Singularity-free Cosmological Models

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Abstract

Singularity-free cosmological models are a recent advance in the field of exact solutions. Previously thought not to exist, these models are globally hyperbolic and obey the strong energy condition everywhere. We describe and summarize the history of these models and their general physical features. The methods by which the singularity-free models evade the general and powerful singularity theorems are discussed. Finally realistic generalisations of the singularity-free models are proposed and examined. Such singularity-free but FLRW-like models may yield a classical way in which to evade the Big Bang singularity of the standard cosmological model.

Declaration of Originality

I declare that the contents of this thesis, except where referenced is the original work of the author.

Michael John Siew Lueng Ashley
November 13, 1996

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Contents

List of Figures

Chapter 1

Introduction

Exact solutions of the Einstein equation can be broadly catagorised into two types. There are:

1. Astrophysical solutions (i.e. the Reissner-Nordström and Kerr space-times).

2. Cosmological solutions (i.e. the Gödel and Friedmann universes).

In the field of exact solutions there are many spacetimes which solve the Einstein equation but are not physically acceptable. Such a spacetime model may propose the existence of unknown forms of matter or simply conflict with experimental evidence. Many of these aphysical solutions exist but are useful only from viewpoint of understanding what types of behaviour Einstein's equations allow. However, there are many complicated exact solutions which are physically useful, and are good approximations to realistic physical situations.

Of the cosmological models to date the most commonly accepted solutions are the well known *Friedmann-Lemaître-Robertson-Walker* (FLRW) family of cosmologies. This family of solutions represent a class of *maximally symmetric perfect fluid* models which have the properties of homogeneity and isotropy. Homogeneity ensures that at any time in the universe the physical conditions at one point in the universe are the same as at any other. Similarly isotropy guarantees that by any local measurement no one direction can be singled out from any other. Particular FLRW models are able to explain for the most part the large-scale observable features of our universe. Apart from the general homogeneity of the universe and the isotropy of the background radiation, the recession of distant galaxies and subsequent red-shift as observed by Hubble is also predicted by this model. These many agreements with observational evidence were seen as a great triumph for this model. However in the late 1960's with the acceptance of the Hawking-Penrose singularity theorems it became increasingly apparent that the existence of a Big-Bang (initial) singularity in these models was unavoidable.

The existence of singularities, cosmological or otherwise, leads to problems of much more complexity than those caused by singularities found in other field theories. Under the conditions of gravitational collapse or in the case of a highly dense primordial universe, the effective length scales of important physical processes enter the quantum regime. Thus describing the physics of these situations requires the use of some form of quantum gravity theory . However present attempts to produce a theory of quantum gravity have yielded few direct results. Initially it was thought that the singularities were due to the high symmetry of exact solutions known at that time. However the singularity theorems, which are topological and quite general in nature, demonstrated that singularities occur in general physical situations and that if singularities were to be avoided extensive changes to Einstein's theory had to be made.

Many theorists motivated by this have appealed to alternate theories of gravity in an attempt to remove cosmological singularities (Jha [1]). The question remains whether singularity-free cosmological solutions exist within the bounds of classical gravity. Indeed exact global solutions have been found which are singularity-free. However the matter in these models may be not be physically reasonable (i.e. obey the strong energy condition). One intriguing example of this is that of a flat vacuum spacetime which continuously develops into a curved space[1] (see Bonnor [2]). Initially it was believed that singularity-free cosmologies did not exist since nobody had found any. Furthermore the singularity theorems seemed to support this. However in 1990, Senovilla [3], found a cosmological model whose curvature invariants were all regular and the matter obeyed the strong energy condition. Later it was determined that the models were in fact null and timelike geodesically complete and obeyed the most stringent of the causality conditions, global hyperbolicity (see Chinea, Fernández-Jambrina and Senovilla [4] and Dadhich and Patel [5]). Later this singularity-free solution was found to be one of an entire class of singularity-free models (Ruiz and Senovilla [6]). All these singularity-free solutions were characterised by cylindrical symmetry and having various types of perfect fluid as the source of curvature. Later work by Patel and Dadhich, showed that these models were stable against heat flow and massless scalar fields [7]. In 1995, Mars [8] obtained the first non-diagonal perfect fluid model and later that year Griffiths and Bičák [9] found a solution which superposed gravitational waves onto this non-diagonal model.

However one particular problem with these cosmologies is that they do not yield any realistic results for the observable properties of the universe. In order to create singularity-free models which possess similar physical properties to the observed universe would require a singularity-free model which has FLRW-like properties. Early in 1996 Senovilla [10] found a family of cosmologies which contain not only all the FLRW models but also all the known diagonal separable metrics found above. Moreover these cosmologies can be selected continuously from the family by a single parameter. Such a family of cosmologies could be used to construct a physically realistic cosmology which would in past epochs have all the properties of the singularity-free models and at later times approach the isotropy and homogeneity of the FLRW models in the limit as time advances. Such cosmologies would agree with all the energy and causality conditions and be singularity-free yet still yield the same agreement with experimental evidence that the FLRW models possess. Ultimately, a well-constructed model of this type would not require a need for quantum gravity anywhere in its dynamics and would constitute a classical means of avoiding the problems associated with initial singularities.

This thesis is composed of four main chapters. The first chapter deals with the important background material on perfect fluids, FLRW models and their method of solution. Also introduced are mathematical methods for analysing the symmetries of a spacetime and how this can be used to create FLRW models in non-standard coordinate systems. Chapter 3 is devoted entirely to the development of basic global analysis and Lorentzian geometry. A knowledge of this is essential to understanding how singularities are defined and the methods by which the singularity theorems are proved. Many of the terms defined in this chapter will be used extensively throughout the remainder of this thesis. Chapter 4 describes the

[1]This spacetime allows the content matter to pass through a stage of negative pressure. Furthermore this stage of energy condition breaking can be made of arbitrarily small duration.

brief history of singularity-free cosmology and in particular aims to give the reader some understanding of the general features of such spacetimes. The remainder of the chapter then goes on in detail to explain how the singularity-free models escape the singularity theorems. The final chapter gives a treatment of Senovilla's continuous family of singularity-free and FLRW metrics as a prelude to the original work in the second part of the chapter dealing with a simple generalisation of the Senovilla model. The work here is aimed at producing a realistic singularity-free cosmology.

In conclusion we find that the question of the existence of realistic singularity-free models is open and that continuing work resulting from the solution of perfect fluid criterion will allow an analytic solution to be found.

Chapter 2

Concepts in Cosmology

When applying general relativity to the problem of producing exact cosmological solutions it is important to realise that the Einstein equation alone does not contain enough information to restrict a solution. Often one must apply some philosophical preference for the symmetry structure or the source of curvature of the solution. In the case of the former one might consider only those models which have symmetries that could be physically reasonable. Similarly a choice of source for the model may be dependent on the complexity of the model or indeed whether it is to be filled with radiation or matter. Hence the primary purpose of this chapter is to define information that is specific to exact solutions dealt with in cosmology. In the process some of the major concepts and mathematical properties used to classify exact solutions will be used. The classification of solutions is vital in providing a logical structure to the many known solutions and to supply criteria by which exact solutions may be related. The various relevant types of spacetime symmetry will be examined. It is important to note that in general relativity coordinate freedom allows a spacetime to be equally well-described by many types of coordinate systems and hence symmetries or other spacetime features may be hidden whenever a solution is found in a poor choice of coordinates. However other fundamental mathematical techniques allow these concepts to be precisely stated and easily verified.

2.1 Perfect Fluids

Since the Einstein equation is often impossible to solve for a general source of curvature many approximations may be made. On the distance scales relevant to cosmology the relative dimensions of galaxies to the observable universe are so small that it is possible to consider that the universe is filled with a "fluid" of galaxies, groups of galaxies, etc .... These objects can to first approximation be assumed to not interact except gravitationally. This leads to a particularly important idealisation of the source; the *perfect fluid.* Perfect fluids are so termed because they possess no transport processes such as viscosity and heat flow. The directional pressures measured by an observer moving along with the flow of a perfect fluid are isotropic. Consequently a good example of a perfect fluid is an ideal gas. Therefore in the frame of an observer moving with the perfect fluid the stress-energy tensor for the fluid would be of the form $T_{ab} = \mathrm{diag}(\rho, p, p, p)$ where ρ is the energy density and p is the fluid pressure. Note that the condition of isotropic pressure constrains the diagonal spatial elements to be equal to one another. Hence the stress-energy tensor can be written in the form[1]:

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b) \tag{2.1}$$

or in frame independent notation:

$$\mathbf{T} = (\rho + P)\mathbf{u} \otimes \mathbf{u} + P\mathbf{g} \tag{2.2}$$

where $\mathbf{u}$ is the 4-velocity of the fluid with respsect to the observer. Note that the divergence-free condition $\nabla \cdot \mathbf{T} = 0 \equiv \nabla^a T_{ab} = 0$ gives the following equations of motion:

$$\nabla_{\mathbf{u}}\rho + (\rho + P)\nabla \cdot \mathbf{u} = 0 \tag{2.3}$$
$$(P + \rho)\nabla_{\mathbf{u}}\mathbf{u} + (\mathbf{g} + \mathbf{u} \otimes \mathbf{u})\nabla P = 0 \tag{2.4}$$

Taking the flat spacetime limit we obtain the following equations:

$$u^a \partial_a \rho + (\rho + P)\partial^a u_a = 0 \tag{2.5}$$
$$(P + \rho)u^a \partial_a u_b + (\eta_{ab} + u_a u_b)\partial^a P = 0 \tag{2.6}$$

Finally taking the non-relativistic limit $P \ll \rho$, $u^a = (1, \vec{u})$ and $u\frac{dP}{dt} \ll |\vec{\nabla} P|$ gives:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{2.7}$$
$$\rho \left\{ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right\} = -\vec{\nabla} P \tag{2.8}$$

which are respectively the conservation of mass equation and the Euler equation (conservation of momentum) from classical fluid mechanics.

If we define a congruence of inertial observers with parallel 4-velocities, $\mathbf{v}$ so that $\partial_b v^a = 0$ then by (2.2) the 4-vector, J_a defined by:

$$J_a = -T_{ab} v^b \tag{2.9}$$

represents the energy-momentum flux density of the perfect fluid source as measured by the congruence of observers. Hence in special relativity the conservation equation $\nabla \cdot \mathbf{T} = 0$ implies that:

$$\nabla \cdot \mathbf{J} = \nabla_a J^a = 0 \tag{2.10}$$

which implies both the conservation of energy and momentum. However in curved spacetime this is not true. This is due to the fact that there do not always exist observers with $\nabla_b v_a = 0$. Requiring $\nabla_b v_a = 0$ or merely $\nabla_{(b} v_{a)} = 0$ would allow such a conservation equation to hold. However the second of these relations is Killing's equation (see (2.11)) which represents the symmetries of the spacetime. However a general spacetime may admit no symmetries at all. This can be understood in physical terms since in general asymmetric spaces gravitational waves and curvature can remove energy from a system and conservation will not strictly hold. This problem is related to the issue of the non-localisability of energy in general relativity (see Misner, Thorne and Wheeler [11] p.466).

[1] The term $g_{ab} + u_a u_b$ is a projection operator onto the vector subspace perpendicular to u_a.

2.1.1 Perfect Fluid Barotropes

There are many classes of perfect fluid stress-energy tensors. This is because we have not specified what exact functional dependencies or equations of state the density and pressure of the fluid will obey. This is subsequently a problem of thermodynamics. Perfect fluids can be endowed with equations of state that restrict the dynamics of the fluid.

Definition 2.1 (Barotropic equation of state) If a perfect fluid obeys an equation of state of the form $P = P(\rho)$ then the perfect fluid is termed a *barotrope*. If in addition $P = \gamma\rho$, where γ is a constant, $0 \leqslant \gamma \leqslant 1$ then the perfect fluid is said to obey the γ-law equation of state.

Barotropes obeying the γ-law obey a very simple equation of state where the density and pressure are directly proportional much like the equation of state of an ideal gas under isothermal processes. Within the class of γ-law barotropes there are some special cases:

$\gamma = 0$ (*Dust case*) Equation of state: $P = 0$. There is no fluid pressure. The particles in the perfect fluid exert no pressure on each other and do not interact except gravitationally.

$\gamma = \frac{1}{3}$ (*Incoherent radiation case*) Equation of state: $3P = \rho$. This form of the equation of state is so-called because it represents a random superposition of waves of a massless field (i.e. photons from an electromagnetic field).

$\gamma = 1$ (*Stiff fluid case*) Equation of state: $P = \rho$. In a stiff fluid the speed of sound is 1. Hence the term 'stiff'.

From the last point we obtain the condition that γ can never be greater than 1. Otherwise the speed of sound in the medium would be greater than the speed of light. In general it is not necessary for a perfect fluid to obey an equation of state. Often complicated equations of state can be deduced for matter by the Einstein equation.

2.2 Homogeneity and Isotropy

Definition 2.2 (Isometry) A diffeomorphism[2], $\phi : M \to M$ is an isometry of a spacetime if $\phi^*\mathbf{g} = \mathbf{g}$.

From the definition of the Lie derivative (A.10) it is easy to see that if a vector field $\mathbf{v}$ generates a one-parameter group of isometries (i.e. ${\phi_t}^*\mathbf{g} = \mathbf{g}, \forall t$) then $\pounds_{\mathbf{v}}\mathbf{g} = 0$. From (A.13) we find that:

$$\begin{aligned} \pounds_{\mathbf{v}}\mathbf{g} &= \pounds_{\mathbf{v}} g_{ab} \\ &= v^c \nabla_c g_{ab} + g_{cb} \nabla_a v^c + g_{ac} \nabla_b v^c \\ &= \nabla_a v_b + \nabla_b v_a \end{aligned} \tag{2.11}$$

The last line of (2.11) holding as ∇ is the covariant derivative compatible with $\mathbf{g}$ (i.e. $\nabla \mathbf{g} \equiv 0$). Therefore if $\pounds_{\mathbf{v}}\mathbf{g} = 0$ for some vector $\mathbf{v}$ then $\mathbf{v}$ is the generator of an isometry.

[2]See appendix A.

Definition 2.3 (Killing Vectors) A vector field $\boldsymbol{\xi}$ is termed *Killing vector field* if it obeys the Killing equation:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{2.12}$$

Hence Killing vectors represent isometries of the metric. It can also be shown that if $\mathbf{v}$ and $\mathbf{w}$ are both Killing vector fields then their commutator $[\mathbf{v}, \mathbf{w}]$ is also a Killing vector.

Examples of Killing Vectors in various spaces

Example 1 (Minkowski space) In Minkowski space, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, there are 10 linearly independent Killing vectors. These Killing vectors correspond to:

(i) the four generators of translations $\boldsymbol{\partial}_\mathbf{t}$, $\boldsymbol{\partial}_\mathbf{x}$, $\boldsymbol{\partial}_\mathbf{y}$ and $\boldsymbol{\partial}_\mathbf{z}$,

(ii) the three generators of rotation $[\boldsymbol{\partial}_\mathbf{x}, \boldsymbol{\partial}_\mathbf{y}]$, $[\boldsymbol{\partial}_\mathbf{y}, \boldsymbol{\partial}_\mathbf{z}]$ and $[\boldsymbol{\partial}_\mathbf{x}, \boldsymbol{\partial}_\mathbf{z}]$,

(iii) the three generators of boosts $\mathbf{v}_\mathbf{x}$, $\mathbf{v}_\mathbf{y}$ and $\mathbf{v}_\mathbf{z}$.

Note that under all these transformations the form of the metric remains unchanged.

Example 2 (Schwartzschild spacetime) Schwartzschild spacetime, $ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta) d\phi^2$, admits two linearly independent Killing vector fields, $\boldsymbol{\partial}_t$ and $\boldsymbol{\partial}_\phi$. Hence time translations and ϕ rotations have no effect on the metric. This can be seen quite clearly since none of the metric functions have t or ϕ dependence.

2.2.1 Maximally symmetric spaces

One important question that remains to be answered is that in a manifold of dimension n exactly how many Killing fields are allowed to exist.

Let $\boldsymbol{\xi}$ be any Killing vector. Starting with the definition of the Riemann tensor in a coordinate basis we obtain:

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = R_{abc}{}^d \xi_d$$

Using the Killing equation $\nabla_a \nabla_b \xi_c = -\nabla_b \nabla_a \xi_c$ we get:

$$\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a = R_{abc}{}^d \xi_d$$

Permuting indices and summing $(abc) + (bca) - (cab)$ we obtain:

$$2\nabla_b \nabla_c \xi_a = -2R_{cab}{}^d \xi_d$$

Finally applying the identity $R_{[abc]}{}^d = 0$ we find that:

$$\nabla_a \nabla_b \xi_c = -R_{bca}{}^d \xi_d \tag{2.13}$$

From identity (2.13) we notice that a Killing field is completely determined by the values of ξ^a and $\nabla_a \xi_b$ at some point $p \in M$. Therefore for a given $(\xi^a, \nabla_a \xi_b)$ at $p \in M$ the value of the Killing field and its derivative at some point q is given by integrating the set of ordinary differential equations:

$$v^a \nabla_a \xi_b = v^a \Omega_{ab} \tag{2.14}$$
$$v^a \nabla_a \Omega_{bc} = -R_{cba}{}^d \xi_d v^a \tag{2.15}$$

where $\mathbf{v}$ is the tangent vector to any curve between p and q and $\Omega_{ab} = \nabla_a \xi_b$.

There are two immediate consequences of (2.14) and (2.15). The first is that if a Killing field and its derivative vanish at some point then the Killing field vanishes everywhere. Secondly, since the ξ^a and Ω_{ab} are independently specifiable[3] there can be at most $n + \frac{1}{2}n(n-1) = \frac{n}{2}(n+1)$ linearly independent Killing fields. Therefore in relativity the maximum number of linearly independent Killing fields[4] allowed for a spacetime is ten. This is the same as the case above for Minkowski spacetime. Since Minkowski spacetime is the simplest spacetime it is not entirely unexpected that it possesses the greatest allowed number of symmetries. In practice every spacetime is not maximally symmetric since (2.14) and (2.15) constitute a set of integrability conditions for the Killing fields. Spaces that admit the maximum allowed number of linearly independent Killing vector fields are termed *maximally symmetric*. This restriction provides stringent restictions on the form of the Riemann tensor in general relativity. In fact the Riemann tensor must obey the condition (Clarke and De Felice [12] p.127).

$$R_{abcd} = \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{2.16}$$

where R is constant. Contraction of (2.16) implies that:

$$R_{ab} = \frac{R}{n} g_{ab} \tag{2.17}$$

It should also be noted that the curvature scalar of such a space is constant. A space which obeys (2.16) has[5] constant sectional curvature. Hence all maximally symmetric spaces are *spaces of constant curvature*. If we choosing $R = 0$ for a maximally symmetric space we trivially obtain the Minkowski spacetime.

Definition 2.4 (Spatial Homogeneity) A spacetime is *spatially homogeneous* if there exists a one-parameter family of space-like hypersurfaces Σ_t foliating the spacetime such that for each t and for any points p, $q \in \Sigma_t$ there exists an isometry of the spacetime metric, g_{ab} which maps p into q.

This situation is shown in Figure 2.1. In essence (spatial) homogeneity implies that the spacetime can be "sliced up" or foliated by space-like hypersurfaces where every point on each slice has the same physical conditions as any other. These hypersurfaces may each represent the state of the universe at some instant with respect to observers moving with the

[3]Note that due to the Killing equation Ω_{ab} is antisymmetric.

[4]It is important to note that the set of all Killing fields on a spacetime form a Lie algebra with the binary operation of commutation. This set may not represent all symmetries of the spacetime however. For example, reflection symmetry in a plane is not generated by Killing vector fields and so the full group of isometries of a spacetime may be larger than just those generated by Killing vector fields.

[5]See Schur's Lemma (Kobayashi and Nomizu [13] p.202).

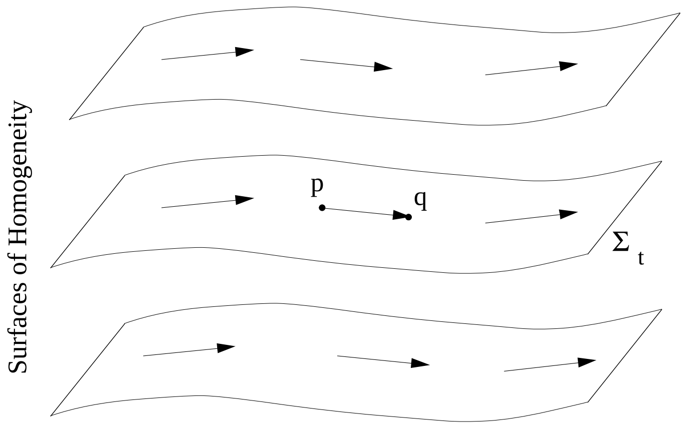

Figure 2.1: Homogeneity of a spacetime.

cosmological fluid. The identical nature of physical conditions at each point is guaranteed by the existence of an isometry between any two points on a surface of homogeneity. Such a constraint on a spacetime is in fact very strong. For example at every point on such a hypersurface the density and pressure must be the same. Similarly the curvature of spacetime must be the same.

Definition 2.5 (Isotropy) A spacetime $(M, \mathbf{g})$ is termed *spatially isotropic* if throughout the spacetime there exists a vector field $\mathbf{u}$ whose integral curves form a congruence of timelike paths such that for any point $p \in M$ and any two unit vectors $\mathbf{s_1}$, $\mathbf{s_2} \in T_p(M)$ orthogonal to $\mathbf{u}(p)$ there exists an isometry of $\mathbf{g}$ which leaves p and $\mathbf{u}$ unchanged but rotates $\mathbf{s_1}$ into $\mathbf{s_2}$.

This definition is depicted in Figure 2.2

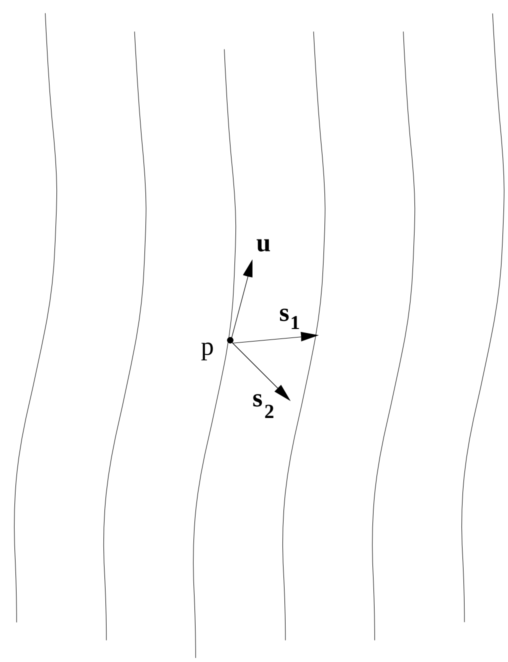

Figure 2.2: Definition of isotropy in a spacetime.

The definition of (spatial) isotropy requires that at any point in spacetime and for any observer 'moving along' with the cosmological fluid (i.e the timelike congruence of paths above) no particular spatial direction can be distinguished from any other direction by local measurements. Moreover the surfaces of homogeneity of a spacetime, Σ_t, must be orthogonal to the tangent vectors, $\mathbf{u}$, of the world lines of isotropic observers. If this were not the case then, assuming that the surfaces of homogeneity and the isotropic observers are unique[6], the projections of the isotropic observer's tangent vector field onto the tangent space of Σ_t at that point would provide a geometrically prefered spatial direction for each observer, contradicting the isotropy of the spacetime. Isotropy at every point of a spacetime implies that the space is spatially homogeneous. This is straightforward since isotropy guarantees the equivalence of all directions in every tangent space of every point of the surfaces of homogeneity, Σ_t. Hence every point of the surface is equivalent[7]. Minkowski spacetime is a simple example of an isotropic spacetime. If however a weak magnetic field existed in Minkowski space, isotropy would be broken since a local physical measurement could distinguish a particular spatial direction as being special. Note however that in this case the space would remain spatially homogeneous. There are also complicated cosmological solutions that are spatially homogeneous but not isotropic. For example, both the Kantowski-Sachs models and the Bianchi metrics[8] possess many different types of homogeneity but are not isotropic.

2.3 FLRW models

The FLRW (Friedmann-Lemaître-Robertson-Walker) Models are a class of perfect fluid cosmological models that are characterised by being spatially homogeneous and isotropic at every point. The need for spatial homogeneity is motivated by the philosophical prejudice that no point of the universe should possess special physical properties. In particular there should be no point of the universe which has any special symmetry. Copernicus was the first to propose this type of idea. The Copernican universe assumed the Earth did not hold a special reference frame in the then known universe. Indeed, the non-existence of special reference frames motivated the construction of relativity in it present form. So an FLRW model is one that agrees with the so-called Copernican Principle - that there exist no special reference frames in the universe. The motivation for isotropy is an extension of this principle. Isotropy guarantees the equivalence of spatial directions for any observer and so the Copernican principle is extended to also include that there are no favoured orientations or directions in the universe. These two restrictions require that the congruence of isotropic observers associated with the cosmological fluid be geodesic, shear-free and vorticity-free. The first condition can be deduced from the fact that non-geodesic observers would be able to detect a force that could be used to define a spatial reference direction. Similarly vorticity and shear would also define special spatial orientations in any hypersurface of homogeneity.

From the discussion developed in Section 2.2 we note that these two restrictions are very strong and in addition to the Einstein equation provide enough information to create models for the universe. It is important to note that the hypersurfaces of homogeneity are 3-

[6] Even if the choice of isotropic observers and surfaces of homogeneity are not unique it is still possible to show that isotropic observers orthogonal to a particular set of homogeneous surfaces exist.

[7] Another way to view this is that between any two points in the spatial subspace of an isotropic observer there always exists an rotation isometry about a third point that maps the two points into each other. Hence isotropy implies homogeneity.

[8] For brief overview of these models see MacCallum [14] p.533.

dimensional and hence allow a maximum of six linearly independent Killing vector fields. In this particular case, isotropy and homogeneity imply that there are six linearly independent Killing vector fields on the hypersurface. Three of the Killing fields correspond to the translation invariance of the metric as guaranteed by spatial homogeneity and the final three correspond to rotational invariance about any point in the surface with respect to the rotation group $SO(3)$. Hence the hypersurfaces of homogeneity are spaces of maximal symmetry. From the development of the properties of maximally symmetric spaces in Section 2.2.1 the hypersurfaces of homogeneity must be 3-surfaces of constant scalar curvature $K = {R^{(3)}}_\alpha{}^\alpha$. Consequently there are only three options that can be chosen for such a 3-surface, Σ_t:

$K > 0$ Σ_t possesses positive constant curvature and hence is homeomorphic to the 3-sphere, S^3 and has metric:

$$d\sigma^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.18}$$

Such hypersurfaces will be compact and "closed" representing a finite universe.

$K = 0$ The Σ_t possesses constant zero curvature, is homeomorphic to $\mathbf{R}^3$ and possesses metric:

$$d\sigma^2 = d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.19}$$

$K < 0$ Σ_t is homeomorphic to a 3-hyperboloid, H^3 and possesses metric:

$$d\sigma^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\phi^2) \tag{2.20}$$

The hypersurfaces, Σ_t are 'open'.

where χ, θ and ϕ are the standard coordinates for all the 3-geometries above.

Hence the spatial metric can be written in general as:

$$d\sigma^2 = d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.21}$$

where:

$$f(\chi) = \begin{cases} \sin\chi & \text{if } k = K/|K| = 1, \\ \chi & \text{if } k = K/|K| = 0, \\ \sinh\chi & \text{if } k = K/|K| = -1. \end{cases}$$

The corresponding spacetime metric for these cases can all be written in the form:

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \tag{2.22}$$

where $a(t)$ is called the *cosmological scale factor* of the model. Calculating the components of the Einstein tensor in the comoving rest frame of the cosmological fluid ($\boldsymbol{\omega}^{\hat{t}} = \boldsymbol{d}t$, $\boldsymbol{\omega}^{\hat{\chi}} = a(t)\boldsymbol{d}\chi$, $\boldsymbol{\omega}^{\hat{\theta}} = a(t)f(\chi)\boldsymbol{d}\theta$, $\boldsymbol{\omega}^{\hat{\phi}} = a(t)f(\chi)\sin\theta\boldsymbol{d}\phi$) results in a diagonal tensor whose components are:

$$G_{\hat{t}\hat{t}} = \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} \tag{2.23}$$

$$G_{\hat{\chi}\hat{\chi}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \tag{2.24}$$

Since the metric guarantees the isotropy of physical measurements, the spatial part of the Einstein tensor must be some multiple of the identity map so that the stress-energy tensor has isotropic pressures. Similarly the off-diagonal components and the time-space components must also be zero or the vector $G_{ab}u^b$ would possess a spatial component and isotropy would be violated. Therefore the results (2.23) and (2.24) agree with these predictions.

If we now assume that the source of curvature is a general perfect fluid then $T_{ab} = \mathrm{diag}(\rho, P, P, P)$ in the comoving reference frame and the Einstein equation reduces to the following evolution equations for the FLRW cosmology, namely:

$$3\frac{\dot{a}^2}{a^2} = \rho - \frac{3k}{a^2} \tag{2.25}$$

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3P) \tag{2.26}$$

in units where $8\pi G = c = 1$. One should note that if the time coordinate t is reparametrised so that the metric takes on the form:

$$ds^2 = a^2(t)(-dt^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)) \tag{2.27}$$

then the dynamical equations of motion become:

$$\rho = \frac{3}{a^2}\left(\frac{\dot{a}^2}{a^2} - k\right) \tag{2.28}$$

$$P = -\frac{1}{a^2}\left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - k\right) \tag{2.29}$$

2.3.1 FLRW Models in Non-standard coordinates

It will be useful, especially in dealing with the family of models which are described in Chapter 5, to analyse FLRW models in coordinate systems other than those which have been used previously. In the standard treatment of a FLRW model coordinates on a 3-sphere or 3-hyperboloid are chosen. However for the purposes of producing FLRW-like singularity-free models we need to consider coordinates which possess cylindrical symmetry.

First let us assume that the spacetime admits a two-dimensional Abelian symmetry group (i.e. that there exist two spacelike commutative Killing vector field $\boldsymbol{\partial_z}$ and $\boldsymbol{\partial_\phi}$ say). Also let us assume that the space contains a foliation of 3-surfaces which contain the orbits of the symmetries. Using coordinate freedom we define a coordinate t, such that $t =$ constant is the equation of the spaces contained in the foliation. Finally we make this coordinate a time coordinate.

Hence the spacetime metric under such conditions is:

$$ds^2 = -dt^2 + h_{\alpha\beta}dx^\alpha dx^\beta \tag{2.30}$$

where $h_{\alpha\beta}$ depends only on t and some other coordinate x^α which we will denote r.

If we now assume that $h_{\alpha\beta}$ is diagonal and that the curvature of the 3-spaces is constant due to homogeneity and isotropy then the curvature tensor must be of the form (2.16). Assuming a perfect fluid as the source the Einstein equation yields a set of differential equations for the metric functions. Solving this set of equations we obtain an FLRW metric in cylindrical coordinates.

This will now be done for the metric[9]:

$$ds^2 = -dt^2 + A^2(r)dr^2 + B^2(r)d\phi^2 + C^2(r)dz^2 \tag{2.31}$$

Since some coordinate freedom still remains, we choose r such that $A(r) = 1$. Calculating the Einstein tensor in the comoving frame we obtain:

$$G_{\hat{a}\hat{b}} = \begin{pmatrix} \frac{-B''}{B} - \frac{C''}{C} - \frac{B'C'}{BC} & 0 & 0 & 0 \\ 0 & \frac{B'C'}{BC} & 0 & 0 \\ 0 & 0 & \frac{C''}{C} & 0 \\ 0 & 0 & 0 & \frac{B''}{B} \end{pmatrix} \tag{2.32}$$

where primes have been used to denote derivatives with respect to r. To enforce isotropy on the space the three pressures must be equal. Equating $p_{\hat{r}}$ and $p_{\hat{\phi}}$ we find that:

$$\frac{B'}{B} = \frac{C''}{C'} \tag{2.33}$$

which has solution $C' = K_1 B$ for $K_1 B(r) \neq 0$.

Similarly equating the pressures $p_{\hat{r}}$ and $p_{\hat{z}}$ we obtain the equation:

$$\frac{C'}{C} = \frac{B''}{B} \tag{2.34}$$

which has solution $B' = K_2 C$ for $K_2 C(r) \neq 0$. Combining these two solutions we find that:

$$\frac{C''}{K_1} = K_2 C \tag{2.35}$$

Using the integrating factor $2C'$ we obtain:

$$(C')^2 = \underbrace{K_1 K_2}_{M} C^2 + N \tag{2.36}$$

Taking the explicit choice $K_1 = 1$ and assuming that the above differential equation holds, the metric can be rewritten in the form:

$$ds^2 = -dt^2 + dr^2 + \Sigma'^2 d\phi^2 + \Sigma^2 dz^2 \tag{2.37}$$

where Σ obeys the differential equation:

$$\Sigma'^2 = M\Sigma^2 + N \tag{2.38}$$

After rescaling the t coordinate the metric takes the form:

$$ds^2 = T^2(t)(-dt^2 + dr^2 + \Sigma'^2 d\phi^2 + \Sigma^2 dz^2) \tag{2.39}$$

This metric represents a maximally symmetric, isotropic, perfect fluid solution[10] in the new cylindrical coordinates. The metric family (2.39) with restriction (2.38) will be useful in considering singularity-free models and general anisotropic and inhomogeneous models which have the FLRW models as their homogeneous and isotropic limit.

[9] t has been chosen so that there is no time dependence (i.e. the FLRW scale factor $T(t) = 1$).

[10] This can be easily verified by examining the Einstein tensor of this spacetime.

Chapter 3

A Brief Introduction to Global Analysis

An understanding of global analysis is essential to any consideration of singularities in general relativity. The failure of timelike geodesics to be extended arbitrarily is a global property of a manifold as any particular geodesic may pass through many 'coordinate patches' or charts. Much of relativity is done *locally*. The Einstein equation is a local equation because $T_{\alpha\beta}$ and $G_{\alpha\beta}$ are local properties of the matter/energy and curvature at an event in spacetime. Hence the Einstein equation (in units where $G = c = 1$:

$$\mathbf{G}(p) = 8\pi\mathbf{T}(p), \quad \text{for all} \quad p \in M \tag{3.1}$$

is a statement of a local restriction[1] applied globally over a spacetime. A result such as this may then provide restrictions on the topology of the manifold underlying the spacetime. Since extended geometric properties also rely on global structure, global analysis is also the analysis of the geometry of a spacetime in the large. The methods by which local and global properties are related and compared are the subject of global analysis.

The aim of this chapter is to introduce those concepts which are fundamental to global analysis. Concepts such as 'spacetime' and 'causality' are made more precise by writing them in the language of differential topology. This process allows many basic physical concepts to be exactly stated and more readily verified. However the ultimate aim of this treatment of global Lorentzian geometry is to show under what conditions we expect singularities to occur. This is tackled using the tools developed in this chapter.

3.1 A Mathematical Model of Spacetime

A spacetime is often described in many basic texts as being simply a collection of events[2] which are characterised by what occurs at each point. This collection of points must have additional topological properties for it to constitute a spacetime which is physically reasonable.

Definition 3.1 (Spacetime) A *spacetime* is a pair $(M, \mathbf{g})$, where:

[1]This restriction is in fact motivated by local energy conservation $\nabla \cdot \mathbf{T}(p) = \nabla \cdot \mathbf{G}(p) \equiv 0, \forall p \in M$.
[2]see Misner, Thorne and Wheeler [11] p.5 for details.

(i) M is a C^∞ Hausdorff, locally Euclidean, second countable topological space without boundary,

(ii) M is connected and 4-dimensional, and

(iii) $\mathbf{g}$ is a non-degenerate, symmetric tensor of Lorentz signature.

Condition (i) implies that M is a smooth[3] topological space and that it obeys the Hausdorff separation property. The Hausdorff condition states that for any two distinct[4] points of the topological space p, $q \in M$ there exist neighbourhoods of those points U and V such that $p \in U$, $q \in V$ and $U \cap V = \phi$. If the manifold were not Hausdorff then there would exist two distinct points which could not be separated by disjoint open neighbourhoods. This would then make non-sensical the notion of such a pair of points being physically distinct events. Second countable topological spaces have a countable base of subsets which generate the topology by arbitrary unions. More importantly if M is second countable then we can cover M using a locally finite, countable family of coordinate patches which have compact closure. Finally the condition that M be without boundary requires the spacetime to not have an 'edge' to it. This is reasonable both from the pathological physical properties such an edge would have and the fact that none have been observed experimentally.

Condition (ii) requires that M cannot be written as the union of two disjoint open sets. If this were not the case then the spacetime could be constructed from independent parts between which no physical interaction could take place. From a philosophical point of view our universe *must* be connected. Similarly the dimensionality requirement is chosen to agree with our current theories of space and time.

Condition (iii) requires a metric to exist on the spacetime such that there exists a set of four C^∞ basis vector fields, $\mathbf{e}_a$, for which the metric provides the constant inner product[5] $g_{ab} = diag(-1, +1, +1, +1)$. The existence of such a basis set guarantees that at each point of the spacetime there always exists a local Lorentz frame in which the spacetime appears locally flat and that these frames mesh together smoothly.

Note also that together (i) and (iii) imply that M is paracompact[6]. This is important as it implies that (Kobayashi and Nomizu [13]) M admits a Riemannian Metric. Those familiar with Riemannian manifolds will note that many of the concepts of Lorentzian geometry are identical. However the one significant difference between the two types of geometry is that no version of the Hopf-Rinow theorem holds for Lorentzian spacetimes. The Hopf-Rinow theorem (Beem and Ehrlich [17] p.4) guarantees the equivalence of metric and geodesic completeness for Riemannian manifolds. Hence a similar version for spacetime would solve the problem of considering geodesic completeness and hence the question of singularities. However since the Lorentzian metric function is not positive definite and does not supply a way of defining open neighbourhoods this method of attacking singularities is doomed to fail and alternate methods of determining geodesic incompleteness must be used.

[3]Smooth will imply C^∞ unless otherwise stated.

[4]The term *distinct* will mean that $p \neq q$.

[5]or the negative of this as some conventions allow.

[6]A topological space $(X, \mathcal{T})$ is paracompact if every open cover $\{O_a\}$ of X has a locally finite refinement $\{V_b\}$. See Wald [15] p.427 and VonWestenholz [16] p.12.

3.2 Causal Structure

Causal structure is the study of how the structure of a spacetime affects all possible causal links between events in that spacetime. Locally the causal structure of spacetime is qualitatively similar to that of flat Minkowski space. However a spacetime may not be based on any standard or trivial topology. The presence of boundary points posed by singularities and the deformations and shearing of light cones can complicate the causal structure of a spacetime. The purpose of this section is to introduce the definitions and basic properties of the objects that will be used in dealing with the singularity theorems. At this stage there is no need to impose the Einstein equation on the properties of the metric. Indeed, apart from certain restrictions the spacetimes will be very general. However many of the examples and counter-examples to situations found in this section may seem very contrived. This is in fact not so as in most cases the examples given represent simplified situations of aberrant behaviour that can occur in less artificial exact solutions.

Definition 3.2 (Timelike, Null, Spacelike) Let $(M, \mathbf{g})$ be a spacetime and $x \in M$. Then any tangent vector $X \in T_x(M)$ is termed:

- *spacelike*, if $\mathbf{g}_x(X, X) > 0$,
- *timelike*, if $\mathbf{g}_x(X, X) < 0$, or
- *null*, if $\mathbf{g}_x(X, X) = 0$.

The above definition is in direct analogy with special relativity.

Definition 3.3 (Time-orientability) A spacetime is said to be *time-orientable* if a consistent continuous choice of one vector from the set of timelike vectors in $T_p(M)$ can be made for all points of M.

Clearly a spacetime will be orientable if a there exists a global non-vanishing timelike vector field on it. The converse is also true:

Theorem 3.1 Let $(M,\mathbf{g})$ be time orientable. Then there exists a (highly non-unique) smooth non-vanishing timelike vector field on M (Wald [15] p.189).

Proof. (Wald [15] p.189) Using the paracompactness of M there must exist a smooth Riemannian metric h_{ab}. Let v_α be the set of vectors such that $h_{ab}v^a_\alpha v^b_\alpha = 1$ for all α. Then at each point $p \in M$ there exists a unique future-directed timelike vector t^a chosen from the v^a that minimises the value of $g_{ab}v^a v^b$. This defines a smooth vector field all over M as h_{ab} is smooth and moreover since h_{ab} was chosen arbitrarily there are other smooth non-vanishing timelike vector fields. $\square$

Spacetimes that are not time orientable, such as that shown in Figure 3.1, have the undesirable property that a consistent global definition of going 'backwards and forwards' in time cannot be made[7]. Consequently for the remainder of the chapter it is assumed that $(M, \mathbf{g})$ is a spacetime with a fixed time-orientation.

[7]If a spacetime is not time orientable there always exists a twofold unwrapping of that spacetime which is time orientable (Penrose [18] p.2).

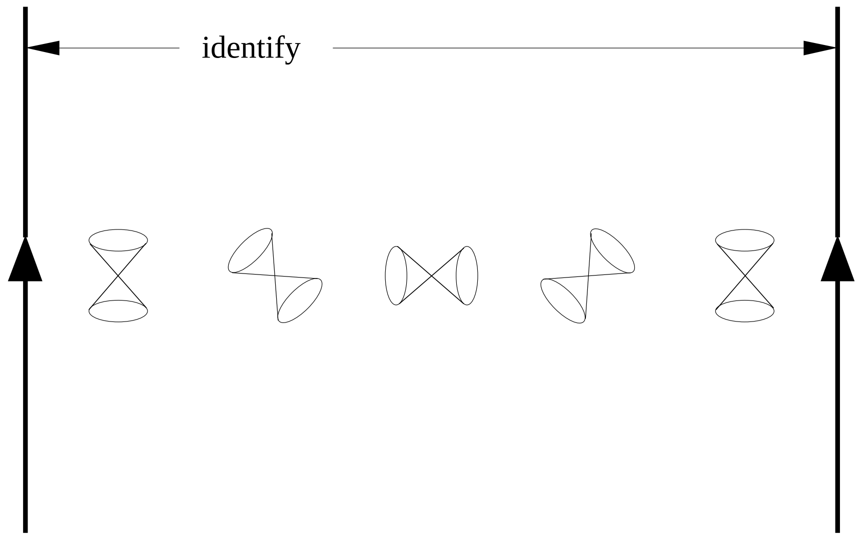

Figure 3.1: An example of a non-time orientable spacetime.

Definition 3.4 (Path or curve) A *path* or *curve* is a continuous map $\gamma : I \to M$ where $I \subset \mathbb{R}$ is a connected subset with more than one point. Furthermore a path is termed *smooth* if γ is smooth with non-vanishing derivative $\dot{\gamma}$.

Definition 3.5 (Endpoint) Let $\gamma : I \to M$ be a path and let $a = \inf I$, $b = \sup I$[8]. Then the $x \in M$ is an *endpoint* if for all sequences $\{x_i\} \in I$:

- $x_i \longrightarrow a$ implies $\gamma(x_i) \longrightarrow x$ (*past endpoint*), or
- $x_i \longrightarrow b$ implies $\gamma(x_i) \longrightarrow x$ (*future endpoint*).

A path can be simply defined as *timelike*, *null*, or *spacelike* depending on whether $\dot{\gamma}$, the tangent vector to the path, is timelike, null or spacelike along all of the path. Paths which are either timelike or null are often termed *causal*. Hence causal paths represent the trajectories[9] of matter or energy. It is convenient to require that all timelike and causal curves contain their endpoints. This implies that timelike curves must remain smooth and timelike at their endpoints. This requirement guarantees that a particle can physically attain the spacetime coordinates of the endpoint. Causal curves without a future endpoint must extend indefinitely into the future. Curves of this type are termed *future-endless*. *Past-endless* curves are similarly defined. Causal curves with no past or future endpoints are simply termed *endless*.

3.2.1 Compactness

Definition 3.6 (Directed Set) A *directed set* Λ is a nonempty set with a relation $\preccurlyeq$ such that:

(i) $\lambda \preccurlyeq \lambda$, for all $\lambda \in \Lambda$. (*reflexive*)

[8]this includes the cases where $a = -\infty$ and/or $b = \infty$.

[9]A particle's path in spacetime could also be thought of as the history of the particle.

(ii) if $\lambda \preccurlyeq \mu$ and $\mu \preccurlyeq \nu$ then $\lambda \preccurlyeq \nu$, for all λ, μ, $\nu \in \Lambda$. (*transitive*)

(iii) if λ, $\mu \in \Lambda$ then there exists a $\nu \in \Lambda$ such that $\lambda \preccurlyeq \nu$ and $\mu \preccurlyeq \nu$. (*directive*)

Definition 3.7 (Net) A *net* in M (with domain Λ) is a map $x : \Lambda \to M$ where Λ is a directed set. A net is often denoted $\{x_\lambda\}_{\lambda \in \Lambda}$.

Nets are an extension of the concept of sequences. In fact a net with $\Lambda = \mathbb{N}$ and the standard binary relation $\leqslant$ is a sequence. A net differs from a sequence chiefly because the index set Λ does not have to be countable.

Definition 3.8 (Limit point) We say $x \in M$ is a *limit point* of the net $\{x_\lambda\}_{\lambda \in \Lambda}$ if and only if for every neighbourhood $U \subset M$ of x, there exists some $\lambda_0 \in \Lambda$ such that $\lambda \succcurlyeq \lambda_0$ implies $x_\lambda \in U$. In this case $\{x_\lambda\}_{\lambda \in \Lambda}$ is said to converge to x.

Note the that this definition is very similar to that for the convergence of a sequence of points.

Some important results involving nets and limit points

(i) If X is a Hausdorff topological space then any net in X has at most one limit point.

(ii) If A is a subset of a topological space X then $\overline{A}$, the closure of A, is set of limit points of all possible nets in A.

Another important concept is that of a *cluster* or *accumulation point.*

Definition 3.9 (Accumulation point) A point $x \in M$ is said to be a *cluster* or *accumulation point* of the net $\{x_\lambda\}_{\lambda \in \Lambda}$ if and only if for all neighbourhoods U of x and any $\lambda_0 \in \Lambda$ there exists some $\lambda \succcurlyeq \lambda_0$ such that $x_\lambda \in U$.

This definition differs in a subtle way from the definition of a limit point. Although a limit point is always an accumulation point the converse is not true. For a point to be a limit point of a net, the net must 'go into and stay inside' every neighbourhood of the point. However a net need only keep 're-entering' every neighbourhood of an accumulation point. This property will be very important in considering the topological property of compactness.

Definition 3.10 (Compactness) A topological space X is said to be *compact* if every open cover of X has a finite subcover.

Definition 3.11 (Equivalent definition of compactness) A topological space X is said to be *compact* if every net in X has an accumulation point.

The equivalence of these two definitions of compactness is taken up in Geroch [19] p.166.

Some important results involving compactness

(i) The extreme value theorem: any continuous real function defined on some compact set has a maximum and a minimum.

(ii) Continuous functions map compact sets to compact sets.

(iii) Compact subsets of Hausdorff topological spaces are closed.

Since causal curves will play an important part in the proofs that follow it is useful to define terms similar to those above but involving sequences of curves.

Definition 3.12 (Limit point of a sequence of curves) Let $\{\gamma_n\}$ be a sequence of curves. A point $p \in M$ is a *limit point* of $\{\gamma_n\}$ if for any open neighbourhood U of p there is a N such that γ_n intersects U for all $n > N$.

Definition 3.13 (Limit curve) A curve γ is termed a *limit curve* of $\{\gamma_n\}$ if every point of γ is a limit point of $\{\gamma_n\}$.

Definition 3.14 (Accumulation point of a sequence of curves) Let $\{\gamma_n\}$ be a sequence of curves. A point $p \in M$ is a *accumulation point* of $\{\gamma_n\}$ if every open neighbourhood of p intersects infinitely many γ_n.

Definition 3.15 (Accumulation Curve) A curve γ is an *accumulation curve* of $\{\gamma_n\}$ if there is some subsequence $\{\gamma'_n\}$ for which γ is a limit curve. Hence if γ is a accumulation curve then each point of that curve is an accumulation point.

The most significant result of the above definitions is given in the following theorem:

Theorem 3.2 Let $\{\gamma_n\}$ be a sequence of future inextendible causal curves which have an accumulation point p. Then there exists a future inextendible causal curve γ which is an accumulation curve of $\{\gamma_n\}$ and which passes through p.

Sketch of Proof. Choosing a convex normal neighbourhood[10], N, containing p we define a sphere $B_r \subset N$ of Riemannian normal coordinate radius R. Choosing a subsequence of $\{\gamma_n\}$ that converges to p we use the compactness of B_r to guarantee that there exists a sub-subsequence which converges to a point on this sphere. We then examine all the spheres of radius $1/n$ where $n \in \mathbb{N}$ and continue extract the limit points on each sphere and the sub-sequences converging to these points for each n in turn. Taking the closure of the set of limit points it can be shown that it defines a C^1 causal limit curve γ of the sequence $\{\gamma_n\}$. We can then go to the endpoint of γ on B_r and repeat the above construction. This process can be repeated indefinitely and so γ can be made future inextendible. $\square$

The technical details of this important theorem can be found in Hawking and Ellis [20] p.185.

3.2.2 Domains of Influence

Definition 3.16 Let p, $q \in M$. Then p *chronologically precedes* q if there exists a future-directed time-like curve beginning at p and ending at q. This is also denoted $p \ll q$.

[10] A convex normal neighbourhood is one in which the exponential map $exp_p(\mathbf{v}) : T_pM \to M$ is well-defined. The map sends a vector from $T_p(M)$ to a point 1 parameter unit along the geodesic starting from p with tangent vector $\mathbf{v}$. See Penrose [18] p.4 and Misner, Thorne and Wheeler [11] p.285.

Definition 3.17 Let $p, q \in M$. p *causally precedes* q if there exists a future-directed causal curve beginning at p and ending at q. This is also denoted $p \prec q$.

These definitions both imply that events at point p can be used to affect events at point q. In the case $p \ll q$ matter can be used to transmit information (e.g. throwing tennis balls) whereas in the case $p \prec q$ any form of communication is allowed (i.e. radio transmission). The symbols $\gg$ and $\succ$ are trivially defined from the statements above. Simple ordering relations also exist.

$$p \ll q \implies p \prec q \tag{3.2}$$
$$p \ll q, q \ll r \implies p \ll r \tag{3.3}$$
$$p \prec q, q \prec r \implies p \prec r \tag{3.4}$$

We also find that:

$$p \ll q \prec r \implies p \ll r \tag{3.5}$$
$$p \prec q \ll r \implies p \ll r \tag{3.6}$$

Definition 3.18 Given an event $p \in M$ the *chronological future* $I^+(p)$, *chronological past* $I^+(p)$, *causal future* $J^+(p)$, and *causal past* are defined as

$$I^+(p) = \{q \in M | q \gg p\}, \quad (\textit{chronological future}) \tag{3.7}$$
$$I^-(p) = \{q \in M | q \ll p\}, \quad (\textit{chronological past}) \tag{3.8}$$
$$J^+(p) = \{q \in M | q \succ p\}, \quad (\textit{causal future}) \tag{3.9}$$
$$J^-(p) = \{q \in M | q \prec p\}, \quad (\textit{causal past}) \tag{3.10}$$

The chronological and causal futures represent the sets of points of the spacetime that can be affected by events at p. Similarly points in the past of p can affect events at p. These definitions can be expanded to entire sets in the obvious way.

Definition 3.19 The *chronological* and *causal futures of a set* $S \subset M$ are defined by

$$I^+(S) = \bigcup_{p \in S} I^+(p) = \{q \in M : q \gg p \text{ for some } s \in S\} \quad \text{and} \tag{3.11}$$
$$J^+(S) = \bigcup_{p \in S} J^+(p) = \{q \in M : q \succ p \text{ for some } p \in S\} \tag{3.12}$$

respectively.

Definitions for the pasts of a set S can be obtained analogously. The set $I^+(p)$ can be shown to be open and consequently $I^+(S)$ is open for any $S \subset M$. Also it can also be shown that $I^+(S) = I^+(\overline{S})$ and $I^+(S) = I^+(I^+(S)) \subset J^+(S) = J^+(J^+(S))$. Similarly one might think that $J^+(p)$ is always closed or that $J^+(p) = \overline{I^+(p)}$ however these statements are in fact false. Figure 3.2 depicts a spacetime where both statements are not true. The 'shadow' cast by the removed point on the boundary of $I^+(p)$ prevents causal curves from extending further along the boundary. Hence the boundary does not form part of $J^+(S)$, and $J^+(S)$ must not be closed. These spacetimes can also be simply constructed for the causal and chronological futures of a sets.

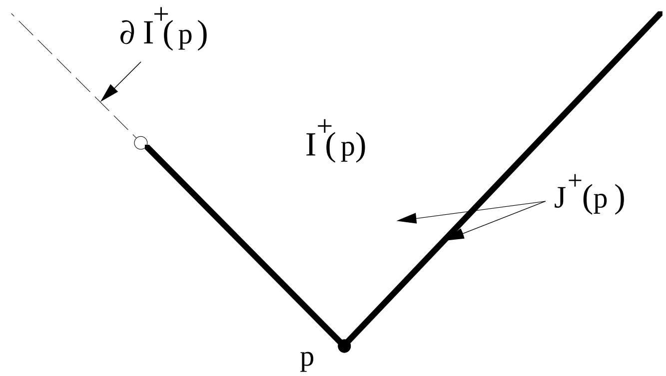

Figure 3.2: An example of a spacetime where $J^+(p) \neq \overline{I^+(p)}$.

3.2.3 Causality Conditions

Since causality is such an important concept in the physical sciences it is vital to characterise the many different notions used to define it in precise mathematical language. The lack of a fixed causal relationship between events can allow many paradoxical circumstances to occur. The problem arises mainly when a particle/observer is able is able to return to some initial event by following a timelike path. Such an observer for example would be able to influence their past and hence change history. Stated in another way an event would precede itself; clearly a pathological property. These closed causal paths are the global objects that must not be allowed to exist in the spacetime[11]. When a spacetime contains no closed timelike curves it is termed *chronological*. Similarly when a spacetime excludes closed causal curves it is termed *causal*. However excluding just these objects from the spacetime may not be sufficient to exclude ambiguous behaviour. Figure 3.3 depicts a spacetime which does not contain closed timelike or null curves but does allow a causal path to pass arbitrarily close to itself. Spacetimes which do not allow violations of this kind are termed *strongly causal*.

Definition 3.20 (Strong Causality) A spacetime is said to obey the *strong causality condition* if for all $p \in M$ and every neighbourhood U of p there exists a neighbourhood $V \subset U$ of p such that no causal curve intersects V more than once.

Let $\tilde{\mathbf{g}}$ be defined by

$$\tilde{g}_{ab} = g_{ab} - t_a t_b \tag{3.13}$$

where t_a is a continuous non-vanishing timelike vector field. Note that $\tilde{\mathbf{g}}$ is also a Lorentz metric. However the lightcones defined by $\tilde{\mathbf{g}}$ are also 'bigger' than those defined by $\mathbf{g}$ i.e. every causal vector of $\mathbf{g}$ is a timelike vector of $\tilde{\mathbf{g}}$ (see Figure 3.4).

Definition 3.21 (Stable Causality) A spacetime $(M, \mathbf{g})$ is termed *stably causal* if there exists a continuous non-vanishing timelike vector field $\mathbf{t}$ such that the spacetime $(M, \tilde{\mathbf{g}})$ (with $\tilde{\mathbf{g}}$ defined as in (3.13) above) has no closed timelike curves.

Definition 3.22 (Time-function) A *time-function* $\tau : M \to \mathbb{R}$ is a smooth scalar field

[11] It is presently a point of contention whether causality violations should be allowed in a spacetime or not. Many researchers are considering the implications and the consistency of relativity in such cases.

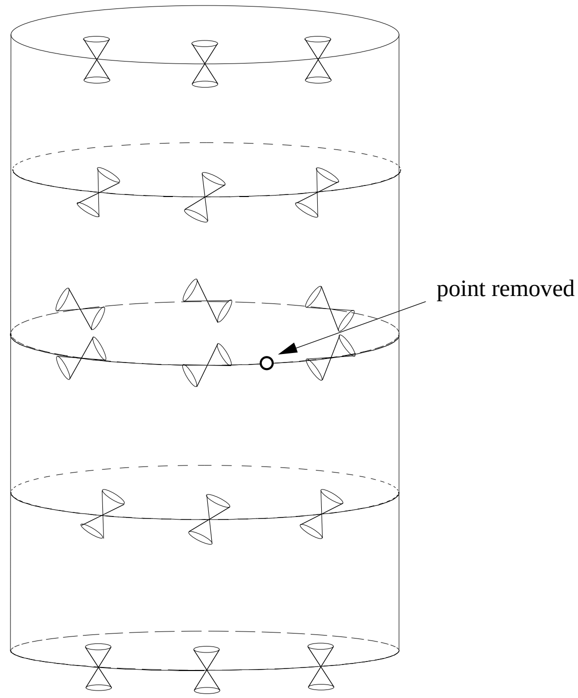

Figure 3.3: An example of a spacetime which does not allow the existence of closed timelike curves. Such a spacetime is causal but not strongly causal.

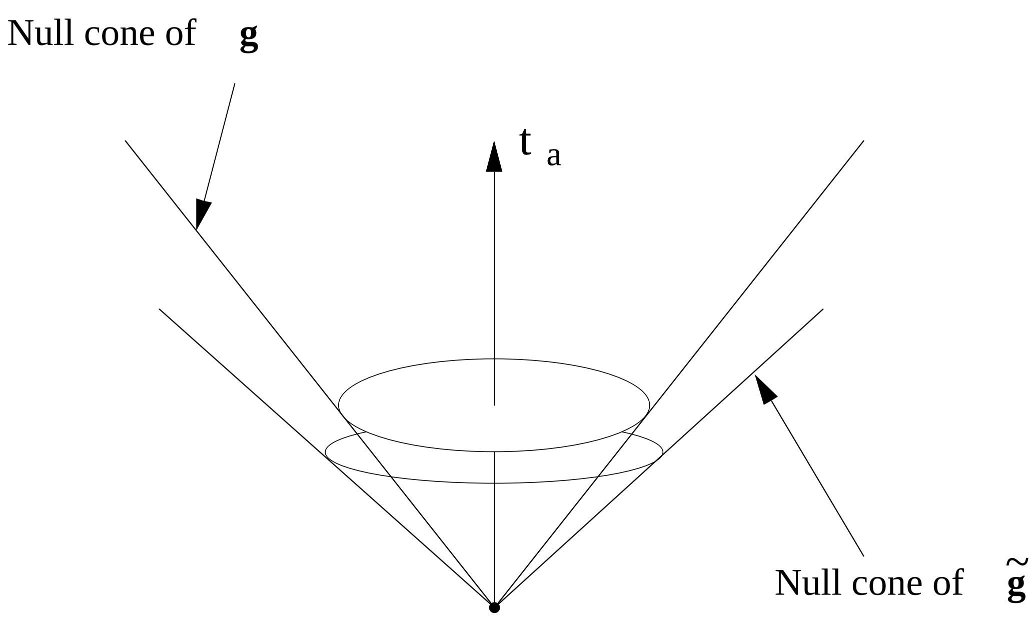

Figure 3.4: In a stably causal spacetime the light cones can be broadened in the direction of some timelike vector and the spacetime still remain causal.

whose gradient is a strictly timelike vector. This definition can be extended to C^0 functions by requiring them to be strictly increasing along each future timelike direction.

Definition 3.23 (Slice) A *slice* is a spacelike, three dimensional submanifold of the spacetime which is closed with respect to M.

Not all spacetimes admit slices. The Gödel spacetime is one well-known example[12]. However stably causal spacetimes allow slices passing through any point of the spacetime to exist. This can be proved in a straightforward manner by using the fact that stable causality allows the existence of a scalar time function over the spacetime. Hence the 3-dimensional, spacelike submanifolds t = constant will be the required slices.

Definition 3.24 (Achronal set) Let S be a closed set. Then S is termed *achronal* if no two points of S can be joined by a causal path.

An achronal set, then has no points which precede another point of the set. However it is possible to have slices which are not achronal (as in Figure 3.5)[13].

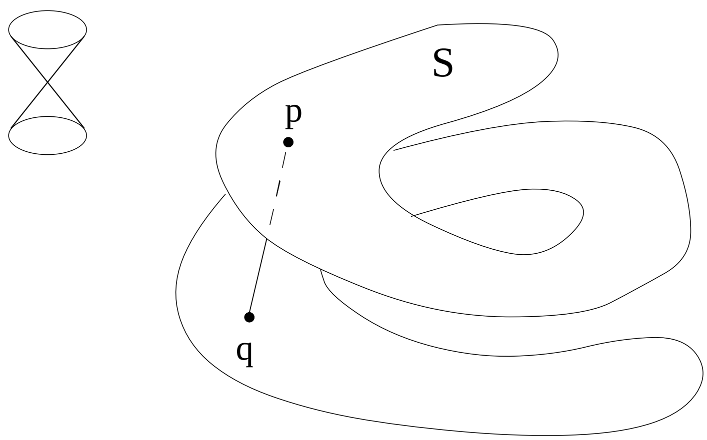

Figure 3.5: An example of a non-achronal slice.

Definition 3.25 (Edge of a set) The *edge* of a closed, achronal set S is defined as the set of points $p \in S$ such that for every open neighbourhood U of p there exists points q and $r \in U$ such that

(1) $q \in I^+(p)$,

(2) $r \in I^-(p)$,

(3) there exists a timelike curve γ from r to q where $\gamma \cap S = \phi$.

This definition is represented graphically in Figure 3.6.

[12]This spacetime is very badly behaved as there are closed timelike curves through every point of the spacetime. Details can be found in Hawking and Ellis [20] p.170.

[13]Not all slices require an edge to escape self-intersection. One example is a hyperplane on a Minkowski space with spatial toroidal boundary conditions. Such a hyperplane can be constructed with no edge, and since it 'wraps' around the spacetime achronality is violated.

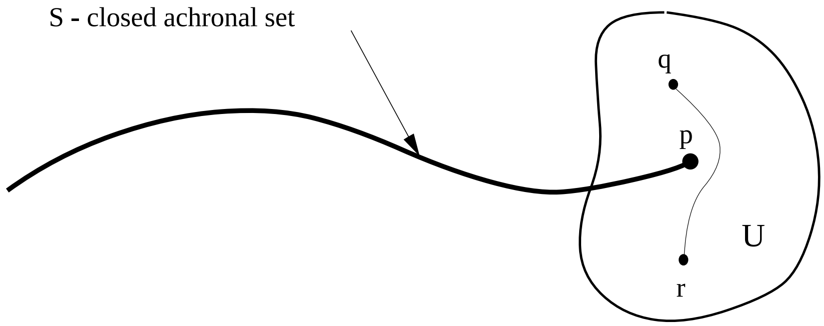

Figure 3.6: The definition of an edge.

Definition 3.26 (Domain of Dependence) Let $S \subset M$ be a closed achronal set. The *future*, *past* and *total domains of dependence* are defined as

$$D^+(S) = \{p \in M | \text{every past-endless causal path from } p \text{ meets } S\} \tag{3.14}$$

$$D^-(S) = \{p \in M | \text{every future-endless causal path from } p \text{ meets } S\} \tag{3.15}$$

$$D(S) = \{p \in M | \text{every endless causal path from } p \text{ meets } S\} \tag{3.16}$$

Clearly $D(S) = D^+(S) \cup D^-(S)$.

Examples of domains of dependence of a various sets are shown in Figures 3.7 and 3.8. The concept of domain of dependence is related to the question of what points of spacetime are 'completely determined' by the physical conditions defined in some region. Points outside the $D(S)$ may have their physics only partially defined by events on the set S. It should be noted that excised points or regions cast 'shadows'. These 'shadows' represent sections of the spacetime which would be affected by information or events originating in or near the excised region. So if for example some sort of black hole singularity of the spacetime were at the hole shown in Figure 3.8 then the shadow cast beyond it would be the set of events affected for example by Hawking radiation or the physical properties (mass, angular momentum, charge) of the black hole.

There are many results which explain how the physical data specified on some achronal slice S determine the physics of domain of dependence (see Hawking and Ellis [20] p.226-55 and Wald [15] p.243-68 for an introduction). One example is the forward time propagation of Maxwell field data from the slice. Given the values and derivatives of the electromagnetic fields on S and that the fields obey the Maxwell equations then there exists a unique smooth solution to the Maxwell equations in the domain of dependence of S. Hence the Cauchy initial value problem for a Maxwell field is well-posed in the domain of dependence.

It is also possible to recast the Einstein equation into a so-called $3+1$ formalism. This approach splits the Einstein equation into spatial and temporal parts. When self-consistent initial data is provided then there exists a metric satisfying the Einstein equation throughout the domain of dependence. This solution is consistent with the initial data set specified on S and the solution is defined everywhere in $D(S)$. Finally it is possible to choose a solution to the $3+1$ formalism that is maximal, implying that any other solution satisfying the constraints can be embedded within the maximal solution i.e. the maximal solution is unique and hence well-posed physically[14].

[14]This is only strictly true when the equations of the matter fields are specified and even then we only have this result for vacuum, scalar field and Maxwell field sources.

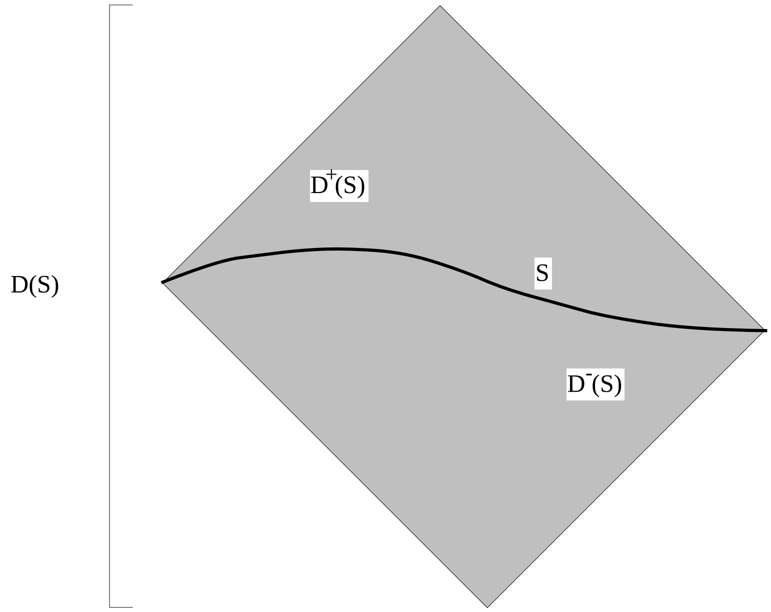

Figure 3.7: The domain of dependence of a closed achronal set S.

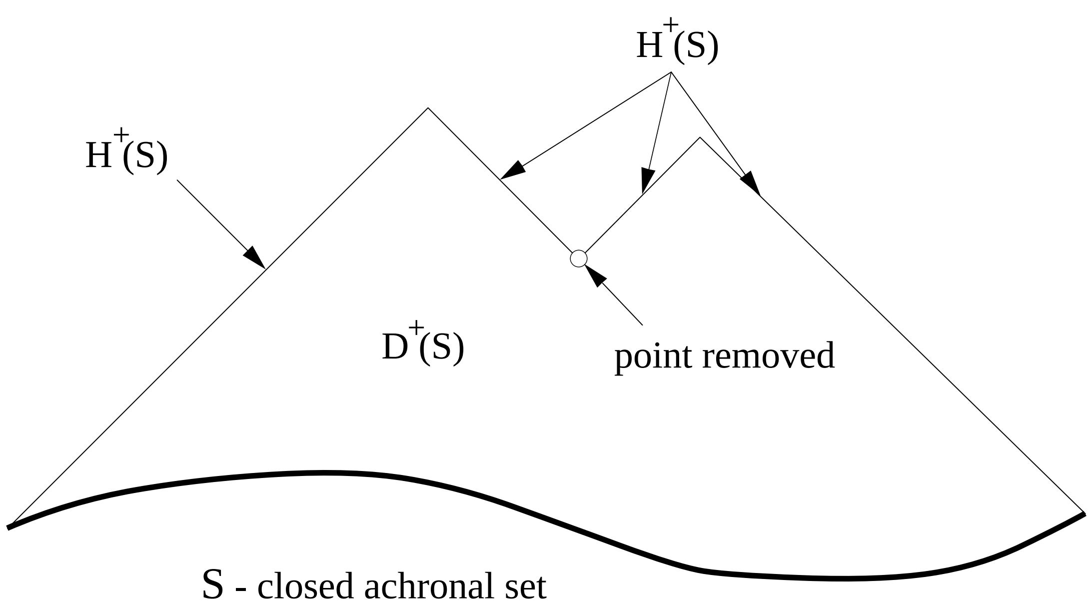

Figure 3.8: The effect of a ‘hole’ on the domain of dependence.

Figure 3.7 also demonstrates another important definition. It is useful to know the properties of the set of points which are in domain of dependence but which precede no other points of the future domain of dependence. These points form the future Cauchy horizon.

Definition 3.27 The *future*, *past* and *total Cauchy horizons* of some closed achronal set $S \subset M$ are defined as

$$H^+(S) = \{p \in M | p \in D^+(S) \quad \text{and} \quad I^+(p) \cap D^+(S) = \phi\} \tag{3.17}$$
$$H^-(S) = \{p \in M | p \in D^-(S) \quad \text{and} \quad I^-(p) \cap D^-(S) = \phi\} \tag{3.18}$$
$$H(S) = H^+(S) \cup H^-(S) \tag{3.19}$$

respectively.

The above definition can be neatly restated as

$$H^\pm(S) = D^\pm(S) - I^\mp(D^\pm(S)) \tag{3.20}$$

The domains of dependence and Cauchy horizons of a closed achronal set S have the following properties (similar ones holding for the past domains and horizons):

(1) $D^+(S)$ is closed,

(2) $H^+(S)$ is closed,

(3) $S \subset D^+(S)$,

(4) $x \in D^+(S) \implies I^-(x) \cap J^+(S) \subset D^+(S)$,

(5) $\partial D^+(S) = H^+(S) \cup S$,

(6) $\partial D(S) = H(S)$,

(7) $I^+(H^+(S)) = I^+(S) - D^+(S)$,

(8) $\text{int}[D^+(S)] = I^+(S) \cap I^-(D^+(S))$.

The Cauchy horizon if it is not empty is always null. In fact there always exists a past directed inextendible null geodesic within the Cauchy horizon.

Theorem 3.3 Let S be an achronal slice and $p \in H^+(S)$. Then there exists a maximally extended past-directed null geodesic λ such that $\lambda \subset H^+(S)$.

Proof. Choose a point $p \in H^+(S)$ and a sequence $\{p_i\}$ of points that converge to p from the future. Since $p_i \notin D^+(S)$ for any i there is a sequence of timelike past-directed curves $\{\gamma_i\}$ with endpoints p_i respectively such that all the curves in the sequence can be maximally extended and all curves do not intersect S. Now take the accumulation curve γ of the sequence, which must be causal and past-directed. γ cannot be timelike otherwise there would be some i for which γ_i intersects S. Hence γ is a null geodesic because no point of γ is in $I^-(p)$, and since $p \in D^+(S)$, γ is a subset of $D^+(S)$. If however there were a point of $D^+(S)$ were in $I^+(\gamma)$ then for some i, γ_i would intersect S and hence γ must lie in $H^+(S)$. □

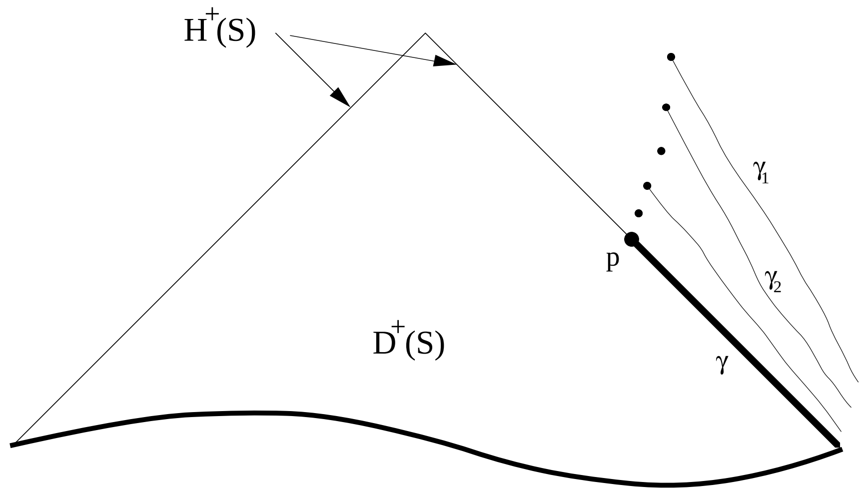

Figure 3.9: Construction of a null geodesic on the Cauchy horizon.

Also there are two important results about domains of dependence and maximal geodesics.

Theorem 3.4 Let S be an achronal slice and let p and q be points in $\mathrm{int}[D(S)]$, p preceding q, then the space of causal curves between p and q, $C(p,q)$, is compact[15]. Moreover the closure of $I^+(p) \cap I^-(q)$ is compact.

A proof of these facts are found in Wald [15] p.206.

Corollary 3.4.1 Between any two points p and q in $int[D(S)]$, $p \prec q$ there exists a timelike geodesic of maximal length.

Proof. This is easily determined from the compactness of $C(p,q)$. We can define on $C(p,q)$ a Lorentzian arc length function[16]:

$$L(\gamma) = \int_{t_i}^{t_f} \sqrt{-g(\dot{\gamma}(t), \dot{\gamma}(t))}dt = \int_{t_i}^{t_f} \sqrt{-g_{ab}\dot{\gamma}^a(t)\dot{\gamma}^b(t)}dt \tag{3.21}$$

This function must attain a maximum and hence within the domain of dependence there must be a maximal geodesic between p and q. This result can be extended to show that for any $p \in \mathrm{int}[D(S)]$ there exists a timelike geodesic of maximal length from p to S. □

We will use these two results above to prove some simple singularity theorems.

As can be seen from the above properties, the domain of dependence and the Cauchy horizon seem to be regions which obey our notions of what a physically realistic portion of spacetime is. It would seem that the 'nicest' type of spacetime is one which contains a surface which defines physical conditions everywhere. This leads to the concept of *global hyperbolicity*.

Definition 3.28 (Global Hyperbolicity) A spacetime $(M, \mathbf{g})$ is termed *globally hyperbolic* if there exists an achronal slice S such that $D(S) = M$.

[15]This is true in the compact-open (i.e. C^0) topology, but not for instance the C^1 topology.

[16]It is important to note that this function is not continuous. However it is upper semi-continuous in the C^0 topology (Hawking and Ellis [20] p.214, and Beem and Ehrlich [17] p.135-140).

Consequently edge$[S] = \phi$, as if there were an edge, there would be points in M that are not in $D(S)$. The existence of a Cauchy surface is in fact a very strong condition on a spacetime. It turns out that all globally hyperbolic spacetimes must be stably causal and hence strongly causal. Moreover globally hyperbolic spacetimes must have a Cauchy surface through every point of the spacetime and these Cauchy surfaces must all be homeomorphic to one another. In conclusion global hyperbolicity ensures that a spacetime has a causal structure free of abnormalities[17]. This is due to restricting the spacetime to be more than causal but by ensuring that it is entirely deterministic as well. The relative strengths of the causality conditions considered in this section are displayed in Figure 3.10.

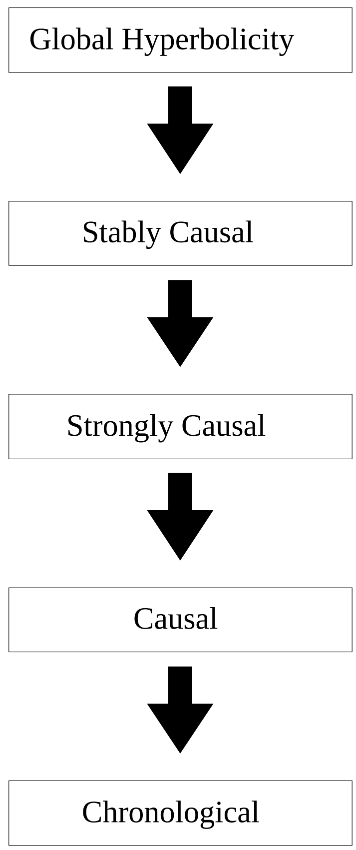

Figure 3.10: The relative hierachy of causality conditions. Arrows denote implication.

3.2.4 Compactness and Causality

The compactness or local compactness of spaces is something that is often taken for granted in physics. The use of the Lagrangian formulation requires local compactness to extract extremal paths from the first variation of physical quantities. However the assumption of global compactness of spacetimes is in fact quite useless. Apart from the fact that not all compact spacetimes admit metrics of Lorentzian signature (whereas non-compact ones do [18])

[17] Global hyperbolicity however does not rule out that there be achronal slices of different topology, take for example De Sitter spacetime.

[18] See Geroch and Horowitz [21] p.219.

compactness also allows spacetimes to possess causality violations. The following theorem from Beem and Ehrlich [17] p.58 states:

Theorem 3.5 Any compact spacetime $(M, \mathbf{g})$ contains a closed timelike curve and thus fails to be chronological.

Proof. This can be simply understood using our previous definition of compactness. If we assume that M is compact and define an arbitrary timelike vector field on M then we can choose some timelike curve which is an integral curve of the vector field. We now choose a sequence of points along this curve and since M is compact there is an accumulation point q say. Hence by the definition of accumulation point in every neighbourhood of q there would an infinite number of points from the sequence. Hence our timelike curve must pass though every neighbourhood of q more than once leading to causality violation. However since the timelike vector field was arbitrary any timelike curve will violate causality. □

It is mainly due to the above theorem that compact spacetimes will not be important to the later treatment of the singularity theorems.

3.3 Singularities

Singularities are found in many different field theories. One natural way of defining a singularity is that a singularity is a point where some physical quantity becomes infinite. For example in electromagnetism the electric field of a point charge could be considered singular at the point $r = 0$ as at this point the field becomes infinite. However a point charge is just a concept in electromagnetism. No point charges actually exist and so there is no real physical problem associated with it. In relativity there are many different ways of characterising the meaning of singularity. As in electromagnetics we could try and define a some type of field variable which may then become infinite. However relativity does not provide any unique choice of field variable to use. One method of attack is to consider field quantities based on the tidal forces that gravity exerts on test objects. Some of the objects that would appear good candidates would be:

(1) components of the Riemann tensor, R_{abcd},

(2) components of the Ricci tensor, $R_{ab} = R_{acb}{}^{c}$,

(3) components of the Weyl tensor, C_{abcd},

(4) the Kretschmann scalar, $R_{abcd}R^{abcd}$,

(5) the curvature scalar, $R = R_{ab}{}^{ab}$,

(6) the sectional curvature (see Beem and Ehrlich [17] p.29).

Infinities in any or all of these objects would descibe the singular mechanism of infinite tidal forces. It was this notion of curvature singularity that led to the Friedmann, Schwarzschild and other spacetimes being considered singular. Initially it was thought that these singularities were simply due to the simple symmetrical structure of the exact solutions that were known. It was well known that similar symmetrical situations in Newtonian gravity[19]also caused singularities. Another concept used to determine singularities originated in the problem of finding the boundary points of a spacetime. There have been many types of boundary construction used to determine and classify boundary points (e.g. the b-boundary (Schmidt 1971), the c-boundary (Geroch, Kronheimer and Penrose 1972) and most recently the a-boundary (Scott and Szekeres 1994)). One common theme to the boundary constructions however is the notion of path incompleteness or inextendibility. This entails that certain physically important paths or curves (i.e. timelike paths with bounded acceleration or geodesics) in the spacetime can not be extended to all values of their parameter. This would be the case for an observer falling into a black hole and having his existence terminated after only a finite time. Similarly we can look at past inextendibility of a curve as would be the case for an initial singularity such as in the Big Bang. The particular definition of a singular spacetime used for this section will be based on geodesic incompleteness:

Definition 3.29 (Geodesic Incompleteness) A spacetime is said to be *timelike/null incomplete* if it contains a maximally extended timelike/null geodesic whose affine parameter does not assume values in the full range $(-\infty, \infty)$.

The singularity theorems stated later in this section guarantee the existence of singularities by showing that given certain causal properties of the spacetime and properties of the stress-energy tensor for the source, causal geodesics cannot be extended to arbitrary values of their affine parameter. Such a case is clearly singular since any free-falling observer (represented by a timelike geodesic) would experience only a finite time. Moreover the singularity theorems are topological in nature. Apart from the physically reasonable energy conditions on the matter the properties of the Einstein equation are not really used. This implies that the presence of singularities are closely related to the qualitative properties of gravity and that any theory of gravity without singularities would be quite different. Before going on to prove some weaker versions of the singularity theorems some knowledge of the energy restrictions on matter is required. This is detailed in the next section.

3.3.1 Energy Conditions

It is reasonable to assume that the stress-energy tensor will obey certain inequalities by virtue of the fact that it must represent real matter. Various types of stress-energy tensor can be classified according to the type of matter-energy they represent and mathematically by their eigenvectors (see Hawking and Ellis[20]). In all generality the stress-energy tensor will be composed of many different contributions from all types of fields[20]. In this general case it would be very difficult to classify the stress-energy tensor even given the dynamical equations for each field. One of the most important relations we expect the stress-energy to obey is that the local energy density measured by any observer be non-negative. Since $T_{ab}v^a v^b$ is the energy density as measured by an observer with 4-velocity v^a the *weak energy condition* can be defined.

[19]If a collapsing, spherically symmetric, nonrotating shell of dust is released from rest a singularity will form at $r = 0$ when all the matter reaches that point.

[20]Any type of field possessing energy-momentum must be included i.e. electromagnetic, scalar fields, neutrinos, etc.

Definition 3.30 (Weak Energy Condition) For all timelike vectors $\mathbf{v} \in T_p(M)$ and at all points $p \in M$, $\mathbf{T}(\mathbf{v}, \mathbf{v}) \geq 0$.

Note that by continuity the above inequality will also be true for any null vector $\mathbf{w} \in \mathbf{T}_p(M)$.

The local energy flow vector measured by an observer with 4-velocity v^a is $T_{ab}v^b$. We expect that this local energy flow vector is causal. Requiring the stress-energy tensor to obey this and the weak energy condition is termed the *dominant energy condition.*

Definition 3.31 (Dominant Energy Condition) For all timelike $\mathbf{v}$, $\mathbf{T}(\mathbf{v}, \mathbf{v}) \geqslant 0$ and $T_{ab}v^b$ is causal.

This leads to an equivalent statement of the dominant energy condition as an inequality between components of the stress-energy tensor. i.e.

$$T^{00} \geqslant |T^{ab}|,$$

for all a, b.

Hence the dominant energy condition implies that in any orthonormal basis the energy dominates the other components of the stress-energy tensor.

The final important energy condition used in dealing with the singularity theorems is the *strong energy condition.*

Definition 3.32 (Strong Energy Condition) For all timelike $\mathbf{v}$, $\mathrm{Ric}(\mathbf{v}, \mathbf{v}) \geqslant 0$.

Since the Ricci tensor, R_{ab}, is a measure of the curvature, the strong energy condition essentially states that the curvature encountered by an observer with 4-velocity $\mathbf{v}$ is always attractive and that some congruence of timelike geodesics cannot expand. Using the Einstein equation it is possible to rephrase the strong energy condition as:

$$T(\mathbf{v}, \mathbf{v}) = T_{ab}v^a v^b \geqslant -\frac{1}{2}T \tag{3.22}$$

where $T = T_a{}^a$ is the trace of the stress-energy tensor. The inequality above implies that for matter/energy to violate the strong energy condition immense negative pressures are required to make the trace strongly negative. However these conditions have never been observed in the universe.

A geometrical condition which is useful in this study of singularities is the *generic condition.*

Definition 3.33 (Generic Condition) A spacetime is said to satisfy the *generic condition* if every non-spacelike geodesic (with tangent vector $\mathbf{K}$) posseses a point at which $K_{[a}R_{b]cd[e}K_{f]}K^cK^d$ is nonzero.

This condition ensures (roughly speaking) that all non-spacelike geodesics of the spacetime encounter 'effective curvature' at some stage in their path. In this sense effective implies that neighbouring geodesics will eventually be refocussed somewhere in the spacetime. This condition seems reasonable as any metric which has symmetries allowing a K to exist such that $K_{[a}R_{b]cd[e}K_{f]}K^cK^d = 0$ can always be perturbed so that this is not the case.

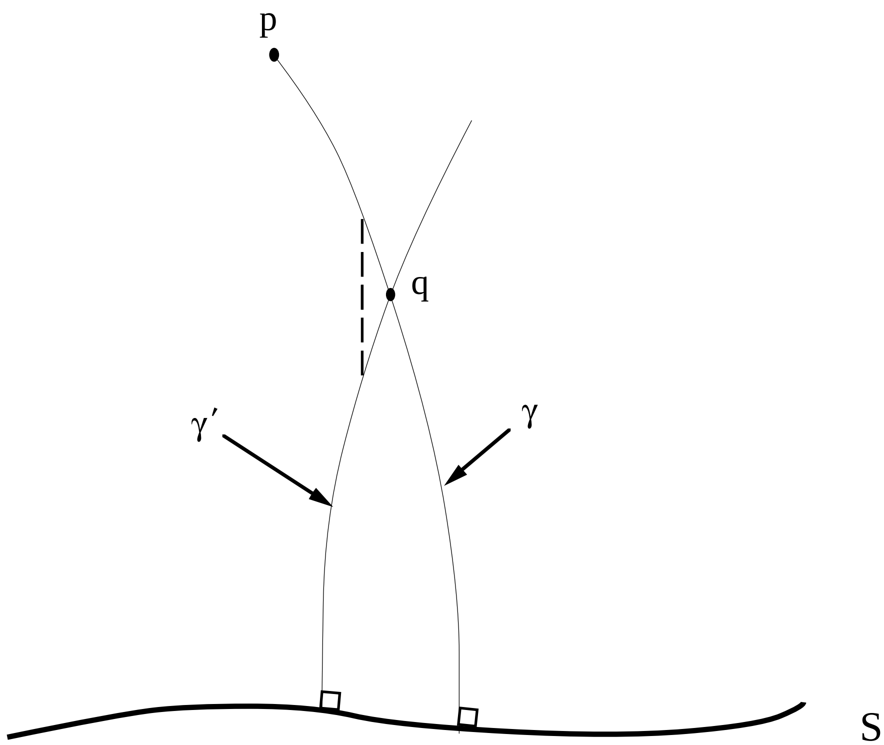

Figure 3.11: A geodesic that is crossed by another geodesic cannot be maximal.

3.3.2 Maximal Geodesics and Conjugacy

The proofs of the singularity theorems that follow will require some basic relationships between the curvature tensor and its effects on neighbouring geodesics. Most importantly we will need to show when a causal geodesic is not *maximal*. A geodesic between a point p and and achronal slice S is maximal if it possesses the longest Lorentzian length of any curve between the two. However it is important to notice that if a geodesic is crossed by another then it cannot be maximal[21]. The the point at which the second geodesic intersects the original geodesic is termed a *conjugate point*[22]. This situation is depicted in Figure 3.11. Suppose that γ is a maximal geodesic between p and S. If there exists a second geodesic γ' intersecting the first at a point q say then it is possible to construct a new geodesic, normal to S having the same endpoint p by going first along the geodesic γ' to q and then along the second from q to p. This broken geodesic may be 'lengthened' by rounding off the corner as shown by the dotted line. Hence the original geodesic could not have been maximal. The non-maximality of geodesics will be used with the earlier theorems guaranteeing the existence of maximal geodesics to obtain a contradiction leading to geodesic incompleteness. The only problem is that we have no criteria for examining when geodesics cross. This issue is explored next.

[21] See Hawking and Penrose [22] p.10 for an interesting Riemannian analogy.

[22] A *cut point* is defined as the point along a geodesic at which the geodesic ceases to be maximal. However not all conjugate points are cut points. Taking the example of a cylinder, the cut point of some particular point and chosen geodesic is the point of the geodesic on the opposite side of the cylinder. However there are no conjugate points since any two geodesics from one point never focus.

A brief analysis of the convergence of geodesics

Let $(M, \mathbf{g})$ be a time-orientented spacetime and let $S \subset M$ be a slice. In some neighbourhood of U of S we choose a function $t : U \to \mathbb{R}$ where:

(1) $t(p) = 0$, for all $p \in S$, and

(2) $\nabla_a t$ be unit length, future-directed and timelike for all $p \in U$.

We now define $t^a = \nabla^a t = g^{ab}\nabla_b t$. Since $t^a t_a = -1$, $\nabla(t^a t_a) = 0$ being the gradient of a constant function. This implies that $t^m \nabla^a t_m = 0$. Using the fact that:

$$\begin{aligned}\nabla_m \nabla_a t &= \nabla_m t_{,a}\\ &= t_{,am} - \Gamma^b_{\ am} t_{,b}\\ &= t_{,ma} - \Gamma^b_{\ ma} t_{,b} \quad (\nabla \text{ is torsionless and } t \in C^2(M, \mathbb{R}))\\ &= \nabla_a \nabla_m t \end{aligned} \tag{3.23}$$

we can show that $t^m \nabla_m t^a = \nabla_{\mathbf{t}} \mathbf{t} = 0$. Therefore $\mathbf{t}(p)$ is a unit geodesic vector field pointing normally from S. Defining the convergence of the field as $c = -\nabla_m t^m$ we obtain:

$$\begin{aligned} t^a \nabla_a c &= -t^a \nabla_a \nabla_b t^b\\ &= R_{ab} t^a t^b - t^a \nabla_b \nabla_a t^b \quad (\text{Using } [\nabla_a \nabla_b - \nabla_b \nabla_a] t^b = -R_{ab} t^b)\\ &= R_{ab} t^a t^b + (\nabla_b t_a)(\nabla^a t^b) \quad (\text{Using } \nabla_b (t^a \nabla_a t^b) = 0) \end{aligned} \tag{3.24}$$

Since $\nabla_a t_b$ is symmetric (i.e. obeys (3.23)) and has vanishing contraction with t^a, $\nabla_a t_b$ is a symmetric three dimensional tensor in the subspace orthogonal to t^a. Moreover, in that space the metric is positive definite hence $(\nabla_b t_a)(\nabla^a t^b)$ is at least one-third the trace of $\nabla_b t_a$ squared[23]. Therefore if we require that the matter in the spacetime obey the strong energy condition then:

$$t^a \nabla_a c \geqslant \frac{1}{3} c^2 \tag{3.25}$$

Note that in the case of equality this equation becomes:

$$\frac{d}{d\tau} c = c^2/3 \tag{3.26}$$

where $d/d\tau$ is the tangent vector to the geodesic. This is simply solved to yield that

$$c(\tau) = \frac{3c_0}{3 - c_0 \tau} \tag{3.27}$$

where $c = c_0 > 0$ is the convergence when $\tau = 0$. This implies that the convergence becomes infinite at parameter value $\tau = 3/c_0$, and hence all geodesics must cross each other within parameter distance $3/c_0$ along the geodesic. One can see from (3.25) that a positive convergence of the geodesics increases the rate of convergence of those same geodesics. This ultimately leads to a 'runaway' effect that forces all causal geodesics to cross within finite affine parameter.

[23]This can be determined more easily if one considers $(\nabla_b t_a)(\nabla^a t^b)$ to be the trace of the square of the matrix $(\nabla^a t^b)$.

General form of the geodesic analysis: The Raychaudhuri equation

The section above dealt with the particular case of timelike geodesics. However the analysis can be expanded to include all causal geodesics. Defining the spatial metric[24] perpendicular to the tangent to the curve, h_{ab}, as:

$$h_{ab} = g_{ab} + \xi_a \xi_b \tag{3.28}$$

the expansion, shear and vorticity are then defined as:

$$\theta = \nabla_c \xi^a g^{cb} h_{ab} \quad (\textit{expansion}) \tag{3.29}$$

$$\sigma_{ab} = \nabla_{(b}\xi_{a)} - \frac{1}{3}\theta h_{ab} \quad (\textit{shear}) \tag{3.30}$$

$$\omega_{ab} = \nabla_{[b}\xi_{a]} \quad (\textit{twist} \text{ or } \textit{vorticity}) \tag{3.31}$$

$$\tag{3.32}$$

Hence $\nabla_b \xi_a$ can be decomposed as:

$$\nabla_b \xi_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \tag{3.33}$$

where the shear is trace-less and symmetric and the vorticity is trace-less and antisymmetric. Using these definitions

$$\xi^c \nabla_c \theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\xi^c \xi^d \quad \text{(Raychaudhuri's equation)} \tag{3.34}$$

where ξ^a is a causal, unit length (i.e. $\xi^a \xi_a = -1$), geodesic vector field. This equation was first found by Raychaudhuri [23] in 1955. A proof of the Raychaudhuri equation can also be found in Wald [15] p.217. Note that the Raychaudhuri equation retains the major features of the initial treatment. Moreover it shows that while shear helps to create greater convergence vorticity does not. The effects of vorticity can be explained qualitatively. As vorticity causes geodesics to rotate spatially a centrifugal force is produced. This attempts to expand the rotating congruence of geodesics.

3.3.3 Some Simple Singularity Theorems

The singularity theorems that follow all have the same essential structure.

(i) An energy condition on the source of the curvature,

(ii) a causality condition for the spacetime,

(iii) a condition (topological) guaranteeing that the curvature is strong enough to trap the geodesics in some region.

[24] $h^a{}_b = g^{ac}h_{cb}$ is the projection operator onto the subspace of $T_p(M)$ orthogonal to unit vector $\xi(p)$. Note that the contraction of h_{ab} with ξ^a yields zero i.e. $h_{ab}\xi^a = (g_{ab} + \xi_a\xi_b)\xi^a = \xi_b - \xi_b = 0$ since ξ_a is of unit length.

The first requirement combined with the third, guarantee that geodesics in the spacetime cross or become conjugate. The second condition usually requires the spacetime to have maximal timelike or null geodesics. However these two conditions are incompatible and the result is geodesic incompleteness. To see this in action, three example theorems are proved below.

Theorem 3.6 Let $(M, \mathbf{g})$ be a time-oriented, globally hyperbolic spacetime obeying the strong energy condition. Let $(M, \mathbf{g})$ also possess a Cauchy surface S, where the convergence of the geodesics normal to S, is bounded below by $c_0 > 0$. Then $(M, \mathbf{g})$ is geodesically incomplete.

Proof. This will be proved by contradiction. Suppose that the spacetime were geodesically complete. We then choose a geodesic from S and go along it some affine parameter length greater than $3/c_0$; we define this as point p. Since $(M, \mathbf{g})$ is globally hyperbolic, p is in the future domain of dependence of the Cauchy surface, S. Hence by Theorem 3.4.1 there will exist a maximal geodesic between p and S. Since we have chosen p in the manner above the length of this maximal geodesic exceeds $3/c_0$. Moreover this maximal geodesic must be normal to S otherwise the curve could be altered to increase the length. However the convergence on S has lower bound c_0, hence by parameter value $3/c_0$ along the constructed maximal geodesic some other geodesic originating from S must cross and intersect it. This contradicts the fact that the constructed geodesic is maximal and hence the assumption is false and the spacetime is geodesically incomplete. □

As we can see above, all of the three conditions play a part in producing geodesic incompleteness. The strong energy condition guarantees that geodesics converge. Global hyperbolicity guarantees the existence of maximal geodesics to all portions of the spacetime. Finally, the lower bound on the convergence guarantees the conjugacy of geodesics and hence the failure of geodsics to be maximal. Although the singularity theorem above is very simple it is also quite weak. This weakness originates from two of the hypotheses. Firstly, it must be noted that the condition of global hyperbolicity is perhaps not completely physically realistic. We are unsure of the causal structure of the universe and global hyperbolicity is a stringent restriction. Secondly, the requirement that the convergence of geodesics be bounded below on Cauchy surface S is strong since c_0 must be greater than zero for all points of the Cauchy surface. No regions where c_0 disobeys this requirement (i.e. where matter is expanding) are allowed. Notice that since S really is a snapshot of the universe at one time that the convergence requirement is really another way of guaranteeing that at some time the entire universe is contracting. In realistic terms, we have no guarantee that our universe is expanding everywhere (i.e. the time-reversed situation).

The next theorem attempts to relax the condition of global hyperbolicity. It turns out that to do this more than one extra condition is required to replace global hyperbolicity.

Theorem 3.7 Let $(M, \mathbf{g})$ be stably causal time-oriented spacetime which admits a compact slice S (i.e. S is closed, spacelike sub-manifold). If the normals from S are everywhere converging (i.e. they have a lower bound $c \geqslant c_0 > 0$) then $(M, \mathbf{g})$ must be timelike geodesically incomplete.

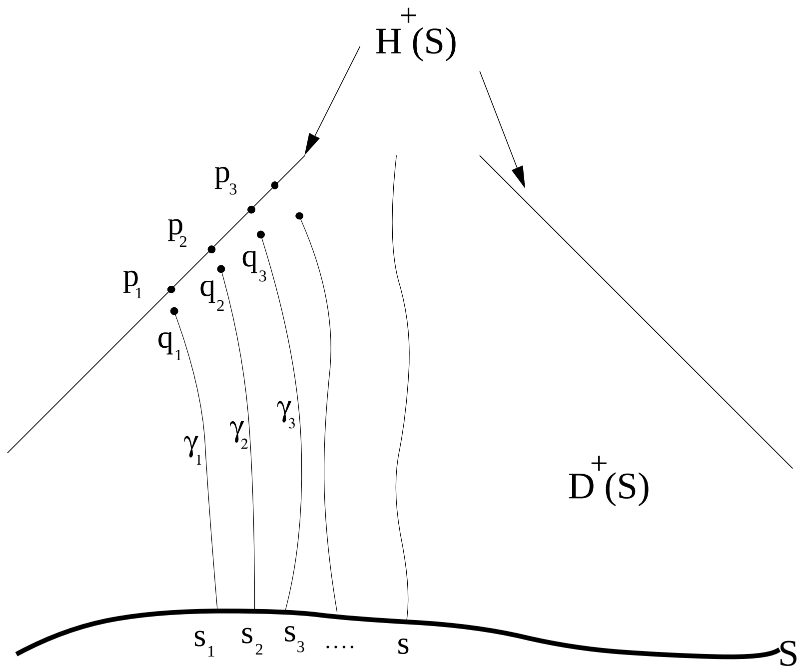

Figure 3.12: Diagram of the proof of theorem 3.6.

Proof. Suppose first that the spacetime is geodesically complete. We now consider the domain of dependence $D(S)$ and its future horizon $H^+(S)$. The future Cauchy horizon must either be compact or non-compact. Suppose first that $H^+(S)$ is compact. Due to theorem 3.3 there must pass through every point of the horizon a past directed maximally extended null geodesic which lies entirely within $H^+(S)$. However since the Cauchy horizon is compact this curve must return arbitrarily near to itself in direct violation of stable causality. Hence $H^+(S)$ cannot be compact. Since $H^+(S)$ is not compact there must be some sequence of points, $\{p_i\}$ say within $H^+(S)$ which does not have an accumulation point. We now displace these points slightly into the past and obtain a new sequence of points $\{q_i\}$ in the interior of $D^+(S)$ which must also have no accumulation point by construction. This is displayed in Figure 3.13. Since every point from $\{q_i\}$ is in $D^+(S)$, from each point q_i in the sequence there exists a maximal past-directed timelike geodesic which intersects S. This then defines a set of curves γ_i which each intersect S normally at some point s_i say. Since the s_i define a sequence of points on S, and by the compactness of S this sequence must have an accumulation point s. On S we already have a lower bound c_0 for the convergence of the normal timelike geodesics hence no geodesic can have length greater than $3/c_0$ otherwise we could find an accumulation point for the q_i[25] and hence for the sequence $\{p_i\}$. This leads to a contradiction as the curves cannot be both maximal and the spacetime geodesically complete. □

The next proof drops the condition that the convergence of geodesics emanating normally from S is bounded below. However another condition, the generic condition is required. It turns out that to make use of this condition we need to use the fact that in a spacetime that obeys the generic condition and the strong energy condition nearby causal geodesics must

[25] It is important to remember that if S is compact and $\Gamma = \{\gamma | \gamma \text{ is timelike and unit speed}\}$ then $K = \{\gamma(t) | \gamma \in \Gamma,\ |t| \leqslant a, \text{ and } \gamma(0) \in S\}$ is also compact.

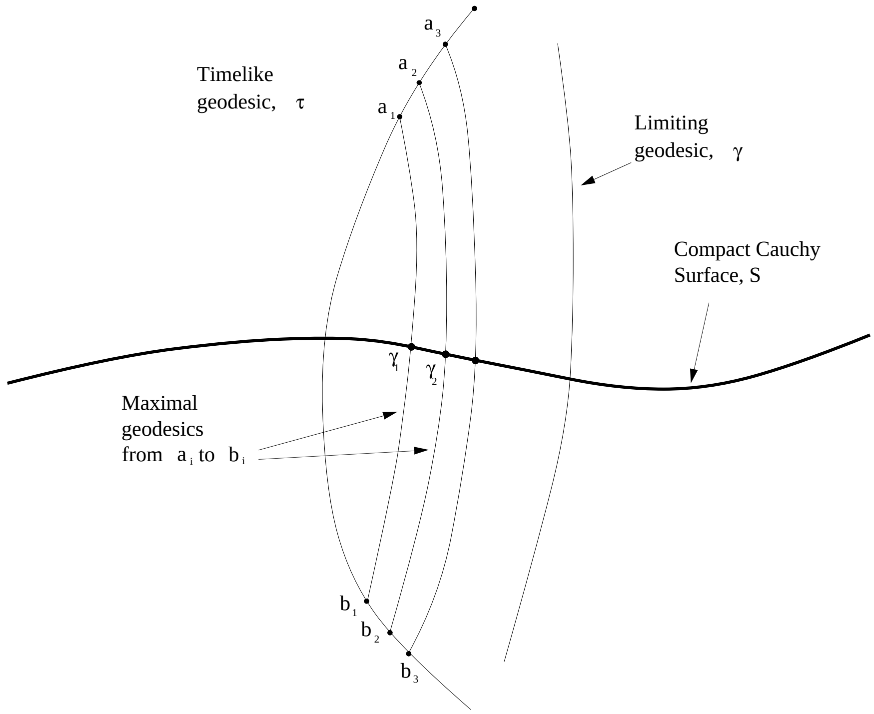

Figure 3.13: Diagram of the proof of theorem 3.8.

intersect each other more than once (see Wald [15] p.227-232).

Theorem 3.8 Let $(M, \mathbf{g})$ be generic and obey the strong energy condition. If $(M, \mathbf{g})$ possesses a compact Cauchy surface S then $(M, \mathbf{g})$ is geodesically incomplete.

Proof. Suppose that the spacetime is geodesically complete and let τ be a timelike geodesic in the spacetime which intersects S. We now choose a sequence of points $\{a_i\}$ on $\tau \cap D^+(S)$ such that each subsequent point is in the future of the previous one and the sequence extends indefinitely into the future. A similar sequence $\{b_i\}$ is chosen on $\tau \cap D^-(S)$. As S is a Cauchy surface there exists a timelike maximal geodesic γ_i which extends between each pair of points a_i, b_i. Since each a_i is in the future domain of dependence and b_i is in the past domain of dependence every geodesic intersects S. Each geodesic, γ_i can then be characterised completely by its point of intersection with S and the tangent vector of the geodesic at that point. The compactness of S guarantees that these intersection points must have an accumulation point and that there must be some limiting tangent direction which must be causal. Let γ be the limit curve through the accumulation point on S. γ will be a complete geodesic (by supposition). However since the spacetime obeys the generic and strong energy conditions some nearby geodesic must meet γ more than once. However this will also imply that there is some curve γ_i which also intersects the same nearby geodesic twice since γ is a limit curve of $\{\gamma_i\}$. But the γ_i are all maximal so we obtain a contradiction and hence $(M, \mathbf{g})$ is geodesically incomplete. □

In the above proof it is possible to eliminate the assumption of global hyperbolicity. The slightly less restrictive condition of stable causality can be assumed and also obtain geodesic incompleteness with all other conditions the same.

3.3.4 Singularity theorems

The purpose of this section is to state those singularity theorems that will be important to the later analysis of the singularity-free cosmological models.

Additional Notes and Definitions

Some specific terms are used in the following versions of the original singularity theorems. These are explained below.

Definition 3.34 (Null Second Fundamental Forms) Let $\mathcal{S}$ be a spacelike two-surface (i.e. an embedded two-dimensional submanifold) and let ${N_1}^a$ and ${N_2}^a$ be two null vectors normal to $\mathcal{S}$ normalised so that ${N_1}^a {N_2}^a g_{ab} = -1$. Similarly introduce two spacelike unit vectors X^a and Y^a orthogonal to each other and to ${N_1}^a$, ${N_2}^a$. The hence X^a and Y^a lie in the tangent space of the spacelike two-surface and the set of vectors $\{{N_1}^a, {N_2}^a, X^a, Y^a\}$ comprise a pseudo-orthonormal basis for $\mathcal{S}$. The two *second fundamental forms* of $\mathcal{S}$ are defined as:

$$ {}_n\hat{\chi}_{ab} = -N_{nc;d}(X^c X_a + Y^c Y_a)(X^d X_b + Y^d Y_b) \tag{3.35}$$

where n takes the values 1, 2.

This definition can be viewed as the extrinsic curvature of the spacelike two-surface in the following way. The second fundamental form, ${}_n\hat{\chi}_{ab}(p)$ is a map that takes in some vector at $T_p(M)$ and first finds the projection of that vector onto the tangent space $T_p(\mathcal{S})$, it then produces the directional derivative of the chosen null unit normal vector ${N_n}^a$ along that projection. Hence the null second fundamental forms map vectors into the infinitesimal change in the null normal vector in the direction of the vector projection on that two-surface.

Definition 3.35 (Closed Trapped Surfaces) A *closed trapped surface* $\mathcal{T}$ is a C^2 closed, compact, spacelike two-surface without boundary such that the two families of null geodesics orthogonal to $\mathcal{T}$ are converging at $\mathcal{T}$ (i.e. ${}_1\hat{\chi}_{ab}g^{ab}$ and ${}_2\hat{\chi}_{ab}g^{ab}$ are both negative where ${}_1\hat{\chi}_{ab}$ and ${}_2\hat{\chi}_{ab}$ are the two null second fundamental forms of $\mathcal{T}$).

The above definition makes sense since if both second null fundamental forms are negative then both inward and outward directed normal light beams from the surface are converging. Similarly if both forms are positive then there is some trapped surface for past directed beams (see Hawking and Ellis [20] p.2-3) .

Definition 3.36 (Horismos) The *future* and *past horismos*, $E^+(S)$ and $E^-(S)$ of the set S are defined as:

$$ E^+(S) = J^+(S) - I^+(S) \tag{3.36}$$
$$ E^-(S) = J^-(S) - I^+(S) \tag{3.37}$$

respectively.

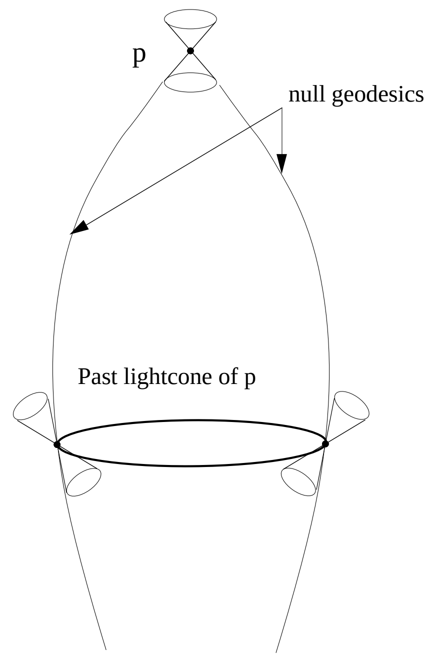

Figure 3.14: An example of a point with a converging past light cone.

The Singularity Theorems

Theorem 3.9 (Hawking and Penrose 1970) Spacetime $(M, \mathbf{g})$ is not timelike and null geodesically complete if:

(1) $R_{ab}K^aK^b \geqslant 0$ for every non-spacelike vector $\mathbf{K}$.

(2) The generic condition is satisfied, i.e. every non-spacelike geodesic contains a point at which $K_{[a}R_{b]cd}K_{[e}K_{f]}K^cK^d \neq 0$, where $\mathbf{K}$ is the tangent vector to the geodesic.

(3) The chronology condition holds on M (i.e. there are no closed timelike curves).

(4) There exists at least one of the following:

 (i) a compact achronal set without edge,

 (ii) a closed trapped surface,

 (iii) a point p such that on every past (or every future) null geodesic from p the divergence $\hat{\theta}$ of the null geodesics from p becomes negative (i.e. the null geodesics from p are focussed by the matter or curvature and start to reconverge as in Figure 3.14).

An alternative version of the theorem is that the following three conditions cannot all hold:

Theorem 3.10 (a) every inextendible non-spacelike geodesic contains a pair of conjugate points;

(b) the chronology condition holds on M;

(c) there is an achronal set S such that $E^+(S)$ or $E^-(S)$ is compact[26].

In the above theorem it is possible to replace conditions (2) and (3) by the following:

(2) S is a Cauchy surface for M.

(3) $\chi^a{}_a$ is bounded away from zero on S.

This version of the theorem will be used later in section 4.2.

Theorem 3.11 (Tipler 1977) Suppose that $(M, \mathbf{g})$ contains a maximal Cauchy surface S. Then there is at least one timelike geodesic which is incomplete to the future of S, and at least one timelike geodesic which is incomplete to the past of S, provided:

(1) the Einstein equation holds on $(M, \mathbf{g})$;

(2) the strong energy condition holds on $(M, \mathbf{g})$;

(3) there exist positive constants a, b such that

$$\left|\int_0^a (T_{ab} - \frac{1}{2}g_{ab}T)v^a v^b dt\right| = \left|\int_0^a R_{ab}v^a v^b dt\right| \geqslant b \tag{3.38}$$

for *every* timelike geodesic segment $\gamma \cap J^+(S)$ *and* $\gamma \cap J^-(S)$, where γ is a geodesic intersecting S orthogonally and the proper time t along γ is zero at S.

The following theorem due to Penrose [24] is one of the earliest general results on geodesic completeness which did not equire specific symmetry conditions.

Theorem 3.12 (Penrose 1965) A spacetime $(M, \mathbf{g})$ cannot be null geodesically complete if:

(i) $R_{ab}K^a K^b \geqslant 0$ for all null vectors $\mathbf{K}$.

(ii) there is a non-compact Cauchy surface Σ in M.

(iii) there is a closed trapped surface $\mathcal{T}$ in M.

3.3.5 Concluding remarks

This chapter is not intended to be a thorough treatment of global analysis and Lorentzian geometry. It was produced to aid the reader to quickly obtain a feeling for the subject and its methods. In this sense I have followed the respective treatments of Geroch and Horowitz [21], and Wald [15]. Many of the concepts dealt with in this chapter will be essential in the discussion of the singularity-free cosmological models in the next chapter. The simple theorems proved here have been chosen to elucidate the structure of the more complicated proofs found in Hawking and Ellis [20], Penrose [18], Beem and Ehrlich [17], and Tipler [25]. These sources should be consulted for a more in depth study of the standard singularity theorems. These texts will also provide more thorough treatments of other uses of global analysis and differential topology as used in relativity.

[26] in such cases S is termed *future trapped* or *past trapped* respectively.

Chapter 4

Singularity-free Models

The presence of singularities in the solutions of general relativity have always lead to significant problems of interpretation. Not only can exact solutions possess singular behaviour in objects such as the Ricci or Weyl curvature but other more pathological singular behaviour can occur. In fact general relativity allows singularities that can be directional in nature, as opposed to the 'black hole' type of singularity which is familiar to most and commonly encountered in astrophysical work. In addition the singularities of general relativity cannot be so easily dismissed as those that occur in electrodynamics or even Newtonian gravity. The field of a point sources in Newtonian gravity and electrodynamics both allow a $1/r$ potential singularity. We are able to explain away this problem since point sources do not really exist and the physical situations that are modelled by point sources really can be done (although by more complicated means) using extended sources; we define a 'density of source' and obtain the solutions through the Poisson equation. In Einsteinian gravity there are no simple solutions like this that can be applied, since in general relativity a singular point may represent a piece of the spacetime fabric integral to the spacetime as a whole but which has aberrant physical properties. Hence singularities may be some essential irregularity of the manifold or of the coordinates used to describe the spacetime. In the latter case choices of good global coordinates (e.g. the Kruskal-Szekeres extension) allows the solution to be extended beyond any of these 'removable' singularities. In these situations the singularity is not really an essential feature of the spacetime but simply a place where the coordinate system does not apply. The previous case produces a much more difficult problem. Such essential singularities stretch classical general relativity to the point where the significant length scale of the physics is so small that only a quantum version of gravity has any hope of fully describing the situation there. It is not surprising that the problems posed by singularities have been of sufficient severity that many theoreticians have appealed to alternate theories of gravity in an attempt to remove them.

The singularity-free models that are discussed in chapter this have specific properties that set them apart from other types of singularity-free models found in the literature:

(i) These models do not use alternate theories of gravity but simply use classical general relativity (as does the standard FLRW model). This choice is important if we eventually wish to create singularity-free models which may be realistic enough to represent our universe.

(ii) The singularity-free models are all orthogonally transitive G_2 cosmologies. This means that the symmetry group of the model is a 2-dimensional Lie algebra. The singularity-free metrics spacetimes considered here possess two spacelike commuting Killing vectors which for these models are $\boldsymbol{\partial}_z$ and $\boldsymbol{\partial}_\phi$. In particular these Killing vectors are mutually orthogonal and are hypersurface orthogonal; therefore the singularity-free models are cylindrically symmetric. Note also that the Killing vectors must commute otherwise the commutator will provide another Killing vector field and so the spacetime will not contain just two Killing symmetries. These models are therefore inhomogeneous and anisotropic and therefore have much more behavioural freedom.

(iii) All the models possess perfect fluid sources. Since the FLRW models also possess perfect fluid sources and since perfect fluids are a good general approximation to the matter of the universe only singularity-free models with at least a perfect fluid limit are any use in creating realistic singularity-free models.

(iv) The solutions all obey the strong energy condition and are globally hyperbolic. These two conditions are (by the arguments of the previous chapter) required elements of any undeniably singularity-free realistic cosmological model. Hence if we wish to have singularity-free models which do not circumvent the singularity theorems by proposing exotic matter or strange causal structure these requirements must not be dropped.

(vi) Finally the solutions have no provision for the internal formation of black holes. This is an obvious point but also implies that the models are not physically realistic. We now expect black holes to exist in the universe from substantial observational evidence, but it is not so certain that the universe was created in a Big Bang type initial singularity. This point will be raised again in the following chapter.

This chapter is structured in the following way. First a brief overview of the history of exact singularity-free solutions will be made. Then the simplest of these models will be examined to distinguish the major physical properties of the singularity-free models. Since the singularity theorems are not obeyed by these cosmologies some time will be spent examining the ways in which the singularity-free models escape the conclusions of the theorems. From this it will be concluded that the models escape the theorems by never allowing the geodesics in the spacetime to be trapped. Finally a summary of the present situation in singularity-free cosmology and the open questions that singularity-free cosmology has raised will be discussed.

4.1 History of the Singularity-free Models

4.1.1 Senovilla's original Singularity-free model

The very first singularity-free model of this type was found by Senovilla in 1990 [3]. The metric for this model is:

$$ds^2 = e^{2f}(-dt^2 + dr^2) + G(qd\phi^2 + q^{-1}dz^2) \tag{4.1}$$

where G, q and f are the following functions of t and r:

$$e^f = [AC(at) + BS(at)]^2 C(3ar) \tag{4.2}$$
$$G = [AC(at) + BS(at)]S(3ar)C^{-2/3}(3ar) \tag{4.3}$$
$$q = [AC(at) + BS(at)]^3 S(3ar) \tag{4.4}$$

where a, A and B are arbitrary constants and the expressions $S(x)$ and $C(x)$ have been used as shorthand expressions for $\sinh(x)$ and $\cosh(x)$ respectively. Choosing these constants appropriately and simplifying the metric we obtain:

$$ds^2 = C^4(at)C^2(3ar)(-dt^2 + dr^2) + \frac{1}{9a^2}C^4(at)S^2(3ar)C^{-2/3}(3ar)d\phi^2 + C^{-2}(at)C^{-2/3}(3ar)dz^2 \tag{4.5}$$

This solution has a perfect fluid source with the radiation dominated equation of state, $p = \rho/3$.

The Weyl scalars for the spacetime are:

$$\Phi_0 = 3a^2[C(2at)C(3ar) + S(2at)S(3ax)]C^{-6}(at)C^{-3}(3ar) \tag{4.6}$$

$$\Phi_2 = a^2[C(2at) - S^2(3ax)]C^{-6}(at)C^{-4}(3ar) \tag{4.7}$$

$$\Phi_4 = 3a^2[C(2at)C(3ax) - S(2at)S(3ar)]C^{-6}(at)C^{-3}(3ar) \tag{4.8}$$

Hence in general the spacetime is of Petrov Type I.

The non-vanishing components of the acceleration, expansion and shear for the fluid are:

$$a_1 = -3aS(3ar)C^{-2}(at)C^{-2}(3ar) \tag{4.9}$$

$$\theta = 3aS(at)C^{-3}(at)C^{-1}(3ar) \tag{4.10}$$

$$\begin{aligned}\sigma_{33} &= -2\sigma_{11} = -2\sigma_{22} \\ &= 2aS(at)C^{-3}(at)C^{-1}(3ar)\end{aligned} \tag{4.11}$$

From the expression for the acceleration above it is obvious that there is no vorticity. Evaluating the Einstein tensor, the pressure takes the form:

$$p = \rho/3 = 15a^2C^{-4}(at)C^{-4}(3ar) \tag{4.12}$$

where units $8\pi G = c = 1$ have been used. Note that all the above quantities were determined in the perfect fluid's comoving reference frame and for the Weyl tensor the natural null frame was used.

From the relations above the pressure and density of the model are both finite for all coordinate values. The singularity-free nature of the solution was deduced at this preliminary stage only by the fact that the Weyl scalars and the Ricci tensor were all regular in the limits $r \longrightarrow \infty$ and $t \longrightarrow \pm\infty$. This was determined using the fact that the Weyl and Ricci invariants can all be constructed from the Weyl scalars the pressure, p, and the density, ρ. Therefore all the curvature invariants are regular. There is also evidence of the fact that the hypersurface, $r = 0$ is special. This corresponds to a coordinate singularity however this is of no consequence as all curvature invariants are finite and hence regular along this axis. This feature is also important since the model has a discrete symmetry $r \to -r$ about this axis corresponding to the fact that the solution is cylindrically symmetric[1].

[1]It should also be noted that when $r = 0$ the spacetime is of Petrov type D.

One particularly important feature of this model is that the perfect fluid does not follow geodesics. The acceleration in such a model corresponds to a pressure acting against the effects of gravity, but no contribution to this process is made by vorticity. This simple fact allows the models to escape one of the simplest singularity theorems due to Raychaudhuri [23] (see Section 4.2).

Later this solution was found by Ruiz and Senovilla [6] in 1992 to be just one of a whole family of singularity-free solutions contained within a general class of inhomogeneous perfect-fluid solutions. This singularity-free family has the form:

$$\begin{aligned} ds^2 &= \cosh^{1+n}(at)\cosh^{n-1}(nar)\left[-dt^2 + \frac{\sinh^2(nar)}{P^2}dr^2\right] \\ &+ \cosh^{1+n}(at)\frac{P^2}{n^2a^2L^2\cosh^{(n-1)/n}(nar)}d\phi^2 + \frac{\cosh^{1-n}(at)}{\cosh^{(n-1)/n}(nar)}dz^2 \end{aligned}$$

where $L \equiv K - \dfrac{K-1}{2n}$ and $P^2 = \cosh^2(nar) + (K-1)\cosh^{(2n-1)/n}(nar) - K$.

Early in the same year Chinea, Fernández-Jambrina and Senovilla [4] showed that the spacetime given by (4.5) was in fact geodesically complete.

The geodesic equations for the spacetime as given in [4] are:

$$\begin{aligned} \ddot{t} + 2aT(at)(\dot{t}^2+\dot{r}^2) + 6aT(3ar)\dot{t}\dot{r} + (2/9a)T(at)S^2(3ar)C^{-8/3}(3ar)\dot{\phi}^2 \\ - aC^{-7}(at)S(at)C^{-8/3}(3ar)\dot{z}^2 = 0 \end{aligned} \quad (4.13)$$

$$\begin{aligned} \ddot{r} + 3aT(3ar)(\dot{t}^2+\dot{r}^2) + 4aT(at)\dot{t}\dot{r} - (1/9a)S(3ar)C^{-5/3}(3ar)[3 - T^2(3ar)]\dot{\phi}^2 \\ + aC^{-6}(at)S(3ar)C^{-11/3}(3ar)\dot{z}^2 = 0 \end{aligned} \quad (4.14)$$

$$(9a^2)^{-1}C^4(at)S^2(3ar)C^{-2/3}(3ar)\dot{\phi} = K \quad (4.15)$$

$$C^{-2}(at)C^{-2/3}(3ar)\dot{z} = L \quad (4.16)$$

$$C^4(at)C^2(3ar)(-\dot{t}^2+\dot{r}^2) + (9a^2)^{-1}C^4(at)S^2(3ar)C^{-2/3}(3ar)\dot{\phi}^2 + C^{-2}(at)C^{-2/3}(3ar)\dot{z}^2 = -\delta \quad (4.17)$$

where $T(x) \equiv \tanh(x)$ and K and L are constants of the motion due to the two Killing vectors in the spacetime. δ is a final constant of the motion such that:

$$\delta = \begin{cases} 1 & \text{if the geodesic is timelike, and} \\ 0 & \text{if the geodesic is null.} \end{cases}$$

A brief inspection of the above equations verifies that all the coefficients are nonsingular. Thus we expect the geodesic solutions of the differential equations to exist and be unique. Given that the first derivatives of the coordinate functions of the geodesics (i.e. the tangent vectors of any geodesic) are non-singular and bounded we expect to show that the geodesics in this family are complete. This result is proved rigorously in [4] and [5]. In both of these sources various families of geodesics are reviewed. It is important to note that the only way to guarantee the timelike and null geodesic completeness of a spacetime is to investigate the completeness of all families of causal geodesics within it.

Qualitative Physical Behaviour of the Senovilla's Model

From the the previous expressions for the Weyl scalars and the Einstein tensor it follows that this cosmology approaches that of flat space as $t \to -\infty$. As t increases the cosmological fluid begins to contract under the acceleration and the density increases. During this time there is a positive shear of the fluid congruence in the r and ϕ directions but the z direction has negative shear. At $t = 0$ the shear and the expansion vanish. Moreover at $t = 0$ the density attains its greatest value. As t now increases the whole process occurs in reverse with the shears changing sign. Just after the $t = 0$ the cosmology is inflationary (expanding) and the density begins to increase again. Again as $t \to \infty$ the spacetime approaches flatness and the kinematic properties, Weyl scalars etc. all tend to zero.

4.1.2 Singularity-Free Models Incorporating Heat Flow, Viscosity and Massless Scalar Fluids.

The next major development in the area of singularity-free solutions was the discovery (by Patel and Dadhich [7]) of singularity-free models which possess heat flow. Heat flow can be introduced into the stress-energy tensor of the source by including a convection term:

$$T_{ik} = (\rho + p)u_i u_k - p g_{ik} + \underbrace{2u_{(i}q_{k)}}_{\text{convection term}} \tag{4.18}$$

where $u_i u^i = 1$ and q_i is the heat flow vector obeying $q_i u^i = 0$. Hence the heat flow must be in the orthogonal subspace to the fluid flow vector. This model reduces to Senovilla's original perfect fluid model in the limit as $\mathbf{q} \longrightarrow \mathbf{0}$.

Patel and Dadhich also examined the effects of viscous drag on singularity-free models but found that the viscosity cannot be made non-negative for all times. Other models were also found. In particular a family of models which incorporate massless scalar fields allow the equation of state to be of the for the form, $\rho = \mu p$ where $4 \geqslant \mu \geqslant 3$. One of the most relevant results however was that by applying a simple inhomogenisation and anisotropisation of an open FLRW model it is possible to obtain the metric for the singularity-free family of perfect fluid models (4.13) mentioned in [6].

The Inhomogenisation and Anisotropisation Procedure of Dadhich, Patel and Tikekar

An FLRW model with negative curvature takes the form:

$$ds^2 = dt^2 - T^2(t)\left(\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{4.19}$$

transforming this into cylindrical coordinates,

$$r \longrightarrow (\sinh^2 z + \bar{r}^2 \cosh^2 z)^{\frac{1}{2}},\ \tan\theta \longrightarrow \frac{\bar{r}}{\sinh z\sqrt{1+\bar{r}^2}} \tag{4.20}$$

the metric takes the form:

$$ds^2 = dt^2 - T^2(t)\left(\frac{d\bar{r}^2}{1+\bar{r}^2} + (1+\bar{r}^2)dz^2 + \bar{r}^2 d\phi^2\right) \tag{4.21}$$

making the additional change of coordinate $m\bar{r} = \sinh(m\hat{r})$ and dropping the hats we obtain:

$$ds^2 = dt^2 - T^2(t)(dr^2 + C^2(mr)dz^2 + m^{-2}S^2mrd\phi^2) \tag{4.22}$$

where $S(u) \equiv \sinh u$ and $C(u) \equiv \cosh u$. The 'inhomogenisation' and 'anisotropisation' procedure consists of taking different powers of $T(t)$ and $C(mr)$. Therefore the metric has the form:

$$ds^2 = T(t)^{2\alpha}C^{2a}(mr)(dt^2 - dr^2) - T(t)^{2\beta}C^{2b}(mr)dz^2 - m^{-2}S^2(mr)T(t)^{2\gamma}C^{2c}(mr)d\phi^2 \tag{4.23}$$

where coordinate freedom allows a change of time coordinate so that $g_{tt} = |g_{rr}|$. The above metric implies that the fluid velocity 1-form for such a model is $\tilde{\mathbf{u}} = T^{\alpha}(t)\cosh^{a}(mr)\boldsymbol{dt}$. However the isotropy of the fluid pressures requires that $T(t) = \cosh(kt)$ and the condition $\alpha = \gamma$. The term $m^{-2}S^2(mr)$ does not participate in the inhomogenisation and anisotropisation process since, but serves merely to guarantee elementary flatness[2] near the axis $r = 0$ and that the ϕ coordinate has 2π periodicity. In order for (4.23) to correspond to a singularity-free metric we require that both the Weyl and Ricci tensors be regular at infinity. This restriction enforces that $\alpha = \gamma$. Together with the isotropy of pressure these two restrictions yield only two possible singularity-free circumstances:

(i) $b = c$, $\alpha = \gamma$, $\alpha + \beta = 1$, $a = -b/(1+2b)$, $k = (1+2b)m$,

(ii) $b + c = 1$, $\alpha = \gamma$, $\alpha + \beta = 1$, $a = -b(1-b)$, $k = 2m$.

The first case yields singularity-free cosmologies that do not obey the γ-law of state in general, however choosing $b = -1/3$ gives the radiation-dominated matter cosmology (4.5) of Senovilla. The second case yields stiff fluid solutions, however the matter free limit of these stiff fluid models yield two possible singularity-free vacuum solutions. This anisotropisation and inhomogenisation procedure tends to imply that there may be some deeper relationship between the open FLRW models and the singularity-free models. It turns out that they can be extracted from a single family of cosmological metrics. These metrics and their importance in producing a realistic singularity-free cosmology are the subjects of the following chapter. A significant result of the above analysis from [28] is that the singularity-free models in [6] are the only possible models which are cylindrically symmetric and have separable metric functions.

4.1.3 Non-diagonal Singularity-free models

At the beginning of 1995, Mars [8] found a singularity-free solution which has a non-diagonal metric. The symmetry structure of this metric is similar to the previous singularity-free

[2] Elementary flatness is concerned with the notion of a space being locally Lorentzian in a neighbourhood of a point. Take for example the manifold of a cone with the point at the tip removed. Every point of this manifold is flat, since no tearing is required to produce a cone from a flat sheet of paper. However if we draw a circle with centre at the removed point we find that the ratio of the circumference to the radius is less than 2π in the limit $r \longrightarrow 0$. This example is clearly not Euclidean even though the Riemann tensor vanishes everywhere. Such spaces are said to violate elementary flatness. This particular example of a 'conical' singularity is associated with the excision and 'reglueing' of some region of flat space. Hence this property is directly related to the global topology of the underlying manifold. Additional information on elemetary flatness can be found in Kramer et al. [26] p. 192 and Synge [27] p.273 and p.323.

models since it is invariant under an Abelian group of symmetries. However unlike the previous models the Killing vectors corresponding to these symmetries are not hypersurface orthogonal and therefore the metric is non-diagonal. Explicitly the metric of the spacetime is:

$$ds^2 = e^{\alpha a^2 r^2} \cosh(2at)(-dt^2 + dr^2) + r^2 \cosh(2at) d\phi^2 + frac1 \cosh(2at)(dz + ar^2 d\phi)^2 \tag{4.24}$$

where α and a are arbitrary constants and the coordinates have range: $-\infty < t,\ z < \infty$, $0 \leqslant r,\ 0 \leqslant \phi \leqslant 2\pi$. The solution possesses an axis of symmetry at $r = 0$ and obeys the elementary flatness condition. From these considerations the r coordinate can be interpreted to be the usual cylindrical radial coordinate. The model is a perfect fluid model with fluid velocity:

$$\mathbf{u} = \frac{e^{-(1/2)\alpha a^2 r^2}}{\cosh^{1/2}(2at)} \boldsymbol{\partial}_t \tag{4.25}$$

and so the expressions for the acceleration, expansion, and shear are:

$$\mathbf{a} = \alpha a^2 r \frac{e^{-\alpha a^2 r^2}}{\cosh(2at)} \boldsymbol{\partial}_r \tag{4.26}$$

$$\theta = \frac{a e^{-(1/2)\alpha a^2 r^2} \sinh(2at)}{\cosh^{3/2}(2at)} \tag{4.27}$$

$$\sigma_{11} = \sigma_{22} = -\frac{\sigma_{33}}{2} = \frac{2}{3}\theta \tag{4.28}$$

Note that the fluid congruence has no vorticity.

From evaluation of the Einstein tensor the density and pressure are given by:

$$p = \rho = \frac{a^2(\alpha - 1) e^{-\alpha a^2 r^2}}{\cosh(2at)} \tag{4.29}$$

The pressure and density will be positive if and only if $\alpha > 0$. It is well known that stiff fluid solutions can be obtained from vacuum solutions[3] using the procedure of Wainright, Ince and Marshman [29]. The vacuum solution that yields this family is contained within (4.24) as the case $\alpha = 1$. For non-zero a the Killing vectors of the spacetime are $\boldsymbol{\partial}_z$ and $\boldsymbol{\partial}_\phi$ as in the previous models but when $\alpha = 0$ there is a four dimensional group of symmetries transitive on a family of 3-dimensional spacelike hypersurfaces. Moreover this symmetry group also contains a 3-dimensional Bianchi type II subgroup[4]. The result of this is the model is locally rotationally symetric for $\alpha = 0$.

An examination of the Weyl scalars yield that the Weyl tensor is regular at infinity and as we have found in every other singulrity-free cosmology the general Petrov type is I except for points on the axis of symmetry. Combined with the regularity of the Einstein tensor at infinity Mars deduces the model has no curvature singularities. The models are also stably

[3]Since stiff fluids have the speed of sound in the medium equal to the speed of light the features of its dynamical equations are the same as those of the gravitational field. This procedure can be used to create stiff fluid solutions from vacuum solutions.

[4]See [26] Chapter 8 and in particular p.97.

causal and globally hyperbolic. Mars also states that an analysis of the geodesic equations shows this model to be timelike and null geodesically complete and hence is singularity-free.

Griffiths and Bičák [9] also deal with this same family of metrics. They state that this model had been obtained previously by Letelier [30] in 1979 but that the only complete analysis of the properties of this model were done by Mars. It seems then that the first singularity-free model of this type may well have been found over 10 years before Senovilla's original result. The remainder of Griffiths and Bičák's paper is dedicated to producing a solution composed of cylindrical gravitational and acoustic waves on a Letelier-Mars background spacetime. However we will not be considering these non-diagonal metrics from now on.

4.2 Evasion of the Singularity Theorems

It is instructive to view the methods in which the singularity-free solutions evade the singularity theorems. Since the models all obey the strong energy condition and are all globally hyperbolic the solutions satisfy all the physical prerequisites for representing realistic cosmologies. The strong energy condition guarantees that the matter content of these models will behave similarly to that which we find in our observed universe. Similarly the global hyperbolicity of such spacetimes give it the strictest causal structure of which we have no evidence to the contrary. Hence of the three conditions given in the singularity theorems of Chapter 3 the only condition which is not satisfied (and the only condition that can be at fault if we wish to obtain a physically realistic model) is the existence of some geodesic trapping criterion. Note from the previous chapter that with the other two types of condition certain trapping arguments guarantee geodesic incompleteness. One particular property of these models which may be an essential feature of singularity-free cosmologies is the presence of non-geodesic motion. It has been proved by Raychaudhuri that irrotational, expanding, geodesic models must possess an initial singularity if the strong energy condition holds. However in these spacetimes the fluid is almost always under the influence of an oppositely directed acceleration. Such an acceleration can be associated with the presence of a spacelike pressure gradient which generally acts against the force of gravity and prevents the existence of conditions extreme enough to produce trapped surfaces. In connection with the lack of such a trapping condition being satisfied by the singularity-free cosmologies we will examine in detail the counterexamples to the existence of various trapped surfaces in the particular model given in [4].

Theorem 3.10 is one of the simplest of the singularity theorems. It requires the existence of a Cauchy surface which has the trace of its second fundamental form bounded away from zero. However the trace of the second fundamental form for any of the Cauchy surfaces $t =$ constant, is given simply by the expansion of the fluid congruence θ. In particular:

$$\chi^{a}{}_{a} = \theta = 3aS(at)C^{-3}(at)C^{-1}(3ar) \tag{4.30}$$

From this expression we note that for all t, $\theta \to 0$ as $r \to \infty$. So these Cauchy surfaces in fact do not have $\chi^{a}{}_{a}$ bounded away from zero.

Another way to view this is that if we choose any Cauchy surface, $t = t_1$ (constant) which has $\theta_{t_1} < 0$ then there must be conjugate points to the Cauchy surface along every normal geodesic from the Cauchy surface by at least affine parameter value $-3/\theta_{t_1}$. But we have no bounds on θ_{t_1} away from zero hence the parameter distances between the conjugate points and the Cauchy surface have no upper bound. Hence there always exists a maximal geodesic normal to the Cauchy surface through every point of the future of the surface.

One interesting additional property possessed by the Cauchy surface Σ, $t=0$ is that it is a maximal spacelike hypersurface. Maximal surfaces possess a second fundamental form[5] which is traceless. From the results developed in (Tipler [25]) this implies that the spacetime is both causally and time symmetric. This requires that the spacetimes have the property that $I^+(S)$, $J^+(S)$ and $D^+(S)$ are isometric to $I^-(S)$, $J^-(S)$ and $D^-(S)$ respectively. In the same paper it is proved that causally symmetric spacetimes obey a different type of singularity theorem. This theorem was introduced as Theorem 3.11 in Chapter 3. Theorem 3.11 guarantees the existence of a singularity in causally symmetric spacetimes by requiring that the spacetime possess a non-compact, maximal Cauchy surface, Σ, obey the strong energy condition and finally a condition on the Ricci tensor of the spacetime. This condition was that there must exist b,$c>0$ such that:

$$\left|\int_0^b R_{ad}v^a v^d d\tau\right| \geqslant c$$

is satisfied by all timelike geodesics possessing tangent vector, $\mathbf{v}$ which are normal to Σ and intersect Σ at affine parameter value $\tau=0$. The result of the above theorem is future and past geodesic incompleteness. The first two conditions of the theorem are satisfied by the singularity-free spacetime since the Cauchy surface Σ above is clearly not compact and the source obeys the strong energy condition. However the condition on the Ricci tensor is not satisfied. This is because if we choose any two constants b and $c>0$ the value of the integral must be positive but it does not possess a bound above zero. We can always find some timelike geodesics with a large initial r such that the integral is as small as we like and so we can construct cases with the integral smaller than any $c>0$.

One of the first singularity theorems is due to Penrose. Theorem 3.12 requires that the strong energy condition hold, that there are non-compact Cauchy surfaces in the spacetime and that a closed trapped surface exists. The first two conditions are satisfied and it remains to prove that the third condition is not. First, in order to obtain a contradiction we assume that the spacetime does possess some closed trapped surface, $\mathcal{T}$. Since the surface is compact there must exist some point $p\in M$ such that the coordinate function $r(p)$ is a maximum. This point, p, possesses a normal vector which must therefore be some linear combination of the basis vectors $\boldsymbol{\partial}_\mathbf{r}$ and $\boldsymbol{\partial}_\mathbf{t}$. However the traces of the two null second fundamental forms[6], ${}_\pm\hat{\chi}_{ab}$, at p have the form:

$$\begin{aligned}{}_\pm\hat{\chi}^b{}_b &= {}_\pm\hat{\chi}_{ab}g^{ab} \\ &= -g^{zz}n_{z,z}(p) - g^{\phi\phi}n_{\phi,\phi} + 2^{-1/2}aC^{-2}(at)S^{-1}(3ar)[-T(at)T(3ar)\pm 3\mp 2T^2(3ar)]\end{aligned} \tag{4.31}$$

where $T(x)=\tanh(x)$. The functions $n_{z,z}(p)$, $n_{\phi,\phi}$ must be positive for outgoing normals and negative for ingoing normals at point p by construction of the null normal pseudobasis hence:

$${}_+\hat{\chi}^b{}_b \geqslant 2^{-1/2}aC^{-2}(at)S^{-1}(3ar)[1-T(at)T(3ar)]>0 \tag{4.32}$$

$${}_-\hat{\chi}^b{}_b \leqslant 2^{-1/2}aC^{-2}(at)S^{-1}(3ar)[1+T(at)T(3ar)]<0 \tag{4.33}$$

[5]This is also known as the extrinsic curvature.

[6]See definitions 3.34 and 3.35.

by taking bounds on the square bracketed term and using $0 \leqslant T^2(x) < 1$. So the two null second fundamental forms have opposite sign ; a contradiction. Therefore the 'trapped' surface is in fact not trapped.

A more physically intuitive example of the above result can be interpreted though. If we choose any closed compact 2-surface in the spacetime then it must intersect the ingoing and outgoing radial geodesics somewhere. But at every point of the spacetime there exist ingoing radial geodesics which are contracting and the outgoing radial geodesics which are expanding hence it is impossible to contain the radial geodesics in the chosen 2-surface.

The final and strongest of the theorems presented in Chapter 3 was Theorem 3.9. Since the spacetime obeys the condition $R_{ab}v^a v^b > 0$ for all null $\mathbf{v}$, the singularity-free model [4] is also generic[7]. Hence of the conditions given in Theorem 3.9 only the conditions 4 has not been shown for the spacetime. Condition 4 allows for three separate possibilities which all lead to geodesic incompleteness. These were:

(i) a closed trapped surface,

(ii) a compact achronal set without an edge,

(iii) M contains a point such that on every past (or future) null geodesic from the point the divergence of those null geodesics becomes negative.

(i) has already been shown not to hold in the singularity-free model. So it remains to show that the other two conditions do not hold. (iii) will be dealt with first. Through every point of the spacetime there are future directed radial null geodesics which diverge if outgoing and past directed radial null geodesics which diverge if ingoing. Hence there can be no points with property (iii). The family of null geodesics with $z =$ constant can also be used to construct a similar situation. Since these geodesics have bounded coordinate function $r(p)$ these null geodesics can never converge with the radial ones. The final possibility is (ii). Intuitively it is obvious that the there are no compact achronal sets without edge particularly because all the singularity-free models are open and any edgeless achronal set we choose to construct will not be bounded or compact. However a rigorous proof as found in [4] follows:

Proof. Let S be some edgeless achronal set of the manifold and q be a point in S. Using the radial geodesics of the spacetime we can always find points $q_+ \in I^+(S)$ and $q_- \in I^-(S)$ such that $r(q_-) = r(q_+) > r(q)$. However by construction $q_+ \in I^+(q_-)$ and $r(q_-) = r(q_+)$ hence there is a worldline of the cosmological fluid which passes passes through q_+ and q_-. Since S is edgeless then this worldline must intersect S at a point $\tilde{q}$ where $r(\tilde{q}) = r(q_-) = r(q_+) < r(q)$. Hence if we choose any point p on the edgeless achronal set S then we can always find another point $q \in S$ such that $r(p) < r(q)$ (i.e. S is not bounded[8]). Therefore S is not compact. $\square$

A diagrammatic version of the above construction is found in Figure 4.1.

In [4] it is stated that there are other singularity theorems in Hawking and Ellis [20] relevant to the cosmologies above. However the conditions in these theorems (namely theorems 3 and 4 p.271-2) are more restrictive than those in the theorems already considered. Since

[7]Note that the above condition is more stringent than the strong energy condition. It is stated in Beem and Ehrlich ([17] p.309) that if $\mathrm{Ric}(v, v) > 0$ for all null vectors $\mathbf{v} \in T(M)$ then every inextendible geodesic of $(M, \mathbf{g})$ satisfies the generic condition.

[8]Therefore we can construct a new net which extends off to unbounded r coordinate values and hence has no accumulation point.

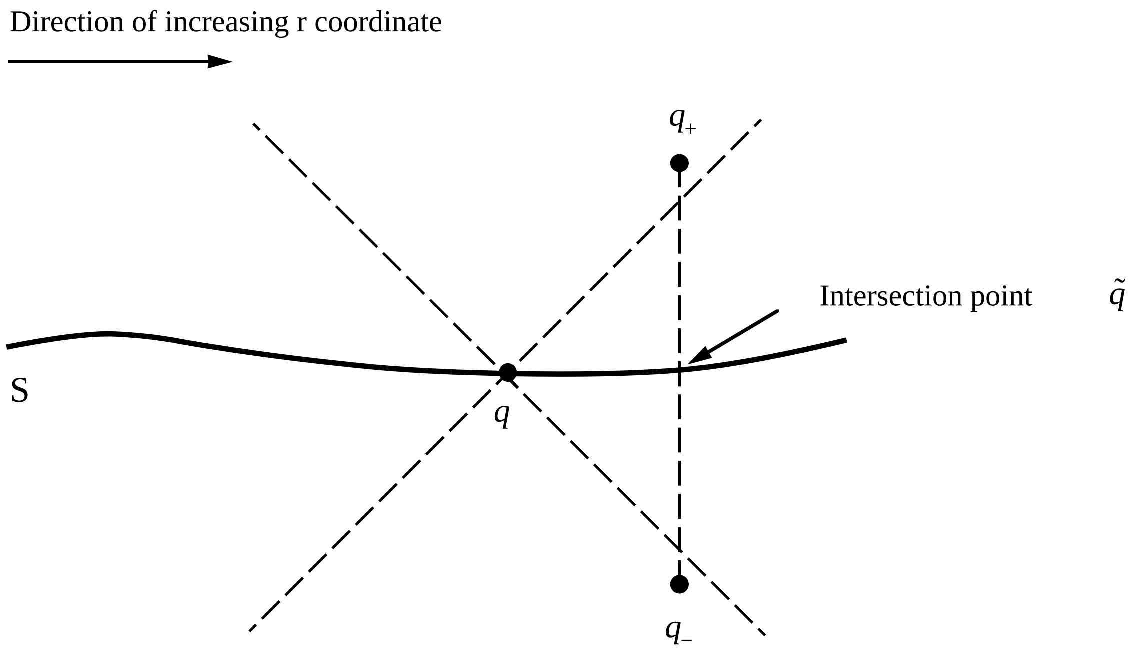

Figure 4.1: Construction of $\tilde{q}$.

it was already proved that the conditions of Theorem 3.9 are not true this must also apply to these other theorems. Using this argument the authors of [4] believe that their treatment of the singularity theorem violations are complete.

It must be remarked though that the singularity theorems are of themselves not complete and so guaranteeing that all the singularity theorems are not satisfied does not guarantee geodesic completeness. Thus it is important that geodesic completeness always be verified by examining the complete family of causal geodesics of a spacetime. Nevertheless an analysis of the failure of the singularity theorems is essential to gaining an insight into the essential structure that the singularity-free models possess. It seems that the common theme to the violations of the singularity theorems above is that the spacetimes never obey a condition that allows geodesics to be trapped in a region. The energy and causality conditions of the theorems are always obeyed as would be expected of any physically reasonable cosmology. Moreover the solutions obey the generic condition and so are quite general.

4.3 The Future of Singularity-free cosmology

This chapter was intended to be a summary of the properties and structure of singularity-free solutions which are known at present. The general qualitative properties of the cosmologies presented in this chapter are the same as those singularity-free models within the family of cosmologies presented in the following chapter. Correspondingly a knowledge of the cosmologies in this chapter is essential to understanding the problem of creating realistic singularity-free models.

An an area of exact solutions the field of singularity-free cosmology is in its infancy. The existence of singularity-free solutions was only shown relatively recently and there are few singularity-free solutions of any sort in the field.

Mathematically these cosmologies pose many interesting questions. Within the family of general inhomogeneous cosmological models found in [6] there are models with many different types of singularity as well as those models without singularity. Every singularity-free model in the family above has cylindrical symmetry, but every model in the family which does not posssess this symmetry has singularities. A more striking fact is that all the singularity-free models found uptil now possess cylindrical symmetry. It has been suggested [4], [28] that the lapse of symmetry in going from a spherical case to a cylindrical case may under certain circumstances be what enables a cosmology to be geodesically complete. Principal idea is that the increasing lack of symmetry allows any particular model more freedom. This may then allow global condition which cause timelike and null geodesics to never be trapped in a region. One intriguing question is whether this could be exploited to obtain singularity-free cosmologies which are less symmetric than the cylindrical models. We are however quite far from answering this question as the number of models available for analysis is quite small.

Originally there was doubt whether singularity-free models were stable against the presence of viscosity, heat flow and the like. However due to work by Dadhich, Patel and Tikekar, [28], [5] we now have singularity-free models possessing heat flow and those which couple to massless scalar fields. The stability of these models is in fact related with the question of whether the singularity-free models are substantive or whether, in a hypothetical space of metrics they comprise a set of measure zero. The existence of Senovilla's family of models developed in the next chapter partially address this problem. At present, no concrete theory of quantum gravity exists and so the problem of dealing with the physical conditions present at the Big Bang is at present not completely solvable. However with no quantum theory of gravity, singularity-free cosmology may be the only way to investigate the complete behaviour of the universe. A more relevant development however would be the construction of a realistic singularity-free cosmological model which agrees with current observational data. A realistic singularity-free model would be an alternative to the standard model, and because of its global regularity have a distinct advantages over the standard model. In particular the ability to determine properties throughout the entire history of the model even to the initial inflationary stage. However even in the standard model there are conflicts between theoretical models and observed data (i.e. the need for large quantities of 'dark matter', and the results of recent cosmological observations such as COBE.). Recently inflation models have become popular candidates to solve some problems, but the problems posed by primordial black holes still remain. Singularity-free cosmologies may well be realistic candidates for classically avoiding the Big Bang.

Chapter 5

FLRW-like Singularity-free cosmological models

In the previous chapter the structure and properties of the newly-found singularity-free cosmological models were discussed. In particular the ways in which these models escape the trapping conditions of the singularity theorems. Apart from obeying all energy conditions and being globally hyperbolic these solutions are geodesicaly complete and correspondingly agree with all the qualitative causal and source properties expected of a realistic cosmological model. However there is one major fault with any of these cosmologies that cannot be easily resolved in any of them. All of the singularity-free solutions do not agree with the presently observed approximate isotropy of the universe and correspondingly its homogeneity. The models encountered so far are all G_2 cosmologies and hence do not possess enough symmetries to be isotropic. Moreover there is no means within the models to make them more symmetric by reducing the anisotropy.

In early 1996 Senovilla published a paper that began to tackle these problems. The paper (Senovilla [10]) presents an entire family of cosmological models which contain within them all the present G_2 diagonal singularity-free models found but additionally the family also contains FLRW models. Moreover a cosmology can be continuously selected from the family by a single real parameter. Such a family could then be used to construct a singularity-free cosmological model that develops more FLRW-like as cosmic time passes on.

The purpose of this chapter is to introduce this family of cosmological models and describe the salient features and properties of the cosmological models contained within the family. Using these results we will then formulate some methods of attack on the problem and analyse the difficulties found in trying to formulate a singularity-free cosmological model from a suitable generalisation of the metric of this family. Finally the aims and direction of future research into this problem will be stated.

5.1 Senovilla's family of singularity-free and FLRW metrics

The metric for Senovilla's family of models is:

$$ds^2 = T^{2(1+n)}\Sigma^{2n(n-1)}(-d\tau^2 + dr^2) + T^{2(1+n)}\Sigma^{2n}\Sigma'^2 d\phi^2 + T^{2(1-n)}\Sigma^{2(1-n)}dz^2 \tag{5.1}$$

where $T(\tau)$ is an arbitrary function of the time coordinate, τ, n is a non-negative constant and $\Sigma(r)$ is a function of r which obeys the ordinary differential equation:

$$\Sigma'^2 - \left(M\Sigma^2 + N - nK\Sigma^{2(1-2n)}\right) = 0 \tag{5.2}$$

where M, N and K are arbitrary constants.

It is important to note that the metric given in (5.1) possesses the two global Killing vector fields $\boldsymbol{\partial}_z$ and $\boldsymbol{\partial}_\phi$ and that it is a solution of Einstein's equation for anisotropic fluids in general. Given any arbitrary function $\Sigma(r)$ the above metric possesses a diagonal Einstein tensor with anisotropic pressures. Condition (5.2) is a necessary but not sufficient condition for the metric family (5.1) to possess a perfect fluid stress-energy tensor. Hence the FLRW models within (5.1) will always obey the differential equation (5.2). It is instructive to examine the solutions that are produced by this differential equation for varying M, N and K.

5.1.1 Solutions of $\Sigma'^2 - \left(M\Sigma^2 + N - nK\Sigma^{2(1-2n)}\right) = 0$

Case $nK = 0$

The differential equation takes the form:

$$\Sigma'^2 = M\Sigma^2 + N \tag{5.3}$$

Dealing with the derivative of this equation we obtain:

$$\Sigma'' = M\Sigma \tag{5.4}$$

This is a standard second order equation with general solution:

$$\Sigma(r) = Ae^{\sqrt{M}r} + Be^{-\sqrt{M}r} \tag{5.5}$$

$$\tag{5.6}$$

Therefore Σ can be rewritten as in terms of trigonometric and hyperbolic functions as:

$$\Sigma(r) = \begin{cases} K\cos(\sqrt{|M|}r) \quad \text{or} \quad K\sin(\sqrt{|M|}r) & \text{if } M \leqslant 0, \text{ or} \\ K\cosh(\sqrt{M}r) \quad \text{or} \quad K\sinh(\sqrt{M}r) & \text{if } M \geqslant 0, \end{cases} \tag{5.7}$$

with K an arbitrary constant. Note however that when $M = 0$ we obtain the linear solution $\Sigma(r) = \sqrt{N}r + A$, with A an arbitrary constant. Using (5.6) and substituting into (5.3) we obtain:

$$\Sigma'^2 = M\Sigma^2 - 2MAB \tag{5.8}$$

Hence $N = -2MAB$ if M is non-zero. If $AB \geqslant 0$ then $\Sigma(r)$ is either a cosine or hyperbolic cosine function. Similarly if $AB \leqslant 0$ then $\Sigma(r)$ is either a sine or hyperbolic sine function. Finally this implies that:

$$\Sigma(r) = \begin{cases} K\cosh(\sqrt{M}r) & \text{if } M \geqslant 0 \text{ and } N \leqslant 0, \\ K\sinh(\sqrt{M}r) & \text{if } M \geqslant 0 \text{ and } N \geqslant 0, \\ K\cos(\sqrt{|M|}r) & \text{if } M \leqslant 0 \text{ and } N \geqslant 0, \\ K\sin(\sqrt{|M|}r) & \text{if } M \leqslant 0 \text{ and } N \leqslant 0. \end{cases} \tag{5.9}$$

Finally if $N = 0$ then we obtain an exponential solution $\Sigma(r) = Ae^{\pm\sqrt{M}r}$.

From this initial analysis of the solutions of (5.2) we see that the effect of the sign of M is to select whether $\Sigma(r)$ is either a hyperbolic function or a standard trigonometric function. Similarly the sign of N then determines whether $\Sigma(r)$ is the cosine or sine version of the function.

Case $N = 0$, $n \neq 0$

In this case the differential equation takes the form:

$$\Sigma'^2 - \left(M\Sigma^2 - nK\Sigma^{2(1-2n)}\right) = 0 \tag{5.10}$$

This case can be explicitly solved using the substitution:

$$\Sigma(r) = \Xi^{\frac{1}{2n}}(2nr) \tag{5.11}$$

The resulting differential equation for $\Xi(x)$ is then:

$$\left(\frac{d\Xi}{dx}\right)^2 = M\Xi^2 - nK \tag{5.12}$$

Hence following the treatment above, $\Xi(x)$ is hyperbolic if $M \geqslant 0$, linear if $M = 0$ and trigonometric if $M \leqslant 0$. In particular for the case $M \geqslant 0$, the hyperbolic function is either $\sinh(\sqrt{M}r)$ or $\cosh(\sqrt{M}r)$ dependent on the sign of K. Finally the required function $\Sigma(r)$ can be obtained by substituting $\Xi(x)$ into (5.11) with $x = 2nr$. Shown explicitly,

$$\Sigma(r) = \begin{cases} K\cosh^{\frac{1}{2n}}(2\sqrt{M}nr) & \text{if } M \geqslant 0 \text{ and } K \geqslant 0, \\ K\sinh^{\frac{1}{2n}}(2\sqrt{M}nr) & \text{if } M \geqslant 0 \text{ and } K \leqslant 0, \\ K\cos^{\frac{1}{2n}}(2\sqrt{|M|}nr) & \text{if } M \leqslant 0 \text{ and } K \leqslant 0, \\ K\sin^{\frac{1}{2n}}(2\sqrt{|M|}nr) & \text{if } M \leqslant 0 \text{ and } K \geqslant 0. \end{cases} \tag{5.13}$$

The general behaviour of $\Sigma(r)$

In the general case where M, N and K can assume arbitrary values Senovilla suggests that a 'kinematic' approach be taken to examine the qualitative behaviour of the solutions of (5.2) instead of solving it explicitly. If Σ'^2 is considered to be the square of a 'coordinate velocity' of Σ then using a conservation of energy approach we can consider the effective potential,$V(\Sigma)$ to be:

$$V(\Sigma) = -(M\Sigma^2 + N - nK\Sigma^{2(1-2n)}) \leqslant 0 \tag{5.14}$$

By examining the extrema and zeros of $V(\Sigma)$ we can determine the properties of $V(\Sigma)$ and hence the behaviour of Σ. This type of analysis yields the following general results:

- When $M < 0$, Σ behaves like a trigonometric function which can be chosen to be a cosine plus an arbitrary constant.

- When $M > 0$ then Σ behaves like a hyperbolic function (sinh or cosh) or an exponential, depending on the values of N, K, and n.

The most important consequence of the second case is that the function $\Sigma(r)$ is never vanishes. Otherwise the metric would become degenerate and lead to possible singularity formation. However the function Σ' may become zero for certain values of Σ. This is derived from the kinematic analysis which yields zeros of Σ' at the same places as $V(\Sigma)$ possesses its zeros. Since Σ' is a coefficent of one of the terms in metric (5.1) we would expect that this could cause a singularity also. The terms for the Einstein and Weyl tensors will show that this is not true in general. In fact $\Sigma' = 0$ defines the axis of cylindrical symmetry of the model. It is possible to make this axis regular by requiring that the space satisfy the requirement of elementary flatness in its neighbourhood.

5.1.2 The Einstein and Weyl tensors for the metric

If we take the natural comoving basis,

$$\begin{aligned} \boldsymbol{\omega}^{\hat{\tau}} &= T^{1+n}\Sigma^{n(n-1)}\boldsymbol{d}\tau, & \boldsymbol{\omega}^{\hat{r}} &= T^{1+n}\Sigma^{n(n-1)}\boldsymbol{d}r, \\ \boldsymbol{\omega}^{\hat{\phi}} &= T^{1+n}\Sigma^{n}\Sigma'\boldsymbol{d}\phi, & \boldsymbol{\omega}^{\hat{z}} &= T^{1-n}\Sigma^{1-n}\boldsymbol{d}z \end{aligned} \tag{5.15}$$

the Einstein tensor of the spacetime has the form:

$$G_{\hat{\tau}\hat{\tau}} = \frac{[(n+1)(n-3)+2]\dfrac{\Sigma''}{\Sigma} - (n+1)(n-3)\dfrac{\dot{T}^2}{T^2} - \dfrac{\Sigma'''}{\Sigma'}}{T^{2(1+n)}\Sigma^{2n(n-1)}} \tag{5.16}$$

$$\begin{aligned} G_{\hat{r}\hat{r}} &= \frac{(n-1)^2\dfrac{\Sigma''}{\Sigma'} - (n^2-2n-1)\dfrac{\dot{T}^2}{T^2} - 2\dfrac{\ddot{T}}{T}}{T^{2(1+n)}\Sigma^{2n(n-1)}} \\ &= G_{\hat{\phi}\hat{\phi}} \end{aligned} \tag{5.17}$$

$$G_{\hat{z}\hat{z}} = \frac{\dfrac{\Sigma'''}{\Sigma'} + n(n+2)\dfrac{\Sigma''}{\Sigma} - (n-1)^2\dfrac{\dot{T}^2}{T^2} - 2(n+1)\dfrac{\ddot{T}}{T}}{T^{2(1+n)}\Sigma^{2n(n-1)}} \tag{5.18}$$

Differentiating (5.2) a few times we obtain relations for Σ''' and Σ'' in terms of Σ':

$$\Sigma'' = M\Sigma + nK(2n-1)\Sigma^{1-4n} \tag{5.19}$$

$$\Sigma''' = M\Sigma' + nK(2n-1)(1-4n)\Sigma^{-4n}\Sigma' \tag{5.20}$$

where we have assumed $\Sigma' \neq 0$. Substituting (5.19) and (5.20) into the terms of the Einstein tensor and simplifying yields:

$$\rho = G_{\hat{\tau}\hat{\tau}} = \frac{(2n-1)(n-1)(n+3)nK + \Sigma^{4n}(n+1)(n-3)\left(M - \frac{\dot{T}^2}{T^2}\right)}{T^{2(1+n)}\Sigma^{2n(n+1)}} \tag{5.21}$$

$$p_r = G_{\hat{r}\hat{r}} = \frac{(2n-1)(n-1)^2 nK + \Sigma^{4n}\left((n-1)^2 M - [(n+1)(n-3)+2]\frac{\dot{T}^2}{T^2} - 2\frac{\ddot{T}}{T}\right)}{T^{2(1+n)}\Sigma^{2n(n+1)}} \tag{5.22}$$

$$p_z = G_{\hat{z}\hat{z}} = \frac{(2n-1)(n-1)^2 nK + \Sigma^{4n}\left((n+1)^2 M - (n+1)(n-1)\frac{\dot{T}^2}{T^2} - 2(n+1)\frac{\ddot{T}}{T}\right)}{T^{2(1+n)}\Sigma^{2n(n+1)}} \tag{5.23}$$

Hence the stress-energy tensor has the form $T_{ab} = \mathrm{diag}(\rho, p_r, p_r, p_z)$ where units have been chosen such that $8\pi G = c = 1$. It should be noted that the above Einstein tensor possesses no singular behaviour since $T(\tau)$ and $\Sigma(r)$ are both non-vanishing. If we equate the two pressures p_r and p_z we obtain a criterion for the cosmology to possess a perfect fluid source. hence:

$$p_r = p_z = p \Longleftrightarrow n\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2} - 2M\right) = 0 \tag{5.24}$$

Thus the source will be a perfect fluid if $n = 0$ or $\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2} - 2M\right) = 0$. This differential equation can be solved by the substitution $X(\tau) = T^2(\tau)$. Thus:

$$\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2} - 2M\right) = 0 \Longleftrightarrow \ddot{X} = 4MX \tag{5.25}$$

T solutions of this ordinary differential equation are of the form:

$$T^2(\tau) = \begin{cases} A\cosh(2\sqrt{(M)}\tau) + B\sinh(2\sqrt{(M)}\tau) & \text{if } M > 0, \\ A\tau + B & \text{if } M = 0, \\ A\cos(2\sqrt{(|M|)}\tau) + B\sin(2\sqrt{|}M|\tau) & \text{if } M < 0, \end{cases}$$

where A and B are arbitrary constants.

The Weyl scalars for the spacetime are:

$$\Phi_1 = \Phi_3 = 0,$$

$$3\Phi_2 = \frac{n\Sigma^{2n(1-n)}}{T^{2(n+1)}}\left[n(n^2-1)\frac{K}{\Sigma^{4n}} - \frac{\ddot{T}}{2T} + \frac{2n+1}{2}\frac{\dot{T}^2}{T^2} - Mn - (n-1)\frac{3N}{2\Sigma^2}\right],$$

$$\Phi_0 - \Phi_4 = 2n(n^2-1)\frac{\Sigma^{2n(1-n)}}{T^{2(n+1)}}\left[\frac{(M\Sigma^2 + N - nK\Sigma^{2(1-2n)})^{1/2}}{\Sigma}\frac{\dot{T}}{T}\right],$$

$$\Phi_0 + \Phi_4 = \frac{n\Sigma^{2n(1-n)}}{T^{2(n+1)}}\left[(2n+1)\frac{\dot{T}^2}{T^2} - \frac{\ddot{T}}{T} + 2M(n^2-n-1) + (2n-1)(n-1)\frac{N}{\Sigma^2}\right] \tag{5.26}$$

The first point to note is that the Weyl scalars are all regular. Hence there are no Weyl tensor singularities. As can be readily determined from the Weyl scalars[1] of the family of models is Petrov Type I in general. However if $n = 1$ or $\Sigma'(r) = 0$ then $\Phi_0 = \Phi_4 = 3\Phi_2$ and the Petrov type is D. Finally if $n = 0$ all the Weyl scalars vanish. Hence for the case $n = 0$ the Weyl tensor is identically zero and the spacetime is conformally flat.

5.1.3 Physical Properties of the Family of Cosmologies

The fluid velocity 1-form of the source within these models is given by

$$\tilde{\mathbf{u}} = -\omega^{\hat{t}} = T^{1+n}\Sigma^{n(n-1)}\boldsymbol{d}\tau \tag{5.27}$$

Hence the only non-vanishing component of the acceleration is:

$$a_{\hat{r}} = n(n-1)\Sigma^{n(n-1)}\frac{\Sigma'}{T^{1+n}\Sigma} \tag{5.28}$$

Using the definitions of expansion, shear and vorticity from chapter 3 we can decompose this into:

$$\theta = (n+3)\Sigma^{n(1-n)}\frac{\dot{T}}{T^{n+2}} \tag{5.29}$$

$$\sigma_{\hat{r}\hat{r}} = \sigma_{\hat{\phi}\hat{\phi}} = -\frac{\sigma_{\hat{z}\hat{z}}}{2} = \frac{2n}{3}\Sigma^{n(n-1)}\frac{\dot{T}}{T^{n+2}} \tag{5.30}$$

It is important to note that when $n = 0$ the acceleration and shear both vanish leading to geodesic fluid motion. Moreover from the analysis of the perfect fluid criteria this is also the condition required for the fluid to be perfect. Hence the metric for the case $n = 0$ is an FLRW model. Substituting $n = 0$ into (5.1) and (5.2) the family collases down to models of the form:

$$ds^2 = T^2(-d\tau^2 + dr^2 + \Sigma'^2 d\phi^2 + \Sigma^2 dz^2) \tag{5.31}$$

$$\Sigma'^2 = M\Sigma^2 + N \tag{5.32}$$

[1] See Kramer, Stephani, MacCallum and Herlt [26] p.43-65 for specific details on the Petrov classification of spacetimes.

The reader should note that these models are the familiar FLRW models in cylindrical coordinates that were found at the end of chapter 2 (i.e. (2.39) and (2.38). Using the expressions for the density and pressure in such a model we find that:

$$\rho_{FLRW} = \frac{3}{T^2}\left(\frac{\dot{T}^2}{T^2} - M\right) \tag{5.33}$$

$$p_{FLRW} = -\frac{1}{T^2}\left(2\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} - M\right) \tag{5.34}$$

which are (2.28) and (2.29), the dynamical equations for the FLRW cosmology. Hence the negative of the sign of M is the curvature index k for the FLRW models and so the cosmologies will be open, closed or flat dependent on whether M is positive, negative or flat respectively[2]. The most important consequence of the above results is that the metric family (5.1) with restriction(5.2) contain all the FLRW models in its homogeneous and isotropic limit.

However the most outstanding fact about the family of models (5.1) is that all the known G_2 diagonal separable singularity-free perfect-fluid models which satisfy the energy conditions can be found within it. These cosmologies are obtained by an explicit choice of T which obeys (5.24) and an appropriate choice of Σ. The singularity-free metric given in Ruiz and Senovilla [6] can be reproduced by substituting:

$$M > 0, \quad T^2(\tau) = \cosh(2\sqrt{M}\tau), \quad \Sigma^{2n} = \cosh(2n\sqrt{M}\bar{r}), \quad N = nK - M \tag{5.35}$$

and using the change of coordinate:

$$dr^2 = \frac{M\sinh^2(2n\sqrt{M}\bar{r})d\bar{r}^2}{M\cosh^2(2n\sqrt{M}\bar{r}) + N\cosh^{(2n-1)/n}(2n\sqrt{M}\bar{r}) - nK} \tag{5.36}$$

Substituting the equation of state for radiation-dominated matter, $p = \rho/3$, and the condition $M = 3K$ we can obtain the first singularity-free model by Senovilla [3] (4.5).

Contained within (5.1) there are many other singularity-free models. However the only ones of importance are those that obey the strong energy condition and have non-negative energy density. With the strong energy restriction the only singularity-free models are those with:

$$M > 0, K \geqslant 0, n > 1, T \neq 0 \tag{5.37}$$

$$\frac{(T^{n-1})^{\cdot\cdot}}{T^{n-1}} \leqslant (n-1)^2 M \tag{5.38}$$

The first condition implies that the singularity-free models are all open. The second implies that N can always be chosen such that $\Sigma(r)$ behaves like a hyperbolic cosine function. However in order to guarantee non-negative energy density we additionally require:

$$n = 3 \quad \text{or} \quad \left[n > 3 \quad \text{and} \quad \frac{\dot{T}^2}{T^2} \leqslant M \right] \tag{5.39}$$

[2]From the previous kinematic analysis of the form of $\Sigma(r)$ for different values of M it seems that M also defines whether the general cosmological models in the family are open or closed.

Finally if we wish the dominant energy condition to apply we obtain the restriction:

$$0 < 2M \leqslant \frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2} \leqslant 2M + (n-3)\left(M - \frac{\dot{T}^2}{T^2}\right) \tag{5.40}$$

The most significant result from the above three restrictions is that a singularity-free model obeying the strong and dominant energy conditions can only have one rebound time. The rebound time is defined by the turning point of the cosmological scale factor $T(\tau)$ i.e. where $\dot{T} = 0$. Hence models which pass through many cycles of expansion and contraction are not allowed under these conditions. Therefore from the general qualitative features of singularity-free models investigated in the previous chapter the expansion phase of the model occurs after the rebound time and this phase has no end. It is useful to examine the importance of the free cosmology selecting parameter n. A calculation of the relative shear of the models, σ/θ, where $2\sigma^2 = \sigma_{ab}\sigma^{ab}$, yields:

$$\frac{\sigma}{\theta} = \frac{2}{\sqrt{3}}\frac{n}{n+3} \tag{5.41}$$

Hence the parameter n determines the anisotropy of the model. When $n = 0$ the model is shearless and so we obatin the FLRW case as expected. In the limit $n \longrightarrow \infty$ the relative shear approaches the value $2/\sqrt{3}$. Also the relative shear is constant for any choice of the other free functions and is completely independent of the cosmological scale factor, $T(\tau)$. It is useful to compare the above result to the inhomogenisation and anisotropisation procedure introduced in Chapter 4. The procedure involved the raising of metric functions to arbitrary powers in order to create anisotropy. The selection constant n is also present as a power of the separated metric functions and so the operation of including these extra degrees of freedom is aptly named.

5.2 Producing Realistic Singularity-free Models

Senovilla in [10] has stated that a realistic singularity-free cosmology should obey the relation:

$$T(\tau) \sim \begin{cases} T_{SF} & \text{around } \tau = 0 \text{ with any n,} \\ T_{FLRW} & \text{for } \tau > \bar{\tau} \text{ with } n = 0, \end{cases} \tag{5.42}$$

where $\bar{\tau}$ is some fixed time, T_{FLRW} is the scale factor of the FLRW models which best describes the present universe and T_{SF} is any function $T(\tau)$ which leads to a singularity-free scale factor. Such a function must also be smooth, non-vanishing and possess a local minimum at $\tau = 0$. With such a function the above criterion sets the rebound time of the cosmology at $\tau = 0$. Senovilla then states that if we are not to violate the energy conditions then we must consider cases where n varies. For example we could choose the form of $n(\tau)$ such that:

$$n(\tau) = \begin{cases} n_0 > 1 & \text{for small } \tau, \\ 0 & \text{for } \tau > \bar{\tau}. \end{cases} \tag{5.43}$$

The restriction $n_0 > 1$ guarantees that at $\tau = 0$ the cosmology starts singularity-free. One particular choice would be $n_0 = 3$ to agree with the equation of state of the radiation-filled singularity-free perfect fluid model given in Ruiz and Senovilla [6]. However Senovilla states that this is in fact not right as if the model is FLRW-like at any time (i.e. the condition that $n(\tau) = 0$ for some $\tau > \bar{\tau}$) then there must exist closed trapped surfaces and hence by the singularity theorems the model must either be singular or violate the strong energy condition. Senovilla cites that the only way to escape this situation is to allow n to vary smoothly not only with τ but also with r. Using a careful choice of $n(\tau, r)$ a universe could be constructed which has regions with $n = 0$ for $\tau > \bar{\tau}$ and other regions with $n \neq 0$ for $\tau > \bar{\tau}$ such that the FLRW regions are not so large as to allow closed trapped surfaces. Such a cosmology might then be nonsingular and satisfy energy conditions. Moreover Senovilla states that since the particle horizon has an estimated distance similar to the radius of the trapped spheres around us it is possible that we live in a FLRW regions contained within a larger non-FLRW universe.

However in this treatment given by Senovilla [10] there is a major flaw in his argument of completely abandoning the case $n(\tau)$. First of all the above criterion (5.43) for a realistic singularity-free cosmology is in fact not as general as it could be. Instead of requiring that the model be exactly FLRW-like after some time it seems reasonable that a realistic singularity-free model may only approach the FLRW case in the limit $\tau \to \infty$. Such a cosmological model would then not possess a closed trapped surface or compact achronal slice without an edge and hence be geodesically complete. Moreover adjusting the 'decay' time of such a cosmology may enable the model to successfully account for the isotropy and homogeneity of today's observed universe.

5.2.1 Generalisations of the metric

In order to apply the above concepts to the models it is necessary to generalise the metric (5.1) obtained by Senovilla and the differential equation for $\Sigma(r)$ (5.2). The choice of generalisation however is not unique. The following transformations:

$$n, \Sigma(r) \longrightarrow \bar{n}(\tau), \bar{\Sigma}(r) \tag{5.44}$$
$$n, \Sigma(r) \longrightarrow \bar{n}(\tau), \bar{\Sigma}(r, \tau) \tag{5.45}$$
$$n, \Sigma(r) \longrightarrow \bar{n}(r, \tau), \bar{\Sigma}(r) \tag{5.46}$$
$$n, \Sigma(r) \longrightarrow \bar{n}(r, \tau), \bar{\Sigma}(r, \tau) \tag{5.47}$$

are all reasonable to make. Moreover it is obvious from the structure of the metric (5.1) that all these generalisations retain the cylindrical symmetry of the metric since $\boldsymbol{\partial}_z$ and $\boldsymbol{\partial}_\phi$ are both Killing vector fields for this class of metrics.

It is also important to consider the effect such generalisations make on the differential equation (5.2). Since we wish to have an equation of the form (5.2) as the restrictive equation on Σ in the case n constant there are many options to choose from. There is of course the straightforward generalisation of (5.2) obtaioned by simply replacing the n and Σ with its respective partner. This was the first method applied. We will show however that this is in fact not the best choice of generalisation and an alternative method will be defined.

Proceding with this approach[3] only two of the above possible generalisation are consistent with the new form of the differential equation. These are the models with $[n(\tau), \Sigma(r, \tau)]$ and

[3]Note that in the following calculations the overbars will be omitted. Similarly dots will be used to represent partial derivatives with respect to τ and primes will be used for partial derivatives with respect to r.

$[n(r,\tau),\Sigma(r,\tau)]$. The reason for this is that since n has now become a function and is no longer constant it is possible for Σ to have a non-trivial dependence on τ and therefore only generalisations with $\Sigma(r,\tau)$ are acceptable. Hence the two generalised differential equations are:

$$\Sigma'^2(r,\tau) = \begin{cases} M\Sigma^2(r,\tau) + N - n(\tau)K\Sigma^{2(1-2n(\tau))}(r,\tau) & \text{for } n = n(\tau),\ \Sigma = \Sigma(r,\tau), \\ \\ M\Sigma^2(r,\tau) + N - n(r,\tau)K\Sigma^{2(1-2n(r,\tau))}(r,\tau) & \text{for } n = n(r,\tau),\ \Sigma = \Sigma(r,\tau). \end{cases} \tag{5.48}$$

Since the case $[n(r,\tau),\Sigma(r,\tau)]$ produces a very complicated Einstein tensor it will not be considered for the remainder of this section.

The first thing that should be noted about the differential equation $\Sigma'^2(r,\tau) = M\Sigma^2(r,\tau) + N - n(\tau)K\Sigma^{2(1-2n(\tau))}(r,\tau)$ is that the same methods which were used to produce the Einstein tensor in the form of equations (5.21), (5.22) and (5.23) will not work with this metric. A brief look at the structure of the Einstein tensor in this case (see Appendix C) will show that the equations for the components contain various mixed derivatives involving $\dot{\Sigma}$, $\dot{\Sigma}'$ and so on. Substitution of the derivatives into the equation will lead to an expression which can be written in terms of $\dot{\Sigma}$ and Σ' however it cannot be further reduced without some knowledge of either $\dot{\Sigma}$ or the form of $n(\tau)$ which give $\Sigma(r,\tau)$ its τ dependence. These expressions are also of great complexity[4]. If one takes the case where $\dot{n} = 0 \implies \dot{\Sigma}(r,\tau) = 0$ many term vanish from the terms of the Einstein tensor and the terms collapse to give (5.18). Since many terms in the Einstein tensor are of the form $\ddot{\Sigma}'/\Sigma'$, Σ'''/Σ', $\dot{\Sigma}'/\Sigma'$ it appears that the Einstein tensor could be greatly simplifyed if separation of variables were used to decompose $\Sigma(r,\tau)$ into τ-dependent and r-dependent parts. This is in principle true however the differential equation is trivially separable (since there are no τ-derivatives of Σ in the differential equation). Hence the equation can actually be solved as before but with $n(\tau)$ replacing n in all the solutions found in Section 5.1.1. If we require Σ to be decomposable in the form $\Sigma(r,\tau) = X(\tau)Y(r)$ then we may actually reduce the number of solutions we obtain. This in fact happens to be the case. Explicitly substituting $\Sigma(r,\tau) = X(\tau)Y(r)$ into (5.48) we can only obtain a separation when $K = 0$: In this case (5.48) takes the form:

$$Y'^2(r) - MY^2(r) = \frac{N}{X^2(\tau)} = \text{constant} \tag{5.49}$$

Since there are no τ-derivatives the equation for $X(\tau)$ is trivial and the only class of functions of τ allowed are constant functions. Therefore we defeat the purpose of even attempting such an analysis. This has three direct consequences. The first is that this method of generalising the differential equation is flawed by not allowing enough freedom to take place. This arises as a failure of the separation of variables technique. Secondly there is no general method of writing the Einstein tensor as a relation involving K, M, N, and Σ. Additional information is required to obtain the τ-dependence of Σ. Finally it implies that there may be more merit than originally thought in examining all the metric generalisations again but with a different criterion other than differential equations such as (5.48) or (5.48). In fact many possible arbitrary choices could have be made for the differential equation all of which would reduce to (5.2) in the limit of constant N. The correct criterion can be determined from the source

[4]Without the help of the algebraic computing package Maple V and the relativity package GRTensorII many of the calculations in this chapter woud have been very long and error prone. Symbolic computing was an essential tool in examining the $[n(\tau),\Sigma(r,\tau)]$ case.

of the original equation (5.2). Originally (5.2) was proposed since it was a necessary but not sufficient condition to require that the fluid source behaves as a perfect fluid. This is not surprising since a similar derivation involving equating the fluid pressures allowed us to obtain the form of a cylindrically symetric FLRW model in Section 2.3.1. Hence the problem in all four proposed metric generalisations is to obtain criteria for the model to possess a perfect fluid source. This is also important for achieving the final aim of investigating such models, which is obtaining FLRW-like models which are singularity-free and have perfect fluid source. With this in mind let us now examine the Einstein tensor of the simplest of the models. This can be found in Appendix A. We can guarantee that the model possess no radial momentum flux if the component $G_{\hat{r}\hat{t}} = 0$. Therefore this occurs provided:

$$\dot{n}(n-1)\left[\frac{\Sigma'}{\Sigma}\ln T - \frac{\Sigma''}{\Sigma'}\ln\Sigma\right] = 0 \tag{5.50}$$

Hence either n is a constant or $T(\tau)$ and $\Sigma(r)$ obey the condition:

$$\ln T = \frac{\Sigma\Sigma''}{\Sigma'^2}\ln\Sigma \tag{5.51}$$

However the left hand side depends only on the coordinate τ while the right handside depends only on the coordinate r. Hence the functions $T(\tau)$ and $\Sigma(r)$ would both have to be constant. This would give a metric containing exponential functions but it is yet to be shown that the curvature invariants of such are space are regular for this particular $\Sigma' = 0$ case. Nevertheless this class of metrics will not yield singularity-free models of the Senovilla family. Thus we will discard this result and assume that $G_{\hat{r}\hat{t}} \neq 0$ in general.

Equating $G_{\hat{r}\hat{r}}$ with $G_{\hat{\phi}\hat{\phi}}$ we obtain the first condition:

$$\dot{n}^2 + (n-1)\left[\ddot{n} + 2\dot{n}\frac{\dot{T}}{T}\right] = 0 \tag{5.52}$$

Similarly equating $G_{\hat{\phi}\hat{\phi}}$ with $G_{\hat{z}\hat{z}}$ and $G_{\hat{r}\hat{r}}$ with $G_{\hat{z}\hat{z}}$ we obtain the additional conditions:

$$\left[4\dot{n}\frac{\dot{T}}{T} - 2\ddot{n}\right](\ln\Sigma + \ln T) + 4\dot{n}\frac{\dot{T}}{T} = \frac{\Sigma'''}{\Sigma'} + (4n-1)\frac{\Sigma''}{\Sigma} + 2n\left[\frac{\dot{T}^2}{T^2} + \frac{\ddot{T}}{T}\right] \tag{5.53}$$

$$\left[4\dot{n}\frac{\dot{T}}{T} - 2\ddot{n}\right](\ln\Sigma + \ln T) + 4\dot{n}\frac{\dot{T}}{T} + 2\dot{n}^2\ln\Sigma = \frac{\Sigma'''}{\Sigma'} + (4n-1)\frac{\Sigma''}{\Sigma} + 2n\left[\frac{\dot{T}^2}{T^2} + \frac{\ddot{T}}{T}\right] \tag{5.54}$$

The choice of any two of the three equations, (5.52), (5.53) and (5.54) produces a set of non-linear coupled differential equations[5] which guarantee that the source of curvature is a perfect fluid. Solving this set of equations is the next step in producing a solution, however as any preliminary investigation will show, this is very difficult. Extracting solutions of this equation will be the task in producing a singularity-free cosmological model.

[5]In general these equations constitute a set of non-linear coupled *partial* differential equations.

5.3 Future Development of this work

5.3.1 Case $n(\tau)$, $\Sigma(r)$

Once a perfect fluid condition has been obtained the constraints that energy conditions put on the functions of the metric and on any singularity-free models within this family of metrics can be determined. Then the behaviour of the timelike and null geodesics within the models can be examined and the model analysed for geodesic completeness. Given no other difficulties above it is hoped that a singulrity-free solution decaying into an FLRW model can be found by a suitable choice of function $n(\tau)$. There are some potential problems that could arise with producing such models from the Senovilla family of cosmologies (5.1). Primarily, if the general behaviour of the new cosmologies is related and not all that different to the models with n a constant then it could be expected that problems could occur when the function $n(\tau) < 1$ for some τ. In the Senovilla family of models this condition allowed the stress-energy tenosr to violate the strong energy condition. This could potentially occur in the new family of models when the function $n(\tau)$ is changing slowly. In such a case $\dot{n}$ and $\ddot{n}$ will both be close to zero and provided the behaviour of the models in this limit is stable this may allow strong energy violations to occur. However it is expected that this type of simple behaviour will not occur for more general cases and the form of the function $n(\tau)$ is likely to very singificant in determining the energy properties of the matter in the spacetime when $\dot{n}$ is larger. This is further justified by the fact that these models all contain non-vanishing $G_{\hat{r}\hat{t}}$. This term represents radial momentum transfer in the model and it is directly proportional to $\dot{n}$. Hence an increased $\dot{n}$ implies an increased flux of momentum which interferes with the slow τ-development of a model. The role of this flux in any singularity-free model is at present only understood in terms of heat flow. We already know that singularity-free models are stable against heat flow and one possible development of this model could be to compare this family of models to the solutions found by Patel and Dadhich [7].

5.3.2 Other models

We should also note that the above generalisations of the metric are guaranteed to contain singularity-free models since we can always choose n a constant and regain the original Senovilla model. Using the restrictions posed by the perfect fluid requirement these models could be analysed for analogous behaviour to the one above. In practice the perfect fluid restriction would be rather difficult to examine analytically given the great complexity of the Einstein tensor. If no physically realistic simplifying assumptions can be made then it may be necessary to investigate these models numerically.

5.3.3 Concluding Remarks

The above models have not been investigated to any great extent however there is promise in obtaining an analytical solution of the $[n(\tau), \Sigma(r)]$ case. More generally MacCallum [14] and Misner [11] have already stated the great importance of studying inhomogeneous and anisotropic cosmologies and there is great potential in the field of singularity-free cosmology to obtain an alternate standard model of our universe. However the question of the existence of realistic singularity-free cosmolgies is still an open one. It is hoped that this is another important step in attaining that goal.

Appendix A

Mappings of Manifolds and Lie Derivatives

Definition A.1 (Diffeomorphism) A map $\Phi : M \to N$ where M and N are both smooth mainfolds is termed a *diffeomorphism* if:

(i) Φ is in $C^\infty(M, N)$,

(ii) Φ is one-to-one and onto,

(iii) Φ^{-1} exists and is $C^\infty(N, M)$.

If a diffeomorphism exists between two differentiable manifolds M and N then the two manifolds are termed *diffeomorphic*. For example a sphere and a sphere rotated about an axis are diffeomorphic under the rotation transformation. Similarly a 2-sphere is diffeomorphic to any smooth deformed version of a 2-sphere. By smooth it is implied that the deformation does not produce any 'pinched' or spiked regions. As a result of this there exists no diffeomorphism between the 2-sphere, S^2 and the standard torus T^2 since any diffeomorphism would have to drive a hole in the sphere changing its open set structure and hence its topology. From this fact alone the genus of a differentiable manifold is preserved by diffeomorphisms. One example of a diffeomorphism of Minkowski spacetime into itself is a *boost*. Diffeomorphisms will prove to be important in the work that follows.

Definition A.2 (One-parameter group of diffeomorphisms) A one-parameter group of diffeomorphisms ϕ_t is a C^∞ mapping from $\mathbb{R} \times M \to M$ such that:

1. for fixed $t_0 \in \mathbb{R}$, $\phi_{t_0} : M \to M$ is a diffeomorphism and,
2. $\forall s, t \in \mathbb{R}$, $\phi_t \circ \phi_s = \phi_{s+t}$.

A vector field can be associated with this group of diffeomorphisms by choosing some fixed point $p \in \mathbf{M}$ and then $\phi_t(p) : \mathbb{R} \to \mathbf{M}$ is a curve (called an *orbit*) which passes through p when $t = 0$. The set of all orbits in $\mathbf{M}$ is termed a *congruence*. Within the congruence no two orbits can intersect as then the one-parameter group of maps would not be diffeomorphisms[1] If we define $\mathbf{v}\mid_p$ to be the tangent vector to the orbit through p and

[1]Intersection implies that the inverse map ϕ^{-1} is not one-to-one and hence the Jacobian of the mapping would be singular.

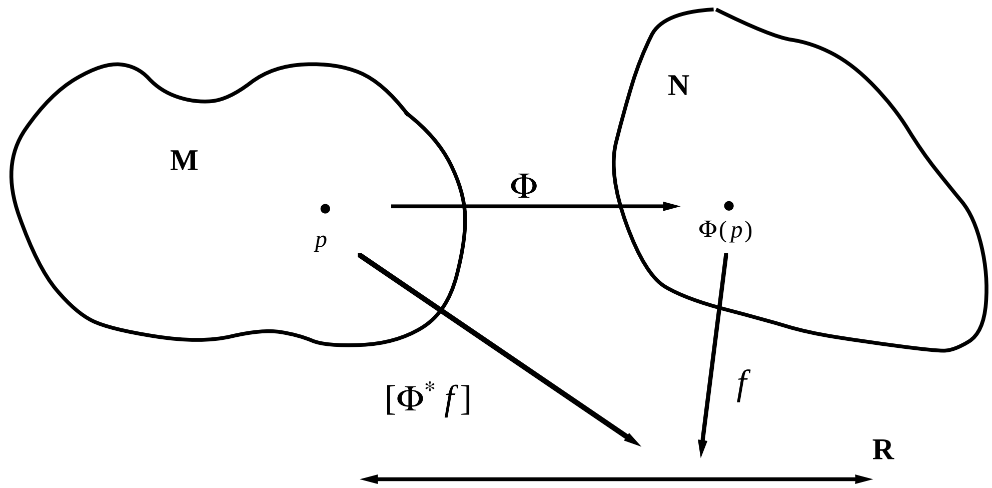

Figure A.1: Φ^* maps real functions defined on N to real functions defined on M.

allow p to vary over all $\mathbf{M}$ we obtain a vector field whose integral curves are the congruence by construction. This vector field $\mathbf{v}$ is said to be the *infinitesimal generator* of ϕ_t. Note that the second condition implies that $\phi_0 = id(M)$ and also guarantees that if two points p,$q \in \mathbf{M}$ lie on the same integral curve then the definition of the vector field is consistent. An example of a one-parameter group of diffeomorphisms is again the rotation map applied to a sphere with $t = \theta$ the angle of rotation. In this example the orbits would be circles like those of latitude on a globe. The collection of all such orbits would be a congruence. The vector field generating the diffeomorphism would be the set of tangent vectors to the orbits. Finally note that no two orbits intersect.

The map $\Phi^* : C^k(N) \to C^k(M)$ of any mapping $\Phi : M \to N$ is defined by:

$$[\Phi^* f](p) = f(\Phi(p)) = f \circ \Phi \tag{A.1}$$

where $f : N \to \mathbb{R}$ is any real function defined on the manifold N. The action of Φ^* is to 'pull back' functions defined on N and redefine them on M. This action is depicted graphically in Figure A.1. Using this map it is possible a define a map, $\Phi_* : T_p(M) \to T_{\Phi(p)}(N)$, to move vector fields from one manifold to another. Φ_* is defined by the relation:

$$[\Phi_* \mathbf{w}](f)|_{\Phi(p)\in N} = \mathbf{w}(f \circ \Phi) = \mathbf{w}([\Phi^* f])|_{p\in M} \tag{A.2}$$

where $\mathbf{w} \in T_p(M)$ (see Figure A.2). Hence the map Φ_* creates a vector field, $[\Phi_* \mathbf{w}]$ on $T_{\Phi(p)}(N)$ whose action on any real function f is the same as $\mathbf{w}$ acting on the pulled back version of f. Consequently since the action of a 1-form on a vector field is a real function, Φ^* can be used to map 1-forms between manifolds.

$$\langle [\Phi^* \boldsymbol{\sigma}], \mathbf{w} \rangle = \langle \boldsymbol{\sigma}, [\Phi_* \mathbf{w}] \rangle \tag{A.3}$$

where $\sigma \in T^*_{\Phi(p)}$. Hence Φ^* is also a map from $T^*_{\Phi(p)}(N)$ to $T^*_p(M)$ (See Figure A.3). . Note that if Φ is a diffeomorphism then Φ^{-1} is also defined, hence):

$$\langle \boldsymbol{\mu}, \mathbf{v} \rangle|_{\Phi(p)\in N} = \langle [\Phi^* \boldsymbol{\mu}], [\Phi^{-1}{}_* \mathbf{v}] \rangle|_{p\in M} \tag{A.4}$$

$$= \langle [\Phi^{-1*} \boldsymbol{\mu}], [\Phi_* \mathbf{v}] \rangle|_{\Phi(p)\in N} \tag{A.5}$$

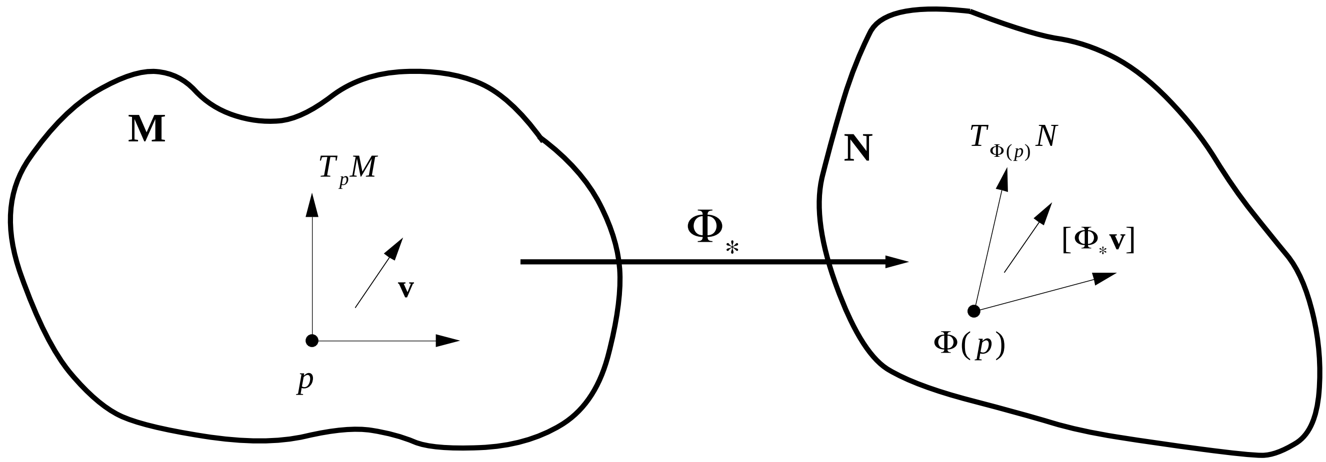

Figure A.2: Φ_* maps vector fields on M into vector fields in N.

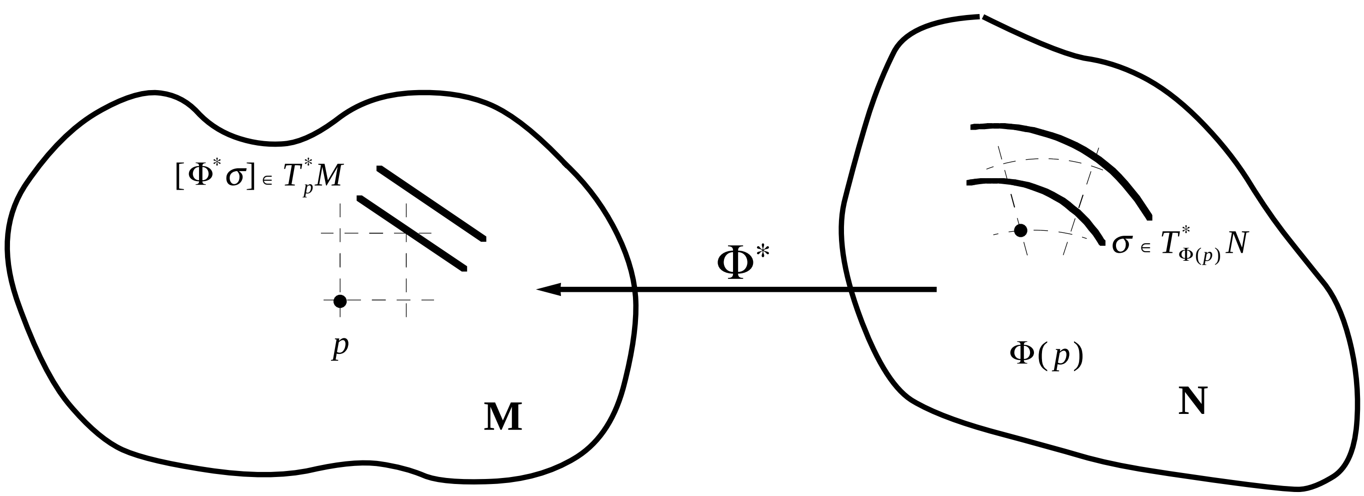

Figure A.3: Φ^* also maps 1-forms from one manifold to another.

for all $\boldsymbol{\mu} \in T_p^*(N)$ and $\mathbf{v} \in T_p(N)$. However since the inner product is just a function it is preserved under (A.1) and using (A.3) we obtain $\Phi^* = \Phi^{-1}{}_*$ and $\Phi_* = \Phi^{-1*}$. Therefore if ϕ is a diffeomorphism vectors and 1-forms can be mapped using only Φ_* and Φ_*^{-1}. In a straightforward way these two maps can be extended to map tensors of arbitrary rank from one manifold to another. For example if $\mathbf{T}$ is a tensor of rank $\binom{n}{m}$ then $[\Phi^*\mathbf{T}]$ is defined as:

$$(\Phi^* T)(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n, \mathbf{v}_1, \ldots, \mathbf{v}_m) = T(\Phi^*\boldsymbol{\mu}_1, \ldots, \Phi^*\boldsymbol{\mu}_n, \Phi_*^{-1}\mathbf{v}_1, \ldots, \Phi_*^{-1}\mathbf{v}_n) \tag{A.6}$$

$$\begin{aligned}(\Phi^* T)^{b_1\ldots b_n}{}_{a_1\ldots a_m}(\mu_1)_{b_1}\ldots(\mu_n)_{b_n}(v_1)^{a_1}\ldots(v_m)^{a_m} \\ = T^{b_1\ldots b_n}{}_{a_1\ldots a_m}(\Phi^*\mu_1)_{b_1}\ldots(\Phi^*\mu_n)_{b_n}(\Phi_*^{-1}v_1)_{a_1}\ldots(\Phi_*^{-1}v_m)_{a_m}\end{aligned}$$

Let the neighbourhoods of p and $\Phi(p)$ have local coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ respectively. Then the 1-form $\boldsymbol{\mu}$ can be written as $\boldsymbol{\mu} = \mu_a(y)dy^a$ and $\Phi^*\boldsymbol{\mu} = \tilde{\mu}(x)_b dx^b$ in the respective coordinate bases. Hence $\Phi^*\boldsymbol{\mu}$ and $\tilde{\mu}_b$ can be easily determined using the standard transformations of differentials:

$$\Phi^*\boldsymbol{\mu} = \underbrace{\mu_a(y(x))\frac{\partial y^a}{\partial x^b}}_{\tilde{\mu}_k}\boldsymbol{dx}^b \tag{A.8}$$

Similarly for a vector field, $\mathbf{v} = v^a(y)\frac{\partial}{\partial y^a}$, we obtain the action of Φ^* on the vector field as:

$$\Phi^*\mathbf{v} = v^a(y(x))\frac{\partial x^b}{\partial y^a}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^b} \tag{A.9}$$

Using (A.9) and (A.8) with (A.7) above the result of mapping any tensor can be determined.

Using the mapping ϕ_t^* it is possible to transport tensors along the orbits of the congruence. Naturally from this a derivative can be defined:

Definition A.3 (Lie Derivative) The *Lie derivative* of tensor, $\mathbf{T}$ with respect to a vector, $\mathbf{v}$ is defined by

$$\pounds_{\mathbf{v}}\mathbf{T} = \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*\mathbf{T} - \mathbf{T}\right] \tag{A.10}$$

The Lie derivative is a linear operator and obeys standard differential identities like the Leibnitz rule etc. From the definition above and relation (A.7) the Lie derivative of any tensor can be obtained. For a scalar function f, $\pounds_{\mathbf{v}} f \equiv \nabla_{\mathbf{v}} f$. So the Lie derivative corresponds to the directional derivative for scalar functions. In the case of a 1-form σ the Lie derivative takes the form:

$$\pounds_{\mathbf{v}}\boldsymbol{\sigma} = \pounds_{\mathbf{v}}\sigma_a = v^b\nabla_b\sigma_a + \sigma_b\nabla_a v^b \tag{A.11}$$

Similarly for a vector field $\mathbf{w}$:

$$\pounds_{\mathbf{v}}\mathbf{w} = \pounds_{\mathbf{v}}w^a = v^b\nabla_b w^a - w^b\nabla_b v^a = [\mathbf{v}, \mathbf{w}] \tag{A.12}$$

From the above two results we obtain for a tensor field of valence $\binom{n}{m}$:

$$\begin{aligned}\pounds_{\mathbf{v}}\mathbf{T} &= \pounds_{\mathbf{v}}T^{a_1\ldots a_n}{}_{b_1\ldots b_m} \\ &= v^c\nabla_c T^{a_1\ldots a_n}{}_{b_1\ldots b_m} - \sum_{i=1}^{n} T^{a_1\ldots c\ldots a_n}{}_{b_1\ldots b_m}\nabla_c v^{a_i} + \sum_{j=1}^{m} T^{a_1\ldots a_n}{}_{b_1\ldots c\ldots b_m}\nabla_{b_j}v^c\end{aligned} \tag{A.13}$$

Appendix B

The Einstein Tensor for the case $n(\tau)$ and $\Sigma(r)$

Metric:

$$ds^2 = T^{2(1+n(\tau))}\Sigma^{2n(\tau)(n(\tau)-1)}(r)(-d\tau^2 + dr^2) + T^{2(1+n(\tau))}\Sigma^{2n(\tau)}(r)\Sigma'^2(r)d\phi^2 \\ + T^{2(1-n(\tau))}\Sigma^{2(1-n(\tau))}(r)dz^2 \quad \text{(B.1)}$$

Using the comoving basis:

$$\boldsymbol{\omega}^{\hat{t}} = T^{1+n(\tau)}(\tau)\Sigma^{n(\tau)(n(\tau)-1)}(r)\boldsymbol{d}\tau, \qquad \boldsymbol{\omega}^{\hat{r}} = T^{1+n(\tau)}(\tau)\Sigma^{n(\tau)(n(\tau)-1)}(r)\boldsymbol{d}r,$$
$$\boldsymbol{\omega}^{\hat{\phi}} = T^{1+n(\tau)}(\tau)\Sigma^{n(\tau)}(r)\Sigma'(r)\boldsymbol{d}\phi, \qquad \boldsymbol{\omega}^{\hat{z}} = T^{1-n(\tau)}(\tau)\Sigma^{n(\tau)-1}(r)\boldsymbol{d}z$$

gives the following expressions for the Einstein tensor where $\Delta = T^{2(1+n)}\Sigma^{2n(n-1)}(r)$.

$$G_{\hat{t}\hat{t}} = \frac{1}{\Delta}\Bigg[[(n+1)(n-3)+1]\frac{\Sigma''}{\Sigma} - (n+1)(n-3)\frac{\dot{T}^2}{T^2} - \frac{\Sigma'''}{\Sigma'} \\ - \left(\dot{n}^2(\ln\Sigma + \ln T)^2 + 2\dot{n}(n-1)\frac{\dot{T}}{T}(\ln T - \ln\Sigma)\right)\Bigg] \quad \text{(B.2)}$$

$$G_{\hat{t}\hat{r}} = \frac{2\dot{n}(n-1)}{\Delta}\left[\frac{\Sigma'}{\Sigma}\ln T - \frac{\Sigma''}{\Sigma'}\ln\Sigma\right] \quad \text{(B.3)}$$

$$G_{\hat{r}\hat{r}} = \frac{1}{\Delta}\Bigg[(n-1)^2\frac{\Sigma''}{\Sigma} - \frac{\dot{T}^2}{T^2}[(n+1)(n-3)+2] - 2\frac{\ddot{T}}{T} \\ - \left(\dot{n}^2(\ln\Sigma + \ln T)^2 + 2\dot{n}(n-1)\frac{\dot{T}}{T}(\ln T - \ln\Sigma)\right)\Bigg] \quad \text{(B.4)}$$

$$G_{\hat{\phi}\hat{\phi}} = \frac{1}{\Delta}\Bigg[(n-1)^2\frac{\Sigma''}{\Sigma} - \frac{\dot{T}^2}{T^2}[(n+1)(n-3)+2] - 2\frac{\ddot{T}}{T} - 2[\dot{n}^2 + \ddot{n}(n-1)]\ln\Sigma \\ - \left(\dot{n}^2(\ln\Sigma + \ln T)^2 + 2\dot{n}(n-1)\frac{\dot{T}}{T}(\ln T + \ln\Sigma)\right)\Bigg] \quad \text{(B.5)}$$

$$G_{\hat{z}\hat{z}} = \frac{1}{\Delta}\Bigg[\frac{\Sigma'''}{\Sigma'} + n(2+n)\frac{\Sigma''}{\Sigma} - (n+1)(n-1)\frac{\dot{T}^2}{T^2} - 2(n+1)\frac{\ddot{T}}{T} - 2\ddot{n}(n\ln\Sigma + \ln T) - 2\dot{n}^2\ln\Sigma$$
$$- \left(\dot{n}^2(\ln\Sigma + \ln T)^2 + 2\dot{n}\frac{\dot{T}}{T}[(n+1)(\ln\Sigma + \ln T) + 2]\right)\Bigg] \quad \text{(B.6)}$$

Appendix C

The Einstein Tensor for the case $n(\tau)$ and $\Sigma(r,\tau)$

Metric:

$$ds^2 = T^{2(1+n(\tau))}(\tau)\Sigma^{2n(\tau)(n(\tau)-1)}(r,\tau)(-d\tau^2+dr^2) + T^{2(1+n(\tau))}(\tau)\Sigma^{2n(\tau)}(r,\tau)\Sigma'^2(r,\tau)d\phi^2 \\ + T^{2(1-n(\tau))}(\tau)\Sigma^{2(1-n(\tau))}(r,\tau)dz^2 \quad \text{(C.1)}$$

Using the comoving basis:

$$\boldsymbol{\omega}^{\hat{t}} = T^{1+n(\tau)}(\tau)\Sigma^{n(\tau)(n(\tau)-1)}(r,\tau)\boldsymbol{d}\tau, \qquad \boldsymbol{\omega}^{\hat{r}} = T^{1+n(\tau)}(\tau)\Sigma^{n(\tau)(n(\tau)-1)}(r,\tau)\boldsymbol{d}r,$$

$$\boldsymbol{\omega}^{\hat{\phi}} = T^{1+n(\tau)}(\tau)\Sigma^{n(\tau)}(r,\tau)\Sigma'(r,\tau)\boldsymbol{d}\phi, \qquad \boldsymbol{\omega}^{\hat{z}} = T^{1-n(\tau)}(\tau)\Sigma^{n(\tau)-1}(r,\tau)\boldsymbol{d}z.$$

gives the following expressions for the Einstein tensor where $\Delta = T^{2(1+n)}\Sigma^{2n(n-1)}(r,\tau)$.

$$G_{\hat{t}\hat{t}} = \frac{1}{\Delta}\Bigg[\left(\frac{\Sigma''}{\Sigma} - \frac{\Sigma'''}{\Sigma'}\right) + 2\frac{\dot{T}}{T}\left(\frac{\dot{\Sigma}'}{\Sigma'} + \frac{\dot{\Sigma}}{\Sigma}\right) + (n+1)(n-3)\left(\frac{\Sigma''}{\Sigma} - \frac{\dot{T}^2}{T^2}\right) + (n-1)^2\frac{\dot{\Sigma}\dot{\Sigma}'}{\Sigma\Sigma'} \\ + 2(n-1)\dot{n}\left(\frac{\dot{\Sigma}'}{\Sigma'}\ln\Sigma - \frac{\dot{\Sigma}}{\Sigma}\ln T + \frac{\dot{T}}{T}(\ln\Sigma - \ln T)\right) - \dot{n}^2(\ln T + \ln\Sigma)^2\Bigg] \quad \text{(C.2)}$$

$$G_{\hat{t}\hat{r}} = \frac{1}{\Delta}\Bigg[2(n-1)\dot{n}\left(\frac{\Sigma'}{\Sigma}\ln T - \frac{\Sigma''}{\Sigma'}\ln\Sigma\right) - \frac{\dot{\Sigma}'}{\Sigma}[(n+1)(n-3)+2] \\ + \left(\frac{\dot{\Sigma}''}{\Sigma'} + \frac{\Sigma''\dot{\Sigma}}{\Sigma'\Sigma}(2n-n^2)\right)\Bigg] \quad \text{(C.3)}$$

$$G_{\hat{r}\hat{r}} = \frac{1}{\Delta}\Bigg[(n-1)^2\frac{\Sigma''}{\Sigma} + [(n+1)(n-3)+2]\left(\frac{\dot{\Sigma}'\dot{\Sigma}}{\Sigma'\Sigma} - \frac{\dot{T}^2}{T^2}\right) - 2\frac{\dot{T}}{T}\left(\frac{\dot{\Sigma}'}{\Sigma'} + \frac{\dot{\Sigma}}{\Sigma}\right) - \left(\frac{\ddot{\Sigma}'}{\Sigma'} + \frac{\ddot{\Sigma}}{\Sigma} + 2\frac{\ddot{T}}{T}\right) \\ + 2(n-1)\dot{n}\left(\frac{\dot{\Sigma}'}{\Sigma'}\ln\Sigma - \frac{\dot{\Sigma}}{\Sigma}\ln T + \frac{\dot{T}}{T}(\ln\Sigma - \ln T)\right) - \dot{n}^2(\ln\Sigma + \ln T)^2\Bigg] \quad \text{(C.4)}$$

$$G_{\hat{\phi}\hat{\phi}} = \frac{1}{\Delta}\Bigg[(n-1)^2\left(\frac{\Sigma''}{\Sigma} - 2\frac{\dot{T}\dot{\Sigma}}{T\Sigma} - \frac{\ddot{\Sigma}}{\Sigma}\right) + 2\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2}[(n+1)(n-3)+2] - 2\ddot{n}(n-1)\ln\Sigma$$
$$- 2(n-1)\dot{n}\left[\left(\frac{\dot{T}}{T} + \frac{\dot{\Sigma}}{\Sigma}\right)(\ln T + \ln\Sigma) + 2\frac{\dot{\Sigma}}{\Sigma}\right] - \dot{n}^2\left[(\ln T + \ln\Sigma)^2 + 2\ln\Sigma\right]\Bigg] \quad (C.5)$$

$$G_{\hat{z}\hat{z}} = \frac{1}{\Delta}\Bigg[\left(\frac{\Sigma'''}{\Sigma'} - \frac{\ddot{\Sigma}'}{\Sigma''}\right) - \dot{n}^2[(\ln\Sigma + \ln T)^2 + 2\ln\Sigma] - (n+1)(n-1)\frac{\dot{T}^2}{T^2}$$
$$- 2\dot{n}\left[\frac{\dot{\Sigma}'}{\Sigma'}(\ln\Sigma + \ln T) - \frac{\dot{\Sigma}}{\Sigma}n(2 + \ln\Sigma + \ln T) + \frac{\dot{T}}{T}[(\ln T + \ln\Sigma)(n+1) + 2]\right]$$
$$+ n^2\left(\frac{\Sigma''}{\Sigma} - 2\frac{\dot{T}\dot{\Sigma}}{T\Sigma} - \frac{\ddot{\Sigma}}{\Sigma}\right) + 2n\left(\frac{\Sigma''}{\Sigma} - \frac{\dot{T}\dot{\Sigma}}{T\Sigma} - \frac{\dot{\Sigma}'\dot{T}}{\Sigma'T} - \frac{\dot{\Sigma}'\dot{\Sigma}}{\Sigma'\Sigma} - \frac{\ddot{T}}{T}\right) - \left(2\frac{\dot{\Sigma}'\dot{T}}{\Sigma'T} + 2\frac{\ddot{T}}{T}\right)\Bigg] \quad (C.6)$$

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