

ESSAYS ON CIVIL LITIGATION AND CONTEST THEORY

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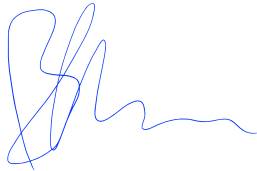
Declaration

This thesis by compilation comprises four research papers, and an introduction and a conclusion. This thesis is within the 100,000 word limit.

I certify that three research papers (chapters 2, 3 and 4) are co-authored with Dr. José A. Rodrigues-Neto. The modeling work was split evenly and undertaken in meetings over the last four years. The research topic, selection of methodology, connection with the literature (including the literature reviews), and production and presentation of slides were entirely my own work.

I further certify that one research paper (chapter 5) and the introduction and conclusion were entirely my own work.

All research papers (chapters 2- 5) are manuscripts in preparation for submission.



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Abstract

This thesis primarily develops novel game-theoretic tools, and applies these tools to study civil litigation in common law jurisdictions — the process through which a private plaintiff seeks judicial remedies against a private defendant.

First, this thesis models civil litigation as a simultaneous-move contest between two litigants, each of whom exerts costly efforts to maximize her monetary payoff. A success function describes the litigants' respective probabilities of success based on their efforts and exogenous relative advantages. Instead of having a functional form, the success function satisfies general and intuitive assumptions which capture frequently-used functional forms. Another generalization is the cost-shifting rule, which allows the winner to recover an exogenous proportion of her litigation costs from the loser. There exists a unique Nash equilibrium with positive efforts. In equilibrium, more cost shifting makes the outcome of the case more predictable, but typically increases the litigants' collective expenditure.

Second, further developing the litigation game, this thesis allows monetary and emotional variables to motivate a plaintiff and a defendant to exert costly efforts; the emotional variables capture their relational emotions toward each other, and a non-monetary joy of winning. In equilibrium, negative relational emotions (but not positive joy of winning) amplify the effects of cost shifting. Negative relational emotions increase the equilibrium relative effort and probability of success of the more advantageous litigant.

The novel tools developed to study litigation have broader implications. Generalizing the litigation games is a contest game of complete information, in which two players simultaneously spend to compete for a prize. They have potentially different probability-of-success functions and spillovers. Each success function satisfies general and intuitive assumptions without having a particular functional form. Applications of this game capture optimism and pessimism in military conflicts, and asymmetric R&D contests.

Finally, in addition to litigation efforts and costs, the great variety of legal remedies affects substantive behaviors. This thesis will reveal that, when private actions produce social harm, the actor has incentives to take a socially optimal action if she owes liabilities that optimize externalities. The proposed theory of externalities optimization generalizes the existing theory of externalities internalization, and explains apparently-unrelated rules that optimize incentives when complete internalization fails. Illustrating the proposed theory is an innovative model in which the actor owes a restitutionary liability to disgorge some of her private gain *and* a liability to compensate for some of the social harm.

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1 Introduction

A focus of law and economics research is civil litigation in common law jurisdictions — the process through which a person or entity — the plaintiff — seeks judicial remedies against another person or entity — the defendant. Civil litigation is costly to the private litigants and to the public. A litigant's costs include paying for lawyers, conducting discovery of evidence, researching the law, preparing and making legal arguments, and other costly activities taken to maximize her own payoff. The costs to the public include providing courts, judges, and effectuating judicial rulings coercively when the losing litigant resists enforcement. US\$1 million is the median annual litigation spend of various companies across the globe, according to a recent survey of corporate counsel.¹ The protection of rights and enforcement of duties under substantive laws — such as property law, contract law and tort law — depends on the accurate and efficient operation of the litigation system. Moreover, at stake are values that inhere in the rule of law, especially predictability in the judicial determination of litigated cases. Legal predictability is attractive to the liberal ideal of the rule of law because it gives fair notice to individuals of the legal consequences of their choices and holds public officials accountable for their exercises of public powers.² Understanding the economics of civil litigation and the strategic interaction of litigants is thus fundamental to the functioning of a society governed by the rule of law.

To be addressed in this thesis are several important questions that the vast literature on the economics of civil litigation has left unresolved.³ The first question concerns the **cost-shifting rules** that modern judicial systems apply to allocate litigation costs between the litigants. Influencing the strategic interaction of the litigants, the cost-shifting rule affects the predictability and accuracy of the outcome of the case, as well as the costs spent on litigation. On one end of the costs-shifting spectrum is the traditional **American rule** which requires that each litigant bears her own costs. On the other end is the traditional **English rule** which requires that the loser pays the winner's costs. In modern times, most legal systems across the globe apply intermediate cost-shifting rules that operate

¹See the 2016 *Annual Litigation Trends Survey* by Norton Rose Fulbright, an international law firm, at <http://www.nortonrosefulbright.com/files/20160915-2016-litigation-trends-annual-survey-142485.pdf>.

²For a survey of the philosophical literatures on the rule of law and its relationship with legal predictability and accessibility, see *The Rule of Law* (June 22, 2016), Stanford Encyclopaedia of Philosophy, <https://plato.stanford.edu/entries/rule-of-law/>. For a survey of the seminal papers on the complex relationship between high probability suits induced by cost shifting and the substantive behaviors giving rise to the dispute, see Katz and Sanchirico (2012), pp. 278-86.

³For surveys, see, for example, Katz and Sanchirico (2012), Spier (2007) and section 2.1 in chapter 2 of this thesis.

somewhere in between the extremes; even American and English jurisdictions now apply intermediate cost-shifting rules (see Katz and Sanchirico 2012, pp. 273-75, and section 2.1 in chapter 2 of this thesis). Yet only a small literature has studied intermediate cost-shifting rules, and its conclusions depend on restrictive assumptions.⁴ Hence the first question for this thesis is: *How do intermediate and extreme cost-shifting rules affect litigation efforts and costs, as well as other inherent values of the rule of law, especially legal predictability and accuracy?* Chapter 2 will answer this question.

Second, while litigation models premised on pure self interest and monetary preferences aptly describe commercial litigation, they do not capture well-documented behavioral traits that emotional litigants tend to exhibit. Litigation is essentially a contest, and participants in contest experiments consistently exert significantly greater efforts than equilibrium predictions based on pure self interest and monetary preferences. A natural explanation is that, in addition to the monetary outcome of winning, participants tend to consider non-monetary and relative outcomes (see, for example, Mago, Samak, and Sheremeta 2016, Price and Sheremeta 2011, Sheremeta 2010). Moreover, litigants involved in divorce and inheritance disputes often exhibit spiteful behaviors. For example, an American judge recalled a divorce case in which the husband spent millions just to keep the wife from having a painting that was sold for less than half a million (Duncan 2007 p. 125). These empirical findings and observations raise the second question for this thesis: *How do these different emotional preferences — namely, preferences for the non-monetary value of winning and for relative outcomes — affect litigation outcomes?* Chapter 3 will answer this question.

To answer these questions, this thesis will develop and apply novel techniques in contest theory — a class of game-theoretic models in which each player spends to increase her probability of winning a prize. Contest-theoretic models aptly describe civil litigation because the litigants' incentives to make costly efforts are driven by the monetary or non-monetary benefits of winning. Moreover, recent developments in contest theory have captured spillovers (see, for example, Chowdhury and Sheremeta 2011a, Siegel 2009, 2010, and section 4.1 in chapter 4), and cost shifting in civil litigation essentially creates spillovers by shifting some or all of the winner's litigation costs to the loser. In addition, the literature has introduced emotional variables into simple contest models (see Dechenaux, Kovenock, and Sheremeta 2015 at pp. 614-616 and section 3.1 in chapter 3), inviting an analysis of emotions in civil litigation.

⁴Section 2.1 in chapter 2 contains a survey of the literature on cost shifting.

A game-theoretic approach to studying civil litigation also complements other methodologies. The present approach will focus on capturing the litigants' incentives, how they interact, and how they respond to changes in legal, monetary and emotional variables. This approach will complement empirical approaches by offering insights that are not restricted by concerns of external validity and selection bias. Overcoming concerns of selection bias is important because reliable sources of litigation data — for example, actual reported cases — are often distorted by the litigants' (endogenous) decisions to sue or settle. In addition, the present game-theoretic approach complements doctrinal analysis of specific statutes and cases by offering a general theory of litigation that accounts for a broader range of variables affecting litigation incentives.

Moreover, a contest model has a success function that maps the players' strategies into their respective probabilities of winning and losing. The existing contest models typically assume the success function takes a specific functional form (see Serena and Corchón's 2017 survey). While the chosen functional form may appropriately capture some types of contests in the real world, it may poorly capture other contests. It can be hard to ascertain which of the existing functional forms is "ideal". Moreover, while specifying a "nice" functional form can simplify the solution process, the resulting positive predictions and policy recommendations may not be robust to alternative functional forms. To ensure that the implications of the contest model do not depend on the modeler's idiosyncrasies, the theoretical foundations of the contest model should be sufficiently robust. This thesis therefore aims to answer a theoretical problem that is not confined to civil litigation: *Can a contest model be sufficiently general to capture a large class of success functions, in order to give rise to robust positive predictions and policy recommendations?* Chapter 4 will answer this question.

Finally, in addition to questions of litigation efforts and costs, how the great variety of legal remedies affects substantive economic behaviors has fascinated economists and lawyers. Behaviors that generate private gain to the actor but impose social harm are ubiquitous. Modern law and economics commenced with Coase's (1960) analysis of such a problem of externalities. Absent reputational concerns or other-regarding preferences, the actor has perverse incentives to take socially suboptimal actions without regard to the resulting social harm. The standard legal remedy internalizes externalities completely; a contractual or tortious remedy can make the actor pay for the social harm arising from her actions, and thereby removes her perverse incentives. Yet there are many obstacles to complete internalization of externalities. For instance, transaction costs and asymmetric

information may prevent a socially optimal contract, and imperfection in the judicial process may hinder efforts to measure social harm with a sufficient degree of accuracy (see section 5.1 in chapter 5). The final problem for this thesis is: *How can legal remedies generate socially optimal incentives when insurmountable obstacles prevent complete internalization of externalities?* Chapter 5 will answer this question with a decision-theoretic model.

Chapters 2, 3 and 4 are primarily of interest to economists and technically-inclined lawyer-economists, and chapter 5 to generalist lawyer-economists. While chapters 2, 3 and 4 take axiomatic approaches to develop contest theory and elicit its implications for civil litigation and beyond, chapter 5 is drafted with the minimum level of technicality needed to facilitate an intuitive analysis. Hence, when compared to chapters 2, 3 and 4, chapter 5 adopts a less mathematically demanding methodology, but has a more detailed discussion of legal doctrine.

2 A Robust Theory of Incentives in Civil Litigation: Non-Specified Probability-of-Success Functions and Arbitrary Cost-Shifting Rules

2.1 Introduction

A civil lawsuit typically has two opposing litigants: a plaintiff who seeks judicial remedies at the expense of a defendant. Participation in litigation is costly. A litigant's costs include paying for lawyers, conducting discovery of evidence, researching the law, preparing and making legal arguments, and other costly activities taken to maximize her own payoff. The amount of costs involved in running a lawsuit is a major concern to litigants and the society at large. A recent survey of corporate counsel from various companies across the globe reports the median annual litigation spend to be US\$1 million.⁵ The protection of rights and enforcement of duties under substantive laws — such as property, contracts and torts — depends on the ability of the judicial system to operate fairly and efficiently. Understanding the economics of civil litigation and the strategic interaction of litigants is thus fundamental to the functioning of a society governed by the rule of law.

A litigation model typically includes a **success function** that maps litigation efforts to the litigants' respective probabilities of success. The existing litigation models typically assume the success function takes a specific functional form (see Katz and Sanchirico 2012 and the discussion below). While the chosen functional form may appropriately capture some judicial systems in the real world, it may poorly capture other judicial systems. It can be hard to ascertain which of the existing functional forms is “ideal”. Moreover, while specifying a “nice” functional form can simplify the solution process, the resulting positive predictions and policy recommendations may not be robust to alternative functional forms. To ensure that the implications of the litigation model do not depend on the modeler's idiosyncrasies, the theoretical foundations of the model should be sufficiently robust.

We therefore propose a litigation model that imposes general and intuitive assumptions on its success function *without* specifying its functional form. A success function that satisfies general assumptions has broader economic implications than a success function that takes a specific functional form. Departing from the standard practice of specifying

⁵See the 2016 *Annual Litigation Trends Survey* by Norton Rose Fulbright, an international law firm, at <http://www.nortonrosefulbright.com/files/20160915-2016-litigation-trends-annual-survey-142485.pdf>.

a functional form, the present model captures a large class of success functions, including those frequently used in the existing literature, as well as their convex combinations.⁶ The present generalization of success function thus enables us to formulate a robust theory of incentives in civil litigation. The generality of the present model can facilitate verification of whether the positive predictions and normative recommendations obtained in the existing literature (see below) remain valid under weaker assumptions. The present model also can give rise to novel positive predictions and normative recommendations that the existing, more specialized models cannot obtain.

Modern judicial systems apply a **cost-shifting rule** to allocate litigation costs between the winner and loser of a civil suit.⁷ Litigants in American jurisdictions also may contract for the application of a particular cost-shifting rule (see Prescott, Spier, and Yoon 2014, pp. 66-67, 123). Influencing the strategic interaction of the litigants, the cost-shifting rule affects the predictability and accuracy of the outcome of the case, as well as the costs spent on litigation. On one end of the costs-shifting spectrum is the traditional **American rule** which requires that each litigant bears her own costs. On the other end is the traditional **English rule** which requires that the loser pays the winner's costs. However, unlike the typical litigation model in the existing literature, most modern legal systems across the globe apply intermediate cost-shifting rules that operate somewhere in between the extremes (Katz and Sanchirico 2012, pp. 273-75). Hence, to capture a great diversity of judicial systems, we construct a general model of litigation to analyze intermediate and extreme cost-shifting rules.

The present litigation model is a simultaneous-move game of complete information with two risk-neutral players — a plaintiff and a defendant — each of whom exerts costly effort to maximize her own payoff. A generally-formulated success function describes the litigants' respective posterior probabilities of success based on their efforts and an exogenous prior. The prior reflects the relative advantages of the litigants. The defendant pays a monetary judgment sum to the plaintiff if and only if the plaintiff wins. A cost-shifting rule allows the winner to recover an exogenous proportion of her litigation costs from the loser. Characterizing the cost-shifting rule as a fixed proportion of costs recoverable captures the extreme American and English rules as well as the intermediate rules that shift a part of the winner's costs.

The present formulation of success function is novel and general; it does not take

⁶Subsection 2.8.1 will reveal that the ability to capture the convex combinations of different success functions enables the model to cover cases where the identity of the judge is uncertain.

⁷Unless stated otherwise, litigation costs in this paper include attorneys' fees.

a functional form, but satisfies general assumptions (Assumptions 1-6 in section 2.2). Roughly, under these assumptions: a litigant's posterior probability of success is unaffected by a mere change in her label as "plaintiff" or "defendant", or by proportionate changes in efforts; her posterior probability of success is strictly increasing with her prior probability of success, and is strictly increasing with her effort at a diminishing rate; the curvature of the success function is sufficiently small compared to the cost function's — ensuring the quasiconcavity of the litigant's payoff function; in the special case where the cost function is linear and the English rule applies to allow full recovery of the winner's costs from the loser, a litigant cannot win almost surely by exerting infinitely more effort than the other litigant does. These assumptions capture a large class of success functions; in particular, they capture the Tullock success function — which is the standard in rent-seeking and contest-theory literatures — that expresses a contestant's probability of success as the ratio of her effort relative to total efforts.

The **Litigation Game** that we construct has a unique Nash equilibrium with positive effort levels. In equilibrium, the relatively more advantageous litigant is more likely to win, and her equilibrium probability of success increases if the proportion of costs recoverable increases. Hence more cost shifting reduces uncertainty, making the outcome of the case more predictable ex ante. Moreover, litigation efforts may cause **distortion** in the sense of driving posterior probabilities of success away from the prior. If this obfuscatory effect disappears when the litigants exert equal efforts, then more cost shifting necessarily increases distortion in equilibrium.

Influencing litigation efforts in equilibrium, the cost-shifting rule also affects the costs of exerting such efforts. Call the sum of both litigants' costs **litigation expenditure**. If the relative advantages of the litigants are sufficiently balanced or the cost function is sufficiently convex,⁸ then more cost shifting increases litigation expenditure in equilibrium.⁹ In other cases, how cost shifting affects litigation expenditure depends on the curvatures of the success function and cost function. Intuitively, an increase in the proportion of costs recoverable raises the stakes by widening the difference in monetary outcome between winning and losing. More cost shifting also reduces the expected marginal cost of exerting

⁸Section 2.7 precisely defines the notions of sufficiently balanced advantages and sufficiently convex cost functions.

⁹Baye, Kovenock, and Vries (2005) and Klemperer (2003), Appendix 1 proved a largely similar result with an auction-theoretic model in which two symmetric litigants have private information over their own types and the highest spender wins. A difference is expected litigation costs under the English rule (the loser bears all of the winner's costs) are unbounded in their models, but are bounded in our model. Remark 5 will reveal the present Assumption 5 ensures bounded litigation costs and, together with other assumptions, guarantees the existence and uniqueness of an equilibrium with finite efforts.

effort (Katz and Sanchirico 2012, p. 275). Unless she has very poor prospects, a litigant has incentives to exert more effort in equilibrium to take advantage of a more generous cost-shifting rule.

Even though in many jurisdictions most civil lawsuits settle before the court gives judgment (Hodges, Vogenauer, and Tulibacka 2010, p. 165), they settle in the shadow of the law; that is, the litigants reach a settlement with an expectation of what the outcome would be if the court were to adjudicate their dispute. Studying litigation outcomes in the absence of a settlement is thus absolutely essential to understanding settlement negotiations. The (equilibrium) litigation expenditure that arises in the present Litigation Game is the surplus that the litigants would share in a pre-game that models their settlement negotiations, and the litigants' (equilibrium) payoffs in the Litigation Game are their outside options in that pre-game. This paper thus provides the parameters for future research projects that more comprehensively study settlement negotiations.¹⁰

The choice of cost-shifting rules affects the policy goals of reducing litigation expenditure and improving legal predictability and accuracy. This paper establishes that a more generous cost-shifting rule improves legal predictability in equilibrium, but increases litigation expenditure in cases with balanced advantages or sufficiently convex cost functions. This paper also identifies sufficient conditions for concluding that distortion to the litigants' relative advantages is monotonic with the proportion of costs recoverable. How distortion affects legal accuracy depends on the extent to which relative advantages reflect the inherent merits of the case.

The present generalization of success function overcomes many limitations that arise from specifying the functional form of the success function in a litigation model. First, empirical work on litigation needs to wrestle with the problem of selection bias that arises from decisions to file or settle suits. That problem hinders efforts to ascertain empirically which of the existing functional forms (for example, compare the success functions in Plott 1987 and Carbonara, Parisi, and von Wangenheim 2015) prevails. By comparison, positive predictions arising from the present axiomatized success function extend as far as the generality and defensibility of its assumptions. Secondly, the present generalization of success function expands the range of real-world scenarios beyond those captured by the existing functional forms. Consider a scenario in which the litigants exert efforts before they observe the identity of the judge who is assigned to their case. Suppose each of the potential judges rules according to a different success function. Assuming the litigants

¹⁰Subsection 2.9 and section 2.10 contain discussions of future research directions.

have a common prior probability for each judge being assigned to their case, we prove that a special case of the Litigation Game captures this scenario.

This paper builds on the vast body of literature on the economics of cost shifting, the seminal papers in which are surveyed by Katz and Sanchirico (2012).¹¹ To our best knowledge, only a few authors have considered intermediate cost shifting when litigation efforts are endogenous. Some authors formulate that an exogenous *quantity* marks the limit below which the winner's costs are fully recoverable and above which her costs are fully unrecoverable (for example, Hyde and Williams 2002, Carbonara et al. 2015, Farmer and Pecorino 2016, pp. 214-15, Dari-Mattiacci and Saraceno 2017, Appendix D.2). Some other authors formulate that an exogenous *proportion* of the winner's costs is recoverable (for example, Plott 1987, Hause 1989, Gong and McAfee 2000, Luppi and Parisi 2012). The proportion formulation often predicts that more cost shifting increases litigation costs in equilibrium, a result that is consistently verified in the empirical literature (most recently, Fenn, Grembi, and Rickman's 2017 natural experiment from the United Kingdom). We further develop the proportion formulation due to its greater generality,¹² and generalizes the success functions and cost functions used by these authors. However, we do not consider optimism or divergent beliefs regarding probabilities of success, while some authors do (for example, Hause 1989, Hyde and Williams 2002). Nor do we consider cost-shifting rules that are one-sided or conditional on the margin of victory (for example, Bebchuk and Chang 1996). Moreover, the success function in our model is exogenously given, while the success functions in Skaperdas and Vaidya's (2012) litigation model (with no cost shifting) are derived from the inference process of a Bayesian judge.

We analyze intermediate cost-shifting rules mainly because they represent the reality in modern judicial systems, including the modern American and English systems. After the American Revolution, American jurisdictions eventually departed from the English position of shifting the "necessary" or "reasonable" costs of conducting litigation, for reasons including a failure to increase statutory caps on costs recoverable and distrust of lawyers. The general application of the American rule was subject to exceptions

¹¹See, for example, Spier (2007) (at pp. 300-05) for an earlier survey.

¹²That the proportion formulation of cost-shifting rules generalizes the American and English rules (by setting the proportion to zero or one) implies it generalizes the quantity formulation in the following sense. Suppose, under the quantity formulation, litigation costs in an equilibrium fall below the limiting quantity. Then infinitesimal changes in efforts affect each litigant's payoff in the same way as if the English rule were applied, and applying the English rule would induce the same equilibrium. Alternatively, suppose equilibrium litigation costs under the quantity formulation exceed the limiting quantity, then the following steps would obtain the same equilibrium from applying the American rule: include in the (fixed) judgment sum in dispute the part of costs that does not exceed the limiting quantity; and apply the American rule to render fully unrecoverable the part of costs that exceeds the limiting quantity.

including bad faith proceedings which are considered unwarranted, baseless or vexatious, and contempt proceedings enforcing prior judgments. The U.S. Federal Rules of Civil Procedure now provide for shifting of costs other than lawyers' fees by default, and allow for shifting of lawyers' fees in narrow circumstances.¹³ In addition, most non-American jurisdictions now apply intermediate cost-shifting rules (Katz and Sanchirico 2012, pp. 273-75). There is a recoverability gap in many jurisdictions that allow for cost shifting by judicial discretion; respondents to a 2009 survey reveal this gap to be 25 percent of the winner's actual costs in England, 30-45 percent in Australia, and 33-50 percent in Singapore (Hodges et al. 2010, p. 20). Hence the heavily-analyzed "American rule" (no shifting of the winner's costs) and "English rule" (full shifting of the winner's costs) should be understood as ideal extremes that bound the cost-shifting spectrum.

In addition to realism, a model that permits intermediate cost shifting reveals surprising results that a comparison of the extreme American and English rules cannot obtain. For instance, a comparison of (equilibrium) litigation expenditures under the American and English rules obtains the result that more cost shifting generates additional incentives to spend in the litigated cases (see generally Katz and Sanchirico 2012, p. 275). The present analysis reveals that this result does *not* always hold once we account for intermediate cost shifting. For instance, in cases where the litigants have extremely asymmetric relative advantages and insufficiently convex cost functions, an increase in cost shifting that amounts to a small departure from the American rule may *decrease* litigation expenditure in equilibrium (see section 2.7, especially the discussion of Figure 4). Thus the conventional wisdom that more cost shifting necessarily encourages spending in the litigated cases is not robust to more general formulations of the litigation model.

This paper builds on the literature on contest theory. Tullock (1980) formulates the standard success function, which gives a contestant's probability of success as the ratio of her own effort relative to the aggregate effort of all contestants. Lazear and Rosen (1981) studied optimal labor contracts based on rank-order tournaments, and Lazear (1989) considered sabotage in a competitive workplace. Cornes and Hartley (2005), (2012) refine the Tullock contest model to incorporate risk aversion and general technologies. Einy, Haimanko, Moreno, Sela, and Shitovitz (2015) prove the existence of pure-strategy Bayesian-Nash equilibria in the Tullock contest model with incomplete information. Serena and Corchón (2017) offer a recent survey of the contest-theory literature, while Konrad

¹³See the U.S. Supreme Court's interpretation of rule 54(d)(1) of the Federal Rules of Civil Procedure in the case of *Marx v. General Revenue Corporation*, 133 S. Ct. 1166, 1175 (2013).

(2009) and Vojnović (2016) provide textbook treatments.

Section 2.2 constructs a model of litigation that simultaneously generalizes the success function, cost function and cost-shifting rule. A large class of success functions and cost functions, including those commonly used to study litigation, are special cases of the present assumptions. Moreover, the model's formulation of cost-shifting rules covers the extreme ones as well as the intermediate ones which operate in the vast majority of jurisdictions across the globe. Section 2.3 finds and characterizes the unique nontrivial Nash equilibrium, making explicit and quantifying the strategic aspects of civil litigation. Section 2.4 studies how the equilibrium relative efforts and probabilities of success change in response to infinitesimal changes in, respectively, the litigants' relative advantages, the cost-shifting rule, and the cost function. Sections 2.5-2.7 respectively analyze how cost shifting affects legal predictability, distortion to relative advantages, and litigation expenditure. Section 2.8 illustrates how our litigation model captures uncertainty regarding the identity of the judge, and the frequently-used Tullock contest model. Section 2.10 concludes with a discussion of the normative implications and limitations of our positive predictions. Appendix A.1 contains all proofs. Appendix A.2 contains calculations that facilitate the presentation of examples.

2.2 The Litigation Game

The **Litigation Game** is a simultaneous-move game of complete information characterized by two players, Plaintiff and Defendant, their common set of actions $\mathbb{R}_+$ and respective payoff functions $u_P, u_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$. Their payoff functions and all exogenous parameters are common knowledge. Each player's payoff is her expected monetary outcome in litigation; she is implicitly assumed to be risk neutral.¹⁴

Plaintiff and Defendant respectively exert $e_P, e_D \geq 0$ levels of effort. Let each litigant's cost of exerting effort be given by a homogeneous cost function $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with an exogenous degree of homogeneity $k \geq 1$ that satisfies additional assumptions to be set out below. Given a pair of efforts (e_P, e_D) , the judicial process determines whether, under the law, Defendant is to transfer a judgment sum 1 to Plaintiff.¹⁵ This transfer takes place with (posterior) probability given by a success function $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ satisfying assumptions

¹⁴The simplifying assumption of risk neutrality is particularly apt to describe litigants which are large corporations, each holding a portfolio of lawsuits, and the dispute between them does not concern a sum that is large relative to their wealth.

¹⁵The assumption that the judgment sum is equal to 1 is made without loss of generality because the exact sum merely scales the litigants' effort levels in equilibrium. See subsection 2.8.2.

to be set out below.

The function $\theta(\cdot)$ has an exogenous parameter $0 < \mu < 1$ which represents Plaintiff's prior probability of success. Defendant's prior probability of success is $1 - \mu$. Plaintiff (respectively, Defendant) is **relatively more advantageous** if $\mu > 0.5$ ($\mu < 0.5$). Relative advantages reflect institutional factors that do not vary with litigation efforts but influence the outcome of the case. These factors may reflect the inherent merits of the case. These factors also may reflect the judge's consideration of some salient but legally irrelevant characteristic of the case (see, for example, Bordalo, Gennaioli, and Shleifer 2015). Moreover, the judge may rely on her own personal and professional experiences, and may have her own biases and policy preferences (see, for example, Gennaioli and Shleifer 2007, 2008). The litigants take these exogenous factors as given.

An exogenous parameter $0 \leq \lambda \leq 1$ characterizes the applicable **cost-shifting rule**, where the value of λ satisfies additional assumptions to be set out below. This parameter is the proportion of the winner's costs recoverable from the loser. Important instances of all cost-shifting rules are the extremes. The **English rule** characterized by $\lambda = 1$ allows for full recovery of the winner's costs from the loser, whereas the **American rule** characterized by $\lambda = 0$ allows for no recovery.

Remark 1. *We interpret litigation efforts and probabilities of success as follows. Suppose, given the facts that characterize the relevant dispute and given the litigation efforts, a random variable will realize one of two outcomes — “Plaintiff wins” or “Defendant wins” — at the end of the litigation process. Before such realization, the litigants first exert some minimum sunk efforts to initiate the litigation process, and then exert additional efforts to influence the realization of the outcome. Sunk efforts are referable to activities to acquire knowledge of the “rules of the game”, commence legal proceedings and present the bare minimum “amounts” of evidence and legal arguments to obtain a judicial ruling; these activities may involve initial consultations with lawyers, filing the required documents and giving notice to the interested persons. Additional efforts refer to activities beyond the bare minimum, such as conducting extensive discovery, adducing voluminous evidence and making lengthy legal arguments.¹⁶ Plaintiff's (respectively, Defendant's) prior probability of success μ (respectively, $1 - \mu$) is the probability that the outcome realizes in her favor*

¹⁶For simplicity, we assume there is no agency cost in the relationship between each litigant and her lawyer(s). An extension of the Litigation Game, which is beyond the scope of this paper, may model the principal-agent relationship between a litigant and her lawyer. See Spier (2007) (at pp. 307-311) for a survey of the law-and-economics literature on the lawyer-client relationship. See also Baumann and Friehe (2012a) for a Tullock contest model that captures the interaction of cost-shifting rules and contingent-fee arrangements.

conditional on exertion of sunk efforts and no additional efforts. The variables e_P and e_D are the additional efforts that the litigants exert to influence the realization of the outcome. The facts, sunk efforts and the practical operation of the judicial system affect the prior probabilities of success, but additional efforts do not affect these probabilities. Plaintiff's (respectively, Defendant's) posterior probability of success θ (respectively, $1 - \theta$) is the probability that the outcome realizes in her favor after exertion of sunk efforts and additional efforts. Because the Litigation Game is a model of the litigants' strategic interaction after their exertion of sunk efforts, we call variables e_P and e_D "efforts" and drop the "additional" label for simplicity.

We now state assumptions to guarantee equilibrium existence and uniqueness. On its subdomain $\mathbb{R}_{++}^2$, the success function $\theta(\cdot)$ is twice continuously differentiable and satisfies Assumptions 1-6, where Assumptions 5 and 6 also constrain the degree of homogeneity k of the cost function and the proportion λ of costs recoverable.¹⁷

Assumption 1. *Holding the efforts and the prior constant, whether a litigant is labeled "Plaintiff" or "Defendant" does not affect her posterior probability of success. Formally, $\theta(e_1, e_2; \mu_0) = 1 - \theta(e_2, e_1; 1 - \mu_0)$, for any real numbers $e_1, e_2 > 0$ and $0 < \mu_0 < 1$.*

Assumption 2. *Holding the prior constant, proportionate changes in effort levels do not affect Plaintiff's posterior probability of success. Formally, $\theta(e_P, e_D; \mu) = \theta(xe_P, xe_D; \mu)$, for all scalar $x > 0$.*

Assumption 1 requires a litigant's posterior probability of success to be unaffected by merely changing her label from "Plaintiff" — whose effort, prior and posterior probabilities of success are respectively denoted e_P, μ, θ — to "Defendant" — whose effort, prior and posterior probabilities of success are respectively denoted $e_D, 1 - \mu, 1 - \theta$. The parameter μ captures any asymmetry between the litigants that does not vary with their litigation efforts.¹⁸ Assumption 2 further requires Plaintiff's probability of success to be unaffected by proportionate changes in effort levels.

Assumption 3. *Holding the efforts constant, Plaintiff's posterior probability of success is strictly increasing with her prior probability of success. Formally, $\frac{\partial \theta}{\partial \mu} > 0$.*

¹⁷These assumptions do not impose restrictions in cases where one litigant spends zero effort. Hence these assumptions can capture success functions under which a litigant has a positive probability of losing even though the other litigant spends zero effort. For an example of such a success function, see θ_L defined by (11) in subsection 2.6.2. For a real-life example, see rule 65 of the U.S. Federal Rules of Civil Procedure.

¹⁸For example, if the burden or standard of proof that a litigant bears is dependent on her label as "Plaintiff" or "Defendant", then the parameter μ captures such asymmetry.

Assumption 4. *Holding the prior and Defendant's effort constant, Plaintiff's posterior probability of success is strictly increasing with and weakly concave in her effort. Formally, $\frac{\partial \theta}{\partial e_P} > 0$ and $\frac{\partial^2 \theta}{\partial e_P^2} \leq 0$.*

Assumption 3 requires that holding all else constant, an increase in Plaintiff's prior probability of success strictly increases her posterior probability of success. The strict inequality in Assumption 3 captures the intuition that Plaintiff is more likely to win if she becomes relatively more advantageous. Assumption 4 further requires that holding all else constant, more effort by Plaintiff strictly increases her posterior probability of success but at a diminishing rate. The strict inequality in Assumption 4 ensures that Plaintiff does not make costly effort in vain.

Remark 2. *Assumption 3 reflects the observation that as the litigants' relative advantages (which the exogenous prior captures) play a greater role in determining the outcome of the case, the relatively more advantageous litigant is more likely to succeed. However, the judge must attribute some weight to the litigants' efforts; she sees their evidence and hears their arguments. The judge cannot ignore litigation efforts because in an adversarial system of civil litigation, which the Litigation Game aims to capture, greater constitutional and moral principles mandate that litigants be given an opportunity to present their case and have their arguments heard. The judge also may have to give adequate reasons.¹⁹ Assumption 4 thus reflects the observation that the litigants' participation in the litigation process is not in vein.*

Assumption 5. *For the interested pair of cost-shifting rule characterized by λ and cost function characterized by k , the following condition holds*

$$\frac{\frac{\partial^2}{\partial e_P^2} \left(\frac{\theta}{1-\lambda\theta} \right)}{\frac{\partial}{\partial e_P} \left(\frac{\theta}{1-\lambda\theta} \right)} < \frac{C''(e_P)}{C'(e_P)}. \quad (1)$$

Assumption 5 requires that the curvature of the ratio $\theta/(1 - \lambda\theta)$ — being Plaintiff's distorted²⁰ posterior probability of success — be small when compared to the curvature

¹⁹An early constitutional protection of procedural fairness was clause 39 of the Magna Carta 1215. Modern constitutional protections include the Due Process Clause of the Fifth and Fourteenth Amendments to the United States Constitution, and articles 6 and 45 of the European Convention on Human Rights and Fundamental Freedoms.

²⁰The ratio $\theta/(1 - \lambda\theta)$ is distorted by the cost-shifting rule λ in sense that it lies between Plaintiff's posterior probability of success θ and her relative posterior probability of success $\theta/(1 - \theta)$; that is, $\theta \leq \theta/(1 - \lambda\theta) \leq \theta/(1 - \theta)$.

of the cost function, in the precise sense described by condition (1). This technical assumption ensures that Plaintiff's payoff function is strictly quasiconcave in her own effort.²¹

Assumption 6. *If the cost function is linear and the English rule applies to allow full recovery of the winner's costs from the loser, then as she exerts infinitely more effort than Defendant does, Plaintiff's probability of success does not approach 1. Formally, that $k = \lambda = 1$ implies $\lim_{e_D/e_P \rightarrow 0} \theta < 1$.*

In the special case where the marginal cost of exerting effort is constant and the English rule applies, Assumption 6 prevents Plaintiff from winning almost surely (and recovering all her costs almost surely) by exerting infinitely more effort than Defendant does. This technical assumption prevents Plaintiff from incurring explosive litigation costs under the expectation that all her costs are borne by Defendant.

Remark 3. *As an alternative to Assumption 6 and the assumption that the cost-shifting rule is a (positive) proportion (that is, $0 \leq \lambda \leq 1$), the following condition (A6') is also sufficient for the existence and uniqueness of a Nash equilibrium with positive efforts (see Proposition 1 in section 2.3):*

$$-\infty < \lambda < \frac{1}{\lim_{e_D/e_P \rightarrow 0} \theta}. \quad (\text{A6'})$$

If the success function θ satisfies $\lim_{e_D/e_P \rightarrow 0} \theta < 1$,²² then adopting condition (A6') would allow the Litigation Game to capture a cost-shifting rule that requires the loser to reimburse more than the winner's costs (that is, $\lambda > 1$). Adopting (A6') also would capture a cost-shifting rule that requires the winner to reimburse the loser (that is, $\lambda < 0$).

However, we are not aware of real-world cost-shifting rules that satisfy $\lambda < 0$ or $\lambda > 1$. Adopting condition (A6') also would complicate the proof of the present equilibrium existence and uniqueness result. Hence we do not adopt condition (A6').

Plaintiff and Defendant respectively have payoff functions $u_P, u_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by

$$u_P = \theta[1 - (1 - \lambda)C(e_P)] - (1 - \theta)[C(e_P) + \lambda C(e_D)] \quad (2)$$

$$u_D = -\theta[1 + C(e_D) + \lambda C(e_P)] - (1 - \theta)(1 - \lambda)C(e_D). \quad (3)$$

²¹Remark 5 will discuss the extent to which Assumption 5 guarantees equilibrium existence in existing models of litigation using the Tullock success function.

²²One such success function is θ_L defined by (11) in subsection 2.6.2.

Plaintiff's payoff u_P is the weighted average of her monetary outcome in the event that she wins, $1 - (1 - \lambda)C(e_P)$, and her monetary outcome in the event that she loses, $-C(e_P) - \lambda C(e_D)$. Weights θ and $1 - \theta$ are her probabilities of winning and losing respectively.

Defendant's payoff u_D is the weighted average of her monetary outcome in the event that she loses, $-1 - C(e_D) - \lambda C(e_P)$, and her monetary outcome in the event that she wins, $-(1 - \lambda)C(e_D)$. Weights θ and $1 - \theta$ are respectively her probabilities of losing and winning.

The exogenous parameters and the litigants' payoff functions are common knowledge between them. The cost-shifting rule (λ) is common knowledge because it represents matters of law and community values. The prior probability of success μ is also common knowledge between the litigants because they have had the opportunity to observe the true facts and circumstances of the case as well as the institutional factors of the judicial system. Similarly, the degree of homogeneity of the cost function (k) is common knowledge because it reflects legal services commonly available in the market. To focus on the study of litigation efforts and probabilities of success, further assume there is no settlement or risk of default.

The solution concept adopted is a Nash equilibrium that is nontrivial in the sense of comprising positive efforts by both litigants. A pair of positive efforts denoted (e_P^*, e_D^*) is a nontrivial Nash equilibrium if given the other player's effort, each player chooses an effort to maximize her payoff.

To facilitate presentation, let $\Lambda \subset [0, 1]$ represent a collection of cost-shifting rules that shift some $\lambda \in \Lambda$ proportion of the winner's costs to the loser. Let $K \subset [1, +\infty)$ represent a collection of homogeneous cost functions of some degree $k \in K$. Let $\Theta(\Lambda, K)$ denote the set of twice continuously differentiable functions $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfy Assumptions 1-6 when the applicable pair of cost-shifting rule and cost function is characterized by some $(\lambda, k) \in \Lambda \times K$. Then $\Theta([0, 1], [1, +\infty))$ denotes the set of twice continuously differentiable functions that satisfy Assumptions 1-6 for all $0 \leq \lambda \leq 1$ and all $k \geq 1$. Unless stated otherwise, all lemmas, propositions and corollaries assume the success function $\theta \in \Theta(\{\lambda\}, \{k\})$ where (λ, k) characterizes the pair of cost-shifting rule and cost function that applies to the case between the litigants.

Remark 4. *The Litigation Game captures uncertainty regarding the identity of the judge who will be assigned to the litigants' case. Consider a modified model in which a judge*

chosen from a finite collection of $n \geq 1$ judges will hear and decide the case. A success function $\theta_i(\cdot) \in \Theta(\{\lambda\}, \{k\})$ characterizes judge $i \in \{1, 2, \dots, n\}$. Judge i rules in favor of Plaintiff with posterior probability $\theta_i(e_P, e_D; \mu)$ and in favor of Defendant with posterior probability $1 - \theta_i(e_P, e_D; \mu)$. Plaintiff and Defendant exert efforts before they observe the identity of the chosen judge. They assign a common prior probability $p_i \geq 0$ to judge i being chosen, where $\sum_{i=1}^n p_i = 1$. The prior belief $(p_1, p_2, \dots, p_n)$ is common knowledge between the litigants. Subsection 2.8.1 will reveal a special case of the Litigation Game that adopts the success function $\theta = \sum_{i=1}^n p_i \theta_i$ captures this modified model.²³

2.3 Equilibrium Existence and Uniqueness

This section proves the existence and uniqueness of a nontrivial Nash equilibrium. Lemma 1 allows any nontrivial Nash equilibrium to be characterized by a system of first order conditions (hereinafter, FOCs). Appendix A.1 contains all proofs.

Lemma 1. *Each litigant's payoff function is strictly quasiconcave in her own effort.*

Lemma 1 implies given the other litigant's effort, a litigant's FOC characterizes her best reply.²⁴ A pair of positive efforts $(e_P^*, e_D^*) \in \mathbb{R}_{++}^2$ constitutes a Nash equilibrium if and only if it satisfies system (4):

$$\begin{cases} \frac{\partial u_P}{\partial e_P} = \frac{\partial \theta}{\partial e_P} [1 + \lambda C(e_P) + \lambda C(e_D)] - (1 - \lambda \theta) C'(e_P) = 0 \\ \frac{\partial u_D}{\partial e_D} = \frac{\partial (1 - \theta)}{\partial e_D} [1 + \lambda C(e_P) + \lambda C(e_D)] - (1 - \lambda(1 - \theta)) C'(e_D) = 0. \end{cases} \quad (4)$$

System (4) reveals how the presence of cost shifting affects the litigants' incentives to exert costly efforts. For instance, holding Defendant's effort e_D fixed, more cost shifting (λ increases) increases Plaintiff's marginal benefits of exerting effort e_P by shifting a greater proportion of her costs $C(e_P)$ into the "prize" of winning: $1 + \lambda C(e_P) + \lambda C(e_D)$. This shift also decreases her marginal costs of exerting effort: $(1 - \lambda \theta) C'(e_P)$. The same observations apply to Defendant's incentives to exert costly effort when we hold Plaintiff's effort fixed.

Lemma 2 finds a unique, positive effort ratio which will be used to characterize the nontrivial Nash equilibrium. To simplify notation, define an auxiliary variable $s = e_D/e_P$

²³For real-life examples of this modified model, see Wallace, Mack, and Roach Anleu (2014), especially pp. 687-89.

²⁴Theorem 8 of Diewert, Avriel, and Zang (1981) holds any local maximizer of a strictly quasiconcave function is the unique global maximizer.

whenever Plaintiff's effort $e_P > 0$; s is the ratio of Defendant's effort relative to Plaintiff's. Assumption 2 implies that for any two pairs of positive efforts $(e_P, e_D), (e'_P, e'_D) \in \mathbb{R}_{++}^2$ such that $e_D/e_P = e'_D/e'_P$, the success function satisfies $\theta(e_P, e_D; \mu) = \theta(e'_P, e'_D; \mu)$. By a slight abuse of notation, denote $\theta(s; \mu) = \theta(e_P, e_D; \mu)$, $\theta_s = \frac{\partial}{\partial s} \theta(s; \mu)$ and $\theta_{ss} = \frac{\partial^2}{\partial s^2} \theta(s; \mu)$.

Lemma 2. *There exists a unique positive effort ratio $s^* > 0$ satisfying*

$$s^* = \left[\frac{1 - \lambda \theta(s^*; \mu)}{1 - \lambda(1 - \theta(s^*; \mu))} \right]^{1/k}. \quad (5)$$

The value of s^ satisfies the following properties:*

1. *If the American rule applies (that is, $\lambda = 0$), then $s^* = 1$.*
2. *If the cost-shifting rule allows the winner to recover at least some costs from the loser (that is, $\lambda > 0$) and Plaintiff's prior probability of success $\mu > 0.5$ (respectively, $= 0.5, < 0.5$), then $s^* < 1$ (respectively, $= 1, > 1$).*

Proposition 1 establishes the existence and uniqueness of a nontrivial Nash equilibrium. It also characterizes the litigants' relative efforts in equilibrium.

Proposition 1. *There exists a unique Nash equilibrium with positive efforts (e_P^*, e_D^*) , which is characterized by²⁵*

$$e_P^* = [C(1)[ks^{*k-1}[1 - \lambda(1 - \theta(s^*; \mu))]/(-\theta_s(s^*; \mu)) - \lambda(1 + s^{*k})]]^{-1/k}, \quad e_D^* = s^* e_P^*$$

where Lemma 2 gives s^* .

This Nash equilibrium satisfies the following properties:

1. *If the American rule applies or relative advantages are equal, then the litigants exert the same levels of effort in equilibrium. Formally, $\lambda = 0$ or $\mu = 0.5$ implies $e_P^* = e_D^*$. Moreover, that relative advantages are equal implies posterior probabilities of success are equal in equilibrium. Formally, $\mu = 0.5$ implies $\theta(s^*; \mu) = 0.5$.*
2. *If the cost-shifting rule allows the winner to recover at least some costs from the loser, then the relatively more advantageous litigant exerts relatively more effort and has a relatively greater posterior probability of success in equilibrium. Formally, $\lambda > 0$ and $\mu > 0.5$ (respectively, $\mu < 0.5$) implies $e_P^* > e_D^*$ and $\theta(s^*; \mu) > 0.5$ ($e_P^* < e_D^*$ and $\theta(s^*; \mu) < 0.5$).*

²⁵Parts 5 and 9 of Lemma 9, a technical lemma contained in Appendix A.1, respectively imply that both the numerator and denominator of e_P^{*k} are positive.

Proposition 1 reveals that the applicable cost-shifting rule and the prior determine the litigants' relative efforts in the nontrivial Nash equilibrium. That the American rule applies to deny the winner of any recovery of her costs is sufficient to induce equal equilibrium efforts. If the cost-shifting rule allows at least some recovery and she is relatively more advantageous (respectively, relatively less advantageous), then Plaintiff's equilibrium effort is greater than (smaller than) Defendant's. The litigants exert equal efforts in equilibrium if their relative advantages are equal.²⁶

All subsequent analyses of equilibrium properties are referable to the unique nontrivial Nash equilibrium characterized by Proposition 1. We are not interested in any equilibrium that is trivial in the sense that at least one litigant exerts zero effort.

As a preliminary to subsequent discussions of the equilibrium implications of variations in exogenous parameters, Corollary 1 ensures that a nontrivial Nash equilibrium actually exists within the interested range of exogenous parameters.

Corollary 1. *Consider a success function $\theta \in \Theta(\{\bar{\lambda}\}, \{\underline{k}\})$ and a pair of cost-shifting rule and success function characterized by some $0 \leq \bar{\lambda} \leq 1$ and $\underline{k} \geq 1$. There exists a unique nontrivial Nash equilibrium under the same success function θ and any pair of cost-shifting rule and cost function characterized by $\lambda \leq \bar{\lambda}$ and $k \geq \underline{k}$.*

Given a success function $\theta \in \Theta(\{\bar{\lambda}\}, \{\underline{k}\})$ where $\bar{\lambda}$ characterizes the most generous cost-shifting rule and $\underline{k}$ the least convex cost function which arouse our interest, Corollary 1 enables us to analyse equilibrium properties for all combinations of cost-shifting rule $\lambda \leq \bar{\lambda}$ and cost function $k \geq \underline{k}$. Corollary 1 also enables us to analyse how equilibrium properties respond to variations in the applicable pair of cost-shifting rule and cost function, to the extent that such variations do not extend beyond the scope of $[0, \bar{\lambda}] \times [\underline{k}, +\infty)$.²⁷ It follows that if $\theta \in \Theta(\{1\}, \{1\})$, then we can analyse equilibrium properties for all cost-shifting rules and cost functions.

Remark 5. *The existing literature finds that the English rule ($\lambda = 1$) often does not admit a nontrivial Nash equilibrium. Using the following Tullock success function $\theta_T : \mathbb{R}_+^2 \rightarrow [0, 1]$ where*

$$\theta_T(e_P, e_D; \mu) = \begin{cases} \frac{\mu e_P}{\mu e_P + (1-\mu)e_D} & \text{if } e_P + e_D \neq 0 \\ \mu & \text{otherwise,} \end{cases} \quad (6)$$

²⁶Corollary 2 in section 2.4 will reveal that more cost shifting increases the equilibrium relative effort and probability of success of the relatively more advantageous litigant.

²⁷By a slight abuse of notation, let $[0, \bar{\lambda}] = \{0\}$ if $\bar{\lambda} = 0$.

Farmer and Pecorino (1999) (at pp. 281-82) and Carbonara et al. (2015) (at pp. 8-9) showed that the English rule induces a nontrivial Nash equilibrium if and only if litigation efforts are insufficiently influential on posterior probabilities of success.²⁸ The conditions guaranteeing equilibrium existence in their models are special cases of Assumption 5 in the present Litigation Game. Some algebra using Appendix A.2 reveals θ_T satisfies

$$\frac{\frac{\partial^2}{\partial e_P^2} \left(\frac{\theta_T}{1-\lambda\theta_T} \right)}{\frac{\partial}{\partial e_P} \left(\frac{\theta_T}{1-\lambda\theta_T} \right)} \leq 0 \leq \frac{C''(e_P)}{C'(e_P)}$$

where the first weak inequality holds strictly if $\lambda < 1$ and with equality if $\lambda = 1$, and the second weak inequality holds strictly if $k > 1$ and with equality if $k = 1$. Hence condition (1) in Assumption 5 is satisfied if and only if $k > 1$ or $\lambda < 1$; that is, $\theta_T \in \Theta([0, 1), [1, +\infty)) \cup \Theta([0, 1], (1, +\infty))$, but $\theta_T \notin \Theta(\{1\}, \{1\})$. If the cost-shifting rule permits less than full recovery ($\lambda < 1$) or the cost function is strictly convex ($k > 1$), then Proposition 1 proves the existence and uniqueness of a nontrivial Nash equilibrium when θ_T operates.

Assumption 5 does not cover the litigation models of Hause (1989), Hyde and Williams (2002). Their models allow for generally-formulated success functions and divergent beliefs regarding posterior probabilities of success, but assume the English rule induces a Nash equilibrium.²⁹ With a minor modification to introduce divergent beliefs, the Tullock success function θ_T provides a counter-example that satisfies the conditions imposed by Hause (1989) or Hyde and Williams (2002) but does not induce a nontrivial Nash equilibrium under the English rule.

2.4 Comparative Statics

This section calculates the equilibrium effects of variations in the prior, cost-shifting rule and degree of homogeneity of the cost function. To facilitate presentation, let $\theta^* = \theta(e_P^*, e_D^*; \mu)$ denote Plaintiff's posterior probability of success in the nontrivial Nash equilibrium, and call it her **equilibrium probability of success**. Defendant's equilibrium probability of success is $1 - \theta^*$. Call $s^* = e_D^*/e_P^*$ Defendant's **equilibrium relative effort**. Plaintiff's equilibrium relative effort is $1/s^*$.

²⁸Using a special case of the success function θ_L given by (11) in subsection 2.6.2, Plott (1987) (at p. 189) also proved the English rule induces a nontrivial Nash equilibrium if and only if the litigants' efforts do not completely determine posterior probabilities of success.

²⁹Hause (1989), pp. 165-66, and Hyde and Williams (2002), pp.137-39, 147-48.

Corollary 2 reveals how equilibrium relative efforts and probabilities of success respond to infinitesimal variations in the applicable cost-shifting rule.

Corollary 2. *Consider the nontrivial Nash equilibrium (e_P^*, e_D^*) .*

1. *Suppose Plaintiff's is relatively more advantageous. Then her equilibrium relative effort and probability of success are increasing with the proportion of costs recoverable. Formally, $\mu > 0.5$ implies $\frac{d(1/s^*)}{d\lambda} > 0$ and $\frac{d\theta^*}{d\lambda} > 0$.*
2. *Suppose the relative advantages are equal. Then each litigant's equilibrium relative effort and probability of success do not change with the proportion of costs recoverable. Formally, $\mu = 0.5$ implies $\frac{ds^*}{d\lambda} = 0$ and $\frac{d\theta^*}{d\lambda} = 0$.*
3. *Suppose Defendant's is relatively more advantageous. Then her equilibrium relative effort and probability of success are increasing with the proportion of costs recoverable. Formally, $\mu < 0.5$ implies $\frac{ds^*}{d\lambda} > 0$ and $\frac{d(1-\theta^*)}{d\lambda} > 0$.*

Corollary 2 proves that the relative advantages of the litigants determine how equilibrium relative efforts and probabilities of success respond to infinitesimal variations in the applicable cost-shifting rule. If one litigant is relatively more advantageous (that is, $\mu \neq 0.5$), then parts 1 and 3 prove that she exerts relatively more effort in equilibrium. Parts 1 and 3 also reveal that more cost shifting increases the equilibrium probability of success of the relatively more advantageous litigant. Intuitively, more cost shifting incentivizes the relatively more advantageous litigant — who has better prior prospects of winning — to exert relatively more effort. Then that both relative advantages and relative effort are in favor of the relatively more advantageous litigant; a greater equilibrium probability of success for her follows.

Corollary 3. *Consider the nontrivial Nash equilibrium (e_P^*, e_D^*) .*

1. *If the American rule applies, then an increase in a litigant's prior probability of success does not affect her equilibrium relative effort, but increases her equilibrium probability of success. Formally, $\lambda = 0$ implies $\frac{d(1/s^*)}{d\mu} = 0$, $\frac{d\theta^*}{d\mu} > 0$ for Plaintiff and $\frac{ds^*}{d(1-\mu)} = 0$, $\frac{d(1-\theta^*)}{d(1-\mu)} > 0$ for Defendant.*
2. *If the cost-shifting rule allows the winner to cover at least some costs from the loser, then a litigant's relative effort and equilibrium probability of success are increasing with her prior probability of success. Formally, $\lambda > 0$ implies $\frac{d(1/s^*)}{d\mu} > 0$, $\frac{d\theta^*}{d\mu} > 0$ for Plaintiff and $\frac{ds^*}{d(1-\mu)} > 0$, $\frac{d(1-\theta^*)}{d(1-\mu)} > 0$ for Defendant.*

Corollary 3 proves that the equilibrium effects of changes in a litigant's relative advantages depends on the applicable cost-shifting rule. Part 1 proves that under the American rule (that is, $\lambda = 0$), becoming more advantageous does not incentivize a litigant to exert relatively more effort in equilibrium. Nonetheless, the increase in her prior probability of success (which represents her relative advantages) has a direct effect that improves her equilibrium probability of success. Part 2 proves that if cost shifting takes place (that is, $\lambda > 0$), becoming more advantageous incentivizes a litigant to exert relatively more effort in equilibrium. Then the increase in her prior probability of success directly improves her equilibrium probability of success, and indirectly does so through increasing her relative effort.

Remark 6. *Corollary 3 implies that in equilibrium, each of the effort ratio s^* and the posterior probability θ^* is a bijective function of the prior probability μ . This result does not suggest that the judge, who does not know μ , has enough information to infer it and decide the case without giving weight to litigation efforts. As discussed in Remark 1, we interpret the prior and litigation efforts as influencing the posterior probabilities that a random variable — representing the practical operation of the judicial system — realizes one of two values: “Plaintiff wins” or “Defendant wins”. Consistently with reality, the judge observes the realized value of this random variable, but does not observe the effort ratio or posterior probability. Hence the judge has insufficient information to infer μ .*

Corollary 4. *Consider the nontrivial Nash equilibrium (e_P^*, e_D^*) .*

1. *If the American rule applies or the relative advantages are equal, then changes in the degree of homogeneity of the cost function does not affect each litigant's relative effort and probability of success in equilibrium. Formally, if $\lambda = 0$ or $\mu = 0.5$, then $\frac{ds^*}{dk} = 0$ and $\frac{d\theta^*}{dk} = 0$.*
2. *If the cost-shifting rule allows the winner to recover at least some costs from the loser and one litigant's case is relative more advantageous, then that litigant's equilibrium relative effort and probability of success are decreasing with the degree of homogeneity of the cost function. Formally, that $\lambda > 0$ and $\mu > 0.5$ (respectively, $\mu < 0.5$) implies $\frac{d(1/s^*)}{dk} < 0$ and $\frac{d\theta^*}{dk} < 0$ (respectively, $\frac{ds^*}{dk} < 0$ and $\frac{d(1-\theta^*)}{dk} < 0$).*

Corollary 4 proves the effects of changes in the degree of homogeneity (that is, k) of the cost function depends on the applicable cost-shifting rule. The value of k corresponds to the convexity of the cost function; as k increases, the cost function becomes more convex.

Part 1 proves changes in k do not affect equilibrium relative efforts or probabilities of success in cases where the American rule applies ($\lambda = 0$) or neither litigant is relatively more advantageous ($\mu = 0.5$). In other cases, a greater k incentivizes the relatively more advantageous litigant to exert relatively *less* effort in equilibrium, which indirectly decreases her equilibrium probability of success. Intuitively, each litigant's equilibrium effort typically falls below the judgment sum 1. As the cost function becomes more convex (k increases), a chosen level of costs “produces” more effort, and the magnitude of the additional effort “production” *falls* when the chosen level of costs increases. An increase in k thus has a milder impact on the incentives of the relatively more advantageous litigant, because she incurs a relatively greater level of costs in equilibrium; the results in part 2 reflect this milder impact on her incentives to exert costly effort.

2.5 Cost Shifting Affects Legal Predictability

This section reveals how changes in the applicable cost-shifting rule affects legal predictability in equilibrium. Consider two arbitrary cost-shifting rules $0 \leq \lambda_1, \lambda_2 \leq 1$ and a success function $\theta \in \Theta(\{\lambda_1, \lambda_2\}, \{k\})$. Corollary 1 proves the existence and uniqueness of a nontrivial Nash equilibrium under each of these cost-shifting rules. Let θ_1^*, θ_2^* denote Plaintiff's equilibrium probabilities of success under λ_1, λ_2 respectively. We say the cost-shifting rule λ_2 makes the outcome of the case **more predictable** than the cost-shifting rule λ_1 does if and only if

$$|\theta_2^* - 0.5| > |\theta_1^* - 0.5|. \quad (7)$$

Intuitively, the worst scenario for legal predictability occurs when the litigants win with equal probabilities in equilibrium. Then θ_2^* is better for legal predictability than θ_1^* is in the sense that θ_2^* is further away from 0.5 than θ_1^* is. Changing the applicable cost-shifting rule from λ_1 to λ_2 improves legal predictability from Plaintiff's perspective by changing her equilibrium probability of success from θ_1^* to θ_2^* . The same reasoning applies to legal predictability from Defendant's perspective; condition (7) is equivalent to $|(1 - \theta_2^*) - 0.5| > |(1 - \theta_1^*) - 0.5|$, where $1 - \theta_2^*, 1 - \theta_1^*$ are Defendant's equilibrium probabilities of success under λ_2, λ_1 respectively.

In an equal-advantages case ($\mu = 0.5$), part 2 of Corollary 2 renders trivial the question whether more cost shifting improves or impairs legal predictability. This is because variations in the cost-shifting rule does not affect equilibrium relative efforts or

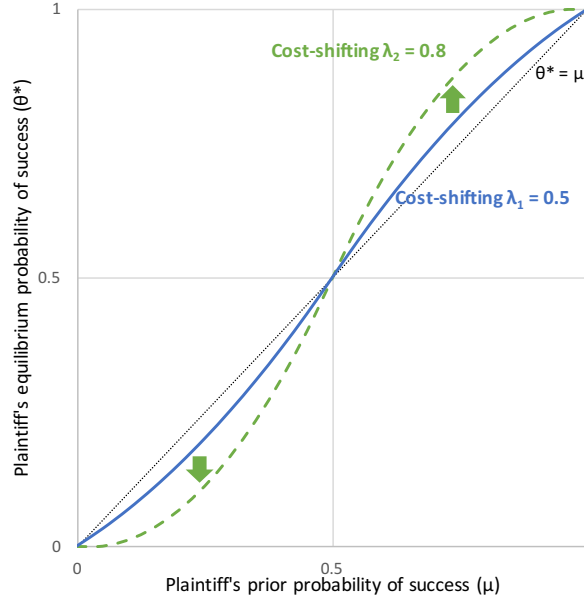


Figure 1: Plaintiff's equilibrium probabilities of success as functions of her prior probability of success under two cost-shifting rules, $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$, in the $\mathbb{T}_1$ Game.

probabilities of success. Intuitively, no litigant has an advantage over the other. Any variation in the cost-shifting rule affects the incentives of both litigants equally, and in equilibrium they exert equal efforts and win with equal probabilities.

Applying the results in Corollary 2 to cases in which one litigant is relatively more advantageous, Corollary 5 reveals how variations in cost shifting affects legal predictability.

Corollary 5. *Consider two cost-shifting rules $0 \leq \lambda_1 < \lambda_2 \leq 1$, where the success function $\theta \in \Theta(\{\lambda_2\}, \{k\})$ and Plaintiff's equilibrium probability of success is θ_1^* under λ_1 and θ_2^* under λ_2 . If one litigant is relatively more advantageous, then increasing the applicable cost-shifting rule from λ_1 to λ_2 makes the outcome of the case more predictable. Formally, $\mu \neq 0.5$ implies $|\theta_2^* - 0.5| > |\theta_1^* - 0.5|$.*

Corollary 5 proves that if one litigant is relatively more advantageous ($\mu \neq 0.5$), then more cost shifting improves legal predictability in equilibrium. Intuitively, the relatively more advantageous litigant is more likely to win in equilibrium (according to Proposition 1), and more cost shifting ($\lambda_1 \rightarrow \lambda_2$) incentivizes her further to increase her equilibrium effort relative to the other litigant's (according to Corollary 2). This increase in her relative effort further increases her equilibrium probability of success, thereby improving legal predictability in her favor ($|\theta_2^* - 0.5| > |\theta_1^* - 0.5|$).

Figure 1 illustrates Corollary 5 with a special case of the Litigation Game, called the $\mathbb{T}_1$ Game, that adopts a linear cost function ($k = 1$) and the Tullock success function θ_T given by (6) in Remark 5. Figure 1 plots the relationship between Plaintiff's equilibrium

probability of success θ^* and prior probability of success μ under two cost-shifting rules $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$. The blue solid curve (respectively, green dashed curve) depicts θ^* as a function of μ when $\lambda_1 = 0.5$ (respectively, $\lambda_2 = 0.8$). For all $\mu \neq 0.5$, the value of θ^* on the green dashed curve is further away from 0.5 compared to that on the blue solid curve.

2.6 Cost Shifting Distorts Relative Advantages

This section explores how changes in the applicable cost-shifting rule influence the extent to which the litigants' relative advantages affects their equilibrium probabilities of success. Define **distortion** $\Delta : (0, 1) \times [0, 1] \times [1, +\infty) \rightarrow \mathbb{R}$ by the magnitude of the difference between Plaintiff's equilibrium probability of success θ^* and prior probability of success μ

$$\Delta(\mu, \lambda, k) = |\theta^* - \mu|, \quad (8)$$

where θ^* is a function of μ , the applicable cost-shifting rule λ and the degree of homogeneity k of the cost function. Distortion from Defendant's perspective is $\Delta(\mu, \lambda, k) = |1 - \theta^* - (1 - \mu)|$, which is the magnitude of the difference between her equilibrium probability of success $1 - \theta^*$ and prior probability of success $1 - \mu$.

The present notion of distortion captures an argument, by Hirshleifer and Osborne (2001) (pp. 185-86) and others, that “justice” requires a litigant's equilibrium probability of success to match the inherent merits of her case, as reflected by her relative advantages. Intuitively, distortion measures the extent to which litigation efforts drive equilibrium probabilities of success away from the litigants' relative advantages, as captured by the prior. A large (respectively, small) distortion means that, compared to relative advantages, litigation efforts have a significant (insignificant) influence on equilibrium probabilities of success. If changing a cost-shifting rule increases (respectively, decreases) distortion, then this change increases (decreases) the influence that litigation efforts have on equilibrium probabilities of success.

Our previous assumptions are not sufficient for answering the question how cost shifting affects distortion in every case. That question is trivial in equal-advantages cases ($\mu = 0.5$) because, as part 2 of Corollary 2 proves, each litigant's equilibrium probability of success is not affected by any variation in the cost-shifting rule. However, if the relative advantages are unequal ($\mu \neq 0.5$), how cost shifting affects distortion is not immediately

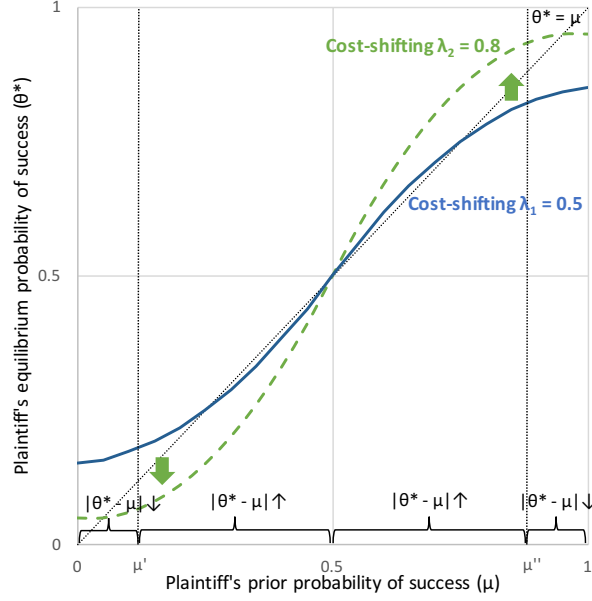


Figure 2: Plaintiff's equilibrium probabilities of success as functions of her prior probability of success under two cost-shifting rules, $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$.

clear. To illustrate potential complexities, consider Figure 2, which depicts for a special case of the Litigation Game the relationship between Plaintiff's equilibrium probability of success θ^* and her prior probability of success μ under two cost-shifting rules, $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$. The blue solid curve (respectively, green dashed curve) depicts θ^* as a function of μ when $\lambda_1 = 0.5$ (respectively, $\lambda_2 = 0.8$). For any case characterized by a μ satisfying $\mu' < \mu < 0.5$ or $0.5 < \mu < \mu''$, the value of θ^* on the green dashed curve is further away from μ compared to that on the blue solid curve. In these cases, increasing the cost-shifting rule from λ_1 to λ_2 increases distortion. For any case characterized by a μ satisfying $\mu < \mu'$ or $\mu > \mu''$, the value of θ^* on the green dashed curve is closer to μ compared to that on the blue solid curve. In these cases, increasing the cost-shifting rule from λ_1 to λ_2 decreases distortion. For a case characterized by $\mu = \mu'$ or $\mu = \mu''$, increasing the cost-shifting rule from λ_1 to λ_2 does not affect distortion.

2.6.1 When More Cost Shifting Increases Distortion

We now propose additional conditions that are sufficient to answer the question whether more cost shifting increases or decreases distortion in cases where one litigant is relatively more advantageous.

Assumption 7. *If litigation efforts are positive and equal, then Plaintiff's posterior probability of success equals her prior probability of success. Formally, $e_P = e_D > 0$ implies $\theta(e_P, e_D; \mu) = \mu$.*

Assumption 7 imposes a condition in respect of all positive effort levels $e_P, e_D > 0$, not just the equilibrium pair of efforts (e_P^*, e_D^*) . It requires that a litigant's posterior probability of success accurately reflects her relative advantages if litigation efforts are equal. Intuitively, under Assumption 7, equal efforts do not distort the litigants' relative advantages. Satisfaction of Assumption 7 does not depend on the applicable cost-shifting rule or cost function.

Adding Assumption 7, Proposition 2 proves the relatively more advantageous litigant has an equilibrium probability of success that is no smaller than her prior probability of success. To facilitate presentation, let Θ_7 denote the set of twice continuously differentiable functions $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfy Assumption 7.

Proposition 2. *Suppose the success function $\theta \in \Theta(\{\lambda\}, \{k\}) \cap \Theta_7$ and one litigant is relatively more advantageous (that is, $\mu \neq 0.5$). In the nontrivial Nash equilibrium, the equilibrium probability of success of the relatively more advantageous litigant is no smaller than her prior probability of success. Her equilibrium probability of success is greater than her prior probability of success if the cost-shifting rule makes at least some costs recoverable. Formally:*

1. *That $\mu > 0.5$ implies $\theta^* \geq \mu$, holding strictly if $\lambda > 0$.*
2. *That $\mu < 0.5$ implies $1 - \theta^* \geq 1 - \mu$, holding strictly if $\lambda > 0$.*

Corollary 6. *Consider two cost-shifting rules $0 \leq \lambda_1 < \lambda_2 \leq 1$, where the success function $\theta \in \Theta(\{\lambda_2\}, \{k\}) \cap \Theta_7$. If one litigant is relatively more advantageous, then increasing the applicable cost-shifting rule from λ_1 to λ_2 increases distortion in equilibrium. Formally, $\mu \neq 0.5$ implies $\Delta(\mu, \lambda_2, k) > \Delta(\mu, \lambda_1, k)$.*

Using the results in Proposition 2, Corollary 6 proves that adding Assumption 7 is sufficient for concluding that in any unequal-advantages case, more cost shifting increases distortion in equilibrium. Intuitively, Assumption 7 ensures that in any unequal-advantages case and under any cost-shifting rule, the equilibrium probability of success of the more advantageous litigant is no smaller than her prior probability of success. Then allowing for more cost shifting increases her relative effort (according to Corollary 2), which further pushes her equilibrium probability of success above her prior probability of success.

Figure 1 illustrates Corollary 6 using the $\mathbb{T}_1$ Game, which satisfies Assumption 7. Figure 1 plots the relationship between Plaintiff's equilibrium probability of success θ and her prior probability of success μ under two cost-shifting rules $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$.

The blue solid curve (respectively, green dashed curve) depicts θ^* as a function of μ when $\lambda_1 = 0.5$ ($\lambda_2 = 0.8$). For all $\mu \neq 0.5$, the value of θ^* on the green dashed curve is further away from μ compared to that on the blue solid curve.

2.6.2 When More Cost Shifting Decreases Distortion

We now propose an alternative assumption that imposes a sufficient condition for ensuring that more cost shifting decreases distortion in unequal-advantages cases. To facilitate presentation, denote $\theta_\mu = \frac{\partial \theta}{\partial \mu}$, $\theta_{\mu\mu} = \frac{\partial^2 \theta}{\partial \mu^2}$, $\theta_{ss} = \frac{\partial^2 \theta}{\partial s^2}$ and $\theta_{s\mu} = \frac{\partial^2 \theta}{\partial s \partial \mu}$, and define functions $\alpha, \beta : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ by

$$\begin{aligned}\alpha(s; \mu, \lambda, k) &= k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s \\ \beta(s; \mu, \lambda, k) &= \frac{k(2 - \lambda)^2 s^k}{(1 + s^k)^2} + \lambda(2 - \lambda)s\theta_s\end{aligned}$$

where $\theta \in \Theta(\{\lambda\}, \{k\})$, and μ, λ, k are exogenous parameters in functions α, β .³⁰

Assumption 8. *If Plaintiff is relatively more advantageous (that is, $\mu > 0.5$) and her effort is no less than some positive effort by Defendant (that is, $0 < s \leq 1$), then at least one of the following conditions holds:*

$$\alpha^2 \theta_{\mu\mu} - 2\lambda(2 - \lambda)\alpha s \theta_\mu \theta_{s\mu} \geq -\lambda^2(2 - \lambda)^2 s \theta_\mu^2 \left\{ s \theta_{ss} + \left[1 - \frac{k\lambda(2\theta - 1)}{2 - \lambda} \right] \theta_s \right\} \quad (9)$$

or

$$\beta^2 \theta_{\mu\mu} - 2\lambda(2 - \lambda)\beta s \theta_\mu \theta_{s\mu} \geq -\lambda^2(2 - \lambda)^2 s \theta_\mu^2 \left\{ s \theta_{ss} + \left[1 - \frac{k(1 - s^k)}{1 + s^k} \right] \theta_s \right\}. \quad (10)$$

Given a pair of cost-shifting rule λ and a cost function k , Assumption 8 imposes restrictions on the first, second and cross derivatives of the success function $\theta(s; \mu)$.³¹ As Proposition 3 will prove, adding Assumption 8 ensures that Plaintiff's equilibrium probability of success is convex (respectively, concave) in her prior probability of success when her case is relatively more (respectively, less) advantageous. To facilitate presentation, let $\Theta_8(\Lambda, K)$ denote the set of twice continuously differentiable functions $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfy Assumption 8 when the applicable cost-shifting rule and cost function are characterized by $\lambda \in \Lambda \subset [0, 1]$ and $k \in K \subset [0, +\infty)$ respectively.

³⁰Part 9 of Lemma 9, a technical lemma in Appendix A.1, implies $\alpha, \beta > 0$.

³¹Part 6 of Lemma 9, a technical lemma in Appendix A.1, reveals that Assumptions 1, 3 and 4 imply $\theta_\mu > 0$, $\theta_s < 0$ and $\theta_{ss} \geq 0$. Assumption 8 imposes additional restrictions in respect of their magnitude.

Proposition 3. *Suppose the success function $\theta \in \Theta(\{\lambda\}, \{k\}) \cap \Theta_8(\{\lambda\}, \{k\})$ and one litigant is relatively more advantageous (that is, $\mu \neq 0.5$). If her case is relatively more (respectively, less) advantageous, then Plaintiff's equilibrium probability of success is a convex (respectively, concave) function of her prior probability of success. The convexity (respectively, concavity) is strict if condition (9) or (10) holds strictly. Formally:*

1. *That $\mu > 0.5$ implies $\frac{d^2\theta^*}{d\mu^2} \geq 0$, holding strictly if condition (9) or (10) holds strictly.*
2. *That $\mu < 0.5$ implies $\frac{d^2\theta^*}{d\mu^2} \leq 0$, holding strictly if condition (9) or (10) holds strictly.*

Corollary 7. *Consider two cost-shifting rules $0 \leq \lambda_1 < \lambda_2 \leq 1$, where the success function $\theta \in \Theta(\{\lambda_2\}, \{k\}) \cap \Theta_8(\{\lambda_1, \lambda_2\}, \{k\})$. If one litigant is relatively more advantageous, then reducing the cost-shifting rule from λ_2 to λ_1 increases distortion in equilibrium. Formally, $\mu \neq 0.5$ implies $\Delta(\mu, \lambda_2, k) < \Delta(\mu, \lambda_1, k)$.*

Using the results in Proposition 3, Corollary 7 proves that adding Assumption 8 is sufficient for concluding that in any unequal-advantages case, reducing cost shifting increases distortion in equilibrium. Intuitively and as confirmed by Proposition 3, Assumption 8 imposes conditions on the success function to ensure that in any unequal-advantages case and under any cost-shifting rule, the equilibrium probability of success of the more advantageous litigant is no greater than her prior probability of success. Then reducing cost shifting, which decreases her relative effort (according to Corollary 2), further pushes her equilibrium probability of success below her prior probability of success. This in turn increases distortion.

Figure 3 illustrates Corollary 7 using a special case of the Litigation Game, called the $\mathbb{L}_k$ Game, that adopts a strictly convex cost function (that is, $k > 1$) and the following linear success function $\theta_L : \mathbb{R}_+^2 \rightarrow [0, 1]$

$$\theta_L(e_P, e_D; \mu) = \begin{cases} \mu\eta + (1 - \eta)\frac{e_P}{e_P + e_D} & \text{if } e_P + e_D \neq 0 \\ \mu & \text{otherwise} \end{cases} \quad (11)$$

where an exogenous weight $0 < \eta < 1$ determines the relative influences of the prior and of the litigation efforts on the posterior probabilities of success.³² The $\mathbb{L}_k$ Game satisfies Assumption 8 under any pair of cost-shifting rule and cost function.³³ Figure 3 depicts

³²An increase in η represents an increase in the relative weight that the judicial process gives to the relative advantages, and a corresponding decrease in the relative weight that it gives to the litigants' relative effort level.

³³Using Appendix A.2, some algebra reveals that $\frac{\partial^2 \theta_L}{\partial^2 \mu} = \frac{\partial^2 \theta_L}{\partial s \partial \mu} = 0$ for all μ , and that $\mu > 0.5$ and

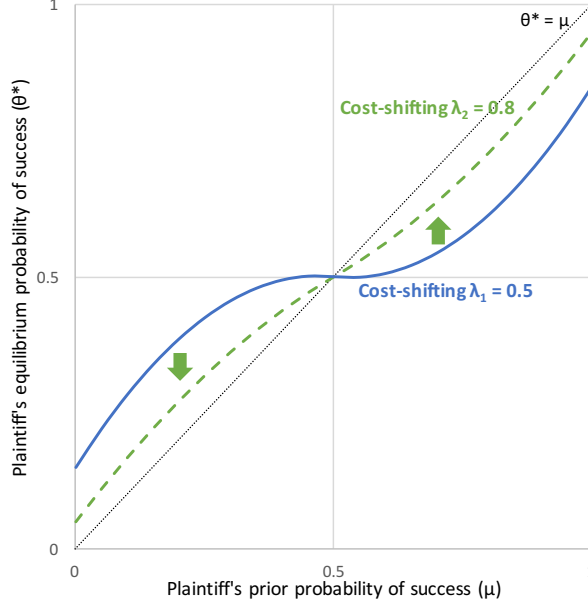


Figure 3: Plaintiff's equilibrium probabilities of success as functions of her prior probability of success under two cost-shifting rules, $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$, in the $\mathbb{L}_k$ Game.

the relationship between Plaintiff's equilibrium probability of success θ^* and her prior probability of success μ under two cost-shifting rules $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$. The blue solid curve (respectively, green dashed curve) depicts θ^* as a function of μ when $\lambda_1 = 0.5$ (respectively, $\lambda_2 = 0.8$). For all $\mu \neq 0.5$, the value of θ^* on the green dashed curve is closer to μ compared to that on the blue solid curve.

2.7 Cost Shifting Affects Expenditure in Litigated Cases

This section considers the effects of cost shifting on litigation costs in the nontrivial Nash equilibrium (e_P^*, e_D^*) . **Litigation expenditure** (in equilibrium), denoted C^* , is defined as the sum of Plaintiff and Defendant's respective litigation costs in equilibrium

$$C^* = C(e_P^*) + C(e_D^*). \quad (12)$$

The present definition of litigation expenditure only represents the litigation costs borne by those litigants who proceed to litigation. This definition does not include the public costs borne by the judicial system or the society at large, such as the costs of providing

$0 < s \leq 1$ imply

$$s \frac{\partial^2 \theta_L}{\partial s^2} + \left[1 - \frac{k(1-s^k)}{1+s^k} \right] \frac{\partial \theta_L}{\partial s} \geq 0,$$

holding strictly if $k > 1$. Hence condition (10) holds for all $\lambda \in [0, 1]$ and all $k \geq 1$, implying that $\theta_L \in \Theta_8([0, 1], [1, +\infty))$.

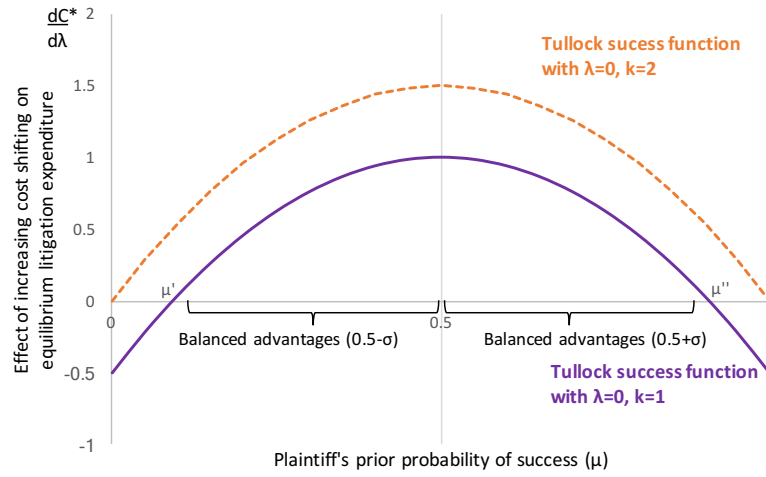


Figure 4: How equilibrium litigation expenditure responds to more cost shifting in the $\mathbb{T}_1$ Game and $\mathbb{T}_2$ Game, where the American rule ($\lambda = 0$) is the baseline rule.

judges to adjudicate cases, running and maintaining courts and enforcing judgments. Nor does this definition attempt to capture how private (and public) litigation costs change in response to decisions to bring suit, contest suit, or settle. A more comprehensive (and complex) model that includes the society's perspective on the costs and benefits of litigation is required to resolve issues regarding the optimal balance between the litigants' private interests and the interests of the society. These issues, and those that section 2.10 below will identify, cannot be resolved without a comprehensive and robust analysis of how cost-shifting rules affect private litigation expenditure. The present section offers that analysis.

Assumptions 1-6 are not sufficient for answering the question whether more cost shifting increases litigation expenditure in every case. To see this, consider Figure 4. For each value of Plaintiff's prior probability of success μ and under the American rule (that is, $\lambda = 0$), Figure 4 depicts how litigation expenditure responds to infinitesimally more cost shifting (that is, $\frac{dC^*}{d\lambda}$). The purple solid curve represents the $\mathbb{T}_1$ Game, which has a linear cost function characterized by $k = 1$. The orange dashed curve represents the $\mathbb{T}_2$ Game, which has a strictly convex cost function characterized by $k = 2$. Each of these Games adopts the Tullock success function θ_T given by (6). Consider the $\mathbb{T}_1$ Game first. In cases characterized by sufficiently balanced relative advantages (here, cases with μ satisfying $\mu' < \mu < \mu''$), more cost shifting increases litigation expenditure (that is, $\frac{dC^*}{d\lambda} > 0$). In cases characterized by extreme relative advantages (here, cases with $\mu < \mu'$ or $\mu > \mu''$), more cost shifting *decreases* litigation expenditure (that is, $\frac{dC^*}{d\lambda} < 0$).³⁴ In

³⁴In the $\mathbb{T}_1$ Game, the result that (equilibrium) litigation expenditure decreases with the cost-shifting rule in extreme cases differs from what is often found in the existing literature. The reason for the present result

borderline cases characterized by $\mu = \mu'$ or $\mu = \mu''$, more cost shifting does not affect litigation expenditure (that is, $\frac{dC^*}{d\lambda} = 0$). However, in the $\mathbb{T}_2$ Game, more cost shifting increases litigation expenditure in *all* cases.

2.7.1 Sufficiently Balanced Relative Advantages or Sufficiently Convex Cost Functions

Motivated by the special cases depicted in Figure 4, this subsection considers the effect of cost shifting on (equilibrium) litigation expenditure in cases characterized by sufficiently balanced relative advantages or sufficiently convex cost functions. As a preliminary, Corollary 8 characterizes the sufficient and necessary condition for litigation expenditure to be increasing with the proportion of costs recoverable.

Corollary 8. *Consider the nontrivial Nash equilibrium. Litigation expenditure C^* is increasing with the cost-shifting rule λ if and only if the following condition holds:*

$$-\frac{(2\theta - 1)s\theta_{ss}}{\theta_s} > (2\theta - 1) \left[1 - \frac{k\lambda(2\theta - 1)}{2 - \lambda} \right] + \frac{\alpha(2 - \lambda)s\theta_s}{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)]} - \frac{\alpha}{2 - \lambda} \quad (13)$$

where $s = s^*$ given by Lemma 2.

Corollary 8 identifies condition (13) as the sufficient and necessary condition for more cost shifting to increase litigation expenditure. Condition (13) requires that the relative curvature of the success function θ with respect to effort ratio (that is, $-\frac{\theta_{ss}}{\theta_s}$) to be sufficiently large in equilibrium. Corollaries 9 and 10 will use condition (13) and Lemma 2 to ascertain how cost shifting affects litigation expenditure given sufficiently balanced advantages or sufficiently convex cost functions.

To facilitate presentation, define a function $\sigma : [0, 1] \times [1, +\infty) \rightarrow (0, 0.5]$ by³⁵

$$\sigma(\lambda, k) = \max \{ \mu \in [0, 1] \mid \theta^* \leq (3 - \lambda)/(4 - \lambda) \} - 0.5.$$

Corollary 9. *Consider two cost-shifting rules $0 < \lambda_1 < \lambda_2 \leq 1$, where the success function $\theta \in \Theta(\{\lambda_2\}, \{k\})$ and (equilibrium) litigation expenditure is denoted C_1^* under*

is that neither the Litigation Game nor the proof of its equilibrium assumes $\frac{de_P}{d\lambda} = 0$, while some authors do (for example, Fenn et al. 2017 at 147-48).

³⁵To see that the function $\sigma(\cdot)$ exists and $0 < \sigma(\lambda, k) \leq 0.5$, first fix a pair of cost-shifting rule λ and cost function k and use part 1 of Proposition 1 to obtain that $\mu = 0.5$ implies $\theta^* = 0.5$ in equilibrium, which in turn implies $\theta^*(4 - \lambda) > 3 - \lambda$. Then the property $\frac{d\theta^*}{d\mu} > 0$ from Corollary 3 implies a one-to-one relationship between μ and θ^* . Hence there exists at most one $0.5 < \mu'' \leq 1$ that induces $\theta^*(4 - \lambda) = 3 - \lambda$, and that all $0.5 < \mu < \min\{1, \mu''\}$ induces $\theta^*(4 - \lambda) < 3 - \lambda$.

λ_1 and C_2^* under λ_2 . Suppose relative advantages are sufficiently balanced in the precise sense of $0.5 - \sigma(\lambda_2, k) \leq \mu \leq 0.5 + \sigma(\lambda_2, k)$. Then increasing the proportion of costs recoverable from λ_1 to λ_2 increases litigation expenditure. Formally, $0.5 - \sigma(\lambda_2, k) \leq \mu \leq 0.5 + \sigma(\lambda_2, k)$ implies $C_2^* > C_1^*$.

Corollary 9 proves that if relative advantages of the litigants are sufficiently balanced, then more cost shifting increases litigation expenditure. Intuitively, more cost shifting increases litigation expenditure if both litigants exert more efforts in equilibrium, or if one litigant's exertion of additional effort is not offset by a more rapid reduction in effort by the other litigant. More cost shifting reduces a litigant's expected marginal cost by allowing a greater recover of her costs if she wins. By increasing the recoverable-costs part of the "prize", more cost shifting also widens the difference in monetary outcome between winning and losing. A litigant must have very poor prospects of success to reduce equilibrium effort — which further harms her prospects of success — in order to save costs. In cases characterized by sufficiently balanced relative advantages, no litigant has very poor prospects of success. Hence, in these cases, more cost shifting incentivizes the litigant collectively to exert more equilibrium efforts. The function $\sigma(\cdot)$ defines what is required for relative advantages to be "sufficiently balanced" in this sense. As a function of the applicable cost-shifting rule λ and cost function k , $\sigma(\cdot)$ marks the upper and lower bounds within which the prior — being the parameter that represents relative advantages — is considered sufficiently balanced.

Corollary 10. Consider two cost-shifting rules $0 \leq \lambda_1 < \lambda_2 \leq 1$, where the success function $\theta \in \Theta(\{\lambda_2\}, \{k\})$ and (equilibrium) litigation expenditure is denoted C_1^* under λ_1 and C_2^* under λ_2 . If the cost function is sufficiently convex in the sense that its degree of homogeneity $k \geq 2$, then increasing the proportion of costs recoverable from λ_1 to λ_2 increases litigation expenditure. Formally, $k \geq 2$ implies $C_2^* > C_1^*$.

Corollary 10 proves that if the cost function is sufficiently convex, then more cost shifting increases litigation expenditure (in equilibrium). This holds even in extreme cases which fall outside the scope of Corollary 9 due one litigant having very favorable prior probability of success. Hence Corollaries 9 and 10 together provide general conditions under which more cost shifting increases litigation expenditure. This result expands a finding in the existing literature, that the English rule (full recovery of the winner's costs) encourages greater legal expenditure in litigated cases than the American rule (no recovery

of the winner's costs) does.³⁶

2.7.2 Extreme Relative Advantages and Insufficiently Convex Cost Functions

As a result of Corollaries 9 and 10, only in exceptional cases characterized by very one-sided prior and insufficiently convex cost functions may it be possible for litigation expenditure to be nonincreasing with the proportion of costs recoverable. We now propose an additional condition, captured by Assumption 9, that is sufficient for concluding that even in these exceptional cases, more cost shifting increases litigation expenditure.

Assumption 9. *Suppose the prior is very favorable to Plaintiff in the sense that $\mu > 0.5 + \sigma(\lambda, k)$, and the cost function is insufficiently convex in the sense that $k < 2$. If Plaintiff's effort is no less than some positive effort by Defendant (that is, $0 < s \leq 1$), then one of the following condition holds:*

$$-\frac{(2\theta - 1)s\theta_{ss}}{\theta_s} > (2\theta - 1) \left[1 - \frac{k\lambda(2\theta - 1)}{2 - \lambda} \right] + \frac{\alpha(2 - \lambda)s\theta_s}{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)]} - \frac{\alpha}{2 - \lambda} \quad (14)$$

or

$$-\frac{(2\theta - 1)s\theta_{ss}}{\theta_s} > (2\theta - 1) \left[1 - \frac{k(1 - s^k)}{1 + s^k} \right] + \frac{\beta(1 + s^k)^2 s\theta_s}{k(2 - \lambda)s^k} - \frac{\beta}{2 - \lambda}. \quad (15)$$

Assumption 9 requires the relative curvature of the success function with respect to effort ratio (that is, $-\frac{\theta_{ss}}{\theta_s}$) to be sufficiently large. For example, the $\mathbb{L}_k$ Game, which adopts the success function θ_L defined in (11) in subsection 2.6.2, satisfies Assumption 9.³⁷

To facilitate presentation, let $\Theta_9(\Lambda, K)$ denote the set of twice continuously differentiable functions $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfy Assumption 9 when the applicable cost-shifting rule and cost function are characterized some $\lambda \in \Lambda \subset [0, 1]$ and some $k \in K \subset [0, +\infty)$ respectively.

Proposition 4. *Consider two cost-shifting rules $0 \leq \lambda_1 < \lambda_2 \leq 1$, where the success function $\theta \in \Theta(\{\lambda_2\}, \{k\}) \cap \Theta_9([\lambda_1, \lambda_2], \{k\})$ and (equilibrium) litigation expenditure is*

³⁶For example, Braeutigam, Owen, and Panzar (1984), Katz (1987) and Plott (1987).

³⁷Using Appendix A.2, some algebra will reveal that $\mu \geq 0.5$ and $0 < s \leq 1$ implies

$$-s \frac{\partial^2 \theta_L}{\partial s^2} \bigg/ \frac{\partial \theta_L}{\partial s} \geq 1 - \frac{k(1 - s^k)}{1 + s^k}.$$

The property $\theta_s < 0$ from Lemma 9, a technical Lemma in Appendix A.1, implies $\theta_L \in \Theta_9([0, 1], [1, +\infty))$. Hence the $\mathbb{L}_k$ Game satisfies Assumption 9 under any cost-shifting rule $0 \leq \lambda \leq 1$ and any cost function $k \geq 1$.

C_1^* under λ_1 and C_2^* under λ_2 . Then increasing the proportion of costs recoverable from λ_1 to λ_2 increases litigation expenditure. Formally, $\theta \in \Theta(\{\lambda_2\}, \{k\}) \cap \Theta_9([\lambda_1, \lambda_2], \{k\})$ implies $C_2^* > C_1^*$.

Proposition 4 proves that adding Assumption 9 is sufficient for concluding that in all cases, more cost shifting increases litigation expenditure. This holds even if one litigant has very favorable relative advantages and the cost function is insufficiently convex.

2.8 Remarks on Generality

2.8.1 Multiple Judges

This subsection demonstrates that the Litigation Game as formulated in section 2.2 captures uncertainty in respect of the judge who hears and decides the case.

Consider the following modification of the Litigation Game, called the **Litigation Game with Multiple Judges**. Suppose that a judge chosen from a finite collection of $n \geq 1$ judges will hear the dispute. A judge denoted $i \in \{1, 2, \dots, n\}$ rules in favor of Plaintiff with posterior probability $\theta_i(e_P, e_D; \mu)$ where $\theta_i \in \Theta(\{\lambda\}, \{k\})$, and rules in favor of Defendant with probability $1 - \theta_i(e_P, e_D; \mu)$. Before the identity of the judge is revealed, the litigants observe Plaintiff's relative advantages μ , the cost function k and cost-shifting rule λ , and exert effort levels e_P, e_D . It is common knowledge that the litigants assign the prior probability $0 \leq p_i \leq 1$ to judge i being chosen, where $\sum_{i=1}^n p_i = 1$. Plaintiff and Defendant's payoff functions are respectively $\tilde{u}_P, \tilde{u}_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by

$$\tilde{u}_P = \mathbb{E}\{\theta_i[1 - (1 - \lambda)C(e_P)] - (1 - \theta_i)[C(e_P) + \lambda C(e_D)]\},$$

$$\tilde{u}_D = \mathbb{E}\{-\theta_i[1 + C(e_D) + \lambda C(e_P)] - (1 - \theta_i)(1 - \lambda)C(e_D)\}$$

where $\mathbb{E}$ is the expectation operator with respect to $(p_i)_{i=1}^n$.

Remark 7. *In reality, some courts disclose the identity of the judicial officer randomly assigned to the case only late in the litigation process, sometimes on the day of the hearing. This practice is justified on grounds including promotion of judicial independence and impartiality as well as discouragement of "judge shopping".³⁸ Under this practice, the litigants only find out about the identity of the judge after they have exerted significant*

³⁸For example, some courts in Australia and Europe follow this practice. See, generally, Wallace et al. (2014), especially pp. 687-89.

litigation efforts. The Litigation Game with Multiple Judges captures this practice by specifying that the litigants exert efforts before they observe the identity of the judge.

The Litigation Game with Multiple Judges is captured by a special case of the original Litigation Game formulated in section 2.2. To see this, construct a success function $\theta = \sum_{i=1}^n p_i \theta_i$. Lemma 3 establishes that $\theta \in \Theta(\{\lambda\}, \{k\})$.

Lemma 3. *Consider a finite collection of success functions $\theta_1, \theta_2, \dots, \theta_n \in \Theta(\{\lambda\}, \{k\})$. If a success function θ is their convex combination, then $\theta \in \Theta(\{\lambda\}, \{k\})$. Formally, for weights $p_1, p_2, \dots, p_n \geq 0$ satisfying $\sum_{i=1}^n p_i = 1$, that $\theta_1, \theta_2, \dots, \theta_n \in \Theta(\{\lambda\}, \{k\})$ and $\theta = \sum_{i=1}^n p_i \theta_i$ implies $\theta \in \Theta(\{\lambda\}, \{k\})$.*

An application of Lemma 3 proves that a special case of the original Litigation Game that adopts the success function $\theta = \sum_{i=1}^n p_i \theta_i$ falls within the scope of Assumptions 1-6.

Now, some algebra using Plaintiff's payoff in the Litigation Game with Multiple Judges and the linearity of the expectation operator reveals

$$\begin{aligned} \tilde{u}_P &= \mathbb{E}\{\theta_i[1 + \lambda C(e_P) + \lambda C(e_D)] - C(e_P) - \lambda C(e_D)\} \\ &= \sum_{i=1}^n (p_i \theta_i)[1 + \lambda C(e_P) + \lambda C(e_D)] - C(e_P) - \lambda C(e_D) \\ &= \theta[1 + \lambda C(e_P) + \lambda C(e_D)] - C(e_P) - \lambda C(e_D) = u_P \end{aligned}$$

where u_P is Plaintiff's payoff (given by (2)) in the (original) Litigation Game formulated in section 2.2. Similarly, obtain $\tilde{u}_D = u_D$ (given by (3)) for Defendant, where u_D is Defendant's payoff in the Litigation Game.

Hence the Litigation Game with Multiple Judges is a special case of the (original) Litigation Game that adopts the success function $\theta = \sum_{i=1}^n p_i \theta_i$. Then the Litigation Game with Multiple Judges has a unique nontrivial Nash equilibrium as established by Proposition 1, and attracts the previous analyses of equilibrium properties of the Litigation Game.

2.8.2 Arbitrary Judgment Sum

This subsection demonstrates that the Litigation Game as formulated in section 2.2 captures any positive judgment sum.

Consider the following modification of the Litigation Game, called the **Litigation Game with Arbitrary Judgment Sum**. Suppose the judgment sum which may be

awarded to Plaintiff is characterized by an exogenous parameter $J > 0$. The cost function $\bar{C} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is homogenous of degree $\bar{k} \geq 1$. Plaintiff and Defendant respectively have payoff functions $\bar{u}_P, \bar{u}_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ where

$$\begin{aligned}\bar{u}_P &= \theta[J - (1 - \lambda)\bar{C}(e_P)] - (1 - \theta)[\bar{C}(e_P) + \lambda\bar{C}(e_D)], \\ \bar{u}_D &= -\theta[J + \bar{C}(e_D) + \lambda\bar{C}(e_P)] - (1 - \theta)(1 - \lambda)\bar{C}(e_D).\end{aligned}$$

The Litigation Game with Arbitrary Judgment Sum is captured by the original Litigation Game as formulated in subsection 2.2. To see this, consider the original Litigation Game with the judgment sum 1 and a cost function $C(\cdot)$ defined by the degree of homogeneity $k = \bar{k}$ and $C(1) = \bar{C}(1)/J$. From equation (2), Plaintiff's payoff function in this game is:

$$\begin{aligned}u_P &= \theta[1 - (1 - \lambda)C(e_P)] - (1 - \theta)[C(e_P) + \lambda C(e_D)] \\ u_P &= \theta \left[1 - \frac{(1 - \lambda)\bar{C}(1)e_P^k}{J} \right] - (1 - \theta) \left[\frac{\bar{C}(1)e_P^k}{J} + \frac{\lambda\bar{C}(1)e_D^k}{J} \right] \\ \Leftrightarrow Ju_P &= \theta[J - (1 - \lambda)\bar{C}(1)e_P^k] - (1 - \theta)[\bar{C}(1)e_P^k + \lambda\bar{C}(1)e_D^k] = \bar{u}_P\end{aligned}$$

where $\bar{u}_P$ is Plaintiff's payoff function in the Litigation Game with Arbitrary Judgment Sum. Similar steps establish $Ju_D = \bar{u}_D$, where u_D is Defendant's payoff function in the original Litigation Game.

Hence each litigant's payoff function in the Litigation Game with Arbitrary Judgment Sum is a positive affine transformation of her payoff function in the original Litigation Game that adopts the judgment sum 1 and the cost function $C(\cdot)$ characterized by $k = \bar{k}$ and $C(1) = \bar{C}(1)/J$. Then the Litigation Game with Arbitrary Judgment Sum has a unique nontrivial Nash equilibrium as established by Proposition 1, and attracts the same analyses of equilibrium properties as those that apply to the original Litigation Game.

2.8.3 How Changes in Relative Advantages Affect Expenditure

A conventional wisdom in the existing literature on Tullock contest models is a more asymmetric contest reduces rent dissipation, thereby decreases the total costs of exerting efforts.³⁹ This subsection shows that this conventional wisdom does not necessary

³⁹For such a result in a Tullock contest with asymmetric technologies, see Cornes and Hartley (2005) (at pp. 940-41). For such a result in a model of litigation, see Carbonara et al. (2015) (at pp. 6-7).

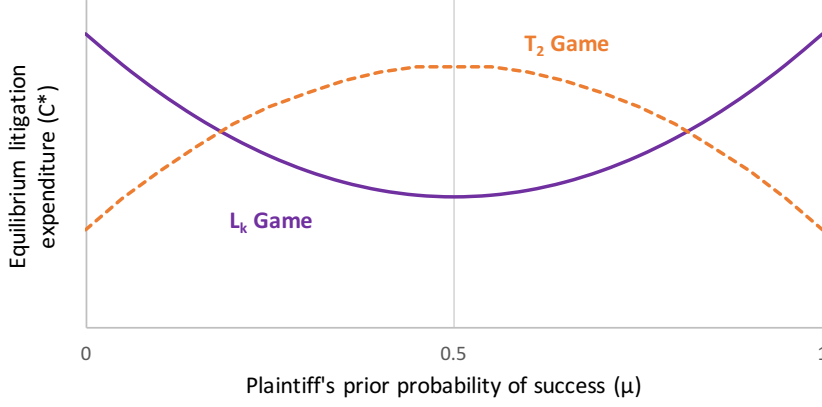


Figure 5: Litigation expenditure as a function of Plaintiff's prior probability of success in the $\mathbb{L}_k$ Game and $\mathbb{T}_2$ Game.

hold in the Litigation Game. It further proposes a sufficient condition that ensures this conventional wisdom holds.

In the Litigation Game, the contest between the litigants becomes more asymmetric as the prior becomes more one-sided. The notion of litigation expenditure sums their total costs of exerting efforts (see section 2.7). Our previous assumptions are not sufficient for answering the question whether litigation expenditure increases or decreases when the prior becomes more one-sided. Figure 5 illustrates potential complexities. Figure 5 depicts litigation expenditure (C^*) as a function of Plaintiff's prior probability of success μ in the $\mathbb{L}_k$ Game and in another special case of the Litigation Game, called the $\mathbb{T}_2$ Game, that adopts the Tullock success function θ_T given by (6) and a homogeneous cost function of degree $k = 2$. The purple solid curve represents the $\mathbb{L}_k$ Game and the orange dashed curve the $\mathbb{T}_2$ Game. In the $\mathbb{L}_k$ Game, litigation expenditure increases when the prior becomes more one-sided. By contrast, in the $\mathbb{T}_2$ Game, litigation expenditure decreases when the prior becomes more one-sided.

Motivated by Figure 5, we propose Assumption 10 as a sufficient condition for ensuring that litigation expenditure increases when the prior becomes more one-sided

Assumption 10. *Suppose Plaintiff's case is relatively more advantageous (that is, $\mu > 0.5$) and her effort is no less than Defendant's positive effort (that is, $0 < s \leq 1$). Then at least one of the following conditions holds:*

$$\lambda(2 - \lambda)\theta_\mu \left\{ s\theta_{ss} + \left[1 - \frac{k(1 - s^k)}{(1 + s^k)} \right] \theta_s \right\} \geq \beta\theta_{s\mu} \quad (16)$$

or

$$\lambda(2 - \lambda)\theta_\mu \left\{ s\theta_{ss} + \left[1 - \frac{k\lambda(2\theta - 1)}{2 - \lambda} \right] \theta_s \right\} \geq \alpha\theta_{s\mu}. \quad (17)$$

Assumption 10 imposes restrictions on the curvature of the success function θ with respect to Defendant's relative effort s and Plaintiff's prior probability of success parameter μ . For example, the success function θ_L given by (11), which the $\mathbb{L}_k$ Game adopts, satisfies condition (16) strictly (respectively, with equality) if the applicable pair of cost-shifting rule and cost function satisfies $\lambda > 0$ and $k > 1$ (respectively, $\lambda = 0$ or $k = 1$).

Proposition 5 proves adding Assumption 10 is sufficient for concluding that as the prior becomes more favorable to one litigant, litigation expenditure does not decrease. To facilitate presentation, let $\Theta_{10}(\{\lambda\}, \{k\})$ denote the set of twice continuously differentiable functions $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfy Assumption 10 when the applicable pair of cost-shifting rule and cost function is characterized by (λ, k) .

Proposition 5. *Suppose the success function $\theta \in \Theta(\{\lambda\}, \{k\}) \cap \Theta_{10}(\{\lambda\}, \{k\})$. Then as the prior becomes more favorable to one litigant, (equilibrium) litigation expenditure does not decrease; it increases if condition (16) or (17) holds strictly. Formally, that $\mu > 0.5$ (respectively, $\mu < 0.5$) implies $\frac{dC^*}{d\mu} \geq 0$ (respectively, $\frac{dC^*}{d\mu} \leq 0$), holding strictly if condition (16) or (17) holds strictly.*

Proposition 5 identifies a class of success functions — those that satisfy Assumption 10 in addition to Assumptions 1-6 — under which litigation expenditure weakly decreases when the prior becomes more balanced. Any one of these success functions violates the conventional wisdom that a more asymmetric contest decreases the total costs of exerting efforts.

We now propose alternative conditions the satisfaction of which is sufficient to uphold this conventional wisdom. Consider Assumption 11, which imposes restrictions on the curvature of the success function θ with respect to Defendant's relative effort s and Plaintiff's prior probability of success μ .

Assumption 11. *Suppose Plaintiff's case is relatively more advantageous (that is, $\mu > 0.5$) and her effort is no less than some positive effort by Defendant (that is, $0 < s \leq 1$). Then at least one of the following conditions holds:*

$$\lambda(2 - \lambda)\theta_\mu \left\{ s\theta_{ss} + \left[1 - \frac{k(1 - s^k)}{(1 + s^k)} \right] \theta_s \right\} \leq \beta\theta_{s\mu} \quad (18)$$

or

$$\lambda(2 - \lambda)\theta_\mu \left\{ s\theta_{ss} + \left[1 - \frac{k\lambda(2\theta - 1)}{2 - \lambda} \right] \theta_s \right\} \leq \alpha\theta_{s\mu}. \quad (19)$$

Assumption 11 captures the Tullock success function θ_T given by (6). Some algebra reveals that θ_T satisfies condition (19) strictly (respectively, with equality) if the applicable cost function is characterized by $\lambda < 1$ (respectively, $\lambda = 1$)

Proposition 6 proves that adding Assumption 10 is sufficient for concluding that as the prior becomes more favorable to one litigant, litigation expenditure does not increase. To facilitate presentation, let $\Theta_{11}(\{\lambda\}, \{k\})$ denote the set of twice continuously differentiable functions $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfy Assumption 10 when the applicable pair of cost-shifting rule and cost function is characterized by (λ, k) .

Proposition 6. *Suppose the success function $\theta \in \Theta(\{\lambda\}, \{k\}) \cap \Theta_{11}(\{\lambda\}, \{k\})$. Then as the prior becomes more favorable to one litigant, (equilibrium) litigation expenditure does not increase; it decreases if condition (18) or (19) holds strictly. Formally, if $\mu > 0.5$ (respectively, $\mu < 0.5$), then $\frac{dC^*}{d\mu} \leq 0$ (respectively, $\frac{dC^*}{d\mu} \geq 0$), holding strictly if condition (18) or (19) holds strictly.*

Proposition 6 proves that if the success function satisfies Assumption 10 in addition to Assumptions 1-6, then litigation expenditure weakly increases when the prior becomes more balanced. Such a success function respects the conventional wisdom that a more asymmetric contest decreases the total costs of exerting efforts.

2.9 Remarks on Settlement

The Litigation Game can offer valuable insights on the litigants' choices between settlement and litigation. To provide a comprehensive analysis of litigation efforts in a general and robust model, the present paper largely abstracts away from pre-litigation behaviors and the society's perspective on litigation. For instance, outside the present scope is a comprehensive treatment of settlement negotiations to divide the saved litigation expenditure. Assuming that litigation takes place, Corollaries 9-10 and subsection 2.8.3 reveal how changes in the parameters affect litigation expenditure in equilibrium. Using these results, this section will offer several remarks on settlement. These remarks are necessarily tentative because the present focus is on litigation rather than pre-litigation behaviors, and because we make the assumptions that the litigants are risk-neutral and non-

emotional players in a one-shot game (see section 2.2). We also assume that the litigants have sufficiently large budgets, so their equilibrium efforts are the result of unconstrained optimization.⁴⁰

2.9.1 Settlement Range

If the litigants can negotiate for a settlement before they decide whether to litigate, then the Litigation Game gives the range of acceptable settlement amounts. Assuming that Plaintiff settles if she is indifferent between settling or litigating, the lower bound of the range of acceptable settlement offers occurs when Defendant makes a take-it-or-leave-it offer which is equal to Plaintiff's equilibrium payoff in proceeding to litigation, that is, $u_P(e_P^*, e_D^*; \mu, \lambda, k)$. The upper bound occurs when Plaintiff makes a take-it-or-leave-it offer which is equal to the magnitude of Defendant's equilibrium payoff in litigation, that is, $-u_D(e_P^*, e_D^*; \mu, \lambda, k)$. The range of mutually acceptable settlement amounts is the closed interval $[u_P(e_P^*, e_D^*; \mu, \lambda, k), -u_D(e_P^*, e_D^*; \mu, \lambda, k)]$. The length of this interval is

$$-u_D(e_P^*, e_D^*; \mu, \lambda, k) - u_P(e_P^*, e_D^*; \mu, \lambda, k) = -U^* = C^* > 0.$$

Hence settlement and avoidance of litigation will generate surplus to the litigants. Corollaries 9-10 reveal that more cost shifting increases litigation expenditure (C^*) at least in cases where the litigants' relative advantages are sufficiently balanced or the cost function is sufficiently convex. In these cases, more cost shifting by increasing the size of the surplus arising from settlement disincentivizes the litigants from bringing their case to litigation. In other cases, how cost shifting affects the settlement surplus would depend on the specific properties of the success function and of the cost function (see subsections 2.7.2). How the size of the surplus is shared between the litigants would depend on their relative bargaining powers.

2.9.2 Relative Merits of Suits that Proceed to Litigation

A frequently-obtained result in the existing literature is that by increasing the stakes of proceeding to litigation, cost shifting can discourage unmeritorious suits. Underlying this result is the assumption that a greater litigation expenditure dampens incentives to proceed to litigation (see Spier 2007 at pp. 264-265, Katz and Sanchirico 2012 at pp. 278-280).

⁴⁰Budget constraints sometimes can give contestants incentives to settle their contests by making side-payments, even when those side-payments are not enforceable in a court of law. Compare Beviá and Corchón (2010) and Kimbrough and Sheremeta (2013).

Maintaining this assumption, this subsection will reveal the functional form of the success function also matters. Following the typical practice in the existing contest models of litigation (for example, Farmer and Pecorino 1999 at pp. 272-274, Carbonara et al. 2015 at pp. 118-120), let the relative-advantages parameter μ in the Litigation Game represent Plaintiff's relative merits, and $1 - \mu$ Defendant's relative merits. As μ approaches 0.5, the case becomes more balanced.

Subsection 2.8.3 considers the relationship between μ and (equilibrium) litigation expenditure, C^* . If the litigants can avoid proceeding to litigation, then they collectively save C^* . Depicted in Figure 5 is the $\mathbb{T}_2$ Game, in which C^* increases when the case becomes more balanced (μ approaches 0.5). This suggests that the litigants have less incentives to proceed to litigation when the case becomes more balanced. Thus, adopting the Tullock success function given by (6) and a quadratic cost function ($k = 2$), the $\mathbb{T}_2$ Game suggests that relatively extreme cases (μ approaches 0 or 1) are more likely to proceed to litigation.

However, also depicted in Figure 5 is the $\mathbb{L}_k$ Game, in which C^* increases when the case becomes more *extreme* (μ approaches 0 or 1). Thus, adopting the linear success function given by (11), the $\mathbb{L}_k$ Game suggests that relatively balanced (μ approaches 0.5) are more likely to proceed to litigation. This result, which holds in the special case of a quadratic cost function ($k = 2$), is in stark contrast to what the $\mathbb{T}_2$ Game suggests.

Hence how the litigants' relative merits affect their incentives to proceed to litigation critically depends on the functional form of the success function. This observation highlights the need for a robust model of litigation and the value of predictions that are premised on general assumptions rather than particular functional forms.

2.9.3 Incentives to File or Defend the Case

A litigant's incentives to file or defend a case can depend on her **participation constraint**, in the sense of proceeding to litigation gives her a better payoff than not filing or defending the case at all (Farmer and Pecorino 1999 at p. 276, Carbonara et al. 2015 at pp. 120, 125). In the Litigation Game, if we assume that Plaintiff obtains zero payoff upon not filing her case, then her participation constraint is

$$u_P(e_P^*, e_D^*; \mu, \lambda, k) > 0, \quad (20)$$

which captures the intuition that she would proceed to litigation only if she could not do better by not filing her case. Similarly, if we assume that Defendant can pay the judgment sum 1 upon not defending the case against her, then her participation constraint is

$$u_D(e_P^*, e_D^*; \mu, \lambda, k) > -1, \quad (21)$$

which captures the intuition that she would proceed to litigation only if she could not do better by not defending the case.

Specifying the Tullock success function θ_T given by (6), Farmer and Pecorino (1999) and Carbonara et al. (2015) reveal that how cost shifting affects the litigants' incentives to proceed to litigation critically depends on their relative merits and legal technologies. To use the notation of the present Litigation Game, let the relative-advantages parameter μ and the degree of homogeneity k of the cost function respectively capture relative merits and legal technologies. Among the findings of Farmer and Pecorino (at pp. 279-280) is that for some values of k , under the American rule ($\lambda = 0$), Plaintiff would file suit only if μ is sufficiently large, and Defendant would defend only if μ is sufficiently small. They (at p. 284) predict similar outcomes under the English rule ($\lambda = 1$) for some other values of k . Moreover, Carbonara et al. (2015) discover a similar result under intermediate cost-shifting rules that limit the quantity of costs recoverable.⁴¹ They (at pp. 132-133) establish that as the limiting quantity increases, fewer cases characterized by an extreme μ would proceed to litigation, and whether litigation would eventually cease depends on k . A well-known result following from these findings is that in many cases, more cost shifting tends to disincentivise the litigation of "one-sided" cases characterized by extreme relative merits.⁴² Dari-Mattiacci and Saraceno (2017) recently obtain a similar result with a settlement model in which each litigant has private information about her own evidence.

Without specifying the success function, section 2.7 of this paper reaches a similar conclusion. To see this, suppose the litigants' participation constraints (20) and (21) are satisfied under some initial proportion of cost-shifting, $\lambda = \lambda_0$ where $0 \leq \lambda_0 \leq 1$. Then an infinitesimal increase in λ (from λ_0 to $\lambda_0 + \delta\lambda$ for a very small $\delta\lambda > 0$) would not lead to violation of the participation constraints. Corollary 9 predicts that for cases with sufficiently balanced relative merits (μ sufficiently close to 0.5), the increase in λ would lead to greater litigation expenditure (C^* increases). This encourages the litigants to settle

⁴¹The quantity formulation of cost-shifting rules is a special case of the proportion formulation in the Litigation Game. See footnote 12.

⁴²See Carbonara et al. (2015) (at pp. 134-137) for a discussion of the normative aspects of this result.

rather than litigate. Corollary 10 makes a similar prediction if k is sufficiently high. These predictions confirm the robustness of the well-known result that more cost shifting often discourages the litigation of cases with extreme relative merits.

2.10 Normative Discussion

This section discusses the normative implications and limitations of our positive predictions regarding how cost shifting affects legal predictability, accuracy and expenditure. Fixing the pre-litigation behaviors (for example, the injurious activity in a tort case), section 2.2 develops a contest model of litigation to analyze the litigants' strategic interaction under different cost-shifting rules governing the allocation of litigation costs. This Litigation Game generalizes large classes of success functions, cost functions and cost-shifting rules. In particular, the present characterization of cost-shifting rules covers the extreme ones that shift either all or none of the winner's costs to the loser, as well as the intermediate ones that shift a proportion of such costs. Premising on the unique nontrivial Nash equilibrium that Section 2.3 finds and characterizes, the positive predictions of the Litigation Game are general and thus facilitate a normative analysis of whole classes of judicial systems. Yet the Litigation Game is about efforts to litigate; it does not model every behavior that is or should be subject to legal regulation. The following thus elicits the normative implications of cost shifting to the extent that litigation efforts are the only variables, and discusses how these results facilitate future research into other normatively relevant variables.

First, cost shifting affects the policy objective of improving predictability in the judicial determination of litigated cases. This policy is particularly relevant to commercial litigants. Vague standards (such as "reasonableness"), the open texture of language and judicial discretion are among the factors that render imperfectly predictable the application of substantive law in a case and therefore the outcome of the case. From a utilitarian perspective, improved legal predictability better enables individuals and businesses to make plans for the future. Legal predictability is also attractive to the liberal ideal of the rule of law because it gives fair notice to individuals of the legal consequences of their choices and holds public officials accountable for their exercises of public powers.⁴³

⁴³For a survey of the philosophical literatures on the rule of law and its relationship with legal predictability and accessibility, see *The Rule of Law* (June 22, 2016), Stanford Encyclopaedia of Philosophy, <https://plato.stanford.edu/entries/rule-of-law/>. For a survey of the seminal papers on the complex relationship between high probability suits induced by cost shifting and the substantive behaviors giving rise to the dispute, see Katz and Sanchirico (2012), pp. 278-86.

Fixing the pre-litigation behaviors, Corollary 5 in section 2.5 establishes that more cost shifting unambiguously drives equilibrium litigation efforts to improve legal predictability. This result reveals the desirability of cost shifting to the extent that litigation efforts are the only variables and the society aims to improve legal predictability. Future research may consider a pre-game in which the litigants choose their pre-litigation behaviors, and study how more cost shifting by improving legal predictability in the Litigation Game affects those choices.

Secondly, there is a complex relationship between cost-shifting rules and the policy objective of deciding cases accurately to reflect their inherent merits. Using a standard model of tort liability for harmful activities, Kaplow and Shavell (1996) illustrate that greater accuracy ameliorates the misalignment between the levels of precautions that informed injurers take and the magnitude of the harm that they are likely to generate.⁴⁴ However, they also reveal that greater accuracy has limited incentive-alignment effects when the injurers are not informed. They fix the amount of litigation costs and vary the pre-litigation behaviors (levels of precautions and decisions to learn about harm), while we fix the pre-litigation behaviors and vary litigation costs. We nonetheless reach a similarly complex conclusion regarding how cost-shifting rule by determining equilibrium litigation efforts (and costs) affects accuracy. Assuming that a litigant's prior probability of success (that is, her relative advantages before exerting efforts, see section 2.2) accurately reflects the inherent merits of her case,⁴⁵ section 2.6 reveals that the properties of the success function determine whether more cost shifting increases or decreases the difference between her prior and equilibrium probabilities of success. That difference, which we call distortion (see (8) in section 2.6), measures accuracy in the expected outcome of the case. Section 2.6 also provides exact sufficient conditions (Assumptions 7, 8) for ensuring that distortion is monotonic with the proportion of costs recoverable. Future research may seek empirical evidence on whether a particular judicial system satisfies one of these conditions to ascertain the relationship between cost shifting and accuracy in that system.

Thirdly, the present results regarding litigation efforts under different cost-shifting rules do not resolve questions concerning social welfare or incentives to settle or proceed to trial. Landes (1971), Posner (1972) and Gould (1973) found that settlement decisions depend on risk preferences and disagreements on the likelihood of success. Katz and

⁴⁴See also Hirshleifer and Osborne (2001) (pp. 185-86), arguing that "justice" requires a litigant's equilibrium probability of success to match her merits, as reflected by her relative advantages.

⁴⁵The existing models of cost shifting generally adopt this assumption (for example, Plott 1987, p. 188, Katz 1988, p. 129-30, Gong and McAfee 2000, p. 223, Carbonara et al. 2015, p. 5).

Sanchirico (2012) (pp. 278-86) surveyed the seminal contributions on the relationship between cost shifting and decisions to bring suit or settle. Many contributions account for information asymmetry, but make the assumption that once a case proceeds to trial, individual or total litigation efforts and costs do not vary with the extent of cost shifting.⁴⁶ While the present paper challenges that assumption,⁴⁷ it does not account for information asymmetry.⁴⁸ Moreover, Shavell (1997) observes that a private litigant's decisions to bring suit, settle or incur litigation costs are socially suboptimal due to two externalities: the negative externality arising from her lack of incentives to consider the costs that she exerts on others, such as the other litigant or the state; and the positive externality arising from her lack of incentives to consider the social benefits of litigation, such as deterrence of future injuries. In particular, he uses examples with fixed litigation costs to illustrate that a suboptimal amount of suits arise under the English rule (which shifts all of the winner's costs to the loser). In a similar vein, Spier (1997) reveals the social suboptimality of private incentives to settle suits against negligent injurers, but she (at pp. 620-21) also finds conditions under which the English rule outperforms the American rule (no shifting of the winner's costs). Section 2.9 reaches similarly complex conclusions, applying the Litigation Game with endogenous litigation costs and proportionate cost-shifting rules. If we account for the possibility of settlement, then in cases where the litigants' relative advantages are sufficiently balanced or the cost function is sufficiently convex, more cost shifting by increasing the size of the surplus arising from settlement incentivizes the litigants to settle. However, this result may not hold in the other cases (see section 2.7). Moreover, section 2.9 reveals that the relationship between the litigants' relative advantages and their incentives to proceed to litigation critically depends on the functional form of the success function.

Finally, future research may introduce additional features to the present model of litigation. For example, introducing divergence in the litigants' valuation of the judgment sum would enable an analysis of the judge's decisions regarding both the winner of the

⁴⁶For example, Bebchuk (1984), p. 406; Bebchuk and Chang (1996), pp. 376, 378-81; Klement and Neeman (2005), p. 289; Klerman and Lee (2014), pp. 216, 224; Polinsky and Rubinfeld (1998), pp. 524; Reinganum and Wilde (1986), p. 559; Spier (1994), p. 200; Talley (1995), p. 473. Assuming total litigation costs are fixed, Anderlini, Felli, and Immordino (2018) recently prove that the choice of cost-shifting rule does not affect whether a given suit is settled or litigated.

⁴⁷See Corollaries 9, 10 and Proposition 4.

⁴⁸See Dari-Mattiacci and Saraceno (2017), especially Appendix D.2, for a settlement model that accounts for information asymmetry without assuming fixed total litigation costs. These authors use the Tullock success function, and formulate the cost-shifting rule to impose an exogenous cap on the quantity of costs recoverable. We capture a class of success functions, and formulate the cost-shifting rule as an exogenous proportion of costs recoverable. For a comparison of the proportion and quantity formulations of cost-shifting rules, see footnote 12 and accompanying text.

case and the magnitude of the judgment sum. Introducing budget constraints also would enable an analysis of the implications of divergence in the litigants' wealth levels or provisions of legal aid. Moreover, it may be fruitful to study cost-shifting rules with the incomplete-information Tullock contest model developed by Einy et al. (2015).

3 Varieties of Emotions in Civil Litigation: A Robust Theory

3.1 Introduction

A civil lawsuit typically involves a plaintiff who seeks judicial remedies at the expense of a defendant. Civil lawsuits generate private and social benefits, such as enforcing the substantive law, guiding future conduct and deterring future injuries (Shavell 1997). Civil lawsuits also impose enormous costs on the litigants and on the public, such as the costs of hiring lawyers, discovering evidence, providing judges, and running courts.⁴⁹ Moreover, some litigants incur legal costs that well exceed the monetary value of the subject of the dispute.⁵⁰

Economic analysis of litigation efforts and costs typically employs a contest model with rational and self-interested contestants.⁵¹ However, contest experiments consistently suggest that, rather than being purely self-interested, contestants tend to consider relative and non-monetary payoffs.⁵² Litigation models may give rise to misleading predictions and policy recommendations if they neglect well-documented behavioral traits. This shortcoming may undermine our understanding of the civil justice system, which is fundamental to the functioning of a society governed by the rule of law.

A contest model of civil litigation typically specifies two players — a plaintiff and a defendant — who simultaneously exert costly efforts to maximize their respective payoffs. Based on the litigants' efforts and an exogenous parameter reflecting their (prior) relative advantages, a **success function** gives their respective (posterior) probabilities of success. The defendant transfers a judgment sum to the plaintiff if and only the plaintiff wins. A **cost-shifting rule** specifies the extent to which the loser pays the winner's litigation costs. To our best knowledge, chapter 2 of this thesis offers the most general contest model of litigation. Further extending that model, we introduce non-monetary and two different kinds of emotional preferences, and reveal that these preferences have very different

⁴⁹For example, a recent survey reports that various companies across the globe spend on average \$US 1 million on litigation per annum. See the 2016 *Annual Litigation Trends Survey* by Norton Rose Fulbright, an international law firm, at <http://www.nortonrosefulbright.com/files/20160915-2016-litigation-trends-annual-survey-142485.pdf>.

⁵⁰For example, an American judge recalled a divorce case in which the husband spent millions just to keep the wife from having a painting that was sold for less than half a million. See Duncan (2007) p. 125.

⁵¹See the subsequent summary of the literature on cost shifting and Katz and Sanchirico's (2012) survey of seminal papers. Other important applications of contest models include optimal labor contracts (see, for example, Lazear and Rosen 1981).

⁵²See the survey in Dechenaux et al. (2015) (at pp. 614-616).

implications on (equilibrium) litigation outcomes.

In the present **Emotional Litigation Game**, each litigant acts to maximize an emotional payoff that represents her expectations regarding her monetary outcome, her non-monetary **joy of winning** and her negative or positive **relational emotions** toward the other litigant. The joy of winning arises from winning the lawsuit, while negative (respectively, positive) relational emotions arise from harming (benefiting) the other litigant. To capture a great diversity of judicial systems, this game adopts general formulations of the success function, litigation cost function and cost-shifting rule. There exists a unique Nash equilibrium with positive effort levels.

The two different forms of non-monetary considerations — relational emotions and joy of winning — have different implications on the equilibrium properties of the Emotional Litigation Game. Intuitively, a greater joy of winning directly increases the litigant's marginal benefits of exerting costly efforts to increase her probability of success; more negative relational emotions generate similar direct effects because the litigant has a heightened desire to harm her adversary. However, unlike changes in the joy of winning, changes in relational emotions have indirect effects in cases where a cost-shifting rule operates to shift some or all of the winner's costs to the loser. Cost shifting creates externalities (in expectation) because, when she chooses her effort level, a litigant expects that with a positive probability some or all of her costs are borne by her adversary. More *negative* relational emotions indirectly *amplify* such externalities because a litigant derives a greater value from inflicting expected costs on her adversary. Moreover, more negative relational emotions (or more cost shifting) heighten incentives to exert efforts in an *asymmetric* manner; the litigant with stronger relative advantages experiences *greater* increases in incentives to exert efforts, because her expected reward from doing so is greater than the weaker litigant's. Formalizing these observations, we prove that more negative relational emotions increase the equilibrium relative effort and probability of success of the relatively more advantageous litigant. Except in rare circumstances, more negative relational emotions also increase the litigants' total litigation costs in equilibrium.

Drastically different normative implications arise from the subtle differences between relational and outcome-dependent emotions. Our equilibrium analysis suggests that the presence of relational emotions typically strengthens the cost-shifting rule, while the presence of joy of winning has no such effect. Hence, as section 3.5 will further elaborate, to understand and optimize cost shifting in civil litigation requires taking into account and responding differently to these two forms of emotions. While both outcome-dependent

and relational emotions typically increase costs in litigated cases, only relational emotions interact with the cost-shifting rule.

Unlike most other contest models of litigation, the present model gives rise to conclusions that do *not* depend on the specific functional form of the success function, the degree of homogeneity of the cost function, or the extent of cost shifting. Building upon the axiomatization effort of chapter 2 of this thesis, the present model imposes general and reasonable assumptions on the success function without specifying its functional form. Subsuming oft-used functional forms, these assumptions accommodate judges with very different styles and ways of aggregating the litigants' (prior) relative advantages and litigation efforts. These assumptions also capture uncertainty regarding the identity or decisionmaking style of the judge.⁵³

The literature on the economics of civil litigation is vast, and Sanchirico (2012) contains recent surveys of the seminal contributions.⁵⁴ Specifying extreme cost-shifting rules that either shift all or none of the winner's costs to the loser, Braeutigam et al. (1984) and Katz (1987) were among the early proponents of applying contest models to study litigation efforts and costs. In reality, intermediate cost shifting is the norm (Hodges et al. 2010, p. 20), and authors that applied contest models to study intermediate cost shifting include Carbonara et al. (2015), Farmer and Pecorino (2016), Gong and McAfee (2000), Hause (1989), Hyde and Williams (2002), Luppi and Parisi (2012) and Plott (1987), while Baye et al. (2005) and Klemperer (2003) applied auction-theoretic models. More recently, chapter 2 of this thesis axiomatized the contest model to study intermediate cost shifting under general formulations of the success function and cost function. While the success function in most models is exogenously given, the success functions in Skaperdas and Vaidya's (2012) litigation model (with no cost shifting) are derived from the inference process of a Bayesian judge.

We take the unorthodox step to incorporate emotional considerations into a litigation model in order to capture well-documented behavioral traits. As Millner and Pratt (1989) first observed and Dechenaux et al. (2015) (at pp. 614-616) recently surveyed, a variety of contest experiments consistently reveal that subjects typically exert significantly greater efforts than the equilibrium predictions of contest models based on pure self interest. An

⁵³More precisely, if a finite number of success functions satisfy the present assumptions, then their convex combination also satisfies the present assumptions. Thus these assumptions capture the scenario in which each of the potential judges rules according to a different success function and the litigants have a common prior probability for each judge being assigned to their case. For a more detailed discussion, see chapter 2 of this thesis, subsection 2.8.1.

⁵⁴An earlier survey is in Spier (2007).

explanation is, in addition to the monetary outcome of winning, subjects tend to consider non-monetary and relative outcomes (see, for example, Mago et al. 2016, Price and Sheremeta 2011, Sheremeta 2010).⁵⁵ The present Emotional Litigation Game introduces the notions of joy of winning and relational emotions to capture, respectively, the non-monetary value of winning and concerns over relative outcomes. Moreover, the Tullock contest experiments conducted by Herrmann and Orzen (2008) and Fonseca (2009) suggest that the subjects' spiteful preferences to harm their adversaries explain their over-exertion of efforts. For our purposes, their results are most relevant because the Tullock contest model is widely used to study civil litigation. In addition, many real-life litigants exhibit spiteful behaviors,⁵⁶ and in severe cases, courts have sanctioned vexatious litigants who repeatedly brought frivolous lawsuits to harass their adversaries.⁵⁷ Our equilibrium analysis thus pays close attention to those special cases involving litigants who have negative relational emotions to harm each other.

Interdependent preferences are also prevalent in non-contest situations, such as conspicuous consumption (Veblen 1899) and ultimatum and dictator games (see, for example, the recent survey in Dhami 2016, ch. 5).⁵⁸ For instance, Cameron (1999) conducted high-stake experiments in Indonesia to confirm the frequently-obtained result that participants in ultimatum games tend to realize much fairer distribution of surplus than the equilibrium prediction based on rational and self-interested agents. Inequality aversion is one of the explanations that Fehr and Schmidt (1999) captured with their well-known formulation of utility functions. Using a variety of games, Charness and Rabin (2002) presented experimental results supporting alternative explanations based on concerns for joint surplus and reciprocity. While inequality aversion and concerns for joint surplus are not our focus, the present model nonetheless may be interpreted to capture these preferences (see Remark 10 in section 3.2).

⁵⁵Alternative explanations for the over-extension of efforts include risk aversion, endowment effect as well as probability distortion, mistakes, judgmental biases, and problems with the experimental design. See Sheremeta (2013), Chowdhury and Moffatt (2017) and the papers surveyed by Dechenaux et al. (2015) at p. 617. See also Eisenkopf, Friehe, Wohlschlegel, et al. (2018) for an experiment which does not find that negative emotions have an impact on efforts or on decisions to initiate the contest.

⁵⁶In fact, an American judge observed that the adversarial litigation system — which is prevalent in common law jurisdictions — heightens antagonism and angst in divorce cases. See Duncan (2007) p. 11.

⁵⁷For example, U.S. federal trial courts may prohibit vexatious litigants from filing lawsuits without prior permission. See the opinion of the U.S. Court of Appeals, Second Circuit in the case of *Safir v. U.S. Lines, Inc.* 792 F.2d 19 (2d Cir. 1986). Anglo-Australian courts also have a similar power. See, for example, section 8 of the Vexatious Proceedings Act 2008 (NSW) and the opinion of the English Court of Appeal in the case of *Bhamjee v Forsdick & Ors* (No 2) [2003] EWCA Civ 1113. A list of vexatious litigants in England is available at <https://www.gov.uk/guidance/vexatious-litigants>.

⁵⁸Earlier surveys of the literature on interdependent preferences include Camerer and Thaler (1995) and Sobel (2005).

The present model does not cover some forms of interdependent preferences or informational structures that other authors have captured. Rabin (1993) constructed a complete-information model that endogenously generates intentions-based reciprocity. Bolton and Ockenfels (2000) offered an incomplete-information model that captures concerns for relative outcomes. Capturing preferences for reciprocity, Segal and Sobel (2007) presented a representation theorem for games with players who have preferences over strategies in addition to outcomes. Segal and Sobel (2008) used their model to introduce a condition for ensuring that a player is more likely to be kind to an opponent who treats him nicely. Pollak (1976) examined demand behaviors with a model in which interdependent preferences operate through past consumption.

The present paper builds upon a small literature that explores the role of emotions in civil litigation. To our best knowledge, Huang and Wu (1992) first considered the effect of emotions on pretrial bargaining and on decisions to bring suit or settle, while Baumann and Friehe (2012b) first studied emotions in a contest model with endogenous litigation efforts. Using a Tullock contest model with no cost shifting, Baumann and Friehe (2012b) showed that the equilibrium implications of introducing outcome-dependent emotions are similar to raising the stakes (see Baumann and Friehe 2012b at pp. 196, 203-04).⁵⁹ Extending Plott's (1987) litigation model, Angenendt (2014) considered the how loss aversion affects well-known predictions regarding extreme cost-shifting rules. We complement their work by using a generally-formulated litigation model; by introducing intermediate cost shifting and relational emotions; and by revealing that relational emotions and outcome-dependent emotions have drastically different equilibrium implications.⁶⁰ However, the present model does *not* subsume Baumann and Friehe's (2012b) model or analysis. Unlike us, Baumann and Friehe (2012b) (at. pp.196, 202-12) allowed the litigants to have asymmetric outcome-dependent emotions, and they used the functional forms of the contest success function and emotions to study incentives to sue and accuracy in adjudication. Moreover, the present model does not capture how emotional preferences affect incentives to settle a dispute, while a body of empirical literature uses bargaining models to explore that issue (for example, see Farmer and Tiefenthaler 2001 and the papers surveyed in Baumann and Friehe 2012b at pp. 197-99).

Section 3.2 constructs the Emotional Litigation Game to introduce emotional variables into a general contest model that captures whole classes of success functions, cost func-

⁵⁹The present Emotional Litigation Game captures outcome-dependent emotions as "the joy of winning". See section 3.2.

⁶⁰See especially Corollary 11 in section 3.3 and the normative discussion in section 3.5.

tions and cost-shifting rules. Section 3.3 proves the existence and uniqueness of a Nash equilibrium with positive efforts, and reveals the different roles of relational emotions and joy of winning. Section 3.4 reveals how changes in relational emotions or joy of winning affect equilibrium outcomes. In particular, more negative relational emotions distort equilibrium outcomes in favor of the relatively more advantageous litigant, but typically increase the litigants' total costs. Section 3.5 concludes with a discussion of normative implications and future research directions. Appendix A.3 contains all proofs.

3.2 The Emotional Litigation Game

The **Emotional Litigation Game** is a simultaneous-move game of complete information characterized by two risk-neutral players, Plaintiff and Defendant, their common set of actions $\mathbb{R}_+$, and their payoff functions $\check{u}_P, \check{u}_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$. Each payoff function has monetary and non-monetary components, including the joy of winning and the player's emotions regarding the other player. The payoff functions and all exogenous parameters are common knowledge.

Plaintiff and Defendant simultaneously and respectively exert $e_P, e_D \geq 0$ levels of efforts. Giving each litigant's monetary cost of exerting effort is a homogeneous cost function $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with an exogenous degree of homogeneity $k \geq 1$, where k satisfies additional assumptions to be set out below. An exogenous parameter $0 < \mu < 1$ represents Plaintiff's prior probability of success; Defendant's prior probability of success is $1 - \mu$. Plaintiff (respectively, Defendant) is **relatively more advantageous** if $\mu > 0.5$ ($\mu < 0.5$).⁶¹ Given a prior parameter μ and a pair of efforts (e_P, e_D) , the judicial process with probability $\theta(e_P, e_D; \mu)$ requires Defendant to transfer a judgment sum of 1 to Plaintiff, where the success function $\theta : \mathbb{R}_+^2 \rightarrow [0, 1]$ satisfies additional assumptions to be set out below. Upon determination of the outcome of the case, a **cost-shifting rule** requires the loser to pay an exogenous $0 \leq \check{\lambda} \leq 1$ proportion of the winner's costs, where $\check{\lambda}$ satisfies additional assumptions to be set out below. In particular, $\check{\lambda} = 0$ characterizes the **American rule** that allows for no recovery, and $\check{\lambda} = 1$ the **English rule** that allows for full recovery. Containing these monetary variables are Plaintiff and Defendant's respective

⁶¹Relative advantages reflect institutional factors that do not vary with litigation efforts but influence the outcome of the case. See Remark 1 in chapter 2 of this thesis for a detailed discussion of relative advantages and litigation efforts.

monetary payoffs $u_P, u_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by

$$u_P = \theta[1 - (1 - \check{\lambda})C(e_P)] - (1 - \theta)[C(e_P) + \check{\lambda}C(e_D)] \quad (22)$$

$$u_D = -\theta[1 + C(e_D) + \check{\lambda}C(e_P)] - (1 - \theta)(1 - \check{\lambda})C(e_D). \quad (23)$$

Each litigant's monetary payoff is her expected monetary outcome. Plaintiff's monetary payoff u_P is the weighted average of her monetary outcome in the event that she wins, $1 - (1 - \check{\lambda})C(e_P)$, and her monetary outcome in the event that she loses, $-C(e_P) - \check{\lambda}C(e_D)$. Weights θ and $1 - \theta$ are respectively her probabilities of winning and losing. Similarly, Defendant's monetary payoff u_D is the weighted average of her monetary outcome in the event that she loses, $-1 - C(e_D) - \check{\lambda}C(e_P)$, and her monetary outcome in the event that she wins, $-(1 - \check{\lambda})C(e_D)$. Weights θ and $1 - \theta$ are respectively her probabilities of losing and winning.

Remark 8. *Litigation efforts and probabilities of success have the following interpretation. Suppose, given the facts that characterize the relevant dispute and given the litigation efforts, a random variable will realize one of two outcomes — “Plaintiff wins” or “Defendant wins” — at the end of the litigation process. Before such realization, the litigants first exert some minimum sunk efforts to initiate the litigation process, and then exert additional efforts to influence the realization of the outcome. “Sunk efforts” capture activities to acquire knowledge of the “rules of the game”, commence legal proceedings and present the minimum “amounts” of evidence and arguments to obtain a judicial ruling. “Additional efforts” refer to activities beyond the bare minimum, such as conducting extensive discovery, adducing voluminous evidence and making lengthy arguments. A litigant's prior probability of success (μ for Plaintiff and $1 - \mu$ for Defendant) is the probability that the outcome realizes in her favor conditional on exertion of sunk efforts and no additional efforts. The facts, sunk efforts and the practical operation of the judicial system affect her prior probability of success. The litigant's posterior probability of success (θ for Plaintiff and $1 - \theta$ for Defendant) is the probability that the outcome realizes in her favor after exertion of sunk efforts and additional efforts. Because the Emotional Litigation Game models the litigants' strategic interaction after their exertion of sunk efforts, we call variables e_P and e_D “efforts” and drop the “additional” label for simplicity.*

Instead of acting solely to maximize her monetary payoff, each litigant acts to maximize an **emotional payoff** that includes the following non-monetary variables. In addition to

any monetary transfer, the winner derives an exogenous value $v \geq 0$, called her **joy of winning**.⁶² Moreover, each litigant derives value from her feelings about the other litigant's outcome; an exogenous $\xi < 1$ captures such **relational emotions**, meaning that each litigant is indifferent between one unit of her own monetary payoff (or joy of winning) and ξ units of the other litigant's. Containing these non-monetary variables and the monetary payoffs are Plaintiff and Defendant's respective emotional payoffs $\check{u}_P, \check{u}_D : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by⁶³

$$\check{u}_P = u_P + \theta v + \xi[u_D + (1 - \theta)v] \quad (24)$$

$$\check{u}_D = u_D + (1 - \theta)v + \xi[u_P + \theta v]. \quad (25)$$

Plaintiff's emotional payoff $\check{u}_P$ sums her monetary payoff u_P , her expected joy of winning θv and her relational emotions regarding Defendant's outcome, $\xi[u_D + (1 - \theta)v]$. Similarly, Defendant's emotional payoff $\check{u}_D$ sums her monetary payoff u_D , her expected joy of winning $(1 - \theta)v$ and her relational emotions regarding Plaintiff's outcome, $\xi[u_P + \theta v]$.

Remark 9. *While we interpret v and ξ to represent the joy of winning and relational emotions respectively, they can capture non-emotional preferences. For instance, v can represent a litigant's monetary benefits of winning beyond the judgment sum, such as the monetary value of having a reputation as a strong litigant. Similarly, ξ can capture monetary spillovers arising from litigation efforts, such as the damages caused by negative publicity. Moreover, as Remark 10 will explain, v and ξ can capture concerns for relative payoff, inequality or joint surplus.*

The empirical literature on contests suggests that relational emotions are typically negative ($\xi < 0$).⁶⁴ Under this specification, each litigant has competitive preferences; her emotional payoff increases when the other litigant's monetary payoff or expected joy of winning decreases. To our best knowledge, there is no empirical literature measuring emotional variables in litigated cases, but the specification of $\xi < 0$ is intuitively appealing given the adversarial nature of civil litigation, especially in common law jurisdictions.

⁶²See section 3.1 for a discussion of the experimental literature supporting the existence of non-monetary utilities of winning, which we call joy of winning to avoid confusion with relational emotions.

⁶³For simplicity, we formulate each litigant's emotional payoff as an independent function of strategies (e_P, e_D) directly, rather than as an interdependent function of her monetary payoff and her opponent's emotional payoff. An alternative approach is to start with a system of interdependent payoff functions, and use it to induce independent payoff functions. For conditions that enable the alternative approach, see Bergstrom (1999).

⁶⁴See section 3.1.

However, the Emotional Litigation Game also allows for the possibility of atypical cases characterized by $0 < \xi < 1$.⁶⁵

Remark 10. *In addition to the joy of winning and relational emotions, the Emotional Litigation Game can capture a broad variety of non-monetary preferences that are typically present in contests or in bargaining and other more "cooperative" games (see section 3.1).*

First, assuming zero joy of winning ($v = 0$) and specifying $\xi > -1$, a rearrangement of the litigants' emotional payoffs (24) and (25) gives

$$\frac{\check{u}_P}{1 + \xi} = u_P - \left(\frac{\xi}{1 + \xi} \right) (u_P - u_D), \quad \frac{\check{u}_D}{1 + \xi} = u_D - \left(\frac{\xi}{1 + \xi} \right) (u_D - u_P)$$

where the function $\check{u}_P/(1 + \xi)$ is a strictly positive affine transformation of Plaintiff's emotional payoff function $\check{u}_P$ in (24); these two functions thus represent the same underlying preferences. (An analogous logic applies to Defendant.) Hence the Emotional Litigation Game captures concerns for relative payoffs by interpreting the weight $\xi/(1 + \xi)$ as the monetary value of relative payoffs. If $-1 < \xi < 0$, then each litigant's emotional payoff increases when her relative payoff increases; that is, she has competitive preferences. If $\xi > 0$, then each litigant's emotional payoff decreases when her relative payoff increases; that is, she has "spiteful" preferences (see, for example, Fehr and Schmidt 1999, Herrmann and Orzen 2008, and Chowdhury, Jeon, and Ramalingam 2018).

Secondly, assuming $v = 0$ and specifying $0 < \xi < 1$, a rearrangement of the litigants' emotional payoffs (24) and (25) gives

$$\check{u}_P = (1 - \xi)u_P + \xi(u_P + u_D), \quad \check{u}_D = (1 - \xi)u_D + \xi(u_P + u_D)$$

where each litigant's emotional payoff is a weighted average of her monetary payoff and joint surplus. Hence the Emotional Litigation Game captures concerns for joint surplus by interpreting the weight ξ as the extent to which each litigant values joint surplus.

Thirdly, the mere presence of the joy of winning — without concerns for relative outcomes — may be sufficient to explain the over-exertion of efforts in contest experiments (see, for example, Sheremeta 2010, pp. 738-739 and the papers surveyed there). The Emotional Litigation Game captures this possibility by specifying $v > 0$ and $\xi = 0$.

⁶⁵We exclude the possibility that each litigant values the other litigant's monetary payoff (or expected joy of winning) more than her own; that is, $\xi \geq 1$. Some algebra using equations (24) and (25) will reveal that, in the limiting case of $\xi = 1$, each litigant acts to minimize total litigation costs. An examination of the first order conditions in system (28) in section 3.3 will also reveal that allowing for $\xi \geq 1$ would render the model uninteresting, because each litigant would only have incentives to exert zero effort in any equilibrium.

Moreover, the Emotional Litigation Game captures loss aversion — a well-known phenomenon in contests (see, recently, Chowdhury et al. 2018). To see this, specify a constant $\alpha > 1$, and rearrange the litigants' emotional payoffs (24) and (25) to obtain

$$\check{u}_P - \frac{\alpha v}{1 + \alpha} - \frac{\xi \alpha v}{1 + \alpha} = u_P + \theta \left[\frac{v}{1 + \alpha} \right] - (1 - \theta) \left[\frac{\alpha v}{1 + \alpha} \right] + \xi \left[u_D + (1 - \theta) \left[\frac{v}{1 + \alpha} \right] - \theta \left[\frac{\alpha v}{1 + \alpha} \right] \right] \quad (26)$$

$$\check{u}_D - \frac{\alpha v}{1 + \alpha} - \frac{\xi \alpha v}{1 + \alpha} = u_D + (1 - \theta) \left[\frac{v}{1 + \alpha} \right] - \theta \left[\frac{\alpha v}{1 + \alpha} \right] + \xi \left[u_P + \theta \left[\frac{v}{1 + \alpha} \right] - (1 - \theta) \left[\frac{\alpha v}{1 + \alpha} \right] \right], \quad (27)$$

where equation (26) is a strictly positive affine transformation of Plaintiff's emotional payoff function $\check{u}_P$ in (24); thus they represent the same preferences. (The same logic applies to Defendant.) An examination of the right-hand side of equation (26) (respectively, (27)) reveals that Plaintiff (Defendant) has loss aversion: her “joy of winning” parameter here is $v/(1 + \alpha)$, which is strictly smaller than her “pain of losing” $\alpha v/(1 + \alpha)$. Hence the Emotional Litigation Game captures loss aversion by interpreting $\alpha/(1 + \alpha)$ as the extent to which loss is valued relatively more than gain.

However, the Emotional Litigation Game does not differentiate between advantageous and disadvantageous inequality aversion, in the sense that a player's degree of inequity aversion depends on whether her payoff is greater or smaller than her opponent's. For two-player symmetric Tullock contests that differentiate between advantageous and disadvantageous inequality aversion, see Herrmann and Orzen (2008) and Chowdhury et al. (2018).

We now state assumptions to guarantee equilibrium existence and uniqueness.⁶⁶ On its subdomain $\mathbb{R}_{++}^2$, the success function $\theta(\cdot)$ is twice continuously differentiable and satisfies the following Assumptions 12-17, where Assumptions 16 and 17 also constrain the degree of homogeneity k of the cost function, the proportion $\check{\lambda}$ of costs recoverable and the extent of relational emotions ξ .

Assumption 12. Holding the efforts and the prior constant, whether a litigant is labeled “Plaintiff” or “Defendant” does not affect her posterior probability of success. Formally, $\theta(e_1, e_2; \mu_0) = 1 - \theta(e_2, e_1; 1 - \mu_0)$, for any positive real numbers $e_1, e_2 > 0$ and $0 < \mu_0 < 1$.

Assumption 13. Holding the prior constant, proportionate changes in effort levels do not

⁶⁶For conditions that give rise to multiple equilibria in Tullock contests, see Chowdhury and Sheremeta (2011b).

affect Plaintiff's posterior probability of success. Formally, $\theta(e_P, e_D; \mu) = \theta(xe_P, xe_D; \mu)$, for all scalar $x > 0$.

Assumption 14. Holding the prior and Defendant's effort constant, Plaintiff's posterior probability of success is strictly increasing with and weakly concave in her effort. Formally, $\frac{\partial \theta}{\partial e_P} > 0$ and $\frac{\partial^2 \theta}{\partial e_P^2} \leq 0$.

Assumption 15. Holding the efforts constant, Plaintiff's posterior probability of success is strictly increasing with her prior probability of success. Formally, $\frac{\partial \theta}{\partial \mu} > 0$.

Assumptions 12-15 capture intuitions regarding the properties of reasonable success functions. Assumption 12 requires a litigant's posterior probability of success to be unaffected by merely changing her label from "Plaintiff" — whose effort, prior and posterior probabilities of success are respectively denoted e_P, μ, θ — to "Defendant" — whose effort, prior and posterior probabilities of success are respectively denoted $e_D, 1 - \mu, 1 - \theta$. In other words, the parameter μ captures any asymmetry that does not vary with litigation efforts.⁶⁷ Under Assumption 13, proportionate changes in effort levels do not vary Plaintiff's probability of success. Assumptions 14-15 further require that an increase in Plaintiff's prior probability of success or effort strictly increases her posterior probability of success.

Assumption 16. Interdependence in payoffs is limited in the following precise sense:

1. The cost-shifting rule $\check{\lambda}$ and relational emotions ξ satisfy $\check{\lambda}(1 - \xi) \leq 1$.
2. Suppose $\check{\lambda}(1 - \xi) = 1$ and the cost function is linear, $k = 1$. Then Plaintiff does not win almost surely by exerting infinitely more effort than Defendant does. Formally,

$$\check{\lambda}(1 - \xi) = 1 \quad \text{implies} \quad \lim_{e_P/e_D \rightarrow +\infty} \theta < 1.$$

Assumption 16 restricts the combined "strength" of the cost-shifting rule $\check{\lambda}$ and relational emotions ξ . Part 1 ensures that each litigant's marginal costs of exerting effort are always positive; allowing for $\check{\lambda}(1 - \xi) > 1$ might induce zero marginal costs for some extremely asymmetric effort pairs. Part 2 ensures that, in special cases involving "strong" negative relational emotions and cost shifting (precisely, $\check{\lambda}(1 - \xi) = 1$) and a

⁶⁷For instance, if being called "Plaintiff" requires a litigant to discharge a more onerous burden of proof than if she were called "Defendant", then the parameter μ should reflect such asymmetry.

linear cost function ($k = 1$), Plaintiff does not have incentives to make explosive efforts ($e_P/e_D \rightarrow +\infty$) with an expectation that she almost surely inflicts strong negative relational emotions *and* unbounded litigant costs on Defendant.⁶⁸

Assumption 17. *For the interested triple of parameters $(\check{\lambda}, k, \xi)$, the following condition holds*

$$\frac{\frac{\partial^2}{\partial e_P^2} \left(\frac{\theta}{1 - \check{\lambda}(1 - \xi)\theta} \right)}{\frac{\partial}{\partial e_P} \left(\frac{\theta}{1 - \check{\lambda}(1 - \xi)\theta} \right)} < \frac{C''(e_P)}{C'(e_P)}.$$

Assumption 17 is a technical assumption that ensures Plaintiff's emotional payoff is strictly quasiconcave in her own effort.⁶⁹ It ensures that the curvature of Plaintiff's distorted posterior probability of success — $\theta/(1 - \check{\lambda}(1 - \xi)\theta)$ — be small compared to the curvature of the cost function.

The exogenous parameters and the litigants' payoff functions are common knowledge between them. To focus on the study of litigation efforts and probabilities of success, further assume there is no settlement or risk of default.

The solution concept adopted is a pure-strategy Nash equilibrium that is non-trivial in the sense of comprising positive efforts by both litigants. A pair of positive efforts is such an equilibrium if given the other litigant's effort, each litigant chooses an effort to maximize her emotional payoff.

3.3 Equilibrium: Existence, Uniqueness, and the Role of Emotional Variables

This section proves the existence and uniqueness of a non-trivial Nash equilibrium in the Emotional Litigation Game. It then presents our main result revealing how the presence of relational emotions and joy of winning affects equilibrium efforts.

Lemma 4 allows any equilibrium to be characterized by a system of first order conditions (FOCs). Appendix A.3 contains all proofs.

Lemma 4. *Each litigant's emotional payoff function is strictly quasiconcave in her own effort.*

⁶⁸System (28) to be stated below will formalize these intuitive observations.

⁶⁹See the Proof of Lemma 4 in Appendix A.3.

Lemma 4 implies given the other litigant's effort, a litigant's FOC characterizes her best reply.⁷⁰ A substitution exercise using equations (22)-(25) reveals that a pair of positive efforts $(e_P, e_D) \in \mathbb{R}_{++}^2$ constitutes a Nash equilibrium if and only if it satisfies system (28):

$$\begin{cases} 0 = \frac{\partial \theta}{\partial e_P}(1 - \xi)[1 + \nu + \check{\lambda}C(e_P) + \check{\lambda}C(e_D)] - [1 - \check{\lambda}(1 - \xi)\theta]C'(e_P) \\ 0 = \frac{\partial(1-\theta)}{\partial e_D}(1 - \xi)[1 + \nu + \check{\lambda}C(e_P) + \check{\lambda}C(e_D)] - [1 - \check{\lambda}(1 - \xi)(1 - \theta)]C'(e_D). \end{cases} \quad (28)$$

System (28) reveals how the cost-shifting rule $\check{\lambda}$, relational emotions ξ and joy of winning ν affect a litigant's incentives to exert costly effort. For instance, holding Defendant's effort e_D fixed, more cost shifting ($\check{\lambda}$ increases) increases Plaintiff's marginal benefits of exerting effort e_P by shifting a greater proportion of her costs — $C(e_P)$ — into the "prize" of winning: $(1 - \xi)[1 + \nu + \check{\lambda}C(e_P) + \check{\lambda}C(e_D)]$. This shift also reduces her marginal costs of exerting effort: $[1 - \check{\lambda}(1 - \xi)\theta]C'(e_P)$. Performing a similar role as the cost-shifting rule, more negative relational emotions (ξ decreases) increase Plaintiff's marginal benefits of exerting effort by scaling up the "prize" of winning. A decreased ξ also reduces her marginal costs of exerting effort. By comparison, a greater joy of winning ν increases Plaintiff's marginal benefits of exerting effort, but does *not* affect her marginal costs. The same observations apply to Defendant's incentives to exert costly effort when Plaintiff's effort is fixed. These intuitive observations have profound equilibrium implications, as Corollary 11 below will reveal.

Lemma 5 finds a unique, positive effort ratio which will be used to characterize the nontrivial Nash equilibrium. To simplify notation, define an auxiliary variable $s = e_D/e_P$ whenever Plaintiff's effort $e_P > 0$; s is the ratio of Defendant's effort relative to Plaintiff's. Assumption 13 implies that for any two pairs of positive efforts $(e_P, e_D), (e'_P, e'_D) \in \mathbb{R}_{++}^2$ satisfying $e_D/e_P = e'_D/e'_P$, the success function satisfies $\theta(e_P, e_D; \mu) = \theta(e'_P, e'_D; \mu)$. By a slight abuse of notation, denote $\theta(s; \mu) = \theta(e_P, e_D; \mu)$ and $\theta_s = \frac{\partial}{\partial s}\theta(s; \mu)$.

Lemma 5. *There exists a unique positive effort ratio $s^* > 0$ that satisfies*

$$s^* = \left[\frac{1 - \check{\lambda}(1 - \xi)\theta(s^*; \mu)}{1 - \check{\lambda}(1 - \xi)[1 - \theta(s^*; \mu)]} \right]^{1/k}. \quad (29)$$

Proposition 7 establishes the existence and uniqueness of a non-trivial Nash equilibrium. It also characterizes the litigants' relative efforts in equilibrium.

⁷⁰Theorem 8 of Diewert et al. (1981) holds any local maximizer of a strictly quasiconcave function is the unique global maximizer.

Proposition 7. *There exists a unique Nash equilibrium with positive efforts (e_P^*, e_D^*) , which is characterized by*

$$e_P^* = \left[\frac{C(1)}{(1+\nu)(1-\xi)} \left[\frac{ks^{*k-1}[1 - \check{\lambda}(1-\xi)(1-\theta(s^*; \mu))]}{-\theta_s(s^*; \mu)} - \check{\lambda}(1-\xi)(1+s^{*k}) \right] \right]^{-1/k}$$

$$e_D^* = s^* e_P^*$$

where Lemma 5 gives s^* .⁷¹

Proposition 7 finds and characterizes the unique non-trivial Nash equilibrium of the Emotional Litigation Game; all subsequent references to the Game's equilibrium refer to this non-trivial Nash equilibrium. Although the expressions for the equilibrium efforts are complicated, Lemma 5 immediately reveals that the application of the American rule ($\check{\lambda} = 0$) leads to equal equilibrium efforts ($s^* = e_D^*/e_P^* = 1$). Under other cost-shifting rules, $s^* = 1$ also holds in the limit when relational emotions $\xi \rightarrow 1$. Remark 11 below will reveal the equilibrium relative efforts under different cost-shifting rules.

To simplify subsequent discussion and attract the comparative-static analysis in chapter 2 of this thesis that covers all parameters *except* the joy of winning (ν) and relational emotions (ξ), Corollary 11 below will reveal the exact roles that ν and ξ play in equilibrium. To facilitate presentation, fix and suppress the relative-advantages parameter μ and the cost function $C(\cdot)$. Let $\mathbb{G}(\xi, \nu, \check{\lambda})$ denote the Emotional Litigation Game when a generic triple $\xi, \nu, \check{\lambda}$ of parameters respectively capture the relational emotions, the joy of winning, and the cost-shifting rule. Using Proposition 7, let $e_P^*(\xi, \nu, \check{\lambda})$ and $\theta^*(\xi, \nu, \check{\lambda})$ respectively denote Plaintiff's equilibrium effort and probability of success in $\mathbb{G}(\xi, \nu, \check{\lambda})$. Similarly, let $e_D^*(\xi, \nu, \check{\lambda})$ and $s^*(\xi, \nu, \check{\lambda}) = e_D^*(\xi, \nu, \check{\lambda})/e_P^*(\xi, \nu, \check{\lambda})$ denote Defendant's equilibrium (absolute) effort and relative effort. Finally, in the special case of $\nu = \xi = 0$, call the game a **Monetary Litigation Game**; intuitively, the litigants act only to maximize their monetary payoffs (see equations (22), (23)).

Presenting our main result, Corollary 11 relates the equilibrium of an Emotional Litigation Game to the equilibrium of a Monetary Litigation Game with a *different* cost-shifting rule characterized by $\check{\lambda}(1 - \xi)$.

Corollary 11. *Consider the equilibrium $(e_P^*(\xi, \nu, \check{\lambda}), e_D^*(\xi, \nu, \check{\lambda}))$ of the Emotional Litigation Game $\mathbb{G}(\xi, \nu, \check{\lambda})$ and the equilibrium $(e_P^*(0, 0, \check{\lambda}(1 - \xi)), e_D^*(0, 0, \check{\lambda}(1 - \xi)))$ of the*

⁷¹Section 3.4 will reveal that more negative relational emotions or more cost shifting increases the equilibrium relative effort and probability of success of the relatively more advantageous litigant.

Monetary Litigation Game $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$. Each litigant's equilibrium effort in $\mathbb{G}(\xi, \nu, \check{\lambda})$ is $(1 + \nu)^{1/k}(1 - \xi)^{1/k}$ times her equilibrium effort in $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$; the litigant has the same relative effort and posterior probability of success in these equilibria. Formally,

$$\begin{aligned} e_P^*(\xi, \nu, \check{\lambda}) &= (1 + \nu)^{1/k}(1 - \xi)^{1/k} e_P^*(0, 0, \check{\lambda}(1 - \xi)), \\ e_D^*(\xi, \nu, \check{\lambda}) &= (1 + \nu)^{1/k}(1 - \xi)^{1/k} e_D^*(0, 0, \check{\lambda}(1 - \xi)), \end{aligned}$$

and

$$s^*(\xi, \nu, \check{\lambda}) = s^*(0, 0, \check{\lambda}(1 - \xi)), \quad \theta^*(\xi, \nu, \check{\lambda}) = \theta^*(0, 0, \check{\lambda}(1 - \xi)).$$

With normative implications to be discussed in section 3.5, Corollary 11 reveals the *different* implications that relational emotions and joy of winning have on equilibrium efforts. Suppose relational emotions are negative in the Emotional Litigation Game $\mathbb{G}(\xi, \nu, \check{\lambda})$; that is, $\xi < 0$. Such negative relational emotions affect each litigant's equilibrium effort directly and indirectly. Indirectly, negative relational emotions render each litigant's equilibrium effort in $\mathbb{G}(\xi, \nu, \check{\lambda})$ to be *proportionate* to her equilibrium effort in the Monetary Litigation Game $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$, in which the cost-shifting rule is scaled *up* by $(1 - \xi)$. Directly, each litigant's equilibrium effort in $\mathbb{G}(\xi, \nu, \check{\lambda})$ is also scaled up by $(1 - \xi)^{1/k}$ compared to her equilibrium effort in $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$. The opposite direct and indirect effects arise in the presence of positive relational emotions, $0 < \xi < 1$. However, only relational emotions have both such direct and indirect effects on equilibrium efforts. A positive joy of winning $\nu > 0$ directly scales up equilibrium efforts by $(1 + \nu)^{1/k}$, but does *not* vary the effects of cost shifting.

Hence, unlike the joy of winning, the presence of relational emotions modifies the effects of cost shifting. For instance, equilibrium litigation efforts given negative relational emotions ($\xi < 0$) and the cost-shifting rule $\check{\lambda}$ are enlargements of equilibrium litigation efforts given pure self interest ($\xi = 0$) and a greater cost-shifting rule $\check{\lambda}(1 - \xi)$; in other words, the presence of negative relational emotions strengthens the cost-shifting rule. The opposite is true in the presence of positive emotion, $0 < \xi < 1$. Because it is typical to have negative relational emotions $\xi < 0$ in a litigated case (see section 3.1), the "true" effects of cost shifting are greater than what they would be if the litigants were purely self-interested. Section 3.5 will further develop the normative implications of this result.

Moreover, a litigant has the same relative effort and probability of success in the

equilibrium of the Emotional Litigation Game $\mathbb{G}(\xi, \nu, \check{\lambda})$ and in the equilibrium of the Monetary Litigation Game $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$. This is because her (absolute) efforts in these equilibria are proportional, so are the other litigant's. Thus each litigant's relative effort is the same in these equilibria. This immediately implies she has the same probability of success in these equilibria because the success function θ only (directly) depends on relative advantages and relative efforts.

Corollary 11 reveals a bijective relationship between each litigant's equilibrium efforts in the Emotional Litigation Game $\mathbb{G}(\xi, \nu, \check{\lambda})$ and the Monetary Litigation Game $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$. For that reason, call $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$ the **transformed Monetary Litigation Game** of $\mathbb{G}(\xi, \nu, \check{\lambda})$. Except in the special case of $\check{\lambda} = 0$, $\xi = 0$ or $\nu = 0$, transforming $\mathbb{G}(\xi, \nu, \check{\lambda})$ to $\mathbb{G}(0, 0, \check{\lambda}(1 - \xi))$ requires changing three parameters: the joy of winning from ν to 0, the relational emotions from ξ to 0, and the cost-shifting rule from $\check{\lambda}$ to $\check{\lambda}(1 - \xi)$. Section 3.4 below will use this property to simplify the comparative-statics analysis.

Remark 11. *The Litigation Game that chapter 2 of this thesis presented is a Monetary Litigation Game in the present sense. Thus the present Corollary 11 attracts their description of equilibrium outcomes (see chapter 2, Proposition 1) to the present Emotional Litigation Game:*

1. *If the American rule applies or relative advantages are equal, then the litigants exert the same levels of effort in equilibrium. Formally, $\check{\lambda} = 0$ or $\mu = 0.5$ implies $e_P^*(\xi, \nu, \check{\lambda}) = e_D^*(\xi, \nu, \check{\lambda})$. Moreover, equal relative advantages imply equal equilibrium probabilities of success. Formally, $\mu = 0.5$ implies $\theta^*(\xi, \nu, \check{\lambda}) = 0.5$.*
2. *Suppose the cost-shifting rule allows the winner to recover a positive proportion of her costs from the loser. Then in equilibrium, the relatively more advantageous litigant exerts relatively more effort and has a relatively greater probability of success. Formally, $\check{\lambda} > 0$ and $\mu > 0.5$ (respectively, $\mu < 0.5$) implies $e_P^*(\xi, \nu, \check{\lambda}) > e_D^*(\xi, \nu, \check{\lambda})$ and $\theta^*(\xi, \nu, \check{\lambda}) > 0.5$ ($e_P^*(\xi, \nu, \check{\lambda}) < e_D^*(\xi, \nu, \check{\lambda})$ and $\theta^*(\xi, \nu, \check{\lambda}) < 0.5$).*

3.4 Comparative Statics

This section considers the equilibrium implications of variations in the parameters of the Emotional Litigation Game.

3.4.1 Relative Efforts and Probabilities of Success

This subsection reveals the different implications that relational emotions and joy of winning have on equilibrium efforts and probabilities of success. Corollary 12 first considers the implications arising from changes in relational emotions.

Corollary 12. *Consider the equilibrium of the Emotional Litigation Game $\mathbb{G}(\xi, v, \check{\lambda})$.*

1. *Suppose the American rule applies or the litigants' relative advantages are equal. Then each litigant's equilibrium relative effort and probability of success do not change as relational emotions ξ change. Formally, $\check{\lambda} = 0$ or $\mu = 0.5$ implies $\frac{d}{d\xi}s^*(\xi, v, \check{\lambda}) = 0$ and $\frac{d}{d\xi}\theta^*(\xi, v, \check{\lambda}) = 0$.*
2. *Suppose the cost-shifting rule allows the winner to recover a positive proportion of her costs from the loser. Suppose further that Plaintiff is relatively more advantageous. Then more negative relational emotions (ξ decreases) increase Plaintiff's equilibrium relative effort and probability of success. Formally, $\check{\lambda} > 0$ and $\mu > 0.5$ imply $\frac{d}{d\xi}s^*(\xi, v, \check{\lambda}) > 0$ and $\frac{d}{d\xi}\theta^*(\xi, v, \check{\lambda}) < 0$.*
3. *Suppose the cost-shifting rule allows the winner to recover a positive proportion of her costs from the loser. Suppose further that Defendant is relatively more advantageous. Then more negative relational emotions (ξ decreases) increase Defendant's equilibrium relative effort and probability of success. Formally, $\check{\lambda} > 0$ and $\mu < 0.5$ imply $\frac{d}{d\xi}s^*(\xi, v, \check{\lambda}) < 0$ and $\frac{d}{d\xi}\theta^*(\xi, v, \check{\lambda}) > 0$.*

Corollary 12 reveals how variations in relational emotions ξ affect equilibrium effort ratios and probabilities of success. Consider cases in which the cost-shifting rule allows the winner to recover at least some of her litigation costs from the loser ($\check{\lambda} > 0$) and one litigant is relatively more advantageous ($\mu \neq 0.5$). Then as relational emotions become more negative (ξ decreases), that litigant exerts relatively more effort and enjoys a greater probability of success in equilibrium.

Corollary 13. *Consider the equilibrium of the Emotional Litigation Game $\mathbb{G}(\xi, v, \check{\lambda})$. Variations in the joy of winning v do not affect each litigant's relative effort and probability of success. Formally, $\frac{d}{dv}s^*(\xi, v, \check{\lambda}) = 0$ and $\frac{d}{dv}\theta^*(\xi, v, \check{\lambda}) = 0$.*

Corollaries 12 and 13 reveal that relational emotions and joy of winning have very different effects on equilibrium relative efforts and probabilities of success. A greater joy of winning v affects the litigants in a symmetric manner; to the same extent for both litigants,

a greater ν increases the marginal benefits of exerting effort to increase the probability of success. Corollary 13 thus reveals that equilibrium efforts and probabilities of success remain constant when ν increases. By comparison, relational emotions ξ interact with the cost-shifting rule (see Corollary 11). Amplifying the effects of cost shifting, more negative relational emotions (ξ decreases) affect the litigants' incentives asymmetrically. Corollary 12 confirms that such asymmetric effects distort equilibrium efforts and probabilities of success in favor of the relatively more advantageous litigant.

3.4.2 Monetary Consequences

This subsection considers how changes in relational emotions or joy of winning affect the litigants' monetary payoffs and costs, assuming that these changes do not stop the case from proceeding to litigation. Only private costs and benefits are considered here.⁷²

Define (equilibrium) **litigation expenditure** $C^*(\xi, \nu, \check{\lambda})$ of the Emotional Litigation Game $\mathbb{G}(\xi, \nu, \check{\lambda})$ as the sum of Plaintiff and Defendant's respective litigation costs:

$$C^*(\xi, \nu, \check{\lambda}) = C(e_P^*(\xi, \nu, \check{\lambda})) + C(e_D^*(\xi, \nu, \check{\lambda})). \quad (30)$$

Changes in ξ or ν affect each litigant's incentives to exert efforts in equilibrium. Changes in equilibrium efforts then affect litigation expenditure.

Corollary 14 below will reveal that more negative relational emotions (ξ decreases) typically increase litigation expenditure. To facilitate presentation, define a function $\sigma : (-\infty, 1) \times [0, 1] \times [1, +\infty) \rightarrow (0, 0.5]$ by

$$\sigma(\xi, \check{\lambda}, k) = \max \left\{ \mu \in [0, 1] \mid \theta(s^*; \mu) \leq (3 - \check{\lambda}(1 - \xi))/(4 - \check{\lambda}(1 - \xi)) \right\} - 0.5,$$

where Lemma 5 gives the equilibrium effort ratio s^* , which is a function of all parameters — including the degree of homogeneity k of the cost function — except the joy of winning parameter ν (see Corollary 13). The function $\sigma(\cdot)$ chooses the maximum prior parameter μ that induces an equilibrium probability $\theta(s^*; \mu)$ no greater than a value between $2/3$ and $3/4$ — where that value depends on the cost-shifting rule $\check{\lambda}$ and relational emotions ξ — and deducts 0.5 .⁷³

⁷²A study of social costs and benefits of litigation is beyond the scope of this paper. See Shavell (1997), Spier (1997) for discussions of the social suboptimality of private incentives to litigate or settle.

⁷³To see that the function σ exists and satisfies $0 < \sigma \leq 0.5$, suppose $\mu = 0.5$ and fix all other parameters. Remark 11 confirms that the symmetry between the litigants implies equal (equilibrium efforts and) probabilities of success, $\theta(1; 0.5) = 0.5$. Then the properties $\frac{d}{d\mu} \theta(s^*; \mu) > 0$ (from Remark

Corollary 14. *Consider the equilibrium of the Emotional Litigation Game $\mathbb{G}(\xi, \nu, \check{\lambda})$. If any one of the following sufficient conditions holds, then as relational emotion become marginally more negative, litigation expenditure increases; that is, $\frac{d}{d\xi}C^*(\xi, \nu, \check{\lambda}) < 0$.*

1. *The American rule applies, $\check{\lambda} = 0$.*
2. *The relative advantages of the litigants are sufficiently balanced, in the precise sense of $0.5 - \sigma(\xi, \check{\lambda}, k) \leq \mu \leq 0.5 + \sigma(\xi, \check{\lambda}, k)$.*
3. *The cost function $C(\cdot)$ is sufficiently convex, in the precise sense of $k \geq 2$.*

Corollary 14 offers sufficient conditions for concluding that in equilibrium, more negative relational emotions (ξ decreases) lead to a greater litigation expenditure. Intuitively and as confirmed by the expressions for the litigants' emotional payoffs (in equations (24), (25)), more negative relational emotions directly increase the emotional reward of winning, heightening incentives to exert costly efforts. Corollary 11 also reveals that more negative relational emotions strengthen the effects of cost shifting; this indirectly affects each litigant's incentives to exert costly efforts. In equilibrium, the indirect effect on a litigant's effort may or may not be in the same direction as the direct effect. However, litigation expenditure — which sums the litigants' costs of exerting efforts in equilibrium — increases if both litigants exert more efforts, or if one litigant's exertion of additional effort is not offset by a more rapid reduction in effort by the other litigant. Corollary 14 identifies sufficient conditions for concluding that overall, the direct and indirect effects of more negative relational emotions increase litigation expenditure.

Part 1 of Corollary 14 reveals that one such sufficient condition is the application of the American rule to allow for no recovery of the winner's costs from the loser ($\check{\lambda} = 0$). Intuitively, the absence of cost shifting removes the indirect effect that more negative relational emotions (decreased ξ) have on equilibrium efforts. Independently of the cost-shifting rule, more negative relational emotions directly increase each litigant's equilibrium effort (see Corollary 11). In the absence of any countervailing indirect effect arising from the "scaling up" of the cost-shifting rule, a greater litigation expenditure follows.

Parts 2 and 3 of Corollary 14 utilize Corollaries 8 and 9 in chapter 2 of this thesis, which contain sufficient conditions for concluding that more *cost shifting* increases litigation expenditure in a Monetary Litigation Game. Intuitively, more cost shifting generally

12) and $0.5 < 2/3 \leq (3 - \check{\lambda}(1 - \xi))/(4 - \check{\lambda}(1 - \xi))$ imply there exists some $\mu' \in (0.5, 1]$ satisfying $0.5 < \theta(s^*; \mu') \leq (3 - \check{\lambda}(1 - \xi))/(4 - \check{\lambda}(1 - \xi))$. The value of σ is uniquely determined by the maximum of all such μ' .

increases litigation expenditure. This is because more cost shifting reduces each litigant's expected marginal cost of exerting effort by shifting away a greater proportion of her costs if she wins. More cost shifting also widens the difference in monetary outcome between winning and losing by increasing the recoverable-costs component of the "prize" of winning. These observations and their results apply to the present analysis because the present Corollary 11 reveals that relational emotions and the cost-shifting rule reinforce each other. Part 2 of the present Corollary 14 proves that if relative advantages of the litigants are sufficiently balanced, then more negative relational emotions (ξ decreases) increase litigation expenditure. A litigant must have very poor prospects of success to reduce equilibrium effort — which further harms her prospects of success — in order to save costs. In cases characterized by sufficiently balanced relative advantages, no litigant has very poor prospects of success. Hence, in these cases, a decreased ξ incentivizes the litigants collectively to exert more equilibrium efforts, leading to a greater litigation expenditure. The function σ defines what is required for relative advantages to be "sufficiently balanced" in this sense; it marks the upper and lower bounds within which the relative-advantages parameter μ is sufficiently balanced. Moreover, part 3 of Corollary 14 proves that if the cost function is sufficiently convex ($k \geq 2$), then a decreased ξ increases litigation expenditure. This holds even in extreme cases falling outside the scope of part 1 or 2.

Corollary 14 thus provides general sufficient conditions for concluding that more negative relational emotions lead to a greater litigation expenditure (in equilibrium). To fall outside the scope of any one of these sufficient conditions, a case needs to meet *all* of the following requirements: the American rule does not apply, $\check{\lambda} > 0$; the relative advantages of the litigants are sufficiently extreme in the sense of $\mu < 0.5 - \sigma$ or $\mu > 0.5 + \sigma$; and the cost function is insufficiently convex in the sense of $k < 2$.⁷⁴

Corollary 15 reveals how changes in the joy of winning affect (individual) monetary payoffs and (collective) litigation expenditure. To facilitate presentation, let $u_P^*(\xi, \nu, \check{\lambda})$ and $u_D^*(\xi, \nu, \check{\lambda})$ respectively denote Plaintiff and Defendant's equilibrium monetary payoffs (see equations (22), (23)).

Corollary 15. *Consider the equilibrium of the Emotional Litigation Games $\mathbb{G}(\xi, \nu, \check{\lambda})$.*

1. *A greater joy of winning ν decreases each litigant's monetary payoff. Formally,*

$$\frac{d}{d\nu} u_P^*(\xi, \nu, \check{\lambda}) < 0 \text{ and } \frac{d}{d\nu} u_D^*(\xi, \nu, \check{\lambda}) < 0.$$

⁷⁴See chapter 2 of this thesis for extreme examples that fall outside the scope of Corollary 14.

2. A greater v leads to a greater litigation expenditure. Formally, $\frac{d}{dv}C^*(\xi, v, \check{\lambda}) > 0$.

Compared to Corollary 14, Corollary 15 establishes a more general result regarding the monetary implications of changes in the joy of winning v : a greater v increases (collective) litigation expenditure in equilibrium, even in exceptional cases falling outside the scope of Corollary 14.⁷⁵ A greater v increases the marginal benefits of exerting costly efforts to win *without* affecting the equilibrium probabilities of success (see Corollary 13). However, unlike relational emotions (ξ), v does not interact with the cost-shifting rule (see Corollary 11); no variable in this model offsets the heightened incentives to exert costly efforts arising from a greater v . Hence a lower individual monetary payoff and a greater litigation expenditure follow.

Remark 12. *Corollary 11 implies that changes in the proportion of costs recoverable $\check{\lambda}$, Plaintiff's relative advantages μ or the cost function C have the same equilibrium implications in the Emotional Litigation Game and in its transformed Monetary Litigation Game. Chapter 2 of this thesis offered a detailed analysis of these equilibrium implications in a Monetary Litigation Game. A brief summary of the relevant corollaries is as follows:*

1. *In cases where one litigant is relatively more advantageous ($\mu \neq 0.5$), an increase in $\check{\lambda}$ increases that litigant's equilibrium relative effort and probability of success. However, there is no such equilibrium effect if the relative advantages are equal ($\mu = 0.5$).*
2. *In cases where the cost-shifting rule allows for recovery ($\check{\lambda} > 0$), an increase in μ increases Plaintiff's equilibrium relative effort and probability of success. However, if the American rule applies ($\check{\lambda} = 0$), then an increase in μ does not affect Plaintiff's equilibrium relative effort, but it increases her equilibrium probability of success.*
3. *In cases where one litigant is relatively more advantageous ($\mu \neq 0.5$) and the cost-shifting rule allows recovery ($\check{\lambda} > 0$), an increase in k decreases that litigant's equilibrium relative effort and probability of success. In the other cases ($\mu = 0.5$ or $\check{\lambda} = 0$), an increase in k has no such equilibrium effect.*

Moreover, more cost shifting ($\check{\lambda}$ increases) generally leads to a more predictable outcome, in the sense of driving the equilibrium probability of success $\theta(s^*; \mu)$ closer to 1 (respectively, 0) if Plaintiff (Defendant) is relatively more advantageous, $\mu > 0.5$

⁷⁵Corollary 15 reinforces an existing finding of Baumann and Friehe (2012b) (at pp. 196, 203-04) that the equilibrium implications of introducing outcome-dependent emotions is similar to raising the stakes.

(< 0.5). However, a greater $\check{\lambda}$ typically increases litigation expenditure. Depending on the properties of the success function and cost function, a greater $\check{\lambda}$ also may increase or decrease accuracy in outcome, which is measured by the difference between the prior and equilibrium probabilities of success, $|\theta(s^*; \mu) - \mu|$.

3.5 Discussion

Focusing on issues pertaining to civil litigation,⁷⁶ this section discusses some normative implications of the Emotional Litigation Game and future research directions.

3.5.1 Cost Shifting

There is a debate about the positive implications and normative merits of cost-shifting rules. For instance, using a generalized contest model with purely self-interested litigants, chapter 2 of this thesis extended the result — first observed by Braeutigam et al. (1984), Katz (1987) and Plott (1987) — that more cost shifting tends to increase costs in litigated cases and distort litigation outcomes in favor of the litigant with stronger prior advantages. Moreover, models based on purely self-interested litigants with non-common priors (for example, Shavell 1982) or information asymmetry (see footnote 84) reveal that decisions to file suit or settle also depend on cost shifting rules. However, the divergence of private and social incentives to litigate or settle implies that cost shifting itself is unlikely to be sufficient to induce the socially optimally number of suits (see Shavell 1997, Spier 1997).

Most relevant to the debate about cost-shifting rules is the following result. Suppose negative relational emotions — meaning a litigant derives value from harming her adversary — are present. Corollary 11 in section 3.3 reveals the result that these negative relational emotions *amplify* the effects of cost shifting. This result reveals that models based on purely self-interested litigants typically *underestimate* the full effects of cost shifting. The presence of preferences to harm an adversary is intuitively plausible in many litigated cases and frequently observed in contest experiments (see section 3.1). Thus a nominally low-powered cost-shifting rule tends to have the practical implications of a higher-powered cost-shifting rule. If the lawmaker (or the judge, when she has discretion over cost shifting) aims to effectuate a particular extent of cost shifting, then in the presence of significant negative relational emotions she should stipulate a nominally weak

⁷⁶See, for example, Quiggin (1993) for an analysis of the implications of incorporating social preferences into welfare analysis.

cost-shifting rule.⁷⁷ As a real-world example, while many common law jurisdictions apply high-powered cost-shifting rules by default, judges often exercise their discretion to effectuate low-powered cost-shifting rules in cases involving emotionally-charged litigants with intertwined and conflicted interests, such as succession disputes.⁷⁸ Our analysis reveals the behavioral-economic foundations of that judicial practice, and supports it. Moreover, a lawmaker or judge who aims to minimize the impact of negative relational emotions should implement the American rule — that each litigant bears her own costs. This is because negative relational emotions *amplify* the effects of cost shifting, but there is “nothing” to amplify if cost shifting is zero.⁷⁹

On the other hand, our analysis does not support adjusting the cost-shifting rule in response to a strong non-monetary joy of winning, which a litigant obtains upon winning the lawsuit rather than upon harming her adversary. A real-life example concerns disputes over properties that have non-monetary value to the litigants.⁸⁰ Unlike relational emotions, such outcome-dependent value does not modify the effects of cost shifting (see Corollary 11 in section 3.3). Moreover, if the judge can observe whether emotions are present in individual cases while the lawmaker cannot, then the present analysis also reveals an advantage of conferral of judicial discretion over cost shifting: it enables the judge to adjust the cost-shifting rule to account for the presence or absence of different types of emotions.

3.5.2 Mediation

Mediation has arisen as a popular alternative to formal adjudication by a court of law. Mediation typically involves an independent specialist who facilitates discussions between the disputants and helps them reach an agreement regarding their dispute. Although participation in mediation is typically voluntary, some jurisdictions make it a mandatory prerequisite to obtaining a formal court hearing. For example, in Australia and California, mediation is usually compulsory for disputes concerning parenting arrangements after

⁷⁷More precisely, in the presence of negative relational emotions $\xi < 0$, to effectuate the effects of a cost-shifting rule characterized by the loser bearing $0 < \lambda \leq 1$ proportion of the winner’s costs requires stipulating a weaker cost-shifting rule characterized by $\check{\lambda} = \lambda/(1 - \xi)$.

⁷⁸For example, see the opinion of Justice Gaudron in the Australian case of *Singer v Berghouse* (1993) 114 ALR 521, [6].

⁷⁹More precisely, following on from footnote 77, if $\lambda = 0$, then $\check{\lambda} = 0$ regardless of ξ .

⁸⁰For example, according to the Restatement (Second) of Contracts, section 360, comment e (American Law Institute, 1981), in American common law jurisdictions “land has long been regarded as unique and impossible of duplication by the use of any amount of money”. Restatements are authoritative statements of American law.

a divorce.⁸¹ In New South Wales and Ontario, mediation is compulsory for common succession disputes.⁸² Aside from adopting less formal procedures to reduce legal costs, mediation is typically designed to reduce acrimony between the disputants.⁸³

The present analysis provides behavioral-economic justifications for using mediation to resolve disputes involving emotionally-charged litigants, such as divorce and succession disputes. Mediation can be modeled as a contest in which the disputants exert costly efforts to increase their respective shares of the surplus arising from not proceeding to formal adjudication. To the extent that mediation achieves its stated goal of reducing acrimony, the present analysis reveals a corresponding reduction in the disputants' costs; Corollary 14 reveals that more positive relational emotions typically reduce costs, so does a reduced (non-monetary) joy of winning according to Corollary 15. These results together suggest that, *ceteris paribus*, mediation by reducing emotional motivations typically leads to positive monetary consequences for the disputants.

3.5.3 Unresolved Questions: Filing, Settlement, and Causes of Emotional Considerations

An important qualification of the present analysis is that decisions to file suit or settle are beyond the scope of the Emotional Litigation Game. Katz and Sanchirico (2012) (at pp. 278-87) and Spier (2007) surveyed the vast economic literature on filing and settlement decisions. Shavell (1982) and others employed non-common-prior models to analyze these decisions under extreme cost-shifting rules that shift either all or none of the winner's costs to the loser. An alternative approach uses contract-theoretic models to explain settlement outcomes when the litigants have asymmetric information.⁸⁴ These models typically assume the litigants are purely self-interested. Moreover, the empirical work of Farmer and Tiefenthaler (2001) and others (see the survey in Baumann and Friehe 2012b at pp. 197-99) employ bargaining models to explore how emotions affect settlement outcomes. To account for emotional preferences in a general, contest-theoretic framework, future research may modify the present Emotional Litigation Game to an extensive-form game that models pre-litigation filing or settlement decisions. The litigants' emotional

⁸¹See subsection 60I(7) of the (Australian) Family Law Act 1975 (Cth) and section 3170 of the California Family Code (2005).

⁸²See Supreme Court of New South Wales Practice Note No. SC EQ 7, 7 and Ontario Rules of Civil Procedure rule 75.1.

⁸³ See, for example, subsection 3161(a) of the California Family Code (2005).

⁸⁴For example, Bebchuk (1984), Bebchuk and Chang (1996), Klement and Neeman (2005), Polinsky and Rubinfeld (1998), Reinganum and Wilde (1986), Spier (1994) and Talley (1995)).

payoffs in the equilibrium of the Emotional Litigation Game are their outside payoffs for failing to settle in the pre-litigation stages of such an extensive-form game. Through changing these outside payoffs, variations in emotional variables in litigation (see section 3.4) may have filing or settlement implications.

The present analysis also does not consider the causes of emotional motivations. The present Emotional Litigation Game directly includes emotional variables in individual payoffs, and offers experimental findings as evidence (see section 3.1). This exogenous approach reveals the profound implications of these variables, but is silent on their causes. There are strong evolutionary foundations for the prevalence of social behaviors that appear inconsistent with pure self interest (see, for example, Bergstrom and Stark 1993).⁸⁵ Postlewaite (1998) discussed the advantages and disadvantages of different approaches to incorporating social variables. In particular, while there is a strong evolutionary argument for the exogenous approach, the alternative approach of endogenously deriving individual concerns for social variables is better at explaining how standard economic variables give rise to social concerns. For instance, Cole, Mailath, and Postlewaite (1992) constructed a matching model in which non-market decisions endogenously generate concerns for relative outcomes, and applied that model to explain cross-country differences in economic growth rates. Charness and Rabin (2002) (in appendix 1) and Rabin (1993) offered models that endogenously generate intentions-based reciprocity. Sano (2014) applied Rabin's (1993) approach to study a symmetric Tullock contest with a linear cost function. Future research may generate intentions-based reciprocity in the present contest model in order to capture whole classes of success functions, cost functions and asymmetric relative advantages.

⁸⁵Bergstrom (2002) surveyed a variety of game-theoretic tools used to model the evolution of social behaviors.

4 Contests with Non-Specified Success Functions and Non-linear Spillovers

4.1 Introduction

A contest is a game in which each player incurs expenses to increase her probability of winning a prize. Real-world scenarios answering that description are ubiquitous. A presidential candidate spends on political campaigns to increase her chance of winning the election. A litigant incurs legal expenses to increase her chance of winning the lawsuit. Firms make costly R&D expenses to be the first to obtain a patent and launch an innovative product. Armies employ soldiers and buy weapons to win the war. Contest models have facilitated economic analysis of these and many other scenarios.

A critical component of a contest model is a **success function** that maps the players' strategies into their respective probabilities of winning and losing. The existing contest models typically assume the success function takes a specific functional form (see Serena and Corchón's 2017 survey). While the chosen functional form may appropriately capture some types of contests in the real world, it may poorly capture other contests. It can be hard to ascertain which of the existing functional forms is "ideal". Moreover, while specifying a "nice" functional form can simplify the solution process, the resulting positive predictions and policy recommendations may not be robust to alternative functional forms. To ensure that the implications of the contest model do not depend on the modeler's idiosyncrasies, the theoretical foundations of the contest model should be sufficiently robust.

We therefore propose a general theory of contests. Instead of assuming a particular functional form for the success function, we build a robust theory that allows for a large class of success functions. Imposing only general and reasonable assumptions on the success function *without* specifying its functional form expands the descriptive scope of the model to cover a whole class of contests. Economic analysis premised on a success function that satisfies general assumptions has broader implications than those premised on a success function that takes a specific functional form. The proposed model captures a large class of success functions, including functions frequently used in the existing contest literature, and their convex combinations. Ensuring that the proposed assumptions are closed under finite convex combinations enables the present model to capture uncertainty regarding the success functions themselves. Consider a scenario in which the players do not know which one of a finite collection of success functions will operate to determine

their respective probabilities of winning and losing. Suppose the probabilities that each player assigns to these success functions are common knowledge. Subsection 4.6.1 reveals this scenario falls within the proposed model.

The existing contest models also typically assume that the players have homogeneous success functions (see Serena and Corchón's 2017 survey). While this assumption captures the standard case in which the players believe their probabilities of success sum to 1 and that fact is common knowledge, this assumption does not cover some well-documented behavioral traits. In particular, a body of empirical literature finds that judgments of probabilities commonly exhibit optimism or pessimism (see, for example, Kahneman and Tversky 1977, Radcliffe and Klein 2002). Contest models that assume homogeneous success functions can fail to cover common and important real-world phenomena, and thus give rise to insufficiently robust conclusions.

We introduce and axiomatize a model that allows the players to have heterogeneous success functions. Our formulation covers the standard case in which the probabilities of success functions sum to 1 and that fact is common knowledge. As section 4.4 will illustrate, the present formulation also captures the non-standard case of players collectively showing optimism (they believe their probabilities of success sum to more than 1) or pessimism (they believe their probabilities of success sum to less than 1). In the non-standard case, each player thinks that she holds correct beliefs and that her opponent is the only optimistic or pessimistic player.

Many real-world contests, such as R&D and litigation, have players whose strategies generate **spillovers** (or externalities) to each other. The existing literature often assumes that spillovers are linear functions of the players' strategies (see, for example, Baye, Kovenock, and De Vries 2012, Chowdhury and Sheremeta 2011a). While assuming linear spillovers has simplification benefits, this assumption can result in poor approximations of spillover functions with high curvatures. Relaxing the linearity assumption, the present model allows spillovers to be homogeneous functions of the players' strategies. In fact, the present model goes further to allow a player to generate spillovers that have *different* degrees of homogeneity in affecting herself and in affecting her opponent (see section 4.2). The degrees of homogeneity characterize the returns to scale in generating spillovers. Thus, by permitting non-linear spillovers and cross-player differences in returns to scale, the present formulation advances the realism and robustness of contest theory.

In the **Contest Game** set up in section 4.2, two risk-neutral players simultaneously choose expenses to compete for a prize. Their expenses are inputs in their potentially

different success functions, which do not take a functional form. Their expenses also may generate potentially different spillovers to the winner or the loser of the contest. We propose general and reasonable assumptions that roughly require: the players' expenses to be similarly effective in affecting probabilities and in generating spillovers; each player's success function to be increasing with her expenses and sufficiently concave; the spillovers to be sufficiently small so that they alone do not incentivize a player to spend. The players may have different beliefs regarding the success functions, but the parameters and payoff functions are in common knowledge. Section 4.3 proves the existence of a nontrivial Nash equilibrium comprising positive expenses by both players.

The present assumptions capture well-documented behavioral traits that, to our best knowledge, fall outside the scope of existing contest models. For example, the Conquest Game set up in section 4.4 specializes the Contest Game to capture optimism and pessimism in military decisionmaking. In the Conquest Game, the winner takes the resources of both players less the dissipation arising from their military expenses and any additional destruction. The players believe that their respective probabilities of success may *not* sum to 1. In the unique equilibrium, the relatively more optimistic player spends relatively *less*, but she believes that she is more likely to win. As the players become collectively more optimistic, their total equilibrium expenses decrease.

The present assumptions subsume oft-used success functions, in particular, the ratio-form Tullock success function that expresses a player's probability of winning as her share of total expenses. Adopting an asymmetric Tullock success function and assuming the players believe their respective probabilities of success sum to 1, the R&D Game set up in section 4.5 specializes the Contest Game to capture asymmetric advantages in R&D contests. Total equilibrium expenses decrease if the loser receives more spillovers. Moreover, changes in spillovers have a greater impact on total equilibrium expenses if the players' relative advantages become more balanced. This result suggests that legal mechanisms that change R&D spillovers — such as intellectual property law — are more effective in scenarios involving similarly competitive R&D firms than in scenarios involving a dominant firm.

These applications of the Contest Game confirm the benefits of generalizing contest models. Adopting an asymmetric Tullock success function, the R&D Game suggests that the effectiveness of legal mechanism to alter R&D spillovers depends on the balancedness or extremity of the R&D contest. This novel result illustrates that a general contest model, such as the present one, can give rise to novel positive predictions and normative

recommendations that the existing, more specialized models cannot obtain (see subsection 4.5.2). Moreover, the generality of the present model can facilitate verification of whether the positive predictions and normative recommendations obtained in the existing literature remain valid under weaker assumptions. Illustrating this point is a modification of the R&D Game that reveals, for some non-Tullock success function, the conventional wisdom that a more balanced contest leads to greater incentives to spend *does* not hold (see subsection 4.5.3).

Contest theory traces back to Tullock's (1967, 1980) and Krueger's (1974) analyses of rent-seeking behaviors. Cornes and Hartley (2005), (2012) and others refined Tullock's model to introduce risk aversion and general technologies. Menezes and Quiggin (2010) proved that the standard Tullock contest is strategically equivalent to an oligopsonistic market in which expenditure is the strategic variable. Baye and Hoppe (2003) established that many innovation tournaments and patent-race games are strategically equivalent to rent-seeking contests, including the standard Tullock contest. Building on Katz (1987) and others, chapter 2 of this thesis constructed a contest model of civil litigation that captures whole classes of success functions and cost-shifting rules that shift a proportion of the winner's legal costs to the loser. Serena and Corchón (2017) offered a recent survey of the contest-theory literature, while Konrad (2009) and Vojnović (2016) provided textbook treatments. Moreover, an extensive experimental literature has developed to document loss aversion and social preferences in contests (recently, Chowdhury et al. 2018), as well as a variety of other non-standard behavioral traits (see Dechenaux et al.'s 2015 survey).

Taking an axiomatic approach to capture both standard and non-standard behavioral traits, this paper builds upon efforts to microfound success functions, especially those originating from Tullock's ratio-form success function.⁸⁶ Among the seminal work are Skaperdas's (1996) axiomatization of ratio-form success functions and Clark and Riis's (1998) extension to allow for asymmetric players. Critical to the ratio form is an assumption of independence of irrelevant alternatives (IIA), which, in a two-player model, requires a player to win almost surely if she is the only player who incurs positive expenses. That assumption is not imposed on the present Contest Game (see section 4.4 and subsection 4.6.3). The present axiomatization thus offers microfoundations for more flexible success functions, allowing the theory to match a wide range of contests. Indeed, the present

⁸⁶Serena and Corchón (2017) contains a survey of alternative approaches to microfound success functions, such as deriving them as the optimal choice of a contest designer (for example, Corchón and Dahm 2010, Polishchuk and Tonis 2013), exposing their stochastic components (for example, Dixit 1987, Jia 2008), and laying their Bayesian foundations (Skaperdas and Vaidya 2012).

axiomatization captures success functions that violate IIA, such as those proposed by Plott (1987) and Beviá and Corchón (2015). However, the present assumptions do not capture difference-form success functions that map differences in the players' strategies into probabilities (for example, Hirshleifer 1989a, Che and Gale 2000).⁸⁷ Moreover, the present assumptions do not allow a player's strategy to be a n -tuple vector of non-negative real numbers, while Rai and Sarin's (2009) axiomatization effort does.⁸⁸

Parallel to the Tullock tradition of contest models — in which no player wins a prize almost surely (in equilibrium) no matter how much she spends — are all-pay or rank-order contests (or auctions) — in which the highest spending player wins the highest prize almost surely. Baye, Kovenock, and De Vries (1996) first characterized equilibria for this class of games, and Konrad (2009) and Serena and Corchón (2017) provided recent surveys. Applications of all-pay contests include political lobbying (for example, Hillman and Riley 1989, Baye, Kovenock, and De Vries 1993, Che and Gale 1998), R&D (for example, Che and Gale 2003), litigation (for example, Klemperer 2003, Baye et al. 2005), and school tracking (for example, Xiao 2016). Studying moral-hazard problems faced by firms, Lazear and Rosen (1981) compared workers' incentives under rank-based compensation and under output-based compensation, and Akerlof and Holden (2012) characterized the optimal rank-based compensation structure. To advance their goals, all-pay contest designers can optimally determine the number and distribution of prizes (Moldovanu and Sela 2001), exploit the contestants' concerns for relative ranking (Moldovanu, Sela, and Shi 2007), or introduce insurance to reimburse the losers (Minchuk and Sela 2017). While the present effort clearly does not attempt to develop all-pay contests, we nonetheless share the same ambition as authors who generalize all-pay contests. In particular, Baye et al. (2012) offer equilibrium characterization of a simultaneous-move, two-player rank-order contests with complete information, in which each player generates affine spillovers that depend on her rank. Siegel (2009, 2010) generalizes all-pay contests to allow for arbitrary cost functions, while Xiao's (2016) model permits convex prize sequences. Moreover, Olszewski and Siegel (2016) approximate the equilibrium outcomes of all-pay contests with a large number of asymmetric players who may have complete or incomplete information. Xiao (2017) recently builds upon Siegel (2009, 2010) to introduce performance spillovers in

⁸⁷Corchón's (2007) survey revealed the extent to which difference-form success functions admit a Nash equilibrium in common scenarios.

⁸⁸A contest model in which a player's strategy is a vector of two variables can capture her spending of resources to increase her probability of success and to sabotage her opponent. For a survey of sabotage in contests, see Chowdhury and Gürtler (2015).

the sense of a player's strategy entering into the other players' cost functions. Sacco and Schmutzler (2008), Bos and Ranger (2014) and Chowdhury (2017) also model all-pay auctions with spillovers in the sense of the winner's bid size affecting her prize of winning.

Section 4.2 constructs the Contest Game. Section 4.3 finds and characterizes an equilibrium with positive expenses by both players. Section 4.4 applies the Contest Game to model optimism and pessimism in military decisionmaking, assuming the players believe their respective probabilities of success may *not* sum to 1. Section 4.5 applies the Contest Game to model asymmetric advantages in R&D contests, in which the probabilities of success sum to 1 but the players are asymmetrically productive. Section 4.6 reveals the extent to which the Contest Game captures: uncertainty regarding the success functions; homogeneous expenses and spillovers; and Tullock contest models. Section 4.7 explores future research directions. Appendix A.4 contains all proofs.

4.2 The Contest Game

The **Contest Game** is a simultaneous-move game of complete information characterized by two risk-neutral players 1 and 2, their common action space $\mathbb{R}_+$, and their payoff functions U_1, U_2 . The payoff functions and parameters are common knowledge.

Let $i \in \{1, 2\}$ represent a generic player and $j \in \{1, 2\} \setminus \{i\}$ her opponent. Players i, j simultaneously choose $e_i, e_j \geq 0$ levels of expenses. Player i values the prize at $v_i > 0$. A twice-continuously-differentiable **success function** $\theta_i : \mathbb{R}_+^2 \rightarrow [0, 1]$ that satisfies additional assumptions to be set out below gives $\theta_i(e_i, e_j)$ as player i 's probability of winning a prize according to her own belief; she believes that her opponent j wins the prize with complementary probability $1 - \theta_i$. Thus player i believes her and the opponent j 's respective probabilities of winning sum to 1.

Remark 13. *It is possible to have $\theta_1 + \theta_2 \neq 1$. We interpret the scenario of $\theta_1 + \theta_2 \neq 1$ as capturing non-common beliefs regarding a "true" success function. Suppose there is a collection of success functions containing θ_1, θ_2 . Before the players choose their expenses, Nature chooses one of these success functions to determine the outcome of the contest. At the time of choosing their respective expenses, both players know the collection of success functions, but they do not observe Nature's choice. Player 1 assigns probability 1 to the event of Nature choosing θ_1 , while player 2 assigns probability 1 to Nature choosing θ_2 . See subsection 4.6.1 for an alternative scenario.*

Both the winner and the loser may experience spillovers. Exogenous scalars $w_{ii}, l_{ii} \in \mathbb{R}$

respectively characterize the **winner's spillovers** and **loser's spillovers** that player i 's expenses e_i generates to herself. Player i receives (self-generated) winner's spillovers $w_{ii}e_i$ if she wins, and loser's spillovers $l_{ii}e_i$ if she loses.

Player i also may receive winner's and loser's spillovers generated by her opponent j 's expenses e_j . Exogenous scalars $w_{ij}, l_{ij} \in \mathbb{R}$ and $k_i > 0, i \neq j$, characterize these spillovers. Player i receives winner's spillovers $w_{ij}e_j^{k_i}$ if she wins, and loser's spillovers $l_{ij}e_j^{k_i}$ if she loses. The exponent k_i characterizes the returns of e_j in generating spillovers to player i .⁸⁹ Example 1 contains a case where $k_i \neq 1$. Call $e_j^{k_i}$ **player j 's effective expenses affecting player i** , and e_i **player i 's effective expenses affecting herself**. Thus player i receives self-generated, intra-player spillovers ($w_{ii}e_i$ and $l_{ii}e_i$) and opponent-generated, inter-player spillovers ($w_{ij}e_j^{k_i}$ and $l_{ij}e_j^{k_i}$).⁹⁰

Player i 's **payoff** is $U_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by

$$U_i(e_i, e_j) = \theta_i(e_i, e_j) \left[v_i + w_{ii}e_i + w_{ij}e_j^{k_i} \right] + [1 - \theta_i(e_i, e_j)] \left[l_{ii}e_i + l_{ij}e_j^{k_i} \right] - e_i, \quad (31)$$

which is her expected outcome measured in monetary terms. She believes that if she wins, then she receives the sum of her prize value and her winner's spillovers, $v_i + w_{ii}e_i + w_{ij}e_j^{k_i}$. She believes that if she loses, then she receives the sum of her loser's spillovers, $l_{ii}e_i + l_{ij}e_j^{k_i}$. Weights $\theta_i(e_i, e_j)$, $1 - \theta_i(e_i, e_j)$ are respectively her probabilities of winning and losing according to her belief. Because she incurs expenses e_i whether she wins or loses, the probability weights do not scale e_i . This specification for the payoff function assumes additive separability of expenses, spillovers and prizes. This means that expenses, spillovers and prizes are perfect substitutes and the players are risk neutral.

We now state and impose the following Assumptions 18-23 to guarantee equilibrium existence.

Assumption 18. *Player i 's success function satisfies $\theta_i(e_i, e_j) = \theta_i(x^{k_i}e_i, xe_j)$ for any scalar $x > 0$.*

Assumption 18 requires player i to believe that the same ratio of effective expenses — $e_i/e_j^{k_i}$ and $x^{k_i}e_i/(xe_j)^{k_i}$ for $x > 0$ — leads to the same probabilities of winning.⁹¹ Intuitively, Assumption 18 requires e_i and e_j to be similarly effective in affecting prob-

⁸⁹As subsection 4.6.2 will reveal, the present specification that each player's expenses linearly produce spillovers to herself captures the more general case of these spillovers being homogeneous functions.

⁹⁰Sections 4.4 and 4.5 will utilize the distinction between these two notions of spillovers to capture real-world scenarios.

⁹¹Section 4.3 will offer diagrammatic representations of Assumption 18.

abilities of winning and in producing spillovers. Because scalars 1 and k_i respectively characterize the returns of e_i and e_j in generating spillovers to player i , Assumption 18 ensures that 1 and k_i also respectively characterize the returns of e_i and e_j in affecting her belief regarding her probability of winning; player i 's belief regarding how expenses affect her spillovers and probability of winning is thus consistent in this sense. Moreover, we aim to define a class of success functions that includes Tullock success functions, because they are the standard in contest theory (see section 4.1 and subsection 4.6.3). In the special case of $k_i = 1$, Assumption 18 reduces to the homogeneity of degree zero property, which is satisfied by Tullock success functions.

Assumption 19. *Player i 's success function θ_i satisfies the following properties:*

1. *Her marginal probability of success with respect to her own expenses e_i is positive and non-increasing, and is bounded above. Formally, for a positive upper bound $\bar{b}$ which may depend on her opponent's expenses e_j ,*

$$0 < \frac{\partial \theta_i}{\partial e_i} \leq \bar{b}, \quad \frac{\partial^2 \theta_i}{\partial e_i^2} \leq 0.$$

2. *Her marginal probability of success with respect to her opponent's effective expenses $e_j^{k_i}$ is non-decreasing, and is bounded below. Formally, for a lower bound $\underline{b}$ which may depend on e_i ,*

$$\frac{\partial \theta_i}{\partial e_j^{k_i}} \geq \underline{b}, \quad \frac{\partial^2 \theta_i}{\left(\partial e_j^{k_i}\right)^2} \geq 0.$$

Together with Assumption 22 below, the upper and lower bounds that Assumption 19 impose prevent player i from have incentives to make unbounded expenses.

Define **spillover differentials** $\delta_{ii}, \delta_{ij} \in \mathbb{R}$ by $\delta_{ii} = w_{ii} - l_{ii}$ and $\delta_{ij} = w_{ij} - l_{ij}$ respectively. The value $\delta_{ii}e_i$ (respectively, $\delta_{ij}e_j^{k_i}$) is the difference between the winner's and loser's spillovers of player i arising from her own expenses e_i (her opponent's expenses e_j). Rearrange player i 's payoff U_i defined by (31) to obtain

$$U_i(e_i, e_j) = \theta_i(e_i, e_j) \left[v_i + \delta_{ii}e_i + \delta_{ij}e_j^{k_i} \right] - (1 - l_{ii})e_i + l_{ij}e_j^{k_i}. \quad (32)$$

Equation (32) expresses player i 's payoff as her expected benefits of winning — the weighted sum of her prize value and spillover differentials, $\theta_i(e_i, e_j) \left[v_i + \delta_{ii}e_i + \delta_{ij}e_j^{k_i} \right]$ —

less her *unweighted* expenses $(1 - l_{ii})e_i$, and plus a term $l_{ij}e_j^{k_i}$ that does not vary with her own expenses e_i . The loser's spillovers $l_{ii}e_i$ reduce her unweighted expenses because $\delta_{ii}e_i$ gives the *additional* spillovers of winning that her own expenses produce.

Assumption 20. *Player i believes her spillovers are sufficiently small, in the following sense:*

$$\delta_{ii}\theta_i(e_i, e_j) + \left[\delta_{ii}e_i + \delta_{ij}e_j^{k_i} \right] \frac{\partial \theta_i}{\partial e_i} < 1 - l_{ii}, \quad (33)$$

where the inequality holds strictly in the limit when $e_i \rightarrow 0+$.

Assumption 20 ensures that player i would have no incentives to make positive expenses if the (non-spillover) prize of winning had zero value to her ($v_i = 0$). Inequality (33) comes from taking the partial derivative of her payoff U_i in equation (32) with respect to her own expenses e_i , and assuming $v_i = 0$. The left-hand side of inequality (33) sums her marginal (expected) benefits of making positive expenses, while the right-hand side her marginal unweighted expenses. Inequality (33) ensures her marginal unweighted expenses to be greater than her marginal benefits if $v_i = 0$.

Assumption 21. *Player i 's success function θ_i is sufficiently concave in the following sense:*

$$\frac{\frac{\partial^2}{\partial e_i^2} \left[\frac{\theta_i(e_i, e_j)}{1 - l_{ii} - \delta_{ii}\theta_i(e_i, e_j)} \right]}{\frac{\partial}{\partial e_i} \left[\frac{\theta_i(e_i, e_j)}{1 - l_{ii} - \delta_{ii}\theta_i(e_i, e_j)} \right]} < 0.$$

Assumption 21 is a technical assumption that ensures player i 's payoff is strictly quasiconcave in her own expenses.

Assumption 22. *Player i believes that her success function and self-generated spillovers satisfy the following properties:*

1. *Her self-generated spillovers do not exceed her expenses. Formally, $1 \geq \max\{w_{ii}, l_{ii}\}$.*
2. *In the special case of her self-generated winner's (respectively, loser's) spillovers covering her expenses, she does not win (lose) almost surely by making infinitely more (less) effective expenses than her opponent j does. Formally,*

$$1 = w_{ii} \quad \Rightarrow \quad \lim_{e_i/e_j^{k_i} \rightarrow +\infty} \theta_i(e_i, e_j) < 1,$$

$$1 = l_{ii} \quad \Rightarrow \quad \lim_{e_i/e_j^{k_i} \rightarrow 0+} \theta_i(e_i, e_j) > 0.$$

Assumption 22 prevents player i from recouping all expenses by generating spillovers alone. Part 1 bounds her marginal benefit of generating spillovers above by 1, her marginal cost of spending. In the special case where her spillovers cover her expenses if she wins (respectively, loses), part 2 prevents her from recovering all expenses almost surely by making infinitely more (less) effective expenses than her opponent does. Assumption 22 is trivially satisfied if her expenses strictly exceed her spillovers, $1 > \max\{w_{ii}, l_{ii}\}$.

Assumption 23. *The players' expenses are insufficiently effective in affecting each other, in the sense of $k_1 k_2 \leq 2$.*

Assumption 23 restricts the effectiveness of the players' expenses in affecting each other. Assumption 23 is trivially satisfied in the special case where each player's expenses linearly affect her opponent, $k_1 = k_2 = 1$. In general, the product of k_1 and k_2 must be sufficiently small (in the sense of $k_1 k_2 \leq 2$) to satisfy Assumption 23. Intuitively, Assumption 23 facilitates equilibrium existence by ruling out significant divergence in relative effective expenses. In all that follows, the previous assumptions hold.

The solution concept adopted is a Nash equilibrium that is **nontrivial** in the sense of comprising positive expenses by both players.

4.3 Equilibrium

This section proves the existence of a nontrivial Nash equilibrium, and provides a characterization of it.

Lemmas 6 and 7 together allow each player's best response to be characterized by her FOC. Appendix A.4 contains all proofs.

Lemma 6. *Player i 's payoff function U_i is strictly quasiconcave in her own expenses, e_i .*

Lemma 7. *Suppose some $e'_i > 0$ satisfies the FOC for player i 's payoff function U_i restricted to a function of one variable, her expenses e_i . Then e'_i is a global maximum for the same restriction of U_i .*

Lemma 6 implies a player's locally-optimal strategy is globally-optimal, holding her opponent's strategy fixed. Lemma 7 then allows a player's FOC to characterize her optimal

strategy. Thus Lemmas 6 and 7 together imply that a pair of positive strategies constitutes a nontrivial Nash equilibrium if and only if it satisfies the following system of FOCs:

$$\begin{cases} \frac{\partial U_1}{\partial e_1} = \frac{\partial \theta_1}{\partial e_1} \left[v_1 + \delta_{11}e_1 + \delta_{12}e_2^{k_1} \right] + \delta_{11}\theta_1(e_1, e_2) - (1 - l_{11}) = 0 \\ \frac{\partial U_2}{\partial e_2} = \frac{\partial \theta_2}{\partial e_2} \left[v_2 + \delta_{21}e_1^{k_2} + \delta_{22}e_2 \right] + \delta_{22}\theta_2(e_2, e_1) - (1 - l_{22}) = 0. \end{cases} \quad (34)$$

where the first two terms in each player's FOC are her marginal benefits (weighted by her probability of winning), and the last term her marginal expenses (unweighted).

We now use Assumption 18 to obtain and characterize the relative effective expenses in any potential equilibrium. Assumption 18 implies that, given a positive constant r_{ii} , player i believes she has the same probability of success $\theta_i(e_i, e_j)$ under any pair of positive expenses (e_i, e_j) satisfying $e_i/e_j^{k_i} = r_{ii}$; some algebra using Assumption 18 reveals

$$\theta_i(e_i, e_j) = \theta_i(r_{ii}, 1) = \theta_i\left(1, r_{ii}^{-1/k_i}\right). \quad (35)$$

If $e_i, e_j > 0$, then $r_{ii} = e_i/e_j^{k_i}$ captures player i 's effective expenses *ratio* affecting her, while $r_{ij} = 1/r_{ii}$ captures her opponent j 's effective expenses ratio affecting player i . A pair of positive constants (r_{ii}, k_i) thus characterizes a class of positive expenses pairs inducing the same probability of success according to player i 's belief. Slightly abusing notation, we utilize equation (35) to denote

$$\theta_i(r_{ii}) = \theta_i(r_{ii}, 1), \quad \theta_i^{(i)}(r_{ii}) = \frac{\partial}{\partial r_{ii}} \theta_i(r_{ii}).$$

Example 1. *Illustrating Assumption 18 is the following modification of the Tullock success function:*

$$\theta_1(e_1, e_2) = \begin{cases} \frac{e_1}{e_1 + e_2^{k_1}} & \text{if } e_1 + e_2 > 0, \\ \frac{1}{2} & \text{otherwise,} \end{cases} \quad (36)$$

where k_1 also captures the effectiveness of player 2's expenses in producing spillovers to player 1 (see player 1's payoff as defined by equation (31)).

Figure 6 depicts three classes of positive expenses pairs, where each class induces the same effective expenses ratio and probabilities according to player 1's success function defined by (36). The black solid curve captures the class inducing $r_{11} = 1$ and $\theta_1 = \frac{1}{2}$ when $k_1 = 2$; this is the case of player 1 believing that, given the returns to her expenses are

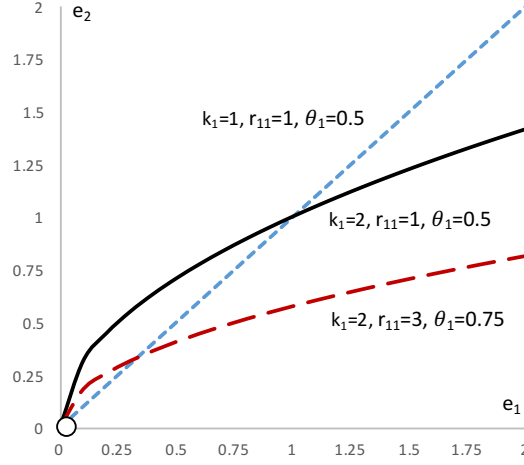


Figure 6: Positive expenses pairs that induce the same probabilities according to success function (36).

half of player 2's, her effective relative expenses ratio is 1 and her probability of success $\frac{1}{2}$. The blue dotted curve represents the class of positive expenses pairs inducing $r_{11} = 1$ and $\theta_1 = \frac{1}{2}$ when $k_1 = 1$, while the red dashed curve the class inducing $r_{11} = 3$ and $\theta_1 = \frac{3}{4}$ when $k_1 = 2$.

Lemma 8 will find a pair of positive effective expenses ratios which will be used to characterize a nontrivial Nash equilibrium. For each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, define a function $\phi_{ij} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ by⁹²

$$\phi_{ij}(r_{ii}) = \frac{v_i \theta_i^{(i)}(r_{ii})}{1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}) - (\delta_{ii} r_{ii} + \delta_{ij}) \theta_i^{(i)}(r_{ii})}. \quad (37)$$

Lemma 8. *There exists a pair of positive constants (r_{11}^*, r_{22}^*) that satisfies the following properties:*

1. *If $k_1 k_2 = 1$, then*

$$r_{11}^* r_{22}^{*k_1} = 1, \quad [r_{11}^* \phi_{12}(r_{11}^*)]^{k_2} = \phi_{21}(r_{22}^*).$$

2. *If $k_1 k_2 \neq 1$, then*

$$r_{11}^{*k_1 k_2} r_{22}^{*k_1} = [\phi_{12}(r_{11}^*)]^{1-k_1 k_2}, \quad r_{22}^{*k_1 k_2} r_{11}^{*k_2} = [\phi_{21}(r_{22}^*)]^{1-k_1 k_2}.$$

Proposition 8 will reveal that the constant r_{11}^* (respectively, r_{22}^*) that Lemma 8 charac-

⁹²Lemma 12 in Appendix A.4 proves that function ϕ_{ij} is strictly positive.

terizes is player 1's (player 2's) relative expenses ratio affecting her in a nontrivial Nash equilibrium.⁹³

Proposition 8. *The Contest Game has a nontrivial Nash equilibrium $(e_1^*, e_2^*) \in \mathbb{R}_{++}^2$ characterized by*

$$e_1^* = r_{11}^* \phi_{12}(r_{11}^*) \quad e_2^* = r_{22}^* \phi_{21}(r_{22}^*)$$

where Lemma 8 defines $(r_{11}^*, r_{22}^*) \in \mathbb{R}_{++}^2$.

Proposition 8 finds and characterizes a nontrivial Nash equilibrium (e_1^*, e_2^*) , which is a function of the pair of positive expenses ratios (r_{11}^*, r_{22}^*) that Lemma 8 characterizes. Upon specifying the functional forms of the success functions, an application of Lemma 8 and Proposition 8 will give close-form expressions for the equilibrium expenses; sections 4.4 and 4.5 will specify the functional forms of the success functions to study military conflicts and R&D.

Corollary 16. *If Lemma 8 defines a unique pair of positive constants, then Proposition 8 characterizes the unique nontrivial Nash equilibrium of the Contest Game.*

Corollary 16 proves that a sufficient condition for the uniqueness of a nontrivial Nash equilibrium is Lemma 8 finding a unique pair (r_{11}^*, r_{22}^*) . Hence adding assumptions to restrict the success functions and spillover functions to ensure a unique (r_{11}^*, r_{22}^*) will guarantee equilibrium uniqueness.⁹⁴

Example 2. *To illustrate closed-form expressions for an equilibrium of the Contest Game, consider a special case with the following features:*

1. *Player 1 values the (non-spillover) prize at $v_1 = 1$, while player 2 values it at v_2 .*⁹⁵
2. *Both players' expenses e_1, e_2 generate no spillovers, $w_{ii} = l_{ii} = w_{ij} = l_{ij} = 0$.*
3. *If $e_1 + e_2 > 0$, then players 1 and 2's respective success functions θ_1, θ_2 have the following functional forms*

$$\theta_1(e_1, e_2) = \frac{e_1}{e_1 + e_2}, \quad \theta_2(e_2, e_1) = \frac{e_2}{e_1^2 + e_2}.$$

⁹³The proof of Proposition 8 reveals that $\phi_{12}(r_{11}^*)$ (respectively, $\phi_{21}(r_{22}^*)$) is player 2's (player 1's) relative expenses ratio affecting player 1 (player 2) in equilibrium.

⁹⁴For an example of a generally-formulated contest model of civil litigation that has a unique nontrivial Nash equilibrium, see chapter 2 of this thesis. For conditions under which multiple equilibrium exist in Tullock contests, see Chowdhury and Sheremeta (2011b).

⁹⁵Assumption $v_1 = 1$ is not a mere normalization assumption. This is due to the non-linearity of beliefs.

If $e_1 + e_2 = 0$, then $\theta_1(e_1, e_2) = \theta_2(e_2, e_1) = \frac{1}{2}$.

4. Players 1 and 2's payoff functions U_1, U_2 are respectively

$$U_1(e_1, e_2) = \theta_1(e_1, e_2) - e_1, \quad U_2(e_2, e_1) = \theta_2(e_2, e_1)v_2 - e_2.$$

To find a nontrivial Nash equilibrium directly would require solving the following system (38) of equations that has a three-degree polynomial:

$$\begin{cases} e_1^3 - 2(1 + \sqrt{v_2})e_1^2 + (2 + v_2 + 2\sqrt{v_2})e_1 - \sqrt{v_2} = 0 \\ e_2 = (\sqrt{v_2} - e_1)e_1. \end{cases} \quad (38)$$

Alternatively, applying Lemma 8 and Proposition 8 finds a nontrivial Nash equilibrium indirectly. To see this, first apply Lemma 8 to find a unique pair of positive constants (r_{11}^*, r_{22}^*) satisfying⁹⁶

$$\begin{cases} \sqrt{v_2}r_{11}^{*3} + 2(\sqrt{v_2} - 1)r_{11}^{*2} + (\sqrt{v_2} - 2)r_{11}^* - 1 = 0 \\ r_{22}^* = (1 + r_{11}^{*-1})^2. \end{cases} \quad (39)$$

An application of Proposition 8 then finds the corresponding nontrivial Nash equilibrium:

$$e_1^* = \frac{r_{11}^*}{(1 + r_{11}^*)^2}, \quad e_2^* = v_2 \left(\frac{r_{11}^*(1 + r_{11}^*)}{r_{11}^{*2} + (1 + r_{11}^*)^2} \right)^2.$$

Figure 7 depicts the equilibrium expenses under different values of v_2 . The red dotted and purple solid curves respectively indicate players 1 and 2's equilibrium expenses. For instance, if $v_2 = 25/16$, then $r_{11}^* = 1$, $r_{22}^* = 4$, and $e_1^* = e_2^* = 1/4$.

4.4 Application: Optimism and Pessimism in Military Conflicts

This section applies the Contest Game to study optimism and pessimism in military conflicts, in particular, the scenario of two warring states spending resources in order to conquer each other. In this scenario, each state makes costly military expenses with

⁹⁶The coefficients in the three-degree polynomial in system (39) change sign exactly once, because it is impossible to have $\sqrt{v_2} - 1 < 0$ and $\sqrt{v_2} - 2 > 0$. Then Descartes's sign rule implies that this polynomial has a unique real root.

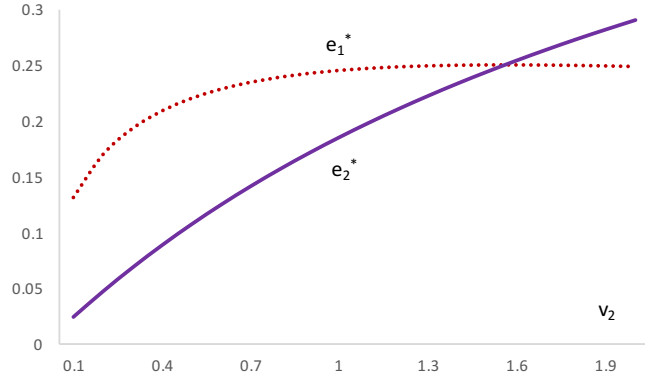


Figure 7: Equilibrium expenses in Example 2.

its own resources, and the winner keeps its own remaining resources and takes over the loser's remaining resources. Dissipating the resources that the winner obtains, the loser's expenses create negative (cross-player) spillovers. Hirshleifer (1989b) studied these military conflicts with Tullock contest models, and Garfinkel and Skaperdas (2000) used a two-period Tullock model to reveal that a player may make military expenses in order to weaken her opponent in the future. Introducing private information regarding the valuation of the resources at stake, Corchón and Yıldızparlak (2013) considered the equilibrium properties of a two-period Tullock model in which the declaration of war in the first period signals information. This subsection introduces a (one-period) contest model that captures optimism and pessimism, which are well-documented behavioral traits (see, for example, Kahneman and Tversky 1977; Weinstein 1980; Radcliffe and Klein 2002; Puri and Robinson 2007). Military texts and history have long recognized the importance of accounting for optimism and pessimism in military decisionmaking.⁹⁷

4.4.1 Setup

Construct the **Conquest Game** by specializing the following features of the Contest Game:

1. The players value the (non-spillover) prize of winning equally; $v_1 = v_2 = v$ for a constant $v > 0$. The prize is interpreted as the sum of the players' individual pre-war resources, $0.5v$. The winner will obtain v by conquest.

⁹⁷For example, The Art of War by Sun Tzu, dating back to the fifth century BC, recommends “[i]f your opponent is temperamental, seek to irritate him. Pretend to be weak, that he may grow arrogant.” Joachim Peiper, a famous World War II military commander, was also famously optimistic.

2. Player i 's success function θ_i has the following functional form:⁹⁸

$$\theta_i(e_i, e_j) = \begin{cases} \eta\mu_i + (1 - \eta)\frac{e_i}{e_1 + e_2} & \text{if } e_1 + e_2 > 0, \\ \mu_i & \text{otherwise,} \end{cases} \quad (40)$$

where constants $\mu_i, \eta \in (0, 1)$ respectively capture player i 's belief regarding her relative advantages in winning the war, and the weight she assigns to μ_i rather than her share of total expenses, $e_i/(e_1 + e_2)$.

3. A constant $\lambda \geq 1$ sums the marginal costs of player i 's expenses and of any additional destruction of the pre-war resources; the marginal cost of such destruction is $\lambda - 1$. Formally, $k_i = 1$, $w_{ii} = 1 - \lambda$, $w_{ij} = -\lambda$, $l_{ii} = 1$ and $l_{ij} = 0$.

4. Player i 's payoff function U_i is given by

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)[v - \lambda e_i - \lambda e_j].$$

In the Conquest Game, the players simultaneously spend e_1, e_2 to determine who will obtain their combined pre-war resources less the dissipation arising from their military expenses and any additional destruction: $v - \lambda e_1 - \lambda e_2$. Being conquered by the winner, the loser obtains no resources and loses all her remaining resources.⁹⁹

The specifications in part 3 serves the following functions. First, the specification $k_i = 1$ ensures the players' military expenses e_i, e_j to be equally effective in producing spillovers to themselves and to each other; this specification removes cross-player asymmetries in producing spillovers. Secondly, the specifications $w_{ii} = 1 - \lambda$ and $l_{ii} = 1$ together simplify calculation and presentation by letting one parameter — λ — to capture how e_i affects player i 's prize of conquest in the event of her victory. Similarly, the specifications $w_{ij} = -\lambda$ and $l_{ij} = 0$ together simplify calculation and presentation by letting λ also to capture how e_j affects player i 's prize of conquest in the event of her victory.

⁹⁸In the special case of $\mu_1 = \mu_2$, the success function defined by (40) takes the relative-difference form that Beviá and Corchón (2015) proposed to combine the desirable properties of ratio-form and difference-form success functions. Assuming $\mu_1 = \mu_2$, function (40) is also similar to Nitzan's (1991) sharing rule in collective contests and the success function that Balart, Chowdhury, and Troumpounis (2017) proposed to use in individual contests. Moreover, assuming $\mu_1 = \mu_2 = 0.5$, function (40) specializes to the one that Plott (1987) applied to study cost-shifting rules in civil litigation. However, while function (40) is confined to two players, the sharing rule in Nitzan (1991) and the success function in Beviá and Corchón (2015) and Balart et al. (2017) can be extended to more than two players.

⁹⁹For example, this specification captures the Norman conquest of England in 1066, as well as the Mongol conquests in the thirteenth century.

4.4.2 Equilibrium

Corollary 17 characterizes the unique nontrivial equilibrium of the Conquest Game.

Corollary 17. *The Conquest Game has a unique nontrivial Nash equilibrium (e_1^*, e_2^*) , which is characterized by*

$$e_1^* = \frac{v(1-\eta)(1-\eta+\eta\mu_2)}{\lambda[2(1-\eta)+\eta(\mu_1+\mu_2)]^2}, \quad e_2^* = \frac{v(1-\eta)(1-\eta+\eta\mu_1)}{\lambda[2(1-\eta)+\eta(\mu_1+\mu_2)]^2}.$$

In equilibrium, players 1 and 2 respectively believe that their probabilities of winning are θ_1^ , θ_2^* given by*

$$\theta_1^* = \eta\mu_1 + \frac{(1-\eta)(1-\eta+\eta\mu_2)}{2(1-\eta)+\eta(\mu_1+\mu_2)}, \quad \theta_2^* = \eta\mu_2 + \frac{(1-\eta)(1-\eta+\eta\mu_1)}{2(1-\eta)+\eta(\mu_1+\mu_2)}.$$

Corollary 17 reveals that the equilibrium expenses differ to the extent of the relative advantages parameters μ_1, μ_2 in the numerator. In the special case of $\mu_1 = \mu_2$, the players' equilibrium expenses are equal. The remainder of this subsection will analyze the equilibrium implications of changes in μ_1, μ_2 .

Intuitively, the parameter μ_i measures player i 's individual tastes for probabilities. When μ_i increases while the pair of expenses (not necessarily the equilibrium pair) is fixed, player i believes that her probability of winning increases. For this reason, call player i **relatively more optimistic ex ante** if $\mu_i > \mu_j$. Conversely, call player i **relatively more pessimistic ex ante** if $\mu_i < \mu_j$.

Corollary 18. *Consider the nontrivial Nash equilibrium of the Conquest Game.*

1. *The player who is relatively more optimistic (respectively, pessimistic) ex ante spends relatively less (more). Formally, $e_i^* > e_j^*$ if and only if $\mu_i < \mu_j$.*
2. *The player who is relatively more optimistic (respectively, pessimistic) ex ante believes in a relatively greater (smaller) probability of winning. Formally, $\theta_i^* < \theta_j^*$ if and only if $\mu_i < \mu_j$.*

Part 1 of Corollary 18 reveals that the player who is relatively more pessimistic ex ante incurs relatively *more* expenses in equilibrium. Intuitively, she does so to offset her relative disadvantages. Part 2 reveals that her relatively more expenses only partially offsets her relative disadvantages; she still believes in a smaller probability of winning compared to her opponent. The converse is true for the player who is relatively more optimistic ex ante.

Corollary 18 adds to the existing analyses of asymmetries in military spending. In particular, using a Tullock success function, Skaperdas and Syropoulos (1997) constructed a two-player model in which each player allocates her resources between military spending and useful production (see Garfinkel and Skaperdas 2007 for a simplified version). One of their equilibrium results is that the player who is relatively less effective in useful production tends to spend relatively more on her military; she is thus more likely to win (Skaperdas and Syropoulos 1997 pp. 105-106, Garfinkel and Skaperdas 2007 pp. 665-667). Unlike their model, the present Conquest Game does not account for differences in productiveness, but captures differences in individual tastes for probabilities with a different success function (see (40)). Without allowing for differences in productiveness, Corollary 18 nonetheless reveals optimism and pessimism *ex ante* as a source of asymmetry in military spending.

To measure the players' collective tastes for probabilities, define the **degree of collective optimism** as $\sigma = \mu_1 + \mu_2 - 1$. Some algebra using equation (40) obtains

$$\theta_1(e_1, e_2) + \theta_2(e_2, e_1) - 1 = \eta\sigma, \quad (41)$$

for all pairs of positive expenses, not just the equilibrium pair. The value of $\theta_1(e_1, e_2) + \theta_2(e_2, e_1)$ sums the players' respective probabilities of winning according to their own beliefs; $\theta_1(e_1, e_2) + \theta_2(e_2, e_1) = 1$ if and only if these beliefs coincide.

A non-zero degree of optimism captures collective optimism or pessimism. When $\sigma > 0$, equation (41) reveals that the probabilities of winning sum to $\theta_1 + \theta_2 > 1$; this captures the scenario of the players being **collectively optimistic**. When $\sigma < 0$, the probabilities of winning sum to $\theta_1 + \theta_2 < 1$; this captures the scenario of the players being **collectively pessimistic**. Because the sum of the players' probabilities of winning is increasing with σ , the players become collectively more optimistic (respectively, collectively more pessimistic) if σ increases (decreases).

Corollary 19. *In the nontrivial Nash equilibrium of the Conquest Game, the players collectively make more (respectively, less) expenses if their degree of optimism σ decreases (increases). Formally,*

$$\frac{d}{d\sigma}(e_1^* + e_2^*) < 0.$$

A surprising result, Corollary 19 reveals that more collective optimism (respectively,

pessimism) decreases (increases) total expenses in equilibrium. Intuitively, as the players become collectively more optimistic, they believe that they can incur less expenses to offset the influence of their relative advantages. The converse is true if the players become collectively more pessimistic.

The result that total equilibrium expenses change in response to optimism or pessimism affects the players' total equilibrium payoffs ex ante, denoted U^* , where¹⁰⁰

$$U^* = U_1(e_1^*, e_2^*) + U_2(e_2^*, e_1^*) = (\theta_1^* + \theta_2^*) [v - \lambda(e_1^* + e_2^*)]. \quad (42)$$

Corollary 20. *In the nontrivial Nash equilibrium of the Conquest Game, as the players' degree of optimism increases, the sum of their ex-ante payoffs increases. Formally,*

$$\frac{dU^*}{d\sigma} > 0.$$

Corollary 20 reflects the direct and indirect effects that collective optimism σ has on total ex-ante payoffs U^* . Directly, a greater σ increases U^* by raising the players' total probabilities of success in equilibrium, $\theta_1^* + \theta_2^*$, according to their own beliefs (see equation (41)). Indirectly, a greater σ decreases the players' total expenses in equilibrium (see Corollary 19); indicating a smaller dissipation of resources arising from military expenses, this indirect effect increases U^* . Thus both the direct and indirect effects of an increase in σ increase U^* .

4.5 Application: R&D Spillovers with Asymmetric Advantages

Returning to the standard case of probabilities of success summing to 1, this section applies the Contest Game to model spillovers in R&D. For reasons including information sharing and imperfect protection of intellectual property, firms may benefit from each other's R&D expenses. D'Aspremont and Jacquemin (1988), Kamien, Muller, and Zang (1992), and Hartwick (1984) were among the first to offer economic analysis of these positive spillovers (or externalities). Using a Tullock contest model with multiple symmetric players, Chung (1996) captured R&D spillovers with a concave prize function that is increasing with total expenses. More recently, Chowdhury and Sheremeta (2011a) constructed a Tullock

¹⁰⁰The value U^* is an ex-ante notion because it sums the players' equilibrium payoffs in expectation, before the winner of the war is determined. After the winner is determined, her ex post payoff is $v - \lambda(e_1^* + e_2^*)$ while the loser's is 0. The sum of payoffs ex-post is thus $v - \lambda(e_1^* + e_2^*)$, which is smaller than (respectively, equal to, greater than) U^* if and only if the degree of optimism $\sigma > 0$ ($= 0$, < 0).

contest model with two symmetric players and a prize function that is linearly separable in each player's expenses; their model captured R&D spillovers by specifying that the prize is increasing with each player's expenses. Building upon their effort, the following will reveal the implications of R&D spillovers when the players have asymmetric advantages.

4.5.1 Setup

Construct the **R&D Game** by specializing the following features of the Contest Game:

1. The players value the (non-spillover) prize of winning equally; $v_1 = v_2 = v$ for a constant $v > 0$.
2. Player 1's success function θ_1 has the following Tullock form:

$$\theta_1(e_1, e_2) = \begin{cases} \frac{\mu e_1}{\mu e_1 + (1-\mu)e_2} & \text{if } e_1 + e_2 > 0, \\ \mu & \text{otherwise,} \end{cases} \quad (43)$$

where the constant $\mu \in (0, 1)$ captures player 1's relative advantages in R&D.¹⁰¹

3. Player 2's success function is $\theta_2 = 1 - \theta_1$. The value $1 - \mu$ captures her relative advantages.
4. Player i 's own expenses give rise to no spillovers to herself; $w_{ii} = l_{ii} = 0$.
5. The players linearly generate the same non-negative spillovers to each other, where these spillovers satisfy Assumption 20. Formally, $k_i = 1$, $w_{ij} = w$ and $l_{ij} = l$ for constants $w, l \geq 0$ satisfying $w - l < \min\{\mu^{-1}(1 - \mu), (1 - \mu)^{-1}\mu\}$.
6. Player i 's payoff function U_i is

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)[v + we_j] + [1 - \theta_i(e_i, e_j)]le_j - e_i. \quad (44)$$

In the R&D Game, the players simultaneously make R&D expenses e_1, e_2 to compete for the prize v . For instance, the prize may be a successful patent. The players' expenses generate non-negative spillovers w, l to each other; the winner and loser receive different spillovers if $w \neq l$. The players are symmetric except in respect of their relative advantages in R&D, measured by μ . Player 1's (respectively, player 2's) probability of winning increases (decreases) with μ .

¹⁰¹The special case of $\mu = 0.5$ captures the success function in the two-player Tullock model with positive spillovers that Chowdhury and Sheremeta (2011a) (at p. 418) constructed.

The specifications in part 5 serves the following functions. First, the specification $k_i = 1$ ensures the players' R&D expenses e_i, e_j to be equally effective in producing R&D spillovers to each other; this specification removes cross-player asymmetries in producing spillovers to each other. Secondly, the specifications $w_{ij} = w$ and $l_{ij} = l$ together simplify presentation by removing the subscript ij , which, in more general settings, captures cross-player differences in producing spillovers to each other (see section 4.2). Thirdly, as an implication of Assumption 20, the specification $w - l < \min\{\mu^{-1}(1 - \mu), (1 - \mu)^{-1}\mu\}$ ensures that equilibrium expenses are positive and bounded above (see Corollary 21 in subsection 4.5.2). This specification requires that the winner's spillovers w be not too much greater than the loser's spillovers l , in order to prevent differences in spillovers to incentivize explosive R&D expenses.

4.5.2 Equilibrium

Corollary 21 provides a closed-form expression for the unique nontrivial equilibrium of the R&D Game. The rest of this subsection will analyze the properties of this equilibrium.

Corollary 21. *The R&D Game has a unique nontrivial Nash equilibrium (e_1^*, e_2^*) , which is characterized by*

$$e_1^* = e_2^* = \frac{v\mu(1 - \mu)}{1 - (w - l)\mu(1 - \mu)}.$$

In equilibrium, players 1 and 2's probabilities of winning are respectively $\theta_1^ = \mu$, $\theta_2^* = 1 - \mu$.*

Let $\delta = w - l$ denote the (cross-player) spillover differential. Our specification ensures that δ is bounded above by $\min\{\mu^{-1}(1 - \mu), (1 - \mu)^{-1}\mu\}$, and does not impose a lower bound; δ potentially may be negative. To player i , δ interacts with her opponent's effort e_j . Suppose e_j increases by one unit. Then δ is the spillover premium that player i receives from winning rather than losing. In other words, δ is the per-unit spillover gain of winning. Thus δ is a part of the stakes of the contest.

Let $\tau = \mu(1 - \mu)$ measure the balancedness of the players' relative advantages; τ increases (respectively, decreases) if their relative advantages become more balanced (extreme), that is, $|\mu - 0.5|$ decreases (increases).

Corollary 22 considers how changes in δ and τ affect equilibrium expenses in the R&D Game.

Corollary 22. *In the R&D Game, a greater spillover differential increases total expenses in equilibrium. Formally,*

$$\frac{d}{d\delta}(e_1^* + e_2^*) > 0. \quad (45)$$

Moreover, when the relative advantages of the players become more extreme (that is, τ decreases), the magnitude of the resulting increase in total equilibrium expenses becomes smaller, and diminishes in the limit. Formally,

$$\frac{d}{d\tau}\left(\frac{d}{d\delta}(e_1^* + e_2^*)\right) > 0, \quad (46)$$

$$\lim_{\tau \rightarrow 0+}\left(\frac{d}{d\delta}(e_1^* + e_2^*)\right) = 0. \quad (47)$$

Inequality (45) in Corollary 22 first reveals that a greater spillover differential δ — for instance, due to a player receiving *smaller* loser's spillovers (l decreases) or greater winner's spillovers (w increases) from her *opponent* — heightens the players' collective incentives to spend on R&D. Chowdhury and Sheremeta (2011a) (at p. 418) found such a result in a symmetric Tullock contest, and the present Corollary 22 confirms the result continues to hold in the presence of asymmetric advantages.

To see the intuition underlying inequality (45), rewrite player i 's payoff function in equation (44) as

$$U_i(e_i, e_j) = \theta_i(e_i, e_j)[v + \delta e_j] - e_i + l e_j, \quad (48)$$

where the component $v + \delta e_j$ sums player i 's total marginal benefits of winning rather than losing the R&D contest; δ is a part of that component. Suppose δ increases. In response to the resulting increase in the stakes of the contest, player i has incentives to increase her R&D expenses, e_i . A greater δ and a greater e_i then increase the stake of the contest from player j 's perspective, heightening her incentives to spend as well. As inequality (45) confirms, this chain reaction continues until the players reach an equilibrium with greater collective expenses than before the increase in δ .

Now consider the case of asymmetric advantages ($\mu \neq 0.5$). Inequality (46) in Corollary 22 reveals that when the players' relative advantages become more extreme (τ decreases), changes in the spillover differential δ have a smaller impact on incentives to spend on R&D. Measuring how δ affects total equilibrium expenses at the margin, the value

of the derivative $\frac{d}{d\delta}(e_1^* + e_2^*)$ decreases when τ decreases, according to inequality (46). Thus inequality (46) reveals that changes in δ have a smaller impact on R&D expenses as τ decreases. Indeed, equation (47) further reveals that in the limit when τ approaches 0 — indicating an extreme contest in which one player has absolute advantages over her opponent — the value of $\frac{d}{d\delta}(e_1^* + e_2^*)$ approaches zero. In other words, as the contest becomes very imbalanced, changes in δ have a very small impact on incentives to spend on R&D.

Corollary 22 has policy implications. Suppose a social planner wishes to increase R&D spending. Intellectual property law is among the legal mechanisms that she may use to alter the extent of R&D spillovers. For example, a race to invent a new product is a R&D contest, and strengthening patent protection can reduce the extent to which the loser can profit from the winner's invention. In the language of the present R&D Game, smaller loser's spillovers (l decreases) reflect strengthened protection of patents (or other intellectual properties). Ceteris paribus, a smaller l leads to a greater spillover differential δ . This results in greater R&D expenses in equilibrium, as inequality (45) in Corollary 22 reveals and the above discussion explains. Corollary 22 thus offers guidance on how to affect incentives to spend on R&D.

Moreover, Corollary 22 suggests the *effectiveness* of policy interventions to affect R&D spending depends on the relative advantages of the players involved. As inequality (46) reveals and the above discussion explains, when the relative advantages of the players become more balanced (τ increases), changes in the spillover differential δ have a greater impact on incentives to spend. Changes in δ capture policy interventions to alter R&D spillovers, for example, strengthening intellectual property protection to decrease the loser's spillovers l . Thus, in terms of affecting incentives to spend on R&D, policy interventions that change R&D spillovers are more effective in scenarios involving R&D firms that are similarly competitive than in scenarios involving a dominant firm.

Let U_i^* denote player i 's payoff in equilibrium. Corollary 23 considers how changes in parameters affect the players' total payoffs in equilibrium, $U_1^* + U_2^*$. Their total payoffs do not account for the public benefits of R&D.

Corollary 23. *Consider the equilibrium of the R&D Game.*

1. *If the winner's spillovers increase, then total payoffs increase. Formally,*

$$\frac{d}{dw}(U_1^* + U_2^*) > 0.$$

2. Suppose the winner's spillovers are sufficiently small (respectively, large). Then as the loser's spillovers increase, total payoffs increase. Formally,

$$\begin{aligned} w < \frac{1 + 2\mu(1 - \mu)}{2\mu(1 - \mu)} & \Rightarrow \frac{d}{dl}(U_1^* + U_2^*) > 0 \\ w = \frac{1 + 2\mu(1 - \mu)}{2\mu(1 - \mu)} & \Rightarrow \frac{d}{dl}(U_1^* + U_2^*) = 0 \\ w > \frac{1 + 2\mu(1 - \mu)}{2\mu(1 - \mu)} & \Rightarrow \frac{d}{dl}(U_1^* + U_2^*) < 0. \end{aligned}$$

3. Suppose total spillovers are sufficiently small (respectively, large). Then as the relative advantages of the players become more balanced, total payoffs decrease (increase). Formally,

$$\begin{aligned} w + l < 2 & \Rightarrow \frac{d}{d\tau}(U_1^* + U_2^*) < 0 \\ w + l = 2 & \Rightarrow \frac{d}{d\tau}(U_1^* + U_2^*) = 0 \\ w + l > 2 & \Rightarrow \frac{d}{d\tau}(U_1^* + U_2^*) > 0. \end{aligned}$$

Part 1 of Corollary 23 reveals that more (cross-player) winner's spillovers (w) lead to greater total payoffs in equilibrium, while part 2 reveals that how changes in (cross-player) loser's spillovers (l) affect total equilibrium payoffs depend on w ; if w is small (respectively, large) in the sense described in part 2, then total payoffs in equilibrium increase (decrease) with l .

Part 3 of Corollary 23 reveals that how changes in the players' relative advantages affect their total payoffs in equilibrium depend on total (cross-player) spillovers, $w + l$. If $w + l$ is sufficiently small (respectively, large) in the sense described in part 3, then total payoffs in equilibrium decrease (increase) as the relative advantages become more balanced, that is, τ increases.

4.5.3 Robustness of the Tullock Success Function

A conventional wisdom in the literature on Tullock contest models is that as the relative advantages of the players become more balanced, their total expenses increase.¹⁰² To highlight the importance of robust formulation of success functions, this subsection reveals

¹⁰²For such a result in a Tullock contest with asymmetric technologies, see Cornes and Hartley (2005) (pp. 940-41). For such a result in a model of litigation, see Carbonara et al. (2015) (pp. 6-7). For a recent paper that shows this result may not hold in a general model of litigation, see chapter 2 of this thesis.

that the conventional wisdom has limitations.

First, Corollary 24 confirms that the conventional wisdom holds in the R&D Game, which adopts the Tullock success function.

Corollary 24. *In the R&D Game, as the relative advantages of the players become more balanced, their total expenses in equilibrium increase. Formally,*

$$\frac{d}{d\tau}(e_1^* + e_2^*) > 0.$$

Corollary 24 reveals that the players have heightened incentives to spend on R&D as their relative advantages, measured by τ , become more balanced.

Now construct an **Alternative R&D Game** by modifying the R&D Game only in respect of the success functions. In the Alternative R&D Game, let players 1 and 2's respective success functions θ_1, θ_2 take the following forms:

$$\theta_1(e_1, e_2) = \begin{cases} \eta\mu + (1 - \eta)\frac{e_1}{e_1 + e_2} & \text{if } e_1 + e_2 > 0, \\ \mu & \text{otherwise,} \end{cases} \quad (49)$$

$$\theta_2(e_2, e_1) = 1 - \theta_1(e_1, e_2),$$

where constants $\mu, \eta \in (0, 1)$ are respectively player 1's relative advantages in R&D, and the weight that she assigns to μ rather than to her share of total expenses, $e_1/(e_1 + e_2)$. As some algebra will reveal, the value $1 - \mu$ captures player 2's relative advantages, and η is the weight that she assigns to $1 - \mu$ rather than to her share of total expenses, $e_2/(e_1 + e_2)$.

In the Alternative R&D Game, player i believes that her probability of winning is what her opponent j believes to be her probability of losing; formally, $\theta_i = 1 - \theta_j$. Thus, unlike the Conquest Game (see section 4.4), the players in the Alternative R&D Game are neither collectively optimistic nor collectively pessimistic. Instead, the success function adopted in the Alternative R&D Game captures the intuition that a player's probability of winning the R&D contest can be the weighted average of her relative advantages and her share of total expenses.

Corollary 25 finds and characterizes the unique nontrivial Nash equilibrium of the Alternative R&D Game.

Corollary 25. *The Alternative R&D Game has a unique nontrivial Nash equilibrium*

(e_1^*, e_2^*) , which is characterized by

$$e_1^* = e_2^* = \frac{(1 - \eta)v}{4 - (1 - \eta)\delta}.$$

In equilibrium, players 1 and 2's probabilities of winning are respectively $\theta_1^* = (\mu - 0.5)\eta + 0.5$, $\theta_2^* = (0.5 - \mu)\eta + 0.5$.

Using the close-form expression for the nontrivial Nash equilibrium of the Alternative R&D Game, Corollary 26 ascertains how total expenses in equilibrium respond to changes in the relative advantages of the players, and in the spillovers.

Corollary 26. *Consider the equilibrium of the Alternative R&D Game.*

1. *As the relative advantages of the players change, their total expenses remain the same. Formally,*

$$\frac{d}{d\tau}(e_1^* + e_2^*) = 0.$$

2. *As the spillover differential increases, total expenses increase. Formally,*

$$\frac{d}{d\delta}(e_1^* + e_2^*) > 0. \tag{50}$$

A comparison of Corollary 24 (for the original R&D Game) and the present Corollary 26 first reveals that, due to the adoption in the Alternative R&D Game of the success function (49), changes in the players' relative advantages *no longer* affect their total expenses in equilibrium. Thus the conventional wisdom regarding Tullock success functions does *not* apply to the Alternative R&D Game; in this Game, when the contest becomes more balanced (τ increases), total expenses do not change. This result identifies a limitation of Tullock success functions, and highlights the importance of inquiries into the robustness of contest models.

Moreover, a comparison of inequality (45) in Corollary 22 (for the original R&D Game) and inequality (50) in Corollary 26 (for the Alternative R&D Game) suggests that comparative statics regarding the spillover differential δ are robust. Under the Tullock success function adopted in the original R&D Game and under the success function (49) adopted in the Alternative R&D Game, a greater δ heightens incentives to spend on R&D. This result suggests that equilibrium predictions and policy recommendations regarding δ are robust to different formulations of success functions.

4.6 Remarks on Generality

4.6.1 Uncertain Success Functions

This subsection will illustrate that the Contest Game captures uncertainty regarding the success functions. The approach taken is to establish equivalence conditions, which approach is in the same spirit as Baye and Hoppe (2003) and Chowdhury and Sheremeta (2015).

Construct the **Uncertain Contest Game** by making the following modifications to the Contest Game:

1. At the time of making expenses, player i is uncertain as to which one of a finite collection of $N \geq 1$ success functions $\{\theta_{i(1)}, \theta_{i(2)}, \dots, \theta_{i(N)}\}$ will determine her probability of winning. Each $\theta_{i(z)}$, $z \in \{1, \dots, N\}$, satisfies Assumptions 18-23.
2. It is common knowledge that player i assigns the prior probability $p_{i(z)} \geq 0$ to $\theta_{i(z)}$ being operative, where $\sum_{z=1}^n p_{i(z)} = 1$.
3. Player i 's payoff function is $\tilde{U}_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$\tilde{U}_i(e_i, e_j) = \sum_{z=1}^n p_{i(z)} \left(\theta_{i(z)}(e_i, e_j) \left[v_i + w_{ii}e_i + w_{ij}e_j^{k_i} \right] + [1 - \theta_{i(z)}(e_i, e_j)] \left[l_{ii}e_i + l_{ij}e_j^{k_i} \right] - e_i \right). \quad (51)$$

Player i 's payoff in the present Uncertain Contest Game is the weighted average of her expected monetary outcome in the original Contest Game; the weights are her prior probabilities regarding the operative success function. She assigns the prior probability $p_{i(z)}$ to the event that the success function $\theta_{i(z)}$ will operate to determine her probability of winning. In that event, she obtains her expected monetary outcome in the Contest Game given the success function $\theta_{i(z)}$.

The remainder of this subsection will reveal that the present Uncertain Contest Game is strategically equivalent to the original Contest Game. To see this, construct a success function θ_i by

$$\theta_i(e_i, e_j) = \sum_{z=1}^n p_{i(z)} \theta_{i(z)}(e_i, e_j). \quad (52)$$

Some algebra comparing player i 's payoff U_i (defined by (31)) in the Contest Game and her payoff $\tilde{U}_i$ (defined by (51)) in the Uncertain Contest Game reveals that these payoffs

are equal:

$$\begin{aligned}\tilde{U}_i(e_i, e_j) &= \sum_{z=1}^n p_{i(z)} \theta_{i(z)}(e_i, e_j) \left[v_i + w_{ii} e_i + w_{ij} e_j^{k_i} \right] + \left(1 - \sum_{z=1}^n p_{i(z)} \theta_{i(z)}(e_i, e_j) \right) \left[l_{ii} e_i + l_{ij} e_j^{k_i} \right] - e_i \\ \tilde{U}_i(e_i, e_j) &= \theta_i(e_i, e_j) \left[v_i + w_{ii} e_i + w_{ij} e_j^{k_i} \right] + [1 - \theta_i(e_i, e_j)] \left[l_{ii} e_i + l_{ij} e_j^{k_i} \right] - e_i = U_i.\end{aligned}\quad (53)$$

Proposition 9 will establish that the success function θ_i defined by (52) satisfies Assumptions 18-23.

Proposition 9. *Consider a finite collection of success functions $\{\theta_{i(1)}, \theta_{i(2)}, \dots, \theta_{i(N)}\}$. If each one of them satisfies Assumptions 18-23, then their convex combination also satisfies Assumptions 18-23.*

An application of Proposition 9 reveals that adopting for the original Contest Game the success function θ_i defined by (52) satisfies Assumptions 18-23. Hence Proposition 8 finds and characterizes a nontrivial Nash equilibrium in the Contest Game given the success function θ_i defined by (52). This equilibrium is also a nontrivial Nash equilibrium in the Uncertain Contest Game, due to the equality of payoffs established in (53).

Proposition 9 thus reveals that the original Contest Game captures not only a large class of success functions that appear in the contest theory literature, but also captures the class of convex combinations of these success functions as well. This attests to the generality and robustness of the original Contest Game and of its equilibrium predictions and normative implications.

4.6.2 Homogeneous Expenses and Spillovers

This subsection will illustrate that the Contest Game captures expenses and spillovers that are homogeneous functions.

Construct the **Homogeneous Contest Game** by modifying the Contest Game as follows:

1. Let $y_i \geq 0$ be player i 's choice variable, which is interpreted as her **effort**. A strictly increasing homogeneous function $E_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of degree $k_{ii} > 0$ gives $E_i(y_i)$ as her **costs** of exerting effort.¹⁰³ Let $\theta_i(y_i, y_j)$ denote what she believes to be her probability of winning when she and her opponent j respectively exert y_i, y_j levels of efforts.

¹⁰³We assume the fixed cost of exerting effort is zero. For a discussion of the relevance of fixed costs, see Remark 1 in chapter 2 of this thesis.

2. Homogeneous functions $W_{ii}, L_{ii} : \mathbb{R}_+ \rightarrow \mathbb{R}$ of degree k_{ii} respectively give the winner's and loser's spillovers of player i arising from her own effort y_i ; W_{ii}, L_{ii} have the same degree of homogeneity as E_i . Let the function $\Delta_{ii} = W_{ii} - L_{ii}$ capture the spillover differential arising from y_i and affecting player i .
3. Homogeneous functions $W_{ij}, L_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}$ of degree $k_{ij} > 0$ respectively give the winner's spillovers and loser's spillovers of player i arising from her opponent j 's effort y_j ; these functions may not have the the same degree of homogeneity as player i 's cost function E_i or her opponent j 's cost function E_j . Let the function $\Delta_{ij} = W_{ij} - L_{ij}$ capture the spillover differential arising from y_j and affecting player i .
4. Player i 's payoff is $\widehat{U}_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ given by

$$\widehat{U}_i(y_i, y_j) = \theta_i(y_i, y_j) [v_i + W_{ii}(y_i) + W_{ij}(y_j)] + [1 - \theta_i(y_i, y_j)] [L_{ii}(y_i) + L_{ij}(y_j)] - E_i(y_i). \quad (54)$$

Proposition 10 proves that a change of variables reveals the present Homogeneous Contest Game is strategically equivalent to the original Contest Game.¹⁰⁴

Proposition 10. *Suppose the spillover parameters in the original Contest Game are*

$$w_{ii} = \frac{W_{ii}(1)}{E_i(1)}, \quad l_{ii} = \frac{L_{ii}(1)}{E_i(1)}, \quad w_{ij} = \frac{W_{ij}(1)}{E_j(1)^{\frac{k_{ij}}{k_{jj}}}}, \quad l_{ij} = \frac{L_{ij}(1)}{E_j(1)^{\frac{k_{ij}}{k_{jj}}}}, \quad k_i = \frac{k_{ij}}{k_{jj}}. \quad (55)$$

Then a pair of positive efforts (y_1^, y_2^*) is a nontrivial Nash equilibrium of the Homogeneous Contest Game if and only if it relates to a nontrivial Nash equilibrium (e_1^*, e_2^*) of the original Contest Game via the following transformation:*

$$(y_1^*, y_2^*) = \left(\left(\frac{e_1^*}{E_1(1)} \right)^{\frac{1}{k_{11}}}, \left(\frac{e_2^*}{E_2(1)} \right)^{\frac{1}{k_{22}}} \right). \quad (56)$$

Equation (56) in Proposition 10 characterizes a bijection between a nontrivial Nash equilibrium of the Homogeneous Contest Game $((y_1^*, y_2^*))$ and a nontrivial Nash equilibrium of the original Contest Game $((e_1^*, e_2^*))$. Intuitively, player i faces the same incentives

¹⁰⁴Chowdhury and Sheremeta (2015) offered strategic equivalence conditions and results among two-player symmetric Tullock contests with linear cost and spillover functions. Building upon their efforts, the present section 4.6.2 presents equivalence conditions and results that allow for non-specified success functions, homogeneous cost and spillover functions, and asymmetries therein.

for choosing expenses e_i in the original Contest Game and choosing effort y_i^* in the Homogeneous Contest Game; equation (56) implies $e_i^* = E_i(y_i^*)$. The original Contest Game thus captures the players' incentives when their cost functions and spillover functions are homogeneous as well. This result further attests to the generality and robustness of the original Contest Game and of its equilibrium properties.

4.6.3 Relationship with Tullock Contests

This subsection considers the extent to which the Contest Game generalizes contest models based on the Tullock success function.

Consider a success function $\rho_i : \mathbb{R}_+^n \rightarrow [0, 1]$ that gives $\rho_i(e)$ as the probability of success of player $i \in \{1, \dots, n\}$, $n > 1$, where $e = (e_1, \dots, e_n)$ is the vector of strategies. The following Assumption 24 restricts ρ .

Assumption 24. *The success function ρ satisfies the following properties:*

1. $1 > \rho_i(e) \geq 0$ and $\sum_i \rho_i(e) = 1$; if $e_i > 0$ then $\rho_i(e) > 0$.
2. $\rho_i(e)$ is strictly increasing in e_i and nonincreasing in e_j , $j \neq i$.
3. (Independence of irrelevant alternatives.) $\rho_i(e_1, \dots, e_{k-1}, 0, e_{k+1}, \dots, e_n) = \frac{\rho_i(e)}{1 - \rho_k(e)}$ for every $i \neq k$.
4. (Homogeneity of degree zero.) $\rho_i(e) = \rho_i(xe)$ for every i , every $x > 0$ where $xe = (xe_1, \dots, xe_i, \dots, xe_n)$.

Building on the seminal work of Skaperdas (1996), Clark and Riis (1998) proved that Assumption 24 holds if and only if ρ takes the following asymmetric Tullock form:

$$\rho_i(e) = \frac{z_i e_i^\gamma}{\sum_{j=1}^n z_j e_j^\gamma},$$

where constants $\gamma, z_i, z_j > 0$.

Because the present Contest Game is a two-player model, it does not generalize n -player contest models based on Assumption 24. By comparison, the Contest Game permits individualized success functions (see section 4.4) and differences in returns to expenses (see Assumption 18), while these behavioral traits respectively violate properties 1 and 4 of Assumption 24.

Confined to the scenario of two players sharing the same success function, having equal returns to expenses and generating zero spillovers, Assumption 24 is a special case

of Assumptions 18-23 of the Contest Game. The opposite is not true. To see this, consider the success function of the Conquest Game (see equation (40) in section 4.4), which falls within the scope of Assumptions 18-23. If $\mu_1 = \mu_2$, $e_1 > 0$ and $e_2 = 0$, then player 1's probability of winning is

$$\theta_1(e_1, 0) = 1 - (1 - \mu_1)\eta,$$

which violates property 3 of Assumption 24, because $\theta_1(e_1, 0) < 1 = \theta_1(e_1, e_2)/[1 - \theta_2(e_2, e_1)]$ for any $e_2 > 0$.

4.7 Discussion

Extending the descriptive scope of contest theory, the present Contest Game allows for general and individualized success functions and spillovers. Future research may proceed in several directions.

First, the Contest Game is a two-player model; future research may introduce more than two players. Although two-player contests are common, many real-life contests — such as the U.S. presidential primaries and multi-state wars — have more than two contestants. Group contests such as team sports also have more than two players (see Serena and Corchón's 2017 survey, pp. 25-27). It may be fruitful to explore the implications of general and individualized success functions and spillovers in multi-player contests.

Secondly, extending the Contest Game to multiple periods may capture intertemporal incentives and information revelation over time. These factors have implications in military contests (see Garfinkel and Skaperdas 2000 and Corchón and Yıldızparlak 2013). Many other real-life scenarios also answer the description of dynamic contests (see Serena and Corchón's 2017 survey, pp. 20-23); elimination tournaments and sports leagues are among the prominent examples. Giving a dynamic structure to the Contest Game may reveal how intertemporal incentives and information revelation over time may affect the differences in the players' beliefs regarding the success functions.

Thirdly, future research may modify the Contest Game to incorporate private information. The players in the present Contest Game may have non-common beliefs regarding the success functions, but their common knowledge includes their payoff functions and the parameters characterizing their spillovers. It may be fruitful to explore the role of spillovers by building upon models that assume common priors but permit private information. For example, Einy et al. (2015) proved the existence of a pure-strategy Bayesian-Nash

equilibrium in Tullock contests with private information.

Fourthly, future research may consider modifying the Contest Game to account for the possibility of a draw. Generalizing Skaperdas (1996) and Clark and Riis (1998), Blavatskyy (2010) axiomatized a Tullock contest that permits a draw. Chowdhury (2017) characterized equilibria for an all-pay auction in which the highest bid may fail to win the prize, and the prize value may depend on the bid level. Many real-life contests, such as individual matches in the group stage of the FIFA World Cup, can result in no winner. Modifying the Contest Game to permit a draw may capture these contests without specifying the functional form of the success function.

Finally, future research may build upon the present efforts to develop contest models that capture well-documented behavioral traits. Optimism and pessimism are prominent behavioral traits that fall within the scope of the present Contest Game (see section 4.4). However, the present assumptions do not aim to capture some behavioral traits that consistently appear in contest experiments, such as preferences for relative outcomes (see Dechenaux et al.'s 2015 survey at pp. 614-616). Future research may modify the Contest Game to incorporate these behavioral traits.

5 Restitution for Wrongs: A Theory of Externalities Optimization

5.1 Introduction

5.1.1 Intuition and Contributions

Actions that generate private gain to the actor but impose social harm on someone else are ubiquitous. Coase's (1960) analysis of these problems of **externalities** gave birth to modern law and economics. Consider Coase's hypothetical of a cattle-raiser whose herding activities generate profits to herself but damage a neighboring farmer's crops. Absent reputational concerns or other-regarding preferences, socially suboptimal outcomes may arise because the cattle-raiser has perverse incentives to over-herd without regard to the farmer's harm. When insurmountable obstacles prohibit a contract between the cattle-raiser and the farmer, the standard legal solution to achieving social optimality is to internalize externalities completely: Holding the cattle-raiser legally liable for all of the harm that she imposes on the farmer removes the incentives to over-herd. Example 3 offers a numerical illustration of the problem of externalities and its standard solution.

Example 3. *Suppose the cattle-raiser in Coase's hypothetical is deciding whether to add one extra cattle to her herd. Doing so would cost her \$1 and would allow her to gain \$2. However, adding the extra cattle would impose an additional harm valued at \$h to the farmer's crops; the externalities here sum to $-\$h$ at the margin. Assume both parties value money equally. Adding the extra cattle is socially optimal if the cattle-raiser's marginal net gain ($\$2 - \$1 = \$1$) exceeds the farmer's marginal harm ($\$h$), but is socially suboptimal otherwise. Formally, adding the extra cattle is socially optimal if and only if $\$1 \geq \h .*

The problem of externalities is, in the absence of some legal duty, the cattle-raiser has incentives to add the extra cattle regardless of the farmer's harm. The standard, non-contractual solution to this problem is to internalize externalities. For instance, in the absence of administrative costs, a strict liability rule that holds the cattle-raiser liable for \$h would incentivize her to add the extra cattle if and only if $\$1 \geq \h . The externalities-internalization theory thus generates socially optimal incentives on the part of the cattle-raiser.

The problem that animates this paper is: When insurmountable obstacles prevent

complete internalization of externalities, there is no well-established theory to generate socially optimal incentives. Using a modification of the standard tort model, this paper proposes a more general theory to generate optimal incentives: The optimization of externalities. The proposed theory generalizes the externalities-internalization theory, and achieves social optimality even when complete internalization of externalities fails. Example 4 offers a preview of the proposed theory.

Example 4. *Consider Example 3 again but now assume that insurmountable obstacles prevent: A contract between the cattle-raiser; and a money judgment from compensating more than one quarter of the farmer's harm ($\$0.25h$ at the margin). Complete internalization of externalities is unattainable because three quarters of the farmer's harm ($\$0.75h$ at the margin) would always enter into the calculation of externalities. For instance, a strict liability rule that holds the cattle-raiser liable for $\$0.25h$ would not give her socially-optimal incentives, because she has incentives to add the extra cattle when $\$h > \$1 \geq \$0.25h$.*

There nonetheless remain many legal solutions to the problem of externalities.¹⁰⁵ As the literature has recognized,¹⁰⁶ one simple solution is to impose on the cattle-raiser both a liability to pay $\$0.25h$ and a liability to transfer three quarters of her net gain — $\$0.75$ at the margin — to the farmer. Under these liabilities, the cattle-raiser's marginal utility is $\$0.25 - \$0.25h$; she has incentives to add the extra cattle if $\$1 \geq \h , and she has no incentives to do so otherwise. These liabilities thus generate socially optimal incentives on the part of the cattle-raiser, even though there often remain non-zero externalities netting to $\$0.75 - \$0.75h$ at the margin. In fact, the cattle-raiser's choice optimizes net externalities; she has incentives to add the extra cattle if and only if $\$0.75 \geq \$0.75h$.

There also exist legal solutions that generate socially optimal incentives without imposing any liability for the farmer's harm. One such solution imposes on the cattle-raiser a liability to transfer $0.5h$ proportion of her gross gain ($\$2$ if she adds the extra cattle) to the farmer.¹⁰⁷ Under this liability, the cattle-raiser's marginal utility is $\$2 \times (1 - 0.5h) - \$1 = \$1 - \h , and she has incentives to add the extra cattle if and only if $\$1 \geq \h . Because marginal net externalities sum to $\$2 \times 0.5h - \$h = 0$, her choice also

¹⁰⁵Section 5.3 will reveal that the proposed externalities-optimization theory usually describes uncountably many legal solutions to the problem of externalities.

¹⁰⁶See, for example, Huang (2016) at pp. 1611-12. See generally subsection 5.1.2 for a discussion of the existing literature. See Corollary 27 in subsection 5.3.1 for a formal description of optimal liability to transfer net gain.

¹⁰⁷Such a liability arises in the law of restitution. See generally section 5.4. See Corollary 28 in subsection 5.3.1 for a formal description of optimal liability to transfer gross gain.

optimizes net externalities.

The intuition underlying the proposed externalities-optimization theory is most apparent when a tiny change in the externalities-generating action results in a tiny change in the actor's utility.¹⁰⁸ As the sum of individual utilities, social welfare is also the sum of externalities and non-externalities; the actor's own utility comprises non-externalities. When the actor marginally changes her action, the corresponding *marginal* change in social welfare equals the sum of the marginal change in externalities and the marginal change in non-externalities. The actor has incentives to take an action that leads to *zero* marginal non-externalities; that action maximizes her utility by equating her own marginal benefits with her own marginal costs. Thus *that* action equates marginal social welfare with marginal externalities. This implies that the actor's utility-maximizing action leads to zero marginal social welfare if it leads to zero marginal externalities. In other words, that action optimizes social welfare if it optimizes externalities.

Thus the law may generate socially optimal incentives by altering the respective sizes of externalities and non-externalities. The law may reduce negative externalities by holding the actor liable for some or all of the social harm arising from her action. The law also may create positive externalities by shifting some or all of the actor's (gross or net) gain to others. In other words, the law may "divide" the social welfare "pie" into one "slice" of externalities and one "slice" of non-externalities, and the law may arbitrarily set the respective "slice sizes". Thus the law may achieve an optimal "division", so that when the actor optimizes the "slice size" of non-externalities, her action simultaneously optimizes the "slice size" of externalities. Complete internalization of externalities is just one such optimal "division"; it sets the "slice size" of externalities to zero for all actions, so the actor's utility-maximizing action trivially optimizes that "slice size".

To capture this intuition, section 5.2 makes a modification to the standard tort model: The law may impose on the actor a restitutionary liability to disgorge some proportion of her (gross or net) gain *and* a liability to compensate for some proportion of the social harm. Section 5.3 proves that there usually are uncountably many combinations of restitutionary and compensatory liabilities that optimize externalities. Section 5.3 also proves that, regardless of whether social optimality requires zero or positive action, liabilities that optimize externalities incentivize the actor to take the socially optimal action. Moreover, if social optimality demands positive action, then optimization of externalities is a necessary

¹⁰⁸More precisely, suppose the actor's utility function is differentiable. The externalities-optimization theory does *not* depend that assumption. See appendix A.5.5.

condition for social optimality.

The proposed externalities-optimization theory achieves social optimality in cases where complete internalization of externalities is unattainable. Section 5.3 proves that, in these cases, if some positive action is socially optimal, then the law must leave the actor with some, but not all, of her private gain. The intuition underlying this result is most apparent in a scenario where the law cannot hold the actor liable for any of the social harm. In this scenario, if the law leaves the actor with no gain, then she has no incentives to take any positive action. In contrast, if the law allows the actor to keep all of her gain, then she has perverse incentives to over-act without regarding to the resulting social harm. It follows that the law must allow the actor to keep an intermediate amount of her gain in order to give her incentives to take a positive action that is socially optimal. However, in an alternative scenario where the law holds the actor liable for all social harm, the law must allow her to keep all of her private gain in order to incentivize a positive, socially optimal action. Thus social optimality requires the imposition of restitutionary liability only in cases of incomplete compensation of the social harm.

Not all liabilities that optimize externalities generate the same administrative costs, in particular, information costs. Subsection 5.3.3 reveals the exact information that a judge (or social planner) would need to have in order to implement different types of externalities-optimizing liabilities. A liability that completely internalizes externalities generates the information costs of ascertaining the level of the social harm. In contrast, any other liability that optimizes externalities requires information on the level of the private gain and, potentially, how the underlying gain function changes at the margin. Thus a liability that completely internalizes externalities often generates smaller information costs than any other liability that optimizes externalities. Moreover, if social optimality obviously requires zero action, then there is no need to incur the information costs of externalities optimization because it is a sufficient condition, but not a necessary condition, for social optimality. For example, maximum deterrence via complete restitution is also socially optimal, and generates smaller information costs than intermediate restitution does.

Section 5.4 applies the theory of externalities optimization to explain the cardinal principles of the American law of restitution and unjust enrichment, focusing on its application to wrongful actions. Wrongful actions typically expose the actor to liability in the law of torts to compensate the victim for some or all of the resulting harm. Some wrongful actions, such as a breach of fiduciary duty, expose the actor to liability in restitution to transfer some or all of her wrongful gain to the victim; her liability in

restitution may be exclusive of, or an alternative to, her liability to compensate. As section 5.4 will elaborate, standards resembling “equity” and “conscience” underly the law of restitution and limit the extent of restitutionary liability or remedy. The law of restitution also employs peculiar defenses and limiting principles that lack obvious counterparts in the law of torts. Moreover, there is a tendency to allow restitution of the wrongful gain only if complete compensation of the wrongful harm is unattainable.

The theory of externalities optimization explains a great extent of these cardinal principles of the law governing restitution for wrongs. First, subsection 5.3.3 reveals that it is usually less informationally demanding to achieve social optimality by completely internalizing externalities rather than by optimizing externalities. This observation explains the law’s tendency to limit restitution of the wrongful gain to cases of *incomplete* compensation of the wrongful harm; these are cases of incomplete internalization of externalities. Second, Corollary 28 in section 5.3 reveals that, in cases where the socially optimal action is positive but complete compensation is unattainable, it is socially optimal to disgorge an *intermediate* amount of the actor’s gain. Equitable standards and peculiar defenses and limiting principles may effectuate such intermediate restitution. However, in cases where it is obviously socially optimal to take no action, there is no need to incur the information costs of optimizing externalities. In these cases, the law of restitution should disgorge all wrongful gain, and should not let standards and limiting principles to dilute its ex-ante deterrence effect.

The remainder of this section connects the proposed externalities-optimization theory to the existing literature. Using the model that section 5.2 constructs, section 5.3 establishes the theory of externalities optimization as a general approach to generating socially optimal incentives. Section 5.4 applies this theory to explain the cardinal principles of the law governing restitution for wrongs in the United States. Section 5.5 concludes with a discussion of future research directions. The appendices reveal the externalities-optimization theory remains valid under various model modifications.

5.1.2 Relationship with Existing Literature

This paper builds upon the vast literature on the divergence of private and social interests in the presence of externalities. The problem of externalities has been well-known to economists since Pigou’s (1920) analysis of optimal taxation, and to lawyers since Coase’s (1960) analysis of assignment of property rights under different specifications re-

garding transaction costs (broadly defined). Assuming transaction costs are negligible and information is symmetric, the Coase Theorem holds that any initial assignment of property rights leads to a contract that internalizes externalities to generate socially optimal incentives (Stigler 1966 pp. 112-113). Thereafter, scholars have applied the externalities-internalization theory to explain and evaluate the cardinal principles of contract law (for example, Birmingham 1970, Goetz and Scott 1977, Shavell 1980a), tort law (for example, Brown 1973, Calabresi 1970, Shavell 1980b),¹⁰⁹ criminal law (for example, Becker 1968), and many other areas of law (for example, Posner 1972). This paper generalizes the externalities-internalization theory. The proposed externalities-optimization theory solves the problem of externalities in cases where complete internalization is unattainable, and subsumes the externalities-internalization theory in cases where complete internalization is attainable.

The externalities-optimization theory also explains several apparently-unrelated legal rules that generate socially optimal incentives without completely internalizing externalities. Polinsky and Rubinfeld (2003) proposed a sharing rule to address a lawyer's suboptimal incentives to act for her client when their contingency fee agreement only gives the lawyer a proportion of her client's judgment sum in the event of winning the case. Their proposed rule incentivizes the lawyer to act optimally by shifting away some of her cost of acting so that the remaining proportion of her cost matches her proportion of the judgment sum. Focusing on cases where the plaintiff's entitlement is uncertain and she may seek a preliminary injunction to enjoin a potential breach by the defendant, Brooks and Schwartz (2005) revealed that the plaintiff's incentives to seek the injunction are optimal if, in the event that she loses (at the conclusion of the case), she is liable for the defendant's costs of complying with any injunction sought. Considering scenarios in which the total social harm produced by multiple injurers is verifiable but the individual social harm is not verifiable, Marco, Van Woerden, and Woodward (2009) proposed a cost-sharing rule under which each injurer bears the same proportion of the total private cost (of all injurers' actions) and of the total social harm.¹¹⁰ Aiming to retain optimal

¹⁰⁹See also Dari-Mattiacci (2004), who revealed that traditional tort liability may generate suboptimal incentives in cases where an injurer produces a negative externality to a victim *and* a positive externality to a gainer. He further used the externalities-internalization theory to argue that these three parties' respective liabilities should be decoupled so that each party bears the positive and negative externalities produced by her activity. By comparison, the present paper demonstrates how to generate optimal incentives when the externalities-internalization theory fails.

¹¹⁰Marco et al. (2009) reacted to Cooter and Porat (2007). Considering similar scenarios, Cooter and Porat (2007) proposed a modified negligence rule to hold each injurer individually liable for any total social harm in excess of its socially optimal level. Their proposal generates optimal incentives in a Nash equilibrium because, when other injurers act optimally, an injurer's marginal liability for over-acting matches

incentives when social harm is hard to internalize completely, Huang (2016) proposed to utilize an observation that an injurer has the same incentives under (i) an optimal harm-based liability that is imposed on her almost surely; and (ii) a probabilistic mixture of *that* liability and a restitutionary liability that disgorges her net gain from acting.

The rules that these authors have proposed achieve social optimality by aligning private and social incentives at the margin: At the time of choosing her action, the actor expects that her marginal liability equals the marginal (net) externalities arising from her action. From an ex-ante perspective, these proposals do *not* completely internalize externalities because the actor is not liable for all externalities arising from her action. Instead, these proposals *optimize* externalities: The actor's utility function and the net externalities function are proportionate to the social welfare function, so her utility-maximizing action simultaneously maximizes net externalities and social welfare.¹¹¹ The externalities-optimization theory thus connects these proposals, and offers a unifying explanation of why they generate socially optimal incentives without completely internalizing externalities.

The law-and-economics literature on restitution is thin relative to that on other traditional areas of law. Levmore (1990) (especially at pp. 710-12) informally argued that a restitutionary liability that completely disgorges an injurer's gross gain from failing to take precautions overdeters her, while disgorgement of her net gain results in underdeterrence. Polinsky and Shavell (1994) further established the social suboptimality of a complete restitutionary liability in a one-player tort model that accounts for errors in observing the wrongful gain and harm. Using a two-player model where each player makes decisions affecting her own utility and the probability and consequences of their interaction, Bar-Gill and Porat (2014) showed that a complete restitutionary liability gives the injurer socially suboptimal incentives. Using a one-player model, Cooter and Porat (2015) proposed a negligence rule with limited disgorgement damages to incentivize the socially optimal level of care. These limited damages are designed to offset the negligent injurer's expected gain from taking suboptimal care. Cooter and Porat (2015) built upon their previous idea in more specific contexts (see Cooter and Porat 2006, Cooter and Porat 2014, ch. 10, pp. 179-180), and rebutted the alternative argument that applying a multiplier to restitutionary liability is generally impractical (for example, Levmore 1990 at p. 713).

The model that section 5.2 constructs (and its modifications in the appendices) builds

the marginal increase in the total social harm.

¹¹¹See Corollary 27 in subsection 5.3.1 for a formal statement. The mathematical foundation is that if a function has a maximizer, then a strictly increasing transformation of the function — for example, multiplying the function with a positive constant — has the same maximizer.

upon the existing models of restitution, especially Cooter and Porat's (2015) and Huang's (2016). With some exceptions (for example, Huang 2016), the existing models focus on pure restitution or pure compensation, that is, a wrongful action attracts either a liability in restitution or a liability to compensate, but not both. Using a modification of the standard tort model, the present model captures a mixture or combination of both restitution and compensation. Thus the present model allows for great flexibility in adjusting positive externalities (by arbitrarily specifying the extent of restitution) and negative externalities (by arbitrarily specifying the extent of compensation). That flexibility also enables the present model to capture a prominent characteristic of the American law of restitution: That standards and judicial discretion give rise to ex-ante uncertainty regarding whether a wrongful action attracts restitution or compensation ex post (see subsection 5.4.1). Appendix A.5.2 further reveals that the externalities-optimization theory continues to hold when the victim has a choice between restitution and compensation; existing models do not capture that choice. Moreover, although the optimal restitutionary liabilities that section 5.3 proposes are different from what Cooter and Porat (2015) proposed,¹¹² the present proposal reinforces their idea that some intermediate restitutionary liability is socially optimal. The externalities-optimization theory also offers an alternative interpretation for the rule they proposed.¹¹³

In common law jurisdictions, private law often imposes tortious liability to discourage actions that generate negative externalities, but it rarely imposes restitutionary liability to encourage actions that generate *positive* externalities.¹¹⁴ As a result, an actor may have insufficient incentives to take socially beneficial actions when insurmountable obstacles (such as prohibitive transaction costs) prevent her from contracting with the recipient(s) of these benefits. To remedy this deficiency, Porat (2009) proposed a limited expansion of the

¹¹²More precisely, in section 5.3, optimal intermediate restitutionary liabilities induce the actor to take an action that satisfies the first order condition of her differentiable utility function (see condition (60) in section 5.2). By comparison, optimal disgorgement damages according to Cooter and Porat (2015) (at p. 260) give the injurer a utility function that is *not* differentiable at her chosen action.

¹¹³ More precisely, in Cooter and Porat's (2015) model and under their proposed rule, net externalities sum to $B(x^0) - B(x) - p(x)L$, where x is a negligent injurer's chosen level of care, $B(x)$ is her cost of taking care, $p(x)$ is the probability of an accident, L is the fixed loss that the accident causes, and x^0 is the socially optimal level of care (that minimizes the sum of the cost of taking care and the social harm, $B(x) + p(x)L$). Functions B and p satisfy $B' > 0$, $B'' = 0$, $p' < 0$ and $p'' < 0$. The injurer has incentives to choose the socially optimal level of care under their proposed rule (see Cooter and Porat 2015 p. 260), and her choice also optimizes net externalities.

The same reasoning also reveals that the externalities-optimization theory explains those situations in which traditional negligence rules induce socially optimal outcomes (see, for example, Brown 1973, Shavell 1980b).

¹¹⁴See generally Levmore (1985), discussing various positive explanations of private law's asymmetric imposition of tortious and restitutionary liabilities.

law of restitution to incentivize the creation of public goods. More recently, Porat and Scott (2018) proposed a limited right in restitution as a background rule for benefits-and-costs sharing among the members of a fragile business network. While this paper focuses on negative externalities (see section 5.2) and takes a largely positive approach, its normative implication is largely consistent with the view of Porat (2009) and Porat and Scott (2018): A limited right in restitution is often socially optimality. Moreover, the present focus on negative externalities is without loss of generality; appendix A.5.1 modifies the model set up in section 5.2 to capture actions that generate positive externalities. Thus this paper also provides a formal framework to support the informal analyses of Porat (2009) and Porat and Scott (2018).

However, this paper assumes the absence of a contract between the actor and the victim; high transaction costs (Coase 1960) or asymmetric information (Myerson and Satterthwaite 1983) may justify this assumption. Thus neither the present model(s) nor the externalities-optimization theory adds to the literature on a contractual party's liability in restitution. For instance, Levmore (1993) explored the merits of imposing restitutionary liability on a contractual party who fails to take cost-benefit-justified actions. In a contract-theoretic framework, Brooks and Stremitzer (2012) proved that allowing rescission and restitution as a remedy for (some) breach of contract solves the hold-up problem arising from non-contractible investments that are relationship-specific and cooperative.¹¹⁵

5.2 The Model

5.2.1 Setup

The present **Model** modifies the standard model for intentional torts. There is one utility-maximizing decisionmaker, called the **actor**, who chooses an action level $x \in \mathbb{R}_+$. The actor believes that a real-valued function G satisfying $G' > 0$ and $G'' \leq 0$ gives $G(x)$ as her **gain** arising from choosing x . She believes that her cost of choosing x is to $C(x)$, where C is a real-valued function satisfying $C' > 0$ and $C'' \geq 0$. She also believes that her choice x generates **harm** $H(x)$ to some **victim**, where H is a real-valued function

¹¹⁵Brooks and Stremitzer (2012) used the incomplete contracts/property rights theory — a Nobel prize-winning theory that Grossman and Hart (1986) and Hart and Moore (1988) have developed for understanding the boundary of the firm and a broad range of economic phenomena. For an introduction to the incomplete contracts/property rights theory, see, for example, Aghion and Holden (2011) and Holden (2017). See Holden and Malani (2014) for a discussion of real-world contracts and legal doctrines that implement theoretical solutions to the hold-up problem arising from non-contractible investments that are relationship-specific and selfish.

satisfying $H' > 0$ and $H'' \geq 0$.¹¹⁶

A triple of proportions $(\delta, \gamma, \lambda) \in [0, 1] \times [0, 1] \times [0, 1]$ describes what the actor believes to be the operation of the law in allocating, between her and the victim, the gain, harm and cost arising from the chosen action. Call δ, γ, λ the **gain-allocation rule**, **cost-allocation rule** and **harm-allocation rule** respectively. The law allocates to the actor $1 - \delta$ proportion of the gain $G(x)$, $1 - \gamma$ proportion of the cost $C(x)$, and λ proportion of the harm $H(x)$. The law allocates to the victim the remaining δ proportion of the gain, γ proportion of the cost, and $1 - \lambda$ proportion of the harm. Describing legal consequences by a triple of allocation rules is the sole modification of the standard tort model.

Under an arbitrary triple of allocation rules $(\delta, \gamma, \lambda)$, the actor's utility is

$$A(x) = (1 - \delta)G(x) - (1 - \gamma)C(x) - \lambda H(x), \quad (57)$$

which is the portion of gain that she expects to keep $(1 - \delta)G(x)$, less the portions of cost and harm that she expects to bear, $(1 - \gamma)C(x)$ and $\lambda H(x)$.

The actor's utility function A may not match the **social welfare** function, denoted S . The social welfare arising from an action x is the value $S(x)$, which deducts from the resulting gain all the resulting private cost and social:

$$S(x) = G(x) - C(x) - H(x). \quad (58)$$

To ensure the social welfare function S has a unique optimizer, assume that as the action x becomes very large, the marginal gain eventually falls below the sum of the marginal cost and the marginal social harm.¹¹⁷ To ensure that the actor's utility function A has an optimizer, confine attention to triples of allocation rules under which she does not have incentives to take explosively large actions.¹¹⁸

Representing the operation of the law, the gain-, cost- and harm-allocation rules may leave room for externalities.¹¹⁹ Positive externalities arise if the gain-allocation rule shifts

¹¹⁶The Model set up in section 5.2 captures negative externalities. Appendix A.5.1 will modify the Model to capture positive externalities.

¹¹⁷Formally, assume: $G'' < 0$, $C'' > 0$, or $H'' > 0$; and there exists some action $\bar{x} > 0$ such that $G'(\bar{x}) \leq C'(\bar{x}) + H'(\bar{x})$. Appendix A.5.5 will demonstrate that this assumption is without loss of generality. Moreover, the assumption that the socially optimal action is unique is made to facilitate proofs of the welfare results. These results do not depend on the uniqueness assumption.

¹¹⁸Formally, confine attention to triples of allocation rules in $\{(\delta, \gamma, \lambda) \in [0, 1] \times [0, 1] \times [0, 1] \mid \exists \bar{x} > 0 \text{ s.t. } A'(\bar{x}) \leq 0\}$.

¹¹⁹Section 5.4 will consider the extent to which the gain-, cost- and harm-allocation rules capture legal doctrine.

some or all of the actor's gain to the victim, formally, $\delta > 1$. Negative externalities arise if the harm-allocation rule fails to shift to the actor all of the victim's harm, that is, $\lambda < 1$. Negative externalities also arise if the cost-allocation rule shifts to the victim some or all of the actor's cost, formally, $\gamma > 1$. Thus the harm-allocation rule generates negative externalities by passively failing to transfer some of the victim's harm to the actor, while the cost-allocation rule generates negative externalities by actively transferring some of the actor's cost to the victim. Let a function V capture the **net externalities**:

$$V(x) = \delta G(x) - \gamma C(x) - (1 - \lambda)H(x), \quad (59)$$

where V is also the victim's utility.

5.2.2 Solution

The actor has an utility-maximizing action,¹²⁰ denoted $x^\diamond$, that solves her first order condition:

$$A'(x^\diamond) \leq 0. \quad (60)$$

There is a unique socially optimal action, denoted x^* , that satisfies the first order condition for the social welfare function S :

$$S'(x^*) \leq 0. \quad (61)$$

Section 5.3 will reveal the conditions under which the actor's chosen action is socially optimal.

5.2.3 Remarks on Assumptions

To facilitate presentation and focus on analyzing the different roles of restitution and compensation, the Model adopts several assumptions that are often without loss of generality.

First, there is an assumption that none of the gain-allocation rule (δ), cost-allocation rule (γ) and harm-allocation rule (λ) varies with the action (x). Appendix A.5.3 will reveal that the welfare results in section 5.3 remain valid without that assumption. Moreover, the

¹²⁰The actor has a unique utility-maximizing action if we add assumptions to ensure that her utility function is strictly concave. The present Model does not add such assumptions because, in reality, the law may give her a utility function that is zero everywhere; formally, $(\delta, \gamma, \lambda) = (1, 1, 0)$ implies $A(x) = 0$. See section 5.4 and Huang (2016).

welfare results in section 5.3 do *not* depend on the assumption $\lambda \leq 1$; these results continue to hold if $\lambda > 1$, that is, a liability to pay punitive damages as a multiple of the wrongful harm. Similarly, appendix A.5.2 reveals that the welfare results in section 5.3 continue to hold in cases where the victim has a choice between restitution and compensation.

Second, the Model suppresses the role of the victim in acting to modify the consequences of the actor's actions. This approach reflects the reality that the types of wrongs giving rise to restitution in American law usually involve an inactive victim. For example, the actor may be a trustee who commits a breach of the duty of loyalty against a beneficiary — the victim — who is unable to monitor the trustee (see Sitkoff 2011). However, suppressing the victim's actions excludes cases in which she has an active role (for example, Shavell 1980b, Bar-Gill and Porat 2014). Appendix A.5.4 will reveal that the welfare results in section 5.3 continue to hold when the victim actively affects the gain and harm arising from the actor's actions.

Third, the present formulation of functions G , C and H accounts for uncertainty regarding, respectively, the gain, cost and harm arising from the action x . For instance, let these functions take the following probabilistic forms:

$$G(x) = p(x)g \qquad H(x) = q(x)h \qquad C(x) = r(x)c$$

where $g, h, c > 0$ are constants, and $p(x), q(x), r(x) \in (0, 1)$ are respectively the probabilities that the gain g , the harm h and the cost c realize according to the actor's belief at the time of choosing her action x . Imposing the appropriate assumptions on the derivatives of p, q and r will bring these probabilistic gain, harm and costs functions within the scope of the Model.¹²¹ Thus the Model also captures liability for accidents.¹²² However, the Model does not capture laws that transfer realized values — g, h and c in this example — without regard to their probabilities of realization — p, q, r here.

Fourth, the present Model assumes that the actor generates social harm, rather than social benefit. Appendix A.5.1 will reveal the main welfare results in section 5.3 continue to hold under the alternative specification that the actor generates social benefit.

Fifth, there is an assumption that the actor's utility function and the social welfare function are concave and differentiable in her action. As section 5.3 will make apparent,

¹²¹More precisely, assume $p', q', r' > 0$; $p'' \leq 0, q'' \geq 0, r'' \geq 0$, with at least one holding strictly; and $p'(\bar{x})g \leq q'(\bar{x})h + r'(\bar{x})c$ for some $\bar{x} > 0$.

¹²²For a survey of economic models of liability for accidents, see Shavell (2007b).

these technical assumptions are made to simplify the proofs of the welfare results.¹²³ Appendix A.5.5 will show that the underlying intuition of the proposed externalities-optimization theory does not depend on these assumptions.

However, there is a loss of generality in excluding administrative costs and the possibility of contracting to vary or avoid the law. Administrative costs are assumed to be independent or sunk costs that do not affect the actor's incentives at the time of choosing her action. Such an assumption also rules out cases where litigation costs vary with the allocation rules.¹²⁴ Subsection 5.3.3 will consider information costs. The possibility of contracting is left for future research.

Moreover, neither the Model nor its modifications in the appendices accounts for the law's expressiveness.

5.3 Welfare Analysis

The welfare results to be stated in this section follow from expressing the social welfare function S as the sum of the actor's utility A and net externalities V :

$$S(x) = \underbrace{(1 - \delta)G(x) - (1 - \gamma)C(x) - \lambda H(x)}_{\text{The actor's utility } A(x)} + \underbrace{\delta G(x) - \gamma C(x) - (1 - \lambda)H(x)}_{\text{Net externalities } V(x)}$$

$$S(x) = A(x) + V(x), \tag{62}$$

which implies

$$S'(x^\diamond) = A'(x^\diamond) + V'(x^\diamond). \tag{63}$$

Equation (63) captures the intuition underlying all subsequent welfare results. The gain-allocation rule (δ), cost-allocation rule (γ) and harm-allocation rule (λ) partition social welfare (S) into the actor's utility (A) and net externalities (V). The actor's choice ($x^\diamond$) necessarily optimizes her utility according to her first order condition (60). If her choice further optimizes net externalities, then optimization of social welfare follows. In other words, a positive action that maximizes the actor's individual utility leads to alignment of *marginal* social welfare with *marginal* net externalities; formally, if $A'(x^\diamond) =$

¹²³In particular, these technical assumptions allow first order conditions to characterize the actor's utility-maximizing action and the socially optimal action.

¹²⁴For models of endogenous litigation costs, see the surveys in Spier (2007) and Katz and Sanchirico (2012), and, more recently, chapter 2 of this thesis.

0, then equation (63) implies $S'(x^\diamond) = V'(x^\diamond)$. Similarly, optimization of net externalities leads to alignment of marginal social welfare with the actor's marginal utility; formally, if $V'(x^\diamond) = 0$, then equation (63) implies $S'(x^\diamond) = A'(x^\diamond)$.

The remainder of this section will consider separately cases in which the socially optimal action is positive, $x^* > 0$; and cases in which $x^* = 0$.

5.3.1 When Social Optimality Requires Positive Action

Proposition 11 formally states the main result.

Proposition 11. *Suppose the socially optimal action is positive, $x^* > 0$. Then the actor's chosen action $x^\diamond$ is socially optimal if and only if the law optimizes net externalities.*

Formally, suppose $x^ > 0$. Then $x^\diamond = x^*$ if and only if $x^\diamond$ defined by condition (60) satisfies*

$$\delta G'(x^\diamond) = \gamma C'(x^\diamond) + (1 - \lambda)H'(x^\diamond). \quad (64)$$

Proof

Assume $x^* > 0$. Then $S'(x^*) = 0$ follows from the first order condition (61) for social welfare.

For one direction, assume $x^\diamond = x^*$. This implies $A'(x^\diamond) = 0$ and $S'(x^\diamond) = 0$ (due to first order conditions (60), (61) respectively). Equation (63) then implies $V'(x^\diamond) = 0$. Condition (64) immediately follows from the definition of V in equation (59).

For the other direction, assume $x^\diamond$ satisfies condition (64). The actor's first order condition (60) implies $A'(x^\diamond) \leq 0$. There are two possibilities:

1. Suppose $A'(x^\diamond) = 0$. Then equation (63) and condition (64) imply $S'(x^\diamond) = A'(x^\diamond) + V'(x^\diamond) = 0$. The first order condition (61) for social welfare is satisfied. The strict concavity of S thus implies the uniqueness of its optimizer, $x^\diamond = x^*$.
2. Suppose, for a contradiction, $A'(x^\diamond) < 0$. Then $x^\diamond = 0$, and equation (63) and condition (64) imply $S'(0) = A'(0) + V'(0) < 0$. But $x^* > 0$ and $0 = S'(x^*) > S'(0)$, a contradiction to $S'' < 0$ (the strict concavity of S). $\square$

The left-hand side of condition (64) sums the marginal positive externalities that the actor generates, while the right-hand side sums the marginal negative externalities. Proposition 11 proves that the actor has socially optimal incentives if positive and negative externalities are aligned at the margin — that is, net externalities are optimized. Moreover,

assuming the socially optimal action is positive, Proposition 11 proves that optimization of net externalities is also a necessary condition for social optimality. Thus, in this Model, condition (64) captures all allocation rules that generate socially optimal incentives.

Proposition 11 generalizes the externalities-internalization approach to inducing social optimality in the presence of externalities. In the present Model, complete internalization of externalities occurs under the triple $(\delta, \gamma, \lambda) = (0, 0, 1)$; in other words, the gain- and cost-allocation rules allow the actor to keep all of her gain and cost, while the harm-allocation rule shifts to her all the harm that she imposes on the victim. As it is well understood (see section 5.1), this triple of allocation rules equates the actor's utility with social welfare ($I = S$), leading to social optimality in her choice of action ($x^\diamond = x^*$). Proposition 11 explains this triple of allocation rules as a special case of a significantly more general condition of (net) externalities optimization: Because this triple induces zero net externalities ($V = 0$), the actor's chosen action trivially optimizes net externalities.

Proposition 11 usually reveals uncountably many triples of gain-, cost- and harm-allocation rules that induce the socially optimal action. Many such triples do *not* completely internalize externalities. The remainder of this subsection will describe some of these optimal triples.

Corollary 27 (Optimal restitution of net gain). *Suppose the socially optimal action is positive ($x^* > 0$). The actor's chosen action $x^\diamond$ is socially optimal if the law shares some proportion of the actor's net gain with the victim and shares the complementary proportion of the victim's harm with the actor.*

Formally, $x^\diamond = x^ > 0$ if*

$$\delta = \gamma = 1 - \lambda < 1. \quad (65)$$

Proof

Assume $x^* > 0$. The triple described by condition (65) gives the actor the following utility:

$$A(x) = \lambda G(x) - \lambda C(x) - \lambda H(x) = \lambda S(x),$$

which implies she chooses $x^\diamond$ defined by $A'(x^\diamond) = \lambda S'(x^\diamond) = 0$. Then $\lambda > 0$ implies $S'(x^\diamond) = 0$, which satisfies the first order condition (61) for social welfare. The strict concavity of S implies the uniqueness of its optimizer, $x^\diamond = x^*$. $\square$

Corollary 27 reveals that restitution of (a proportion of) net gain — the law shifts to the victim the same proportion of the actor’s private gain and cost, $\delta = \gamma < 1$ — may induce social optimality *without* complete internalization of externalities.¹²⁵ Social optimality arises if the actor also bears the complementary proportion of the victim’s harm, $\lambda = 1 - \delta$. Intuitively, the law gives the actor a utility function that is *proportionate* to the social welfare function, $A = \lambda S$; the law also gives rise to a net externalities function that is proportionate to the social welfare function, $V = (1 - \lambda)S$. Then the actor acts to induce zero marginal utility if and only her action also induces zero marginal social welfare and zero marginal (net) externalities:

$$A'(x^\diamond) = 0 \quad \Leftrightarrow \quad S'(x^\diamond) = 0 \quad \Leftrightarrow \quad V'(x^\diamond) = 0.$$

In other words, the law shares social welfare between the actor and the victim in a proportionate way, and *any* such sharing incentivizes the actor to take the socially optimal action.

Moreover, even restitution of gross gain may induce social optimality. Before Corollary 28 states the general result, Example 5 illustrates the conditions under which restitution of gross gain may be socially optimal.

Example 5. Suppose the wrongful action x generates gain $G(x) = 4\sqrt{x}$ and cost $C(x) = x$ to the actor and harm $H(x) = x$ to the victim. A substitution exercise using the first order condition (61) for social welfare reveals that the socially optimal action is $x^* = 1$.¹²⁶ Let the actor bear all of her private cost ($\gamma = 0$).¹²⁷ The pair of gain- and harm-allocation rules $(\delta, \lambda) = (0.25, 0.5)$ is one of the many pairs that induce the actor to choose $x^\diamond = x^* = 1$. Under this pair, the choice $x^\diamond = 1$ satisfies her first order condition (60), that is, $A'(1) = 0.75G'(1) - 0.5H'(1) - C'(1) = 0.75 \times 2 \times 1^{-0.5} - 0.5 \times 1 - 1 = 0$. Notice that the pair $(\delta, \lambda) = (0.25, 0.5)$ does not internalize all externalities; when $x^\diamond = 1$, net externalities sum to $V(1) = 0.25G(1) - 0.5H(1) = 0.5$. This is the optimized value of V because $V'(1) = 0.25G'(1) - 0.5H'(1) = 0.25 \times 2 \times 1^{-0.5} - 0.5 = 0$.

Figure 8 depicts the social welfare function S (the black dotted line), the actor’s utility function A (the green solid line) and the net externalities function V (the red solid line) under the pair $(\delta, \lambda) = (0.25, 0.5)$. As Figure 8 reveals, the actor’s choice $x^\diamond = 1$ satisfies

¹²⁵Corollary 27 follows the same intuition as those underlying the socially optimal rules that Polinsky and Rubinfeld (2003) and Huang (2016) proposed. See subsection 5.1.2 for a discussion of their proposals.

¹²⁶More precisely, $S'(x) = G'(x) - C'(x) - H'(x) = 2x^{-0.5} - 1 - 1$, and $S'(x^*) = 0$ implies $x^* = 1$.

¹²⁷See footnote 141 for a U.S. case recognizing that a securities law violator typically cannot offset her restitutionary liability with her private cost.

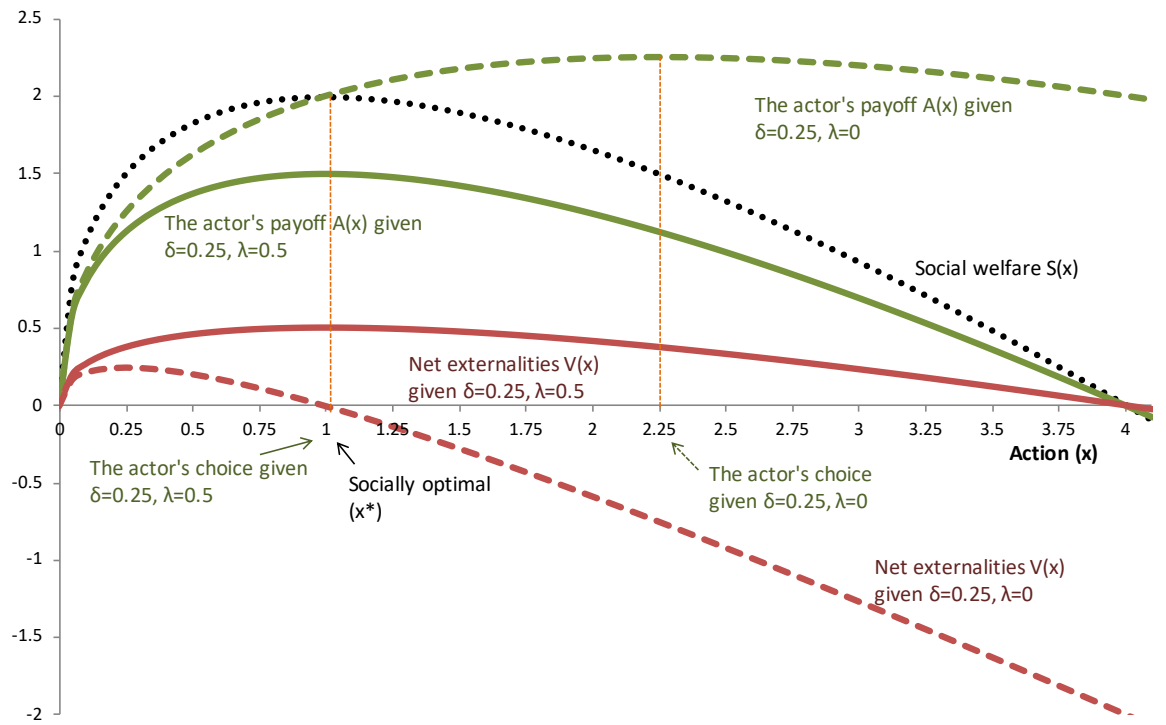


Figure 8: The implications of two pairs of gain- and harm-allocation rules in Example 5, where the actor bears all of her private cost ($\gamma = 0$).

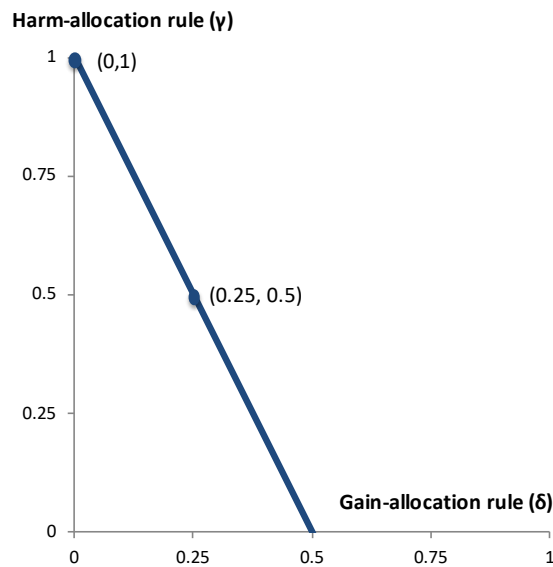


Figure 9: Optimal pairs of gain- and harm-allocation rules for Example 5, where the actor bears her private cost ($\gamma = 0$).

the first order conditions of S , A and V .

Pairs of gain- and harm-allocation rules that fail to optimize net externalities leads to a suboptimal action. One such suboptimal pair is $(\delta, \lambda) = (0.25, 0)$. Under this pair, the actor's first order condition (60) leads to the choice $x^\diamond = 2.25$,¹²⁸ which well exceeds the socially optimal action $x^* = 1$. The action $x^\diamond = 2.25$ also fails to optimize net externalities. Under the pair $(\delta, \lambda) = (0.25, 0)$, the net-externalities function is $V(x) = 0.25G(x) - H(x) = \sqrt{x} - x$, the optimizer of which is 0.25 rather than 2.25.

Figure 8 also depicts the actor's utility function A (the green dashed line) and the net externalities function V (the red dashed line) under the pair $(\delta, \lambda) = (0.25, 0)$. In this case, the actor's choice $x^\diamond = 2.25$ optimizes A , but it does not optimize S or V .

The blue solid line in Figure 9 depicts the set of pairs of gain- and harm-allocation rules that induce the socially optimal action.¹²⁹ This set contains the externalities-internalization pair $(\delta, \lambda) = (0, 1)$, the pair $(\delta, \lambda) = (0.25, 0.5)$ and uncountably many other pairs.

Figure 8 also suggests that, when the actor fails to bear all private cost and all social harm ($\lambda < 1$ or $\gamma > 0$), it is socially optimality to have an *intermediate* gain-allocation rule, $\delta \in (0, 1)$. Under such an intermediate gain-allocation rule, both the actor and the victim receive a positive proportion of the wrongful gain. Corollary 28 formalizes this observation.¹³⁰

Corollary 28 (Optimal restitution). *Suppose the actor's chosen action is positive and socially optimal ($x^\diamond = x^* > 0$). Then:*

1. *The gain-allocation rule takes an intermediate form ($\delta \in (0, 1)$) if and only if the cost- and harm-allocation rules fail to hold the actor liable for all of her private cost and all of the social harm ($\gamma > 0$ or $\lambda < 1$).*

In particular, if the actor bears all of her private cost and none of the social harm ($\gamma = \lambda = 0$), then the gain-allocation rule takes the following intermediate form:

$$\delta = \frac{H'(x^\diamond)}{G'(x^\diamond)} = 1 - \frac{C'(x^\diamond)}{G'(x^\diamond)} = \frac{H'(x^\diamond)}{H'(x^\diamond) + C'(x^\diamond)}. \quad (66)$$

¹²⁸More precisely, $A'(x) = 0.75G'(x) - C'(x) = 1.5x^{-0.5} - 1 = 0$, and $A'(x^\diamond) = 0$ implies $x^\diamond = 2.25$.

¹²⁹Formally, this optimal set is $\{(\delta, \lambda) \in [0, 1] \times [0, 1] | \lambda = 1 - 2\delta\}$.

¹³⁰Subsection 5.3.3 will consider the critical role of information costs.

2. *The gain-allocation rule allows the actor to keep all of her private gain ($\delta = 0$) if and only if the cost- and harm-allocation rules hold her liable for all of her private cost and all of the social harm ($\gamma = 0$ and $\lambda = 1$).*

Proof

Assume $x^\diamond = x^* > 0$. Then $S'(x^\diamond) = 0$ due to condition (61), and Proposition 11 implies $x^\diamond$ is induced by a triple of rules $(\delta, \gamma, \lambda)$ that satisfies condition (64). These imply

$$\delta G'(x^\diamond) - \gamma C'(x^\diamond) - (1 - \lambda)H'(x^\diamond) = 0 = G'(x^\diamond) - H'(x^\diamond) - C'(x^\diamond),$$

a rearrangement of which gives

$$\delta = \frac{G'(x^\diamond) - (1 - \gamma)C'(x^\diamond) - \lambda H'(x^\diamond)}{G'(x^\diamond)}.$$

For one direction, consider two cases:

1. Suppose $\lambda < 1$ or $\gamma > 0$. Then $S'(x^\diamond) = 0$, $C' > 0$, $H' > 0$, $\lambda \geq 0$ and $\gamma \leq 1$ imply $0 = G'(x^\diamond) - C'(x^\diamond) - H'(x^\diamond) < G'(x^\diamond) - (1 - \gamma)C'(x^\diamond) - \lambda H'(x^\diamond) < G'(x^\diamond)$. Then $\delta \in (0, 1)$.

Condition (66) follows from assuming $\lambda = \gamma = 0$ and using $S'(x^\diamond) = 0$.

2. Suppose $\lambda = 1$ and $\gamma = 0$. Then condition (64) implies $\delta = 0$.

For the other direction, suppose $\delta \in (0, 1)$ (respectively, $\delta = 0$) and, for a contradiction, $\lambda = 1$ and $\gamma = 0$ (respectively, $\lambda < 1$ or $\gamma > 0$). Then case 2 (respectively, case 1) above leads to a contradiction. $\square$

The intuition underlying part 1 of Corollary 28 is most apparent in a simple scenario in which the actor retains all of her private cost ($\gamma = 0$) and does not bear any of the social harm ($\lambda = 0$). If the gain-allocation rule allows her to keep all of her gain ($\delta = 0$), then she has perverse incentives to over-act without regard to the resulting harm. By comparison, if the gain-allocation rule disgorges all of her gain ($\delta = 1$), then she has no incentives to take positive action. Thus, to incentivize her to take a positive action that is socially optimal ($x^\diamond = x^* > 0$), the gain-allocation rule has to give her an expectation of receiving some, but not all, of the wrongful gain. The same intuition explains the social optimality of intermediate restitution when the actor bears some, but not all, of the social harm ($\lambda \in (0, 1)$).

However, if the cost- and harm-allocation rules hold the actor liable for all of her private cost ($\gamma = 0$) and social harm ($\lambda = 1$), then part 2 of Corollary 28 reveals that the gain-allocation rule must allow her to keep all of her private gain ($\delta = 0$) in order to incentivize a positive, socially optimal action. This result is the externalities-internalization theory; the triple $(\delta, \gamma, \lambda) = (0, 0, 1)$ removes all externalities. Thus social optimality requires the imposition of restitutionary liability only in cases where complete compensation of the social harm is unattainable; as soon as the law can shift all wrongful harm to the actor, social optimality stops requiring any disgorgement of her private gain.

Corollary 28 does *not* cover cases in which the socially optimal action is zero, $x^* = 0$. In these cases, any onerous liability that disincentivizes the actor from acting positively would be socially optimal. As subsection 5.3.2 will elaborate, this intuition explains the scope and limitations of externalities optimization more generally.

5.3.2 When Social Optimality Requires No Action

Suppose the socially optimal action is zero, $x^* = 0$. Optimization of net externalities remains a sufficient condition for social optimality, but is no longer a necessary condition. Proposition 12 formally states this result.

Proposition 12. *Suppose the socially optimal action is zero, $x^* = 0$. Then the actor's chosen action $x^\diamond$ is socially optimal if the gain-, cost- and harm-allocation rules optimize net externalities.*

Formally, suppose $x^ = 0$. Then $x^\diamond = x^*$ if $x^\diamond$ defined by condition (60) satisfies*

$$\delta G'(x^\diamond) \leq \gamma C'(x^\diamond) + (1 - \lambda)H'(x^\diamond). \quad (67)$$

Proof

Assume $x^* = 0$ and condition (67) holds. Suppose, for a contradiction, that the actor's choice $x^\diamond > 0$. Then her first order condition (60) holds with equality, $A'(x^\diamond) = 0$, implying

$$S'(x^\diamond) = V'(x^\diamond) \leq 0,$$

where the (first) equality follows from equation (63), and the (last) weak inequality follows from condition (67). Hence $x^\diamond$ optimizes the social welfare function S due to the satisfaction of its first order condition (61). But the strict concavity of S implies the uniqueness of its optimizer $0 = x^* = x^\diamond$, a contradiction. $\square$

In cases where the socially optimal action is zero, Proposition 12 confirms that the actor continues to have socially optimal incentives when net externalities are optimized. However, in these cases, optimization of net externalities is *not* the only approach to achieve social optimality; any triple of gain-, cost- and harm-allocation rules that discourages the actor from taking positive action is socially optimal. Example 6 illustrates this point.

Example 6. *Suppose the wrongful action x generates gain $G(x) = \ln(x + 1)$ and cost $C(x) = x$ to the actor, and harm $H(x) = x$ to the victim. Consider a triple of gain-, cost- and harm-allocation rules $(\delta, \gamma, \lambda) = (1, 0, 0.5)$. Under this triple, the social welfare function S , actor's utility function A , the net externalities function V and their respective derivatives are*

$$\begin{aligned} S(x) &= \ln(x + 1) - 2x, & A(x) &= -1.5x, & V(x) &= \ln(x + 1) - 0.5x, \\ S'(x) &= (x + 1)^{-1} - 2, & A'(x) &= -1.5, & V'(x) &= (x + 1)^{-1} - 0.5. \end{aligned}$$

An application of the first order conditions (60), (61) reveals the actor chooses the socially optimal action, $x^ = x^\diamond = 0$. However, $x^\diamond = 0$ violates the first order condition (67) for $V(x)$.*

From now on, refer to conditions (64) and (67) (in Propositions 11, 12 respectively) collectively as the **Externalities-Optimization Principle**. This choice of terminology reflects the finding that, regardless of whether the socially optimal action is positive or zero, optimizing net externalities is sufficient for social optimality. Section 5.4 will explore the extent to which the Externalities-Optimization Principle explains or supports the American law of restitution for wrongs.

5.3.3 Information Costs

The preceding analysis does not consider administrative costs, in particular, information costs to the court, the litigants and their lawyers. Consideration of information costs offers guidance regarding which one of the triples of gain-, cost- and harm-allocation rules should be implemented in a given class of cases.¹³¹ This subsection offers an

¹³¹Consideration of administrative costs other than information costs also may shed light on which triple(s) of allocation rules should be implemented in a given class of cases. For instance, Kaplow and Shavell (1996), Dari-Mattiacci (2005), and Shavell (2007a) (at pp. 79-83, 115-18, 131-32) have explored how error costs in respect of the determination of tort liability or the assessment of compensatory damages affect incentives to take precautions. To focus on analyzing information costs, this subsection ignores the other administrative costs; hence the present arguments rise no higher than suggesting that differences in information costs *affect*,

intuitive analysis to reveal that, even though many triples of allocation rules satisfy the Externalities-Optimization Principle, they typically do not generate the same information costs. The objective is to ascertain the different kinds of information that may be required to implement various triples of allocation rules; how such information is obtained is beyond the present scope. To this end, assume that any required information is available at a cost, which cost may be prohibitively high. However, if such cost is incurred, then the required information is obtained with certainty.

To consider the interesting cases first, assume the socially optimal activity level is positive ($x^* > 0$). This subsection will conclude with dropping that assumption.

Let a function $L(\delta, \gamma, \lambda)$ describe the actor's **liability** under an arbitrary triple of gain-, cost- and harm-allocation rules $(\delta, \gamma, \lambda)$:

$$L(\delta, \gamma, \lambda) = \delta G(x^\diamond) - \gamma C(x^\diamond) + \lambda H(x^\diamond), \quad (68)$$

where the actor's choice $x^\diamond$ depends on $(\delta, \gamma, \lambda)$ via her first order condition (60).

Equation (68) reveals what a judge (or a social planner) would need to know if she were to set the actor's liability $L(\delta, \gamma, \lambda)$ in order to generate socially optimal incentives. Choosing a triple of allocation rules in accordance with the Externalities-Optimization Principle is sufficient and necessary for social optimality (see Proposition 11) when $x^* > 0$. For instance, equation (68) reveals that, to implement the externalities-internalization triple $(\delta, \gamma, \lambda) = (0, 0, 1)$, the judge would set $L(0, 0, 1) = H(x^\diamond)$. Hence, she would need to know enough about the value $H(x^\diamond)$; thus the information costs of ascertaining $H(x^\diamond)$ arise.

The judge would need different information if she were to implement an optimal gain-allocation rule *without* shifting any of the victim's harm to the actor and any of the actor's cost to the victim. Formally, the judge chooses a triple of allocation rules $(\delta, \gamma, \lambda) = (\delta^*, 0, 0)$ such that δ^* satisfies condition (66) in Corollary 28, which is a special case of the Externalities-Optimization Principle. Equation (68) reveals the judge would set the actor's liability to $L(\delta^*, 0, 0) = \delta^* G(x^\diamond)$. The actor's utility-maximizing choice $x^\diamond$ depends on δ^* , which is an intermediate value (see Corollary 28) that potentially depends on how functions G , H or C change at the margin when her action changes; how C , C or H changes at the margin may be relevant because it may affect $x^\diamond$ through the actor's first

rather than conclusively determine, which triple(s) of allocation rules should be applied to a given class of cases. Alternatively, consideration of information costs would be paramount if one were to interpret the present arguments as premised on an assumption that, when applied to the relevant class of cases, one triple of allocation rules $(\delta_1, \gamma_1, \lambda_1)$ generates greater information costs than another triple $(\delta_2, \gamma_2, \lambda_2)$ does if and only if the other administrative costs are no smaller under $(\delta_1, \gamma_1, \lambda_1)$ than under $(\delta_2, \gamma_2, \lambda_2)$.

Allocation rules	The actor's liability	Information required
$(0, 0, 1)$	$H(x^\diamond)$	$H(x^\diamond)$
$(\delta^*, 0, 0)$ given $\delta^* \in (0, 1)$	$\delta^* G(x^\diamond)$	$G(x^\diamond)$, maybe $G'(x^\diamond)$, $C'(x^\diamond)$, $H'(x^\diamond)$
$(\delta^{**}, \gamma^{**}, \lambda^{**})$ given $\delta^{**}, \gamma^{**}, \lambda^{**} \in (0, 1)$	$\delta^{**} G(x^\diamond) - \gamma^{**} C(x^\diamond)$ $+ \lambda^{**} H(x^\diamond)$	$H(x^\diamond)$, $C(x^\diamond)$, $G(x^\diamond)$, maybe $G'(x^\diamond)$, $C'(x^\diamond)$, $H'(x^\diamond)$

Table 1: The information costs of implementing three different triples of gain-, cost- and harm-allocation rules that satisfy the Externalities-Optimization Principle.

order condition (60). In other words, the judge would need to know enough about $G(x^\diamond)$, and she might need to know enough about $G'(x^\diamond)$, $C'(x^\diamond)$ or $H'(x^\diamond)$. Thus the information costs of ascertaining these values arise.

Moreover, different information costs may arise if the judge were to choose a triple of intermediate allocation rules, $(\delta^{**}, \gamma^{**}, \lambda^{**})$ with $\delta^{**}, \gamma^{**}, \lambda^{**} \in (0, 1)$, in accordance with the Externalities-Optimization Principle. In this case, the judge would need to know the values $G(x^\diamond)$, $C(x^\diamond)$ and $H(x^\diamond)$, and she might need to know how functions G , H or C change at the margin.¹³² Thus implementing $(\delta^{**}, \gamma^{**}, \lambda^{**})$ gives rise to the information costs of ascertaining these values.

As Table 1 summarizes, the externalities-internalization triple $(0, 0, 1)$ typically imposes smaller information costs than the triple of intermediate allocation rules, $(\delta^{**}, \gamma^{**}, \lambda^{**})$; both triples generate the information costs of ascertaining the value $H(x^\diamond)$, but $(\delta^{**}, \gamma^{**}, \lambda^{**})$ also generates the information costs of ascertaining $G(x^\diamond)$ and $C(x^\diamond)$, and potentially $G'(x^\diamond)$, $H'(x^\diamond)$, $C'(x^\diamond)$. For similar reasons, the triple $(\delta^{**}, \gamma^{**}, \lambda^{**})$ may generate greater information costs than the triple $(\delta^*, 0, 0)$ does. Moreover, whether information costs are greater under $(0, 0, 1)$ or $(\delta^*, 0, 0)$ depends on whether it is more costly to ascertain $H(x^\diamond)$; or $G(x^\diamond)$ and, potentially, $G'(x^\diamond)$, $C'(x^\diamond)$ or $H'(x^\diamond)$.¹³³ Thus the main advantage of choosing the externalities-internalizing triple $(0, 0, 1)$ over the other externalities-optimizing triples is the avoidance of the information costs of ascertaining marginal changes in the gain, harm or cost functions.

To conclude the present discussion of information costs, consider the possibility that it is socially optimal to take zero action ($x^* = 0$). Proposition 12 reveals that the court would not need to induce social optimality by optimizing net externalities, *if it already knew the*

¹³²To see that implementing $(\delta^{**}, \gamma^{**}, \lambda^{**})$ may *not* require knowledge of $G'(x^\diamond)$, $C'(x^\diamond)$ and $H'(x^\diamond)$, suppose the judge effectuates socially-optimal restitution of net gain in accordance with Corollary 27. Condition (65) in Corollary 27 does *not* depend on these derivatives, and any triple of allocation rules that satisfies this condition only generates the information costs of ascertaining $G(x^\diamond)$, $C(x^\diamond)$ and $H(x^\diamond)$.

¹³³Notice that ascertaining the level of function H evaluated at $x^\diamond$ — that is, $H(x^\diamond)$ — is not the same task as ascertaining the rate of change of H at $x^\diamond$ — that is, $H'(x^\diamond)$.

socially optimal action is zero. With that knowledge, the court to deter a positive action could apply to the actor any triple of onerous allocation rules that typically generates small information costs. For example, the court could apply $\delta = 1$, $\gamma = 0$, and an arbitrary λ . However, unless the case is obvious or complete internalization of externalities is attainable ($\lambda = 1$), the court would need to consider marginal changes in the gain, cost and harm functions in order to determine whether social optimality requires zero or positive action *in the first place* (see equation (58) and condition 61). In other words, unless the case is obvious or complete internalization is attainable, the court to ascertain what social optimality requires would have already generated information costs that are similar to those arising from implementing optimal intermediate liabilities. Thus the preceding analysis of information costs continues to apply to cases where it is not obvious whether social optimality requires zero or positive action.

5.4 Doctrinal Application: Restitution for Wrongs

This section offers an economic theory of restitution for wrongs in American law. At the highest level of abstraction, the American law of restitution aims to reverse unjust enrichment in accordance with the dictates of “equity and good conscience”.¹³⁴ Judicial discretion guided by these **equitable standards** typifies this area of law. To the extent of their application to *profitable* wrongs, equitable standards inform and shape the finer rules and standards governing restitutionary liability and remedy.¹³⁵ The archetypal class of wrongs attracting restitutionary liability consists of breaches of fiduciary duties or similar duties arising in a relationship of trust and confidence. Liability for these wrongs typically arises without regard to notice or fault on the part of the breaching actor.¹³⁶ The victims of these wrongs are often vulnerable persons (such as minors) or persons who are unable to monitor the actor (see Sitkoff 2011). Conscious interference with rights to property is another wrong that attracts restitutionary liability.¹³⁷ Equitable standards further shape the rules and standards governing defenses to restitutionary liability as well as the availability and form of restitutionary remedy (see subsection 5.4.2). Remedies in restitution may take the form of money judgment or rights in property. Depending on equitable considerations, the actor may be liable to disgorge some or all of her wrongful gain, or compensate some

¹³⁴Restatement (Third) of Restitution and Unjust Enrichment §1 comments a, b (Am. Law Inst. 2011).

¹³⁵Restatement (Third) of Restitution and Unjust Enrichment §3 (Am. Law Inst. 2011).

¹³⁶Restatement (Third) of Restitution and Unjust Enrichment §§43, 51(4) (Am. Law Inst. 2011).

¹³⁷Restatement (Third) of Restitution and Unjust Enrichment §§40-42 (Am. Law Inst. 2011).

or all of the victim's harm.¹³⁸

This section will argue that the Externalities-Optimization Principle explains three distinguishing characteristics of the law governing restitution for wrongs: Equitable standards; comparison of inequitable conducts; and the gap-filling role of restitution. I will also argue in favor of a strong restitutionary liability in cases where social optimality obviously requires no action.

Two preliminary observations must be made. First, claims for restitution typically arise in cases where compensatory damages do not (or cannot) remedy all wrongful harm.¹³⁹ The Model captures these cases by the specification that the harm-allocation rule fails to shift all wrongful harm to the actor, $\lambda < 1$. The remainder of this section, except subsection 5.4.3, will adopt this specification.

Second, the present economic theory relies upon an ex-ante notion of social welfare: The social welfare arising from the wrongful action according to the actor's expectation at the time of so acting (see function S defined by equation (58)). This social welfare criterion may appear incompatible with the dictates of "equity" and "conscience" — the standards that underlie the law of restitution. However, such apparent incompatibility does not devalue the present economic theory as a *functional* theory of the law. Moreover, subsection 5.4.1 will argue the social welfare criterion is largely consistent with equitable standards.

5.4.1 Equitable Standards and Discretion

Standards in terms of "equity" and "conscience" guide the operation of the law governing restitution for wrongs, in particular, the exercise of judicial discretion to award, modify or withhold remedies. These equitable standards impose little restriction on the facts and circumstances that the court may consider.¹⁴⁰ This subsection argues in favor of interpreting and applying equitable standards in accordance with the Externalities-Optimization Principle.

First, the plain meaning of equitable standards describes other-regarding behaviors, especially those exhibiting inequality aversion and preferences for reciprocity fairness.

¹³⁸Restatement (Third) of Restitution and Unjust Enrichment §§51, 61 (Am. Law Inst. 2011).

¹³⁹Restatement (Third) of Restitution and Unjust Enrichment §4 comment e (Am. Law Inst. 2011). See also Huang (2016) (at pp. 1624-28), proposing to disgorge a proportion of the wrongful gain when imperfect public or private enforcement leads to incomplete compensation of the harm; Cooter and Porat (2015) (at p. 254), proposing to impose a negligence rule with limited disgorgement damages (described in footnote 113 of this paper) when these damages are easier to measure than, and greater than, traditional compensatory damages.

¹⁴⁰See, for example, *Jones v. Star Credit Corp.*, 298 NYS 2d 264 (N.Y. Sup. Ct. 1969).

Other-regarding behaviors are well-documented in psychology and in behavioral economics. Experiments consistently reveal behaviors that lead to much fairer outcomes than the equilibrium predictions of models premised on pure self interest (see, for example, the recent survey in Dhami 2016, ch. 5). Inequality aversion is one of the explanations for these experimental results (see, for example, Fehr and Schmidt 1999, Fershtman, Hvide, and Weiss 2003 and Chowdhury et al. 2018). Preferences for reciprocity fairness also explain these results (see, for example, Rabin 1993, Charness and Rabin 2002). For the reasons that follow, interpreting equitable standards as describing other-regarding behaviors connects these standards to the social welfare criterion.

Intuitively, the actions of an other-regarding actor are approximately socially optimal because these actions account for some or all of the harm (or benefit) that she imposes on others. To formalize this intuition using the notations of the Model, fix a positive constant $\psi > 0$ and let a function O represent the preferences of an other-regarding actor:

$$O(x) = A(x) + \psi V(x),$$

where, as in the Model, A is the utility of a purely self-regarding actor (see equation (57)) and $V(x)$ the victim's utility (see equation (59)). The coefficient ψ captures the extent to which the other-regarding actor values the victim's outcome. As ψ approaches 1, the utility function O approaches the social welfare function S (see equation (62)); the other-regarding actor's choice of action thus approaches the socially optimal action, x^* defined by condition (61). She chooses x^* if $\psi = 1$, which means the actor values her own outcome as much as the victim's.

Thus, to the extent that equitable standards describe the behaviors of an other-regarding actor, these standards demand a self-regarding actor to take other-regarding actions. Because these actions are approximately socially optimal, the social welfare criterion is a good proxy for equitable standards. Conversely, equitable standards are a good proxy for the social welfare criterion. The coefficient ψ reflects the judicial application of equitable standards, as constrained by established doctrine and precedent; ψ is therefore a measure of the extent to which equitable standards approximate the social welfare criterion. Then the Externalities-Optimization Principle — which induces social optimality according to Propositions 11 and 12 — reflects and formalizes the demands of equitable standards.

Second, even if equitable standards do not describe the behaviors of an other-regarding actor, their application ex post may give rise to an expectation of *intermediate* restitutionary

liability *ex ante*. To see this, suppose the discretionary application by judges of equitable standards sometimes achieves complete restitution and sometimes results in no restitution. Suppose further that the actor bears her private cost ($\gamma = 0$).¹⁴¹ Then, at the time choosing her action, the actor expects intermediate restitution. In other words, the gain-allocation rule (δ) in the Model may be interpreted as the probability of complete restitution. For instance, under this interpretation, the triple $(\delta, 0, 0)$ where $\delta \in (0, 1)$ gives the actor an expected restitutionary liability of $L(\delta, 0, 0) = \delta G(x^\diamond)$; this triple captures the actor's *ex-ante* expectation regarding the *ex-post* exercise of judicial discretion to impose a complete restitutionary liability or no liability. Similarly, under the triple $(\delta, 0, \lambda)$ where $\delta \in (0, 1)$ and $\delta = 1 - \lambda$, the actor expects a mixture of liabilities to disgorge and to compensate, $L(\delta, 0, \lambda) = \delta G(x^\diamond) + \lambda H(x^\diamond)$; this triple captures the actor's *ex-ante* expectation regarding the *ex-post* exercise of judicial discretion to award a complete restitutionary remedy or a complete compensatory remedy. In the same vein, the triple $(\delta, 0, \lambda)$ where $0 < \delta + \lambda < 1$ captures the actor's *ex-ante* expectation regarding the *ex-post* exercise of judicial discretion to award a complete restitutionary remedy, a complete compensatory remedy, or no remedy.

Moreover, as Huang (2016) (at pp. 1626-27) has discussed and Example 7 will consider, there is Supreme Court authority for applying equitable standards to grant *both* an intermediate restitutionary remedy *and* an intermediate compensatory remedy. An *ex-post* imposition of such combined remedies clearly gives the actor an *ex-ante* expectation of a similar combination.

Example 7. *In Kansas v. Nebraska*,¹⁴² Nebraska profited from a knowing breach of Kansas's water rights under a settlement agreement between them. Finding sufficient flexibility in equitable standards, the Supreme Court upheld the simultaneous imposition on Nebraska of an intermediate restitutionary liability to disgorge a small amount of its wrongful gain, and an intermediate compensatory liability to pay damages for an agreed amount of Kansas's loss.¹⁴³ As Huang (2016) has observed (at p. 1627), the compensatory liability in this case is intermediate because it is referable to a conceded amount of Kansas's loss.

Because this case concerns a profitable breach of contract, I do not read it as holding

¹⁴¹ See, for example, *S.E.C. v. Brown*, 658 F.3d 858, 860-61 (8th Cir. 2011) (recognizing "the overwhelming weight of authority hold[s] that securities law violators may not offset their disgorgement liability with business expenses.") (internal citations omitted).

¹⁴² 135 S.Ct. 1042 (2015).

¹⁴³ *Kansas v. Nebraska*, 135 S.Ct. 1042, 1051, 1053, 1058 (2015).

that profitable wrongs may attract combined restitutionary and compensatory remedies. This paper concerns profitable wrongs only (see section 5.1). There is nonetheless enough obiter to suggest that equitable standards — whether applied to remedy breaches of tortious or contractual duties — are sufficiently flexible to permit judicial imposition of combined, intermediate restitutionary and compensatory remedies.

However, it is unrealistic to expect courts or legislatures to state rules with the mathematical precision of the Externalities-Optimization Principle (see especially condition (64)). In cases where the socially optimal action is positive ($x^* > 0$) and the actor does not bear all of her private cost or social harm ($\gamma > 0$ or $\lambda < 1$), Corollary 28 reveals that to be socially optimal, the gain-allocation rule must take an intermediate form. Subsection 5.3.3 reveals that even in the stylized Model, consideration of information costs complicates the task of achieving optimal intermediate restitution.¹⁴⁴ That task only becomes more informationally demanding (and analytically difficult) in reality. Thus the high information costs of promulgating rules ex ante may explain the adoption of equitable standards to govern liability in restitution.¹⁴⁵ These standards incentivize approximately socially optimal actions to the extent that their ex-post application gives the actor an ex-ante expectation that her liability approximately reflects what the Externalities-Optimization Principle requires. Hence the present arguments do *not* suggest that every application of equitable standards ex post generates socially optimal incentives ex ante; I only argue that, in cases where $\gamma > 0$ or $\lambda < 1$, incentives may be better optimized by equitable standards that lead to an intermediate restitutionary remedy than by rules that only lead to an incomplete compensatory remedy.

The preceding arguments on the basis of information costs also apply when it is not obvious that social optimality requires zero action. Although the Externalities-Optimization Principle only supplies a sufficient condition for social optimality (see Proposition 12), the information costs of ascertaining whether social optimality requires zero or positive action still arise (see subsection 5.3.3). However, in cases where it is obviously socially optimal to take zero action ($x^* = 0$), there is no need to incur the information costs of achieving intermediate restitution according to the Externalities-Optimization Principle.

¹⁴⁴Considerations of information costs are critical to the present arguments. Otherwise, in a hypothetical world where administrative costs (including information costs) are nil and the actor is risk neutral, Corollary 27 implies that the judge to achieve social optimality can just flip a coin to decide between allowing restitution of net gain and allowing complete compensation; that is, $\delta = \gamma = 1 - \lambda$, where each multiplier is interpreted as a probability.

¹⁴⁵The disadvantages of adopting these standards instead of rules include high costs of compliance and of adjudication. See Kaplow (1992).

In these case, maximum deterrence via complete restitution of (gross) gain ($\delta = 1$ and $\gamma = 0$) is among the socially optimal liabilities; it deters the actor from acting positively. Consideration of information costs thus suggests that there is no need to apply standards to dilute the extent of restitution in expectation.

5.4.2 Competing Equities

The law governing restitution for wrongs has peculiar defenses and limiting principles. In particular, the victim's own inequitable conduct ("unconscionability" or "unclean hands") limits her restitutionary remedy or removes the actor's restitutionary liability.¹⁴⁶ Thus a claim in restitution is subject to an analysis of **competing equities**: A comparison of the actor's inequitable conduct and the victim's. Example 8 illustrates the competing-equities analysis.¹⁴⁷

Example 8. *In Salomon Smith Barney Inc. v. Vockel*,¹⁴⁸ a brokerage firm (the victim) sued a former employee (the actor) for acting upon a solicitation by her new employer to pass on some confidential information of the victim-firm. In the past, the victim-firm had obtained confidential information by soliciting employees of other firms. The court denied the victim-firm's present claim for equitable relief on the basis of "unclean hands". It the court's own words, "[the victim-firm] has not shown that it has come into this court with clean hands. In fact, the opposite has been established. Accordingly, as a court sitting in equity, we will not aid a wrongdoer." ¹⁴⁹

The Externalities-Optimization Principle explains and partially supports the competing-equities analysis. Appendix A.5.4 reveals that the Principle continues to hold when the victim takes actions that are similar to those that the actor takes in the present Model (see section 5.2). Thus, in cases where social optimality may require positive action, limiting restitution on the basis of competing equities allows for approximation of optimal intermediate restitution (see Corollary 28). However, in cases where taking no action is obviously socially optimal, there is no need to incur the information costs of achieving optimal intermediate restitution. In these cases, maximum deterrence is socially optimal, and consideration of information costs suggests that the extent of the actor's liability in restitution should not be limited on the basis of competing equities.

¹⁴⁶Restatement (Third) of Restitution and Unjust Enrichment §§63, 70 (Am. Law Inst. 2011).

¹⁴⁷For more examples, see Restatement (Third) of Restitution and Unjust Enrichment §63 illustrations (Am. Law Inst. 2011).

¹⁴⁸137 F. Supp.2d 599 (E.D. Pa. 2000).

¹⁴⁹Salomon Smith Barney Inc. v. Vockel, 137 F. Supp.2d 599, 604 (E.D. Pa. 2000).

5.4.3 The Gap-Filling Role of Restitution

This subsection concludes the present doctrinal analysis by addressing a controversy in the law of restitution: Whether restitution should be available *only if* compensatory damages do not or cannot remedy the wrongful harm. If this restriction applies, then restitution is unavailable in cases where complete compensation is attainable, and may become available only in cases of incomplete compensation. This restriction thus limits restitution to a gap-filling role. American courts disagree on whether to impose this restriction.¹⁵⁰ The Restatements generally take the position that, if the case permits both restitution and compensation, then the claimant should be afforded great latitude to choose between these remedies.¹⁵¹

The present economic theory generally contradicts the Restatements' position. Appendix A.5.2 reveals that the Externalities-Optimization Principle continues to hold when the victim has a choice between restitution and compensation. Thus the analysis of information costs in subsection 5.3.3 continues to apply. Subsection 5.3.3 reveals that, within the set of liabilities that satisfy the Externalities-Optimization Principle, the liability that completely compensates the victim often imposes the least information costs. Because the interests of a private claimant imperfectly align with the society's interests (see Shavell 1997), her choice between disgorgement and compensation may not minimize information costs for the society. Hence consideration of information costs suggests that complete compensation should be preferred if it is attainable.

5.5 Conclusion

To achieve social optimality in the presence of externalities, this paper proposes a theory of externalities optimization. The proposed theory generalizes the well-established theory of externalities internalization in cases where complete internalization is attainable. The proposed theory further achieves social optimality in cases where complete internalization is unattainable. A modification of the standard tort model captures the law's ability to optimize externalities; instead of assuming either a liability to compensate for the wrongful harm or a liability to disgorge the wrongful gain, section 5.2 constructs a model that permits a mixture or combination of both liabilities. The model thus allows for flexibility

¹⁵⁰See the survey in *In re Processed Egg Products Antitrust Litigation*, 851 F.Supp.2d 867, 914-918 (E.D. Pa. 2012). See also Laycock (1990).

¹⁵¹Restatement (Third) of Restitution and Unjust Enrichment §4 comment e (Am. Law Inst. 2011); Restatement (Third) of Trusts §95 comment e (Am. Law Inst. 2012).

in imposing different liabilities simultaneously to create positive externalities and reduce negative externalities. As section 5.3 reveals, that flexibility permits the law to achieve social optimality by optimizing (net) externalities. The appendices show that the theory of externalities optimization continues to hold under various model modifications.

Section 5.4 applies the proposed externalities-optimization theory to explain the cardinal principles of the American law of restitution, focusing on its application to wrongful actions that generate social harm. In particular, the underlying standards of “equity” and “conscience” tend to limit the actor’s restitutionary liability to disgorge her wrongful gain. Such a limited form of restitutionary liability may approximately meet the demands of the externalities-optimization theory. Moreover, many courts tend to confine restitution to cases where complete compensation is unattainable. This tendency reflects an intuitive observation that a liability that completely internalizes externalities often generates smaller information costs than any other liability that optimizes externalities (see subsections 5.3.3, 5.4.3).

A limitation of this paper arises from the assumption that the actor and the victim do not enter into a contract that governs the wrongful action. High transaction costs (Coase 1960) or asymmetric information (Myerson and Satterthwaite 1983) may prevent such a contract. Thus neither the present model nor its modifications in the appendices captures restitution as a consequence of contract disaffirmance (see Brooks and Stremitzer 2011, 2012). Future research may relax the no-contract assumption, and consider whether the externalities-optimization theory extends to contractual scenarios. Relaxing that assumption also may enable an economic analysis of the mechanisms for modifying fiduciary duties and authorizing their breaches (see Conaglen 2011). Moreover, the present, ex-ante notion of social welfare reflects the actor’s belief regarding the value of the victim’s harm at the time of acting (see section 5.2); there is no opportunity for updating her belief. Contract negotiation may reveal information regarding the victim’s own valuation of harm, and introducing the opportunity to contract may enable an analysis of an ex-post notion of social welfare that incorporates such new information.

Future research also may modify the present model to introduce multiple victims with competing interests. This modification will capture bankruptcy cases, in which the bankrupt actor’s limited-liability constraint forces the law to allocate among her victims (as her creditors) the wrongful gain and harm arising from her actions.¹⁵² In these cases,

¹⁵²To some extent, the model in section 5.2 captures the doctrine of equitable subordination in U.S. bankruptcy law. That doctrine allows a court to subordinate the claims of a senior creditor (the actor) whose

bankruptcy law may apply a priority rule to rank the victims' respective claims, while the law of restitution may limit the claims in restitution.¹⁵³

actions harm a junior creditor (the victim). Breaches of fiduciary duties, or gross and egregious misconduct, may lead to equitable subordination. See, for example, *In Re Yellowstone Mountain Club, LLC*, 436 B.R. 598 (Bankr. D. Mont. 2010).

¹⁵³Restatement (Third) of Restitution and Unjust Enrichment §§56, 60, 61 (Am. Law Inst. 2011).

A Appendices

A.1 Proofs for Chapter 2

This appendix contains all proofs for chapter 2.

Lemma 9 is a technical lemma which will be used to prove other Lemmas, Propositions and Corollaries.

Lemma 9. *On the subdomain $\mathbb{R}_{++}^2$, the success function $\theta(\cdot)$ satisfies the following properties:*

1. $\mu > 0.5$ (respectively, $= 0.5$, < 0.5) and $e_P = e_D$ imply $\theta > 0.5$ ($= 0.5$, < 0.5).
2. $\frac{\partial}{\partial e_D}(1 - \theta) > 0$, $\frac{\partial^2}{\partial e_D^2}(1 - \theta) \leq 0$.
- 3.

$$\frac{\frac{\partial^2}{\partial e_D^2}\left(\frac{1-\theta}{1-\lambda(1-\theta)}\right)}{\frac{\partial}{\partial e_D}\left(\frac{1-\theta}{1-\lambda(1-\theta)}\right)} < \frac{C''(e_D)}{C'(e_D)}. \quad (69)$$

4. $k = \lambda = 1 \Rightarrow \lim_{s \rightarrow +\infty} 1 - \theta < 1$.
- 5.

$$\begin{aligned} \frac{\partial s}{\partial e_P} &= -\frac{s}{e_P}, & \frac{\partial s}{\partial e_D} &= \frac{s}{e_D}, & \frac{\partial \theta}{\partial e_P} &= -\frac{s\theta_s}{e_P}, & \frac{\partial}{\partial e_D}(1 - \theta) &= -\frac{s\theta_s}{e_D}, \\ \frac{\partial^2 \theta}{\partial e_P^2} &= \frac{s^2\theta_{ss}}{e_P^2} + \frac{2s\theta_s}{e_P^2}, & \frac{\partial^2}{\partial e_D^2}(1 - \theta) &= -\frac{s^2\theta_{ss}}{e_D^2}, & \frac{\partial^2 \theta}{\partial e_P \partial e_D} &= -\frac{s(\theta_s + s\theta_{ss})}{e_P e_D}. \end{aligned}$$

6. $\theta_s < 0$, $\theta_{ss} \geq 0$.
7. $s\theta_{ss} < -(k+1)\theta_s - \frac{2\lambda s\theta_s^2}{(1-\lambda\theta)}$.
8. $s\theta_{ss} > (k-1)\theta_s + \frac{2\lambda s\theta_s^2}{[1-\lambda(1-\theta)]}$.
9. $-\lambda(2-\lambda)s\theta_s < k(1-\lambda\theta)[1-\lambda(1-\theta)]$.

Proof of Lemma 9

Part 1

Let $e_P = e_D = e_1$ for some arbitrary $e_1 > 0$. First suppose $\mu = 0.5$. Then use Assumption 1 to obtain

$$\theta(e_1, e_1; 0.5) = 1 - \theta(e_1, e_1; 1 - 0.5) = 1 - \theta(e_1, e_1; 0.5)$$

which implies $\theta(e_1, e_1, \mu) = 0.5$. The results for $\mu > 0.5$ and $\mu < 0.5$ follow from Assumption 3.

Parts 2-4

Fix μ and $e_P = e_1$ for some arbitrary $e_1 > 0$. Then Assumption 1 implies $1 - \theta(e_1, e_D, \mu) = \theta(e_D, e_1; 1 - \mu)$. Hence

$$\frac{\partial}{\partial e_D}(1 - \theta(e_1, e_D; \mu)) = \frac{\partial}{\partial e_D}\theta(e_D, e_1; 1 - \mu) > 0$$

where the inequality follows from Assumption 4.

A similar approach establishes $\frac{\partial^2}{\partial e_D^2}(1 - \theta(e_P, e_D; \mu)) \leq 0$, condition (69), and part 4.

Part 5

The chain rule and some algebra will give

$$\begin{aligned} \frac{\partial s}{\partial e_P} &= \frac{\partial}{\partial e_P}\left(\frac{e_D}{e_P}\right) = -\frac{e_D}{e_P^2} = -\frac{s}{e_P} & \frac{\partial s}{\partial e_D} &= \frac{\partial}{\partial e_D}\left(\frac{e_D}{e_P}\right) = \frac{1}{e_P} = \frac{s}{e_D} \\ \frac{\partial \theta}{\partial e_P} &= \frac{\partial \theta}{\partial s} \frac{\partial s}{\partial e_P} = -\frac{\partial \theta}{\partial s} \frac{s}{e_P} & \frac{\partial(1 - \theta)}{\partial e_D} &= -\frac{\partial \theta}{\partial s} \frac{\partial s}{\partial e_D} = -\frac{\partial \theta}{\partial s} \frac{s}{e_D} \\ \frac{\partial^2 \theta}{\partial e_P^2} &= \frac{\partial}{\partial e_P}\left(\frac{\partial \theta}{\partial s} \frac{\partial s}{\partial e_P}\right) = \frac{\partial^2 \theta}{\partial s^2} \left(\frac{\partial s}{\partial e_P}\right)^2 + \frac{\partial \theta}{\partial s} \frac{\partial^2 s}{\partial e_P^2} = \frac{s^2}{e_P^2} \frac{\partial^2 \theta}{\partial s^2} + \frac{2s}{e_P^2} \frac{\partial \theta}{\partial s} \\ \frac{\partial^2(1 - \theta)}{\partial e_D^2} &= -\frac{\partial^2 \theta}{\partial s^2} \frac{1}{e_P^2} = -\frac{\partial^2 \theta}{\partial s^2} \frac{s^2}{e_D^2} \\ \frac{\partial^2 \theta}{\partial e_P \partial e_D} &= \frac{\partial}{\partial e_D}\left(-\frac{\partial \theta}{\partial s} \frac{e_D}{e_P^2}\right) = -\frac{\partial^2 \theta}{\partial s^2} \frac{1}{e_P} \frac{e_D}{e_P^2} - \frac{\partial \theta}{\partial s} \frac{1}{e_P^2} = -\frac{s(\theta_s + s\theta_{ss})}{e_P e_D} = \frac{\partial^2 \theta}{\partial e_D \partial e_P}, \end{aligned}$$

where the last equality uses Young's Theorem.

Part 6

The chain rule and the properties $\partial \theta / \partial e_P > 0$, $s / e_P > 0$ imply $\theta_s < 0$. The expression of $\frac{\partial^2}{\partial e_D^2}(1 - \theta)$ in part 5 and the property $\frac{\partial^2}{\partial e_D^2}(1 - \theta) \leq 0$ from part 2 imply $\theta_{ss} \geq 0$.

Part 7

Apply Euler's theorem for homogeneous functions to obtain

$$C(e_P) = e_P^k C(1), \quad C'(e_P) = \frac{k}{e_P} C(e_P) = k e_P^{k-1} C(1), \quad C''(e_P) = k(k-1) e_P^{k-2} C(1).$$

Some algebra will reveal that condition (1) holds if and only if

$$\frac{(1 - \lambda \theta) \partial^2 \theta / \partial e_P^2 + 2 \lambda (\partial \theta / \partial e_P)^2}{\partial \theta / \partial e_P} < \frac{(1 - \lambda \theta)(k-1)}{e_P}.$$

Then use part 5 to obtain

$$\frac{(1 - \lambda\theta)s\theta_{ss} + 2\theta_s(1 - \lambda\theta) + 2\lambda s\theta_s^2}{-\theta_s} < (1 - \lambda\theta)(k - 1)$$

$$\Leftrightarrow s\theta_{ss} < -(k + 1)\theta_s - \frac{2\lambda s\theta_s^2}{(1 - \lambda\theta)}.$$

Part 8

Some algebra will reveal that condition (69) holds if and only if

$$\frac{-[1 - \lambda(1 - \theta)]\partial^2\theta/\partial e_D^2 + 2\lambda(\partial\theta/\partial e_D)^2}{-\partial\theta/\partial e_D} < \frac{[1 - \lambda(1 - \theta)](k - 1)}{e_D}.$$

Then use part 5 to obtain

$$\frac{-[1 - \lambda(1 - \theta)]s\theta_{ss} + 2\lambda s\theta_s^2}{-\theta_s} < [1 - \lambda(1 - \theta)](k - 1)$$

$$\Leftrightarrow (k - 1)\theta_s + \frac{2\lambda s\theta_s^2}{[1 - \lambda(1 - \theta)]} < s\theta_{ss}.$$

Part 9

Using $\theta_s < 0$, some algebra will derive the result from parts 7 and 8. $\square$

Proof of Lemma 1

This proof establishes the result for Plaintiff. Defendant's result follows symmetric steps. This proof takes the following steps: (i) establish that if Plaintiff's FOC holds at a pair of efforts, then her SOC is negative at that pair; (ii) using the results established in step (i), a theorem by Diewert et al. (1981) proves that Plaintiff's payoff function is strictly quasiconcave in her own effort.

Step (i)

Take the partial derivatives of Plaintiff's payoff function in (2) with respect to her effort e_P to obtain

$$\frac{\partial u_P}{\partial e_P} = \frac{\partial \theta}{\partial e_P} [1 + \lambda C(e_P) + \lambda C(e_D)] - (1 - \lambda\theta)C'(e_P) \quad (70)$$

$$\frac{\partial^2 u_P}{\partial e_P^2} = \frac{\partial^2 \theta}{\partial e_P^2} [1 + \lambda C(e_P) + \lambda C(e_D)] + 2\lambda \frac{\partial \theta}{\partial e_P} C'(e_P) - (1 - \lambda\theta)C''(e_P). \quad (71)$$

Suppose Plaintiff's FOC holds, then some algebra using equation (70) reveals

$$1 + \lambda C(e_P) + \lambda C(e_D) = \frac{(1 - \lambda\theta)C'(e_P)}{\partial\theta/\partial e_P}.$$

A substitution exercise using equation (71) gives

$$\begin{aligned}\frac{\partial^2 u_P}{\partial e_P^2} &= \frac{\partial^2 \theta}{\partial e_P^2} \left[\frac{(1 - \lambda\theta)C'(e_P)}{\partial \theta / \partial e_P} \right] + 2\lambda \frac{\partial \theta}{\partial e_P} C'(e_P) - (1 - \lambda\theta)C''(e_P) \\ &= C'(e_P)(1 - \lambda\theta) \left[\frac{(1 - \lambda\theta) \frac{\partial^2 \theta}{\partial e_P^2} + 2\lambda \left(\frac{\partial \theta}{\partial e_P} \right)^2}{(1 - \lambda\theta) \frac{\partial \theta}{\partial e_P}} - \frac{C''(e_P)}{C'(e_P)} \right] < 0\end{aligned}$$

where the last inequality uses condition (1) in Assumption (5).

Step (ii)

Corollary 9.3 of Diewert et al. (1981) holds that a twice continuously differentiable function f defined on an open S is strictly quasiconcave if and only if $x^0 \in S$, $v^T v = 1$ and $v^T \nabla f(x^0)v = 0$ implies $v^T \nabla^2 f(x^0)v < 0$; or $v^T \nabla^2 f(x^0)v = 0$ and $g(t) \equiv f(x^0 + tv)$ does not attain a local minimum at $t = 0$. We apply their result.

Fix Defendant's effort $e_D = e_1$ for some arbitrary $e_1 > 0$, and consider Plaintiff's payoff function $u_P(\cdot)$. Suppose $e_P > 0$, $v^T v = 1$ and

$$0 = v^T \nabla u_P(e_P, e_1; \mu, \lambda, k)v = v^T \frac{\partial}{\partial e_P} u_P(e_P, e_1; \mu, \lambda, k)v.$$

That $v^T v = 1$ implies $v \neq 0$. Hence

$$\frac{\partial}{\partial e_P} u_P(e_P, e_1; \mu, \lambda, k) = 0.$$

Then step (i) proves:

$$\frac{\partial^2}{\partial e_P^2} u_P(e_P, e_1; \mu, \lambda, k) < 0$$

where

$$\frac{\partial^2}{\partial e_P^2} u_P(e_P, e_1; \mu, \lambda, k) = \nabla^2 u_P(e_P, e_1; \mu, \lambda, k).$$

That $v \neq 0$ implies $v^T \nabla^2 u_P(e_P, e_1; \mu, \lambda, k)v < 0$. Hence an application of Corollary 9.3 of Diewert et al. (1981) proves Plaintiff's payoff function is strictly quasiconcave in her own effort. $\square$

Proof of Lemma 2

Define a function $h : \mathbb{R}_{++} \rightarrow \mathbb{R}$ by:

$$h(s) = 1 - \lambda\theta - s^k[1 - \lambda(1 - \theta)]. \quad (72)$$

The first two steps of this proof establishes the existence of an s^* such that $h(s^*) = 0$, and its value relative to 0.5, in the following two cases: (i) $\mu = 0.5$ or $\lambda = 0$; and (ii) $\mu > 0.5$ and $\lambda > 0$. (The case of $\mu < 0.5$ and $\lambda > 0$ follows similar steps as case (ii).) The third step establishes uniqueness.

Step (i)

Suppose $\mu = 0.5$. Then part 1 of Lemma 9 implies that choosing $s^* = 1$ induces $\theta = 0.5 = 1 - \theta$. Hence $h(1) = 0$.

Now suppose $\lambda = 0$. Then choosing $s^* = 1$ induces $h(s^*) = 0$.

Step (ii)

Suppose $\mu > 0.5$ and $\lambda > 0$. Define a new function $h_1(s)$ by:

$$h_1(s) = h(s)/s^k = \frac{1 - \lambda\theta}{s^k} - [1 - \lambda(1 - \theta)]. \quad (73)$$

Part 1 of Lemma 9 implies that $s = 1$ induces $\theta(1; \mu) > 0.5 > 1 - \theta(1; \mu)$. Some algebra and the property $\lambda > 0$ give:

$$1 - \lambda\theta(1; \mu) < 1 - \lambda(1 - \theta(1; \mu)) \quad \Leftrightarrow \quad h_1(1) < 0.$$

Now, consider the limit of $h_1(s)$ as s approaches 0

$$\lim_{s \rightarrow 0} h_1(s) = \lim_{s \rightarrow 0} \left(\frac{1 - \lambda\theta}{s^k} - [1 - \lambda(1 - \theta)] \right) = \lim_{s \rightarrow 0} \left(\frac{1 - \lambda\theta}{s^k} \right) - 1 + \lim_{s \rightarrow 0} [\lambda(1 - \theta)].$$

Consider two scenarios:

1. Suppose $\lim_{s \rightarrow 0}(\lambda\theta) < 1$. This implies $\lim_{s \rightarrow 0} h_1(s) = +\infty > 0$.
2. Suppose $\lim_{s \rightarrow 0}(\lambda\theta) = 1$. Then Assumption 6 implies $k > 1$. An application of the L'Hospital's Rule obtains:

$$\lim_{s \rightarrow 0} \left(\frac{1 - \lambda\theta}{s^k} \right) = \lim_{s \rightarrow 0} \left(\frac{\frac{\partial}{\partial s}(1 - \lambda\theta)}{\frac{\partial}{\partial s}(s^k)} \right) = \lim_{s \rightarrow 0} \left(\frac{-\lambda\theta_s}{ks^{k-1}} \right) = +\infty$$

where the last equality uses the properties $k > 1$, $\theta_s < 0$ and $\theta_{ss} \geq 0$. These results imply $\lim_{s \rightarrow 0} h_1(s) = +\infty > 0$.

Using the results $h_1(1) < 0$ and $\lim_{s \rightarrow 0} h_1(s) > 0$, the intermediate value theorem implies that there exists some $0 < s^* < 1$ such that $h_1(s^*) = 0$. Then use the definition of $h_1(\cdot)$ in equation (73) to obtain $h(s^*) = 0$.

Step (iii)

The function $h(\cdot)$ is continuously differentiable. Differentiate it to obtain:

$$h'(s) = -\lambda(1 + s^k)\theta_s - ks^{k-1}[1 - \lambda(1 - \theta)] = -\frac{ks^k[1 - \lambda(1 - \theta)] + \lambda(1 + s^k)s\theta_s}{s}.$$

Steps (i)-(ii) prove the existence of some $s^* > 0$ that satisfies $h(s^*) = 0$. Choose one such s^* and consider $s = s^*$. Then some algebra gives

$$s^k = \frac{1 - \lambda\theta}{1 - \lambda(1 - \theta)} \quad (1 + s^k) = \frac{2 - \lambda}{1 - \lambda(1 - \theta)}$$

and a substitution exercise reveals

$$h'(s) = -\frac{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s}{s[1 - \lambda(1 - \theta)]} < 0$$

where the last inequality uses part 9 of Lemma 9. Hence $h'(s) < 0$ whenever $h(s) = 0$.

Now suppose, for a contradiction, that there exist two different $s^{*'} > s^* > 0$ satisfying: $h(s^{*'}) = h(s^*) = 0$; and $h(s') \neq 0$ for all $s^* < s' < s^{*'}$. Then that $h'(s^*), h'(s^{*'}) < 0$ implies for some very small $\epsilon > 0$, we have $h(s^* + \epsilon) < 0$ and $h(s^{*' - \epsilon}) > 0$. Then the intermediate value theorem implies there exists some $s'' > 0$ satisfying $s^* < s'' < s^{*'}$ and $h(s'') = 0$, a contradiction. Hence there exists at most one s^* satisfying $h(s^*) = 0$. $\square$

Proof of Proposition 1

This proof will first establish that the pair (e_P^*, e_D^*) satisfies both Plaintiff and Defendants' FOCs in system (4), thereby characterizing a Nash equilibrium. It will then prove the other direction and uniqueness. An application of Lemma 2 gives the relative levels of e_P^*, e_D^* , and an application of Corollary 3 the size of $\theta(e_P^*, e_D^*; \mu)$ relative to 0.5.

Step (i)

Let $s = s^*$ and obtain, from the expression for e_P^* :

$$e_P^{*k} = \frac{-\theta_s}{C(1)[ks^{k-1}[1 - \lambda(1 - \theta)] + \lambda(1 + s^k)\theta_s]} = \frac{-s\theta_s}{C(1)[k(1 - \lambda\theta) + \lambda s(1 + s^k)\theta_s]}$$

where the last equality multiplies both the numerator and the denominator by s and uses Lemma 2. Then more algebra reveals

$$\frac{-s\theta_s}{e_P^*} = C(1)[k(1 - \lambda\theta) + \lambda s(1 + s^k)\theta_s]e_P^{*k-1}$$

$$\begin{aligned}
\frac{-s\theta_s}{e_P^*} &= C(1)(1 - \lambda\theta)ke_P^{*k-1} + C(1)\lambda s(1 + s^k)\theta_s e_P^{*k-1} \\
\frac{-s\theta_s}{e_P^*} - C(1)\lambda s(1 + s^k)\theta_s e_P^{*k-1} &= C(1)(1 - \lambda\theta)ke_P^{*k-1} \\
\frac{-s\theta_s}{e_P^*} [1 + \lambda C(1)(1 + s^k)e_P^{*k}] &= C(1)(1 - \lambda\theta)ke_P^{*k-1} \\
\frac{-s\theta_s}{e_P^*} [1 + \lambda C(1)e_P^{*k} + \lambda C(1)s^k e_P^{*k}] &= (1 - \lambda\theta)C(1)ke_P^{*k-1} \\
\frac{\partial \theta}{\partial e_P} [1 + \lambda C(e_P^*) + \lambda C(e_D^*)] &= (1 - \lambda\theta)C'(e_P^*)
\end{aligned}$$

where the last equality uses the properties that $C(\cdot)$ is homogeneous of degree k , $s^k e_P^{*k} = e_D^{*k}$, and $\frac{\partial \theta}{\partial e_P} = \frac{-s\theta_s}{e_P^*}$ from Lemma 9. Hence the pair (e_P^*, e_D^*) satisfies Plaintiff's FOC.

Now consider the expression for e_D^*

$$e_D^{*k} = s^k e_P^{*k} = \frac{-s^{k+1}\theta_s}{C(1)[ks^k[1 - \lambda(1 - \theta)] + \lambda s(1 + s^k)\theta_s]}$$

a rearrangement of which gives:

$$\begin{aligned}
\frac{-s\theta_s}{e_D^*} &= C(1)[ks^k[1 - \lambda(1 - \theta)] + \lambda s(1 + s^k)\theta_s] \frac{e_D^{*k-1}}{s^k} \\
\frac{-s\theta_s}{e_D^*} &= [1 - \lambda(1 - \theta)]C(1)ke_D^{*k-1} + \lambda C(1)s(1 + s^k)\theta_s \frac{e_D^{*k-1}}{s^k} \\
\frac{-s\theta_s}{e_D^*} - \lambda C(1)s(1 + s^k)\theta_s \frac{e_D^{*k-1}}{s^k} &= [1 - \lambda(1 - \theta)]C(1)ke_D^{*k-1} \\
\frac{-s\theta_s}{e_D^*} \left[1 + \lambda C(1) \frac{(1 + s^k)e_D^{*k}}{s^k} \right] &= [1 - \lambda(1 - \theta)]C(1)ke_D^{*k-1} \\
\frac{-s\theta_s}{e_D^*} \left[1 + \lambda C(1) \left(\frac{e_D^{*k}}{s^k} + e_D^{*k} \right) \right] &= [1 - \lambda(1 - \theta)]C(1)ke_D^{*k-1} \\
\frac{\partial(1 - \theta)}{\partial e_D} [1 + \lambda C(e_P^*) + \lambda C(e_D^*)] &= (1 - \lambda\theta)C'(e_D^*)
\end{aligned}$$

where the last equality uses the properties that $C(\cdot)$ is homogeneous of degree k , $s^k e_P^{*k} = e_D^{*k}$, and $\frac{\partial(1 - \theta)}{\partial e_D} = \frac{-s\theta_s}{e_D^*}$ from Lemma 9. Hence the pair (e_P^*, e_D^*) satisfies Defendant's FOC.

Step (ii)

Suppose $(e'_P, e'_D) \in \mathbb{R}_{++}^2$ is a Nash equilibrium with positive efforts. Denote $s' = e'_D/e'_P$.

Some algebra reveals

$$e_P'^k = \frac{(e'_P + e'_D)^k}{(1 + s')^k} \quad e_D'^k = \frac{s'^k(e'_P + e'_D)^k}{(1 + s')^k}.$$

Substituting these into Plaintiff and Defendant's FOCs in system (4), some algebra reveals:

$$\begin{aligned} \left. \frac{-(1+s)^k s \theta_s}{C(1)[k(1-\lambda\theta) + \lambda s(1+s^k)\theta_s]} \right|_{s=s'} &= (e'_P + e'_D)^k \\ &= \left. \frac{-(1+s)^k s \theta_s}{C(1)[ks^k[1-\lambda(1-\theta)] + \lambda s(1+s^k)\theta_s]} \right|_{s=s'} \end{aligned}$$

where the first equality (respectively, second equality) is derived from Plaintiff's (Defendant's) FOC. Then some algebra using the equality of both sides will reveal that $s = s'$ induces $1 - \lambda\theta = s^k[1 - \lambda(1 - \theta)]$. Hence the uniqueness limb of Lemma 2 implies $s' = s^*$.

Then some algebra using the definition of e_P^* in Proposition 1 obtains:

$$(1 + s^*)e_P^* = \left[\frac{-(1+s)^k s \theta_s}{C(1)[ks^k[1 - \lambda(1 - \theta)] + \lambda s(1+s^k)\theta_s]} \right]^{1/k} \bigg|_{s=s^*=s'} = e'_P + e'_D$$

where the properties $e'_P + e'_D = (1 + s')e'_P$ and $s' = s^*$ imply $e'_P = e_P^*$. Similarly, use the properties $e'_P + e'_D = e'_D(1 + s')/s'$ and $s' = s^*$ to obtain $e'_D = e_D^*$. $\square$

Lemmas 10 and 11 are technical lemmas on equilibrium properties, which will be used to prove subsequent propositions and corollaries.

Lemma 10. *Let $(e_P, e_D) = (e_P^*, e_D^*)$, the nontrivial Nash equilibrium characterized by Proposition 1. Let $s = s^*$ given by Lemma 2 and $\theta = \theta^*$ (Plaintiff's equilibrium probability of success). Denote $C^* = C(e_P^*) + C(e_D^*)$ and $\gamma = \frac{s^k}{(1+s^k)^2}$. The following properties hold:*

1.

$$\frac{\lambda(2\theta - 1)}{2 - \lambda} = \frac{1 - s^k}{1 + s^k} \quad (1 - \lambda\theta)[1 - \lambda(1 - \theta)] = (2 - \lambda)^2 \gamma.$$

2.

$$C^* = \frac{-(2 - \lambda)s\theta_s}{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s} = - \left[\lambda + \frac{(2 - \lambda)k\gamma}{s\theta_s} \right]^{-1}.$$

3.

$$\frac{d\gamma}{d\lambda} = \frac{k\gamma(1 - s^k)}{s(1 + s^k)} \frac{ds}{d\lambda} \quad \frac{d\gamma}{d\mu} = \frac{k\gamma(1 - s^k)}{s(1 + s^k)} \frac{ds}{d\mu} \quad \frac{d\gamma}{dk} = - \frac{\ln(s)s^k(1 - s^k)}{(1 + s^k)^3}.$$

Proof of Lemma 10

Part 1

Using Lemma 2, some algebra will give these results.

Part 2

Apply Lemma 9 to system (4) to obtain:

$$-\frac{s\theta_s}{e_P^*}[1 + \lambda C^*] = k(1 - \lambda\theta)C(1)e_P^{*k-1}, \quad -\frac{s\theta_s}{e_D^*}[1 + \lambda C^*] = k[1 - \lambda(1 - \theta)]C(1)e_D^{*k-1}.$$

Using the homogeneity of $C(\cdot)$ and Lemma 2, some algebra will give the result.

Part 3

some algebra reveals

$$\frac{d\gamma}{d\lambda} = \frac{ks^{k-1}(1 - s^k)}{(1 + s^k)^3} \frac{ds}{d\lambda} = \frac{k\gamma(1 - s^k)}{s(1 + s^k)} \frac{ds}{d\lambda}.$$

Similarly for $\frac{d\gamma}{d\mu}$ and $\frac{d\gamma}{dk}$.

□

Lemma 11. *Consider two cases that differ only in respect of Plaintiff's prior probability of success; it is μ in one case and $\mu' = 1 - \mu$ in the other case. Suppose (e_P^*, e_D^*) (respectively, $(e_P^{*'}, e_D^{*'})$) is the nontrivial Nash equilibrium in the case characterized by μ (respectively, μ'). Then Plaintiff's equilibrium effort in one case equals to Defendant's equilibrium effort in the other case. Formally, $e_P^* = e_D^{*'} and $e_D^* = e_P^{*'}$.$*

Proof of Lemma 11

To facilitate presentation and for the purpose of this proof only, let u_{P1} (respectively, u_{P2}) denote the partial derivative of Plaintiff's payoff given by (2) with respect to its first (second) argument, namely, Plaintiff's effort (Defendant's effort). Similarly, let u_{D1} (respectively, u_{D2}) denote the partial derivative of Defendant's payoff given by (3) with respect to its first (second) argument, namely, Plaintiff's effort (Defendant's effort).

In respect of the case characterized by μ , fix arbitrary real numbers $e_1, e_2 > 0$ and consider generic efforts e_P, e_D taken by Plaintiff and Defendant respectively. Some algebra using Assumption 1 reveals

$$\begin{cases} u_P(e_P, e_2; \mu, \lambda, k) = u_D(e_2, e_P; \mu', \lambda, k) + 1 \\ u_D(e_1, e_D; \mu, \lambda, k) = u_P(e_D, e_1; \mu', \lambda, k) - 1, \end{cases}$$

which implies

$$\begin{cases} u_{P1}(e_P, e_2; \mu, \lambda, k) = u_{D2}(e_2, e_P; \mu', \lambda, k) \\ u_{D2}(e_1, e_D; \mu, \lambda, k) = u_{P1}(e_D, e_1; \mu', \lambda, k). \end{cases} \quad (74)$$

That the pair of positive real numbers (e_P^*, e_D^*) is the nontrivial Nash equilibrium in the case characterized by μ is equivalent to

$$\begin{cases} u_{P1}(e_P^*, e_D^*; \mu, \lambda, k) = 0 \\ u_{D2}(e_P^*, e_D^*; \mu, \lambda, k) = 0. \end{cases} \quad (75)$$

Then by choosing real numbers $e_1 = e_P^*, e_2 = e_D^*$ in system (74), a substitution exercise using systems (74) and (75) reveals

$$\begin{cases} u_{P1}(e_P^*, e_D^*; \mu, \lambda, k) = u_{D2}(e_D^*, e_P^*; \mu', \lambda, k) = 0 \\ u_{D2}(e_P^*, e_D^*; \mu, \lambda, k) = u_{P1}(e_D^*, e_P^*; \mu', \lambda, k) = 0. \end{cases}$$

Hence the pair of positive real numbers (e_D^*, e_P^*) is the nontrivial Nash equilibrium in the case characterized by μ' . Then the uniqueness limb of Proposition 1 implies $e_P^* = e_D^{*'} and $e_D^* = e_P^{*'}$, where $(e_P^{*'}, e_D^{*'})$ is the nontrivial Nash equilibrium in the case characterized by μ' . $\square$$

Proof of Corollary 1

This proof establishes that $\Theta(\{\bar{\lambda}\}, \{\underline{k}\}) \subset \Theta([0, \bar{\lambda}], \{\underline{k}, +\infty\})$; that is, if the success function θ satisfies Assumptions 1-6 for a cost function of homogeneous of degree $\underline{k}$ and the cost-shifting rule $\bar{\lambda}$, then θ also satisfies Assumptions 1-6 for any arbitrary pair of cost-shifting rule $\lambda \leq \bar{\lambda}$ and cost function $k \geq \underline{k}$. Then an application of Proposition 1 to the pair λ, k gives the result.

Suppose the success function $\theta \in \Theta(\{\bar{\lambda}\}, \{\underline{k}\})$ and choose an arbitrary pair of λ, k satisfying $\lambda \leq \bar{\lambda}$ and $k \geq \underline{k}$. Let $\underline{C}(\cdot)$ denote the homogeneous cost function characterized by $\underline{k}$. Satisfaction of Assumptions 1-4 does not depend on the values of λ, k . Assumption 6 is either satisfied if $\lambda = \bar{\lambda} = k = \underline{k} = 1$, or not applicable otherwise. It remains to check that θ satisfies Assumption 5 for the pair λ, k .

Some algebra using the property that θ satisfies Assumption 5 under the pair $\bar{\lambda}, \underline{k}$

reveals

$$\frac{\underline{\underline{C''}}(e_P)}{\underline{\underline{C'}}(e_P)} > \frac{\frac{\partial^2}{\partial e_P^2} \left(\frac{\theta}{1-\bar{\lambda}\theta} \right)}{\frac{\partial}{\partial e_P} \left(\frac{\theta}{1-\bar{\lambda}\theta} \right)} = \frac{(1-\bar{\lambda}\theta) \frac{\partial^2 \theta}{\partial e_P^2} + 2\bar{\lambda} \left(\frac{\partial \theta}{\partial e_P} \right)^2}{\frac{\partial \theta}{\partial e_P}} \geq \frac{(1-\lambda\theta) \frac{\partial^2 \theta}{\partial e_P^2} + 2\lambda \left(\frac{\partial \theta}{\partial e_P} \right)^2}{\frac{\partial \theta}{\partial e_P}}$$

where the last weak inequality uses the properties $\lambda \leq \bar{\lambda}$, $1 - \bar{\lambda}\theta \leq 1 - \lambda\theta$ and Assumption 4. Then the result that θ satisfies Assumption 5 under (λ, k) follows from some algebra revealing

$$\frac{\frac{\partial^2}{\partial e_P^2} \left(\frac{\theta}{1-\lambda\theta} \right)}{\frac{\partial}{\partial e_P} \left(\frac{\theta}{1-\lambda\theta} \right)} = \frac{(1-\lambda\theta) \frac{\partial^2 \theta}{\partial e_P^2} + 2\lambda \left(\frac{\partial \theta}{\partial e_P} \right)^2}{\frac{\partial \theta}{\partial e_P}} \quad \frac{\underline{\underline{C''}}(e_P)}{\underline{\underline{C'}}(e_P)} \leq \frac{C''(e_P)}{C'(e_P)}.$$

The choice of λ, k was arbitrary; hence θ satisfies Assumptions 1-6 for any pair of cost-shifting rule $\lambda \leq \bar{\lambda}$ and cost function $k \geq \underline{k}$. $\square$

Proof of Corollary 2

This proof establishes the result for the case of $\mu > 0.5$. Similar steps establish the results for $\mu = 0.5$ and $\mu < 0.5$.

Let $s = s^*$ and $\theta = \theta^*$. Lemma 2 and Proposition 1 prove that in the nontrivial Nash equilibrium

$$s^k = \frac{1 - \lambda\theta}{1 - \lambda(1 - \theta)}.$$

Take the total derivative of both sides with respect to λ

$$\begin{aligned} k s^{k-1} \frac{\partial s}{\partial \lambda} &= \frac{\left(-\theta - \lambda \theta_s \frac{\partial s}{\partial \lambda} \right) [1 - \lambda(1 - \theta)] - \left(-(1 - \theta) + \lambda \theta_s \frac{\partial s}{\partial \lambda} \right) (1 - \lambda\theta)}{[1 - \lambda(1 - \theta)]^2} \\ k s^{k-1} [1 - \lambda(1 - \theta)]^2 \frac{\partial s}{\partial \lambda} &= -\theta [1 - \lambda(1 - \theta)] + (1 - \theta)(1 - \lambda\theta) \\ &\quad - \lambda [1 - \lambda(1 - \theta) + 1 - \lambda\theta] \theta_s \frac{\partial s}{\partial \lambda} \\ k s^k [1 - \lambda(1 - \theta)]^2 \frac{\partial s}{\partial \lambda} &= s(1 - 2\theta) - \lambda(2 - \lambda) s \theta_s \frac{\partial s}{\partial \lambda} \\ k(1 - \lambda\theta) [1 - \lambda(1 - \theta)] \frac{\partial s}{\partial \lambda} &= s(1 - 2\theta) - \lambda(2 - \lambda) s \theta_s \frac{\partial s}{\partial \lambda} \end{aligned}$$

where the last step uses Lemma 2. Then some algebra reveals

$$\frac{\partial s}{\partial \lambda} = \frac{s(1 - 2\theta)}{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda) s \theta_s} \quad (76)$$

where part 9 of Lemma 9 proves $k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s > 0$.

Now, take the total derivative of θ with respect to λ

$$\frac{d\theta}{d\lambda} = \theta_s \frac{ds}{d\lambda} \quad (77)$$

where Lemma 9 proves $\theta_s < 0$.

Suppose $\mu > 0.5$. Then Proposition 1 proves $s \leq 1$. Part 1 of Lemma 9 and the property $\theta_s < 0$ together prove $\theta > 0.5$. From equation (76), that $\theta > 0.5$ implies $\frac{\partial s}{\partial \lambda} < 0$, which is equivalent to $\frac{\partial(1/s)}{\partial \lambda} > 0$. Then equation (77) implies $\frac{d\theta}{d\lambda} > 0$. $\square$

Proof of Corollary 3

In this proof, let $s = s^*$ and $\theta = \theta^*$. Lemma 2 and Proposition 1 prove that in the nontrivial Nash equilibrium

$$s^k = \frac{1 - \lambda\theta}{1 - \lambda(1 - \theta)}.$$

Differentiate both sides respect to μ

$$k s^{k-1} \frac{ds}{d\mu} = \frac{-\lambda \frac{d\theta}{d\mu} [1 - \lambda(1 - \theta)] - \lambda \frac{d\theta}{d\mu} (1 - \lambda\theta)}{[1 - \lambda(1 - \theta)]^2} = -\frac{\lambda(2 - \lambda)}{[1 - \lambda(1 - \theta)]^2} \frac{d\theta}{d\mu}$$

where taking the total derivative of θ with respect to μ reveals

$$\frac{d\theta}{d\mu} = \theta_s \frac{ds}{d\mu} + \frac{\partial \theta}{\partial \mu}. \quad (78)$$

Then a substitution exercise reveals

$$\begin{aligned} \frac{ds}{d\mu} \left[k s^{k-1} + \frac{\lambda(2 - \lambda)\theta_s}{[1 - \lambda(1 - \theta)]^2} \right] &= -\frac{\lambda(2 - \lambda)}{[1 - \lambda(1 - \theta)]^2} \frac{\partial \theta}{\partial \mu} \\ \frac{ds}{d\mu} \left[\frac{k s^k [1 - \lambda(1 - \theta)]^2 + \lambda(2 - \lambda)s\theta_s}{s[1 - \lambda(1 - \theta)]^2} \right] &= -\frac{\lambda(2 - \lambda)}{[1 - \lambda(1 - \theta)]^2} \frac{\partial \theta}{\partial \mu} \\ \frac{ds}{d\mu} [k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s] &= -\lambda(2 - \lambda)s \frac{\partial \theta}{\partial \mu} \end{aligned}$$

where the last step applies Lemma 2. Then some algebra reveals

$$\frac{ds}{d\mu} = \frac{-\lambda(2 - \lambda)s \frac{\partial \theta}{\partial \mu}}{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s} \quad (79)$$

where part 9 of Lemma 9 proves $k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s > 0$ and Assumption 3 holds $\frac{\partial \theta}{\partial \mu} > 0$. Hence $\frac{ds}{d\mu} \leq 0$, holding strictly if $\lambda > 0$. Then an application of the chain

rule gives the results with respect to s^* .

Now, using equations (78) and (79), some algebra reveals

$$\begin{aligned}
\frac{d\theta}{d\mu} &= \frac{\partial\theta}{\partial\mu} - \frac{\lambda(2-\lambda)s\theta_s \frac{\partial\theta}{\partial\mu}}{k(1-\lambda\theta)[1-\lambda(1-\theta)] + \lambda(2-\lambda)s\theta_s} \\
&= \frac{\partial\theta}{\partial\mu} \left(1 - \frac{\lambda(2-\lambda)s\theta_s}{k(1-\lambda\theta)[1-\lambda(1-\theta)] + \lambda(2-\lambda)s\theta_s} \right) \\
&= \frac{\partial\theta}{\partial\mu} \left(\frac{k(1-\lambda\theta)[1-\lambda(1-\theta)]}{k(1-\lambda\theta)[1-\lambda(1-\theta)] + \lambda(2-\lambda)s\theta_s} \right)
\end{aligned} \tag{80}$$

which implies $\frac{d\theta}{d\mu} > 0$. Then an application of the chain rule gives the results with respect to θ^* . $\square$

Proof of Corollary 4

Consider the nontrivial Nash equilibrium, where $s = s^*$ given by Lemma 2 and $\theta = \theta^*$.

Totally differentiate θ with respect to k to obtain

$$\frac{d\theta}{dk} = \theta_s \frac{ds}{dk}$$

where Lemma 9 proves $\theta_s < 0$, and some algebra using equation (5) obtains:

$$\begin{aligned}
k \ln s &= \ln(1-\lambda\theta) - \ln(1-\lambda(1-\theta)) \\
\frac{d}{dk}(k \ln s) &= \frac{d}{dk}(\ln(1-\lambda\theta) - \ln(1-\lambda(1-\theta))) \\
\ln s + \frac{k}{s} \frac{ds}{dk} &= -\frac{\lambda\theta_s}{1-\lambda\theta} \frac{ds}{dk} - \frac{\lambda\theta_s}{1-\lambda(1-\theta)} \frac{ds}{dk} \\
\frac{ds}{dk} &= \frac{-\ln s}{\frac{k}{s} + \frac{\lambda\theta_s}{1-\lambda\theta} + \frac{\lambda\theta_s}{1-\lambda(1-\theta)}} \\
\frac{ds}{dk} &= \frac{-(1-\lambda\theta)(1-\lambda(1-\theta))s \ln s}{k(1-\lambda\theta)(1-\lambda(1-\theta)) + \lambda(2-\lambda)\theta_s}
\end{aligned}$$

where part 9 of Lemma 9 proves the denominator in the right-hand side is positive.

There are three cases:

1. If $\lambda = 0$ or $\mu = 0.5$, then Proposition 1 proves $s = 1$ for all $k \geq 1$, implying $\ln(s) = 0$. Hence $\frac{ds}{dk} = 0$, which implies $\frac{d\theta}{dk} = 0$.
2. If $\lambda > 0$ and $\mu > 0.5$, then Proposition 1 proves $s < 1$, implying $\ln(s) < 0$. Then $\frac{ds}{dk} > 0$. This and the property $\theta_s < 0$ (from Lemma 9) imply $\frac{d\theta}{dk} < 0$.
3. If $\lambda > 0$ and $\mu < 0.5$, then Proposition 1 proves $s > 1$, implying $\ln(s) > 0$. Then $\frac{ds}{dk} < 0$. This and $\theta_s < 0$ imply $\frac{d\theta}{dk} > 0$. $\square$

Proof of Corollary 5

Corollary 1 proves the nontrivial Nash equilibrium exists for all $\lambda \in [0, \lambda_2]$. Hence Corollary 2 applies to all $\lambda \in [\lambda_1, \lambda_2] \subset [0, \lambda_2]$.

Consider the nontrivial Nash equilibrium, where $\theta = \theta^*$. Suppose $\mu > 0.5$ (respectively, $\mu < 0.5$), then Proposition 1 and part 1 (respectively, part 3) of Corollary 2 prove that $\theta^* > 0.5$ and $d\theta^*/d\lambda > 0$ (respectively, $\theta^* < 0.5$ and $d\theta^*/d\lambda < 0$). Hence that $\lambda_2 > \lambda_1$ implies $\theta_2 > \theta_1 > 0.5$ (respectively, $\theta_2 < \theta_1 < 0.5$). $\square$

Proof of Proposition 2

In this proof, let $s = s^*$ given by Lemma 2. This proof will first prove the case of $\mu > 0.5$, and then the case of $\mu < 0.5$.

Case (i)

Suppose $\mu > 0.5$. Then Proposition 1 proves that $s^* \leq 1$, holding strictly if $\lambda > 0$. The property $\theta_s < 0$ (from Lemma 9) implies $\theta(1, \mu) \leq \theta(s^*, \mu) = \theta^*$, where the weak inequality holds strictly if $\lambda > 0$. Then use Assumption 7 to obtain:

$$\theta^* \geq \theta(1, \mu) = \mu$$

where the weak inequality holds strictly if $\lambda > 0$.

Case (ii)

Suppose $\mu < 0.5$. Use Assumption 7 to obtain:

$$1 - \mu = \theta(1; 1 - \mu) = 1 - \theta(1; \mu) \quad (81)$$

where the last equality applies Assumption 1.

Now, Proposition 1 proves that $s^* \geq 1$, holding strictly if $\lambda > 0$. The property $\theta_s < 0$ (from Lemma 9) implies $1 - \theta(1, \mu) \leq 1 - \theta(s^*, \mu) = 1 - \theta^*$, where the weak inequality holds strictly if $\lambda > 0$. Then use (81) to obtain

$$1 - \theta^* \geq 1 - \theta(1, \mu) = 1 - \mu$$

where the weak inequality holds strictly if $\lambda > 0$. $\square$

Proof of Corollary 6

Corollary 1 proves the nontrivial Nash equilibrium exists for all $\lambda \in [0, \lambda_2]$. Hence Corollary 2 applies to all $\lambda \in [\lambda_1, \lambda_2] \subset [0, \lambda_2]$. Then use part 1 (respectively, part 2) of Proposition 2 and part 1 (respectively, part 3) of Corollary 2 to obtain that $\mu > 0.5$

(respectively, $\mu < 0.5$) implies $\theta_2 > \theta_1 \geq \mu$ (respectively, $\mu \geq \theta_1 > \theta_2$). $\square$

Proof of Proposition 3

Part I

Suppose $\mu > 0.5$ and consider the nontrivial Nash equilibrium, where $s = s^*$ given by Lemma 2 and $\theta = \theta^*$. Denote an auxiliary variable $\gamma = \frac{s^k}{(1+s^k)^2}$. Lemma 10 proves $(1 - \lambda\theta)[1 - \lambda(1 - \theta)] = (2 - \lambda)^2\gamma$. A substitution exercise using equation (80) in the proof of Corollary 3 reveals

$$\begin{aligned}\frac{d\theta}{d\mu} &= \frac{\partial\theta}{\partial\mu} \left(\frac{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)]}{k(1 - \lambda\theta)[1 - \lambda(1 - \theta)] + \lambda(2 - \lambda)s\theta_s} \right) = \frac{\partial\theta}{\partial\mu} \left(\frac{k(2 - \lambda)\gamma}{k(2 - \lambda)\gamma + \lambda s\theta_s} \right) \\ &= \frac{\partial\theta}{\partial\mu} \left(1 + \frac{\lambda s\theta_s}{k(2 - \lambda)\gamma} \right)^{-1}.\end{aligned}$$

Denote $\theta_\mu = \frac{\partial\theta}{\partial\mu}$, $\theta_{\mu\mu} = \frac{\partial^2\theta}{\partial\mu^2}$ and $\theta_{s\mu} = \frac{\partial^2\theta}{\partial s\partial\mu}$. Differentiate both sides of $\frac{d\theta}{d\mu}$ with respect to μ to obtain

$$\begin{aligned}\frac{d^2\theta}{d\mu^2} &= \left(\theta_{\mu\mu} + \theta_{s\mu} \frac{ds}{d\mu} \right) \left(1 + \frac{\lambda s\theta_s}{k(2 - \lambda)\gamma} \right)^{-1} \\ &\quad - \theta_\mu \left(1 + \frac{\lambda s\theta_s}{k(2 - \lambda)\gamma} \right)^{-2} \left[\frac{\lambda \left(\theta_s \frac{ds}{d\mu} + s\theta_{ss} \frac{ds}{d\mu} + s\theta_{s\mu} \right) k(2 - \lambda)\gamma - \lambda s\theta_s k(2 - \lambda) \frac{d\gamma}{d\mu}}{k^2(2 - \lambda)^2\gamma^2} \right] \\ &= \left(\theta_{\mu\mu} + \theta_{s\mu} \frac{ds}{d\mu} \right) \left(\frac{k(2 - \lambda)\gamma}{k(2 - \lambda)\gamma + \lambda s\theta_s} \right) \\ &\quad - \theta_\mu \left(\frac{k(2 - \lambda)\gamma}{k(2 - \lambda)\gamma + \lambda s\theta_s} \right)^2 \left[\frac{\lambda \left(\theta_s \frac{ds}{d\mu} + s\theta_{ss} \frac{ds}{d\mu} + s\theta_{s\mu} \right) k(2 - \lambda)\gamma - \lambda s\theta_s k(2 - \lambda) \frac{d\gamma}{d\mu}}{k^2(2 - \lambda)^2\gamma^2} \right] \\ &= \left(\theta_{\mu\mu} + \theta_{s\mu} \frac{ds}{d\mu} \right) \left(\frac{k(2 - \lambda)\gamma}{k(2 - \lambda)\gamma + \lambda s\theta_s} \right) \\ &\quad - \frac{\lambda \theta_\mu \left[\left(\theta_s \frac{ds}{d\mu} + s\theta_{ss} \frac{ds}{d\mu} + s\theta_{s\mu} \right) k(2 - \lambda)\gamma - s\theta_s k(2 - \lambda) \frac{d\gamma}{d\mu} \right]}{[k(2 - \lambda)\gamma + \lambda s\theta_s]^2}.\end{aligned}$$

Denote an auxiliary variable $X_6 = k(2 - \lambda)\gamma$. Then some algebra using equation (79) in the proof of Corollary 3 gives

$$\frac{ds}{d\mu} = -\frac{\lambda s\theta_\mu}{X_6 + \lambda s\theta_s}. \quad (82)$$

A substitution exercise gives

$$(X_6 + \lambda s\theta_s)^2 \frac{d^2\theta}{d\mu^2} = \left(\theta_{\mu\mu} + \theta_{s\mu} \frac{ds}{d\mu} \right) X_6 (X_6 + \lambda s\theta_s)$$

$$\begin{aligned}
& -\lambda\theta_\mu \left[X_6 s\theta_{s\mu} + X_6(\theta_s + s\theta_{ss}) \frac{ds}{d\mu} - s\theta_s k(2-\lambda) \frac{d\gamma}{d\mu} \right] \\
& = \left(\theta_{\mu\mu} - \frac{\lambda s\theta_{s\mu}\theta_\mu}{X_6 + \lambda s\theta_s} \right) X_6(X_6 + \lambda s\theta_s) \\
& \quad - \lambda\theta_\mu \left[X_6 s\theta_{s\mu} - \frac{X_6(\theta_s + s\theta_{ss})\lambda s\theta_\mu}{X_6 + \lambda s\theta_s} - \frac{s\theta_s k^2(2-\lambda)\gamma(1-s^k)}{s(1+s^k)} \frac{ds}{d\mu} \right]
\end{aligned}$$

where the last step uses the property $\frac{d\gamma}{d\mu} = \frac{k\gamma(1-s^k)}{s(1+s^k)} \frac{ds}{d\mu}$ from Lemma 10 and equation (82).

Hence

$$\begin{aligned}
(X_6 + \lambda s\theta_s)^2 \frac{d^2\theta}{d\mu^2} &= \theta_{\mu\mu} X_6(X_6 + \lambda s\theta_s) - X_6 \lambda s\theta_\mu \theta_{s\mu} \\
&\quad - \lambda\theta_\mu \left[X_6 s\theta_{s\mu} - \frac{X_6(\theta_s + s\theta_{ss})\lambda s\theta_\mu}{X_6 + \lambda s\theta_s} + \frac{X_6 k(1-s^k)\theta_s \lambda s\theta_\mu}{(1+s^k)(X_6 + \lambda s\theta_s)} \right] \\
\frac{(X_6 + \lambda s\theta_s)^2}{X_6} \frac{d^2\theta}{d\mu^2} &= \theta_{\mu\mu}(X_6 + \lambda s\theta_s) - 2\lambda s\theta_\mu \theta_{s\mu} \\
&\quad + \frac{\lambda^2 s\theta_\mu^2}{X_6 + \lambda s\theta_s} \left[\theta_s + s\theta_{ss} - \frac{k(1-s^k)\theta_s}{(1+s^k)} \right] \\
\frac{(2-\lambda)(X_6 + \lambda s\theta_s)^2}{X_6} \frac{d^2\theta}{d\mu^2} &= \theta_{\mu\mu}(k(2-\lambda)^2\gamma + \lambda(2-\lambda)s\theta_s) - 2\lambda(2-\lambda)s\theta_\mu \theta_{s\mu} \\
&\quad + \frac{\lambda^2(2-\lambda)^2 s\theta_\mu^2}{k(2-\lambda)^2\gamma + \lambda(2-\lambda)s\theta_s} \left[\theta_s + s\theta_{ss} - \frac{k(1-s^k)\theta_s}{(1+s^k)} \right] \quad (83)
\end{aligned}$$

where the last step uses the definitions of γ and X_6 . Then using Lemmas 2 and 10, some algebra reveals

$$\begin{aligned}
& \frac{(2-\lambda)(X_6 + \lambda s\theta_s)^2 \frac{d^2\theta}{d\mu^2}}{X_6} \\
&= \theta_{\mu\mu}(k(1-\lambda\theta)[1 - (1-\lambda)\theta] + \lambda(2-\lambda)s\theta_s) - 2\lambda(2-\lambda)s\theta_\mu \theta_{s\mu} \\
&\quad + \frac{\lambda^2(2-\lambda)^2 s\theta_\mu^2}{k(1-\lambda\theta)[1 - (1-\lambda)\theta] + \lambda(2-\lambda)s\theta_s} \left\{ s\theta_{ss} + \left[1 - \frac{k\lambda(2\theta-1)}{2-\lambda} \right] \theta_s \right\} \quad (84)
\end{aligned}$$

where $X_6 > 0$ (due to the properties $\gamma > 0$ and $0 \leq \lambda < 2$), and some algebra using part 9 of Lemma 9 reveals

$$\begin{aligned}
X_6 + \lambda s\theta_s &= \frac{k(2-\lambda)^2\gamma + \lambda(2-\lambda)s\theta_s}{2-\lambda} \\
&= \frac{k(1-\lambda\theta)[1 - (1-\lambda)\theta] + \lambda(2-\lambda)s\theta_s}{2-\lambda} > 0.
\end{aligned}$$

Suppose condition (9) holds. Then $\frac{d^2\theta}{d\mu^2} \geq 0$, holding strictly if condition (9) holds strictly.

Now suppose condition (10) holds. Then some algebra using equation (83) reveals $\frac{d^2\theta}{d\mu^2} \geq 0$, holding strictly if condition (10) holds strictly.

Part 2

Suppose $\mu < 0.5$, and let (e_P^*, e_D^*) denote the nontrivial Nash equilibrium given Plaintiff's prior probability of success is μ . Consider another case that differs only in respect of Plaintiff's prior probability of success, which is given by $\mu' = 1 - \mu$ instead. Let $(e_P^{*'}, e_D^{*'})$ denote the nontrivial Nash equilibrium if Plaintiff's prior probability of success is μ' . Then Lemma 11 proves $e_P^* = e_D^{*'}, e_D^* = e_P^{*'}$, and an application of Assumption 11 reveals

$$\theta(e_P^*, e_D^*; \mu) = 1 - \theta(e_P^{*'}, e_D^{*'}; \mu')$$

where given $\mu' > 0.5$, the proof for part 1 establishes $\theta(e_P^{*'}, e_D^{*'}; \mu')$ is weakly convex in the parameter representing Plaintiff's prior probability of success. Hence an application of the chain rule gives the result. $\square$

Proof of Corollary 7

Corollary 1 proves the nontrivial Nash equilibrium exists for all $\lambda \in [0, \lambda_2]$. Hence Corollary 2 applies to all $\lambda \in [\lambda_1, \lambda_2] \subset [0, \lambda_2]$.

Suppose $\mu > 0.5$. That θ is weakly convex in μ (from part 1 of Proposition 3) and the property $\theta \leq 1$ imply $\theta_2 \leq \mu$. Then use part 1 of Corollary 2 to obtain $\mu \geq \theta_2 > \theta_1$.

Now suppose $\mu < 0.5$. That θ is weakly concave in μ (from part 2 of Proposition 3) and the property $\theta \geq 0$ imply $\theta_2 \geq \mu$. Then use part 3 of Corollary 2 to obtain $\theta_1 > \theta_2 \geq \mu$. $\square$

Proof of Corollary 8

In this proof, let $s = s^*$ given by Lemma 2. Part 2 of Lemma 10 reveals

$$C^* = - \left[\lambda + \frac{(2 - \lambda)k\gamma}{s\theta_s} \right]^{-1}.$$

Differentiate both sides of C^* with respect to λ

$$\begin{aligned} \frac{dC^*}{d\lambda} &= \left[1 + \frac{s\theta_s \left(-k\gamma + (2 - \lambda)k \frac{d\gamma}{d\lambda} \right) - (2 - \lambda)k\gamma(\theta_s + s\theta_{ss}) \frac{ds}{d\lambda}}{s^2\theta_s^2} \right] \left(\lambda + \frac{(2 - \lambda)k\gamma}{s\theta_s} \right)^{-2} \\ \frac{dC^*}{d\lambda} &= \left[s^2\theta_s^2 + \left(-k\gamma + (2 - \lambda)k \frac{d\gamma}{d\lambda} \right) s\theta_s - (\theta_s + s\theta_{ss})(2 - \lambda)k\gamma \frac{ds}{d\lambda} \right] s^{-2}\theta_s^{-2} C^{*2} \end{aligned}$$

$$\begin{aligned}
\frac{s^2\theta_s^2}{C^{*2}} \frac{dC^*}{d\lambda} &= s^2\theta_s^2 + \left(-k\gamma + (2-\lambda)k \frac{d\gamma}{d\lambda}\right) s\theta_s - (\theta_s + s\theta_{ss})(2-\lambda)k\gamma \frac{ds}{d\lambda} \\
\frac{s^2\theta_s^2}{C^{*2}} \frac{dC^*}{d\lambda} &= s^2\theta_s^2 - k\gamma s\theta_s + \frac{(2-\lambda)(1-s^k)k^2\gamma\theta_s}{(1+s^k)} \frac{ds}{d\lambda} - (\theta_s + s\theta_{ss})(2-\lambda)k\gamma \frac{ds}{d\lambda} \\
\frac{s^2\theta_s^2}{C^{*2}} \frac{dC^*}{d\lambda} &= s^2\theta_s^2 - k\gamma s\theta_s - k\gamma(2-\lambda) \frac{ds}{d\lambda} \left(\theta_s + s\theta_{ss} - \frac{k(1-s^k)\theta_s}{1+s^k} \right) \quad (85)
\end{aligned}$$

where the second last equality uses Lemma 10, and equation (76) in the proof of Corollary 2 reveals

$$\frac{\partial s}{\partial \lambda} = \frac{s(1-2\theta)}{k(1-\lambda\theta)[1-\lambda(1-\theta)] + \lambda(2-\lambda)s\theta_s}.$$

Then a substitution exercise using Lemma 10 and equation (85) gives the result. $\square$

Proof of Corollary 9

To facilitate presentation, define a function $g(\mu, \lambda, k)$ by

$$g(\mu, \lambda, k) = s^2\theta_s^2 - k\gamma s\theta_s - k\gamma(2-\lambda) \frac{ds}{d\lambda} \left(\theta_s + s\theta_{ss} - \frac{k(1-s^k)\theta_s}{1+s^k} \right) \quad (86)$$

where $s = s^*$ given by Lemma 2. Using equation (85), some algebra reveals that condition (13) is equivalent to $g(\mu, \lambda, k) > 0$.

Corollary 1 proves the nontrivial Nash equilibrium exists for all $\lambda \in [0, \lambda_2]$. Hence Corollary 8 applies to all $\lambda \in [\lambda_1, \lambda_2] \subset [0, \lambda_2]$. This proof will establish that for an arbitrary $\lambda \in [\lambda_1, \lambda_2]$, $g(\mu, \lambda, k) > 0$ in each of the following cases: (i) $\mu = 0.5$; (ii) $0.5 < \mu \leq 0.5 + \sigma(\lambda_2, k)$; (iii) $0.5 - \sigma(\lambda_2, k) \leq \mu < 0.5$. Then an application of Corollary 8 gives the result.

Case (i)

Suppose $\mu = 0.5$ and consider the nontrivial Nash equilibrium, where $s = s^*$ given by Lemma 2 and $\theta = \theta^*$. Then Corollary 2 proves $\frac{ds}{d\lambda} = 0$. Hence the property $\theta_s < 0$ from Lemma 9 implies $g(\mu, \lambda, k) > 0$.

Case (ii)

Suppose $0.5 < \mu \leq 0.5 + \sigma(\lambda_2, k)$ and consider the nontrivial Nash equilibrium, where $s = s^*$ given by Lemma 2 and $\theta = \theta^*$. Use equation (76) from the proof of Corollary 2 and part 2 of Lemma 10 to obtain

$$-k\gamma(2-\lambda) \frac{\partial s}{\partial \lambda} = \frac{k\gamma(2-\lambda)s(2\theta-1)}{k(1-\lambda\theta)(1-\lambda(1-\theta)) + \lambda(2-\lambda)s\theta_s} = \frac{k\gamma(2-\lambda)s(2\theta-1)}{k(2-\lambda)^2\gamma + \lambda(2-\lambda)s\theta_s}$$

$$= \frac{s(2\theta - 1)[k(2 - \lambda)\gamma + \lambda s\theta_s - \lambda s\theta_s]}{(2 - \lambda)[k\gamma(2 - \lambda) + \lambda s\theta_s]} = \frac{s(2\theta - 1)(1 + \lambda C^*)}{2 - \lambda}.$$

Hence

$$\begin{aligned} g(\mu, \lambda, k) &= s^2\theta_s^2 - k\gamma s\theta_s + \frac{s(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)} \left(s\theta_{ss} + \left(1 - \frac{k(1 - s^k)}{1 + s^k} \right) \theta_s \right) \\ &= s^2\theta_s^2 - k\gamma s\theta_s + \frac{(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)} \left(1 - \frac{k(1 - s^k)}{1 + s^k} \right) s\theta_s + \frac{(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)} s^2\theta_{ss} \end{aligned}$$

where the properties $\theta > 0.5$ (from Proposition 1) and $s\theta_{ss} > (k - 1)\theta_s + \frac{2\lambda s\theta_s^2}{[1 - \lambda(1 - \theta)]}$ (from Lemma 9) imply

$$\begin{aligned} g(\mu, \lambda, k) &> s^2\theta_s^2 - k\gamma s\theta_s + \frac{(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)} \left(1 - \frac{k(1 - s^k)}{1 + s^k} \right) s\theta_s \\ &\quad + \frac{(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)} \left((k - 1)s\theta_s + \frac{2\lambda s^2\theta_s^2}{[1 - \lambda(1 - \theta)]} \right) \\ &= s^2\theta_s^2 - k\gamma s\theta_s + \frac{2ks^k(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)(1 + s^k)} s\theta_s \\ &\quad + \frac{(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)} \left(\frac{2\lambda s^2\theta_s^2}{[1 - \lambda(1 - \theta)]} \right). \end{aligned}$$

Now, some algebra reveals

$$\begin{aligned} 1 + \lambda C^* &= 1 - \frac{\lambda s\theta_s}{k\gamma(2 - \lambda) + \lambda s\theta_s} = \frac{k\gamma(2 - \lambda)}{k\gamma(2 - \lambda) + \lambda s\theta_s} \\ &= \frac{-s\theta_s k\gamma(2 - \lambda)}{-s\theta_s[k\gamma(2 - \lambda) + \lambda s\theta_s]} = \frac{k\gamma(2 - \lambda)C^*}{-s\theta_s}. \end{aligned}$$

Hence

$$\begin{aligned} g(\mu, \lambda, k) &> s^2\theta_s^2 - k\gamma s\theta_s \\ &\quad + \frac{2ks^k(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)(1 + s^k)} s\theta_s + \frac{(2\theta - 1)k\gamma(2 - \lambda)C^*}{-s\theta_s(2 - \lambda)} \left(\frac{2\lambda s^2\theta_s^2}{[1 - \lambda(1 - \theta)]} \right) \\ &= s^2\theta_s^2 - k\gamma s\theta_s + \frac{2ks^k(2\theta - 1)(1 + \lambda C^*)}{(2 - \lambda)(1 + s^k)} s\theta_s - \frac{2k\gamma(2\theta - 1)\lambda C^*}{1 - \lambda(1 - \theta)} s\theta_s \\ &= s^2\theta_s^2 - k\gamma s\theta_s + \frac{2ks^k(2\theta - 1)}{(2 - \lambda)(1 + s^k)} s\theta_s \\ &\quad + 2ks\theta_s\lambda C^*(2\theta - 1) \left[\frac{s^k}{(2 - \lambda)(1 + s^k)} - \frac{\gamma}{1 - \lambda(1 - \theta)} \right] \end{aligned}$$

where some algebra using the property $s^k = \frac{1 - \lambda\theta}{1 - \lambda(1 - \theta)}$ from Lemma 2 and the definition of

γ (that $\gamma = s^k / (1 + s^k)^2$) reveals

$$\frac{s^k}{(2 - \lambda)(1 + s^k)} = \frac{\gamma(1 + s^k)}{(2 - \lambda)} = \frac{\gamma(1 - \lambda(1 - \theta) + 1 - \lambda\theta)}{(2 - \lambda)[1 - \lambda(1 - \theta)]} = \frac{\gamma}{1 - \lambda(1 - \theta)}.$$

Hence

$$\begin{aligned} g(\mu, \lambda, k) &> s^2\theta_s^2 - k\gamma s\theta_s + \frac{2ks^k(2\theta - 1)}{(2 - \lambda)(1 + s^k)}s\theta_s = s^2\theta_s^2 - ks\theta_s \left[\gamma - \frac{2s^k(2\theta - 1)}{(2 - \lambda)(1 + s^k)} \right] \\ &= s^2\theta_s^2 - ks\theta_s \left[\frac{s^k}{(1 + s^k)^2} - \frac{2s^k(2\theta - 1)}{(2 - \lambda)(1 + s^k)} \right] \\ &= s^2\theta_s^2 - \frac{ks^{k+1}\theta_s}{(1 + s^k)} \left[\frac{1}{1 + s^k} - \frac{2(2\theta - 1)}{2 - \lambda} \right] \\ &= s^2\theta_s^2 - \frac{ks^{k+1}\theta_s}{(1 + s^k)} \left[\frac{1 - \lambda(1 - \theta)}{2 - \lambda} - \frac{2(2\theta - 1)}{2 - \lambda} \right] \\ &= s^2\theta_s^2 - \frac{ks^{k+1}\theta_s[3 - \lambda - \theta(4 - \lambda)]}{(1 + s^k)(2 - \lambda)} \\ &= -s\theta_s \left[-s\theta_s + \frac{ks^k[3 - \lambda - \theta(4 - \lambda)]}{(1 + s^k)(2 - \lambda)} \right] \\ \Rightarrow g(\mu, \lambda, k) &> -s\theta_s \left[-s\theta_s + \frac{k(1 - \lambda\theta)}{(2 - \lambda)^2(4 - \lambda)} \left(\frac{3 - \lambda}{4 - \lambda} - \theta \right) \right]. \end{aligned} \quad (87)$$

Now, some algebra reveals

$$\frac{d}{d\lambda} \left(\frac{3 - \lambda}{4 - \lambda} - \theta \right) = -\frac{1}{(4 - \lambda)^2} - \frac{d\theta}{d\lambda} < 0$$

where the last inequality uses part 1 of Corollary 2 and the assumption $\mu > 0.5$. Hence, for all $\lambda \in [\lambda_1, \lambda_2]$,

$$\frac{3 - \lambda}{4 - \lambda} - \theta \geq \frac{3 - \lambda_2}{4 - \lambda_2} - \theta|_{\lambda=\lambda_2} \geq 0$$

where the last inequality uses the definition of $\sigma(\lambda_2, k)$. Then the property $\theta_s < 0$ from Lemma 9 and inequality (87) imply $g(\mu, \lambda, k) > 0$ for all $\lambda \in [\lambda_1, \lambda_2]$.

Case (iii)

Suppose $0.5 - \sigma \leq \mu < 0.5$. Let (e_p^*, e_D^*) denote the nontrivial Nash equilibrium given Plaintiff's prior probability of success is μ . Consider another case that differs only in respect of Plaintiff's prior probability of success, which is given by $\mu' = 1 - \mu$ instead. Let (e_p^*, e_D^*) denote the nontrivial Nash equilibrium if Plaintiff's prior probability of success

is μ' . Then Lemma 11 proves $e_P^* = e_D^{*'}, e_D^* = e_P^{*'}$. Hence

$$C(e_P^*) + C(e_D^*) = C(e_P^{*'}) + C(e_D^{*'}). \quad (88)$$

Some algebra will reveal that the properties $\mu' = 1 - \mu$ and $0.5 - \sigma \leq \mu < 0.5$ imply $0.5 < \mu' \leq 0.5 + \sigma$. Hence the proof for case (ii) establishes that the right hand side of equation (88), being the litigation expenditure given Plaintiff's prior probability of success is μ' , is increasing with λ . Then the left hand side of equation (88), being the litigation expenditure given Plaintiff's prior probability of success is μ , is also increasing with λ . $\square$

Proof of Corollary 10

Corollary 1 proves the nontrivial Nash equilibrium exist for all $\lambda \in [0, \lambda_2]$. Hence Corollary 8 applies to all $\lambda \in [\lambda_1, \lambda_2] \subset [0, \lambda_2]$.

Suppose the homogeneous cost function is of degree $k \geq 2$ and consider the nontrivial Nash equilibrium, where $s = s^*$ given by Lemma 2. Choose an arbitrary $\lambda \in [\lambda_1, \lambda_2]$. If $0.5 - \sigma(\lambda, k) \leq \mu \leq 0.5 + \sigma(\lambda, k)$, then inequality (87) in the proof of Corollary 9 proves the result. There are two remaining cases: $\mu > 0.5 + \sigma(\lambda, k)$; and $\mu < 0.5 - \sigma(\lambda, k)$. In respect of the case of $\mu > 0.5 + \sigma(\lambda, k)$, this will proof establish $C_2^* > C_1^*$ by showing that $\frac{dC^*}{d\lambda} > 0$ for $\lambda \in [\lambda_1, \lambda_2]$. Then the case of $\mu < 0.5 - \sigma(\lambda, k)$ follows from steps similar to those for case (iii) in the proof of Corollary 9.

Suppose $\mu > 0.5 + \sigma(\lambda, k)$. If $-(2 - \lambda)^2 s \theta_s + k(1 - \lambda \theta)[3 - \lambda - \theta(4 - \lambda)] \geq 0$, then some algebra using inequality (87) in the proof of Corollary 9 gives the result. The rest of this proof supposes $-(2 - \lambda)^2 s \theta_s + k(1 - \lambda \theta)[3 - \lambda - \theta(4 - \lambda)] < 0$; using Lemma 2, it is equivalent to

$$s \theta_s > k \gamma - \frac{2k(2\theta - 1)s^k}{(2 - \lambda)(1 + s^k)}. \quad (89)$$

Using equation (76) from the proof of Corollary 2 and part 2 of Lemma 10, some algebra reveals

$$(2 - \lambda) \frac{\partial s}{\partial \lambda} = \frac{(2 - \lambda)s(1 - 2\theta)}{k(2 - \lambda)^2 \gamma + \lambda(2 - \lambda)s \theta_s} = \frac{s(1 - 2\theta)}{k \gamma (2 - \lambda) + \lambda s \theta_s}.$$

Then substitute into the definition of $g(\mu, \lambda, k)$ (given by (86) in the proof of Corollary

9) to obtain

$$g(\mu, \lambda, k) = s^2\theta_s^2 - k\gamma s\theta_s + \left(\frac{k\gamma(2\theta - 1)}{k\gamma(2 - \lambda) + \lambda s\theta_s} \right) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right).$$

Define an auxiliary variable $X_4 = k\gamma(2 - \lambda) + \lambda s\theta_s$. Some algebra using the property $\theta_s < 0$ from Lemma 9 and inequality (89) reveals

$$\begin{aligned} X_4 g(\mu, \lambda, k) &= [s^2\theta_s^2 - k\gamma s\theta_s][k\gamma(2 - \lambda) + \lambda s\theta_s] + k\gamma(2\theta - 1) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right) \\ &\geq (s^2\theta_s^2 - k\gamma s\theta_s) \left[k\gamma(2 - \lambda) + \lambda k\gamma - \frac{2k\lambda(2\theta - 1)s^k}{(2 - \lambda)(1 + s^k)} \right] \\ &\quad + k\gamma(2\theta - 1) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right) \end{aligned}$$

where the inequality holds strictly if $\lambda > 0$. Then some algebra using the properties $\lambda(2\theta - 1)/(2 - \lambda) = (1 - s^k)/(1 + s^k)$ (from Lemma 10) and $\gamma = s^k/(1 + s^k)^2$ (the definition of the auxiliary variable γ) reveals

$$\begin{aligned} X_4 g(\mu, \lambda, k) &\geq (s^2\theta_s^2 - k\gamma s\theta_s) \left[2k\gamma - \frac{2k(1 - s^k)s^k}{(1 + s^k)^2} \right] \\ &\quad + k\gamma(2\theta - 1) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right) \\ &= (s^2\theta_s^2 - k\gamma s\theta_s) [2k\gamma - 2(1 - s^k)k\gamma] \\ &\quad + k\gamma(2\theta - 1) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right) \\ &= 2k\gamma s^k (s^2\theta_s^2 - k\gamma s\theta_s) + k\gamma(2\theta - 1) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right) \\ \Leftrightarrow \frac{X_4 g(\mu, \lambda, k)}{k\gamma} &\geq 2s^{k+2}\theta_s^2 - \frac{2ks^{2k+1}\theta_s}{(1 + s^k)^2} + (2\theta - 1) \left(s\theta_s + s^2\theta_{ss} - \frac{k(1 - s^k)s\theta_s}{1 + s^k} \right) \\ &= 2s^{k+2}\theta_s^2 + (2\theta - 1)s^2\theta_{ss} \\ &\quad - \frac{s\theta_s}{(1 + s^k)^2} \left[2ks^{2k} + (2\theta - 1)(1 + s^k)^2 \left(\frac{k(1 - s^k)}{1 + s^k} - 1 \right) \right] \end{aligned}$$

where some algebra reveals $1 = 2\theta - 1 + 2(1 - \theta)$. Then the property $s > 0$ from Proposition 1, and the properties $\theta_s < 0$ and $\theta_{ss} \geq 0$ from Lemma 9 imply

$$\begin{aligned} &\frac{X_4 g(\mu, \lambda, k)}{k\gamma} \\ &> - \frac{s\theta_s}{(1 + s^k)^2} \left[4k(1 - \theta)s^{2k} + 2(2\theta - 1)ks^{2k} + (2\theta - 1)(1 + s^k)^2 \left(\frac{k(1 - s^k)}{1 + s^k} - 1 \right) \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{s\theta_s}{(1+s^k)^2} \left\{ 4k(1-\theta)s^{2k} + (2\theta-1) \left[2ks^{2k} + k(1-s^k)(1+s^k) - (1+s^k)^2 \right] \right\} \\
&= -\frac{s\theta_s}{(1+s^k)^2} \left\{ 4k(1-\theta)s^{2k} + (2\theta-1) \left[2ks^{2k} + k(1-s^{2k}) - 1 - 2s^k - s^{2k} \right] \right\} \\
&= -\frac{s\theta_s}{(1+s^k)^2} \left\{ 4k(1-\theta)s^{2k} + (2\theta-1) \left[(k-2)(1+s^{2k}) + (1-s^k)^2 \right] \right\}.
\end{aligned}$$

Then the properties $\theta_s < 0$ (from Lemma 9) and $X_4 > 0$ (due to Lemma 10 and part 9 from Lemma 9) and $k\gamma > 0$, and the assumption $k \geq 2$ imply $g(\mu, \lambda, k) > 0$, which is equivalent to $\frac{dC^*}{d\lambda} > 0$. $\square$

Proof of Proposition 4

Corollary 1 proves the nontrivial Nash equilibrium exist for all $\lambda \in [0, \lambda_2]$. Hence Corollary 8 applies to all $\lambda \in [\lambda_1, \lambda_2] \subset [0, \lambda_2]$.

Choose an arbitrary $\lambda \in [\lambda_1, \lambda_2]$ and consider the nontrivial Nash equilibrium, where $s = s^*$ given by Lemma 2. If $0.5 - \sigma(\lambda, k) \leq \mu \leq 0.5 + \sigma(\lambda, k)$, then inequality (87) in the proof of Corollary 9 proves the result. If $k \geq 2$, then Corollary 10 establishes the result. There are two remaining cases: $k < 2$ and $\mu > 0.5 + \sigma(\lambda, k)$; $k < 2$ and $\mu < 0.5 - \sigma(\lambda, k)$. In respect of the case of $k < 2$ and $\mu > 0.5 + \sigma(\lambda, k)$, this will proof establish $C_2^* > C_1^*$ by showing that $\frac{dC^*}{d\lambda} > 0$ for $\lambda \in [\lambda_1, \lambda_2]$. Then the case of $k < 2$ and $\mu < 0.5 - \sigma(\lambda, k)$ follows from steps similar to those for case (iii) in the proof of Corollary 9.

Suppose $k < 2$ and $\mu > 0.5 + \sigma(\lambda, k)$. Suppose condition (15) holds. Then some algebra using equation (85) in the proof of Corollary 8 proves the result.

Now suppose condition (14) holds. Then some algebra using Lemma 2 and equation (85) gives the result. $\square$

Proof of Lemma 3

This proof applies the principle of finite induction. The result clearly holds for the base case $\theta = \theta_1$. The rest of this proof assumes the result holds for all integer value $j \leq n-1$, and proves that it holds for $j = n$. To facilitate presentation, define auxiliary variables $0 \leq a, \bar{a} \leq 1$ and a success function $\theta_I : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by

$$a = p_1 + p_2 + \dots + p_{n-1} \quad \bar{a} = 1 - a \quad \theta_I = \sum_{i=1}^{n-1} \left(\frac{p_i}{a} \theta_i \right)$$

where $p_1, p_2, \dots, p_{n-1}$ are weights assigned to $\theta_1, \theta_2, \dots, \theta_{n-1}$ respectively. The assumption that the result holds for $j = n-1$ implies $\theta_I \in \Theta(\{\lambda\}, \{k\})$.

Assumption 1

Consider three arbitrary real numbers $e_1, e_2 > 0$ and $0 < \mu_0 < 1$. The definition of θ

and the property that θ_I, θ_n satisfy Assumption 1 imply

$$\begin{aligned}\theta(e_1, e_2; \mu_0) &= a\theta_I(e_1, e_2; \mu_0) + \bar{a}\theta_n(e_1, e_2; \mu_0) = a[1 - \theta_I(e_2, e_1; 1 - \mu_0)] \\ &\quad + \bar{a}[1 - \theta_n(e_2, e_1; 1 - \mu_0)] \\ &= 1 - [a\theta_I(e_2, e_1; 1 - \mu_0) + \bar{a}\theta_n(e_2, e_1; 1 - \mu_0)] = 1 - \theta(e_2, e_1; 1 - \mu_0).\end{aligned}$$

Assumption 2

For any $x > 0$, use the definition of θ and the property that θ_I, θ_n satisfy Assumption 2 to obtain

$$\begin{aligned}\theta(xe_P, xe_D, \mu) &= a\theta_I(xe_P, xe_D; \mu) + \bar{a}\theta_n(xe_P, xe_D; \mu) \\ &= a\theta_I(e_P, e_D; \mu) + \bar{a}\theta_n(e_P, e_D; \mu) = \theta(e_P, e_D; \mu).\end{aligned}$$

Assumptions 3-4

Using the linearity of differentiation, some algebra will establish the result.

Assumption 5

Using the property that θ_I, θ_n satisfy condition (1), some algebra reveals

$$a \frac{\partial^2 \theta_I}{\partial e_P^2} + a \frac{2\lambda \left(\frac{\partial \theta_I}{\partial e_P} \right)^2}{(1 - \lambda \theta_I)} < a \frac{\partial \theta_I}{\partial e_P} \frac{C''(e_P)}{C'(e_P)}, \quad \bar{a} \frac{\partial^2 \theta_n}{\partial e_P^2} + \bar{a} \frac{2\lambda \left(\frac{\partial \theta_n}{\partial e_P} \right)^2}{(1 - \lambda \theta_n)} < \bar{a} \frac{\partial \theta_n}{\partial e_P} \frac{C''(e_P)}{C'(e_P)}.$$

Summing these inequalities give

$$\begin{aligned}a \frac{\partial^2 \theta_I}{\partial e_P^2} + \bar{a} \frac{\partial^2 \theta_n}{\partial e_P^2} + a \frac{2\lambda \left(\frac{\partial \theta_I}{\partial e_P} \right)^2}{(1 - \lambda \theta_I)} + \bar{a} \frac{2\lambda \left(\frac{\partial \theta_n}{\partial e_P} \right)^2}{(1 - \lambda \theta_n)} &< a \frac{\partial \theta_I}{\partial e_P} \frac{C''(e_P)}{C'(e_P)} + \bar{a} \frac{\partial \theta_n}{\partial e_P} \frac{C''(e_P)}{C'(e_P)} \\ \Leftrightarrow \frac{\partial^2 \theta}{\partial e_P^2} + a \frac{2\lambda \left(\frac{\partial \theta_I}{\partial e_P} \right)^2}{(1 - \lambda \theta_I)} + \bar{a} \frac{2\lambda \left(\frac{\partial \theta_n}{\partial e_P} \right)^2}{(1 - \lambda \theta_n)} &< \frac{\partial \theta}{\partial e_P} \frac{C''(e_P)}{C'(e_P)}\end{aligned} \tag{90}$$

where the last step uses the definition of θ and the linearity of differentiation.

Now, some algebra reveals

$$\begin{aligned}0 &\leq a\bar{a} \left[\frac{\partial \theta_I}{\partial e_P} (1 - \lambda \theta_n) - \frac{\partial \theta_n}{\partial e_P} (1 - \lambda \theta_I) \right]^2 \\ \Leftrightarrow 2a\bar{a} \frac{\partial \theta_I}{\partial e_P} \frac{\partial \theta_n}{\partial e_P} (1 - \lambda \theta_I)(1 - \lambda \theta_n) &\leq a\bar{a} \left(\frac{\partial \theta_I}{\partial e_P} \right)^2 (1 - \lambda \theta_n)^2 + a\bar{a} \left(\frac{\partial \theta_n}{\partial e_P} \right)^2 (1 - \lambda \theta_I)^2\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow 2a\bar{a}\frac{\partial\theta_I}{\partial e_P}\frac{\partial\theta_n}{\partial e_P} + a^2\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + \bar{a}^2\left(\frac{\partial\theta_n}{\partial e_P}\right)^2 \leq \frac{a\bar{a}\left(\frac{\partial\theta_I}{\partial e_P}\right)^2(1-\lambda\theta_n)^2 + a\bar{a}\left(\frac{\partial\theta_n}{\partial e_P}\right)^2(1-\lambda\theta_I)^2}{(1-\lambda\theta_I)(1-\lambda\theta_n)} \\
&\quad + a^2\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + \bar{a}^2\left(\frac{\partial\theta_n}{\partial e_P}\right)^2 \\
&\Leftrightarrow \frac{a^2\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + 2a\bar{a}\frac{\partial\theta_I}{\partial e_P}\frac{\partial\theta_n}{\partial e_P} + \bar{a}^2\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{a(1-\lambda\theta_I) + \bar{a}(1-\lambda\theta_n)} \leq \frac{a(1-\lambda\theta_n)\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + \bar{a}(1-\lambda\theta_I)\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{(1-\lambda\theta_I)(1-\lambda\theta_n)} \\
&\Leftrightarrow \frac{\left[a\frac{\partial\theta_I}{\partial e_P} + \bar{a}\frac{\partial\theta_n}{\partial e_P}\right]^2}{1-\lambda(a\theta_I + \bar{a}\theta_n)} \leq \frac{a(1-\lambda\theta_n)\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + \bar{a}(1-\lambda\theta_I)\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{(1-\lambda\theta_I)(1-\lambda\theta_n)} \\
&\Leftrightarrow \frac{\left(\frac{\partial\theta}{\partial e_P}\right)^2}{1-\lambda\theta} \leq \frac{a(1-\lambda\theta_n)\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + \bar{a}(1-\lambda\theta_I)\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{(1-\lambda\theta_I)(1-\lambda\theta_n)}
\end{aligned}$$

where the last step uses the definition of θ and the linearity of differentiation. Hence

$$\frac{\left(\frac{\partial\theta}{\partial e_P}\right)^2}{1-\lambda\theta} \leq \frac{a(1-\lambda\theta_n)\left(\frac{\partial\theta_I}{\partial e_P}\right)^2 + \bar{a}(1-\lambda\theta_I)\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{(1-\lambda\theta_I)(1-\lambda\theta_n)} = a\frac{\left(\frac{\partial\theta_I}{\partial e_P}\right)^2}{1-\lambda\theta_I} + \bar{a}\frac{\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{1-\lambda\theta_n}$$

which implies

$$\frac{2\lambda\left(\frac{\partial\theta}{\partial e_P}\right)^2}{(1-\lambda\theta)} \leq a\frac{2\lambda\left(\frac{\partial\theta_I}{\partial e_P}\right)^2}{(1-\lambda\theta_I)} + \bar{a}\frac{2\lambda\left(\frac{\partial\theta_n}{\partial e_P}\right)^2}{(1-\lambda\theta_n)}.$$

Then use inequality (90) to obtain

$$\frac{\partial^2\theta}{\partial e_P^2} + \frac{2\lambda\left(\frac{\partial\theta}{\partial e_P}\right)^2}{(1-\lambda\theta)} < \frac{\partial\theta}{\partial e_P} \frac{C''(e_P)}{C'(e_P)}$$

where some algebra will reveal that it implies θ satisfies condition (1).

Assumption 6

Suppose $k = \lambda = 1$. Use the definition of θ and the properties of limits to obtain

$$\lim_{s \rightarrow 0} \theta = \lim_{s \rightarrow 0} (a\theta_I + \bar{a}\theta_n) = a\left(\lim_{s \rightarrow 0} \theta_I\right) + \bar{a}\left(\lim_{s \rightarrow 0} \theta_n\right) < a + \bar{a} = 1$$

where the last inequality uses the property that θ_I, θ_n satisfy Assumption 6. $\square$

Proof of Proposition 5

Part 1

Suppose $\mu > 0.5$. Using part 2 of Lemma 10, some algebra reveals

$$\frac{dC^*}{d\mu} = \left\{ s\theta_s(2-\lambda)k \frac{d\gamma}{d\mu} - (2-\lambda)k\gamma \left[(\theta_s + s\theta_{ss}) \frac{ds}{d\mu} + s\theta_{s\mu} \right] \right\} s^{-2} \theta_s^{-2} \left[\lambda + \frac{(2-\lambda)k\gamma}{s\theta_s} \right]^{-2}$$

where Lemma 10 gives

$$\frac{d\gamma}{d\mu} = \frac{k\gamma(1-s^k)}{s(1+s^k)} \frac{ds}{d\mu}.$$

Then a substitution exercise reveals

$$\begin{aligned} [(2-\lambda)k\gamma + \lambda s\theta_s]^2 \frac{dC^*}{d\mu} &= \frac{\theta_s(2-\lambda)k^2\gamma(1-s^k)}{(1+s^k)} \frac{ds}{d\mu} - (2-\lambda)k\gamma \left[(\theta_s + s\theta_{ss}) \frac{ds}{d\mu} + s\theta_{s\mu} \right] \\ \frac{[(2-\lambda)k\gamma + \lambda s\theta_s]^2 \frac{dC^*}{d\mu}}{(2-\lambda)k\gamma} &= \frac{k(1-s^k)\theta_s}{(1+s^k)} \frac{ds}{d\mu} - (\theta_s + s\theta_{ss}) \frac{ds}{d\mu} - s\theta_{s\mu} \\ &= \frac{ds}{d\mu} \left[\frac{k(1-s^k)\theta_s}{(1+s^k)} - \theta_s - s\theta_{ss} \right] - s\theta_{s\mu} \\ &= -\frac{\lambda s\theta_{\mu}}{k(2-\lambda)\gamma + \lambda s\theta_s} \left[\frac{k(1-s^k)\theta_s}{(1+s^k)} - \theta_s - s\theta_{ss} \right] - s\theta_{s\mu} \quad (91) \end{aligned}$$

where the last step uses equation (82) in the proof of Proposition 3.

Suppose condition (16) holds. Then using the properties $(2-\lambda)k\gamma + \lambda s\theta_s > 0$ (due to part 9 of Lemma (9) and $\gamma > 0$, equation (91) reveals $\frac{dC^*}{d\mu} \geq 0$, holding strictly if condition (16) holds strictly.

Now suppose condition (17) hold. Then using Lemma (10), some algebra will reveal that $\frac{dC^*}{d\mu} \geq 0$, holding strictly if condition (17) holds strictly.

Part 2

Suppose $\mu < 0.5$. Let (e_P^*, e_D^*) denote the nontrivial Nash equilibrium given Plaintiff's prior probability of success is μ . Consider another case that differs only in respect of Plaintiff's prior probability of success, which is given by $\mu' = 1 - \mu$ instead. Let $(e_P^{*'}, e_D^{*'})$ denote the nontrivial Nash equilibrium if Plaintiff's prior probability of success is μ' . Then Lemma 11 proves $e_P^* = e_D^{*'}, e_D^* = e_P^{*'}$. Hence

$$C(e_P^*) + C(e_D^*) = C(e_P^{*'}) + C(e_D^{*'})$$

where the proof for part 1 establishes that the right hand side is convex in Plaintiff's prior probability of success μ' . Then an application of the chain rule gives the result. $\square$

Proof of Proposition 6

Using equation (91) in the proof of Proposition 5, some algebra will give the result. $\square$

A.2 Derivatives of Illustrative Success Functions for Chapter 2

This Appendix calculates the partial, second partial and cross derivatives for the illustrative success functions θ_T in (6) and θ_L in (11).

The Tullock success function θ_T satisfies the following properties:

$$\begin{aligned}\frac{\partial \theta_T}{\partial e_P} &= \frac{\mu(1-\mu)e_D}{[\mu e_P + (1-\mu)e_D]^2} & \frac{\partial^2 \theta_T}{\partial e_P^2} &= \frac{-2\mu^2(1-\mu)e_D}{[\mu e_P + (1-\mu)e_D]^3} \\ \frac{\partial \theta_T}{\partial \mu} &= \frac{s}{[\mu + (1-\mu)s]^2} & \frac{\partial^2 \theta_T}{\partial \mu^2} &= \frac{2(s-1)s}{[\mu + (1-\mu)s]^3} \\ \frac{\partial \theta_T}{\partial s} &= \frac{-\mu(1-\mu)}{[\mu + (1-\mu)s]^2} & \frac{\partial^2 \theta_T}{\partial s^2} &= \frac{2\mu(1-\mu)^2}{[\mu + (1-\mu)s]^3} \\ \frac{\partial^2 \theta_T}{\partial \mu \partial s} &= \frac{\mu - (1-\mu)s}{[\mu + (1-\mu)s]^3} & \frac{\partial \theta_T}{\partial s} + s \frac{\partial^2 \theta_T}{\partial s^2} &= \frac{\mu(1-\mu)[(1-\mu)s - \mu]}{[\mu + (1-\mu)s]^3}.\end{aligned}$$

The linear success function θ_L satisfies the following properties:

$$\begin{aligned}\frac{\partial \theta_L}{\partial e_P} &= \frac{(1-\eta)e_D}{[e_P + e_D]^2} & \frac{\partial^2 \theta_L}{\partial e_P^2} &= \frac{-2(1-\eta)e_D}{[e_P + e_D]^3} \\ \frac{\partial \theta_L}{\partial \mu} &= \eta & \frac{\partial^2 \theta_L}{\partial \mu^2} &= 0 \\ \frac{\partial \theta_L}{\partial s} &= \frac{-(1-\eta)}{(1+s)^2} & \frac{\partial^2 \theta_L}{\partial s^2} &= \frac{2(1-\eta)}{(1+s)^3} \\ \frac{\partial \theta_L}{\partial \mu \partial s} &= 0 & \frac{\partial \theta_L}{\partial s} + s \frac{\partial^2 \theta_L}{\partial s^2} &= \frac{(s-1)(1-\eta)}{(1+s)^3}.\end{aligned}$$

A.3 Proofs for Chapter 3

This appendix contains all proofs for chapter 3.

Proof of Lemma 4

This proof establishes the result for Plaintiff. Defendant's result follows symmetric steps. This proof takes the following steps: (i) establish that if Plaintiff's FOC holds at a pair of efforts, then her SOC is negative at that pair; (ii) using the results established in step (i), a theorem by Diewert et al. (1981) proves that Plaintiff's payoff function is strictly quasiconcave in her own effort.

Step (i)

Take the partial derivatives of Plaintiff's emotional payoff function in (24) with respect

to her effort e_P to obtain

$$\frac{\partial \check{u}_P}{\partial e_P} = \frac{\partial \theta}{\partial e_P} (1 - \xi) [1 + v + \check{\lambda} C(e_P) + \check{\lambda} C(e_D)] - [1 - \check{\lambda}(1 - \xi)\theta] C'(e_P) \quad (92)$$

$$\begin{aligned} \frac{\partial^2 \check{u}_P}{\partial e_P^2} &= \frac{\partial^2 \theta}{\partial e_P^2} (1 - \xi) [1 + v + \check{\lambda} C(e_P) + \check{\lambda} C(e_D)] + 2\check{\lambda}(1 - \xi) \frac{\partial \theta}{\partial e_P} C'(e_P) \\ &\quad - [1 - \check{\lambda}(1 - \xi)\theta] C''(e_P). \end{aligned} \quad (93)$$

Suppose Plaintiff's FOC holds, then some algebra using equation (92) reveals

$$(1 - \xi) [1 + v + \check{\lambda} C(e_P) + \check{\lambda} C(e_D)] = \frac{[1 - \check{\lambda}(1 - \xi)\theta] C'(e_P)}{\partial \theta / \partial e_P}.$$

A substitution exercise using equation (93) gives

$$\begin{aligned} \frac{\partial^2 \check{u}_P}{\partial e_P^2} &= \frac{\partial^2 \theta}{\partial e_P^2} \left[\frac{[1 - \check{\lambda}(1 - \xi)\theta] C'(e_P)}{\partial \theta / \partial e_P} \right] + 2\check{\lambda}(1 - \xi) \frac{\partial \theta}{\partial e_P} C'(e_P) - [1 - \check{\lambda}(1 - \xi)\theta] C''(e_P) \\ &= C'(e_P) [1 - \check{\lambda}(1 - \xi)\theta] \left[\frac{[1 - \check{\lambda}(1 - \xi)\theta] \frac{\partial^2 \theta}{\partial e_P^2} + 2\check{\lambda}(1 - \xi) \left(\frac{\partial \theta}{\partial e_P} \right)^2}{[1 - \check{\lambda}(1 - \xi)\theta] \frac{\partial \theta}{\partial e_P}} - \frac{C''(e_P)}{C'(e_P)} \right] < 0 \end{aligned}$$

where the last inequality uses Assumptions (16), (17).

Step (ii)

Corollary 9.3 of Diewert et al. (1981) holds that a twice continuously differentiable function f defined on an open S is strictly quasiconcave if and only if $y^0 \in S$, $w^T w = 1$ and $w^T \nabla f(y^0) w = 0$ implies $w^T \nabla^2 f(y^0) w < 0$; or $w^T \nabla^2 f(y^0) w = 0$ and $g(z) \equiv f(y^0 + zw)$ does not attain a local minimum at $z = 0$. We apply their result.

Fix Defendant's effort $e_D = e_1$ for some arbitrary $e_1 > 0$, and consider Plaintiff's emotional payoff function $\check{u}_P(\cdot)$. Suppose $e_P > 0$, $w^T w = 1$ and

$$0 = w^T \nabla \check{u}_P(e_P, e_1) w = w^T \frac{\partial}{\partial e_P} \check{u}_P(e_P, e_1) w.$$

That $w^T w = 1$ implies $w \neq 0$. Hence $\frac{\partial}{\partial e_P} \check{u}_P(e_P, e_1) = 0$. Then step (i) proves

$$0 > \frac{\partial^2}{\partial e_P^2} \check{u}_P(e_P, e_1) = \nabla^2 \check{u}_P(e_P, e_1).$$

That $w \neq 0$ implies $w^T \nabla^2 \check{u}_P(e_P, e_1) w < 0$. Hence an application of Corollary 9.3 of Diewert et al. (1981) proves $\check{u}_P$ is strictly quasiconcave in e_P . $\square$

Proof of Lemma 5

That the success function θ and the cost-shifting rule $\check{\lambda}$ satisfy Assumptions 12-17 in the present Emotional Litigation Game implies θ and a different cost-shifting rule defined by $\lambda = \check{\lambda}(1 - \xi)$ satisfy Assumptions 1-6 on the Litigation Game constructed in chapter 2, section 2.2. Lemma 2 in section 2.3 proves that there exists a unique real number s^* satisfying

$$s^{*k} = \frac{1 - \lambda\theta(s^*; \mu)}{1 - \lambda(1 - \theta(s^*; \mu))}.$$

Choosing $s = s^*$ gives the result for the present Emotional Litigation Game. $\square$

Proof of Proposition 7

This proof will first establish that the pair (e_P^*, e_D^*) satisfies both Plaintiff and Defendant's FOCs in system (28), thereby characterizing a Nash equilibrium. It will then prove the other direction and uniqueness.

Step (i)

Let $s = s^*$ and use the expression for e_P^* to obtain

$$\begin{aligned} e_P^{*k} &= \frac{-(1 + \nu)(1 - \xi)s\theta_s}{C(1)[ks^k[1 - \check{\lambda}(1 - \xi)(1 - \theta)] + \check{\lambda}(1 - \xi)(1 + s^k)s\theta_s]} \\ &= \frac{-(1 + \nu)(1 - \xi)s\theta_s}{C(1)[k[1 - \check{\lambda}(1 - \xi)\theta] + \check{\lambda}(1 - \xi)(1 + s^k)s\theta_s]} \end{aligned}$$

where the last equality uses the property $1 - \check{\lambda}(1 - \xi)\theta = s^k[1 - \check{\lambda}(1 - \xi)(1 - \theta)]$ from Lemma 5. Then

$$\begin{aligned} \frac{-(1 + \nu)(1 - \xi)s\theta_s}{e_P^*} &= C(1)[k[1 - \check{\lambda}(1 - \xi)\theta] + \check{\lambda}(1 - \xi)(1 + s^k)s\theta_s]e_P^{*k-1} \\ \frac{-(1 + \nu)(1 - \xi)s\theta_s}{e_P^*} &= C(1)[1 - \check{\lambda}(1 - \xi)\theta]ke_P^{*k-1} + C(1)\check{\lambda}(1 - \xi)(1 + s^k)s\theta_s e_P^{*k-1} \\ \frac{-(1 + \nu)(1 - \xi)s\theta_s}{e_P^*} - C(1)\check{\lambda}(1 - \xi)(1 + s^k)s\theta_s e_P^{*k-1} &= C(1)[1 - \check{\lambda}(1 - \xi)\theta]ke_P^{*k-1} \\ \frac{-(1 - \xi)s\theta_s}{e_P^*} [1 + \nu + \check{\lambda}C(1)(1 + s^k)e_P^{*k}] &= C(1)[1 - \check{\lambda}(1 - \xi)\theta]ke_P^{*k-1} \\ \frac{\partial \theta}{\partial e_P}(1 - \xi)[1 + \nu + \check{\lambda}C(e_P^*) + \check{\lambda}C(e_D^*)] &= [1 - \check{\lambda}(1 - \xi)\theta]C'(e_P^*) \end{aligned}$$

where the last equality uses the properties that $C(\cdot)$ is homogeneous of degree k , $s^k e_P^{*k} = e_D^{*k}$. Hence the pair (e_P^*, e_D^*) satisfies Plaintiff's FOC.

Now consider the expression for e_D^*

$$e_D^{*k} = s^k e_P^{*k} = \frac{-(1+\nu)(1-\xi)s^{k+1}\theta_s}{C(1)[ks^k[1-\check{\lambda}(1-\xi)(1-\theta)] + \check{\lambda}(1-\xi)(1+s^k)s\theta_s]}$$

a rearrangement of which gives:

$$\begin{aligned} \frac{-(1+\nu)(1-\xi)s\theta_s}{e_D^*} &= C(1)[ks^k[1-\check{\lambda}(1-\xi)(1-\theta)] + \check{\lambda}(1-\xi)(1+s^k)s\theta_s] \frac{e_D^{*k-1}}{s^k} \\ \frac{-(1+\nu)(1-\xi)s\theta_s}{e_D^*} &= [1-\check{\lambda}(1-\xi)(1-\theta)]C(1)ke_D^{*k-1} + \check{\lambda}(1-\xi)C(1)(1+s^k)s\theta_s \frac{e_D^{*k-1}}{s^k} \\ \frac{-(1+\nu)(1-\xi)s\theta_s}{e_D^*} - \check{\lambda}(1-\xi)C(1)(1+s^k)s\theta_s \frac{e_D^{*k-1}}{s^k} &= [1-\check{\lambda}(1-\xi)(1-\theta)]C(1)ke_D^{*k-1} \\ \frac{-(1-\xi)s\theta_s}{e_D^*} \left[1+\nu + \check{\lambda}C(1)\left(\frac{e_D^{*k}}{s^k} + e_D^{*k}\right) \right] &= [1-\check{\lambda}(1-\xi)(1-\theta)]C(1)ke_D^{*k-1} \\ \frac{\partial(1-\theta)}{\partial e_D}(1-\xi)[1+\nu + \check{\lambda}C(e_P^*) + \check{\lambda}C(e_D^*)] &= [1-\check{\lambda}(1-\xi)(1-\theta)]C'(e_D^*) \end{aligned}$$

which implies the pair (e_P^*, e_D^*) satisfies Defendant's FOC.

Step (ii)

Suppose $(e'_P, e'_D) \in \mathbb{R}_{++}^2$ is a Nash equilibrium with positive efforts. Denote $s' = e'_D/e'_P$.

Some algebra reveals

$$e_P'^k = \frac{(e'_P + e'_D)^k}{(1+s')^k} \quad e_D'^k = \frac{s'^k(e'_P + e'_D)^k}{(1+s')^k}.$$

Substituting these into Plaintiff and Defendant's FOCs in system (28), some algebra reveals:

$$\begin{aligned} &\left. \frac{-(1+\nu)(1-\xi)s(1+s)^k\theta_s}{C(1)[k[1-\check{\lambda}(1-\xi)\theta] + \check{\lambda}(1-\xi)(1+s^k)s\theta_s]} \right|_{s=s'} = (e'_P + e'_D)^k \\ &= \left. \frac{-(1+\nu)(1-\xi)s(1+s)^k\theta_s}{C(1)[ks^k[1-\check{\lambda}(1-\xi)(1-\theta)] + \check{\lambda}(1-\xi)(1+s^k)s\theta_s]} \right|_{s=s'} \end{aligned}$$

where the first equality (respectively, second equality) is derived from Plaintiff's (Defendant's) FOC. Then some algebra using the equality of both sides will reveal that $s = s'$ induces $1 - \check{\lambda}(1-\xi)\theta = s^k[1 - \check{\lambda}(1-\xi)(1-\theta)]$. Hence the uniqueness limb of Lemma 5 implies $s' = s^*$.

Then obtain from the definition of e_P^* in Proposition 7

$$\begin{aligned} e'_P + e'_D &= \left[\frac{-(1+\nu)(1-\xi)s(1+s)^k \theta_s}{C(1)[ks^k[1-\check{\lambda}(1-\xi)(1-\theta)] + \check{\lambda}(1-\xi)(1+s^k)s\theta_s} \right]^{1/k} \Big|_{s=s^*=s'} \\ &= (1+s^*)e_P^* \end{aligned}$$

where the properties $e'_P + e'_D = (1+s')e'_P$ and $s' = s^*$ imply $e'_P = e_P^*$. Similarly, use the properties $e'_P + e'_D = e'_D(1+s')/s'$ and $s' = s^*$ to obtain $e'_D = e_D^*$. $\square$

Proof of Corollary 11

An application of Lemma 5 and Proposition 7 gives the result. $\square$

Proof of Corollary 12

To facilitate presentation, define a function $\lambda = \check{\lambda}(1-\xi)$. An application of Corollary 11 and the chain rule reveals

$$\frac{d}{d\xi} s^*(\xi, \nu, \check{\lambda}) = \frac{d}{d\xi} s^*(0, 0, \lambda) = \frac{d\lambda}{d\xi} \frac{d}{d\lambda} s^*(0, 0, \lambda) = -\check{\lambda} \frac{d}{d\lambda} s^*(0, 0, \lambda).$$

Then part 1 follows from letting $\check{\lambda} = 0$, and parts 2-3 an application of Corollary 2 in chapter 2, section 2.4. $\square$

Proof of Corollary 13

Lemma 5 reveals that the equilibrium effort ratio s^* does not depend on the value of joy of winning ν . The success function θ also does not depend on ν . $\square$

Proof of Corollary 14

Using equation (30), Corollary 11 and the homogeneity of the cost function $C(\cdot)$, some algebra obtains

$$\begin{aligned} C^*(\xi, \nu, \check{\lambda}) &= C(e_P^*(\xi, \nu, \check{\lambda})) + C(e_D^*(\xi, \nu, \check{\lambda})) \\ &= C((1+\nu)^{1/k}(1-\xi)^{1/k} e_P^*(0, 0, \check{\lambda}(1-\xi))) \\ &\quad + C((1+\nu)^{1/k}(1-\xi)^{1/k} e_D^*(0, 0, \check{\lambda}(1-\xi))) \\ &= (1+\nu)(1-\xi)C(1)[e_P^*(0, 0, \check{\lambda}(1-\xi))]^k + C(1)(1-\xi)[e_D^*(0, 0, \check{\lambda}(1-\xi))]^k \\ &= (1+\nu)(1-\xi)C^*(0, 0, \check{\lambda}(1-\xi)) \end{aligned} \tag{94}$$

where $C^*(0, 0, \check{\lambda}(1-\xi))$ is the (equilibrium) litigation expenditure in the transformed Monetary Litigation Game with a different cost-shifting rule defined by $\lambda = \check{\lambda}(1-\xi)$.

Then an application of the product rule and the chain rule reveals

$$\begin{aligned}
\frac{d}{d\xi}C^*(\xi, v, \check{\lambda}) &= \frac{d}{d\xi}[(1+v)(1-\xi)C^*(0, 0, \lambda)] \\
&= (1+v) \left[(1-\xi) \frac{d}{d\xi}C^*(0, 0, \lambda) - C^*(0, 0, \lambda) \right] \\
&= (1+v) \left[(1-\xi) \frac{d\lambda}{d\xi} \frac{d}{d\lambda}C^*(0, 0, \lambda) - C^*(0, 0, \lambda) \right] \\
&= -(1+v) \left[C^*(0, 0, \lambda) + (1-\xi)\check{\lambda} \frac{d}{d\lambda}C^*(0, 0, \lambda) \right]. \tag{95}
\end{aligned}$$

Then part 1 follows from letting $\check{\lambda} = 0$ and noting $C^*(0, 0, \lambda) > 0$. Parts 2 and 3 respectively follow from Corollaries 8 and 9 in chapter 2, section 2.7. $\square$

Proof of Corollary 15

Part 1

The proof for this part will establish the result for Plaintiff's monetary payoff; similar steps gives the result for Defendant's.

Using equations (22), (94) and Corollary 11, some algebra reveals

$$\begin{aligned}
\frac{d}{dv}u_P^*(\xi, v, \check{\lambda}) &= \frac{d}{dv} \left[\theta^*(\xi, v, \check{\lambda})[1 + \check{\lambda}C^*(\xi, v, \check{\lambda})] - C(e_P^*(\xi, v, \check{\lambda})) - \check{\lambda}C(e_D^*(\xi, v, \check{\lambda})) \right] \\
&= \frac{d}{dv} \left[\theta^*(\xi, v, \check{\lambda})[1 + \check{\lambda}(1+v)(1-\xi)C^*(0, 0, \check{\lambda}(1-\xi))] \right] \\
&\quad - \frac{d}{dv} \left[(1+v)(1-\xi)[C(e_P^*(0, 0, \check{\lambda}(1-\xi))) + \check{\lambda}C(e_D^*(0, 0, \check{\lambda}(1-\xi)))] \right] \\
&= [1 + \check{\lambda}(1+v)(1-\xi)C^*(0, 0, \check{\lambda}(1-\xi))] \frac{d}{dv}\theta^*(\xi, v, \check{\lambda}) \\
&\quad - \theta^*(\xi, v, \check{\lambda})\check{\lambda}(1-\xi)C^*(0, 0, \check{\lambda}(1-\xi)) \\
&\quad - (1-\xi)[C(e_P^*(0, 0, \check{\lambda}(1-\xi))) + \check{\lambda}C(e_D^*(0, 0, \check{\lambda}(1-\xi)))]
\end{aligned}$$

where Corollary 13 reveals $\frac{d}{dv}\theta^*(\xi, v, \check{\lambda}) = 0$. Hence $\frac{d}{dv}u_P^*(\xi, v, \check{\lambda}) < 0$.

Part 2

The result follows from differentiating both sides of equation (94) with respect to v . $\square$

A.4 Proofs for Chapter 4

This appendix contains all proofs for chapter 4. Lemma 12 is a technical lemma that will facilitate calculations. For each $i \in \{1, 2\}$, define functions $\theta_i^{(j)}, \phi_{ii} : \mathbb{R}_{++} \rightarrow \mathbb{R}$ by

$$\theta_i^{(j)}(r_{ij}) = \frac{\partial}{\partial r_{ij}} \theta_i(r_{ij}^{-1}) \quad (96)$$

$$\phi_{ii}(r_{ij}) = \frac{-v_i \theta_i^{(j)}(r_{ij})}{1 - l_{ii} - \delta_{ii} \theta_i(r_{ij}^{-1}) + (\delta_{ii} + \delta_{ij} r_{ij}) r_{ij} \theta_i^{(j)}(r_{ij})}. \quad (97)$$

Lemma 12. *Whenever expenses $e_i, e_j > 0$ for $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, the following properties hold:*

1. *The expenses ratios r_{ii}, r_{ij} and success function θ_i satisfy*

$$\begin{aligned} \frac{\partial r_{ii}}{\partial e_i} &= \frac{r_{ii}}{e_i}, & \frac{\partial r_{ij}}{\partial e_i} &= \frac{-r_{ij}}{e_i}, & \frac{\partial \theta_i}{\partial e_i} &= \frac{r_{ii} \theta_i^{(i)}}{e_i} = \frac{-r_{ij} \theta_i^{(j)}}{e_i}, \\ \theta_i^{(i)} &> 0, & \theta_i^{(j)} &< 0, & r_{ii} \theta_i^{(i)} &= -r_{ij} \theta_i^{(j)}. \end{aligned}$$

2. $1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}) > [\delta_{ii} r_{ii} + \delta_{ij}] \theta_i^{(i)}$, where the inequality holds strictly in the limit when $r_{ii} \rightarrow 0$.

3. $1 - l_{ii} - \delta_{ii} \theta_i(r_{ij}^{-1}) > -[\delta_{ii} + \delta_{ij} r_{ij}] r_{ij} \theta_i^{(j)}$.

4. $r_{ii} \phi_{ij}(r_{ii}) = r_{ij} \phi_{ii}(r_{ij})$.

5. $1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}) > 0$, and

$$\lim_{r_{ii} \rightarrow 0+} (1 - l_{ii} - \delta_{ii} \theta_i(r_{ii})) \in (0, +\infty), \quad \lim_{r_{ij} \rightarrow 0+} \left(1 - l_{ii} - \delta_{ii} \theta_i(r_{ij}^{-1})\right) \in (0, +\infty).$$

6. $\frac{\partial \theta_i^{(i)}}{\partial r_{ii}} \leq 0$, and $\lim_{r_{ii} \rightarrow 0+} \theta_i^{(i)} \in (0, +\infty)$.

7. $\frac{\partial \theta_i^{(j)}}{\partial r_{ij}} \geq 0$, and $\lim_{r_{ij} \rightarrow 0+} \theta_i^{(j)} \in (-\infty, 0)$.

8. $\phi_{ij}(r_{ii}) > 0$, and $\lim_{r_{ii} \rightarrow 0+} \phi_{ij}(r_{ii}) \in (0, +\infty)$.

9. $\phi_{ii}(r_{ij}) > 0$, and $\lim_{r_{ij} \rightarrow 0+} \phi_{ii}(r_{ij}) \in (0, +\infty)$.

Proof of Lemma 12

Part 1

The chain rule and some algebra will give

$$\frac{\partial r_{ii}}{\partial e_i} = \frac{\partial}{\partial e_i} \left(\frac{e_i}{e_j^{k_i}} \right) = \frac{1}{e_j^{k_i}} = \frac{r_{ii}}{e_i}, \quad \frac{\partial r_{ij}}{\partial e_i} = \frac{\partial}{\partial e_i} \left(\frac{e_j^{k_i}}{e_i} \right) = -\frac{e_j^{k_i}}{e_i^2} = -\frac{r_{ij}}{e_i},$$

and, using equation (35),

$$\frac{\partial \theta_i}{\partial e_i} = \frac{\partial \theta_i}{\partial r_{ii}} \frac{\partial r_{ii}}{\partial e_i} = \frac{\partial \theta_i}{\partial r_{ii}} \frac{r_{ii}}{e_i}, \quad \frac{\partial \theta_i}{\partial e_i} = \frac{\partial \theta_i}{\partial r_{ij}} \frac{\partial r_{ij}}{\partial e_i} = -\frac{\partial \theta_i}{\partial r_{ij}} \frac{r_{ij}}{e_i}.$$

Then an application of the chain rule using these results and the property $\frac{\partial \theta_i}{\partial e_i} > 0$ from Assumption 19 gives $\theta_i^{(i)} > 0$ and $\theta_i^{(j)} < 0$.

Parts 2-3

Some algebra using Assumption 20 and part 1 gives the result.

Part 4

Some algebra using part 1 gives the result.

Part 5

Some algebra using Assumption 22 and part 1 gives the result.

Part 6

Using part 1 and equation (35), some algebra reveals

$$\theta_i(e_i, e_j) = e_j^{-2k_i} \frac{\partial \theta_i^{(i)}}{\partial r_{ii}} \leq 0$$

where the last inequality comes from Assumption 19; this implies $\frac{\partial \theta_i^{(i)}}{\partial r_{ii}} \leq 0$. This result, the property $\theta_i^{(i)} = e_j^{k_i} \frac{\partial \theta_i}{\partial e_i} > 0$ (from part 1) and the upper-bound aspect of Assumption 19 implies $\lim_{r_{ii} \rightarrow 0+} \theta_i^{(i)} \in (0, +\infty)$.

Part 7

Conceive a function $\hat{e}_j(r_{ij}) = r_{ij}^{1/k_i}$ and use equations (35), (96) to obtain

$$\theta_i^{(j)} = \frac{\partial}{\partial r_{ij}} \theta_i(1, r_{ij}^{1/k_i}) = \left[\frac{\partial}{\partial r_{ij}} \left(\hat{e}_j^{k_i} \right) \right] \left[\frac{\partial}{\partial \hat{e}_j^{k_i}} \theta_i(1, \hat{e}_j) \right] = \frac{\partial}{\partial \hat{e}_j^{k_i}} \theta_i(1, \hat{e}_j)$$

where the second last equality follows from the chain rule. Then

$$\lim_{r_{ij} \rightarrow 0+} \theta_i^{(j)} = \lim_{\hat{e}_j^{k_i} \rightarrow 0+} \left(\frac{\partial}{\partial \hat{e}_j^{k_i}} \theta_i(1, \hat{e}_j) \right)$$

$$\frac{\partial \theta_i^{(j)}}{\partial r_{ij}} = \left(\frac{\partial}{\partial r_{ij}} \left(\hat{e}_j^{k_i} \right) \right) \left(\frac{\partial^2}{\partial \left(\hat{e}_j^{k_i} \right)^2} \theta_i(1, \hat{e}_j) \right) = \frac{\partial^2}{\partial \left(\hat{e}_j^{k_i} \right)^2} \theta_i(1, \hat{e}_j).$$

The results then follow from Assumption 19 and the property $\theta_i^{(j)} < 0$ in part 1.

Part 8

Parts 2 and 6 respectively prove that the denominator and numerator of $\phi_{ij}(r_{ii})$ (defined by equation (37)) are positive; thus $\phi_{ij}(r_{ii}) > 0$. Then some algebra using part 6 and the properties of limits reveals

$$\lim_{r_{ii} \rightarrow 0+} \phi_{ij}(r_{ii}) = \frac{v_i \left(\lim_{r_{ii} \rightarrow 0+} \theta_i^{(i)}(r_{ii}) \right)}{\lim_{r_{ii} \rightarrow 0+} \left(1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}) - \delta_{ij} \theta_i^{(i)}(r_{ii}) \right)}$$

where parts 5 and 2 respectively prove the numerator and denominator of the right-hand side are (strictly) positive and bounded above, giving the result.

Part 9

Parts 3 and 7 respectively prove that the denominator and numerator of $\phi_{ii}(r_{ij})$ (defined by equation (97)) are positive; thus $\phi_{ii}(r_{ij}) > 0$. Then some algebra using part 6 and the properties of limits reveals

$$\lim_{r_{ij} \rightarrow 0+} \phi_{ii}(r_{ij}) = \frac{-v_i \lim_{r_{ij} \rightarrow 0+} \theta_i^{(j)}(r_{ij})}{\lim_{r_{ij} \rightarrow 0+} \left(1 - l_{ii} - \delta_{ii} \theta_i(r_{ij}^{-1}) \right)}$$

where parts 5 and 7 respectively prove the denominator and numerator of the right-hand side are positive and bounded above, giving the result. $\square$

Proof of Lemma 6

Take the partial derivatives of player i 's payoff function in equation (32) with respect to her expenses e_i to obtain

$$\frac{\partial U_i}{\partial e_i} = \frac{\partial \theta_i}{\partial e_i} \left[v_i + \delta_{ii} e_i + \delta_{ij} e_j^{k_i} \right] + \delta_{ii} \theta_i(e_i, e_j) - (1 - l_{ii}) \quad (98)$$

$$\frac{\partial^2 U_i}{\partial e_i^2} = \frac{\partial^2 \theta_i}{\partial e_i^2} \left[v_i + \delta_{ii} e_i + \delta_{ij} e_j^{k_i} \right] + 2 \delta_{ii} \frac{\partial \theta_i}{\partial e_i}. \quad (99)$$

Supposing player i 's FOC holds and using equation (98), (99), a substitution exercise

reveals

$$\begin{aligned}\frac{\partial^2 U_i}{\partial e_i^2} &= \frac{\partial^2 \theta_i}{\partial e_i^2} \left[\frac{(1 - l_{ii} - \delta_{ii} \theta_i(e_i, e_j))}{\partial \theta_i / \partial e_i} \right] + 2\delta_{ii} \frac{\partial \theta_i}{\partial e_i} \\ &= (1 - l_{ii} - \delta_{ii} \theta_i(e_i, e_j)) \left[\frac{(1 - l_{ii} - \delta_{ii} \theta_i(e_i, e_j)) \frac{\partial^2 \theta_i}{\partial e_i^2} + 2\delta_{ii} \left(\frac{\partial \theta_i}{\partial e_i} \right)^2}{(1 - l_{ii} - \delta_{ii} \theta_i(e_i, e_j)) \frac{\partial \theta_i}{\partial e_i}} \right] < 0\end{aligned}\quad (100)$$

where the last inequality follows from some algebra using Assumption 21 and part 5 of Lemma 12.

Corollary 9.3 of Diewert et al. (1981) holds that a twice continuously differentiable function f defined on an open S is strictly quasiconcave if and only if (i) $x^0 \in S$, $v^T v = 1$ and $v^T \nabla f(x^0) v = 0$ implies $v^T \nabla^2 f(x^0) v < 0$; or (ii) $v^T \nabla^2 f(x^0) v = 0$ and $g(t) \equiv f(x^0 + tv)$ does not attain a local minimum at $t = 0$. Fixing player j 's expenses e_j , inequality (100) implies player i 's payoff function U_i is strictly quasiconcave in her expenses e_i . $\square$

Proof of Lemma 7

Fix an arbitrary $e'_j > 0$, and let $U_i(\cdot, e'_j)$ denote the player i 's payoff function U_i restricted to one variable, e_i . Suppose there exists some $e'_i > 0$ that satisfies the FOC for $U_i(\cdot, e'_j)$. Then the proof of Lemma 6 (see inequality (100)) proves that e'_i also satisfies the SOC for $U_i(\cdot, e'_j)$. Hence e'_i is a local maximum of $U_i(\cdot, e'_j)$. Then Lemma 6 implies e'_i is a global maximum of $U_i(\cdot, e'_j)$. $\square$

Proof of Lemma 8

This proof will consider three different cases: (i) $k_1 k_2 = 1$; (ii) $k_1 k_2 < 1$; (iii) $1 < k_1 k_2 \leq 2$ (Assumption 23 imposes the upper bound $2 \geq k_1 k_2$).

Case (i): $k_1 k_2 = 1$

Use equations (37) and (97) to define functions $F, \bar{F} : \mathbb{R}_{++} \rightarrow \mathbb{R}$ by

$$\begin{aligned}F(r_2) &= r_2 \phi_{11}(r_2) - \left[\phi_{21} \left(r_2^{1/k_1} \right) \right]^{1/k_2}, \\ \bar{F}(r_1) &= \phi_{12}(r_1) - r_1 \left[\phi_{22} \left(r_1^{1/k_1} \right) \right]^{1/k_2},\end{aligned}\quad (101)$$

where $r_2, r_1 > 0$ are positive real numbers. Some algebra using parts 1,4 of Lemma 12 and the specification $r_1 = 1/r_2$ reveals

$$r_2 F(r_2) = \bar{F}(r_1) \quad (102)$$

Some algebra using parts 8-9 in Lemma 12 and the properties of limits obtains

$$\lim_{r_2 \rightarrow 0+} F(r_2) = - \lim_{r_2 \rightarrow 0+} \left[\phi_{21} \left(r_2^{1/k_1} \right) \right]^{1/k_2} < 0,$$

$$\lim_{r_1 \rightarrow 0+} \bar{F}(r_1) = \lim_{r_1 \rightarrow 0+} \phi_{12}(r_1) > 0.$$

Hence there exist sufficiently small positive real numbers $\epsilon_1, \epsilon_2 > 0$ satisfying

$$0 > F(\epsilon_2), \quad (103)$$

$$0 < \bar{F}(\epsilon_1) = \epsilon_1^{-1} F(\epsilon_1^{-1}), \quad (104)$$

where the last equality uses equation (102).

Conditions (103), (104) and the intermediate value theorem imply there exists some positive real number $r_2^* \in (\epsilon_2, \epsilon_1^{-1})$ such that $F(r_2^*) = 0$. Choosing $r_{11}^* = 1/r_2^*$, $r_{22}^* = r_2^{*k_1}$ gives the result in part 1 of Lemma 8.

Case (ii): $k_1 k_2 < 1$

Define functions $R_{22}, R_{21}, G, \bar{G} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ by

$$R_{22}(r_{12}) = r_{12}^{(2-k_1 k_2)/k_1} [\phi_{11}(r_{12})]^{(1-k_1 k_2)/k_1}, \quad (105)$$

$$R_{21}(r_{11}) = r_{11}^{k_2} [\phi_{12}(r_{11})]^{(k_1 k_2 - 1)/k_1}, \quad (106)$$

$$G(r_{12}) = r_{12}^{(k_1 k_2 - 1)k_2} [\phi_{11}(r_{12})]^{(k_1 k_2 - 1)k_2} - [\phi_{21}(R_{22}(r_{12}))]^{k_1 k_2 - 1}, \quad (107)$$

$$\bar{G}(r_{11}) = r_{11}^{(1-k_1 k_2)k_2} [\phi_{12}(r_{11})]^{(k_1 k_2 - 1)(2-k_1 k_2)/k_1} - [\phi_{22}(R_{21}(r_{11}))]^{k_1 k_2 - 1}, \quad (108)$$

where $r_{12}, r_{11} > 0$ are positive real numbers and part 9 of Lemma 12 imply

$$R_{21}(r_{11}) > 0. \quad (109)$$

Some algebra using parts 1,4 of Lemma 12 and the specification $r_{12} = 1/r_{11}$ reveals

$$R_{21}(r_{11}) = [R_{22}(r_{12})]^{-1}, \quad (110)$$

$$\frac{r_{12}^{k_2}}{[R_{22}(r_{12})]^{k_1 k_2}} = r_{12}^{(k_1 k_2 - 1)k_2} [\phi_{11}(r_{12})]^{(k_1 k_2 - 1)k_2}, \quad (111)$$

$$\frac{[R_{21}(r_{11})]^{2-k_1 k_2}}{r_{11}^{k_2}} = r_{11}^{(1-k_1 k_2)k_2} [\phi_{12}(r_{11})]^{(k_1 k_2 - 1)(2-k_1 k_2)/k_1}, \quad (112)$$

$$G(r_{12}) = [R_{21}(r_{11})]^{2(k_1 k_2 - 1)} \bar{G}(r_{11}), \quad (113)$$

where the last equality follows from some algebra commencing with substituting equations (112), (113) into equations (107), (108) respectively.

Some algebra using parts 8-9 of Lemma 12, the specification $k_1 k_2 \leq 2$ and the properties of limits obtains

$$\lim_{r_{12} \rightarrow 0+} R_{22}(r_{12}) \in [0, +\infty), \quad (114)$$

$$\lim_{r_{11} \rightarrow 0+} R_{21}(r_{11}) = 0. \quad (115)$$

Using these limit properties, those in parts 8-9 of Lemma 12, and the specification $k_1 k_2 < 1$, some algebra obtains

$$\lim_{r_{12} \rightarrow 0+} G(r_{12}) = +\infty, \quad \lim_{r_{11} \rightarrow 0+} \bar{G}(r_{11}) < 0,$$

which respectively imply the existence of some small positive real numbers $\alpha_1, \alpha_2 > 0$ such that

$$0 < G(\alpha_2), \quad (116)$$

$$0 > \bar{G}(\alpha_1) = [R_{21}(\alpha_1)]^{2(k_1 k_2 - 1)} G(\alpha_1^{-1}), \quad (117)$$

where the last equality follows from equation (113).

Conditions (109), (116), (117) and the intermediate value theorem imply there exists some positive real number $r_{12}^* \in (\alpha_1^{-1}, \alpha_2)$ such that $G(r_{12}^*) = 0$. Choosing $r_{11}^* = 1/r_{12}^*$ and $r_{22}^* = R_{22}(r_{12}^*)$ as defined by (105) and using equations (107) and (111), some algebra gives the result in part 2 of Lemma 8 when $k_1 k_2 < 1$.

Case (iii): $1 < k_1 k_2 \leq 2$

Consider functions $R_{22}(r_{12})$, $R_{21}(r_{11})$, $G(r_{12})$ and $\bar{G}(r_{11})$ respectively defined by equations (105)-(108). These functions continue to satisfy conditions (109)-(117) under the specification $1 < k_1 k_2 \leq 2$. Using conditions (116)-(117), parts 8-9 of Lemma 12, and the specification $1 < k_1 k_2 \leq 2$, some algebra obtains

$$\lim_{r_{12} \rightarrow 0+} G(r_{12}) < 0, \quad \lim_{r_{11} \rightarrow 0+} \bar{G}(r_{11}) = +\infty,$$

which respectively imply the existence of some small positive real numbers $\beta_1, \beta_2 > 0$

such that

$$0 > G(\beta_2), \quad (118)$$

$$0 < \bar{G}(\beta_1) = [R_{21}(\beta_1)]^{2(k_1 k_2 - 1)} G(\beta_1^{-1}), \quad (119)$$

where the last equality follows from equation (113).

Conditions (109), (118), (119) and the intermediate value theorem imply there exists some positive real number $r_{12}^* \in (\beta_2, \beta_1^{-1})$ such that $G(r_{12}^*) = 0$. Choosing $r_{11}^* = 1/r_{12}^*$ and $r_{22}^* = R_{22}(r_{12}^*)$, some algebra gives the result in part 2 of Lemma 8 when $1 < k_1 k_2 \leq 2$. $\square$

Proof of Proposition 8

This proof will establish that the characterizations of e_1^*, e_2^* in this Proposition simultaneously satisfy the players' FOCs (system (34)) in two different cases: (i) $k_1 k_2 = 1$; (ii) $k_1 k_2 \neq 1$.

Case (i): $k_1 k_2 = 1$

Assuming $k_1 k_2 = 1$, some algebra reveals that, for each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, part 1 of Lemma 8 is equivalent to

$$r_{ii}^* r_{jj}^{*k_i} = 1 \quad (120)$$

$$r_{ii}^* \phi_{ij}(r_{ii}^*) = \left[\phi_{ji}(r_{jj}^*) \right]^{k_i}. \quad (121)$$

To establish FOC satisfaction for player i , use the expression for e_j^* and equations (120)-(121) to obtain

$$\begin{aligned} e_j^{*k_i} &= r_{jj}^{*k_i} \left[\phi_{ji}(r_{jj}^*) \right]^{*k_i} = r_{jj}^{*k_i} \left[r_{ii}^* \phi_{ij}(r_{ii}^*) \right]^{*k_i k_j} = r_{jj}^{*k_i} r_{ii}^{*k_i k_j} \left[\phi_{ij}(r_{ii}^*) \right]^{*k_i k_j} = \phi_{ij}(r_{ii}^*) \\ e_j^{*k_i} &= e_i^* / r_{ii}^* \end{aligned} \quad (122)$$

where the second last equality uses part 1 of Lemma 8 and the assumption $k_1 k_2 = 1$, and the last equality uses the expression for e_i^* .

Then some algebra using equation (37) and the expression for e_i^* obtains

$$\frac{v_i \theta_i^{(i)}(r_{ii}^*) r_{ii}^*}{1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}^*) - (\delta_{ii} r_{ii}^* + \delta_{ij}) \theta_i^{(i)}(r_{ii}^*)} = e_i^* \quad (123)$$

$$v_i \theta_i^{(i)}(r_{ii}^*) r_{ii}^* + (\delta_{ii} r_{ii}^* + \delta_{ij}) \theta_i^{(i)}(r_{ii}^*) e_i^* = [1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}^*)] e_i^* \quad (124)$$

$$\theta_i^{(i)}(r_{ii}^*) r_{ii}^* \left[v_i + (\delta_{ii} r_{ii}^* + \delta_{ij}) \frac{e_i^*}{r_{ii}^*} \right] = [1 - l_{ii} - \delta_{ii} \theta_i(r_{ii}^*)] e_i^* \quad (125)$$

$$\frac{\theta_i^{(i)}(r_{ii}^*)r_{ii}^*}{e_i^*} \left[v_i + \delta_{ii}e_i^* + \frac{\delta_{ij}e_i^*}{r_{ii}^*} \right] = 1 - l_{ii} - \delta_{ii}\theta_i(r_{ii}^*) \quad (126)$$

$$\frac{\partial \theta_i}{\partial e_i} \Big|_{(e_i, e_j) = (e_i^*, e_j^*)} \left[v_i + \delta_{ii}e_i^* + \delta_{ij}e_j^{*k_i} \right] = 1 - l_{ii} - \delta_{ii}\theta_i(e_i^*, e_j^*) \quad (127)$$

where the last equality uses equations (35), (122) and part 1 of Lemma 12, establishing player i 's FOC in system (34).

Case (ii): $k_1k_2 \neq 1$

Assuming $k_1k_2 \neq 1$, some algebra reveals that, for each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, part 2 of Lemma 8 is equivalent to

$$r_{ii}^{*k_i k_j} r_{jj}^{*k_i} = [\phi_{ij}(r_{ii}^*)]^{1-k_i k_j} \quad (128)$$

$$r_{jj}^{*k_i k_j} r_{ii}^{*k_j} = [\phi_{ji}(r_{jj}^*)]^{1-k_i k_j}. \quad (129)$$

Then use the expression for e_j^* and equation (129) to obtain

$$\begin{aligned} e_j^{*k_i} &= r_{jj}^{*k_i} [\phi_{ji}(r_{jj}^*)]^{*k_i} = r_{jj}^{*k_i} [r_{jj}^{*k_i k_j} r_{ii}^{*k_j}]^{k_i/(1-k_i k_j)} = (r_{jj}^*)^{k_i + [k_i^2 k_j / (1-k_i k_j)]} (r_{ii}^*)^{k_i k_j / (1-k_i k_j)} \\ &= (r_{jj}^*)^{k_i / (1-k_i k_j)} (r_{ii}^*)^{k_i k_j / (1-k_i k_j)} = \phi_{ij}(r_{ii}^*) = e_i^* / r_{ii}^* \end{aligned}$$

where the second last equality uses equation (128), and the last equality uses the expression for e_i^* . Then the steps establishing equations (123)-(127) prove FOC satisfaction for player $i \in \{1, 2\} \setminus \{j\}$. $\square$

Proof of Corollary 16

Suppose Lemma 8 defines a unique pair (r_{11}^*, r_{22}^*) . Suppose, for a contradiction, there exists a nontrivial Nash equilibrium of the Contest Game, denoted $(\bar{e}_1, \bar{e}_2)$, such that $(\bar{e}_1, \bar{e}_2) \neq (e_1^*, e_2^*)$ given by Proposition 8.

Define auxiliary constants $\bar{r}_{11} = \bar{e}_1 / \bar{e}_2^{k_1}$ and $\bar{r}_{22} = \bar{e}_2 / \bar{e}_1^{k_2}$. Using the property that $(\bar{e}_1, \bar{e}_2)$ satisfies the FOC in system (34) and reversing the steps establishing equations (123)-(127), some algebra obtains

$$\bar{e}_1 = \bar{r}_{11} \phi_{12}(\bar{r}_{11}) \quad (130)$$

$$\bar{e}_2^{k_1} = \phi_{12}(\bar{r}_{11}) \quad (131)$$

$$\bar{e}_2 = \bar{r}_{22} \phi_{21}(\bar{r}_{22}) \quad (132)$$

$$\bar{e}_1^{k_2} = \phi_{21}(\bar{r}_{22}). \quad (133)$$

Then some algebra comparing equation (130) with equation (133), and equation (131) with equation (132), obtains

$$\phi_{21}(\bar{r}_{22}) = \bar{r}_{11}^{k_2} [\phi_{12}(\bar{r}_{11})]^{k_2} \quad (134)$$

$$\phi_{12}(\bar{r}_{11}) = \bar{r}_{22}^{k_1} [\phi_{21}(\bar{r}_{22})]^{k_1} \quad (135)$$

where a substitution exercise using these equations reveals

$$\bar{r}_{22}^{k_1 k_2} \bar{r}_{11}^{k_2} = [\phi_{21}(\bar{r}_{22})]^{1-k_1 k_2} \quad (136)$$

$$\bar{r}_{11}^{k_1 k_2} \bar{r}_{22}^{k_1} = [\phi_{12}(\bar{r}_{11})]^{1-k_1 k_2}. \quad (137)$$

Consider two different cases: (i) $k_1 k_2 \neq 1$; (ii) $k_1 k_2 = 1$.

Case (i): $k_1 k_2 \neq 1$

Equations (136) and (137), the assumption $k_1 k_2 \neq 1$ and the assumption that Lemma 8 defines a unique pair (r_{11}^*, r_{22}^*) imply $(\bar{r}_{11}, \bar{r}_{22}) = (r_{11}^*, r_{22}^*)$. Then an application of Proposition 8 obtains $(\bar{e}_1, \bar{e}_2) = (e_1^*, e_2^*)$, a contradiction.

Case (ii): $k_1 k_2 = 1$

The assumption $k_1 k_2 = 1$ and equation (136) imply

$$\bar{r}_{11} \bar{r}_{22}^{k_1} = 1. \quad (138)$$

A substitution exercise using equations (135) and (138) and the assumption $k_1 k_2 = 1$ obtains

$$[\bar{r}_{11} \phi_{12}(\bar{r}_{11})]^{k_2} = \phi_{21}(\bar{r}_{22}). \quad (139)$$

Equations (138) and (139), the assumption $k_1 k_2 = 1$ and the assumption that Lemma 8 defines a unique pair (r_{11}^*, r_{22}^*) imply $(\bar{r}_{11}, \bar{r}_{22}) = (r_{11}^*, r_{22}^*)$. Then an application of Proposition 8 obtains $(\bar{e}_1, \bar{e}_2) = (e_1^*, e_2^*)$, a contradiction. $\square$

Proof of Corollary 17

Following similar steps in the proof of Corollary 21, some algebra reveals $F(r_2) = 0$ and $r_2 > 0$ if and only if

$$r_2 = \frac{\theta_1}{\theta_2} = \frac{\eta\mu_1 + \frac{1-\eta}{1+r_2}}{\eta\mu_2 + \frac{(1-\eta)r_2}{1+r_2}} \quad (140)$$

Some algebra using equation (40) reveals equation (140) is equivalent to

$$[1 - \eta(1 - \mu_2)]r_2^2 - \eta[(1 - \mu_2) - (1 - \mu_1)] - [1 - \eta(1 - \mu_1)] = 0.$$

Using the quadratic formula, some algebra obtains

$$r_2 = \frac{\eta[(1 - \mu_2) - (1 - \mu_1)] \pm \sqrt{\Delta}}{2[1 - \eta(1 - \mu_2)]} \quad (141)$$

where

$$\begin{aligned} \Delta &= \eta^2[(1 - \mu_2) - (1 - \mu_1)]^2 + 4[1 - \eta(1 - \mu_2)][1 - \eta(1 - \mu_1)] \\ &= \eta^2[(1 - \mu_2) - (1 - \mu_1)]^2 + 4[1 - \eta((1 - \mu_2) + (1 - \mu_1))] - 4\eta^2(1 - \mu_2)(1 - \mu_1) \\ &= \eta^2[(1 - \mu_2)^2 - 2(1 - \mu_2)(1 - \mu_1) + (1 - \mu_1)^2 - 4(1 - \mu_2)(1 - \mu_1)] \\ &\quad + 4[1 - \eta((1 - \mu_2) + (1 - \mu_1))] \\ &= \eta^2[(1 - \mu_2)^2 + 2(1 - \mu_2)(1 - \mu_1) + (1 - \mu_1)^2] + 4[1 - \eta((1 - \mu_2) + (1 - \mu_1))] \\ &= \eta^2[(1 - \mu_2) + (1 - \mu_1)]^2 - 4\eta[(1 - \mu_2) + (1 - \mu_1)] + 4 \\ &= [2 - \eta((1 - \mu_2) + (1 - \mu_1))]^2 \end{aligned}$$

where $2 - \eta((1 - \mu_2) + (1 - \mu_1)) > 0$ because $0 \leq \eta, \mu_1, \mu_2, \leq 1$. Substituting back to equation (141) obtains the only positive solution

$$r_2 = \frac{1 - \eta + \eta\mu_1}{1 - \eta + \eta\mu_2}.$$

Then a substitution exercise using Proposition 8 gives the result. $\square$

Proof of Corollary 18

Some algebra using Corollary 17 reveals part 1 and

$$\theta_i^* - \theta_j^* = \frac{(\mu_i - \mu_j)\eta[\eta(\mu_1 + \mu_2) + 1 - \eta]}{2(1 - \eta) + \eta(\mu_1 + \mu_2)}$$

where the right-hand side is negative if and only if $\mu_i < \mu_j$, giving part 2. $\square$

Proof of Corollary 19

Some algebra using Corollary 17 obtains

$$e_1^* + e_2^* = \frac{v(1 - \eta)}{\lambda[2(1 - \eta) + \eta(\mu_1 + \mu_2)]} = \frac{v(1 - \eta)}{\lambda[2 - \eta + \eta\sigma]}$$

then a differentiation exercise gives the result.

Proof of Corollary 20

Some algebra using the product rule and the property $\theta_1^* + \theta_2^* = 1 + \eta\sigma$ from equation (41) reveals

$$\begin{aligned}\frac{dU^*}{d\sigma} &= [v - \lambda(e_1^* + e_2^*)] \frac{d}{d\sigma}(1 + \eta\sigma) - [1 + \eta\sigma] \lambda \frac{d}{d\sigma}(e_1^* + e_2^*) \\ &= [v - \lambda(e_1^* + e_2^*)] \eta - (1 + \eta\sigma) \lambda \frac{d}{d\sigma}(e_1^* + e_2^*)\end{aligned}$$

where Corollary 19 proves $\frac{d}{d\sigma}(e_1^* + e_2^*) < 0$ and some algebra using the properties $0 \leq \mu_1, \mu_2 \leq 1$ and Corollary 17 obtains

$$1 + \eta\sigma > 0, \quad v - \lambda(e_1^* + e_2^*) = \frac{v(1 + \eta\sigma)}{2 - \eta + \eta\sigma} > 0,$$

giving the result. $\square$

Proof of Corollary 21

Denote an auxiliary variable $r_2 = r_{12}$. The assumption $k_1 = k_2$ implies $r_{22} = r_{12} = r_2$. Some algebra reveals that the FOCs in system (34) hold simultaneously if and only if the function $F(r_2) = 0$, where equation (101) defines F . Some algebra further reveals $F(r_2) = 0$ and $r_2 > 0$ if and only if $r_2 = 1$. Then a substitution exercise using Proposition 8 gives the result. $\square$

Proof of Corollary 22

Using Corollary 21, some algebra reveals

$$\begin{aligned}\frac{d}{d\delta}(e_1^* + e_2^*) &= \frac{2v\mu^2(1 - \mu)^2}{[1 - \delta\mu(1 - \mu)]^2} \\ \frac{d}{d(\mu(1 - \mu))} \left(\frac{d}{d\delta}(e_1^* + e_2^*) \right) &= \frac{4v\mu(1 - \mu)}{[1 - \delta\mu(1 - \mu)]^3}.\end{aligned}\tag{142}$$

The result follows from inequality (142) and the assumption $\delta < \min\{\mu^{-1}(1 - \mu), (1 - \mu)^{-1}\mu\}$ (which implies $1 - \delta\mu(1 - \mu) > 0$). $\square$

Proof of Corollary 23

Some algebra using Corollary 21 obtains

$$U_1^* + U_2^* = \frac{v[1 - 2\mu(1 - \mu)(1 - l)]}{1 - (w - l)\mu(1 - \mu)}.$$

Then a calculus exercise reveals

$$\begin{aligned}\frac{d}{dw}(U_1^* + U_2^*) &= \frac{v\mu(1-\mu)}{[1-(w-l)\mu(1-\mu)]^2} \\ \frac{d}{dl}(U_1^* + U_2^*) &= \frac{v\mu(1-\mu)[1+2\mu(1-\mu)(1-w)]}{[1-(w-l)\mu(1-\mu)]^2} \\ \frac{d}{d\tau}(U_1^* + U_2^*) &= \frac{v(w+l-2)}{[1-(w-l)\mu(1-\mu)]^2}\end{aligned}$$

giving the results. $\square$

Proof of Corollary 24

Using Corollary 21, a calculus exercise reveals

$$\frac{d}{d\tau}(e_1^* + e_2^*) = \frac{2v}{[1-\delta\tau]^2} > 0$$

where the last inequality follows from the assumption $\delta < \min\{\mu^{-1}(1-\mu), (1-\mu)^{-1}\mu\}$ and the property $\tau = \mu(1-\mu) \leq 0.25$. $\square$

Proof of Corollary 25

Steps similar to those in the proof of Corollary 21 give the result. $\square$

Proof of Corollary 26

Consider Corollary 25, which finds the close-form expression for the nontrivial Nash equilibrium of the Alternative R&D Game. Part 1 of Corollary 26 follows from the absence of τ — the variable that measures the relative advantages of the players — in the close-form expression. Part 2 of Corollary 26 follows from a calculus exercise using the close-form expression. $\square$

Proof of Proposition 9

This proof applies the principle of finite induction. The result clearly holds for the base case $\theta_i = \theta_{i(1)}$. The rest of this proof assumes the result holds for all integer $z \leq n-1$, and proves that it holds for $z = n$. Define auxiliary constants $0 \leq a, \bar{a} \leq 1$ and a success function $\tilde{\theta} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by

$$a = p_{i(1)} + p_{i(2)} + \dots + p_{i(n-1)}, \quad \bar{a} = 1 - a, \quad \tilde{\theta} = \sum_{z=1}^{n-1} \left(\frac{p_{i(z)}}{a} \theta_{i(z)} \right).$$

Assuming $\tilde{\theta}$, $\theta_{i(n)}$ satisfy Assumption 18 (respectively, 19, 22, 23), some algebra immediately reveals that their linear combination $\theta_i = a\tilde{\theta} + \bar{a}\theta_{i(n)}$ also satisfies Assumption 18 (19, 22, 23). The following will prove that θ_i satisfies the remaining Assumptions 20

and 21.

Assumption 20

Some algebra using the linearity of derivation obtains

$$\begin{aligned}
\frac{\partial \theta_i}{\partial e_i} [\delta_{ii} e_i + \delta_{ij} e_j^{k_i}] + \delta_{ii} \theta_i &= \frac{\partial}{\partial e_i} (a\tilde{\theta} + \bar{a}\theta_{i(n)}) [\delta_{ii} e_i + \delta_{ij} e_j^{k_i}] + \delta_{ii} (a\tilde{\theta} + \bar{a}\theta_{i(n)}) \\
&= a \left(\frac{\partial \tilde{\theta}}{\partial e_i} [\delta_{ii} e_i + \delta_{ij} e_j^{k_i} + \delta_{ii} \tilde{\theta}] \right) + \bar{a} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} [\delta_{ii} e_i + \delta_{ij} e_j^{k_i}] + \delta_{ii} \theta_{i(n)} \right) \\
&< a(1 - l_{ii}) + \bar{a}(1 - l_{ii}) = 1 - l_{ii}
\end{aligned}$$

where the inequality uses the assumption that $\tilde{\theta}, \theta_{i(n)}$ satisfy Assumptions 20. Hence θ_i satisfies Assumption 20.

Assumption 21

Assuming $\tilde{\theta}, \theta_n$ satisfy Assumption (21), some algebra reveals

$$a \frac{\partial^2 \tilde{\theta}}{\partial e_i^2} + \frac{2a\delta_{ii} \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2}{(1 - l_{ii} - \delta_{ii} \tilde{\theta})} < a \frac{\partial \tilde{\theta}}{\partial e_i}, \quad \bar{a} \frac{\partial^2 \theta_{i(n)}}{\partial e_i^2} + \frac{2\bar{a}\delta_{ii} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2}{(1 - l_{ii} - \delta_{ii} \theta_{i(n)})} < \bar{a} \frac{\partial \theta_{i(n)}}{\partial e_i}.$$

Some algebra summing these inequalities and using the definition of θ_i and the linearity of differentiation obtains

$$\frac{\partial^2 \theta_i}{\partial e_i^2} + 2\delta_{ii} \left[\frac{a \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2}{(1 - l_{ii} - \delta_{ii} \tilde{\theta})} + \frac{\bar{a} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2}{(1 - l_{ii} - \delta_{ii} \theta_{i(n)})} \right] < \frac{\partial \theta_i}{\partial e_i}. \quad (143)$$

Some algebra using the non-negative property of squares obtains

$$\begin{aligned}
&a\bar{a} \left[\frac{\partial \tilde{\theta}}{\partial e_i} (1 - l_{ii} - \delta_{ii} \theta_{i(n)}) - \frac{\partial \theta_{i(n)}}{\partial e_i} (1 - l_{ii} - \delta_{ii} \tilde{\theta}) \right]^2 \geq 0 \\
&2a\bar{a} \frac{\partial \tilde{\theta}}{\partial e_i} \frac{\partial \theta_{i(n)}}{\partial e_i} (1 - l_{ii} - \delta_{ii} \tilde{\theta}) (1 - l_{ii} - \delta_{ii} \theta_{i(n)}) \leq a\bar{a} \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2 (1 - l_{ii} - \delta_{ii} \theta_{i(n)})^2 \\
&\quad + a\bar{a} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2 (1 - l_{ii} - \delta_{ii} \tilde{\theta})^2 \\
&2a\bar{a} \frac{\partial \tilde{\theta}}{\partial e_i} \frac{\partial \theta_{i(n)}}{\partial e_i} + a^2 \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2 + \bar{a}^2 \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2 \leq \frac{a\bar{a} \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2 (1 - l_{ii} - \delta_{ii} \theta_{i(n)})^2 + a\bar{a} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2 (1 - l_{ii} - \delta_{ii} \tilde{\theta})^2}{(1 - l_{ii} - \delta_{ii} \tilde{\theta}) (1 - l_{ii} - \delta_{ii} \theta_{i(n)})} \\
&\quad + a^2 \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2 + \bar{a}^2 \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2
\end{aligned}$$

$$\begin{aligned}
& \frac{a^2 \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2 + 2a\bar{a} \frac{\partial \tilde{\theta}}{\partial e_i} \frac{\partial \theta_{i(n)}}{\partial e_i} + \bar{a}^2 \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2}{a(1-l_{ii}-\delta_{ii}\tilde{\theta}) + \bar{a}(1-l_{ii}-\delta_{ii}\theta_{i(n)})} \leq \frac{a(1-l_{ii}-\delta_{ii}\theta_{i(n)}) \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2 + \bar{a}(1-l_{ii}-\delta_{ii}\tilde{\theta}) \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2}{(1-l_{ii}-\delta_{ii}\tilde{\theta})(1-l_{ii}-\delta_{ii}\theta_{i(n)})} \\
& \frac{\left[a \frac{\partial \tilde{\theta}}{\partial e_i} + \bar{a} \frac{\partial \theta_{i(n)}}{\partial e_i} \right]^2}{1-l_{ii}-\delta_{ii}(a\tilde{\theta} + \bar{a}\theta_{i(n)})} \leq \frac{a \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2}{(1-l_{ii}-\delta_{ii}\tilde{\theta})} + \frac{\bar{a} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2}{(1-l_{ii}-\delta_{ii}\theta_{i(n)})} \\
& \frac{\left(\frac{\partial \theta_i}{\partial e_i} \right)^2}{1-l_{ii}-\delta_{ii}\theta_i} \leq \frac{a \left(\frac{\partial \tilde{\theta}}{\partial e_i} \right)^2}{(1-l_{ii}-\delta_{ii}\tilde{\theta})} + \frac{\bar{a} \left(\frac{\partial \theta_{i(n)}}{\partial e_i} \right)^2}{(1-l_{ii}-\delta_{ii}\theta_{i(n)})} \tag{144}
\end{aligned}$$

where the last step uses the definition of θ_i and the linearity of differentiation. Then some algebra using inequalities (143), (144) obtains

$$\frac{(1-l_{ii}-\delta_{ii}\theta_i) \frac{\partial^2 \theta_i}{\partial e_i^2} + 2\delta_{ii} \left(\frac{\partial \theta_i}{\partial e_i} \right)^2}{(1-l_{ii}-\delta_{ii}\theta_i) \frac{\partial \theta_i}{\partial e_i}} < 0,$$

where some further algebra will reveal that it implies θ_i satisfies Assumption 21. $\square$

Proof of Proposition 10

The FOC of player i in the Homogeneous Contest Game is

$$\begin{aligned}
\frac{\partial \widehat{U}_i}{\partial y_i} &= \frac{\partial \theta_i}{\partial y_i} \left[v_i + \Delta_{ii}(1)y_i^{k_{ii}} + \Delta_{ij}(1)y_j^{k_{ij}} \right] - [E_i(1) - L_{ii}(1) - \Delta_{ii}(1)\theta_i] k_{ii} y_i^{k_{ii}-1} = 0, \\
0 &= E'_i(y_i) \left(\frac{\partial \theta_i}{\partial E_i} \left[v_i + \frac{\Delta_{ii}(1)}{E_i(1)} E_i(y_i) + \frac{\Delta_{ij}(1)}{E_j(1)^{\frac{k_{ij}}{k_{jj}}}} [E_j(y_j)]^{\frac{k_{ij}}{k_{jj}}} \right] - \left[1 - \frac{L_{ii}(1)}{E_i(1)} - \frac{\Delta_{ii}(1)}{E_i(1)} \theta_i \right] \right), \tag{145}
\end{aligned}$$

where the last step applies the chain rule and uses the properties of homogeneous functions.

Now, let the spillover parameters in the original Contest Game be those defined by (55). Then a comparison of (145) and system (34) reveals that a pair of positive efforts (y_1^*, y_2^*) satisfies (145) if and only if it relates to a nontrivial Nash equilibrium (e_1^*, e_2^*) of the Contest Game (see Proposition 8) via the transformation in (56).

To see that the pair (y_1^*, y_2^*) satisfies player i 's SOC, some algebra using the chain rule and equation (145) obtains

$$\frac{\partial^2 \widehat{U}_i}{\partial y_i^2} = E''_i(y_i) \left(\frac{\partial \widehat{U}_i}{\partial E_i} \right) + (E'_i(y_i))^2 \left(\frac{\partial^2 \widehat{U}_i}{\partial E_i^2} \right)$$

which is negative when evaluated at (y_1^*, y_2^*) , because $E_i''(y_i^*) \geq 0$, $E_i'(y_i^*) > 0$ and

$$\left. \frac{\partial \widehat{U}_i}{\partial E_i} \right|_{y_i=y_i^*} = \left. \frac{\partial U_i}{\partial e_i} \right|_{e_i=e_i^*} = 0, \quad \left. \frac{\partial^2 \widehat{U}_i}{\partial E_i^2} \right|_{y_i=y_i^*} = \left. \frac{\partial U_i^2}{\partial e_i^2} \right|_{e_i=e_i^*} < 0$$

where U_i is player i 's payoff in the original Contest Game, and the last inequality established in the proof of Lemma 6.

Hence, in the Homogeneous Contest Game, player i by exerting positive effort y_i^* indeed maximizes her payoff $\widehat{U}_i$ when her opponent j exerts positive effort y_j^* . The pair (y_1^*, y_2^*) thus characterizes a nontrivial Nash equilibrium of the Homogeneous Contest Game. $\square$

A.5 Modifications of the Model in Chapter 5

These appendices modify the Model in section 5.2 to demonstrate the scope of the Externalities-Optimization Principle. To simplify presentation, assume the actor bears all of her private cost ($\gamma = 0$).

A.5.1 Positive Externalities

This appendix modifies the Model to capture the actor's direct production of positive externalities. Suppose that, instead of generating social harm $H(x)$, her action x generates social benefit $B(x)$, where B is a real-valued function satisfying $B' > 0$ and $B'' \leq 0$. Suppose further that, instead of a harm-allocation rule (λ) , the law allocates $\beta \in [0, 1]$ proportion of the social benefit to the actor. The actor's utility function becomes

$$\widehat{A}(x) = (1 - \delta)G(x) + \beta B(x) - C(x),$$

where the law continues to disgorge δ proportion of her private gain $G(x)$.

The net externalities and social welfare functions respectively become

$$\widehat{V}(x) = \delta G(x) + (1 - \beta)B(x)$$

$$\widehat{S}(x) = G(x) + B(x) - C(x).$$

To guarantee the social welfare function $\widehat{S}$ has a unique optimizer, assume that as the action x becomes very large, the marginal cost of acting eventually exceeds the marginal

gain and the marginal social benefit.¹⁵⁴

The Externalities-Optimization Principle continues to hold in the present modified model. To see this, observe that the proofs of Proposition 11 and 12 (see section 5.3) do *not* depend on the assumption $H' > 0$ on the harm function H in the (original) Model (see section 5.2). Thus specifying $B = -H$ and $\beta = \lambda$ immediately brings the present modified model within the scope of Propositions 11, 12.

However, Corollary 28 depends on the assumption $H' > 0$, and does not hold in the present model. Under the assumption $B' > 0$ that the present modified model adopts, if the socially optimal action is positive ($x^* > 0$), then the unique pair of optimal allocation rules is $(\delta, \beta) = (0, 1)$. This pair completely internalizes externalities, and is a special case of the Externalities-Optimization Principle.

A.5.2 Endogenous Choice between Restitution and Compensation

This appendix modifies the Model in section 5.2 to capture the victim's optimal choice between compensation and restitution. Modify the Model so that as a result of the victim's choice after the Actor has chosen her action, the actor's liability becomes

$$\tilde{L}(x) = \max\{\delta G(x), \lambda H(x)\},$$

that is, the victim chooses the maximum of restitution ($\delta G(x)$) and compensation ($\lambda H(x)$).

The actor's utility function and the net externalities function respectively become

$$\tilde{A} = G - C - \tilde{L}, \quad \tilde{V} = \tilde{L} - H,$$

while the social welfare function and its unique optimizer continue to be S defined by equation (58) and x^* defined by condition (61), respectively.

To ensure that the actor's utility function $\tilde{A}$ has an optimizer, assume that as her action becomes very large, her private gain does not exceed her private cost.¹⁵⁵ To avoid unnecessary technicalities, assume that if her takes the socially optimal action x^* , then the victim is *not* indifferent between compensation and restitution.¹⁵⁶

The present modified model has welfare results that correspond to those in section 5.3,

¹⁵⁴Formally, $G'' < 0$, $B'' < 0$ or $C'' > 0$; and there exists some $\bar{x} > 0$ such that $G'(\bar{x}) + B'(\bar{x}) \leq C'(\bar{x})$.

¹⁵⁵Formally, assume there exists some $\bar{x} > 0$ such that $G'(\bar{x}) \leq C'(\bar{x})$.

¹⁵⁶More precisely, assume $\delta G(x^*) \neq \lambda H(x^*)$. This assumption ensures that both the actor's utility function $\tilde{A}$ and the net externalities function $\tilde{V}$ are differentiable at x^* .

for the original Model. Because the actor's utility function $\tilde{A}$ remains concave,¹⁵⁷ steps similar to those taken to prove Propositions 11 and 12 obtain the Externalities-Optimization Principle for the present modified model:

1. Assume $x^* > 0$. Then $\tilde{x}^\circ = x^*$ if and only if $\tilde{V}'(\tilde{x}^\circ) = 0$. This result corresponds to Proposition 11.
2. Assume $x^* = 0$. Then $\tilde{x}^\circ = x^*$ if $\tilde{V}'(\tilde{x}^\circ) \leq 0$. This result corresponds to Proposition 12.

A.5.3 Action-Dependent Restitution and Compensation

This appendix modifies the Model to capture rules and standards that adjust the extent of restitution or compensation in accordance with the wrongful action taken. Modify the Model so that, instead of two constants (δ, λ) , two functions $\Delta, \Lambda : \mathbb{R}_+ \rightarrow [0, 1]$ satisfying $\Delta', \Lambda', \Delta'', \Lambda'' \geq 0$ capture the operation of the law. If the wrongful action is x , then the law allocates to the actor $1 - \Delta(x)$ proportion of the resulting gain $G(x)$ and $\Lambda(x)$ proportion of the resulting harm $H(x)$. The law allocates to the victim the remaining $\Delta(x)$ proportion of the gain and $1 - \Lambda(x)$ proportion of the harm. To ensure the actor has a utility-optimizing action, assume that as her action becomes very large, her private gain does not exceed her private cost.¹⁵⁸

The actor's utility function and the net externalities respectively become

$$\begin{aligned}\bar{A}(x) &= [1 - \Delta(x)]G(x) - \Lambda(x)H(x) - C(x) \\ \bar{V}(x) &= \Delta(x)G(x) - [1 - \Lambda(x)]H(x),\end{aligned}$$

while the social welfare function (S defined by equation (58)) remains the same.

The actor's liability is $\bar{L}(x) = \Delta(x)G(x) + \Lambda(x)H(x)$ in the present modified model. The assumption $\Delta' \geq 0$ captures the intuition that, as she takes more wrongful action x , the law cannot allow her to retain a greater share of the resulting gain. The assumption $\Lambda' \geq 0$ captures a similar intuition; as x increases, the law cannot reduce her share of the resulting harm. The assumptions $\Delta'', \Lambda'' \geq 0$ are imposed to guarantee that her utility function $\bar{A}$ is concave, so that her first order condition characterizes a maximizer.

The present modified model has welfare results that are similar those in section 5.3, for

¹⁵⁷To see this, observe that $\tilde{A} = G - C + \min\{-\delta G, -\lambda H\} = \min\{(1 - \delta)G - C, G - C - \lambda H\}$. Because both functions $(1 - \delta)G - C, G - C - \lambda H$ are concave, their minimum $\tilde{A}$ is also concave.

¹⁵⁸Formally, assume there exists some $\bar{x} > 0$ such that $G'(\bar{x}) \leq C'(\bar{x})$.

the original Model. Let $\bar{x}^\diamond$ denote the actor's utility-maximizing action here. Steps similar to those taken to prove Propositions 11 and 12 obtain the Externalities-Optimization Principle for the present modified model:

1. Assume $x^* > 0$. Then $\bar{x}^\diamond = x^*$ if and only if $\bar{V}'(\bar{x}^\diamond) = 0$, where

$$\bar{V}'(\bar{x}^\diamond) = \Delta'(\bar{x}^\diamond)G(\bar{x}^\diamond) + \Delta(\bar{x}^\diamond)G'(\bar{x}^\diamond) + \Lambda'(\bar{x}^\diamond)H(\bar{x}^\diamond) - [1 - \Lambda(\bar{x}^\diamond)]H'(\bar{x}^\diamond).$$

This result corresponds to Proposition 11.

2. Assume $x^* = 0$. Then $\bar{x}^\diamond = x^*$ if $\bar{V}'(\bar{x}^\diamond) \leq 0$. This result corresponds to Proposition 12.

A.5.4 Two-Player Model

This appendix shows that the Externalities-Optimization Principle continues to hold in a two-player model. Calling one of these players the “actor” and the other the “victim” will capture those cases falling within the scope of the competing-equities analysis (see subsection 5.4.2). Examples that fall within the scope of the two-player model include partnerships and joint-ventures, which are subject to fiduciary law and the law of restitution.¹⁵⁹

Consider a simultaneous-move game of complete information, in which two utility-maximizing players respectively choose actions $y, z \in \mathbb{R}_+$. These actions generate a (real-valued) total gain of $\check{G}(y, z)$ and a (real-valued) total harm of $\check{H}(y, z)$. Each player's action increases the total gain at a diminishing rate, formally, $\frac{\partial \check{G}}{\partial y} > 0$, $\frac{\partial \check{G}}{\partial z} > 0$, $\frac{\partial^2 \check{G}}{\partial y^2} \leq 0$, $\frac{\partial^2 \check{G}}{\partial z^2} \leq 0$. Each player's action increases the total harm at an increasing rate, formally, $\frac{\partial \check{H}}{\partial y} > 0$, $\frac{\partial \check{H}}{\partial z} > 0$, $\frac{\partial^2 \check{H}}{\partial y^2} \geq 0$, $\frac{\partial^2 \check{H}}{\partial z^2} \geq 0$.

The player whose action is denoted y incurs the (real-valued) private cost of $\check{C}_y(y)$, where function $\check{C}_y$ satisfies $\check{C}'_y > 0$, $\check{C}''_y \geq 0$. Similarly, the player whose action is denoted z incurs the (real-valued) private cost of $\check{C}_z(z)$, where function $\check{C}_z$ satisfies $\check{C}'_z > 0$, $\check{C}''_z \geq 0$. Let $\check{S}$ denote the social welfare function, where

$$\check{S}(y, z) = \check{G}(y, z) - \check{H}(y, z) - \check{C}_y(y) - \check{C}_z(z).$$

Let a pair of constants $(\sigma, \tau) \in [0, 1] \times [0, 1]$ describe the law's allocation of the total gain and total harm between the players. The law allocates σ proportion of the total

¹⁵⁹See, for example, *Meinhard v. Salmon*, 164 N.E. 545 (N.Y. 1928).

gain and τ proportion of the total harm to the player whose action is denoted y . The law allocates the remaining $1 - \sigma$ proportion of the total gain and the remaining $1 - \tau$ proportion of the total harm to the player whose action is denoted z . The utility functions of these players are respectively

$$\begin{aligned} A_y(y, z) &= \sigma \check{G}(y, z) - \tau \check{H}(y, z) - \check{C}_y(y) \\ A_z(y, z) &= (1 - \sigma) \check{G}(y, z) - (1 - \tau) \check{H}(y, z) - \check{C}_z(z) \end{aligned}$$

where A_y is the utility function of the player whose action is y , and A_z the utility function of the player whose action is z .

It remains to impose assumptions on derivatives to ensure that a pure-strategy Nash equilibrium exists, and that the social welfare function has a unique optimizer.¹⁶⁰

Let $(y^\diamond, z^\diamond)$ denote the pair of individual-utility-optimizing actions comprising the Nash equilibrium. The pair satisfies the following system of first order conditions:

$$\begin{cases} \frac{\partial A_y}{\partial y}(y^\diamond, z^\diamond) = \sigma \frac{\partial \check{G}}{\partial y}(y^\diamond, z^\diamond) - \tau \frac{\partial \check{H}}{\partial y}(y^\diamond, z^\diamond) - \check{C}'_y(y^\diamond) \leq 0 \\ \frac{\partial A_z}{\partial z}(y^\diamond, z^\diamond) = (1 - \sigma) \frac{\partial \check{G}}{\partial z}(y^\diamond, z^\diamond) - (1 - \tau) \frac{\partial \check{H}}{\partial z}(y^\diamond, z^\diamond) - \check{C}'_z(z^\diamond) \leq 0. \end{cases} \quad (146)$$

A pair of actions (y^*, z^*) maximizes social welfare if and only if it satisfies the following system of first order conditions:

$$\begin{cases} \frac{\partial \check{S}}{\partial y}(y^*, z^*) = \frac{\partial \check{G}}{\partial y}(y^*, z^*) - \frac{\partial \check{H}}{\partial y}(y^*, z^*) - \check{C}'_y(y^*) \leq 0 \\ \frac{\partial \check{S}}{\partial z}(y^*, z^*) = \frac{\partial \check{G}}{\partial z}(y^*, z^*) - \frac{\partial \check{H}}{\partial z}(y^*, z^*) - \check{C}'_z(z^*) \leq 0. \end{cases} \quad (147)$$

A substitution exercise using system (146) and evaluating the derivatives of the social welfare function at the pair $(y^\diamond, z^\diamond)$ gives the *marginal* externalities arising from these actions. Let M_y and M_z respectively denote the marginal externalities arising from $y^\diamond, z^\diamond$;

¹⁶⁰More precisely, to guarantee the existence of a pure-strategy Nash equilibrium, assume $\frac{\partial^2 \check{G}}{\partial y^2} < 0$ or $\check{C}''_y > 0$; and $\frac{\partial \check{G}}{\partial y}(\bar{y}, z) \leq \check{C}'_y(\bar{y})$ for some large $\bar{y} > 0$ that does not depend on z . To guarantee the utility function A_z has an optimizer, assume $\frac{\partial^2 \check{G}}{\partial z^2} < 0$ or $\check{C}''_z > 0$; and $\frac{\partial \check{G}}{\partial z}(y, \bar{z}) \leq \check{C}'_z(\bar{z})$ for some large $\bar{z} > 0$ that does not depend on y .

To ensure the social welfare function $\check{S}$ has a unique optimizer, assume its cross-derivative is sufficiently small so that its Hessian is negative semidefinite; formally,

$$\left(\frac{\partial^2 \check{S}}{\partial y \partial z} \right)^2 < \frac{\partial^2 \check{S}}{\partial y^2} \frac{\partial^2 \check{S}}{\partial z^2}.$$

these function are defined by

$$\begin{cases} M_y(y^\diamond, z^\diamond) = \frac{\partial \check{S}}{\partial y}(y^\diamond, z^\diamond) - \frac{\partial A_y}{\partial y}(y^\diamond, z^\diamond) \\ M_z(y^\diamond, z^\diamond) = \frac{\partial \check{S}}{\partial z}(y^\diamond, z^\diamond) - \frac{\partial A_z}{\partial z}(y^\diamond, z^\diamond), \end{cases} \quad (148)$$

where some algebra obtains

$$\begin{cases} M_y(y^\diamond, z^\diamond) = (1 - \sigma) \frac{\partial \check{G}}{\partial y}(y^\diamond, z^\diamond) - (1 - \tau) \frac{\partial \check{H}}{\partial y}(y^\diamond, z^\diamond) \\ M_z(y^\diamond, z^\diamond) = \sigma \frac{\partial \check{G}}{\partial z}(y^\diamond, z^\diamond) - \tau \frac{\partial \check{H}}{\partial z}(y^\diamond, z^\diamond). \end{cases}$$

A comparison of systems (147) and (148) reveals the present two-player model has welfare results that are similar to those in section 5.3, for the original Model. Steps similar to those taken to prove Propositions 11 and 12 obtain the Externalities-Optimization Principle for the present modified model:

1. Assume $y^*, z^* > 0$. Then $(y^\diamond, z^\diamond) = (y^*, z^*)$ if and only if $M_y(y^\diamond, z^\diamond) = M_z(y^\diamond, z^\diamond) = 0$. This result corresponds to Proposition 11.
2. Assume $y^* = 0$ or $z^* = 0$. Then $(y^\diamond, z^\diamond) = (y^*, z^*)$ if $M_y(y^\diamond, z^\diamond) = M_z(y^\diamond, z^\diamond) = 0$. This result corresponds to Proposition 12.

A.5.5 General Technical Foundation

The Model set up in subsection 5.2.1 assumes the actor's utility function and the social welfare function are concave and differentiable in her action. Dropping these assumptions, Proposition 13 in this appendix offers a general technical foundation for the Externalities-Optimization Principle.

Proposition 13. *Consider a continuous function $S : X \rightarrow \mathbb{R}$ with a compact domain X .¹⁶¹ Then $x^* \in X$ maximizes S if and only if there exist two functions $A, V : X \rightarrow \mathbb{R}$ such that $A + V = S$ and x^* maximizes each of A and V .*

Proof

That S is a continuous function on a compact set X guarantees the existence of a maximizer.

1. For one direction, suppose x^* maximizes S . Define functions A, V by $A = V = 0.5S$. These functions satisfy $A + V = S$ and have x^* as a common maximizer.

¹⁶¹Compactness and continuity here follow their respective meanings in topology.

2. For the other direction, suppose there exist two functions A, V satisfying $A + V = S$ and having a common maximizer x^* . Consider an arbitrary $x \in X$. That x^* is a maximizer of A and V implies $A(x^*) \geq A(x)$ and $V(x^*) \geq V(x)$. Then

$$S(x^*) = A(x^*) + V(x^*) \geq A(x) + V(x) = S(x).$$

The choice of x is arbitrary, hence x^* is a maximizer of S . □

The obvious result in Proposition 13 lays a general foundation for the Externalities-Optimization Principle. To see this, interpret function S in Proposition 13 as the social welfare function, the set X as the actor's action space, and functions A, V as her utility function and the net externalities function respectively. So interpreted, Proposition 13 reveals that there exist ways to “slice” S into A and V so that all three functions have a common maximizer; this is the Externalities-Optimization Principle at a high level of generality.

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