# Riemann's Zeta and Two Arithmetic Functions

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*To the creator who gave me this love.*
*Let the path I tread in the land of zeta be yours.*

# Declaration

I have written this thesis during my honours year for submission as my honours thesis. There are references given throughout this thesis when I deem it necessary to give recognition to others. None of the results I give nor the method I use are original discoveries that I claim credit for.

Kirsty Anne Chalker

# Acknowledgments

I shall begin by extending an exuberant 'THANK YOU!' to Dr Tim Trudgian, my supervisor. I am especially grateful for the unquantifiable amount of knowledge I have gained from him, for the risks he has taken in giving me opportunities and backing me on various endeavours and for all of the time he has (so efficiently!) spent. My mathematical journey is more the richer thanks to the mentor Tim has been.

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# Abstract

An intriguing connection exists between the *Riemann zeta-function* $\zeta(s)$ and both the *von Mangoldt function* $\Lambda(n)$ and the *Möbius function* $\mu(n)$. With appropriate information about $\zeta(s)$ this connection provides a door to finding bounds for the sums $\psi(x) - x$ and $M(x)$ that involve these arithmetic functions $\Lambda(n)$ and $\mu(n)$. A bound for $\psi(x)-x$ bounds the error in the well-known Prime Number Theorem. In this thesis we consider a particular approach to finding bounds that is presented in *Problems in Analytic Number Theory* by Murty. Using this approach we prove implicit bounds for $\psi(x) - x$ and $M(x)$ after providing background on the zeta-function, presenting a bird's-eye view of the approach and exploring the history of bounding these sums. The implicit bounds we prove are by no means new. However, we did not come across any explicit form of the bound we prove for $M(x)$ in the literature. Time constraints prevented the completion of such an explicit bound during the course of this thesis, but we do include some details relevant to such a task.

# Contents

# List of Figures

# Describing terms and notation

Below we summarize various terminology and notation that we shall use. For each term and notation we give a description or a reference to a later part of this thesis where a description can be found.

**Standard terminology**

| | |
|---|---|
| Arithmetic function | Any function that has natural numbers as input and complex numbers as output. |
| Implicit bound | A bound of the kind $h(x) = O(f(x))$. This means that $\lvert h(x)\rvert \le Cf(x)$, for all $x \ge \alpha$ for some $\alpha$ and some $C > 0$. |
| Implied constant | The coefficient $C$ that is not included when we write an *implicit bound*, but is included when we write an *explicit bound*. |
| Explicit bound | A bound of the kind $\lvert h(x)\rvert \le Cf(x)$, for all $x \ge \alpha$ for some $\alpha$ and some $C > 0$ when we specify the *implied constant* $C$ and any other constants appearing on the right-hand side. |

**Standard notation**

| | |
|---|---|
| $\mathbb{N}$ | The natural numbers, starting with one. |
| $n$ | An element of $\mathbb{N}$. |
| $p$ | A prime. |
| $\gamma$ | Euler's constant $\gamma = 0.5772\ldots$. |
| $\zeta(s)$ | The Riemann zeta-function. |
| $\rho$ | Any zero of $\zeta(s)$ satisfying $0 < \mathrm{Re}(s) < 1$. |
| $\psi(x)$ | See Definition 1 on page 2. |

## Standard notation

| | |
|---|---|
| $M(x)$ | See Definition 2 on page 2. |
| $\Lambda(n)$ | The von Mangoldt function. See Definition 1 on page 2. |
| $\mu(n)$ | The Möbius function. See Definition 2 on page 2. |
| $\delta(x)$ | See Lemma 12 on page 18. |
| $[x]$ | The integer part or 'floor' of a real number. |
| $\mathrm{Re}(s)$ | The real part $x$ where $s = x + iy$. |
| $\mathrm{Im}(s)$ | The imaginary part $y$ where $s = x + iy$. |
| $\overline{s}$ | The conjugate $x - iy$ where $s = x + iy$. |
| $\mathrm{Res}\,(f(s); s_0)$ | The residue of $f$ at a pole $s_0$. |
| $h(x) = O(f(x))$ | See *implicit bound* above. |
| $h(x) = o(f(x))$ | This means that $\lim_{x\to\infty} h(x)/f(x) = 0$. |
| $h(x) \sim f(x)$ | This means that $\lim_{x\to\infty} h(x)/f(x) = 1$. |

## Non-standard notation

| | |
|---|---|
| $T$ | A parameter introduced in Lemma 9 on page 14. We later define $T$ specifically (see (3.35) on page 39 and (4.25) on 50). |
| $c_i$ | See Table 5.1 in the Appendix. |
| $\tilde{C}_i$ | Lines in the complex plane that together make a closed contour. See Figure 2.5 on page 25. |
| $a$ | A parameter introduced in Perron's formula (2.1) on page 17. We later define $a := 1 - c_1/\log T$ (see Section 3.2.2). |
| $b$ | A parameter introduced in Step 3 of Section 2.3. It is $b := 1 - c_1/\log T$. |
| $\tilde{t}$ | A parameter introduced in (3.4) on page 30. We first use it to specify a zero-free region of $\zeta(s)$. |

# Introduction

The Riemann zeta-function, which is

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\text{ if } \operatorname{Re}(s)>1,$$

is the focus of Riemann's famous 1859 conjecture, the Riemann Hypothesis (RH). In this thesis we put the zeta-function to work, without the assumption of the RH, to prove a bound for two sums involving *arithmetic functions*. However, before we jump into the finer details of this thesis, we shall take a few moments to understand the problem and to motivate why one might be interested in finding such bounds.

There is much intrigue associated with the primes $2, 3, 5, 7, \ldots$ because this somewhat unpredictable set constructs the natural numbers, which are so well-behaved we use them to count. The formalities of this statement are given in the well-known *Fundamental Theorem of Arithmetic*: the natural numbers $1, 2, 3, 4, \ldots$ can each be built by multiplying a unique collection of prime powers $\{p^k\}$, with $p$ being prime and $k$ being a non-negative integer. For example, $15 = 5 \times 3$ and $8 = 2^3$. This theorem is motivation for studying all things related to the primes.

There is another well-known theorem about primes that is directly related to the two sums we bound in this thesis. This theorem gives an approximation to the number of primes up to some positive number $x$ and is called the *Prime Number Theorem* (PNT).

**Prime Number Theorem 1.** *With* $\pi(x)=\sum_{p\leq x}1,$ *$p$ being prime,*

$$\pi(x)\sim\frac{x}{\log x}.$$

The PNT means that $x/\log x$ is an approximation for $\pi(x)$ that gets better and better the larger $x$ is.

Although the above statement is probably the most intuitive form of the PNT, we can also write this theorem using $\psi(x)$.

**Definition 1.** *We define* $\psi(x)$ *by*

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$

*where the von Mangoldt function is defined by*

$$\Lambda(n) = \begin{cases} \log p, & \text{for } n = p^{\beta}, \beta \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$

Using $\psi(x)$ the PNT can be re-stated as follows.

**Prime Number Theorem 2.** *With* $\psi(x) = \sum_{n \le x} \Lambda(n)$,

$$\psi(x) \sim x.$$

We interpret this in the same way we interpreted PNT 1. That is, $x$ is an approximation for $\psi(x)$ that gets better and better the larger $x$ is. (For a proof that PNT 1 and 2 are equivalent see the solution to Exercise 3.1.11 on page 279 of [17].)

Armed with PNT 2 we can now bring our discussion about primes into perspective because $\psi(x) - x$ is one of two sums that we shall bound. Since $\psi(x) \sim x$ is the PNT, a bound for $\psi(x) - x$ bounds the error in the PNT approximation.

In this thesis, we prove the implicit bound

$$\psi(x) - x = O\left(x \exp(-c\sqrt{\log x})\right), \text{ for some } c > 0 \tag{1}$$

using an approach presented in Chapter 4 of Murty's *Problems in Analytic Number Theory*. For $\psi(x) - x$ the work that has already been done by others on finding explicit bounds will most certainly be stronger than what we would obtain, so we call it a day once we prove the implicit bound in (1).

We also use the same approach to bound another sum $M(x)$ involving an arithmetic function $\mu(n)$, which we introduce in the following.

**Definition 2.** *The function*

$$M(x) = \sum_{n \le x} \mu(n),$$

*and involves the Möbius function*

$$\mu(n) = \begin{cases} 1, & \textit{for } n = 1, \\ (-1)^k, & \textit{for } n = p_1 p_2 \dots p_k,\ p_i \neq p_j\ \forall i \neq j, \\ 0, & \textit{otherwise.} \end{cases}$$

The PNT is also related to $M(x)$. We have that $M(x) = o(x)$ by the PNT, as shown by Landau. This fact can be found in [25, p. 1359].

Although the implicit bound

$$M(x) = O\left(x \exp\left(-c\sqrt{\log x}\right)\right), \text{ for some } c > 0 \tag{2}$$

is by no means new, we have not come across any explicit bound of this form in the literature. In this thesis we prove (2). Unfortunately, time constraints meant we were unable to roll out such an explicit bound completely. However, we do include some details that would assist in doing so.

In both (1) and (2) as $c$ gets larger, the bound gets smaller. Hence, when making the bounds explicit one wants to make $c$ as small as possible. We shall see later that this $c$ depends on a parameter $(0 < c_1 < 1)$ that is used to specify a zero-free region of $\zeta(s)$. We will discuss zero-free regions later in this thesis. However, this dependence implies that the greater the improvement we make to particular zero-free regions, the greater the improvement we can make to the bounds. Improving the zero-free regions equates to having greater knowledge on the truth of the RH.

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In Chapter 1 we provide knowledge on the zeta-function that is relevant to this work. In Chapter 2, we prove a preliminary result that is needed and present a bird's-eye view of the approach we have used to prove (1) and (2) above. Chapter 3 is where we prove (1) after exploring the history of the problem of finding a bound for $\psi(x) - x$. Chapter 4 then mimics Chapter 3, but for $M(x)$. In Chapter 5 we answer a particular question about the approach we use and consider what could be further explored. We then tie it all together in the conclusion. Details that would assist with finding explicit bounds can be found in the Appendix.

A description of various terminology and notation is included on page xi.

# Chapter 1

# Meeting Riemann's zeta

The approach we have used to prove a bound for the sums $\psi(x)-x$ and $M(x)$ relies on the Riemann zeta-function $\zeta(s)$. In Section 1.1 below we relate $\Lambda(n)$ and $\mu(n)$ (and so $\psi(x)-x$ and $M(x)$) to $\zeta(s)$. Then in Sections 1.2–1.3 we provide background on the zeta-function that we rely on in the remainder of this thesis.

## 1.1 Relating $\Lambda(n)$ and $\mu(n)$ to zeta

The approach we use to find bounds in Chapters 3 and 4 relies on an interesting connection between $\zeta(s)$ and the functions $\Lambda(n)$ and $\mu(n)$. In Theorems 1 and 2 below we reveal this connection.

**Theorem 1.** *The von Mangoldt function* $\Lambda(n)$ *satisfies*

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}=-\frac{\zeta'}{\zeta}(s)$$

*if* $\mathrm{Re}(s)>1$.

*Proof.* See pages 208–209 of [17].

**Theorem 2.** *The Möbius function* $\mu(n)$ *satisfies*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}$$

*if* $\mathrm{Re}(s)>1$.

*Proof.* See page 209 of [17].

Theorems 1 and 2 are crucial in the approach we use to bound $\psi(x)$ and $M(x)$, but these results alone do not hold the key. We also need certain information about the zeta-function, which we shall present in the next four sections.

## 1.2 A symmetry revealed

In this section we provide facts about the symmetry of $\zeta(s)$.

The Riemann zeta-function is the focus of the Riemann Hypothesis (RH). This hypothesis is the famous conjecture that if the real-part of a zero of $\zeta(s)$ is bounded by zero and one, then this real-part is equal to one-half. Even though the RH is open, we know that the zeros of $\zeta(s)$ in the region $0 < \mathrm{Re}(s) < 1$ come in pairs, one zero below the real axis and one above. This is due to a symmetry $\zeta(s)$ has that comes from the Schwarz Reflection Principle (SRP) [33, p. 222], which we state for $\zeta(s)$ below.

**Lemma 1.** *The zeta-function follows the Schwarz Reflection Principle, namely*

$$\overline{\zeta(s)} = \zeta(\overline{s})$$

*whenever* $\zeta(s)$ *is holomorphic.*

The above lemma implies that any zero of $\zeta(s)$ above the real-axis pairs with one below the real-axis. We give a rigorous form of this statement in the following lemma and illustrate it in Figure 1.1.

**Lemma 2.** *If some* $s \in \mathbb{C}$ *is a zero of* $\zeta(s)$ *we have that* $\overline{s}$ *is also a zero.*

Due to Lemma 2 we now only need to consider zeros of $\zeta(s)$ that are either above or below the real-axis. If we know what zeros are in one of these half-planes, we know what zeros are in the other.

In Figure 1.1 one can see that we not only include the zero $s_1$ and its conjugate $\overline{s_1}$, but also $1 - s_1$ and $1 - \overline{s_1}$. If $s_1$ is a zero of $\zeta(s)$, both $1 - s_1$ and $1 - \overline{s_1}$ are also zeros of $\zeta(s)$. However, this symmetry does not come from Lemma 2, rather it comes from the *functional equation.* We will not go into details about the functional equation here — see, for example, Chapter II of [34].

Later, in Chapters 3 and 4, we shall need that $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$ also follow the SRP. We state this in Lemma 3.

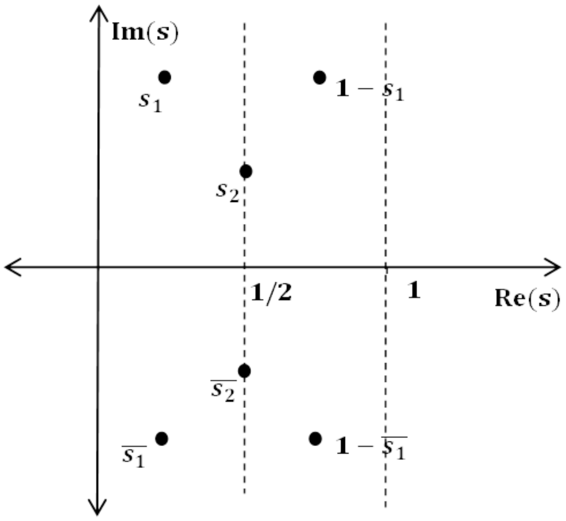

Figure 1.1: The symmetry in the zeros of $\zeta(s)$. We mark each zero with a black dot.

**Lemma 3.** *The functions* $\frac{\zeta'}{\zeta}(s)$ *and* $\frac{1}{\zeta(s)}$ *both follow the Schwarz Reflection Principle, namely*

$$\overline{\frac{\zeta'}{\zeta}(s)} = \frac{\zeta'}{\zeta}(\overline{s}) \text{ and } \frac{1}{\overline{\zeta(s)}} = \frac{1}{\zeta(\overline{s})}$$

*whenever they are holomorphic.*

## 1.3 Identifying the pole

In Chapters 3 and 4 we shall need information about the poles of $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$. In this section we discuss the location of such poles.

In the introduction we provided a definition for the Riemann zeta-function that is only relevant if $\mathrm{Re}(s) > 1$. However, we want the reader to be aware that using analytic continuation one can show that there is just a single point in the complex plane at which $\zeta(s)$ does not exist, due to a simple pole (see, for example, Chapter II of [34]).

**Lemma 4.** *The point* $s = 1$ *is a simple pole of the Riemann zeta-function and*

$$\mathrm{Res}\,(\zeta(s); 1) = 1.$$

*Proof.* See pages 13–15 of [34].

When we find a bound for $\psi(x) - x$ and $M(x)$, in Chapters 3 and 4, we shall need the following lemma.

**Lemma 5.** *The point* $s = 1$ *is a simple pole of* $\frac{\zeta'}{\zeta}(s)$ *and*

$$\operatorname{Res}\left(\frac{\zeta'}{\zeta}(s); 1\right) = -1.$$

*However, the point* $s = 1$ *is not a simple pole of* $\frac{1}{\zeta(s)}$.

*Proof.* We use Lemma 4 to write the zeta-function as

$$\zeta(s) = \frac{f(s)}{(s-1)} \tag{1.1}$$

with a holomorphic $f(s)$ for $s = 1$ and $f(1) \neq 0$. The lemma follows. □

The next lemma gives further information we shall need in Chapters 3 and 4 about the possibility of poles of $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$.

**Lemma 6.** *The* $s \in \mathbb{C}$ *that are zeros of* $\zeta(s)$ *are poles of* $\frac{\zeta'}{\zeta}(s)$ *and* $\frac{1}{\zeta(s)}$.

*Proof.* On assuming $\zeta(s)$ has a zero at $s = \alpha$ and that this zero is of order $n$ we may write the zeta-function as

$$\zeta(s) = f(s)(s-\alpha)^n \tag{1.2}$$

with a holomorphic $f(s)$ for $s = \alpha$ and $f(\alpha) \neq 0$. The lemma follows. □

Other than the zeta-function's simple pole and its zeros there are no other $s \in \mathbb{C}$ that could possibly be poles of $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$.

## 1.4 Discussing zero-free regions

In order to use Lemma 6 in Section 1.3 above we need to know about the subset of the complex plane that is known to contain no zeros of $\zeta(s)$. In this section we reveal such regions.

Despite the RH not yet having a resolution, progress has been made on the zero-free regions of $\zeta(s)$. We shall reveal this progress in four steps below. We shall only concern ourselves with the part of the complex plane satisfying $\operatorname{Re}(s) > 0$ since this is the region that is of interest for finding bounds in

Chapters 3 and 4.

For the first step we determine that the half-plane $\mathrm{Re}(s) > 1$ is zero free. This is not obvious from the definition

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ if } \mathrm{Re}(s) > 1.$$

However, we can use another definition, which we give below.

**Lemma 7.** *The zeta-function can be re-written as*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.3}$$

*which converges if* $\mathrm{Re}(s) > 1$. *This is called Euler's product formula.*

*Proof.* See pages 1–2 of [34].

Without worrying about convergence, we observe that if $(1 - 1/p^s) \neq 0$ for all $p$ then Euler's product formula cannot possibly be equal to zero in this half-plane. It is possible to give a rigorous version of this idea, thus, we have the following lemma.

**Lemma 8.** *The half-plane* $\mathrm{Re}(s) > 1$ *contains no zeros of* $\zeta(s)$.

*Proof.* See pages 1–2 of [34].

For the second step we add the line $\mathrm{Re}(s) = 1$ to the zero-free region. This step was the work of Hadamard and de la Vallée Poussin back in 1896, giving the following theorem.

**Theorem 3.** (Hadamard and de la Vallée Poussin) *The line* $\mathrm{Re}(s) = 1$ *contains no zeros of* $\zeta(s)$.

*Proof.* See Sections 3.1–3.4 of [34].

The zero-free region given by Lemma 8 and Theorem 3 is illustrated in Figure 1.2.

The next two steps are concerned with the part of the complex plane that is called the *critical strip* when one is discussing $\zeta(s)$. This is the region that satisfies $0 < \mathrm{Re}(s) < 1$. We label the critical strip in Figure 1.2.

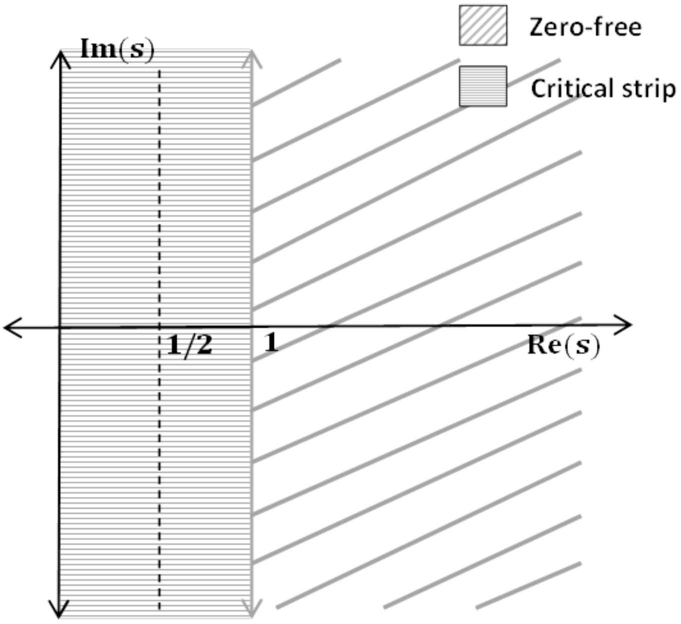

Figure 1.2: Zero-free region: steps 1–2.

Platt found in [21, p. 1532] that all zeros $\rho = \beta + i\gamma$ with $0 < \beta < 1$ and $0 \leq \gamma \leq 3.06 \times 10^{10}$ satisfy the RH, i.e. $\beta = 1/2$.[1] (In [21, p. 1532] it also refers one to [22] for the technique Platt applied.) This also holds for $-3.06 \times 10^{10} \leq \gamma \leq 0$ due to Lemma 1 in Section 1.2. For the third step, we add Platt's result to the zero-free region in Figure 1.3.

For the final step we further add to the zero-free region in the critical strip. There are two main zero-free regions we could use to do this. One of these is called the *classical zero-free region*, namely, the part of the complex plane satisfying

$$\mathrm{Re}(s) \geq 1 - \frac{c_1}{\log \mathrm{Im}(s)}, \text{ and } \mathrm{Im}(s) \geq \tilde{t}, \tag{1.4}$$

under an appropriate choice of $0 < c_1 < 1$ and $\tilde{t} \geq 0$. We illustrate such a zero free region in Figure 1.4. This zero-free region dates back to 1899 and was the discovery of de la Vallée Poussin [3].

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[1] This is the region we shall consider because of the part *interval arithmetic* played in the work of Platt and because of Platt's result having been published. One may also see [12] by Gourdon for another example of work we could consider here. Additionally, there is also such work of Wedeniwski [41] that is referred to in [8, p. 886].

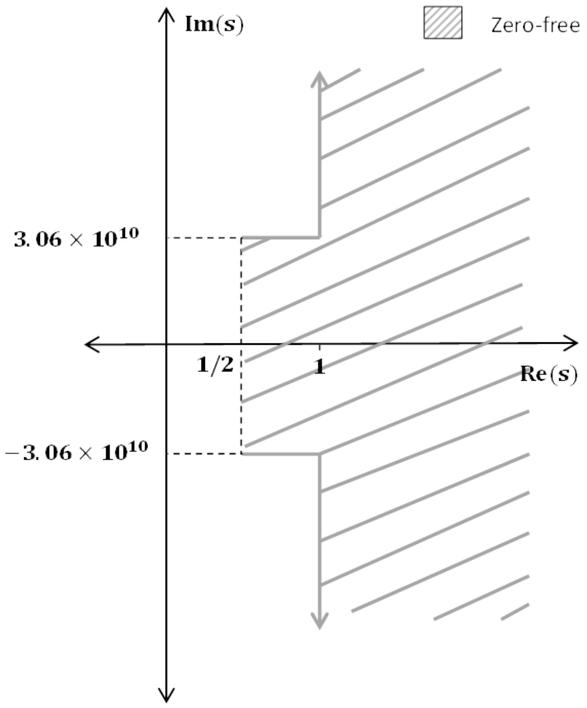

Figure 1.3: Zero-free region: steps 1–3.

As $c_1$ is made smaller in (3.4) the zero-free region gets larger. Over the years significant progress has been made on $c_1$. One can see Table 1 in [16, p. 331] for a summary of the $c_1$ that were obtained before the work of Mossinghoff and Trudgian [16], including the $c_1$ in de la Vallée Poussin's work, $c_1 = 1/30.4679$. Mossinghoff and Trudgian's work is the latest classical zero-free region and gives the following theorem.

**Theorem 4.** (Mossinghoff and Trudgian) *The following region is free from zeros*

$$\operatorname{Re}(s) \geq 1 - \frac{1}{5.573412 \log(\operatorname{Im}(s))} \quad \textit{and} \quad \operatorname{Im}(s) \geq 2.$$

The above theorem also holds if $\operatorname{Im}(s) \leq -2$ due to Lemma 1 in Section 1.2.

Some of the work between that of de la Vallée Poussin and that of Mossinghoff and Trudgian regarding $c_1$ was published in papers that also present bounds for $\psi(x) - x$. For example, see [28, p. 71] and [29, p. 250].

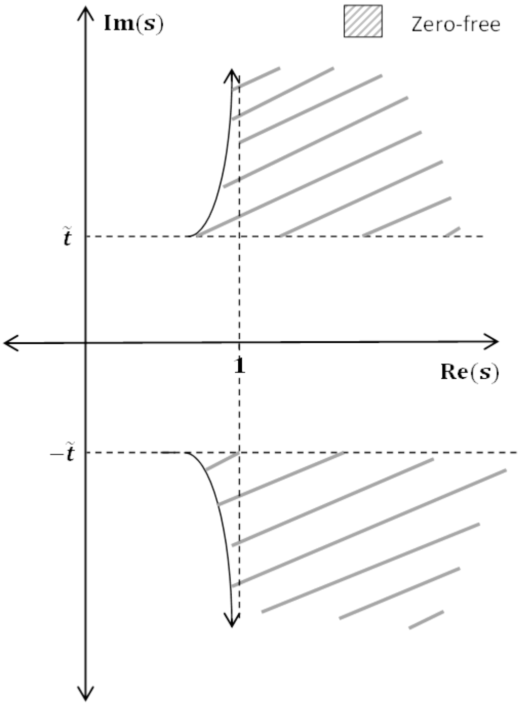

Figure 1.4: A classical zero-free region.

We add the classical zero-free region to the illustration of the zero-free region in Figure 1.5.

Another zero-free region dates back to 1958 and was the discovery of both Vinogradov [38] and Korobov [13]. This region involves the part of the complex plane satisfying

$$\mathrm{Re}(s) > 1 - \frac{c_1}{(\log(\mathrm{Im}(s)))^{2/3}(\log\log(\mathrm{Im}(s)))^{1/3}}, \text{ and } \mathrm{Im}(s) \geq \tilde{t} \tag{1.5}$$

with appropriate $0 < c_1 < 1$ and $\tilde{t} \geq 0$. This is stronger than Theorem 4 for sufficiently large $\mathrm{Im}(s)$. For this zero-free region the work of Ford in [11, p. 4] gave the following theorem

**Theorem 5.** (Ford) *The following region is free from zeros*

$$\mathrm{Re}(s) > 1 - \frac{1}{57.54(\log(\mathrm{Im}(s))^{2/3}(\log\log(\mathrm{Im}(s)))^{1/3}} \textit{ and } \mathrm{Im}(s) \geq 3.$$

The above theorem also holds if $\mathrm{Im}(s) \leq -3$ due to Lemma 1 in Section 1.2.

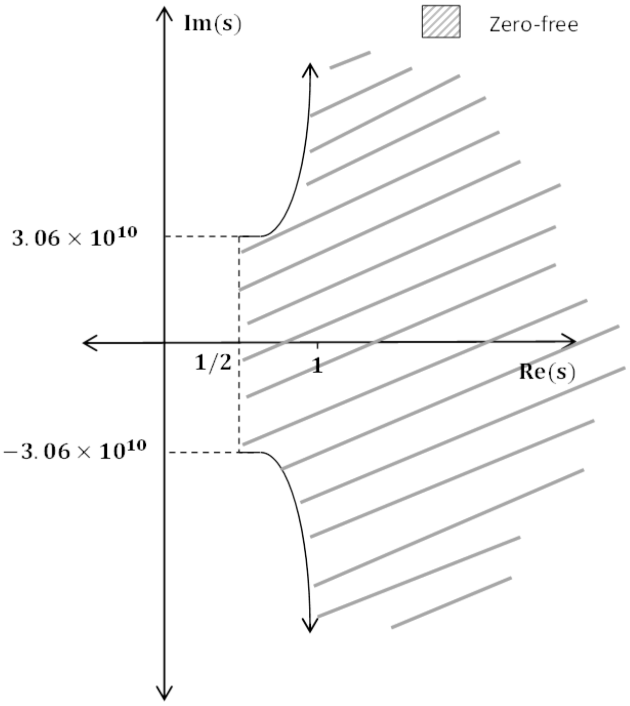

Figure 1.5: Zero-free region: steps 1–4.

Murty [17, pp. 60-62] uses the classical zero-free region to bound $\psi(x) - x$. Whereas, the bound

$$M(x) = \left(x \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})\right)$$

given by Walfisz [40, p. 191] in 1963 was found using the zero-free region in (1.5). An analogous bound for $\psi(x) - x$ also follows from (1.5). However, since $c_1$ is not stated explicitly in this zero-free region this bound could not be completely explicit. Ford [10, p. 566] later applied his explicit zero-free region in Theorem 5, in addition to work by Pintz [20], to find the bound

$$\pi(x) - \mathrm{li}(x) = O\left(x \exp\left(-0.2098(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right)$$

with

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log y} dy.$$

We notice that this bound by Ford is also not completely explicit, but it does contain an explicit constant $c = 0.2098$, whereas Walfisz' bound does not. In the future, one could insert (1.5) in the approach we use to find a bound for

$\psi(x) - x$ and $M(x)$.

Next, we present bounds for $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$ that we shall use later.

## 1.5 A collection of $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$ bounds

When finding bounds for $\psi(x) - x$ and $M(x)$ in Chapters 3 and 4 we shall require bounds for $\frac{\zeta'}{\zeta}(s)$ and $\frac{1}{\zeta(s)}$ respectively. In this section we shall present such bounds. Although we only require implicit bounds, some of the bounds we supply below are explicit. This means that the appropriate bounds would be at hand if one was to make the bounds we shall prove explicit.

One set of bounds we shall use are the work of Trudgian [36] and are given in Theorems 6 and 7 below. The bounds we present in these two theorems are amongst a number of bounds that Trudgian found. See Tables 1 and 2 in [36, pp. 259–260] for other bounds.

**Theorem 6.** (Trudgian) *In the region that satisfies* $\mathrm{Re}(s) \geq 1 - \frac{1}{9\log(\mathrm{Im}(s))}$ *and* $\mathrm{Im}(s) \geq 70.59$, *we have*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq 61.54\log(\mathrm{Im}(s)).$$

**Theorem 7.** (Trudgian) *In the region involved in Theorem 6, we have*

$$\left|\frac{1}{\zeta(s)}\right| \leq 9.6 \times 10^4\log(\mathrm{Im}(s)).$$

We only require the implicit form of the Theorem 6 and 7 bounds. This implicit form of the bounds is old hat. One can, for example, find them on page 60 of [34].

When $0 < \mathrm{Im}(s) < 70.59$ we shall instead use the bounds in the following two lemmas.

**Lemma 9.** *With* $s = 1 - c_1/\log T + it$, $T > \tilde{T}$ *for some* $\tilde{T}$ *and* $c_1 > 0$, *we have*

$$\frac{\zeta'}{\zeta}(s) = O(\log T).$$

*Proof.* Since $s = 1$ is a simple pole of $-\frac{\zeta'}{\zeta}(s)$ we may write $-\frac{\zeta'}{\zeta}(s)$ in the form

$$\frac{\zeta'}{\zeta}(s) = \frac{f(s)}{s-1}$$

with a holomorphic $f(s)$ for $s = 1$ and $f(1) \neq 0$. The lemma follows. ☐

**Lemma 10.** *With $s$ and $T$ as given in Lemma 9 above and $t < \tilde{t}$ for some $\tilde{t}$, we have*

$$\frac{1}{\zeta(s)} = O(1).$$

*Proof.* We start with the bound Titchmarsh gives for $\zeta(s)$ in [34] on page 16, namely

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|).$$

Then, on using the triangle inequality twice we have

$$|\zeta(s)| \geq \frac{1}{|s-1|} - \gamma - O(|s-1|).$$

We then obtain the result by inverting the above inequality and using the restrictions on $s, T$ and $t$ to bound. ☐

Lastly, we shall need Ford's [11, Lemma 3.1] bound.

**Lemma 11.** (Ford) *In the region that satisfies* $1 < \mathrm{Re}(s) \leq 1.06$, *we have*

$$\left|\frac{\zeta'}{\zeta}(s)\right| < \frac{1}{\mathrm{Re}(s) - 1}.$$

# Chapter 2

# Perron's formula

In addition to the zeta-function being paramount to the approach we use for proving bounds, we require a result known as *Perron's formula.* That is, we set $\Phi(x) = \sum_{n\leq x} \phi(n)$ and have $\phi(n)$ being arithmetic. Then with $\sum_{n=1}^{\infty} \phi(n)/n^s$ being uniformly convergent if $\mathrm{Re}(s) > \tilde{\alpha}$ we have that

$$\Phi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left(\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}\right) \frac{x^s}{s}\, ds,\ a > \tilde{\alpha}. \tag{2.1}$$

Using the connection between $\zeta(s)$ and $\Lambda(n)$ that we revealed in Theorem 1 on page 5 we may determine that, with $\Phi(x) = \psi(x)$ and $\phi(n) = \Lambda(n)$, Perron's formula becomes

$$\psi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\, ds,\ a > 1.$$

Similarly, using Theorem 2 on page 5, with $\Phi(x) = M(x)$ and $\phi(n) = \mu(n)$, Perron's formula becomes

$$M(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{\zeta(s)}\, \frac{x^s}{s} ds,\ a > 1.$$

Hence, we notice that Perron's formula provides the machinery needed to transform information about $\zeta(s)$ into information about either $\psi(x)$ or $M(x)$.

## 2.1 Perron's formula proved

In this section we prove (2.1) for a sum $\Phi(x) = \sum_{n\leq x} \phi(n)$ involving an arithmetic function $\phi(n)$.

The proof of the following lemma carries most of the weight in proving Perron's formula. It is a culmination of Exercises 4.1.1–4.1.3 in [17], which are solved on pages 305–307 of [17].

**Lemma 12.** *We have*

$$\delta(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s} ds, \ a > 0 \tag{2.2}$$

*for the function*

$$\delta(x) := \begin{cases} 1, & \textit{for } x > 0, \\ 1/2, & \textit{for } x = 1, \\ 0, & \textit{for } 0 < x < 1. \end{cases}$$

*The contour involved in* (2.2) *is a straight line running from* $-\infty$ *to* $\infty$ *through* $a$*.*

*Proof.* We begin with (2.2) and proceed to show that this satisfies the definition of $\delta(x)$ in each of the cases $x = 1, \ x > 1$ and $0 < x < 1$ in this order.

With $x = 1$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s} ds = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{1}{s} ds.$$

Since $s = a + it$, with $t$ running from $-T$ to $T$,

$$\begin{aligned} \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{1}{s} ds &= \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{a+it}\frac{(a-it)}{(a-it)} dt \\ &= \frac{1}{2\pi}\left(\int_{-T}^{T} \frac{a}{a^2+t^2} dt - i\int_{-T}^{T} \frac{t}{a^2+t^2} dt\right). \end{aligned}$$

Now, from the eveness of the first integral and the oddness of the second integral, and on taking out a factor of $1/a^2$, we find that

$$\frac{1}{2\pi}\left(\int_{-T}^{T} \frac{a}{a^2+t^2} dt - i\int_{-\infty}^{\infty} \frac{t}{a^2+t^2} dt\right) = \frac{1}{\pi a}\int_{0}^{T} \frac{1}{1+t^2/a^2} dt.$$

We then let $y = t/a$, to get

$$\frac{1}{\pi a}\int_{0}^{T} \frac{1}{1+t^2/a^2} dt = \frac{1}{\pi}\int_{0}^{T/a} \frac{1}{1+y^2} dy,$$

and recognise that the last integral above is $\arctan(T/a)$.

Therefore on letting $T \to \infty$, we find that

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{1}{s} ds \to \frac{1}{2}.$$

---

With $x > 1$ we begin by making the contour closed with a particular semicircular piece. We illustrate this in Figure 2.1. The integrand of (2.2) has one pole in this contour, namely $s = 0$. Since $\operatorname{Res}\left(\frac{x^s}{s}; 0\right) = 1$, Cauchy's Theorem gives that

$$\frac{1}{2\pi i}\int_{\tilde{C_1}+\tilde{C_2}} \frac{x^s}{s} ds = 1.$$

Now, if the contribution made to this integral by $\tilde{C_2}$ is zero in the limit with $T \to \infty$ we are done. We prove that this is the case as follows.

Since $s = a + Te^{i\theta}$ with $\theta$ running from $\pi/2$ to $3\pi/2$ and $\varepsilon$ being as small as we want

$$\left|\frac{1}{2\pi i}\int_{\tilde{C_2}} \frac{x^s}{s} ds\right| = O\Bigg(x^a\Bigg(\int_{\pi/2}^{\pi/2+\varepsilon} x^{T\cos\theta} d\theta + \int_{\pi/2+\varepsilon}^{3\pi/2-\varepsilon} x^{T\cos\theta} d\theta + \int_{3\pi/2-\varepsilon}^{3\pi/2} x^{T\cos\theta} d\theta\Bigg)\Bigg). \tag{2.3}$$

Now, since

$$\int_{\pi/2+\varepsilon}^{3\pi/2-\varepsilon} x^{T\cos\theta} d\theta \leq \int_{\pi/2+\varepsilon}^{3\pi/2-\varepsilon} x^{T\cos(\pi/2+\varepsilon)} d\theta,$$

and

$$\int_{\pi/2}^{\pi/2+\varepsilon} x^{T\cos\theta} d\theta \leq \int_{\pi/2}^{\pi/2+\varepsilon} d\theta$$

(similarly for the last integral) we get that (2.3) is bounded by a multiple of

$$x^a\left(2\varepsilon + (\pi - 2\varepsilon)x^{-T\sin\varepsilon}\right).$$

Then, with $T \to \infty$ and $\varepsilon$ being as small as we want, we find that

$$\left|\frac{1}{2\pi i}\int_{\tilde{C_2}} \frac{x^s}{s} ds\right| \to 0$$

with $x > 1$.

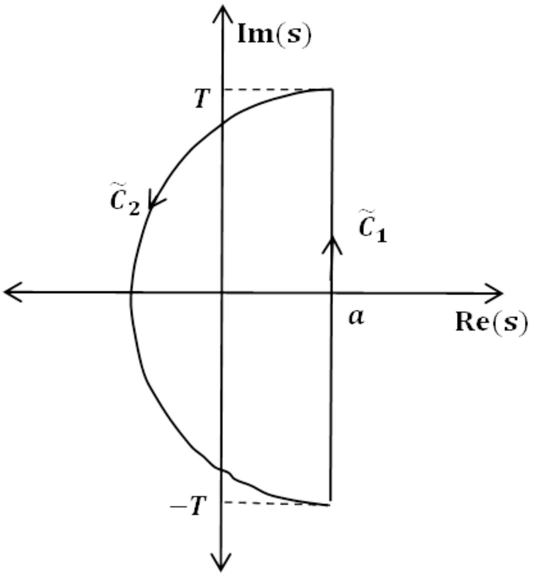

Figure 2.1: The Lemma 12 ($x > 1$) contour.

---

With $0 < x < 1$ we also begin by making the contour closed with a particular semicirclular piece. We illustrate this in Figure 2.2. The integrand of (2.2) does not have any poles within the contour, and so, Cauchy's Theorem gives that the integral is zero, namely

$$\frac{1}{2\pi i}\int_{\tilde{C}_1+\tilde{C}_2}\frac{x^s}{s}ds = 0.$$

Now, if the contribution made to this integral by $\tilde{C}_2$ is zero in the limit with $T \to \infty$ we are done. This is proved in a way analogous to the above. □

We now prove Perron's formula. This next theorem corresponds to Exercise 4.1.5 in [17] which is solved on pages 307–308 of [17].

**Theorem 8.** *We set* $\Phi(x) = \sum_{n\leq x}\phi(n)$ *and have* $\phi(n)$ *being arithmetic. Then with* $\sum_{n=1}^{\infty}\phi(n)/n^s$ *being uniformly convergent if* $\mathrm{Re}(s) > \tilde{\alpha}$ *we have that*

$$\Phi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\right)\frac{x^s}{s}\,ds,\ a > \tilde{\alpha}. \tag{2.4}$$

*We also specify that* $x \notin \mathbb{N}$ *to avoid possible points of discontinuity for* $\Phi(x)$.

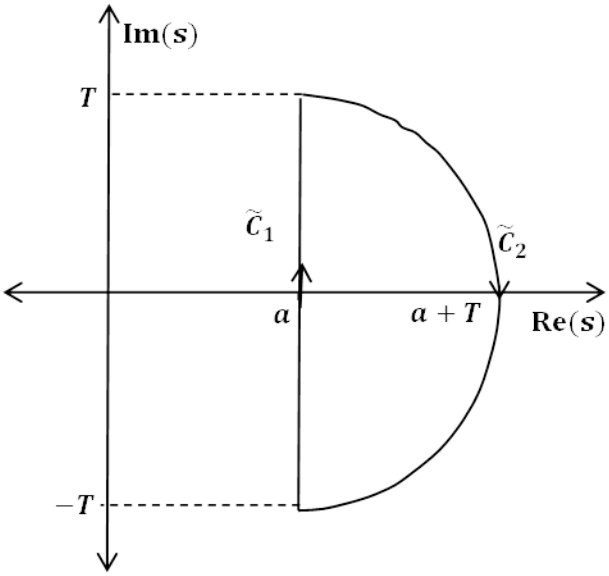

Figure 2.2: The Lemma 12 ($0 < x < 1$) contour.

*Proof.* We begin with $\Phi(x) = \sum_{x \leq n} \phi(n)$ and proceed to obtain the right-hand side of (2.4).

We know that

$$\sum_{n \leq x} \phi(n) = \sum_{n=1}^{\infty} \phi(n) \delta\left(\frac{x}{n}\right) \tag{2.5}$$

because for $n < x$ we have $\delta(x/n) = 1$ and $x \notin \mathbb{N}$. Then, from Lemma 12, (2.5) becomes

$$\frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{a-i\infty}^{a+i\infty} \frac{\phi(n)}{n^s} \frac{x^s}{s} ds.$$

Now, the theorem involves $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$ being uniformly convergent, thus, taking the sum into the integral is allowed. Whence we get Perron's formula. $\square$

The contour involved in (2.4) is a straight line running from $-\infty$ to $\infty$ through $a$ and is illustrated in Figure 2.3.

We now give Perron's formula for $\psi(x)$ and $M(x)$.

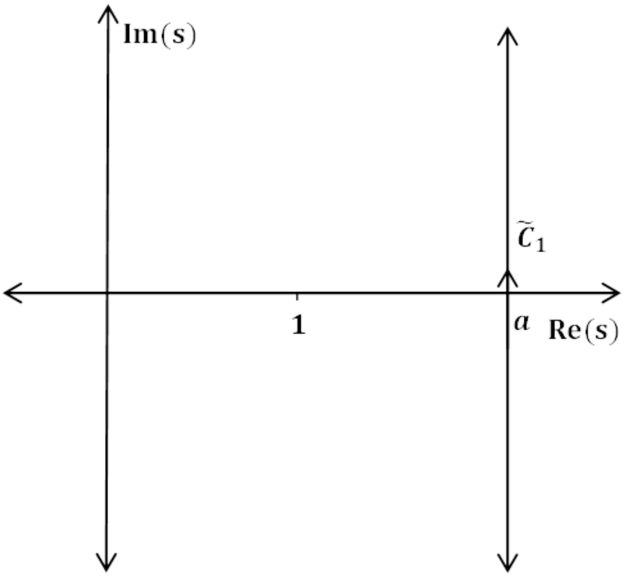

Figure 2.3: The Perron's formula contour.

**Corollary 1.** *We have*

$$\psi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} -\frac{\zeta'}{\zeta}(s)\ ds,\ a > 1.$$

*and*

$$M(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{\zeta(s)}\ ds,\ a > 1.$$

*Proof.* A straightforward application of Theorem 1, 2 and 8. ☐

## 2.2 Altering Perron's formula

In this section we make the contour involved in Perron's formula finite. The following lemma carries most of the weight in making this alteration.

**Lemma 13.** *When $x \neq 1$ we have*

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{x^s}{s} ds - \delta(x)\right| = O\left(x^a \min\left\{1, T^{-1}|\log x|^{-1}\right\}\right),\ x, a, T > 0.$$

*Proof.* See Theorem 4.1.4 on pages 54–56 of [17]. ☐

As a quick side-note, in Lemma 13 the implied constant $1/\pi$ applies if we choose $T^{-1}|\log x/n|^{-1}$ as the minimum and $1/2$ otherwise.

In the next theorem we alter Perron's formula by making the contour it involves finite.

**Theorem 9.** *We have*

$$\Phi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\right)\frac{x^s}{s}ds \\ + O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a|\phi(n)|\min\{1, T^{-1}|\log x/n|^{-1}\}\right). \tag{2.6}$$

*Proof.* We begin with

$$\left|\frac{1}{2\pi i}\left(\int_{a-iT}^{a+iT} - \int_{a-i\infty}^{a+i\infty}\left(\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\right)\frac{x^s}{s}ds\right)\right|. \tag{2.7}$$

Now, $\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}$ being uniformly convergent means taking the sum outside the integral is allowed and we get that (2.7) is bounded above by

$$\sum_{n=1}^{\infty}|\phi(n)|\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{(x/n)^s}{s}ds - \delta(x/n)\right|.$$

Hence we have Theorem 9 due to Lemma 13. □

We illustrate the contour involved in (2.6) in Figure 2.4.

We now apply Theorem 9 to $\psi(x)$ and $M(x)$.

**Corollary 2.** *We have*

$$\psi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds \\ + O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a\Lambda(n)\min\{1, T^{-1}|\log x/n|^{-1}\}\right)$$

*and*

$$\mu(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{1}{\zeta(s)}\frac{x^s}{s}ds \\ + O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a|\mu(n)|\min\{1, T^{-1}|\log x/n|^{-1}\}\right).$$

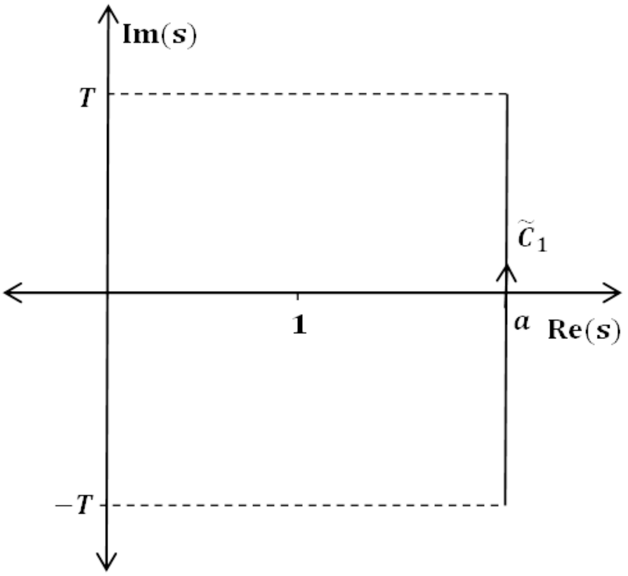

Figure 2.4: The finite contour.

## 2.3 A bird's-eye view of the approach

In this section we preview the approach we use to prove a bound for $\psi(x)-x$ in Chapter 3 and for $M(x)$ in Chapter 4. In either case there are two steps, which we summarize as follows.

***Step* 1: Close the contour.**

We add three pieces to the contour involved in (2.6) of Theorem 9 (see page 23) that make it closed. We add two horizontal pieces $\tilde{C_2}$ and $\tilde{C_4}$ that are $\mathrm{Im}(s) = T$ and $\mathrm{Im}(s) = -T$ for $b \leq \mathrm{Re}(s) \leq a,\ b > 0$, and a vertical piece $\tilde{C_3}$ that satisfies $\mathrm{Re}(s) = b := 1 - c_1/\log T$.

In Section 1.4 we explained that, under an appropriate choice of $c_1$ and with $\mathrm{Im}(s) \geq \tilde{t}$ for some $\tilde{t}$

$$\mathrm{Re}(s) = 1 - \frac{c_1}{\log(\mathrm{Im}(s))}$$

is the boundary of a zero-free region for $\zeta(s)$. We also mentioned in Section 1.4 that Platt found in [21, p. 1532] that all zeros $\rho = \beta + i\gamma$ with $0 < \beta < 1$ and $|\gamma| \leq 3.06 \times 10^{10}$ satisfy the RH. Thus, as long as we make an appropriate choice of $c_1, \tilde{t}$ and $T$ we can make the closed contour surround part of a zero-free region for $\zeta(s)$ — see Section 1.4 for an overview of zero-free regions.

We illustrate the contour after this alteration in Figure 2.5 along with the zero-free region.

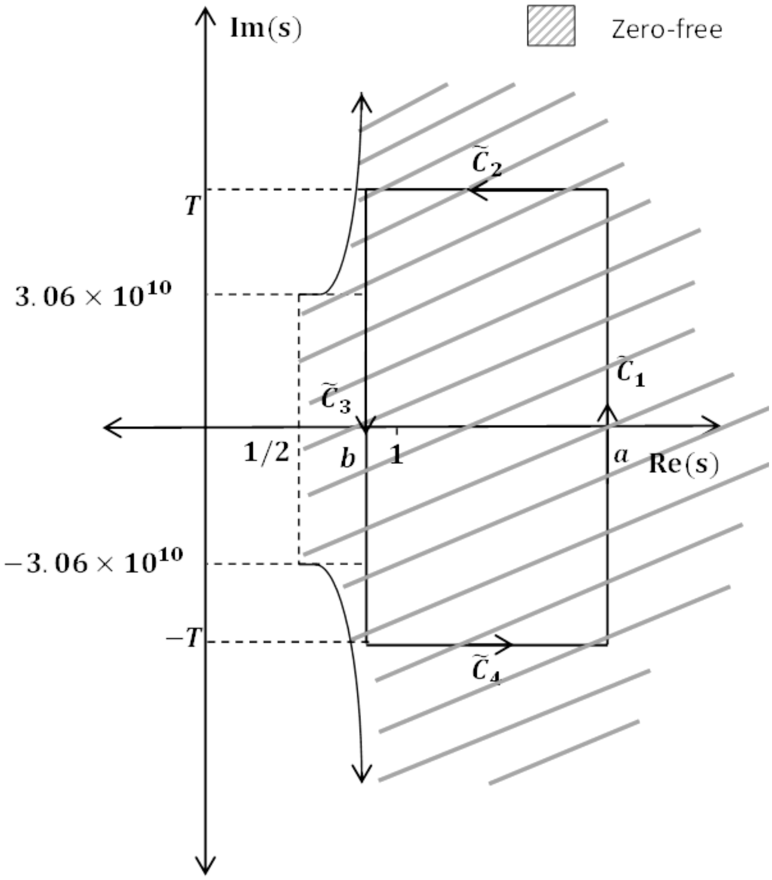

Figure 2.5: The closed contour.

Since we have closed the contour, we may use Cauchy's Theorem to get a value for the integral. If we denote the sum of the residues of the integrand at any poles that are inside the contour by $R$, (2.6) becomes

$$\begin{aligned}\Phi(x) - R = &-\frac{1}{2\pi i}\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}}\left(\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\right)\frac{x^s}{s}ds\\ &+ O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a|\phi(n)|\min\{1, T^{-1}|\log x/n|^{-1}\}\right).\end{aligned}$$

***Step* 2: Find bounds.**

We now find bounds for

$$-\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\right)\frac{x^s}{s}ds \text{ and } \sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a|\phi(n)|\min\{1,T^{-1}|\log x/n|^{-1}\}.$$

---

Since it is possible to bound both terms on the right-hand side of (2.6), a natural question to ask at this step is:

*Why do we bother to make the contour closed?*

We shall give an answer to this question in Chapter 5.

The two steps described above outline the approach we use to prove a bound for each of $\psi(x)-x$ and $M(x)$. However, these same steps are implementable for many sums involving arithmetic functions.

# Chapter 3

# Finding a bound for $\psi(x) - x$

In Section 3.1 of this chapter we consider the history of bounding $\psi(x) - x$ and we give a proof of an implicit bound in Section 3.1.

## 3.1 A history of the problem

In this section we shall give an exploration of the history of bounding $\psi(x)-x$.

Firstly, in Theorem 10 of [31, p. 337] Schoenfeld gives an explicit bound that we would have if and only if someone was to prove the RH. This bound is of the kind

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2 x.$$

with $x \geq 73.2$. Moreover, just by the Prime Number Theorem ($\psi(x) \sim x$) we have that

$$\psi(x) - x = o(x).$$

Hence, we know that any small $\alpha_1$ will be valid in the bound

$$|\psi(x) - x| \leq \alpha_1 x, \ x \geq \alpha_2 \tag{3.1}$$

on making $\alpha_2$ big enough. With this in mind we now explore what has been done without assuming the RH.

The starting point of the remainder of our exploration is the *explicit formula* for $\psi(x)$,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right). \tag{3.2}$$

Intriguingly, one may obtain (3.2) by essentially following what we described in Chapter 2. However, a crucial difference is that one does not make the contour surround a zero-free region when obtaining (3.2) (see [2, Chapter 17]). It may be of interest to see [2, Chapter 18] in which the bound we prove in Section 3.2 is arrived at using (3.2), and [5, Chapter 2] in which the bound

$$|\psi(x) - x| = -\sum_{|\operatorname{Im}(s)|<T} \frac{x^\rho}{\rho} + O\left(\frac{2x\log^2 x}{T}\right)$$

with $x > e^{60}$, $50 < T < x$ is found for $x$ that are halfway between two natural numbers.

Thus far, the history of finding a bound for $\psi(x) - x$ heavily relies on an approach that uses (3.2) or a variation of (3.2). In particular, the approach involves finding an average of (3.2). This appears on page 220 of [27] is

$$\frac{\int_0^h dy_1 \int_0^h dy_2 \ldots \int_0^h \phi(x + y_1 + y_2 + \ldots + y_m) dy_m}{h} + \frac{1}{2}nh^a - zh^{a-1} \tag{3.3}$$

with

$$\phi(x) = \psi(x) - x + \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

We should not concern ourselves with the second and third terms in (3.3) in this case. Rather, we should just recognise that the first term is a sum (in multiple variables) over an interval from 0 to $h$, all divided by the length of the interval $h$. In other words, and roughly speaking, it is an average. This average is then used to find a bound for $\psi(x) - x$.

The paper in which (3.3) appears (namely [27]) and its precurser [26] are the publications that mark the start of the line of enquiry we have introduced above. Other papers that have been published since include [28], [29], [31], [6], [7], [18], [9], [37] and [8]. Advancements in the known zero-free regions of $\zeta(s)$ have acted as key drivers amongst these papers.

It is worth noting that in [9]

$$\mathscr{S}(x) = \sum_{n=1}^{\infty} \Lambda(n) f(n/x)$$

is used. This function is like $\psi(x)$, but with the sum being infinite and weighted by $f$. The authors of [9] express that the use of $\mathscr{S}$ is new, and that it makes for a generalisation of the approach that involves determining an average. Another new choice made in [9] is the use of $N(\sigma_0, T)$ instead of $N(T)$. Both of these functions are sums over a subset of the zeros of $\zeta(s)$ satisfying $0 \leq \mathrm{Im}(s) \leq T$. The zeros summed over in $N(\sigma_0, T)$ also satisfy $\sigma_0 \leq \mathrm{Re}(s) \leq 1$, while those summed over in $N(T)$ satisfy $0 < \mathrm{Re}(s) < 1$.

Another point worth noting is the observation in [8] that extending the height to which the RH is known does not play a very significant role in advancing the bounds. In particular, page 876 of Dusart's paper contains the statement

> The goal of this paper is to show that if the zero-free region has the form (called the de la Vallée Poussin form of the zero-free region) ... then there is no real need of the computable constant $A$ of the Riemann Hypothesis verification ....

This observation of Dusart's is very much apparent in the two tables of bounds for $\psi(x) - x$ on page 887 of his paper. Table 1 is calculated using Platt's work [21] and Table 2 is calculated using Gourdon's work [12]. We recall that Platt found that all zeros of $\zeta(s)$ satisfy the RH when the height of the zero (the imaginary part) is less than some multiple of $10^{10}$. Gourdon, on the other hand found that this is the case when the height of the zero is some multiple of $10^{12}$. However, the Table 1 and 2 bounds are no different whe $x \geq \exp(8000)$.

There is a rather useful Table 1 in [9, p. 2] that collects together various bounds for $\psi(x) - x$. There is a bound of the kind given in (3.1) that is relevant when $x \geq e^{50}$ from each of the papers [27], [28], [29], [6], [7], [18] and [9]. This table also lists the $c_1$ from the classical zero-free region that was used in finding the bounds and a $H$ value. This $H$ value is some height to which the RH is known.

The recent paper by Büthe [1] also presents $\psi(x) - x$ bounds. One of the bounds Büthe lists in Table 1 is

$$|\psi(x) - x| \leq 1.16465 \times 10^{-9} x, \quad (x \geq e^{50}).$$

In comparison, the strongest bound in Table 1 of [9, p. 2] is $2.3643 \times 10^{-9} x$.

With such numerous work on finding bounds for $\psi(x) - x$ already having appeared in the literature one can understand why any endeavour of ours to make the bound we prove in the next section explicit would not end in obtaining anything stronger than what is already known.

## 3.2 An implicit bound

In this section we prove an implicit bound for $\psi(x) - x$. The proof we give is based on the proof of Theorem 4.2.9 in [17, pp. 60–62] with more of the details being included. We have presented a bird's-eye view of the approach we use in Section 2.3.

In his proof of Theorem 4.2.9 Murty finds a weaker bound for $\psi(x) - x$ than the one we find below by employing results that correspond to a zero-free region for $\zeta(s)$ that is smaller than the one we use. Namely, the part of the complex plane satisfying

$$\operatorname{Re}(s) \geq 1 - \frac{c_1}{\log^9 \operatorname{Im}(s)}, \text{ and } \operatorname{Im}(s) \geq \tilde{t}. \tag{3.4}$$

We follow most of the decisions made by Murty in proving the following.

**Theorem 10.** *We have*

$$\psi(x) - x = O(x \exp(-c\sqrt{\log x})), \text{ for some } c > 0, \tag{3.5}$$

*for $x$ that are halfway between two natural numbers.*

We shall prove Theorem 10 in four parts. In Part 1 (Section 3.2.1) we outline some preliminary manipulations one must do. In Parts 2 and 3 (Sections 3.2.2 and 3.2.3) we explain how to find two particular bounds from which we determine our final bound. In Part 4 (Section 3.2.4) we find the final bound. We follow the majority of Murty's notation. We also use some notation that does not appear in Murty's proof.

### 3.2.1 Part 1 of the proof: preliminary manipulations

In this section we consider Perron's formula (see Corollary 1 in Section 2.1),

$$\psi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds, \ a > 1. \tag{3.6}$$

This formula cannot be used when $x \in \mathbb{N}$ because such values are possible points of discontinuity of $\psi(x) = \sum_{n \leq x} \Lambda(n)$. Murty decides to consider $x$ that are halfway between two natural numbers. We shall follow suit with Murty (i.e. $x := [x] + 1/2, \ x > 1$). At the end of this section we shall prove that a bound of this kind may also be used for any positive integer.

The contour involved in (3.6) is a straight line running from $-\infty$ to $\infty$ through $a$ and is illustrated in Figure 2.3 (see page 22). In this part of the proof we make two alterations to this Figure 2.3 contour. These two alterations give a bound for $\psi(x)-x$ and are illustrated in Figure 2.5 (see page 25).

The effect of the first alteration is given in Corrolary 2. This corollary makes the contour finite by defining its endpoints to be $a-iT$ and $a+iT$ for a fixed $T$. Making this alteration introduces an error into the formula for $\psi(x)$.

For the second alteration we add three pieces to the contour, $\tilde{C_2}, \tilde{C_3}$, and $\tilde{C_4}$. (See Step 1 in Section 2.3 for a description of $\tilde{C_2}, \tilde{C_3}$, and $\tilde{C_4}$.) Making this alteration corresponds to adding and subtracting

$$\frac{1}{2\pi i}\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds$$

to the formula for $\psi(x)$. This alteration makes the contour closed, which means Cauchy's Theorem will give a value for the integral

$$\int_{a-iT}^{a+iT} + \int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds. \tag{3.7}$$

This value is determined from the residue of the integrand of (3.7) at the pole $s=1$ (see Lemma 5 on page 8). The integrand also has poles at $s=0$ and any $s \in \mathbb{C}$ that is a zero of $\zeta(s)$ (see Lemma 6 on page 8). However, since these poles are not found in the contour, the residue at $s=1$ is all we require. Thus, from our knowledge that

$$\operatorname{Res}\left(\frac{\zeta'}{\zeta}(s), 1\right) = -1$$

(see Lemma 5 on page 8), (3.7) is

$$\begin{aligned}
-2\pi i \times \operatorname{Res}\left(\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}; 1\right) &= -2\pi i \times \lim_{s\to 1}(s-1)\left(\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s} \\
&= -2x\pi i \times \operatorname{Res}\left(\frac{\zeta'}{\zeta}(s), 1\right) \\
&= 2x\pi i.
\end{aligned}$$

With these alterations (3.6) becomes

$$\begin{aligned}
\psi(x) - x = &-\frac{1}{2\pi i}\left(\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\right) \\
&+ O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a \Lambda(n)\min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\}\right).
\end{aligned} \tag{3.8}$$

The $x$ in (3.8) results from the previous residue calculation and the two expressions on the right are due to the two alterations to the contour.

One now has the task of finding an upper bound for

$$-\frac{1}{2\pi i}\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a \Lambda(n)\min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\}.$$

This shall be our task in the next two sections.

### 3.2.2 Part 2 of the proof: a first step in bounding

In this part of the proof we find a bound for

$$\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a \Lambda(n)\min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\}.$$

Consider first the $|\log x/n|^{-1}$ factor. This factor is very large if $x/n$ gets close to one. To accommodate, we split the sum into three; the sums over $n < x/2$, $n > 3x/2$ and $x/2 \leq n \leq 3x/2$; $|\log x/n|^{-1}$ will not be large in the first two of these regions.

We begin with the sum over $n > 3x/2$,

$$\sum_{n>3x/2}\left(\frac{x}{n}\right)^a \Lambda(n)\min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\}.$$

For this sum $n/x > 3/2$, and so $|\log x/n|^{-1} < (\log 3/2)^{-1}$. Thus,

$$\min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\} < \frac{1}{T}\left(\log\frac{3}{2}\right)^{-1}$$

which implies that

$$\sum_{n>3x/2}\left(\frac{x}{n}\right)^a \Lambda(n)\min\left\{1, T^{-1}\left|\log\frac{x}{n}\right|^{-1}\right\} = O\Big(\sum_{n>3x/2}\left(\frac{x}{n}\right)^a \Lambda(n)\frac{1}{T}\Big). \quad (3.9)$$

Moreover,

$$\sum_{n>3x/2}\left(\frac{x}{n}\right)^a \Lambda(n)\frac{1}{T} \leq \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^a}\frac{x^a}{T}$$

and we have stated that $a > 1$ (see (3.6)), thus

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a} \frac{x^a}{T} = -\frac{\zeta'(a)}{\zeta(a)} \frac{x^a}{T}$$

(see Theorem 1 on page 5). Furthermore,

$$\left| -\frac{\zeta'(a)}{\zeta(a)} \frac{x^a}{T} \right| < \frac{1}{(a-1)} \frac{x^a}{T}$$

by Lemma 11 on page 15. We decide to make $a := 1 + c_1/\log T$. Therefore, (3.9) can be written as

$$O\left( \log T \frac{x^{1+c_1/\log T}}{T} \right). \tag{3.10}$$

The bound for the sum over $n < x/2$ will be identical to (3.10) apart from the implied constant.

We now find a bound for

$$\sum_{x/2 \le n \le 3x/2} \left( \frac{x}{n} \right)^a \Lambda(n) \min\left\{ 1, \frac{1}{T} \left| \log \frac{x}{n} \right|^{-1} \right\}. \tag{3.11}$$

Firstly,

$$\begin{aligned} \left| \log \frac{x}{n} \right| &= \left| -\log \frac{n}{x} \right| \\ &= \left| -\log\left( 1 - \left( 1 - \frac{n}{x} \right) \right) \right|. \end{aligned} \tag{3.12}$$

Moreover, since $-\log(1-(1-n/x))$ and $(1-n/x)$ have identical signs for all $x/2 \le n \le 3x/2$

$$\left| \frac{\log \frac{x}{n}}{1 - n/x} \right| = \frac{-\log(1-(1-n/x))}{(1-n/x)} \tag{3.13}$$

because of (3.12). We also find that, for all $x/2 \le n \le 3x/2$ and $z = 1 - n/x$, $-\log(1-z)/z$ has a positive derivative. Thus, as $-1/2 \le 1 - n/x \le 1/2$, the minimum of (3.13) is found when $1 - n/x = -1/2$. Therefore,

$$\frac{-\log(1-(1-n/x))}{(1-n/x)} > \frac{3}{4},$$

and so

$$\left|\log \frac{x}{n}\right|^{-1} < \frac{4}{3}\frac{x}{|x-n|}. \tag{3.14}$$

Hence,

$$\begin{aligned}\sum_{x/2 \le n \le 3x/2} \left(\frac{x}{n}\right)^a \Lambda(n) \min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\} \\ = O\left(\sum_{x/2 \le n \le 3x/2} \left(\frac{x}{n}\right)^a \Lambda(n) \frac{x}{T}\frac{1}{|x-n|}\right).\end{aligned} \tag{3.15}$$

Continuing on, as $x/2 \le n \le 3x/2$ we have that

$$\begin{aligned}\left(\frac{x}{n}\right)^a \Lambda(n)\frac{x}{T} &\le 2^a \log\left(\frac{3x}{2}\right)\frac{x}{T} \\ &= O\left(\frac{x}{T}\log x\right).\end{aligned} \tag{3.16}$$

We now use (3.16) in (3.15); all that remains is to simplify

$$\begin{aligned}\sum_{x/2 \le n \le 3x/2} \frac{1}{|x-n|} &= \sum_{x/2 \le n \le x} \frac{1}{|x-n|} + \sum_{x \le n \le 3x/2} \frac{1}{|x-n|} \\ &= 2\sum_{x \le n \le 3x/2} \frac{1}{|x-n|} \\ &= O\left(\frac{1}{|x-([x]+1)|} + \ldots + \frac{1}{|x-[3x/2]|}\right)\end{aligned}$$

since we are only considering $x = [x] + 1/2$. Also, with $m := [3x/2] - [x]$ we get

$$\begin{aligned}\frac{1}{|x-([x]+1)|} + \frac{1}{|x-([x]+2)|} + \ldots + \frac{1}{|x-[3x/2]|} \\ = \frac{1}{1/2} + \frac{1}{1/2+1} + \ldots + \frac{1}{1/2+m-1}.\end{aligned}$$

Now, our immediate aim is to find $m-1$. To achieve this we begin with our definition for $m$, which implies that $[x] + m \le 3/2([x] + 1/2)$. From this, we find that $m - 1 \le [x]/2 - 1/4$. We shall use $m - 1 \le [x]/2$, which implies that

$$\frac{1}{1/2} + \frac{1}{1/2+1} + \ldots + \frac{1}{1/2+m-1} = O\left(\sum_{n \le [x]+1} \frac{1}{n}\right).$$

Then, keeping with the standard notation $\gamma$ for *Euler's constant*, (2.2) in [35, p. 261] gives

$$\begin{aligned}\sum_{n\leq[x]+1}\frac{1}{n}&\leq\log([x]+1)+\gamma+\frac{1}{[x]+1}\\&=O(\log[x])\\&=O(\log x).\end{aligned}\tag{3.17}$$

We could have actually arrived at (3.17) in a quicker way. However, the ease at which the extra detail is found (as above) illustrates that turning the implicit bound into an explicit one would be a very much achievable task.

The final bound we have for (3.11) is

$$O\left(\frac{x\log^2 x}{T}\right).\tag{3.18}$$

With (3.10) and (3.18), (3.8) becomes

$$\begin{aligned}\psi(x)-x=-\frac{1}{2\pi i}&\left(\int_{\tilde{C}_2+\tilde{c}_2+\tilde{c}_3}-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\right)\\&\qquad+O\left(\log T\frac{x^a}{T}+\frac{x\log^2 x}{T}\right).\end{aligned}\tag{3.19}$$

### 3.2.3 Part 3 of the proof: a second step in bounding

In this part of the proof we find a bound for

$$\frac{1}{2\pi i}\int_{\tilde{C}_2+\tilde{C}_3+\tilde{C}_4}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds.$$

We do this by finding a bound for the integrals over $\tilde{C}_2$, $\tilde{C}_3$ and $\tilde{C}_4$ separately, then invoking the triangle inequality to combine these bounds.

Firstly,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\tilde{C}_2}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\right|&=\left|\frac{1}{2\pi i}\int_{a+iT}^{b+iT}\frac{\zeta'}{\zeta}(\sigma+iT)\frac{x^{\sigma+iT}}{\sigma+iT}ds\right|\\&\leq\frac{1}{2\pi}\int_b^a\left|\frac{\zeta'}{\zeta}(\sigma+iT)\right|\frac{|x^{\sigma+iT}|}{|\sigma+iT|}d\sigma.\end{aligned}\tag{3.20}$$

With the fact that $|x^{\sigma+iT}| = x^{\sigma}$ and the upper bound $\frac{\zeta'}{\zeta}(\sigma + iT) = O(\log T)$ (see Theorem 7 on page 14) (3.20) is

$$O\left(\frac{\log Tx^{a}}{T\log x}\right). \tag{3.21}$$

The bound for the integral over $\tilde{C_4}$ will be identical to (3.21) and is computed in the same way.

Secondly,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\tilde{C_3}}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\right| &= \left|\frac{1}{2\pi i}\int_{b+iT}^{b-iT}\frac{\zeta'}{\zeta}(b+it)\frac{x^{b+it}}{b+it}ds\right| \\ &\leq \frac{1}{\pi}\int_0^T\left|\frac{\zeta'}{\zeta}(b+it)\right|\frac{x^b}{|b+it|}dt. \end{aligned}\tag{3.22}$$

The change in bounds is possible because $\frac{\zeta'}{\zeta}(s)$ follows the Schwarz Reflection Principle (see Lemma 3 on page 7). Hence, we just calculate one half of the integral and double it. We now consider (3.22) as

$$\frac{1}{\pi}\left(\int_0^{\tilde{t}} + \int_{\tilde{t}}^T\left|\frac{\zeta'}{\zeta}(b+it)\right|\frac{x^b}{|b+it|}dt\right). \tag{3.23}$$

and find a bound for each individual integral in (3.23). There are two reasons for introducing $\tilde{t}$ and breaking the integral up like this. Firstly, we shall be applying two different bounds for $\frac{\zeta'}{\zeta}(s)$ to this integral because one of the bounds from Section 1.5 cannot be use when $0 < \mathrm{Im}(s) < \tilde{t}$ (namely Theorem 6 on page 14). Secondly, if we bounded straight from (3.22) it would be natural to use that $\frac{1}{|b+it|} \leq \frac{1}{t}$ since $t$ is the dominating term in $1/|b+it|$. However, this would result in integrating $1/t$ over an interval starting at 0 where it blows up.

With the bound $\frac{\zeta'}{\zeta}(b+it) = O(\log T)$ (see Theorem 7 on page 14) we have,

$$\begin{aligned}\frac{1}{\pi}\int_{\tilde{t}}^T\left|\frac{\zeta'}{\zeta}(b+it)\right|\frac{x^b}{|b+it|}dt &= O\left(\int_{\tilde{t}}^T\log T\frac{x^b}{t}dt\right) \\ &= O(x^b\log^2 T). \end{aligned}\tag{3.24}$$

With the bound $\frac{\zeta'}{\zeta}(b+it) = O(\frac{1}{|b+it-1|}) = O(\log T)$ (see Lemma 9 on page 14) we have,

$$\begin{aligned}\frac{1}{\pi}\int_0^{\tilde{t}}\left|\frac{\zeta'}{\zeta}(b+it)\right|\frac{x^b}{|b+it|}dt &= O\left(\int_0^{\tilde{t}}\frac{\log T x^b}{b}dt\right)\\ &= O\left(\frac{\log T x^b}{b}\right).\end{aligned}$$

Since $1/b = O(1)$, we sweep this bound under that of (3.24).

Therefore,

$$\left|\frac{1}{2\pi i}\int_{\tilde{C_3}}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\right| = O(x^b\log^2 T). \tag{3.25}$$

With (3.21) and (3.25)

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\\ &\qquad = O\left(\log T\frac{x^{1+c_1/\log T}}{T} + x^{1-c_1/\log T}\log^2 T\right)\end{aligned}$$

and (3.19) becomes

$$\psi(x) - x = O\left(\log T\frac{x^a}{T} + \frac{\log T x^a}{T\log x} + x^b\log^2 T + \frac{x\log^2 x}{T}\right). \tag{3.26}$$

### 3.2.4 Part 4 of the proof: the implicit bound

We now consider (3.26) to determine a final bound for $\psi(x) - x$.

Our aim is to minimize (3.26) with respect to $x$ using an appropriate $T$. A natural way of achieving this is to use the derivative of the right-hand side of (3.26). We shall not use such a technique because the derivative involved will be unwieldy. Instead we use the fact that we have three terms

$$\log T\frac{x^{1+c_1/\log T}}{T}, \tag{3.27}$$

$$\frac{x\log^2 x}{T} \tag{3.28}$$

and

$$\frac{\log T x^a}{T\log x} \tag{3.29}$$

that are decreasing in $T$, and one term

$$x^{1-c_1/\log T} \log^2 T \tag{3.30}$$

that is increasing in $T$. We shall find the $T$ for which one of the decreasing terms balances out the increasing term.

Firstly, we attempt to find the $T$ for which (3.27) balances out (3.30). We have

$$\log T \frac{x^{1+c_1/\log T}}{T} = x^{1-c_1/\log T} \log^2 T$$

which gives

$$x^{2c_1/\log T} = T \log T.$$

We then take logarithms to obtain

$$2c_1 \log x = \log^2 T + \log T \log\log T. \tag{3.31}$$

Secondly, we attempt to find the $T$ for which (3.28) balances out (3.30). We have

$$\frac{(\log x)^2 x}{T} = x^{1-c_1/\log T} \log^2 T,$$

which gives

$$\frac{1}{T \log^2 T} = \frac{x^{-c_1/\log T}}{(\log x)^2}.$$

We then take logarithms to obtain

$$c_1 \log x = \log^2 T + \log T \log\log^2 T - 2 \log T \log\log x. \tag{3.32}$$

Thirdly, we attempt to find the $T$ for which (3.29) balances out (3.30). We have

$$\frac{\log T x^{1+c_1/\log T}}{T \log x} = x^{1-c_1/\log T} \log^2 T,$$

which gives

$$x^{2c_1/\log T} = T \log T \log x.$$

We then take logarithms to obtain

$$2c_1 \log x = \log^2 T + \log T \log\log T + \log T \log\log x. \tag{3.33}$$

The equation

$$2c_1 \log x = \log^2 T \tag{3.34}$$

is close to being (3.31), (3.32) and (3.33). We shall use (3.34) to obtain our definition for $T$

$$T := \exp(\sqrt{2c_1 \log x}). \tag{3.35}$$

Since the definition for $T$ is completely up to our judgement either this is a suitable choice.

We now use (3.35) in each of the four terms on the left-hand side of (3.26) to determine a final bound for $\psi(x) - x$.

Firstly,

$$\begin{aligned}\frac{x^{1+c_1/\log T}\log T}{T} &= \frac{x^{1+c_1/\sqrt{2c_1\log x}}\sqrt{2c_1\log x}}{\exp\sqrt{2c_1\log x}}\\ &= x\exp\left(\left(-\sqrt{2c_1}+\sqrt{\frac{c_1}{2}}\right)\sqrt{\log x}+\log\sqrt{2c_1\log x}\right)\\ &\leq x\exp\left(\left(-\sqrt{\frac{c_1}{2}}+c_2\right)\sqrt{\log x}\right), \end{aligned} \tag{3.36}$$

using the fact that $\log\sqrt{2c_1\log x} \leq c_2\sqrt{\log x}$ for some $c_2$, conditional on $x$ being big enough.

Secondly

$$\begin{aligned}\frac{\log T x^{1+c_1/\log T}}{T\log x} &= \frac{\sqrt{2c_1\log x}x^{1+c_1/\sqrt{2c_1\log x}}}{\exp(\sqrt{2c_1\log x})\log x}\\ &= x\exp\Bigg(\left(-\sqrt{2c_1}+\sqrt{\frac{c_1}{2}}\right)\sqrt{\log x}\\ &\qquad\qquad +\log\sqrt{2c_1\log x}-\log\log x\Bigg)\\ &\leq x\exp\left(\left(-\sqrt{\frac{c_1}{2}}+c_2\right)\sqrt{\log x}\right).\end{aligned}$$

Thirdly,

$$\begin{aligned}x^{1-c_1/\log T}\log^2 T &= x^{1-c_1/\sqrt{2c_1\log x}}(2c_1\log x)\\ &= x\exp\left(-\sqrt{\frac{c_1}{2}}\sqrt{\log x}+\log(2c_1\log x)\right)\\ &\leq x\exp\left(\left(-\sqrt{\frac{c_1}{2}}+c_3\right)\sqrt{\log x}\right).\end{aligned} \tag{3.37}$$

Lastly,

$$\begin{aligned}\frac{(\log x)^2 x}{T} &= \frac{(\log x)^2 x}{\exp(\sqrt{2c_1 \log x})} \\ &= x \exp\left(-\sqrt{2c_1}\sqrt{\log x} + \log(\log x)^2\right) \\ &\leq x \exp\left(\left(-\sqrt{2c_1} + c_4\right)\sqrt{\log x}\right). \qquad (3.38)\end{aligned}$$

We set our definitions for $c_2, c_3$ and $c_4$ in Table 5.1 in the Appendix. By these definitions, one can see that we may have $c_2, c_3$ and $c_4$ being as small as we like, conditional on $x$ being big enough.

Therefore, our final upper bound is

$$\psi(x) - x = O\left(x \exp(-c\sqrt{\log x})\right) \qquad (3.39)$$

with

$$c = \min\left\{\sqrt{\frac{c_1}{2}} - c_2, \sqrt{\frac{c_1}{2}} - c_3, \right\} = \sqrt{\frac{c_1}{2}} - \varepsilon, \qquad (3.40)$$

and $\varepsilon$ being as small as we want conditional on having a big enough $x$.

In (3.39) and (3.40) one is able to see the dependence of our bound on the zero free regions of $\zeta(s)$ since the $c_1$ that is used to specify such a region appears in $c$. As $c$ gets larger (3.39) gets smaller.

□

We now provide the promised extension to Theorem 10 that may be used when $x$ is a positive integer.

**Corollary 3.** *We have*

$$\psi(x) - x = O(x \exp(-\tilde{c}\sqrt{\log x})), \text{ for some } \tilde{c} > 0$$

*for positive integer $x$.*

*Proof.* Since we only considered $x = [x] + 1/2,\ x > 1$ in the previous proof, $x - 1/2$ is a positive integer. Hence, by extending our bound, we mean we require a bound for $|\psi(x - 1/2) - x + 1/2|$. We obtain this bound as follows.

We begin with

$$|\psi(x - 1/2) - x + 1/2| \leq |\psi(x - 1/2) - \psi(x)| + |\psi(x) - x| + 1/2. \qquad (3.41)$$

The first absolute value on the right is equal to

$$\sum_{n\leq x-1/2}\Lambda(n)-\sum_{n\leq x}\Lambda(n)=\Lambda(x-1/2).$$

Now, $\Lambda(x-1/2)\leq\log(x-1/2)$. Thus, (3.41) is bounded by a multiple of

$$\log(x-1/2)+x\exp(-c\sqrt{\log x}).$$

If we now change our definition for $x$ so that it is a positive integer, the above bound can be re-written as

$$\psi(x)-x=O(\log x+(x+1/2)\exp(-c\sqrt{\log(x+1/2)})).$$

Then, since

$$\sqrt{\log(x+1/2)}\geq\sqrt{\log x}$$

with $x\geq 1$ and

$$1/2\exp\left(-c\sqrt{\log x}\right)=O\left(x\exp(-c\sqrt{\log x})\right)$$

we get

$$\psi(x)-x=O\left(\log x+x\exp(-c\sqrt{\log x})\right).$$

Lastly, as $\log x=\exp(\log\log x)$ we have

$$\psi(x)-x=O\left(x\exp\left(-c\sqrt{\log x}+\log\log x\right)\right)=O\left(x\exp\left(-\tilde{c}\sqrt{\log x}\right)\right).$$

$\square$

# Chapter 4

# Finding a bound for $M(x)$

In this chapter we consider the history of bounding $M(x)$ in Section 4.1. Then, in Section 4.2 we find an implicit bound.

## 4.1 A history of the problem

In this section we shall give an exploration of the history of bounding $M(x)$.

Firstly, in [32] on page 141 Soundararajan finds an implicit bound that we would have if the Riemann Hypothesis (RH) was proven, namely

$$M(x) = O\left(x^{1/2}\exp\left(\sqrt{\log x}(\log\log x)^{14}\right)\right).$$

This is a bound of the kind

$$M(x) = O\left(x^{1/2+\varepsilon}\right).$$

Moreover, without using up too much energy and definitely not the RH, one can prove the trivial bound

$$|M(x)| \leq x \tag{4.1}$$

and by the Prime Number Theorem we have that

$$M(x) = o(x).$$

(see the Introduction). Hence, the $x$ on the right-hand side of (4.1) should certainly have a coefficient that is smaller than one. With this in mind, we now explore what has been done without assuming the RH.

A nice starting point for the remainder of our exploration is the website [23] by Ramaré. This website lists various explicit bounds for $M(x)$, but also bounds for the similar function $m(x) = \sum_{n\leq x} \mu(n)/n$. There are three kinds of bounds amongst the ones presented for $M(x)$ on this website. Two of them are

$$|M(x)| \leq \alpha_1 x + \alpha_2 \tag{4.2}$$

with $x \geq \alpha_3$, and

$$|M(x)| \leq \frac{\alpha_1 x}{(\log x)^{\alpha_2}} \tag{4.3}$$

again with $x \geq \alpha_3$. In [25, p. 1359] Ramaré states that "two distinct paths of inquiries have been used for the summatory function $M(x)$ ...." We shall discuss each of these paths below.

In [25, p. 1359] Ramaré attributes one such path to Chebyshev. When referring to this path, Ramaré cites [15], which is a short errata to [14]. Two bounds published by MacLeod in [14] are amongst those listed in [23]. These are

$$|M(x)| < \frac{x}{80} \tag{4.4}$$

with $x \geq 1119$ and

$$|M(x)| < \frac{x}{80} + 5 \tag{4.5}$$

with $x \geq 0$. Another paper Ramaré refers to in [25, p. 1359] when mentioning Chebyshev's idea is [19] by Costa Pereira. A bound from this paper also appears in [23], namely,

$$|M(x)| < \frac{x}{1036} \tag{4.6}$$

with $x \geq 120727$. Each of the bounds (4.4) (4.5) and (4.6) are of the kind given in (4.2). In [23] the bounds listed that are of this kind start with a bound by von Sterneck [39]

$$|M(x)| \leq \frac{1}{9}x + 8$$

with $x \geq 0$ and end with a bound by Cohen et al [4, Theorem 5 bis.], namely

$$|M(x)| \leq \frac{x}{4345}$$

with $x \geq 2160535$.

It is interesting that the other path referred to by Ramaré in [25, p. 1359] that has been used to find bounds for $M(x)$ involves $\psi(x)$. Ramaré attributes this idea to Landau. When referring to this path, Ramaré cites an 1969 paper by Schoenfeld [30] in which the bound

$$|M(x)| \leq \frac{2.9x}{\log x}, \tag{4.7}$$

was proved. This bound is of the kind given in (4.3) and is an improvement on the bounds of the kind given in (4.2). That is, there is some $x$ beyond which this bound outperforms a given bund of the kind in (4.2). To illustrate, on comparing (4.4) and (4.7), we can quickly confirm that for any $x > \exp(232)$ the bound (4.7) is superior to (4.4).

Schoenfeld's paper also presents bounds with a smaller power of $\log x$ in the denominator. For example,

$$|M(x)| < \frac{5.3x}{(\log x)^{10/9}}$$

with $x > 1$.

In addition to referring to Schoenfeld's 1969 paper in [25], on page 1360 Ramaré also refers to one of his own papers, that is [24]. Theorem 1.1 in [24] presents the bound

$$|M(x)| \leq \frac{0.0130 \log x - 0.118}{(\log x)^2} x$$

with $x \geq 1\ 078\ 853$.

The bound we prove in the next section outperforms any bound of the kind given in (4.2) and (4.3) beyond some $x$. This gives motivation for proving such an implicit bound and, furthermore, should motivate any future energies one puts into making this bound explicit.

## 4.2 An implicit bound

We now prove a theorem similar to Theorem 10 (see page 30) for $M(x) = \sum_{n \leq x} \mu(x)$. Namely,

**Theorem 11.** *We have*

$$M(x) = O\left(x \exp(-c\sqrt{\log x})\right), \text{ for some } c > 0, \tag{4.8}$$

*for $x$ that are halfway between two natural numbers.*

As with Theorem 10 we shall prove Theorem 11 in four parts. These four parts correspond to Parts 1–4 (Sections 3.2.1–3.2.4) of the Theorem 10 proof.

### 4.2.1 Part 1 of the proof: preliminary manipulations

We first consider Perron's formula (see Corollary 1 in Section 2.1),

$$M(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{\zeta(s)}\frac{x^s}{s}ds,\ a > 1. \tag{4.9}$$

As in Section 3.2.1 we only consider $x$ that are halfway between two natural numbers (i.e. $x := [x] + 1/2,\ x > 1$). At the end of this section we shall prove that a bound of this kind may also be used for any positive integer.

The contour involved in (4.9) is the one illustrated in Figure 2.3 on page 22. We make this contour finite and closed as we did in the Theorem 10 proof for $\psi(x) - x$.

The integrand of (4.9) does have a pole at $s = 0$ and any $s \in \mathbb{C}$ that is a zero of $\zeta(s)$ (see Lemma 6 on page 8 for this latter fact). However, these poles are not in the contour. Hence,

$$\begin{aligned} M(x) = -\frac{1}{2\pi i}&\left(\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} \frac{1}{\zeta(s)}\frac{x^s}{s}ds\right) \\ &+ O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a |\mu(n)| \min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\}\right). \end{aligned} \tag{4.10}$$

### 4.2.2 Part 2 of the proof: A first step in bounding

We now find a bound for

$$\sum_{n=1}^{\infty}(x/n)^a|\mu(n)| \min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\}.$$

We again split the sum as we did in the Theorem 10 proof. It is not necessary to split the sum in exactly this way. In fact, in his Exercise 4.3.2 solution [17, pp. 314–315] Murty splits the sum in a different way. A point of future consideration could be to investigate the effect of splitting this sum differently.

Beginning with the sum over $n > 3x/2$, we have

$$\sum_{n>3x/2}^{\infty}\left(\frac{x}{n}\right)^a |\mu(n)| \min\left\{1, \frac{1}{T}\left|\log\frac{x}{n}\right|^{-1}\right\} = O\Bigg(\sum_{n>3x/2}\left(\frac{x}{n}\right)^a \frac{1}{T}\Bigg). \tag{4.11}$$

since $|\mu(n)| \leq 1$ and $\min\{1, T^{-1}|\log x/n|^{-1}\} < T^{-1}(\log 3/2)^{-1}$. Moreover,

$$\sum_{n>3x/2} \left(\frac{x}{n}\right)^a \frac{1}{T} \leq \sum_{n=1}^{\infty} \frac{1}{n^a} \frac{x^a}{T}$$

and for (4.8) we stated that $a > 1$, thus

$$\sum_{n=1}^{\infty} \frac{1}{n^a} \frac{x^a}{T} = \zeta(a) \frac{x^a}{T}.$$

Furthermore,

$$\zeta(a) \frac{x^a}{T} = O\left(\frac{1}{(a-1)} \frac{x^a}{T}\right)$$

because of (2.1.16) on page 16 in [34]. Then with $a := 1 + c_1/\log T$, (4.11) is

$$O\left(\log T \frac{x^{1+c_1/\log T}}{T}\right). \tag{4.12}$$

The bound for the sum over $n < x/2$ is identical to (4.12), apart from the implied constant.

---

Now for the other sum,

$$\sum_{x/2 \leq n \leq 3x/2} \left(\frac{x}{n}\right)^a |\mu(n)| \min\left\{1, \frac{1}{T}\left|\log \frac{x}{n}\right|^{-1}\right\}$$
$$= O\left(\sum_{x/2 \leq n \leq 3x/2} \left(\frac{x}{n}\right)^a \frac{x}{T} \frac{1}{|x-n|}\right). \tag{4.13}$$

because $|\mu(n)| \leq 1$ and $|\log x/n|^{-1} < \frac{4x}{3|x-n|}$ — see (3.14) on page 34.

As $2/3 \leq x/n \leq 2$, we have

$$\left(\frac{x}{n}\right)^a \frac{x}{T} \leq 2^a \frac{x}{T} = O\left(\frac{x}{T}\right).$$

Then, since

$$\sum_{x/2 \leq n \leq 3x/2} \frac{1}{|x-n|} = O(\log x)$$

from (3.17) on page 32, (4.13) is

$$O\left(\frac{x\log x}{T}\right). \tag{4.14}$$

With (4.12) and (4.14), (4.10) becomes

$$M(x) = -\frac{1}{2\pi i}\left(\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}}\frac{1}{\zeta(s)}ds\right) + O\left(\log T\frac{x^{1+c_1/\log T}}{T}+\frac{x\log x}{T}\right). \tag{4.15}$$

---

In Exercise 4.3.4 in [17] (which is solved on pages 316–317) Murty gives a weaker bound for $M(x)$ than the one we are proving in this chapter by employing results that correspond to a zero-free region for $\zeta(s)$ that is smaller than the one we use. However, for this part of his proof Murty ends up with

$$\frac{x^{a+\varepsilon}}{T}$$

instead of the two terms

$$\frac{\log Tx^a}{T} \quad \text{and} \quad \frac{x\log x}{T}$$

that we have in (4.15) above. This difference comes Exercise 4.3.2 (which is solved on pages 314–315). One could adopt this different bound, but we shall proceed with the expressions we have proved above.

### 4.2.3 Part 3 of the proof: A second step in bounding

Next, we find a bound for

$$-\frac{1}{2\pi i}\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}}\frac{1}{\zeta(s)}\frac{x^s}{s}ds.$$

Firstly,

$$\left|-\frac{1}{2\pi i}\int_{\tilde{C_2}}\frac{1}{\zeta(s)}\frac{x^s}{s}ds\right| = O\left(\frac{\log Tx^a}{T\log x}\right). \tag{4.16}$$

The bound for the integral over $\tilde{C}_4$ is identical to (4.16). These bounds are computed in the same way as the corresponding bound in the proof of Theorem 10 (see page 36).

Secondly,

$$\left|-\frac{1}{2\pi i}\int_{\tilde{C}_3}\frac{1}{\zeta(s)}\frac{x^s}{s}ds\right| \le \frac{1}{\pi}\left(\int_0^{\tilde{t}}+\int_{\tilde{t}}^{T}\left|\frac{1}{\zeta}(b+it)\right|\frac{x^b}{|b+it|}dt\right).$$

The change in bounds is possible because $1/\zeta(s)$ follows the Schwarz Reflection Principle (see Lemma 3 on page 7). With the bound $\frac{1}{\zeta(b+it)} = O(\log T)$ (see Theorem 7 on page 14) we have that

$$\frac{1}{\pi}\int_{\tilde{t}}^{T}\left|\frac{1}{\zeta(b+it)}\right|\frac{x^b}{|b+it|}dt = O(x^b\log^2 T). \tag{4.17}$$

Also, with the upper bound $\frac{1}{\zeta(b+it)} = O(1)$ (see Lemma 10 on page 15) we have that

$$\begin{aligned}\frac{1}{\pi}\int_0^{\tilde{t}}\left|\frac{1}{\zeta(b+it)}\right|\frac{x^b}{|b+it|}dt &= O\left(\int_0^{\tilde{t}}\frac{x^b}{b}dt\right)\\ &= O(x^b). \end{aligned}\tag{4.18}$$

As (4.18) is bounded by a multiple of $x^b\log^2 T$ we sweep this bound under that of (4.17).

Therefore,

$$\left|-\frac{1}{2\pi i}\int_{\tilde{C}_3}\frac{1}{\zeta(s)}\frac{x^s}{s}ds\right| = O(x^b\log^2 T). \tag{4.19}$$

With (4.16), and (4.19) we have

$$M(x) = O\left(\log T\frac{x^a}{T}+\frac{\log Tx^a}{T\log x}+x^b\log^2 T+\frac{x\log x}{T}\right). \tag{4.20}$$

### 4.2.4 Part 4 of the proof: the final upper bound

We now determine the final bound for $M(x)$.

We aim to minimize (4.20) with respect to $x$ using an appropriate $T$. We use the fact that we have three terms

$$\log T\frac{x^{1+c_1/\log T}}{T}, \tag{4.21}$$

$$\frac{x\log x}{T}, \tag{4.22}$$

and

$$\frac{\log Tx^a}{T\log x} \tag{4.23}$$

that are decreasing in $T$, and one term

$$\log^2 Tx^{1-c_1/\log T} \tag{4.24}$$

that is increasing in $T$. Of (4.21)–(4.24) above, it is only (4.22) that did not appear in Section 3.2.4 of Theorem 10 proof. On finding the $T$ for which (4.22) balances out (4.24) we get

$$c_1\log x = \log^2 T + 2\log T\log\log T - \log T\log\log x.$$

Using an equation that is almost identical to (3.34) in Section 3.2.4, we define $T$ to be

$$T := \exp(\sqrt{c_5c_1\log x}). \tag{4.25}$$

Using $c_5$ instead of 2 in this definition gives more freedom if we were to make this bound explicit.

We now use (4.25) in each of the terms on the right-hand side of (4.20).

The following three bounds are computed in the same way as the corresponding bounds in the proof of Theorem 10 (see Section 3.2.4) except that we have $c_5$ involved instead of 2. We have

$$\frac{x^{1+c_1/\log T}\log T}{T} \le x\exp\left(\left((1-c_5)\sqrt{\frac{c_1}{c_5}}+c_6\right)\sqrt{\log x}\right), \tag{4.26}$$

$$\frac{\log Tx^{1+c_1/\log T}}{T\log x} \le x\exp\left(\left((1-c_5)\sqrt{\frac{c_1}{c_5}}+c_6\right)\sqrt{\log x}\right), \tag{4.27}$$

and

$$x^{1-c_1/\log T}\log^2 T \le x\exp\left(\left(-\sqrt{\frac{c_1}{c_5}}+c_7\right)\sqrt{\log x}\right). \tag{4.28}$$

The bound on (4.22) is

$$\begin{aligned}\frac{x\log x}{T} &= \frac{x\log x}{\exp(\sqrt{c_5c_1\log x})}\\ &= x\exp\left(-\sqrt{c_5c_1}\sqrt{\log x}+\log\log x\right)\\ &\le x\exp\left((-\sqrt{c_5c_1}+c_8)\sqrt{\log x}\right) \end{aligned}\tag{4.29}$$

using the fact that $\log(\log x) \leq c_8\sqrt{\log x}$ for some $c_8$, conditional on $x$ being big enough.

See Table 5.1 of Appendix 2 for our definition of $c_6, c_7$ and $c_8$. By these definitions we may have $c_6, c_7$ and $c_8$ being as small as we like, conditional on having a big enough $x$. Also, we see in (4.26) and (4.27) that $c_5 > 1$ is necessary so that the argument of the exponentials is negative when $x$ is big enough.

Therefore, our final upper bound is

$$\sum_{n\leq x} \mu(n) = O\left(x\exp(-c\sqrt{\log x})\right) \tag{4.30}$$

with

$$c = \min\left\{(c_5-1)\sqrt{\frac{c_1}{c_5}} - c_8, \sqrt{\frac{c_1}{c_5}} - c_6\right\}. \tag{4.31}$$

As $c$ gets larger in (4.30), the bound gets smaller. We can see in (4.31) that this $c$ depends on $c_1$, which is used to specify a zero free regions of $\zeta(s)$.

□

We now provide the promised extension to Theorem 11 that may be used when $x$ is a positive integer.

**Corollary 4.** *We have*

$$M(x) = O\left(x\exp\left(-\tilde{c}\sqrt{\log x}\right)\right), \ \textit{for some } \tilde{c} > 0$$

*for integer $x$.*

*Proof.* Since $x = [x] + 1/2,\ x > 1$, we know that $x - 1/2$ is a positive integer. Hence, we require a bound for $|M(x-1/2)|$. We begin with

$$|M(x-1/2)| \leq |M(x-1/2) - M(x)| + |M(x)| + 1/2.$$

The first absolute value on the right is equal to

$$\sum_{n\leq x-1/2} \mu(n) - \sum_{n\leq x}\mu(n) = \mu(x-1/2). \tag{4.32}$$

Now, $|\mu(x-1/2)| \leq 1$. Thus, (4.32) is bounded by a multiple of

$$3/2 + x\exp(-c\sqrt{\log x}).$$

If we now consider $x$ to be a positive integer we have

$$M(x) = O(x\exp(-\tilde{c}\sqrt{\log x})).$$

□

# Chapter 5

# Looking back at the approach

In this chapter we look back at the approach we have just used to bound $\psi(x)-x$ and $M(x)$ to answer a particular question and consider what could be further explored.

## 5.1 A pertinent question about bounding

Before we make the contour involved in Perron's formula closed we have

$$\begin{aligned}\Phi(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}\right)\frac{x^s}{s}ds \\ +O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^a|\phi(n)|\min\{1,T^{-1}|\log x/n|^{-1}\}\right) \qquad (5.1)\end{aligned}$$

from Theorem 9 in Section 2.2. The integral on the right-hand side of (5.1) looks very much like the integrals we end up bounding in Part 3 of the proofs of Theorems 10 and 11 after making the contour closed. Thus, a very natural question to ask is:

*Why do we bother to make the contour closed?*

In this section we give an answer to this question.

For $\psi(x)$ a straight forward answer is that closing the contour provides the $x$ we need to bound the error in the Prime Number Theorem approximation (which is $\psi(x) \sim x$). However, an answer like this is not apparent in the case of $M(x)$. Another answer that is relevant to both $\psi(x)-x$ and $M(x)$ is that bounding (5.1) does not provide a good result. We explain why this is below using $M(x)$ as an example.

Making the contour closed in the Theorem 11 proof just swaps the integral

$$\int_{a-iT}^{a+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds$$

for the integral

$$\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} \frac{1}{\zeta(s)} ds.$$

Thus, the question becomes:

*How do the bounds on these two integrals compare?*

In the proof of Theorem 11 we obtained the bound

$$\int_{\tilde{C_2}+\tilde{C_3}+\tilde{C_4}} \frac{1}{\zeta(s)} ds = O\left(\frac{\log T x^a}{T \log x} + x^b \log^2 T\right). \tag{5.2}$$

Alternatively,

$$\int_{a-iT}^{a+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds = O(x^a \log^2 T), \tag{5.3}$$

which we obtain by following the kind of process used to bound the integral over $\tilde{C_3}$ (see Part 3 of the Theorem 11 proof). As $a > 1$ and $b < 1$, (5.3) is weaker than (5.2). It is also clear that $M(x) \leq x$ (see (4.1) on page 43). Hence, closing the contour makes sense because without this alteration to the contour even the trivial bound (4.1) outruns the bound we get here.

## 5.2 Suggestions for further exploration

In this section we comment on ways in which the work in this thesis could be continued.

Although the implicit bound we proved for $M(x)$ is not new, we did not come across any explicit bounds of this form in the literature. Moreover, Ramaré expresses that an explicit bound has not been found using the kind of approach we use.

> No one has yet obtained an explicit error term for the function $M$ from the Mellin transform/Perron formula machinery . . . [25, p. 1359].

He also comments that “The implied constants are, however, expected to be too large for any decent use.” However, the bound we give from [36] in Theorem 7 would not have been known to Ramaré when he made this statement. With this result, finding an explicit bound using the approach we have used for $M(x)$ is definitely of high regard. We have included details in the Appendix that would assist one in doing so. If one was to make this bound explicit the next task would be to find out how it performs alongside the bounds in the literature.

We commented in Section 1.4 that one could use the zero-free region discovered by Vinogradov and Korobov rather than the classical one to find a bound. One can see from bounds by Ford and Walfisz we present in Section 1.4 that using this zero-free region has a profound effect on the bound. Incorporating the zero-free region discovered by Vinogradov and Korobov into the approach we have used would be a valuable exercise.

There are also a number of aspects of Part 2 of the proofs of the bounds for $\psi(x) - x$ and $M(x)$ (see Sections 3.2.2 and 4.2.2) that could be further investigated. Firstly, in this part of the proof we split the sum that is involved into three sums over $n < x/2,\ n > 3x/2$ and $x/2 \leq n \leq 3x/2$. The choice to split the sum into these three sums in particular is not necessary; one could make a different decision on how to split this sum. It would be interesting to determine what effect such a change has.

Secondly, in the proof of Theorem 11 we used the trivial bound $|\mu(n)| \leq 1$ to bound

$$\sum \left(\frac{x}{n}\right)^a |\mu(n)| \min\left\{1, 1/T \left|\log \frac{x}{n}\right|^{-1}\right\}.$$

This is certainly not the optimal choice one could make since it is known that $\sum_{n \leq x} |\mu(n)|/x \sim 6/\pi^2$.

Thirdly, at the end of Part 2 of the proof of Theorem 11 we commented on a different bound Murty used. One could go back and incorporate this bound instead of the ones we used.

# Conclusion

We now summarize the main ideas of this thesis.

The focal point of this work is the use of a particular approach to prove implicit bounds for

$$\psi(x) - x = \sum_{n\leq x} \Lambda(n) - x$$

involving the von Mangoldt function $\Lambda(n)$ and

$$M(x) = \sum_{n\leq x} \mu(n)$$

involving the Möbius function $\mu(n)$. Both $\Lambda(n)$ and $\mu(n)$ are arithmetic functions (namely $\Lambda(n) : \mathbb{N} \to \mathbb{C}$ and $\mu(n) : \mathbb{N} \to \mathbb{C}$). Moreover, the well-known Prime Number Theorem is directly related to the sums $\psi(x) - x$ and $M(x)$. The PNT can be written as $\psi(x) \sim x$ (see Prime Number Theorem 2 on page 2) and by the PNT we have that $M(x) = o(x)$ (see the Introduction). Hence, we may use the PNT as motivation for the problem of proving bounds for these sums.

There is an interesting connection between the Riemann zeta-function $\zeta(s)$ and both $\Lambda(n)$ and $\mu(n)$, namely

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) \text{ and } \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

if $\mathrm{Re}(s) > 1$. This connection provides the opportunity to use information about $\zeta(s)$ to investigate $\Lambda(n)$ and $\mu(n)$. We stated this connection in Section 1.1 of Chapter 1, then in the remainder of Chapter 1 we went on to give the particular facts we need about $\zeta(s)$.

In Chapter 2 we firstly proved Perron's formula, that is

$$\Phi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left( \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \right) \frac{x^s}{s} \, ds, \ a > \tilde{\alpha}$$

with $\Phi(x) = \sum_{n \le x} \phi(n)$ (see (2.1) on page 17). We altered this formula in Section 2.2. Perron's formula is part of the heart of the approach we use to bound $\psi(x) - x$ and $M(x)$. It essentially takes in information about the zeta-function and gives back information about either of these sums.

After familiarizing ourselves with Perron's formula, we then previewed the approach we were to use to prove bounds (see Section 2.3). Once we have Perron's formula for $\psi(x) - x$ and $M(x)$ and have made the contour involved finite, this approach involves closing the contour, then finding bounds for what remains.

In exploring the history of bounding $\psi(x) - x$ in Section 3.1 of Chapter 3 we found that it is the explicit formula for $\psi(x)$,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log \left( 1 - \frac{1}{x^2} \right),$$

that is one of the foundations of the work that has already been done. In Section 3.2 we then proved the implicit bound

$$\psi(x) - x = O\left( x \exp(-c\sqrt{\log x}) \right), \text{ for some } c > 0.$$

This implicit bound is not new and even if we were to make it explicit using this approach, we would not arrive at any advancement on bounds that have already been obtained.

In exploring the history of bounding $M(x)$ in Section 4.1 of Chapter 4 we listed three kinds of bounds that others have obtained. We also explained that there are two ways such bounds for $M(x)$ have been found. Interestingly, one of these approaches involves $\psi(x)$. In Section 4.2 we then proved the implicit bound

$$M(x) = O\left( x \exp\left( -c\sqrt{\log x} \right) \right), \text{ for some } c > 0.$$

As with $\psi(x) - x$, this implicit bound is not new, but our exploration of the literature did not run into any explicit bounds of this form. We were unable to make this bound explicit because of the limited time we had. However, this task forms the basis of potential future work.

# Appendix

In this appendix we give details that would assist one in making the bounds proved in Chapters 3 and 4 explicit. Doing so for $\psi(x) - x$ would not result in any advancement, but we expect that such an explicit bound for $M(x)$ would be new.

## Appendix 1: on finding an explicit bound

Below we summarize the five step process for turning the bounds we proved into explicit results.

***Step*** **1: Recover any constants and bounds that were swept aside.**

For example, there are various instances in the proofs of Theorems 10 and 11 (see Chapter 3 and 4) when we use bounds of the form $h(x) = O(f(x))$. The implied constant does not appear in these bounds. For the explicit bound we need such implied constants.

Also, on page 49 we swept the bound in (4.18) (which is a multiple of $x^b$) under that of (4.17) (which is a multiple of $x^b \log^2 T$). The explicit bound may be tighter if we keep such bounds separate.

***Step*** **2: Write the bound out explicitly.**

For example, if we have three terms at the start of Part 4 of our implicit proof the explicit bound would look like

$$\begin{aligned} |M(x)| \leq \alpha_1 x \exp(-\tilde{\alpha_1}\sqrt{\log x}) + \alpha_2 x \exp(-\tilde{\alpha_2}\sqrt{\log x}) \\ + \alpha_3 x \exp\left(-\tilde{\alpha_3}\sqrt{\log x}\right) \end{aligned} \tag{5.4}$$

for some $\alpha_i$ and $\tilde{\alpha_i}$ (with $i = 1, 2, 3$).

***Step* 3: Rearrange the bound.**

Namely, we put the bound in the form

$$|M(x)| \leq C(x) x \exp\left(-c\sqrt{\log x}\right).$$

With reference to the example of Step 2, this involves doing the following.

1. Determine which of the three terms in (5.4) dominates. Since the $\tilde{\alpha_i}$s will have the greatest effect on the behaviour of the bound, this involves calculating
$$c = \min\{\tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\alpha_3}\}.$$
2. Bunch all coefficients and terms, except for the one that dominates, into $C(x)$. For example, if $c = \tilde{\alpha_1}$ for (5.4) we would get
$$C(x) = \alpha_1 + \alpha_2 \exp\left((-\tilde{\alpha_2} + \tilde{\alpha_1})\sqrt{\log x}\right) + \alpha_3 \exp\left((-\tilde{\alpha_3} + \tilde{\alpha_1})\sqrt{\log x}\right).$$

The $C(x)$ in this example illustrates that as $x$ gets larger, $C(x)$ and in turn the bound, gets smaller.

***Step* 4: Substitute the values we have for the constants involved.**

For example, we would substitute the actual value of $c_1$ wherever it appears in the bound. We can get the value of $c_1$ from Theorems 6 and 7 in Chapter 1, namely $c_1 = 1/9$.

***Step* 5: Plug the bound into a computer.**

We want to know how the bound performs alongside the bounds in the literature, so we calculate it for various choices of $x$. On comparing what we get to bounds of the kind given in (4.2) and (4.3) there will be some $x$ beyond which the bound we proved outperforms these other bounds. The aim at this step is to locate such an $x$.

# Appendix 2: definitions

In Table 5.1 below we list definitions for the $c_i$ (with $i = 1, 2, 3 \ldots, 8$) that appear in the proofs of Theorems 10 and 11 of Chapter 3 and 4.

| Constant | Description |
|---|---|
| $c_1$ | A constant that is first used in (3.4) on page 30. We use it to specify a zero-free region of $\zeta(s)$ and it satisfies $0 < c_1 < 1$. Later this constant makes its way into our definition for parameters $a$ and $b$. |
| $c_2$ | A parameter introduced in (3.36) on page 39 that satisfies $\log\sqrt{2c_1\log x} \le c_2\sqrt{\log x}$. We set<br>$$c_2 := \frac{\log\sqrt{2c_1\log x}}{\sqrt{\log x}},$$<br>conditional on $-\sqrt{c_1/2} + c_2 < 0$. |
| $c_3$ | A parameter introduced in (3.37) on page 39 that satisfies $\log(2c_1\log x) \le c_3\sqrt{\log x}$. We set<br>$$c_3 := \frac{\log(2c_1\log x)}{\sqrt{\log x}}$$<br>conditional on $-\sqrt{c_1/2} + c_3 < 0$. |
| $c_4$ | A parameter introduced in (3.38) on page 40 that satisfies $2\log\log x \le c_4\sqrt{\log x}$. We set<br>$$c_4 := \frac{2\log\log x}{\sqrt{\log x}}$$<br>conditional on $-\sqrt{2c_1} + c_4 < 0$. |
| $c_5$ | A constant introduced in our definition for $T$ in (4.25) on page 50, $c_5 > 1$ |

Table 5.1: Descriptions of the constants $c_1$–$c_7$.

| **Constant** | **Description** |
|---|---|
| $c_6$ | A parameter introduced in (4.26) on page 50 that satisfies $\log\sqrt{c_5 c_1 \log x} \le c_6\sqrt{\log x}$. We set<br>$$c_6 := \frac{\log\sqrt{c_5 c_1 \log x}}{\sqrt{\log x}},$$<br>conditional on $(1-c_5)\sqrt{\frac{c_1}{c_5}} + c_6 < 0$. |
| $c_7$ | A parameter introduced in (4.28) on page 50 that satisfies $\log(c_5 c_1 \log x) \le c_7\sqrt{\log x}$. We set<br>$$c_7 := \frac{\log(c_5 c_1 \log x)}{\sqrt{\log x}}$$<br>conditional on $-\sqrt{\frac{c_1}{c_5}} + c_7 < 0$. |
| $c_8$ | A parameter introduced in (4.29) on page 50 that satisfies $\log\log x \le c_8\sqrt{\log x}$. We set<br>$$c_8 = \frac{c_4}{2},$$<br>conditional on $-\sqrt{c_5 c_1} + c_8 < 0$. |

Table 5.1: Descriptions of the constants $c_1$–$c_7$.

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