CORRIGENDUM to Ph.D. Thesis of J. R. LESLIE

NOTATION: T6 means 6th line from the top.
B6 means 6th line counting from the bottom (excluding footnotes).

PAGE	LINE	READS:	SHOULD READ:
4	B1	$P_s(it)$	$e^{-\frac{1}{2}t^2}P_s(it)$
5	B3	$\int_{-2}^{2}$	$\int_{-z}^{z}$
6	B8	for $j + 1$	for $j = 1$
7	T4	$E\vert X_i\vert^{k+2}$	$E\vert X_i\vert^{k+1}$
7	T4	$1,2,\ldots,k+2$	$1,2,\ldots,k+1$
7	T9	conditions	condition
7	B1	$O\left(n^{-\frac{1}{2}(k+2)}\right)$	$o\left(n^{-\frac{1}{2}(k+2)}\right)$
9	T8	$O(\frac{1}{n})$	$o(\frac{1}{n})$
9	T17	Heyde	Heyde [13]
10	T3	$\sqrt{n}\vert\ldots$	$+\sqrt{n}\vert\ldots$
15	T7	If $h'(i)$	If $h'(1)$
15	B4	Harris, (	(Harris,
23	B1	if $\delta>0$ or if r is even and $\delta=0$	if $\delta\geqq 0$ (in particular if r is odd, $\delta=0$ and there exists a $B>0$ such that $P(X < -B) = 0$)
31	B2	$\gamma(\frac{t}{n})$	$\gamma(\frac{t}{\sqrt{n}})$
34	T4	Lemmas 1 and 2	Lemmas 2.5 and 2.6
35	T10	$\} \ \vert < \infty$	$\}\ dt \ \vert < \infty$
37	B4	$\left(\text{or } O(n^{-\frac{1}{2}k})\right)$	$\left\{\text{or } o(n^{-\frac{1}{2}k})\right\}$
38	T1	$\left(\text{or } O(n_i^{-\frac{1}{2}k})\right)$	$\left(\text{or } o(n_i^{-\frac{1}{2}k})\right)$
38	T4	and $_j =$	and $\alpha_j =$

PAGE	LINE	READS:	SHOULD READ:
38	T12	$O(n^{-\delta})$	$O(z^{-\delta})$
39	B5	$dF(x)$	$dF(u)$
41	T1	Moduli signs required outside the integral on the left-hand side and outside the sum of integrals on the right-hand side.	
42	T2 and 4		as $i \to \infty$
42	T14	α_i	α_j
53	T2 and 3	Large brackets should surround $\frac{d}{du}\{u^{k+2}\log(1+u)\}$	
53	T6	$\sum$	$c'\sum$
61	B4	Moduli signs required outside integral $\int^{*}$	
62	T16	$\frac{\nu!}{\ell!(\nu-\ell)!}\,(EY_n)^{\ell-1}$	$\frac{\nu!(-1)^{\ell}}{\ell!(\nu-\ell)!}\,(EY_n)^{\ell-1}$
62	T16	$(EY_n)^{\nu-1}$	$(-1)^{\nu}(EY_n)^{\nu-1}$
63	B8	Therefore	Therefore using (11) and (12)
64	T4	$2s_{\alpha}$	$2s_2$
64	T7	for $\nu = k+1$	for $\nu = k+2$
65	B5	$e^{-7/13}$	$e^{-7/26}$
65	B4	$\leq C(S)$	$\leq c(s)$
66	B8	Delete †	
74	B5	$(\frac{t}{n^n})$	$(\frac{t}{\sigma_n\sqrt{n}})$
79	Footnote	$f_n(t)$	$\overline{f}_n(t)$
81	T8	As Theorem 3.6 is valid for $0 < \delta < 1$, we know EX^2 is finite and so without loss of generality, we will take $EX^2 = 1$.	
81	B6	$\leq \frac{1}{n}\int$	$\leq \frac{1}{\sqrt{n}}\int$
81	B6	$\leq cn^{-1}$	$\leq cn^{-\frac{1}{2}}$
82	T11	N.B. In view of the correction to p.81, we have already taken $EX^2 = 1$	

RATES OF CONVERGENCE TO NORMALITY

by

J. R. Leslie, B.Sc.(Hons.), Sydney

A thesis submitted for the degree of

Doctor of Philosophy

in the Australian National University.

March, 1974.

I, Julian Robert Leslie, declare that this thesis is my own original work save and insofar as where I have expressly acknowledged my debt to other authors.

Julian Robert Leslie
9th March, 1974.

CONTENTS

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The work of Chapters 2 and 4 was carried out jointly with Dr. C. C. Heyde. A large part of the work of Chapter 2 appears in [20] and of Chapter 4 in [19]. Chapters 2A, 3 and 5 are my own work and Chapter 5 appears in a reduced form in [25].

ABSTRACT

This study falls into two parts: in the first part (Chapters 2, 2A and 3) we examine the Chebyshev expansion for the distribution function of normed sums of independent random variables and in the second part (Chapters 4 and 5) we apply some convergence rate results from independent random variable theory to two types of branching process and we obtain some analogues of the classical limit laws.

In particular, in Chapter 2 we derive necessary and sufficient conditions for certain series-type convergence rates for the remainder term $R_{k,n}(x)$ in the Chebyshev approximation to the above mentioned distribution function. These conditions turn out to be pure moment conditions, in contrast to the case of 'order n' convergence rates, treated by I. A. Ibragimov, in which tail-conditions on moments are involved. Chapter 2A shows that the results of Ibragimov may be obtained under slightly weaker conditions and as a corollary to this result, we show how to tie up some loose ends present in Chapter 2.

In Chapter 3, assuming certain moments exist, we derive very precise estimates for the remainder term $R_{k,n}(x)$ and we show that from these estimates, well known results concerning $R_{k,n}(x)$ can be easily derived. Also, without demanding the existence of any moments whatever, we give a new estimate for the remainder term in the central limit theorem which extends a similar result of Osipov and Petrov. Furthermore, we generalise this extension and obtain an estimate for the remainder term in the Chebyshev approximation, again without requiring the existence of any moments. Finally, we show that this latter estimate has, in certain cases, asymptotic behaviour equivalent to that of the remainder term itself.

Chapter 4 provides analogues of the iterated logarithm law in the context of Galton-Watson processes, with or without immigration. Such results had earlier been given under rather severe moment restrictions. In Chapter 4 we show how to remove these restrictions and replace them by a basic second moment condition. We also give a convergence rate for the analogue of the central limit theorem. This rate result plays a vital role in the derivation of the results of Chapter 4.

The final chapter gives both central limit and iterated logarithm analogues for temporally homogeneous Markov branching processes and for the associated increasing process consisting of the number of splits in the original process up to time t.

NOTATION AND ABBREVIATIONS

a.s.	almost surely
(C)	Cramer's condition, see p.1.
c.f.	characteristic function
$c(n) \uparrow$ as $n \uparrow$	$c(n)$ is an increasing function of n
$\xrightarrow{D}$	converges in distribution to
$\overset{D}{=}$	has the same distribution as
d.f.	distribution function
iff	if and only if
$\overline{\lim}$	lim sup
$\underline{\lim}$	lim inf
$\log f(t)$	where $f(t)$ is a characteristic function; we take the principal value of $\log f(t)$ - see [11] p.64.
$\xrightarrow{P}$	converges in probability to
rv	random variable
T.P.A.	Theory of Probability and its Applications
'tail conditions'	on moments is used in a broad sense to include conditions on $\int_{\|u\|>z} u^r dF(u)$ and on $\int_{\|u\|<z} u^s dF(u)$. Lemma 3.7 shows that certain conditions on the former expression induce similar conditions on the latter.
$\cdot$	often, for reasons of clarity, a dot is placed between two expressions whose product is being taken.
$[x]$	integer part of x

CHAPTER 1

INTRODUCTION

1.1. INDEPENDENT SEQUENCES: Let $\{X_i\}$ be a sequence of iid rv's. Throughout this section we shall assume, unless otherwise stated, that $EX_i = 0$ and $EX_i^2 = 1$.

HISTORICAL PERSPECTIVE: In 1901, Lyapounov published his theorem about the convergence of the distribution functions of the normalised sums of a sequence of iid rv's to the normal law. In that same article he gave an estimate of the speed of this convergence. Indeed, upon assuming the existence of the 3rd moment, i.e. $E|X_i|^3 = \beta_3 < \infty$ and taking

$$F_n(x) = P\left\{\frac{X_1 + \ldots + X_n}{\sqrt{n}} < x\right\}$$

he showed that

$$|F_n(x) - \Phi(x)| \leqq C\beta_3 \frac{\log n}{\sqrt{n}}, \quad C \text{ a universal constant.}$$

Some years later (1928) H. Cramér sharpened this result while imposing the additional assumption that $\overline{\lim_{|t|\to\infty}} |f(t)| < 1$ where $f(t) = Ee^{itX}$ (we denote this condition here and elsewhere by (C)) to obtain

$$|F_n(x) - \Phi(x)| \leqq C \frac{\beta_3}{\sqrt{n}}, \quad C \text{ a universal constant.}$$

Berry (1941) and Esseen (1942) independently obtained the same result whilst avoiding the imposition of condition (C).

Ibragimov [22] (1966) relaxed the condition demanding the existence of the 3rd moment and was still able to obtain estimates of the same order as the Berry-Esseen result. Furthermore, he showed that the conditions were also necessary and in addition he gave necessary and sufficient conditions for $|F_n(x) - \Phi(x)|$ to be $O(n^{-\delta/2})$, $0 < \delta < 1$.

His results are given in the following theorem:

THEOREM 1.1 (Ibragimov). In order that $|F_n(x) - \Phi(x)| = O(n^{-\delta/2})$, $0 < \delta \leqq 1$ *it is necessary and sufficient that*

1) $\displaystyle\int_{|x|>z} x^2 dF(x) = O(z^{-\delta})$ as $z \to \infty$

and 2) $\displaystyle\int_{-z}^{z} x^3 dF(x) = O(1)$ as $z \to \infty$ if $\delta = 1$.

We note that the $O(n^{-\frac{1}{2}})$ rate is the best possible in view of the following simple example in which this bound is actually attained. Let the iid sequence $\{X_i\}$ take the values ± 1, each with probability $\frac{1}{2}$. The distribution function of $(X_i + \ldots + X_n)n^{-\frac{1}{2}}$ is a step function with jumps of order of magnitude $\frac{1}{\sqrt{n}}$ while $\Phi(x)$ is of course continuous, hence $\sup_x |F_n(x) - \Phi(x)| = O(n^{-\frac{1}{2}})$. Furthermore, all moments of X_i exist.

1.2. THE CHEBYSHEV EXPANSION FOR THE DISTRIBUTION FUNCTION $F_n(x)$.

With bounds obtained for the difference $|F_n(x) - \Phi(x)|$, interest was next directed towards obtaining some kind of asymptotic expansion for it. The problem is the same as that of finding an asymptotic expansion for $F_n(x)$ in which the first term is $\Phi(x)$. Historically, results of Chebyshev (1859) [5] were used to obtain the desired expansion; Chebyshev had found an expansion for an arbitrary function p(x) in terms of an orthogonal set of polynomials $\{H_k(x)\}$ now known as the Chebyshev-Hermite polynomials. Chevyshev proposed the following expansion

$$p(x) \sim \frac{1}{\sqrt{2\Pi}} \sum_{k=0}^{\infty} C_k \frac{d^k}{dx^k} (e^{-\frac{1}{2}x^2}) = \frac{1}{\sqrt{2\Pi}} \sum_{k=0}^{\infty} (-1)^k C_k e^{-\frac{1}{2}x^2} H_k(x)$$

where $C_k = (-1)^k \int_{\infty}^{\infty} H_k(x) p(x) dx$ (the Fourier coefficients)

and $H_k(x) = (-1)^k e^{\frac{1}{2}x^2} \frac{d^k}{dx^k} e^{-\frac{1}{2}x^2}$, $H_0(x) \equiv 1$.

Since the first term is $\frac{1}{\sqrt{2\Pi}} C_o e^{-\frac{1}{2}x^2}$ this suggested an expansion for an arbitrary probability density (with all moments existing) $p_X(x)$ of a rv X with $EX = 0$ and $EX^2 = 1$. We note that C_k can be written as a linear combination of the first k moments of X, i.e.

$$C_k = \sum_{i=0}^{k} \nu_i \alpha_i, \qquad \alpha_i = EX^2, \quad \nu_i \text{ constants.}$$

It is not difficult to show that in our case $C_o = 1$, $C_1 = 0$ and $C_2 = 0$ giving

$$p_X(x) \sim \frac{1}{\sqrt{2\Pi}} \; e^{-\frac{1}{2}x^2} \left\{1 - C_3 H_3(x) + C_4 H_4(x) + \ldots\right\}$$

Integrating the above expansion, we obtain an expansion for an arbitrary distribution function, all of whose moments exist, namely

$$F_X(x) \sim \Phi(x) + C_3 \Phi^{(3)}(x) + C_4 \Phi^{(4)}(x) + \ldots$$

where

$$\Phi^{(k)}(x) = \frac{(-1)^{k-1}}{\sqrt{2\Pi}} \; H_{k-1}(x) \; e^{-\frac{1}{2}x^2} = \int_{-\infty}^{x} (-1)^k e^{-\frac{1}{2}u^2} \; H_k(u) du$$

We are now very close to the desired expansion. If we consider the rv

$$Y_n = n^{-\frac{1}{2}}(X_1 + X_2 + \ldots + X_n)$$

where the $\{X_i\}$ are iid with $EX_i = 0$ and $EX_i^{\,2} = 1$, we have $EY_n = 0$, $EY_n^{\,2} = 1$ and

$$F_n(x) = F_{Y_n}(x) \sim \Phi(x) + C_{3,n} \Phi^{(3)}(x) + C_{4,n} \Phi^{(4)}(x) + \ldots \tag{1}$$

where the $C_{j,n}$ are linear combinations of the first j moments of Y_n. Remembering that $EX_i = 0$, it is not difficult to show that

$$EY_n^{\,j+2} = \sum_{i=0}^{j} \theta_{i,j} \; n^{-\frac{1}{2}i} \quad \text{where the } \theta_{i,j} \text{ are independent of n.}$$

We can now collect together terms in (1) having the same order of n, which leads to the so-called 'Chebyshev Expansion'

$$F_n(x) \sim \Phi(x) + \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\Pi}} \left\{\frac{Q_1(x)}{n^{\frac{1}{2}}} + \frac{Q_2(x)}{n} + \ldots\right\}$$

where $Q_j(x)$ is a polynomial of degree 3j-1, independent of n and defined in terms of the first $j + 2$ moments of X_i (a precise definition of $Q_j(x)$ is given in Chapter 2).

At this point it is perhaps instructive to note that there is a more direct approach to this expansion. Indeed, without intending this to be a formal derivation, let us assume that the sequence of iid rv's $\{X_i\}$ has finite k^{th} moment. From Taylor's theorem we have for $|t|$ small,

$$Ee^{itX_i} = f(t) = \exp\left\{-\tfrac{1}{2}t^2 + \sum_{s=3}^{k-1} \frac{(it)^s}{s!}\kappa_s + 0\left(|t|^k\right)\right\}$$

Furthermore,

$$Ee^{itY_n} = \left\{f\left(\frac{t}{\sqrt{n}}\right)\right\}^n = \exp\left\{-\tfrac{1}{2}t^2 + \sum_{s=3}^{k-1}\frac{(it)^s}{s!}\frac{\kappa_s}{n^{\frac{1}{2}(s-2)}} + 0\left(\frac{|t|^k}{n^{\frac{1}{2}(k-2)}}\right)\right\}$$

$$= \exp -\tfrac{1}{2}t^2\cdot\exp\left\{\sum_{s=1}^{k-3}\frac{(it)^{s+2}}{(s+2)!}\cdot\kappa_{s+2}\, n^{-\frac{1}{2}s} + 0\left(\frac{|t|^k}{n^{\frac{1}{2}(k-2)}}\right)\right\}$$

By expanding the second exponential term and collecting like powers of n we can write

$$f_n(t) = e^{-\frac{1}{2}t^2}\left(1 + \sum_{s=1}^{k-3} P_s(it)\frac{1}{n^{\frac{1}{2}s}} + m_{k,n}(t)\right)$$

where the $P_s(it)$ are polynomials of degree $3s$ in (it), depending on the first $s+2$ cumulants of X; or, equivalently, on the first $s+2$ moments of X_i. If we note

$$f_n(t) = \int e^{itx}dF_n(x)$$

$$e^{-\frac{1}{2}t^2} = \int e^{itx}d\Phi(x)$$

and

$$e^{-\frac{1}{2}t^2}(it)^j = (-1)^j\int e^{itx}\,d\Phi^{(j)}(x)$$

where $\Phi^{(j)}(x) = \frac{(-1)^{j-1}}{\sqrt{2\Pi}}H_{j-1}(x)e^{-\frac{1}{2}x^2}$,

and thus

$$P_s(it) = \sum_{j=1}^{3s}\alpha_{s,j}(-1)^j\int e^{itx}d\Phi^{(j)}(x),$$

we can immediately extract the following representation:

$$F_n(x) \sim \Phi(x) + \sum_{s=1}^{k-3} \left\{ \sum_{j=1}^{3s} \alpha_{s,j} (-1)^j \, \Phi^{(j)}(x) \right\} n^{-\frac{1}{2}s} + M_{k,n}(x)$$

$$\sim \Phi(x) + \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}x^2} \sum_{s=1}^{k-3} \left\{ \sum_{j=1}^{3s} \alpha_{s,j} (-1) H_{j-1}(x) \right\} n^{-\frac{1}{2}s} + M_{k,n}(x)$$

$$\sim \Phi(x) + \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}x^2} \sum_{s=1}^{k-3} n^{-\frac{1}{2}s} Q_s(x) + M_{k,n}(x)$$

This representation being the first k-2 terms of the Chebyshev series.

1.3. RATE RESULTS FOR THE CHEBYSHEV EXPANSION.

The first convergence rate result for the Chebyshev approximation to $F_n(x)$ was obtained by Cramér (1928 [7]) who showed that if the k+2nd absolute moment ($k \geqq 1$) exists and if condition (C) is satisfied then

$$\sup_x |R_{k-1,n}(x)| \leqq n^{-\frac{1}{2}k} M$$

where

$$R_{k-1,n}(x) = F_n(x) - \Phi(x) - \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}x^2} \sum_{s=1}^{k-1} n^{-\frac{1}{2}s} Q_s(x)$$

and M depending only on k and the distribution function of X_i.

Taking k=1 we obtain essentially the earlier mentioned result of Cramér for $|F_n(x) - \Phi(x)|$. Comparing this with Ibragimov's necessary and sufficient conditions for $|F_n(x) - \Phi(x)|$ to be $O(n^{-\frac{1}{2}})$ suggests that Cramér's Chebyshev expansion rate result could be obtained under more relaxed conditions. It is also possible that the new conditions would be necessary. At this point we shall examine precisely what is meant here by necessary conditions and we will show that this leads to a more general consideration of the problem.

In the present framework, since $Q_j(x)$ is a function of the first j+2 moments, we must at the very least assume the existence of $\lim_{z \to \infty} \int_{-2}^{2} x^{k+2} dF(x)$ as well as of the first k+1 moments in order to define all the $Q_j(x)$'s ($j=1, \ldots, k$) appearing in $R_{kn}(x)$. The problem would then read: assuming $\lim_{z \to \infty} \int_{-z}^{z} x^{k+2} dF(x)$ and $E|X|^j$, $j = 3,4,\ldots,k+1$ exist, find necessary and sufficient conditions for $R_{kn}(x)$ to decrease at a specified rate.

Suppose, however, instead of defining the $Q_j(x)$'s in terms of the moments $\mu_1, \mu_2, \ldots, \mu_{j+2}$, of X_i, we prescribe an artibrary sequence of real numbers $\alpha_1 = 0, \alpha_2 = 1, \alpha_3, \alpha_4, \ldots \alpha_{k+2}$ and on the basis of it we form polynomials $Q_j(x)$ in such a way that their coefficients are expressed in terms of $\alpha_3, \alpha_4, \ldots, \alpha_{k+2}$ in the same way as the classical polynomials $Q_j(x)$ are expressed in terms of $\mu_3, \mu_4, \ldots, \mu_{j+2}$. Now set

$$G_{kn}(x) = \Phi(x) + \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\Pi}} \sum_{s=1}^{k} Q_s(x)n^{-\frac{1}{2}s} .$$

This construction, which henceforth we shall refer to as the 'Ibragimov formulation', raises the following question: if $\sup_x|F_n(x) - G_{kn}(x)|$ decreases at a certain rate, can we make definite statements about the existence of the moments of X_i and as to how these moments relate to the numerical sequence $0, 1, \alpha_3, \alpha_4, \ldots, \alpha_{k+2}$? This problem is considerably more general than the one stated at the end of the previous paragraph. The advantage here, however, is that no moments higher than the second are required to exist in order to formulate the question. Sufficient conditions for a particular convergence rate for $\sup_x|F_n(x) - G_{k,n}(x)|$ can be readily found by requiring $\alpha_j = EX^j$ for j+1, 2, ..., k+2 and thereby reducing the problem to finding sufficient conditions for convergence rates for $\sup_x|R_{kn}(x)|$. It is now natural to ask if these sufficient conditions are also necessary conditions.

The solution to this enlarged problem was found by Ibragimov (1967 [23]) whose results we shall quote in some detail since frequent reference will be made to them in later chapters. The first theorem gives a small order result while the second gives a large order result.

THEOREM 1.2 (Ibragimov). In order that $\sup_x |R_{k,n}(x)| = o(n^{-\frac{1}{2}k})$ *for integer* $k \geqq 1$, *it is necessary and, for distributions satisfying condition* (C), *also sufficient that*

1) $E|X_i|^{k+2} < \infty$ and $\alpha_j = EX_i^j$, $j = 1, 2, \dots, k+2$,

2) $$\int_{|u|>z} |u|^{k+1}\, dF(u) = o(z^{-1}) \quad \text{for } z \to \infty,$$

and 3) $$\lim_{z\to\infty} \int_{-z}^{z} u^{k+2}\, dF(u) = \alpha_{k+2} .$$

THEOREM 1.3 (Ibragimov). In order that $\sup_x |R_{k,n}(x)| = O(n^{-\frac{1}{2}(k+\delta)})$ *for integer* $k \geqq 1$, *it is necessary and, for distributions satisfying conditions* (C), *also sufficient that*

1) $E|X_i|^{k+2} < \infty$ and $\alpha_j = EX_i^j$, $j = 1, 2, \dots, k+2$,

2) $$\int_{|u|>z} |u|^{k+2}\, dF(u) = O(z^{-\delta}), \quad 0 < \delta \leqq 1$$

and 3) $$\lim_{z\to\infty} \int_{|u|<z} u^{k+3}\, dF(u) = O(1) \textit{ when } \delta = 1.$$

Theorem 1.3 generalises *Theorem 1.1*, although the latter does require Cramer's condition (C). To examine condition 2) of these theorems we require the following lemma:

Lemma 1.4 For any rv X,

$$E|X|^r < \infty \iff \sum_{n=1}^{\infty} n^{\frac{1}{2}(r-2)} P\left(|X| > \sqrt{n}\right) < \infty, \qquad r > 0,$$

The proof is elementary and depends on the fact that

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}(r-2)} P\left(|X| > \sqrt{n}\right) < \infty \iff \sum_{k=1}^{\infty} k^{\frac{1}{2}r} P\left(\sqrt{k} < |X| < \sqrt{k+1}\right) < \infty$$

If now 2) of Theorem 1.2 holds, then

$$P\left(|X| > \sqrt{n}\right) \leqq n^{-\frac{1}{2}(k+1)} \int_{|u|>\sqrt{n}} |u|^{k+1}\, dF(u) = o\left(n^{-\frac{1}{2}(k+2)}\right)$$

whence

$$\sum n^{\frac{1}{2}(k-1+\delta')} P\{|X| > \sqrt{n}\} \leqq C\sum n^{-1-\frac{1}{2}(1-\delta')} < \infty$$

for $0 < \delta' < 1$. However, from the lemma,

$$E|X|^{k+1+\delta'} < \infty \iff \sum n^{\frac{1}{2}(k-1+\delta')} P\{|X| > \sqrt{n}\} < \infty$$

thus, condition 2) of Theorem 1.2 $\Rightarrow E|X|^{k+1+\delta'} < \infty, \quad 0 < \delta' < 1$. Similarly, if condition 2) of Theorem 1.3 holds, then $E|X|^{k+2+\delta''} < \infty$ for all $\delta'' < \delta$. So condition 2) of these theorems is not as strict as the pure moment condition $E|X|^{k+2} < \infty$ (Theorem 1.2) or $E|X|^{k+2+\delta} < \infty$ (Theorem 1.3), but is stricter than $E|X|^{k+1+\delta'} < \infty$, $0 < \delta' < 1$ (Theorem 1.2) or $E|X|^{k+2+\delta''}$ $0 < \delta'' < \delta$ (Theorem 1.3).

Finally we examine the role of Cramer's condition: $\lim \sup_{|t|>\alpha} |f(t)| < 1$ for $\alpha > 0$. Characteristic functions can be divided into three disjoint classes: those satisfying (C), those of lattice rv's and those neither of lattice rv's nor satisfying (C). Results of a similar nature to Theorem 1.2 have been obtained for lattice rv's by Osipov (see Petrov [32] p.211). As $F_n(x)$ for a lattice rv has, in general, jumps of order $\frac{1}{\sqrt{n}}$, a discrete component in its corresponding asymptotic expansion had to be included. The construction of this discrete component was facilitated by the regular positioning of the saltus points of $F_n(x)$. When we come to consider the last class of characteristic functions - those neither of lattice rv's nor satisfying condition (C) - we find we are dealing with (non-lattice) rv's whose distribution function F(x) has all of its variation concentrated on a set of measure zero (Cramer [8]). Construction of an expansion for $F_n(x)$ in this case would need to include a function having saltus points co-incident with those of $F_n(x)$. This would of necessity be a very cumbersome expression to cover the general case. Thus, whilst particular cases could be analysed, the general situation is very complex. In conclusion, we give an example from Gnedenko and Kolmogorov [11] p.222 demonstrating that condition (C) cannot be removed from Theorems 1.2 and 1.3.

Let ξ_k take only the values ± 1 and $\pm\sqrt{3}$, each with probability $\frac{1}{4}$, then its distribution is non-lattice and its characteristic function is

$$f(t) = \tfrac{1}{2}(\cos t + \cos t\sqrt{3})$$

which does not satisfy (C). It can be shown that for even n, $F_n(x)$ has a jump at $x = 0$, asymptotically equal to $\frac{2}{\pi n}$. Although all moments of ξ_k are finite, we clearly cannot write the expansion

$$F_n(x) = \Phi(x) + \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\left\{\frac{Q_1(x)}{\sqrt{n}} + \frac{Q_2(x)}{n}\right\} + o(\frac{1}{n})$$

and hence condition (C) cannot be removed from Theorem 1.2 and 1.3.

1.4. RESULTS CONCERNING INDEPENDENT rv's.

From Theorem 1.3 we note that conditions 1), 2) and 3) of that theorem imply $\sum n^{-1+\frac{1}{2}(k+\delta')} \sup_x |R_{kn}(x)| < \infty$ for all δ' such that $0 < \delta' < \delta$. In Chapter 2 we find necessary and sufficient conditions for the convergence of this series. These conditions turn out to be 'pure' moment conditions as opposed to the 'tail' conditions on the moments required for the 'order n' convergence rates. Our results generalise series results for $|F_n(x) - \Phi(x)|$ obtained by Heyde .

Chapter 2A is fairly short and is intended as a supplement to Chapter 2. In it we show that Theorems 1.2 and 1.3 continue to hold if we replace the condition that $n = 1,2,3,\ldots$ by $n = n_1, n_2, \ldots$ where $n_{i+1} > n_i$ and $\frac{n_{i+1}}{n_i} \to C$ for $1 \leqq C < \infty$. We then use this result to show that[†]

$$\sum_{n=1}^{\infty} n^{-1+\frac{1}{2}k} \sup_x |R_{kn}(x)| < \infty \Rightarrow E|X|^{k+2} < \infty \text{ for } k \geqq 1.$$

We now move on to consider very precise bounds for $R_{kn}(x)$ which are interesting in as much as some of the results we have already discussed can very easily be derived from them. The case of $R_{on}(x) = F_n(x) - \Phi(x)$ was examined by Osipov and Petrov [31] from whose

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[†] This result closes a gap present in Chapter 2.

work we can derive the following estimate

$$\sup_x |F_n(x) - \Phi(x)| \leq nP\left[|X| > \tau_n\right] + K_o\left[n^{-\frac{1}{2}} \int_{|u|<\tau_n} u^3 dF(u)\right.$$

$$\left.\sqrt{n}\left|\int_{|u|\leq\tau_n} udF(n)\right| + \int_{|u|>\tau_n} u^2 dF(u)\right]$$

where K_o is an absolute constant and $\{\tau_n\}$ is a sequence of positive real numbers. Taking $\tau_n = \sqrt{n}$ and using Lemma 3.7 of Chapter 3, we can readily obtain the necessity part of Theorem 1.1 for $0 < \delta < 1$. Furthermore, the necessity part of Theorem 2.1 for $0 < \delta < 1$ can also be easily derived (see Heyde [17]).

It would seem likely that a similar estimate could be found for $R_{kn}(x)$, having the same desirable applications. Such bounds were indeed found by Osipov [30] who gave not only bounds uniform in x but also bounds dependent on x. (For some background to non-uniform (in x) convergence rates, the reader is referred to § 3.1).

These estimates, however, have some drawbacks; in particular the case of $\delta = 1$ in both Theorems 1.1 and 1.3 cannot be derived from then, due to the presence of $\int_{|u|<\sqrt{n}} |u|^{k+3} dF(u)$, $(k \geq 0)$.

In Chapter 3, we present a non-uniform estimate of $R_{kn}(x)$ (Theorem 3.3) and its uniform counter-part (Theorem 3.4) which overcome this problem and which lead on to estimates for $R_{kn}(x)$ defined according to the 'Ibragimov formulation'.

So far, $R_{kn}(x)$ has been defined in terms of the moments of X_i and so, finding bounds for it, involves first assuming $E|X_i|^{k+1} < \infty$ and $\lim_{z\to\infty} \int_{-z}^{z} u^{k+2} dF(u) = \alpha_{k+2}$ in order that it may be defined. On the other hand, a bound for $R^*_{kn}(x)$ (being $R_{kn}(x)$ defined according to the 'Ibragimov formulation') will avoid presupposing the existence of these moments. Before discussing this further, we mention a bound due to Osipov and Petrov [31] in which no moments whatever are assumed to exist. Osipov and Petrov give a bound for $\left|P\left\{C_n^{-1}\sum_1^n \xi_i - b_n \leq x\right\} - \Phi(x)\right|$ (see expression (1) of Chapter 3 for the non-uniform case) where $\{\xi_i\}$ is a sequence of independent rv's (none of whose moments, including the first, are assumed to exist), $\{C_n\}$ is a sequence of positive real numbers and $\{b_n\}$ a sequence of real numbers. Heyde [17] examined this bound taking $C_n = \sqrt{n}\,\sigma_n$ where $\sigma_n^2 = \int_{|u|<\sqrt{n}} u^2 dF(u) - \left\{\int_{|u|<\sqrt{n}} u dF(u)\right\}^2$ and $b_n \equiv 0$ with $\{\xi_i\}$ a sequence of iid rv's and ξ_i having expectation zero if $E|\xi_i| < \infty$. He was able to show that the bound itself decreased at a rate of order $n^{-\frac{1}{2}\delta}$, $0 < \delta < 1$ if and only if $\Delta_n(\sqrt{n}\,\sigma_n) = \sup_x |P(n^{-\frac{1}{2}}\sigma_n^{-1}\sum_1^n \xi_i \leq x) - \Phi(x)| = O(n^{-\frac{1}{2}\delta})$. A similar series type convergence rate equivalence was also established and so in these two important cases, $\Delta_n(\sqrt{n}\sigma_n)$ and its bound have equivalent asymptotic behaviour - a most desirable property.

We note, however, that again it was not possible to cover the case of $\delta = 1$ for the order n rate. To rectify this, we give a new uniform bound for $|P(C_n^{-1}\sum_1^n \xi_i - b_n \leq x) - \Phi(x)|$ (expression (2b) of Chapter 3) which allows us to include the $\delta = 1$ case.

Returning now to the question of bounds for $R^*_{kn}(x)$, we go one step further and find bounds for a generalisation of the Osipov-Petrov expression, namely for

$$|P(C_n^{-1}\sum_1^n \xi_i - b_n \leq x) - \Phi(x) - \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\sum_1^k Q_\nu(x)n^{-\nu/2}|$$

the $Q_j(x)$'s being defined in terms of some arbitrary 'moment' sequence $(0, 1, \alpha_3, \alpha_4, \ldots, \alpha_{k+2})$ and $\{\xi_i\}$ being a sequence of iid rv's. It is natural now to ask if this bound (in the uniform case) has the same kind of asymptotic properties as of the bound for $\Delta_n(C_n)$. We find (Theorem 3.6) that for 'order n' convergence rates, providing we assume Cramer's conditions (C), asymptotic equivalence is preserved.

1.5. DEPENDENT SEQUENCES.

The remaining two chapters (4 and 5) of the thesis are devoted to obtaining certain limit results for two kinds of branching processes. Although in both chapters, convergence rates play a vital role, it is the limit results which are most interesting. The process we first deal with is the Galton-Watson process and the second is its continuous time analogue: the time homogeneous Markov branching process.

The Galton-Watson process can be regarded as representing the numbers of individuals in a population at successive generations. At the end of the lifetime of an individual from this population, a random number, ξ , of offspring are produced with distribution

$$P(\xi = k) = a_k, \quad k = 0, 1, 2, \ldots$$

where $a_k \geqq 0$ and $\sum_1^\infty a_k = 1$. All the offspring act independently of each other and at the end of their lifetime (the lifetime of all individuals being the same) each have offspring in accordance with the above distribution. Taking $Z_0 = 1$, Z_n is then the size of the population at the nth generation arising from one individual. In fact, from now on we take $Z_0 = 1$.

If the size of the nth generation is known, the probability law governing later generations does not depend on the sizes of generations preceding the nth and hence $Z_0, Z_1, \ldots$ forms a Markov chain. Furthermore, in view of the assumption that different individuals reproduce independently, if we are given that $Z_n = k$, Z_{n+1} is distributed as the sum of k iid rv's, each distributed like Z_1 (with $Z_0 = 1$). It is this latter property of the Galton-Watson process that provides the key to the results of Chapters 4 and 5. It allows us to overcome the problems imposed by dependence.

It is well known (see Harris [12] Chapter 1) that if $EZ_1 = m$, then $EZ_n = m^n$ and that the behaviour of Z_n as $n \to \infty$ depends on whether m is $\leqq 1$ or > 1. In particular, if $m \leqq 1$, then $Z_n \to 0$ a.s. and if $m > 1$, $Z_n \to 0$ or ∞ with probability q and 1-q respectively where q is the unique non-negative solution (less than 1) of the equation,

$$s = f(s), \quad f(s) = \sum_0^\infty a_k s^k \quad (a_k \text{ as defined earlier}).$$

Further, if $m > 1$ and $EZ_1^2 < \infty$, $W_n = m^{-n} Z_n$ converges a.s. to a non-degenerate rv W having $E\,W = 1$ and $\text{Var } W = (m^2 - m)^{-1} \cdot \text{Var } Z_1 > 0$.

The central limit theorem and the law of the iterated logarithm may be regarded as convergence rate results for the strong law of large numbers (see §4.1). Heyde[15] ,[16] adopted this interpretation with regard to the relation $W_n \to W$ a.s. and proved analogous convergence rate theorems for this strong convergence result. Indeed he showed that, conditional on $Z_n > 0^\dagger$,

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W - W_n) \quad \text{and}$$

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} m^{-\frac{1}{2}j} (m^j - 1)^{-\frac{1}{2}} Z_n^{-\frac{1}{2}} (Z_{n+j} - m^j Z_n), \qquad (\text{fixed } j)$$

are both asymptotically normal $N(0,1)$.

Furthermore, under the additional restriction, $EZ_1^3 < \infty$, a convergence rate for this central limit analogue was provided and using that rate result, the following analogue of the law of the iterated logarithm was established: if $EZ_1^3 < \infty$, conditional on $Z_n > 0^\dagger$,

$$\limsup_{n\to\infty} \frac{Z_{n+r} - m^r Z_n}{(2\sigma_r^2 Z_n \log n)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

$$\limsup_{n\to\infty} \frac{m^n W - Z_n}{\left(2\sigma^2 (m^2 - m)^{-1} Z_n \log n\right)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

with corresponding lim inf results.

These latter results were obtained using the Berry-Esseen bound requiring the imposition of the condition $EZ_1^3 < \infty$. The aim of Chapter 4 is to show how the condition $EZ_1^3 < \infty$ can be replaced by the basic condition $EZ_1^2 < \infty$.

We proceed now to define the Galton-Watson process with immigration for which equivalent results to those mentioned for the standard Galton-Watson process are valid. Let $\{X_n\}$, $X_0 = 1$ be a branching process in which individuals evolve as in a Galton-Watson process and which is subject to an independent immigration component at each generation. Thus, for $n \geq 1$

$^\dagger \sigma^2 = \text{Var } Z_1$, $\sigma_r^2 = \text{Var } Z_r$.

$$X_n = Z_n + U_n^{(1)} + U_n^{(2)} + \ldots + U_n^{(n)}$$

where Z_n is the number of direct descendants of the initial individual and $U_n^{(1)}$, $i = 1, 2, 3, \ldots n$ is the number of descendants of the nth generation from the immigration at the ith.[†] We note that, according to our assumptions, all the foregoing component variables of X_n are independent. Moreover, $\{Z_n\}$, $n \geqq 1$ is an ordinary Galton-Watson process (initiated by a single ancestor). Taking

$$P(\text{Number of offspring of an individual} = k) = a_k$$

and

$$P(\text{Number of new immigrants at any generation} = k) = b_k$$

then from Seneta [34] we have that if $1 < m \equiv \sum_{k=0}^{\infty} ka_k < \infty$, $0 < \lambda \equiv \sum_{k=0}^{\infty} kb_k < \infty$ and $\sum j \log(j)\, a_j < \infty$, $V_n \equiv X_n m^{-n}$ converges a.s. to a proper rv V with finite mean EV and such that $P(V = 0) = 0$.

We now have a setting similar to that of the ordinary Galton-Watson process for obtaining analogues of the central limit theorem associated convergence rates and analogues of the law of the iterated logarithm.

Heyde and Seneta [21] under the assumption that $EZ_1^3 < \infty$ paralleled the results for the ordinary Galton-Watson process quoted above. A further aim of Chapter 4, therefore, is to remove the restriction $EZ_1^3 < \infty$ replacing it with the basic condition $EZ_1^2 < \infty$ as was done for the ordinary Galton-Watson process.

The other branching process with which we shall be concerned is the time (temporally) homogeneous Markov branching process. This process differs from the Galton-Watson process in that no longer is reproduction constrained to occur at fixed discrete times - reproduction can occur continuously in time. We can regard this continuous time process $\{X(t), t \geqq 0\}$ as the total number of individuals at time t in a system where we start with $X(0) = 1$ individuals at $t = 0$. Each individual lives an exponentially distributed length of time with

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[†] Once an immigrant is in the population, its behaviour is indistinguishable from non-immigrant members.

mean a^{-1}, $0 < a < \infty$, and on death splits into a random number of new individuals whose probability generating function we shall denote by $h(z)$. All individuals behave independently of each other and identically. The probability of an individual of age τ dying in the age interval $(\tau, \tau+ d\tau)$ is independent of τ.

Like the Galton-Watson process, the asymptotic behaviour of $X(t)$, $t \to \infty$, depends on whether $h'(1) \leqq 1$ or > 1. If $h'(i) \leqq 1$ $X(t) \to 0$ a.s. and if $h'(1) > 1$ and $h''(1) < \infty$, $X(t)e^{-\lambda t} \to W$ a.s. where W is a non-degenerate rv with $EW = 1$ and $\text{Var } W < \infty$, and $\lambda = a(h'(1) - 1)$ (see Harris [12] Chapter 5). Once more we have the setting for analogues of the central limit theorem and the law of the iterated logarithm. In Chapter 5 such limit results are derived for $\{X(t)e^{-\lambda t} - W\}$ and $\{X(t+r)e^{-\lambda(t+r)} - e^{-\lambda t}X(t)\}$, $r > 0$ under the basic condition $h''(1) < \infty$ or equivalently $EX(1)^2 < \infty$.

From the homogeneous Markov branching process a new increasing process $\{N(t),\ t \geqq 0\}$ can be derived which is not present (except in a trivial sense) in the Galton-Watson process. We define $N(t)$ to be the total number of discontinuities in $X(s)$ for $s \leqq t$. If τ_n is the time of occurrence of the n^{th} change of state (split) of $X(t)$ and if further we rule out the possibility of $X(t)$ replicating itself at a split by taking $P\{X(\tau_n + 0) - X(\tau_n - 0) = 0\} = 0$, then $N(t) = n \Longleftrightarrow \tau_n \leqq t < \tau_{n+1}$. Athreya and Karlin [1] showed that if $\mu = h'(1) > 1$ and $h''(1) < \infty$ then $\lim_{t\to\infty} N(t)\, \mu e^{-\lambda t} = W$ a.s. where $W = \lim_{t\to\infty} X(t)e^{-\lambda t}$ a.s. Moreover, central limit and iterated logarithm analogues were obtained for $\{X(t) - \mu N(t)\}$. We show in Chapter 5 that similar limit results hold also for $\mu N(t)e^{-\lambda t} - W$, $N(t+r)e^{-\lambda(t+2)} - N(t)e^{-\lambda t}$, and $X(t+r) - e^{\lambda r}\mu N(t)$.

The key to these results lies in the presence of an imbedded Galton-Watson process. In fact Harris,(Chapter 5 [12]) $X(n\Delta)$, for $n = 0, 1, 2,$ and Δ any fixed positive real number, forms a supercritical (if $h'(1) > 1$) Galton-Watson process. Hence most of the techniques used in Chapter 4 can be successfully applied in this context.

CHAPTER 2

A UNIFORM CONVERGENCE RATE RESULT

2.1. INTRODUCTION: Let X_i, $i = 1,2,3,\ldots$ be a sequence of iid rv's with $EX_i = 0$ and $EX_i^2 = 1$. Write $F(x)$ for the distribution function and $f(t)$ for the characteristic function of X_i, put $S_n = \sum_{i=1}^{n} X_i$ and set

$$F_n(x) = P\{S_n \leqslant x\sqrt{n}\}.$$

In this chapter we shall obtain a convergence rate result, uniform in x, for a portion of the Chebyshev series expansion of $F_n(x)$. This work corresponds to the Ibragimov [23] results† and is a generalisation of a result of Spitzer [35] namely, that under the assumption $E|X_i|^2 < \infty$, we have

$$|\sum n^{-1}\{P(S_n < 0) - \tfrac{1}{2}\}| < \infty .$$

Spitzer's series was shown to be also absolutely convergent by Rosen [33]. Baum & Katz [2] generalised this by showing that if $E|X_i|^{2+\delta} < \infty$ for $0 \leqslant \delta < 1$, then

$$\sum n^{-1+\delta/2}|P(S_n < 0) - \tfrac{1}{2}| < \infty .$$

Noting that this series is a particular case of the series

$$\sum n^{-1+\delta/2}|P\{S_n \leqslant x\sqrt{n}\} - \Phi(x)|$$

the problem arises as to whether this latter generalisation continues to converge uniformly in x under the Baum & Katz conditions. Heyde [13] completely solved this question with the following theorem:

THEOREM 2.1. (Heyde [13]) *Let $\{X_i\}$ be a sequence of iid rv's with $EX_i = 0$ and $EX_i^2 = 1$. Then*

$$\sum n^{-1+\delta/2} \sup_x |F_n(x) - \Phi(x)| < \infty \quad \text{for} \quad 0 \leqslant \delta < 1$$

iff a) $E|X_i|^{2+\delta} < \infty$ for $0 < \delta < 1$, and

b) $E|X_i|^2 \log(1 + |X_i|) < \infty$ for $\delta = 0$.

From this theorem we see that the Baum & Katz conditions were necessary and sufficient for $0 < \delta < 1$, however, when $\delta = 0$, the stronger

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† already referred to in Chapter 1.

condition $E|X_i|^2 \log(1 + |X_i|) < \infty$ was required.

Comparing these results with those of Ibragimov (Theorem 1.2) where necessary and sufficient conditions are given for $\Delta_n = \sup_x |F_n(x) - \Phi(x)|$ to be $O(n^{-\delta/2})$, $0 < \delta \leqslant 1$, it is striking that series conditions on the Δ_n lead to 'pure' moment conditions, whereas conditions on each Δ_n lead to 'tail conditions' on the moments.

We now consider a generalisation of Theorem 2.1 in which we obtain a corresponding series rate of convergence result for

$$A_{kn} = \sup_x |R_{kn}(x)| = \sup_x |F_n(x) - G_{kn}(x)|$$

where

$$G_{kn}(x) = \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \sum_{s=1}^{k} Q_s(x)\, n^{-s/2},$$

is a given portion of the Chebyshev series defined according to the Ibragimov formulation described in Chapter 1. That is, so as to avoid presupposing the existence of moments of higher order than the second we prescribe an arbitrary numerical sequence $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3, \dots \gamma_{k+2}$. On the basis of this sequence we form polynomials $Q_s(x)$ in such a way that their coefficients are expressed in terms of $\gamma_1, \dots, \gamma_{s+2}$ in the same way as the coefficients of the classical polynomials $Q_s(x)$ are expressed in terms of the cumulants $\kappa_1, \dots, \kappa_{s+2}$ of X_i. Indeed

$$Q_s(x) = -\sum \frac{1}{m_1! \dots m_s!} \left(\frac{\gamma_3}{3!}\right)^{m_1} \dots \left(\frac{\gamma_{s+2}}{(s+2)!}\right)^{m_s} H_{3m_1+\dots+(s+2)m_s-1}(x),$$

where the summation is over all non-negative solutions of $m_1+2m_2+\dots+sm_s = s$ and $H_r(x)$ is the Hermite-Chebyshev polynomial

$$H_r(x) = (-1)^r e^{\frac{1}{2}x^2} \frac{d^r}{dx^r} e^{-\frac{1}{2}x^2}$$

(Petrov [32]). Throughout this chapter the $Q_s(x)$ will be interpreted in this way.

Let $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3, \alpha_4, \dots$ be the 'moment' sequence corresponding to the 'cumulant' sequence $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3, \gamma_4, \dots, \gamma_{k+2}$. We establish the following results.

THEOREM 2.2. In order that

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) - G_{kn}(x)| < \infty \tag{1}$$

where k *is a non-negative integer and* $0 < \delta < 1$, *it is necessary and for* $k = 0$ *or for distributions satisfying* (C) *also sufficient that*

$$E|X_i|^{k+2+\delta} < \infty \quad \text{and} \quad \alpha_j = EX_i^j, \quad j = 1,2,\ldots,k+2. \tag{2}$$

THEOREM 2.3. In order that the relation (1) *hold, where* $0 < \delta < 1$, *it is necessary and for distributions satisfying* (C) *or for* $k = 0$ *also sufficient that*

$$f(t) = \exp\left\{\sum_{s=2}^{k+2} \frac{(it)^s}{s!}\gamma_s + |t|^{k+2}\gamma(t)\right\} \tag{3}$$

where for $A > 0$,

$$\int_0^A \frac{|\gamma(t)|}{t^{1+\delta}}\,dt < \infty \quad . \tag{4}$$

Unfortunately it has not been possible to treat the case $\delta = 0$ in general and then not without certain presuppositions on the existence of moments.[†] In this case we find the following result.

THEOREM 2.4. Suppose $E|X_i|^{k+2} < \infty$ *and* $\alpha_j = EX_i^j$, $j = 1,2,\ldots,k+2$, *where* k *is a non-negative even integer or a non-negative odd integer with* $F(x)$ *such that* $P\{X_i < -B\} = 0$ *for some finite positive constant* B. *Then for* (1) *to hold with* $\delta = 0$, *it is necessary and for* $k = 0$ *or for distributions satisfying* (C) *also sufficient that* $E|X_i|^{k+2}\log(1 + |X_i|) < \infty$.

Remark a) These results appeared in a joint paper of Dr. C. C. Heyde and the author [20] Some time after this paper had been submitted for publication, an article of F. N. Galstian appeared in T.P.A. (1971, 16, No.3, 528-533) giving similar necessary and sufficient conditions to those above but only for symmetric rv's in the classical setting.

Remark b) For the case k odd in Theorem 2.4., it was necessary to consider only 'one-sided' distribution functions since we needed to

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[†] In Chapter 2A this anomaly is removed.

work with $\mathrm{Im}\,\gamma(t)$. This function gives us no information in the general case since it can then vanish identically.

2.2. PRELIMINARY LEMMAS.

LEMMA 2.5. Suppose $E|X_i|^r < \infty$ *for some integer* $r \geq 2$. *Then f(t) is representable in the form*

$$f(t) = \exp\left\{ \sum_{s=2}^{r} \frac{(it)^s}{s!} \kappa_s + |t|^r \gamma(t) \right\} \tag{5}$$

where $\gamma(t) = o(1)$ *as* $t \to 0$ *and* κ_s *denotes the* s-th *cumulant of* X_i. *Furthermore, there exists an* $\varepsilon > 0$ *such that for* $0 < t < \varepsilon$, $|\gamma(t)| > 0$ *or* $|\gamma(t)| \equiv 0$. *If* $|\gamma(t)| \equiv 0$ *for* $0 < t < \varepsilon$, X_i *has a normal distribution.*

Proof. The representation in the form (5) with $\gamma(t) = o(1)$ as $t \to 0$ follows simply from a Taylor expansion of $\log f(t)$ (e.g.[11], p.64).

Next, suppose $\gamma(t) = 0$ for all $t \in \{t_k\}$ where $\{t_k\}$ is a sequence of non-zero real numbers converging to zero. Then,

$$\exp\left\{ \sum_{s=2}^{r} \frac{(it)^s}{s!} \kappa_s + |t|^r \gamma(t) \right\} = \exp\left\{ \sum_{s=2}^{r} \frac{(it)^s}{s!} \kappa_s \right\} \tag{6}$$

for all $t \in \{t_k\}$ and applying Theorem 4.2.1 of Linnik [26], which in this case states that if $\phi(t)$ is regular for $|t| < R$, $(R > 0)$, has no zeros in this circle and has the Hermitian property $\phi(-t) = \overline{\phi(t)}$ and if $\phi_1(t)$ is a characteristic function such that

$$\phi_1(t) = \phi(t) \tag{6a}$$

holds for $t = t_k (k=1,2,\ldots)$ where $\{t_k\}$ is a sequence of non-zero real numbers converging to 0, then (6a) is valid for $|t| < R$. Setting $\phi_1(t)$ to be the L.H.S. of equation (6) and $\phi(t)$ the R.H.S., we find (6) holds for all real t. However, this is impossible unless $r = 2$ since the left hand side of (6) represents a characteristic function and the right hand side does not, in view of Marcinkiewicz's Theorem (e.g. Lukacs [28], p.213), unless $r = 2$. Thus, if $r > 2$ we must be able to choose $\varepsilon > 0$ so that $\gamma(t)$ has no zeros in $(0,\epsilon)$. If $r = 2$, on the other hand, either (6) holds for all t in which case $\gamma(t) \equiv 0$ and X_i has a normal distribution or zero is not a limit point of a sequence of zeros of $\gamma(t)$ and hence we can choose an interval $(0,\varepsilon)$ containing no zeros of $\gamma(t)$.

LEMMA 2.6. Suppose $E|X_i|^r < \infty$ *for some integer* $r \geqq 2$. *Then, f(t) is representable in the form*

$$f(t) = \sum_{s=0}^{r} \frac{(it)^s}{s!} \mu_s + |t|^r \beta(t) \tag{7}$$

where $\beta(t) = o(1)$ *as* $t \to 0$ *and* μ_s *denotes the s-th moment of* X_i. *Furthermore, for any* $A > 0$ *and* $0 \leqq \delta < 1$, *the conditions* $\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty$ *and* $\int_0^A |\gamma(t)| t^{-(1+\delta)} dt < \infty$ *are equivalent,* $\gamma(t)$ *being given by* (5) *and these conditions are in turn equivalent to* $E|X_i|^{r+\delta} < \infty$ *if* $\delta > 0$, $E|X_i|^r \log(1 + |X_i|) < \infty$ *if* r *is even and* $\delta = 0$, *or* r *is odd,* $\delta = 0$ *and a finite* $B > 0$ *exists such that* $P\{X_i < -B\} = 0$.

Proof. Firstly, the representation of f(t) in the form (7) with $\beta(t) = o(1)$ as $t \to 0$ follows simply from a Taylor expansion of f(t). Also, from Lemma 3 of [23] we find that as $t \to 0$,

$$\log f(t) - \sum_{s=2}^{r} \frac{(it)^s}{s!} \kappa_s = f(t) - \sum_{s=0}^{r} \frac{(it)^s}{s!} \mu_s + \frac{\Lambda_{r+1}}{(r+1)!}(it)^{r+1} + O(t^{r+2}) ,$$

where Λ_{k+1} is a constant, so that

$$|t|^r \gamma(t) = |t|^r \beta(t) + \frac{\Lambda_{r+1}}{(r+1)!} (it)^{r+1} + o(t^{r+1}) \tag{8}$$

and we readily deduce the equivalence of

$$\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty \quad \text{and} \quad \int_0^A |\gamma(t)| t^{-(1+\delta)} dt < \infty, \quad 0 \leqq \delta < 1.$$

Now suppose that

$$\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty.$$

This implies

$$\int_0^A |\mathrm{Re}\beta(t)| t^{-(1+\delta)} dt < \infty$$

where Re denotes the real part and hence that

$$\left|\int_0^A \operatorname{Re}\beta(t)t^{-(1+\delta)}dt\right| < \infty.$$

It is with this last condition that we shall work. We have

$$\left|\int_0^A \frac{\operatorname{Re}\beta(t)}{t^{1+\delta}}\,dt\right| = \left|\int_0^A \frac{1}{t^{r+\delta+1}}\left(\int_{-\infty}^{\infty} \operatorname{Re}\left(e^{itx} - \sum_{s=0}^{r}\frac{(itx)^s}{s!}\right)dF(x)\right)dt\right|$$

$$= \left|\int_{-\infty}^{\infty} dF(x)\int_0^A \frac{\operatorname{Re}\left(e^{itx}-\sum_{s=0}^{r}\frac{(itx)^s}{s!}\right)}{t^{r+\delta+1}}\,dt\right| \qquad (9)$$

$$= \left|\int_{-\infty}^{\infty} dF(x)\int_0^A\frac{\left(\cos tx - \sum_{s=0}^{R}(-1)^s\frac{(tx)^{2s}}{(2s)!}\right)}{t^{r+\delta+1}}\,dt\right|$$

where $R = [r/2]$, the integer part of $r/2$. But, after two integrations by parts,

$$\int_0^A \frac{\left(\cos tx - \sum_{s=0}^{R}(-1)^s\frac{(tx)^{2s}}{(2s)!}\right)}{t^{r+\delta+1}}\,dt$$

$$= \frac{-\left(\cos Ax - \sum_{s=0}^{R}(-1)^s\frac{(Ax)^{2s}}{(2s)!}\right)}{(r+\delta)A^{r+\delta}} - \frac{x\left(\sum_{s=0}^{R-1}(-1)^s\frac{(Ax)^{2s+1}}{(2s+1)!} - \sin Ax\right)}{(r+\delta)(r+\delta-1)A^{r+\delta-1}}$$

$$+ \frac{x^2}{(r+\delta)(r+\delta-1)}\int_0^A\frac{\left(\sum_{s=0}^{R-1}(-1)^s\frac{(tx)^{2s}}{(2s)!} - \cos tx\right)}{t^{r+\delta-1}}\,dt,$$

so that, recalling that $E|X_i|^r < \infty$, we must have using (9),

$$\left|\int_{\infty}^{\infty} x^2 dF(x)\int_0^A\frac{\left(\cos tx - \sum_{s=0}^{R-1}(-1)^s\frac{(tx)^{2s}}{(2s)!}\right)}{t^{r+\delta-1}}\,dt\right| < \infty$$

Continuing this reduction, we find ultimately that

$$\left|\int_{-\infty}^{\infty} x^{2R}\,dF(x)\int_0^A\frac{(\cos tx-1)}{t^{r+\delta+1-2R}}\,dt\right| < \infty$$

which transforms to give

$$\int_{-\infty}^{\infty} |x|^{r+\delta}\left(\int_0^{A|x|}\frac{1-\cos u}{u^{r+\delta+1-2R}}\,du\right)dF(x) < \infty .$$

Thus, if r is even, R = r/2 and

$$\int_{-\infty}^{\infty} |x|^{r+\delta}\left(\int_0^{A|x|} \frac{1-\cos u}{u^{1+\delta}}\, du\right) dF(x) < \infty \tag{10}$$

while if r is odd, R = (r - 1)/2 and

$$\int_{-\infty}^{\infty} |x|^{r+\delta}\left(\int_0^{A|x|} \frac{1-\cos u}{u^{2+\delta}}\, du\right) dF(x) < \infty \tag{11}$$

(10) and (11) are clearly equivalent to $E|X_i|^{r+\delta} < \infty$ if $\delta > 0$. On the other hand, when $\delta = 0$, we have for $|x| > 1$

$$\int_0^{A|x|} \frac{1-\cos u}{u}\, du = \int_0^{A} \frac{1-\cos u}{u}\, du + \int_A^{A|x|} \frac{du}{u} - \int_A^{A|x|} \frac{\cos u}{u}\, du \sim \log|x|$$

as $|x| \to \infty$ so that (10) is equivalent to the condition $E|X_i|^r \log(1+|X_i|)<\infty$ when $\delta = 0$. We have thus shown that $\int_0^A |\beta(t)|t^{-(1+\delta)}\, dt < \infty$ implies $E|X_i|^{r+\delta} < \infty$ if $0 < \delta < 1$, $E|X_i|^r \log(1 + |X_i|) < \infty$ if r is even and $\delta = 0$.

Clearly, taking $\delta = 0$ in (11) only gives us $E|X_i|^r < \infty$ which we have already assumed. We now examine the case of r odd and $\delta = 0$ when there exists a constant $B > 0$ such that $P\{X_i < -B\} = 0$. Treating $\int |\mathrm{Im}\beta(t)|t^{-1}\, dt$ in exactly the same way as we did $\mathrm{Re}\beta(t)$ we obtain

$$\int_0^A \left|\frac{\beta(t)}{t}\right| dt < \infty \Rightarrow \left| \int_{-B}^{\infty} x^{r-1} dF(x) \int_0^A (\sin(tx) - tx)t^{-2} dt \right| < \infty .$$

Since $\left|\int_{-B}^{0} x^r dF(x) \int_0^{Ax} (\cos u - 1)u^{-1}\, du\right| < \infty$ we have

$$\left|\int_0^{\infty} x^r\, dF(x) \int_0^{Ax} (\cos u - 1)u^{-1}\, du\right| < \infty$$

This gives us $E|X_i|^r \log(1 + |X|) < \infty$.

Finally, suppose that $E|X_i|^{r+\delta} < \infty$, $\delta > 0$, or that $E|X_i|^r \log(1+|X_i|)<\infty$ if $\delta = 0$, Then,

$$\int_0^A \frac{|\beta(t)|}{t^{1+\delta}}\,dt = \int_0^A \left| f(t) - \sum_{s=0}^{r} \frac{(it)^s}{s!}\,\mu_s \right| t^{-(r+1+\delta)}\,dt$$

$$= \int_0^A \frac{1}{t^{r+1+\delta}} \left| \int_{-\infty}^{\infty} \{e^{itx} - \sum_{s=0}^{r} \frac{(itx)^s}{s!}\}dF(x) \right| dt$$

$$\leq \int_{\infty}^{\infty} \left(\int_0^A \left| \{e^{itx} - \sum_{s=0}^{r} \frac{(itx)^s}{s!}\} t^{-(r+1+\delta)} \right| dt \right) dF(x) \tag{12}$$

$$= \int_{-\infty}^{\infty} |x|^{r+\delta} \left(\int_0^{A|x|} \frac{\left| e^{iu} - \sum_{s=0}^{r} \frac{(iu)^s}{s!} \right|}{u^{r+1+\delta}}\,du \right) dF(x) .$$

But, using the inequalities

$$\left| e^{ix} - \sum_{s=0}^{r} \frac{(ix)^s}{s!} \right| \leqq \frac{|x|^{r+1}}{(r+1)!} \quad \text{for} \; |x| \leqq 1,$$

$$\left| e^{ix} - \sum_{s-2}^{r} \frac{(ix)^s}{s!} \right| \leqq (1 + e)|x|^r \quad \text{for} \; |x| \geqq 1,$$

(for the first of these see e.g. Lemma 1 of [23] while the second is obtained by taking the modulus of each of the terms and bounding this) we have

$$I(|x|) = \int_0^{A|x|} \frac{\left| e^{iu} - \sum_{s=0}^{r} \frac{(iu)^s}{s!} \right|}{u^{r+1+\delta}}\,du$$

$$\leqq \begin{cases} \dfrac{1}{(r+1)!} \displaystyle\int_0^1 \frac{du}{u^\delta} & \text{for } |x| \leqq A^{-1} \\[2ex] \dfrac{1}{(r+1)!} \displaystyle\int_0^1 \frac{du}{u^\delta} + (1 + e) \displaystyle\int_1^{A|x|} \frac{du}{u^{1+\delta}} & \text{for } |x| > A^{-1} . \end{cases}$$

Thus, $I(|x|) \leq c_1$ if $\delta > 0$, $I(|x|) \leq c_2 \log(|x| + 1)$ if $\delta = 0$, where c_1 and c_2 are positive constants and using these results in (12) we have

$$\int_0^A |\beta(t)|\, t^{-(1+\delta)}\,dt < \infty$$

if $\delta > 0$ or if r is even and $\delta = 0$. This completes the proof of the lemma.

Remark c) We note that in the latter part of the above proof in the particular case of $\delta=0$ and when the conditions on k and $F(x)$ in Theorem 2.4 are satisfied, the result holds. However, if X_i is symmetric and r is odd, we can show that $E|X|^r<\infty \Rightarrow \int_0^A|\beta(t)|t^{-1}dt<\infty$, suggesting that to treat the general case of k odd, we require a different criterion from $\int_0^A|\beta(t)|t^{-1}dt$.[†]

LEMMA 2.7. Suppose (1) *holds and* $E|X_i|^r<\infty$ *for some integer* $2\leqq r\leqq k+2$. *Then,* $\alpha_j=EX_i^j$, $j=1,2,\ldots,r$.

Proof. The result of the lemma is true by specification for r=2 and we develop a proof by induction.

Suppose that $E|X_i|^s<\infty$, some $s\geqq 2$ and $\alpha_j=EX_i^j$, $j=\overline{1,s}$. Then, if $E|X_i|^{s+1}<\infty$, let $Q_j^*(x)$, $1\leqq j\leqq s-1$ be the classical Chebyshev polynomials expressed in terms of the cumulants κ_j, $j=1,2,\ldots,s+1$ of X_i and write

$$G^*_{s-1,n}(x)=\Phi(x)+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\sum_{j=1}^{s-1}Q_j^*(x)\frac{1}{n^{j/2}} .$$

We have

$$\sup_x|G_{s-1,n}(x)-G^*_{s-1,n}(x)|\leqq\sup_x|F_n(x)-G_{s-1,n}(x)|+\sup_x|F_n(x)-G^*_{s-1,n}(x)|, \tag{13}$$

and from Theorem 1 of [23] ,

$$n^{(s-1)/2}\sup_x|F_n(x)|=o(1) \tag{14}$$

as $n\to\infty$. Also, from (1), since $\sum n^{-1}f(n)<\infty\Rightarrow \liminf f(n)=0$,

$$\liminf_{n\to\infty} n^{(k+\delta)/2}\sup_x|F_n(x)-G_{kn}(x)|=0,$$

so that

$$\begin{aligned}&\liminf_{n\to\infty} n^{(s-1)/2}\sup_x|F_n(x)-G_{s-1,n}(x)|\\ &\leqq\liminf_{n\to\infty} n^{(s-1)/2}\sup_x|F_n(x)-G_{kn}(x)|+\liminf_{n\to\infty} n^{(s-1)/2}\sup_x|G_{kn}(x)-G_{s-1,n}(x)| \qquad (15)\\ &=0\end{aligned}$$

since

$$G_{kn}(x)-G_{s-1,n}(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\sum_{j=s}^{k}Q_j(x)n^{-j/2} .$$

Consequently, using (14) and (15) in (13),

[†] See remark a) at the end of Chapter 2A

$$\liminf_{n\to\infty} n^{(s-1)/2}\sup_x|G_{s-1,n}(x)-G^*_{s-1,n}(x)| = 0. \tag{16}$$

But,

$$G_{s-1,n}(x)-G^*_{s-1,n}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\{Q_{s-1}(x)-Q^*_{s-1}(x)\}\frac{1}{n^{(s-1)/2}}$$

since $\alpha_j = EX_i^j$, $j=1,2,\dots,s$ implies $Q_j(x) = Q_j^*(x)$, $1\leq j\leq s-2$. and hence (16) can only hold if $Q_{s-1}(x) = Q^*_{s-1}(x)$. This gives $\kappa_{s+1} = \beta_{s+1}$ upon identifying coefficients and hence $\alpha_{s+1} = EX_i^{s+1}$ as required.

2.3. PROOF OF THEOREMS.

We start by proving Theorems 2.2 and 2.3 simultaneously in the following three steps. Firstly we note that the equivalence of (2) and (3),(4) follows immediately from Lemmas 2.5 and 2.6. Next we shall show that (3),(4) ensures (1) under (C) or if k=0 and lastly that (1) ensures (2).

(3),(4) $\Rightarrow$ (1) *under* (C) *or if* $k=0$.

Firstly we note that $G_{kn}(x)$ has a bounded first derivative; let $|G'_{kn}(x)|\leq B$. Then, using a bound due to Esseen, [11] section 39, we have for any $T > 0$,

$$|F_n(x)-G_{kn}(x)|\leq\frac{1}{\pi}\int_{-T}^{T}\left|\frac{f_n(t)-g_{kn}(t)}{t}\right| dt + c\,\frac{B}{T} \tag{17}$$

where $f_n(t)$ and $g_{kn}(t)$ are the characteristic functions corresponding to $F_n(x)$ and $G_{kn}(x)$ respectively and c is a positive constant.

Now, using (3) we have

$$\begin{aligned} f_n(t) &= \{f(\tfrac{t}{\sqrt{n}})\}^n \\ &= \exp\left\{\sum_{s=2}^{k+2}\frac{(it)^s}{s!}\,\frac{\gamma_s}{n^{(s-2)/2}} + n\left|\frac{t}{\sqrt{n}}\right|^{k+2}\gamma(\tfrac{t}{\sqrt{n}})\right\}, \end{aligned}$$

while from Lemma 5 of [23],

$$g_{kn}(t) = \exp\left\{\sum_{s=2}^{k+2}\frac{(it)^s}{s!}\,\frac{\gamma_s}{n^{(s-2)/2}} + D(t)\right\},$$

where $D(t) = O(|n^{-\frac{1}{2}}t|^{k+1})$ as $t\to 0$ so that

$$|f_n(t) - g_{kn}(t)| \leqq \left|\exp\left\{\sum_{s=2}^{k+2} \frac{(it)^s}{s!} \, \frac{\gamma s}{n^{(s-2)/2}}\right\}\right| \left|\exp\left\{n\left|\frac{t}{\sqrt{n}}\right|^{k+2} \gamma\left(\frac{t}{\sqrt{n}}\right)\right\} - 1\right|$$

$$+ \left|\exp\left\{\sum_{s=2}^{k+2} \frac{(it)^s}{s!} \, \frac{\gamma_s}{n^{(s-2)/2}}\right\} - g_{kn}(t)\right|. \qquad (18)$$

Now in view of Lemma 2.5 we may choose $a, 0<a<1$, so small that $\max_{|t|<a} |\gamma(t)| \leqq \frac{1}{4}$. Then for $|t|<a\sqrt{n}$, using the inequality $|e^x - 1| \leqq |x| e^{|x|}$, the first term on the right hand side of (18) is bounded by

$$\exp\left\{\sum_{s=1}^{[\frac{1}{2}(k+2)]} (-1)^s \frac{t^{2s}}{(2s)!} \, \frac{\gamma_{2s}}{n^{s-1}}\right\} \frac{|t|^{k+2}}{n^{k/2}} \left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right| \exp\left\{\frac{|t|^{k+2}}{n^{k/2}} \left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right|\right\}$$

$$\leqq \exp\left\{n\left(\sum_{s=1}^{[\frac{1}{2}(k+2)]} (-1)^s \frac{t^{2s}}{(2s)!} \, \frac{\gamma_{2s}}{n^s} + \frac{1}{4}\left|\frac{t}{\sqrt{n}}\right|^{k+2}\right)\right\} n \left|\frac{t}{\sqrt{n}}\right|^{k+2} \left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right| \qquad (19)$$

$$\leqq \exp(-t^2/8) n \left|\frac{t}{\sqrt{n}}\right|^{k+2} \left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right|$$

for n sufficiently large since $\gamma_2 = 1$. Also, from Lemma 5 of [23] the second term on the right hand side of (18) is bounded by

$$\frac{c}{n^{(k+1)/2}} (|t|^{3(k+1)} + |t|^{k+1}) e^{-t^2/4} \qquad (20)$$

for $|t|<b\sqrt{n}$ when b is sufficiently small, c being a positive constant. Then, choosing $\alpha = \min(a,b)$ and using (18), (19) and (20), we have

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} \left|\frac{f_n(t)-g_{kn}(t)}{t}\right| dt$$

$$\leqq c \sum_{n=1}^{\infty} n^{-1-(1-\delta)/2} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} (|t|^{3k+2} + |t|^k) e^{-t^{2/4}} dt \qquad (21)$$

$$+ \sum_{n=1}^{\infty} n^{-1+\delta/2} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} e^{-t^2/8} |t|^{k+1} \left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right| dt$$

$$\leqq A + \sum_{n=1}^{\infty} n^{(k+\delta)/2} \int_{-\alpha}^{\alpha} |u|^{k+1} |\gamma(u)| e^{-nu^2/8} du,$$

A being a finite constant. Furthermore, using a standard Abelian theorem (e.g. Feller [10] Vol. II p.447), taking $q_n = n^{(k+\delta)/2}$, $\rho = (k+\delta)/2 + 1$, we use the part of the theorem which states that if L(.) is slowly varying at ∞, $0 < \rho < \infty$ and q_n monotone, then

$$q_n \sim \frac{1}{\Gamma(\rho)}\, n^{\rho-1} L(n) \iff \sum_{n=1}^{\infty} q_n s^n \sim \frac{1}{(1-s)^{\rho}}\, L\left(\frac{1}{1-s}\right),\ s \to 1^- .$$

We therefore obtain

$$\lim_{t\uparrow 1}(1-t)^{(k+2+\delta)/2} \sum_{n=1}^{\infty} n^{(k+\delta)/2} t^n = \Gamma\big((k+2+\delta)/2\big)$$

so that for $u \neq 0$ it is possible to choose a constant $K_1 > 0$ such that

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2}\, e^{-nu^2/8} \leqq K_1 (1 - e^{-u^2/8})^{-(k+2+\delta)/2} .$$

Also, $|u| \leqq \alpha < 1$, so that

$$1 - e^{-u^2/8} > \tfrac{1}{8} u^2 (1 - \tfrac{1}{16}\alpha^2)$$

and hence

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2}\, e^{-nu^2/8} \leqq K_2 |u|^{-(k+2+\delta)}$$

for some $K_2 > 0$. Consequently, noting that $|\gamma(t)|$ is symmetric in t, we have

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2} \int_{-\alpha}^{\alpha} |u|^{k+1} |\gamma(u)| e^{-nu^2/8}\, du$$

$$= 2\int_0^{\alpha} u^{k+1} |\gamma(u)| \Big\{ \sum_{n=1}^{\infty} n^{(k+\delta)/2}\, e^{-nu^2/8} \Big\}\, du \qquad (22)$$

$$\leqq 2K_2 \int_0^{\alpha} \frac{|\gamma(u)|}{u^{1+\delta}}\, du < \infty$$

in view of (4).

Next write $T_\alpha = \{t : \alpha\sqrt{n} \leqq |t| \leqq \alpha n^{(k+1)/2}\}$, noting that T_α is empty if $k = 0$. Then, using condition (C), $\max\limits_{|t| \geqq \alpha} |f(t)| = \theta < 1$, so that for t in T_α,

$$|f_n(t)| = \left| f\left(\frac{t}{\sqrt{n}}\right)\right|^n \leqq \theta^n$$

and hence

$$\int_{T\alpha} \left|\frac{f_n(t)}{t}\right| dt \leqq 2\theta^n \int_{\alpha\sqrt{n}}^{\alpha n^{(k+1)/2}} \frac{dt}{t} \sim \theta^n (k+1) \log n$$

as $n \to \infty$. Consequently,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{T\alpha} \left|\frac{f_n(t)}{t}\right| dt < \infty . \qquad (23)$$

Furthermore, by the rules for forming the polynomials $Q_j(x)$, see for example Gnedenko and Kolmogorov [11], Section 38,

$$g_{kn}(t) = e^{-\frac{1}{2}t^2}\left(1 + \sum_{j=1}^{k} P_j(it)n^{-\frac{1}{2}j}\right) .$$

where P_j is a polynomial of degree $3j$ determined from the formal identity

$$\exp\left(\sum_{j=3}^{\infty} \frac{\gamma_j}{j!} \frac{(it)^j}{n^{(j-2)/2}}\right) = 1 + \sum_{j=1}^{\infty} P_j(it)n^{-j/2} .$$

Thus,

$$|g_{kn}(t)| \leqq e^{-\frac{1}{2}t^2} \left(1 + \sum_{j=1}^{3k} a_{nj}|t|^j\right)$$

where the a_{nj} are polynomials in $n^{-\frac{1}{2}}$ which tend to zero as $n \to \infty$. Consequently, we certainly have

$$\int_{T\alpha} \left|\frac{g_{kn}(t)}{t}\right| dt \leqq Ke^{-\alpha^2 n/2} n^{3k(k+1)/2}$$

for K a positive constant which gives

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{T\alpha} \left|\frac{g_{kn}(t)}{t}\right| dt < \infty, \tag{24}$$

and from (23) and (24) we obtain

$$\int_{T\alpha} \left|\frac{f_n(t) - g_{kn}(t)}{t}\right| dt \leqq \int_{T\alpha} \left|\frac{f_n(t)}{t}\right| dt + \int_{T\alpha} \left|\frac{g_{kn}(t)}{t}\right| dt < \infty . \tag{25}$$

The required result (1) then follows, using (17) with $T = \alpha n^{\frac{1}{2}(k+1)}$, in view of (21), (22) and (25).

(1) $\Rightarrow$ (2)

Firstly we symmetrize the X_i's. Consider the sequence Y_i, $i=1,2,3,\ldots$ of independent symmetrized random variables; each Y_i having the distribution of the difference between two independent X_i's. Clearly the Y_i have characteristic function $|f(t)|^2$ and the distribution function of $Z_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} Y_i$ is $F_n(x) * (1 - F_n(-x-0)) = F_n^*(x)$. Write $G_{kn}^*(x)$ for the convolution $G_{kn}(x)*(1 - G_{kn}(-x))$. Then,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n^*(x) - G_{kn}^*(x)|$$

$$= \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x)*(1 - F_n(-x-0)) - G_{kn}(x)*(1 - G_{kn}(-x))| \tag{26}$$

$$\leqq \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x)*(1 - F_n(-x-0)) - G_{kn}(x)*(1 - F_n(-x-0))|$$

$$+ \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |G_{kn}(x)*(1 - F_n(-x-0)) - G_{kn}(x)*(1 - G_{kn}(-x))| < \infty$$

in view of (1).

Next we shall show that (26) implies $E|Y_i|^{k+2} < \infty$ and hence $E|X_i|^{k+2} < \infty$ (Loeve [27], p.263). We note that

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \{1 - G_{kn}^*(x_n)\} < \infty$$

where $x_n = \{(k+\delta+1) \log n\}^{\frac{1}{2}}$, so that from (26),

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} P\left(\left|\sum_{i=1}^{n} Y_i\right| > n^{\frac{1}{2}} x_n\right) < \infty \tag{27}$$

But, for symmetric random variables,

$$P\left(\left|\sum_{i=1}^{n} Y_i\right| > n^{\frac{1}{2}} x_n\right) \geqq \tfrac{1}{2} P\left(\max_{1 \leqq k \leqq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \tag{28}$$

(e.g. [10], Vol.II, p.149) and from Bonferroni's inequalities (e.g. [10] Vol.I, p.110) we have

$$nP(|Y_i| > n^{\frac{1}{2}} x_n)\{1 - \tfrac{1}{2}(n-1)P(|Y_i| > n^{\frac{1}{2}} x_n)\}$$

$$\leqq P\left(\max_{1 \leqq k \leqq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \leqq nP(|Y_i| > n^{\frac{1}{2}} x_n),$$

while $nP(|Y_i| > n^{\frac{1}{2}} x_n) \to 0$ as $n \to \infty$ since $EY_i^2 = 2EX_i^2 < \infty$. Consequently,

$$P\left(\max_{1 \leqq k \leqq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \sim nP(|Y_i| > n^{\frac{1}{2}} x_n) \tag{29}$$

as $n \to \infty$ and hence, from (27), (28) and (29),

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2} P(|Y_i| > n^{\frac{1}{2}} x_n) < \infty . \tag{30}$$

But

$$E|Y_i|^{k+2} = -\int_0^\infty x^{k+2}\, dP(|Y_i|>x)$$

$$\leqq (k+2)\int_0^\infty x^{k+1}\, P(|Y_i|>x)\,dx$$

$$= (k+2)\sum_{n=0}^{\infty}\int_{x_n n^{\frac{1}{2}}}^{x_{n+1}(n+1)^{\frac{1}{2}}} x^{k+1}\, P(|Y_i|>x)\,dx$$

$$\leqq \sum_{n=0}^{\infty} P(|Y_i|>n^{\frac{1}{2}}x_n)\{((n+1)x_{n+1}^2)^{(k+2)/2} - (nx_n^2)^{(k+2)/2}\}$$

$$\leqq c\sum_{n=0}^{\infty} n^{k/2}\,(\log n)^{(k+2)/2}\, P(|Y_i|>n^{\frac{1}{2}}x_n) < \infty$$

in view of (30), c being a positive constant. It then follows that for $\delta>0$ $E|X_i|^{k+2} < \infty$ and an appeal to Lemma 27 gives $\alpha_j = EX_i^j$, $j=1,2,\ldots,k+2$.

Now we note that the characteristic functions of $F_n^*(x)$ and $G_{kn}^*(x)$ are $|f_n(t)|^2$ and $|g_{kn}(t)|^2$ respectively. Then, integrating by parts in the equation

$$|f_n(t)|^2 - |g_{kn}(t)|^2 = \int_{-\infty}^{\infty} e^{itx}\, d\,\{F_n^*(x) - G_{kn}^*(x)\},$$

we obtain

$$-\frac{|f_n(t)|^2 - |g_{kn}(t)|^2}{it} = \int_{-\infty}^{\infty} e^{itx}\{F_n^*(x) - G_{kn}^*(x)\}dx.$$

Also,

$$i\,t\,e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx}\,\frac{x}{\sqrt{2\pi}}\,e^{-x^2/2}\,dx,$$

and we obtain from Parseval's identity (see e.g. [24] p.398)

$$\int_{-\infty}^{\infty}\{|f_n(t)|^2 - |g_{kn}(t)|^2\}e^{-t^2/2}\,dt = \sqrt{2\pi}\int_{-\infty}^{\infty}\{F_n^*(x) - G_{kn}^*(x)\}xe^{-x^2/2}\,dx.$$

Thus from (26)

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2}\left|\int_{-\infty}^{\infty}\{|f_n(t)|^2 - |g_{kn}(t)|^2\}e^{-t^2/2}\,dt\right|$$

$$= \sqrt{2\pi}\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2}\left|\int_{-\infty}^{\infty}\{F_n^*(x) - G_{kn}^*(x)\}xe^{-x^2/2}\,dx\right| \tag{31}$$

$$\leqq 2\sqrt{2\pi}\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2}\,\sup_x|F_n^*(x) - G_{kn}^*(x)| < \infty\,.$$

Furthermore, we note that for $0 < \alpha < \frac{1}{4}$,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_{n^\alpha}^{\infty} e^{-t^2/2} \{|f_n(t)|^2 - |g_{kn}(t)|^2\} dt \right|$$

$$\leqq 2 \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{n^\alpha}^{\infty} e^{-t^2/2} \, dt < \infty,$$

so that in view of (31)

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_0^{n^\alpha} e^{-t^2/2} \{|f_n(t)|^2 - |g_{kn}(t)|^2\} dt \right| < \infty \; . \tag{32}$$

Now from Lemma 5 of [23], letting $R = [\frac{1}{2}(k+2)]$, we have for $t < c'\sqrt{n}$, c' some suitably small positive constant,

$$\left| |g_{kn}(t)|^2 - \exp\left(\sum_{s=1}^{R} (-1)^s \frac{t^{2s} 2\kappa_{2s}}{(2s)!\, n^{s-1}} \right) \right| \leqq \frac{c}{n^{(k+1)/2}} (|t|^{3(k+1)} + |t|^{k+1}) \, e^{-t^2/4} \tag{32a}$$

for some $c>0$ since $\alpha_j = EX_i^j, j=1,2,\ldots,k+2$ ensures $\gamma_j = \kappa_j, j=1,2,\ldots,k+2$.

Thus,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_0^{n^\alpha} \left\{ \exp\left(\sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right) - |g_{kn}(t)|^2 \right\} e^{-t^2/2} \, dt \right| \tag{33}$$

$$\leqq c \sum_{n=1}^{\infty} n^{-1-(1-\delta)/2} \int_0^{n^\alpha} e^{-3t^2/4} (t^{3(k+1)} + t^{k+1}) dt < \infty \; .$$

Also, from Lemma 2.5,

$$|f_n(t)|^2 = \left| f\left(\frac{t}{\sqrt{n}}\right) \right|^{2n} = \exp\left\{ \sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} + 2\frac{|t|^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right\},$$

and using this result in conjunction with (32) and (33),

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2}$$

$$\cdot \left| \int_0^{n^\alpha} e^{-t^2/2} \exp\left(\sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right) \left[1 - \exp\left(2\frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right) \right] dt \right| < \infty. \tag{33a}$$

Now Lemma 2.5 tells us that for n large enough, $\operatorname{Re}\gamma\left(\frac{t}{n}\right)$ will be of constant sign for $0 < t \leqq n^\alpha$ and hence

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2}$$

$$\cdot \int_0^{n^{\alpha}} e^{-t^2/2} \exp\left(\sum_{s=1}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}}\right) \left|1-\exp\left(2\frac{t^{k+2}}{n^{k/2}} \operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)\right| dt < \infty$$

so that

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2}$$

$$\cdot \int_0^{1} \exp\left(-3t^2/2 + \sum_{s=2}^{R} (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}}\right) \left|1-\exp\left(2\frac{t^{k+2}}{n^{k/2}} \operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)\right| dt < \infty$$

which implies

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_0^1 \left|1-\exp\left(2\,\frac{t^{k+2}}{n^{k/2}} \operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)\right| dt < \infty .$$

However, as $n \to \infty$,

$$\left|1 - \exp\left(2\,\frac{t^{k+2}}{n^{k/2}} \operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)\right| = \frac{2|t|^{k+2}}{n^{k/2}} \left|\operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right| (1+o(1)),$$

so that

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \int_0^1 t^{k+2} \left|\operatorname{Re}\gamma\left(\frac{t}{\sqrt{n}}\right)\right| dt < \infty,$$

and, upon making the transformation $u = t/\sqrt{n}$, this yields

$$\sum_{n=1}^{\infty} n^{(k+1+\delta)/2} \int_0^{1/\sqrt{n}} u^{k+2} |\operatorname{Re}\gamma(u)|\, du < \infty. \tag{34}$$

Now $\int_0^{1/\sqrt{x}} u^{k+2} |\operatorname{Re}\gamma(u)|\, du$ is monotone decreasing as x increases so for $X > 1$,

$$\int_1^X x^{(k+1+\delta)/2} \left(\int_1^{1/\sqrt{x}} u^{k+2} |\operatorname{Re}\gamma(u)|\, du\right) dx$$

$$\leqq \sum_{n=1}^{[X]} \int_n^{n+1} x^{(k+1+\delta)/2} \left(\int_1^{1/\sqrt{x}} u^{k+2} |\operatorname{Re}\gamma(u)|\, du\right) dx$$

$$\leqq \sum_{n=1}^{[X]} (n+1)^{(k+1+\delta)/2} \int_1^{1/\sqrt{n}} u^{k+2} |\operatorname{Re}\gamma(u)|\, du$$

$$\leqq c \sum_{n=1}^{[X]} n^{(k+1+\delta)/2} \int_{1}^{1/\sqrt{n}} u^{k+2}|\mathrm{Re}\gamma(u)|\,du,$$

c being a positive constant such that $(n+1)^{(k+1+\delta)/2} < cn^{(k+1+\delta)/2}$ for all positive integral n. Consequently, using (34), we have

$$\int_1^\infty x^{(k+1+\delta)/2}\left(\int_0^{1/\sqrt{x}} u^{k+2}|\mathrm{Re}\gamma(u)|\,du\right)dx < \infty. \tag{35}$$

Now in view of (35) we must have

$$\int_\omega^{2\omega} x^{(k+1+\delta)/2}\left(\int_0^{1/\sqrt{x}} u^{k+2}|\mathrm{Re}\gamma(u)|\,du\right)dx \to 0$$

as $\omega \to \infty$ and

$$\int_\omega^{2\omega} x^{(k+1+\delta)/2}\left(\int_0^{1/\sqrt{x}} u^{k+2}|\mathrm{Re}\gamma(u)|\,du\right)dx$$

$$\geqq \omega^{(k+3+\delta)/2} \int_0^{1/\sqrt{2\omega}} u^{k+2}|\mathrm{Re}\gamma(u)|\,du \geqq 0,$$

so that, putting $v = 1/\sqrt{2\omega}$, we conclude that

$$v^{-(k+3+\delta)} \int_0^v u^{k+2}|\mathrm{Re}\gamma(u)|\,du \to 0 \tag{36}$$

as $v \to 0$. Then, upon making the transformation $v = 1/\sqrt{x}$, (35) becomes

$$\int_0^1\left(\int_0^v u^{k+2}|\mathrm{Re}\gamma(u)|\,du\right)v^{-(k+4+\delta)}\,dv < \infty$$

and, in view of (36), the fact that $\int_0^1 |\mathrm{Re}\gamma(t)|\,t^{-(1+\delta)}dt < \infty$ holds follows immediately from an integration by parts. Lemma 2.6 then enables us to conclude that $E|Y_i|^{k+2+\delta} < \infty$ from which we deduce that $E|X_i|^{k+2+\delta} < \infty$. This completes the proof that (1) $\Rightarrow$ (2) and hence the proof of Theorems 2.2 and 2.3.

For the proof of Theorem 2.4 in the case when k is even, we note firstly that the above proof that (3),(4) $\Rightarrow$ (1) under (c) or if $k = 0$

together with Lemmas 2.5 and 2.6 show that $E|X_i|^{k+2} \log(1+|X_i|) < \infty$ implies (1) with $\delta = 0$ under (C) or if $k = 0$. On the other hand, the above proof that (1) $\Rightarrow$ (2) shows that (1) with $\delta = 0$ implies $\int_0^1 |\mathrm{Re}\,\gamma(t)|\, t^{-1}\, dt < \infty$. Thus, from Lemmas 1 and 2 we have $E|Y_i|^{k+2} \log(1+|Y_i|) < \infty$ from which we deduce the required result that $E|X_i|^{k+2} \log(1+|X_i|) < \infty$.

To prove Theorem 2.4 in the case of k odd and F(x) such that there exists a finite positive constant B with $P\{X_i < -B\} = 0$, we note that Lemmas 2.5 and 2.6 give us

$$f(t) = \exp\left\{\sum_{s=2}^{k+2} \frac{(it)^s}{s!}\kappa_s + |t|^{k+2}\gamma(t)\right\}, \qquad \gamma(t) \to 0 \text{ as } t \to 0$$

$$= e^{\ell_k' + ih_k' + |t|^{k+2}\gamma(t)},$$

and

$$E|X_i|^{k+2}\log(1+|X_i|) < \infty \iff \int_0^A |\beta(t)|\, t^{-1} dt < \infty$$

$$\iff \int_0^A |\mathrm{Im}^+\beta(t)|\, t^{-1}\, dt < \infty \tag{37}$$

where ℓ_k', h_k' are functions of t and are respectively the real and imaginary parts of

$$\sum_{s=2}^{k+2} \frac{(it)^s}{s!}\kappa_s,$$

also we define

$$t^{k+2}\,\mathrm{Im}^+\,\beta(t) = \int_0^\infty b_{k+1}(xt)\,dF(x)$$

where

$$b_{2n}(u) = \sin u - \sum_{s=0}^{n} (-1)^s \frac{u^{2s+1}}{(2s+1)!}.$$

The equivalence (37) follows from the fact that

$$\int_0^A t^{-(k+3)} \left|\int_{-B}^0 b_{k+1}(xt)\,dF(x)\right| = \int_0^A |\beta(t) - \mathrm{Im}^+\beta(t)|\, t^{-1}\, dt, \tag{38}$$

together with the fact that $b_{2n}(u)$ does not change sign for $u > 0$ or for $u < 0$ (e.g. see Lemma 2 [23]) and that

$$|b_{2n}(u)| \leqq \begin{cases} C_1|u|^{2n+3} & \text{if } |u| \leqq 1 \\ C_2|u|^{2n+1} & \text{if } |u| > 1 \end{cases}$$

These give us the following bound for (38) (assume without loss of generality $B > A^{-1}$),

$$C_1 \int_{-B}^{0} \int_{0}^{B^{-1}} t^{-(k+3)}|x|^{k+4}\, t^{k+4}\, dt\, dF(x) + C_2 \int_{-B}^{0} \int_{B^{-1}}^{A} t^{-1}|x|^{k+2}\, dt\, dF(x) < \infty .$$

To establish that (1) $\Rightarrow \int_0^1 \mathrm{Im}^{+}\beta(t)\, t^{-1}\, dt < \infty$ we use expression (31) and the subsequent work with $f_n(t)$ and $g_{kn}(t)$ in place of $|f_n(t)|^2$ and $|g_{kn}(t)|^2$. We obtain, for $0 < \alpha < 1/6$,†

$$\sum_{n=1}^{\infty} n^{-1+k/2} \left| \int_0^{n^{\alpha}} e^{-t^2/2 + \ell_k + ih_k} \left\{1 - \exp\left(\frac{t^{k+2}}{n^{k/2}}\gamma\left(\frac{t}{\sqrt{n}}\right)\right)\right\} \right| < \infty$$

implying that

$$\sum_{n=1}^{\infty} n^{-1} \left| \int_0^{n^{\alpha}} e^{-t^2/2 + \ell_k + ih_k} \cdot t^{k+2} \gamma\left(\frac{t}{\sqrt{n}}\right) dt \right| \leq \infty \qquad (\text{as } k \geq 1)$$

We note that $\alpha < 1/6$, $\frac{t^3}{\sqrt{n}} < n^{-\varepsilon}$ for some $\varepsilon > 0$, $\frac{-ct^2}{2} < \ell_k < 0$ for some $c > 0$, $\cos(h_k) \geqslant \frac{1}{2}$ and $|\sin(h_k)| < C_3 \frac{t^3}{\sqrt{n}}$; all these hold providing $0<t<n^{\alpha}$ and n large

Thus as $|a + ib| \geqq |b|$

$$\sum n^{-1} \left| \int_0^{n^{\alpha}} e^{-t^2/2 + \ell_k} \cdot t^{k+2} \{\cos(h_k) \cdot \mathrm{Im}\gamma\left(\tfrac{t}{\sqrt{n}}\right) + \sin(h_k) \cdot \mathrm{Re}\,\gamma\left(\tfrac{t}{\sqrt{n}}\right)\} dt \right| < \infty$$

But, by the above estimates,

$$\sum n^{-1} \left| \int_0^{n^{\alpha}} e^{-t^2/2 + \ell_k} \cdot t^{k+2} \sin(h_k) \cdot \mathrm{Re}\gamma\left(\tfrac{t}{\sqrt{n}}\right) \right| dt \leqq$$

$$\leqq \sum n^{-1} \cdot C_4\, n^{-\frac{1}{2}} \int_0^{n^{\alpha}} e^{-t^2/2 + \ell_k} \cdot t^{k+5}\, dt < \infty$$

Thus $$\sum n^{-1} \left| \int_0^{n^{\alpha}} e^{-t^2/2 + \ell_k} \cdot t^{k+2} \cos(h_k) \cdot \mathrm{Im}\gamma\left(\tfrac{t}{\sqrt{n}}\right) \cdot dt \right| < \infty$$

† $\ell_k = n \times (\ell_k'$ evaluated at $t = t/\sqrt{n})$, h_k similarly defined.

Also, by the proof of Lemma 2.6, we must therefore have

$$\sum n^{-1} \left| \int_0^{n^\alpha} e^{-t^2/2 + \ell_k} \cdot t^{k+2} \cos(h_k) \operatorname{Im} \beta\left(\frac{t}{\sqrt{n}}\right) dt \right| < \infty$$

However, using the estimates for $b_{k+1}(u)$ above and noting that $\left|\frac{xt}{\sqrt{n}}\right| < 1$,

$$\left| \int_0^{n^\alpha} e^{-t^2/2 + \ell_k} \cos(h_k)\, n^{\frac{1}{2}(k+2)} \int_{-B}^{0} b_{k+1}\left(\frac{xt}{n}\right) \cdot dF(x)\, dt \right|$$

$$\leqq \left| \int_0^{n^\alpha} e^{-t^2/2}\, n^{\frac{1}{2}(k+2)} \int_{-B}^{0} b_{k+1}\left(\frac{xt}{\sqrt{n}}\right) dF(x)\, dt \right| < Cn^{-1} .$$

Thus

$$\sum n^{-1} \left| \int_0^{n^\alpha} e^{-t^2/2 + \ell_k} \cdot \cos(h_k)\ t^{k+2} \operatorname{Im}^+ \beta\left(\frac{t}{\sqrt{n}}\right) dt \right| < \infty$$

But $\operatorname{Im}^+\beta(t)$ does not change sign in $(0,\infty)$, and $\cos(h_k) > \frac{1}{2}$

giving
$$\sum n^{-1} \int_0^1 t^{k+2} \left| \operatorname{Im}^+ \beta\left(\frac{t}{\sqrt{n}}\right) \right| dt < \infty .$$

By the last part of the proof of $(1) \Rightarrow (2)$, we imply from our above series result, that $\int_0^1 |\operatorname{Im}^+\beta(t)|\, t^{-1}\, dt < \infty$. This together with (37) completes the proof of Theorem 2.4. (N.B. From the proof of (3), $(4) \Rightarrow (1)$ and from (37) we have $E|x|^{k+2} \log(1+|x|) < \infty \Rightarrow (1)$).

CHAPTER 2A

THE STRENGTHENING OF A RESULT OF IBRAGIMOV

2A.1. INTRODUCTION: It will be recalled that in the previous chapter we were unable to treat the case of $\delta = 0$ with the same generality as for $0 < \delta < 1$. In particular, we could not establish the existence of $E|X|^{k+2}$ from the convergence of $\sum_{n=1}^{\infty} n^{-\frac{1}{2}k} A_{k,n}$, $k \geqq 1$.

In an attempt to show this, we will modify a result of Ibragimov (of some independent interest) and in the form of a corollary we will show how this anomaly in Chapter 2 is removed. Throughout this chapter we use the notation of Chapter 2.

2A.2. RESULTS: The following theorem generalises Theorem 1.2 and 1.3 by showing these latter results continue to hold when we demand only the existence of a sub-sequence $\{n_i\}$ with $n_i \to \infty$ as $i \to \infty$ and $\frac{n_{i+1}}{n_i} \to c$ $(1 \leqq c < \infty)$. The result is surprising in as much as it is possible to take $n_i = [c^i]$ where c can be greater than 1. Thus it is sufficient to to know that the condition

$$\sup_x |F_n(x) - G_{kn}(x)| = 0\left(n^{-\frac{1}{2}(k+\delta)}\right)$$

is satisfied at a set of geometrically increasing points to ensure that the condition must hold for all n.

As would be expected, the proof of this result follows very closely that of Ibragimov. Let X_i be a sequence of iid rv's with zero means and unit variances.

THEOREM 2A.1. Theorems 1.2 and 1.3 continue to hold if the condition

$$|F_n(x) - G_{kn}(x)| = 0\left(n^{-\frac{1}{2}(k+\delta)}\right) \quad \left(\text{or } 0(n^{-\frac{1}{2}k})\right) \textit{ for all } n$$

is replaced by the following condition: There exists an integer sequence $\{n_i\}$, $n_{i+1} > n_i$, $n_i \to \infty$ *as* $i \to \infty$ *and* $\frac{n_{i+1}}{n_i} \to c$ $(1 \leqq c < \infty)$ *such that*

$$|F_{n_i}(x) - G_{kn_i}(x)| = O\left(n_i^{-\frac{1}{2}(k+\delta)}\right) \quad \left(\text{or } O(n_i^{-\frac{1}{2}k})\right) \qquad (1)$$

where k *is an integer* $\geqq 1$ *and* $0 < \delta \leqq 1$.

COROLLARY 2A.2. If $\sum n^{-1+\frac{1}{2}k}|F_n(x) - G_{kn}(x)| < \infty$ *then* $E|X_i|^{k+2} < \infty$ *and* $_j = EX_i^j$, $j = 1,2,\ldots, k+2$ *for* k *an integer* $\geqq 1$.

PROOF OF THEOREM 2A.1: We need only prove the sufficiency of condition (1) and this will be done by induction on k. Firstly, we prove the following lemma which is, in fact, the above theorem for k = 0 in the large order case.

LEMMA 2A.3. With $\{n_i\}$ *as in Theorem 2A.1, if*

$$|Fn_i(x) - \Phi(x)| = O(n_i^{-\frac{1}{2}\delta}), \qquad 0 < \delta \leqq 1 \qquad (2)$$

then

1) $$\int_{|u|>z} u^2F(u) = O(n^{-\delta})$$

and if $\delta = 1$, *also*

2) $$\int_{|u|<z} u^3dF(u) = O(1)$$

PROOF OF LEMMA: Since $EX_i^2 = 1$, we can write f(t) in the form

$$f(t) = \exp\{-\tfrac{1}{2}t^2\left(1 + \gamma(t)\right)\}$$

with $\lim_{t\to 0} \gamma(t) = 0$ and such that $|\gamma(t)| \neq 0$ for all t in some neighbourhood of the origin $(-\varepsilon,\varepsilon)$. Following Ibragimov and Linnik [24], pp 105-6, we consider first the case f(t) symmetric. From (2) by Parseval's identity we have

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\left(f_{n_i}(t) - e^{-\frac{1}{2}t^2}\right) dt = O(n_i^{-\frac{1}{2}\delta})$$

Thus

$$\int_{-\log n_i}^{\log n_i} e^{-\ t^2}\left(1 - \exp\left(-\tfrac{1}{2}t^2\gamma\left(\frac{t}{\sqrt{n_i}}\right)\right)\right)dt = O(n_i^{-\frac{1}{2}\delta})$$

and as $\gamma(t)$ is real and does not change sign in $0 < t < \varepsilon$,

$$\int_0^1 e^{-t^2}\left[1-\exp\left(-\tfrac{1}{2}t^2\gamma\left(\frac{t}{\sqrt{n_i}}\right)\right)\right]dt = O(n_i^{-\frac{1}{2}\delta})$$

$$\Rightarrow \int_0^1 t^2\gamma\left(\frac{t}{\sqrt{n_i}}\right)dt = O(n_i^{-\frac{1}{2}\delta})$$

$$\Rightarrow \int_0^{1/\sqrt{n_i}} t^2\gamma(t)dt = O\left(n_i^{-\frac{1}{2}(3+\delta)}\right) \qquad (3)$$

Now equation 3.4.11 of Ibragimov and Linnik [24] gives us

$$\int_0^{n_i^{-\frac{1}{2}}} \tfrac{1}{2}t^2|\gamma(t)|dt = n_i^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left[\frac{\sin(un_i^{-\frac{1}{2}})}{un_i^{-\frac{1}{2}}} - 1 + \frac{1}{6}u^2n_i^{-1}\right]dF(u) + O\left(n_i^{-5/2}\right)$$

Thus (3) $\Rightarrow$

$$\int_{\infty}^{\infty}\left[\frac{\sin(un_i^{-\frac{1}{2}})}{un_i^{-\frac{1}{2}}} - 1 + \frac{1}{6}u^2n_i^{-1}\right]dF(u) = O\left(n_i^{-(1+\frac{1}{2}\delta)}\right)$$

But $\frac{\sin u}{u} - 1 + \frac{u^2}{6} \geqq 0$ for all u , hence

$$\int_{|u|>cn_i^{\frac{1}{2}}}\left[\frac{\sin(un_i^{-\frac{1}{2}})}{un_i^{-\frac{1}{2}}} - 1 + \frac{1}{6}u^2n_i^{-1}\right]dF(u) = O\left(n_i^{-(1+\frac{1}{2}\delta)}\right)$$

Taking c = 12, we can make the integrand not less than $\frac{u^2n_i^{-1}}{12}$, thus

$$\int_{|u|>cn_i^{\frac{1}{2}}} u^2dF(x) = O(n_i^{-\frac{1}{2}\delta})$$

Take $cn_i^{\frac{1}{2}} < z \leqq cn_{i+1}^{\frac{1}{2}}$

$$\int_{|u|>z} u^2dF(u) \leqq \int_{|u|>cn_i^{\frac{1}{2}}} u^2dF(u) = O\left(n_i^{-\frac{1}{2}\delta}\right) = O\left[z^{-\delta}\left(\frac{z}{cn_i^{\frac{1}{2}}}\right)^{\delta}\right]$$

But $\frac{z}{cn_i^{\frac{1}{2}}} < \frac{n_{i+1}^{\frac{1}{2}}}{cn_i^{\frac{1}{2}}}$ < some absolute constant by the assumption of the lemma.

Hence $$\int_{|u|>z} u^2 dF(u) = O(z^{-\delta}), \qquad 0 < \delta \leq 1$$

for F(x) a symmetric distribution function.

Using the usual technique for transferring from symmetric rvs over to the general case (see expression (28) of Chapter 2) we find that in general (2) implies

$$\int_{|u|>z} u^2 dF(u) = O(z^{-\delta}), \qquad 0 < \delta \leq 1 \tag{3a}$$

To complete the proof of the lemma it remains to consider the case of $\delta = 1$ for non-symmetric distributions. We thus no longer regard X_i as being necessarily symmetric.

The proof follows its counterpart on p.565 of [22]. Firstly, however, we need to prove that

$$\int_0^x (x - t)t^2 \operatorname{Im} \gamma(t)dt = O(x^5) \quad \text{with } x = n_i^{-\frac{1}{2}} \tag{4}$$

From Lemma 2.4 of [22] we have †

$$\left| \int_0^1 t^2 \gamma\left(\frac{t}{n_i^{\frac{1}{2}}}\right)(1 - t)dt \right| = O(n_i^{-\frac{1}{2}})$$

As $|a + ib| \geq |a|$ or $|b|$,

$$\left| \int_0^1 t^2 \operatorname{Im} \gamma\left(\frac{t}{n_i^{\frac{1}{2}}}\right)(1 - t)dt \right| = O(n_i^{-\frac{1}{2}})$$

from which (4) follows.

Since the work on pp 565-566 of [22] does not require x to be continuous, (indeed x continuous has been used purely for notational convenience) without loss of generality we can put $x = n_i^{-\frac{1}{2}}$.

We can thus immediately extract

$$\int_{-n_i^{\frac{1}{2}}}^{n_i^{\frac{1}{2}}} u^3 dF(u) = O(1) \quad \text{as} \quad i \to \infty$$

As before, take $n_i^{\frac{1}{2}} \leq z \leq n_{i+1}^{\frac{1}{2}}$, then

† From (3a) and p.111 of [24], $\operatorname{Im} \gamma(t) = O\left(|t| \log |t|^{-1}\right)$ and $\operatorname{Re} \gamma(t) = O|t|$.

$$\int_{|u|<z} u^3dF(u) = \int_{|u|<n_i^{\frac12}} u^3dF(u) + \int_{n_i^{\frac12}\le|u|\le z} u^3dF(u)$$

$$\le O(1) + z\int_{|u|>n_i^{\frac12}} u^2dF(u)$$

$$= O(1) + O(zn_i^{-\frac12}) \quad \text{(by first part of proof)}$$

$$= O(1) + O(n_{i+1}^{\frac12}\cdot n_i^{-\frac12})$$

$$= O(1) + O(1) \quad \text{by the definition of } \{n_i\}.$$

Lemma 2A.3. is now established.

Returning now to the proof of the large order case of the theorem; since we know it is true for $k = 0$, we assume it true for k and prove it true for $k + 1$. Again, the proof is identical to that of Ibragimov [23] except we persist in taking $x = n_i^{-\frac12}$. The critical equation 3.9 of [23] becomes

$$\int_{-\infty}^{\infty}\left[\exp(iun_i^{-\frac12}) - \sum_{s=0}^{k+4}\frac{(iun_i^{-\frac12})^s}{s!}\right]\frac{dF(u)}{(iu)^2} + \frac{i^{k+3}n_i^{-\frac12(k+5)}}{(k+5)!}\left(\Lambda_{k+3} - \gamma_{k+3}\right)$$

$$= o\left(n_i^{-\frac12(k+5+\delta)}\right).$$

For k even, using the method and the T_k defined on p.465 of [23], we obtain,

$$\int_{|u|>n_i^{\frac12}T_k} u^{k+2}dF(u) = o\left(n_i^{-\frac12(1+\delta)}\right)$$

$$\Longrightarrow \int_{|u|>z} u^{k+2}dF(u) \le \int_{|u|>n_i^{\frac12}T_k} u^{k+2}dF(u) = o\left[z^{-(1+\delta)}\left(\frac{n_{i+1}}{n_i}\right)^{\frac12(1+\delta)}\right] = o\left(z^{-(1+\delta)}\right),$$

where $n_i^{\frac12}T_k \le z \le n_{i+1}^{\frac12}T_k$.

This relation is equation (3.12) of [23] and hence,

$$\int_{|u|>z} |u|^{k+3}dF(u) = O(z^{-\delta})\,.$$

We similarly obtain $\gamma_{k+3} = \kappa_{k+3}$ and when $\delta = 1$,

$$\int_{|u|<n_i^{\frac{1}{2}}} u^{k+4} dF(u) = O(1)$$

from which we can easily find for all $z > 0$

$$\int_{|u|<z} u^{k+4} dF(u) = O(1)$$

Similarly, when k is odd, the results go through and so we have established the theorem for the larger order case. The smallorder result follows in an identical manner from the corresponding work in [23] . Thus, the theorem is now complete.

PROOF OF COROLLARY 2A.2. From Heyde [17] we know that if $\sum n^{-1}g(n) < \infty$ where $g(n)$ is some positive function of n, then there exists a subsequence $\{n_i\}$ such that $g(n_i) \to 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$.

Setting $g(n) = n^{\frac{1}{2}k} |F_n(x) - G_{kn}(x)|$, we have $\sum n^{-1}g(n) < \infty$ and so $|F_{n_i}(x) - G_{kn_i}(x)| = o(n_i^{-\frac{1}{2}k})$. From Theorem 2A.1 we have

1) $\alpha_i = EX_i^j$, $j = 1, 2, \ldots, k+1$ and $E|X_i|^{k+1} < \infty$

2) $$\int_{|x|>z} |x|^{k+1} dF(x) = o(z^{-1})$$

and 3) $$\int_{-z}^{z} x^{k+2} dF(x) = \alpha_{k+2} \text{ as } z \to \infty$$

From 3) it follows that if k is even, $E|X|^{k+2} < \infty$, as desired. When k is odd, we need to return to the Ibragimov article [23] , this time using his equation (3.6) to prove that[†]

$$\sum n^{-1+\frac{1}{2}k} \left| \int_0^1 \left[1 - \exp\{(it)^{k+2} n^{-\frac{1}{2}k} \gamma_k(\tfrac{t}{n^{\frac{1}{2}}})\}\right](1 - t)dt \right| < \infty, \tag{5}$$

where from 1), 2) 3) and Theorem 3 of [23],

$$f(t) = \exp\left(-\tfrac{1}{2}t^2 + \sum_{s=3}^{k+2} \frac{(it)^s}{s!}\gamma_s + (it)^{k+2} n^{-\frac{1}{2}k}\, \gamma_k(\tfrac{t}{n^{\frac{1}{2}}})\right)$$

[†] There shall be no confusion between γ_k and $\gamma_k(t)$. Also recall that the γ_j are the predetermined sequence and the κ_j are the cumulants of X_i.

$|\gamma_k(t)| = o(1)$ as $t \to 0$ and (N.B. mistake in the English translation of this article).

$$A(t) = \begin{cases} t(1-t)\exp\left\{\frac{1}{2}t^2 - \sum_{j=3}^{k+2} \frac{(it)^j}{j!} \frac{\gamma_j}{n^{\frac{1}{2}(j-2)}}\right\}, & t \in [0,1] \\ 0, \text{ otherwise} \end{cases}$$

From (5) and the fact that $|\gamma(t)| = o(1)$

$$\sum n^{-1+\frac{1}{2}k}\left|\int_0^1 (it)^{k+2} n^{-\frac{1}{2}k} \cdot \gamma_k\left(\frac{t}{n^{\frac{1}{2}}}\right)(1-t)\,dt\right| < \infty$$

As $|a + ib| > |a|$ or $|b|$

$$\sum n^{-1+\frac{1}{2}k}\left|\int_0^1 t^{k+2} n^{-\frac{1}{2}k} \operatorname{Re}\gamma_k\left(\frac{t}{n^{\frac{1}{2}}}\right)(1-t)\,dt\right| < \infty$$

and by expression (3.8) of [23] plus an integration by parts we find

$$\sum n^{-1+\frac{1}{2}(k+4)}\left|\int_{-\infty}^{\infty}\left\{\cos(n^{-\frac{1}{2}}u) - \sum_{s=0}^{\frac{1}{2}(k+3)} \frac{(-1)^s(n^{-\frac{1}{2}}u)^{2s}}{(2s)!}\right\}\frac{dF(u)}{u^2}\right| < \infty \qquad (6)$$

Since the integrand in (6) (call it $b_k(un^{-\frac{1}{2}})\frac{1}{u^2}$) is of constant sign, we have, with T_k defined on p.465 of [23],

$$\sum n^{-1+\frac{1}{2}(k+4)}\left|\int_{|u|>n^{\frac{1}{2}}T_k} b_k(un^{-\frac{1}{2}})\,d\frac{F(u)}{u^2}\right| < \infty$$

and as $b_k(u) > \frac{1}{2}\frac{u^{k+3}}{(k+3)!}$ for $|u| > T_k$,

$$\sum n^{-\frac{1}{2}}\left|\int_{|u|>\sqrt{n}T_k} |u|^{k+1}\,dF(u)\right| < \infty$$

Now $E|X_i|^{k+2} < \infty \iff \sum n^{\frac{1}{2}k} P\{|X_i| > n^{\frac{1}{2}}\} < \infty$ (Lemma 1.4)

but

$$\infty > \sum n^{-\frac{1}{2}} \int_{|u|>n^{\frac{1}{2}}} |u|^{k+1}\,dF(u) \geq \sum n^{\frac{1}{2}k} \int_{|t|>n^{\frac{1}{2}}} dF(u)$$

Thus $E|X_i|^{k+2} < \infty$. Since both odd and even k have now been established, the corollary is proved.

Remark a) In Remark c) at the end of the proof of Lemma 2.6 it is noted that if X_i is symmetric and if $E|X_i|^{k+2} < \infty$ (i,e, $\delta = 0$) for k a positive odd integer, then equation (7) of Chapter 2 holds with $\int_0^A |\gamma(t)| t^{-1}dt < \infty$. Furthermore, from the proof in Chapter 2 that conditions (3) and (4) $\Rightarrow$ (1) under (C) we have that $\int_0^A |\gamma(t)| t^{-1}dt < \infty \Rightarrow$ (1). From corollary 2A.2 we know that (1) $\Rightarrow E|X|^{k+2} < \infty$. Hence, when k is an odd integer and $\delta = 0$, no general result can be obtained, that is, in some cases (1) will be equivalent to $E|X|^{k+2} \log(1 + |X|) < \infty$ whilst in others it will be equivalent to $E|X|^{k+2} < \infty$.

CHAPTER 3

NON-UNIFORM AND UNIFORM BOUNDS FOR $R_{kn}(x)$

3.1. INTRODUCTION: Unless otherwise stated, we take X_i, $i = 1,2,\dots,$ to be a sequence of iid rv's with $EX_i = 0$ and $EX_i^2 = 1$. We define $R_{kn}(x)$ as in Chapter 1, i.e.

$$R_{kn}(x) = F_n(x) - \Phi(x) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \sum_{s=1}^{k} \frac{Q_s(x)}{n^{s/2}}$$

And again, unless we state otherwise, the $Q_s(x)$ will be defined in terms of the moments of X_i and not according to the Ibragimov formulation. We shall denote by $C(k)$ with or without subscripts, a positive constant depending only on k.

So far we have been concerned with convergence rate results for $R_{kn}(x)$ that are independent of x, and indeed in general we have a fairly good idea of how $R_{kn}(x)$ depends on n. The question naturally arises therefore as to how $R_{kn}(x)$ depends on x. Some idea of what happens in the case of $EX^2 < \infty$ is given by the following simple example: taking $EX^2 = 1$, we have $En^{-1}(X_1 + \dots + X_n)^2 = 1$ and so, by Chebyshev's inequality $1 - F_n(x) < x^{-2}$ for $x > 0$ and $F_n(x) < x^{-2}$ for $x < 0$.
These are also valid for $F_n(x)$ replaced by $\Phi(x)$, thus for large x,

$$|F_n(x) - \Phi(x)| < 2x^{-2} .$$

It is clear therefore that for large values of x, uniform estimates ignore a considerable amount of information. One of the first non-uniform estimates for $|F_n(x) - \Phi(x)|$ was found by Esseen (1945) [9] . His estimate is in terms of $\Delta_n = \sup_x |F_n(x) - \Phi(x)|$ and takes the following form

$$|F_n(x) - \Phi(x)| \leqq C \frac{\Delta_n \log \Delta_n^{-1}}{1 + x^2} .$$

Taking $\beta_3 = E|X^3| < \infty$, from Chapter 1[†] $\Delta_n \leqq C\beta_3 n^{-1}$ and thus we have directly from this bound

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[†] The Berry-Esseen result.

$$|F_n(x) - \Phi(x)| < C \frac{n^{-\frac{1}{2}} \log n}{(1 + x^2)} .$$

The development of this topic follows much the same lines as that of the uniform case although no real progress was made until about 1965 when S. V. Nagaev [29] showed that with the same conditions prevailing as for the above bound,

$$|F_n(x) - \Phi(x)| \leqq \frac{C\beta_3}{\left(1 + |x|^3\right)\sqrt{n}}$$

Before moving to estimates of the general remainder term $R_{kn}(x)$ we conclude the treatment of $k = 0$ by giving a bound, due to Osipov and Petrov [31], which does not assume the existence of any moments whatsoever. Bounds can be derived from it for $R_{on}(x)$ but, as we assumed that $EX_i = 0$ and $EX_1^2 < \infty$ when we defined $R_{on}(x)$, this bound is of a much more general nature than simply a bound for $R_{on}(x)$. For convenience we will give the bound for a sequence $\{X_i\}$† of iid rv's rather than just independent rv. For any $a > 0$, b and x,

$$\left| P\left(\frac{1}{a} \sum_{j=1}^{n} X_j - b < x \right) - \Phi(x) \right| \leqq$$

$$\leqq A\left(nP\{|X_i| > \tau_{nx}\} + \frac{|b - \frac{M_n}{a}|}{1 + |x|} + \frac{|1 - \frac{N_n}{a^2}|}{(1 + |x|)^2} + \frac{nE|\overline{X}_i - E\overline{X}_i|^3}{a^3(1 + |x|)^3} \right), \qquad (1)$$

where A is a positive constant, τ_{nx} is a sequence of positive real numbers

$$\overline{X}_j = \begin{cases} X_j, & \text{if } |X_j| < \tau_{nx}, \\ 0, & \text{otherwise}, \end{cases}$$

$$M_n = nE\overline{X}_i \text{ and } N_n = n\left(E\overline{X}_1^2 - (E\overline{X}_1)^2\right) .$$

This is a very good bound in so far as most results of interest concerning $|F_n(x) - \Phi(x)|$ can be obtained from it (see Heyde [17] where the non-uniform counterpart of this bound is discussed - the techniques used there can be applied here if we take $\tau_{nx} = \sqrt{n}(1 + |x|)$).

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† Here, no moments of X_i are assumed to exist.

The general remainder term $R_{kn}(x)$ was considered by Survila [36]; he showed that if Cramer's condition (C) holds and if $E|X|^{k+2} < \infty$, then

$$(1+|x|)^{k+1}\ |R_{kn}(x)| = o(n^{-k/2}).$$

uniformly in x. Bikialis improved this in the case $k = 1$, showing in [3] that for X_i non-lattice and $E|X_i^3| < \infty$,

$$(1+|x|)^3\ |R_{1,n}(x)| = o(n^{-\frac{1}{2}}).$$

Osipov, 1967, [30] considerably extended these results by producing some precise estimates for $R_{kn}(x)$ from which the previous results (assuming (C) holds) readily flow. We shall restate Osipov's results in some detail since his work forms the basis of this chapter. Theorem 3.1 below, is a slightly improved version of Theorem 2 of [30] This version is quoted from V. V. Petrov's book [32] in which the theorem is proved in its entirety (see Theorem 1, Chapter 5, p.197 [32]).

THEOREM 3.1. (Osipov [30]) *Let* $EX_i^2 = 1$ *and* $E|X_i|^{k+2} < \infty$, *for integer* $k \geqq 1$, *then for all* x *and* n,

$$|R_{kn}(x)| \leqq C_1(k)\Big(n^{-\frac{1}{2}k}(1+|x|)^{-(k+2)} \int_{|u|>\sqrt{n}(1+|x|)} |u|^{k+2}dF(u) +$$

$$+\ n^{-(k+1)/2}(1+|x|)^{-(k+3)} \int_{|u|\leqq\sqrt{n}(1+|x|)} |u|^{k+3}dF(u) +$$

$$+\ \Big(\sup_{|t|\geqq\delta}|f(t)| + \frac{1}{2n}\Big)^n \cdot n^{\frac{1}{2}(k+2)(k+3)}\ (1+|x|)^{-(k+3)}\Big),$$

where $\delta = \dfrac{1}{12E|X_i|^3}$.

Remark a) Setting $EX_i^2 = \sigma^2$ (as in Petrov [32]) and substituting in the variables $\frac{X_i}{\sigma}$, we easily recover the dependence on σ^2.

In the original paper, Osipov obtained corresponding uniform bounds for $R_{kn}(x)$. These can be obtained directly from Theorem 3.1 with the exception of the final term. We need only note that

$$\int_{|u|<\sqrt{n}(1+|x|)} |u|^{k+3}dF(u) = \int_{|u|<\sqrt{n}} |u|^{k+3}dF(u) + \int_{\sqrt{n}\leqq|u|<\sqrt{n}(1+|x|)} |u|^{k+3}dF(u).$$

The latter integral is bounded by $\sqrt{n}(1+|x|) \int_{|u|>\sqrt{n}} |u|^{2+k}dF(u)$ and now taking $x = 0$ (being that value of x for which the R.H.S. of the expression in Theorem 3.1 - with the above adjustment - is a maximum) we have

THEOREM 3.2. (Osipov [30]) *If* $EX_i^2 = 1$ *and* $E|X_i|^{k+2} < \infty$ *for some integer* $k \geqq 1$*, then for all* n*,*

$$\sup_x |R_{kn}(x)| \leqq C_2(k)\Big(n^{-\frac{1}{2}k} \int_{|u|>\sqrt{n}} |u|^{k+2}dF(u) +$$

$$+ n^{-\frac{1}{2}(k+1)} \int_{|u|<\sqrt{n}} |u|^{k+3}dF(u) + \Big(\sup_{|t|\geqq\delta} |f(t)| + \frac{1}{2n}\Big)^n (1 + \log n)\Big)$$

where $\delta = (12E|X_1^3|)^{-1}$.

Remark b) Assuming (C), and using Lemma 3.7 with $r = k+2$, $\delta = 0$ and $\ell = k+3$, it easily follows that

$|R_{kn}(x)| = \dfrac{\varepsilon(\sqrt{n}(1+|x|))}{n^{k/2}(1+|x|)^{k+2}}$ and $\sup_x |R_{kn}(x)| = o(n^{-\frac{1}{2}k})$, where $\varepsilon(t)$ is a bounded positive function such that $\varepsilon(t) \to 0$ as $t \to \infty$.

The above remark, combined with the fact that in both these theorems the estimates depend only on 'tail conditions' of the moments, suggest that similar estimates may hold under the milder Ibragimov [23] moment conditions. One of the aims of this chapter will be to obtain these estimates and to examine their ramifications. Our results are set out in Theorems 3.3 and 3.4 and some subsequent remarks. Essentially we obtain Osipov's results whilst demanding the existence of the k + 1st moment and that the k + 2nd moment exist in principal value only.

Unfortunately the estimates we obtain do not allow us to derive results as general as the Ibragimov [23] results since we have had to assume $E|X_i|^{k+1} < \infty$ and $\alpha_j = EX_i^j$, $j = 1,2,\ldots,k+2$, before we can even write down the bounds. To obtain these general results, an estimate is needed for $|R_{kn}(x)|$ which does not assume the existence of any moments other than the second. We naturally think in terms of a generalisation of the Osipov-Petrov bound (1), combined with $R_{kn}(x)$ defined according to the Ibragimov formulation. Indeed, we generalise this setting further by not assuming the existence of any moments and we use it in Theorem 3.5 to give a generalisation of the Osipov-Petrov bound. We provide bounds for

$$A_{kn}(C_n,b_n,x) = \left|P\left(\frac{1}{C_n}\sum_{i=1}^{n} X_i - b_n < x\right) - U_k(x)\right| \tag{2}$$

where $C_n > 0$, $\{b_n\}$ are sequences of real numbers and $U_k(x)$ $\left(= G_{kn}(x)\right.$ of Chapter $\left.2\right)$ is the first k+3 terms of the Chebyshev expansion defined in terms of some arbitrary 'moment' sequence $0,1,\alpha_3,\alpha_4,\ldots$, and of course $U_k(x)$ only depends on $0,1,\alpha_3,\alpha_4,\ldots,\alpha_{k+2}$. The terms in the bound we give are fairly general; however, in a series of remarks following the theorem we show how these can be evaluated.

Finally, in the last theorem of the chapter we show that the uniform version of this bound - like the Osipov-Petrov bound - has, in certain important cases, asymptotic behaviour equivalent to the quantity it bounds.

3.2. RESULTS:

Theorems 3.3 and 3.4 give us essentially Theorems 3.1 and 3.2 under milder assumptions. Unfortunately, we have had to insist that n be large enough to ensure that

$$n^{-\frac{1}{2}(k-1)} \int_{|u|>\sqrt{n}} |u|^{k+1}\, dF(u) < \tfrac{1}{4}\ .$$

In most cases, however, this should be no restriction.

We take $\{X_i\}$ to be a sequence of iid rv's with $EX_i = 0$ and $EX_i^2 = 1$.

THEOREM 3.3. *If* $E|X_i|^{k+1}$ *and* $\lim_{z\to\infty} \int_{-z}^{z} u^{k+2}\, dF(u) = \alpha_{k+2}$ *are finite for integer* $k \geq 1$, *then for all* x *and sufficiently large* n,

$$\begin{aligned}
|R_{kn}(x)| \leq C_3(k) \Big\{ & n^{-\frac{1}{2}(k-1)} (1+|x|)^{-(k+1)} \int_{|u|>\sqrt{n}(1+|x|)} |u|^{k+1}\, dF(u) \; + \\
& + n^{-k/2} (1+|x|)^{-(k+2)} \left| \int^{*}_{|u|>\sqrt{n}(1+|x|)} u^{k+2}\, dF(u) \right| + \\
& + n^{-\frac{1}{2}(k+1)} (1+|x|)^{-(k+3)} \int_{|u|\leq\sqrt{n}(1+|x|)} |u|^{k+3}\, dF(u) \; + \\
& + \Big(\sup_{|t|>\delta} |f(t)| + \frac{1}{2n} \Big)^{n} n^{\frac{1}{2}(k+2)(k+3)} (1+|x|)^{-(k+3)} \Big\}
\end{aligned}$$

where $\delta = (12E|X_i|^3)^{-1}$ *if* $k > 1$ *and* $\delta = \rho$, $0 < \rho < 1$ *if* $k = 1$ *with* ρ *a constant depending on the distribution of* X_i, *and*

$$\int^{*}_{|u|>b} \text{ denoting } \lim_{z\to\infty} \int_{-z}^{z} - \int_{-b}^{b} \; .$$

The next theorem gives the uniform counterparts of Theorem 3.3.

THEOREM 3.4. *Under the conditions of Theorem 3.3 for all* x *and sufficiently large* n,

$$\begin{aligned}
\sup_x |R_{kn}(x)| \leq C_4(k) \Big\{ & n^{-\frac{1}{2}(k-1)} \int_{|u|>\sqrt{n}} |u|^{k+1}\, dF(u) + n^{-\frac{1}{2}k} \left| \int^{*}_{|u|>\sqrt{n}} u^{k+2}\, dF(u) \right| + \\
& + n^{-\frac{1}{2}(k+1)} \int_{|u|\leq\sqrt{n}} |u|^{k+3}\, dF(u) + \Big(\sup_{|t|>\delta} |f(t)| + \frac{1}{2n} \Big)^{n} (1+\log n) \Big\} \; ,
\end{aligned}$$

where δ *is as defined in Theorem 3.3.*

Remark c) In Theorems 3.1-3.4, under condition (C) for n

sufficiently large, $\left(\sup_{|t|>\delta} |f(t)| + \frac{1}{2n}\right) < 1.$

Remark d) Since we have not assumed $E|X_i|^{k+2} < \infty$. the term in Osipov's results involving $\int_{|u|>\sqrt{n}(1+|x|)} |u|^{k+2}\, dF(u)$ appears in our results as the composite term

$$n^{-\frac{1}{2}(k-1)}\left(1+|x|\right)^{-(k+1)} \int_{|u|>\sqrt{n}(1+|x|)} |u|^{k+1}\, dF(u) \;+$$

$$+\; n^{-\frac{1}{2}k}\left(1+|x|\right)^{-(k+2)} \cdot \left| \int^{*}_{|u|>\sqrt{n}(1+|x|)} u^{k+2}\, dF(u) \right| .$$

If $E|X_i|^{k+2}$ is assumed finite, this composite term reduces to that of Osipov.

Remark e) Throughout this remark we assume (C) holds.

i) If $\int_{|u|>z} |u|^{k+1}\, dF(u) = o(z^{-1})$ then

$$\sup_x |R_{kn}(x)| = o(n^{-\frac{1}{2}k}) \quad \text{and} \quad |R_{kn}(x)| = \frac{\varepsilon\left(\sqrt{n}(1+|x|)\right)}{n^{\frac{1}{2}k}(1+|x|)^{k+2}}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$

ii) If $\int_{|u|>z} |u|^{k+1}\, dF(u) = O(z^{-1})$ then

$$\sup_x |R_{k-1,n}(x)| = O(n^{-\frac{1}{2}k}) \quad \text{and} \quad |R_{k-1,n}(x)| = O\left(\frac{1}{n^{\frac{1}{2}k}\left(1+|x|\right)^{k+2}}\right)$$

iii) If $E|X_i|^{k+2} < \infty$ and $\int_{|u|>z} |u|^{k+2}\, dF(u) = O(z^{-\delta})$, $0 < \delta < 1$

then $\sup_x R_{kn}(x) = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$ and $|R_{kn}(x)| = O\left(\frac{1}{n^{\frac{1}{2}(k+\delta)}\left(1+|x|\right)^{k+2+\delta}}\right)$.

These follow easily using Lemma 3.7 below.

In each of the cases *i)*, *ii)* and *iii)*, if we apply theorems 1.3 & 1.4 we find that in the uniform case, the imposed conditions are both necessary and sufficient for their respective order n convergence rates. Thus, in a certain sense, Theorems 3.3 and 3.4 provide optimal estimates for $|R_{kn}(x)|$. Of course *iii)* can also be obtained from Osipov's theorems.

Remark f) If (C) holds, taking $E|X_i|^{k+2+\delta} < \infty$, $0 < \delta < 1$, and $E|X_i|^{k+2} \log\left(1 + |x|\right) < \infty$ for $\delta = 0$,

$$\sum n^{-1+\frac{1}{2}(k+\delta)} \sup_x \left|R_{kn}(x)\right| < \infty .$$

The results of Theorems 2.2 and 2.3 show that these conditions are also necessary for the series to converge providing $0<\delta<1$.

We shall establish the convergence of only the first term on the R.H.S. of the bound in Theorem 3.4 as the same technique is used for establishing the convergence of the remaining terms.

Firstly, we note from Lemma 1.4

that for $0 < \delta < 1$,

$$\sum n^{\frac{1}{2}(k+\delta)} P\left(|X_i| > \sqrt{n}\right) < \infty \iff E|X|^{k+2+\delta} < \infty$$

and that the first term on the R.H.S.of the bound in Theorem 3.4 is less than (using integration by parts)

$$C\left[nP\left(|X_i| > \sqrt{n}\right) + \frac{(k+1)}{n^{\frac{1}{2}(k-1)}} \int_{\sqrt{n}}^{\infty} u^k P\left(|X_i| > u\right) du\right].$$

The second term above can be rewritten as

$$Cn^{-\frac{1}{2}(k-1)} \sum_{j=n}^{\infty} \int_{\sqrt{j}}^{\sqrt{j+1}} u^k P\left(|X_i| > u\right) du$$

which is $\leqq Cn^{-\frac{1}{2}(k-1)} \sum\limits_{j=n}^{\infty} j^{\frac{1}{2}(k-1)} P\left(|X_i| > \sqrt{j}\right)$.

$$\text{Thus } \sum n^{-1+\frac{1}{2}(k+\delta)} \frac{1}{n^{\frac{1}{2}(k-1)}} \int_{|u|>\sqrt{n}} |u|^{k+1} \, dF(u) \leqq \sum n^{\frac{1}{2}(k+\delta)} P\left(|X_i|>\sqrt{n}\right) +$$

$$+ C \sum_{n=1}^{\infty} n^{\frac{1}{2}(\delta-1)} \sum_{j=n}^{\infty} j^{\frac{1}{2}(k-1)} P\left(|X_i| > \sqrt{j}\right)$$

$$< C + C \sum_{j=1}^{\infty} j^{\frac{1}{2}(k-1)} P\left(|X_i| > \sqrt{j}\right) \sum_{n=1}^{j} n^{\frac{1}{2}(\delta-1)}$$

$$< \infty .$$

To treat the case $\delta = 0$, we note that if $E|X|^{k+2} \log \{1 + |X_i|\} < \infty$

$$\begin{aligned} E|X_i|^{k+2} \log \{1 + |X_i|\} &= \int_0^\infty \frac{d}{du} \left(u^{k+2} \log (1+u)\right) P\left(|X_i| > u\right) du \\ &= \sum_{n=1}^\infty \int_{\sqrt{n}}^{\sqrt{n+1}} \frac{d}{du} \left(u^{k+2} \log(1+u)\right) P\left(|X_i| > u\right) du \\ &\geqq \sum_{n=1}^\infty P\left(|X_i| > \sqrt{n+1}\right) u^{k+2} \log(1+u) \Big|_n^{n+1} \\ &\geqq C \sum_{n=1}^\infty n^{\frac{1}{2}k} \log(n)\, P\left(|X_i| > \sqrt{n+1}\right) \\ &\geqq \sum_{n=2}^\infty n^{\frac{1}{2}k} \log(n)\, P\left(|X_i| > \sqrt{n}\right). \end{aligned}$$

So $\sum_{n=1}^\infty n^{\frac{1}{2}k} \log n\, P\left(|X_i| > \sqrt{n}\right) < \infty$. Treating the terms on the R.H.S. of the bound in Theorem 3.4 as was done for the case $0 < \delta < 1$, and using this result, we establish the assertion of the remark.

We now turn to estimating $A_{kn}(c_n, b_n, x)$ as defined by Expression (2). The following theorem generalises the Osipov-Petrov bound (1) above, whose notation we preserve.

THEOREM 3.5. Let X_i *be a sequence of* iid rv's, *then for any integer* $k \geqq 1$,

$$\begin{aligned} &\left|P\left(\frac{1}{C_n} \sum_1^n X_i - b_n \leqq x\right) - U_k(x)\right| \leqq nP\left(|X_i| > \tau_{nx}\right) + \\ &+ c(k)\Bigg[\left(1+|x|\right)^{-(k+3)} n^{-\frac{1}{2}(k+1)} \bigg\{\sigma_n^{-\frac{1}{2}(k+1)(k+2)} \bigg(\int_{|u|<\tau_{nx_o}} |u|^{k+2}\, dF(u)\bigg)^{1+\frac{1}{k}} + \\ &+ \sigma_n^{-(k+3)} \bigg|\overset{*}{\int_{|u|<\tau_{nx_o}}} u^{k+3} dF(u)\bigg|\bigg\} + \left(1+|x_o|\right)^{-(k+4)} \sigma_n^{-(k+4)} n^{-\frac{1}{2}(k+2)} \int_{|u|<\tau_{nx_o}} |u|^{k+4} dF(u) + \\ &+ \bigg\{\sup_{|t|>\delta_n} |f(t)| + 2P\left(|X_i| > \tau_{nx_o}\right) + \frac{1}{2n}\bigg\}^n n^{\frac{1}{2}(k+3)(k+4)} \left(1+|x_o|\right)^{-(k+4)}\Bigg] + \\ &+ \left|U_{k,n}(x_o) - U_{kn}(x)\right| + \left|U_{kn}(x) - U_k(x)\right| \\ &= \theta_{k,n}(C_n, b_n, x) \end{aligned}$$

where $\delta_n = \sigma_n^2(96\, E|\overline{X}_i|^3)^{-1}$, $x_o = px + q$, $p = \dfrac{C_n}{\sqrt{n}\sigma_n}$, $q = \dfrac{C_n b_n - nE\overline{X}_i}{\sqrt{n}\sigma_n}$ $\sigma_n^2 = \text{var}\,\overline{X}_i$, *and* $U_{k,n}(x)$ *is the first* $k + 1$ *terms of the Chebyshev expansion defined in terms of* $E(\overline{X}_i - E\overline{X}_i)^j$, $j = 1, 2, \ldots, k + 2$.

Remark g) From the proof of Theorem 3.3 it is easy to show that

$$|U_{k,n}(x_o) - U_{k,n}(x)| \leqq |\Phi(x_o) - \Phi(x)| +$$

$$+ C_1(k)|x_o - x|\left(1 + |x_o|^{3k+6} + |x|^{3k+6}\right)e^{\frac{1}{2}|x_o^2 - x^2| - \frac{1}{2}x^2} \cdot L_{k+2,n}.$$

where $L_{k+2,n} = n(n\sigma_n)^{-\frac{1}{2}(k+2)} \displaystyle\int_{|u|<\tau_{nx}} |u|^{k+2}\, dF(n)$.

Remark h) From Petrov (p.201,[32]),

$$|U_{k,n}(x) - U_k(x)| \leqq \sum_{\nu=1}^{k+2} n^{-\frac{1}{2}\nu} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\Pi}} |Q_\nu(x) - Q_{\nu,n}(x)|$$

$$= \sum_{\nu=1}^{k+2} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\Pi}} \left| \sum \frac{H_{\nu+2s-1}(x)}{s_1!\ldots s_\nu!(1!)^{s_1}\ldots(\nu!)^{s_\nu}} \sum_{\ell=1}^{\nu} \left(\frac{\gamma_3}{n^{\frac{1}{2}}}\right)^{s_1} \cdots \left(\frac{\gamma_{\ell+1}}{n^{\frac{1}{2}(\ell-1)}}\right)^{s_{\ell-1}} \times \right.$$

$$\left. \times \left\{\left(\frac{\gamma_{\ell+2}}{n^{\frac{1}{2}\ell}}\right)^{s_\ell} - \left(\frac{\gamma_{\ell+2,n}}{\sigma_n^{(\ell+2)} n^{\frac{1}{2}\ell}}\right)^{s_\ell}\right\} \left(\frac{\gamma_{\ell+3,n}}{\sigma_n^{(\ell+3)} n^{\frac{1}{2}(\ell+1)}}\right)^{s_{\ell+1}} \cdots \left(\frac{\gamma_{\nu+2,n}}{\sigma_n^{(\nu+2)} n^{\frac{1}{2}\nu}}\right)^{s_\nu} \right| .$$

where $H_m(x)$ is the m^{th} Chebyshev-Hermite polynomial:

$$H_m(x) = m! \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^k x^{m-2k}}{k!(m-2k)!2^k},$$

$\sigma_n = \text{var}\,\overline{X}_j$, γ_s is the s^{th} 'cumulant' associated with the 'moment' sequence defining U_k, and $\gamma_{s,n}$ is the s^{th} cumulant of the rv $(\overline{X}_i - E\overline{X}_i)$.

The final theorem of the chapter shows that in certain cases $A_{k,n}(C_n) = \sup_x A_{k,n}(C_n, b_n \equiv 0, x)$ and $\Theta_{k,n}(C_n) = \sup_x \Theta_{k,n}(C_n, b_n \equiv 0, x)$ have equivalent asymptotic behaviour. We have taken $b_n \equiv 0$ since we consider a sequence $\{X_i\}$ of iid rv's with $EX_i = 0$ if $E|X_i| < \infty$. Also we take $\tau_{nx} = \sqrt{n}$. Firstly, however, we need to suitably define $\Theta_{k,n}(C_n)$ for $k = 0$. The usual Osipov-Petrov (uniform) bound, as it appears for example in Heyde [17], is not satisfactory since it involves a term in $E|\overline{X}_i|^3$. The bound is too rough if we are imposing the conditions that the third moment exists in principal value only and that

$$\int_{|u|>z} u^2 dF(u) = O(z^{-1})$$ (Lemma 3.7 gives us no information in this case.)

It was for this reason that the case $\delta = 1$ in Theorem 3 of Heyde [17] could not be considered.

The term in $E|\overline{X}_i|^3$ appears in the estimate of $\sup_x \left| P\left\{ \sum_1^n \frac{(\overline{X}_i - E\overline{X}_i)}{\sqrt{n}\,\sigma_n} \leqq x \right\} - \Phi(x) \right|$; we therefore require an estimate that avoids the use of this absolute moment. The following bound has this desired property:

$$\sup_x \left| P\left\{ \sum_1^n \frac{(\overline{X}_i - E\overline{X}_i)}{\sqrt{n}\sigma_n} \leqq x \right\} - \Phi(x) \right| \leqq C\left\{ n^{-\frac{1}{2}} |\alpha_{3,n}| + n^{-1}\alpha_{4,n} + n^{-\frac{1}{2}} d_n^{-1} \right\} \qquad (2a)$$

where $\alpha_{r,n} = EY_n^r$ $(r = 3,4)$, $Y_n = \sigma_n^{-1}(\overline{X}_i - E\overline{X}_i)$, $\overline{F}(u) = P(Y_n \leqq x)$.

and d_n is such that

$$\max_{0<|t|<d_n} \left\{ 2\int_{|ut|>1} u^2 d\overline{F}(u) + \frac{|t|}{3} \left| \int_{|ut|<1} u^3 d\overline{F}(u) \right| + \frac{t^2}{12} \int_{|ut|<1} u^4 d\overline{F}(u) \right\} < \frac{5}{12}$$

We will derive this bound in the proof of the next theorem but in the meantime we define

$$\theta_{o,n}(C_n) = nP(|X_1| > \tau_n) + C\{ n^{-\frac{1}{2}} |\alpha_{3,n}| + n^{-1}\alpha_{4,n} + n^{-\frac{1}{2}} d_n^{-1} +$$

$$+ \sqrt{n}\sigma_n^{-1} |E\overline{X}_1| + |1 - C_n^{-2} n\sigma_n^2| \max(1, C_n^2 n^{-1} \sigma_n^{-2}) \} \qquad (2b)$$

with C an absolute constant and $\{\tau_n\}$ a sequence of positive constants.

We are now in a position to state our last theorem.

THEOREM 3.6. Let $\{X_i\}$ *be a sequence of* iid rv's *such that* $EX_i = 0$ *if* $E|X_i|<\infty$. *If Cramér's condition* (C) *holds for* $k\geq 1$ *and if* $0<\delta\leq 1$, *then for any integer* $k\geq 0$ *the following four conditions are equivalent:*

(i) $E|X_i|^{k+2} < \infty,\ \sigma^2 = EX_i^2,\ \ \alpha_j = \sigma^{-j}\ EX_i^j$ for $j = 0, 1, \ldots, k+2$,

$$\int_{|u|>z} |u|^{k+2} dF(u) = O(z^{-\delta}) \text{ as } z \to \infty$$

and as well for $\delta = 1$, $\displaystyle\int_{|u|<z} u^{k+3} dF(u) = O(1)$.

(ii) $\inf\limits_{C_n} A_{kn}(C_n) = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$ as $n \to \infty$.

(iii) $A_{kn}(\sqrt{n}\sigma_n) = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$ as $n \to \infty$.

(iv) $\Theta_{kn}(\sqrt{n}\,\sigma_n) = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$ as $n \to \infty$ $(\tau_{nx} = \tau_n = \sqrt{n})$.

where $\sigma_n^2 = \text{var}\ \overline{X}_1$.

Since neither Petrov's book [32] nor Osipov's article [30] is available in English, much more detail has been given than would otherwise have been necessary. To establish these theorems we require most of the lemmas used in Petrov [32] to prove Theorem 3.1. Some of these lemmas we quote without alteration whilst others have been changed to suit our purposes.

3.3. SOME LEMMAS. The first lemma is used to derive the uniform bound from its non-uniform counterpart. Its proof is elementary and appears on p.208 of [32] .

LEMMA 3.7. If

$$\int_{|u|>y} |u|^r dF(u) = o(y^{-\delta}),\ r\geq 2,\ \delta\geq 0,$$

then for $\ell > r + \delta$,

$$\int_{|u|\leq z} |u|^{\ell} dF(u) = o(z^{\ell-r-\delta})$$

The corresponding large order result is also valid.

LEMMA 3.8. (p.173 [32]). *Let* $X_i,\dots,X_n$ *be independent* rv's, $EX_j = 0$ *and* $EX_j^2 = \sigma_j^2 < \infty$ $(j = 1,\dots,n)$. *Set*

$$B_n = \sum_{j=1}^{n} \sigma_j^2 \quad \text{and} \quad L_{kn} = B_n^{-\frac{1}{2}k} \sum_{j=1}^{n} E|X_j|^k$$

for $k \geqq 3$. *If* $3 \leqq m \leqq k$ *then* $L_{mn}^{1/(m-2)} \leqq L_{kn}^{1/(k-2)}$.

LEMMA 3.9. (p.174, [32]) *Let the function* $y = y(x)$ *have derivatives of order* $\nu \geqq 1$. *Then*

$$\frac{d^\nu}{dx^\nu} y^n(x) =$$

$$= \nu! \sum_{k=1}^{\min(\nu,n)} {\sum}' \frac{n!}{(n-k)!} y^{n-k}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{1}{m!} \frac{d^m}{dx^m} y(x)\right)^{k_m},$$

where $\sum'$ *denotes summation over all non-negative integer solutions of the following system of equations:*

$$k_1 + 2k_2 + \dots + \nu k_\nu = \nu,$$

$$k_1 + k_2 \dots + k_\nu = k.$$

LEMMA 3.10. (p.175 [32]). *Let* X *be a random variable and* $f(t)$ *its characteristic function. Let* $EX = 0$, $EX^2 = \sigma^2 > 0$, $E|X|^s = \beta_s < \infty$ *for some integer* $s \geqq 3$. *Set* $f_n(t) = \{f(\frac{t}{\sigma\sqrt{n}})\}^n$.

Then, in the interval

$$|t| < \sqrt{n}\left(\frac{\sigma^s}{\beta_s}\right)^{(s-2)^{-1}}$$

we have

$$\left|\frac{d^m}{dt^m}\left(f_n(t) - e^{-\frac{1}{2}t^2}\{1 + \sum_{\nu=1}^{s-3} P_\nu(it)\, n^{-\frac{1}{2}\nu}\}\right)\right| \leqq$$

$$\leqq C(s) \frac{\beta_s}{\sigma^s n^{\frac{1}{2}(s-2)}} \left(|t|^{s-m} + |t|^{3(s-1)+m}\right) e^{-t^2/12},$$

for $m = 0,1,\dots,s-1$. *Here the* $P_\nu(it)$, *already introduced in* Section 2.3 *and well known in asymptotic expansion theory, are polynomials defined in terms of the moments of* X.

LEMMA 3.11.† (p.183, [32]). *Let* X *be a* rv *with* $EX = 0$, $0 < EX^2 = \sigma^2 < \infty$ *and* $P(X < x) = F(x)$. *Let*

$$Y_n = \begin{cases} X, & \text{if } |X| < \sigma\sqrt{n}, \\ 0, & \text{otherwise}, \end{cases}$$

$$V_n(x) = P(Y_n \leqq x),$$

$$Y_{nx} = \begin{cases} X, & \text{if } |X| < \sigma\sqrt{n}(1 + |x|), \\ 0, & \text{otherwise}, \end{cases}$$

and set $Z_{nx} = X - Y_{nx}$.

If $E|X|^k < \infty$ *for any integer* $k \geqq 2$, *then*

$$|F^{*n}(x\sigma\sqrt{n}) - V_n^{*n}(x\sigma\sqrt{n})| \leqq$$

$$\leqq c(k)\left(\frac{E|Z_{n,x}|^k}{\sigma^k n^{\frac{1}{2}(k-2)}(1 + |x|)^k} + \frac{E|Y_{n,x}|^{k+2} - E|Y_n|^{k+2}}{\sigma^{k+2} n^{\frac{1}{2}k}(1 + |x|)^{k+2}}\right)$$

for any n *and* x.

LEMMA 3.12. (p.190, [32]) *Let* G(x) *be a function of bounded variation on the real line and let* g(t) *be its Fourier-Stieltjes transform. Let* $\lim_{|x|\to\infty} G(x) = 0$ *and*

$$\int_{-\infty}^{\infty} |x|^m \, |dG(x)| < \infty$$

for some integer $m \geqq 1$.

Then the function $x^m G(x)$ *is of bounded variation on the real line and we have*

$$(-it)^m \int_{-\infty}^{\infty} e^{itx} \, d\left(x^m G(x)\right) = m! \sum_{\nu=0}^{m} \frac{(-t)^m}{\nu!} \frac{d^\nu}{dt^\nu} g(t).$$

LEMMA 3.13. (p.193, [32]). *Let* F(x) *be a non-decreasing function, let* G(x) *be a differentiable function of bounded variation with* $F(-\infty) = G(-\infty)$ *and* $F(+\infty) = G(+\infty)$ *and let* f(t) *and* g(t) *be the respectiveFourier-Stieltjes transforms. Let*

$$\int_{-\infty}^{\infty} |x|^s \, \left|d\left(F(x) - G(x)\right)\right| < \infty$$

and $|G'(x)| \leqq K(1 + |x|)^{-s}$ $(-\infty < x < \infty)$

---

†This lemma is proved below.

for any integer $s \geqq 2$, *where* K *is a constant. Then*

$$|F(x) - G(x)| \leqq c(s)\left(1 + |x|\right)^{-s}\left(\int_{-T}^{T} \left|\frac{f(t) - g(t)}{t}\right| dt + \right.$$

$$\left. + \int_{-T}^{T} \left|\frac{\delta_s(t)}{t}\right| dt + \frac{K}{T}\right)$$

for any x *and* $T > 1$. *Here*

$$\delta_s(t) = \int_{-\infty}^{\infty} e^{itx} d\left\{x^s\left(F(x) - G(x)\right)\right\},$$

and c(s) *is a positive constant depending only on* s.

Proof of Lemma 3.11. The proof is almost identical to that of the corresponding lemma in Petrov [32] . We will quote some relevant results from his proof and show how these can be used to obtain our lemma.

Set

$$a_n = P\left(\sigma\sqrt{n} \leqq |X| < \sigma\sqrt{n}(1 + |x|)\right),$$

$$N_n(y) = V_n(y) - a_n D(y) , \qquad D(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } y \geqq 0, \end{cases}$$

and $\quad M_n(y) = P(Y_{nx} < y).$

From p.184 of [32] we have

$$\int_{\infty}^{\infty} |y|^m \left|d\left(N_n^{*\ell}(y\sigma\sqrt{n}) - V_n^{*\ell}(y\sigma\sqrt{n})\right)\right| \leqq c(m)na_n,$$

for $\ell = 1,\ldots,n$ and any $m \geqq 0$. Setting $\ell = n$ and $m = k+2$ (instead of $k+1$, as taken by [32]), we obtain for any $x \leqq 0$

$$|N_n^{*n}(x\sigma\sqrt{n}) - V_n^{*n}(x\sigma\sqrt{n})| = \left|\int_{-\infty}^{x} d\left(N_n^{*n}(y\sigma\sqrt{n}) - V_n^{*n}(y\sigma\sqrt{n})\right)\right|$$

$$\leqq \left(1 + |x|\right)^{-(k+2)} \int_{-\infty}^{x} \left(1 + |y|\right)^{k+2} \left|d\left(N_n^{*n}(y\sigma\sqrt{n}) - V_n^{*n}(y\sigma\sqrt{n})\right)\right| \leqq$$

$$\leqq c(k)na_n\left(1 + |x|\right)^{-(k+2)}. \tag{3}$$

Also, inequality (2.25) on p.187 of [32] states:

$$\int_{-\infty}^{\infty} |y|^m \left| d\left(M_n^{*\ell}(y\sigma\surd n) - N_n^{*\ell}(y\sigma\surd n)\right)\right| \leqq c(m)\sigma^{-m} n^{-\frac{1}{2}(m-2)} a_{n,m}$$

for $\ell = 1,\ldots,n$ and $m = 2,3,\ldots,$ where $a_{n,m} = E|Y_{n,x}|^m - E|Y_n|^m$.

So, in the same way as above, we find

$$\left|M_n^{*n}(x\sigma\surd n) - N_n^{*n}(x\sigma\surd n)\right| \leqq c(k)\sigma^{-(k+2)} n^{-\frac{1}{2}k} \quad a_{n,k+2}(1 + |x|)^{-(k+2)} \tag{4}$$

for $x \leqq 0$. From p.190 of [32] we have

$$|F^{*n}(x\sigma\surd n) - M_n^{*n}(x\sigma\surd n)| \leqq \sigma^{-k} n^{-\frac{1}{2}(k-2)}(1 + |x|)^{-k} E|Z_{n,x}|^k \tag{5}$$

for any n and x. Noting that $a_n \leqq (\sigma\surd n)^{-(k+2)} a_{n,k+2}$ and collecting together (3), (4) and (5) we establish the assertion of the lemma for $x \leqq 0$. If we now consider the rv $-X$ we have the lemma true for $x > 0$.

3.4. PROOFS OF THEOREMS. The proof of Theorem 3.3 follows very closely that of Theorem 3.1 on pp. 197-207 of [32]. For convenience, wherever possible, we will preserve the notation of [32]. To be consistent with the results of Chapter 2, we have taken $k \geqq 1$ in contrast to Petrov where $k \geqq 3$. Also, Petrov's $Q_j(x)$ equals $(2\pi)^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}x^2}$ times our $Q_j(x)$ and Petrov's $U_k(x)$ is $U_{k-2}(x)$ according to our definition. We shall need some further notation and auxiliary results.

We recall that we are dealing with a sequence $\{X_i\}$ of iid rv's having zero means and unit variances. Let X be a rv with distribution that of X_1. For $n \geqq 1$, set

$$Y_n = \begin{cases} X, & \text{if } |X| < \surd n, \\ 0, & \text{otherwise,} \end{cases}$$

$Z_n = X - Y_n$, $V_n(x) = P(Y_n < x)$, $\sigma_n^2 = E(Y_n - EY_n)^2$, $W_n(x) = P(Y_n - EY_n < x)$

and $w_n(t) = \int_{-\infty}^{\infty} e^{itx}\, d\,W_n(x)$.

Let further

$$G_n(x) = W_n^{*n}\,(x\sigma_n\sqrt{n}) \quad \text{and} \quad g_n(t) = w_n^n\left(\frac{t}{\sigma_n\sqrt{n}}\right).$$

We define $Q_{\nu,n}(x)$ in terms of the cumulants γ_{jn} of the rv $Y_n - E\,Y_n$. In fact

$$Q_{\nu,n}(x) = -\sum H_{\nu+2s-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}\left(\frac{\gamma_{m+2,n}}{(m+2)!\,\sigma_n^{m+2}}\right)^{k_m},$$

where the summation is over all non-negative integer solutions of the equations

$$k_1 + 2k_2 + \dots + \nu k_\nu = \nu$$
$$k_1 + k_2 + \dots + k_\nu = s$$

We set

$$U_k(x) = \Phi(x) + \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \sum_{\nu=1}^{k} Q_\nu(x) n^{-\frac{1}{2}\nu}$$

$$U_{\ell n}(x) = \Phi(x) + \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \sum_{\nu=1}^{\ell} Q_{\nu,n}(x) n^{-\frac{1}{2}\nu}, \quad \ell = 0,1,\dots$$

$$u_k(t) = \int_{-\infty}^{\infty} e^{itx}\,d\,U_k(x) \quad \text{and} \quad u_{\ell,n}(t) = \int_{-\infty}^{\infty} e^{itx}\,d\,U_{\ell,n}(x).$$

Finally we define

$$L_{\nu,n} = n^{-\frac{1}{2}(\nu-2)}\,E|Y_n|^\nu \qquad (\nu = 1,2,\dots,),$$

$$\Lambda_{\nu,n} = n^{-\frac{1}{2}(\nu-2)}\,E|Z_n|^\nu \qquad (\nu = 1,2,\dots,k+1),$$

$$p_n = \sigma_n^{-1}, \quad \text{and} \quad q_n = -\sigma_n^{-1}\,E\,Y_n\sqrt{n}\,.$$

Our first task is to show that for all n satisfying $\Lambda_{k+1,n} < \frac{1}{4}$, and for all x

$$|U_k(x) - U_{k+1,n}\,(p_n x + q_n)| \leq c(k)\left(\Lambda_{k+1,n} + n^{-\frac{1}{2}k}\int^{*}_{|u|>\sqrt{n}} u^{k+2}\,d\,F(u) + L_{k+3,n}\right)e^{-\frac{1}{4}x^2} \tag{6}$$

The condition $\Lambda_{k+1,n} < \frac{1}{4}$ means that

$$E|Z_n|^{k+1} < \tfrac{1}{4}n^{\frac{1}{2}(k-1)} \tag{6a}$$

From the definition of σ_n it follows that

$$\sigma_n^{\,2} = 1 - EZ_n^2 - (EY_n)^2.$$

Since

$$EZ_n^2 \leqq n^{-\frac{1}{2}(k-1)} \; E|Z_n|^{k+1} < \tfrac{1}{4}$$

and

$$|EY_n| = |EZ_n| \leqq n^{-\frac{1}{2}k} \; E|Z_n|^{k+1} = \frac{1}{\sqrt{n}} \Lambda_{k+1,n}$$

we have

$$\frac{11}{16} < \sigma_n^2 \leqq 1 \tag{7}$$

and

$$1 - \sigma_n^{\ell} \leqq \ell(1 - \sigma_n) \leqq \ell\frac{\Lambda_{k+1,n}}{(1+\sigma_n)} \left(1 + \frac{1}{n}\Lambda_{k+1,n}\right) \leqq \frac{5}{4}\,\ell\Lambda_{k+1,n} \tag{8}$$

for $\ell = 1, 2 \ldots$.

Further $|EY_n| \leqq \frac{1}{\sqrt{n}} EZ_n^2 \leqq n^{-\frac{1}{2}}$

$$E|Y_n|^{\nu} \leqq n^{\frac{1}{2}(\nu-2)} EY_n^2 \leqq n^{\frac{1}{2}(\nu-2)} \qquad (\nu = 2,3,\ldots) \tag{9}$$

Thus

$$|E(Y_n - EY_n)^{\nu} - EY_n^{\nu}| = \left|\sum_{\ell=1}^{\nu} \frac{\nu!\,(-1)^{\ell}}{\ell!\,(\nu-\ell)!} (EY_n)^{\ell}\; EY_n^{\nu-\ell}\right|$$

$$= |EY_n| \left| \sum_{\ell=1}^{\nu-1} \frac{\nu!}{\ell!\,(\nu-\ell)!} (EY_n)^{\ell-1}\; EY_n^{\nu-\ell} + (EY_n)^{\nu-1} \right| \leqq$$

$$\leqq |EY_n| \left(\sum_{\ell=1}^{\nu-1} \frac{\nu!}{\ell!\,(\nu-\ell)!}\; n^{-\frac{1}{2}(\ell-1)+\frac{1}{2}(\nu-\ell-2)} + n^{-\frac{1}{2}(\nu-1)} \right) \leqq$$

$$\leqq n^{-\frac{1}{2}k}\; E|Z_n|^{k+1}\; n^{\frac{1}{2}(\nu-1)} \left(\sum_{\ell=1}^{\nu-1} \frac{\nu!}{\ell!\,(\nu-\ell)!}\; n^{-\ell} + n^{-\ (\nu-1)} \right) \leqq$$

$$\leqq 2^{\nu} n^{-\frac{1}{2}(k-\nu+1)}\; E|Z_n|^{k+1}, \qquad \nu = 1,2,\ldots,k+2.$$

Using the relation

$$EX^{\nu} = EY_n^{\nu} + EZ_n^{\;\nu}$$

we have

$$|EX^{\nu} - EY_n^{\nu}| = |EZ_n^{\nu}| \leqq \begin{cases} n^{\frac{1}{2}(\nu-k-1)} \; E|Z_n|^{k+1}, & \text{for } \nu<k+2, \\ |EZ_n^{\;k+2}| & \text{for } \nu = k+2 \end{cases}$$

and hence

$$|E(Y_n - EY_n)^\nu - EX^\nu| \leqq \begin{cases} (2^\nu+1)n^{\frac{1}{2}(\nu-k-1)}E|Z_n|^{k+1}, & \text{for } \nu < k+2, \\ 2^{k+2}n^{\frac{1}{2}}E|Z_n|^{k+1} + |EZ_n^{k+2}|, & \text{for } \nu = k+2. \end{cases} \tag{10}$$

From (6a),(7) and (9) it follows that

$$E|Y_n - EY_n|^\nu \leqq 2^\nu E|Y_n|^\nu < \left(\frac{2 \times 16}{11}\right)^\nu \sigma_n^\nu n^{\frac{1}{2}(\nu-2)} < 3^\nu \sigma_n^\nu n^{\frac{1}{2}(\nu-2)} \tag{11}$$

for $\nu = 1,2,\ldots, k+2$ and

$$E|X|^\nu = E|Y_n|^\nu + E|Z_n|^\nu \leqq 2n^{\frac{1}{2}(\nu-2)} \tag{12}$$

for $\nu = 1,\ldots,k+1$.

From (7), (8), (10) and (11) we have

$$\begin{aligned} n^{-\frac{1}{2}\nu}|\sigma_n^{-\nu}E(Y_n - EY_n)^\nu - EX^\nu| &\leqq n^{-\frac{1}{2}\nu}|E(Y_n - EY_n)^\nu - EX^\nu| + \\ &\quad + n^{-\frac{1}{2}\nu}|1 - \sigma_n^\nu|E|Y_n - EY_n|^\nu \\ &\leqq \begin{cases} c(\nu)n^{-1}\Lambda_{k+1,n}, & \text{for } \nu < k+2 \\ c(k)n^{-1}\Lambda_{k+1,n} + n^{-\frac{1}{2}(k+2)}|EZ_n^{k+2}| & \text{for } \nu = k+2. \end{cases} \end{aligned}$$

Also for any real a and b and any integer $m \geqq 1$,

$$|a^m - b^m| \leqq m|a - b|\max\left(|a|^{m-1}, |b|^{m-1}\right)$$

Therefore

$$\begin{aligned} &\left|\left(\sigma_n^{-\nu}n^{-\frac{1}{2}\nu}E(Y_n - EY_n)^\nu\right)^\ell - \left(n^{-\frac{1}{2}\nu}EX^\nu\right)^\ell\right| \leqq \\ &\leqq \begin{cases} c(k)\Lambda_{k+1,n}n^{-1} & \text{if } \ell,\nu < k+2 \\ c(k)n^{-1}\left(\Lambda_{k+1,n} + n^{-\frac{1}{2}k}|EZ_n^{k+2}|\right) & \text{if } \nu = k+2 \text{ and } \ell = 1 \end{cases} \end{aligned} \tag{13}$$

From the well known relation expressing cumulants in terms of moments, we have on setting $\alpha_\nu = EX^\nu$, $\nu = 1,\ldots,k+2$ and $\alpha_{n,\nu} = E(Y_n - EY_n)^\nu$, $\nu = 1,2,\ldots$.

$$\left(\frac{\gamma_{\nu,n}}{\sigma_n^{\nu} n^{\frac{1}{2}\nu}} - \frac{\gamma_\nu}{n^{\frac{1}{2}\nu}}\right) = \nu! \sum \frac{(-1)^{s-1}(s-1)!}{s_1!\ldots s_\nu!(1!)^{s_1}\ldots(\nu!)^{s_\nu}} \sum_{\ell=1}^{\nu} \left(\frac{\alpha_{1,n}}{\sigma_n n^{\frac{1}{2}}}\right)^{s_1} \ldots$$

$$\ldots\left(\frac{\alpha_{\ell-1,n}}{\sigma_n^{\ell-1} n^{\frac{1}{2}(\ell-1)}}\right)^{s_{\ell-1}} \times \left(\left(\frac{\alpha_{\ell,n}}{\sigma_n^{\ell} n^{\frac{1}{2}\ell}}\right)^{s_\ell} - \left(\frac{\alpha_\ell}{n^{\frac{1}{2}\ell}}\right)^{s_\ell}\right) \cdot \left(\frac{\alpha_{\ell+1}}{n^{\frac{1}{2}(\ell+1)}}\right)^{s_{\ell+1}} \ldots \left(\frac{\alpha_\nu}{n^{\frac{1}{2}\nu}}\right)^{s_\nu} ,$$

where summation is over all non-negative solutions of the set of equations

$$s_1 + 2s_\alpha + \ldots + \nu s_\nu = \nu$$

and
$$s_1 + s_2 + \ldots + s_\nu = s.$$

Since for each ν, s_ν can only be 0 or 1, we need only treat the case $\nu = k + 2$, $\ell = 1$, in inequality (13). If ℓ had been > 1 for $\nu = k + 1$, we could not have written the constant c(k), since $n^{-\frac{1}{2}k}|EZ_n^{k+2}|$ is not less than some absolute constant.

From the above relation and expressions (11)-(13),

$$\left|\frac{\gamma_{\nu,n}}{\sigma_n^{\nu} n^{\frac{1}{2}\nu}} - \frac{\gamma_\nu}{n^{\frac{1}{2}\nu}}\right| \leqq \begin{cases} \dfrac{c(k)}{n} \Lambda_{k+1,n}, & (\nu = 1,\ldots,k+1) \\[2ex] \dfrac{c(k)}{n} \Lambda_{k+1,n} + n^{-\frac{1}{2}k}|EZ_n^{\ k+2}|, & \text{for } \nu = k+2. \end{cases}$$

It follows that

$$\left|\left(\frac{\gamma_{\nu,n}}{\sigma_n^{\nu} n^{\frac{1}{2}\nu}}\right)^{\ell} - \left(\frac{\gamma_\nu}{n^{\frac{1}{2}\nu}}\right)^{\ell}\right| \leqq \frac{c(k)}{n} \Lambda_{k+1,n} \quad \text{for } \ell,\nu = 1,\ldots,k+1.$$

From the definitions of $Q_\nu(x)$ and $Q_{\nu,n}(x)$ together with the definition of $H_m(x)$ in Remark h), for $\nu \leqq k$ and all x,

$$e^{-\frac{1}{2}x^2} n^{-\frac{1}{2}\nu}|Q_\nu(x) - Q_{\nu,n}(x)| = e^{-\frac{1}{2}x^2} \left| \sum \frac{H_{\nu+2s-1}(x)}{s_1!\ldots s_\nu!(1!)^{s_1}\ldots(\nu!)^{s_\nu}} \right. \times$$

$$\times \sum_{\ell=1}^{\nu} \frac{(\gamma_3)^{s_1}}{n^{\frac{1}{2}s_1}} \ldots \left(\frac{\gamma_{\ell+1}}{n^{\frac{1}{2}(\ell-1)}}\right)^{s_{\ell-1}} \times \left(\left(\frac{\gamma_{\ell+2}}{n^{\frac{1}{2}\ell}}\right)^{s_\ell} - \left(\frac{\gamma_{\ell+2,n}}{\sigma_n^{\ell+2} n^{\frac{1}{2}\ell}}\right)^{s_\ell}\right) \times$$

$$\times \left(\frac{\gamma_{\ell+3,n}}{\sigma_n^{\ell+3} n^{\frac{1}{2}(\ell+1)}}\right)^{s_{\ell+1}} \ldots \left.\left(\frac{\gamma_{\nu+2,n}}{\sigma_n^{\nu+2} n^{\frac{1}{2}\nu}}\right)^{s_\nu} \right| \leqq$$

$$\leqq \begin{cases} c(k)\Lambda_{k+1,n}e^{-\frac{1}{4}x^2}, & \text{for } \nu < k, \\ c(k)\Lambda_{k+1,n}e^{-\frac{1}{4}x^2} + n^{-\frac{1}{2}k}|EZ_n{}^{k+2}|e^{-\frac{1}{4}x^2}, & \text{for } \nu = k \end{cases}$$

Thus $|U_k(x) - U_{k,n}(x)| \leqq \sum_{\nu=1}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}|Q_\nu(x) - Q_{\nu,n}(x)|$

$$\leqq c(k)\left(\Lambda_{k+1,n} + n^{-\frac{1}{2}k}|EZ_n{}^{k+2}|\right) e^{-\frac{1}{4}x^2} \tag{14}$$

for all x.

Next, we bound $|U_{k,n}(x_o) - U_{k,n}(x)|$ where $x_o = p_n x + q_n$.

Now

$$n^{-\frac{1}{2}\nu}|e^{-\frac{1}{2}x_o{}^2}Q_{\nu,n}(x_o) - e^{-\frac{1}{2}x^2}Q_{\nu,n}(x)|$$

$$\leqq |e^{-\frac{1}{2}x_o{}^2} - e^{-\frac{1}{2}x^2}|n^{-\frac{1}{2}\nu}|Q_{\nu,n}(x_o)| + n^{-\frac{1}{2}\nu}e^{-\frac{1}{2}x^2}|Q_{\nu,n}(x_o) - Q_{\nu,n}(x)|$$

also

$$|e^{-\frac{1}{2}x_o{}^2} - e^{-\frac{1}{2}x^2}| \leqq e^{-\frac{1}{2}x^2}|x_o{}^2 - x^2|e^{\frac{1}{2}|x_o{}^2 - x^2|}.$$

Using (7) and (8),

$$|x_o{}^2 - x^2| = \left|\left(\frac{EY_n\sqrt{n}}{\sigma_n}\right)^2 - 2\sigma_n^{-2}\, EY_n\sqrt{n}\, x + (\sigma_n^{-2} - 1)x^2\right|$$

$$\leqq \left(\frac{16}{11}\right)\Lambda_{k+1,n}\left(\Lambda_{k+1,n} + 2|x|\right) + (\sigma_n^{-2} - 1)x^2 .$$

As $\sigma_n^{-2} - 1$ is less than both $\frac{5}{11}$ and $\frac{16}{11}\cdot\Lambda_{k+1,n}$, therefore

$$|e^{-\frac{1}{2}x_o{}^2} - e^{-\frac{1}{2}x^2}| \leqq C\Lambda_{k+1,n}e^{-7/13\, x^2} .$$

Also, using Lemma 3.8 and the fact that† $\frac{\gamma_{s,n}}{\sigma_n^s\, n^{\frac{1}{2}(s-2)}} \leqq C(S)\, L_{s,n}$

$$n^{-\frac{1}{2}\nu}|Q_{\nu,n}(x_o)| \leqq c(\nu)\left(1 + |x_o|^{3\nu-1}\right)\sigma_n^{-(\nu+2)}n^{-\frac{1}{2}\nu}E|Y_n|^{\nu+2} \tag{15}$$

Thus

$$n^{-\frac{1}{2}\nu}|e^{-\frac{1}{2}x_o{}^2} - e^{-\frac{1}{2}x^2}||Q_{\nu,n}(x_o)| \leqq C(\nu)\Lambda_{k+1,n}e^{-\frac{1}{4}x^2} .$$

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†See bottom of p.78

We have also

$$n^{-\frac{1}{2}\nu}|Q_{\nu,n}(x_o) - Q_{\nu,n}(x)| = \left|\sum \left(H_{\nu+2s-1}(x_o) - H_{\nu+2s-1}(x)\right)\prod_{m=1}^{\nu}\frac{1}{k_m!} \times\right.$$

$$\left.\times \left(\frac{\gamma_{m+2,n}}{(m+2)!\,\sigma_n^{m+2}n^{\frac{1}{2}m}}\right)^{k_m}\right|$$

and

$$|H_m(x_o) - H_m(x)| \leqq m!\sum_{j=o}^{[m/2]}\frac{|x_o^{m-2j} - x^{m-2j}|}{j!(m-2j)!2^j}$$

$$\leqq c(m)|x_o - x|\left(1 + |x_o|^{m-1} + |x|^{m-1}\right) .$$

Hence as $|x_o - x| \leqq \frac{5|x|+4}{\sqrt{11}}\Lambda_{k+1,n}$,

$$e^{-\frac{1}{2}x^2}n^{-\frac{1}{2}\nu}|Q_{\nu,n}(x_o) - Q_{\nu,n}(x)| \leqq c(\nu)\Lambda_{k+1,n}n^{-\frac{1}{2}\nu}\sigma_n^{-(\nu+2)}E|Y_n - EY_n|^{\nu+2}e^{-\frac{1}{4}x^2}$$

But from (11) $\frac{E|Y_n-EY_n|^{\nu+2}}{\sigma_n^{\nu+2}n^{\frac{1}{2}\nu}} \leqq c(\nu)$, $\nu = 0, 1, 2, \ldots$, which gives us

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,n^{-\frac{1}{2}\nu}|Q_{\nu,n}(x_o) - Q_{\nu,n}(x)| \leqq c(\nu)\Lambda_{k+1,n}e^{-\frac{1}{4}x^2}$$

for $\nu = 1,2, \ldots, k$.

Combining all these results, we obtain for $\nu = 1,2,\ldots,k$

$$n^{-\frac{1}{2}\nu}|e^{-\frac{1}{2}x_o^2}Q_{\nu,n}(x_o) - e^{-\frac{1}{2}x^2}Q_{\nu,n}(x)| \leqq c(\nu)\Lambda_{k+1,n}e^{-\frac{1}{4}x^2} .^{\dagger}$$

Thus

$$|U_{k,n}(x_o) - U_{k,n}(x)| \leqq c(k)\Lambda_{k+1,n}e^{-\frac{1}{4}x^2} .$$

From (15) we have that

$$n^{-\frac{1}{2}(k+1)}|e^{-\frac{1}{2}x_o^2}Q_{k+1,n}(x_o)| \leqq c(k)n^{-\frac{1}{2}(k+1)}\sigma_n^{-(k+3)}E|Y_n|^{k+3}e^{-\frac{1}{4}x^2}$$

$$= c(k)\,L_{k+3,n}e^{-\frac{1}{4}x^2\dagger} .$$

Continuing now with the proof of the theorem, we note that

$$V_n^{*n}(x\sqrt{n}) = W_n^{*n}(x\sqrt{n} - nEY_n) = G_n(p_nx + q_n)$$

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†This completes the proof of inequality (6).

and so

$$|F_n(x) - U_k(x)| \leqq |F^{*n}(x\sqrt{n}) - V_n^{*n}(x\sqrt{n})| + |G_n(x_o) - U_{k+1,n}(x_o)| + + |U_{k+1,n}(x_o) - U_k(x)| \qquad (16)$$

Further[†]

$$\Lambda_{k+1,n} = n^{-\frac{1}{2}(k-1)} \int_{|u|\geqq\sqrt{n}} |u|^{k+1}\, dF(u) \leqq$$

$$\leqq n^{-\frac{1}{2}(k-1)}\left(n^{-1} \int_{\sqrt{n}<|u|\leqq\sqrt{n}(1+|x|)} |u|^{k+3}\, dF(u) + + \int_{|u|>\sqrt{n}(1+|x|)} |u^{k+1}|\, dF(u)\right) \qquad (17)$$

From (6), (17) and Lemma 3.11, it follows that the first and third parts of inequality (16) do not exceed the bound in Theorem 3.3. It remains, therefore, to estimate

$$|G_n(p_n x + q_n) - U_{k+1,n}(p_n x + q_n)|.$$

In Lemma 3.13 we can set

$$G(x) = U_{k+1,n}(x_o), \quad F(x) = G_n(x_o), \quad s = k+3, \quad T = T_n$$

where

$$T_n = 3^{k+4} B_{k+4,n}^{-1} \text{ and } B_{\nu,n} = \sigma_n^{-\nu} n^{-\frac{1}{2}(\nu-2)} E|Y_n - EY_n|^{\nu}, \quad (\nu = 1,2,\ldots,)$$

From (11), $B_{\nu,n} < 3^{\nu}$ and so $T_n > 1$; also $|U_{k+1,n}(x)| < K(1+|x|)^{-(k+3)}$, for K a constant depending only on k.

Setting

$$\delta_\ell(t) = \int_{-\infty}^{\infty} e^{itx}\, d\{x^\ell (G_n(x) - U_{k+1,n}(x))\} \quad (\ell = 0, 1, \ldots),$$

we obtain from Lemma 3.13

[†] A similar split-up is valid for $|n^{-\frac{1}{2}k} \int^{*}_{|u|>\sqrt{n}} u^{k+2} dF(u)|$.

$$|G_n(x_o) - U_{k+1,n}(x_o)| \leq \frac{c(k)}{(1+|x_o|)^{k+3}} \left\{B_{k+4,n} + \int_{-T_n}^{T_n} \left|\frac{\delta_o(t)}{t}\right| dt \right.$$

$$\left. + \int_{-T_n}^{T_n} \left| \frac{\delta_{k+3}(t)}{t} \right| dt \right\} \qquad (18)$$

At this point we note that if we are only interested in uniform bounds, we could use directly the Esseen bound for $G_n(x_o) - U_{k+1,n}(x_o)$, namely

$$\int_{-T_n}^{T_n} \left|\frac{\delta_o(t)}{t}\right| dt + B_{k+4,n} \qquad (19)$$

We estimate the final integral in (18). From Lemma 3.12 it is sufficient to estimate the integrals

$$I_\nu = \int_{-T_n}^{T_n} \left| \frac{d^\nu}{dt^\nu} \left(g_n(t) - u_{k+1,n}(t)\right) \right| |t|^{\nu-k-4} dt$$

for $\nu = 0, 1, \ldots, k+3$. From Lemma 3.10 it follows that

$$\int_{|t| < B_{k+4,n}^{-1/(k+2)}} \left| \frac{d^\nu}{dt^\nu} \left(g_n(t) - u_{k+1,n}(t)\right)\right| |t|^{\nu-k-4} dt \leq c(k) B_{k+4,n}$$

The same bound holds for

$$\int_{B_{k+4,n}^{-1/(k+2)} \leq |t| \leq T_n} \left| \frac{d^\nu}{dt^\nu} u_{k+1,n}(t)\right| |t|^{\nu-k-4} dt.$$

Therefore

$$I_\nu \leq c(k)\, B_{k+4,n} + \int_{B_{k+4,n}^{-1/(k+2)} \leq |t| < T_n} \left| \frac{d^\nu}{dt^\nu} g_n(t)\right| |t|^{\nu-k-4} dt. \qquad (20)$$

From Lemma 3.9,

$$\frac{d^{\nu}}{dt^{\nu}} g_n(t) = \frac{d^{\nu}}{dt^{\nu}} w_n^n\left(\frac{t}{\sigma_n\sqrt{n}}\right) = \nu\,! \sum_{r=1}^{\min(\nu,n)} \sum{}' \frac{n!}{(n-r)!} w_n^{n-r} \left(\frac{t}{\sigma_n\sqrt{n}}\right) \times$$

$$\times \prod_{m=1}^{\nu} \frac{1}{r_m!} \left[\frac{1}{m!} \frac{d^m}{dt^m} w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right]^{r_m} , \tag{21}$$

where $\sum{}'$ denotes summation over all non-negative integer solutions to

$$r_1 + 2r_2 + \ldots + \nu r_{\nu} = \nu$$

and

$$r_1 + r_2 + \ldots + r_{\nu} = r$$

We now bound $\left|\frac{d^{\nu}}{dt^{\nu}} g_n(t)\right|$ for the case of $k > 1$. The result we obtain could have been extracted directly from the work on pp.204-207 of [32] since this bound depends only on $E|X|^3$ being finite. For the sake of completeness, however, we shall reproduce that work here.

Using (10) and the inequalities

$$\left|\frac{d}{dt} w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right| = \left|E\left(\frac{Y_n - EY_n}{\sigma_n\sqrt{n}} \left(\exp\left\{it \frac{Y_n - EY_n}{\sigma_n\sqrt{n}}\right\} - 1\right)\right|\right|$$

$$\leqq \frac{t}{\sqrt{n}} ,$$

and

$$\left|\frac{d^{\nu}}{dt^{\nu}} w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right| \leqq \frac{1}{n} B_{\nu,n} \quad (\nu = 2,3,\ldots),$$

we obtain

$$\left|\frac{d^{\nu}}{dt^{\nu}} g_n(t)\right| \leqq c(\nu)\left(1 + |t|^{\nu}\right) \left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|^{n-\min(\nu,n)} \qquad (\nu = 1,2,\ldots).$$

Also

$$\left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right) - 1\right| = \left| E\left[\exp\left\{it \frac{Y_n - EY_n}{\sigma_n\sqrt{n}}\right\} - 1 - it\frac{Y_n - EY_n}{\sigma_n\sqrt{n}}\right]\right| \leqq \frac{t^2}{2n} ,$$

and so for $|t| \leqq \sqrt{n}$, $\left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right| \geqq \frac{1}{2}$. Thus for any γ, $0 < \gamma \leqq 1$

$$\sup_{|t| \geqq \gamma\sqrt{n}} \left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right| \geqq \frac{1}{2}$$

and for $\nu = 0, 1, \ldots, k+3$,

$$\left\{ \sup_{|t|>\gamma\sqrt{n}} \left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|\right\}^{n-\min(\nu,n)} \leqq 2^{k+3}\left\{\sup_{|t|\geqq\gamma\sqrt{n}}\left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|\right\}^n$$

Now, as $B_{3,n} \geqq n^{-\frac{1}{2}}$ we have for $\nu = 0, 1, \ldots, k+3$,

$$\int_{B_{3,n}^{-1}<|t|\leqq T_n} \left|\frac{d^\nu}{dt^\nu} g_n(t)\right| |t|^{\nu-k-4}\, dt \leqq$$

$$\leqq c(k)\left\{\sup_{|t|\geqq B_{3,n}^{-1}} \left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|\right\}^n \int_{B_{3,n}^{-1}<|t|\leqq T_n} (1+|t|^\nu)\,|t|^{\nu-k-4}\, dt \leqq$$

$$\leqq c(k)\; a_n^n \; n^{\frac{1}{2}(k+2)(k+3)} \qquad \left(T_n \geqslant n^{\frac{1}{2}(k+2)}\right) \tag{22}$$

where $a_n = \sup_{|t|\geqq(\sigma_n\sqrt{n}B_{3,n})^{-1}} |w_n(t)|$.

From (7) it follows that

$$\sigma_n\sqrt{n}B_{3,n} = \frac{1}{\sigma_n^2}E|Y_n - EY_n|^3 < \frac{16}{11}\cdot 8E|Y_n|^3 < 12E|X|^3$$

Also, since $\Lambda_{k+1,n} < \frac{1}{4}$

$$|w_n(t)| = |Ee^{itY_n}| = \left|\int_{|x|\leqq\sqrt{n}} e^{itx}\,dF(x) + \int_{|x|>\sqrt{n}} dF(x)\right| \leqq$$

$$\leqq |f(t)| + 2\int_{|x|>\sqrt{n}} dF(x) \leqq |f(t)| + \frac{1}{2n} ,$$

it follows that

$$a_n \leqq \sup_{|t|>(12E|X|^3)^{-1}} |f(t)| + \frac{1}{2n} \tag{23}$$

We now show that $|g_n(t)| \leqq e^{-t^2/6}$ for $|t| < B_{3,n}^{-1}$.

Let Z and Z' be iid as the rv $(Y_n - EY_n)$. Then the c.f. of $Z - Z'$ is $|w_n(t)|^2$. Also

$$\begin{aligned}\log|w_n(t)|^2 &= \log\left[1 + \left(|w_n(t)|^2 - 1\right)\right]\\ &= -\left(1 - |w_n(t)|^2\right) - \tfrac{1}{2}\left(1 - |w_n(t)|^2\right)^2 - \ldots\\ &\leqq -1\left(1 - |w_n(t)|^2\right) ,\end{aligned}$$

and

$$1 - \left|w_n\left(\frac{1}{\sigma_n}\right)\right|^2 = t^2 - \int\left(e^{itu} - 1 - itu - \frac{(itu)^2}{2!}\right) d\, \Psi_n(u)$$

where $\Psi_n(u) = P\{Z - Z' < \sigma_n x\}$. Therefore

$$\begin{aligned}
|g_n(t)| &= \exp\{n\log\left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|\} \\
&\leqq \exp\left\{ - \frac{t^2}{2} + \tfrac{1}{2}\int\left(e^{itu} - 1 - itu - \frac{(itu)^2}{2!}\right) d\, \Psi_n(u)\right\} \\
&\leqq \exp\left\{ -\tfrac{1}{2}t^2 + \frac{1}{12}E\left|\frac{Z-Z'}{\sigma_n}\right|^3 \frac{t^3}{\sqrt{n}} \right\}, \qquad (E|Z-Z'|^3 \leqq 2E|Z|(Z-Z')^2 \leqq 4E|Z|^3) \\
&\leqq \exp\left\{ -\tfrac{1}{2}t^2 + \frac{1}{3}E\left|\frac{Z}{\sigma_n}\right|^3 \frac{1}{\sqrt{n}}\, |t|^3\right\} \\
&\leqq \exp\left\{ -\tfrac{1}{2}t^2\left(1 - \frac{2}{3}t\, B_{3,n}\right)\right\} \\
&\leqq \exp\left\{ -\frac{1}{6}t^2\right\} \text{ if } |t| < B_{3,n}^{-1} . \qquad (24)
\end{aligned}$$

From Lemma 3.8 we have

$$B_{k+4,n}^{-(k+2)^{-1}} \leqq B_{3,n}^{-1}$$

Thus from (21)†

$$\int_{B_{k+4,n}^{-(k+2)^{-1}} \leqq |t| < B_{3,n}^{-1}} \left|\frac{d^\nu}{dt^\nu}\, g_n(t)\right| |\, t\, |^{\nu-k-4} dt \leqq c(k) B_{k+4,n}, \quad (\nu=0,\ldots,k+3).$$

And so

$$\int_{-T_n}^{T_n} \left|\frac{\delta_{k+3}(t)}{t}\right| dt \leqq c(k)\left(B_{k+4,n} + a_n^n \; n^{\frac{1}{2}(k+2)(k+3)}\right)$$

follows from (20) and (22). We now estimate $\int_{-T_n}^{T_n} \left|\frac{\delta_o(t)}{t}\right| dt$,

since this is relevant in the uniform case. From the above work (and in particular Lemma 3.10 and (24)) we can readily extract that

$$\int_{-T_n}^{T_n} \left|\frac{\delta_o(t)}{t}\right| dt \leqq c(k) B_{k+4,n} + \int_{B_{3,n}^{-1} < |t| \leqq T_n} \left|\frac{g_n(t)}{t}\right| dt$$

---

†N.B. $0 < h(t) \leqq ce^{-1/6\, t^2} t^\eta \Rightarrow \int_{\alpha<|t|<\beta} h(t)dt \leqq \int_{\alpha<|t|<\beta} c\left(\frac{t}{\alpha}\right)^{k+2} \cdot e^{-\frac{1}{6}t^2} t^\eta dt$

$\leqq c\alpha^{-(k+2)}$

and that (see (22))

$$\int_{B_{3,n}^{-1}<|t|\leqq T_n} \left|\frac{g_n(t)}{t}\right| dt \leqq c(k)\, a_n^n \int_{B_{3,n}^{-1}<|t|\leqq T_n} \frac{1}{|t|}\, dt$$

$$\leqq c(k)\, a_n^n \log n.$$

Hence, from (18) we find

$$|G_n(x_0) - U_{k+1,n}(x_0)| \leqq \frac{c(k)}{(1+|x_0|)^{k+3}} \left[B_{k+4,n} + a_n^n\, n^{\frac{1}{2}(k+2)(k+3)}\right] \tag{25}$$

and from (19)

$$|G_n(x_0) - U_{k+1,n}(x_0)| \leqq c(k)\left(B_{k+4,n} + a_n^n \log n\right) \tag{26}$$

From the definitions of x_0 and using (7)†

$$1 + |x_0| \geqq \tfrac{3}{4} + |x|. \tag{27}$$

Also, as $|Y_n| < \sqrt{n}$

$$B_{k+4,n} = \sigma_n^{-(k+4)}\, n^{-\frac{1}{2}(k+2)}\, E|Y_n - EY_n|^{k+4} \leqq$$

$$\leqq 2^{k+4}\left(\frac{16}{11}\right)^{k+4} n^{-\frac{1}{2}(k+2)}\, E|Y_n|^{k+4} \leqq 3^{k+4}\, n^{-\frac{1}{2}(k+1)}\, E|Y_n|^{k+3} \tag{28}$$

From these estimates together with (16), (17), (23), (25) and (26), we obtain Theorems 3.3 and 3.4 for $k>1$.

To prove the theorems for $k=1$, we must suitably bound

$$\int_{B_{5,n}^{-1/3}<|t|\leqq T_n} \left|\frac{d^\nu}{dt^\nu} g_n(t)\right| |t|^{\nu-5}\, dt$$

for $\nu = 0, 1, \ldots, 4$.

From the derivation of (22) we can write

$$\int_{\gamma\sqrt{n}<|t|\leqq T_n} \left|\frac{d^\nu}{dt^\nu} g_n(t)\right| \cdot |t|^{\nu-5}\, dt \leqq c\left[\sup_{|t|>\gamma\sqrt{n}} \left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|\right]^n n^6$$

where $0<\gamma<1$ and c is an absolute constant. (γ is independent of n).

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† Taking $|x|>\frac{1}{4}$ and remembering that $|EY_n|\sqrt{n}<\frac{1}{4}$, $1+|x_0|=1+\sigma_n^{-1}|x-\sqrt{n}EY_n|\geqq$ $\geqq 1+|x-\sqrt{n}EY_n|\geqq 1+|x|-\sqrt{n}|EY_n|>\frac{3}{4}+|x|$. Also $1+|x_0|\geqq 1$.

Hence $1+|x_0|\geqq \frac{3}{4}+|x|$ for all x.

so, from (7), and from the derivation of (23),

$$\int_{\gamma\sqrt{n}<|t|\leqq T_n} \left|\frac{d^\nu}{dt^\nu} g_n(t)\right| |t|^{\nu-5}\, dt \leqq c\left(\sup_{|t|>\gamma} |f(t)| + \frac{1}{2n}\right)^n n^6 \qquad (29)$$

For the uniform case, it is easily seen that

$$\int_{\gamma\sqrt{n}<|t|<T_n} \left|\frac{g_n(t)}{t}\right| dt \leqq c(k)\left(\sup_{|t|>\gamma} |f(t)| + \frac{1}{2n}\right)^n \log n \qquad (30)$$

It therefore remains to bound $|g_n(t)|$ for $|t| < \gamma\sqrt{n}$ as was done in the case $k > 1$. Again we use the rv $Z - Z'$ and we have

$$\log |w_n(t)|^2 \leqq -\left(1 - |w_n(t)|^2\right)$$

and

$$\begin{aligned} 1 - \left|w_n\left(\frac{t}{\sigma_n}\right)\right|^2 &= t^2 - \int\left(e^{itu} - 1 - itu - \frac{(itu)^2}{2!}\right) d\Psi_n(u) \\ &= t^2 - \int\left(\cos tu - 1 + \frac{(tu)^2}{2}\right) d\Psi_n(u) \\ &\geqq t^2 - \int_{|ut|<1} \left|\cos tu - 1 - \frac{(tu)^2}{2}\right| d\Psi_n(u) - \\ &\quad - \int_{|ut|\geqq 1} \left|\cos(ut) - 1 - \frac{(ut)^2}{2}\right| d\Psi_n(u) \\ &\geqq t^2 - \int_{|ut|<1} \frac{(ut)^4}{4!}\, d\Psi_n(u) - \int_{|ut|\geqq 1} 3u^2t^2\, d\Psi_n(u) \quad . \end{aligned}$$

But, setting $R_n(z) = \int_{|u|>z} u^2 d\Psi_n(u)$, we note that $R_n(z) \to 0$ as $z \to \infty$ and[†]

$$\begin{aligned} t^2 \int_{|ut|<1} u^4 d\Psi_n(u) &\leqq R_n\left(\frac{1}{|t|}\right) + 2t^2 \int_0^{|t|^{-1}} u\, R_n(u)\, du \\ &\leqq R_n\left(\frac{1}{|t|}\right) + 2t^2 \int_0^{|t|^{-\frac{1}{2}}} u\, R_n(u) du + 2t^2 \int_{|t|^{-\frac{1}{2}}}^{|t|^{-1}} u\, R_n(u) du \\ &\leqq R_n\left(\frac{1}{|t|}\right) + 2|t| + R_n\left(|t|^{-\frac{1}{2}}\right). \end{aligned}$$

[†] N.B. $\int_{|ut|<1} u^4 d\Psi_n(u) = -\int_0^{1/|t|} u^2 dR_n(u)$

Thus

$$t^2 \int\limits_{|ut|<1} u^4 d\Psi_n(u) + 3 \int\limits_{|ut|>1} u^2 d\Psi_n(u) \leqq |t| + cR_n\left(|t|^{-\frac{1}{2}}\right)$$

with c an absolute constant. If we can show that

$$\sup_n R_n(z) \to 0 \quad \text{as} \quad z \to \infty$$

we can then choose γ, $0 < \gamma < 1$, independent of n such that for $|t| < \gamma$

$$t^2 \int\limits_{|ut|<1} u^4 d\Psi_n(u) + 3 \int\limits_{|ut|>1} u^2 d\Psi_n(u) < \tfrac{1}{2},$$

uniformly in n.

Suppose $\sup_n R_n(z) \not\to 0$ as $z \to \infty$, then there exists an $\varepsilon > 0$ such that

$$\sup_n R_n(z) > \varepsilon > 0 \quad \text{as} \quad z \to \infty.$$

Since $R_n(z)$ is monotonic decreasing in z and $0 \leqq R_n(z) \leqq 2$ for each n, the above relation implies the existence of some n such that

$$\int\limits_{|u|>z} u^2 d\Psi_n(u) \geqq \varepsilon \qquad \text{for all } z.$$

But $\int_{-\infty}^{\infty} u^2 d\Psi_n(u) = 2$ for all n and for the limiting case of $n \to \infty$ as well. Thus we have an immediate contradiction.

Using the above mentioned γ, we can now write for $|t| < \gamma\sqrt{n}$

$$1 - \left|w_n\left(\frac{t}{\sigma_n}\right)\right|^2 \geqq t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

and so
$$\begin{aligned} |g_n(t)| &= \exp\left\{n \log \left|w_n\left(\frac{t}{n\ n}\right)\right|\right\} \\ &\leqq \exp\left\{-\frac{n}{2}\left(1 - \left|w_n\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|^2\right)\right\} \\ &\leqq e^{-\frac{1}{4}t^2} \quad . \end{aligned}$$

This bound corresponds with (24) and can be used in exactly the same way to show that for $\nu = 0, \ldots, 4$, (also we need (28))

$$\int_{B_{5,n}^{-1/3} \le |t| < \gamma\sqrt{n}} \left|\frac{d^\nu}{dt^\nu} g_n(t)\right| \; |t|^{\nu-5} dt \le c\, n^{-1} E|Y_n|^4 \, .$$

This result combined with (29) and (30) gives us

$$\int_{-T_n}^{T_n} \left|\frac{\delta_4(t)}{t}\right| dt \le c\,(n^{-1}\, E|Y_n|^4 + a_n^n\, n^6)$$

and

$$\int_{-T_n}^{T_n} \left|\frac{\delta_o(t)}{t}\right| dt \le c\,(n^{-1}\, E|Y_n|^4 + a_n^n \;\log n)$$

So, from (18), (19), (27) and (28) we find

$$|G_n(x_o) - U_{2,n}(x_o)| \le \frac{c}{(1 + |x|)^4} \left(n^{-1} E\, |Y_n|^4 + a_n^n\, n^6\right)$$

and

$$|G_n(x_o) - U_{2,n}(x_o)| \le c\left(n^{-1} E|Y_n|^4 + a_n^n \log n\right)$$

as desired.

Finally, let us see what happens when $\Lambda_{k+1,n} \ge \frac{1}{4}$.

$$e^{-\frac{1}{2}x^2}\; n^{-\frac{1}{2}\nu}|Q_\nu(x)| \le c(k)\; n^{-\frac{1}{2}\nu}\; E|X|^{\nu+2}\; e^{-\frac{1}{4}x^2} \le c(k)\left(1 + \Lambda_{k+1,n}\right)\; e^{-\frac{1}{4}x^2}$$

$$\le 5\, c\,(k)\; \Lambda_{k+1,n}\; e^{-\frac{1}{4}x^2} \quad , \qquad \nu \le k - 1,$$

and $|F_n(x) - \Phi(x)| \le c(k)\left(1 + |x|\right)^{-(k+1)}$ by Lemma 5, page 181 of Petrov [32].

However, when we come to bound $Q_k(x)$ we find (see the estimate for $Q_{k+1,n}(x)$ in the proof of Theorem 3.5, pp 78, 79)

$$|e^{-\frac{1}{2}x^2}\; n^{-\frac{1}{2}k} Q_k(x)| \le c(k)\left[n^{-\frac{1}{2}k}|EX_i^{k+2}| + n^{-\frac{1}{2}k}\left(E|X_i|^{k+1}\right)^{(k+1)/k}\right]e^{-\frac{1}{4}x^2}$$

$$\le c(k)\left[n^{-\frac{1}{2}k}|EX_i^{k+2}| + \left(1 + \Lambda_{k+1,n}\right)^{(k+1)/k}\right]e^{-\frac{1}{4}x^2}$$

$$\le c(k)\left[n^{-\frac{1}{2}k}|EX_i^{k+2}| + \left(5\Lambda_{k+1,n}\right)^{(k+1)/k}\right]e^{-\frac{1}{4}x^2}$$

This is clearly not the form required by Theorem 3.3 and for this reason we have to demand $\Lambda_{k+1,n} < \frac{1}{4}$.

This now completes the proof of Theorems 3.3 and 3.4.

3.5 *Proof of THEOREM 3.5*† The L.H.S. of the inequality in Theorem 3.5 can be split up as follows:

$$\left|P\{\frac{1}{C_n} \sum X_i - b_n \leq x\} - U_k(x)\right|$$

$$\leq \left|P\{\frac{1}{C_n} \sum X_i - b_n \leq x\} - P\{\frac{1}{C_n} \sum \overline{X}_i - b_n \leq x\}\right| +$$

$$+ \left|P\{\frac{1}{C_n} \sum \overline{X}_i - b_n \leq x\} - U_{k+1,n}(x_o)\right| + \frac{e^{-\frac{1}{2}x_o^2}}{\sqrt{2\pi}} \left|Q_{k+1,n}(x_o)\right| +$$

$$+ \left|U_{k,n}(x_o) - U_{k,n}(x)\right| + \left|U_{k,n}(x) - U_k(x)\right| .$$

To estimate the first term on the R.H.S. we use the following well known result expressed here as a lemma.

Lemma 3.14. Let $\{Y_i\}$ *be a sequence of rv's, not necessarily independent and let* $\{\ell_i\}$ *and* $\{\lambda_i\}$ *be sequences of real numbers with* $\ell_i < \lambda_i$, $i = 1,2,\ldots$ *Set*

$$\overline{Y}_i = \begin{cases} Y_i & \text{if} \quad \ell_i < Y_i < \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\left|P\{\sum_{i=1}^{n} Y_i \leq x\} - P\{\sum_{i=1}^{n} \overline{Y}_i \leq x\}\right| \leq$$

$$\leq \sum_{i=1}^{n} \{P(Y_i \leq \ell_i) + P(Y_i \geq \lambda_i)\} .$$

Proof of Lemma 3.14. The event $\sum_{i=1}^{n} Y_i < x$ implies the event

$$\left(\sum_{i=1}^{n} \overline{Y}_i < x\right) \cup \left(Y_i \leq \ell_1\right) \cup \left(Y_1 \geq \lambda_1\right) \ldots \cup \left(Y_n \leq \ell_n\right) \cup \left(Y_n \geq \lambda_n\right),$$

and furthermore, the event $\sum_{i=1}^{n} \overline{Y}_i < x$ implies the event

---

†N.B. x_o and σ_n^2 in this section are different from those in previous section. See p.54 for definitions.

$$\left[\sum_{i=1}^{n} Y_i < x\right] U\left(Y_1 \leqq \ell_1\right) U\left(Y_1 \geqq \lambda_1\right) \dots U\left(Y_n \leqq \ell_n\right) U\left(Y_n \geqq \lambda_n\right).$$

The result of the lemma now follows.

From this lemma, using the truncated rv's $\overline{X}_i$ defined in (1), we have,

$$\left|P\left[\frac{1}{C_n}\sum_{1}^{n} X_i - b_n \leqq x\right] - P\left[\frac{1}{C_n}\sum_{1}^{n} \overline{X}_i - b_n \leqq x\right]\right| \leqq nP\left[|X_i| > \tau_{nx}\right]$$

Secondly, we note that for $k \geqq 1$,

$$\left|P\left[\frac{1}{C_n}\sum_{1}^{n} \overline{X}_i - b_n \leqq x\right] - U_{k+1,n}(x_o)\right|$$

$$= \left|P\left(\sum_{1}^{n} \frac{(\overline{X}_i - E\overline{X}_i)}{\sqrt{n}\sigma_n} \leqq x_o\right] - U_{k+1,n}(x_o)\right|$$

where $\sigma_n^2 = \text{var}\ \overline{X}_i$, and if we use the fact that $E|\overline{X}_i|^{k+4} < \infty$, from Theorem 3.1 we find

$$\left|P\left(\sum_{1}^{n} \frac{\overline{X}_i - E\overline{X}_i}{\sqrt{n}\sigma_n} \leqq x_o\right] - U_{k+1,n}(x_o)\right| \leqq c(k)\left[n^{-\frac{1}{2}(k+2)}\left(1 + |x_o|\right)^{-(k+4)}\right.$$

$$\times\ \sigma_n^{-(k+4)} E|\overline{X}_i|^{k+4}$$

$$\left. + \left(\sup_{|t|>\delta_n^1} |\overline{f}(t)| + \frac{1}{2n}\right)^n n^{\frac{1}{2}(k+3)(k+4)}\left(1 + |x_o|\right)^{-(k+4)}\right]$$

where

$$\delta_n^1 = \left[12E|\overline{X}_i - E\overline{X}_i|^3\right]^{-1}\sigma_n^3 \geqq \left(96E|\overline{X}_i|^3\right)^{-1}\sigma_n 3,$$

$$\sigma_n^2 = \text{var}\ \overline{X}_i,$$

and
$$|\overline{f}(t)| = |E \exp it(\overline{X}_i - E\overline{X}_i)\sigma_n^{-1}|$$

$$= |E \exp it\overline{X}_i\sigma_n^{-1}| = \left|\int_{|u|<\tau_{nx}} e^{itu\sigma_n^{-1}}\, dF(u) + P(|X_i| > \tau_{nx})\right|$$

$$\leqq |f(\frac{t}{\sigma_n})| + 2P(|X_i| > \tau_{nx}).$$

To complete the theorem we must bound $|Q_{k+1,n}(x_0)|$. This we do in two stages: firstly we estimate $|\gamma_{k+3,n}|$ and then we use this in estimating $Q_{k+1,n}(x_0)$. From the well known expression for cumulants in terms of moments we have

$$-n^{-\frac{1}{2}(k+1)}\gamma_{k+3,n} = n^{-\frac{1}{2}(k+1)} \sum(-1)^s(s-1)!\; n^{\frac{1}{2}(k+3-2s)} \prod_{r=1}^{k+3} \frac{1}{\ell_r!}\left(\frac{\alpha_{r,n}}{n^{\frac{1}{2}(r-2)}\,r!}\right)^{\ell_r}$$

where summation is over all non-negative integer solutions of

$$\begin{aligned} &\ell_1 + 2\ell_2 + \dots + (k+3)\ell_{k+3} = k+3 \\ &\ell_1 + \ell_2 + \dots + \ell_{k+3} = s \end{aligned} \tag{30a}$$

and $\gamma_{r,n}$ and $\alpha_{r,n}$ are the rth cumulant and moment respectively of $\sigma_n^{-1}(\bar{X}_i - E\bar{X}_i)$.

When $\ell_1 \neq 0$, $\prod_{r=1}^{k+3} \equiv 0^\dagger$, thus

$$n^{-\frac{1}{2}(k+1)}|\gamma_{k+3,n}| = \left|\sum(-1)^s(s-1)!\,n^{(1-s)} \prod_{r=2}^{k+3}\frac{1}{\ell_r!}\left(\frac{\alpha_{r,n}}{n^{\frac{1}{2}(r-2)}\,r!}\right)^{\ell_r}\right|$$

$$\leq \frac{|\alpha_{k+3,n}|}{n^{\frac{1}{2}(k+1)}} + \sum(s-1)!\,n^{(1-s)}\,\frac{1}{\ell_2!}\prod_{r=3}^{k+2}\frac{1}{\ell_r!}\left(\frac{\beta_{r,n}}{n^{\frac{1}{2}(r-2)}r!}\right)^{\ell_r}$$

$$\leq \frac{|\alpha_{k+3,n}|}{n^{\frac{1}{2}(k+1)}} + \sum c(k)n^{-(s-1)}\prod_{r=3}^{k+2} L_{r,n}^{\ell r}, \qquad L_{r,n} = \frac{\beta_{r,n}}{n^{\frac{1}{2}(r-2)}}$$

$$\leq \frac{|\alpha_{k+3,n}|}{n^{\frac{1}{2}(k+1)}} + \sum c(k)n^{-(s-1)}\prod_{r=3}^{k+2} L_{k+2,n}^{k^{-1}(r-2)\ell_r}, \qquad \text{(by Lemma 3.8)}$$

$$\leq \frac{|\alpha_{k+3,n}|}{n^{\frac{1}{2}(k+1)}} + \sum c(k)n^{-(s-1)}\,L_{k+2,n}^{k^{-1}(k+3-2s)}, \qquad \text{by (30a))}$$

and as $s \geq 2$ and $\beta_{k+2,n} \geq 1$,

$$n^{-\frac{1}{2}(k+1)}|\gamma_{k+3,n}| \leq n^{-\frac{1}{2}(k+1)}|\alpha_{k+3,n}| + c(k)n^{-\frac{1}{2}}\,L_{k+2,n}\,.$$

Also $n^{-\frac{1}{2}r}|\gamma_{r+2,n}| \leq c(k)\,L_{r+2,n}$ for $r \leq k$

We now examine $Q_{k+1,n}(x)$.

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† because $\alpha_{1,n} \equiv 0$.

$$n^{-\frac{1}{2}(k+1)}|Q_{k+1,n}(x)| = \sum H_{k+2s}(x) \prod_{m=1}^{k+1} \frac{1}{\ell_m!} \left(\frac{\gamma_{m+2,n}}{(m+2)!\,n^{\frac{1}{2}m}} \right)^{\ell_m}$$

where summation is over the same non-negative integer solutions as above except that $k+3$ is replaced by $k+1$. From the definition of $H_m(x)$ appearing in Remark h), we find

$$n^{-\frac{1}{2}(k+1)}|Q_{k+1,n}(x)| \leqq c(k)\left(1+|x|^{3k+2}\right)\left[n^{-\frac{1}{2}(k+1)}|\alpha_{k+3,n}| + n^{-\frac{1}{2}}L_{k+2,n} + \right.$$
$$\left. + \prod_{m=1}^{k} L_{m+2,n}^{\ell_m}\right]$$

also $\displaystyle\prod_{m=1}^{k} L_{m+2,n}^{\ell_m} \leqq \prod_{m=1}^{k} L_{k+2,n}^{(m\ell_m)/k}$, (by Lemma 3.8)

$$= L_{k+2,n}^{(k+1)/k}$$

Hence $\displaystyle|Q_{k+1,n}(x)| \leqq c(k)\left(1+|x|^{3k+2}\right)\left(n^{-\frac{1}{2}(k+1)}|\alpha_{k+3,n}| + n^{-\frac{1}{2}}L_{k+2,n} + L_{k+2,n}^{(k+1)/k}\right)$

$$\leqq c(k)\left(1+|x|^{3k+2}\right)\left(n^{-\frac{1}{2}(k+1)}|\alpha_{k+3,n}| + L_{k+2,n}^{(k+1)/k}\right) \quad .$$

Theorem 3.5 is now complete.

3.6 *Proof of THEOREM 3.6.* First we must establish inequality (2a). From Lemma 3.10,[†]

$$\left|\bar{f}_n(t) - e^{-\frac{1}{2}t^2}\right| \leqq c\left[n^{-\frac{1}{2}}|\alpha_{3,n}|\,|t|^3\, e^{-\frac{1}{2}t^2} + \frac{\alpha_{4,n}}{n}\left(|t|^4 + |t|^9\right)e^{-t^2/12}\right]$$

for $|t| < \sqrt{n}(\alpha_{4,n})^{-\frac{1}{2}}$. We show that this inequality continues to hold for $\alpha_{4,n}^{-\frac{1}{2}} \leqq \frac{|t|}{\sqrt{n}} < d_n$, with d_n as defined in expression (2a). Since $t^2 \geqq n\alpha_{4,n}^{-1}$ in this interval, we need only show that $|\bar{f}_n(t) - e^{-\frac{1}{2}t^2}| \leqq ct^2e^{-t^2/12}$.

Now setting $\bar{Y}_n = \dfrac{\bar{X}_1 - E\bar{X}_1}{\sigma_n}$, var $\bar{Y}_n = 1$ and so

$$\bar{f}(t) = Ee^{it\bar{Y}_n} = \exp\left[-\tfrac{1}{2}t^2\left(1+\gamma_n(t)\right)\right]$$

where $\gamma_n(t) \to 0$ as $t \to 0$.

[†] Here, $f_n(t)$ is the characteristic function of $\displaystyle\sum_{i=1}^{n} \frac{(\bar{X}_i - E\bar{X}_i)}{\sigma_n\sqrt{n}}$.

For $|t| < 1$, $|\overline{f}(t) - 1| \leq \frac{1}{2}t^2$ so if $|t| < d_n$ (<1),

$$\begin{aligned}\log \overline{f}(t) &= \log \left[1 - \left(1 - \overline{f}(t)\right)\right] \\ &= -\left[1 - \overline{f}(t) + \tfrac{1}{2}\left(1 - \overline{f}(t)\right)^2 + \tfrac{1}{3}\left(1 - \overline{f}(t)\right)^3 + \dots\right] \\ &= \overline{f}(t) - 1 - \tfrac{1}{2}\left(1 - \overline{f}(t)\right)^2 \left[1 + \tfrac{2}{3}\left(1 - \overline{f}(t)\right) + \dots\right]\end{aligned}$$

Thus

$$\begin{aligned}|\gamma_n(t)| &\leq \frac{2}{t^2}\left|\overline{f}(t) - 1 + \frac{t^2}{2}\right| + \frac{1}{t^2}\left|1 - \overline{f}(t)\right|^2 \left[\frac{1}{3} + \frac{2}{3}\left(\frac{1}{1-\frac{1}{2}}\right)\right] \\ &\leq \frac{2}{t^2}\left|\overline{f}(t) - 1 + \frac{t^2}{2}\right| + \frac{5}{12}\end{aligned}$$

But $\overline{f}(t) - 1 + \frac{t^2}{2} = \int_{-\infty}^{\infty} \left[e^{itu} - 1 - itu + \frac{t^2u^2}{2}\right] d\overline{F}(u)$

where $\overline{F}(u) = P(\overline{Y}_n \leq u)$

Hence

$$\left|\overline{f}(t) - 1 + \frac{t^2}{2}\right| = \left| \int_{|ut|<1} \left[e^{itu} - 1 - itu + \frac{t^2u^2}{2} + \frac{it^3u^3}{3!}\right] d\overline{F}(u) - \frac{it^3}{3!}\int_{|ut|<1} u^3 d\overline{F}(u) + \right.$$

$$\left. + \int_{|ut|\geq 1} \left(e^{itu} - 1 - itu + \frac{t^2u^2}{2}\right) d\overline{F}(u) \right|.$$

$$\leq \frac{t^4}{4!}\int_{|ut|<1} u^4 d\overline{F}(u) + \frac{|t|^3}{3!}\left|\int_{|ut|<1} u^3 d\overline{F}(u)\right| + t^2 \int_{|ut|\geq 1} u^2 d\overline{F}(u).$$

From the definition of d_n we have now

$$|\gamma_n(t)| < \frac{5}{12} + \frac{5}{12} = \frac{5}{6}$$

which, for $|t| < d_n\sqrt{n}$, gives

$$\left|\overline{f}_n(t) - e^{-\frac{1}{2}t^2}\right| = \left|e^{-\frac{1}{2}t^2}\left(1 + \gamma_n\frac{(t)}{\sqrt{n}}\right) - e^{-\frac{1}{2}t^2}\right| \quad \text{(now using } |e^x - 1| \leq |x|e^{|x|}\text{)}$$

$$\leq e^{-\frac{1}{2}t^2}\frac{t^2}{2}\left|\gamma_n\left(\frac{t}{\sqrt{n}}\right)\right| e^{\frac{1}{2}t^2\left|\gamma\left(\frac{t}{\sqrt{n}}\right)\right|} \leq \frac{t^2}{2}\exp\left[-\tfrac{1}{2}t^2 + \frac{t^2}{2}\cdot\frac{5}{6}\right] = \frac{t^2}{2}e^{-t^2/12}$$

as required.

It should be noted that d_n can always be chosen such that $d_n \geq \left(5E|Y_n|^3\right)^{-1}$. In view of this, (except for some absolute constants) the Osipov-Petrov bound can be derived from $\Theta_{on}(C_n)$ when τ_n is taken to be $\sqrt{n}$. So, in the case $\tau_n = \sqrt{n}$, Θ_{on} is sharper than the Osipov-Petrov bound, and thus Heyde's results [17] concerning the Osipov-Petrov bound carry over directly to $\Theta_{on}(C_n)$. In particular, for $k = 0$ and $0 < \delta < 1$, Theorem 3.6 is valid. To complete the $k = 0$ case, we need to establish the theorem for $\delta = 1$.

It is clear that for $k = 0$, $\delta = 1$, (iv)$\Longrightarrow$(iii)$\Longrightarrow$ (ii)$\Longrightarrow$ (by Thm. 3.2 Ibragimov [22])(i). So we have only to show (i)$\Longrightarrow$ (iv) to complete the theorem for $k = 0$. This will follow from Lemma 3.7 providing we can establish the existence of a $d > 0$ such that for all large n, $d_n \geq d$. To this end we examine $\int_{|ut|\geq 1} u^2 d\overline{F}(u)$, remembering that $\sigma_n^2 \to 1$ and so for $n > N$, $\sigma_n^2 > \frac{1}{2}$.

$$\int_{|ut|\geq 1} u^2 d\overline{F}(u) = \int_{|ut|\geq 1} u^2 dP\left(\sigma_n^{-1}\left(\overline{X}_i - E\overline{X}_i\right) \leq u\right)$$

$$= \int_{|u-E\overline{X}_1|>|t|^{-1}\sigma_n} \frac{(u-E\overline{X}_i)^2}{\sigma_n^2}\, dP(\overline{X}_i \leq u)$$

$$\leq 2 \int_{|u-E\overline{X}_1|\geq(\sqrt{2}|t|)^{-1}} \left(u^2 - 2uE\overline{X}_i + (E\overline{X}_i)^2\right) dP(\overline{X}_i \leq u)$$

Also $|E\overline{X}_1| \leq \int_{|u|>\sqrt{n}} |u|\, dP(X_1 \leq u) \leq \frac{1}{n}\int_{|u|>\sqrt{n}} u^2 dP(X_1 \leq u) \leq cn^{-1}$

c depending only on the distribution of X_1.

Thus, by taking N sufficiently large with $|t|<1$ (as $d_n < 1$) we have

$$\int_{|ut|\geq 1} u^2 d\overline{F}(u) \leq c\int_{|ut|\geq c} u^2 dP(\overline{X}_i \leq u)$$

$$\leq c\int_{|ut|\geq c} u^2 dP(\overline{X}_i \leq u)$$

$$\leq c|t|$$

again c depending only on the distribution of X_1.

By Lemma 3.7 and (i) we obtain immediately that

$$t^2 \int_{|ut|<1} u^4 d\bar{F}(u) \leqq c|t|$$

and with a little manipulation, that $|t| \left| \int_{|ut|<1} u^3 d\bar{F}(u) \right| \leqq c|t|$ when c depends only on the distribution of X_1.

These results ensure the existence of a $d > 0$ such that $d_n > d$ for all sufficiently large n, thereby completing Theorem 3.6 for $k = 0$.

We can now use induction to establish the theorem for general k. The $k = 0$ case gives us the finiteness of the variance and hence, by our assumptions, $EX_i = 0$. Further, without loss of generality we take EX^2 to be 1.

By observation (iv) => (iii) => (ii). If we can show that (ii) => (i) and (i) => (iv), the theorem will be proved. Thus our first task will be to prove that (ii) => (i). Let us assume the theorem true for $k-1$ and prove true for k ($k \geqq 1$).

Suppose that the sequence of positive real numbers $\{\beta_n\}$ is such that $\inf_{C_n} A_{kn}(C_n) = A_{kn}(\beta_n)$. Then $A_{kn}(\beta_n) = O(n^{-\frac{1}{2}(k+\delta)})$

$$A_{k-1,n}(\beta_n) \leqq A_{kn}(\beta_n) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{Q_k(x)}{n^{\frac{1}{2}k}}$$
$$= O(n^{-\frac{1}{2}k})$$

Hence

$$\inf_{C_n} A_{k-1,n}(C_n) = O(n^{-\frac{1}{2}k}) \tag{31}$$

and so from the induction assumptions, $E|X_i|^{k+1} < \infty$, $\alpha_j = EX_i^j$, $j = 0, 1..., k+1$,

$$\int_{|u|>z} |u|^{k+1} dF(u) = O(z^{-1}) \text{ as } z \to \infty \tag{32}$$

and $$\int_{-z}^{z} u^{k+2} dF(u) = O(1) .$$

By Lemma 2.2 of Ibragimov [22] we can express C_n in the form

$$C_n = \sqrt{\frac{n}{1+\varepsilon_n}}$$

where $\varepsilon_n = O(1)$. For the case $k = 0$, Ibragimov([22] pp.567-9) showed that (ii) $\Rightarrow \varepsilon_n = O(n^{-\frac{1}{2}\delta})$. We now use induction on k to show that $\varepsilon_n = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$. Indeed we could have included the following condition in part (i) of the statement of the theorem,

$$\varepsilon_n = O\left(n^{-\frac{1}{2}(k+\delta)}\right).$$

Hence, from (31)

$$\varepsilon_n = O\left(n^{-\frac{1}{2}k}\right)$$

and so

$$|(1+\varepsilon_n)^{s/2} - 1| = O(n^{-\frac{1}{2}k}), \quad s \geq 2 \tag{33}$$

We are now in a position to employ the technique used by Ibragimov in [22] to show (ii) $\Rightarrow$ (i) when $k = 0$. For $\frac{1}{2} \leq z \leq 2$, set

$$A_z(t) = \begin{cases} (z-t)t \exp\left\{\frac{1}{2}t^2 - \sum\limits_{s=3}^{k+2} \frac{(it)^s}{s!} \frac{\lambda_s}{n^{\frac{1}{2}(s-2)}}\right\} \text{ for } t \in [0,z] \\ 0 \text{ otherwise} \end{cases}$$

where the λ_i's are the 'cumulants' corresponding to the 'moment' sequence $\{0,1,\alpha_3,\alpha_4,\ldots\}$†. It is well known that (see e.g. p.561 [22]).

$$\int_{-\infty}^{\infty} |\tilde{A}(x)|\,dx \leq \infty$$

where $\tilde{A}(x)$ is the Fourier transform of $A_z(t)$. Hence, using Parseval's identity,

$$\left|\int_0^z \frac{h_n(t) - u_k(t)}{it} A_z(t)\,dt\right| \leq A_{kn}(C_n) \int_{-\infty}^{\infty} |\tilde{A}(x)|\,dx$$

$$\leq cA_{kn}(C_n) = O(n^{-\frac{1}{2}(k+\delta)}),$$

where $h_n(t) = \left(f\left(\frac{t}{C_n}\right)\right)^n$, $f(t)$ is the characteristic function of X_i and $u_k(t)$ is the character fn. of $U_k(x)$. Furthermore, from Lemma 5 of [23] for $|t| < \eta\sqrt{n}$ where η is suitably small,

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†These λ_i's are the γ_i's of Chapter 2. To avoid confusion between γ_i and $\gamma(t)$ the γ_i's have been changed to λ_i's.

$$\left|\exp\left(\sum_{s=2}^{k+2}\frac{(it)^s\lambda_s}{s!\,n^{\frac{1}{2}(s-2)}}\right) - u_k(t)\right| \leq \frac{c}{n^{\frac{1}{2}(k+1)}}\left(|t|^{3(k+1)} + |t|^{k+1}\right)e^{-\frac{1}{4}t^2}$$

whence

$$\int_0^z\left\{h_n(t) - \exp\sum_{s=2}^{k+2}\frac{(it)^s\lambda_s}{s!\,n^{\frac{1}{2}(s-2)}}\right\}(it)^{-1}A_z(t)dt = O\left(n^{-\frac{1}{2}(k+\delta)}\right) \qquad (34)$$

From Theorem 4 of Ibragimov [23] and (32)

$$h_n(t) = \exp\left\{n\sum_{s=2}^{k+1}\frac{(it)^s}{s!}\,\frac{\lambda_s}{C_n{}^s} + n\gamma\left(\frac{t}{C_n}\right)\left(\frac{it}{C_n}\right)^{k+2}\right\},\ \gamma(t) = O(1) \text{ as } t\to 0$$

$$= \exp\left\{\sum_{s=2}^{k+1}\frac{(it)^s}{s!}\,\frac{(1+\varepsilon_n)^{s/2}}{n^{(s-2)/2}}\,\lambda_s + \frac{(it)^{k+2}}{n^{\frac{1}{2}k}}\,(1+\varepsilon_n)^{\frac{1}{2}(k+2)}\gamma\left(\frac{t}{C_n}\right)\right\} \qquad (35)$$

We can now write (34) as

$$\int_0^z\left[\exp\left\{\sum_{s=2}^{k+1}\frac{(it)^s}{s!}\,\frac{\lambda s}{n^{\frac{1}{2}(s-2)}}\left((1+\varepsilon_n)^{s/2} - 1\right) + \frac{(it)^{k+2}}{n^{\frac{1}{2}k}}\left[(1+\varepsilon_n)^{\frac{1}{2}(k+2)}\gamma\left(\frac{t}{C_n}\right) - \right.\right.\right.$$

$$\left.\left.\left. - \frac{\lambda_{k+2}}{(k+2)!}\right]\right\} - 1\right] \times (z-t)dt = O\left(n^{-\frac{1}{2}(k+\delta)}\right).$$

Using (33) and recalling that $k \geq 1$, we obtain

$$\int_0^z\left\{-\tfrac{1}{2}t^2\varepsilon_n + \frac{(it)^{k+2}}{n^{\frac{1}{2}k}}\left[(1+\varepsilon_n)^{\frac{1}{2}(k+2)}\gamma\left(\frac{t}{C_n}\right) - \frac{\lambda_{k+2}}{(k+2)!}\right]\right\}(z-t)dt = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$$

that is,

$$-\frac{z^4}{24}\varepsilon_n + \int_0^1\frac{(izt)^{k+2}}{n^{\frac{1}{2}k}}\left[(1+\varepsilon_n)^{\frac{1}{2}(k+2)}\gamma\left(\frac{zt}{C_n}\right) - \frac{\lambda_{k+2}}{(k+2)!}\right](z-zt)d(zt) = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$$

and as $\frac{1}{2} \leq z \leq 2$

$$\frac{\varepsilon_n}{24} + \int_0^1 t^2\left(\frac{(izt)}{n^{\frac{1}{2}}}\right)^k\left[(1+\varepsilon_n)^{\frac{1}{2}(k+2)}\gamma\left(\frac{zt}{C_n}\right) - \frac{\lambda_{k+2}}{(k+2)!}\right](1-t)dt = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$$

Furthermore,

$$\int_0^1 t^2\left(\frac{izt}{n^{\frac{1}{2}}}\right)^k\frac{\lambda_{k+2}}{(k+2)!}\left[(1+\varepsilon_n)^{\frac{1}{2}(k+2)} - 1\right](1-t)dt = O(n^{-k})$$

and as $\gamma(t) = O(1)$ for small t

$$\frac{\varepsilon_n}{24} + \int_0^1 t^2\left(\frac{izt}{C_n}\right)^k\left[\gamma\left(\frac{zt}{C_n}\right) - \frac{\lambda_{k+2}}{(k+2)!}\right](1 - t)dt = O\left(n^{-\frac{1}{2}(k+\delta)}\right) \tag{36}$$

Define

$$\rho(n) = \int_0^1 t^2\left(\frac{it}{n^{\frac{1}{2}}}\right)^k\left[\gamma\left(\frac{t}{n^{\frac{1}{2}}}\right) - \frac{\lambda_{k+2}}{(k+2)!}\right](1 - t)dt$$

then, upon taking $z = (1 + \varepsilon_n)^{-\frac{1}{2}}$ and $\left(2(1 + \varepsilon_n)\right)^{-\frac{1}{2}}$ successively, we find that

$$\frac{\varepsilon_n}{24} + \rho(n) = O\left(n^{-\frac{1}{2}(k+\delta)}\right) \tag{36a}$$

and

$$\frac{\varepsilon_n}{24} + \rho(2n) = O\left(n^{-\frac{1}{2}(k+\delta)}\right) .$$

Hence $|\rho(2n) - \rho(n)| = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$, and as $\rho(n) \to 0$ as $n \to \infty$

$$\rho(2^j) \leqq \sum_{r=j}^{\infty} |\rho(2^{r+1}) - \rho(2^r)|$$

$$\leqq C2^{-\frac{1}{2}j(k+\delta)} \text{ for all sufficiently large } j.$$

We now have $\rho(n) = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$ and from (36a),

$\varepsilon_n = O\left(n^{-\frac{1}{2}(k+\delta)}\right)$. We are now in a position to examine the function $\rho(n)$. From (35) we note that we could have written $(it)^{k+1}\gamma_k(t)$ where $\gamma_k(t) = O(t)$ as $t \to 0$ in place of $(it)^{k+2}\gamma(t)$, so that

$$- \rho(n) = -\int_0^1 t^2\left[\left(\frac{it}{\sqrt{n}}\right)^{k-1}\gamma_k\left(\frac{t}{\sqrt{n}}\right) - \left(\frac{it}{\sqrt{n}}\right)^k\frac{\lambda_{k+2}}{(k+2)!}\right](1 - t)dt$$

$$= n^2\int_0^{n^{-\frac{1}{2}}} (it)^{k+1}\gamma_k(t)(n^{-\frac{1}{2}} - t)dt + \frac{i^k\lambda_{k+2}}{n^{\frac{1}{2}k}(k+4)!}$$

From this last expression we can deduce that in general

$$\int_0^x (x-t)\,(it)^{k+1}\,\gamma_k(t)dt - \frac{i^{k+2}\lambda_{k+2}}{(k+4)!}\,x^{k+4} = O\left(x^{k+4+\delta}\right)$$

as $x \to 0$. This is just expression (3.7) of Ibragimov [23] so we can directly deduce that (1) holds.

It remains to show (i) $\Rightarrow$ (iv). We recall that $\tau_{nx} = \sqrt{n}$ and that here $x_o = x - \sqrt{n}\frac{E\bar{X}_1}{\sigma_n}$, we have

$$|x_o - x| \leqq \frac{\sqrt{n}}{\sigma_n} \int_{|u|>\sqrt{n}} |u|dF(u) \to 0 \text{ as } n \to \infty.$$

$$\leqq cn^{-\frac{1}{2}k} \int_{|u|>\sqrt{n}} |u|^{k+2}dF(u)$$

So, from Remarks g) and h) together with pertinent results in the proof of Theorems 3.3 and 3.4, we have

$$|U_{k,n}(x_o) - U_{k,n}(x)| = O(n^{-\frac{1}{2}(k+\delta)}).$$

and

$$|U_{k,n}(x) - U_k(x)| = O(n^{-\frac{1}{2}(k+\delta)})$$

Also

$$nP\left(|X_i| > \sqrt{n}\right) = n \int_{|u|>\sqrt{n}} dF(u)$$

$$\leqq n^{-\frac{1}{2}k} \int_{|u|>\sqrt{n}} |u|^{k+2}dF(u)$$

$$= O\left(n^{-\frac{1}{2}(k+\delta)}\right)$$

Next, for all sufficiently large n, $\delta_n = \left(100E|\bar{X}_i|^3\right)^{-1}\sigma_n^2 > \frac{1}{2}\left(100E|X_i|^3\right)^{-1}$ which is independent of n and hence with Cramer's condition (c), the term involving f(t) decreases geometrically. Finally, an application of Lemma 3.7 completes the theorem.

CHAPTER 4[†]

SOME CLASSICAL LIMIT ANALOGUES FOR GALTON-WATSON PROCESSES

4.1. INTRODUCTION: In the previous chapters we considered sequences of independent random variables. In this chapter we consider a particular class of dependent sequences, the super-critical Galton-Watson process, and show how some of the results of independent random variable theory can be used to obtain analogues of the classical limit theorems as well as convergence rate results for this process. The chapter is divided into two parts; the first part dealing with the super-critical Galton-Watson process without immigration and the second, with immigration.

Let $Z_o = 1, Z_1, Z_2, \ldots$ denote a super-critical Galton-Watson process as defined in Chapter 1 with $1 < EZ_1 = m$ and $0 < \text{var } Z_1 = \sigma^2 < \infty$. As mentioned in Chapter 1, there exists a non-degenerate random variable W such that $\lim_{n\to\infty} W_n = W$ almost surely, where $W_n = m^{-n} Z_n$. Furthermore, a central limit analogue was established in this context by Heyde [15]. He showed that, conditional on $Z_n > 0$,

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W - W_n) \xrightarrow{D} N(0,1).$$

Although this does not appear to be the classical setting of a central limit theorem, if we note that the usual central limit theorem can be regarded as a convergence rate result for the strong law of large numbers (indeed $\sigma^{-1} n^{\frac{1}{2}} (\mu - n^{-1} S_n) \xrightarrow{D} N(0,1)$ for a sum S_n of iid rv's with mean μ and variance σ^2), the analogy between the Heyde result and the central limit theorem becomes clear.

W. J. Bühler [4] earlier obtained a similar limit result for $Z_{n+j} - m^j Z_n$; namely, that conditional on $Z_n > 0$,

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} m^{-\frac{1}{2}j} (m^j - 1)^{-\frac{1}{2}} Z_n^{-\frac{1}{2}} \left(Z_{n+j} - m^j Z_n \right) \xrightarrow{D} N(0,1).$$

[†] The work of this chapter was carried out jointly with Dr. C. C. Heyde and appears in [19].

Rates of convergence for these central limit analogues under the further restriction that $EZ_1^3 < \infty$, were obtained by Heyde and Brown [18] The proofs hinged on a result of Heyde [15] which states that conditional on $Z_n > 0$, $m^n Z_n^{-\frac{1}{2}}(W - W_n)$ and $Z_n^{-\frac{1}{2}}(Z_{r+n} - m^r Z_n)$ have the same distribution as the sum of Z_n (conditional on $Z_n > 0$) iid rv's which are independent of Z_n and normed by $Z_n^{\frac{1}{2}}$. The restriction $EZ_1^3 < \infty$ allowed the use of the Berry-Esseen bound on the iid rv's. These rate results were then used by Heyde [16] to obtain almost sure convergence results for the Galton-Watson process which are analogues of the law of the iterated logarithm for independent random variables. In this chapter we show how the restriction $EZ_1^3 < \infty$ may be removed. In Section 4.2 we shall obtain convergence rates for the central limit analogues and also iterated logarithm analogues under the basic condition that $EZ_1^2 < \infty$.

In Section 4.4 we shall consider the Galton-Watson process with immigration. The development of the corresponding limit results in this case has followed the pattern described previously for the case without immigration. Indeed, as in Chapter 1, let $\{X_n\}$ be the Galton-Watson process whose offspring distribution has the distribution of Z_1 with $1 < EZ_1 = m$ and $0 < \operatorname{var} Z_1 = \sigma^2 < \infty$ and those immigration distribution has a finite mean. We know from Chapter 1 that under the present conditions $m^{-n} X_n$ converges almost surely to a proper random variable V with finite mean EV and such that $P(V = 0) = 0$. Heyde and Seneta [21] obtained central limit results under $EZ_1^2 < \infty$ for $V - m^{-n} X_n$ and $X_{n+r} - m^r X_n$ as well as rate results and iterated analogues under $EZ_1^3 < \infty$. Again the key to the latter results were the representations for $X_n - m^n V$ and $X_{n+r} - m^r X_n$ as sums of independent random variables. As in Section 4.2 we shall show how to dispense with the moment restriction and will obtain the rate results and iterated logarithm analogues under $EZ_1^2 < \infty$.

4.2. THE PROCESS WITHOUT IMMIGRATION. We shall establish the following theorems: Theorem 4.1, providing convergence rate results, and Theorem 4.2, iterated logarithm analogues.

THEOREM 4.1. Let $1 < m = EZ_1$ *and* $0 < \operatorname{var} Z_1 = \sigma^2 < \infty$. *Then*

$$\sup_x \left| P\left\{ (m^2 - m)^{\frac{1}{2}} \sigma^{-1} u_{Z_n}^{-1} Z_n^{-\frac{1}{2}} m^n (W - W_n) \leqq x \mid Z_n > 0 \right\} - \Phi(x) \right| \leqq c_n$$

and

$$\sup_x \left| P\left(\sigma_r^{-1} v_{Z_n}^{-1} Z_n^{-\frac{1}{2}} \left\{ Z_{n+r} - m^r Z_n \right\} \leqq x \mid Z_n > 0 \right) - \Phi(x) \right| \leqq d_n,$$

where $\{C_n\}$, $\{d_n\}$ *are certain sequences of positive constants satisfying* $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\sum_{n=1}^{\infty} d_n < \infty$. *Here*

$$\sigma_r^2 = \operatorname{var} Z_r = \sigma^2 m^r (m^r - 1)(m^2 - m)^{-1},$$

r *any fixed integer,*

$$u_n = \int_{|x| < \sqrt{n}} x^2 \, dP\left[\sigma^{-1} (m^2 - m)^{\frac{1}{2}} (W - 1) \leqq x \right]$$

$$v_n = \int_{|x| < \sqrt{n}} x^2 \, dP\left[\sigma_r^{-1} \left\{ Z_r - m^r \right\} \leqq x \right]$$

and $\Phi(x)$ *is the distribution function of* N(0,1).

Explicit forms for c_n and d_n can be found by applying the lemma below. We note also that $u_n \uparrow 1$ and $v_n \uparrow 1$ as $n \to \infty$.

THEOREM 4.2. Suppose that $1 < m = EZ_1$ *and* $0 < \operatorname{var} Z_1 = \sigma^2 < \infty$. *Then, on the non-extinction set* $\{W > 0\}$ *we have almost surely*

$$\limsup_{n \to \infty} \frac{Z_{n+r} - m^r Z_n}{\left\{ 2\sigma^2 Z_n \log n \right\}^{\frac{1}{2}}} = 1, \qquad \liminf_{n \to \infty} \frac{Z_{n+r} - m^r Z_n}{\left\{ 2\sigma_r^2 Z_n \log n \right\}^{\frac{1}{2}}} = -1,$$

and

$$\limsup_{n\to\infty} \frac{m^n W - Z_n}{\left[2\sigma^2 (m^2 - m)^{-1} Z_n \log n\right]^{\frac{1}{2}}} = 1, \quad \liminf_{n\to\infty} \frac{m^n W - Z_n}{\left[2\sigma^2 (m^2 - m)^{-1} Z_n \log n\right]^{\frac{1}{2}}} = -1,$$

where r *is any fixed positive integer.*

Theorems 4.1 and 4.2 extend the scope of results given in Heyde & Brown [18] and in Heyde[16] respectively under the additional condition that $EZ_1^3 < \infty$. The form of the bounds obtained in Theorem 4.1 is however, of necessity, much more complicated in this general case. The same situation prevails in the independence case (see [14]). Our Theorem 4.2 preserves exactly the form of the Theorem of [16] under the more general conditions.

In order to establish the above results, we need the following important lemma. The result of the lemma is given in two parts; the first is needed in the present section and the second to obtain corresponding results for the process with immigration in Section 4.4.

LEMMA 4.3. Let ξ_i, $i = 1,2,3,\ldots$ *be independent and identically distributed random variables with* $E(\xi_1) = 0$ *and* $\operatorname{var} \xi_1 = \alpha^2 < \infty$. *Let* N_n *be a positive integer-valued random variable which is independent of the* $\{\xi_i\}$. *Then,*

$$\sup_x \left| P\left(\alpha^{-1} d_{N_n}^{-1} N_n^{-\frac{1}{2}} \left(\xi_1 + \ldots + \xi_{N_n}\right) \leq x\right) - \Phi(x) \right| \tag{1}$$

$$\leqq AE\left(N_n^{-\frac{1}{2}} a_{N_n}\right) + BE\left(N_n^{\frac{1}{2}} b_{N_n}\right) + E\left(N_n c_{N_n}\right)$$

where A, B *are positive constants and*

$$a_n = \int_{|x|<\sqrt{n}} |x|^3 \, dP\left(\alpha^{-1} \xi_1 \leqq x\right), \qquad b_n = \int_{|x|\geqq\sqrt{n}} |x| \, dP\left(\alpha^{-1} \xi_1 \leqq x\right),$$

$$c_n = P\left(\alpha^{-1} |\xi_1| > \sqrt{n}\right), \qquad d_n^2 = \int_{|x|<\sqrt{n}} x^2 \, dP\left(\alpha^{-1} \xi_1 \leqq x\right).$$

If η_n *with* $E|\eta_n| < \infty$ *is a random variable which is independent of the* $\{\xi_i\}$ *and of* N_n, *then for any sequence* $\{\varepsilon_n\}$ *of positive constants with* $\varepsilon_n \to 0$ *as* $n \to \infty$,

$$\sup_x \left| P\left(\alpha^{-1} d_{N_n}^{-1} N_n^{-\frac{1}{2}} \left(\xi_1 + \ldots + \xi_{N_n} + \eta_n\right) \leqq x\right) - \Phi(x) \right| \tag{2}$$

$$\leqq AE\left(N_n^{-\frac{1}{2}} a_{N_n}\right) + BE\left(N_n^{\frac{1}{2}} b_{N_n}\right) + E\left(N_n \, c_{N_n}\right) + \alpha^{-1} \varepsilon_n^{-1} E|\eta_n| E\left(N_n^{-\frac{1}{2}} d_{N_n}^{-1}\right) + \frac{1}{2}\varepsilon_n .$$

Proof of lemma. Let

$$e_n^2 = \int_{|x|<\sqrt{n}} x^2 \, dP\left(\alpha^{-1} \xi_1 \leqq x\right) - \left(\int_{|x|<\sqrt{n}} x \, dP\left(\alpha^{-1} \xi_1 \leqq x\right)\right)^2 .$$

and set

$$\Psi_{N_n} = \alpha^{-1} d_{N_n}^{-1} N_n^{-\frac{1}{2}} \left(\xi_1 + \ldots + \xi_{N_n}\right) \text{ and } \Psi'_{N_n} = e_{N_n}^{-1} d_{N_n} \Psi_{N_n} .$$

We have

$$\sup_x \left| P\left(\Psi_{N_n} \leqq x \mid N_n = k\right) - \Phi(x) \right|$$

$$\leqq \sup_x \left| \left(\Psi'_{N_n} \leqq x \middle| N_n = k\right) - \Phi(x) \right| + \sup_x \left| \Phi(x) - \Phi\left(e_k d_k^{-1} x\right) \right| . \tag{3}$$

Also, using the mean value theorem,

$$\sup_x \left| \Phi(x) - \Phi\left(e_k d_k^{-1} x\right) \right| \leqq c\left(1 - e_k d_k^{-1}\right)$$

$$= c d_k^{-1} \left(e_k + d_k\right)^{-1} \left(\int_{|x|<\sqrt{k}} x \, dP\left(\alpha^{-1} \xi_1 \leqq x\right)\right)^2$$

$$= c d_k^{-1} \left(e_k + d_k\right)^{-1} \left(\int_{|x|\geqq\sqrt{k}} x \, dP\left(\alpha^{-1} \xi_1 \leqq x\right)\right)^2 \tag{4}$$

$$\leqq c_1 b_k^2 \leqq c_1 b_k ,$$

c, c_1 being positive constants. Furthermore, from (22) of Heyde [14] we find that

$$\sup_x \left| P\left(\Psi'_{N_n} \leqq x \mid N_n = k\right) - \Phi(x) \right| \leqq Ak^{-\frac{1}{2}} a_k + B_1 k^{\frac{1}{2}} b_k + kc_k \,, \tag{5}$$

so that using (3), (4) and (5),

$$\sup_x \left| P\left(\Psi_{N_n} \leqq x \mid N_n = k\right) - \Phi(x) \right| \leqq Ak^{-\frac{1}{2}} a_k + Bk^{\frac{1}{2}} b_k + kc_k \,. \tag{6}$$

The result (1) follows readily from (6) and the fact that as the ξ_i are independent of N_n

$$\sum_k P\left(\Psi_k \leqq x\right) \cdot P(N_n = k) = \sum_k P\left(\Psi_k \leqq x \mid N_n = k\right) \cdot P(N_k = k) = P\left(\Psi_{N_n} \leqq x\right),$$

and $EN_n^{-\frac{1}{2}} a_{N_k} = \sum k^{-\frac{1}{2}} a_k P(N_n = k)$.

Inequality (2) is obtained from the well known result[†] below. Indeed setting $\Gamma_{N_n} = \alpha^{-1} d_{N_n}^{-1} N_n^{-\frac{1}{2}} \eta_n$ we note that since the ξ_i and η are independent of each other and of N_n,

$$\sup_x \left| P\left(\Psi_{N_n} + \Gamma_{N_n} \leqq x\right) - \Phi(x) \right| = \sup_x \left| \sum_{k=1}^{\infty} \left[P\left(\Psi_k + \Gamma_k \leqq x\right) - \Phi(x) \right] \cdot P\left(N_n = k\right) \right|$$

$$\leqq \sum_{k=1}^{\infty} \left\{ \sup_x \left| P\left(\Psi_k \leqq x\right) - \Phi(x) \right| + P\left(|\Gamma_k| > \varepsilon_n\right) + \frac{\varepsilon_n}{2} \right\} P\left(N_n = k\right)$$

By the above part of the lemma,

$$\sum_{k=1}^{\infty} \sup_x \left| P\left(\Psi_k \leqq x\right) - \Phi(x) \right| P\left(N_n = k\right) \leqq AE\left(N_n^{-\frac{1}{2}} a_{N_n}\right) + BE\left(N_n^{\frac{1}{2}} b_{N_n}\right) + E\left(N_n c_{N_n}\right)$$

Also, since $E|\eta_n| < \infty$, by Markov's inequality

$$P\left(|\Gamma_k| > \varepsilon_n\right) \leqq \alpha^{-1} \varepsilon_n^{-1} E|\eta_n| \cdot k^{-\frac{1}{2}} d_k^{-\frac{1}{2}}$$

Substituting these estimates into the above inequality, we obtain (2) thereby completing the proof of the lemma.

[†] If X and Y are independent then (see [21]) for any $\varepsilon > 0$,
$\sup_x |P(X+Y \leqq x) - \Phi(x)| \leqq \sup_x |P(X \leqq x) - \Phi(x)| + P(|Y| > \varepsilon) + \frac{1}{2}\varepsilon$.

4.3. PROOF OF THEOREMS.

Proof of Theorem 4.1. Suppose that Z_n^* has the distribution of Z_n conditional on $Z_n > 0$. We firstly note that (see [15], [18]) conditional on $Z_n > 0$, $m^n Z_n^{-\frac{1}{2}}(W - W_n)$ has the same distribution as $(Z_n^*)^{-\frac{1}{2}}\left(U_1 + \ldots + U_{Z_n^*}\right)$, where the U_i are independent of Z_n^* and are independent and identically distributed, each with the distribution of $W - 1$. Also, conditional on $Z_n > 0$, $Z_n^{-\frac{1}{2}}\left(Z_{n+r} - m^r Z_n\right)$ has the same distribution as $(Z_n^*)^{-\frac{1}{2}}\left(V_1 + \ldots + V_{Z_n^*}\right)$ where the V_i are independent of Z_n^* and are independent and identically distributed, each with the distribution of $Z_r - m^r$. We can thus apply the lemma in both cases and obtain bounds which we call c_n, d_n respectively. It remains to show that $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$. We shall indicate the proof for $\sum_{n=1}^{\infty} c_n$; that for $\sum_{n=1}^{\infty} d_n$ follows similarly.

What we have to demonstrate is that $\sum_{n=1}^{\infty} E\left((Z_n^*)^{-\frac{1}{2}} a_{Z_n^*}\right) < \infty$, $\sum_{n=1}^{\infty} E\left((Z_n^*)^{\frac{1}{2}} b_{Z_n^*}\right) < \infty$ and $\sum_{n=1}^{\infty} E\left[Z_n^* c_{Z_n^*}\right] < \infty$ where a_n, b_n, c_n are defined in the lemma with ξ_i having the distribution of $W - 1$. The proofs of the convergence of these three series are similar in form. They depend on results of [14] where it is, in essence, shown in the proof of Theorem 4 that under the conditions of the theorem and if n_k, $k = 1,2,3,\ldots$ is a sequence of integers with $n_k \sim Kc^{2k}$ as $k \to \infty$ $(K > 0,\ c > 1)$, then

$$\sum_{n=1}^{\infty} n^{-3/2} a_n < \infty, \qquad \sum_{n=1}^{\infty} n^{-1/2} b_n < \infty, \qquad \sum_{n=1}^{\infty} c_n < \infty,$$

and $$\sum_{k=1}^{\infty} n_k^{-\frac{1}{2}} a_{n_k} \leqq K_1 < \infty, \quad \sum_{k=1}^{\infty} n_k^{-\frac{1}{2}} b_{n_k} \leqq K_2 < \infty, \quad \sum_{k=1}^{\infty} n_k c_{n_k} \leqq K_3 < \infty$$

for certain K_1, K_2, K_3 independent of K.

For $u > 0$, let

$$a_u = \int_{|x|<\sqrt{u}} |x|^3 dP\left(\sigma^{-1}\left(m^2 - m\right)^{\frac{1}{2}}(W - 1) \leqq x\right).$$

First we deal with the case $P(Z_j = 0) = 0$. What we have to show is

$$\sum_{n=1}^{\infty} E(Z_n^{-\frac{1}{2}} a_{Z_n}) < \infty$$

i.e.

$$E \sum_{n=1}^{\infty} Z_n^{-\frac{1}{2}} a_{Z_n} < \infty .$$

Now $Z_{k+1} \geqq Z_k$ for all k and

$$(\infty >) \sum_{n=1}^{\infty} n^{-3/2} a_n \geqq \sum_{k=0}^{\infty} \sum_{n=Z_k+1}^{Z_{k+1}} n^{-3/2} a_n$$

(where the sum $\sum_{n=Z_k+1}^{Z_{k+1}}$ is understood as being zero if $Z_{k+1} = Z_k$)

$$\geqq \sum_{k=0}^{\infty} Z_{k+1}^{-3/2} (Z_{k+1} - Z_k) a_{Z_k}$$

since $a_n \uparrow$ as $n \uparrow$ and hence

$$\sum_{k=0}^{\infty} E\left[Z_{k+1}^{-3/2} (Z_{k+1} - Z_k) a_{Z_k}\right] < \infty .$$

The required result then follows if we can show that

$$E\left(Z_{k+1}^{-3/2} (Z_{k+1} - Z_k) \middle| \chi_k\right) \geqq c Z_k^{-\frac{1}{2}} \tag{7}$$

for some $c > 0$, χ_k being the σ-field generated by $Z_k, Z_{k-1}, \ldots, Z_1$.

In order to show (7), we need

$$E\left(\frac{Z_k^{\frac{1}{2}}}{Z_{k+1}^{\frac{1}{2}}} - \frac{Z_k^{3/2}}{Z_{k+1}^{3/2}} \middle| \chi_k \right) \geqq c > 0$$

which holds if

$$E\left(\frac{k^{\frac{1}{2}}}{S_k^{\frac{1}{2}}} - \frac{k^{3/2}}{S_k^{3/2}} \right) \geqq c > 0 \tag{8}$$

for all $k \geqq 1$, where $S_k = Z_1^{(1)} + \ldots + Z_1^{(k)}$, the $Z_1^{(i)}$'s being independent and identically distributed, each with the distribution of Z_1. But, using Schwarz's inequality,

$$E\left\{\left(1-\frac{k}{S_k}\right)^{\frac12}\right\} = E\left\{\frac{S_k}{k^{\frac14}}\cdot\frac{k^{\frac14}}{S_k^{\frac14}}\left(1-\frac{k}{S_k}\right)^{\frac12}\right\}$$

$$\leqq \left\{E\left(\frac{S_k^{\frac12}}{k^{\frac12}}\right) E\left(\frac{k^{\frac12}}{S_k^{\frac12}}\left(1-\frac{k}{S_k}\right)\right)\right\}^{\frac12}$$

while

$$\left\{E\left(\frac{S_k^{\frac12}}{k^{\frac12}}\right)\right\}^2 \leqq E\left(\frac{S_k}{k}\right) = m$$

thus

$$E\left(\frac{k^{\frac12}}{S_k^{\frac12}}\left(1-\frac{k}{S_k}\right)\right) \geqq \frac{\left\{E\left(1-\frac{k}{S_k}\right)^{\frac12}\right\}^2}{m^{\frac12}} \tag{9}$$

Furthermore,

$$0 \leqq 1-\frac{k}{S_k} \leqq 1,$$

so that

$$E\left(1-\frac{k}{S_k}\right)^{\frac12} \geqq 1-E\left(\frac{k}{S_k}\right).$$

Using the harmonic mean-arithmetic mean inequality[†],

$$E\,\frac{k}{S_k} = E\,\frac{k}{Z_1+\ldots+Z_1} \leqq E\,\frac{1}{k}\left(\frac{1}{Z_1}+\ldots+\frac{1}{Z_1}\right) = E\left(\frac{1}{Z_1}\right)$$

and since $EZ_1 > 1 \Rightarrow P(Z_1 = 1) < 1$, $E\,\frac{1}{Z_1} = \gamma < 1$.

thus $$E\left(1-\frac{k}{S_k}\right)^{\frac12} \geqq 1-\gamma \geqq c > 0\,. \tag{10}$$

From (8), (9) and (10) we see that (7) holds with

$$c = \frac{1}{m^{\frac12}}\,(1-\gamma)^2$$

and hence the desired result follows.

Now we can consider the case $P(Z_1 = 0) > 0$. As in the Heyde and Brown paper [18] we introduce a related Galton-Watson process $\{Y_n\}$ such that $P(Y_n = 0) = 0$ for each n. Set

$$F_n(s) = \sum_{k=0}^{\infty} P(Z_n = k)s^k$$

[†] For convenience, the superscripts on the $Z_1^{(i)}$'s have been dropped.

and let q be the probability of ultimate extinction of the Z_n process. We define the process $\{Y_n\}$ by

$$H_n(s) = \sum_{k=1}^{\infty} P(Y_n = k)s^k = (1 - q)^{-1}\left[F_n\left(s(1 - q) + q\right) - q\right],$$

and note that upon expanding out,

$$P(Y_n = k) = (1 - q)^{k-1} \sum_{j=k}^{\infty} \binom{j}{k} P(Z_n = j)q^{j-k} \tag{11}$$

What we need to show in this case is

$$\sum_{n=1}^{\infty} E\left(Z_n^{*-\frac{1}{2}} a_{Z_n^*}\right) < \infty ,$$

that is

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} j^{-\frac{1}{2}} a_j \, P(Z_n = j) < \infty . \tag{12}$$

Using the same reasoning as was used to prove that $EX^2 < \infty \Rightarrow \sum n^{-\frac{3}{2}} a_n < \infty$, we can show that $\sum n^{-3/2} a_{rn} < \infty$ for any fixed $r > 0$.

Using this relation in the previous work instead of the particular case $r = 1$ considered above, we easily derive that

$$\sum_{n=1}^{\infty} EZ_n^{-\frac{1}{2}} \, a_{rZ_n} < \infty \quad \text{for any fixed } r > 0, \text{ when } P(Z_n = 0) = 0.$$

Hence, for any fixed $r > 0$, we have

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} j^{-\frac{1}{2}} a_{rj} \, P(Y_n = j) < \infty$$

and hence, using (11)

$$\infty > \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^{-\frac{1}{2}} a_{rk} (1 - q)^{k-1} \sum_{j=k}^{\infty} \binom{j}{k} P(Z_n = j)q^{j-k}$$

$$= (1-q)^{-1} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} P(Z_n = j) \sum_{k=1}^{j} k^{-\frac{1}{2}} a_{rk} \binom{j}{k} (1 - q)^k q^{j-k} . \tag{13}$$

But, if Q_n has a binomial distribution $b(n,p,q)$ where $p = 1 - q$, we have for any $0 < \varepsilon < p$,

$$\sum_{k=1}^{j} k^{-\frac{1}{2}} a_{rk}\binom{j}{k}(1-q)^k q^{j-k} = E\left(Q_j^{-\frac{1}{2}} a_{rQ_j}\right)$$

$$\geqq E\left(Q_j^{-\frac{1}{2}} a_{rQ_j}\{Q_j \geqq jp(1-\varepsilon)\}\right)$$

$$\geqq j^{-\frac{1}{2}} a_{rjp(1-\varepsilon)} P\left(Q_j \geqq jp(1-\varepsilon)\right)$$

Taking $r = \dfrac{1}{p(1-\varepsilon)}$ and noting that $P\{Q_j \geqq jp(1-\varepsilon)\} > c > 0$ for all j we obtain the desired result.

To deal with $\sum_{n=1}^{\infty} E(Z_n^{*\frac{1}{2}} b_{Z_n^*})$ and $\sum_{n=1}^{\infty} E(Z_n^* c_{Z_n^*})$ we essentially use the above technique noting that we must first prove that

$$E\left(\frac{Z_{k+1}^{\frac{1}{2}}}{Z_k^{\frac{1}{2}}} - \frac{Z_k^{\frac{1}{2}}}{Z_{k+1}^{\frac{1}{2}}} \,\middle|\, \chi_k\right) \geqq c > 0 \quad \text{for all } k \tag{14}$$

and

$$E\left(\frac{Z_{k+1}}{Z_k} - 1 \,\middle|\, \chi_k\right) \geqq c > 0 \quad \text{for all } k. \tag{15}$$

Relation (14) is equivalent to showing that for all k

$$E\left(\frac{S_k}{k^{\frac{1}{2}}} - \frac{k^{\frac{1}{2}}}{S_k^{\frac{1}{2}}}\right) \geqq c > 0$$

Now

$$E\left[\left(1 - \frac{k}{S_k}\right)^{\frac{1}{2}}\right] \leqq E\left(\frac{k}{S_k}\right)^{\frac{1}{2}} E\,\frac{S_k^{\frac{1}{2}}}{k^{\frac{1}{2}}}\left(1 - \frac{k}{S_k}\right) \leqq \left(EZ_1^{-1}\right)^{\frac{1}{2}} E\left(\frac{S_k}{k^{\frac{1}{2}}} - \frac{k^{\frac{1}{2}}}{S_k^{\frac{1}{2}}}\right)$$

Thus

$$E\left(\frac{S_k^{\frac{1}{2}}}{k^{\frac{1}{2}}} - \frac{k^{\frac{1}{2}}}{S_k^{\frac{1}{2}}}\right) \geqq \frac{1}{(EZ_1^{-1})^{\frac{1}{2}}}\, E\left[\left(1 - \frac{k}{S_k}\right)^{\frac{1}{2}}\right] \geqq \frac{1}{(EZ_1^{-1})^{\frac{1}{2}}}\left(1 - EZ_1^{-1}\right) \geqq c > 0$$

Relation (15) is equivalent to showing that for all k

$$E\,\frac{S_k}{k} - 1 \geqq c > 0\,.$$

But

$$E\,\frac{S_k}{k} = EZ_1 = m > 1.$$

Hence (14) and (15) are valid. It remains to treat the analogues of (13). We use a slightly different technique; we note that as $c_n \downarrow$ when $n\uparrow$,

$$E\left(Q_n c_{Q_n}\right) \geq c_n E(Q_n) = pnc_n,$$

while $b_n \downarrow$ when $n \uparrow$ so

$$\begin{aligned} E\left\{Q_n^{\frac{1}{2}}\, b_{Q_n}\right\} &\geq b_n\ E\left(Q_n^{\frac{1}{2}}\right) \\ &\geq b_n E\left\{Q_n^{\frac{1}{2}}\left(Q_n > np(1-\varepsilon)\right)\right\} \\ &\geq b_n n^{\frac{1}{2}} p^{\frac{1}{2}} (1-\varepsilon)^{\frac{1}{2}} P\left(Q_n > np(1-\varepsilon)\right) \geq cb_n n^{\frac{1}{2}} \quad . \end{aligned}$$

Proof of Theorem 4.2. This follows the same lines as the proof of the theorem of [16]. However, there is a slight difference in that we establish first the results for $Z_{n+r} - m^r Z_n$ and then show that they are sufficient to establish those for $m^n W - Z_n$, whereas [16] proves part of the $m^n W - Z_n$ result from first principles.

P.T.O.

To begin, we establish that the limsup (for convenience we will write this as $\overline{\lim}$) result for $Z_{n+r} - m^r Z_n$ is at most 1 a.s. That is,

$$P\left[Z_{n+r} - m^r Z_n > (1+\delta)(2\sigma_r^2 Z_n \log n)^{\frac{1}{2}} \text{ infinitely often (i.o.) for any } \delta>0\right] = 0.$$

By the Borel-Cantelli Lemma (Loeve, p.228 [27]) this will be true if

$$\sum_{n=1}^{\infty} P\left[Z_{n+r} - m^r Z_n > (1+\delta)(2\sigma_r^2 Z_n \log n)^{\frac{1}{2}}\right] < \infty.$$

Since $\sum \left[1 - \Phi\left((1+\delta)(2 \log n)^{\frac{1}{2}}\right)\right] < \infty$, from Theorem 4.1

$$\sum_{n=1}^{\infty} P\left[Z_{n+r} - m^r Z_n > (1+\delta)(2\sigma_r^2 v_{Z_n}^2 Z_n \log n)^{\frac{1}{2}}\right] < \infty .$$

This is not quite what we desired, due to the presence of the $v_{Z_n}^2$. However, reversing the above procedure, we can extract the following result,

$$\overline{\lim} \frac{Z_{n+r} - m^r Z_n}{\left(2\sigma_r^2 v_{Z_n}^2 Z_n \log n\right)^{\frac{1}{2}}} \leq 1 \qquad \text{a.s.}$$

Now since $v_{Z_n}^2 \xrightarrow{a.s.} 1$ as $n \to \infty$ we have

$$\overline{\lim} \frac{Z_{n+r} - m^r Z_n}{\left(2\sigma_r^2 Z_n \log n\right)^{\frac{1}{2}}} \leq 1 \qquad \text{a.s.} \tag{16}$$

as required.

To obtain the other half of this $\overline{\lim}$ result we establish it for $r = 1$ and then prove it true for general r. We need to show that

$$P\left[Z_{n+1} - m Z_n > (1-\delta)(2\sigma_1^2 Z_n \log n)^{\frac{1}{2}} \text{ i.o. for any } \delta > 0\right] = 1.$$

This is done via the extended Borel-Cantelli lemma; namely, if G_n is an increasing sequence of σ-fields and if $A_n \in G_n$, then $P(A_n$ i.o.$) = 1$ if and only if

$$\sum_1^\infty P\left(A_n \mid G_{n-1}, G_{n-2}, \ldots\right) = \infty \quad \text{a.s. ([6], p.28).}$$

Setting

$$A_n = \left(Z_{n+1} - m Z_n > (1-\delta)(2\sigma_1^2 Z_n \log n)^{\frac{1}{2}}\right)$$

and

$$G_n = F\left(Z_{n+1}, Z_n, \ldots, Z_2, Z_1\right),$$

$(F(\ldots)$ denoting the σ-field generated by the variables specified$)$ and noting from the proof of Theorem 4.1 that, conditional on $Z_n > 0$, we have the representation for each n

$$Z_n^{-\frac{1}{2}}(Z_{n+1} - mZ_n) = Z_n^{-\frac{1}{2}}(V_1 + \ldots + V_{Z_n}) \quad \text{a.s.}$$

where V_i are independently and identically distributed as $(Z_1 - m)$ and independently of Z_n, we must show that

$$\sum_{n=1}^\infty P\left(Z_{n+1} - mZ_n > (1-\delta)(2\sigma_1^2 Z_n \log n)^{\frac{1}{2}} \;\middle|\; Z_n\right) = \infty. \tag{17}$$

From expression (6) in the proof of the lemma

$$P\left(V_1 + \ldots + V_{Z_n} > (1-\delta)(2\sigma_1^2 Z_n \log n)^{\frac{1}{2}} \;\middle|\; Z_n\right)$$

$$\geqq \left(1 - \Phi\{(1-\delta)(2\log n)^{\frac{1}{2}}\}\right) - AZ_n^{-\frac{1}{2}} a_{Z_n} - BZ_n^{\frac{1}{2}} b_{Z_n} - Z_n c_{Z_n} \tag{18}$$

But $\sum \left(1 - \Phi\{(1-\delta)(2\log n)^{\frac{1}{2}}\}\right) = \infty$ and, from the proof of Theorem 4.1, $\sum EZ_n^{-\frac{1}{2}} a_{Z_n}$, $\sum EZ_n^{\frac{1}{2}} b_{Z_n}$ and $\sum EZ_n c_{Z_n}$ all converge. These latter results imply the a.s. convergence of $\sum Z_n^{-\frac{1}{2}} a_{Z_n}$, $\sum Z_n^{\frac{1}{2}} b_{Z_n}$ and $\sum Z_n c_{Z_n}$.

Thus we have from (18) the required divergent series (17)

Combining this result and (16) we have

$$\overline{\lim} \frac{Z_{n+1} - mZ_n}{(2\sigma_1^2 Z_n \log n)^{\frac{1}{2}}} = 1, \quad \text{a.s.}$$

We now treat the case $r > 1$. Write Z_n^* for Z_{rn}, $n = 0,1,2,\ldots$ It is not difficult to check that $\{Z_n^*\}$ is again a super-critical Galton-Watson process with offspring distribution that of Z_r. Applying the limsup result above for $r = 1$ to the process $\{Z_n^*\}$, we have a.s.,

$$\overline{\lim_{n\to\infty}} \frac{Z_{n+1}^* - m^r Z_n^*}{(2\sigma_r^2 Z_n^* \log n)^{\frac{1}{2}}} = 1, \quad \text{a.s.}$$

that is

$$\overline{\lim_{n\to\infty}} \frac{Z_{nr+r} - m^r Z_{nr}}{(2\sigma_r^2 Z_{nr} \log nr)^{\frac{1}{2}}} = 1, \quad \text{a.s.}$$

But

$$\overline{\lim_{n\to\infty}} \frac{Z_{n+r} - m^r Z_n}{(2\sigma_r^2 Z_n \log n)^{\frac{1}{2}}} \geqq \overline{\lim_{n\to\infty}} \frac{Z_{nr+r} - m^r Z_{nr}}{(2\sigma_r^2 Z_{nr} \log nr)^{\frac{1}{2}}} = 1, \quad \text{a.s.}$$

since the set $\{n+r\}$ contains the set $\{nr+r\}$, $n = 1.2\ldots$.

Combining this result with (16) we obtain the desired result. The liminf case is treated in exactly the same way. We have thus established the first part of Theorem 4.2.

We now show that the results for $Z_{n+r} - m^r Z_n$ are sufficient to establish those for $m^n W - Z_n$.

$$\overline{\lim} \frac{m^n W - Z_n}{(2\sigma_1^2 (m^2-m)^{-1} Z_n \log n)^{\frac{1}{2}}} \leqq \overline{\lim} \frac{m^n W - m^{-r} Z_{n+r}}{(2\sigma_1^2 (m^2-m)^{-1} Z_n \log n)^{\frac{1}{2}}} + $$

$$+ \overline{\lim} \frac{m^{-r}(Z_{n+r} - m^r Z_n)}{(2\sigma_1^2 (m^2-m)^{-1} Z_n \log n)^{\frac{1}{2}}} \qquad \text{a.s.}$$

$$= \frac{1}{m^{r/2}} \overline{\lim} \frac{m^{n+r} W - Z_{n+r}}{\{2\sigma_1^2 (m^2-m)^{-1} Z_{n+r} \log n\}^{\frac{1}{2}}} + \frac{\sigma_r}{\sigma_1 (m^2-m)^{-\frac{1}{2}} m^r} \cdot 1. \quad \text{a.s.} \qquad (19)$$

in view of the already established first part of Theorem 4.2 and the fact that $\frac{Z_n}{m^n} \cdot \frac{m^{n+r}}{Z_{n+r}} \to \frac{W}{W} = 1$ a.s.

Inequality (19') gives us, for all r[†]

$$\left(1-\frac{1}{m^{r/2}}\right)\ \overline{\lim}\ \frac{m^nW-Z_n}{\left(2\sigma_1^2(m^2-m)^{-1}Z_n\log n\right)^{\frac{1}{2}}} \leq \frac{\sigma_r}{\sigma_1(m^2-m)^{-\frac{1}{2}}m^r} = (1-m^{-r})^{\frac{1}{2}} \quad \text{a.s.}$$

and as $\dfrac{(1-m^{-r})^{\frac{1}{2}}}{(1-m^{-r/2})} = \left(\dfrac{1+m^{-r/2}}{1-m^{-r/2}}\right)^{\frac{1}{2}} \geq 1$ for all r and $\to 1$ as $n \to \infty$,

$$\overline{\lim}\ \frac{m^nW-Z_n}{\left(2\sigma_1^2(m^2-m)^{-1}Z_n\log n\right)^{\frac{1}{2}}} \leq 1 \qquad \text{a.s.}$$

Similarly

$$\overline{\lim}\ \frac{Z_n-m^nW}{\left(2\sigma_1^2(m^2-m)^{-1}Z_n\log n\right)^{\frac{1}{2}}} \leq 1 \qquad \text{a.s.} \tag{20}$$

Conversely

$$1 = \overline{\lim}\ \frac{Z_{n+r}-m^rZ_n}{(2\sigma_r^2Z_n\log n)^{\frac{1}{2}}} \qquad \text{a.s.}$$

So, a.s.,

$$\left(1-m^{-r}\right)^{\frac{1}{2}} \leq \overline{\lim}\ m^{-r/2}\frac{Z_{n+r}-m^{n+r}W}{\left(2\sigma_1^2(m^2-m)^{-1}Z_{n+r}\log n\right)^{\frac{1}{2}}} + \overline{\lim}\ \frac{m^nW-Z_n}{\left(2\sigma_1^2(m^2-m)^{-1}Z_n\log n\right)^{\frac{1}{2}}}\ .$$

that is, by (20),

$$\overline{\lim}\ \frac{m^nW-Z_n}{\left(2\sigma_1^2(m^2-m)^{-1}Z_n\log n\right)^{\frac{1}{2}}} \geq (1-m^{-r})^{\frac{1}{2}} - m^{-r/2} \quad \text{a.s., for all } r.$$

So $\overline{\lim}\ \dfrac{m^nW-Z_n}{\left(2\sigma_1^2(m^2-m)^{-1}Z_n\log n\right)^{\frac{1}{2}}} \geq 1$, and with (20), this completes the proof of Theorem 4.2.

[†] Note that $\sigma_r^2 = \sigma_1^2m^r(m^r-1)(m^2-m)^{-1}$

4.4. THE PROCESS WITH IMMIGRATION.

Using the notation of section 4.1, we shall establish the following theorems: (Theorem 4.4 corresponds to Theorem 4.2 in providing convergence rate results and Theorem 4.5 corresponds to Theorem 4.3 and provides iterated logarithm analogues)

THEOREM 4.4. *Let* $1 < m = EZ_1$ *and* $0 < \operatorname{var} Z_1 = \sigma^2 < \infty$. *Then*

$$\sup_x \left| P\left((m^2 - m)^{\frac{1}{2}} \sigma^{-1} u_{X_n}^{-1} X_n^{-\frac{1}{2}} \left(m^n V - X_n \right) \leq x \mid X_n > 0 \right) - \Phi(x) \right| \leq \alpha_n$$

$$\sup_x \left| P\left(\sigma_r^{-1} v_{X_n}^{-1} X_n^{-\frac{1}{2}} \left(X_{n+r} - m^r X_n \right) \leq x \mid X_n > 0 \right) - \Phi(x) \right| \leq \beta_n ,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are certain sequences of positive constants satisfying* $\sum_{n=1}^{\infty} \alpha_n < \infty$ *and* $\sum_{n=1}^{\infty} \beta_n < \infty$. *Here* σ_r, u_n, v_n *are as defined in the statement of Theorem 4.1.*

Explicit forms for α_n and β_n can be found by applying the lemma.

THEOREM 4.5. *Suppose that* $1 < m = EZ_1$ *and* $0 < \operatorname{var} Z_1 = \sigma^2 < \infty$. *Then, with probability one,*

$$\limsup_{n\to\infty} \frac{X_{n+r} - m^r X_n}{\left(2\sigma_r^2 X_n \log n\right)^{\frac{1}{2}}} = 1, \quad \liminf_{n\to\infty} \frac{X_{n+r} - m^r X_n}{\left(2\sigma_r^2 X_n \log n\right)^{\frac{1}{2}}} = -1,$$

$$\limsup_{n\to\infty} \frac{m^n V - X_n}{\left(2\sigma^2 (m^2 - m)^{-1} X_n \log n\right)^{\frac{1}{2}}} = 1, \quad \liminf_{n\to\infty} \frac{m^n V - X_n}{\left(2\sigma^2 (m^2 - m)^{-1} X_n \log n\right)^{\frac{1}{2}}} = -1,$$

where r *is any fixed positive integer.*

Theorems 4.4 and 4.5 extend the scope of results given in Theorems 2 and 3 [21] under the additional condition that $EZ_1^3 < \infty$. Theorem 4.5 preserves exactly the form of Theorem 3 of [21] under the more general conditions.

The proofs of Theorems 4.4 and 4.5 follow the same lines as those of Theorems 4.1 and 4.2, again using the lemma and (6). We make use of the following representations from [21]

$$m^n V - X_n = \left(W^{(1)} - 1\right) + \dots + \left(W^{(X_n)} - 1\right) + I^{(n)} \quad \text{a.s.}$$

where the $\left(W^{(i)} - 1\right)$ are iid as $W - 1$, are independent of $I^{(n)}$ and all are independent of X_n, and

$$X_{n+r} - m^r X_n = \left(Z_r^{(1)} - m^r\right) + \dots + \left(Z_r^{(X_n)} - m^r\right) + Y_{r,n}$$

where $\left(Z_r^{(i)} - m^r\right)$ are iid as $\left(Z_r - m^r\right)$, independently of Y_{rn} and all are independent of X_n. The only real point of difference in the proofs involves showing that we can choose a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n^{-1} E\left(X_n^{-\frac{1}{2}} w_{X_n}^{-1} \mid X_n > 0\right) < \infty$ where w_n is either u_n or v_n.

We know that $u_n \uparrow 1$, $v_n \uparrow 1$ so that $w_n \uparrow 1$ as $n \to \infty$. Thus, conditional on $X_n > 0$, $w_{X_n} \geqq w_1$ and hence

$$\sum_{n=1}^{\infty} \varepsilon_n^{-1} E\left(X_n^{-\frac{1}{2}} w_{X_n}^{-1} \mid X_n > 0\right) \leqq w_1^{-1} \sum_{n=1}^{\infty} \varepsilon_n^{-1} E\left(X_n^{-\frac{1}{2}} \mid X_n > 0\right)$$

$$\leqq w_1^{-1} \sum_{n=1}^{\infty} \varepsilon_n^{-1} \left(E\left(X_n^{-1} \mid X_n > 0\right)\right)^{\frac{1}{2}} .$$

Now, using Lemma 2.3 of [21] we take $\varepsilon_n = \theta^n$ with $0 < \gamma < \theta < 1$, hence

$$\sum_{n=1}^{\infty} \varepsilon_n^{-1} \left(E\left(X_n^{-1} \mid X_n > 0\right)\right)^{\frac{1}{2}} \leqq \sum_{n=1}^{\infty} \theta^{-n} \gamma^n < \infty .$$

Thus, with this choice of ε_n, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and

$\sum_{n=1}^{\infty} \varepsilon_n^{-1} E\left(X_n^{-\frac{1}{2}} w_{X_n}^{-1} \mid X_n > 0\right) < \infty$, as required.

CHAPTER 5

SOME LIMIT THEOREMS FOR MARKOV BRANCHING PROCESSES

5.1. INTRODUCTION:

Chapter 4 dealt with discrete time branching processes, or in other words, branching processes in which the generation times are fixed. Whilst some phenomena fit this situation, the most natural reproductive processes occur continuously in time. For this reason, a continuous time version of branching processes is of some considerable interest.

The Galton-Watson process of the previous chapter is, in fact, a Markov chain in which the transition probabilities are stationary and the number of off-spring of an individual is independent of the other objects present. In this chapter we consider the continuous time extension of the Galton-Watson process: the time homogeneous Markov branching process. This process which we will denote by $\{X(t),\ t \geqq 0\}$, can be regarded as the total number of individuals at time t in a system where we start with $X(0) = 1$, individuals at $t = 0$. Each individual lives an exponentially distributed length of time with mean a^{-1} (say), $0 < a < \infty$, and on death, splits into a random number of new individuals whose generating function we shall denote by $h(z)$. All individuals behave independently of each other and identically. The probability that an individual of age τ, dies in the age interval $(\tau, \tau + d\tau)$ is independent of the age τ.

From Chapter 1 we know that if $h''(1) < \infty$ and $h'(1) > 1$,

$$\lim_{t\to\infty} X(t)e^{-\lambda t} = W \quad \text{a.s.} \tag{1}$$

where W is a non-degenerate random variable with $EW = 1$,

$$EW^2 = \frac{h''(1) - h'(1) + 1}{h'(1) - 1} .$$

and

$$\lambda = a\big(h'(1) - 1\big)$$

This result provides us with a setting similar to that of the previous chapter, for analogues of the classical limit theorems. Part of the aim of this chapter is to obtain these limit results for

$$X(t+r) - e^{\lambda r}X(t), \ r > 0$$

and

$$X(t) - e^{\lambda t}W$$

these being the continuous counterparts of the results obtained for the Galton-Watson process.

We now introduce the process $\{N(t), t \geq 0\}$ defined to be the number of discontinuities of $X(s)$ for $s \leq t$. In the remainder of this section we will examine some possible limit laws for this process and for combinations of it with $X(t)$. To do this, we need a little more notation: let τ_n be the time of the occurrence of the nth split (change of state) of $X(t)$ for $n \geq 1$. We define ξ_n to be $X(\tau_n + 0) - X(\tau_n - 0)$ and since all individuals behave independently of each other and identically, the ξ_n are iid rv's. Putting $p_i = \Pr\{\xi_n = i-1\}$, $i = 0, 1, 2, \ldots$, without loss of generality (see the remarks at the end of the chapter) we set $p_1 = 0$ so as to exclude the case of an individual replicating itself at a split. $N(t)$ now becomes the number of splits of $X(s)$, $s \leq t$. For convenience we take the state space of $X(t)$ to be the positive integers instead of the non-negative integers so as to exclude the possibility of extinction (p_0 is then forced to be zero). Finally, if $u(z)$ is the infinitesimal generating function[†] associated with $X(t)$, then $u(z) = a\big(h(z) - z\big)$ and $h(z) = \sum_{i=2}^{\infty} p_i z^i$, $\sum_{i=2}^{\infty} p_i = 1$. Setting $\mu = h'(1) - 1$ and $\lambda = u'(1)$ (>0), we have $\mu = \lambda a^{-1}$.

In the Galton-Watson context, $N(n)$ becomes the number of discontinuities in Z_j, $j \leq n$ and so $N(n) \leq n$. If, as is the case here, $p_0 = p_1 = 0$, then $N(n) = n$, since Z_n is a strictly increasing sequence.

[†] The infinitesimal generating fn. $\Sigma_0^\infty u_i z^i$ is the gen.fn. for the coeffts. of Δt in $P\big(X(t+\Delta t) - X(t) = i-1\big)$, $i = 0, 1, 2, \ldots$; i.e. $u_i = ap_i$ for $i \neq 1$ and $u_1 = -a$ since $P\big(X(t+\Delta t) - X(t) = 0\big) = 1 - a\Delta t + a\Delta t p_1 = 1 - a\Delta t$.

In the present context, we have from Theorem 4 of Athreya and Karlin [1] that if $h''(1) < \infty$, then

$$\lim_{t\to\infty} \mu N(t)e^{-\lambda t} = W \qquad \text{a.s.} \tag{2}$$

where $W = \lim_{t\to\infty} X(t)e^{-\lambda t}$ and since $p_0 = 0$, $W > 0$ a.s.

N(t) is thus exponential in nature, in contrast to the Galton-Watson case. In the latter, all individuals must reproduce at fixed times whereas in this continuous extension, reproduction is spread out and indeed the probability of 2 or more individuals reproducing in $(t, t+\delta t)$ is $o(\delta t)$.

The limit result (2) provides for N(t) a setting for the classical limit analogues in the same way as (1) did for the process X(t). A further aim of this chapter is to obtain classical limit analogues for

$$N(t+r) - e^{\lambda r} N(t), \quad r > 0,$$

and

$$\mu N(t) - e^{\lambda t}W.$$

Finally, Athreya and Karlin established classical limit analogues for $X(t) - \mu N(t)$. We shall extend these results by obtaining limit results for

$$X(t+r) - e^{\lambda r}\mu N(t), \quad r > 0.$$

The key to establishing these results is the presence of an imbedded Galton-Watson process. This process has already been mentioned in Chapter 1, briefly though, for any fixed $\Delta > 0$ and fixed $t \geqq 0$, $Z_n = X(t + n\Delta)$ forms a Galton-Watson process with $Z_0 = X(t)$ and if $h''(1) < \infty$, $\lim_{n\to\infty} Z_n m^{-n} = W$ a.s. where $m = e^{\lambda\Delta}$. For each of the five random variables under consideration we obtain representations, similar to those of the previous chapter, as sums of iid rv's with the possible addition of a further independent but not identically distributed random variable. The presence of the imbedded Galton-Watson process provides the necessary bridge between the continuous case and the discrete case and using the representations mentioned, the proofs reduce to essentially those of the previous chapter.

5.2. RESULTS. The following two theorems give us analogues of the central limit theorem and of the iterated logarithm law. Similar results were obtained by Heyde[15,16] and in the previous chapter, for the case in which $X(n)$ is a super-critical Galton-Watson process. There is, of course, no real analogue of the split-time process in that context. Also, for the sake of completeness, two results obtained by Athreya and Karlin [1] have been included here.

Theorem 5.1. *If* $EX(1)^2 < \infty$, *then each of the following processes*

(i) $\left(X(t+r) - e^{\lambda r}X(t)\right)\left(\sigma_r^2 X(t)\right)^{-\frac{1}{2}}$, $(r \neq 0)$,

(ii) $\left(X(t) - e^{\lambda t}W\right)\left(\sigma_*^2 X(t)^{-\frac{1}{2}}\right)$,

(iii) $\left(\mu N(t+r) - e^{\lambda r}\mu N(t)\right)\left(d'^2 N(t)\right)^{-\frac{1}{2}}$, $(r \neq 0)$,

(iv) $\left(\mu N(t) - e^{\lambda t}W\right)\left(d_*^2 N(t)\right)^{-\frac{1}{2}}$,

(v) $\left(X(t+r) - e^{\lambda r}\mu N(t)\right)\left(d_r^2 N(t)\right)^{-\frac{1}{2}}$

converges in distribution to $N(0,1)$ *as* $t \to \infty$. *Here*

$$\sigma^2 = \mathrm{var}(\xi), \quad \sigma_*^2 = (\sigma^2/\mu) + \mu, \; \sigma_r^2 = \sigma_*^2 e^{\lambda r}(e^{\lambda r} - 1),$$

$$d_*^2 = \mu\sigma_*^2 + \sigma^2, \qquad d_r^2 = \mu\sigma_r^2 + e^{2\lambda r}\sigma^2, \qquad d_r'^2 = d_r^2 + \sigma^2(2 - 3e^{\lambda r})$$

and r *is a non-negative integer.*

Theorem 5.2. *If* $EX^2(1) < \infty$, *then each of the following processes*

(i) $\left(X(t+r) - e^{\lambda r}X(t)\right)\left(2\sigma_r^2 X(t) \log t\right)^{-\frac{1}{2}}$, $(r \neq 0)$,

(ii) $\left(X(t) - e^{\lambda t}W\right)\left(2\sigma_*^2 X(t) \log t\right)^{-\frac{1}{2}}$,

(iii) $\left(\mu N(t+r) - e^{\lambda r}\mu N(t)\right)\left(2d_r'^2 N(t) \log t\right)^{-\frac{1}{2}}$, $(r \neq 0)$,

(iv) $\left(\mu N(t) - e^{\lambda t}W\right)\left(2d_*^2 N(t) \log t\right)^{-\frac{1}{2}}$,

(v) $\left(X(t+r) - e^{\lambda r}\mu N(t)\right)\left(2d_r^2 N(t) \log t\right)^{-\frac{1}{2}}$

has a lim sup *of* +1 *and a* lim inf *of* −1, *both with probability one as* $t \to \infty$, r *being a non-negative integer.*

Remark a) Since $\lim_{t\to\infty} \mu N(t)e^{-\lambda t} = W$ a.s., it is easily seen that $\lim_{t\to\infty} (\log\log N(t))(\log t)^{-1} = 1$ a.s.

Remark b) The limit results for process (v) of Theorem 5.1 and for process (v) of Theorem 5.2 both with $r = 0$ are quoted directly from Athreya and Karlin, [1] Theorem 4.

5.3. PROOF OF THEOREMS.

Proof of Theorem 5.1. The proofs of the five results in this theorem are almost identical, hence we shall establish the result for process (v) in detail, the remaining proofs being identical in form. Firstly, however, we require the following lemma. We take $r > 0$ throughout the remainder of this chapter.

LEMMA 5.3.

(a) $(X(t+r) - e^{\lambda r}X(t)) = U_{r,1} + U_{r,2} + \dots + U_{r,X(t)}$ a.s.

(b) $X(t) - e^{\lambda t}W = U_1 + U_2 + \dots + U_{X(t)}$ a.s.,

(c) $(\mu N(t+r) - e^{\lambda r}\mu N(t)) = V_{r,1} + V_{r,2} + \dots + V_{r,N(t)} + Y_r$ a.s.,

(d) $\mu N(t) - e^{\lambda t}W = V_1 + V_2 + \dots + V_{N(t)} + Y$ a.s.,

(e) $(X(t+r) - e^{\lambda r}\mu N(t)) = Z_{r,1} + Z_{r,2} + \dots + Z_{r,N(t)} + Y'_r$ a.s.,

where for (a) *and* (b), *the terms on the right-hand side are* iid, *independent of* $X(t)$, *and* $U_{r,i}$ *and* U_i *have distributions the same as* $(X(r) - e^{\lambda r})$ *and* $1 - W$ *respectively; and for* (c), (d) *and* (e), *the first* $N(t)$ *terms are* iid, *independent of* $N(t)$ *and of the last term;* $V_{r,i}$ *has distribution the same as* $(\mu N^{(1)}(r) + \mu N^{(2)}(r) + \dots + \mu N^{(\xi)}(r)) - \mu(e^{\lambda r} - 1)$, *the* ξ *(see introduction) being independent of the* $N^{(i)}(r)$ *all of which are* iid *as* $N(r)$, *similarly* $Z_{r,i}$ *has distribution the same as* $(X^{(1)}(r) + X^{(2)}(r) + \dots + X^{(\xi)}(r) - \mu e^{\lambda r})^{\dagger}$; *finally, the last terms* (Y_r, Y and Y'_r) *are independent of* $N(t)$, Y_r *is distributed as* $\mu N(r)$, Y *as* $-W$ *and* Y'_r *as* $X(r)$.

Remark c) Since $\sigma^2 = h''(1) - h'(1) + 1 - \mu^2$ and $\mu = h'(1) - 1$ by a result of p.103 of [12], $\mathrm{var}(X(r)) = \sigma_r^2$ and exploiting Representation (D)

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† V_i has the same distribution as $\mu - (W^{(1)} + W^{(2)} + \dots + W^{(\xi)})$.

below, we find $\mathrm{var}\big(\mu N(r)\big) = \big(\sigma_*^2 + \mu(e^{\lambda r} - 1)^{-1}\big)(e^{\lambda r} - 1)^2$.

Remark d) $\mathrm{var}(U_{r,i}) = \sigma_r^2$, $\mathrm{var}(U_i) = \sigma_*^2$, $\mathrm{var}(V_i) = d_*^2$, $\mathrm{var}(Z_{r,i}) = d_r^2$ and $\mathrm{var}(V_{r,i}) = d_r'^2$.

Proof of Lemma. These results appear as simple combinations of the following representations.

(A) We know that for t and Δ fixed and $t \geqq 0$ and $\Delta > 0$, $X(t + \Delta n)$ with $n = 0, 1, 2, \ldots$ forms a Galton-Watson process. Thus

$$X(t + r) = X^{(1)}(r) + X^{(2)}(r) + \ldots + X^{(X(t))}(r) \text{ a.s.} \qquad (3)$$

where $X^{(i)}(r)$, $i = 1,2,\ldots, X(t)$, are iid as $X(r)$ and independently of $X(t)$.

(B) Taking limits as $r \to \infty$ while multiplying both sides of (3) by $e^{-\lambda r}$ we obtain

$$e^{\lambda t} W = W^{(1)} + W^{(2)} + \ldots + W^{(X(t))} \qquad \text{a.s.}$$

(C) In view of the Markov property we know that given $N(t)$ and $X(t)$ the distribution of $N(t+r)$, the number of splits up to time $t+r$, depends on the number of objects present at time t (i.e., $X(t)$), on the number of splits up to time t (i.e., $N(t)$), on r and on the fact that the splitting process of any of the $X(t)$ objects from time t to $t+r$ is independent of any of the other splitting processes. Hence we can write

$$N(t+r) = N^{(1)}(r) + N^{(2)}(r) + \ldots + N^{(X(t))}(r) + N(t) \quad \text{a.s.}$$

where the $N^{(i)}(r)$, $i = 1, 2, \ldots, X(t)$, are iid as $N(r)$ and independently of $X(t)$ and $N(t)$.

(D) From [1] we have

$$X(t) = 1 + \xi_1 + \xi_2 + \ldots + \xi_{N(t)} \qquad \text{a.s.}$$

where the ξ_i, $i = 1,2,\ldots,N(t)$, are iid as ξ and independently of $N(t)$. The results of the lemma can now easily be obtained.

Continuing with the proof of Theorem 5.1, we establish the desired limiting distribution of process (v). From the lemma we have

$$E\exp iv\left(\frac{X(t+r)-e^{\lambda r}\mu N(t)}{\left(d_r^2N(t)\right)^{\frac{1}{2}}}\right) = E\exp iv\left(\frac{Z_{r,1}+Z_{r,2}+\ldots+Z_{r,N(t)}+Y'_r}{\left(d_r^2N(t)\right)^{\frac{1}{2}}}\right),$$

and as

$$\frac{Y'_r}{\left(d_r^2N(t)\right)^{\frac{1}{2}}} = \frac{e^{-\lambda r}Y'_r}{e^{\frac{1}{2}\lambda t}}\cdot\left(\frac{e^{\lambda t}}{N(t)}\right)^{\frac{1}{2}}\cdot\frac{1}{(d_r^2e^{-2\lambda r})^{\frac{1}{2}}}$$

also $e^{-\lambda r}Y'_r$ and $e^{\lambda t}N(t)^{-1}$ both converge a.s. ([1] , Theorem 4),

thus $$Y'_r\left(d_r^2N(t)\right)^{-\frac{1}{2}} \to 0$$

in probability as $t \to \infty$. Hence

$$\lim_{t\to\infty} E\exp iv\left(\frac{X(t+r)-e^{\lambda r}\mu N(t)}{\left(d_r^2N(t)\right)^{\frac{1}{2}}}\right) = \lim_{t\to\infty} E\exp iv\left(\frac{Z_{r,1}+Z_{r,2}+\ldots+Z_{r,N(t)}}{\left(d_r^2N(t)\right)^{\frac{1}{2}}}\right).$$

Noticing that $\lim_{n\to\infty}(d_r^2n)^{-\frac{1}{2}}(Z'_{r,1}+Z'_{r,2}+\ldots+Z'_{r,n})$ converges in distribution to N(0,1) for $Z'_{r,i}$ iid and having the same distribution as $Z_{r,i}$ so we have for $\varepsilon > 0$

$$\left|\left(E(\exp iv\, Z'_r(d_r^2n)^{-\frac{1}{2}})\right)^n - e^{-\frac{1}{2}v^2}\right| < \varepsilon$$

for all $n > N$, N sufficiently large; here Z'_r has distribution the same as $Z'_{r,i}$ and hence $Z_{r,i}$. Thus

$$\lim_{t\to\infty} E\exp iv\left(\left(X(t+r)-e^{\lambda r}\mu N(t)\right)\left(d_r^2N(t)\right)^{-\frac{1}{2}}\right)$$

$$= \lim_{t\to\infty}\sum_{j=0}^{N}\left(\left(E\exp(ivZ'_r(d_r^2j)^{-\frac{1}{2}})\right)^j - e^{-\frac{1}{2}v^2}\right)P\left(N(t)=j\right)$$

$$+ \lim_{t\to\infty}\sum_{j=N+1}^{\infty}\left(\left(E\exp(ivZ'_r(d_r^2j)^{-\frac{1}{2}})\right)^j - e^{-\frac{1}{2}v^2}\right)P\left(N(t)=j\right) + e^{-\frac{1}{2}v^2}.$$

But the first two terms on the right-hand side are bounded by $2P\left(N(t) \leqq N\right) + \varepsilon$, and as $N(t) \to \infty$ a.s. the desired result follows.

Proof of Theorem 5.2. The proofs of the limit results for processes (i), (iii) and (v) of Theorem 5.2 are almost identical so we shall establish the results for (v) in detail and indicate the minor alterations required for the remaining two proofs. Finally, we shall show that the limit results for processes (i) and (iii) are sufficient to establish those for the processes (ii) and (iv) respectively. These proofs follow very closely those of similar results in Chapter 4.

To begin, we establish that the lim sup for process (v) is at least 1 a.s. for the case $r = 1$. We do this by proving that

$$P\left[X(n+1) - e^{\lambda}\mu N(n) > (1-\delta)\left(2d_1^2 N(n) \log n\right)^{\frac{1}{2}} \text{ i.o. for any } \delta > 0\right] = 1 \qquad (4)$$

We use the extended Borel-Cantelli lemma already introduced in Section 4.3 and we shall preserve the notation given in that section.

Putting

$$\mathcal{A}_n = \left[X(n+1) - e^{\lambda}\mu N(n) > (1-\delta)\left(2d_1^2 N(n) \log n\right)^{\frac{1}{2}}\right]$$

and

$$\mathcal{G}_n = \mathcal{F}\left[X(n+1), N(n+1), X(n), N(n), \ldots, X(1), N(1)\right],$$

also we have from the lemma

$$X(n+1) - e^{\lambda}\mu N(n) = Z_{1,1} + Z_{1,2} + \cdots + Z_{1,N(n)} + Y_1' \qquad \text{a.s.}$$

and hence (4) is equivalent to

$$\sum P\Big[Z_{1,1} + Z_{1,2} + \cdots + Z_{1,N(n)} + Y_1' > (1-\delta)\left(2d_1^2 N(n) \log n\right)^{\frac{1}{2}} \Big| X(n),N(n)\Big] = \infty \text{ a.s.}$$

Taking $\theta_n = \left(d_1^2\beta_{N(n)}\right)^{-\frac{1}{2}}\left[Z_{1,1} + Z_{1,2} + \cdots + Z_{1,N(n)} + Y_1'\right]$, ($\beta_n$ defined in (6))

$$\sup_x \left|P\left(\theta_n \leq x \mid X(n),N(n)\right) - \Phi(x)\right| \leq \qquad (5)$$

$$\leq \sup_x \left|P\left(\theta_n \leq x, N(n) > 0 \mid X(n),N(n)\right) - \Phi(x)\cdot P\left(N(n) > 0\right)\right| +$$

$$+ \sup_x \left|P\left(\theta_n \leq x, N(n) = 0 \mid X(n),N(n)\right) - \Phi(x)\cdot P\left(N(n) = 0\right)\right|$$

$$= \sup_x \left|P\left(\theta_n \leq x \mid X(n),N(n),N(n) > 0\right) - \Phi(x)\right|\cdot P\left(N(n) > 0\right) +$$

$$+ \sup_x \left|P\left(\theta_n \leq x \mid X(n),N(n) = 0\right) - \Phi(x)\right|\cdot P\left(N(n) = 0\right)$$

$$\leq \sup_x \left| P\left(\theta_n \leq x \mid X(n),N(n),N(n) > 0\right) - \Phi(x)\right| + 2e^{-an}$$

since $P\big(N(n) = 0\big) = P\big(X(n) = 1\big) = e^{-an}$.

From expression (2) of Lemma 4.3, we find

$$\sup_x \left|P\big(\theta_n \leq x \,|\, X(n), N(n), N(n)>0\big) - \Phi(x)\right| \leq AN^{-\frac{1}{2}}(n)\, a_{N(n)} + BN^{\frac{1}{2}}(n)\, b_{N(n)} + $$
$$+ N(n) c_{N(n)} + d_1^{-1} \varepsilon_n^{-1} \{E|e^{-\lambda Y_1'}|\} N^{-\frac{1}{2}}(n) \beta^{-1}_{N(n)} + \tfrac{1}{2}\varepsilon_n. \qquad (6)$$

where ε_n, A, B, d_n, b_n, c_n as defined in Lemma 4.3 and β_n is taken to be the d_n of the same lemma.

As in Chapter 4, we show that

$$\sum \sup_x \left|P\big(\theta_n < n \,|\, X(n), N(n), N(n)>0\big) - \Phi(x)\right| < \infty \quad \text{a.s.}$$

by establishing the convergence of

$$\sum E\big(N^{-\frac{1}{2}}(n)\, a_{N(n)} \,|\, N(n)>0\big), \ \sum E\big(N^{\frac{1}{2}}(n)\, b_{N(n)} \,|\, N(n)>0\big)$$

$$\text{and } \sum E\big(N(n)\, c_{N(n)} \,|\, N(n) > 0\big).$$

To do this, we use essentially the methods of the previous chapter. However, the proof is not obvious and therefore we will establish the convergence of $\sum E\big(N^{-\frac{1}{2}}(n)\, a_{N(n)} \,|\, N(n) > 0\big)$ in some detail; the convergence of the remaining series following easily.

From Chapter 4 we have

$$\sum_{n=1}^{\infty} n^{-3/2}\, a_n = K < \infty$$

Thus, $K = \displaystyle\sum_{k=0}^{\infty} \sum_{n=N(k)+1}^{N(k+1)} n^{-3/2}\, a_n$. (the second summation being zero if $N(k+1) = N(k)$) which is

$$\geq \sum_{k=1}^{\infty} N^{-3/2}(k+1)\big(N(k+1) - N(k)\big) a_{N(k)}$$

giving

$$E\sum_{k=1}^{\infty} N^{-3/2}(k+1)\big(N(k+1) - N(k)\big) a_{N(k)} \leq K$$

that is

$$K \geq \sum_{r=1}^{\infty} E\left\{\sum_{k=1}^{\infty} N^{-3/2}(k+1)\big(N(k+1) - N(k)\big) a_{N(k)} \,|\, N(r)>0, N(r-1)=0\right\} \cdot$$
$$\cdot P\big(N(r)>0, N(r-1)=0\big)$$

$$= \sum_{r=1}^{\infty} E\Big(\sum_{k=r}^{\infty} N^{-3/2}(k+1)\big(N(k+1) - N(k)\big)a_{N(k)}\,\big|\,N(r)>0,N(r-1)=0\Big)P\big(N(r)>0,N(r-1)=0\big)$$

$$= \sum_{k=1}^{\infty}\sum_{r=1}^{k} E\big(N^{-3/2}(k+1)\big(N(k+1) - N(k)\big)a_{N(k)}\,\big|\,N(r)>0,N(r-1)=0\big)\cdot$$

$$\cdot P\big(N(r)>0,N(r-1)=0\big)$$

$$= \sum_{k=1}^{\infty} E\big(N^{-3/2}(k+1)\big(N(k+1) - N(k)\big)a_{N(k)}\,\big|\,N(k) > 0\big)$$

It remains to show that $E\big(N^{-3/2}(k+1)\big(N(k+1)-N(k)\big)N^{\frac{1}{2}}(k)\,|\,N(k)\big) \geqq c_1 > 0$ for then $K \geqq c_1 \sum_{k=1}^{\infty} E\big(N^{-\frac{1}{2}}(k)a_{N(k)}\,|\,N(k)>0\big)$ as required.

Take $N(k) = j$, then by Schwartz inequality,

$$\left[E\left(1 - \frac{j}{N(k+1)}\right)^{\frac{1}{2}}\right]^2 \leqq E\left(\frac{N(k+1)}{j}\right)^{\frac{1}{2}}\cdot E\left(\frac{j}{N(k+1)}\right)^{\frac{1}{2}}\cdot\left(1 - \frac{j}{N(k+1)}\right) \qquad (7)$$

and

$$\left(E\left(\frac{N(k+1)}{j}\right)^{\frac{1}{2}}\right)^2 \leqq E\left(\frac{N(k+1)}{j}\right) = 1 + E\frac{X(k)}{j}\,EN(1)$$ (by representation (C) of Lemma 5.3).

From representation (D), $EX(k) = E(1 + \xi_1 + \ldots + \xi_j) = 1 + j\mu$ which gives

$$E\left(\frac{N(k+1)}{j}\right)^{\frac{1}{2}} \leqq \left(1 + \left(\mu + \frac{1}{j}\right)\mu e^{\lambda}\right)^{\frac{1}{2}} < C_2 \text{ for all } j \qquad (8)$$

Also, $E\left(N^{-1}(k+1)\,|\,N(k) = j\right) = E\,E\left(N^{-1}(k+1)\,|\,N(k) = j, X(k)\right)$

whilst $E\big(N^{-1}(k+1)\,|\,N(k) = j, X(k)\big)$

$$= \sum_{m=j}^{\infty} m^{-1}P\big(j + N^{(1)}(1) + \ldots + N^{(X(k))}(1) = m\,|\,N(k) = j, X(k)\big)$$

$$= \sum_{m=j}^{\infty}\ \sum_{r=0}^{\min(m-j,X(k))} m^{-1}P\Big[j + N^{(1)}(1) + \ldots + N^{(X(k))}(1)$$

$$= m\,|\,N(k) = j, X(k),$$

and exactly r of the $N^{(i)}(1)$ are $> 0\Big]\cdot P\Big[$exactly r of the $N^{(i)}(1)$ are $> 0\,|\,X(k), N(k) = j\Big]$

$$= \sum_{r=0}^{X(k)}\ \sum_{m=j+r}^{\infty} \binom{X(k)}{r} p^r q^{X(k)-r}\cdot m^{-1}P\Big[j + N^{(1)}(1) + \ldots + N^{(r)}(1) = m\,\Big|\,X(k), N(k)=j \text{ and } N^{(i)}(1) > 0 \text{ for } i = \overline{1,r}\Big]$$

(because $N^{(i)}(1)$ are iid as $N(1)$ and are independent of $X(k)$) where

$$p = 1 - q = P(N(1) > 0) = P(X(1) > 1) = 1 - e^{-a},$$

$$= \sum_{r=0}^{X(k)} \binom{X(k)}{r} p^r q^{X(k)-r} \cdot E\left[\left(j + N^{(1)}(1) + \ldots + N^{(r)}(1)\right)^{-1} \middle| N(k) = j, X(k), N^{(i)}(1) > 0 \text{ for } i = \overline{1,r}\right]$$

and by the harmonic mean, arithmetic mean inequality

$$E\left[\left(j + N^{(1)}(1) + \ldots + N^{(r)}(1)\right)^{-1} \middle| N(k) = j, N^{(i)}(1) > 0 \text{ for } i = \overline{1,r}\right]$$

$$\leqq \frac{1}{(r+1)^2}\left[j^{-1} + rE\left(N^{-1}(1) \middle| N(1) > 0\right)\right] \cdot$$

Thus $E\left[N^{-1}(k+1) \middle| N(k) = j, X(k)\right] \leqq \sum_{r=0}^{X(k)} \binom{X(k)}{r} p^r q^{X(k)-r} \cdot \frac{1}{(r+1)^2}\left(j^{-1} + r\gamma_1\right)$

$$\gamma_1 = E\left(N^{-1}(1) \middle| N(1) > 0\right).$$

Using the fact that

$$\sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} \frac{1}{(r+1)^2} \leqq 2(n+1)^{-1}\ (n+2)^{-1}\ p^{-2}$$

and that

$$\sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} \frac{r}{(r+1)^2} \leqq (n+1)^{-1}\ p^{-1}$$

we have

$$E\left(N^{-1}(k+1) \middle| N(k) = j, X(k)\right) \leqq 2X^{-2}(k)(1 - e^{-a})^{-2}\ j^{-1} + X^{-1}(k)\ (1 - e^{-a})^{-1}\gamma_1 \tag{9}$$

$$E\left[1 - \frac{j}{N(k+1)}\right]^{\frac{1}{2}} \geqq 1 - E\left[\frac{j}{N(k+1)}\right] \geqq 1 - C_3 > 0 \quad \text{for all } j.$$

Thus from (7) and (8)

$$E\left(\frac{j}{N(k+1)}\right)^{\frac{1}{2}}\left(1 - \frac{j}{N(k+1)}\right) \geqq \frac{1-C_3}{C_2} = C_1 \quad \text{for all } j$$

as desired.

As in Chapter 4, the convergence of $\sum \varepsilon_n^{-1} E\{N^{-\frac{1}{2}}(n)\beta_{N(n)}^{-1} | N(n) > 0\}$ and $\sum \varepsilon_n$ depends on the existence of a $\gamma, 0<\gamma<1$ such that $E\{N^{-\frac{1}{2}}(n) | N(n)>0\} = O(\gamma^n)$. Such a γ is easily found using the above techniques; briefly, from representation (A) and the harmonic mean inequality,

$$EX^{-1}(n+1) \leqq EX^{-1}(1)EX^{-1}(n) \leqq \left(EX^{-1}(1)\right)^n = \gamma_1{}^n, \qquad 0 < \gamma_1 < 1 \tag{10}$$

and

$$E\left(N^{-1}(n+1) | N(n+1) > 0\right) = P\left(N(n) > 0\right) \cdot E\left(N^{-1}(n+1) | N(n) > 0\right) + $$
$$+ \sum_{j=1}^{\infty} j^{-1} P\left(N(n+1) = j, N(n) = 0\right) . \tag{11}$$

From representation (C),

$$P\left(N(n+1) = j, N(n) = 0\right) = P\left(N(n) + N^{(1)}(1) + \ldots + N^{X(n)}(1) = j | N(n) = 0\right)$$
$$\cdot P\left(N(n) = 0\right)$$

$$= P\ \left(N(1) = j\right) \cdot P\left(X(1) = 1\right) \text{ as } \{N(n) = 0\} \equiv \{X(n) = 1\} \cdot$$

Hence

$$\sum_{j=1}^{\infty} j^{-1} P\left(N(n+1) = j, N(n) = 0\right) \leqq e^{-an} \cdot E\left(N^{-1}(1) | N(1) > 0\right) .$$

From (9),

$$E\left(N^{-1}(n+1) | N(n) > 0\right) \leqq 2(1 - e^{-a})^{-2} E\left(X^{-2}(n)\ N^{-1}(n)\right) | N(n) > 0\right) +$$
$$+ \left(1 - e^{-a}\right)^{-1} EX^{-1}(n) E\left(N^{-1}(1) | N(1)>0\right)$$

Hence, from (10) and (11)

$$E\left(N^{-1}(n+1) | N(n+1)>0\right) \leqq c\left(EX^{-1}(n) + e^{-an}\right) = O(\gamma^n) \text{for some } 0 < \gamma < 1.$$

Continuing with the proof of the theorem we must now have

$$\sum_{n=1}^{\infty} \sup_x \left|P\left(\theta_n > x \mid X(n),N(n)\right) - \left(1 - \Phi(x)\right)\right| < \infty .$$

Taking $x = (1 - \delta)(2 \log n)^{\frac{1}{2}}$ and noting that

$$\sum_{n=2}^{\infty} \{1 - \Phi\left((1 - \delta)(2 \log n)^{\frac{1}{2}}\right)\} = \infty$$

we find, since $\beta_{N(n)} \to 1$ as $n \to \infty$,

$$\overline{\lim}\left(X(t+1) - e^{\lambda}\mu N(t)\right)\left(2d_1^2 N(t) \log t\right)^{-\frac{1}{2}} \geqq 1 \quad \text{a.s.} \qquad (12)$$

To deal with the case $r > 1$, we use the fact that the process $X^*(t)=X(rt)$ is again a temporally homogeneous Markov branching process with mean life length for an individual of $(ar)^{-1}$. Hence, the associated infinitesimal generating function is $u_o(z) = ar\left(h(z) - z\right)$. Putting $\lambda_o = u_o(1)$ $(= r\lambda)$ we obtain from (12)

$$\overline{\lim}\left(X(tr + r) - e^{\lambda r} N(rt)\right)\left(2d_r^2 N(rt) \log rt\right)^{-\frac{1}{2}} \geqq 1 \text{ a.s.}$$

But

$$\overline{\lim}\, \frac{X(t + r) - e^{\lambda r}\mu N(t)}{(2d_r^2 N(t) \log t)^{\frac{1}{2}}} \geqq \overline{\lim}\, \frac{X(tr + r) - e^{\lambda r}\mu N(rt)}{(2d_r^2 N(tr) \log rt)^{\frac{1}{2}}} \geqq 1 \text{ a.s.}$$

Thus, the first part of the proof is finished.

To complete the proof it remains to show

$$\overline{\lim}\left(X(t + r) - e^{\lambda r}\mu N(t)\right)\left(2d_r^2 N(t) \log t\right)^{-\frac{1}{2}} \leqq 1 \text{ a.s.}$$

This will be true if $P(C_n \text{ i.o.}) = 0$ where

$$C_n = \{X(t+r) - e^{\lambda r}\mu N(t) > (1+\delta)\left(2d_r^2 N(t) \log t\right)^{\frac{1}{2}} \text{ for at least one } t\varepsilon(n,n+1]\}$$

We have

$$P(C_n) \leqq P\{\sup_{n<t\leqq n+1} X(t + r) - e^{\lambda r}\mu N(t) > (1 + \delta)\left(2 \log n\ d_r^2 N(t)\right)^{\frac{1}{2}}\}$$

$$= P\{\left(d_r^2 N(t(n))\right)^{-\frac{1}{2}}\left[X\left(t(n)+r\right) - e^{\lambda r}\mu N(t(n))\right] > (1+\delta)(2 \log n)^{\frac{1}{2}}\},$$

where $t(n)$ represents the time of occurrence of the sup for $n<t(n)\leqq n+1$.

From the lemma

$$X(t+r) - e^{\lambda r}\mu N(t) = Z_{r,1} + Z_{r,2} + \dots + Z_{r,N(t)} + Y_r' \quad \text{a.s.},$$

so, using the usual Borel-Cantelli lemmas, we need only establish the convergence of

$$\sum_{i=1}^{\infty} P\{\left(d_r^2 N(t(n))\right)^{-\frac{1}{2}}\left(Z_{r,1} + Z_{r,2} + \dots + Z_{r,N(t(n))} + Y_r'\right) > (1+\delta)(2\log n)^{\frac{1}{2}}\}$$

Once more, referring to Expression (2) of the Lemma 4.3 and noting that $N(n+1) \geqq N(t(n)) \geqq N(n)$, we find[†]

$$\sup\left|P\{\left(d_r^2 N(t(n))\right)^{-\frac{1}{2}}\left(Z_{r,1} + Z_{r,2} + \dots + Z_{r,N(t(n))} + Y_r'\right) > x\} - \left(1 - \Phi(x)\right)\right| \qquad (13)$$

$$\leqq AE\left(N^{-\frac{1}{2}}(n)a_{N(n+1)}\right) + BE\left(N^{\frac{1}{2}}(n+1)b_{N(n+1)}\right) + E\left(N(n+1)c_{N(n+1)}\right)$$

$$+ d_r^{-1}\varepsilon_n^{-1}E|W_r'^{(0)}|E\left(N^{-\frac{1}{2}}(n)\beta_{N(n)}\right) + \tfrac{1}{2}\varepsilon_n.$$

In view of the first part of the proof, it is clear that the sum over n of all but the first term on the right-hand side of (13) converges. But

$$E\left(N^{-\frac{1}{2}}(n)a_{N(n+1)}\right) = EN^{-\frac{1}{2}}(n+1)a_{N(n+1)}e^{\frac{1}{2}\lambda}\left(\frac{\mu N(n+1)e^{-\lambda(n+1)}}{\mu N(n)e^{-\lambda n}}\right)^{\frac{1}{2}},$$

and

$$\lim_{n\to\infty}\left(\frac{\mu N(n+1)e^{-\lambda(n+1)}}{\mu N(n)e^{-\lambda n}}\right)^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

Thus as

$$\sum EN^{-\frac{1}{2}}(n)a_{N(n)} < \infty,$$

we have

$$\overline{\lim}\left(X(t+r) - e^{\lambda r}N(t)\right)\left(2d_r^2 N(t)\log t\right)^{-\frac{1}{2}} \leqq 1 \quad \text{a.s.}$$

Combining (12) and (13), the desired result is obtained. A similar proof establishes the lim inf result.

[†] For convenience we have ignored the 'conditional on $N(n) > 0$' requirement; however, using exactly the same technique as was used in (5), we can deal with the possibility of $N(n) = 0$.

We establish the limit results for (i) and (iii) in exactly the same way using the representations appearing in the Lemma 5.3 and in the case of (i) using Expression (1) of the Lemma 4.3 in place of Expression (2).

Next we show that the limit results for process (i) are sufficient to establish those for (ii).

We have

$$\overline{\lim}\, \frac{X(t) - e^{\lambda t}W}{\left(2\sigma_*^2 X(t) \log t\right)^{\frac{1}{2}}} \leqq \overline{\lim}\, \frac{X(t) - e^{-\lambda r}X(t+r)}{\left(2\sigma_*^2 X(t) \log t\right)^{\frac{1}{2}}} + \overline{\lim}\, \frac{e^{-\lambda r}X(t+r) - e^{\lambda t}W}{\left(2\sigma_*^2 X(t) \log t\right)^{\frac{1}{2}}}$$

$$= (1 - e^{-\lambda r})^{\frac{1}{2}} + e^{-\frac{1}{2}\lambda r}\, \overline{\lim}\, \frac{X(t+r) - e^{\lambda(t+r)}W}{\left(2\sigma_*^2 X(t+r) \log(t+r)\right)^{\frac{1}{2}}} \quad \text{a.s.}$$

since $e^{\lambda r}X(t)\left(X(t+r)\right)^{-1} \to 1$ a.s., and the $(1 - e^{\lambda r})^{\frac{1}{2}}$ follows from (i).

Thus

$$\overline{\lim}\left(X(t) - e^{\lambda t}W\right)\left(2\sigma_*^2 X(t) \log t\right)^{-\frac{1}{2}} \leqq (1 - e^{-\lambda r})^{\frac{1}{2}}(1 - e^{-\frac{1}{2}\lambda r})^{-1}$$

$$= \left(1 + 2(e^{\frac{1}{2}\lambda r} - 1)^{-1}\right)^{\frac{1}{2}} \quad \text{a.s.}$$

for all r. So we have

$$\overline{\lim}\left(X(t) - e^{\lambda t}W\right)\left(2\sigma_*^2 X(t) \log t\right)^{-\frac{1}{2}} \leqq 1 \quad \text{a.s.}$$

Similarly

$$\overline{\lim}\left(e^{\lambda t}W - X(t)\right)\left(2\sigma_*^2 X(t) \log t\right)^{-\frac{1}{2}} \leqq 1 \text{ a.s.} \tag{14}$$

Conversely,

$$1 = \overline{\lim}\left(X(t)e^{\lambda r} - X(t+r)\right)\left(2\sigma_*^2 e^{\lambda r}(e^{\lambda r} - 1)X(t) \log t\right)^{-\frac{1}{2}} \quad \text{a.s.}$$

so

$$(1 - e^{-\lambda r})^{\frac{1}{2}} \leqq \overline{\lim}\, e^{-\frac{1}{2}\lambda r} \frac{e^{\lambda(t+r)}W - X(t+r)}{\left(2\sigma_*^2 X(t+r) \log(t+r)\right)^{\frac{1}{2}}} + \overline{\lim}\, \frac{X(t) - e^{\lambda t}W}{\left(2\sigma_*^2 X(t) \log t\right)^{\frac{1}{2}}} \quad \text{a.s.}$$

From (14) we have

$$\overline{\lim}\left(X(t) - e^{\lambda t}W\right)\left(2\sigma_*^2 X(t) \log t\right)^{-\frac{1}{2}} \geqq (1 - e^{-\lambda r})^{\frac{1}{2}} - e^{-\frac{1}{2}\lambda r} \quad \text{a.s.}$$

for all r and the desired result follows.

Upon noting that $1 \leqq e^{-2\lambda r} d_r'^{2} d_*^{-2} (1 - e^{-\frac{1}{2}\lambda r})^{-1} \leqq 1 + 2(e^{\frac{1}{2}\lambda r} - 1)^{-1}$ for all r and using an argument identical to that of the previous proof we obtain immediately the desired limit results for (iv) from the limit results for process (iii). The proofs are now complete.

Remark e) It is perhaps worth noting that $\left(c_r^2 N(t)\right)^{-\frac{1}{2}}\left(\mu N(t+r) - e^{\lambda r}X(t)\right)$ converges in distribution to N(0,1) and a corresponding law of the iterated logarithm holds where $c_r^2 = d_r^2 - \sigma^2(e^{2\lambda r} + e^{\lambda r} - 2)$.

Remark f) The constraint $p_1 = 0$ is applied without loss of generality in so far as every Markov branching process $\{X(t), t \geqq 0\}$ having $p_1 \neq 0$ and mean life length of a^{-1}, can be associated with a Markov branching process $\{X_o(t), t \geqq 0\}$ having $p'_1 = 0$ and mean life length $\left(a(1-p_1)\right)^{-1}$ and such that $X_o(t)$ and $N_o(t)$ have the same probabilistic behaviour as X(t) and N(t) respectively. Thus many results concerning $X_o(t)$ and $N_o(t)$ hold also for X(t) and N(t). Indeed, we need only form a new process $X_o(t) = X(t)$, $t \geqq 0$, having split-times $\tau_1^*, \tau_2^*, \tau_3^*, \ldots$ where τ_j^* = time of that split (of X(t)) in which $X(\tau+0) > X(\tau)$ for the jth time. The rv's $\tau_{j+1}^* - \tau_j^*$ can easily be shown to be iid as exponential variables of parameter $a(1 - p_1)$.

Remark g) We can extract from inequality (6) the following convergence rate result

$$\sup_x \left| P\left[\left(X(n+1) - e^{\lambda}\mu N(n)\right)\left(2d_1^2\, \beta_{N(n)} N(n) \log n\right)^{-\frac{1}{2}} < x \,\middle|\, X(n), N(n), N(n) > 0\right] - \Phi(x)\right| \leqq \alpha_n$$

where $\sum_{n=1}^{\infty} \alpha_n < \infty$ a.s.

Furthermore, similar convergence rates hold for the four other rv's considered.

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Abbreviations: Z.Wahr. = Zeit. Wahrscheinlichkeitstheorie verw. Gebiete.

T.P.A. = Theory of Probability and its Applications

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