divergent series and for the extrapolation of a velocity formula for small amplitudes of motion.

A FORMULA FOR SUMMING DIVERGENT SERIES

by

J.E. Drummond

1. INTRODUCTION

In classical analysis an infinite series $\sum_0^{\infty} a_n$ is termed divergent if the sequence of its partial sums $\{S_N\}$, where

$$S_N = \sum_{n=0}^N a_n ,$$

is divergent. As is well-known, there is no particular reason why $\lim_{N \rightarrow \infty} S_N$, even when it is convergent, should be taken as the only possible definition of a sum for $\sum a_n$, many other limits being available and appropriate in particular contexts. Generally speaking, however, if $\{S_N\}$ is convergent then all the commonly used sums of the series will coincide with its limit. In this case the sequence $\{S_N\}$ may not afford the best means of determining the sum of the series numerically and we may prefer an alternative method of summation which converges more rapidly.

It is also well-known that an alternating series that is conditionally, but not absolutely convergent can be made to converge in the classical sense to any real number whatever, merely by altering the order of its terms. Between these extremes occur the summation methods that form a sequence of linear combinations (weighted sums) of the terms without altering their order, for instance the Euler transformation given below.

Originally these methods were developed to hasten the convergence of estimates of the sums of slowly convergent series, but when applied to divergent alternating series they are often found to converge, either absolutely or at least asymptotically, to a determinable limit which can meaningfully be accepted as the sums of these series.

For example, given a weakly divergent series with partial sums

$$S_N = \sum_{n=0}^N (-1)^n u_n \quad (u_n \geq 0)$$

in which the ratio u_{n+1}/u_n of successive terms is less than 3, the Euler transformation

$$U_N = - \sum_{n=0}^N \left(-\frac{1}{2}\right)^{n+1} \Delta^n u_0 ,$$

where $\Delta^n u_0$ is the n -th order forward difference of u_0 , provides a rapidly converging sequence whose limit coincides with S_∞ when this exists, and will be a self-consistent definition of a sum of the series when $\{S_N\}$ diverges.

A more widely-applicable general method, due to Aitken (1), may be regarded as replacing the tail of a series, t_n , where

$$t_n = u_n - u_{n+1} + u_{n+2} - \dots ,$$

by a two-term estimate, T_n , where

$$T_n = u_n^2 / (u_n + u_{n+1}) .$$

If the original series is truncated, then each term except the last is split into two parts, giving

$$S_N = u_0 - u_1 + u_2 - u_3 + \dots + (-1)^N u_N$$

$$= [T_0 + (u_0 - T_0)] - [T_1 + (u_1 - T_1)] + \dots + (-1)^{N-1} [T_{N-1} + (u_{N-1} - T_{N-1})] + (-1)^N u_N,$$

and the second part of each term is combined with the first part of the next term to form a new series of terms,

$$T_0 + S_{N-2}^* = T_0 + (u_0 - T_0 - T_1) - (u_1 - T_1 - T_2) + \dots + (-1)^{N-2} (u_{N-2} - T_{N-2} - T_{N-1}).$$

The parts of terms at the end of the truncated series, omitted by this transformation, combine naturally with the other terms already deleted by truncation.

If the series is a convergent geometric progression (G.P.), then $T_n = t_n$, and hence $u_n - T_n - T_{n+1} = 0$ for all n . Hence the Aitken transformation reduces the series to T_0 in one operation. If we wish, we may also use analytic continuation to define T_0 as the geometric sum for a divergent G.P., provided only that $u_{n+1}/u_n \neq 1$.

Now let us describe a series for which the ratio u_{n+1}/u_n tends to a bounded limit, R , when $n \rightarrow \infty$, as "nearly geometric" and a series for which the ratio varies slowly along the series, such as an asymptotic series, as "locally nearly geometric." Then the terms, u_n^* , of the transformed series, where

$$u_n^* = u_n - T_n - T_{n+1},$$

will be smaller than the corresponding terms of the original series, not only for "nearly geometric" but also for "locally nearly geometric" series.

An examination of Table 1 shows that for at least one asymptotic series the Aitken sum of the first few terms appears to converge to the same number as is obtained by other methods.

Other comments about Aitken's method are listed below.

- (1) The transformation may be repeated on the transformed series, S_{N-2}^* , giving a new first term and a new series, S_{N-4}^{**} . The series is reduced by two terms at each transformation. This may be repeated until the series reduces to fewer than two terms or the process becomes unstable.
- (2) We may wish to sum a series in which the ratio of successive terms changes rapidly. This rapid variation usually occurs at the start of a series and the ratio of successive terms settles down to a slow variation further along the series. This may result in an unevenness or instability near the start of the successively transformed series S_{N-2}^* , S_{N-4}^{**} , This can easily be detected if the terms of the series are tabulated. The initial instability does not affect the sum of the series nor the values of the later terms of the transformed series.
- (3) Aitken's transformation does not alter the order of terms so it can be used to speed the convergence of conditionally convergent series.
- (4) We may alter the sum of a divergent series by a biased splitting and regrouping such as

$$\begin{aligned}
 S &= 1 - 1 + 1 - 1 + 1 - \dots \\
 &= \left(\frac{2}{3} + \frac{1}{3} \right) - \left(\frac{1}{3} + \frac{2}{3} \right) + \left(\frac{2}{3} + \frac{1}{3} \right) - \dots \\
 &= \frac{2}{3} + \left(\frac{1}{3} - \frac{1}{3} \right) - \left(\frac{2}{3} - \frac{2}{3} \right) + \left(\frac{1}{3} - \frac{1}{3} \right) - \dots,
 \end{aligned}$$

where the positive and negative terms are split in different ways. However, both the Euler and the Aitken transformation are not sign-dependent so they do not have any obvious bias which might affect the sum of a series.

A second general method, due to S. Lubkin (3), is to use a three-term approximant, L_n , to the tail, t_n , of a series, where

$$L_n = \frac{u_n (1 + u_{n+1}/u_n)}{1 + 2u_{n+1}/u_n + u_{n+1}/u_{n-1}}$$

This is used in the same way as T_n to transform a series and effectively speeds the convergence of a slowly convergent series of positive terms. However, on divergent alternating series, two applications of Lubkin's transformation, using up six terms of a series, do not reduce the size of the remaining terms as much as three applications of Aitken's method. Nevertheless, L_n can be written as a weighted mean of T_n and $u_{n-1}^{-1}T_{n-1}$. Hence Aitken and Lubkin sums of alternating series are the same.

A third general set of formulae, called " e_n processes," was derived by D. Shanks (5), assuming that each term of a series may be represented by a sum of n exponentials. The formulae were demonstrated on a number of convergent and divergent series. They are not sign-dependent, so appear to be unbiased and e_1 is Aitken's two-term formula. While e_n is exact for a series whose terms are a combination of n or fewer exponentials, one of Shanks's examples shows that for other series e_2 does not reduce the size of the transformed terms as much as a double use of e_1 , both using up $\frac{4}{4}$ terms. Shanks's formulae are complicated but P. Wynn (7) gives a recurrence relation for calculating the successive e_n 's.

The author's (2) own weighted Euler sum is another formula which is relatively simple and uses the reciprocals of the terms for weights in the Euler sum. It is identical with Aitken's or Lubkin's formula when two or three terms are used. It appears to sum divergent alternating series to the same sum as Aitken's formula, but the rate of convergence is slower.

On the other hand, it does not appear to suffer seriously from initial instability and is convenient when a reasonably large number of terms of the series is available.

A different method again for summing series is the use of Padé approximants (4,6) which are compared in section 3 (vi) with the new formula.

While there are better specialist formulae, Aitken's (1) formula seems to be the best, unbiased general formula for summing a general divergent alternating series when we have a limited number of terms available, and we assume only that the later terms of the transformed series alternate .

The formula which follows is a weighted sum which appears to be more powerful initially in speeding the convergence or reducing the size of terms than Aitken's repeated formula; it is a general formula; it is not sign-dependent so appears to be unbiased; it may be calculated either directly or from a recurrence relation with comparable labour to that for Aitken's formula and while it may, like other formulae, suffer from initial instability, this can easily be observed and avoided when it occurs.

2. THE FORMULA

Let S_{mn} denote the sequence

$$S_{mn} = u_m - u_{m+1} + \dots + (-1)^{n-m} u_n \quad (n \geq m).$$

Let R_r be the ratio of two successive terms, i.e. $R_r = u_{r+1}/u_r$. Then an estimate of $S_{0,\infty}$, using the terms u_m to u_n as weights, is

$$T_{mn} = P_{mn}/Q_{mn},$$

where

$$P_{mn} = S_{0n} + (R_m + R_{m+1} + \dots + R_{n-1})S_{0,n-1} + (R_m^2 + R_m R_{m+1} + \dots + R_{n-2}^2)S_{0,n-2} + \dots + R_m^{n-m} S_{0m}$$

and

$$Q_{mn} = 1 + (R_m + R_{m+1} + \dots + R_{n-1}) + (R_m^2 + R_m R_{m+1} + \dots + R_{n-2}^2) + \dots + R_m^{n-m}.$$

Recurrence relations for generating P_{mn} and Q_{mn} are

$$P_{mm} = S_{0m} \quad Q_{mm} = 1$$

$$P_{mn} = P_{m+1,n} + R_m P_{m,n-1}$$

$$Q_{mn} = Q_{m+1,n} + R_m Q_{m,n-1}.$$

3. PROPERTIES OF THE FORMULA

(i) The estimators, T_{mn} , are weighted means of the truncated sums $S_{0n}, S_{0,n-1}, \dots, S_{0m}$ with weights

$$1, (R_m + R_{m+1} + \dots + R_{n-1}), (R_m^2 + R_m R_{m+1} + \dots + R_{n-2}^2), \dots, R_m^{n-m}.$$

If this is compared with the author's (2) other weighted Euler sum, whose weights were $1, \binom{n-m}{1} R_{n-1}, \binom{n-m}{2} R_{n-1}^2, \dots, R_{n-1}^{n-m}$, we see that the weighting function for divergent series uses the ratios of terms earlier in the series.

(ii) The first estimator of $S_{0\infty}$ is the trivial S_{0m} and the second estimator is the Aitken formula or Shanks's e_1 . Thus $T_{mm} = S_{0m}$ and with a little rearranging the second estimator gives

$$\begin{aligned} T_{m,m+1} &= \frac{S_{0m} + (-1)^{m+1} u_{m+1} + R_m S_{0m}}{1 + R_m} = S_{0m} + (-1)^{m+1} \frac{u_m u_{m+1}}{u_m + u_{m+1}} \\ &= S_{0,m-1} + (-1)^m \frac{u_m^2}{u_m + u_{m+1}}, \end{aligned}$$

which we recognise as Aitken's transformation.

(iii) If the ratios R_r are constant and equal to R , / then P_{mn} and Q_{mn} are binomials in $(1+R)$ and

$$T_{mn} = S_{0,m-1} + (-1)^m u_m \frac{(1+R)^{n-m-1}}{(1+R)^{n-m}}.$$

Proof: If $R_r = R$ for all values of r , then $Q_{mn} = (1+R)^{n-m}$.

Similarly,

$$\begin{aligned} P_{mn} &= S_{0,m-1} Q_{mn} + (-1)^m u_m \{ (1-R_m + \dots + (-1)^{n-m-1} R_m R_{m+1} \dots R_{n-1}) + \dots + R_m^{n-m} \} \\ &= S_{0,m-1} Q_{mn} + (-1)^m u_m (1+R)^{n-m-1} . \end{aligned}$$

$$\therefore T_{mn} = S_{0,m-1} + (-1)^m u_m \frac{(1+R)^{n-m-1}}{(1+R)^{n-m}} .$$

(iv) T_{0n} is a general formula in the sense that if the R_r are small and slowly varying and if T_{0n} is expanded as a power series in the R_r starting from u_0 , then the $(n+1)$ largest terms will coincide exactly with the first $(n+1)$ terms of $S_{0\infty}$ and the larger of the later terms form an approximate G.P.. Similarly, this holds if the R_r are large and slowly varying and $(-1)^n (S_n - T_{0n})$ is expanded as an inverse power series starting backwards from u_n .

Proof: If the R_r are small then $S_{0\infty}$ may be written as

$$u_0 (1 - R_0 + R_0 R_1 - R_0 R_1 R_2 + \dots)$$

and

$$T_{01} = \frac{u_0}{1+R_0} = u_0 (1 - R_0 + R_0^2 - R_0^3 + \dots) .$$

If, on the other hand, the R_r are large then

$$-S_1 = u_1 \left(1 - \frac{1}{R_0} \right)$$

and

$$-S_1 + T_{01} = u_1 \left(1 - \frac{1}{R_0} + \frac{1}{R_0^2} - \frac{1}{R_0^3} + \dots \right) .$$

Hence the two largest terms of our two-term approximant and the original series agree exactly whether the series is convergent or divergent. Also the other large terms will approximately agree if R_r varies slowly along the series.

If T_{0n} , as defined in the formula, is expanded as a power series, using R_0 to R_{n-1} , where R_0 to R_{n-1} are small and slowly varying, and ^{if} the first $(n+1)$ terms of T_{0n} are the same as in the original series and the remainder can be summed, then

$$T_{0n} = \frac{P_{0n}}{Q_{0n}} = u_0(1 - R_0 + \dots (-1)^n R_0 \dots R_{n-1} + (-1)^{n+1} R^{n+1} / (1+R)),$$

using an order of magnitude expression for the remainder in which R is comparable with the R_r . Also

$$T_{1,n+1} = \frac{P_{1,n+1}}{Q_{1,n+1}} = u_0(1 - R_0 + \dots + (-1)^{n+1} R_0 \dots R_n + (-1)^{n+2} R^{n+2} / (1+R)).$$

If R^* is any positive number, a weighted mean is given by

$$\frac{P_{1,n+1} + R^* P_{0n}}{Q_{1,n+1} + R^* Q_{0n}} = u_0 \left(1 - R_0 + \dots + (-1)^{n+1} R_0 \dots R_n + (-1)^{n+2} \frac{\frac{R^{n+2}}{1+R} Q_{1,n+1} - \frac{R^* R^{n+1}}{1+R} Q_{0n} + R_0 \dots R_n R^* Q_{0n}}{Q_{1,n+1} + R^* Q_{0n}} \right).$$

Hence if R^* is comparable with the members of the slowly varying set, R_0 to R_n , and if $Q_{1,n+1} \doteq Q_{0n}$, then the first $(n+2)$ terms of the expansion agree exactly with the first $(n+2)$ terms of the original series and the larger of the later terms form an approximate G.P.. Similarly, ^{this holds} for the expansion in reciprocal powers. Finally, T_{01} is a general formula. Hence, using the recurrence formulae, T_{02} and T_{03} and so on are all general formulae, in the sense that the $(n+1)$ largest terms of T_{0n} and the original series coincide exactly and the other larger terms form comparable G.P.s.,

(v) The best choice of R^* for T_{mn} for a divergent, almost geometric, alternating series is R_m .

Justification: The error in a formula which fits $(n-m)$ terms of a series exactly is a fraction of the smallest term used, which is the first term for a divergent series, and the fraction will not change rapidly along the series if R_m varies slowly with m . Hence the ratio of errors in $T_{m,n-1}$ and $T_{m+1,n}$ is approximately $-R_m$, so we can substantially reduce the error by combining them in the ratio $R^*:1$, where R^* is close to R_m .

We also wish not to waste terms of the series, so in constructing T_{mn} from $T_{m,n-1}$ and $T_{m+1,n}$ we will only construct R^* from u_m to u_n or R_m to R_{n-1} . In addition we wish to avoid the growth of rounding off errors in calculations, so we want a positive denominator. This means that we cannot use forms like $(2R_m - R_{m+1})$, which approximates R_{m-1} but introduces negative terms into the denominator, so our only choice for a divergent series is the single term $R^* = R_m$. This choice of weight also requires that $Q_{m,n-1} \div Q_{m+1,n}$. It can be seen in tables 2 and 3 that this soon ceases to be true at the start of the series, but the repeated transformations keep feeding in information from later terms of the series, where the transformation is better, thus damping down initial instability.

On the other hand, if we had a convergent series, we should use the ratio of last terms, i.e. R_{n-1} , in combining $T_{m,n-1}$ and $T_{m+1,n}$ to form T_{mn} . Hence if, after several transformations of a weakly divergent series, our residuals formed a convergent series, then this process would become less powerful. The second numerical example exhibits a rapid initial convergence of this type, which peters out.

(vi) The formula in section 2 is similar to the Padé approximants (4,6). It contains more terms and contains no negative terms in the denominator.

The Padé approximants for a series $F(z) = \sum_{r=0}^{\infty} a_r z^r$ are the ratios

of two polynomials. The $[L, M]$ Padé approximant to $F(z)$ is the quotient of two polynomials $P_L(z)$ and $Q_M(z)$, of degree L and M respectively, i.e.,

$$[L, M] \equiv \frac{P_L(z)}{Q_M(z)} \equiv \frac{p_0 + p_1 z + \dots + p_L z^L}{1 + q_1 z + \dots + q_M z^M},$$

chosen so that the power series expansion of $[L, M]$ for small z agrees exactly with the terms of $F(z)$ up to z^{L+M} .

Now, if $S_{0\infty}$ is a power series in z , with $u_r = a_r z^r$, then $R_r = a_{r+1} z / a_r$, so P_{mn} is a polynomial of degree n or $(n-1)$ and Q_{mn} is a polynomial of degree $(n-m)$ and we have deduced in section 2(iv) that T_{mn} agrees exactly with the terms of $S_{0\infty}$ up to z^n .

Hence the first row of the Padé table $[n, 0]$ and the T_{nn} are the same truncated sums of the series up to z^n . The second row of the Padé table $[n-1, n]$ are the first Aitken or Shanks e_1 transformation of the series, using the last two terms for the transformation. Thereafter the number of terms in T_{mn} grows at twice the rate of the number of terms in $[L, M]$ when we match more terms of the original series.

Thus if we use 4 terms of the series

$$S_{\infty} = u_0 (1 - R_0 + R_0 R_1 - R_0 R_1 R_2 + \dots) ,$$

then

$$[1,2] = u_0 \frac{(R_0 - R_1) + (-R_0^2 + 2R_0 R_1 - R_1 R_2)}{(R_0 - R_1) + R_1(R_0 - R_2) + R_0 R_1(R_1 - R_2)} ,$$

$$\text{and } T_{03} = u_0 \frac{1 + (R_1 + R_2) + (R_0 R_1 + R_1^2 - R_0 R_2)}{1 + (R_0 + R_1 + R_2) + (R_0^2 + R_0 R_1 + R_1^2) + R_0^3} .$$

The similarities between the formulae are that they are both quotients of two polynomials and fit four terms of the original series exactly and appear to converge with comparable rapidity. The differences are that the Padé approximants may be found from a relationship between five approximants, whereas the P_{mn} and Q_{mn} are obtained from simple recurrence relations. Also, the denominators of the Padé approximants contain negative terms, so they must be calculated more accurately to avoid rounding-off errors.

4. NUMERICAL EXAMPLES

(i) The sum of alternating factorials.

If we have an asymptotic series whose terms decrease at first, then as long as the terms are becoming smaller using a summation formula is no better than evaluating one more term of the series, since the truncation error is comparable with the first term omitted. Hence it only becomes profitable to use a summation formula on the terms which are not rapidly converging.

As an example we take

$$I = \int_1^{\infty} e^{1-x} dx/x = 0.596347361 \dots$$

$$= 0! - 1! + 2! - 3! + 4! - \dots$$

In table 1 we use Aitken's method repeatedly, first calculating

$$T_n = u_n^2 / (u_n + u_{n+1}) \text{ and the transformed series } u'_n = u_n - T_n - T_{n+1}$$

then repeating this process.

The numbers left over at the start of each transformation are marked with a star and later added.

TAKE IN TABLE 1

Using 8 terms of the original series

$$I = .5000000 + .0833333 + .0115377 + .0013372 + \dots$$

the truncation error at this stage being .0001392. The new series ceases to be monotonic or becomes unstable after using 14 terms of the original series.

In table 2 we list n , u_n , R_n and T_{nn} ($= S_{0n}/1$) in the first four columns and calculate P_{mn}/Q_{mn} using the recurrence relations.

TAKE IN TABLE 2

The fractions are all estimates of I which improve towards the top right corner of table 2. To compare them with the results using Aitken's formula, we write the sequence T_{01} , T_{03} , T_{05} and T_{07} as a series

Table 1. Repeated Aitken sum of factorials

n	u_n	T_n	u'_n	T'_n	u''_n	T''_n	u'''_n	T'''_n
0	1	$\frac{1}{2}^*$						
1	-1	$-\frac{1}{3}$	$\frac{1}{2.3}$	$\frac{4^*}{2.3.8}$				
2	2	$\frac{2}{4}$	$\frac{-2}{3.4}$	$\frac{-2.5}{3.4.15}$	$\frac{2.16}{3.4.8.14}$	$.0115377^*$		
3	-6	$-\frac{6}{5}$	$\frac{6}{4.5}$	$\frac{6.6}{4.5.22}$	$\frac{-6.26}{4.5.14.22}$	$-.0093617$	$.0029101$	$.0013372^*$
4	24	$\frac{24}{6}$	$\frac{-24}{5.6}$	$\frac{-24.7}{5.6.32}$	$\frac{24.38}{5.6.22.32}$	$.0125397$	$-.0034232$	
5	-120	$-\frac{120}{7}$	$\frac{120}{6.7}$	$\frac{120.8}{6.7.44}$	$\frac{-120.52}{6.7.32.44}$			
6	720	$-\frac{720}{8}$	$\frac{-720}{7.8}$					
7	-5040							

Table 2. T_{mn} for the sum of factorials

n	u_n	R_n	T_{nn}	$T_{n,n+1}$	$T_{n,n+2}$	$T_{n,n+3}$	$T_{n,n+4}$	$T_{n,n+5}$	$T_{n,n+6}$	$T_{n,n+7}$
0	1	1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{9}{15}$	$\frac{31}{52}$	$\frac{121}{203}$	$\frac{523}{877}$	$\frac{2469}{4140}$
1	-1	2	$\frac{0}{1}$	$\frac{2}{3}$	$\frac{6}{10}$	$\frac{22}{37}$	$\frac{90}{151}$	$\frac{402}{674}$	$\frac{1946}{3263}$	
2	2	3	$\frac{2}{1}$	$\frac{2}{4}$	$\frac{10}{17}$	$\frac{46}{77}$	$\frac{222}{372}$	$\frac{1142}{1915}$		
3	-6	4	$\frac{-4}{1}$	$\frac{4}{5}$	$\frac{16}{26}$	$\frac{84}{141}$	$\frac{476}{799}$			
4	24	5	$\frac{20}{1}$	$\frac{0}{6}$	$\frac{20}{37}$	$\frac{140}{235}$				
5	-120	6	$\frac{-100}{1}$	$\frac{20}{7}$	$\frac{40}{50}$					
6	720	7	$\frac{620}{1}$	$\frac{-80}{8}$						
7	-5040		$\frac{-4420}{1}$							

$$I = .5000000 + .1000000 - .0039409 + .0003177 - \dots ,$$

the truncation error at this stage being $-.0000294$. This series is more rapidly convergent than Aitken's and does not become unstable until 18 terms of the original series are used. Even if we had used weights from the other end of the series, namely $R^* = R_{n-1}$, giving the author's weighted Euler sum, we still get a convergent process. Using the same terms we obtain

$$I = .50000 + .08824 + .00684 + .00100 + \dots .$$

(ii) A nearly geometric series.

The following is a demonstration that T_{mn} is useful for divergent, alternating, almost geometric series in which R_n increases with n .

Almost geometric, divergent series may be divided into two groups, one in which the ratio of successive terms rises to a constant limit greater than 1 and the other where the ratio of successive terms decreases to a constant limit greater than 1.

If the series is a binomial with exponent (-1) or a G.P. such as $S = 1-2+4-8+ \dots$, then all the formulae give the sum in one operation as $1/3$ without remainder.

As an example of a series in which R_n increases we take the logarithmic series.

The function $\ln(1+z)$ has a logarithmic singularity in the complex plane at $z = -1$ so we cannot expect to represent it as a rational function for all z . Even so we still obtain considerable convergence before the T_{mn}

become unstable. As an example, we use

$$1.7918 \dots = \ln(1+5) = 5 - 5^2/2 + 5^3/3 - 5^4/4 + 5^5/5 - \dots$$

In table 3, the values of n , u_n , R_n and S_{1n} or T_{nn} are listed in the first four columns and the recurrence relation is used to generate the higher order T_{mn} values.

TAKE IN TABLE 3

The successive sums for 2, 4 and 6 terms using Aitken's formula are 1.4286, 1.7368 and 1.7862, while for comparison T_{01} , T_{03} and T_{05} are 1.4286, 1.7487 and 1.7934. If further terms are calculated, Aitken's formula continues to provide a convergent sequence but the new formula provides a sequence that overshoots and then oscillates.

The corresponding Padé approximants using the same terms are [1,2], [2,3] and [3,4]. These give 1.4286, 1.7213 and 1.7787, which are comparable but not quite as good as the repeated Aitken and the T_{mn} approximations.

As a second example of a series with increasing R_n we take the binomial series for the inverse square root of 9:

$$(1+8)^{-1/2} = 1-4+24-160+1120-8064+ \dots = 1/3 = .333 \dots$$

TAKE IN TABLE 4

The details for T_{mn} are given in table 4 and the sums for 2, 4 and 6 terms to three decimal places are :

Repeated Aitken	.200	.299	.328
Padé [n,n+1]	.200	.294	.323
$T_{0,2n-1}$	.200	.298	.332

As an example of a series with decreasing R_n we take a binomial with power $-3/2$:

$$(1+8)^{-3/2} = 1-12+120-1120+10080 - 88704 + \dots = .037$$

Table 3. T_{mn} for a nearly geometric series

n	u_n	R_n	T_{nn}	$T_{n,n+1}$	$T_{n,n+2}$	$T_{n,n+3}$	$T_{n,n+4}$	$T_{n,n+5}$
0	5	$\frac{5}{2}$	$\frac{5}{1}$	$\frac{5}{3.5}$	$\frac{21.667}{13.083}$	$\frac{90.764}{51.903}$	$\frac{386.14}{216.55}$	$\frac{1688.3}{941.4}$
1	$-\frac{25}{2}$	$\frac{10}{3}$	$-\frac{7.5}{1}$	$\frac{9.1667}{4.3333}$	$\frac{36.597}{19.194}$	$\frac{159.23}{86.79}$	$\frac{722.99}{400.03}$	
2	$\frac{125}{3}$	$\frac{15}{4}$	$\frac{34.167}{1}$	$\frac{6.0417}{4.75}$	$\frac{37.240}{22.812}$	$\frac{192.22}{110.71}$		
3	$-\frac{625}{4}$	$\frac{20}{5}$	$-\frac{122.08}{1}$	$\frac{14.583}{5}$	$\frac{52.569}{25.167}$			
4	$\frac{3125}{5}$	$\frac{25}{6}$	$\frac{502.92}{1}$	$-\frac{5.7639}{5.1667}$				
5	$-\frac{15625}{6}$		$-\frac{2101.25}{1}$					

TABLE 4. Calculation of an inverse square root

n	u_n	R_n	T_{nn}	$T_{n,n+1}$	$T_{n,n+2}$	$T_{n,n+3}$	$T_{n,n+4}$	$T_{n,n+5}$
0	1	4	$\frac{1}{1}$	$\frac{1}{5}$	$\frac{7}{27}$	$\frac{47}{157.7}$	$\frac{316.7}{988}$	$\frac{2173}{6552}$
1	-4	6	$\frac{-3}{1}$	$\frac{3}{7}$	$\frac{19}{149/3}$	$\frac{128.7}{357.1}$	$\frac{906.0}{2601}$	
2	24	$20/3$	$\frac{21}{1}$	$\frac{1}{23/3}$	$\frac{44/3}{532/9}$	$\frac{134.0}{458.3}$		
3	-160	7	$\frac{-139}{1}$	$\frac{8}{8}$	$\frac{181/5}{321/5}$			
4	1120	$36/5$	$\frac{981}{1}$	$\frac{-99/5}{41/5}$				
5	-8064		$\frac{-7083}{1}$					

The new formula is inferior to the older formulae on this example.

The sums using different formulae on 6 terms are:

Repeated Aitken	.0385
Padé [3,4]	.0383
T_{05}	.0551

(iii) A velocity formula for an inextensible sheet oscillating sinusoidally in a viscous liquid.

If σ is the angular frequency, k is the wave number and b is the amplitude, then the velocity, v , of the sheet relative to the fluid is given by

$$v = \frac{\sigma}{2k} \left[(bk)^2 - \frac{19}{16} (bk)^4 + \frac{41}{32} (bk)^6 - \frac{16913}{12288} (bk)^8 + \dots \right]$$

Two transformations of this series are

$$T_{13} = \frac{\sigma}{2k} \frac{b^2 k^2 + \frac{288875}{299136} b^4 k^4 - \frac{10143611}{90937344} b^6 k^6}{1 + \frac{644099}{299136} b^2 k^2 + \frac{1681}{1444} b^4 k^4}$$

and

$$T_{03} = \frac{\sigma}{2k} \frac{b^2 k^2 + \frac{644099}{299136} b^4 k^4 + \frac{106369861}{90937344} b^6 k^6}{1 + \frac{999323}{299136} b^2 k^2 + \frac{356313}{92416} b^4 k^4 + \frac{6859}{4096} b^6 k^6}$$

The original series appears to have a radius of convergence $bk \doteq .93$ and if we want a result, say, to 3% accuracy the truncated power series may be used for bk up to 0.6. When the series is transformed, T_{03} and T_{13} agree to better than 3% for amplitudes up to 1.1, which is beyond the original radius of convergence.

5. CONCLUSION

The choice of weight $R^* = R_m$ in section 3 (v) was made to reduce the error for a divergent series. When a new sequence or series is constructed using the transformation of section 2 and the original series is locally nearly geometric, the new terms are smaller than the original terms. If the later terms of the original series are more nearly geometric than the earlier terms, then the fractional reduction is greater further along the series, so the ratio of successive terms is also reduced.

If the weights, R_m , that we use in the transformation decrease along the original series, then in the second transformation we will use weights which are larger than ideal and so cause a slow convergence. On the other hand, if the ratios, R_m , increase along the series and each transformation reduces the length of the series by one term, then we consistently use the smaller weights available. Hence for a while at least the new formula remains efficient.

An examination of the numerical examples shows that for the asymptotic series the transformation may be repeated 9 times and gives a sum accurate to 9 decimal places before it starts to show an instability or loss of efficiency. In the two examples of geometric series with increasing ratio, the instability starts earlier but the transformation is efficient until it becomes unstable and the instability can easily be observed by tabulating a few adjacent terms in the table of T_{mn} . It then appears that the formula of section 2 is useful for divergent series in which the ratio of the terms increases along the series.

The final example is a series which is truncated because of the difficulty of calculating later terms. A transformation of this series increases both its accuracy and range.

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SUMMING DIVERGENT SERIES

J.E. DRUMMOND

1. SUMMING SERIES. Summing series is one of the skills of the practical mathematician. If the series is rapidly convergent and sufficient terms are available then the series can be summed to any desired accuracy, but if the series is slowly convergent or very few terms are available then there are a number of transformations available to speed the convergence of series.

When the same transformations are applied to divergent series the divergent series are also found to transform into less rapidly divergent or into convergent series. Furthermore, these transformed series appear to give meaningful numbers.

Hence a practical mathematician wishing to calculate some quantity need not spend a lot of time looking for a rapidly convergent or even a slowly convergent series. Instead, he merely needs to find a convenient series (convergent or divergent) which represents his quantity then transform it to a more rapidly convergent series.

In the following I wish to briefly describe several very powerful transformations of series, then discuss a few difficulties and dangers that occur in the use of divergent series before finally urging mathematicians to make more use of divergent series.

2. POWERFUL NON LINEAR TRANSFORMATIONS OF SERIES. Without wishing to cover the subject exhaustively I must first say that there are many special techniques for transforming series. There are also a number of comparison techniques and linearly weighted means which were invented by Euler, Cesàro, Voronoi, Holder and others. These latter methods are described in G.H. Hardy's book "Divergent Series" and Karl Zeller's book "Theorie der Limitierungsverfahren". Of these, the Euler sum is probably best known and appears in books on numerical methods.

A different technique called TERM SPLITTING (1) has been described by me and consists of splitting each term of a series into two parts, then recombining the second part of each term with the first part of the next term. If we can find a splitting rule which makes the first part of a split term approximately equal to the tail of the series starting with that term then we get almost perfect cancellation of all but the first part of the first term which is approximately equal to the sum of the series.

If, for example, we start with an alternating power series,
 $a_0 - a_1x + a_2x^2 + \dots$, and split each term in the ratio $1:x$ then regroup, we get a new series

$$\frac{1}{1+x} \{ a_0 + (a_0 - a_1)x - (a_1 - a_2)x^2 + \dots \}.$$

If we split repeatedly we obtain an Euler transformation with weights $1:x$, namely

$$\sum b_n x^n / (1+x)^n \quad \text{where} \quad b_n = (-1)^n \Delta^n a_0 = a_0 - \binom{n}{1} a_1 + \binom{n}{2} a_2 - \dots$$

A better procedure is to split the terms of an alternating series $(u_0 - u_1 + u_2 - \dots)$ in the ratio of successive terms to get a new series after recombining, namely

(1) The Australian Mathematics Teacher, Vol 30, No 5 (1974) pp 179-183.

$$\frac{u_0^2}{u_0 + u_1} + \sum_{n=0}^{\infty} (-1)^n \left(\frac{u_n u_{n+1}}{u_n + u_{n+1}} - \frac{u_{n+1}^2}{u_{n+1} + u_{n+2}} \right).$$

The terms of this new series may be split repeatedly in the same way to give a non linear transformation used by A.C. Aitken (2) and more recently described by Daniel Shanks (3) as an e_1 process.

To compare the weighted Euler and Aitken sums we take the slowly convergent series $(4 - 4/3 + 4/5 - \dots)$ for π . If we start with 10 terms it is best to repeatedly halve the last seven terms to obtain the Euler sum 3.14161.. whereas with Aitken's formula we obtain a sum 3.14159265... With more terms greater accuracy may be obtained.

As an example of a divergent series $\ln 5$ may be written as $(4 - 4^2/2 + 4^3/3 - \dots)$. With 10 terms it is best to use a weighted Euler transformation with weights 1:4 starting at the third term to obtain a sum 1.598... while the Aitken sum of the same ten terms is 1.609432... and $\ln 5 = 1.609437..$ The Aitken sum may also be used for asymptotic series. Thus 10 terms of the series $\sum (-1)^n n!$ sum to 0.59634.. as compared with the value 0.59635.. for $e \int_1^{\infty} e^{-t}/t dt$. For the best Euler sum we stop at half the smallest term giving an estimate of $1/2$ for this series.

There are a number of other contributors to very powerful transformations of whom I shall only mention three, H. Padé (4), S. Lubkin (5) and P. Wynn (6), whose methods are comparable with Aitken's and Shanks's and very much more powerful than the general linear transformations.

3. AXIOMS FOR DIVERGENT SERIES. If we are given the first few terms of a series and know nothing about the later terms then we cannot sum the series or even approximate to the sum. If, however, we may say that the series is transformable to a new series that is summable then we say that the original series is summable in the sense of the transformation (e.g. the series $\sum (-1)^n n!$ has an Aitken sum which approximates to $e \int_1^{\infty} e^{-t}/t dt$ to 4 figure accuracy when 10 terms are used).

If these transformations of divergent series are to be useful they must first be consistent. Hence they must at least satisfy some simple criteria. G.H. Hardy gives three axioms that they must satisfy:

- A. If $\sum a_n = s$ then $\sum k a_n = ks$
- B. If $\sum a_n = s$ and $\sum b_n = t$ then $\sum (a_n + b_n) = s + t$
- C. If $a_0 + a_1 + \dots = s$ then $a_1 + \dots = s - a_0$

To these may be added a regularity condition, namely that the transformation must not alter the sum of a convergent series. Also a combination of axioms B and C is equivalent to a statement that the transformation of an alternating series is unbiased

(2) Proc. Roy. Soc. Edin. 46 (1925-26) pp 289-305.

(3) J. Math. Phys. 34 (1955) pp 1-42.

(4) Padé approximants (1892) are described by H.S. Wall "Analytical Theory of Continued Fractions".

(5) J.R.N.B.S. 48 (1952) pp 228-254.

(6) Math. Tables Aids Comp. 10(1956) pp 91-96.

towards the positive or negative terms. Thus a transformation independent of sign must satisfy the equation $(a_0 - a_1 + a_2 - a_3 + \dots) + (A - a_0 + a_1 - a_2 + \dots) = A$. The requirement that the transformations of alternating series be unbiased also follows from axiom A when $k = -1$.

4. ANALYTIC CONTINUATION AND THE REGULARITY CONDITION. One source of divergent series is a power series outside its radius of convergence. If such a series is transformed into another series, say by Euler's transformation, then both series will have the same value when the first series is convergent, the later terms of the series tend to zero and the weighting ratio, $1:x$, is positive. As an example we take the power series for $\ln(1+x)$, namely

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

which converges for $|x| < 1$.

The weighted Euler transformation of the series using a constant weighting ratio $1:x$ is

$$\frac{x}{1+x} + \frac{1}{2} \left(\frac{x}{1+x} \right)^2 + \frac{1}{3} \left(\frac{x}{1+x} \right)^3 + \dots$$

which converges if $R(x) > -\frac{1}{2}$. If this latter series is constructed from the former by repeated term splitting then we can easily show that it has the same value as the original series on the real line $0 < x < 1$, and less easily prove agreement over the whole common region of convergence. The Euler transformation of the original series is an analytic continuation of the original series over the new domain of convergence outside the original circle of convergence.

Similarly, Aitken's transformation does not alter the value of a convergent series. Hence Aitken's transformation satisfies the regularity condition and, if the original series is a power series, the Aitken sum will be an analytic continuation of it outside the original circle of convergence.

5. UNBIASED TRANSFORMATIONS. I wish to demonstrate with examples that if a transformation is biased or favours terms of one sign this may alter the calculated sum of a series. On the other hand, it may be possible to write a series in a way consistent with its origin so that one or more of the standard transformations give a sum consistent with the origin of the series.

As an example, if the series $(1 - 1 + 1 - 1 + \dots)$ is the analytic continuation of the power series $(1 - x + x^2 - x^3 + \dots)$ when $x = 1$ or if the series is summed by any one of a number of standard transformations its sum is $\frac{1}{2}$. If we use an obviously biased transformation in which we split the positive terms in the ratio $1:2$ and the negative terms in the ratio $2:1$ and regroup, then we obtain the transformation from

$$S = \frac{1+2}{3} - \frac{2+1}{3} + \dots \quad \text{to} \quad \frac{1}{3} + \frac{2-2}{3} - \frac{1-1}{3} + \dots = \frac{1}{3}$$

$$\text{while } -S = -\frac{2+1}{3} + \frac{1+2}{3} - \dots \text{ goes to } -\frac{2}{3} - \frac{1-1}{3} + \frac{2-2}{3} - \dots = -\frac{2}{3}.$$

This transformation has a periodicity 2 and favours the negative terms in both cases. The two values for S are contradictory and the transformation does not satisfy Hardy's axioms.

On the other hand if the series can be associated with the Fourier series $(2/\sqrt{3}) \sum \sin(2n\pi/3)$ or is the analytic continuation of a different power series $(1 - x + x^3 - x^4 + x^6 - x^7 + \dots)$ which sums to $1/(1+x+x^2)$ then we should write the series in the form $S' = 1 - 1 + 0 + 1 - 1 + 0 + \dots$ and use a transformation which does not have the period 3. If we split in half and regroup then

$$S' = \frac{1}{2} + 0 - \frac{1}{2} + \frac{1}{2} + 0 - \dots = \frac{1}{2} - \frac{1}{2} S' \quad \therefore S' = \frac{1}{3}, \text{ whereas the same transformation}$$

$$\text{gives us } S'' = 1 + 0 - 1 + 1 + 0 - 1 + 1 + \dots = \frac{2}{3} = \lim_{x \rightarrow 1} (1 - x^2 + x^3 - x^5 + \dots).$$

I conclude that not only should the transformation be unbiased or aperiodic or have a different period from the series or vary continuously along the series but the mechanism for producing the series should behave similarly.

6. ERRORS OF LOWER ORDER. In summing asymptotic series which have no circle of convergence we meet with a difficulty of deciding what quantity the series represents. Thus when we attempt to represent a function as a series of inverse powers of x terms like e^{-x} are smaller than any inverse power of x when x is large, so may be unobserved.

An example of this is the solution of the differential equation

$$-\frac{dy}{dx} + y = A/x.$$

The general solution is

$$y = ke^x + Ae^x \int_x^c e^{-t}/t dt,$$

while the asymptotic series solution is

$$y = A\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots\right).$$

The general term, ke^x , is of lower order than any reciprocal power of x when x is large negative so is not detected in the asymptotic series if x is negative. Near $x = 0$, y is unbounded. When x is positive the general term will dominate the other terms unless $k = 0$.

If we transform the asymptotic series by using Aitken's or Lubkin's or Padé's transform when x is positive we find that, depending on the number of terms used, our approximants appear to approximate closely to the integral

$$e^x \int_x^\infty e^{-t}/t dt \quad \text{with } k = 0 \text{ and } c = \infty.$$

It appears from this example that the powerful transformations mentioned above are mutually consistent in dealing with an asymptotic series but that by themselves they cannot inform us about any exponentially small term associated with the series or with the differential equation.

7. CONCLUSION. The summation of divergent series is a useful technique that should be known to every practical mathematician. If we have a power series we can transform it to another series convergent outside the original circle of convergence. Furthermore if the terms of the series are complicated we sometimes need only find a few terms then convert them into a more rapidly convergent series.

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