CARDINAL NUMBERS WITH PARTITION PROPERTIES

by

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STATEMENT OF ORIGINALITY

The results presented in this thesis are my own except where the contrary is stated.

Neil H. Williams

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PREFACE

During the time that the work for this thesis was done, I held a Commonwealth Postgraduate award, which was supplemented by the Australian National University. The period July, 1967 to June, 1968 was spent at the University of California at Los Angeles; I record my gratitude to both universities involved for making it possible for me to spend this portion of my course at U.C.L.A.

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CONTENTS

INTRODUCTION

This discussion divides into two quite separate parts, chapters I - IV and chapter V. The first four chapters are concerned with a generalization of the partition properties $\kappa \to (\eta_\ell)^n_{\ell<\nu}$ and $\kappa \to (\eta)^{<\omega}_\nu$ extensively studied by Erdös and his collaborators (see [5] and [6] for further references).

Chapter I is devoted to establishing notation, the definition of these properties and immediate consequences of these definitions. A brief mention of a few well known results concerning the Erdös cardinals is included. Chapters II and IV are concerned with a generalization of the relation $\kappa \to (\eta_\ell)^n_{\ell<\nu}$, a generalization obtained by considering not partitions of the n-element subsets of κ, but rather partitions of certain finite sequences of n-element subsets. This is in fact a special case of the polarized partition relation defined in [6, p.100]. Chapter II contains a general method for obtaining positive relations from Erdös relations. An apparently different, although in many cases equivalent, form of the definition is discussed. Consideration is given to extending the generalization to cover partitions of the infinite sequences of finite subsets of a given set. In the course of this, the answer is provided to a problem of Moschovakis ([13, P2061]). Chapter IV deals directly with the question of extending the methods and results of [6] to the more general situation. Chapter III deals with an analogous generalization of the property $\kappa \to (\eta)^{<\omega}_\nu$.

In Chapter V, forcing (see [3]) is developed for classes of conditions and theorems of Easton [4] and Jensen [9] are re-proved. The original intention was to use similar methods to obtain corresponding results for cardinals having the generalized form of the Ramsey property, $\kappa \to (\kappa)^{<\omega}$. Subsequently, it was found that no cardinals rejoice in these properties (see §III·2).

Since the development of the forcing method is done in a manner slightly different to that in [4], and a couple of points of minor independent interest appear, the chapter is included anyway.

CHAPTER I. INTRODUCTORY

§1. Notation

The set theory in which this thesis takes place is the system ZF of Zermelo-Fraenkel, together with the axiom of choice, AC. (See, for example, [12] or [19] for details.) The generalized continuum hypothesis is abbreviated GCH.

Set inclusion is written $\subseteq$, with $\subset$ denoting proper inclusion. For a set A and formula Φ of ZF, the set of those members of A which have the property Φ is denoted $\{x \in A ; \Phi\}$. Sequences are written $\langle x_\alpha ; \alpha \in A\rangle$, except that for finite n an ordered n-tuple is usually written $\langle x_1,\ldots,x_n\rangle$. The concatenation of two sequences u and v is represented as $u^\frown v$. The domain, Dx, and the range, Rx, of x are respectively $\{y ; \exists z(\langle y,z\rangle \in x)\}$ and $\{z ; \exists y(\langle y,z\rangle \in x)\}$. The restriction, $A\upharpoonright B$, of A to B is $\{\langle x,y\rangle \in A ; x \in B\}$. The set of all functions with domain x and range contained in y is denoted ${}^x y$. The cartesian product of a family $\{x_i ; i \in I\}$ is written $\Pi\{x_i ; i \in I\}$. The powerset of x is Px, and $x-y$ denotes set-difference.

Ordinals are defined in some standard manner (e.g. [17]) so that if α is an ordinal, $\alpha = \{\beta ; \beta$ is an ordinal and $\beta < \alpha\}$. The initial ordinals, i.e. ordinals having power larger than that of any of their members, are identified with the cardinals. The sequence of infinite cardinals is $\omega_0, \omega_1, \omega_2, \ldots$. I shall usually write ω in place of ω_0. Ordinal sum and ordinal product are denoted by $+$ and $\cdot$ respectively. Cardinal sum is $+_c$ and cardinal product is indicated either by $\times_c$ or by juxtaposition. Unless the contrary is stated, κ^λ means cardinal exponentiation and $\sum$ means the infinite sum of cardinal numbers. If β is a limit ordinal (in particular, a cardinal) $Cf(\beta)$ is defined, as in [1], to be the smallest cardinal κ such that β is the union of κ smaller ordinals. This is related to the cofinality index of [23]. For κ a cardinal, κ^+ is the cardinal next after κ, and κ^- is the cardinal immediately before κ if there is such a cardinal, whereas otherwise $\kappa^- = \kappa$. A cardinal κ is called regular if $Cf(\kappa) = \kappa$, and singular if not regular. A cardinal κ is termed strongly inaccessible if κ is regular and $\lambda < \kappa \Rightarrow 2^\lambda < \kappa$. It is measurable if the powerset of κ supports a non-trivial, κ-additive, two-valued measure.

The cardinality of a set x is denoted $|x|$. The set $\{y \in Px ; |y| = \kappa\}$ of exactly κ-element subsets of x (for κ a cardinal) is denoted $[x]^\kappa$. Similarly, $[x]^{<\kappa}$, $[x]^{\leqq\kappa}$ and $[x]^{\geqq\kappa}$ stand for the sets of subsets of x which have cardinality less than κ, at most κ and at least κ. In particular, $[x]^{<\omega}$ is the set of finite subsets of x. The symbol y^{*x} is used for $\{f \in {}^x([y]^{<\omega}) ; \forall a,b \in x(|f(a)| = |f(b)|)\}$, the set of mappings from x to the finite subsets of y such that all function values have the same cardinality.

A relation $<$ is said to be well founded when, if a is any set, then $a \neq \emptyset \Rightarrow \exists x \in a\ \forall y \in a(y \not< x)$. Definitions by transfinite recursion over a well founded relation will frequently be used (see [14]). If A is a set of ordinals, $\sup(A)$, the strict supremum of A, is defined by: $\sup(A) = \bigcup\{\alpha+1 ; \alpha \in A\}$, and $\min(A)$ is the least member of A. For any set x, the rank $rk(x)$ of x is defined inductively as: $rk(x) = \sup\{rk(y) ; y \in x\}$.

The cardinal beths are denoted $\underline{2}_\alpha(\kappa)$, (where α is an ordinal and κ a cardinal), and are defined inductively by

$$\underline{2}_o(\kappa) = \kappa \quad , \quad \underline{2}_{\alpha+1}(\kappa) = 2^{\underline{2}_\alpha(\kappa)} \quad , \quad \alpha = \cup\alpha > 0 \Rightarrow \underline{2}_\alpha(\kappa) = \cup\{\underline{2}_\beta(\kappa) \ ; \ \beta < \alpha\} \ .$$

Beth α , written $\underline{2}_\alpha$, stands for $\underline{2}_\alpha(\omega)$.

If A is a set well ordered by a relation $<$, then the lexicographic ordering $<<$ on PA (or on ${}^A2$) is defined so that $a << b$ iff the first point, under $<$, where a and b differ is a member of b (or is given the value 1 by b).

Let $\mathcal{L}$ be a first order language. Let τ be a formula (or term) of $\mathcal{L}$ and let A be a set or class of ZF . Then τ_A or $(\tau)_A$ denotes the relativization of τ to A , i.e. the result of replacing in τ (or in the definition of τ) all occurrences of $\forall$ by $\forall \cdot \in A$ and all occurrences of $\exists$ by $\exists \cdot \in A$. If $\Phi(v_o,\ldots,v_n)$ is a formula of $\mathcal{L}$ with exactly the free variables shown, if $\underline{\underline{A}}$ is a model of $\mathcal{L}$ and if $a_o,\ldots,a_n$ are points in the underlying set of $\underline{\underline{A}}$, then $\underline{\underline{A}} \models \Phi[a_o,\ldots,a_n]$ expresses that the formula Φ is true in $\underline{\underline{A}}$ at the point $\langle a_o,\ldots,a_n\rangle$.

A partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of a set A into ν parts is a decomposition $A = \cup\{\Delta_\ell \ ; \ \ell < \nu\}$ where no Δ_ℓ is empty. Given such a partition, if $a,b \in A$ then $a \equiv b \ (\text{mod}\ \Delta)$ indicates that there is some class Δ_ℓ which has both a and b as members. The partition is called disjoint if the Δ_ℓ are pairwise disjoint.

Finally, a word on the conventions concerning variables. Unless the contrary is stated or implied, ι , κ , λ denote infinite cardinals. Other small Greek letters denote ordinals, as also frequently do i,j,k,ℓ . Natural numbers, usually non-zero, are represented by m,n . In the first four chapters, p,q,r,s also represent arbitrary natural numbers. Logical formulae are represented by Φ , Ψ and logical variables by u,v . For partitions of some set, Δ , Γ , Λ will be used. Capital Roman letters and the remaining small Roman letters denote arbitrary sets.

§2. Definitions

The partition relations $\kappa \to (\eta_\ell \,;\, \ell < \nu)^n$ and $\kappa \to (\eta)_\nu^{<\omega}$, defined below, have been extensively studied in the literature, in particular in [6].

2.1 DEFINITION $\quad \kappa \to (\eta_\ell \,;\, \ell < \nu)^n$ if for all partitions $\Delta = \{\Delta_\ell \,;\, \ell < \nu\}$ of $[\kappa]^n$ into ν parts, there exist $\ell < \nu$ and $H \subseteq \kappa$ having order type η_ℓ (in the ordering $<$ amongst ordinals), such that $[H]^n \subseteq \Delta_\ell$.

2.2 DEFINITION $\quad \kappa \to (\eta)_\nu^{<\omega}$ if for all partitions $\Delta = \{\Delta_\ell \,;\, \ell < \nu\}$ of $[\kappa]^{<\omega}$ into ν parts, there is a set $H \subseteq \kappa$ having order type η, which is homogeneous for Δ, in the sense that for all $n < \omega$ there is $\ell < \nu$ for which $[H]^n \subseteq \Delta_\ell$.

The partition relations that I shall be considering are generalizations of these, obtained by considering not partitions of finite subsets, but rather partitions of finite sequences of finite subsets.

2.3 DEFINITION $\quad \kappa \to {}^m(\eta_\ell \,;\, \ell < \nu)^n$ if for all partitions $\Delta = \{\Delta_\ell \,;\, \ell < \nu\}$ of ${}^m([\kappa]^n)$ into ν parts, there are $\ell < \nu$ and a sequence $H_1, \ldots, H_m$ (where each $H_i \subseteq \kappa$ and has order type η_ℓ), such that $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_\ell$.

2.4 DEFINITION $\quad \kappa \to {}^m(\eta)_\nu^{<\omega}$ if for all partitions $\Delta = \{\Delta_\ell \,;\, \ell < \nu\}$ of κ^{*m} into ν parts, there is a sequence $H_1, \ldots, H_m$ (where each $H_i \subseteq \kappa$ and has order type η), which is homogeneous for Δ, in the sense that for all $n < \omega$ there is $\ell < \nu$ for which $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_\ell$.

Thus the properties 2.1 and 2.2 correspond, respectively, to the special cases of 2.3 and 2.4 for which $m = 1$. The special case of 2.3 in which $\eta_\ell = \eta$ for all $\ell < \nu$ will be written $\kappa \to {}^m(\eta)^n_\nu$. The property $\kappa \to {}^m(\eta_\ell \ ; \ \ell < \nu \ , \ \theta_k \ ; \ k < \mu)^n$ has its obvious meaning.

Property 2.3 itself is a special case of the general polarized partition property defined, but not discussed, in [6]. This may be expressed as follows.

2.5 DEFINITION
$$\begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_m \end{pmatrix} \to \begin{pmatrix} \eta_{1\ell} \\ \vdots \\ \eta_{m\ell} \end{pmatrix}^{n_1,\dots,n_m}_{\ell<\nu}$$

if for all partitions $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $[\kappa_1]^{n_1} \times\dots\times [\kappa_m]^{n_m}$ into ν parts, there are $\ell < \nu$ and a sequence $H_1,\dots,H_m$ (where for each i, $H_i \subseteq \kappa_i$ and has order type $\eta_{i\ell}$) such that $[H_1]^{n_1} \times\dots\times [H_m]^{n_m} \subseteq \Delta_\ell$.

Thus property 2.3 is that case of 2.5 in which $n_1 = \dots = n_m = n$, $\kappa_1 = \dots = \kappa_m = \kappa$ and $\eta_{1\ell} = \dots = \eta_{m\ell} = \eta_\ell$ for each $\ell < \nu$. Reference will rarely be made to the general property in the sequel, although some of the results do extend to it, and a few of the proofs do actually establish a stronger property of the general relation.

A symbol similar to that used in any of 2.1-2.5, but with $\to$ replaced by $\not\to$, indicates that the appropriate property fails to hold.

§3. Simple observations

Before a more detailed discussion of the various partition relations, a few simple remarks are in order (see also [11]).

The use of the ordinal ν to index the classes of the partition Δ is not essential. Any set with the same power as ν will serve equally well, and the truth or falsity of the relation will remain unchanged. In particular, the ordinals η_ℓ for $\ell < \nu$ in the relation $\kappa \to {}^m(\eta_\ell\ ;\ \ell < \nu)^n$ may be permuted without effecting the relation.

If κ has a partition property and λ is any cardinal at least as big as κ, then λ has that same property.

If κ enjoys a partition property with ν classes and μ is any ordinal smaller than ν, then the corresponding property with μ classes also holds for κ, since a partition with a small number of classes can be extended to a partition with a larger number of classes by adding superfluous parts (of, for example, one element).

If κ has a partition property in which the homogeneous sets have order type η_ℓ (or η), then for any ordinals $\theta_\ell \leqq \eta_\ell$ (or $\theta \leqq \eta$) the corresponding property with η_ℓ replaced by θ_ℓ (or η by θ) also holds for κ.

If $\kappa \to {}^m(\eta)^{<\omega}_\nu$, then also $\kappa \to {}^m(\eta)^n_\nu$ for any $n < \omega$.

If κ has a partition property involving sequences of length m, and if $m' \leqq m$, then κ has the corresponding property for sequences of length m'. (Given any partition involving sequences of length m', choose a partition of m-length sequences for which membership of any partition class depends on only the first m' places of the sequences involved.)

In the case that all the η_ℓ of 2.3 are cardinal numbers (and similarly for the η of 2.4), rather than requiring that the homogeneous sets have order type exactly η_ℓ, it suffices to specify that their power be η_ℓ. Thus, when the η_ℓ are cardinals, $\kappa \to {}^m(\eta_\ell\ ;\ \ell < \nu)^n$ if and only if for all partitions ${}^m([\kappa]^n) = \cup\{\Delta_\ell\ ;\ \ell < \nu\}$ there are $\ell < \nu$ and a sequence $H_1,\ldots,H_m$ from $[\kappa]^{\eta_\ell}$ such that $[H_1]^n \times\ldots\times [H_m]^n \subseteq \Delta_\ell$. The "only if" is clear. And if $H_1,\ldots,H_m \in [\kappa]^{\eta_\ell}$ then by cutting down $H_1,\ldots,H_m$, if necessary, it may be assumed that each H_i has order type exactly η_ℓ , so the "if" part holds also. Frequent use of this equivalence will be made in the following chapters, without further comment.

The relations 2.3 and 2.4 are of interest only if the following conditions hold (see [6, p.98] for a discussion of the degenerate cases for $m = 1$):

$$m \geqq 1 \ , \quad n \geqq 1 \ , \quad 2 \leqq \nu \leqq \kappa \ , \quad \omega \leqq \eta \leqq \kappa \ , \quad n < \eta_\ell \leqq \kappa \text{ (for all } \ell < \nu) \ .$$

Thus, unless the contrary is stated, these conditions are assumed to be in force.

The definitions 2.1-2.5 are expressed as properties of cardinal numbers. They could equally well be expressed as properties of arbitrary sets of the appropriate power. For example, $\kappa \to {}^m(\eta)^{<\omega}_\nu$ is equivalent to the following:

For any sequence of sets $S_1,\ldots,S_m$ where each S_i has power κ and is well ordered by a relation $<_i$, let there be given any partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $\cup\{[S_1]^n \times\ldots\times [S_m]^n \ ; \ n < \omega\}$. Then there is a sequence $H_1,\ldots,H_m$ where each $H_i \subseteq S_i$ and has order type η in $<_i$, such that for each $n < \omega$ there is $\ell < \nu$ for which $[H_1]^n \times\ldots\times [H_m]^n \subseteq \Delta_\ell$.

There is an obvious extension of the relation 2.3 to the case that n is an infinite cardinal. A well known example of Sierpinski [21] shows that any such relation with infinite n is false. Similarly, there are obvious extensions of 2.3 and 2.4 to the case that m is an infinite ordinal. These will be discussed in §II.3, where it will be shown that they, likewise, are false.

§4. Comments on the Erdös relations

In this section, I mention, very briefly, a few well known properties of the relations 2.1 and 2.2. The properties are either for later use, or to illustrate the surprising effects that partition cardinals can cause.

The first are Ramsey's theorems [16]:

4.1 THEOREM For all $n, r \in \omega$, $\omega \to (\omega)^n_r$.

4.2 THEOREM For all $\ell, n, r \in \omega$ there is $k = k(\ell, n, r) \in \omega$ such that $k \to (\ell)^n_r$.

The next few results may all be found in [6].

4.3 THEOREM If $\nu < \mathrm{Cf}(\kappa)$, then $\kappa^+ \to (\kappa)^2_\nu$.

4.4 THEOREM If $\alpha \geqq 0$ and $\nu < \mathrm{Cf}(\omega_\alpha)$, then $\omega_{\alpha+(n-1)} \to (\omega_\alpha)^n_\nu$.

4.5 THEOREM If $n \geqq 3$ and κ is not inaccessible, then $\kappa \not\to (\kappa, n+1)^n$.

Theorems 4.3-4.5 depend on the GCH. Without assuming the GCH, one can show (see, for example [15]):

4.6 THEOREM $\underline{2}_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$.

Let us turn now to the property $\kappa \to (\eta)^{<\omega}_\nu$. It has long been known that $\omega \not\to (\omega)^{<\omega}_2$. In fact (see [22, p.84]):

4.7 THEOREM If η is a limit ordinal and κ is least such that $\kappa \to (\eta)_2^{<\omega}$, then κ is strongly inaccessible.

Since it is consistent with ZF + AC to assume that there are no inaccessible cardinals, it follows that the existence of a cardinal with the property $\kappa \to (\eta)_\nu^{<\omega}$ cannot be proved in ZF + AC. Whenever I discuss such cardinals, I shall tacitly assume that they do in fact exist.

A frequently used theorem of Rowbottom [18] states the following:

4.8 THEOREM Let η be a limit ordinal. Then if $\kappa \to (\eta)_2^{<\omega}$, also $\kappa \to (\eta)_{2^\omega}^{<\omega}$.

In fact, Silver has shown [22, p.83] that if κ is least such that $\kappa \to (\eta)_2^{<\omega}$ for limit ordinal η, then $\kappa \to (\eta)_\nu^{<\omega}$ for any $\nu < \kappa$.

The existence of cardinals with these partition properties has a remarkable bearing on the size of the constructible universe of Gödel [7]. Work by Rowbottom, Gaifman and Silver has shown that if there is a cardinal κ with the property $\kappa \to (\omega_1)_2^{<\omega}$, then only a countable number of subsets of ω are constructible. (And much more. See [22] for full details.) Silver shows that cardinals κ with the property $\kappa \to (\omega)_2^{<\omega}$, however, are too small to have this effect - if their existence is consistent with ZF + AC, then it is also consistent with ZF + "V = L" (where "V = L" is Gödel's axiom of constructibility).

Cardinals κ such that $\kappa \to (\kappa)_2^{<\omega}$ are called Ramsey cardinals. It is well known:

4.9 THEOREM If κ is Ramsey, then κ is strongly inaccessible.

In §III.2, it will be shown that the Ramsey cardinals are, in a sense, the largest cardinals with any of the partition properties that are under consideration. They are by no means the largest cardinals whose existence has been postulated. If κ is measurable, then κ is Ramsey. And in fact, from [18],

4.10 THEOREM If κ is measurable, then there is a Ramsey cardinal λ such that $\lambda < \kappa$.

I conclude this section by mentioning a characterization of the property $\kappa \to (\eta)_2^{<\omega}$ in terms of models.

4.11 THEOREM Let η be a limit ordinal. Then the property $\kappa \to (\eta)_2^{<\omega}$ is equivalent to the following:

Take any model $\underset{=}{A} = \langle A, <, \ldots \rangle$ of a language having at most countably many relation and function symbols, for which $<$ well orders A . Then for any $A' \in [A]^{\kappa}$ there is a set $H \subseteq A'$, having order type η , which is homogeneous for $\underset{=}{A}$, in the sense that for any formula $\Phi(v_o, \ldots, v_n)$ in the language of $\underset{=}{A}$, given two increasing sequences $a_o < \ldots < a_n$ and $b_o < \ldots < b_n$ from H ,

$$\underset{=}{A} \models \Phi[a_o, \ldots, a_n] \iff \underset{=}{A} \models \Phi[b_o, \ldots, b_n] .$$

CHAPTER II. THE RELATION $\kappa \to {}^m(\eta_\ell \, ; \, \ell < \nu)^n$

§1. Some positive and negative results

The following simple observation forms one of the most general methods I have for obtaining positive relations.

1.1 THEOREM Let ν be a cardinal and for $\ell < \nu$ let η_ℓ be ordinals. Put $\eta = \sup\{\eta_\ell \, ; \, \ell < \nu\}$. Suppose κ is a cardinal such that $\kappa \to (\eta \cdot m)^{mn}_\nu$. Then $\kappa \to {}^m(\eta_\ell \, ; \, \ell < \nu)^n$.

PROOF: Let $\Delta = \{\Delta_\ell \, ; \, \ell < \nu\}$ be any partition of ${}^m([\kappa]^n)$. Choose any partition $\Gamma = \{\Gamma_\ell \, ; \, \ell < \nu\}$ of $[\kappa]^{mn}$ which satisfies:
if $\alpha_1 < \alpha_2 < \dots < \alpha_{mn} < \kappa$ then for all $\ell < \nu$,

$$\{\alpha_1, \alpha_2, \dots, \alpha_{mn}\} \in \Gamma_\ell \iff \langle \{\alpha_1, \dots, \alpha_n\}, \dots, \{\alpha_{mn-n+1}, \dots, \alpha_{mn}\} \rangle \in \Delta_\ell \; .$$

Since $\kappa \to (\eta \cdot m)^{mn}_\nu$, there is $H \subseteq \kappa$ having order type $\eta \cdot m$ such that $[H]^{mn} \subseteq \Gamma_\ell$ for some $\ell < \nu$. Divide H into m pieces, $H_1, \dots, H_m$, each having order type η, such that $\sup(H_i) \leqq \min(H_{i+1})$ for each i. However, then $[H_1]^n \times \dots \times [H_m]^n \subseteq \Delta_\ell$, and since $\eta_\ell \leqq \eta$ it follows that $\kappa \to {}^m(\eta_\ell \, ; \, \ell < \nu)^n$.

Coupled with the results mentioned in §I.4, this leads for example to the following corollaries:

1.2 COROLLARY For all $\ell, m, n, r \in \omega$ there is $k = k(\ell, m, n, r) \in \omega$ such that $k \to {}^m(\ell)^n_r$.

1.3 COROLLARY If $\alpha \geqq 0$ and $\eta_\ell < \omega_{\alpha+1}$ for $\ell < \omega_\alpha$, then $\omega_{\alpha+mn} \rightarrow {}^m(\eta_\ell \,;\, \ell < \omega_\alpha)^n$.
In particular, $\omega_{\alpha+mn} \rightarrow {}^m(\omega_\alpha)^n_{\omega_\alpha}$.

Corollary 1.3 assumes the GCH. Without the GCH, one has

1.4 COROLLARY If $\alpha \geqq 0$, $\eta_\ell < \omega_{\alpha+1}$ for $\ell < \omega_\alpha$, and provided $mn > 1$, then $\beth_{mn-1}(\omega_\alpha)^+ \rightarrow {}^m(\eta_\ell \,;\, \ell < \omega_\alpha)^n$.

Theorem I.4.1 shows that there exist cardinals κ with the property $\kappa \rightarrow {}^m(\kappa)^n_\nu$ for $m = 1$. When $m > 1$, this is not the case. The following theorem shows that even the weakest such property fails to hold.

1.5 THEOREM For all κ, $\kappa \not\rightarrow {}^2(\kappa)^1_2$.

PROOF: Take any cardinal κ, and define a partition $\Delta = \{\Delta_o, \Delta_1\}$ of $[\kappa]^1 \times [\kappa]^1$ as follows:

$$\langle\{\alpha\},\{\beta\}\rangle \in \Delta_o \iff \alpha \leqq \beta \quad ; \quad \langle\{\alpha\},\{\beta\}\rangle \in \Delta_1 \iff \beta \leqq \alpha \quad .$$

Let H_o, H_1 be any pair homogeneous for Δ. Suppose, say, that $[H_o]^1 \times [H_1]^1 \subseteq \Delta_o$. If $\beta = \min(H_1)$ and $\alpha \in H_o$ then $\alpha \leqq \beta$. But $\beta \in \kappa$, and so $|H_o| < \kappa$. Similarly, if $[H_o]^1 \times [H_1]^1 \subseteq \Delta_1$ then $|H_1| < \kappa$. Thus no pair homogeneous for Δ can have $|H_o| = |H_1| = \kappa$. This proves the theorem.

This same partition shows that there is little future in trying to strengthen the property $\kappa \rightarrow {}^m(\eta_\ell \,;\, \ell < \nu)^n$ by demanding that all the sets in a homogeneous sequence should be the same. For suppose H, H is a pair homogeneous for Δ. Then for all $\alpha \in H$ it must be that $\alpha \leqq \min(H)$. Hence $|H| = 1$.

§2. The property $\kappa \xrightarrow{w} {}^{m}(\eta)^{n}_{\nu}$

The proof of theorem 1.1 used a sequence of homogeneous sets in which the supremum of one was no larger than the minimum of the next. Provided that the partition has a sufficiently large number of classes, this may always be assumed for the relation $\kappa \to {}^{m}(\eta)^{n}_{\nu}$. I prove it only in the case that η is a limit ordinal.

2.1 THEOREM Let $\kappa \to {}^{m}(\eta)^{n}_{\nu}$ where ν is infinite and η is a limit ordinal. Then for any partition $\Delta = \{\Delta_{\ell}\ ;\ \ell < \nu\}$ of ${}^{m}([\kappa]^{n})$ there is a sequence $H_1, \dots, H_m$ homogeneous for Δ (where each $H_i \subseteq \kappa$ and has order type η) for which there is a permutation σ of $\{1, \dots, m\}$ such that $\sup(H_{\sigma(i)}) \leqq \min(H_{\sigma(i+1)})$ for each i.

PROOF: If $m = 1$ there is nothing to show; so suppose $m > 1$. Put $k = m!$ and let σ_j for $j < k$ enumerate all permutations of $\{1, \dots, m\}$. Let a partition $\Delta = \{\Delta_{\ell}\ ;\ \ell < \nu\}$ of ${}^{m}([\kappa]^{n})$ be given. Define a partition $\Gamma = \{\Gamma_{\ell}\ ;\ \ell < \nu \cdot k\}$ of ${}^{m}([\kappa]^{n})$ to refine Δ as follows:
for $j < k$, $\ell < \nu$ and $a_1, \dots, a_m \in [\kappa]^{n}$,

$$\langle a_1, \dots, a_m \rangle \in \Gamma_{\nu \cdot j+\ell} \iff \langle a_1, \dots, a_m \rangle \in \Delta_{\ell} \text{ and}$$
$$\forall i \in \{2, \dots, m\}[\min(a_{\sigma_j(i-1)}) \leqq \min(a_{\sigma_j(i)})] .$$

Since $\nu \geqq \omega$, it follows that $|\nu \cdot k| = |\nu|$ and so by the property $\kappa \to {}^m(\eta)^n_\nu$ there are $\ell' < \nu \cdot k$ and a sequence $H_1, \ldots, H_m$ (each $H_i \subseteq \kappa$ and having order type η) such that $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Gamma_{\ell'}$. Suppose $\ell' = \nu \cdot j + \ell$ (where $0 \leqq \ell < \nu$). Then $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_\ell$. Further, $\sup(H_{\sigma_j(i-1)}) \leqq \min(H_{\sigma_j(i)})$ for each $i \in \{\overset{2}{\cancel{1}}, \ldots, m\}$. For take $\alpha \in H_{\sigma_j(i-1)}$ and $\beta \in H_{\sigma_j(i)}$. Since η is a limit ordinal, choose $a \in [H_{\sigma_j(i-1)}]^n$ and $b \in [H_{\sigma_j(i)}]^n$ such that $\alpha = \min(a)$ and $\beta = \min(b)$. Then it follows from the definition of $\Gamma_{\ell'} = \Gamma_{\nu \cdot j + \ell}$ that $\alpha = \min(a) \leqq \min(b) = \beta$. This suffices to prove 2.1.

The property $\kappa \to {}^m(\eta)^n_\nu$ is concerned with sequences of finite subsets. Theorem 1.5 shows that this condition is too strong to allow for a generalization of the property $\kappa \to (\kappa)^2_2$. Consequently, let us consider a weakening of $\kappa \to {}^m(\eta)^n_\nu$ which ignores the ordering in the sequence.

2.2 DEFINITION $\kappa \overset{w}{\to} {}^m(\eta)^n_\nu$ if for all partitions $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $[[\kappa]^n]^m$ into ν parts, there are $\ell < \nu$ and a sequence $H_1, \ldots, H_m$ (where each $H_i \subseteq \kappa$ and has order type η) such that $|\cup\{H_i \ ; \ i < m\}| \geqq m$ and

(1) $\forall i \in \{1, \ldots, m\}(a_i \in [H_i]^n)$ and $|\{a_1, \ldots, a_m\}| = m \Rightarrow \{a_1, \ldots, a_m\} \in \Delta_\ell$.

(Such a sequence $H_1, \ldots, H_m$ will be called weakly homogeneous for Δ . The requirement $|\cup\{H_i \ ; \ i < m\}| \geqq m$ is to ensure that the condition (1) is non-vacuous.)

That 2.2 is certainly no strengthening of the original property is shown by the following theorem.

2.3 THEOREM If $\kappa \to {}^m(\eta)^n_\nu$ then $\kappa \xrightarrow{w} {}^m(\eta)^n_\nu$.

PROOF: Take any partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $[[\kappa]^n]^m$. Choose any partition $\Gamma = \{\Gamma_\ell \ ; \ \ell < \nu\}$ of ${}^m([\kappa]^n)$ such that:
if $a_1, \ldots, a_m \in [\kappa]^n$ are all distinct, then for all $\ell < \nu$

$$\langle a_1, \ldots, a_m \rangle \in \Gamma_\ell \iff \{a_1, \ldots, a_m\} \in \Delta_\ell \ .$$

Then any sequence homogeneous for Γ is weakly homogeneous for Δ . This establishes 2.3.

The surprising fact about 2.2 is that when $n > 1$ and ν is infinite, then 2.2 is equivalent to $\kappa \to {}^m(\eta)^n_\nu$. The following lemma is needed (the analogue of 2.1).

2.4 LEMMA Let $\kappa \xrightarrow{w} {}^m(\eta)^n_\nu$ where $1 < n < \eta$ and ν is infinite. Then for any partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $[[\kappa]^n]^m$ there is a sequence $H_1, \ldots, H_m$ (each $H_i \subseteq \kappa$ and having order type η) weakly homogeneous for Δ , for which there is a permutation σ of $\{1, \ldots, m\}$ such that $\sup(H_{\sigma(i)}) \leqq \min(H_{\sigma(i+1)})$ for each i .

PROOF: If $m = 1$ there is nothing to be shown; so suppose $m > 1$. Consider any set $\{\{\alpha_1, \ldots, \alpha_n\}, \ldots, \{\alpha_{mn-n+1}, \ldots, \alpha_{mn}\}\} \in [[\kappa]^n]^m$. We may suppose that $\alpha_1 < \cdots < \alpha_n \ ; \ldots ; \ \alpha_{mn-n+1} < \cdots < \alpha_{mn}$ and also that $\{\alpha_1, \ldots, \alpha_n\} \ll \cdots \ll \{\alpha_{mn-n+1}, \ldots, \alpha_{mn}\}$, (where $\ll$ is the lexicographic ordering). Let L_j for $j < k$ enumerate all possible arrangements of the ordering of the α_i amongst themselves (for example, $\alpha_n = \alpha_{n+1}$, $\alpha_{2n} < \alpha_{2n+1}$,..., $\alpha_{mn-n} < \alpha_{mn-n+1}$). Then certainly $k < \omega$.

Take any partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $[[\kappa]^n]^m$. Define a partition $\Gamma = \{\Gamma_\ell \ ; \ \ell < \nu \cdot k\}$ of $[[\kappa]^n]^m$ to refine Δ as follows: for $j < k$, $\ell < \nu$ and $a_1,\ldots,a_m \in [\kappa]^n$ with $|\{a_1,\ldots,a_m\}| = m$,

$\{a_1,\ldots,a_m\} \in \Gamma_{\nu\cdot j+\ell} \iff \{a_1,\ldots,a_m\} \in \Delta_\ell$ and the elements of $a_1,\ldots,a_m$, written out in the order in which they occur, fall under the ordering L_j .

– However, if $\Gamma_{\nu\cdot j+\ell} = \emptyset$, exclude it from Γ.

Since $|\nu| \geq \omega$, it follows that $|\nu\cdot k| = |\nu|$. Hence there are $\ell' < \nu\cdot k$ and a sequence $H_1,\ldots,H_m$ (each $H_i \subseteq \kappa$ and having order type η) such that

$$\forall i \in \{1,\ldots,m\}(a_i \in [H_i]^n) \text{ and } |\{a_1,\ldots,a_m\}| = m \Rightarrow \{a_1,\ldots,a_m\} \in \Gamma_{\ell'} .$$

Thus $H_1,\ldots,H_m$ is weakly homogeneous for Δ , and further, if $\ell' = \nu\cdot j + \ell$ (where $0 \leq \ell < \nu$) then the elements of any distinct sets $a_1,\ldots,a_m$ (where each $a_i \in [H_i]^n$) are arranged in the order L_j . Since $\eta > n$, this can happen only if there is a permutation σ of $\{1,\ldots,m\}$ such that for each i , if $\alpha \in H_{\sigma(i)}$ and $\beta \in H_{\sigma(i+1)}$ then $\alpha < \beta$. This establishes the result.

2.5 THEOREM Let $n > 1$ and $\nu \geq \omega$. Then if $\kappa \overset{w}{\to} {}^m(\eta)^n_\nu$, also $\kappa \to {}^m(\eta)^n_\nu$.

PROOF: Consider a sequence $\langle a_1,\ldots,a_m\rangle$ where $a_1,\ldots,a_m$ are distinct elements from $[\kappa]^n$. Put $k = m!$ and let L_j for $j < k$ enumerate all possible arrangements of $a_1,\ldots,a_m$ under the lexicographic ordering.

Let any partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of ${}^m([\kappa]^n)$ be given. Define a partition $\Gamma = \{\Gamma_\ell \ ; \ \ell < \nu\cdot k\}$ of $[[\kappa]^n]^m$ such that: for $j < k$, $\ell < \nu$ and $a_1,\ldots,a_m \in [\kappa]^n$ with $|\{a_1,\ldots,a_m\}| = m$,

$$\{a_1,\ldots,a_m\} \in \Gamma_{\nu\cdot j+\ell} \iff \langle a'_1,\ldots,a'_m\rangle \in \Delta_\ell ,$$

where $\langle a'_1,\ldots,a'_m\rangle$ is that sequence with entries in the order L_j built from $a_1,\ldots,a_m$.

By the property $\kappa \overset{w}{\to} {}^m(\eta)^n_\nu$, there are $j < k$ and $\ell < \nu$, together with a sequence $H_1,\ldots,H_m$ (each $H_i \subseteq \kappa$ and having order type η) such that for distinct sets $a_1 \in [H_1]^n,\ldots,a_m \in [H_m]^n$ always $\{a_1,\ldots,a_m\} \in \Gamma_{\nu\cdot j+\ell}$. When $\eta > n$ (and otherwise the theorem is trivial), by Lemma 2.4 it may be assumed further that there is some permutation σ of $\{1,\ldots,m\}$ for which $\sup(H_{\sigma(i)}) \leqq \min(H_{\sigma(i+1)})$. Take distinct sets $a_1 \in [H_1]^n,\ldots,a_m \in [H_m]^n$. Then always $\langle a'_1,\ldots,a'_m\rangle \in \Delta_\ell$, where $\langle a'_1,\ldots,a'_m\rangle$ is $a_1,\ldots,a_m$ arranged in the lexicographic order L_j. By the additional property of $H_1,\ldots,H_m$, these sets can be renamed $H'_1,\ldots,H'_m$ so that always if $a'_i \in [H'_i]^n$ then $\langle a'_1,\ldots,a'_m\rangle$ is already in the order L_j. Thus the sequence $H'_1,\ldots,H'_m$ is homogeneous for Δ. This proves 2.5.

Theorem 2.5 excludes the case $n = 1$. In this case, however:

2.6 THEOREM Let η be a limit ordinal, and suppose $\kappa \to (\eta)^m_\nu$. Then $\kappa \overset{w}{\to} {}^m(\eta)^1_\nu$.

PROOF: Identify $[\kappa]^m$ and $[[\kappa]^1]^m$. Take any partition $\Delta = \{\Delta_\ell \,;\, \ell < \nu\}$ of $[[\kappa]^1]^m$. Since $\kappa \to (\eta)^m_\nu$ there is $H \subseteq \kappa$, having order type η, homogeneous for Δ, as a partition of $[\kappa]^m$. Enumerate H in increasing order, $H = \{h_\alpha \,;\, \alpha < \eta\}$. For $i = 1,\ldots,m$ put $H_i = \{h_{m\cdot\alpha+i-1} \,;\, \alpha < \eta\}$. Then each H_i has order type η, and $H_1,\ldots,H_m$ is weakly homogeneous for Δ, as a partition of $[[\kappa]^1]^m$. This establishes the theorem.

Thus, by Theorem I.4.1, there are cardinals κ for which $\kappa \overset{w}{\to} {}^2(\kappa)^1_2$, as opposed to the result of Theorem 1.5. The negative result is not far off, however.

2.7 THEOREM For no κ is it true that $\kappa \overset{w}{\to} {}^2(\kappa)^2_2$.

PROOF: Given κ, consider the disjoint partition $\Delta = \{\Delta_o,\Delta_1\}$ of $[[\kappa]^2]^2$ defined by

$$\{\{\alpha,\beta\},\{\gamma,\delta\}\} \in \Delta_o \iff \alpha < \beta < \gamma < \delta \ .$$

Take $H_o, H_1 \subseteq \kappa$, both having order type κ. Then each is cofinal in the other. Thus given $\alpha,\beta \in H_o$, there are $\gamma,\delta \in H_1$ such that $\alpha,\beta < \gamma,\delta$ so that $\{\{\alpha,\beta\},\{\gamma,\delta\}\} \in \Delta_o$. Equally well, given $\alpha \in H_o$ there are $\gamma,\delta \in H_1$ such that $\alpha < \gamma,\delta$ and then there is $\beta \in H_o$ such that $\gamma,\delta < \beta$. Then $\{\{\alpha,\beta\},\{\gamma,\delta\}\} \in \Delta_1$. Thus Δ has no weakly homogeneous sequence H_o, H_1 where both H_o and H_1 have order type κ.

§3. <u>The case $m = \omega$</u>

The definitions I.2.3 and I.2.4 have obvious extensions to the case of infinite m . In this section, it will be shown that all non-trivial such relations are false, by establishing that for no κ does $\kappa \to {}^{\omega}(2)^{1}_{2}$ hold. This supplies an answer to an open problem of Moschovakis [13, P2061], who asks if $\omega_2 \to {}^{\omega}(2)^{1}_{2}$. This same problem has been answered recently also by Kunen, with a different example. The first result involving infinite m was obtained by D.A. Martin, who showed that no κ has the property $\kappa \to {}^{\omega_1}(\omega_1)^{<\omega}_{2}$. Martin's result follows from ours, for the relation $\kappa \not\to {}^{\omega}(2)^{1}_{2}$ certainly implies $\kappa \not\to {}^{\omega}(\omega)^{<\omega}_{2}$.

The extension of 2.2 to infinite m will be considered also.

Several lemmas are needed. The first is a theorem (unpublished) of Galvin.

3.1 LEMMA For any infinite set S , there is a function f mapping $[S]^{\omega}$ into 2^{ω} which has the following property:
for any $t \in 2^{\omega}$ and any two sequences $\langle a_i ; i < \omega\rangle$, $\langle b_i ; i < \omega\rangle$ of disjoint finite subsets of S such that $a_i \cap b_i = \emptyset$ and $|\bigcup\{a_i \cup b_i ; i < \omega\}| = \omega$, there is a sequence $\langle c_i ; i < \omega\rangle$, where $c_i \in \{a_i, b_i\}$, such that $|\bigcup\{c_i ; i < \omega\}| = \omega$ and $f(\bigcup\{c_i ; i < \omega\}) = t$.

The next lemma gives a similar result for a mapping of the infinite sequences of finite subsets. The proof is a trivial modification of Galvin's proof of Lemma 3.1.

3.2 LEMMA For any infinite set S , there is a function f mapping ${}^{\omega}([S]^{<\omega})$ into 2^{ω} which has the following property:
for any $t \in 2^{\omega}$ and any two sequences $\langle a_i ; i < \omega\rangle$, $\langle b_i ; i < \omega\rangle$ of finite subsets of S such that $a_i \cap b_i = \emptyset$ and $|\bigcup\{a_i \cup b_i ; i < \omega\}| = \omega$,

there is a sequence $\langle c_i ; i < \omega \rangle$, where $c_i \in \{a_i, b_i\}$, such that $|\bigcup\{c_i ; i < \omega\}| = \omega$ and $f(\langle c_i ; i < \omega \rangle) = t$.

PROOF: If $|S| \leq 2^\omega$, there are 2^ω such pairs $\langle a_i ; i < \omega \rangle$, $\langle b_i ; i < \omega \rangle$. Each pair gives rise to 2^ω suitable sequences $\langle c_i ; i < \omega \rangle$, and it follows by a diagonalization argument that the required function f can be defined. Hence the lemma holds if $|S| \leq 2^\omega$.

Now let S be arbitrary. For each $A \in [S]^\omega$, choose f_A mapping ${}^\omega([A]^{<\omega})$ into 2^ω to satisfy the requirements of the lemma for the set A . Let $[S]^\omega$ be well ordered, and for each $A \in [S]^\omega$, let A' be the first set in the well ordering for which $|A \cap A'| = \omega$. Take a sequence $\langle c_i ; i < \omega \rangle \in {}^\omega([S]^{<\omega})$. Put $C = \bigcup\{c_i ; i < \omega\}$. In the case $|C| = \omega$, define

$$f(\langle c_i ; i < \omega \rangle) = f_{C'}(\langle c_i \cap C' ; i < \omega \rangle) .$$

If $|C| < \omega$, the value of $f(\langle c_i ; i < \omega \rangle)$ is immaterial. We show that the function f so defined has the required property.

Take two sequences $\langle a_i ; i < \omega \rangle$, $\langle b_i ; i < \omega \rangle$ of finite subsets of S such that $a_i \cap b_i = \emptyset$ and $|\bigcup\{a_i \cup b_i ; i < \omega\}| = \omega$. Put $A = \bigcup\{a_i \cup b_i ; i < \omega\}$, so $A \in [S]^\omega$. Put $a_i' = a_i \cap A'$ and $b_i' = b_i \cap A'$. Then $|\bigcup\{a_i' \cup b_i' ; i < \omega\}| = |A \cap A'| = \omega$. Thus for any $t \in 2^\omega$, there is a sequence $\langle d_i ; i < \omega \rangle$ where $d_i \in \{a_i', b_i'\}$ such that $|\bigcup\{d_i ; i < \omega\}| = \omega$ and $f_{A'}(\langle d_i ; i < \omega \rangle) = t$. Choose a sequence $\langle c_i ; i < \omega \rangle$ such that $c_i \in \{a_i, b_i\}$ and if $c_i' = c_i \cap A'$ then $c_i' = d_i$. Then $|\bigcup\{c_i ; i < \omega\}| = \omega$, and $f_{A'}(\langle c_i' ; i < \omega \rangle) = t$. However, $(\bigcup\{c_i ; i < \omega\})' = A'$, since $|A' \cap \bigcup\{c_i ; i < \omega\}| = |\bigcup\{c_i' ; i < \omega\}| = \omega$

and $\bigcup\{c_i \; ; \; i < \omega\} \subseteq A$. Thus $f(\langle c_i \; ; \; i < \omega\rangle) = f_{A'}(\langle c'_i \; ; \; i < \omega\rangle) = t$. This completes the proof of the lemma.

3.3 LEMMA For any set S , there is a function g mapping $[S]^{<\omega}$ into 2 which has the following property:
for any $t \in 2$ and any sequence $\langle\{\alpha_i,\beta_i\} \; ; \; i < \omega\rangle$ such that $\{\alpha_i,\beta_i\} \in [S]^2$ and $|\bigcup\{\{\alpha_i,\beta_i\} \; ; \; i < \omega\}| < \omega$, there is a sequence $\langle\gamma_i \; ; \; i < \omega\rangle$ such that $\gamma_i \in \{\alpha_i,\beta_i\}$ and $g(\{\gamma_i \; ; \; i < \omega\}) = t$.

PROOF: Define g mapping $[S]^{<\omega}$ into 2 by:

$$\text{for } a \in [S]^{<\omega} , \quad g(a) = |a| \pmod 2 .$$

To show that g , so defined, has the required property, given any suitable sequence $\langle\{\alpha_i,\beta_i\} \; ; \; i < \omega\rangle$, it suffices to find sequences $\langle\gamma_i \; ; \; i < \omega\rangle$ and $\langle\delta_i \; ; \; i < \omega\rangle$ such that $\gamma_i,\delta_i \in \{\alpha_i,\beta_i\}$ and $|\{\delta_i \; ; \; i < \omega\}| = |\{\gamma_i \; ; \; i < \omega\}| + 1$.

Since $|\bigcup\{\{\alpha_i,\beta_i\} \; ; \; i < \omega\}| < \omega$, there must be some α_i or β_i occurring in more than one pair - say for instance $\alpha_0 = \alpha_1$. For $i > 1$, choose $\zeta_i \in \{\alpha_i,\beta_i\}$ such that $\zeta_i \neq \beta_0$. Define the sequences $\langle\gamma_i \; ; \; i < \omega\rangle$ and $\langle\delta_i \; ; \; i < \omega\rangle$ as follows:

$$\gamma_0 = \alpha_0 , \; \delta_0 = \beta_0 ; \quad \gamma_1 = \delta_1 = \alpha_1 ; \quad \gamma_i = \delta_i = \zeta_i \quad \text{for } i > 1 .$$

Then $|\{\delta_i \; ; \; i < \omega\}| = |\{\gamma_i \; ; \; i < \omega\}| + 1$. The lemma is proved.

3.4 THEOREM For all κ , $\kappa \nrightarrow {}^{\omega}(2)^1_2$.

PROOF: Let κ be given. Let f be a function as provided by Lemma 3.2 for the set κ, and let g be a function as provided by Lemma 3.3 for κ. Identify ${}^{\omega}([\kappa]^1)$ and ${}^{\omega}\kappa$. Let $\Delta = \{\Delta_0, \Delta_1\}$ be that disjoint partition of ${}^{\omega}\kappa$ defined by:

$$\langle\alpha_i \,;\, i < \omega\rangle \in \Delta_0 \iff (|\{\alpha_i \,;\, i < \omega\}| = \omega \text{ and } f(\langle\{\alpha_i\} \,;\, i < \omega\rangle) = 0)$$
$$\text{or } (|\{\alpha_i \,;\, i < \omega\}| < \omega \text{ and } g(\{\alpha_i \,;\, i < \omega\}) = 0) \,.$$

Consider any sequence $\langle H_i \,;\, i < \omega\rangle$ where each $H_i \in [\kappa]^2$, say $H_i = \{\alpha_i, \beta_i\}$. If $|\bigcup\{H_i \,;\, i < \omega\}| < \omega$, by Lemma 3.3 there are $\langle\gamma_i \,;\, i < \omega\rangle$, $\langle\delta_i \,;\, i < \omega\rangle \in \Pi\{H_i \,;\, i < \omega\}$ such that $g(\{\gamma_i \,;\, i < \omega\}) = 0$ and $g(\{\delta_i \,;\, i < \omega\}) = 1$. If $|\bigcup\{H_i \,;\, i < \omega\}| = \omega$, apply Lemma 3.2 to the sequences $\langle\{\alpha_i\} \,;\, i < \omega\rangle$ and $\langle\{\beta_i\} \,;\, i < \omega\rangle$ to find $\langle\gamma_i \,;\, i < \omega\rangle$, $\langle\delta_i \,;\, i < \omega\rangle \in \Pi\{H_i \,;\, i < \omega\}$ such that $f(\langle\{\gamma_i\} \,;\, i < \omega\rangle) = 0$ and $f(\langle\{\delta_i\} \,;\, i < \omega\rangle) = 1$. Thus no such sequence $\langle H_i \,;\, i < \omega\rangle$ can be homogeneous for Δ. The theorem follows.

Let us turn now to the case of weak homogeneity. The definition reads:

3.5 DEFINITION $\kappa \overset{w}{\to} {}^{\omega}(\eta)^n_\nu$ if for all partitions $\Delta = \{\Delta_\ell \,;\, \ell < \nu\}$ of $[[\kappa]^n]^\omega$ into ν parts, there are $\ell < \nu$ and a sequence $\langle H_i \,;\, i < \omega\rangle$ (where each $H_i \subseteq \kappa$ and has order type η) such that $|\bigcup\{H_i \,;\, i < \omega\}| \geqq \omega$ and

$$\forall i \in \omega(a_i \in [H_i]^n) \text{ and } |\{a_i \,;\, i < \omega\}| = \omega \implies \{a_i \,;\, i < \omega\} \in \Delta_\ell \,.$$

A negative result comes from Galvin's theorem, 3.1.

3.6 THEOREM For no κ is it true that $\kappa \overset{w}{\rightarrow} {}^{\omega}(\omega)^1_2$.

PROOF: Let κ be given. Let f be a function mapping $[\kappa]^\omega$ into 2^ω as given by Lemma 3.1 for the set κ . Identify $[[\kappa]^1]^\omega$ and $[\kappa]^\omega$. Let $\Delta = \{\Delta_o, \Delta_1\}$ be that disjoint partition of $[\kappa]^\omega$ defined by:

$$\text{for } A \in [\kappa]^\omega \ , \quad A \in \Delta_o \iff f(A) = 0 \ .$$

Consider any sequence $\langle H_i \ ; \ i < \omega \rangle$ where each $H_i \in [\kappa]^\omega$. For each $i < \omega$, choose $\{\alpha_i, \beta_i\} \in [H_i - \cup\{\{\alpha_j, \beta_j\} \ ; \ j < i\}]^2$. Then the sequences $\langle \{\alpha_i\} \ ; \ i < \omega \rangle$ and $\langle \{\beta_i\} \ ; \ i < \omega \rangle$ satisfy the conditions imposed by Lemma 3.1, and so there are $\gamma_i, \delta_i \in \{\alpha_i, \beta_i\}$ such that $|\{\gamma_i \ ; \ i < \omega\}| = |\{\delta_i \ ; \ i < \omega\}| = \omega$, for which $f(\{\gamma_i \ ; \ i < \omega\}) = 0$ and $f(\{\delta_i \ ; \ i < \omega\}) \neq 0$. Thus no such sequence $\langle H_i \ ; \ i < \omega \rangle$ can be weakly homogeneous for Δ . This proves the proposition.

This leaves open the question:

3.7 PROBLEM Is there κ such that $\kappa \overset{w}{\rightarrow} {}^{\omega}(2)^1_2$ is true? In particular, does ω_o have this property?

CHAPTER III. THE RELATION $\kappa \to {}^m(\eta)^{<\omega}_{\nu}$

§1. Equivalent forms of the definition

Definition I.2.4 is restricted to a consideration of those sequences of finite subsets of κ for which all entries in the sequence have the same power. Provided that η is a limit ordinal, this restriction may be ignored. Theorem 1.2 contains the result. Its proof depends, however, on the following lemma (the analogue of Rowbottom's theorem, I.4.8).

1.1 LEMMA Let η be a limit ordinal, and suppose $\kappa \to {}^m(\eta)^{<\omega}_2$. Then $\kappa \to {}^m(\eta)^{<\omega}_{2^\omega}$.

PROOF: Let $\kappa \to {}^m(\eta)^{<\omega}_2$ for some limit ordinal η . Take any disjoint partition $\Delta = \{\Delta_f \; ; \; f \in {}^{\omega}2\}$ of κ^{*m}. Choose some fixed enumeration of $\omega \times \omega$ which has the property that if $\langle p,q\rangle$ is the n-th pair then always $p,q \leq n$. Select some partition $\Gamma = \{\Gamma_o, \Gamma_1\}$ of κ^{*m} which satisfies, for each n :

if $\alpha_o(i) < \cdots < \alpha_n(i) < \kappa$ for $i = 1,\ldots,m$, and if $\langle p,q\rangle$ is the n-th pair of $\omega \times \omega$ in the chosen enumeration, then for $j < 2$

$$\langle\{\alpha_o(1),\ldots,\alpha_n(1)\},\ldots,\{\alpha_o(m),\ldots,\alpha_n(m)\}\rangle \in \Gamma_j \iff$$

$$\exists f(\langle\{\alpha_o(1),\ldots,\alpha_p(1)\},\ldots,\{\alpha_o(m),\ldots,\alpha_p(m)\}\rangle \in \Delta_f \text{ and } f(q) = j).$$

Take any sequence $H_1,\ldots,H_m$ homogeneous for Γ , as given by the property $\kappa \to {}^m(\eta)^{<\omega}_2$. Then $H_1,\ldots,H_m$ is also homogeneous for Δ , as follows.

For each i , take any $\{\alpha_o(i),\ldots,\alpha_p(i)\},\{\beta_o(i),\ldots,\beta_p(i)\} \subseteq H_i$, and it may as well be assumed that each set is listed in increasing order.

Take any $q < \omega$, and let n be such that $\langle p,q \rangle$ is the n-th pair. Since η is a limit ordinal, there are $\gamma_{p+1}(i) < \ldots < \gamma_n(i)$ in H_i such that $\alpha_p(i), \beta_p(i) < \gamma_{p+1}(i)$. Suppose $\langle \{\alpha_o(i),\ldots,\alpha_p(i)\} ; i = 1,\ldots,m \rangle \in \Delta_f$ and $\langle \{\beta_o(i),\ldots,\beta_p(i)\} ; i = 1,\ldots,m \rangle \in \Delta_g$. (Since Δ is disjoint, f and g are unique.) Then by the homogeneity of $H_1,\ldots,H_m$ for Γ, when $j < 2$:

$$f(q) = j \iff \langle \{\alpha_o(i),\ldots,\alpha_p(i),\gamma_{p+1}(i),\ldots,\gamma_n(i)\} ; i = 1,\ldots,m \rangle \in \Gamma_j$$

$$\iff \langle \{\beta_o(i),\ldots,\beta_p(i),\gamma_{p+1}(i),\ldots,\gamma_n(i)\} ; i = 1,\ldots,m \rangle \in \Gamma_j$$

$$\iff g(q) = j .$$

Thus $f(q) = g(q)$ for all $q < \omega$, and so $f = g$. Hence $H_1,\ldots,H_m$ is homogeneous for Δ. This proves the lemma.

1.2 THEOREM Let η be a limit ordinal; let $\kappa \to {}^m(\eta)^{<\omega}_{\nu}$. Then for any partition $\Delta = \{\Delta_\ell ; \ell < \nu\}$ of ${}^m([\kappa]^{<\omega} - \{\emptyset\})$ there is a sequence $H_1,\ldots,H_m$ where each $H_i \subseteq \kappa$ and has order type η, which is homogeneous for Δ in the extended sense, i.e. for each sequence $n_1,\ldots,n_m$ where each $n_i > 0$, there is $\ell < \nu$ for which $[H_1]^{n_1} \times \ldots \times [H_m]^{n_m} \subseteq \Delta_\ell$.

PROOF: Let a partition $\Delta = \{\Delta_\ell ; \ell < \nu\}$ of ${}^m([\kappa]^{<\omega} - \{\emptyset\})$ be given. For each $n > 0$, put

$$F_n = \{f ; f \text{ maps } \{1,\ldots,m\} \text{ into } \{1,\ldots,n\}\} .$$

For each $f \in F_n$, define a partition $\Gamma(f) = \{\Gamma_\ell(f) ; \ell < \nu\}$ of ${}^m([\kappa]^n)$ by:

if $\alpha_1(i) < \ldots < \alpha_n(i) < \kappa$ for $i = 1,\ldots,m$ then

$$\langle\{\alpha_1(i),\ldots,\alpha_n(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Gamma_\ell(f) \iff$$

$$\langle\{\alpha_1(i),\ldots,\alpha_{f(i)}(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Delta_\ell\ .$$

Then $\Gamma = \{\Gamma_\ell(f)\ ;\ f \in \cup\{F_n\ ;\ n < \omega\}$ and $\ell < \nu\}$ is a partition of κ^{*m}. Now $|F_n| < \omega$ for all n, and so $|\Gamma| \leqq \omega|\nu|$. Since $\kappa \to {}^m(\eta)^{<\omega}_\nu$, also $\kappa \to {}^m(\eta)^{<\omega}_{\omega|\nu|}$ - either trivially if $|\nu| \geqq \omega$, or by Lemma 1.1 if $|\nu| < \omega$. Thus there is a sequence $H_1,\ldots,H_m$ (each $H_i \subseteq \kappa$ and having order type η) which is homogeneous for Γ. This same sequence $H_1,\ldots,H_m$ is also homogeneous for Δ in the extended sense, as follows.

Take any sequence $n_1,\ldots,n_m$ where each $n_i > 0$. Let $n = \max\{n_1,\ldots,n_m\}$, and define $f \in F_n$ by $f(i) = n_i$. For each i, take sets $\alpha_1(i) < \cdots < \alpha_{f(i)}(i)$ and $\beta_1(i) < \cdots < \beta_{f(i)}(i)$ from H_i. Since η is a limit ordinal, there are $\gamma_{f(i)+1}(i) < \cdots < \gamma_n(i)$ in H_i with $\alpha_{f(i)}(i),\beta_{f(i)}(i) < \gamma_{f(i)+1}$. But now for any $\ell < \nu$, by the homogeneity of $H_1,\ldots,H_m$ for Γ,

$$\langle\{\alpha_o(i),\ldots,\alpha_{f(i)}(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Delta_\ell$$

$$\iff \langle\{\alpha_o(i),\ldots,\alpha_{f(i)}(i),\gamma_{f(i)+1}(i),\ldots,\gamma_n(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Gamma_\ell(f)$$

$$\iff \langle\{\beta_o(i),\ldots,\beta_{f(i)}(i),\gamma_{f(i)+1}(i),\ldots,\gamma_n(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Gamma_\ell(f)$$

$$\iff \langle\{\beta_o(i),\ldots,\beta_{f(i)}(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Delta_\ell\ .$$

Thus $H_1,\ldots,H_m$ is indeed homogeneous for Δ in the extended sense. This concludes the proof.

In analogy with Theorem I.4.11, there is a formulation of the property $\kappa \to {}^m(\eta)^{<\omega}_2$ in terms of models. This is as follows.

1.3 THEOREM Let η be a limit ordinal. Then the property $\kappa \to {}^{m}(\eta)_2^{<\omega}$ is equivalent to the following:

Take any model $\underline{\underline{A}} = \langle A, <_1, \ldots, <_m, \ldots \rangle$ of a language having at most countably many relation and function symbols, for which there are sets $A_1, \ldots, A_m \subseteq A$ such that $A_1 \times \ldots \times A_m \subseteq A$ and A_i is well ordered by $<_i$, for each i. Take any sequence $A_1', \ldots, A_m'$ where $A_i' \in [A_i]^{\kappa}$. Then there is a sequence $H_1, \ldots, H_m$ (where always $H_i \subseteq A_i'$ and has order type η in $<_i$), which is homogeneous for $\underline{\underline{A}}$, in the sense that for any formula $\Phi(v_o, \ldots, v_n)$ in the language of $\underline{\underline{A}}$, given for each $i = 1, \ldots, m$ sequences $a_o(i) <_i \ldots <_i a_n(i)$ and $b_o(i) <_i \ldots <_i b_n(i)$ from H_i,

$$\underline{\underline{A}} \models \Phi[\langle a_o(1), \ldots, a_o(m) \rangle, \ldots, \langle a_n(1), \ldots, a_n(m) \rangle]$$

$$\iff \underline{\underline{A}} \models \Phi[\langle b_o(1), \ldots, b_o(m) \rangle, \ldots, \langle b_n(1), \ldots, b_n(m) \rangle] .$$

PROOF: The proof is a simple extension of the proof of Theorem I.4.11, so I shall mention it but briefly.

Suppose first that $\kappa \to {}^{m}(\eta)_2^{<\omega}$, and let $\underline{\underline{A}}$, $A_1, \ldots, A_m$ and $A_1', \ldots, A_m'$ be as called for by the theorem. We may as well suppose that $A_1' = \ldots = A_m' = \kappa$. Define a function F with domain κ^{*m} by, if $a_o(i) <_i \ldots <_i a_n(i)$ in A_i' for each i, then

$$F(\langle \{a_o(i), \ldots, a_n(i)\} ; i = 1, \ldots, m \rangle)$$

$$= \{\Phi ; \underline{\underline{A}} \models \Phi[\langle a_o(1), \ldots, a_o(m) \rangle, \ldots, \langle a_n(1), \ldots, a_n(m) \rangle]\} .$$

Then F maps κ^{*m} into a set of power at most 2^{ω}, namely the set of all sets of formulae of the language of $\underline{\underline{A}}$. Consider F as defining a partition of κ^{*m} . An application of Lemma 1.1 gives a sequence $H_1,\ldots,H_m$ (each $H_i \subseteq A_i$ and having order type η in $<_i$) homogeneous for F . Then $H_1,\ldots,H_m$ is also homogeneous for $\underline{\underline{A}}$.

Conversely, suppose the condition stated in the theorem holds. Take any disjoint partition $\Delta = \{\Delta_0,\Delta_1\}$ of κ^{*m} . For each $n \geqq 0$ define $\Delta(n)$ by, if $\alpha_0(i) < \ldots < \alpha_n(i) < \kappa$ for $i = 1,\ldots,m$ then

$$\langle\langle\alpha_0(1),\ldots,\alpha_0(m)\rangle,\ldots,\langle\alpha_n(1),\ldots,\alpha_n(m)\rangle\rangle \in \Delta(n)$$

$$\Longleftrightarrow \quad \langle\{\alpha_0(1),\ldots,\alpha_n(1)\},\ldots,\{\alpha_0(m),\ldots,\alpha_n(m)\}\rangle \in \Delta_0 \ .$$

Let $\mathcal{L}$ be a language with binary relations $\underset{\sim}{\in}$, $\underset{\sim}{<}$ and relations $\underset{\sim}{\Delta}(1),\underset{\sim}{\Delta}(2),\ldots$ where $\underset{\sim}{\Delta}(n)$ is $(n+1)$-ary. Form the model $\underline{\underline{A}} = \langle\kappa \cup {}^m\kappa, \in, <, \Delta(1), \Delta(2),\ldots\rangle$ of $\mathcal{L}$. Then there is a sequence $H_1,\ldots,H_m$ where each $H_i \subseteq \kappa$ and has order type η (under $<$), which is homogeneous for $\underline{\underline{A}}$. Then $H_1,\ldots,H_m$ is also homogeneous for Δ . This completes the proof.

There is also a characterization of the property $\kappa \to {}^m(\eta)_2^{<\omega}$, where η is a limit ordinal, in terms of sequences $H_1,\ldots,H_m$ homogeneous in the extended sense for particular types of models. This is a combination of Theorems 1.2 and 1.3, and is sufficiently clear not to need stating.

§2. Some order properties

Let us first mention a negative result. It is immediate from Theorem II.1.5 that if $m > 1$ then $\kappa \not\to {}^m(\kappa)_2^{<\omega}$. Thus there is no useful extension of the Ramsey property.

Along the lines of Theorem II.1.1, there is a method for obtaining positive results of the form $\kappa \to {}^m(\eta)_\nu^{<\omega}$. The proof is analogous to the proof of Theorem II.1.1.

2.1 THEOREM Let ν be a cardinal and η an ordinal. Suppose κ is a cardinal such that $\kappa \to (\eta\cdot m)_\nu^{<\omega}$. Then $\kappa \to {}^m(\eta)_\nu^{<\omega}$.

Thus in particular, if $\kappa \to (\omega_1)_\nu^{<\omega}$ then $\kappa \to {}^m(\omega)_\nu^{<\omega}$ for all m . (See, however, the comments after Theorem I.4.7.) Also if κ is Ramsey, i.e. if $\kappa \to (\kappa)_2^{<\omega}$, then $\kappa \to {}^m(\eta)_2^{<\omega}$ for all m and all $\eta < \kappa$.

Having now seen that it is reasonable to assume that these cardinals exist, let us examine the order in which they occur, and how they fit amongst the cardinals with the properties $\kappa \to (\eta)_\nu^{<\omega}$. For simplicity, in the rest of this section I shall consider only the case $\nu = 2$, and write $\kappa \to {}^m(\eta)^{<\omega}$ in place of $\kappa \to {}^m(\eta)_2^{<\omega}$.

2.2 THEOREM Let ζ and η be infinite ordinals and suppose $\eta\cdot m < \zeta$. Let κ be the least cardinal such that $\kappa \to {}^m(\eta)^{<\omega}$, and let λ be any cardinal such that $\lambda \to (\zeta)^{<\omega}$. Then $\kappa < \lambda$.

PROOF: Let κ and λ be as mentioned in the theorem. By Theorem 2.1, $\kappa \leqq \lambda$. Suppose that in fact $\kappa \to (\zeta)^{<\omega}$, and seek a contradiction.

Now $|\beta| < \kappa$ for any $\beta \in \kappa$, and so there is a partition $\Delta(\beta) = \{\Delta_o(\beta), \Delta_1(\beta)\}$ of β^{*m} which has no homogeneous sequence $H_1,\ldots,H_m$ where each H_i has order type at least η . Take any partition $\Gamma = \{\Gamma_o,\Gamma_1\}$ of $[\kappa]^{<\omega}$ which has the following property: for all n , if $\beta_1(1) < \dots < \beta_n(1) < \beta_1(2) < \dots < \beta_n(2) < \dots < \beta_1(m) < \dots < \beta_n(m) <$ $< \beta < \kappa$, then

$$\{\beta_1(1),\dots,\beta_n(m),\beta\} \in \Gamma_o \iff \langle\{\beta_1(i),\dots,\beta_n(i)\} \; ; \; i = 1,\dots,m\rangle \in \Delta_o(\beta) \quad .$$

By virtue of the assumption $\kappa \rightarrow (\zeta)^{<\omega}$, there is $H \subseteq \kappa$, having order type ζ , which is homogeneous for Γ . However, for each $\alpha \in H$, let $\{\alpha' \in H \; ; \; \alpha' < \alpha\}$ be split into m sets $H_1(\alpha),\dots,H_m(\alpha)$ all having the same order type, such that $\sup(H_i(\alpha)) \leq \min(H_{i+1}(\alpha))$. (Any elements of $\{\alpha' \in H \; ; \; \alpha' < \alpha\}$ left over may be ignored.) Then $H_1(\alpha),\dots,H_m(\alpha)$ is a sequence homogeneous for $\Delta(\alpha)$, and so each $H_i(\alpha)$ has order type less than η . From this it follows that the order type of $\{\alpha' \in H \; ; \; \alpha' < \alpha\}$ is smaller than $\eta \cdot m$. Hence the order type of H does not exceed $\eta \cdot m$. This contradicts the order type of H being ζ . The proof is complete.

The following lemma is required in order to establish the effect of changing the value of m .

2.3 LEMMA Let $\kappa \rightarrow {}^m(\eta)^{<\omega}_\nu$ where η is a limit ordinal. Then for any partition $\Delta = \{\Delta_\ell \; ; \; \ell < \nu\}$ of κ^{*m} there is a sequence $H_1,\dots,H_m$ homogeneous for Δ , (where each $H_i \subseteq \kappa$ and has order type η), for which there is a permutation σ of $\{1,\dots,m\}$ such that $\sup(H_{\sigma(i)}) \leq \min(H_{\sigma(i+1)})$ for each i .

The proof is analogous to that of Theorem II.2.1, using Lemma 1.1 in the case that ν is finite.

2.4 THEOREM Let η be a limit ordinal. Suppose κ is the least cardinal such that $\kappa \to {}^{m}(\eta)^{<\omega}$ and let λ be any cardinal with the property $\lambda \to {}^{m+1}(\eta)^{<\omega}$. Then $\kappa < \lambda$.

PROOF: Let κ and λ be as mentioned in the theorem. Clearly $\kappa \leq \lambda$. Suppose that κ does have also the property $\kappa \to {}^{m+1}(\eta)^{<\omega}$, and seek a contradiction.

As in the proof of 2.2, for each $\beta < \kappa$ there is a partition $\Delta(\beta) = \{\Delta_o(\beta), \Delta_1(\beta)\}$ of β^{*m} which has no homogeneous sequence $H_1,\ldots,H_m$ where each H_i has order type at least η . Choose a partition $\Gamma = \{\Gamma_o,\Gamma_1\}$ of κ^{*m+1} which satisfies the following condition:
if $\beta_1(i) < \ldots < \beta_n(i)$ for $i = 1,\ldots,m+1$, and there is some j for which $\beta_1(j) > \max\{\beta_n(i) \ ; \ i \in \{1,\ldots,m+1\} - \{j\}\}$, then

$$\langle\{\beta_1(i),\ldots,\beta_n(i)\} \ ; \ i \in \{1,\ldots,m+1\}\rangle \in \Gamma_o$$

$$\Longleftrightarrow \quad \langle\{\beta_1(i),\ldots,\beta_n(i)\} \ ; \ i \in \{1,\ldots,m+1\} - \{j\}\rangle \in \Delta_o(\beta_1(j)) \ .$$

Let $H_1,\ldots,H_{m+1}$ be any sequence homogeneous for Γ , where each $H_i \subseteq \kappa$ and has order type η . By Lemma 2.3, it may be assumed further that in some reordering the H_i are increasing - suppose in fact that always $\sup(H_i) \leq \min(H_{i+1})$. Then if $\alpha_o,\alpha_1,\ldots$ is an enumeration of H_{m+1} in increasing order, it follows that if $\alpha_1(i) < \ldots < \alpha_n(i)$ are arbitrary from H_i (for each i), then

$$\langle\{\alpha_1(i),\ldots,\alpha_n(i)\} \ ; \ i = 1,\ldots,m\rangle \in \Delta_o(\alpha_o)$$

$$\Longleftrightarrow \quad \langle\{\alpha_1(i),\ldots,\alpha_n(i)\} \ ; \ i = 1,\ldots,m\rangle^\frown\langle\{\alpha_o,\ldots,\alpha_{n-1}\}\rangle \in \Gamma_o \ .$$

Hence the sequence $H_1,\ldots,H_m$ is homogeneous for $\Delta(\alpha_o)$, and each $H_i \subseteq \alpha_o$ and has order type η . However, this contradicts the choice of the partition $\Delta(\alpha_o)$. The theorem is thus proved.

Theorems 2.2 and 2.4 suffice to determine completely the ordering of the smallest cardinals with the properties $\kappa \to {}^m(\eta)^{<\omega}$ for η an ordinal power of ω . The least cardinals with the following properties form a strictly increasing sequence:

$\kappa \to (\omega)^{<\omega}$, $\kappa \to {}^2(\omega)^{<\omega}$, $\kappa \to {}^3(\omega)^{<\omega}$,..., $\kappa \to (\omega\cdot\omega)^{<\omega}$, $\kappa \to {}^2(\omega\cdot\omega)^{<\omega}$, $\kappa \to {}^3(\omega\cdot\omega)^{<\omega}$,..., $\kappa \to (\omega_1)^{<\omega}$, $\kappa \to {}^2(\omega_1)^{<\omega}$, $\kappa \to {}^3(\omega_1)^{<\omega}$,..., $\kappa \to (\kappa)^{<\omega}$.

In the case that η is not a power of ω , some questions remain. The first such questions are the following:

2.5 PROBLEM Let κ be the least cardinal with the property $\kappa \to (\omega\cdot 2)^{<\omega}$, let λ be the least cardinal with the property $\lambda \to {}^2(\omega)^{<\omega}$ and let ι be the least cardinal with the property $\iota \to {}^3(\omega)^{<\omega}$. Is it true that $\lambda < \kappa$? Or that $\kappa < \iota$?

§3. An inaccessibility result

In this section, it will be established that the least cardinal κ for which $\kappa \to {}^m(\eta)^{<\omega}$, where η is a cardinal, is strongly inaccessible. This result follows from Theorems 3.1 and 3.2 below. Their proofs follow the corresponding proofs in [15] for the case $m = 1$.

3.1 THEOREM Let η be a cardinal, and let κ be the least cardinal for which $\kappa \to {}^m(\eta)^{<\omega}$. Then κ is regular.

PROOF: The proof is sufficiently well illustrated by the case $m = 2$. So let κ be least such that $\kappa \to {}^2(\eta)^{<\omega}$, and suppose that in fact κ is singular. Hence there are pairwise disjoint sets $A_i \in [\kappa]^{<\kappa}$ for $i \in I$, where $|I| < \kappa$, such that $\kappa = \bigcup\{A_i \; ; \; i \in I\}$. Since κ is least such that $\kappa \to {}^2(\eta)^{<\omega}$, for all $i,j \in I$ there are disjoint partitions

$$\begin{aligned}
\Delta(i,j) &= \{\Delta_o(i,j), \Delta_1(i,j)\} \text{ of } [A_i]^{<\omega} \times [A_j]^{<\omega}, \\
\Delta(i) &= \{\Delta_o(i), \Delta_1(i)\} \text{ of } [A_i]^{<\omega} \times [I]^{<\omega}, \\
\Delta'(j) &= \{\Delta'_o(j), \Delta'_1(j)\} \text{ of } [I]^{<\omega} \times [A_j]^{<\omega}, \text{ and} \\
\Delta &= \{\Delta_o, \Delta_1\} \text{ of } [I]^{<\omega} \times [I]^{<\omega} ;
\end{aligned}$$

none of which possess a homogeneous pair H_o, H_1 with $|H_o| = |H_1| \geq \eta$.

Define a partition $\Gamma = \{\Gamma_o, \ldots, \Gamma_7\}$ of $[\kappa]^{<\omega} \times [\kappa]^{<\omega}$ as follows: if $a,b \in [\kappa]^{<\omega}$ and $a,b \neq \emptyset$ then for $\ell = 0,1$

$$\langle a,b\rangle \in \Gamma_\ell \iff \exists i,j \in I[a \subseteq A_i \text{ and } b \subseteq A_j \text{ and } \langle a,b\rangle \in \Delta_\ell(i,j)],$$

$$\begin{aligned}\langle a,b\rangle \in \Gamma_{2+\ell} \iff & [|b| = 1 \text{ or } \nexists j \in I(b \subseteq A_j)] \text{ and } \exists i \in I[a \subseteq A_i \text{ and} \\ & \langle a, \{j \in I \; ; \; b \cap A_j \neq \emptyset\}\rangle \in \Delta_\ell(i)],\end{aligned}$$

$$\begin{aligned}\langle a,b\rangle \in \Gamma_{4+\ell} \iff & [|a| = 1 \text{ or } \nexists i \in I(a \subseteq A_i)] \text{ and } \exists j \in I[b \subseteq A_j \text{ and} \\ & \langle\{i \in I \; ; \; a \cap A_i \neq \emptyset\}, b\rangle \in \Delta'_\ell(j)],\end{aligned}$$

$$\begin{aligned}\langle a,b\rangle \in \Gamma_{6+\ell} \iff & [|a| = 1 \text{ or } \nexists i \in I(a \subseteq A_i)] \text{ and } [|b| = 1 \text{ or } \nexists j \in I(b \subseteq A_j)] \\ & \text{and } \langle\{i \in I \; ; \; a \cap A_i \neq \emptyset\}, \{j \in I \; ; \; b \cap A_j \neq \emptyset\}\rangle \in \Delta_\ell .\end{aligned}$$

Because $\kappa \to {}^2(\eta)^{<\omega}$, by Lemma 1.1 and Theorem 1.2, there is a pair H,K (where $H,K \subseteq \kappa$ with $|H| = |K| = \eta$) which is homogeneous for Γ in the extended sense. Suppose that for some $i \in I$ there are distinct elements $h_o, h_1 \in H \cap A_i$. Choose any $k \in K$. Then for arbitrary $h \in H - \{h_o,h_1\}$,

$$\langle\{h_o,h_1\},\{k\}\rangle \equiv \langle\{h_o,h\},\{k\}\rangle \pmod{\Gamma} \quad .$$

By the definition of Γ_ℓ , if $\langle\{h_o,h_1\},\{k\}\rangle \in \Gamma_\ell$ then $\ell \leqq 3$, and so $\{h_o,h\} \subseteq A_{i'}$, for some $i' \in I$. Since $h_o \in A_i$ and the A_i are pairwise disjoint, $i' = i$ and so $h \in A_i$. Thus $H \subseteq A_i$. Similarly, if for some $j \in I$ there are distinct elements $k_o,k_1 \in K \cap A_j$, then it follows that $K \subseteq A_j$.

There are now four cases to consider.

1. There are $i,j \in H$ for which $|H \cap A_i| \geqq 2$ and $|K \cap A_j| \geqq 2$. In this case $H \subseteq A_i$, $K \subseteq A_j$ and H , K is homogeneous for $\Delta(i,j)$.

2. There is $i \in I$ for which $|H \cap A_i| \geqq 2$ and yet $|K \cap A_j| \leqq 1$ for all $j \in I$. Here $H \subseteq A_i$, $|\{j \in I \ ; K \cap A_j \neq \emptyset\}| = |K|$ and $H,\{j \in I \ ; K \cap A_j \neq \emptyset\}$ is a pair homogeneous for $\Delta(i)$.

3. There is $j \in I$ for which $|K \cap A_j| \geqq 2$ and yet $|H \cap A_i| \leqq 1$ for all $i \in I$. Then $K \subseteq A_j$, $|\{i \in I \ ; H \cap A_i \neq \emptyset\}| = |H|$ and $\{i \in I \ ; H \cap A_i \neq \emptyset\}$, K is a homogeneous pair for $\Delta'(j)$.

4. $|H \cap A_i| \leqq 1$ and $|K \cap A_i| \leqq 1$ for all $i \in I$. In this case, $|\{i \in I \ ; H \cap A_i \neq \emptyset\}| = |H|$, $|\{j \in I \ ; K \cap A_j \neq \emptyset\}| = |K|$ and $\{i \in I \ ; H \cap A_i \neq \emptyset\},\{j \in I \ ; K \cap A_j \neq \emptyset\}$ is a pair homogeneous for Δ .

By the choice of the various partitions, each of the cases 1-4 leads to the conclusion $|H|,|K| < \eta$. This contradicts that $|H| = |K| = \eta$. This contradiction establishes the result.

3.2 THEOREM Let η be a cardinal, and suppose $\kappa \not\to {}^{m}(\eta)^{<\omega}$. Then also $2^\kappa \not\to {}^{m}(\eta)^{<\omega}$.

PROOF: Suppose $\kappa \not\to {}^m(\eta)^{<\omega}$. By Lemma 1.1, it suffices to show that $2^\kappa \not\to {}^m(\eta)^{<\omega}_\omega$.

Let $\Delta = \{\Delta_0, \Delta_1\}$ be a partition of κ^{*m} which has no homogeneous sequence $H_1, \ldots, H_m$ where each $H_i \in [\kappa]^\eta$. Let $\Delta^* = \{\Delta^*_\ell \ ; \ \ell < \omega\}$ be a partition of $\cup\{{}^m({}^n\kappa) \ ; \ 1 \leq n < \omega\}$ which refines Δ , in the sense that if

$$\langle\langle\alpha_1(i),\ldots,\alpha_n(i)\rangle \ ; \ i = 1,\ldots,m\rangle \equiv \langle\langle\beta_1(i),\ldots,\beta_n(i)\rangle \ ; \ i = 1,\ldots,m\rangle \ (\text{mod } \Delta^*),$$

then also

$$\langle\{\alpha_1(i),\ldots,\alpha_n(i)\} \ ; \ i = 1,\ldots,m\rangle \equiv \langle\{\beta_1(i),\ldots,\beta_n(i)\} \ ; \ i = 1,\ldots,m\rangle \ (\text{mod } \Delta).$$

Let Δ^* have the further property that if

$$\langle\langle\alpha_1(i),\ldots,\alpha_n(i)\rangle \ ; \ i = 1,\ldots,m\rangle \equiv \langle\langle\beta_1(i),\ldots,\beta_n(i)\rangle \ ; \ i = 1,\ldots,m\rangle \ (\text{mod } \Delta^*),$$

then for each i , the ordering arrangement of $\alpha_1(i),\ldots,\alpha_n(i)$ is the same as that of $\beta_1(i),\ldots,\beta_n(i)$.

Let $\ll$ be the lexicographic ordering of ${}^\kappa 2$. For distinct elements $x,y \in {}^\kappa 2$, define $\delta(x,y)$ to be the least $\alpha < \kappa$ for which $x(\alpha) \neq y(\alpha)$. For $x,y,z \in {}^\kappa 2$, the following simple properties hold:

(1) if $x \ll y \ll z$ then $\delta(x,y) \neq \delta(y,z)$ and $\delta(x,z) = \min\{\delta(x,y),\delta(y,z)\}$,

(2) if $x \ll y \ll z$ and $\delta(x,y) < \delta(y,z)$ then $\delta(x,y) = \delta(x,z)$.

(3) if $x \ll y \ll z$ and $\delta(x,y) > \delta(y,z)$ then $\delta(x,z) = \delta(y,z)$.

Choose a partition $\Gamma = \{\Gamma_\ell \ ; \ \ell < \omega\}$ of $({}^{\kappa}2)^{*m}$ which satisfies: if $x_o(i) \ll \ldots \ll x_n(i)$ for $i = 1,\ldots,m$ then for each $\ell < \omega$

$$\langle\{x_o(i),\ldots,x_n(i)\} \ ; \ i = 1,\ldots,m\rangle \in \Gamma_\ell$$

$$\Longleftrightarrow \ \langle\langle\delta(x_o(i),x_1(i)),\ldots,\delta(x_{n-1}(i),x_n(i))\rangle \ ; \ i = 1,\ldots,m\rangle \in \Delta^*_\ell \ .$$

Let $H_1,\ldots,H_m$ be homogeneous for Γ (each $H_i \subseteq {}^{\kappa}2$). For each i , take $x_o(i) \ll x_1(i) \ll x_2(i)$ and $y_o(i) \ll y_1(i) \ll y_2(i)$ from H_i . Then

$$\langle\{x_o(i),x_1(i),x_2(i)\} \ ; \ i = 1,\ldots,m\rangle \equiv \langle\{y_o(i),y_1(i),y_2(i)\} \ ; \ i = 1,\ldots,m\rangle \pmod{\Gamma},$$

and so

$$\langle\langle\delta(x_o(i),x_1(i)),\delta(x_1(i),x_2(i))\rangle \ ; \ i = 1,\ldots,m\rangle \equiv$$
$$\langle\langle\delta(y_o(i),y_1(i)),\delta(y_1(i),y_2(i))\rangle \ ; \ i = 1,\ldots,m\rangle \pmod{\Delta^*}.$$

Hence by the choice of Δ^* , and in virtue of (1):

for each i either

(4) for all $x_o(i) \ll x_1(i) \ll x_2(i)$ in H_i , $\delta(x_o(i),x_1(i)) < \delta(x_1(i),x_2(i))$,

or

(5) for all $x_o(i) \ll x_1(i) \ll x_2(i)$ in H_i , $\delta(x_o(i),x_1(i)) > \delta(x_1(i),x_2(i))$.

Thus by (2) or (3), for each i either

(6) for all $x_o(i) \ll x_1(i) \ll x_2(i)$ in H_i , $\delta(x_o(i),x_1(i)) = \delta(x_o(i),x_2(i))$,

or

(7) for all $x_o(i) \ll x_1(i) \ll x_2(i)$ in H_i, $\delta(x_o(i),x_2(i)) = \delta(x_1(i),x_2(i))$.

For $i = 1,\ldots,m$ put

$$K_i = \{\delta(x_o(i),x_1(i)) \; ; \; x_o(i),x_1(i) \in H_i \text{ and } x_o(i) \ll x_1(i)\} .$$

Then by (6) and (7), there is a one:one correspondence between H_i and K_i, so $|H_i| = |K_i|$. Further, the sequence $K_1,\ldots,K_m$ is homogeneous for Δ. This may be seen as follows.

Take $\delta(x_{oo}(i),x_{o1}(i)) < \ldots < \delta(x_{no}(i),x_{n1}(i))$ from K_i, where for $j = 0,\ldots,n$ we may assume that $x_{jo}(i) \ll x_{j1}(i)$. Then if (4) holds,

(8) $$x_{oo}(i) \ll \ldots \ll x_{no}(i) \ll x_{n1}(i) ,$$

and if (5) holds,

(9) $$x_{no}(i) \ll x_{n1}(i) \ll \ldots \ll x_{o1}(i) .$$

For suppose, say, (4) holds and yet $x_{1o}(i) \ll x_{oo}(i)$. Then $\delta(x_{1o}(i),x_{oo}(i)) < \delta(x_{oo}(i),x_{o1}(i))$; however $\delta(x_{1o}(i),x_{oo}(i)) = \delta(x_{1o}(i),x_{11}(i))$, contradicting that $\delta(x_{oo}(i),x_{o1}(i)) < \delta(x_{1o}(i),x_{11}(i))$.

Thus if (4) holds, by (6) and (8)

$$\langle\delta(x_{oo}(i),x_{o1}(i)),\delta(x_{1o}(i),x_{11}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i))\rangle$$
$$= \langle\delta(x_{oo}(i),x_{1o}(i)),\delta(x_{1o}(i),x_{2o}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i))\rangle,$$

and if (5) holds, by (7) and (9),

$$\langle \delta(x_{no}(i),x_{n1}(i)),\delta(x_{n-1\ o}(i),x_{n-1\ 1}(i)),\ldots,\delta(x_{oo}(i),x_{o1}(i))\rangle$$
$$= \langle \delta(x_{no}(i),x_{n1}(i)),\delta(x_{n1}(i),x_{n-1\ 1}(i)),\ldots,\delta(x_{11}(i),x_{o1}(i))\rangle.$$

Let $s_i(\delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i)))$ be $\langle \delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i))\rangle$ if (4) holds, or $\langle \delta(x_{no}(i),x_{n1}(i)),\ldots,\delta(x_{oo}(i),x_{o1}(i))\rangle$ if (5) holds. Then by (8) or (9), there are $x_o(i) \ll \ldots \ll x_{n+1}(i)$ in H_i such that

$$s_i(\delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i))) = \langle \delta(x_o(i),x_1(i)),\ldots,\delta(x_n(i),x_{n+1}(i))\rangle.$$

Thus, for all $\ell < \omega$,

$$\langle s_i(\delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i)))\ ;\ i = 1,\ldots,m\rangle \in \Delta^*_\ell$$
$$\Longleftrightarrow \quad \langle \{x_o(i),\ldots,x_{n+1}(i)\}\ ;\ i = 1,\ldots,m\rangle \in \Gamma_\ell .$$

From the homogeneity of $H_1,\ldots,H_m$ for Γ , it now follows that all such sequences $\langle s_i(\delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i)))\ ;\ i = 1,\ldots,m\rangle$ are in the same class of Δ^* , and so since Δ^* refines Δ , all sequences $\langle \{\delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i))\}\ ;\ i = 1,\ldots,m\rangle$ where $\{\delta(x_{oo}(i),x_{o1}(i)),\ldots,\delta(x_{no}(i),x_{n1}(i))\} \in [K_i]^{n+1}$ are in the same class of Δ . Thus the sequence $K_1,\ldots,K_m$ is indeed homogeneous for Δ .

This means, from the choice of Δ , that $|K_i| < \eta$ for some i . Then $|H_i| < \eta$, and so it follows that Γ is a partition of ${}^{\kappa}2$ which has no homogeneous sequence $H_1,\ldots,H_m$ where each $|H_i| = \eta$. Thus $2^\kappa \not\to {}^m(\eta)^{<\omega}_\omega$, and the proof is complete.

The main obstacle in establishing Theorems 3.1 and 3.2 for the more general case that η is a limit ordinal (as has been done for the case $m = 1$ - see Theorem I.4.7) seems to lie in answering the following question.

3.3 PROBLEM Let η be a limit ordinal, let $m > 1$ and let κ be the least cardinal such that $\kappa \to {}^{m}(\eta)^{<\omega}$. Take any $\nu < \kappa$. Is it true that $\kappa \to {}^{m}(\eta)_{\nu}^{<\omega}$?

As mentioned in §I.4, Silver has given a positive answer in the case $m = 1$.

§4. The constructible universe

This section is devoted to showing that the methods used by Silver [22, pp.87-90] admit an obvious extension leading to the following theorem.

4.1 THEOREM Let η be a constructibly countable ordinal and suppose that $ZF + \exists\kappa\ (\kappa \to {}^{m}(\eta)^{<\omega})$ is a consistent theory. Then so also is $ZF + "V = L" + \exists\kappa\ (\kappa \to {}^{m}(\eta)^{<\omega})$.

The case when κ has the property $\kappa \to {}^{m}(\eta)^{<\omega}$ in virtue of having the property $\kappa \to (\eta \cdot m)^{<\omega}$ is already covered by Silver's result. However, since $\kappa \to {}^{m}(\eta)^{<\omega}$ is possibly a weaker property than $\kappa \to (\eta \cdot m)^{<\omega}$ (see Problem 2.5), the theorem retains some value. The proof comes immediately from Theorem 4.3 below. The following definitions and lemma will be needed.

Let η be an ordinal, let f be a bijection from ω to η, and suppose κ is a cardinal with $\Delta = \{\Delta_o,\Delta_1\}$ a partition of κ^{*m}. Define

$$T_m(\Delta,f) \;=\; \{\langle k_1,\ldots,k_m\rangle \;;\; \exists n < \omega \;\forall i \in \{1,\ldots,m\}$$
$$(k_i \text{ is an order preserving map from } \{f(p)\;;\;p<n\} \text{ to } \kappa)$$
$$\text{and } Rk_1,\ldots,Rk_m \text{ is a homogeneous sequence for } \Delta\}\;.$$

For $\langle h_1,\ldots,h_m\rangle$ and $\langle k_1,\ldots,k_m\rangle$ from $T_m(\Delta,f)$, define

$$\langle h_1,\ldots,h_m\rangle < \langle k_1,\ldots,k_m\rangle \iff \forall i \in \{1,\ldots,m\}(k_i \subset h_i)\;.$$

4.2 LEMMA There is a sequence $H_1,\ldots,H_m$ (where each $H_i \subseteq \kappa$ and has order type η) which is homogeneous for Δ if and only if $<$ is not well founded.

PROOF: Suppose a sequence $H_1,\ldots,H_m$ where each $H_i \subseteq \kappa$ and has order type η, is homogeneous for Δ. Enumerate H_i in increasing order by the function $h_i = \{\langle\alpha\,,\,h_i(\alpha)\rangle\;;\;\alpha<\eta\}$. For each $n<\omega$, put $h_{in} = h_i \restriction \{f(p)\;;\;p<n\}$. Then the sequence $\langle\langle h_{1n},\ldots,h_{mn}\rangle\;;\;n<\omega\rangle$ forms an infinite descending $<$-chain, and so $<$ is not well founded.

On the other hand, suppose $\langle\langle h_{1n},\ldots,h_{mn}\rangle\;;\;n<\omega\rangle$ forms an infinite descending $<$-chain. Then for each i, it follows that $h_i = \bigcup\{h_{in}\;;\;n<\omega\}$ is a function, and since there are infinitely many sets $\{f(p)\;;\;p<n\}$ contained in Dh_i, it must be that $Dh_i = \eta$. Thus since h_i remains order preserving, Rh_i has order type η. However, $Rh_1,\ldots,Rh_m$ is a sequence homogeneous for Δ. This proves the lemma.

4.3 THEOREM Let $\langle M, \in \rangle$ be a transitive model of ZF, let $\eta \in M$ be an ordinal such that $M \models (\eta$ is countable), and let κ be a cardinal in M such that $\kappa \to {}^{m}(\eta)^{<\omega}$. Then in fact $M \models (\kappa \to {}^{m}(\eta)^{<\omega})$.

PROOF: Since $\eta \in M$ is countable in M, there is a function $f \in M$ such that $M \models (f$ maps ω onto $\eta)$. Then indeed, f maps ω onto η. Take any $\Delta \in M$ for which $M \models (\Delta = \{\Delta_o, \Delta_1\}$ is a partition of κ^{*m}). Since $m < \omega$, it follows that $(\kappa^{*m})_M = \kappa^{*m}$, and so Δ really is a partition of κ^{*m}.

Form $T_m(\Delta, f)$. Observe that $T_m(\Delta, f) = (T_m(\Delta, f))_M$, so that $T_m(\Delta, f) \in M$. Hence since M is transitive, $< \, = (<)_M$, so that also $< \, \in M$. Since $\kappa \to {}^{m}(\eta)^{<\omega}$, there is a sequence $H_1, \ldots, H_m$ homogeneous for Δ, where each $H_i \subseteq \kappa$ and has order type η. Thus $<$ is not well founded. However, "not well founded" is absolute for M (see [22, p.89]), and so since $< \, \in M$, also $M \models (<$ is not well founded). Hence by Lemma 4.2, applied inside M, it follows that $M \models$ (there is a sequence $H_1, \ldots, H_m$ homogeneous for Δ, where each $H_i \subseteq \kappa$ and has order type η). Thus $M \models (\kappa \to {}^{m}(\eta)^{<\omega})$. This concludes the proof.

CHAPTER IV. MORE ON THE RELATION $\kappa \to {}^m(\eta_\ell \,;\, \ell < \nu)^n$

§1. The ramification lemma

Corollary II.1.3 gives the result that provided all $\eta_\ell < \omega_{\alpha+1}$ then $\omega_{\alpha+mn} \to {}^m(\eta_\ell \,;\, \ell < \omega_\alpha)^n$. This can hardly be the best possible. In the present chapter, the methods of [6] will be extended to the case $\kappa \to {}^m(\eta_\ell \,;\, \ell < \nu)^n$ in an attempt to improve this result. In most cases, the improvements gained are but marginal. It is clear that a new, more powerful, method is required before significant further progress can be made.

The following notation will be in force throughout this chapter. The class of all functions with domain σ and taking ordinal values will be denoted by SEQ_σ . If $X \in SEQ_\sigma$ for a non-limit ordinal σ , say $\sigma = \tau + 1$, then $X \restriction \tau$ will be abbreviated to $\bar{X}$. Thus $X = \bar{X} \cup \{\langle \tau, X(\tau) \rangle\}$. The GCH will be assumed wherever it leads to a simplification in the result. Theorems reached with its aid will be marked (*) .

Many of the positive results obtained in [6] depend on the following lemma, the Ramification Lemma.

1.1 LEMMA Let $\rho > 0$ be a limit ordinal. For all sequences $X \in SEQ_\sigma$ and $Y \in SEQ_{\sigma+1}$ where $\sigma < \rho$ let there be given sets $S(Y)$, $F(X)$ and an ordinal $n(X)$. Let a set $S = S(\emptyset)$ be given. Put

$$N = \{X \in SEQ_\sigma \,;\, \sigma \leqq \rho \text{ and } \forall \tau < \sigma(X(\tau) < n(X \restriction \tau))\} ,$$

and for $X \in SEQ_\sigma$ define

$$S'(X) = S \cap \bigcap\{S(X \upharpoonright \tau + 1) \ ; \ \tau < \sigma\} .$$

Suppose that whenever $\sigma < \rho$ and $X \in N \cap SEQ_\sigma$ then

(a) $S'(X) = F(X) \cup \bigcup\{S(Y) \ ; \ Y \in SEQ_{\sigma+1}$ and $\bar{Y} = X$ and $Y(\sigma) < n(X)\}$,

(b) $F(X) \cap \bigcup\{S(Y) \ ; \ Y \in SEQ_{\sigma+1}$ and $\bar{Y} = X$ and $Y(\sigma) < n(X)\} = \emptyset$.

Under these conditions,

(i) $\tau < \sigma < \rho$ and $X \in N \cap SEQ_\sigma \Rightarrow F(X) \cap F(X \upharpoonright \tau) = \emptyset$,

(ii) $S = \bigcup\{F(X) \ ; \ \exists\sigma < \rho(X \in N \cap SEQ_\sigma)\} \cup \bigcup\{S'(X) \ ; \ X \in N \cap SEQ_\rho\}$,

(iii) Suppose $\omega \leqq \kappa \leqq |S|$, $|\rho| < Cf(\kappa)$ and $|F(X)| < \kappa$ whenever $X \in N$.

For $\sigma < \rho$ let there be given cardinals λ_σ such that $\lambda_\sigma^{|\sigma|} < Cf(\kappa)$, and suppose that

$$\tau < \sigma < \rho \text{ and } X \in N \cap SEQ_\sigma \Rightarrow |n(X \upharpoonright \tau)| \leqq \lambda_\sigma .$$

Then there is $X \in N \cap SEQ_\rho$ for which $S'(X)$ is non-empty.

(iv) Suppose κ is strongly inaccessible. Let $|S| \geqq \kappa$, $|\rho| < \kappa$ and assume that for $\sigma < \rho$,

$$X \in N \cap SEQ_\sigma \Rightarrow |F(X)| < \kappa \text{ and } |n(X)| < \kappa .$$

Then there is $X \in N \cap SEQ_\rho$ for which $S'(X)$ is non-empty.

The proof is in [6], pp. 103-105.

The system $\underline{\underline{R}}$ of sets N, $F(X)$ and $S(X)$ is called a ramification system on S of length ρ. In all references to the Ramification Lemma, the symbols ρ, N, $S(X)$, ... are to have the significance ascribed to them in that lemma. In applications, the ramification system $\underline{\underline{R}}$ will be constructed inductively. To do this, it is sufficient to assume for any $\sigma < \rho$ that $S'(X)$ has already been defined for some fixed $X \in SEQ_\sigma$, and to define $n(X)$, $F(X)$ and each $S(Y)$ for $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < n(X)$.

As a first application of this principle, the following theorem will be established.

1.2 THEOREM Let $\lambda_1 \geqq \omega$ and λ_2 be cardinals such that $\lambda_1 \to (\eta_\ell \cdot 2\ ;\ \ell < \nu)^1$ and $\lambda_2 \to {}^2(\theta_k\ ;\ k < \mu)^1$. Suppose κ is a regular cardinal such that λ_1^+, $\lambda_2 \leqq \kappa$, and $\iota < \lambda_1 \Rightarrow (|\nu|\lambda_2^-)^\iota < \kappa$. Then $\kappa \to {}^2(\theta_k\ ;\ k < \mu\ ,\ \eta_\ell\ ;\ \ell < \nu)^1$.

PROOF: Consider first the case where the η_ℓ are all cardinals. Suppose $\kappa \times \kappa$ is partitioned,

$$\kappa \times \kappa = \cup\{\Delta_k\ ;\ k < \mu\} \cup \cup\{\Gamma_\ell\ ;\ \ell < \nu\}.$$

It may be assumed for all $H_o, H_1 \subseteq \kappa$ with $|H_o|, |H_1| \geqq \lambda_2$ that $H_o \times H_1 \nsubseteq \cup\{\Delta_k\ ;\ k < \mu\}$, since otherwise the theorem follows from the property $\lambda_2 \to {}^2(\theta_k\ ;\ k < \mu)^1$. Thus it suffices to find $\ell < \nu$ and $H_o, H_1 \subseteq \kappa$ such that H_o, H_1 have power η_ℓ and $H_o \times H_1 \subseteq \Gamma_\ell$.

Define a ramification system $\underline{\underline{R}}$ on κ of length $\rho = \lambda_1$ as follows. Take $\sigma < \rho$ and $X \in SEQ_\sigma$. Suppose $S'(X)$ has already been defined, and consider two cases.

Case 1: σ even

If for no $x \in S'(X)$ is it the case that $\langle x,x\rangle \in \bigcup\{\Delta_k ; k < \mu\}$, choose $x \in S'(X)$ and put $F(X) = \{x\}$. Otherwise, choose $F(X) \subseteq S'(X)$ maximal with the property $F(X) \times F(X) \subseteq \bigcup\{\Delta_k ; k < \mu\}$. In either case, $|F(X)| < \lambda_2$. Choose $Q(X) \subseteq S'(X)$ maximal such that $F(X) \subseteq Q(X)$ and $Q(X) \times F(X) \subseteq \bigcup\{\Delta_k ; k < \mu\}$, except that if there is no $x \in S'(X)$ with $\langle x,x\rangle \in \bigcup\{\Delta_k ; k < \mu\}$ then the requirement $F(X) \subseteq Q(X)$ is to be ignored.

Then if $y \in S'(X) - Q(X)$, there is $x \in F(X)$ such that $\langle y,x\rangle \notin \bigcup\{\Delta_k ; k < \mu\}$, by the maximality of $Q(X)$. Thus there is a decomposition

$$S'(X) - (Q(X) \cup F(X)) = \bigcup\{S(Y) ; \bar{Y} = X \text{ and } Y(\sigma) < n'(X)\},$$

where given $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$, for some $x(Y) \in F(X)$ and $\ell(Y) < \nu$,

$$y \in S(Y) \iff \langle y, x(Y)\rangle \in \Gamma_{\ell(Y)} .$$

Further, $|n'(X)| \leqq |\nu| \times_c |F(X)| \leqq |\nu|\bar{\lambda}_2$. Finally, if $\bigcap\{S(Y) ; \bar{Y} = X$ and $Y(\sigma) < n'(X)\} - F(X) = S'(X) - F(X)$, put $n(X) = n'(X)$; otherwise $n(X) = n'(X) + 1$ and $S(X \cup \{\langle\sigma,n'(X)\rangle\}) = Q(X) - F(X)$.

Case 2: σ odd, say $\sigma = \zeta + 1$

If $X(\zeta) < n'(\bar{X})$, put $F(X) = \emptyset$, $n(X) = 1$ and $S(X \cup \{\langle\sigma,0\rangle\}) = S'(X)$. If $X(\zeta) = n'(\bar{X})$, still put $F(X) = \emptyset$. However, in this case $S'(X) \subseteq Q(\bar{X}) - F(\bar{X})$. Hence by the maximality of $F(\bar{X})$, for any $y \in S'(X)$ either $\langle y,y\rangle \notin \bigcup\{\Delta_k ; k < \mu\}$ or there is $x \in F(\bar{X})$ such that $\langle x,y\rangle \notin \bigcup\{\Delta_k ; k < \mu\}$. Thus there is a decomposition

$$S'(X) = \cup\{S(Y) \; ; \; \bar{Y} = X \text{ and } Y(\sigma) < n'(X)\} ,$$

where given Y such that $\bar{Y} = X$, for some $x(Y) \in F(\bar{X})$ and $\ell(Y) < \nu$,

$$y \in S(Y) \iff \langle x(Y), y\rangle \in \Gamma_{\ell(Y)} \text{ or } \langle y,y\rangle \in \Gamma_{\ell(Y)} .$$

Again, $|n'(X)| \leq |\nu| \times_c |F(\bar{X})| \leq |\nu|\lambda_2^-$. Put $n(X) = n'(X)$.

This completes the definition of the ramification system $\underline{\underline{R}}$. Further, Lemma 1.1(iii) applies to $\underline{\underline{R}}$. Hence there is a sequence $X \in N \cap SEQ_\rho$ for which $S'(X) \neq \emptyset$. Choose such a sequence X . For each $\sigma < \rho$, put $x_\sigma = x(X \restriction \sigma + 1)$ whenever $x(X \restriction \sigma + 1)$ is defined. By Lemma 1.1(i), if $\tau + 1 < \sigma$ then $x_\tau \neq x_\sigma$. Now it follows

(1) $\tau < \sigma < \rho$ and $X(2\cdot\tau) \neq n'(X \restriction 2\cdot\tau) \Rightarrow \langle x_{2\cdot\sigma}, x_{2\cdot\tau}\rangle \in \Delta_{\ell(X\restriction 2\cdot\tau+1)}$,

for $x_{2\cdot\sigma} \in F(X \restriction 2\cdot\sigma) \subseteq S'(X \restriction 2\cdot\sigma) \subseteq S(X \restriction 2\cdot\tau + 1)$. Also

(2) $\tau < \sigma < \rho$ and $X(2\cdot\tau) = n'(X \restriction 2\cdot\tau) \Rightarrow \langle x_{2\cdot\tau+1}, x_{2\cdot\sigma+1}\rangle \in \Delta_{\ell(X\restriction 2\cdot\tau+2)}$,

for $x_{2\cdot\sigma+1} \in F(X \restriction 2\cdot\sigma) \subseteq S'(X \restriction 2\cdot\sigma) \subseteq S(X \restriction 2\cdot\tau + 2)$.

For $\tau < \rho$, either $X(2\cdot\tau) = n'(X \restriction 2\cdot\tau)$ or $X(2\cdot\tau) \neq n'(X \restriction 2\cdot\tau)$. Hence there is $H \subseteq \rho$ with $|H| = \rho$ such that either

(3) $\tau \in H \Rightarrow X(2\cdot\tau) \neq n'(X \restriction 2\cdot\tau)$, or

(4) $\tau \in H \Rightarrow X(2\cdot\tau) = n'(X \restriction 2\cdot\tau)$.

Let H be partitioned, $H = \cup\{\Lambda_\ell \; ; \; \ell < \nu\}$ where

$$\Lambda_\ell = \{\tau \in H \ ; \ \ell(X \upharpoonright 2\cdot\tau + 1) = \ell\}$$

if (3) holds, or if (4) holds

$$\Lambda_\ell = \{\tau \in H \ ; \ \ell(X \upharpoonright 2\cdot\tau + 2) = \ell\} .$$

Since by assumption $\lambda_1 \to (\eta_\ell\cdot 2 \ ; \ \ell < \nu)^1$ and $|H| = \rho = \lambda_1$, there are $I \subseteq H$ and $\ell < \nu$ such that I has order type $\eta_\ell\cdot 2$ and $I \subseteq \Lambda_\ell$. Enumerate I as $\{\tau(\alpha) \ ; \ \alpha < \eta_\ell\cdot 2\}$, in increasing order. If (3) holds, put

$$H_o = \{x_{2\cdot\tau(\alpha)} \ ; \ \eta_\ell \leqq \alpha < \eta_\ell\cdot 2\} , \quad H_1 = \{x_{2\cdot\tau(\alpha)} \ ; \ \alpha < \eta_\ell\} ,$$

whereas if (4) holds, put

$$H_o = \{x_{2\cdot\tau(\alpha)+1} \ ; \ \alpha < \eta_\ell\} , \quad H_1 = \{x_{2\cdot\tau(\alpha)+1} \ ; \ \eta_\ell \leqq \alpha < \eta_\ell\cdot 2\} .$$

Then H_o, H_1 both have power η_ℓ, and it follows from (1) and (2) that $H_o \times H_1 \subseteq \Gamma_\ell$.

This proves the theorem when the η_ℓ are all cardinals. A slight modification of the argument above allows the x_σ to be chosen so that $\tau + 1 < \sigma \Rightarrow x_\tau < x_\sigma$. The general case then follows.

1.3 COROLLARY (*) Let $\alpha \geqq 0$, $\nu < Cf(\omega_\alpha)$ and $\eta_\ell < Cf(\omega_\alpha)$ for $\ell < \nu$. Then $\omega_{\alpha+1} \to {}^2(\omega_{\alpha+1} , \eta_\ell \ ; \ \ell < \nu)^1$.

PROOF: Put $\lambda_1 = Cf(\omega_\alpha)$ and $\lambda_2 = \omega_{\alpha+1}$. Then λ_1 is regular, so that $\lambda_1 \to (\lambda_1)^1_\nu$ and hence, in particular, $\lambda_1 \to (\eta_\ell\cdot 2 \ ; \ \ell < \nu)^1$. Since

$\lambda_1 \leqq \omega_\alpha$, if $\iota < \lambda_1$ then

$$(|\nu|\lambda_2^-)^\iota \leqq (Cf(\omega_\alpha) \times_C \omega_\alpha)^\iota = (\omega_\alpha)^\iota = \omega_\alpha < \omega_{\alpha+1} .$$

Further $\lambda_2 \rightarrow {}^2(\lambda_2)^1_1$. Thus Theorem 1.2 applies with $\kappa = \omega_{\alpha+1}$. This yields the result.

A similar result can be reached for inaccessible cardinals.

1.4 THEOREM (*) Let κ be strongly inaccessible. Take $\nu < \kappa$ and for each $\ell < \nu$ let $\eta_\ell < \kappa$. Then $\kappa \rightarrow {}^2(\kappa, \eta_\ell \ ; \ \ell < \nu)^1$.

PROOF: The proof is very similar to that of Theorem 1.2. Put $\lambda_2 = \kappa$, so $\lambda_2 \rightarrow {}^2(\kappa)^1_1$. Put $\lambda_1 = (\sum \{|\eta_\ell|^+ \ ; \ \ell < \nu\})^+$, so $\lambda_1 < \kappa$. Then $\lambda_1 \rightarrow (|\eta_\ell|^+ \ ; \ \ell < \nu)^1$, so certainly $\lambda_1 \rightarrow (\eta_\ell \cdot 2 \ ; \ \ell < \nu)^1$.

Suppose $\kappa \times \kappa$ is partitioned, $\kappa \times \kappa = \Delta \cup \bigcup\{\Gamma_\ell \ ; \ \ell < \nu\}$. The proof now becomes almost identical to that of Theorem 1.2, except that the application of Lemma 1.1(iii) is replaced by an appeal to Lemma 1.1(iv).

As another application of the ramification method, I shall establish a theorem of which the case $m = 1$ is the Stepping-up Lemma of [6, p.107]. The conclusion obtained for $m > 1$ has not, however, the strength of the case $m = 1$. This means that the most powerful method in [6] of generating new relations from known relations loses much of its force in the course of this generalization.

1.5 THEOREM Let $m, n_1, \ldots, n_m \geqq 1$. Let λ be a cardinal such that

(1) $$\begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix} \to \begin{pmatrix} \eta_{1\ell} \\ \vdots \\ \eta_{m\ell} \end{pmatrix}^{n_1,\ldots,n_m}_{\ell < \nu} ,$$

and suppose κ is such that $|\nu|^\lambda < \mathrm{Cf}(\kappa)$. Then

$$\begin{pmatrix} \kappa \\ \vdots \\ \kappa \end{pmatrix} \to \begin{pmatrix} \eta_{1\ell} + 1 \\ \eta_{2\ell} \\ \vdots \\ \eta_{m\ell} \end{pmatrix}^{n_1+1,n_2,\ldots,n_m}_{\ell < \nu} .$$

PROOF: The method is sufficiently well illustrated by the case $m = 2$. So suppose

$$\begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \to \begin{pmatrix} \eta_{1\ell} \\ \eta_{2\ell} \end{pmatrix}^{n,p}_{\ell < \nu} ,$$

with $n,p \geqq 1$, and take κ so that $|\nu|^\lambda < \mathrm{Cf}(\kappa)$.

Let any disjoint partition $\Delta = \{\Delta_\ell \ ; \ \ell < \nu\}$ of $[\kappa]^{n+1} \times [\kappa]^p$ be given. Define inductively a ramification system $\underline{\underline{R}}$ on κ of length $\rho = \lambda \cdot 2$, together with elements $x(X)$ for certain $X \in \mathrm{SEQ}_\sigma$ with $\sigma < \rho$. Then $\rho < \mathrm{Cf}(\kappa)$. Let $\sigma < \rho$ and suppose $S'(X)$ has already been defined for some $X \in \mathrm{SEQ}_\sigma$. If $|S'(X)| < \kappa$, put $F(X) = S'(X)$ and $n(X) = 0$. Otherwise, choose $x(X) \in S'(X)$ such that $\tau < \sigma \Rightarrow x(X \restriction \tau) < x(X)$, and put $F(X) = \{x(X)\}$. Place $G(X) = \cup\{F(X \restriction \tau) \ ; \ \tau \leqq \sigma\}$. Define a partition $\Gamma(X)$ of $S'(X) - F(X)$ as follows:

$y \equiv z \pmod{\Gamma(X)} \iff$

$$\forall a \in [G(X)]^n \ \forall b \in [G(X)]^p (\langle a \cup \{y\}, b\rangle \equiv \langle a \cup \{z\}, b\rangle \pmod{\Delta}) .$$

Put $n(X) = |\Gamma(X)|$, and let $S(Y)$, for $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < n(X)$, range over the classes of $\Gamma(X)$. This defines $\underline{\underline{R}}$.

For $\sigma < \rho$, put $\lambda_\sigma = |\nu|^\mu$ where $\mu = |\sigma + 1|^{\overset{np}{\cancel{n+p}}}$. Then $|n(X \restriction \tau)| \leq \lambda_\sigma$ whenever $X \in SEQ_\sigma$ and $\tau < \sigma < \rho$. Moreover, from the choice of κ it follows that $\lambda_\sigma^{|\sigma|} < Cf(\kappa)$ when $\sigma < \rho$. Hence Lemma 1.1(iii) applies to $\underline{\underline{R}}$, and so there is $X \in N \cap SEQ_\rho$ for which $S'(X) \neq \emptyset$. Choose such a sequence X . Then for no $\sigma < \rho$ can it be that $|S'(X \restriction \sigma)| < \kappa$. Put $x_\sigma = x(X \restriction \sigma)$. Then $\tau < \sigma \leq \rho \Rightarrow x_\tau < x_\sigma$. Further, for any $\tau < \rho$, if $y,z \in S'(X \restriction \tau)$ then

$$y \equiv z \pmod{\Gamma(X \restriction \tau)} \iff$$

$$\forall a \in [\{x_\alpha \; ; \; \alpha \leq \tau\}]^n \; \forall b \in [\{x_\alpha \; ; \; \alpha \leq \tau\}]^p (\langle a \cup \{y\}, b\rangle \equiv \langle a \cup \{z\}, b\rangle \pmod{\Delta}) \; .$$

Now if $\tau < \sigma \leq \rho$ then $x_\sigma \in F(X \restriction \sigma) \subseteq S'(X \restriction \sigma) \subseteq S(X \restriction \tau + 1)$, and so for $\tau_1 < \cdots < \tau_n < \tau < \rho$ and $\sigma_1 < \cdots < \sigma_p < \tau$,

$$\langle \{x_{\tau_1}, \ldots, x_{\tau_n}, x_\tau\}, \{x_{\sigma_1}, \ldots, x_{\sigma_p}\}\rangle \equiv \langle \{x_{\tau_1}, \ldots, x_{\tau_n}, x_\rho\}, \{x_{\sigma_1}, \ldots, x_{\sigma_p}\}\rangle \pmod{\Delta} \; .$$

Put $K_1 = \{x_\sigma \; ; \; \lambda \leq \sigma < \lambda \cdot 2\}$ and $K_2 = \{x_\sigma \; ; \; \sigma < \lambda\}$. Then $|K_1| = |K_2| = \lambda$. Moreover, $[K_1]^n \times [K_2]^p = \cup\{\Delta'_\ell \; ; \; \ell < \nu\}$ where for $\ell < \nu$,

$$\Delta'_\ell \;=\; \{\langle a,b\rangle \in [K_1]^n \times [K_2]^p \; ; \; \langle a \cup \{x_\rho\}, b\rangle \in \Delta_\ell\} \; .$$

By the relation (1), there are $\ell < \nu$ and $H_1 \subseteq K_1$, $H_2 \subseteq K_2$ such that the order type of H_1 is $\eta_{1\ell}$, the order type of H_2 is $\eta_{2\ell}$ and

$[H_1]^n \times [H_2]^p \subseteq \Delta'_\ell$. But then $[H_1 \cup \{x_\rho\}]^{n+1} \times [H_2]^p \subseteq \Delta_\ell$. This proves 1.5.

1.6 COROLLARY (*) Suppose $\omega_\alpha \rightarrow {}^m(\eta_\ell \ ; \ \ell < \nu)^n$. Then $\omega_{\alpha+2m} \rightarrow {}^m(\eta_\ell + 1 \ ; \ \ell < \nu)^{n+1}$.

PROOF: Since $\omega_\alpha \rightarrow {}^m(\eta_\ell \ ; \ \ell < \nu)^n$, certainly $|\nu| \leqq \omega_\alpha$. Take any $\beta \geqq \alpha$. Then, by virtue of the GCH , it follows that $|\nu|^{\omega_\beta} \leqq \omega_\alpha{}^{\omega_\beta} = \omega_{\beta+1} < Cf(\omega_{\beta+2})$. Thus Theorem 1.5 may be applied m successive times to prove the corollary.

In the case $m = 1$, the proof of Theorem 1.5 leads to the better conclusion that from $\omega_\alpha \rightarrow (\eta_\ell \ ; \ \ell < \nu)^n$ follows $\omega_{\alpha+1} \rightarrow (\eta_\ell + 1 \ ; \ \ell < \nu)^{n+1}$ (see [5, p.468]).

§2. Negative relations

In this section, some cases where a partition property fails to hold will be discussed. In particular, a method will be given whereby a negative relation for a cardinal κ may be extended to a negative relation for 2^κ .

Theorem II.1.5, that $\kappa \not\rightarrow {}^2(\kappa)^1_2$ is the first negative result.

2.1 THEOREM (*) Provided $\kappa > \omega$, then $\kappa^+ \not\rightarrow {}^2(\kappa^+,\kappa)^1$.

This follows from Theorem 43 of [6]. The proof will not be given here.

As in §III.3, let $\ll$ be the lexicographic ordering of ${}^{\kappa}2$, and for $x \neq y$ in ${}^{\kappa}2$, define $\delta(x,y)$ to be the least $\alpha < \kappa$ for which $x(\alpha) \neq y(\alpha)$.

2.2 THEOREM Suppose $\lambda < \kappa \Rightarrow 2^{\lambda} < \kappa$. Then

$$\begin{pmatrix} 2^{\kappa} \\ 2^{\kappa} \end{pmatrix} \not\to \begin{pmatrix} \kappa & 2 \\ 2 & \kappa \end{pmatrix}^{2,2}$$

PROOF: Define a disjoint partition $[{}^{\kappa}2]^2 \times [{}^{\kappa}2]^2 = \Delta_o \cup \Delta_1$ by

$$\langle \{x_o,x_1\},\{y_o,y_1\} \rangle \in \Delta_o \iff \delta(x_o,x_1) \leqq \delta(y_o,y_1) .$$

Take any $y_o,y_1 \in {}^{\kappa}2$ with $y_o \neq y_1$. Then $\delta(y_o,y_1) < \kappa$. However, if $x_o,x_1 \in {}^{\kappa}2$ are such that $\delta(x_o,x_1) \leqq \delta(y_o,y_1)$, then x_o and x_1 differ no later than at $\delta(y_o,y_1)$. Hence if $H \subseteq {}^{\kappa}2$ has the property

$$\{x_o,x_1\} \in [H]^2 \Rightarrow \delta(x_o,x_1) \leqq \delta(y_o,y_1) ,$$

then $|H| \leqq 2^{|\delta(y_o,y_1)+1|} < \kappa$. Thus there are no $H_o \in [{}^{\kappa}2]^{\kappa}$, $H_1 \in [{}^{\kappa}2]^2$ such that $[H_o]^2 \times H_1 \subseteq \Delta_o$. Similarly, there are no $H_o \in [{}^{\kappa}2]^2$, $H_1 \in [{}^{\kappa}2]^{\kappa}$ such that $H_o \times [H_1]^2 \subseteq \Delta_1$. This proves Theorem 2.2.

2.3 COROLLARY (*) If $\kappa^- = \kappa$, then $\kappa^+ \not\to {}^2(\kappa)^2_2$.

A similar construction may be used to show:

2.4 THEOREM Suppose $\lambda < \kappa \Rightarrow 2^{\lambda} < \kappa$. Then

$$\binom{2^{\kappa}}{\kappa} \not\to \binom{\kappa\ \ 2}{1\ \ \kappa}^{2,1} .$$

Let X be a set and $<$ a binary relation on X . In the notation of [6], if $x_o,\ldots,x_n \in X$ then $\{x_o,\ldots,x_n\}_<$ denotes the set $\{x_o,\ldots,x_n\}$ and, in addition, expresses that $x_o < x_1 < \ldots < x_n$.

Choose a well ordering $<$ of ${}^{\kappa}2$ which has order type 2^{κ} . For the rest of this section, let $n \geq 3$ be fixed. We shall need the following sets:

$$K_{o1} = \{\{x_1,\ldots,x_n\}_< \subseteq {}^{\kappa}2 \ ; \ x_1 \ll x_2 \text{ and } x_2 \gg x_3\} ,$$

$$K_{1o} = \{\{x_1,\ldots,x_n\}_< \subseteq {}^{\kappa}2 \ ; \ x_1 \gg x_2 \text{ and } x_2 \ll x_3\} ,$$

$$L_o = \{\{x_1,\ldots,x_n\}_< \subseteq {}^{\kappa}2 \ ; \ x_1 \ll \ldots \ll x_n\} ,$$

$$L_1 = \{\{x_1,\ldots,x_n\}_< \subseteq {}^{\kappa}2 \ ; \ x_1 \gg \ldots \gg x_n\} ,$$

$$L = L_o \cup L_1 ,$$

$$P_o = \{\{x_1,\ldots,x_n\}_< \subseteq L \ ; \ \delta(x_1,x_2) < \ldots < \delta(x_{n-1},x_n)\} .$$

The next three lemmas occur in §13 of [6].

2.5 LEMMA Let $X \subseteq {}^{\kappa}2$. Then if $[X]^n \subseteq L$, either $[X[^n \subseteq L_o$ or $[X]^n \subseteq L_1$.

PROOF: The lemma is trivial if $|X| \leq n$, so assume $|X| \geq n+1$. Suppose there are $a,b \in [X]^n$ such that $a \notin L_o$ and $b \notin L_1$. Choose $x \in X - a$, and let $\{x\} \cup a \cup b = \{x_1,\ldots,x_p\}_<$, so $p \geq n+1$. Since

$[X]^n \subseteq L$ always $\{x_{r+1},\ldots,x_{r+n}\} \in L$ for $r+n \leq p$. Thus $x_{r+1} \ll \ldots \ll x_{r+n}$ or $x_{r+1} \gg \ldots \gg x_{r+n}$ for $r+n \leq p$, and so it follows that $x_1 \ll \ldots \ll x_p$ or $x_1 \gg \ldots \gg x_p$. Hence $a,b \in L_o$ or $a,b \in L_1$, contrary to the initial assumption. This proves 2.5.

2.6 LEMMA Let $X \in [{}^{\kappa}2]^{\lambda}$ for some $\lambda \geq \omega$, and suppose $[X]^n \cap K_{o1} = \emptyset$ or $[X]^n \cap K_{1o} = \emptyset$. Then there is $X' \in [X]^{\lambda}$ such that $[X']^n \subseteq L_o$ or $[X']^n \subseteq L_1$.

2.7 LEMMA Let $X \in [{}^{\kappa}2]^{\geq\lambda}$ for some regular cardinal λ, and suppose $[X]^n \subseteq L_o$ or $[X]^n \subseteq L_1$. Then there is $X' \in [X]^{\lambda}$ with $[X']^n \subseteq P_o$.

The proofs of Lemmas 2.6 and 2.7 may be found in §§13.7 and 13.8 of [6].

2.8 LEMMA Let $X \in [{}^{\kappa}2]^{\geq\omega}$. Then there is $Y \in [X]^{\omega}$ such that $[Y]^n \subseteq P_o$.

PROOF: Let X be given, and define a partition $[X]^2 = \Delta_o \cup \Delta_1$ by

$$\{x,y\}_< \in \Delta_o \iff x \ll y \;;\quad \{x,y\}_< \in \Delta_1 \iff x \gg y \,.$$

Since $\omega \to (\omega)^2_2$, there is $X' \in [X]^{\omega}$ such that $[X']^2 \subseteq \Delta_o$ or $[X']^2 \subseteq \Delta_1$. Thus if $X' = \{x_r ; r < \omega\}$, listed in increasing $<$-order, either $x_o \ll x_1 \ll x_2 \ll \ldots$ or $x_o \gg x_1 \gg x_2 \gg \ldots$. Hence $p < q < r < \omega \implies \delta(x_p,x_q) \neq \delta(x_q,x_r)$.

Define a partition $[X']^3 = \Gamma_o \cup \Gamma_1$ by

$$\{x,y,z\}_< \in \Gamma_o \iff \delta(x,y) < \delta(y,z) \ ; \quad \{x,y,z\}_< \in \Gamma_1 \iff \delta(x,y) > \delta(y,z) \ .$$

Since $\omega \to (\omega)^3_2$, there is $Y \in [X']^\omega$ such that $[Y]^3 \subseteq \Gamma_o$ or $[Y]^3 \subseteq \Gamma_1$. Thus if $Y = \{y_r \ ; \ r < \omega\}$, listed in increasing $<$-order, either $\delta(y_o,y_1) < \delta(y_1,y_2) < \delta(y_2,y_3) < \cdots$ or $\delta(y_o,y_1) > \delta(y_1,y_2) > \delta(y_2,y_3) > \cdots$. However, the second possibility gives an infinite descending sequence of ordinals. Thus $\delta(y_o,y_1) < \delta(y_1,y_2) < \cdots$, and so $[Y]^n \subseteq P_o$. This establishes the lemma.

2.9 THEOREM Let $n \geqq 3$. Let η_ℓ for $\ell < \nu$ be cardinals such that $\eta_o, \eta_1 \geqq \omega$ and η_o is regular. If κ is a cardinal such that $\kappa \not\to {}^m(\eta_\ell \ ; \ \ell < \nu)^{n-1}$, then $2^\kappa \not\to {}^m(\eta_\ell \ ; \ \ell < \nu)^n$.

PROOF: The result is trivial if any $\eta_\ell < n$, so suppose all $\eta_\ell \geqq n$. The method of proof is sufficiently illustrated by the case $m = 2$. So choose a disjoint partition

$$[\kappa]^{n-1} \times [\kappa]^{n-1} = \cup\{\Delta_\ell \ ; \ \ell < \nu\}$$

where for each $\ell < \nu$ there are no $H_o, H_1 \in [\kappa]^{\eta_\ell}$ such that $[H_o]^{n-1} \times [H_1]^{n-1} \subseteq \Delta_\ell$. Put

$$\Delta'_\ell = \{\langle\{x_1,\ldots,x_n\}_< \ , \ \{y_1,\ldots,y_n\}_<\rangle \in P_o \times P_o \ ;$$
$$\langle\{\delta(x_1,x_2),\ldots,\delta(x_{n-1},x_n)\},\{\delta(y_1,y_2),\ldots,\delta(y_{n-1},y_n)\}\rangle \in \Delta_\ell\} \ .$$

Define a disjoint partition $\Gamma = \{\Gamma_\ell \,;\, \ell < \nu\}$ of $[{}^\kappa 2]^n \times [{}^\kappa 2]^n$ as follows:

$$\Gamma_1 = (K_{o1} \times P_o) \cup (P_o \times K_{o1}) \cup \Delta_1' \,,$$
$$\Gamma_\ell = \Delta_\ell' \quad \text{for} \quad 2 \leqq \ell < \nu \,.$$

Let $X,Y \in [{}^\kappa 2]^{\eta_o}$ be such that $[X]^n \times [Y]^n \subseteq \Gamma_o$. By Lemma 2.8, there is $y \in [Y]^n$ for which $y \in P_o$. Since $[X]^n \times \{y\} \subseteq \Gamma_o$, it must be that $[X]^n \cap K_{o1} = \emptyset$. Thus from Lemma 2.6, there is $X' \in [X]^{\eta_o}$ such that $[X']^n \subseteq L_o$ or $[X']^n \subseteq L_1$. But then, in virtue of Lemma 2.7, there is $X'' \in [X]^{\eta_o}$ such that $[X'']^n \subseteq P_o$. Similarly, $[Y]^n \cap K_{o1} = \emptyset$, and there is $Y'' \in [Y]^{\eta_o}$ such that $[Y'']^n \subseteq P_o$. Thus

$$[X'']^n \times [Y'']^n \subseteq (P_o \times P_o) \cap \Gamma_o = \Delta_o' \,.$$

Now let $X,Y \in [{}^\kappa 2]^{\eta_1}$ be such that $[X]^n \times [Y]^n \subseteq \Gamma_1$. Then $[X]^n \subseteq K_{o1} \cup P_o$, and so $[X]^n \cap K_{1o} = \emptyset$. Hence by Lemma 2.6, there is $X' \in [X]^{\eta_1}$ such that $[X']^n \subseteq L_o$ or $[X']^n \subseteq L_1$. Thus $[X']^n \subseteq P_o$. Similarly, there is $Y' \in [Y]^{\eta_1}$ such that $[Y']^n \subseteq P_o$. Hence

$$[X']^n \times [Y']^n \subseteq (P_o \times P_o) \cap \Gamma_1 = \Delta_1' \,.$$

Thus, to prove the theorem, it suffices to show, for any $\ell < \nu$:

(1) If $X,Y \subseteq {}^\kappa 2$ are such that $[X]^n \times [Y]^n \subseteq \Delta_\ell'$ and X , Y have the same order type η under $<$, where η is a limit ordinal, then $\eta < \eta_\ell$.

Take such sets X , Y and let $X = \{x_\sigma ; \sigma < \eta\}$, $Y = \{y_\sigma ; \sigma < \eta\}$, listed in increasing $<$-order. Put $C = \{\delta(x_\sigma, x_{\sigma+1}) ; \sigma < \eta\}$ and $D = \{\delta(y_\sigma, y_{\sigma+1}) ; \sigma < \eta\}$. Since $[X]^n, [Y]^n \subseteq P_o$, it follows that $\delta(x_\sigma, x_{\sigma+1}) < \delta(x_{\sigma+1}, x_{\sigma+2})$ and $\delta(y_\sigma, y_{\sigma+1}) < \delta(y_{\sigma+1}, y_{\sigma+2})$ for $\sigma < \eta$. Hence C and D both have order type η .

Take $c \in [C]^{n-1}$ and $d \in [D]^{n-1}$. It follows as in the proof of Theorem III.3.2 that there are $\sigma_1 < \dots < \sigma_n < \eta$ and $\tau_1 < \dots < \tau_n < \eta$ such that $c = \{\delta(x_{\sigma_1}, x_{\sigma_2}), \dots, \delta(x_{\sigma_{n-1}}, x_{\sigma_n})\}$ and $d = \{\delta(y_{\tau_1}, y_{\tau_2}), \dots, \delta(y_{\tau_{n-1}}, y_{\tau_n})\}$. Hence, by the definition of Δ'_ℓ , it follows that $\langle c,d\rangle \in \Delta_\ell$. Thus $[C]^{n-1} \times [D]^{n-1} \subseteq \Delta_\ell$, and so, since C and D have the same power, $|C| < \eta_\ell$. Thus $\eta < \eta_\ell$. This proves (1), and completes the proof of the theorem.

Theorem II.1.5 yields that $\kappa \not\to {}^2(\kappa)^1_2$ (for any κ), and so certainly $\kappa \not\to {}^2(\kappa)^2_2$. Thus by Theorem 2.9, $\kappa^+ \not\to {}^2(\kappa)^3_2$. If $\kappa^- = \kappa$, applying Theorem 2.9 to Corollary 2.3(*) leads to the stronger result that $\kappa^{++} \not\to {}^2(\kappa)^3_2$. Likewise, Theorem 2.1 gives that $\kappa^+ \not\to {}^2(\kappa^+,\kappa)^1$, so that $\kappa^+ \not\to {}^2(\kappa^+,\kappa)^2$. Then 2.9 yields that $\kappa^{++} \not\to {}^2(\kappa^+,\kappa)^3$.

§3. The special case $n = 1$

In the case $n = 1$, some particular constructions are available which lead to positive relations. The results in this and the following sections have their genesis in part II of [6] , where relations of the form

$$\begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \to \begin{pmatrix} \kappa_1 & \kappa_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{1,1}$$

are discussed extensively.

3.1 LEMMA Let $p < \omega$, and suppose

(1) $$\binom{\kappa}{\kappa^+} \to \binom{\eta_\ell}{\zeta_\ell}^{1,1}_{\ell < \nu} .$$

Then the following relation holds:

$$\binom{\kappa}{\kappa^+} \to \binom{\kappa}{p}\binom{\eta_\ell}{\zeta_\ell}^{1,1}_{\ell < \nu} .$$

PROOF: Use induction on p. If $p = 0$ the lemma is trivial. Assume the result true for some $q < \omega$. Take a disjoint partition

$$\kappa \times \kappa^+ = \Delta \cup \bigcup\{\Delta_\ell \; ; \; \ell < \nu\} ,$$

and suppose that for $\ell < \nu$ there are no sets $H_o \subseteq \kappa$, $H_1 \subseteq \kappa^+$ having order types η_ℓ and ζ_ℓ respectively, for which $H_o \times H_1 \subseteq \Delta_\ell$. By the inductive hypothesis, there must then be sets $S \in [\kappa]^\kappa$ and $T \in [\kappa^+]^q$ such that $S \times T \subseteq \Delta$.

For $\beta < \kappa^+$, put $Q(\beta) = \{\alpha \in \kappa \; ; \; \langle\alpha,\beta\rangle \in \Delta\}$. If there is $\beta \in \kappa^+ - T$ such that $|S \cap Q(\beta)| = \kappa$, then $(S \cap Q(\beta)) \times (T \cup \{\beta\}) \subseteq \Delta$, and the result follows. So suppose that $|S \cap Q(\beta)| < \kappa$ for all $\beta \in \kappa^+ - T$. We shall show that a contradiction results.

For $\kappa' < \kappa$, put $T(\kappa') = \{\beta \in \kappa^+ - T \; ; \; |S \cap Q(\beta)| = \kappa'\}$. There must be some $\lambda < \kappa$ with $|T(\lambda)| = \kappa^+$. Write S as a disjoint union, $S = \bigcup\{S_\mu \; ; \; \mu < \lambda^+\}$, where each $|S_\mu| = \kappa$. If $\beta \in T(\lambda)$, then $|S \cap Q(\beta)| = \lambda < \lambda^+$, and hence there is $\mu(\beta) < \lambda^+$ for which $S_{\mu(\beta)} \cap Q(\beta) = \emptyset$. Since $\kappa^+ \to (\kappa^+)^1_\kappa$, there are $Y \in [T(\lambda)]^{\kappa^+}$ and

$\mu < \lambda^+$ such that $\beta \in Y \Rightarrow \mu(\beta) = \mu$. Then $S_\mu \cap Q(\beta) = \emptyset$ for all $\beta \in Y$. Hence $S_\mu \times Y \subseteq \bigcup\{\Delta_\ell \ ; \ \ell < \nu\}$. Now $|S_\mu| = \kappa$ and $|Y| = \kappa^+$. Thus by the relation (1), for some $\ell < \nu$ there are $H_o \subseteq S_\mu$ and $H_1 \subseteq Y$, having order types η_ℓ and ζ_ℓ respectively, for which $H_o \times H_1 \subseteq \Delta_\ell$. This contradicts the assumed property of the partition.

Thus the induction step is completed, and the lemma proved.

3.2 LEMMA(*) Let κ be a singular cardinal with the property

(1) $$\binom{\kappa}{\kappa^+} \rightarrow \binom{\eta_\ell}{\zeta_\ell}^{1,1}_{\ell < \nu} .$$

For some $\iota \geqq \kappa$ let there be given a partition

$$\iota \times \kappa^+ = \Delta \cup \bigcup\{\Delta_\ell \ ; \ \ell < \nu\}$$

such that for no $\ell < \nu$ are there sets $H_o \subseteq \iota$ and $H_1 \subseteq \kappa^+$, having order types η_ℓ and ζ_ℓ respectively, for which $H_o \times H_1 \subseteq \Delta_\ell$. Let λ be any regular cardinal with $Cf(\kappa) < \lambda < \kappa$. Let $A \in [\iota]^\kappa$ and $B \subseteq \kappa^+$. Then there is a set $A^* \subseteq [A]^\lambda$ with $|A^*| \leq \kappa$ and there is a map f from A^* to the subsets of B such that $a \in A^* \Rightarrow a \times f(a) \subseteq \Delta$, and $|B - \bigcup\{f(a) \ ; \ a \in A^*\}| \leq \kappa$.

PROOF: Take $A \in [\iota]^\kappa$ and $B \subseteq \kappa^+$. For $\beta < \kappa^+$ put $Q(\beta) = \{\alpha < \iota \ ; \ \langle\alpha,\beta\rangle \in \Delta\}$. Let $B_1 = \{\beta \in B \ ; \ |A \cap Q(\beta)| < \lambda\}$ and $B_2 = B - B_1$. Lemma 3.1 applied to the relation (1) yields in particular

$$\binom{\kappa}{\kappa^+} \to \left(\binom{\kappa}{1}\binom{\eta_\ell}{\zeta_\ell}_{\ell<\nu}\right)^{1,1} .$$

Now $A \times B_1 \subseteq \Delta \cup \bigcup\{\Delta_\ell \; ; \; \ell < \nu\}$. Thus if $|B_1| = \kappa^+$, either there is $\beta \in B_1$ such that $|A \cap Q(\beta)| = \kappa$ (contrary to the definition of B_1), or there are $\ell < \nu$ and $K_o \subseteq A$, $K_1 \subseteq B_1$ such that the order type of K_o is η_ℓ , the order type of K_1 is ζ_ℓ and $K_o \times K_1 \subseteq \Delta_\ell$ (contrary to the choice of the partition). Hence it must be the case that $|B_1| \leq \kappa$.

Choose cardinals $\kappa_\sigma < \kappa$ for $\sigma < Cf(\kappa)$ such that $\kappa = \sum \{\kappa_\sigma \; ; \; \sigma < Cf(\kappa)\}$. There is a disjoint partition $A = \bigcup\{A_\sigma \; ; \; \sigma < Cf(\kappa)\}$ where $|A_\sigma| = \kappa_\sigma$. Put $A^* = \bigcup\{[A_\sigma]^\lambda \; ; \; \sigma < Cf(\kappa)\}$, so $|A^*| \leq \sum \{\kappa_\sigma^{\ \lambda} \; ; \; \sigma < Cf(\kappa)\} \leq \kappa$. For $a \in A^*$, define $f(a) = \{\beta \in B_2 \; ; \; a \subseteq A \cap Q(\beta)\}$, so $a \times f(a) \subseteq \Delta$ and f maps A^* into PB . Further, for each $\beta \in B_2$ there is $a \in A^*$ for which $\beta \in f(a)$, since otherwise $|A_\sigma \cap Q(\beta)| < \lambda$ for all $\sigma < Cf(\kappa)$, and so by the regularity of λ and the definition of B_2 ,

$$\lambda \leq |A \cap Q(\beta)| = \sum \{|A_\sigma \cap Q(\beta)| \; ; \; \sigma < Cf(\kappa)\} < \lambda .$$

Hence $B_2 = \bigcup\{f(a) \; ; \; a \in A^*\}$. This leads to the required result.

The following theorem is Corollary 17 of [6]. For later reference, I include the proof.

3.3 THEOREM (*) For all infinite κ , $\kappa^+ \to {}^2(\kappa)^1_2$.

PROOF: Let a partition $\kappa^+ \times \kappa^+ = \Delta_o \cup \Delta_1$ be given. Assume for all

$H_o, H_1 \in [\kappa^+]^\kappa$ that $H_o \times H_1 \nsubseteq \Delta_1$, and seek $H_o, H_1 \in [\kappa^+]^\kappa$ such that $H_o \times H_1 \subseteq \Delta_o$.

For $\alpha, \beta < \kappa^+$ define

$$P_o(\alpha) = \{\beta \in \kappa^+ ; \langle\alpha,\beta\rangle \in \Delta_o\} ; \quad Q_o(\beta) = \{\alpha \in \kappa^+ ; \langle\alpha,\beta\rangle \in \Delta_o\} .$$

Then if $A \in [\kappa^+]^\kappa$ and $B \subseteq \kappa^+$, since $A \times (B - \cup\{P_o(\alpha) ; \alpha \in A\}) \subseteq \Delta_1$ it follows that $|B - \cup\{P_o(\alpha) ; \alpha \in A\}| < \kappa$. Similarly, if $B \in [\kappa^+]^\kappa$ and $A \subseteq \kappa^+$, then $|A - \cup\{Q_o(\beta) ; \beta \in B\}| < \kappa$.

Suppose first that κ is regular. Define a ramification system $\underline{\underline{R}}$ on κ^+ of length $\rho = \kappa$ as follows. Take $\sigma < \kappa$ and $X \in SEQ_\sigma$. Suppose $S'(X)$ has already been defined. If $|S'(X)| \leqq \kappa$, put $F(X) = S'(X)$ and $n(X) = 0$. If $|S'(X)| = \kappa^+$, put $n(X) = \kappa$ and consider two cases.

Case 1: σ is even

Choose $R(X) \in [S'(X)]^\kappa$ and for $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$, let $\alpha(Y)$ be the $Y(\sigma)$-th element of $R(X)$. Put

$$F(X) = R(X) \cup (S'(X) - \cup\{P_o(\alpha(Y)) ; \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}) .$$

Then $S'(X) - F(X) = \cup\{S(Y) ; \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}$, where for Y such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$,

$$S(Y) = (S'(X) \cap P_o(\alpha(Y))) - F(X) .$$

Case 2: σ is odd

Choose $R(X) \in [S'(X)]^\kappa$ and for Y such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$, let $\beta(Y)$ be the $Y(\sigma)$-th element of $R(X)$. Define

$$F(X) = R(X) \cup (S'(X) - \bigcup\{Q_o(\beta(Y)) \ ; \ \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}) \ .$$

Then $S'(X) - F(X) = \bigcup\{S(Y) \ ; \ \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}$, where for Y such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$,

$$S(Y) = (S'(X) \cap Q_o(\beta(Y))) - F(X) \ .$$

By definition, $|R(X)| = \kappa$. From an earlier remark, it thus follows that $|F(X)| = \kappa$. Hence the Ramification Lemma, 1.1(iii), applies to $\underline{\underline{R}}$. This yields a sequence $X \in N \cap SEQ_\kappa$ where each $X(\sigma) < \kappa$, such that $S'(X) \neq \emptyset$. Choose and fix such a sequence X. Put $\alpha_\sigma = \alpha(X \upharpoonright \sigma + 1)$, $\beta_\sigma = \beta(X \upharpoonright \sigma + 1)$ for $\sigma < \kappa$. Then either α_σ or β_σ is defined for all $\sigma < \kappa$.

Put $H_o = \{\alpha_{2\cdot\sigma} \ ; \ \sigma < \kappa\}$ and $H_1 = \{\beta_{2\cdot\sigma+1} \ ; \ \sigma < \kappa\}$. Since $\alpha(X \upharpoonright \tau + 1) \in F(X \upharpoonright \tau)$ and the $F(X \upharpoonright \tau)$ are pairwise disjoint (by Lemma 1.1(i)), it follows that $|H_o| = \kappa$. Similarly, $|H_1| = \kappa$. Moreover, $H_o \times H_1 \subseteq \Delta_o$. For let σ be even, τ be odd. It suffices to show that $\langle\alpha_\sigma, \beta_\tau\rangle \in \Delta_o$. Suppose $\sigma < \tau$. Then

$$\beta_\tau = \beta(X \upharpoonright \tau + 1) \in R(X \upharpoonright \tau) \subseteq S'(X \upharpoonright \tau) \subseteq S(X \upharpoonright \sigma + 1) \ .$$

Thus $\langle\alpha_\sigma, \beta_\tau\rangle \in \{\alpha_\sigma\} \times S(X \upharpoonright \sigma + 1) \subseteq \{\alpha_\sigma\} \times P_o(\alpha_\sigma) \subseteq \Delta_o$. Similarly, $\langle\alpha_\sigma, \beta_\tau\rangle \in \Delta_o$ if $\tau < \sigma$. This proves the theorem if κ is regular.

For the case of singular κ, put $\rho = Cf(\kappa)$ and choose regular cardinals κ_σ for $\sigma < \rho$ such that $\rho < \kappa_o < \kappa_1 < \kappa_2 \cdots$ and $\kappa = \sum \{\kappa_\sigma \ ; \ \sigma < \rho\}$. Note that trivially

$$\binom{\kappa}{\kappa^+} \to \binom{\kappa}{\kappa}^{1,1} \quad ,$$

and so Lemma 3.2 may be applied to the partition $\kappa^+ \times \kappa^+ = \Delta_o \cup \Delta_1$.

Define a ramification system $\underline{\underline{R}}$ on κ^+ of length ρ as follows. Take $\sigma < \rho$ and $X \in SEQ_\sigma$. Suppose $S'(X)$ has already been defined. If $|S'(X)| \leqq \kappa$, put $F(X) = S'(X)$ and $n(X) = 0$. If $|S'(X)| = \kappa^+$, put $n(X) = \kappa$ and consider two cases.

Case 1: σ is even

Choose $R(X) \in [S'(X)]^\kappa$. Apply Lemma 3.2 with $A = R(X)$, $B = S'(X)$, $\lambda = \kappa_\sigma$ and $\Delta = \Delta_o$ to obtain a set $A^*(X)$ and a map f_X with the properties stated in the lemma. Write $A^*(X) = \{a(Y) \; ; \; \bar{Y} = X$ and $Y(\sigma) < \kappa\}$. Define

$$F(X) \quad = \quad R(X) \cup (S'(X) - \cup\{f_X(a(Y)) \; ; \; \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}) \; .$$

Then $S'(X) - F(X) = \cup\{S(Y) \; ; \; \bar{Y} = X$ and $Y(\sigma) < \kappa\}$, where for Y such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$,

$$S(Y) \quad = \quad f_X(a(Y)) - F(X) \; .$$

Case 2: σ is odd

Choose $R(X) \in [S'(X)]^\kappa$. A symmetrical argument shows that there is $B^*(X) \subseteq [R(X)]^{\kappa_\sigma}$ with $|B^*(X)| \leqq \kappa$ and there is a map g_X from $B^*(X)$ to the subsets of $S'(X)$ such that $b \in B^*(X) \Rightarrow g_X(b) \times b \subseteq \Delta_o$, and $|S'(X) - \cup\{g_X(b) \; ; \; b \in B^*(X)\}| \leqq \kappa$. Write $B^*(X) = \{b(Y) \; ; \; \bar{Y} = X$ and $Y(\sigma) < \kappa\}$, and define

$$F(X) = R(X) \cup (S'(X) - \bigcup\{g_X(b(Y)) \ ; \ \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}) .$$

Then $S'(X) - F(X) = \bigcup\{S(Y) \ ; \ \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}$, where for Y such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$,

$$S(Y) = g_X(b(Y)) - F(X) .$$

This completes the definition of $\underline{\underline{R}}$.

Then, as in the regular case, there is a sequence $X \in N \cap SEQ_{\kappa\rho}$ where each $X(\sigma) < \kappa$, such that $S'(X) \neq \emptyset$. Put $a_\sigma = a(X \restriction \sigma + 1)$ and $b_\sigma = b(X \restriction \sigma + 1)$ for $\sigma < \rho$. Put $H_o = \bigcup\{a_{2\cdot\sigma} \ ; \ \sigma < \rho\}$ and $H_1 = \bigcup\{b_{2\cdot\sigma+1} \ ; \ \sigma < \rho\}$. Then $|H_o| = |H_1| = \kappa$. Thus, to prove the theorem, it suffices to show that $H_o \times H_1 \subseteq \Delta_o$. Let σ be even, τ be odd. We shall show that $a_\sigma \times b_\tau \subseteq \Delta_o$. Suppose $\sigma < \tau$. Then

$$b_\tau = b(X \restriction \tau + 1) \subseteq R(X \restriction \tau) \subseteq S'(X \restriction \tau) \subseteq S(X \restriction \sigma + 1) \subseteq f_{X\restriction\sigma}(a_\sigma) .$$

Thus $a_\sigma \times b_\tau \subseteq a_\sigma \times f_{X\restriction\sigma}(a_\sigma) \subseteq \Delta_o$. Similarly, also if $\tau < \sigma$ then $a_\sigma \times b_\tau \subseteq \Delta_o$. This completes the proof for singular κ , and concludes the proof of 3.3.

The following lemma could appropriately be called the Stepping-out Lemma. Working from Theorem 3.3, it yields a result for $m > 2$.

3.4 LEMMA (*) Let $m \geq 1$ and suppose $\omega_{\alpha+1} \to {}^2(\omega_\alpha)^1_\nu$. Then $\omega_{\alpha+m} \to {}^{m+1}(\omega_\alpha)^1_\nu$.

PROOF: By induction on m. The case $m = 1$ is true by assumption. The method of the inductive step is shown sufficiently well for $m = 2$.

Put $\omega_\alpha = \kappa$. Take any partition $\Delta = \{\Delta_\ell \; ; \; \ell < \nu\}$ of ${}^3(\kappa^{++})$. Define a ramification system $\underline{\underline{R}}$ on ${}^3(\kappa^{++})$ of length $\rho = \kappa^+$ as follows. Take $\sigma < \kappa^+$ and $X \in SEQ_\sigma$. Suppose $S'(X)$ has already been defined. If $S'(X) = \emptyset$, put $F(X) = \emptyset$ and $n(X) = 0$. Otherwise, choose $\langle x(X), y(X), z(X) \rangle \in S'(X)$ and put $F(X) = \{\langle x(X), y(X), z(X) \rangle\}$. Place

$$G(X) = \{x \in \kappa^{++} \; ; \; \exists \tau \leq \sigma(x = x(X \upharpoonright \tau) \text{ or } x = y(X \upharpoonright \tau) \text{ or } x = z(X \upharpoonright \tau))\} .$$

Define a partition $\Gamma(X)$ of $S'(X) - F(X)$ by:

$$\langle x_1, y_1, z_1 \rangle \equiv \langle x_2, y_2, z_2 \rangle \pmod{\Gamma(X)} \iff$$

$$\forall u, v \in G(X)(\langle x_1, u, v \rangle \equiv \langle x_2, u, v \rangle \pmod{\Delta} \text{ and}$$

$$\langle u, y_1, v \rangle \equiv \langle u, y_2, v \rangle \pmod{\Delta} \text{ and } \langle u, v, z_1 \rangle \equiv \langle u, v, z_2 \rangle \pmod{\Delta}) .$$

Put $n(X) = |\Gamma(X)|$. For $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < n(X)$, let $S(Y)$ range over the classes of $\Gamma(X)$. This defines $\underline{\underline{R}}$.

For $\sigma < \rho$, put $\lambda_\sigma = |\nu|^{|\sigma|+\omega}$. Then $|n(X \upharpoonright \tau)| \leq \lambda_\sigma$ whenever $X \in SEQ_\sigma$ and $\tau < \sigma < \rho$. Moreover, if $\sigma < \rho$ then $\lambda_\sigma^{|\sigma|} \leq |\nu|^\kappa \leq (\kappa^+)^\kappa = \kappa^+ < Cf(\kappa^{++})$, for certainly $|\nu| \leq \kappa^+$. Hence Lemma 1.1(iii) applies to $\underline{\underline{R}}$, and so there is $X \in N \cap SEQ_\rho$ for which $S'(X) \neq \emptyset$. Choose such a sequence X. Then $S'(X \upharpoonright \sigma) \neq \emptyset$ for all $\sigma < \rho$. Thus we may define $x_\sigma = x(X \upharpoonright \sigma)$, $y_\sigma = y(X \upharpoonright \sigma)$ and $z_\sigma = z(X \upharpoonright \sigma)$ for $\sigma < \rho = \kappa^+$. Then by Lemma 1.1(i), $\langle x_\sigma, y_\sigma, z_\sigma \rangle \neq \langle x_\tau, y_\tau, z_\tau \rangle$ for any

$\sigma < \tau < \kappa^+$. Hence there is $H \in [\kappa^+]^{\kappa^+}$ such that $\{\sigma,\tau\} \in [H]^2 \Rightarrow x_\sigma \neq x_\tau$, or $\{\sigma,\tau\} \in [H]^2 \Rightarrow y_\sigma \neq y_\tau$, or $\{\sigma,\tau\} \in [H]^2 \Rightarrow z_\sigma \neq z_\tau$.

Suppose in fact that $\{\sigma,\tau\} \in [H]^2 \Rightarrow x_\sigma \neq x_\tau$. (The situation is analogous in the other cases.) Take $\mu,\sigma < \tau \leqq \rho$, and put $\zeta = \max\{\mu,\sigma\} + 1$. Then

$$\langle x_\tau, y_\tau, z_\tau\rangle \in F(X \restriction \tau) \subseteq S'(X \restriction \tau) \subseteq S(X \restriction \zeta) .$$

Thus $\langle x_\tau, y_\tau, z_\tau\rangle$ and $\langle x_\rho, y_\rho, z_\rho\rangle$ are in the same class $S(X \restriction \zeta)$ of $\Gamma(X \restriction \zeta - 1)$. Hence $\langle x_\tau, x_\mu, x_\sigma\rangle \equiv \langle x_\rho, x_\mu, x_\sigma\rangle \pmod{\Delta}$.

Define a partition $\Lambda = \{\Lambda_\ell \,;\, \ell < \nu\}$ of $H \times H$ by, for $\ell < \nu$,

$$x_\mu, x_\sigma \in H \Rightarrow (\langle x_\mu, x_\sigma\rangle \in \Lambda_\ell \iff \langle x_\rho, x_\mu, x_\sigma\rangle \in \Delta_\ell) .$$

Since $|H| = \kappa^+$ and $\kappa^+ \to {}^2(\kappa)^1_\nu$, there are $\ell < \nu$ and $H_1, H_2 \in [H]^\kappa$ such that $H_1 \times H_2 \subseteq \Lambda_\ell$. Further, there is $H_o \in [H]^\kappa$ such that if $x_\tau \in H_o$ and $x_\mu \in H_1 \cup H_2$ then $\mu < \tau$. Then $H_o \times H_1 \times H_2 \subseteq \Delta_\ell$. This completes the proof of 3.4.

3.5 COROLLARY (*). If $m \geqq 1$ then $\omega_{\alpha+m} \to {}^{m+1}(\omega_\alpha)^1_2$.

Theorem II.1.1 applied to the best possible relation $\omega_{\alpha+m} \to (\omega_{\alpha+1})^m_2$ yields that $\omega_{\alpha+m} \to {}^m(\omega_\alpha)^1_2$. Hence 3.5 is a slight strengthening of this result. I have been unsuccessful in improving 3.5 further. Since $\omega_\alpha \not\to {}^m(\omega_\alpha)^1_2$ for $m > 1$, the most simple open questions are seen to be:

3.6 PROBLEM (*) Is $\kappa^+ \to {}^3(\kappa)^1_2$ true? Is $\kappa^{++} \to {}^4(\kappa)^1_2$ true?

The method of Lemma 3.4 shows that a positive answer to the second question is implied by a positive answer to the first.

The question of extending 3.3 to partitions involving more than two classes is also unsolved in general. Inspection of the proof leads to the following observation.

3.7 THEOREM (*) Suppose the relation

(1) $$\binom{\kappa}{\kappa^+} \rightarrow \binom{\kappa}{\kappa}^{1,1}_{\nu}$$

to hold. Then $\kappa^+ \rightarrow {}^2(\kappa)^1_{1+\nu}$.

PROOF: Take a partition

(2) $$\kappa^+ \times \kappa^+ = \Delta \cup \bigcup\{\Delta_\ell \ ; \ \ell < \nu\} ,$$

and assume for all $H_o, H_1 \in [\kappa^+]^\kappa$ that $H_o \times H_1 \not\subseteq \Delta_\ell$ for any $\ell < \nu$. We must find $H_o, H_1 \in [\kappa^+]^\kappa$ such that $H_o \times H_1 \subseteq \Delta$.

For $\alpha, \beta < \kappa^+$ define

$$P(\alpha) = \{\beta \in \kappa^+ \ ; \ \langle\alpha,\beta\rangle \in \Delta\} \ ; \quad Q(\beta) = \{\alpha \in \kappa^+ \ ; \ \langle\alpha,\beta\rangle \in \Delta\} .$$

Note that if $A \in [\kappa^+]^\kappa$ and $B \subseteq \kappa^+$ then

$$A \times (B - \bigcup\{P(\alpha) \ ; \ \alpha \in A\}) \subseteq \bigcup\{\Delta_\ell \ ; \ \ell < \nu\} .$$

Thus if $|B - \bigcup\{P(\alpha) \ ; \ \alpha \in A\}| = \kappa^+$, by the relation (1) there are $\ell < \nu$ and $H_o \in [A]^\kappa$, $H_1 \in [B - \bigcup\{P(\alpha) \ ; \ \alpha \in A\}]^\kappa$ such that $H_o \times H_1 \subseteq \Delta_\ell$.

This contradicts the choice of the partition (2), and so $|B - \cup\{P(\alpha) \; ; \; \alpha \in A\}| \leqq \kappa$. Similarly, if $A \subseteq \kappa^+$ and $B \in [\kappa^+]^\kappa$ then $|A - \cup\{Q(\beta) \; ; \; \beta \in B\}| \leqq \kappa$. The proof of 3.7 for regular κ now proceeds as did the proof of 3.3.

In the event that κ is singular, the proof goes as before, by applying Lemma 3.2 to the relation (1) and partition (2). Thus Theorem 3.7 is established.

The relation (1) above is proved in [6], Theorem 42(*), for the case $\nu = 2$ and $Cf(\kappa) = \omega$. In the following section, this result will be extended to cover the situation $\nu < \omega$ and $Cf(\kappa) = \omega$. To the best of my knowledge, when $Cf(\kappa) > \omega$ the truth of the relation (1) is still open.

§4. Results for cardinals cofinal with ω

The following sequence of lemmas is required in order to discuss the relation (1) of the previous theorem for the case $Cf(\kappa) = \omega$. The main result of this section is expressed in Theorem 4.11 below.

4.1 LEMMA Let $\kappa \geqq \omega$ and $Cf(\lambda) > \kappa$. Let $p \in \omega$ and suppose

(1) $$\binom{\kappa}{\lambda} \rightarrow \binom{\eta_\ell}{\zeta_\ell}^{1,1}_{\ell < \nu} .$$

Then it follows that

(2) $$\binom{\kappa}{\lambda} \rightarrow \left(\begin{matrix} p \\ \lambda \end{matrix} \binom{\eta_\ell}{\zeta_\ell}_{\ell < \nu}\right)^{1,1} .$$

PROOF: By induction on p. The case $p = 0$ is trivial. Assume (2) holds with $p = q$. Take a disjoint partition

$$\kappa \times \lambda = \Delta \cup \bigcup\{\Delta_\ell \; ; \; \ell < \nu\} ,$$

and suppose that for $\ell < \nu$ there are no sets $H_0 \subseteq \kappa$, $H_1 \subseteq \lambda$ having order types η_ℓ, ζ_ℓ respectively, for which $H_0 \times H_1 \subseteq \Delta_\ell$. By the inductive hypothesis, there must then be sets $S \in [\kappa]^q$ and $T \in [\lambda]^\lambda$ such that $S \times T \subseteq \Delta$.

Put $P(\alpha) = \{\beta \in \lambda \; ; \; \langle \alpha, \beta \rangle \in \Delta\}$. Then

$$(\kappa - S) \times (T - \bigcup\{P(\alpha) \; ; \; \alpha \in \kappa - S\}) \subseteq \bigcup\{\Delta_\ell \; ; \; \ell < \nu\} .$$

Now $|\kappa - S| = \kappa$, and so by the relation (1) and the choice of the partition, $|T - \bigcup\{P(\alpha) \; ; \; \alpha \in \kappa - S\}| < \lambda$. Hence $|T \cap \bigcup\{P(\alpha) \; ; \; \alpha \in \kappa - S\}| = \lambda$. Since $\mathrm{Cf}(\lambda) > \kappa$, there must be $\alpha \in \kappa - S$ for which $|T \cap P(\alpha)| = \lambda$. But

$$(S \cup \{\alpha\}) \times (T \cap P(\alpha)) \subseteq \Delta ,$$

and so since $|S \cup \{\alpha\}| = q + 1$, the induction step is complete. This establishes Lemma 4.1.

The following two lemmas are quoted without proof. They follow from [6], Lemma 8(*) and Lemma 3A(*), respectively.

4.2 LEMMA(*) Let κ and λ be infinite cardinals with $\mathrm{Cf}(\kappa) \neq \mathrm{Cf}(\lambda)$. Let $|A| = \kappa$, $|B| = \kappa^+$ and for each $b \in B$ let there be given a set $A_b \in [A]^{\geq \lambda}$. Then there is $B' \in [B]^{\kappa^+}$ such that $|\bigcap\{A_b \; ; \; b \in B'\}| \geq \lambda$.

4.3 LEMMA(*) Let κ be singular, and let the cardinals κ_σ for $\sigma < \mathrm{Cf}(\kappa)$ be such that $\sigma < \tau < \mathrm{Cf}(\kappa) \Rightarrow \kappa_\sigma < \kappa_\tau < \kappa$, and $\kappa = \sum \{\kappa_\sigma ; \sigma < \mathrm{Cf}(\kappa)\}$. Let $\nu < \kappa$ and suppose a disjoint partition is given, $\kappa \times \kappa = \cup\{\Delta_\ell ; \ell < \nu\}$. Then there are sets A_σ, B_σ and ordinals $h(\sigma,\tau) < \nu$ such that

$$A_\sigma \in [\kappa]^{\kappa_\sigma} \; ; \quad B_\sigma \in [\kappa]^{\kappa_\sigma} \; ; \quad \tau \neq \sigma \Rightarrow A_\sigma \cap A_\tau = B_\sigma \cap B_\tau = \emptyset \; ;$$

$$\sigma,\tau < \mathrm{Cf}(\kappa) \quad \Rightarrow \quad A_\sigma \times B_\tau \subseteq \Delta_{h(\sigma,\tau)} \; .$$

4.4 LEMMA(*) Let κ be infinite, and suppose

$$(1) \qquad \binom{\kappa}{\kappa^+} \rightarrow \binom{\eta_\ell}{\zeta_\ell}^{1,1}_{\ell < \nu} \; .$$

Then for any $\lambda < \kappa$,

$$(2) \qquad \binom{\kappa}{\kappa^+} \rightarrow \left(\begin{matrix}\lambda \\ \kappa^+\end{matrix}\binom{\eta_\ell}{\zeta_\ell}_{\ell < \nu}\right)^{1,1} \; .$$

PROOF: The case $\kappa = \omega$ is immediate from Lemma 4.1, so suppose $\kappa > \omega$. Define ι as follows: if $\mathrm{Cf}(\kappa) = \kappa$ then $\iota = \lambda$; otherwise $\iota = \lambda^+ +_C (\mathrm{Cf}(\kappa))^+$. In either case, $\lambda \leq \iota < \kappa$ and $\mathrm{Cf}(\iota) \neq \mathrm{Cf}(\kappa)$.

Take any partition, $\kappa \times \kappa^+ = \Delta \cup \cup\{\Delta_\ell ; \ell < \nu\}$. To establish (2), we must find sets $H_o \in [\kappa]^\lambda$, $H_1 \in [\kappa^+]^{\kappa^+}$ such that $H_o \times H_1 \subseteq \Delta$, or else $\ell < \nu$ and sets $H_o \subseteq \kappa$, $H_1 \subseteq \kappa^+$, having order types η_ℓ, ζ_ℓ respectively, such that $H_o \times H_1 \subseteq \Delta_\ell$.

Put $T = \{\beta \in \kappa^+ ; |Q(\beta)| \geq \iota\}$, where $Q(\beta) = \{\alpha \in \kappa ; \langle\alpha,\beta\rangle \in \Delta\}$. Suppose $|T| = \kappa^+$. Then by Lemma 4.2, there is $T' \in [T]^{\kappa^+}$ such that $|\bigcap\{Q(\beta) ; \beta \in T'\}| \geq \iota$. Since $\bigcap\{Q(\beta) ; \beta \in T'\} \times T' \subseteq \Delta$, there is nothing more to be shown. So consider the case when $|T| < \kappa^+$. Then $|\kappa^+ - T| = \kappa^+$. Since $\iota^+ \leq \kappa$, there is a disjoint partition $\kappa = \bigcup\{S_\mu ; \mu < \iota^+\}$ where $|S_\mu| = \kappa$ for each $\mu < \iota^+$. However, $|Q(\beta)| < \iota$ for $\beta \in \kappa^+ - T$. Hence for each $\beta \in \kappa^+ - T$, there is $\mu(\beta) < \iota^+$ for which $S_{\mu(\beta)} \cap Q(\beta) = \emptyset$. Moreover, there must be $\mu < \iota^+$ and $T_1 \in [\kappa^+ - T]^{\kappa^+}$ for which $\beta \in T_1 \Rightarrow \mu(\beta) = \mu$. Now $S_\mu \cap Q(\beta) = \emptyset$ for $\beta \in T_1$ and so $S_\mu \times T_1 \subseteq \bigcup\{\Delta_\ell ; \ell < \nu\}$. Since $|S_\mu| = \kappa$ and $|T_1| = \kappa^+$, the result follows from the relation (1). This concludes the proof.

4.5 THEOREM(*). If $Cf(\kappa) = \omega$, then for any $p < \omega$,

$$\binom{\kappa}{\kappa^+} \to \left(\kappa \atop \kappa^+ \binom{\kappa}{\omega}_p\right)^{1,1} .$$

PROOF: By induction on p . The case $p = 0$ is trivial. Assume the result is true for some $q < \omega$. Take a partition

$$\kappa \times \kappa^+ = \Delta \cup \bigcup\{\Delta_\ell ; \ell \leq q\} ,$$

and suppose that there are no sets $H_o \in [\kappa]^\kappa$, $H_1 \in [\kappa^+]^{\kappa^+}$ with $H_o \times H_1 \subseteq \Delta$, nor sets $H_o \in [\kappa]^\kappa$, $H_1 \in [\kappa^+]^\omega$ with $H_o \times H_1 \subseteq \Delta_\ell$ for any $\ell \in \{1,\ldots,q\}$. We must find sets $H_o \in [\kappa]^\kappa$, $H_1 \in [\kappa^+]^\omega$ with $H_o \times H_1 \subseteq \Delta_o$.

Write $\kappa = \sum\{\kappa_r ; r < \omega\}$ where $\kappa_r < \kappa$ for $r < \omega$. Define inductively sets D_r, $B_r \in [\kappa^+]^{\kappa^+}$, $A_r \in [\kappa]^\kappa$ and $y_r \in B_r$ for $r < \omega$ as follows:

By Lemma 4.4 and the inductive hypothesis, choose

$$D_o \in [\kappa]^{\kappa_o} \text{ and } B_o \in [\kappa^+]^{\kappa^+} \text{ such that } D_o \times B_o \subseteq \Delta_o .$$

By Lemma 3.1 and the inductive hypothesis, choose

$$A_o \in [\kappa]^\kappa \text{ and } y_o \in B_o \text{ such that } A_o \times \{y_o\} \subseteq \Delta_o .$$

Generally, by Lemma 4.4 and the inductive hypothesis, choose

$$D_r \in [A_{r-1}]^{\kappa_r} \text{ and } B_r \in [B_{r-1} - \{y_{r-1}\}]^{\kappa^+} \text{ such that } D_r \times B_r \subseteq \Delta_o .$$

By Lemma 3.1 and the inductive hypothesis, choose

$$A_r \in [A_{r-1}]^\kappa \text{ and } y_r \in B_r \text{ such that } A_r \times \{y_r\} \subseteq \Delta_o .$$

Put $H_o = \cup\{D_r ; r < \omega\}$ and $H_1 = \{y_r ; r < \omega\}$. Then $|H_o| = \kappa$ and $|H_1| = \omega$. Note that if $r \leq s$, then $D_r \times \{y_s\} \subseteq D_r \times B_s \subseteq D_r \times B_r \subseteq \Delta_o$. And if $r > s$, then $D_r \times \{y_s\} \subseteq A_s \times \{y_s\} \subseteq \Delta_o$. Hence $H_o \times H_1 \subseteq \Delta_o$, and the induction step is complete. This proves Theorem 4.5.

4.6 COROLLARY(*). If $Cf(\kappa) = \omega$, then for any $p,q < \omega$ and any $\lambda < \kappa$,

$$\begin{pmatrix}\kappa\\ \kappa^+\end{pmatrix} \rightarrow \left(\begin{pmatrix}\kappa\\ \kappa^+\end{pmatrix}\begin{pmatrix}\lambda\\ \kappa^+\end{pmatrix}_p \begin{pmatrix}\kappa\\ \omega\end{pmatrix}_q\right)^{1,1} .$$

Perusal of the proofs of the preceding results shows that the GCH is not required for the special case $\kappa = \omega$. Thus we obtain

4.7 COROLLARY For any $p < \omega$,

$$\binom{\omega}{\omega_1} \rightarrow \left(\begin{matrix}\omega \\ \omega_1\end{matrix}\ \binom{\omega}{\omega}_p\right)^{1,1} .$$

4.8 LEMMA(*) Let $\kappa \geqq \omega$ and $p < \omega$. Take any partition

$$\kappa \times \kappa = \Delta \cup \bigcup\{\Delta_\ell \ ; \ \ell < p\} .$$

If there are no sets $H_o, H_1 \in [\kappa]^\kappa$ with $H_o \times H_1 \subseteq \Delta$, then for some $\ell < p$ there are $H \in [\kappa]^\kappa$ and $\alpha \in \kappa$ such that either $H \times \{\alpha\} \subseteq \Delta_\ell$ or else $\{\alpha\} \times H \subseteq \Delta_\ell$.

PROOF: By an obvious induction. The case $p = 1$ comes as a special case of Theorem 38(*) of [6].

4.9 THEOREM(*) Suppose κ is singular; let $\lambda < \kappa$ and $p < \omega$. Then

(1) $$\binom{\kappa}{\kappa^+} \rightarrow \binom{\kappa}{\lambda}_p^{1,1} .$$

PROOF: By induction on p. The case $p = 1$ is trivial, so suppose the result is true for some q with $1 \leqq q < \omega$.

We may suppose that $Cf(\kappa) < \lambda < \kappa$, and that λ is regular. Put $\rho = Cf(\kappa)$. Choose cardinals κ_σ for $\sigma < \rho$ such that $\kappa = \sum\{\kappa_\sigma \ ; \ \sigma < \rho\}$

and $\lambda < \kappa_\sigma < \kappa_\tau < \kappa$ when $\sigma < \tau < \rho$. Take any disjoint partition

(2) $$\kappa \times \kappa^+ = \cup\{\Delta_\ell \ ; \ \ell \leq q\} ,$$

and suppose that whenever $1 \leq \ell \leq q$ then there are no sets $H_o \in [\kappa]^\kappa$ and $H_1 \in [\kappa^+]^\lambda$ for which $H_o \times H_1 \subseteq \Delta_\ell$. We must find $H_o \in [\kappa]^\kappa$ and $H_1 \in [\kappa^+]^\lambda$ such that $H_o \times H_1 \subseteq \Delta_o$.

Suppose that ι is regular with $\rho < \iota < \kappa$. Take any $A \in [\kappa]^\kappa$ and $B \in [\kappa^+]^{\kappa^+}$. Then by Lemma 3.2, there are $A^* \subseteq [A]^\iota$ with $|A^*| \leq \kappa$ and a map f from A^* to the subsets of B such that

(3) $$a \in A^* \Rightarrow a \times f(a) \subseteq \Delta_o \quad ; \quad |B - \cup\{f(a) \ ; \ a \in A^*\}| \leq \kappa .$$

Define inductively a ramification system $\underline{\underline{R}}$ on κ^+ of length ρ as follows. Take $\sigma < \rho$ and $X \in SEQ_\sigma$. Suppose that $S'(X)$ has already been defined. If $|S'(X)| \leq \kappa$, put $F(X) = S'(X)$ and $n(X) = 0$. If $|S'(X)| = \kappa^+$, put $n(X) = \kappa$ and choose $R(X) \in [S'(X)]^\kappa$. Then ~~by~~ by applying (3) with $A = \kappa$, $B = S'(X) - R(X)$ and $\iota = \kappa_\sigma$, one can find a set $A^*(X) \subseteq [\kappa]^{\kappa_\sigma}$ with $|A^*(X)| \leq \kappa$, and a map f_X from $A^*(X)$ to PB , which together satisfy the appropriate form of (3). Write $A^*(X) = \{a(Y) \ ; \ \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}$. For $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < \kappa$, put $S(Y) = f_X(a(Y))$. Define

$$F(X) = R(X) \cup (S'(X) - \cup\{S(Y) \ ; \ \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}) .$$

This defines $\underline{\underline{R}}$. Further

$$|F| = |R| +_C |(S' - R) - \bigcup\{S(Y) \; ; \; \bar{Y} = X \text{ and } Y(\sigma) < \kappa\}|$$

$$= |R| +_C |B - \bigcup\{f(a) \; ; \; a \in A^*\}| \; ,$$

and so $|F| = \kappa$, by (3). Hence Lemma 1.1(iii) applies to $\underline{\underline{R}}$, so choose a sequence $X \in N \cap SEQ_\rho$ such that $S'(X) \neq \emptyset$. Then for each $\sigma < \rho$, it must be that $|S'(X \restriction \sigma)| = \kappa^+$, and so always $R(X \restriction \sigma)$ is defined. Choose a bijection $g_{X\restriction\sigma}$ from $[\kappa]^{\kappa_\sigma}$ onto $[R(X \restriction \sigma)]^{\kappa_\sigma}$. Put

$$g(\sigma) = g_{X\restriction\sigma}(a(X \restriction \sigma + 1)) \; ;$$

then always $g(\sigma) \in [R(X \restriction \sigma)]^{\kappa_\sigma}$. Put $A = \bigcup\{a(X \restriction \sigma + 1) \; ; \; \sigma < \rho\}$ and $B = \bigcup\{g(\sigma) \; ; \; \sigma < \rho\}$. Then $A \in [\kappa]^\kappa$ and $B \in [\kappa^+]^\kappa$. Moreover, if $\sigma < \tau < \rho$ then

$$g(\tau) \subseteq R(X \restriction \tau) \subseteq S'(X \restriction \tau) \subseteq S(X \restriction \sigma + 1) \; .$$

However, $a(X \restriction \sigma + 1) \times S(X \restriction \sigma + 1) \subseteq \Delta_0$ by (3), and so

(4) $$\sigma < \tau < \rho \;\Rightarrow\; a(X \restriction \sigma + 1) \times g(\tau) \subseteq \Delta_0 \; .$$

The partition (2) restricts to a partition of $A \times B$. Hence by Lemma 4.3, there are sets $A_\sigma \in [A]^{\kappa_\sigma}$, $B_\sigma \in [B]^{\kappa_\sigma}$ and numbers $h(\sigma,\tau) \leq q$ such that if $\sigma < \tau < \rho$ then $A_\sigma \cap A_\tau = B_\sigma \cap B_\tau = \emptyset$, and also

(5) $$\sigma,\tau < \rho \;\Rightarrow\; A_\sigma \times B_\tau \subseteq \Delta_{h(\sigma,\tau)} \; .$$

For $\ell \leq q$, put $\Delta'_\ell = \{\langle\sigma,\tau\rangle \in \rho \times \rho \; ; \; h(\sigma,\tau) = \ell\}$. Thus $\rho \times \rho = \bigcup\{\Delta'_\ell \; ; \; \ell \leq q\}$. Apply Lemma 4.8 to this partition of $\rho \times \rho$.

There are three cases to consider.

Case 1: There are $H \in [\rho]^{\rho}$, $\tau < \rho$ and ℓ with $1 \leqq \ell \leqq q$ such that $h(\sigma,\tau) = \ell$ for all $\sigma \in H$. Put $A' = \bigcup\{A_{\sigma} ; \sigma \in H\}$; then $|A'| = \kappa$, and $A' \times B_{\tau} \subseteq \Delta_{\ell}$ by (5). Since $\lambda < \kappa_{\tau}$, this contradicts the choice of the partition (2).

Case 2: There are $H \in [\rho]^{\rho}$, $\sigma < \rho$ and ℓ with $1 \leqq \ell \leqq q$ such that $h(\sigma,\tau) = \ell$ for all $\tau \in H$. Put $B' = \bigcup\{B_{\tau} ; \tau \in H\}$; then $|B'| = \kappa$, and $A_{\sigma} \times B' \subseteq \Delta_{\ell}$ by (5). Choose $\alpha \in A_{\sigma}$. Then $\{\alpha\} \times B' \subseteq \Delta_{\ell}$. Further, there is $\sigma_1 < \rho$ such that $\alpha \in a(X \restriction \sigma_1 + 1)$. Hence it follows from (4) that $B' \subseteq \bigcup\{g(\tau) ; \tau \leqq \sigma_1\}$. However, this yields the contradiction

$$\kappa = |B'| \leqq \sum \{|g(\tau)| ; \tau \leqq \sigma_1\} = \sum \{\kappa_{\tau} ; \tau \leqq \sigma_1\} < \kappa .$$

Case 3: This case must prevail. There are $K_0, K_1 \in [\rho]^{\rho}$ such that $h(\sigma,\tau) = 0$ for $\sigma \in K_0$ and $\tau \in K_1$. Put $H_0 = \bigcup\{A_{\sigma} ; \sigma \in K_0\}$ and $H_1 = \bigcup\{B_{\tau} ; \tau \in K_1\}$. Then $H_0 \in [\kappa]^{\kappa}$, $H_1 \in [\kappa^+]^{\kappa}$, and $H_0 \times H_1 \subseteq \Delta_0$ by (5). Since $\lambda < \kappa$, the induction step is complete. This proves Theorem 4.9.

4.10 LEMMA(*) Let κ be uncountable with $Cf(\kappa) = \omega$. Take $p,q < \omega$ with $q \geqq 1$. Suppose for all $\iota < \kappa$ that

$$(1) \qquad \binom{\kappa}{\kappa^+} \to \left(\binom{\kappa}{\kappa}_p \binom{\kappa}{\iota}_q\right)^{1,1} .$$

Then for all $\lambda < \kappa$,

(2) $$\binom{\kappa}{\kappa^+} \to \left(\binom{\kappa}{\kappa}_{p+1} \binom{\kappa}{\lambda}_{q-1}\right)^{1,1} .$$

PROOF: Let $\lambda < \kappa$ be given. Choose cardinals $\kappa_r < \kappa$ for $r < \omega$ such that $\kappa = \sum \{\kappa_r ; r < \omega\}$ and $\lambda < \kappa_o < \kappa_1 < \cdots$. Take any partition

(3) $$\kappa \times \kappa^+ = \cup\{\Delta_\ell ; \ell < p + 1\} \cup \cup\{\Gamma_k ; k < q - 1\} ,$$

and suppose that for all $\ell < p$ there are no sets $H_o \in [\kappa]^\kappa$, $H_1 \in [\kappa^+]^\kappa$ with $H_o \times H_1 \subseteq \Delta_\ell$, and also that for all $k < q - 1$ there are no sets $H_o \in [\kappa]^\kappa$, $H_1 \in [\kappa^+]^\lambda$ with $H_o \times H_1 \subseteq \Gamma_k$. To establish (2) we must find sets $H_o \in [\kappa]^\kappa$, $H_1 \in [\kappa^+]^\kappa$ such that $H_o \times H_1 \subseteq \Delta_p$.

From the relation (1), it follows that

$$\binom{\kappa}{\kappa^+} \to \left(\binom{\kappa}{\kappa}_{p} \binom{\kappa}{\lambda}_{q-1}\right)^{1,1} ,$$

and so by Lemma 4.4, we obtain, for any $\iota < \kappa$,

(4) $$\binom{\kappa}{\kappa^+} \to \left(\binom{\kappa}{\kappa}_{p} \begin{matrix}\iota\\ \kappa^+\end{matrix} \binom{\kappa}{\lambda}_{q-1}\right)^{1,1} .$$

Take any $r < \omega$ and any sets $A \in [\kappa]^\kappa$ and $B \in [\kappa^+]^{\kappa^+}$. By (4) and the choice of the partition (3), there are A' , B' such that

(5) $$A' \in [A]^{\kappa_r} , \quad B' \in [B]^{\kappa^+} , \quad A' \times B' \subseteq \Delta_p .$$

Since $\kappa_r < \kappa$, by (1) and the choice of the partition (3), there are A'' , B'' such that

(6) $$A'' \in [A]^{\kappa} \quad , \quad B'' \in [B]^{\kappa_r} \quad , \quad A'' \times B'' \subseteq \Delta_p \ .$$

Use induction over $r < \omega$ to define the sets $A_r^* \in [\kappa]^{\kappa}$, $B_r^* \in [\kappa^+]^{\kappa^+}$, A_r and B_r as follows. Put $A_o^* = \kappa$ and $B_o^* = \kappa^+$. Suppose $A_r^* \in [\kappa]^{\kappa}$ and $B_r^* \in [\kappa^+]^{\kappa^+}$ have already been defined for some $r < \omega$. By (5), choose A_r and B_{r+1}^* so that

$$A_r \in [A_r^*]^{\kappa_r} \quad , \quad B_{r+1}^* \in [B_r^*]^{\kappa^+} \quad , \quad A_r \times B_{r+1}^* \subseteq \Delta_p \ .$$

By (6), choose A_{r+1}^* and B_r so that

$$A_{r+1}^* \in [A_r^*]^{\kappa} \quad , \quad B_r \in [B_{r+1}^*]^{\kappa_r} \quad , \quad A_{r+1}^* \times B_r \subseteq \Delta_p \ .$$

Then for all $r < \omega$, the following hold:

$$A_{r+1}^* \subseteq A_r^* \ , \quad B_{r+1}^* \subseteq B_r^* \ , \quad |A_r^*| = \kappa \ , \quad |B_r^*| = \kappa^+ \ , \quad |A_r| = \kappa_r \ , \quad |B_r| = \kappa_r \ .$$

Put $A = \cup\{A_r \ ; \ r < \omega\}$ and $B = \cup\{B_r \ ; \ r < \omega\}$. Then $A \in [\kappa]^{\kappa}$ and $B \in [\kappa^+]^{\kappa}$. Thus to prove the lemma, it suffices to show

(7) $$A \times B \subseteq \Delta_p \ .$$

Take $r, s < \omega$. We must show that $A_r \times B_s \subseteq \Delta_p$. However, if $r \leqq s$ then $B_s \subseteq B_{s+1}^* \subseteq B_{r+1}^*$, and so $A_r \times B_s \subseteq A_r \times B_{r+1}^* \subseteq \Delta_p$. If $r > s$ then $A_r \subseteq A_r^* \subseteq A_{s+1}^*$, and so $A_r \times B_s \subseteq A_{s+1}^* \times B_s \subseteq \Delta_p$. This establishes (7), and completes the proof.

4.11 THEOREM(*) Suppose $\mathrm{Cf}(\kappa) = \omega$. Then for any $p < \omega$

(1) $$\binom{\kappa}{\kappa^+} \to \binom{\kappa}{\kappa}^{1,1}_p .$$

PROOF: The case $\kappa = \omega$ is a consequence of Corollary 4.7, so suppose $\kappa > \omega$. Then by Theorem 4.9, for all $\lambda < \kappa$,

$$\binom{\kappa}{\kappa^+} \to \binom{\kappa}{\lambda}^{1,1}_p .$$

Repeated applications of Lemma 4.10 now yield the result.

The result of Theorem 4.11 is clearly the best possible, for if $\mathrm{Cf}(\kappa) = \omega$ then it is easy to see that

$$\binom{\kappa}{\kappa^+} \not\to \binom{\kappa}{1}^{1,1}_\omega .$$

There is the following corollary to Theorem 4.11. (Compare with Corollary 3.5.)

4.12 COROLLARY(*) Suppose $\mathrm{Cf}(\kappa) = \omega$. Then $\kappa^+ \to {}^2(\kappa)^1_p$ for any $p < \omega$.

Thus for partitions of $\kappa^+ \times \kappa^+$, the most simple open questions are seen to be:

4.13 PROBLEM(*) Is $\kappa^+ \to {}^2(\kappa)^1_3$ true if $\mathrm{Cf}(\kappa) > \omega$? Is $\kappa^+ \to {}^2(\kappa)^1_\omega$ true if $\mathrm{Cf}(\kappa) = \omega$?

I cannot answer this problem even for $\kappa = \omega_0$ or $\kappa = \omega_1$.

I conclude this section with a further application of the relation (1) of Theorem 4.11.

4.14 **THEOREM(*)** Suppose either one of the following conditions is satisfied:

(1) $$\kappa \to (\kappa)^2_\nu ,$$

(2) $$\binom{\kappa}{\kappa^+} \to \binom{\kappa}{\kappa}^{1,1}_\nu .$$

Then the following relation holds.

(3) $$\binom{\kappa^{++}}{\kappa^{++}} \to \binom{\kappa}{\kappa}^{2,1}_\nu .$$

PROOF: For the purposes of this proof, if γ is an ordinal number, let γ^* represent the ordinal sum $\sum \{\beta \; ; \; \beta < \gamma\}$, with the convention that $0^* = 0$.

Take any partition $[\kappa^{++}]^2 \times \kappa^{++} = \cup\{\Delta_\ell \; ; \; \ell < \nu\}$. Define inductively a ramification system $\underline{\underline{R}}$ on κ^{++} of length $\rho = \kappa^+$, together with elements $x(X), y(X) \in \kappa^{++}$; the latter whenever $X \in SEQ_\sigma$ for σ such that there is $\gamma < \kappa^+$ with $\sigma = 2\cdot\gamma^*$ or $\sigma = 2\cdot\gamma^* + 1$, respectively.

Choose $x(\emptyset) \in \kappa^{++}$ and put $F(\emptyset) = \{x(\emptyset)\}$. Put $n(\emptyset) = \nu$ and for $\alpha < \nu$ define $S(\{\langle 0,\alpha\rangle\}) = \kappa^{++} - F(\emptyset)$. For $X \in SEQ_1$, choose $y(X) \in S'(X)$. Put $F(X) = \{y(X)\}$ and $n(X) = \nu$. For $\alpha < \nu$ define $S(X \cup \{\langle 1,\alpha\rangle\}) = S'(X) - F(X)$.

Suppose $1 < \sigma < \kappa^+$, take $X \in SEQ_\sigma$ and suppose $S'(X)$ has already been defined. If $S'(X) = \emptyset$, put $F(X) = \emptyset$ and $n(X) = 0$. Otherwise, let σ' be such that either $\sigma = 2\cdot\sigma'$ or $\sigma = 2\cdot\sigma' + 1$. Let γ be least such that $(\gamma+1)^* > \sigma'$. Then there is $\zeta < \gamma$ for which $\sigma' = \gamma^* + \zeta$. Consider two cases.

<u>Case 1</u>: $\sigma = 2\cdot\sigma'$

If $\zeta = 0$, choose $x(X) \in S'(X)$ and put $F(X) = \{x(X)\}$; otherwise $F(X) = \emptyset$. In either event, put $n(X) = \nu$ and for $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < \nu$, put

$$S(Y) = \{z \in S'(X) - F(X) ; \langle\{z, x(X \restriction 2\cdot\gamma^*)\}, y(X \restriction 2\cdot\zeta^* + 1)\rangle \in \Delta_{Y(\sigma)}\} .$$

<u>Case 2</u>: $\sigma = 2\cdot\sigma' + 1$

If $\zeta = 0$, choose $y(X) \in S'(X)$ and put $F(X) = \{y(X)\}$; otherwise $F(X) = \emptyset$. In either event, put $n(X) = \nu$. For $Y \in SEQ_{\sigma+1}$ such that $\bar{Y} = X$ and $Y(\sigma) < \nu$, put

$$S(Y) = \{z \in S'(X) - F(X) ; \langle\{x(X \restriction 2\cdot\zeta^*), x(X \restriction 2\cdot\gamma^*)\}, z\rangle \in \Delta_{Y(\sigma)}\} .$$

Since clearly $\nu < \kappa$, Lemma 1.1(iii) applies to $\underline{\underline{R}}$. Choose a sequence $X \in N \cap SEQ_\rho$ such that $S'(X) \neq \emptyset$. For $\sigma < \kappa^+$, put $x_\sigma = x(X \restriction 2\cdot\sigma^*)$ and $y_\sigma = y(X \restriction 2\cdot\sigma^* + 1)$, noting that x_σ and y_σ are always defined. Moreover,

(4) $$\mu < \sigma < \tau < \kappa^+ \Rightarrow \langle\{x_\mu, x_\sigma\}, y_\tau\rangle \in \Delta_{X(2\cdot(\sigma^*+\mu)+1)} ,$$

(5) $\tau < \mu < \sigma < \kappa^+ \Rightarrow \langle\{x_\mu, x_\sigma\}, y_\tau\rangle \in \Delta_{X(2\cdot(\mu^*+\tau))}$.

For suppose $\mu < \sigma < \tau < \kappa^+$. Then

$$y_\tau \in F(X \restriction 2\cdot\tau^* + 1) \subseteq S'(X \restriction 2\cdot\tau^* + 1) \subseteq S(X \restriction 2\cdot(\sigma^* + \mu) + 2) \quad ,$$

and so (4) follows. Similarly, if $\tau < \mu < \sigma < \kappa^+$, then

$$x_\sigma \in F(X \restriction 2\cdot\sigma^*) \subseteq S(X \restriction 2\cdot(\mu^* + \tau) + 1) \quad ,$$

and so (5) follows.

Suppose relation (1) holds. Define a partition $[\kappa]^2 = \cup\{\Gamma_\ell \; ; \; \ell < \nu\}$ by, for $\ell < \nu$,

$$\{\mu,\sigma\}_< \in \Gamma_\ell \iff X(2\cdot(\sigma^* + \mu) + 1) = \ell \; .$$

Then there is $H \in [\kappa]^\kappa$ such that for some $\ell < \nu$,

$$\{\mu,\sigma\}_< \in [H]^2 \Rightarrow X(2\cdot(\sigma^* + \mu) + 1) = \ell \; .$$

Put $H_o = \{x_\sigma \; ; \; \sigma \in H\}$ and $H_1 = \{y_\tau \; ; \; \kappa \leqq \tau < \kappa\cdot 2\}$. Then by Lemma 1.1(i), $|H_o| = |H_1| = \kappa$, and it follows from (4) that $[H_o]^2 \times H_1 \subseteq \Delta_\ell$. Thus the relation (3) holds.

On the other hand, suppose relation (2) is satisfied. Define a partition $(\kappa^+ - \kappa) \times \kappa = \cup\{\Lambda_\ell \; ; \; \ell < \nu\}$ by, for $\ell < \nu$,

$$\langle\mu,\tau\rangle \in \Lambda_\ell \iff X(2\cdot(\mu^* + \tau)) = \ell \quad .$$

Then there are $K_o \in [\kappa^+ - \kappa]^\kappa$ and $K_1 \in [\kappa]^\kappa$ such that $K_o \times K_1 \subseteq \Delta_\ell$ for some $\ell < \nu$. Put $H_o = \{x_\mu ; \mu \in K_o\}$ and $H_1 = \{y_\tau ; \tau \in K_1\}$. Then $|H_o| = |H_1| = \kappa$, and it follows from (5) that $[H_o]^2 \times H_1 \subseteq \Delta_\ell$. Thus the relation (3) again holds. This completes the proof.

In the case that relation (2) holds, an alternative proof, similar to that of the Stepping-out Lemma, 3.4, may be given.

In view of Theorem 4.11, from Theorem 4.14 comes:

4.15 COROLLARY(*) If $Cf(\kappa) = \omega$, then the relation (3) of Theorem 4.14 holds for any $\nu < \omega$.

Since, assuming the GCH, for any $\nu < \kappa^+$ the relation $\kappa^{+++} \to (\kappa^+)^3_\nu$ holds, the method of Theorem II.1.1 shows that, for $\nu \leq \kappa$,

(6) $$\begin{pmatrix} \kappa^{+++} \\ \kappa^{+++} \end{pmatrix} \to \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}^{2,1}_\nu .$$

Thus, for a few special cases, a better result is at hand. I do not know if (6) always holds with κ^{+++} replaced by κ^{++}.

CHAPTER V. FORCING WITH A CLASS OF CONDITIONS

This chapter is devoted to a discussion of the forcing technique of Cohen [3], as applied to the case when the collection of forcing conditions is a proper class in the ground model. The present formulation was inspired by a construction used by Shoenfield [20] for a set of conditions. (A related exposition occurs in Chapter 9 of his book [19].) The resulting construction is rather different from that of the standard approach (for example [4]) or from the development through Boolean valued models. The advantages are, perhaps, not very great (apart from the interest of seeing that it can be done at all). A few minor simplifications result. The construction will be used to re-prove, in a common setting, theorems of Easton and Jensen.

§1. The basic construction

Let M be a countable, transitive, standard model of $ZF + AC$, (a CTM for short). By a *class in M* is meant a subset X of M such that for some formula $\Phi(v,v_1,\dots,v_n)$ of $ZF + AC$ with at most the free variables shown, and for some $a_1,\dots,a_n \in M$,

$$X = \{a \in M \ ; \ M \models \Phi[a,a_1,\dots,a_n]\} .$$

Let C be any class in M such that $\emptyset \in C$. Henceforth, the letters p,q,r,s will denote members of C. They are to be called *conditions*, and if $p \supseteq q$ then we say that p *extends* q. Write $p \between q$, and say that p and q are *compatible*, if there is r such that $r \supseteq p$ and $r \supseteq q$.

For a condition p and an ordinal α, put

$$(p)_\alpha = \{x \in p \ ; \ rk(x) < \alpha\} .$$

We shall assume that C is closed under the operations $(\cdot)_\alpha$, and that if $rk(q) \leqq \alpha$ and $(p)_\alpha \subseteq q$ then there is $r \in C$ such that $p \cup q \subseteq r$. The following simple properties of $(\cdot)_\alpha$ are all that will be required:

(i) $$p \supseteq (p)_\alpha ,$$

(ii) $$p \supseteq q \;\Rightarrow\; (p)_\alpha \supseteq (q)_\alpha \quad ,$$

(iii) $$(p)_\alpha \not\perp (q)_\alpha \;\Rightarrow\; p \not\perp (q)_\alpha \quad ,$$

(iv) $$\alpha \leqq \beta \Rightarrow ((p)_\alpha)_\beta = ((p)_\beta)_\alpha = (p)_\alpha \quad ,$$

(v) $$rk(p) \leqq \alpha \;\Rightarrow\; (p)_\alpha = p \quad ,$$

(vi) $$\{p \; ; \; p = (p)_\alpha\} \in M \quad .$$

The use of set inclusion to define "p extends q" and the use of the rank function in defining the operations $(\cdot)_\alpha$ are both in the interests of simplicity. The following construction is valid for any pre-ordered class (with greatest element) of M, provided that suitable operations $(\cdot)_\alpha$ are available.

Note that although possibly $C \notin M$, for all p it is the case that $\{q \in C \; ; \; p \supseteq q\} \in M$.

Make the definition, for each set x,

$$\gamma(x) \quad = \quad \sup\{rk(y) \; ; \; y \in Rx\} \quad .$$

For $x,y \in M$ and $p \in C$, define a relation ϵ_p by:

$$x \epsilon_p y \iff rk(p) < \gamma(y) \text{ and } \forall q \supseteq p(rk(q) < \gamma(y) \Rightarrow \langle x,q\rangle \in y) \quad .$$

Thus if $x \in_p y$, then $\langle x,p\rangle \in y$. And if $x \in_p y$ and $q \supseteq p$, then provided $rk(q) < \gamma(y)$ also $x \in_q y$. Further, if $\gamma(y) \leqq \alpha$ then by virtue of (v) above, $x \in_p y \Rightarrow x \in_{(p)_\alpha} y$. Observe that if $x \in_p y$ then $rk(x) < rk(y)$, and so $\in_p$ is a well founded relation.

For any non-empty set $G \subseteq C$ (where G need not necessarily be a class in M), define a relation $\in_G$ by:

$$x \in_G y \iff \exists p \in G(x \in_p y) \quad .$$

Then again $\in_G$ is rank decreasing, and so well founded.

Collapse the structure $\langle M, \in_G\rangle$ to a transitive structure by a collapsing map $\bar{\cdot}^G$, defined by induction on $\in_G$: for $x \in M$,

$$\bar{x}^G = \{\bar{y}^G \,;\, y \in_G x\} \, .$$

Put $M[G] = \{\bar{x}^G \,;\, x \in M\}$, and consider the structure $\langle M[G], \in\rangle$.

1.1 LEMMA

(a) $M[G]$ is transitive.

(b) If $\emptyset \in G$, then $M \subseteq M[G]$.

(c) If $\emptyset \in G$, then M and $M[G]$ have the same ordinals.

(d) If $\forall p \in G \, \forall q \subseteq p(q \in G)$, then G is a class in $M[G]$.

PROOF:

(a) By the definition of $M[G]$.

(b) For $x \in M$, define by $\in$-induction inside M,

$$\hat{x} = \{\hat{y} \,;\, y \in x\} \times \{p \,;\, rk(p) \leqq \gamma(x)\} \, ,$$

so that $\hat{x} \in M$. Use $\in$-induction to show that $\bar{\hat{x}} = x$. (When there can be no confusion, I write $\bar{\ }$ for $\bar{\ }^G$.) So suppose for all $y \in x$ that $\bar{\hat{y}} = y$. Now $\bar{\hat{x}} = \{\bar{z} \; ; \; z \in_G \hat{x}\} = \{\bar{z} \; ; \; \exists p \in G(z \in_p \hat{x})\}$. But if $\exists p \in G(z \in_p \hat{x})$, there is p such that $\langle z,p \rangle \in \hat{x}$ and so $z = \hat{y}$ for some $y \in x$. And if $y \in x$, then $\hat{y} \in D\hat{x}$ and so for any q , it follows that $\langle \hat{y},q \rangle \in \hat{x}$ provided only that $rk(q) \leqq \gamma(x)$. In particular, $\hat{y} \in_{\emptyset} \hat{x}$, so since $\emptyset \in G$ the condition $\exists p \in G(\hat{y} \in_p \hat{x})$ is satisfied. Thus $\bar{\hat{x}} = \{\bar{\hat{y}} \; ; \; y \in x\} = \{y \; ; \; y \in x\} = x$. Hence $\bar{\hat{x}} = x$ for all $x \in M$, and thus $M \subseteq M[G]$.

(c) Use $\in_G$-induction to show that $rk(\bar{x}) \leqq rk(x)$. Now $rk(\bar{x}) = \sup\{rk(\bar{y}) \; ; \; y \in_G x\} \leqq \sup\{rk(y) \; ; \; y \in_G x\} \leqq rk(x)$, since if $y \in_G x$ then $rk(y) < rk(x)$. Hence if α is an ordinal in $M[G]$, say $\alpha = \bar{x}$, then $\alpha \leqq rk(x)$ and $rk(x) \in M$, so $\alpha \in M$. And by (b), if α is an ordinal in M , then certainly $\alpha \in M[G]$. Thus, since "ordinal" is absolute for transitive sets, the ordinals of M and of $M[G]$ are the same.

(d) For each ordinal $\alpha \in M[G]$, put $G(\alpha) = \{w \in G \; ; \; rk(w) < \alpha\}$. We show that each $G(\alpha) \in M[G]$. Given α , put $x_\alpha = \{\langle \hat{p},q \rangle \; ; \; q \supseteq p \text{ and } rk(q) < \alpha\}$, so $x_\alpha \in M$. Then $\gamma(x_\alpha) = \sup\{rk(q) \; ; \; rk(q) < \alpha\} \leqq \alpha$. Hence

$$\begin{aligned}
z \in_G x_\alpha &\iff \exists p \in G(rk(p) < \alpha \text{ and } \forall q \supseteq p(rk(q) < \alpha \Rightarrow \langle z,q \rangle \in x_\alpha)) \\
&\iff \exists p \in G(rk(p) < \alpha \text{ and } \forall q \supseteq p(rk(q) < \alpha \Rightarrow \exists r(q \supseteq r \text{ and } z = \hat{r}))) \\
&\iff \exists p \in G(rk(p) < \alpha \text{ and } \exists r(p \supseteq r \text{ and } z = \hat{r})) \\
&\iff \exists r \in G(rk(r) < \alpha \text{ and } z = \hat{r}) .
\end{aligned}$$

The last equivalence uses the condition assumed satisfied by G . Thus

$$\bar{x}_\alpha = \{\bar{z} \ ; \ z \in_G x_\alpha\} = \{\bar{\hat{r}} \ ; \ r \in G \text{ and } rk(r) < \alpha\}$$
$$= \{r \ ; \ r \in G \text{ and } rk(r) < \alpha\} .$$

Hence $\bar{x}_\alpha = G(\alpha)$, and so $G(\alpha) \in M[G]$. It is now immediate that G is a class in $M[G]$, since

$$p \in G \iff \exists\alpha(\ \alpha \text{ is an ordinal and } p \in G(\alpha)\ ) .$$

This completes the proof of Lemma 1.1.

1.2 DEFINITION A non-empty subset G of C is called C-generic over M if

(1) $$p \supseteq q \text{ and } p \in G \Rightarrow q \in G ,$$

(2) $$p,q \in G \Rightarrow \exists r \in G(r \supseteq p \text{ and } r \supseteq q) ,$$

(3) If X is a class in M such that $\forall p\, \exists q \supseteq p(q \in X)$, then $G \cap X \neq \emptyset$.

A class X with the property $\forall p\, \exists q \supseteq p(q \in X)$ will be called a dense class.

It is easily seen that (3) of Definition 1.2 is equivalent to:

(3') For all classes X in M there is $p \in G$ such that
$$p \in X \text{ or } \forall q \supseteq p(q \notin X) .$$

This leads to:

1.3 THEOREM For any $p \in C$, there is a class $G \subseteq C$ which is C-generic over M and has p as a member.

PROOF: Since M is countable and the language is countable, the classes in M can be enumerated, $\{X_n ; n \in \omega\}$ say. Define a sequence of conditions $\langle p_n ; n \in \omega \rangle$ so that $p_o = p$ and

$$p_{n+1} \supseteq p_n \quad ; \quad p_{n+1} \in X_n \quad \text{or} \quad \forall q \supseteq p_{n+1}(q \notin X_n) \ .$$

Since either $\exists q \supseteq p_n(q \in X_n)$ or $\forall q \supseteq p_n(q \notin X_n)$, such a p_{n+1} can always be found. Put $G = \{q ; \exists n \in \omega(p_n \supseteq q)\}$. Then G is generic over M — conditions (1) and (2) of the definition are immediate, and the p_n were chosen so as to make condition (3') hold. This proves 1.3.

A class X in M is called dense below p, for some $p \in C$, if $\forall q \supseteq p \, \exists r \supseteq q(r \in X)$. Frequent use will later be made of:

1.4 LEMMA Let G be C-generic over M, let $p \in G$ and let X be a class in M dense below p. Then $X \cap G \neq \emptyset$.

PROOF: The class $X' = X \cup (C - \{q \in C ; q \not\Phi p\})$ is in M, and it is easily seen that X' is dense. Hence $G \cap X' \neq \emptyset$, by (3) of Definition 1.2. From (2) of that definition, it follows that $G \subseteq \{q \in C ; q \not\Phi p\}$, and so $G \cap X \neq \emptyset$, as asserted.

§2. The forcing relation defined

2.1 DEFINITION The forcing language $\mathcal{L}_M$ over M consists of:

the usual language of set theory (using, say, the logical symbols $\&$, $\neg$, $\exists$, $\in$, $=$ for and , not , there exists , belongs to , equals) together with constants $\underset{\sim}{m}$ for each $m \in M$ and bounded quantifiers $\exists \cdot \in \underset{\sim}{m}$ for each $m \in M$.

Let C be a class of conditions in M , and let $G \subseteq C$. Form an interpretation of $\mathcal{L}_M$ in $M[G]$ as follows: $\in$ is interpreted as $\in$, $=$ as $=$, $\underset{\sim}{m}$ as $\bar{m}$ and $\exists v \in \underset{\sim}{m}$ as $\exists x(x \in \bar{m}$ and $\cdot)$. If Φ is a sentence of $\mathcal{L}_M$, write $M[G] \vdash \Phi$ if Φ is true in $M[G]$ under this interpretation. If Φ is a sentence of set theory, $M[G] \models \Phi$ is to mean that Φ is true in the structure $\langle M[G], \in \rangle$.

For Φ a formula of $\mathcal{L}_M$, write Φ' for that formula of set theory which results from Φ by carrying out this interpretation in $M[G]$. Then if $\Phi(v)$ is a formula of $\mathcal{L}_M$ with one free variable, $M[G] \models \Phi'[\bar{a}] \iff M[G] \models (\Phi(\underset{\sim}{a}))'$. Hence $M[G] \vdash \exists v(\Phi(v)) \iff$ $\exists \bar{a}(M[G] \models \Phi'[\bar{a}]) \iff \exists a \in M(M[G] \models (\Phi(\underset{\sim}{a}))') \iff \exists a \in M(M[G] \vdash \Phi(\underset{\sim}{a}))$, and likewise $M[G] \vdash \exists v \in \underset{\sim}{a}(\Phi(v)) \iff M[G] \models \exists v \in \bar{a}(\Phi'(v)) \iff$ $\exists \bar{x} \in \bar{a}(M[G] \models \Phi'[\bar{x}]) \iff \exists x \in_G a(M[G] \vdash \Phi(\underset{\sim}{x}))$.

Henceforth, the letters a , b , c will be used to range over M .

A formula Φ of $\mathcal{L}_M$ will be called *bounded* if it contains no occurrences of the unbounded quantifiers $\exists$ or $\forall$.

2.2 DEFINITION If Φ is a bounded sentence of $\mathcal{L}_M$, define $\mathrm{Od}(\Phi)$, the *order of* Φ , by

$$\mathrm{Od}(\Phi) = \omega^2 \cdot \alpha + \omega \cdot \beta + \gamma \text{ , where}$$

γ is the number of occurrences of $\neg$, $\&$ and the bounded quantifiers $\exists \cdot \in \underset{\sim}{a}$ in Φ ,

α is $\max(\{rk(a) + 1 \; ; \; \underset{\sim}{a}$ occurs as a constant in $\Phi\}$

$\cup \{rk(a) \; ; \; \exists \cdot \in \underset{\sim}{a}$ occurs in $\Phi\})$,

$$\beta \text{ is } \begin{cases} 1 & \text{if subformulae } \underset{\sim}{a} \in \underset{\sim}{b} \text{ or } \underset{\sim}{a} = \underset{\sim}{b} \text{ or } \underset{\sim}{b} = \underset{\sim}{a} \text{ occur in } \Phi \text{ with} \\ & \alpha = rk(a) + 1 \text{ ,} \\ 0 & \text{otherwise .} \end{cases}$$

(All logical symbols in Φ are to be considered as defined in terms of $\neg$, $\&$, and $\exists \cdot \in \underset{\sim}{a}$ for $a \in M$. Both $\in$ and $=$ are to be taken as primitive. Here ω^2 is ordinal exponentiation.)

2.3 DEFINITION For a condition p , define $Od(p)$, the *order of p* , to be

$$Od(p) = \omega^2 \cdot rk(p) \text{ .}$$

2.4 DEFINITION If Φ is a bounded sentence of $\mathcal{L}_M$ and $p \in C$, define $p \Vdash \Phi$, *p forces Φ* , by induction on $\max\{Od(p), Od(\Phi)\} + Od(\Phi)$, in the following manner:

(1) $$p \Vdash \underset{\sim}{a} \in \underset{\sim}{b} \iff p \Vdash \exists v \in \underset{\sim}{b}(v = \underset{\sim}{a}) \text{ ,}$$

(2) $$p \Vdash \underset{\sim}{a} = \underset{\sim}{b} \iff p \Vdash (\forall v \in \underset{\sim}{a}(v \in \underset{\sim}{b}) \; \& \; \forall v \in \underset{\sim}{b}(v \in \underset{\sim}{a})) \text{ ,}$$

(3) $$p \Vdash \exists v \in \underset{\sim}{a}(\Phi(v)) \iff \exists b \, \exists q \subseteq p(b \in_q a \text{ and } p \Vdash \Phi(\underset{\sim}{b})) \text{ ,}$$

(4) $$p \Vdash \neg\Phi \iff \forall q \not\subseteq p(Od(q) \leq Od(\Phi) \Rightarrow \text{not}(q \Vdash \Phi)) \text{ ,}$$

(5) $$p \Vdash \Phi \; \& \; \Psi \iff p \Vdash \Phi \text{ and } p \Vdash \Psi \text{ .}$$

Gödel number the language $\mathcal{L}_M$ in some reasonable manner. Then the set of (Gödel numbers of) constants of $\mathcal{L}_M$ is a class in M, the set of (Gödel numbers of) formulae of $\mathcal{L}_M$ is a class in M, and so on. Then for bounded sentences Φ of $\mathcal{L}_M$, the relation $Od(\Phi) = \delta$ can be expressed inside M. Further, $\{\langle p,\Phi\rangle ; \max\{Od(p),Od(\Phi)\} + Od(\Phi) < \delta\}$ is a set in M. Thus the whole Definition 2.4 can be given inside M. Hence there is a formula $\Pi(v_o,v_1)$ of set theory such that, for bounded sentences Φ of $\mathcal{L}_M$ and $p \in C$,

$$p \Vdash \Phi \iff \Pi(p,\Phi) \iff \Pi_M(p,\Phi) .$$

For a more detailed discussion on this point, see for example [8].

2.5 LEMMA If Φ is a bounded sentence of $\mathcal{L}_M$, then

$$p \Vdash \Phi \text{ and } q \supseteq p \implies q \Vdash \Phi .$$

PROOF: By induction on $\max\{Od(p),Od(\Phi)\} + Od(\Phi)$, as in Definition 2.4. All cases are clear.

2.6 LEMMA If Φ is a bounded sentence of $\mathcal{L}_M$ with $Od(\Phi) < \omega^2\cdot(\alpha+1)$, then

$$p \Vdash \Phi \iff (p)_\alpha \Vdash \Phi .$$

PROOF: By induction, as in Definition 2.4. Cases (1), (2) and (5) are immediate. Case (4) is also simple: if $Od(\neg\Phi) < \omega^2\cdot(\alpha+1)$ then also $Od(\Phi) < \omega^2\cdot(\alpha+1)$. Thus if $Od(q) \leqq Od(\Phi)$, then $rk(q) \leqq \alpha$ and so in view of properties (i), (iii) and (v) of $(\cdot)_\alpha$, it follows that $q \not\perp p \iff q \not\perp (p)_\alpha$. Case (4) is now obvious.

Suppose Φ is $\exists v \in \underset{\sim}{a}(\Psi(v))$. Then $\omega^2 \cdot rk(a) \leq Od(\Phi) < \omega^2 \cdot (\alpha+1)$, so that $rk(a) \leq \alpha$ and hence $\gamma(a) \leq \alpha$. Thus if $b \in_q a$, also $b \in_{(q)_\alpha} a$. Together with properties (i) and (ii) of $(\cdot)_\alpha$ and the inductive hypothesis, this yields the equivalences:

$$p \Vdash \Phi \iff \exists b\, \exists q(p \supseteq q \text{ and } b \in_q a \text{ and } p \Vdash \Psi(\underset{\sim}{b}))$$

$$\iff \exists b\, \exists q((p)_\alpha \supseteq (q)_\alpha \text{ and } b \in_{(q)_\alpha} a \text{ and } (p)_\alpha \Vdash \Psi(\underset{\sim}{b}))$$

$$\iff (p)_\alpha \Vdash \Phi .$$

This completes the proof of 2.6.

2.7 LEMMA If Φ is a bounded sentence of $\mathcal{L}_M$, then

$$p \Vdash \neg \Phi \iff \forall q \supseteq p(\text{not}(q \Vdash \Phi)) .$$

PROOF: Suppose first that $\forall q \supseteq p(\text{not}(q \Vdash \Phi))$. Take any q compatible with p , so there is r which extends both p and q . Since r extends p , then r does not force Φ . However, if $q \Vdash \Phi$ then by Lemma 2.5, also $r \Vdash \Phi$. Thus no q compatible with p can force Φ . Hence, it follows that $p \Vdash \neg \Phi$.

On the other hand, let $Od(\Phi) = \omega^2 \cdot \alpha + \omega \cdot \beta + \gamma$, and suppose $p \Vdash \neg \Phi$. Since $Od((q)_\alpha) \leq \omega^2 \cdot \alpha \leq Od(\Phi)$, it follows from (4) of Definition 2.4 that $\forall q \not\perp p(\text{not}((q)_\alpha \Vdash \Phi))$. Then Lemma 2.6 yields that $\forall q \not\perp p(\text{not}(q \Vdash \Phi))$, and so in particular, $\forall q \supseteq p(\text{not}(q \Vdash \Phi))$. This proves 2.7.

The definition of forcing has now to be extended to cover also unbounded sentences Φ of $\mathcal{L}_M$. A finite number of applications of the following rules reduce $p \Vdash \Phi$ to $q \Vdash \Psi$ where Ψ is a bounded formula:

(3') $\quad p \Vdash \exists v \in \underset{\sim}{a}(\Phi(v)) \iff \exists b\ \exists q \subseteq p(b \in_q a \text{ and } p \Vdash \Phi(\underset{\sim}{b}))$,

(4') $\quad p \Vdash \neg\Phi \iff \forall q \supseteq p(\text{not}(q \Vdash \Phi))$,

(5') $\quad p \Vdash \Phi_1 \,\&\, \Phi_2 \iff p \Vdash \Phi_1 \text{ and } p \Vdash \Phi_2$,

(6') $\quad p \Vdash \exists v(\Phi(v)) \iff \exists b(p \Vdash \Phi(\underset{\sim}{b}))$.

Take $p \Vdash \Phi$ to be that formula which results after the successive reductions have been carried out. Then for any fixed unbounded sentence Φ of $\mathcal{L}_M$, there is a formula $\Pi_\Phi(v_o)$ of set theory such that, for $p \in C$,

$$p \Vdash \Phi \iff \Pi_\Phi(p) \iff (\Pi_\Phi(p))_M .$$

2.8 LEMMA For all sentences Φ of $\mathcal{L}_M$,

$$p \Vdash \Phi \text{ and } q \supseteq p \implies q \Vdash \Phi .$$

PROOF: It is clear that the lemma holds over the reductions in rules (3')-(6'). Once a bounded formula is reached, Lemma 2.5 applies to give the result.

2.9 LEMMA For all p and Φ, not both $p \Vdash \Phi$ and $p \Vdash \neg\Phi$ hold.

PROOF: Immediate from rule (4') and Lemma 2.7.

2.10 LEMMA Let Φ be a sentence of $\mathcal{L}_M$, and let G be C-generic over M. Then there is $p \in G$ such that either $p \Vdash \Phi$ or $p \Vdash \neg \Phi$.

PROOF: By property (3) of Definition 1.2, it suffices to show that $D = \{p \in C \,;\, p \Vdash \Phi \text{ or } p \Vdash \neg \Phi\}$ is a dense class in M. Certainly D is a class in M. And if $q \in C$, then by whichever of Lemma 2.7 or rule (4') applies, either $q \Vdash \neg \Phi$ or there is $p \supseteq q$ such that $p \Vdash \Phi$. In either case, there is $p \supseteq q$ with $p \in D$, so that D is dense. Lemma 2.10 follows.

The forcing relation may be used to discover properties of the set $M[G]$. The following theorem, the Cohen Truth Lemma, shows how.

2.11 THEOREM Let M be a CTM, let C be a class of forcing conditions and let G be C-generic over M. Then, for all sentences Φ of $\mathcal{L}_M$,

$$M[G] \vdash \Phi \iff \exists p \in G(p \Vdash \Phi) .$$

PROOF: By induction on $\mathrm{Od}(\Phi)$ if Φ is bounded, and if Φ is unbounded by induction on the number of applications of rules (3')-(6') required to reach a bounded formula. Cases 3 and 6 below depend on remarks made at the beginning of this section.

<u>Case 1</u>: $M[G] \vdash \underset{\sim}{a} \in \underset{\sim}{b} \iff M[G] \models \bar{a} \in \bar{b} \iff M[G] \models \exists x \in \bar{b}(x = \bar{a})$ (since $M[G]$ is transitive), $\iff M[G] \vdash \exists v \in \underset{\sim}{b}(v = \underset{\sim}{a})$
$\iff \exists p \in G(p \Vdash \exists v \in \underset{\sim}{b}(v = \underset{\sim}{a}))$ (by the inductive hypothesis),
$\iff \exists p \in G(p \Vdash \underset{\sim}{a} \in \underset{\sim}{b})$.

Case 2: $M[G] \vdash \underset{\sim}{a} = \underset{\sim}{b} \iff \exists p \in G(p \Vdash \underset{\sim}{a} = \underset{\sim}{b})$, similarly.

Case 3: $M[G] \vdash \exists v \in \underset{\sim}{a}(\Phi(v)) \iff \exists x \in_G a(M[G] \vdash \Phi(\underset{\sim}{x}))$
$\iff \exists x\, \exists q \in G\, \exists r \in G(x \in_q a$ and $r \Vdash \Phi(\underset{\sim}{x}))$ (by the inductive hypothesis),
$\iff \exists p \in G\, \exists x\, \exists q(p \supseteq q$ and $x \in_q a$ and $p \Vdash \Phi(\underset{\sim}{x}))$ (using (1) and (2) of Definition 1.2 and Lemma 2.8), $\iff \exists p \in G(p \Vdash \exists v \in \underset{\sim}{a}(\Phi(v)))$.

Case 4: $M[G] \vdash \neg \Phi \iff$ not$(M[G] \vdash \Phi) \iff \forall p \in G(not(p \Vdash \Phi))$ (by the inductive hypothesis). However, if $\forall p \in G($not$(p \Vdash \Phi))$ then $\exists q \in G(q \Vdash \neg \Phi)$ by Lemma 2.10. And if $\exists q \in G(q \Vdash \neg \Phi)$ then $\forall p \in G($not$(p \Vdash \Phi))$, for otherwise, in view of Lemma 2.8 and Definition 1.2(2), there is $r \in G$ such that $r \Vdash \Phi$ and $r \Vdash \neg \Phi$, in contradiction to Lemma 2.9. Thus $M[G] \vdash \neg \Phi \iff \exists q \in G(q \Vdash \neg \Phi)$, as required.

Case 5: $M[G] \vdash \Phi \,\&\, \Psi \iff M[G] \vdash \Phi$ and $M[G] \vdash \Psi$
$\iff \exists p \in G(p \Vdash \Phi)$ and $\exists q \in G(q \Vdash \Psi) \iff \exists r \in G(r \Vdash \Phi$ and $r \Vdash \Psi)$ (using Lemma 2.8 and Definition 1.2(2)) , $\iff \exists r \in G(r \Vdash \Phi \,\&\, \Psi)$.

Case 6: $M[G] \vdash \exists v(\Phi(v)) \iff \exists b(M[G] \vdash \Phi(\underset{\sim}{b})) \iff \exists b\, \exists p \in G(p \Vdash \Phi(\underset{\sim}{b}))$
$\iff \exists p \in G(p \Vdash \exists v(\Phi(v)))$.

This completes the proof of Theorem 2.11.

§3. The axioms of set theory in M[G]

In this section, it will be shown that, under certain hypotheses, the set $M[G]$ for a generic class G forms a model of ZF + AC . This cannot be shown in general, but certain further restrictions have to be imposed on the class of conditions. A little further progress can be made before this becomes necessary, however.

So let M be a CTM, let C be a class of forcing conditions as in §1, and let G be C-generic over M. For this section, let also x, y, z as well as a, b, c,... range over M. Then $\bar{x}$, $\bar{a}$,... range over $M[G]$.

Since $M[G]$ is transitive, the axiom of extensionality is satisfied in $M[G]$. Since $M[G]$ is a set, it is well founded and so the axiom of regularity is true in $M[G]$. The axiom of infinity holds since $\omega \in M[G]$, for $\omega \in M$ and $M \subseteq M[G]$.

3.1 THEOREM If $\Phi(v)$ is a bounded formula of $\mathcal{L}_M$, then the subset axiom corresponding to $\Phi'(v)$ is valid in $M[G]$.

PROOF: We need to show:

$$\forall\bar{a}\ \exists\bar{z}\ \forall\bar{x}(\bar{x} \in \bar{z} \iff \bar{x} \in \bar{a} \text{ and } M[G] \models \Phi'[\bar{x}]) .$$

Given a, let δ be least such that

$$\omega^2\cdot\delta \geq \max\{\omega^2\cdot\gamma(a), \sup\{Od(\Phi(\underset{\sim}{x})) ; x \in Da\}\} .$$

Then $\delta \in M$, since both a and Da are in M. Put

$$z = \{\langle x,p\rangle ; \exists q \subseteq p(x \in_q a) \text{ and } rk(p) \leq \delta \text{ and } p \Vdash \Phi(\underset{\sim}{x})\} .$$

Then $z \in M$. It suffices to show that $\bar{z}$ is the required set.

Suppose $\bar{x} \in \bar{z}$; to show $\bar{x} \in \bar{a}$ and $M[G] \models \Phi'[\bar{x}]$. It is sufficient to show that if $x \in_G z$, then $x \in_G a$ and $M[G] \vdash \Phi(\underset{\sim}{x})$.

If $x \in_G z$, there is $p \in G$ such that $x \in_p z$, so in particular $\langle x,p\rangle \in z$. Thus $p \Vdash \Phi(\underset{\sim}{x})$ and $x \in_q a$ for some $q \subseteq p$. But then also $q \in G$. Hence $x \in_G a$ and, in virtue of Theorem 2.11, also $M[G] \vdash \Phi(\underset{\sim}{x})$.

Now let $\bar{x} \in \bar{a}$ and $M[G] \models \Phi'[\bar{x}]$, so we may suppose that $x \in_G a$ and $M[G] \vdash \Phi(\underset{\sim}{x})$. Thus there is $p \in G$ such that $x \in_p a$ and, again by Theorem 2.11, there is $q \in G$ such that $q \Vdash \Phi(\underset{\sim}{x})$. Take $r \in G$ such that r extends both p and q , and consider $r' = (r)_\delta$. By Lemma 2.8, $r \Vdash \Phi(\underset{\sim}{x})$. Now since $x \in_p a$ it follows that $x \in Da$, and so $Od(\Phi(\underset{\sim}{x})) < \omega^2\cdot(\delta+1)$. Hence $r' \Vdash \Phi(\underset{\sim}{x})$, by Lemma 2.6. Moreover, $rk(p) < \delta$ since $x \in_p a$ and $\gamma(a) \leqq \delta$. Since $r \supseteq p$, it follows that $r' = (r)_\delta \supseteq (p)_\delta = p$. Thus $r' \Vdash \Phi(\underset{\sim}{x})$, $rk(r') \leqq \delta$ and $\exists p \subseteq r'(x \in_p a)$; in other words, $\langle x,r'\rangle \in z$.

Further, if $s \supseteq r'$ with $rk(s) < \gamma(z)$ then also $s \supseteq p$ and $s \Vdash \Phi(\underset{\sim}{x})$. Since $\gamma(z) \leqq \delta+1$, also $rk(s) \leqq \delta$, so that $\langle x,s\rangle \in z$. Hence $x \in_{r'} z$. Since $r \in G$ and $r \supseteq r'$, also $r' \in G$. Thus $x \in_G z$, so $\bar{x} \in \bar{z}$ as required. This completes the proof.

3.2 THEOREM The axioms of pair set and union set hold in $M[G]$.

PROOF: To establish the pair set axiom, let x and y be given. We must show that $\{\bar{x},\bar{y}\} \in M[G]$. Define

$$a = \{x,y\} \times \{\emptyset\} ,$$

so $a \in M$. However, $\bar{a} = \{\bar{x},\bar{y}\}$, for

$$\bar{z} \in \bar{a} \iff z \in_G a \iff \exists p \in G(z \in_p a) \iff z \in \{x,y\} ,$$

using that $\emptyset \in G$. Hence $\{\bar{x},\bar{y}\} \in M[G]$.

For the union set axiom, in view of Theorem 3.1, it suffices to show:

$$\forall \bar{a}\ \exists \bar{b}\ \forall \bar{x}(\exists \bar{y} \in \bar{a}(\bar{x} \in \bar{y}) \Rightarrow \bar{x} \in \bar{b}) .$$

Given $\bar{a}$, suppose $\bar{x}$ is such that $\exists \bar{y} \in \bar{a}(\bar{x} \in \bar{y})$. Then we may as well suppose $\exists y \in_G a(x \in_G y)$, and so $\exists y\ \exists p \in G\ \exists q \in G(x \in_p y$ and $y \in_q a)$. Hence $x \in D(\bigcup Da)$. Put

$$b = D(\bigcup Da) \times \{\emptyset\} ;$$

then $b \in M$. Moreover, b has the required property, for

$$\bar{x} \in \bar{b} \iff x \in D(\bigcup Da) .$$

This completes the proof.

3.3 THEOREM Suppose that the replacement schema holds in $M[G]$. Then also the axiom of choice is valid in $M[G]$.

PROOF: It was shown in Lemma 1.1 that G is a class in $M[G]$. Thus, using the replacement schema, $a \div^G$ may be defined inside $M[G]$ much as before:

if $u,v \in M[G]$ and $p \in C$, define in $M[G]$

$$u \in_p v \iff rk(p) < \gamma(\cancel{u}\,v) \text{ and } \forall q \supseteq p(rk(q) < \gamma(v) \Rightarrow \langle u,q\rangle \in v) ,$$

$$u \in_G v \iff \exists p \in G(u \in_p v) ,$$

$$\bar{v}^G = \{\bar{u}^G ; u \in_G v\} .$$

Then if $v \in M$, this gives the same result as before, since if $v \in M$ and $u \in_G v$ then $u \in Dv$, so that $u \in M$.

Now take $\bar{a} \in M[G]$. Every member of $\bar{a}$ is $\bar{x}$ for some $x \in Da$. However, Da is well ordered in M because the axiom of choice holds in M . Since $M \subseteq M[G]$, this well ordering in fact lies in $M[G]$, so Da is well ordered in $M[G]$. Thus $\bar{a}$ can be well ordered in $M[G]$, since $\bar{a}$ is contained in the image of Da under $\bar{\cdot}^G$, and $\bar{\cdot}^G$ can be defined in $M[G]$. This establishes Theorem 3.3.

In order to establish the truth of the power set axiom and the replacement schema in $M[G]$, further restrictions have to be imposed on the class C of conditions. The remainder of this chapter owes much to lectures by C.C. Chang [2].

Henceforth, let C have the following properties (a)-(d) .

(a) For all ordinals ν in M , there are operations $\cdot \rceil \nu$ and $\cdot \rfloor \nu$ on C which are M definable, and satisfy

(i) $$p = p \rfloor \nu \cup p \rceil \nu ; \quad p \rfloor \nu \cap p \rceil \nu = \emptyset ,$$

(ii) $$p = \bigcup\{p_i ; i \in I\} \Rightarrow p \rceil \nu = \bigcup\{p_i \rceil \nu ; i \in I\} \text{ and}$$
$$p \rfloor \nu = \bigcup\{p_i \rfloor \nu ; i \in I\} ,$$

(iii) $\nu \leqq \tau \Rightarrow (p \rfloor \tau) \rfloor \nu = p \rfloor \nu$ and $(p \rceil \nu) \rceil \tau = p \rceil \tau$,

(iv) $\nu < \tau \Rightarrow p \rfloor \nu \subseteq p \rfloor \tau$ and $p \rceil \tau \subseteq p \rceil \nu$,

(v) if $\langle p_\tau \; ; \; \tau < \sigma \rangle$ is a sequence from C such that $\nu < \tau < \sigma \Rightarrow p_\tau \supseteq p_\nu \rceil \nu$, then $\bigcup \{p_\nu \rceil \nu \; ; \; \nu < \sigma\} \in C$,

(vi) $\exists q(q \rceil \nu \supset p \rceil \nu)$.

(b) For all $\alpha \in M$, $\{p \rfloor \alpha \; ; \; p \in C\} \in M$.

(c) There is a function class F definable in M which maps the ordinals of M into the infinite cardinals of M , and has the properties:

(i) $\alpha \leqq F(\alpha)$; $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;

$\alpha = \bigcup \alpha > 0 \Rightarrow F(\alpha) = \sum \{F(\beta) \; ; \; \beta < \alpha\}$,

(ii) if $b \subseteq \{p \rceil \alpha \; ; \; p \in C\}$ is a pairwise ~~in~~compatible set having, in M , power at most $F(\alpha)$, then $\bigcup b \in C$,

(iii) if α is not a limit ordinal greater than 0 , then $\{p \rfloor \alpha \; ; \; p \in C\}$ satisfies, in M , the $F(\alpha)^+$-chain condition, i.e. every set of pairwise incompatible elements from $\{p \rfloor \alpha \; ; \; p \in C\}$ has cardinality strictly less than $F(\alpha)^+$.

(d) If two conditions p and q are compatible, or if $p = p \rfloor \alpha$ and $q = q \rceil \beta$ where $\alpha \leq \beta$, then $p \cup q \in C$.

The $p \urcorner \nu$ and $p \lrcorner \nu$ divide each condition p into disjoint sections, the "part above ν" and the "part below ν". Property (a) is of local nature. It provides that the sections should fit together smoothly. Property (c), on the other hand, imposes global restrictions on the sections.

The result of imposing these added properties on the class C of conditions must now be investigated, unfortunately at some length. The following definitions are needed.

3.4 DEFINITION For $p \in C$ and Φ a sentence of $\mathcal{L}_M$, say p *decides* Φ, and write $p \parallel \Phi$, if either $p \Vdash \Phi$ or $p \Vdash \neg \Phi$.

3.5 DEFINITION For any condition p, any subset A of C and any set Ω of sentences of $\mathcal{L}_M$, say p and A *predecide* Ω, and write $p{:}A \Vdash \Omega$, if

$$\forall q \supseteq p \ \forall \Phi \in \Omega \ \exists r \in A (r \not\perp q \text{ and } r \cup p \parallel \Phi) .$$

Thus for any $\Phi \in \Omega$, if for some $q_\Phi \supseteq p$ it is the case that $q_\Phi \parallel \Phi$ then there is $r_\Phi \in A$ such that $r_\Phi \cup p$ decides Φ in the same way as q_Φ. Note that since $r \not\perp q$ and $q \supseteq p$, also $r \not\perp p$ so that by (d) above, $r \cup p \in C$.

For any $B \subseteq C$, put $B \lrcorner \alpha = \{p \lrcorner \alpha \ ; \ p \in B\}$ and $B \urcorner \alpha = \{p \urcorner \alpha \ ; \ p \in B\}$.

The next three lemmas take place inside M .

3.6 LEMMA For α not an infinite limit ordinal, p a condition and Φ a sentence of $\mathcal{L}_M$, there are $q \supseteq p$ and $a \subseteq C \lrcorner \alpha$ such that

$$|a| \leq F(\alpha) \quad ; \quad q \lrcorner \alpha = p \lrcorner \alpha \quad ; \quad q : a \Vdash \{\Phi\} .$$

PROOF: Take $b \subseteq C$ maximal with respect to the properties:

(1) $$r \in b \Rightarrow r \parallel \Phi$$

(2) $$\{r,s\} \in [b]^2 \Rightarrow (r \urcorner \alpha \supset s \urcorner \alpha \text{ or } s \urcorner \alpha \supset r \urcorner \alpha) \text{ and not}(r \lrcorner \alpha \,\phi\, s \lrcorner \alpha) .$$

Since $C \lrcorner \alpha$ satisfies the $F(\alpha)^+$-chain condition, it follows from (2) that $|b| \leq F(\alpha)$. Since $B \urcorner \alpha$ is pairwise compatible, $\bigcup(b \urcorner \alpha) \in C$ by property (c,ii). Take $q_1 \in C \urcorner \alpha$ such that $q_1 \supset \bigcup(b \urcorner \alpha)$, as given by (a,vi). Put $q = p \cup q_1$, so $q \lrcorner \alpha = p \lrcorner \alpha$. The result will follow if it can be shown that $q : b \lrcorner \alpha \Vdash \{\Phi\}$.

Take any $r \supseteq q$ and seek $s \in b \lrcorner \alpha$ such that $s \,\phi\, r$ and $s \cup q \parallel \Phi$. We may assume $r \parallel \Phi$, for always there is $r' \supseteq r$ such that $r' \parallel \Phi$, and if $s' \in b \lrcorner \alpha$ is such that $s' \,\phi\, r'$ and $s' \cup q \parallel \Phi$, also $s' \,\phi\, r$. Note that $r \urcorner \alpha \notin b \urcorner \alpha$, since $r \urcorner \alpha \supseteq q \urcorner \alpha \supseteq q_1$ and q_1 was chosen so that $q_1 \supset q'$ for all $q' \in b \urcorner \alpha$.

If $r \lrcorner \alpha \in b \lrcorner \alpha$ then putting $s = r \lrcorner \alpha$ gives the result. For then certainly $s \,\phi\, r$. And there is $q' \in b$ such that $r \lrcorner \alpha = q' \lrcorner \alpha$, so since $q \urcorner \alpha \supset q' \urcorner \alpha$ it follows that $s \cup q \supset q'$; yet since $q' \in b$

then $q' \parallel \Phi$ in virtue of (1). Thus $s \cup q \parallel \Phi$, as required.

On the other hand, suppose $r \upharpoonright \alpha \notin b \upharpoonright \alpha$. As above, $r \urcorner \alpha \supset t \urcorner \alpha$ for any $t \in b$. Since $r \notin b$, by the maximality of b there must be $s' \in b$ for which $r \upharpoonright \alpha \, \phi \, s' \upharpoonright \alpha$. Then $r \, \phi \, s' \upharpoonright \alpha$. Further, $s' \parallel \Phi$ because $s' \in b$, and so since $s' \upharpoonright \alpha \cup q \supseteq s' \upharpoonright \alpha \cup q_1 \supseteq s'$, in fact $(s' \upharpoonright \alpha \cup q) \parallel \Phi$. Hence taking $s = s' \upharpoonright \alpha$ satisfies the requirements. This completes the proof.

3.7 LEMMA For α not an infinite limit ordinal, p a condition and Ω a set of power at most $F(\alpha)$ of sentences of $\mathcal{L}_M$, there are $q \supseteq p$ and $a \subseteq C \upharpoonright \alpha$ such that

$$|a| \leq F(\alpha) \quad ; \quad q \upharpoonright \alpha = p \upharpoonright \alpha \quad ; \quad q{:}a \Vdash \Omega \ .$$

PROOF: Well order Ω as $\{\Phi_\nu \, ; \, \nu < F(\alpha)\}$. Use transfinite induction to find for each $\nu < F(\alpha)$ a condition $q_\nu \supseteq p$ and a set $a_\nu \subseteq C \upharpoonright \alpha$ with $|a_\nu| \leq F(\alpha)$, which together satisfy

$$q_\nu \upharpoonright \alpha = p \upharpoonright \alpha \quad ; \quad q_\nu{:}a_\nu \Vdash \{\Phi_\nu\} \quad ; \quad \tau \leq \nu \Rightarrow q_\nu \supseteq q_\tau \ .$$

To find q_0 and a_0, apply Lemma 3.6 to p and Φ_0. Suppose q_τ and a_τ have been suitably defined. Apply Lemma 3.6 to q_τ and $\Phi_{\tau+1}$ to find $q' \supseteq q_\tau$ and $a' \subseteq C \upharpoonright \alpha$ with $|a'| \leq F(\alpha)$ such that $q' \upharpoonright \alpha = q_\tau \upharpoonright \alpha = p \upharpoonright \alpha$ and $q'{:}a' \Vdash \{\Phi_{\tau+1}\}$. Thus take $q_{\tau+1} = q'$ and $a_{\tau+1} = a'$.

If $\tau = \cup\tau > 0$ where q_σ and a_σ have already been defined for all $\sigma < \tau$, put $r = \cup\{q_\sigma \urcorner \alpha ; \sigma < \tau\}$. By property (c,ii) of C, it follows that $r \in C$. Apply Lemma 3.6 to $p \downharpoonright \alpha \cup r$ and Φ_τ, to produce $q' \supseteq p \downharpoonright \alpha \cup r$ and $a' \subseteq C \downharpoonright \alpha$ with $|a'| \leq F(\alpha)$ such that $q' \downharpoonright \alpha = (p \downharpoonright \alpha \cup r) \downharpoonright \alpha = p \downharpoonright \alpha$ and $q' : a' \Vdash \{\Phi_\tau\}$. Moreover, if $\sigma < \tau$ then $q' \supseteq p \downharpoonright \alpha \cup r \supseteq p \downharpoonright \alpha \cup q_\sigma \urcorner \alpha = q_\sigma$ and so we may take $q_\tau = q'$ and $a_\tau = a'$. This completes the definition.

Now put $s = \cup\{q_\nu \urcorner \alpha ; \nu < F(\alpha)\}$, so again by (c,ii), $s \in C$. Define $q = p \downharpoonright \alpha \cup s$ and $a = \cup\{a_\nu ; \nu < F(\alpha)\}$; then q and a have the required properties. Clearly $a \subseteq C \downharpoonright \alpha$, and also $|a| \leq F(\alpha)$. If $\nu < F(\alpha)$, then $q \supseteq p \downharpoonright \alpha \cup q_\nu \urcorner \alpha = q_\nu \supseteq p$, and $q \downharpoonright \alpha = p \downharpoonright \alpha$. And if $r \supseteq q$ then for any Φ_ν, since also $r \supseteq q_\nu$, there is $s' \in a_\nu$ such that $r \phi s$ and $s' \cup q_\nu \parallel \Phi_\nu$. But $s' \cup q \supseteq s' \cup q_\nu$ and so also $s' \cup q \parallel \Phi_\nu$. Thus there is $s' \in a$ such that $r \phi s'$ and $s' \cup q \parallel \Phi_\nu$. Hence $q : a \Vdash \Omega$. This concludes the proof of Lemma 3.7.

3.8 LEMMA For α a non-zero limit ordinal, p a condition and Ω a set of power at most $F(\alpha)$ of sentences of $\mathcal{L}_M$, there are $q \supseteq p$ and $a \subseteq C \downharpoonright \alpha$ such that

$$|a| \leq F(\alpha) \quad ; \quad q : a \Vdash \Omega .$$

PROOF: By virtue of property (c,i) of F, write $\Omega = \cup\{\Omega_\nu ; \nu < \alpha\}$ where $|\Omega_\nu| \leq F(\nu)$ for each $\nu < \alpha$. Transfinite induction will be used to find for each $\nu < \alpha$ a condition q_ν and a set $a_\nu \subseteq C \downharpoonright \nu{+}1$ with $q_o \supseteq p$ and $|a_\nu| \leq F(\nu{+}1)$, which satisfy

$$q_\nu \rfloor \nu \subseteq p \rfloor 0 \cup \bigcup\{q_\tau \rceil \tau \ ; \ \tau < \nu\} \ ,$$

$$\tau < \nu \Rightarrow q_\nu \supseteq q_\tau \rceil \tau \ ,$$

$$q_\nu : a_\nu \Vdash \Omega_\nu \ .$$

To find q_0 and a_0, apply Lemma 3.7 to p and Ω_0. Suppose q_τ and a_τ have already been suitably defined. Apply Lemma 3.7 to q_τ and $\Omega_{\tau+1}$ to obtain $q' \supseteq q_\tau$ and $a' \subseteq C \rfloor \tau+1$ with $|a'| \leq F(\tau+1)$ such that $q' \rfloor \tau+1 = q_\tau \rfloor \tau+1$ and $q' : a' \Vdash \Omega_{\tau+1}$. But now by properties (a,i,iii),

$$q' \rfloor \tau+1 = (q_\tau \rceil \tau) \rfloor \tau+1 \cup q_\tau \rfloor \tau \subseteq p \rfloor 0 \cup \bigcup\{q_\sigma \rceil \sigma \ ; \ \sigma < \tau+1\} \ .$$

And if $\sigma < \tau+1$ then $q' \supseteq q_\tau \supseteq q_\sigma \rceil \sigma$, using (a,i) when $\sigma = \tau$. Hence it suffices to define $q_{\tau+1} = q'$ and $a_{\tau+1} = a'$.

If $\tau = \bigcup\tau > 0$ where q_σ and a_σ have already been defined for all $\sigma < \tau$, put

$$q^*_\tau = p \rfloor 0 \cup \bigcup\{q_\sigma \rceil \sigma \ ; \ \sigma < \tau\} \ .$$

Then $q^*_\tau \in C$, by properties (a,v) and (d). Apply Lemma 3.7 to q^*_τ and Ω_τ, noting that $|\Omega_\tau| \leq F(\tau) < F(\tau+1)$. This yields $q' \supseteq q^*_\tau$ and $a \subseteq C \rfloor \tau+1$ with $|a| \leq F(\tau+1)$, such that $q' \rfloor \tau+1 = q^*_\tau \rfloor \tau+1$ and $q' : a' \Vdash \Omega_\tau$. But then $q' \rfloor \tau = q^*_\tau \rfloor \tau \subseteq q^*_\tau = p \rfloor 0 \cup \bigcup\{q_\sigma \rceil \sigma \ ; \ \sigma < \tau\}$. And if $\sigma < \tau$ then $q' \supseteq q^*_\tau \supseteq q_\sigma \rceil \sigma$, by the definition of q^*_τ. Thus $q_\tau = q'$ and $a_\tau = a'$ is a suitable definition.

By another application of (a,v), if

$$q^* = p \downharpoonright 0 \cup \bigcup\{q_\nu \upharpoonright \nu ; \nu < \alpha\} ,$$

then $q^* \in C$. Also $q^* \supseteq p \downharpoonright 0 \cup q_o \upharpoonright 0 \supseteq p \downharpoonright 0 \cup p \upharpoonright 0 = p$. Further, if $\nu < \alpha$ then $q^* \supseteq q_\nu$, since

$$q_\nu = q_\nu \upharpoonright \nu \cup q_\nu \downharpoonright \nu \subseteq p \downharpoonright 0 \cup \bigcup\{q_\tau \upharpoonright \tau ; \tau \leq \nu\} \subseteq q^* .$$

Define $a^* = \bigcup\{a_\nu ; \nu < \alpha\}$, so $a^* \subseteq C \downharpoonright \alpha$ and $|a^*| \leq \sum \{F(\nu+1) ; \nu < \alpha\} = F(\alpha)$. Finally, we show $q^*:a^* \Vdash \Omega$.

Take $r \supseteq q^*$ and $\Phi \in \Omega$, and seek $s \in a^*$ such that $s \phi r$ and $s \cup q^* \parallel \Phi$. There is $\nu < \alpha$ with $\Phi \in \Omega_\nu$. Since $r \supseteq q^*$, also $r \supseteq q_\nu$. Because $q_\nu:a_\nu \Vdash \Omega_\nu$, there is $s \in a_\nu$ such that $r \phi s$ and $s \cup q_\nu \parallel \Phi$. But then also $s \cup q^* \parallel \Phi$. Thus the choice $q = q^*$ and $a = a^*$ suffices to establish Lemma 3.8.

These lemmas combine to yield:

3.9 THEOREM Let G be C-generic over M for a class C satisfying the conditions (a)-(d). For any set $\Omega \in M$ of power (in M) at most $F(\alpha)$ of sentences of $\mathcal{L}_M$, there are $p \in G$ and $a \subseteq C \downharpoonright \alpha$ with $|a| \leq F(\alpha)$ such that $p:a \Vdash \Omega$.

PROOF: Lemmas 3.7 and 3.8 imply that

$$\{q ; \exists a \subseteq C \downharpoonright \alpha(|a| \leq F(\alpha) \text{ and } q:a \Vdash \Omega\}$$

is a dense class. It is definable in M , so the theorem follows from property (3) of a generic class.

This theorem is fundamental to the next two lemmas, which are aimed directly towards establishing the replacement schema.

3.10 LEMMA Let G be C-generic over M for a class C satisfying the properties (a)-(d). For any formula $\Phi(v,v_1,\ldots,v_n)$ of $\mathcal{L}_M$ and any $a \in M$, there is $b \in M$ such that

$$M[G] \vdash \forall v_1,\ldots,v_n \in \underset{\sim}{a}(\exists v(\Phi(v,v_1,\ldots,v_n)) \iff \exists v \in \underset{\sim}{b}(\Phi(v,v_1,\ldots,v_n))) .$$

PROOF: Only the implication from left to right need be established. Let a and Φ be given. Put

$$\Omega = \{"\exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))" \ ; \ x_1,\ldots,x_n \in Da\} ,$$

where $"\Psi"$ is (the Gödel number of) the formula Ψ, considered as an object in M. Then $\Omega \in M$, so by Theorem 3.9 there are $p \in G$ and $A \subseteq C$, with $A \in M$, such that $p{:}A \Vdash \Omega$.

For $x_1,\ldots,x_n \in Da$ and $r \in A$, make the definition:

$F_\Phi(x_1,\ldots,x_n,r)$ is the least β for which there is x with $rk(x) = \beta$ such that $r \cup p \Vdash \Phi(\underset{\sim}{x},\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n)$, provided $r \cup p \in C$ and $r \cup p \Vdash \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$; otherwise $F_\Phi(x_1,\ldots,x_n,r) = 0$.

Since if $q \Vdash \exists v(\Phi(v))$, then $q \Vdash \Phi(\underset{\sim}{b})$ for some b, it follows that F_Φ is well defined. Put

$$\delta = \sup\{F_\Phi(x_1,\ldots,x_n,r) \; ; \; x_1,\ldots,x_n \in Da \text{ and } r \in A\} \; ;$$

then it follows that $\delta \in M$. Define also

$$b = \{x \; ; \; rk(x) < \delta\} \times \{\emptyset\} .$$

Then $b \in M$. If for some $x_1,\ldots,x_n \in Da$ and some $r \in A$, $r \cup p \Vdash \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$ then $r \cup p \Vdash \Phi(\underset{\sim}{x},\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n)$ for some $x \in Db$. Moreover, if $x \in Db$ then $x \in_\emptyset b$, and $r \cup p \supseteq \emptyset$ so that in this case $r \cup p \Vdash \exists v \in \underset{\sim}{b}(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$. Thus b has the property: for all $x_1,\ldots,x_n \in Da$ and all $r \in A$,

$$r \cup p \in C \text{ and } r \cup p \Vdash \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$$
$$\Rightarrow \; r \cup p \Vdash \exists v \in \underset{\sim}{b}(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n)) .$$

Suppose $\bar{x}_1,\ldots,\bar{x}_n \in \bar{a}$ are such that $\exists \bar{x}(M[G] \models \Phi'[\bar{x},\bar{x}_1,\ldots,\bar{x}_n])$, where we may as well assume that $x_1,\ldots,x_n \in Da$. By Theorem 2.11, there is $q \in G$ for which $q \Vdash \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$, and we may suppose that $q \supseteq p$. It is sufficient to show

$$(1) \qquad \forall r \supseteq q[\text{not}(r \Vdash \neg(\exists v \in \underset{\sim}{b}(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))))] ,$$

for then there must be $r \in G$ with $r \supseteq q$ such that $r \Vdash \exists v \in \underset{\sim}{b}(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$. But then $\exists \bar{x} \in \bar{b}(M[G] \models \Phi'[\bar{x},\bar{x}_1,\ldots,\bar{x}_n])$, and this leads to the desired result.

To show (1), take $r \supseteq q$. Then $r \supseteq p$, so since $p;A \Vdash\!\!= \Omega$ and "$\exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$" is in Ω, there is $s \in A$ such that $r \phi s$

and $s \cup p \| \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$. Now $q \not\perp s$ since $r \not\perp s$ and $r \supseteq q$, and so $q \not\perp (s \cup p)$. However, $q \Vdash \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$, and so by Lemmas 2.8 and 2.9, it cannot be the case that $s \cup p \Vdash \neg(\exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n)))$; hence $s \cup p \Vdash \exists v(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$. By virtue of the property enjoyed by b , this means that $s \cup p \Vdash \exists v \in \underset{\sim}{b}(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$. Since also $r \not\perp (s \cup p)$ it follows that it cannot be the case that $r \Vdash \neg(\exists v \in \underset{\sim}{b}(\Phi(v,\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n))$. This proves (1), and completes the proof of Lemma 3.10.

3.11 LEMMA Let C have the properties (a)-(d); let G be C-generic over M . For any formula $\Phi(v_1,\ldots,v_n)$ of $\mathcal{L}_M$ and any $a \in M$ there is a bounded formula $\Phi_o(v_1,\ldots,v_n)$ such that

$$M[G] \vdash \forall v_1,\ldots,v_n \in \underset{\sim}{a}(\Phi(v_1,\ldots,v_n) \iff \Phi_o(v_1,\ldots,v_n)) .$$

PROOF: By induction on the complexity of Φ . Only the case when Φ is $\exists v(\Psi)$ requires comment. However, given $a \in M$, use Lemma 3.10 to find $b \in M$ such that

$$M[G] \vdash \forall v_1,\ldots,v_n \in \underset{\sim}{a}(\Phi(v_1,\ldots,v_n) \iff \exists v \in \underset{\sim}{b}(\Psi(v,v_1,\ldots,v_n))) .$$

Choose $c \in M$ such that $\bar{c} = \bar{a} \cup \bar{b}$. By the inductive hypothesis applied to $\Psi(v,v_1,\ldots,v_n)$, there is a bounded formula $\Psi_o(v,v_1,\ldots,v_n)$ such that

$$M[G] \vdash \forall v,v_1,\ldots,v_n \in \underset{\sim}{c}(\Psi(v,v_1,\ldots,v_n) \iff \Psi_o(v,v_1,\ldots,v_n)) .$$

In particular, for all $\bar{x}_1,\ldots,\bar{x}_n \in \bar{a}$,

$$M[G] \models \Phi'[\bar{x}_1,\ldots,\bar{x}_n] \iff \exists\bar{x} \in \bar{b}(M[G] \models \Psi'[\bar{x},\bar{x}_1,\ldots,\bar{x}_n])$$
$$\iff \exists\bar{x} \in \bar{b}(M[G] \models \Psi'_o[\bar{x},\bar{x}_1,\ldots,\bar{x}_n]) .$$

Hence $\exists v \in \underset{\sim}{b}(\Psi_o(v,v_1,\ldots,v_n))$ is a formula with the desired property. This proves 3.11.

3.12 THEOREM Let C rejoice in the properties (a)-(d) ; let G be C-generic over M . Then the replacement schema holds in $M[G]$.

PROOF: Let $\Phi(u,v)$ be any formula of set theory. It suffices to show:

$$\forall\bar{a}[\forall\bar{x} \in \bar{a}\ \exists\bar{c}\ \forall\bar{y}(\Phi_{M[G]}(\bar{x},\bar{y}) \Rightarrow \bar{y} \in \bar{c})$$
$$\Rightarrow \exists\bar{b}\ \forall\bar{y}(\exists\bar{x} \in \bar{a}(\Phi_{M[G]}(\bar{x},\bar{y})) \iff \bar{y} \in \bar{b})] .$$

Let a be given, and suppose $\Phi(u,v)$ is such that $\forall\bar{x} \in \bar{a}\ \exists\bar{c}\ \forall\bar{y}(\Phi_{M[G]}(\bar{x},\bar{y}) \Rightarrow \bar{y} \in \bar{c})$. By Lemma 3.10, choose z with the property

$$\forall\bar{x} \in \bar{a}[\exists\bar{c}\ \forall\bar{y}(\Phi_{M[G]}(\bar{x},\bar{y}) \Rightarrow \bar{y} \in \bar{c}) \iff \exists\bar{c} \in \bar{z}\ \forall\bar{y}(\Phi_{M[G]}(\bar{x},\bar{y}) \Rightarrow \bar{y} \in \bar{c})] .$$

Thus we have $\forall\bar{x} \in \bar{a}\ \exists\bar{c} \in \bar{z}\ \forall\bar{y}(\Phi_{M[G]}(\bar{x},\bar{y}) \Rightarrow \bar{y} \in \bar{c})$. In view of Theorem 3.2, there is $b' \in M$ such that $\bar{b}' = \bigcup\bar{z}$. Hence for all $\bar{y}$, it follows that $\exists\bar{x} \in \bar{a}(\Phi_{M[G]}(\bar{x},\bar{y})) \Rightarrow \bar{y} \in \bar{b}'$.

By Lemma 3.11, there is a bounded formula $\Psi(v)$ for which $\forall\bar{y} \in \bar{b}'(\exists\bar{x} \in \bar{a}(\Phi_{M[G]}(\bar{x},\bar{y})) \iff \Psi_{M[G]}(\bar{y}))$. Apply the bounded subset schema, Theorem 3.1, to $\Psi(v)$ and the set $\bar{b}'$ to obtain a set $\bar{b}$ such that

$$\forall \bar{y}(\bar{y} \in \bar{b} \iff \bar{y} \in \bar{b}' \text{ and } \Psi_{M[G]}(\bar{y})) \ .$$

This $\bar{b}$ is the set required, for if $\bar{y} \in \bar{b}$ then $\Psi_{M[G]}(\bar{y})$ and $\bar{y} \in \bar{b}'$, so that $\exists \bar{x} \in \bar{a}(\Phi_{M[G]}(\bar{x},\bar{y}))$; on the other hand, if $\exists \bar{x} \in \bar{a}(\Phi_{M[G]}(\bar{x},\bar{y}))$ then $\bar{y} \in \bar{b}'$, and so $\Psi_{M[G]}(\bar{y})$, with the result that $\bar{y} \in \bar{b}$. This completes the proof of Theorem 3.12.

3.13 THEOREM Let C enjoy the properties (a)-(d); let G be C-generic over M. Then the power set axiom is true in $M[G]$.

PROOF: Let a be given. We must find $\bar{b}$ for which

$$\bar{x} \subseteq \bar{a} \implies \bar{x} \in \bar{b} \ .$$

Put $w = \{y \in M \ ; \ rk(y) < rk(a)\}$, so $w \in M$ and hence also $w \in M[G]$. For each x, put $\Omega_x = \{"\underset{\sim}{z} \in \underset{\sim}{x}" \ ; \ z \in w\}$, so always $\Omega_x \in M$. By Theorem 3.9, there are $p_x \in G$ and $A_x \in M$ for which $p_x : A_x \mathrel{|}\!\models \Omega_x$. Further, there is $\beta \in M$ such that always $A_x \subseteq C \upharpoonright \beta$. Put $A = C \upharpoonright \beta$; by virtue of property (b) of C, then $A \in M$. Moreover, $p_x : A \mathrel{|}\!\models \Omega_x$ for all x.

Make the definition

$$K(u,x) \iff \exists p \in G(p \Vdash \underset{\sim}{x} \subseteq \underset{\sim}{a} \text{ and } p : A \mathrel{|}\!\models \Omega_x \text{ and }$$
$$u = \{\langle r,z\rangle \in A \times w \ ; \ r \cup p \Vdash \underset{\sim}{z} \in \underset{\sim}{x}\}) \ .$$

Then if $\bar{x} \subseteq \bar{a}$, there is $u \in M$ such that $K(u,x)$. For if $\bar{x} \subseteq \bar{a}$, by Theorem 2.11 there is $q \in G$ such that $q \Vdash \underset{\sim}{x} \subseteq \underset{\sim}{a}$, and q may be chosen so that $q \supseteq p_x$. Put $u = \{\langle r,z\rangle \in A \times w \ ; \ r \cup q \Vdash \underset{\sim}{z} \in \underset{\sim}{x}\}$;

then $u \in M$ and $K(u,x)$.

Also, if $K(u,x)$ and $K(u,y)$ then $\bar{x} \subseteq \bar{a}$ and $\bar{x} = \bar{y}$. For if $K(u,x)$ and $K(u,y)$, there are $p,q \in G$ with the appropriate properties for x and y , respectively. Then $p \Vdash \underset{\sim}{x} \subseteq \underset{\sim}{a}$, and so $\bar{x} \subseteq \bar{a}$. Take $\bar{z} \in \bar{x}$. Then $\bar{z} \in \bar{a}$; hence $z \in Da$ and so $rk(z) < rk(a)$. Choose $r \in G$ such that

$$r \supseteq p,q \quad ; \quad r \Vdash \underset{\sim}{z} \in \underset{\sim}{x} \quad ; \quad r \,\|\, \underset{\sim}{z} \in \underset{\sim}{y} .$$

Since $r \supseteq p$ and $p : A \models \Omega_x$, there is $s \in A$ such that $s \,\phi\, r$ and $s \cup p \,\|\, \underset{\sim}{z} \in \underset{\sim}{x}$. Hence $s \cup p \Vdash \underset{\sim}{z} \in \underset{\sim}{x}$, and so $\langle s,z \rangle \in u$. But since $K(u,y)$, it follows that $s \cup q \Vdash \underset{\sim}{z} \in \underset{\sim}{y}$. Now $r \,\|\, \underset{\sim}{z} \in \underset{\sim}{y}$, so since $r \,\phi\, s \cup q$ it must be the case that $r \Vdash \underset{\sim}{z} \in \underset{\sim}{y}$. Thus $\bar{z} \in \bar{y}$, and so $\bar{x} \subseteq \bar{y}$. Similarly, $\bar{y} \subseteq \bar{x}$. Hence $\bar{x} = \bar{y}$ as claimed.

Let k be defined by

$$k \;=\; \{\bar{x} \;;\; \exists u(u \in P_M(A \times w) \;\text{ and }\; K(u,x))\} .$$

Since G is a class in $M[G]$, it follows that $K(u,x)$ can be defined in $M[G]$. Further, $P_M(A \times w) \in M \subseteq M[G]$; (P_M is the power set operation in M). Hence, it follows from the replacement schema in $M[G]$ that $k \in M[G]$. However, from the remarks above, we have that if $\bar{x} \subseteq \bar{a}$ then $\bar{x} \in k$. Hence the power set axiom holds in $M[G]$.

Thus, whenever the class of forcing conditions satisfies the properties (a)-(d), $M[G]$ for any C-generic class G over M is a model of $ZF + AC$. The assumptions (a)-(d) have further consequences in such a generic model, as the following two lemmas show.

3.14 LEMMA Let C have the properties (a)-(d); let G be C-generic over M. For all limit ordinals $\sigma \in M$, if $(\mathrm{Cf}(\sigma))_M > F(\alpha)$ then also $(\mathrm{Cf}(\sigma))_{M[G]} > F(\alpha)$.

PROOF: Suppose $(\mathrm{Cf}(\sigma))_M > F(\alpha)$, and let f be a function in $M[G]$ mapping $F(\alpha)$ into σ. We must show that $\bigcup \mathrm{R} f \neq \sigma$.

Let $\dot{f} \in M$ be such that $\bar{\dot{f}} = f$. By Theorem 2.11, there is $p' \in G$ such that $p' \Vdash (\dot{f}$ is a function mapping $F(\hat{\underset{\sim}{\alpha}})$ into $\hat{\underset{\sim}{\sigma}}$). (Of course, here and henceforth, this means $p' \Vdash \Psi$, where Ψ is the appropriate formal sentence of $\mathcal{L}_M$.) Put

$$\Omega = \{\text{"}\exists v \in \hat{\underset{\sim}{\sigma}}(\dot{f}(\hat{\underset{\sim}{\tau}}) = v)\text{"} ; \tau \in F(\alpha)\},$$

so $\Omega \in M$ and $M \models (|\Omega| = F(\alpha))$. Hence by Theorem 3.9, there is $p \in G$ and $a \subseteq C$ with $|a| \leq F(\alpha)$ in M, such that $p{:}a \Vdash\!\models \Omega$.

For $r \in a$ and $\tau \in F(\alpha)$, define a function g by

$$g(r,\tau) = \begin{cases} \text{least } \nu < \sigma \text{ such that } r \cup p \Vdash \underset{\sim}{\dot{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\nu}}, \text{ if there is such a } \nu; \\ 0 \text{ otherwise}. \end{cases}$$

Then $g \in M$. Since $M \models (|a \times F(\alpha)| = F(\alpha)$ and $\mathrm{Cf}(\sigma) > F(\alpha))$, if δ is defined by

$$\delta = \bigcup\{g(r,\tau) ; r \in a \text{ and } \tau \in F(\alpha)\},$$

then $\delta \in M$ and in fact $\delta < \sigma$. We shall show

$$(1) \qquad \forall\tau \in F(\alpha)\ \forall\nu(\delta \leqq \nu < \sigma \ \Rightarrow\ M[G] \models f(\tau) \neq \nu)\ ;$$

from this it follows that $M[G] \models \bigcup Rf \neq \sigma$, and the lemma will be proved.

Suppose there is $\tau \in F(\alpha)$ and ν with $\delta \leqq \nu < \sigma$ such that $M[G] \models f(\tau) = \nu$. Then there must be $q \in G$ for which $q \Vdash \dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\nu}}$, and we may suppose that $q \supseteq p', p$. Thus also $q \Vdash (\mathring{\underset{\sim}{f}}$ is a function). Since $p{:}a \Vdash \Omega$, there is $r \in a$ such that $r \Phi q$ and $r \cup p \parallel \exists v \in \hat{\underset{\sim}{\sigma}}(\dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = v)$. Now $q \Vdash \dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\nu}}$, and $\hat{\nu} \in_{\emptyset} \hat{\sigma}$, so that $q \Vdash \exists v \in \hat{\underset{\sim}{\sigma}}(\dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = v)$. However, q is compatible with $r \cup p$, and so $r \cup p \Vdash \exists v \in \hat{\underset{\sim}{\sigma}}(\dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = v)$. Thus by Definition 2.4, $r \cup p \Vdash \dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\nu}}$ for some $\nu < \sigma$. Hence by the definition of δ , there is $\gamma < \delta$ such that $r \cup p \Vdash \dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\gamma}}$. But now, $r \cup q \Vdash (\dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\nu}}$ and $\dot{\underset{\sim}{f}}(\hat{\underset{\sim}{\tau}}) = \hat{\underset{\sim}{\gamma}}$ and $\dot{\underset{\sim}{f}}$ is a function), since $q \supseteq p$. Hence if H is any generic class with $r \cup q \in H$,

$$M[H] \models (\dot{f}^{H}(\tau) = \nu \text{ and } \dot{f}^{H}(\tau) = \gamma \text{ and } \dot{f}^{H} \text{ is a function}) .$$

Thus $\nu = \gamma$. This is impossible, however, since $\gamma < \delta$ and $\delta \leqq \nu < \sigma$. Thus (1) must hold; and the proof is complete.

3.15 LEMMA Let C rejoice in properties (a)-(d) , let G be C-generic over M . Then $(|PF(\alpha)|)_{M[G]} \leqq (|C \restriction \alpha \times F(\alpha)|^{F(\alpha)})_M$.

PROOF: Given α , put $w = \{x \in M\ ;\ rk(x) \leqq rk(C \restriction \alpha \times F(\alpha) \times F(\alpha))\}$, so $w \in M$ by virtue of property (b) of C . Thus also $w \in M[G]$.

For $p \in C$ and $u \subseteq C$, for each x define

$$g_x(p,u) = \{\langle r,\tau\rangle \in u \times F(\alpha) \; ; \; r \cup p \in C \text{ and } r \cup p \Vdash \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}\} ,$$

so if $u \in M$ then always $g_x(p,u) \in M$. Put also

$$\Omega_x = \{"\hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}" \; ; \; \tau \in F(\alpha)\} ,$$

so $\Omega_x \in M$ and $M \models |\Omega_x| = F(\alpha)$. Define

$$K(x,y) \iff \exists p \in G \; \exists u \subseteq C \restriction \alpha \; (u \in w \text{ and } p{:}u \Vdash \Omega_x \text{ and } y = g_x(p,u)$$
$$\text{and } \exists f \in w \; (f \text{ is an injection from } u \text{ to } F(\alpha))) .$$

Then if $\bar{x} \subseteq F(\alpha)$, there is $y \in w$ with $y \subseteq C \restriction \alpha \times F(\alpha)$ such that $K(x,y)$ and $M \models |y| \leq F(\alpha)$. For, by Theorem 3.9, there is $p \in G$ and $a \subseteq C \restriction \alpha$ with $|a| \leq F(\alpha)$ in M for which $p{:}a \Vdash \Omega_x$. But then if $y = g_x(p,a)$, it follows that $K(x,y)$ and y has the other properties mentioned.

Further, if $\bar{x},\bar{x}' \subseteq F(\alpha)$ are such that $K(x,y)$ and $K(x',y)$, then $\bar{x} = \bar{x}'$. For suppose $K(x,y)$ and $K(x',y)$, so there are appropriate $p,p' \in G$ and $u,u' \subseteq C \restriction \alpha$ for x and x', respectively. Let $\tau \in \bar{x}$. Choose $q \in G$ such that

$$q \supseteq p,p' \quad ; \quad q \Vdash \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x} \quad ; \quad q \parallel \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}' .$$

Since $q \supseteq p$ and $p{:}u \Vdash \Omega_x$, there is $r \in u$ such that $q \,\phi\, r$ and $p \cup r \parallel \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}$. Because $q \,\phi\, p \cup r$ and $q \Vdash \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}$, it must be that $p \cup r \Vdash \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}$, and so $\langle r,\tau\rangle \in y$. But then $r \cup p' \Vdash \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}'$, since also $y = g_{x'}(p',u')$. However, $q \,\phi\, r \cup p'$ and q was chosen so that $q \parallel \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}'$. Hence $q \Vdash \hat{\underset{\sim}{\tau}} \in \underset{\sim}{x}'$, and so $\tau \in \bar{x}'$. Thus $\bar{x} \subseteq \bar{x}'$, and similarly $\bar{x}' \subseteq \bar{x}$.

Now define

$$k = \{\langle \bar{x}, v\rangle \; ; \; \bar{x} \subseteq F(\alpha) \text{ and } v \subseteq C \restriction \alpha \times F(\alpha) \text{ and } K(x,v)$$
$$\text{and } \exists f \in w(f \text{ is an injection from } v \text{ to } F(\alpha))\} \,.$$

Since $w \in M[G]$ and G is a class in $M[G]$, it follows that k can be defined in $M[G]$. Since $M[G] \models k \subseteq PF(\alpha) \times P(C \restriction \alpha \times F(\alpha))$, in fact $k \in M[G]$. Moreover, $Dk = P_{M[G]}F(\alpha)$. Also k is a one:one relation, by the one:one property of $K(x,y)$ established above. Since $Rk \subseteq (\{y \subseteq C \restriction \alpha \times F(\alpha) \; ; \; |y| \leq F(\alpha)\})_M$,

$$M[G] \models |PF(\alpha)| \leq |(\{y \subseteq C \restriction \alpha \times F(\alpha) \; ; \; |y| \leq F(\alpha)\})_M| \,.$$

Hence $(|PF(\alpha)|)_{M[G]} \leq (|C \restriction \alpha \times F(\alpha)|^{F(\alpha)})_M$. This proves Lemma 3.15.

The results of this section are summarized by the theorem:

3.16 THEOREM Let M be a CTM . Let C be a class in M with $\{x \in p \; ; \; rk(x) < \alpha\} \in C$ whenever $p \in C$, and assume C enjoys the properties (a)-(d) . Let G be a C-generic class over M . Then $M[G]$ is a CTM , extending M and having the same ordinals as M , such that the following further properties hold:

(1) $(Cf(\sigma))_M > F(\alpha) \Rightarrow (Cf(\sigma))_{M[G]} > F(\alpha)$, for all limit ordinals $\sigma \in M$,

(2) $$(|PF(\alpha)|)_{M[G]} \leq (|C \restriction \alpha \times F(\alpha)|^{F(\alpha)})_M \,.$$

§4. Applications

The machinery developed in the earlier sections of this chapter will now be used to establish the theorems of Easton and Jensen, mentioned above.

4.1 THEOREM (Easton [4]) Let M be a CTM in which the GCH is true. Let K be a function class in M, mapping the ordinals into the ordinals, with the properties (in M):

(1) $$\alpha \leqslant \beta \Rightarrow K(\alpha) \leqslant K(\beta) \quad ,$$

(2) $$\alpha \leqq \beta \Rightarrow \mathrm{Cf}(\omega_\alpha) < \mathrm{Cf}(\omega_{K(\beta)}) \quad .$$

Then there is a CTM, N, which is an extension of M and has the same ordinals and cardinals as M, such that

(3) $$N \models \forall\alpha(\omega_\alpha \text{ is regular} \Rightarrow 2^{\omega_\alpha} = \omega_{K(\alpha)}) \quad .$$

Note 1 Easton proved this theorem for M a model of Gödel-Bernays set theory. In his context, K could be any class in the Gödel sense, rather than only a definable class, as is the meaning of the word in ZF set theory.

Note 2 Provided that the definition of K is of the appropriate form, Theorem 4.1 leads to a relative consistency result. If K is absolute from M to N (in particular, if K is absolute for all transitive models of ZF) then, in view of the well known results of Cohen [3], it follows that:

If the theory ZF is consistent, then so is the theory
$ZF + AC + \forall\alpha(\omega_\alpha \text{ is regular} \Rightarrow 2^{\omega_\alpha} = \omega_{K(\alpha)})$.

Theorem 4.1 will be proved by constructing a suitable set of forcing conditions, and considering the generic extension which results. So let M be a CTM in which the GCH is true.

Define, in M, a class C by:

$p \in C$ if and only if the following four conditions are met.

(α) p is a set of quadruples $\langle i,\xi,\alpha,\eta\rangle$ where $i \in 2$, α is an ordinal, $\xi < \omega_\alpha$ and $\eta < \omega_{K(\alpha)}$,

(β) $\langle i,\xi,\alpha,\eta\rangle \in p \Rightarrow \langle 1-i,\xi,\alpha,\eta\rangle \notin p$,

(γ) ω_α is regular $\Rightarrow |p \cap (2 \times \omega_\alpha \times \{\alpha\} \times \omega_{K(\alpha)})| < \omega_\alpha$,

(δ) ω_α is singular $\Rightarrow p \cap (2 \times \omega_\alpha \times \{\alpha\} \times \omega_{K(\alpha)}) = \emptyset$.

Thus if $p \in C$, then $\{x \in p \,;\, rk(x) < \alpha\} \in C$ for all α. If $p,q \in C$ and p,q are compatible, then certainly $p \cup q \in C$.

For each $p \in C$ and each ordinal ν, define

$$p \rfloor \nu = \{\langle i,\xi,\alpha,\eta\rangle \in p \,;\, \alpha \leq \nu\} ,$$

$$p \urcorner \nu = \{\langle i,\xi,\alpha,\eta\rangle \in p \,;\, \alpha > \nu\} .$$

Define a function class F in M, by

$$F(\alpha) = \omega_\alpha .$$

Then all the properties (a)-(d) are satisfied. Only property (c,iii) requires comment. So suppose ν is not a non-zero limit ordinal, and take $B \subseteq C \restriction \nu$ such that B is pairwise incompatible. We seek to show that $|B| \leq \omega_\nu$ in M. The following proof takes place in M.

Define a mapping f into B as follows: for each $p \in C \restriction \nu$, if there is $p' \in B$ with $p' \supseteq p$, let $f(p)$ be such a p'; otherwise $p \notin Df$. For $p \in C$, put

$$Ep = \{\langle i,\alpha,\eta\rangle \in p ; \exists\xi(\langle i,\xi,\alpha,\eta\rangle \in p)\} .$$

Define a sequence B_β for $\beta < \omega_\nu$ by:

$$B_0 = \emptyset ,$$

$$B_{\beta+1} = B_\beta \cup \{f(p) ; Ep \subseteq \bigcup\{Ep' ; p' \in B_\beta\}\} ,$$

$$\beta = \bigcup\beta > 0 \Rightarrow B_\beta = \bigcup\{B_\gamma ; \gamma < \beta\} .$$

Then the equality $B = \bigcup\{B_\beta ; \beta < \omega_\nu\}$ holds. Clearly $\bigcup\{B_\beta ; \beta < \omega_\nu\} \subseteq B$, so take $p \in B$ and try to show that $p \in \bigcup\{B_\beta ; \beta < \omega_\nu\}$. Define

$$p^* = \{\langle i,\xi,\gamma,\eta\rangle \in p ; \langle i,\gamma,\eta\rangle \in \bigcup\{Ep' ; p' \in \bigcup\{B_\beta ; \beta < \omega_\nu\}\}\} .$$

Note that

$$(4) \qquad p \in C \restriction \nu \Rightarrow |p| < \omega_\nu ,$$

for if $p \in C \restriction 0$, then $|p| < \omega_0$; otherwise $|p| \leq \sum\{\omega_\alpha ; \alpha < \nu\} < \omega_\nu$, since here ν is a non-limit ordinal. Thus, in particular, $|Ep^*| < \omega_\nu$. Hence there is $\beta < \omega_\nu$ such that $Ep^* \subseteq \bigcup\{Ep' ; p' \in B_\beta\}$, since ω_ν is regular. Because $p \supseteq p^*$ and $p \in B$, it follows that $p^* \in Df$ and hence $f(p^*) \in B_{\beta+1}$. But then

$$f(p^*) \supseteq p^* \supseteq \{\langle i,\xi,\gamma,\eta\rangle \in p \; ; \; \langle i,\gamma,\eta\rangle \in Ef(p^*)\} \quad ,$$

since $f(p^*) \in B_{\beta+1}$. Hence $f(p^*)$ and p agree where they are both defined, and so $f(p^*) \cup p \in C$. Further, $f(p^*) \cup p$ extends both $f(p^*)$ and p, so that $f(p^*)$ and p are compatible. However, both $f(p^*)$ and p are in B, and B is pairwise incompatible. Hence $f(p^*) = p$, and so $p \in \bigcup\{B_\beta \; ; \; \beta < \omega_\nu\}$. Thus indeed, $B = \bigcup\{B_\beta \; ; \; \beta < \omega_\nu\}$.

By employing the GCH, induction will be used to show:

$$\beta < \omega_\nu \quad \Rightarrow \quad |B_\beta| \leqq \omega_\nu \quad .$$

The successor case alone is non-trivial, so take $\gamma < \omega_\nu$ and suppose $|B_\gamma| \leqq \omega_\nu$. It follows from (4) that $|\bigcup\{Ep' \; ; \; p' \in B_\gamma\}| \leqq \omega_\nu$, and hence $|\{Ep \; ; \; Ep \subseteq \bigcup\{Ep' \; ; \; p' \in B_\gamma\}\}| \leqq \omega_\nu^{\smile\omega_\nu} = \omega_\nu$. Put $D_2 p = \{\xi \; ; \; \exists i,\alpha,\eta(\langle i,\xi,\alpha,\eta\rangle \in p)\}$, then $D_2 p \subseteq \omega_\nu$ for $p \in C\!\downarrow\!\nu$. Hence $|\{D_2 p \; ; \; p \in C\!\downarrow\!\nu\}| \leqq \omega_\nu^{\smile\omega_\nu} = \omega_\nu$. Thus $|\{p \in C\!\downarrow\!\nu \; ; \; Ep \subseteq \bigcup\{Ep' \; ; \; p' \in B_\gamma\}\}| \leqq \omega_\nu\omega_\nu = \omega_\nu$. Hence $|B_{\gamma+1}| \leqq \omega_\nu$. This completes the inductive argument.

It now follows that $|B| \leqq \sum\{|B_\beta| \; ; \; \beta < \omega_\nu\} \leqq \omega_\nu$. This establishes that $C\!\downarrow\!\nu$ satisfies the ω_ν^{+}-chain condition.

Take a C-generic class G over M. Then Theorem 3.16 applies, and so $M[G]$ is a CTM extending M, with the same ordinals as M, and such that the properties (1) and (2) of Theorem 3.16 hold. Use induction to show that the cardinals of M and $M[G]$ are the same, i.e. for all α ,

$$(\omega_\alpha)_M = (\omega_\alpha)_{M[G]} \quad .$$

Certainly ω is the first cardinal of both. The induction step for α a limit ordinal is clear. So suppose $(\omega_\beta)_M = (\omega_\beta)_{M[G]}$. Now $(\mathrm{Cf}((\omega_{\beta+1})_M))_M > (\omega_\beta)_M = F(\beta)$, and so $(\mathrm{Cf}((\omega_{\beta+1})_M))_{M[G]} > F(\beta) = (\omega_\beta)_M = (\omega_\beta)_{M[G]}$. From this it is immediate that $(\omega_{\beta+1})_M = (\omega_{\beta+1})_{M[G]}$. Thus the cardinals of M and $M[G]$ are the same.

Theorem 4.1 will thus be proved if (3) can be verified for $M[G]$. So let ω_α be regular in $M[G]$. (No ambiguity results from using ω_α to denote the α-th cardinal of either M or $M[G]$.) By (2) of Theorem 3.16,

$$(5) \qquad (|P\omega_\alpha|)_{M[G]} \leq (|C \rfloor \alpha \times \omega_\alpha|^{\omega_\alpha})_M \; .$$

However, $|C \rfloor \alpha|$ can be estimated in M . Since clearly if $p \in C \rfloor \alpha$ then $|p| \leq \omega_\alpha$, it follows in view of property (2) of K and the GCH that, in M ,

$$|C \rfloor \alpha| \leq |P2 \times [\omega_\alpha]^{\leq \omega_\alpha} \times [\alpha+1]^{\leq \omega_\alpha} \times [\omega_{K(\alpha)}]^{\leq \omega_\alpha}| = \omega_{\alpha+1}\omega_{K(\alpha)} \; .$$

However, from property (2) it follows that $\alpha+1 \leq K(\alpha)$, so $|C \rfloor \alpha| \leq \omega_{K(\alpha)}$. Again using (2) and the GCH , this gives

$$(6) \qquad (|C \rfloor \alpha \times \omega_\alpha|^{\omega_\alpha})_M \leq (\omega_{K(\alpha)})_M \; .$$

Hence, combining (5) and (6), $(|P\omega_\alpha|)_{M[G]} \leqq \omega_{K(\alpha)}$.

Finally, to obtain the equality $(|P\omega_\alpha|)_{M[G]} = \omega_{K(\alpha)}$ demanded by (3), we must show $(|P\omega_\alpha|)_{M[G]} \geqq \omega_{K(\alpha)}$. However, the choice of C causes there to be at least $\omega_{K(\alpha)}$ subsets of ω_α in $M[G]$. For $\eta < \omega_{K(\alpha)}$, define

$$a_\eta = \{\xi \ ; \ \exists p \in G(\langle 1,\xi,\alpha,\eta\rangle \in p)\} .$$

Then for all η , it follows that $a_\eta \in M[G]$ and $a_\eta \subseteq \omega_\alpha$. Moreover,

$$\zeta \neq \eta \Rightarrow a_\zeta \neq a_\eta ,$$

for given $\zeta < \eta < \omega_{K(\alpha)}$, put

$$D = \{p \in C \ ; \ \exists\xi(\langle 0,\xi,\alpha,\zeta\rangle \in p \text{ and } \langle 1,\xi,\alpha,\eta\rangle \in p)\} .$$

Then D is a dense class in M . (Given $q \in C$, since $|\{\langle\xi,\eta\rangle \ ; \ \exists i(\langle i,\xi,\alpha,\eta\rangle \in q)\}| < \omega_\alpha$ there is $\beta < \omega_\alpha$ such that for $i = 0,1$:

$$\langle i,\beta,\alpha,\zeta\rangle \notin q \ ; \ \langle i,\beta,\alpha,\eta\rangle \notin q .$$

But then if $p = q \cup \{\langle 0,\beta,\alpha,\zeta\rangle , \langle 1,\beta,\alpha,\eta\rangle\}$ then $p \supseteq q$ and $p \in D$.) Thus $D \cap G \neq \emptyset$, and hence $a_\zeta \neq a_\eta$. This establishes that $(|P\omega_\alpha|)_{M[G]} \geqq \omega_{K(\alpha)}$.

Thus (3) has been shown to hold in $M[G]$, and the proof of Theorem 4.1 is complete.

4.2 THEOREM (Jensen [9]) Let M be a CTM. There is a CTM, N, which is an extension of M, which has the same ordinals as M and in which the GCH is true, such that every measurable cardinal of M is still a measurable cardinal in N.

Note This leads to the following relative consistency result: If the theory ZF + AC + $\exists\kappa$ (κ is a measurable cardinal) is consistent, then so is the theory ZF + AC + GCH + $\exists\kappa$ (κ is a measurable cardinal).

Let M be a CTM. The extension N of Theorem 4.2 will again be constructed as a generic extension for a suitable class of conditions.

Let the class C be defined in M by:
$p \in C$ if and only if the following three requirements are satisfied.

(α) p is a set of triples $\langle \xi,\alpha,\eta \rangle$ where α is an ordinal, $\xi < \underline{2}_\alpha^+$ and $\eta < \underline{2}_{\alpha+1}$,

(β) $|p \cap (\underline{2}_\alpha^+ \times \{\alpha\} \times \underline{2}_{\alpha+1})| \leq \underline{2}_\alpha$,

(γ) $\langle \xi,\alpha,\eta \rangle \in p$ and $\langle \xi,\alpha,\eta' \rangle \in p \Rightarrow \eta = \eta'$.

For each $p \in C$ and each ordinal ν, define

$$p \rfloor \nu = \{\langle \xi,\alpha,\eta \rangle \in p \,;\, \alpha < \nu\},$$

$$p \rceil \nu = \{\langle \xi,\alpha,\eta \rangle \in p \,;\, \alpha \geqq \nu\}.$$

Define a function class F in M by

$$F(\alpha) = \underline{2}_\alpha .$$

Then the conditions (a)-(d) are fulfilled by C and F . Again, only property (c,iii) deserves mention. This is established in the same manner as it was for the forcing conditions of Theorem 4.1. In fact, if ω is replaced by $\underline{2}$ throughout, the various definitions phrased for triples rather than for quadruples, and the exponentiation of beths appealed to in place of the GCH , then the earlier proof requires no further change.

Take a C-generic class G over M . Then Theorem 3.16 applies, and so M[G] is a CTM , extending M and with the same ordinals as M .

Furthermore, the GCH is true in M[G] . For the course of this verification, $\underline{2}_o$, $\underline{2}_1$, $\underline{2}_2$,... is the sequence of beths in M , and ω_o , ω_1 , ω_2 ,... is the sequence of infinite cardinals in M[G] . Transfinite induction will be used to show

(7) $$(|\underline{2}_\alpha|)_{M[G]} = \omega_\alpha .$$

Again, only the successor case requires comment. So suppose $(|\underline{2}_\beta|)_{M[G]} = \omega_\beta$. Now $(\underline{2}_\beta^+)_M \leq \underline{2}_{\beta+1}$, so certainly $(|(\underline{2}_\beta^+)_M|)_{M[G]} \leq (|\underline{2}_{\beta+1}|)_{M[G]}$. In fact

(8) $$(|(\underline{2}_\beta^+)_M|)_{M[G]} = (|\underline{2}_{\beta+1}|)_{M[G]} ,$$

for $\{\langle\xi,\eta\rangle ; \exists p \in G(\langle\xi,\beta,\eta\rangle \in p)\}$ is a surjection in M[G] from $(\underline{2}_\beta^+)_M$ onto $\underline{2}_{\beta+1}$ - it is a function by property (γ) of C , because any two elements of G are compatible; it has domain $(\underline{2}_\beta^+)_M$ and range $\underline{2}_{\beta+1}$ since both $\{p \in C ; \exists\eta(\langle\xi,\beta,\eta\rangle \in p)\}$ for fixed $\xi \in (\underline{2}_\beta^+)_M$ and

$\{p \in C \; ; \; \exists\xi(\langle \xi, \beta, \eta \rangle \in p)\}$ for fixed $\eta \in \underline{2}_{\beta+1}$ are dense classes in M, and so meet G.

Now $(\mathrm{Cf}((\underline{2}^+_\beta)_M))_M > \underline{2}_\beta = F(\beta)$, and so by (1) of Theorem 3.16, $(\mathrm{Cf}((\underline{2}^+_\beta)_M))_{M[G]} > \underline{2}_\beta$. Hence $(|(\underline{2}^+_\beta)_M|)_{M[G]} = (|\underline{2}_\beta|^+)_{M[G]}$. But in view of the inductive hypothesis, $(|\underline{2}_\beta|^+)_{M[G]} = (\omega^+_\beta)_{M[G]} = \omega_{\beta+1}$. Thus $(|\underline{2}_{\beta+1}|)_{M[G]} = \omega_{\beta+1}$, using (8). This completes the inductive step. Thus (7) is established.

It is now easy to show that always $2^{\omega_\alpha} = \omega_{\alpha+1}$ in $M[G]$. Observe that

$$(|C \rfloor \alpha|)_M \leq (|[\underline{2}^+_\alpha]^{\leq \underline{2}_\alpha} \times [\alpha]^{\leq \underline{2}_\alpha} \times [\underline{2}_{\alpha+1}]^{\leq \underline{2}_\alpha}|)_M = \underline{2}_{\alpha+1} .$$

By Theorem 3.16(2),

$$(|P\underline{2}_\alpha|)_{M[G]} \leq (|C \rfloor \alpha \times \underline{2}_\alpha|^{\underline{2}_\alpha})_M .$$

Hence $(|P\underline{2}_\alpha|)_{M[G]} \leq (|\underline{2}_{\alpha+1}|)_{M[G]} = \omega_{\alpha+1}$, and so $(2^{\omega_\alpha})_{M[G]} \leq \omega_{\alpha+1}$. Thus $M[G] \models 2^{\omega_\alpha} = \omega_{\alpha+1}$; and the GCH is indeed true in $M[G]$.

Only the fate of a measurable cardinal remains to be discussed. So let κ be a measurable cardinal in M. Thus there is, in M, a non-principal, κ-complete ultrafilter D on $P\kappa$, and D may be chosen to be normal (see, for example, [10, §1]). (That D is κ-complete means if $X \in [P\kappa]^{<\kappa}$ and $\bigcup X \in D$ then some $x \in X$ is in D. To say that D is normal means for any f mapping κ to κ such that $\{\alpha \in \kappa \; ; \; f(\alpha) < \alpha\} \in D$, there is $\nu < \kappa$ such that $\{\alpha \in \kappa \; ; \; f(\alpha) = \nu\} \in D$.)

Since κ measurable implies that κ is a limit beth (in fact, κ is inaccessible), it follows that κ is still a cardinal in $M[G]$.

Define in $M[G]$:

$$D^* = \{u \subseteq \kappa \ ; \ \exists x \in D(x \subseteq u)\} \ .$$

We shall show that D^* is a non-principal κ-complete ultrafilter in $M[G]$. This will prove Theorem 4.2.

Clearly D^* is a filter, with $D \subseteq D^*$ and $\emptyset \notin D^*$. Suppose D^* has been shown to be κ-complete, i.e.

(9) $\quad \sigma < \kappa$ and $\kappa = \bigcup\{z(\nu) \ ; \ \nu < \sigma\} \ \Rightarrow \ \exists \nu < \sigma(z(\nu) \in D^*)$.

Since $\emptyset \notin D^*$, it follows from (9) that D^* is prime. It is now immediate that D^* is non-principal, i.e. that $\bigcap D^* \notin D^*$, for $\kappa - \bigcap D \subseteq \kappa - \bigcap D^*$, and so $\kappa - \bigcap D^* \in D^*$ since $\kappa - \bigcap D \in D$. Thus only the verification of (9) remains.

Suppose there is $\sigma < \kappa$ such that $M[G] \models (\kappa = \bigcup\{z(\nu) \ ; \ \nu < \sigma\})$ for some function $z \in M[G]$ which maps σ into $P\kappa$. Let $\dot{z} \in M$ be such that $\bar{\dot{z}} = z$. Then, in particular, $M[G] \models \text{not}(\text{not}(\forall x \in \bar{\hat{\kappa}} \ \exists y \in \bar{\hat{\sigma}}(x \in \bar{\dot{z}}(y))))$. By Theorem 2.11, there must be $p \in G$ such that

(10) $$p \Vdash \neg\neg \forall u \in \underset{\sim}{\hat{\kappa}} \ \exists v \in \underset{\sim}{\hat{\sigma}}(u \in \underset{\sim}{\dot{z}}(v)) \ .$$

Apply Definition 2.4 and Lemma 2.7 to (10). For any p , observe that $\exists q \subseteq p(b \in_q \hat{a})$ if and only if $b = \hat{c}$ for some $c \in a$. Thus it follows that (10) is equivalent to:

(11) $\forall p' \supseteq p\ \forall\alpha \in \kappa\ \exists\nu \in \sigma\ \exists p'' \supseteq p'(p'' \Vdash \neg\neg(\hat{\underset{\sim}{\alpha}} \in \dot{\underset{\sim}{z}}(\hat{\underset{\sim}{\nu}})))$.

Take any $q \supseteq p$. We shall seek $r \supseteq q$ such that

(12) $\exists\nu \in \sigma\ \exists x \in D\ \forall\alpha \in x(r \Vdash \neg\neg(\hat{\underset{\sim}{\alpha}} \in \dot{\underset{\sim}{z}}(\hat{\underset{\sim}{\nu}})))$.

For (12) is equivalent to

$$r \Vdash \exists u \in \hat{\underset{\sim}{\sigma}}\ \exists v \in \hat{\underset{\sim}{D}}(v \subseteq \dot{\underset{\sim}{z}}(u)) ,$$

and so $\{r ; r \Vdash \exists u \in \hat{\underset{\sim}{\sigma}}\ \exists v \in \hat{\underset{\sim}{D}}(v \subseteq \dot{\underset{\sim}{z}}(u))\}$ is dense below p . Hence, in view of Lemmas 1.4 and 2.11,

$$M[G] \models \exists\nu \in \sigma\ \exists x \in D(x \subseteq z(\nu)) ,$$

and so it follows from the definition of D^* that

$$\exists\nu \in \sigma(z(\nu) \in D^*) .$$

This will justify (9).

The remainder of the argument takes place in M .

Use induction to construct a sequence $\langle q_\alpha ; \alpha < \kappa\rangle$ of conditions and a sequence $\langle \nu_\alpha ; \alpha < \kappa\rangle$ of ordinals less than σ such that

$$q_\alpha \supseteq q \quad ; \quad \beta < \alpha \Rightarrow q_\alpha \supseteq q_\beta \restriction \beta \quad ; \quad q_\alpha \Vdash \neg\neg(\hat{\underset{\sim}{\alpha}} \in \dot{\underset{\sim}{z}}(\underset{\sim}{\hat{\nu}_\alpha})) .$$

Suppose $\gamma < \kappa$ is given such that q_β and ν_β have already been defined for all $\beta < \gamma$. By properties (a,v) and (d) of C , it follows that $q \cup \bigcup\{q_\beta \restriction \beta ; \beta < \gamma\}$ is a condition. Since $q \supseteq p$, (11) may be applied with $p' = q \cup \bigcup\{q_\beta \urcorner \beta \,;\, \beta < \gamma\}$ and $\alpha = \gamma$. This yields suitable q_γ and ν_γ.

Define f mapping $\kappa \times \kappa$ into κ as follows:

$$f(\alpha,\beta) = \begin{cases} 0 \text{ , if there is no } \nu \leqq \alpha,\beta \text{ such that } q_\alpha \upharpoonright \nu = q_\beta \upharpoonright \nu \,; \\ \\ \text{greatest } \nu \leqq \alpha,\beta \text{ such that } q_\alpha \upharpoonright \nu = q_\beta \upharpoonright \nu \text{ , otherwise .} \end{cases}$$

Note that when ν is a limit ordinal, if $p \upharpoonright \tau = p' \upharpoonright \tau$ for all $\tau < \nu$, then $p \upharpoonright \nu = p' \upharpoonright \nu$. Thus $f(\alpha,\beta)$ is well defined.

From the fact that D is normal, it follows (non-trivially) that there is $x \in D$ such that either

(13) $$\exists \rho < \kappa \; \forall \alpha,\beta \in x (f(\alpha,\beta) = \rho) \text{ , or}$$

(14) $$\forall \alpha,\beta \in x (f(\alpha,\beta) = \min\{\alpha,\beta\}) \text{ .}$$

Suppose (13) holds, so whenever $\rho < \alpha < \beta < \kappa$ and $\alpha,\beta \in x$ then $q_\alpha \upharpoonright \rho{+}1 \neq q_\beta \upharpoonright \rho{+}1$. Thus $|C \upharpoonright \rho{+}1| \geqq \kappa$, since if $x \in D$ then $|x| = \kappa$. However, as calculated earlier, $|C \upharpoonright \rho{+}1| \leqq \underline{2}_{\rho+2}$. Since κ is inaccessible and $\rho < \kappa$, it follows that $\underline{2}_{\rho+2} < \kappa$, so $|C \upharpoonright \rho{+}1| < \kappa$. This contradiction shows that (14) must hold.

Thus if $\alpha,\beta \in x$ then $\alpha < \beta \Rightarrow q_\alpha \upharpoonright \alpha = q_\beta \upharpoonright \alpha$. From this it follows that $\bigcup\{q_\alpha \upharpoonright \alpha \,;\, \alpha \in x\}$ is a condition. Furthermore, by the choice of the conditions q_α also $\bigcup\{q_\alpha \urcorner \alpha \,;\, \alpha \in x\}$ is a condition. Hence if

$$r = \bigcup\{q_\alpha \upharpoonright \alpha \,;\, \alpha \in x\} \cup \bigcup\{q_\alpha \urcorner \alpha \,;\, \alpha \in x\} = \bigcup\{q_\alpha \,;\, \alpha \in x\} \text{ ,}$$

then $r \in C$. Moreover, if $\alpha \in x$ then $r \supseteq q_\alpha$ and so $r \Vdash \neg\neg(\underset{\sim}{\hat{\alpha}} \in \dot{\underset{\sim}{z}}(\hat{\nu}_{\underset{\sim}{\alpha}}))$. Define, for $\nu < \sigma$,

$$a(\nu) = \{\alpha \in x \,;\, r \Vdash \neg\neg(\underset{\sim}{\hat{\alpha}} \in \dot{\underset{\sim}{z}}(\underset{\sim}{\hat{\nu}}))\} .$$

Then certainly $x = \bigcup\{a(\nu) \,;\, \nu < \sigma\}$. Now $x \in D$, $\sigma < \kappa$ and D is κ-complete. Hence there is $\nu' < \sigma$ such that $a(\nu') \in D$. Since $r \supseteq q$, it follows that the choice $\nu = \nu'$ and $x = a(\nu')$ suffices to satisfy (12). This completes the proof of Theorem 4.2.

<u>Note 1</u> The weak forcing relation $p \Vdash^* \Phi$, defined by

$$p \Vdash^* \Phi \iff p \Vdash \neg\neg\Phi ,$$

(which was in fact used in the last section of the proof above) has the desirable property that if $\Phi \iff \Psi$, then $p \Vdash^* \Phi \iff p \Vdash^* \Psi$, for all p. In particular, $p \Vdash^* \Phi \iff p \Vdash^* \neg\neg\Phi$. This not true in general for the relation $p \Vdash \Phi$.

<u>Note 2</u> Jensen has established the following further properties of the model $M[G]$ constructed during the proof of Theorem 4.2:

(1) If κ is a Ramsey cardinal in M, then κ is still a Ramsey cardinal in $M[G]$.

(2) If η is a limit ordinal and $\overset{M}{\cancel{M[G]}} \models$ (κ is the least cardinal such that $\kappa \to (\eta)_2^{<\omega}$), then also $M[G] \models \kappa \to (\eta)_2^{<\omega}$.

<u>Note 3</u> Silver has obtained the relative consistency result which stems from Theorem 4.2 by a completely different method. He shows that if D is

a normal, κ-complete, non-principal ultrafilter for a cardinal κ , then $L(D)$, the universe of sets constructible from D , is a model of $ZF + AC + GCH$ in which κ remains measurable.

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Cardinal numbers with partition properties.

Corrigenda.

p 16. Definition 2.2.

The requirement $|\cup\{H_i ; i < m\}| \geq m$ does not achieve its stated aim. It should be replaced by:

$$\forall i \in \{1,\ldots,m\}\ \exists a_i \in H_i (|\{a_1,\ldots,a_m\}| = m) .$$

A similar change should be made to Definition 3.5 on p 24.

p 51. Proof of Theorem 1.5.

The element x_ρ occurring on p 52, ℓ 8 has not been defined. Clearly, one must choose $x_\rho \in S'(X)$. However, there is then no guarantee that $x_\tau < x_\rho$ for $\tau < \rho$. This can be ensured by modifying the definition of $F(X)$ when constructing the ramification system $\underline{\underline{R}}$. Consequently, the definition on p 51 should be changed to read:

If $S'(X) = \phi$, put $F(X) = \phi$ and $n(X) = 0$. Otherwise, choose $x(X) \in S'(X)$ and put $F(X) = \{x \in S'(X) ; x \leq x(X)\}$. Place $G(X) = \{x(X \restriction \tau) ; \tau \leq \sigma\}$.

p 121. Proof of Theorem 4.1.

The class C of conditions defined here fails to satisfy property (a,v) of p 103 if ω_σ is inaccessible. This may be overcome as follows.

Notice that Theorems 3.12 and 3.13 may be proved on the strength of Lemma 3.7, without appeal to Lemma 3.8. Hence if a class C of forcing conditions satisfies the properties (a) - (d), with the exception of (a,v), and if G is C-generic over M, then M[G] is still a model of ZF + AC. Furthermore, Theorems 3.14 and 3.15 may be proved for all α to which Lemma 3.7 applies. Hence for this particular choice of C, Theorem 3.16 is valid when the properties (1) and (2) are restricted to hold only when ω_α is regular and accessible. This suffices to prove (3) of Theorem 4.1 whenever ω_α is regular and accessible.

To obtain Theorem 4.1 as stated, a different choice of C is required. Modify the definition by replacing clause (γ) by

(γ') ω_α is regular $\Rightarrow |p \cap (2 \times \omega_\alpha \times (\alpha + 1) \times \omega_{K(\alpha)})| < \omega_\alpha$.

Property (a,v) still fails; however now $C\downharpoonright\alpha$ satisfies the $F(\alpha)^+$-chain condition whenever ω_α is regular. This means that Lemma 3.7 holds whenever ω_α is regular, and as above, this yields Theorem 4.1.