Bekenstein–Hawking Entropy in Classical and Quantum Gravity

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To you

Declaration

The work in this thesis is my own except where otherwise stated.

Peter Ilyk

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Abstract

We outline the background and classical derivation of the black hole entropy formula as described in Bekenstein 1973 [Bek73]. This construction was highly physical and lacked sufficient mathematical rigour, which we use to motivate transitioning to loop quantum gravity to obtain a rigorous microscopic origin of this entropy. We recast general relativity into a form that allows us to perform deformation quantisation. Upon quantising, we describe the Hilbert space of the theory following Ashtekar et al. [ABK00]. This allows us to obtain the black hole entropy formula of quantum gravity by counting certain basis states of this Hilbert space, proving several claims made by the authors. We finally outline some directions for future research and propose an alternative direction for the derivation based on concepts from knot theory and homology.

Contents

Notation and terminology

Throughout the thesis, we will assume all manifolds are smooth unless otherwise specified.

Notation

$\hbar \approx 1.054535 \times 10^{-34}\mathrm{Js}$	Reduced Planck constant
$G \approx 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$	Newton's gravitational constant
$c \approx 2.99 \times 10^8\mathrm{ms}^{-1}$	Speed of light in vacuum
$\ell_p \approx 10^{-35}\mathrm{m}$	Planck length
$\kappa = \frac{8\pi G}{c^3}$	Gravitational coupling constant
$k_B = 1.381 \times 10^{-23} JK^{-1}$	Boltzmann's Constant
β	Barbero-Immirzi parameter
$\mathcal{M}$	4-dimensional Lorentzian spacetime manifold
$J^{\pm}(p)$	Causal future/past of an event $p \in \mathcal{M}$
$I^{\pm}(p)$	Chronological future/past of an event $p \in \mathcal{M}$
Σ	Cauchy hypersurface
$\mu, \nu, \rho ... = 0, 1, 2, 3$	Tensorial spacetime indices
$a, b, c, ...$	Penrose abstract tensor indices OR Tensorial spatial indices
$I, J, K, ... = 0, 1, 2, 3$	Vector indices for fibres of the Minkowski bundle
$i, j, k, ... = 1, 2, 3$	Vector indices for $\mathbb{R}^3$
$\epsilon_{a_1 ... a_D}$	Levi-Civita totally antisymmetric symbol in D dimensions
Γ_P	Symplectic Banach manifold of field configurations
$\Omega^k(M, \mathfrak{g})$	Space of Lie algebra valued k-forms on the base space of a principal G-bundle

Leg	Legendre transform
$H^*(X;R)$	Singular cohomology of a topological space X with coefficients in a ring R
τ_i	Lie algebra generators for $\mathfrak{so}(3)$
$\mathcal{B}$	Borel σ-algebra
$\mathcal{D}'(U)$	Space of tempered distributions over $U \subset \mathbb{R}^n$.
$\mathcal{I}$	Indexing set
$\mathcal{L}\{-\}$	Laplace transform
$\hat{\rho}$	Density operator
$\mathcal{E}$	The space of all edges on a manifold M
γ	Geodesic OR oriented finite semianalytic graph
Cyl^∞	C^*-algebra of complex-valued cylindrical functions
$\mathrm{Spec}(\mathfrak{B})$	Spectrum of a unital Banach algebra $\mathfrak{B}$

Terminology

Natural units	Throughout this thesis, unless we restore units or state otherwise, we will assume natural units, i.e. $\hbar = c = G = 1$.
Smearing field	Borrowed from physics, we write this to mean the natural pairing $\phi(f) := \langle \phi, f \rangle_{L^1(\mathbb{R}^d)}$ of an operator-valued distribution ϕ with a test function f.

Introduction

For as long as humanity has existed, we have questioned the nature and structure of the world around us. The breakthroughs of mathematics and science have led us closer to a complete understanding of the universe, yet we still lack a complete picture. The solution to this seems embedded in the marriage of general relativity and quantum mechanics. While this remains unresolved, examination of key concepts in both fields has led to promising developments.

In 1973, mathematical physicists Jacob D. Bekenstein and Stephen Hawking made a remarkable discovery - black holes, as described by general relativity, exhibit a type of information entropy [Bek73].

Theorem. *The entropy of a black hole in general relativity is given by*

$$S_{bh} = \frac{A}{4\pi\hbar G}, \tag{1}$$

where A is the area of the black hole event horizon.

This result gives a correspondence between the area of a black hole's event horizon and thermodynamic entropy, implying a connection between general relativity and statistical mechanics, and consequently quantum mechanics. This inference led physicists to attempt to describe a microscopic origin for this entropy, resulting in advancements in string theory, quantum gravity and the development of entanglement entropy in quantum field theory.

This thesis is dedicated to obtaining the microscopic explanation for the above theorem, which was obtained by Abhay Ashtekar, Jerzy Lewandowski, Thomas Thiemann et al. in 1997 ([Ash86],[LOST06],[Thi07]) through the theory of loop quantum gravity – a mathemtically rigorous quantum theory of gravity constructed directly from general relativity, rather than attempting to incorporate perturbatively as is done in string theory of quantum field theory.

More formally, loop quantum gravity recasts the mathematical structures of general relativity into the language of fibre bundles and operator theory. This is significant, as the original derivation proposed by Bekenstein and Hawking for (1) was based on several physical assumptions and was not entirely rigorous. We aim to explore the classical theory of black hole entropy (Chapters 1-3), and see how this can lead us to a rigorous derivation of the entropy formula of (1) based on loop quantum gravity (Chapters 4-7).

Aside from the physics, the techniques of loop quantum gravity are of interest to mathematicians. On the geometric side, these are directly related to Chern-Simons theory and topological quantum field theory, which may allow for new perspectives in these areas. On the representation theoretic side, certain basis states (so-called *spin networks*) of Hilbert spaces in loop quantum gravity have been used in constructing new Skein modules and certain character varieties, which are of central importance to knot theory and low-dimensional topology. We now give a chapter-wise outline of this thesis.

0.1 Outline of the Thesis

In Chapter 1, we give a summary of tools and objects, principal bundle and operator theory, which we will consistently utilise throughout this thesis. We will provide many examples of objects introduced here, several of which will be utilised later in the thesis.

In Chapter 2, we explore several key concepts in general relativity, mainly causality and singularities. We conclude by defining black holes and their properties.

In Chapter 3, we focus on describing the derivation of the black hole entropy formula uncovered by Jacob Bekenstein. We describe the exact physical derivation provided by Bekenstein and discuss its implications and issues, motivating the passage to quantum gravity.

In Chapter 4, we generalise the notion of a black hole's boundary (the *event horizon*) to one which utilises local spacetime structure, allowing for a notion of a 'black hole at equilibrium' (an *isolated horizon*). We develop additional tools from fibre bundle theory before rigorously defining isolated horizons, and recast our classical theory from Chapter 2 into this language, a foundational step in the transition to loop quantum gravity.

In Chapter 5, we construct a quantum theory of gravity through deformation quantisation in the general setting (in contrast to traditional geometric quantisation). We construct our classical algebra of states before obtaining a quantum algebra, using operator theory and functional analysis. We conclude by describing the specific Hilbert space of states that will be utilised in the quantisation of black hole entropy.

In Chapter 6, we explicitly describe quantum states associated with the isolated horizon and the surrounding space, before implementing physical constraints to obtain our Hilbert space of physical states.

Finally, in Chapter 7, we utilise this Hilbert space to derive the quantum black hole entropy formula, reducing it to a counting problem [ABK00]. We explicitly show that this agrees with the result obtained by Bekenstein and Hawking classically. We conclude with a discussion of this result and propose some directions for future research.

We will assume the reader has a solid background in differential geometry, gauge theory and abstract algebra, as well as some familiarity with basic functional analysis, algebraic topology and complex geometry/analysis. We also recommend knowledge of general relativity equivalent to that of a first course, as well as basic quantum/statistical mechanics.

0.2 Statement of Originality

Unless stated otherwise, we do not claim the original ideas and methods used throughout this thesis, and we credit these to their respective sources. All figures are original unless otherwise stated. All exposition is original.

Much of the content of Chapter 1 follows [Ham17],[Thi07]. Example 1.34 and the proof of Lemma 1.23 is original. Chapter 2 largely follows [Wal84] and [Cho15]. Examples 2.9, 2.23 and Lemma 2.15 are original. Chapter 4 draws mostly from [Thi07], [Tec19] and [AK04]. Examples 4.6, 4.17,4.18, the proof of Lemma 4.8 and Proposition 4.19 are original. Chapter 5 follows [LOST06] and [Thi07]. The proofs of Theorem 5.5 and Lemma 5.11 are original. Chapter 6 follows very closely to [ABK00]. The proof of Proposition 6.19 is original. Chapter 7.1 also follows [ABK00]. We frame three claims from the source as Lemmas 7.15, 7.16, 7.17. In particular, the proofs of these are original, addressing claims the authors assume and skim over in the paper. Lemma 7.16 elaborates further on the authors' arguments and provides additional details not present in the paper, thus offering an extension to the literature. This makes Chapter 7 one of the most substantial. The content of 7.2 is original.

Chapter 1

Preliminaries

This chapter collates the geometric and analytical tools that we will use to describe the underlying geometry of general relativity and the quantisation of gravity. We begin with the most basic structure of general relativity - a 4-dimensional Lorentzian manifold known as a spacetime. We build on this by formally introducing principal bundles, and some key examples (particularly spin bundles) that underpin the geometric structure of loop quantum gravity. Finally, we summarise some essential ideas from operator algebras and Gel'fand theory, which play a crucial role in our later algebraic quantisation of general relativity.

The purpose of this chapter is to clearly state and describe the central objects and mathematical machinery of geometry and analysis, easing the reader into the advanced constructions that follow and enabling the quantisation of Bekenstein–Hawking entropy. Section 1.1 follows no particular reference, Sections 1.2-1.3 loosely follow the parts of [Ham17] and [Fri00], and Section 1.4 follows Chapter 27 of [Thi07].

1.1 Pseudo–Riemannian Geometry

We begin with a generalised variant of the Riemannian manifold from differential geometry:

Definition 1.1. Let M be a differentiable manifold equipped with a metric tensor $g : T_xM \times T_xM \to \mathbb{R}$ for $x \in M$. We say M is a **pseudo–Riemannian manifold** if g is non-degenerate, smooth and symmetric.

Comparing this to the usual definition of a Riemannian manifold, we have relaxed the positive-definite condition for g. The reason for this becomes clear when we want to classify so-called *timelike* and *spacelike* geodesics.

Definition 1.2. The **signature** of a metric tensor $g \in \Gamma(M, (TM \otimes TM)^*)$ is a function $\sigma : \Gamma(M, (TM \otimes TM)^*) \to \mathbb{Z}$ defined as the signature of its quadratic form Q associated to g, where $Q(v) := g(v, v)$* for $v \in T_xM$. That is, the triple (p, r, q), where p, r, q denote the number of positive, zero and negative eigenvalues, respectively. Since the metric is assumed

*This is basis-independent due to Sylvester's Law of Inertia to be non-degenerate, we will omit the r and simply write (p, q). In some cases, we may write $M^{p,q}$ to denote the signature of the metric on M explicitly.

If the metric g on a pseudo–Riemannian manifold M has signature (p, q), it is common in mathematics literature to say M itself has signature (p, q), referring to the metric signature.

Remark 1.3. Sometimes in physics literature, the signature may be denoted as

$$(\underbrace{+, ..., +}_{p}, \underbrace{0, ..., 0}_{r}, \underbrace{-, ..., -}_{q}),$$

which shows explicitly the sign of the eigenvalues.

Definition 1.4. Let (M, g) be a pseudo–Riemannian manifold of dimension n. We say (M, g) is **Lorentzian** if g has signature $(n-1, 1)$.

Definition 1.5. A **spacetime** $(\mathcal{M}, g, \nabla)$ is a connected 4-dimensional oriented Lorentzian manifold $(\mathcal{M}, g)$ of signature $(3, 1)$ with Levi-Civita connection ∇ of g on $\mathcal{M}$. We call a point $p \in \mathcal{M}$ an **event**.

We define the spacetime to be a Lorentzian manifold rather than a Riemannian one because we can separate space and time. This is desirable because these notions should be physically distinguished from one another (for example, one can physically travel in a loop in space, but the same is not true for time). More precisely, we distinguish space and time by classifying vectors $X \in T_p\mathcal{M}$ for $p \in \mathcal{M}$:

Definition 1.6. Let $(\mathcal{M}, g)$ be a spacetime and $X \in T_p\mathcal{M}$ for $p \in \mathcal{M}$. We say X is

- *timelike* if $g(X, X) < 0$
- *null* if $g(X, X) = 0$
- *spacelike* if $g(X, X) > 0$

This definition allows us to classify smooth curves $\lambda : [0, 1] \to \mathcal{M}$. We say that a curve is **timelike/spacelike/null** if every tangent vector to λ is timelike/spacelike/null for every $t \in [0, 1]$. Furthermore, we say λ is **causal** if every tangent vector is timelike OR null for all $t \in [0, 1]$.

Example 1.7. Let $\mathcal{M} = \mathbb{R}^{d-1,1} \cong \mathbb{R}^d$ and let $\eta = ds^2 = -dx_1^2 + \sum_{i=2}^{d-1} dx_i^2$. This is the *Minkowski metric* in local coordinates $(x_1, ..., x_d)$. The triple $(\mathbb{R}^{d-1,1}, \eta, \nabla)$, where ∇ is the usual Levi-Civita connection on $\mathbb{R}^d$ known as *Minkowski spacetime*. This is an example of a flat spacetime, which is the spacetime upon which special relativity is based.

Example 1.8. Anti–de Sitter spacetime $\mathrm{AdS_{p+q}}$ is a spacetime of signature (p, q) isometrically embedded in the space $\mathbb{R}^{p+1,q}$, with local coordinates $(x_1, ..., x_{p+1}, y_1, ..., y_q)$ satisfying

$$\sum_{j=1}^{p+1} x_j^2 - \sum_{i=1}^{q} y_i^2 = -R^2, \tag{1.1}$$

for some $R \neq 0$. In the case of the spacetime $\mathbb{R}^{d-1,2}$ with coordinates $(x_0, x_1, ..., x_{d-1}, x_d)$, we have $\mathrm{AdS_{d-1+1}}$ is the submanifold solving

$$\sum_{i=1}^{d-1} x_i^2 - (x_d)^2 - (x_0)^2 = -R^2. \tag{1.2}$$

In particular, $\mathrm{AdS_{d-1+1}} \cong \mathbb{R}^{d-1} \times S^1$. $\mathrm{AdS_{d-1+1}}$ describes spacetimes with negative sectional curvature. In cosmology, such spacetimes correspond to negative cosmological constants.

Notation 1.9. In the above discussions, we have expressed the metric tensor in local coordinates, where the Greek indices indicated basis components. This notation, while convenient in low dimensions, becomes extremely cumbersome when performing even the simplest operations (such as contraction) with higher-order tensors. To mitigate this, we will use so-called abstract index notation, wherein a tensor T of type (k, l) will be expressed as $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$, where the Latin indices denote the k contravariant and l covariant components respectively. Note these do not denote the basis component, but rather the types of objects the tensor *acts* on. As an example, the metric tensor in Minkowski spacetime, in local coordinates, is denoted as

$$\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu,$$

whereas in abstract index notation, we would denote this as η_{ab}, and we can read off exactly the type of tensor and how it behaves without introducing a basis.

Equivalently, one can think of this as 'labelling' each slot of a tensor with a Latin index. For η (a $(0, 2)$ tensor), it has 2 slots $\eta = \eta(-, -)$. We will utilise this notation in Chapters 2-3 as we will deal with multiple tensors, and drop it from Chapter 4 onwards.

1.2 Elements of Fibre Bundle Theory

We invite the reader to recall the basic definition of a fibre bundle:

Definition 1.10. Let E, F, B be connected topological spaces (known as the *total space, base space* and *fibre* respectively). A **fibre bundle** is defined by the quadruple (E, B, π, F), where $\pi : E \to B$ is a surjection that satisfies the so-called local triviality condition: Let $U \subset B$ be an open neighbourhood of $x \in B$. Then the local triviality condition states there exists a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ for which the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{proj}"] \\ U & \end{tikzcd}$$

We call the pair $\{U, \phi\}$ a **local trivialisation** of E.

This object is ubiquitous in differential geometry. In fact, we can build on this structure further by considering the *group action* by a topological group. This leads us to a key construction:

Definition 1.11. Let M be a topological space and let G be a topological group. A **principal G-bundle** is a a fibre bundle $P \xrightarrow{\pi} M$ for which the fibres are locally homeomorphic to G, along with a free and transitive right action on P by G

$$\begin{aligned} P \times G &\to P \\ (p, g) &\mapsto pg. \end{aligned}$$

We note that, similar to vector bundles, principal bundles admit local triviality. Given $U \subset M$, we have there exists a local trivialisation

$$\Psi : \pi^{-1}(U) \to U \times G, \tag{1.3}$$

with $\Psi(v) = (\pi(v), \bar{g}(v))$ where $v \in \pi^{-1}(U) \subset P$ and $\bar{g} : \pi^{-1}(U) \to G$ is a G-equivariant map into the group G. This allows group action on $\bar{g}$ by G to act in the usual way.

Conversely, given a principal bundle P, this can descend to a vector bundle:

Definition 1.12. Let $P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G over a finite dimensional vector space V. The **associated bundle** is the vector bundle given by

$$P \times_\rho V = \frac{P \times V}{(p, v) \sim (p \cdot g, \rho(g^{-1})v)}, \tag{1.4}$$

with each fibre isomorphic to V.

We now give some essential examples of principal G-bundles for this thesis.

Example 1.13. Let $E \to M$ be a rank k vector bundle over a manifold M. We define a *frame* at $x \in M$ to be the set of all ordered bases of the fibre E_x. This is naturally in bijection with the set of linear isomorphisms $T : \mathbb{R}^k \to E_x$. Denote the collection of all frames at x as $F_x(M)$. There is a continuous right action of $F_x(M)$ by $\mathrm{GL}(k, \mathbb{R})$ by the composition $T \cdot g : \mathbb{R}^k \to E_x$. Then the **frame bundle** is a principal $\mathrm{GL}(k, \mathbb{R})$-bundle over M is the disjoint union of all frames, denoted by $F(M) := \coprod_{x \in M} F_x(M)$.

Example 1.14. Let $P \to M$ be a principal G-bundle with G a Lie group. From Lie theory, we know for all g we have the action $\mathrm{Ad}_g : G \to G$ given $\mathrm{Ad}_g(h) = g^{-1}hg$ for $h \in G$. We also have the so-called *adjoint action* $\mathrm{ad}_g : \mathfrak{g} \to \mathfrak{g}$ on the Lie algebra. As an action, $\mathrm{ad}_g \in \mathrm{GL}(\mathfrak{g})$. This therefore gives us a representation $\mathrm{ad}_g : G \to \mathrm{GL}(\mathfrak{g})$. With this, we can form an associated vector bundle $\mathrm{ad}P := P \times_{\mathrm{ad}} \mathfrak{g}$ known as the **adjoint bundle**.

We can also recognise that the adjoint action defines an automorphism of G. This allows us to similarly form the associated bundle of groups $\mathrm{Ad}P := P \times_{\mathrm{Ad}} G$.

As with vector bundles, we can also equip principal bundles with a connection. This is a crucial structure for the bundles we will be examining, especially in later chapters.

Recall from differential geometry that given a Lie group G and manifold M, the *fundamental vector field* $\tilde{X}_p$ for $p \in M$ is the vector field associated with an element of the Lie algebra $\mathfrak{g}$. Specifically

$$\tilde{X}_p := \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX), \tag{1.5}$$

for $X \in \mathfrak{g}$. In particular, we have a Lie algebra homomorphism

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{X}(M) \\ X &\mapsto \tilde{X}. \end{aligned}$$

With this, we can define the connection one-form on a principal bundle:

Definition 1.15. Let $P \xrightarrow{\pi} M$ be a principal G-bundle. A **connection one-form** on P is a one form $\omega \in \Omega^1(P, \mathfrak{g})$ which satisfies the following:

1. $R_g^* \omega = \mathrm{Ad}_{g^{-1}}(\omega)$ for all $g \in G$
2. $\omega(\tilde{X}) = X$, where $\tilde{X} \in \mathfrak{X}(M)$ is the fundamental vector field associated to $X \in \mathfrak{g}$

Example 1.16. Let $M = \{*\}$ consist of a single point. Then we have the trivial principal bundle $G \to \{*\}$ which is simply the total space G. If G is a Lie group, there exists a unique connection one-form $\omega_G \in \Omega^1(G, \mathfrak{g})$ identifying the tangent space at each point with its Lie algebra, known as the *Maurer-Cartan form*. In particular, if G is a matrix Lie group, the Maurer-Cartan form is given by $\omega_G = g^{-1}dg$.

1.3 Spin Geometry

In this section, we introduce another type of principal bundle with its own special geometry, known as a spinor bundle. Such a bundle encapsulates the symmetries of geometric objects known as spinors (introduced later on in this section), which prove extremely useful in extending the geometric structure of our spacetime. We begin with the definition of a spin structure in the most general case:

Definition 1.17. Let $P \xrightarrow{\pi} M$ be a principal $SO(p,q)$–bundle over an $p+q = n$–dimensional space and orientable pseudo–Riemannian manifold M. A **spin structure** on P is a pair (Q, Ψ), for which Q is a $\mathrm{Spin}(p,q)$-principal bundle and $\Psi : Q \to P$ a 2-fold covering map, with respect to the double covering $\sigma : \mathrm{Spin}(p,q) \to SO(p,q)$, such that the following diagram commutes.

$$\begin{tikzcd}
Q \times \mathrm{Spin}(p,q) \arrow[r] \arrow[dd, "\Psi\times\sigma"] & Q \arrow[dr, "\tilde{\pi}"] \arrow[dd, "\Psi"] & \\
& & M \\
P \times SO(p,q) \arrow[r] & P \arrow[ur, "\pi"] &
\end{tikzcd}$$

Definition 1.18. Two spin structures (Q, Ψ), (Q', Ψ') are said to be **equivalent** if there exists a $\mathrm{Spin}(p, q)$-equivarient bundle isomorphism $\varphi : Q \to Q'$ such that the following diagram

$$\begin{tikzcd} Q \arrow[rr, "\varphi"] \arrow[dr, "\Psi"'] & & Q' \arrow[dl, "\Psi"] \\ & P & \end{tikzcd}$$

commutes

For completeness, the following lemma gives us the necessary condition for a principal bundle to admit a spin structure:

Lemma 1.19. *P admits a spin structure if and only if the second Stiefel–Whitney class $w_2(P) = 0$, where $w_i(P)$ is an element of the cohomology ring $H^i(M; \mathbb{Z}/2)$.*

For this thesis, we assume all our principal bundles obey this condition. For greater depth on characteristic classes and spin structures, we direct the reader to [Fri00],[Hat03].

We can see that many familiar manifolds of interest may be equipped with a spin structure. Below is a low-dimensional example in the Riemannian case:

Example 1.20. Let M $= S^1$ be orientated counterclockwise with the metric given by the usual Euclidean inner product, inherited from its embedding in $\mathbb{C} \cong \mathbb{R}^2$. Let $P \to S^1$ be the frame bundle $F(TS^1)$.

Since S^1 is of dimension 1, there is only one positively oriented tangent vector in T_zS^1 for all $z \in S^1$. Therefore, we have only a single choice of a positively oriented orthonormal frame, which determines the frame bundle.

$$F(S^1) \cong S^1.$$

As $SO(1) = \{1\}$, we have that by the action of σ: $\mathrm{Spin}(1) = \{\pm 1\} \cong \mathbb{Z}_2$. From here, two different spin structures arise. The first is $Q := S^1 \times \mathbb{Z}/2$ with $\Psi(z, q) = x$

The second is $Q' := \frac{S^1 \times \mathbb{Z}/2}{\sim}$, with equivalence given by $(\pm 1, 0) \sim (\mp 1, 2\pi)$, and $\Psi'([z, q]) = z$

From the above, one can observe that Q is a covering space of P, with a lift given by Ψ. Effectively, we have, in a sense, extended our principal bundle to be 'double covered' in the same way $\mathrm{Spin}(p, q)$ double covers $SO(p, q)$. The upshot is that we can now extend our fields on M to also include spinor fields, which are a fundamental object of study in theoretical physics. To describe these, let us define the spin representation:

Definition 1.21. The **spin representation** is a faithful representation of the spin group given by

$$\rho : \mathrm{Spin}(p, q) \to \mathrm{Aut}(\Delta_n), \tag{1.6}$$

where Δ_n is the Clifford module $(Cl(p, q))$ of n-spinors, which is isomorphic to $\mathbb{C}^{2k} = \bigotimes_{j=1}^{k} \mathbb{C}^2$ for $n = p + q = 2k, 2k + 1$ where $k \in \mathbb{N}$.

Elements of Δ_n are known as *spinors*. In many physical applications, they are represented as complex-valued vectors. This construction extends to the spin structure we defined above as follows:

Definition 1.22. Suppose (Q, Ψ) is a spin structure on a principal $SO(n)$-bundle $P \to M$. A **spinor bundle** is the associated complex vector bundle

$$S := Q \times_\rho \Delta_n, \tag{1.7}$$

which admits a Hermetian metric. An element $\psi \in \Gamma(S)$ is known as a **spinor field.**

Now that we have spinor fields included in the description of our bundle, it is natural also to obtain a covariant derivative that acts on spinors. As it turns out, we can construct this out of the connection one-forms that are already present on our principal bundle.

Recall the connection one-form ω from the previous section. We have that P is a subbundle of the frame bundle, since our structure group has been reduced from $GL(p,q)$ to $SO(p,q)$, and hence $\omega \in \Omega^1(P, \mathfrak{so}(p,q))$. Consider the homomorphism $\lambda : \mathrm{Spin}(p,q) \to SO(p,q)$ defined by sending $\pm\mathrm{id}_{\mathrm{Spin}(p,q)} \mapsto \mathrm{id}_{SO(p,q)}$ (since it is a double cover). This induces an isomorphism of Lie algebras

$$\lambda_* : \mathfrak{spin}(p,q) \cong \mathfrak{so}(p,q). \tag{1.8}$$

Using this, we can 'lift' our original connection one-form ω to obtain a connection one-form on the spinor bundles:

Lemma 1.23.

$$A := (\sigma_*)^{-1} \circ (\Psi^*\omega) \in \Omega^1(Q, \mathfrak{spin}(p,q)), \tag{1.9}$$

*is a connection one-form on Q known as the **spin connection**, where $\Psi^*\omega = \omega \circ d\Psi$, where $d\Psi : TQ \to TP$ is the tangent map of Ψ.*

Proof. To prove this, we need to check that this satisfies the conditions of a connection one-form as in Definition 1.15. We will mainly use the equivariance of the spin structure map Ψ.

Firstly, let $X \in TQ$ and $g \in \mathrm{Spin}(p,q)$. We obtain:

$$\begin{aligned} R_g^* A(X) = A(R_{*g}X) &= (\sigma_*)^{-1} \circ \omega(\Psi_*(R_{*g}X)) \\ &= (\sigma_*)^{-1} \circ \omega(R_{*\sigma(g)}\Psi_*(X)) \\ &= (\sigma_*)^{-1} \circ \omega((Ad_{\sigma(g)^{-1}}\Psi_*(X)) \\ &= (\sigma_*)^{-1} \circ Ad_{\sigma(g)^{-1}} \circ \omega((\Psi_*(X)) \\ &= \mathrm{Ad}_{g^{-1}} \circ A(X). \end{aligned} \tag{1.10}$$

In the second line, we used the equivariance of Ψ, the definition of the adjoint representation in terms of the pushforward of R_g in the third line, and the equivariance of ω and Ad in the fourth and fifth lines. This proves the first condition. The second condition follows similarly, using the definition of the fundamental vector field. □

Similar to usual connection one-forms on P, we can pull back A by a local frame on the spinor bundle S associated to Q to obtain a local description of A:

Definition 1.24. Let $s \in \Gamma(S)$ be a spinor on $U \subset M$. The **associated local spin connection** is defined by the pullback

$$s^*A \in \Omega^1(U, \mathfrak{spin}(p,q)). \tag{1.11}$$

Example 1.25. Consider a principal $SO(3)$-bundle $F_{SO}(M) \to M$. Let $s : U \to S$ be a section of the spinor bundle. Using the exterior covariant derivative, as well as lifting $\sigma_*^{-1} : \mathfrak{so}(3) \to \mathfrak{spin}(3) \cong \mathfrak{su}(2)$, we obtain the local spin connection to be

$$s^*A = \omega_{ij}\sigma_*^{-1}(\tau^i \wedge \tau^j), \tag{1.12}$$

where ω_{ij} is the connection one-form associated to a frame $e : U \to F_{SO}(M)$ and τ^i are Pauli spin matrices, which form a basis for $\mathfrak{so}(3)$.

We have all the required tools of differential geometry. Next, we describe some essential tools from operator algebras.

1.4 Operator Algebras

The purpose of this section is to outline some tools from operator theory to allow a mathematical description of the construction of the quantum *-algebra associated with the configuration space defined in Chapter 4. In particular, we introduce Banach and C*-algebras in full generality, followed by spectra, as well as a topology introduced by Gel'fand on the spectrum of certain Banach algebras. For greater depth, we direct the reader to [KR97].

Definition 1.26. An **involution** on an algebra $\mathcal{A}$ is a map $* : \mathcal{A} \to \mathcal{A}$ defined by $a \mapsto a^*$ which, for all $z_1, z_2, \in \mathbb{C}$, $a, b \in \mathcal{A}$, satisfies the following properties:

1. $(z_1 a + z_2 b)^* = \bar{z}_1 a^* + \bar{z}_2 b^*$
2. $(ab)^* = b^* a^*$
3. $(a^*)^* = a$

By equipping $\mathcal{A}$ with an involution, we call the pair $(\mathcal{A}, *)$ a ***-algebra**

As with metric spaces, we can equip algebras with a notion of magnitude. In particular, equipping an algebra $\mathcal{A}$ with a norm $\|.\| : \mathcal{A} \to \mathbb{R}_{\geq 0}$ gives us the following:

Definition 1.27. A **normed algebra** is an algebra $\mathcal{A}$ equipped with a norm $\|.\|$. If $\mathcal{A}$ is a *-algebra, then we require $\|a^*\| = \|a\|$. If $\mathcal{A}$ is unital, we require $\|1\| = 1$.

Naturally, a norm induces a metric $d(a,b) := \|a - b\|,\ a, b \in \mathcal{A}$. If $\mathcal{A}$ is complete under this norm, we obtain a *Banach algebra*.

Definition 1.28. A **C*-algebra** $\mathfrak{B}$ is a Banach *-algebra such that, for all $b \in \mathfrak{B}$

$$\|b^*b\| = \|b\|^2 \tag{1.13}$$

Such algebras are prevalent in the mathematical foundations of quantum mechanics. One can observe that the defining properties of a C*-algebra closely resemble properties of operators in quantum mechanics, i.e. operators on Hilbert spaces. In particular, since unbounded operators do not form an algebra, we have C*-algebras define the algebra of bounded linear operators on Hilbert spaces, with $\|.\|$ playing the role of the operator norm.

Accordingly, we can consider a generalisation of the spectrum of an operator on a Hilbert space:

Definition 1.29. Suppose $\mathfrak{B}$ is a unital Banach algebra and let $b \in \mathfrak{B}$. The **spectrum** $\sigma(b)$ of b is defined as the set

$$\sigma(b) := \{\lambda \in \mathbb{C} \mid (b - \lambda 1) \text{ is not invertible}\}. \tag{1.14}$$

Recall that a unital algebra contains a multiplicative identity element. We also have a notion for the spectrum of the entire algebra. This is really a special case of equivalence classes of *-representations of algebras on Hilbert spaces:

Definition 1.30. The **spectrum** of a unital Banach algebra $\mathfrak{B}$ is the set $\mathrm{Spec}(\mathfrak{B})$ of all non-zero *-homomorphisms $\chi : \mathfrak{B} \to \mathbb{C}$. The homomorphism χ is known as a *character*.

This is equivalently the set of irreducible representations of the algebra over $\mathbb{C}$. In particular, this may be thought of as the 'points' in the underlying Hilbert space which are acted on by the algebra.

Given a unital Banach algebra, we have a natural topology induced by the norm. From this, it is natural to ask about the topology on $\mathrm{Spec}(\mathfrak{B})$. Of course, we have the norm topology induced by the inner product on $\mathbb{C}$. However, if our Banach algebra is also commutative, we can endow it with another topology:

Definition 1.31. Let $\mathfrak{B}$ be a commutative unital Banach algebra. The **Gel'fand toplogy** on $\mathrm{Spec}(\mathfrak{B})$ is the weak *-topology induced by the topological dual $\overline{\mathfrak{B}}$ on $\mathrm{Spec}(\mathfrak{B})$. That is, the weakest topology such that all functions $z : \overline{\mathfrak{B}} \to \mathbb{C}$ given by $\phi \mapsto \phi(z)$ are continuous.

With this topology, $\mathrm{Spec}(\mathfrak{B})$ is known as the **Gel'fand spectrum**.

Note in the above definition, we identified $\mathfrak{B} \cong \overline{\overline{\mathfrak{B}}}$, the double dual.

In this topology, the spectrum of a commutative unital Banach algebra turns into a compact Hausdorff topological space. As outlined in Chapter 4, this allows us to do functional analysis and measure theory on the Gel'fand spectrum, as a result of the following theorem, relating linear functionals on compact Hausdorff spaces to Borel measures:

Theorem 1.32. ***(Riesz–Markov–Kakutani)****: Let X be a locally compact Hausdorff topological space and ψ a positive linear functional on compactly supported functions on X (Denoted by $C_0(X)$). Then there exists a unique positive Borel measure μ on X such that, for* *all* $f \in C_0(X)$:

$$\psi(f) = \int_X f d\mu \tag{1.15}$$

This turns $(X, \mathcal{B}, \mu)$ *into a complete measure space.*

Furthermore, we obtain an isomorphism between $\mathfrak{B}$ and the space of linear functionals on Spec($\mathfrak{B}$). This is known as the *Gel'fand transformation*:

Definition 1.33. The **Gel'fand transformation**, is defined by

$$\begin{aligned} \bigwedge : \mathfrak{B} &\to \overline{\mathrm{Spec}(\mathfrak{B})} \\ b &\mapsto \hat{b} \end{aligned} \tag{1.16}$$

where $\hat{b}(\chi) := \chi(b)$

Let us provide an example of some of these concepts:

Example 1.34. Let $\mathfrak{B} = \Delta_n \subset M_n(\mathbb{C})$, the subalgebra of diagonal complex $n \times n$ matrices. Note this is a commutative C*-algebra. Then define characters $\chi_j : \Delta_2 \to \mathbb{C}$ by

$$\chi_j(\mathrm{diag}(\alpha_1, ..., \alpha_n)) = \alpha_j \qquad j \in \{1, ..., n\}$$

It is clear these are *-homomorphisms, and so $\mathrm{Spec}(\Delta_n) = \{\chi_j\}_{j=1}^n$. One can see that the Gel'fand spectrum will be the discrete set $\{1, ..., n\}$, since this is topologically equivalent in the weak *-topology. Furthermore, we can define a Gel'fand transform as from $\Delta_n \xrightarrow{\cong} C(\{1, ..., n\})$ by $\hat{M}(\chi_j) = \chi_j(M)$, where $M \in \Delta_n$

We have now introduced all the relevant mathematical preliminaries to unveil black holes in classical and quantum theory. The remainder of this thesis will focus on defining and examining the phenomenology of black holes in both these regimes. In the next chapter, we begin with the classical theory of general relativity to obtain our most intuitive and physical understanding of black holes and their properties.

Chapter 2

Black Holes in Classical General Relativity

This chapter develops the mathematical ideas of general relativity used to understand black holes geometrically. It introduces fundamental concepts (such as causality), which are further expanded to include isolated horizons in loop quantum gravity (Ch 5,6,7) and the singularity theorems of Penrose and Hawking. We conclude with a formal description of black holes and the key properties that they exhibit. We closely follow the treatments of the concepts given in [Wal84].

Throughout the remainder of this thesis, we will also be assuming our spacetimes are *time-orientable*. For our purposes, we understand this as follows: A spacetime $(\mathcal{M}, g)$, is said to be time orientable if for all $p \in \mathcal{M}$ there are two connected components of timelike vectors in $T_p\mathcal{M}$ where each component is an equivalence class (where $X \sim Y$ if $g(X, Y) < 0$ for all $X, Y \in T_p\mathcal{M}$), known as *future/past*-directed classes of vectors. *

2.1 Causality

Let us recall the central object underpinning all of general relativity (GR) - the Einstein Field Equations (EFEs):

Definition 2.1. Let $(\mathcal{M}, g)$ be a spacetime and let T be a $(0, 2)$-tensor field known as the *stress energy tensor*. The **Einstein Field Equations** are a set of coupled non-linear PDEs for the metric tensor, given by

$$R_{ab} - \frac{1}{2}Rg_{ab} - 8\pi T_{ab} = 0, \tag{2.1}$$

in abstract index notation. Here, T represents matter distributions in spacetime, R_{ab} the Ricci curvature, and R the scalar curvature. In particular, a vacuum spacetime corresponds

*This description can be formalised compactly - a spacetime is said to be time orientable if its frame bundle $F(\mathcal{M})$ can be reduced to a $O^+(3,1)$-bundle, where $O^+(3,1)$ denotes the subgroup of orientation-preserving orthogonal transformations, known as the *orthochronous Lorentz group*.

to $T = 0$. †

Accordingly, in every spacetime we consider, the metric solves the EFEs.

In determining the dynamics of the spacetime, one is often led to solving an initial value problem for the EFEs. From typical PDE theory, one considers initial data related to the problem to enable one to find a solution. In our present case, we need to specify what we mean by 'initial data'. In general relativity, this intuitively means choosing an instant in time. Such data is encoded in a geometric object known as a *Cauchy surface*. Let us recall some definitions:

Definition 2.2. The **causal future** $J^+(p)$ of a point $p \in \mathcal{M}$ is defined as the set of events that can be reached in a finite time by a future-directed causal curve emanating from p

$$\begin{aligned} J^+(p) := \{q \in \mathcal{M} \mid & \text{ there exists future-directed causal curve} \\ & \lambda \text{ such that } \lambda(0) = p,\ \lambda(1) = q\}. \end{aligned} \tag{2.2}$$

The **chronological future** $I^+(p)$ is defined identically by replacing 'causal' with 'timelike'.

Analogously, one can also define the **causal/timelike pasts** $J^-(p)$, $I^-(p)$ by replacing 'future-directed' with 'past-directed'.

Definition 2.3. Let $\lambda : [0, 1] \to \mathcal{M}$ be a future-directed causal curve. We say $p \in \mathcal{M}$ is a **future endpoint** of λ if for every neighbourhood $U \ni p$ there exists t' such that $\lambda(t) \in U$ for all $t > t'$. If no such p exists, we say λ is **inextendible.**

In particular, Definition 2.2 provides a notion of separation between past and future events from a given point p. We remark that in the setting of Minkowski spacetime, the causal future and pasts form the *light cone* of special relativity. From here, we generalise the standard notion of simultaneity:

Definition 2.4. A subset $S \subset \mathcal{M}$ is **achronal** if there do not exist $p, q \in S$ such that $q \in I^+(p)$, i.e. $I^+(p) \cap S = \emptyset$.

Definition 2.5. The **edge** of a closed achronal set S is the set of points $p \in S$ such that every open neighbourhood $U \ni p$ contains a point $q \in I^+(p)$, a point $r \in I^-(p)$ and a timelike curve λ from r to q which does not intersect S

These types of sets allow us to establish the following definition:

Definition 2.6. Let S be a closed, achronal set with or without an edge. Denote the set of past (future) inextendible causal curves through p as $\mathfrak{C}^\mp(p)$. The **future (past) Cauchy development** $D^+(S)$ is

$$D^\pm(S) := \left\{p \in M \mid \lambda(t) \cap S \neq \emptyset \text{ for all } \lambda \in \mathfrak{C}^\mp(p)\right\}. \tag{2.3}$$

†In index-free notation, the EFEs may be written as $\mathrm{Ric(g)} - \frac{1}{2}Rg = 0$. This more clearly shows $g \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ as a field satisfying field equations in the mathematical sense.

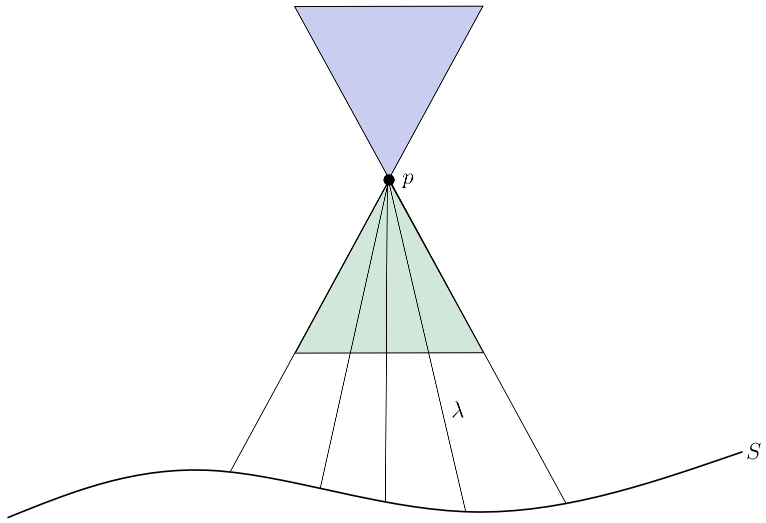


Figure 2.1: An example of a point $p \in D^+(S)$ for some one-dimensional surface S.

Physically, one can interpret this as being the collection of points whose causal trajectory originated from, or will intersect S. As Wald [Wal84] puts it "any signal sent to $p \in D^+(S)$ must have registered on S".

Definition 2.7. If $\Sigma \subset \mathcal{M}$ is a closed achronal set satisfying $D^+(\Sigma) \cup D^-(\Sigma) = \mathcal{M}$, Σ is called a **Cauchy surface.**

The concept of a Cauchy surface extends to arbitrary pseudo-Riemannian manifolds. However, for a 4-dimensional spacetime, every Cauchy surface is characterised by the following theorem ([Wal84] Theorem 8.3.1):

Theorem 2.8. *Let $S \subset \mathcal{M}$ be closed, non-empty, and achronal with edge $(S) = \emptyset$. Then S is a three-dimensional, embedded C^0 submanifold of $\mathcal{M}$.*

It follows from this definition that Cauchy surfaces are edgeless. Accordingly, Theorem 2.8 tells us *every* Cauchy surface is an embedded C^0 three-submanifold of M.

Example 2.9. For simplicity, let us consider the case of a Cauchy surface in a 2-dimensional Lorentzian manifold (so Theorem 2.8 is not applicable). Let $(\mathcal{M}, g) = (\mathbb{R}^{1,1}, \eta)$. Given global coordinates (t, x), the surface defined by $\Sigma = (t = \text{const}, x)$ defines a Cauchy surface. In particular, this is edgeless and partitions the two-dimensional plane into regions lying above and below the line $t = \text{const}$.

Definition 2.10. A spacetime $(\mathcal{M}, g)$ is said to be **globally hyperbolic** if it possesses a Cauchy surface.

The entire history and future of a globally hyperbolic universe can therefore be predicted or retrodicted from conditions on Σ. Hence, they give a well-defined notion of a 'snapshot' of time, and may be utilised in solving the EFEs as noted previously.

For our purposes, the following theorem of Geroch, Bernal, and Sanchez [Ger70],[BS03] will be of utmost importance in the quantisation of gravity.

Theorem 2.11. *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with Cauchy surface Σ. Then there exists a global time function t such that every surface of constant t (given by Σ_t). Hence, the slices Σ_t foliate $\mathcal{M}$, and $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$.*

With our new generalised notion of simultaneity, as well as a well-defined 'initial data set' given by Cauchy surfaces, we can begin our central discussion of black holes in spacetimes.

2.2 Singularities

In general relativity, spacetimes are described by a manifold with a globally defined metric. One may find, however, that there are regions where the metric is undefined regardless of the choice of coordinates, as is the case for the Schwarzschild solution to the EFEs when $r = 0$. As such, for the metric to be defined globally, one cannot assume these points are even part of the spacetime. Such points are known vaguely as *singularities.*

One solution to dealing with such points would be to consider these regions as boundary regions of $\mathcal{M}$. However, prescribing these regions as elements of a topological space or as the singular boundary of a manifold to include them in spacetime raises issues with coordinate definitions and gives rise to pathological topological properties. As a result, singularities are mathematically ill-defined, despite being genuine features of the spacetime. However, we will see that they are at detectable since there will exist geodesics which are inextendible in at least one direction but with finite affine parameter (such geodesics are called *incomplete*). This is the content of Penrose and Hawking's singularity theorems. Therefore, we can set out conditions for the existence of singularities.

The purpose of this section will be to discuss and provide the necessary conditions for the existence of singularities in a spacetime, using a singularity theorem of Penrose and Hawking. The discussion in this section will closely follow Chapter 9 of [Wal84].

We begin with the following:

Definition 2.12. Let $\mathcal{M}$ be a spacetime manifold and $O \subset \mathcal{M}$ open. A **congruence** in O is a family of curves such that through each $p \in O$ there passes precisely one curve in this family.

Remark 2.13. All tangents to a congruence yield a vector field in O and conversely every continuous vector field generates a congruence of curves. A congruence is *smooth* if the vector field is smooth

In particular, a congruence describes a set of integral curves of a non-vanishing vector field, often taken to be either timelike or null. Physically, one interprets such timelike curves as being the set of world lines emanating from observers at a particular point, and null congruences as being families of freely propagating light rays.

Given a congruence of unit timelike geodesics ξ^a (geodesics satisfying $g(\xi, \xi) = -1$), define the tensor field

$$B_{ab} := \nabla_b \xi_a. \tag{2.4}$$

Such a field is of interest to us, since it allows us to examine the behaviour of congruences in different regions of the spacetime. To demonstrate how this is done, recall from the previous section that any globally hyperbolic spacetime is diffeomorphic to $\mathbb{R} \times \Sigma$ for a Cauchy surface Σ. Given the spacetime metric g and a unit timelike vector field ξ^a, we can define a spatial metric on Σ by 'projecting' g onto it. In particular, define the spatial metric

$$h_{ab} = g_{ab} + \xi_a \xi_b. \tag{2.5}$$

From this, we can define various quantities:

Definition 2.14. The **expansion** θ, **shear** σ_{ab} and **vorticity** ω_{ab} of a geodesic congruence are the quantities

$$\theta = B^{ab} h_{ab} \tag{2.6}$$

$$\sigma_{ab} = B_{(ab)} - \frac{1}{3}\theta h_{ab} \tag{2.7}$$

$$\omega_{ab} = B_{[ab]} \tag{2.8}$$

By utilising these quantities, one can express the tensor field B_{ab} in Equation 2.4 as

$$B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \tag{2.9}$$

For our purposes, we omit the details and redirect the reader to [Wal84]. By taking the covariant derivative of this along the timelike geodesic ξ^a and taking the trace, one can obtain the so-called *Rayachaudhuri equation.*

$$\xi^c \nabla_c \theta = -\frac{1}{3}\theta^2 + \mathrm{Tr}(\omega^2 - \sigma^2) - R_{cd}\xi^c \xi^d \tag{2.10}$$

This equation can be physically interpreted as describing the evolution of the expansion of geodesic congruences. We will not dwell on this equation; however, the final term on the right-hand side is of interest. In particular, it must be negative; otherwise, this would imply the existence of negative energy physically:

Lemma 2.15. *(Strong energy condition) Let ξ^a be a unit timelike vector in a geodesic congruence and suppose g_{ab} satisfies the Einstein equations. Then $R_{ab}\xi^a\xi^b \geq 0$ if $T_{ab}\xi^a\xi^b \geq -\frac{1}{2}T$*

Proof. This is a matter of applying the Einstein equation to the left-hand side, as is done in Equation (9.2.15) in [Wal84]. Indeed, if we recall that the curvature scalar is related to the trace of the stress energy tensor by $R = -8\pi T$, then we can rewrite the left-hand side using the EFEs for g_{ab} as

$$R_{ab}\xi^a\xi^b = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)\xi^a\xi^b = 8\pi\left(T_{ab}\xi^a\xi^b + \frac{1}{2}T\right), \tag{2.11}$$

since $g_{ab}\xi^a\xi^b = -1$ by assumption. Enforcing this to be nonnegative proves the claim immediately. $\square$

As a corollary to this lemma, we may devise some further conditions on geodesic congruences. While of little purpose mathematically, these constrain the types of metrics permitted in physical spacetimes.

Definition 2.16. A spacetime is said to satisfy the **timelike/null generic condition** if each timelike/null geodesic possesses at least one point where $R_{abcd}\xi^a\xi^c \neq 0$.[‡]

We need to introduce one more object before we can state the singularity theorem: a trapped surface.

Definition 2.17. A compact codimension 2 smooth spacelike submanifold $T \subset \mathcal{M}$ having the property that the expansion θ of both congruences of (incoming and outgoing) future directed null geodesics orthogonal to T is everywhere negative is called a **trapped surface**.

One can infer intuitively that this is a region of spacetime for which future-directed light rays (null geodesics) converge towards each other, moving along the null normals of T. In particular, every null geodesic in the congruence converges to a point, in effect 'trapping' it in and around the surface.

Accordingly, we can now state the singularity theorem, as established by Penrose and Hawking from several other stronger theorems devised by the pair et al. [HP70] ([Wal84] Theorem 9.5.4):

Theorem 2.18. *Suppose $\mathcal{M}$ satisfies four conditions:*

1. $R_{ab}v^av^b \geq 0$ *for all timelike and null geodesics* v^a*, as will be in the case* $\mathcal{M}$ *is globally hyperbolic and satisfies the strong energy condition.*
2. *The timelike and null generic conditions are satisfied*
3. *No closed timelike curves exist*
4. *At least one of the following hold: (a)* $\mathcal{M}$ *possesses a compact achronal set without edge (*$\mathcal{M}$ *is a closed universe), (b)* $\mathcal{M}$ *possesses a trapped surface, (c) there exists a point* $p \in \mathcal{M}$ *such that the expansion of the future (or past) directed null geodesics emanating from* p *become negative along each geodesic in this congruence.*

Then $\mathcal{M}$ *must contain at least one incomplete timelike or null geodesic*

The incomplete geodesic in the result of the theorem represents a singularity, and hence this theorem provides the criteria required for a spacetime to contain a singularity. While it provides no means of pinpointing its location, the results suggest that our physical universe is singular. Indeed, observational evidence of black holes in recent decades allows us to confirm this reasoning.

We have established the fundamental result on the existence of singularities in spacetimes. The singularity theorems of Penrose and Hawking show that, under mild physical assumptions

[‡]As is remarked in [Car19], "These fancy conditions simply serve to exclude very special metrics for which the curvature consistently vanishes in some directions".

of our spacetime, there exist incomplete geodesics which correspond to the formation of a singularity. This encourages further investigation - particularly, we are prompted to ask where these singularities actually reside and how they are (not) hidden from observers? The latter question is the content of the so-called *cosmic censorship conjectures* [GH79], which are interesting in their own right; however, we will focus on classifying singularities based on the existence of incomplete geodesics. Since Theorem 2.18 only guarantees existence, additional work must be done to determine where regions containing singularities lie. This is the content of the following section.

2.3 Black Holes

Informally, a black hole can be thought of as a region in spacetime that cannot influence the 'infinite future' of light rays. This implies a black hole region needs to be sufficiently 'isolated' from the remainder of spacetime. In treating it like this, however, we are treating null infinity as a point in our spacetime. To formalise this, we need to give a precise notion of taking limits towards infinity. The most common way of achieving this is by constructing a conformal compactification (i.e., an embedding of our non-compact $\mathcal{M}$ into a larger spacetime as a dense open subset via a conformal map) of $\mathcal{M}$. Under some conditions, we obtain a well-defined notion of 'going to infinity' and 'flatness at infinity' for our physical spacetime $(\mathcal{M}, g)$. Such a spacetime is called *asymptotically flat*. Rather than providing a full, detailed definition (for complete details, see [AH78] and [Wal84]), we will give a more streamlined one.

Definition 2.19.

- We define the **conformal compactification** of a non-compact spacetime $\mathcal{M}$ as an embedding via a conformal (i.e. locally angle-preserving) transformation into a Lorentzian manifold $\tilde{\mathcal{M}}$ as a dense compact subset. We call $(\tilde{\mathcal{M}}, \tilde{g})$ the **unphysical spacetime**.

- A vacuum spacetime $(\mathcal{M}, g)$ is called **asymptotically flat at null and spatial infinity** if there exists a conformal compactification $(\tilde{\mathcal{M}}, \tilde{g})$, where $\tilde{g}$ smooth everywhere except possibly at i^0 where it is $C^{>0}$, and a conformal isometry $\psi : \mathcal{M} \to \psi[\mathcal{M}] \subset \tilde{\mathcal{M}}$ with conformal factor Ω (so we obtain $\tilde{g}_{ab} = \Omega^2 \psi^* g_{ab}$ in $\psi[M]$ which satisfies reality conditions as outlined in [AH78].

Example 2.20. (Conformal compactification of Minkowski space) Let $(\mathcal{M} = \mathbb{R}^{3,1}, \eta)$ be Minkowski spacetime. From Cartesian coordinates, one can re-express the metric components in usual spherical coordinates as

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.12}$$

For a fixed θ, ϕ, null curves occur along $dt^2 = dr^2$, and it can be shown that for null geodesics, there is a linear relationship between t, r and the affine parameter of the curves. To obtain the conformal completion, we can choose new coordinates (v, w, θ, ϕ) on $\mathcal{M}$, given by $v = t + r$, $w = t - r$. These are known as *advanced* and *retarded* null coordinates, respectively.

For fixed θ, ϕ, v (w) are constant along incoming (outgoing) null geodesics. So the surfaces defined by $\{v = \text{const}\}$, $\{w = \text{const}\}$ are null surfaces. The metric in (2.12) in these new coordinates becomes

$$ds^2 = -dwdv + \frac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.13}$$

Applying a further change of coordinates $(v, w, \theta, \phi) \mapsto (p, q, \theta, \phi)$ given by

$$\begin{aligned} &\tan p = v \qquad \tan q = w \\ &-\frac{\pi}{2} < p < \frac{\pi}{2} \qquad -\frac{\pi}{2} < q < \frac{\pi}{2}. \end{aligned} \tag{2.14}$$

Using this and computing the differentials dv, dw in terms of the new coordinates, one can show that the metric now takes the form

$$ds^2 = \sec^2 p \sec^2 q \left(-dpdq + \frac{1}{4}\sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{2.15}$$

This now gives us a conformal relationship between the Minkowski metric and an 'unphysical' metric $\bar{g}$:

$$\eta = \Omega^2 \bar{g}, \tag{2.16}$$

with $\bar{g}$ given by $d\bar{s}^2 = -4dpdq + \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)$ and $\Omega^2 = \frac{1}{2}\sec^2 p \sec^2 q$. By further changing coordinates to $T = p + q$, $R = p - q$, we obtain a natural Lorentz metric over a new 'unphysical' spacetime $\tilde{\mathcal{M}}$

$$d\bar{s}^2 = -dT^2 + dR^2 + \sin^2 R(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.17}$$

We can observe that all of Minkowski spacetime is now defined in the region $T \pm R \in (-\pi, \pi)$. This is exactly the conformal compactification of $(\mathcal{M}, \eta)$ into a larger spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ (where $\tilde{\mathcal{M}}$ is diffeomorphic to $\mathbb{R} \times S^3$. Note $\tilde{\mathcal{M}}$ need not be compact itself). The portion of Minkowski space embedded in the larger space is shown in the diagram in Figure 2.2.

Observe in Figure 2.2, the boundary of the spacetime can be divided into five portions: We have $\mathscr{I}^{\pm}$, representing *future/past null infinity*; $i^{\pm}$ representing *future/past timelike infinity*; and i^0 representing *spatial infinity* (which is regarded as a single point).

This is significant: conformal compactifications allow us to regard infinity as a point. In particular, it makes sense to talk about the causal future/past of infinity, and this is precisely what lets us define a black hole.

Concretely, we define a black hole as a region that cannot influence the causal past of $\mathscr{I}^+$:

Definition 2.21. Let $(\mathcal{M}, g)$ be an asymptotically flat spacetime with conformal compactification in $(\tilde{M}, \tilde{g})$. We say $\mathcal{M}$ is **strongly asymptotically predictable** if in the unphysical spacetime there exists and open region $\tilde{V} \subset \tilde{\mathcal{M}}$, $\overline{\mathcal{M} \cap J^-(\mathscr{I}^+)} \subset \tilde{V}$ such that $(\tilde{V}, \tilde{g})$ is globally hyperbolic (recalling that J^- denotes the causal past (Definition 2.2.))

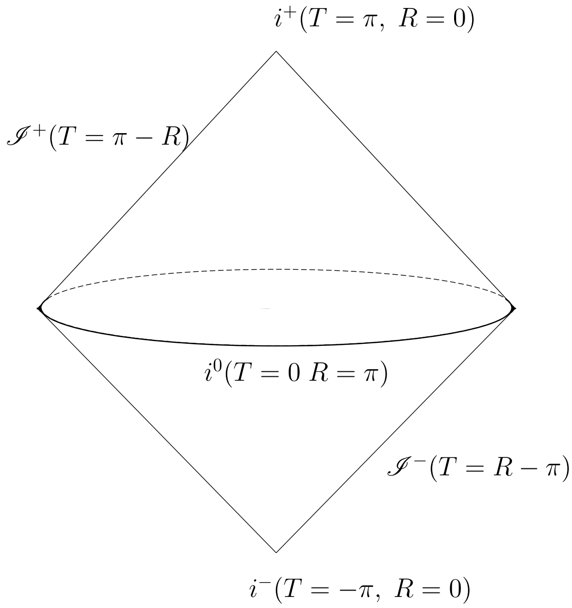


Figure 2.2: The region of the unphysical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ representing Minkowski spacetime, given by a null double cone.

Definition 2.22. The **black hole region** B of a strongly asymptotically predictable spacetime $(\mathcal{M}, g)$ is the region

$$B := \mathcal{M} \setminus (\mathcal{M} \cap J^-(\mathscr{I}^+)). \tag{2.18}$$

The **event horizon** of a black hole region is the boundary $H := \partial B = \mathcal{M} \cap \partial J^-(\mathscr{I}^+)$

Specifically, this is the region of our spacetime which *not* contained in the causal past of future null infinity. Equivalently, an event lies in a black hole region if and only if no future-directed causal curves reach $\mathscr{I}^+$. Physically, signals emanating from within B can never reach an external observer at any event in the spacetime. This has significant implications for spacetimes such as the one in which we exist. The singularity theorems and cosmic censorship conjectures suggest that singularities are hidden behind black hole regions and are inaccessible to external observers.

We will now give a prototypical example of a black hole region, arising from the Schwarzschild solution:

Example 2.23. Consider Minkowski spacetime $(\mathbb{R}^{3,1}, \eta)$. In the coordinates we introduced in Example 2.20, the Schwarzschild metric components (describing an isolated, spherical star of mass M) take the form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.19}$$

Given a null radial curve $r(v)$ parameterised by an affine parameter t, we can set the metric component $ds^2 = 0$ to obtain the null geodesics:

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dv}{dt}\right)^2 + 2\frac{dv}{dt}\frac{dr}{dt} = 0. \tag{2.20}$$

For future-directed null curves, we have that $\frac{dv}{dt} > 0$, By an application of the chain rule, this reduces to

$$\frac{dr}{dv} = \frac{1}{2}\left(1-\frac{2M}{r}\right). \tag{2.21}$$

For $r < 2M$, the left-hand side is negative, so any future-directed outgoing null curves within the horizon $r = 2M$, $r(v)$ is decreasing towards the singularity $r = 0$. Furthermore, any points on the curve beginning at points $r < 2M$ can never reach arbitrarily large r. Hence, such points cannot be in the region $J^-(\mathscr{I}^+)$. This means our black hole region and event horizons are

$$B := \{p \in \mathcal{M} \mid r(p) < 2M\} \qquad H := \{r = 2M\}. \tag{2.22}$$

Along with our general definition, we can also have more interesting types of black holes that exhibit rotation and charge. These are described by a generalisation of the Schwarzschild metric known as the *Kerr metric*. We direct the reader to [Cho15] for further details.

2.3.1 Properties of Black Holes

Our current definition provides us with the global structure of a black hole, however we cannot say anything about it at some instant in time (i.e. over a Cauchy surface Σ_t for some fixed t). Hence, we wish to extend our current notion of black hole regions. Let $(\mathcal{M}, g)$ be strongly asymptotically predictable with $\tilde{V} \supset \overline{M \cap J^-(\mathscr{I}^+)}$ with a black hole region B. If Σ is a Cauchy surface for $\tilde{V}$, then $\Sigma \cap B$ is the black hole region at time Σ. Each connected component $\mathscr{B}$ of $\Sigma \cap B$ is a **black hole at time** Σ. The number of black holes will vary as our choice of Cauchy surface varies with time (i.e., with respect to the foliation Σ_t). But as the following theorem shows, black holes may never disappear or bifurcate ([Wal84] Theorem 12.2.1).

Theorem 2.24. *Let (M, g) be strongly asymptotically predictable and Σ_1 and Σ_2 Cauchy surfaces for $\tilde{V}$, $\Sigma_2 \subset I^+(\Sigma_1)$. If $\mathscr{B}_1$ is the connected component of $B \cap \Sigma_1$, then $J^+(\mathscr{B}_1) \cap \Sigma_2 \neq \emptyset$ and is contained within a single connected component of $B \cap \Sigma_2$*

By definition, we need to know the entire time evolution of $(\mathcal{M}, g)$ to determine whether a black hole will be present. Therefore, it is helpful to develop criteria for the existence of black holes without knowledge of the global evolution. Indeed, we know from the Singularity Theorem 2.18 that trapped surfaces are related to singularities, and hence black holes ([Wal84] Proposition 12.2.2).

Theorem 2.25. *Let $(\mathcal{M}, g)$ be strongly asymptotically predictable with $R_{ab}k^a k^b \geq 0$ for all null k (i.e., weak or strong energy condition is satisfied). Suppose $\mathcal{M}$ contains a trapped surface T. Then $T \subset B$, for a black hole region B.*

This means all trapped surfaces in physically reasonable spacetimes are contained within black holes. This is significant, as it aligns with observational evidence, indicating that any matter entering a black hole region cannot escape by the definition of a trapped surface.

Finally, we arrive at a key result concerning the black hole event horizon. Given a black hole region B and a connected component $B \cap \Sigma$ at time $\Sigma_t = \Sigma$, it can be shown [Wal84] that the intersection $\mathscr{H} = H \cap \Sigma$ of the event horizon with a Cauchy surface is a codimension 2 submanifold of Σ. The following theorem, proved by Hawking in [Haw72], demonstrates that the area of $\mathscr{H}$ is monotone increasing with time, which is the key to unlocking the deep and interconnected relationship between general relativity, statistical mechanics, and quantum mechanics [Haw72],[Wal84]:

Theorem 2.26. ***(Hawking Black Hole Area Theorem):*** *Suppose $(\mathcal{M}, g)$ contains a black hole region and $R_{ab}k^a k^b \geq 0$ for all null k. Let Σ_1, Σ_2 be spacelike Cauchy surfaces for the globally hyperbolic region $\tilde{V}$ and $\Sigma_2 \subset I^+(\Sigma_1)$. Set $\mathscr{H}_1 = H \cap \Sigma_1$, $\mathscr{H}_2 = H \cap \Sigma_2$. Then the area $A(\mathscr{H}_2) \geq A(\mathscr{H}_1)$*

This concludes our introduction to black holes in general relativity. We have rigorously defined the notion of causality in general relativity and established the criteria for a spacetime to admit singularities and, hence, black hole regions, culminating in the celebrated area theorem. Through the work of Jacob Bekenstein, we find a profound corollary of this theorem. Namely, the theorem implies black holes obey a set of laws that are in bijection with the laws of thermodynamics. In the following chapter, we will see that black holes have a well-defined notion of entropy.

Chapter 3

Black Hole Thermodynamics

In this chapter, we use the black hole area theorem to glimpse at the interplay between statistical mechanics and general relativity – two very distinct fields on the surface.

Let us begin our discussion with an overview of general black holes. Black holes may exhibit physical properties such as charge and rotation, which may be described by the so-called *Kerr metric* (for details see Chapter 12 of [Wal84]). This metric can be thought of as 'parameterising' every possible type of black hole, allowing one to derive numerous quantities from it. In particular, we can use it to obtain a black hole's mass.

Definition 3.1. Let Σ be a Cauchy surface intersecting the event horizon H f a black hole B, such that $\Sigma \cap H \cong S^2$, which forms the boundary of Σ. The **mass** of the black hole is determined by

$$M := 2\int_{\Sigma}\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)n^a\xi^b d\text{vol} + \frac{\kappa}{4\pi}A + 2\Omega_H J_H, \tag{3.1}$$

where $T = -\frac{R}{8\pi}$, n^a is the unit normal to Σ, ξ^b a timelike Killing field, κ is the gravitational coupling constant, $A = \int_{\Sigma\cap H}\epsilon_{ab}$ is the horizon area, $\Omega_H = \frac{a}{r_+ + a}$ is a constant relating to the horizon radius r_+ and angular acceleration a, J_H is the black hole's angular momentum.

A full derivation of this may be found in Wald Chapter 12 [Wal84]. In particular, the differential mass of a rotating black hole to its area and angular momentum:

Theorem 3.2. *Given the mass M of a black hole B*

$$\delta M = \frac{k_S}{8\pi}\delta A + \Omega_H \delta J_H, \tag{3.2}$$

where $\Omega_H = \frac{a}{r_+ + a}$ is a constant relating to the horizon radius r_+ and angular acceleration a, J_H is the black hole's angular momentum and k_S is the surface gravity, which is constant on the horizon.

Readers familiar with statistical mechanics [*] may observe the resemblance of Theorem 3.2 with the first law of thermodynamics. This is no coincidence, as Bekenstein and Hawking

[*]For a quick review on statistical mechanics, we direct the reader to Appendix B.

[Bek73], [Haw75] have shown in their seminal papers. This chapter explores this analogy, concluding with a physically motivated proof that black holes exhibit entropy, related to their area. We closely follow the arguments as presented in [Bek73] while elaborating on some of the details presented.

3.1 The Macroscopic Black Hole Entropy

In this section, we utilise our general result for the variational mass of a black hole in Theorem 3.2 to obtain an expression for the associated entropy. At the outset, it may seem like Theorem 3.2 is not sufficient to calculate *any* thermodynamic properties, since we do not have any actual thermodynamic quantities present in the above equation. However, one can see that Theorem 3.2 closely resembles the thermodynamic expression for variational energy in a closed system:

Definition 3.3. Consider an isolated thermodynamic system with total energy U. Then the **first law of thermodynamics** is

$$dU = TdS - PdV, \tag{3.3}$$

where T is the temperature, S the entropy, P the pressure and V the volume of the system.

Bekenstein [Bek73] made the connection between the classical thermodynamic law and the variational black hole mass. In particular, the first term on the right-hand side of each expression suggests a connection between the area and entropy.

Indeed, Bekenstein originally established this connection utilising the Shannon entropy of information theory:

Definition 3.4. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be probability space, where $\Omega \subset \mathbb{R}$. Let $E \subset \mathcal{B}$ be a measurable subset. Let X be a discrete random variable distributed according to $p : \mathcal{X} \to [0, 1]$ such that $p(x) = \mathbb{P}(X = x)$. Then the **Shannon entropy** of X is given by

$$S_{\mathbb{P}}(X) = -\sum_{x \in E} p(x) \log_2 p(x). \tag{3.4}$$

In this form, one unit of entropy is known as a **bit**.

This quantity measures the average uncertainty associated with the outcome of a particular event in a probability space. This, in fact, also has a thermodynamic interpretation - the uncertainty as to which all the microstates of a thermodynamic system are compatible with the macroscopic parameters. This means that, given a system with a countable number Ω of microstates (the multiplicity), each with a probability p_i of occupying a state i, then each p_i is compatible with a given macrostate if the partition function used to obtain the macroscopic variables (such as temperature, energy, magnetisation, etc.) using the mircrostates describes the macrostate. Bekenstein realised that this description applies to black holes defined by the Kerr metric analogously, as they can be characterised macroscopically by three parameters: namely, their mass, charge, and angular momentum. By analogy, he concluded that there must exist 'internal configurations' within the object which are all compatible with the macrostate of the black hole. In particular, these internal configurations need not be uniform for all black holes and depend on their physical formation process. Hence, the black hole entropy is interpreted as measuring the inaccessibility of information within the trapped surface defining the black hole.

As such, the entropy of a black hole, described by the microstates and macrostates, would not be the thermal entropy within the interior, but rather an equivalence class of black holes of identical mass, charge and angular momentum[Bek73]. With these physical motivations, Bekenstein then sought to derive a formula for black hole entropy. We outline the argument as proposed in [Bek73].

3.2 Derivation of the Entropy Formula

By comparing (3.2) and (3.3), and noting the correspondence between variational mass and the first law of thermodynamics, a candidate for black hole entropy is the area term in Theorem (3.2). This is due to the Black Hole Area Theorem 2.26, which gives us the total variation $\delta A \geq 0$. The second law of thermodynamics also states $\delta S \geq 0$, and this proves to be a judicious choice.

The next assumption, based on Theorem 2.26, is that the entropy of the black hole is described by a monotonically increasing function f of its area, in particular

$$S_{BH} := f(A). \tag{3.5}$$

For physical reasons, the simplest choice of monotone increasing function is $f(A) = \alpha A$, where α is some real positive constant. We are then left to find a value for α. We begin by identifying the minimum possible increase in the black hole's area. To do so, we interpret black hole entropy to be a measure of the trapped surface's internal information. As such, it should also reflect a change in information obtained by absorbing a particle (i.e, a particle travelling along a causal geodesic intersecting the black hole region). According to the definition of Shannon entropy and our analogy, the internal configuration of the black hole must gain at least one bit of information. Therefore, we can adopt this as a sort of 'toy model' to identify the area increase.

We assume the particle itself is a physical object, rather than a point along a causal geodesic. In particular, we assume the particle is spherical with a rest mass m and radius d. As shown in Appendix A of [Bek73], the minimum increase ΔA_{min} in the black hole area is

$$\Delta A_{\text{min}} = 2md. \tag{3.6}$$

Here, d must satisfy $d \geq \min(\hbar m^{-1}, 2m)$. This bound is related to the Compton wavelength λ and gravitational radius R of a physical particle, respectively. In particular, $\lambda = \hbar m^{-1}$ is larger than $R = 2m$ if $m \leq \sqrt{\frac{\hbar}{2}}$ and vice versa if $m > \sqrt{\frac{\hbar}{2}}$. These two inequalities then imply that the area increase must satisfy

$$2md \geq 2\hbar. \tag{3.7}$$

This gives us precisely the minimum area increase possible.

From our prior reasoning relating bits of information with internal configurations, we equate the change in entropy from a test particle with one bit of information. In particular, the entropy is assumed to be a monotone increasing function of the area, so $\frac{df}{dA} = \alpha$. So hence

$$(\Delta S_{\text{min}}) = \alpha \Delta A_{\text{min}} = 2\hbar \frac{df}{dA} = 1. \tag{3.8}$$

Solving the first-order differential equation gives us

$$f(A) = \frac{A}{2\hbar}. \tag{3.9}$$

This now suggests that our constant $\alpha = \frac{1}{2\hbar}$.

Upon completely restoring units to the equation, and inserting a factor of $\frac{1}{4\pi}$ to account for the so-called 'rationalised area' [Bek73], we obtain the final form of the black hole entropy:

$$S_{BH} = \frac{A}{4\pi\hbar} \frac{k_B c^3}{G}. \tag{3.10}$$

This is the celebrated *Bekenstein-Hawking Entropy* of a black hole.

The work we have done in this section, utilising the geometric descriptions provided in Chapter 2, now extends our knowledge of the physical properties of black holes. In particular, we see that they behave similarly to thermodynamic systems (in the information-theoretic sense, rather than a true thermodynamic sense). Physically, this gives physicists a familiar framework to work with when describing or observing black hole dynamics.

While this description may be sufficient physically, mathematically, more work is required to demonstrate that the entropy of a black hole, as defined geometrically in Chapter 2, can be described in this manner. In particular, the argument and derivation we have described in this section rely heavily on the physical properties of matter in general relativity, and much of the framework lacks mathematical rigour. This is in contrast to much of the spacetime geometry described in Chapter 2.

As a result, we seek an alternate pathway to obtain (3.10). For the remainder of this thesis, we will utilise the mathematical tools and framework provided by loop quantum gravity to rederive the entropy formula. Not only will this provide us with a solid mathematical description of the Bekenstein–Hawking entropy, but it will also offer a microscopic description that aligns with a quantised theory of gravity. This is significant in that it reconciles a classical macroscopic quantity with its microscopic analogue in the quantised theory. Furthermore, this provides more information on the unification of general relativity with quantum theory, which is an active and open area of research in theoretical and mathematical physics.

Chapter 4

Isolated Horizons

We begin our alternative pathway with an introduction to a generalised type of event horizons known as *isolated horizons*. We will view this through the Hamiltonian framework of GR developed by Arnowitt, Deser, and Misner (known as the *ADM formalism*). With this, our theory of classical GR will be recast in the language of fibre bundles and gauge theory, giving us richer geometrical structures to work with. In particular, we will redefine our dynamical variables to be connection one-forms and frame fields, and outline how this changes our classical picture of general relativity. This will lead us to a consideration of loop quantum gravity (LQG) - an ambitious theory designed to provide a quantum theory of gravity that is independent of an explicit description of the spacetime metric.

Notation 4.1. For the remainder of this thesis, we will drop the abstract index notation utilised in Chapters 2 and 3. From now on, lowercase Latin indices ($a, b, c, ... = 1, 2, 3$) starting from a will denote tensorial spatial indices, unless otherwise stated.

4.1 Elements of The ADM Formalism

This section is defines the necessary components of the ADM formalism that we need to begin our passage into LQG. This is based on Chapter 1.1-1.2 of [Thi07]. Recall from Theorem 2.11 that a globally hyperbolic spacetime manifold $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$, for a Cauchy surface Σ such that we get a foliation of $\mathcal{M}$ by hypersurfaces Σ_t. In particular, we have a diffeomorphism $X : \mathbb{R} \times \Sigma \to \mathcal{M}$ where $(t, x) \mapsto X(t, x)$. This allows us to describe each slice Σ_t by the image

$$\Sigma_t := X(t, \Sigma). \tag{4.1}$$

This gives us a $1 + 3$ decomposition of our spacetime into purely spacelike and timelike components. Now, it is of interest to see how each of these slices 'change' depending on an observer's position on Σ as they move through time.

Definition 4.2. Let X be the diffeomorphism as above, and fix $(t, x) \in \mathbb{R} \times \Sigma$ where (x^μ)

are local coordinates on some $U \subset \mathbb{R} \times \Sigma$. Define the **deformation vector field** by

$$T^\mu(X) := \left(\frac{\partial X}{\partial t}\right)\Bigg|_{X=X(t,x)} = N(X)n^\mu(X) + N^\mu(X), \tag{4.2}$$

where N, N^μ are the **lapse** and **shift vector field** respectively, and n^μ is an normal vector to Σ_t.

The figure below (deriving from Figure 1.1 of [Thi07]) geometrically describes the lapse function and shift vectors. In particular, they measure how much a point x on some slice Σ_{t_1} shifts in the global frame of the spacetime after some elapsed time $t_2 - t_1$.

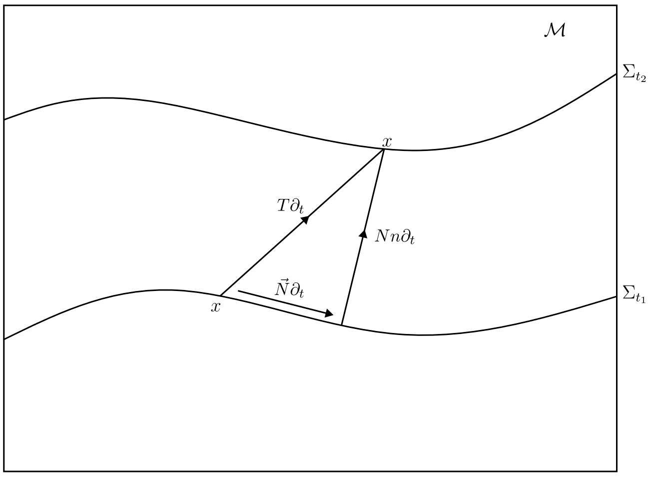


Figure 4.1: Geometric interpretation of N^μ, N.

A key property [Thi07] is $N > 0$ everywhere, to ensure future directedness of the foliation.

Our final piece we need from the ADM formalism is a more explicit form for our spatial metric on Σ, as we first described in (2.5). More specifically, we have

Definition 4.3. The **first fundamental form** (or **spatial metric)** on each slice Σ is the tensor field

$$q := g - sn \otimes n, \tag{4.3}$$

where $s = g(n, n)$.

Observe that q is purely spatial, for contraction with n (a purely timelike vector) causes q to vanish:

$$q_{\mu\nu}n^\nu = g_{\mu\nu}n^\nu - sn_\mu n_\nu n^\nu = n_\mu - s^2 n_\mu.$$

For a unit timelike normal, we have $s = 1$, so this clearly vanishes. With this convention, we can now express the components of q using purely spatial indices $(a, b, ...)$.

The introduction of the lapse and shift vector fields, along with the decomposition of the metric above, allows one to recast the Lagrangian density in the Einstein–Hilbert action into a Hamiltonian density:

Definition 4.4. Given a globally hyperbolic spacetime $(\mathcal{M}, g)$ with local coordinates (x^μ), the **Einstein–Hilbert action** of general relativity is defined by the functional

$$S_{EH} := \frac{1}{2\kappa} \int_{\mathcal{M}} \mathcal{L} \, d^4x, \tag{4.4}$$

where $\mathcal{L} = \sqrt{|\det g|} R : T\mathcal{M} \to \mathbb{R}$ is the Lagrangian density, with R is the curvature scalar associated to g. The Hamiltonian density is defined by the Legendre transform of $\mathcal{H} := \mathrm{Leg}\mathcal{L}$, where $H : T^*\mathcal{M} \to \mathbb{R}$. The functional integral of this density is the **ADM action**:

$$S_{ADM} := \int_{\mathbb{R}} dt \int_{\Sigma} \mathcal{H}(q_{\mu\nu}, N, N^\mu) d^3x, \tag{4.5}$$

where $\mathcal{H}$ is a functional of the first fundamental form, lapse, and shift vectors.

For an explicit description of the ADM action, as well as the Hamiltonian density $\mathcal{H}$, we direct the reader to [Thi07]. Otherwise, this is all the machinery we require from the ADM formalism. The following section discusses isolated horizons in general relativity, which serve as a natural transition point between classical and quantum general relativity. This discussion is based on a similar discussion found in [Abh04], supplemented by [AK04] and [Abh00a].

4.2 Geometry of Isolated Horizons

As is the starting point for classical black holes, we will attempt to derive a quantised entropy formula for the simplest case: A non-rotating, uncharged black hole region. To do this, we need our black hole to be sufficiently isolated from the influence of external matter. We can impose this via boundary conditions on the horizon, requiring that the horizon itself is time independent and leaving the external spacetime dynamical. This intrinsically describes the horizon based purely on local spacetime structure. Accordingly, we need not consider the entire history of spacetime to locate such a horizon. Such a horizon is a generalisation of an event horizon, known as an *isolated horizon*, and this section is dedicated to describing the geometry of such horizons.

We begin our discussion by considering a null 3-dimensional spacelike submanifold $\Delta \subset \mathcal{M}$. Since Δ is a null surface by definition, it has an intrinsic degenerate metric q of signature $(+, +, 0)$ (it has a degenerate eigenvalue) by our prior discussion. In particular, this means that q is not necessarily invertible. However, we can define a 'weak' inverse q^{ab} as follows

Definition 4.5. Let q_{ab} be the components of a degenerate metric q on a spacelike submanifold. Then $q_w^{-1} := q^{ab}$ is a **weak inverse** for q_{ab} if it satisfies

$$q_{ab} = q_{ac} q^{cd} q_{db}. \tag{4.6}$$

Example 4.6. Suppose $\{u, x^1, x^2\}$ are local coordinates on some chart U on Δ. Then, with respect to the basis $\{\partial_u, \partial_{x^1}, \partial_{x^2}\}$ of $T_p\Delta$ for $p \in U$, $q_{ab} = \mathrm{diag}(0, 1, 1)$ is a degenerate metric.

In particular, the kernel consists of Span$\{\partial_u\}$. A weak inverse q_w^{-1} has components

$$q^{ab} = \mathrm{diag}(1, 1, 1).$$

By matrix multiplication, one can easily check $q_{ik}q^{kl}q_{lj} = q_{ij}$

In particular, such a weak inverse always exists pointwise over Δ. Note that this inverse is not unique: If l^a is a null normal to Δ and V^b is a tangent vector field, then $q^{ab} \mapsto q^{ab} + \ell^{(a}V^{b)}$ also satisfies the condition for a weak inverse, as the contraction $q_{am}l^m = 0$, since ℓ^a is null. Using this weak inverse, (2.6) gives

$$\theta(l) := q^{ab}\nabla_a \ell_b. \tag{4.7}$$

Definition 4.7. Let $(\mathcal{M}, g)$ be a (1+3) dimensional spacetime. A null 3-dimensional spacelike submanifold Δ is a **non-expanding horizon** if the following hold:

1. Δ is a null surface and is homeomorphic to $S^2 \times \mathbb{R}$
2. The expansion θ of any null normal vector ℓ on Δ vanishes
3. The Einstein field equations are satisfied on Δ, such that $T^a_b \ell^b$ is future directed and causal, for stress-energy tensor T_{ab} on Δ and ℓ^b a future directed null normal to Δ.

Let us unpack this definition. Condition 1, which is the so-called 'definiteness' condition, in the sense that horizons diffeomorphic to $S^2 \times \mathbb{R}$ are the only physically reasonable/desirable ones. Condition 2 ensures that any cross-section of Δ is a marginally trapped surface (a trapped surface for which the expansion θ is identically 0.). Condition 3 constrains which matter fields are present in our spacetime, and in fact follows from the dominant energy condition.

This is a desirable construction physically, as the flux of any matter fields across the surface vanishes, ensuring that the black holes remain at equilibrium. It also gives desirable boundary conditions, which we will see later in our discussion.

To characterise the geometry of Δ, we need to equip it with a connection. A natural choice would be the induced connection inherited from the spacetime connection ∇ on $\mathcal{M}$ (and, therefore, compatible with it). However, since q is degenerate, there exist infinitely many connections compatible with ∇. However, from our conditions on Δ, we have the following to allow us to obtain a unique torsion-free derivative operator on Δ:

Lemma 4.8. *Let ∇ be the Levi-Civita connection in spacetime $\mathcal{M}$. Then Δ inherits a unique torsion-free derivative operator D from ∇ if and only if every null normal vector ℓ satisfies $\varphi^*(\nabla_b \ell_a) = 0$, where $\varphi : \Delta \to \mathcal{M}$ is an embedding and ℓ_b are the components of the associated one-form.*

Proof. The 'only if' part of this proof is based on the proof of Lemma 4.1 of [Don12]. Firstly, suppose $\varphi^*(\nabla_b \ell_a) = 0$. Suppose also that W is any one form on $\mathcal{M}$ with pullback under φ

$$\varphi^*(W_a) = \omega_a.$$

We would like to define our derivative operator as $DW := \varphi^*(\nabla W)$. The uniqueness and torsion-free conditions are inherited from the Levi-Civita connection. We are left to show that this is well-defined. In particular, let U be another one-form on $\mathcal{M}$ given by

$$U_a = W_a + f\ell_a. \tag{4.8}$$

Now we claim that $\varphi^*(\ell_a) = 0$. Let $v \in T_p\Delta$ for some point $p \in \Delta$. Then, since ℓ is a null normal, we have the inner product

$$\ell_a v^a = q(v, \ell) = \ell_a(v^a) = 0,$$

since $\ell \perp v$ for all $v \in T_p\Delta$. In particular, this means the associated one-form ℓ_a vanishes when evaluated on all tangent vectors. Therefore, its pullback must be identically zero $\varphi^*(\ell_a) = 0$.

This means that U as defined in (4.8) has the same pullback as W, since $\varphi^*(U_a) = \varphi^*(W_a) + \varphi^*(f\ell_a) = \omega_a$. Now, to show the well-definedness of D, consider

$$\begin{aligned} D(U_a - W_a) &= \varphi^*(\nabla_b(U_a - W_a)) \\ &= \varphi^*(\nabla_b(f\ell_a)) \\ &= \varphi^*(\nabla_b f)\varphi^*(\ell_a) + f\varphi^*(\nabla_b \ell_a) \\ &= 0. \end{aligned} \tag{4.9}$$

where the final equality follows by assumption. Hence D is well-defined as required.
On the other hand, suppose such a D exists. Then we have from Section 2.1.1 of [AK04] that since ℓ is expansion-free, there exists a unique connection one-form ω on Δ such that $D_a\ell^b = \omega_a\ell^b$. From this, we have that the Lie derivative of the induced metric along ℓ satisfies

$$\mathcal{L}_\ell(q_{ab}) = 2\nabla_a\ell_b = 0, \tag{4.10}$$

which implies $\nabla_a\ell_b = 0$, as required. □

Remark 4.9. We see in (4.10) above that since the Lie derivative vanishes, ℓ is a Killing field for the intrinsic metric. It is said to be a symmetry of q, as Killing fields are defined to be infinitesimal generators of isometries. *

We can now describe the horizon Δ in a 'self-contained' way, in the sense that we have a description of the submanifold in terms of its intrinsic properties, in particular the intrinsic metric q and derivative operator D. We refer to the pair (q_{ab}, D) as the *geometry* of Δ.

The conditions listed above for a non-expanding horizon are, unfortunately, insufficient to describe physical situations (see [AK04],[Don12]). Although ℓ is a symmetry for q, for physical applications, we would like it also to be a symmetry of the derivative operator D. It unfortunately fails to be such on an arbitrary non-expanding horizon [Abh00a],[AK04]. This motivates a 'stronger' construction for physical horizons:

*Recall a Killing vector field X on a (pseudo–) Riemannian manifold (M, g) is one for which the Lie derivative $\mathcal{L}_X g = 0$

Definition 4.10. Let ℓ, ℓ' be null normals on a non-expanding horizon Δ. We say $\ell \sim \ell'$ if $\ell' = c\ell$ for some non-zero constant factor c

It is straightforward for the reader to check that $\sim$ is an equivalence relation and defines an equivalence class of null normals $[\ell]$. We are now able to define the isolated horizon:

Definition 4.11. An **isolated horizon (IH)** $(\Delta, [\ell])$ is a pair consisting of a non-expanding horizon Δ with an equivalence class of null normals $[\ell]$ for which any representative ℓ is a symmetry for the geometry of Δ. (So that $\mathcal{L}_\ell q_{ab} = [\mathcal{L}_\ell, D] = 0$ on Δ).

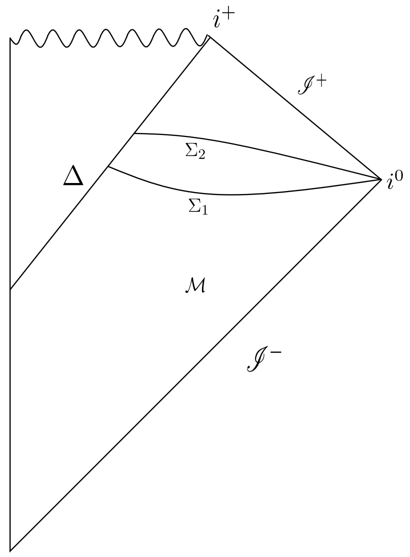


Figure 4.2: An Penrose diagram depicting an isolated horizon Δ, represented by the region adjacent to the Cauchy surface Σ. The intersections $\Sigma \cap \Delta$ are both topologically S^2.

Let us fix a representative $\ell \in [\ell]$. From this, we see isolated horizons are defined entirely locally on Δ and are hence independent of any symmetries of the spacetime $\mathcal{M}$. Therefore, one can intuitively think of an isolated horizon as a non-expanding horizon that is independent of time, and therefore represents a true black hole horizon in equilibrium.

As with the flat spacetime symmetries described by the Poincaré group and the asymptotically flat spacetime symmetries by the BMS group, we wish to characterise the symmetry group associated to isolated horizons.

Definition 4.12. A **symmetry** of the isolated horizon (Δ, ℓ) is a spacetime diffeomorphism of Δ to itself. †

The symmetry group of an isolated horizon is, by construction, not as straightforward as that of an arbitrary submanifold in $\mathcal{M}$: As we approach infinity, the symmetry group approaches a global symmetry group (such as the Poincaré group for Minkowski spacetime) since the metric tensor approaches a universal fixed metric. However, isolated horizons do indeed have local symmetry, which may be classified into three universality classes [Abh04],[AK04].

†In particular, such symmetries must preserve q and D.

These correspond to I. Spherical isolated horizon geometry, II. Axis-symmetric isolated horizon geometry and III. Diffeomorphism group of ℓ^α.

In the following section, we will introduce a framework to count entropy states in Type I horizons, which may also be extended to Type II horizons (see Section 8.3 of [Abh04]). The treatment of Type III horizons is an ongoing area of research in LQG, due to the lack of underlying assumptions on their geometric symmetry (unlike the Type I and II cases).

4.3 Intermezzo: Connection formalism

Before we can delve into the quantisation of black hole entropy, we need additional preliminaries. We will, in a sense, recast our original formalism of general relativity into the so-called *connection formalism* of general relativity. This will allow us to describe our spacetime as a principal G-bundle, providing us with the necessary structure to quantise the fields describing the geometry of the spacetime. There are multiple formalisms of general relativity, the most common being the affine formalism in terms of the metric tensor and Levi-Civita connection, which has been our approach thus far. The connection formalism recasts general relativity into a theory defined in terms of frame fields and spin connection, which are now the dynamical variables of the theory. Here, we will introduce the basic notions required to discuss the quantisation of gravity and will follow the formulation described in [Tec19]. For a detailed discussion on general relativity in terms of covielbeins from the ground up, the reader is directed to Chapter 2 of [Abh04] and [Ash86].

We begin our discussion by defining a particular choice of frame on our spacetime, known as a *covielbein*[‡].

Definition 4.13. Let (M, g) an n-dimensional Lorentzian manifold of signature $(n-1, 1)$ and let (V, η) be an n-dimensional vector space with a bilinear form $\eta = \mathrm{diag}(-1, ..., 1)$ (a copy of the Minkowski metric on V). A **covielbein** at $x \in U \subset M$ is a linear isometry

$$
{}_xe : T_xU \to V, \tag{4.11}
$$

with the property that ${}_xe^*\eta = \eta_{IJ\,x}e^I{}_xe^J = g$. Here, the capital Latin indices denote coordinates on V.

Equivalently, the collection $({}_xe^I)$ is an orthonormal basis for the dual bundle fibre T^*_xU. One can also infer that a **vielbein** is the dual object of a covielbein. The idea of introducing this vector space (V, η) is to construct an associated bundle that has a fibrewise metric η as follows:

Definition 4.14. The **covielbein bundle** $F^*_{SO}(M)$ is an $SO(n-1, 1)$-principal bundle, where each fibre $F^*_{xSO}(M)$ is the set of all covielbein fields at $x \in U \subset M$. Dually, we have the **vielbein bundle** $F_{SO}(M)$ being the principal bundle of all vielbein fields at $x \in U$

[‡] Also known as a *coframe* in other literature

Definition 4.15. The **Minkowski bundle** $\mathcal{V}$ is the associated bundle:

$$\mathcal{V} := F_{SO}(M) \times_\rho V, \tag{4.12}$$

where ρ is the fundamental representation of $SO(n-1,1)$.

Remark 4.16. If M is parallelisable, every (co)vielbein extends to a global section of TM (T^*M), which means that $e^* : TM \to M \times \mathcal{V}$ is the identity map. This implies, since every covierbein corresponds to a choice of ordered basis globally, that e^* may be identified with an element of $\Omega^1(M, V)$. In particular, if v_I is an orthonormal basis for V, then $e^I v_I \in \Omega^1(M, V)$, with $e^I \in \Omega^1(M)$ and

$$g = \eta_{IJ} e^I e^J. \tag{4.13}$$

Physicists usually refer to the fibre V as the *internal space*. In particular, it may be intuitively thought of as a copy of Minkowski spacetime glued onto each $x \in M$ to recover local inertial measurements in spacetime. Definition 4.13 (and consequently (4.13)) also imply that ${}_x e$ is a solder form for the metric:

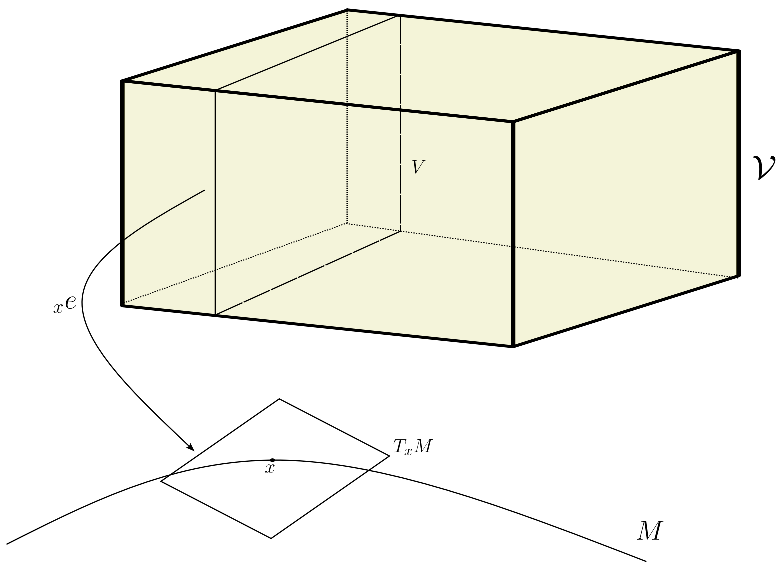


Figure 4.3: A diagram showing the covielbein ${}_x e$ 'gluing' the fibre V of the Minkowski bundle onto the tangent space $T_x M$

.

In the case that M is not parallelisable, it should absolutely be emphasised that ${}_x e^I$ does *not* provide a canonical isomorphism between $\mathcal{V}$ and TM. What this means is that (e_I) is not a basis for $T_x M$ for any x; rather, it is what is known as an *anholonomic basis* (a basis for the fibre $T_x M$ that has non-vanishing Lie bracket).

Example 4.17. Let $(\mathcal{M}, g)$ be a spacetime with $\mathcal{M} \cong \mathbb{R} \times \Sigma$, where the Cauchy surface Σ is a compact and orientable 3-manifold (as is the case in the ADM formalism of general relativity). Then $\mathcal{M}$ is parallelisable, and hence the covielbein fields $e : TM \to \mathcal{V}$ are bundle isomorphisms. Since $n = 4$, the covielbein fields are known as *tetrads*, or *covierbeins*. These will be our main fields of consideration in the connection formalism of general relativity [Abh04],[Thi07].

Example 4.18. In a similar fashion, the pullback of a covielbein field to the Cauchy surface described in the above example yields a *codreibein*, or *triad*, which are bundle isomorphisms $T\Sigma \to \mathcal{W}$. Here $\mathcal{W} \subset \mathcal{V}$ is a subbundle of $\mathcal{V}$ where each fibre is isomorphic to $\mathbb{R}^3$ with the Euclidean metric. In this case, the structure group now reduces to $SO(3)$, rather than $SO(3,1)$, since Σ is a Riemannian manifold with positive definite metric q. Parallelisability Σ also gives $e^i v_i \in \Omega^1(\Sigma, W)$, $e^i \in \Omega^1(\Sigma)$ and $q = \delta_{ij} e^i e^j$. In particular, e is also a solder form for q.

The above example will be essential for us later in the next section in defining the canonical phase space of connections. This constitutes one half of the theory, the other half coming from a spin connection on our Cauchy surface $\Sigma \subset \mathcal{M}$.

Building upon the above example, consider a principal $SO(3)$-bundle $F_{SO}(\Sigma) \to \Sigma$ and $e : U \subset \Sigma \to F_{SO}(\Sigma)$ a dreibein field, equivalently an isomorphism of $T\Sigma \xrightarrow[e]{\sim} F_{SO}(\Sigma) \times_\rho W$. A connection one-form $\omega_{SO} \in \Omega^1(F_{SO}(\Sigma), \mathfrak{so}(3))$. Then upon pulling back by e, we get the local form $e^*\omega_{SO} \in \Omega^1(U, \mathfrak{so}(3))$. This description of ω_{SO} is in terms of the dreibein field, which suggests, by the isomorphism to the associated bundle, we can express ω_{SO} in terms of the internal space W. This turns out to be the case:

Proposition 4.19. *$\Lambda^2 W$ is isomorphic to $\mathfrak{so}(3)$, with isomorphism induced by the Euclidean metric δ.*

Proof. This is an adapted version of the proof described in [Tec19]. Let v_1, v_2, v_3 be a basis for W. By taking exterior products, we obtain a basis for $\Lambda^2 W$ explicitly as $\{v_1 \wedge v_2, v_2 \wedge v_3, v_3 \wedge v_1\}$. These map bijectively onto the basis elements of $\mathfrak{so}(3)$, explicitly:

$$\tau_i^j = v_i \wedge v_k \delta^{kj}. \tag{4.14}$$

Omitting an explicit calculation, this gives us the desired isomorphism $\Lambda^2 W \cong \mathfrak{so}(3)$ □

Remark 4.20. As shown in [Tec19], this isomorphism also generalises to the fibres of the Minkowski bundle $\Lambda^2 V \cong \mathfrak{so}(3,1)$.

As a corollary, the pullback of the connection one-form

$$e^*\omega_{SO} = \omega_a{}^i{}_j = e^b_j \nabla_a e^i_b, \tag{4.15}$$

is a one-form taking values in the exterior algebra $\Lambda^2 W$.

Notation 4.21. We remind the reader that lowercase Latin indices starting from i $(i, j, k, ... = 1, 2, 3)$ denote vector indices in $\mathbb{R}^3$ so as not to be confused with tensorial spacetime indices $(a, b, c, ... = 1, 2, 3)$ starting from a.

Now, we define the spin connection associated with ω_{SO}.

To do so, by Lemma 1.19 we can 'lift' the connection one-form ω_{SO} to a Spin(3)-principal bundle $(F_{\text{Spin}}(\Sigma), \Psi)$. In particular, we see that the following diagram commutes.

$$\begin{tikzcd} F_{\text{Spin}}(\Sigma) \arrow[r, "\Psi"] & F_{SO}(\Sigma) \\ U \arrow[u, "s"] \arrow[r, "\text{id}"] & U \arrow[u, "e"'] \end{tikzcd}$$

where $s : U \to F_{\text{Spin}}(\Sigma)$ is a local spin frame. From this, we observe $\Psi \circ s = e$. Given the well-known result $\text{Spin}(3) \cong SU(2)$, and $\mathfrak{spin}(3) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$, we define a *spin connection* $\Gamma \in \Omega^1(F_{\text{Spin}}(\Sigma), \mathfrak{spin}(3))$ by

$$\Gamma := \sigma_*^{-1} \circ \Psi^* \omega_{SO}, \tag{4.16}$$

where $\sigma : SU(2) \to SO(3)$ is the usual double-cover map. Locally, one can determine that this spin connection has components defined by the pullback

$$\begin{aligned} s^*\Gamma &= \sigma_*^{-1} \circ e^* \omega_{SO} \in \Omega^1(U, \mathfrak{su}(2)) \\ \Gamma^i_a : &= \frac{1}{2} \epsilon^i{}_{jk} \omega^{jk}_{aSO}. \end{aligned} \tag{4.17}$$

With this, we now have our so-called configuration variables (Γ, e) for the connection formalism, consisting of the spin connection Γ and the dreibein e on the Cauchy surface Σ. It has been shown that using these variables, the physical dynamics of the spacetime (in particular, those governed by the Einstein-Hilbert action) are identical to those in the affine formalism [Abh04] [Thi07]. We will not delve into precisely why this is true, as it is well-documented in the literature; instead, we will focus on further developing the machinery provided by this formalism. These new variables will play a crucial role in developing the classical mechanics of black hole horizons.

Our next step is to obtain a quantum theory based on these objects, which will be the subject of the following chapters.

4.4 Classical Theory on Isolated Horizons

We begin with a classical treatment of isolated horizons, but expressed in the language of the connection formalism, rather than the affine formalism. We will not provide a whole systematic derivation of the classical framework (this is well-documented in, say, Chapter 4 of [Thi07], [Abh00a], [LW90], among other literature) and focus on the aspects core to describing black hole entropy. We will follow the work done in [AK04], [Abh04], and [Cor09].

Recall our classical configuration variables (Γ, e) from prior. To describe Type 1 IHs, we need to 'refine' them; in particular, we aim to rework them into a form that is more 'compatible' with the Hamiltonian framework in the ADM formalism of relativity.

Definition 4.22. Let $(\mathcal{M}, g)$ be a spacetime where $\mathcal{M} \cong \mathbb{R} \times \Sigma$ and let $P \to \Sigma$ be an $SU(2)$-

principal bundle. The **canonical variables** of general relativity comprise of an $SU(2)$-connection one-form $A \in \Omega^1(P, \mathfrak{su}(2))$ and a *densitised dreibein field* $E : U \subset \Sigma \to P$ taking values in fibres of the associated bundle $P \times_\rho W$, where $W \cong \mathbb{R}^3$. Locally, these have components

$$A^i_a := \Gamma^i_a - \beta K^i_a \quad ; \quad E^a_i := \sqrt{q} e^a_i, \tag{4.18}$$

where $K^i_a = (q^\alpha_a q^b_\beta \nabla_\alpha n^\beta) e^i_b$ is the extrinsic curvature of Σ, n the unit normal to Σ, $a, i \in \{1, 2, 3\}$ are indices describing coordinates on a chart U and fibre W respectively, and $\alpha, \beta \in \{0, 1, 2, 3\}$ describe the coordinates of Σ embedded in $\mathcal{M}$. The parameter β is a non-zero real number known as the *Barbero–Immirzi parameter* (or sometimes just the Immirzi parameter).

The pair (A, E) was first introduced and described by Ashtekar [Ash86] to simplify the physical constraint equations derived from the constraint algebra in the ADM formalism. Indeed, these objects form much of the foundation of quantum gravity.

Note that in our construction of (A, E), we have suppressed the spatial dependence $A(x), E(x)$ for $x \in U$. Under the inclusion into the whole spacetime manifold, for fixed t, the fields $(A(t, x), E(t, x))$ are points in an infinite-dimensional symplectic manifold Γ_P known as the *phase space*.

Proposition 4.23. *Fix $t \in \mathbb{R}$. Then the space*

$$\Gamma_P := \{(A(t, x), E(t, x)) \in \Omega^1(U, \mathfrak{su}(2)) \times \Gamma(P \times_\rho W)\}, \tag{4.19}$$

is a Banach manifold with symplectic structure given by

$$\Omega_P[(\delta A, \delta\Theta), (\delta A', \delta\Theta')] := \frac{1}{8\pi G} \int_\Sigma \mathrm{Tr}[\delta A \wedge \delta\Theta' - \delta A' \wedge \delta\Theta], \tag{4.20}$$

where $\Theta_{abi} := \frac{1}{6\beta} \epsilon_{abc} E^c_i$ is the conjugate momentum of E (a byproduct of a Legendre transform [Gao12]) and $(\delta A, \delta\Theta), (\delta A', \delta\Theta') \in T\Gamma_P$ are any two elements in the tangent space of Γ_P.

To be more precise, one must specify the Banach space that the manifold is modelled over. For the purposes of this thesis, we will assume it is already equipped with an appropriate structure (further details may be found in [Thi07], [CM74]).

It is routine in Hamiltonian field theories to obtain a set of constraints for the classical theory. Loosely, given a symplectic manifold M for which the Hamiltonian field theory is based on, a *constraint* is a function $f \in C^\infty(M)$ vanishing on certain states (known as physically allowed states). We say the collection of constraints f_α is *first class* if their Poisson bracket $\{f_\alpha, f_\beta\}$ vanishes everywhere [§].

In our context, in passing to the connection formalism, one obtains three constraints: The *Gauss, Diffeomorphism and Hamiltonian Constraints*. These, respectively, are interpreted as

[§]Since Hamiltonian functionals may be obtained through Legendre transforms of Lagrangians, one obtains constraints provided the Legendre transform fails to be surjective. For further detail, we direct the reader to Chapter 24 of [Thi07].

generating $SU(2)$ transformations on Γ_P, generating spatial diffeomorphisms on Σ, and generating temporal translations. For our purposes, we will focus on the Hamiltonian constraint, since this is the most significant regarding isolated horizons (For complete descriptions of the other constraints, see [Abh04]).

Definition 4.24. The **Hamiltonian constraint** of general relativity is a functional on Σ given by

$$H[N] := \int_\Sigma N(x) H^E(x)\, d^3x, \tag{4.21}$$

where N is the lapse function from Section 4.1 and $H^E(x)$ a function of the form

$$H^E := \frac{\epsilon^{ij}{}_k E^a_i E^b_j}{\sqrt{\det q}} F^k_{ab} \;+\; (\text{terms involving } K \text{ and } \beta), \tag{4.22}$$

Note we leave the second term of the expression for H^E loose, since functionally it will not be important for our later discussions. Note also that some authors tend to write H^E to be the Hamiltonian constraint, rather than the full smeared functional of N. By considering the pairs (A, E) for which (4.21) vanishes, we obtain a submanifold of Γ_P corresponding to physically allowed configurations.

Building on this, one can reformulate the condition that the induced derivative operator on Δ vanishes on null normals (see Definition 4.11) in terms of the Ashtekar configuration variables [Abh00b],[A. 98]. In this formulation, the boundary condition becomes a condition on the pullback of the curvature F^i_{ab} of the connection A^i_a onto the intersection $S = \Delta \cap \Sigma \cong S^2$.

Definition 4.25. The **horizon boundary condition** on S is the requirement that the curvature 2-form F of A satisfies

$$\varphi^* F^i_{ab} = -\frac{2\pi\beta^2}{A_S} \varphi^* \Theta^i_{ab}, \tag{4.23}$$

where $\varphi : \Delta \to \mathcal{M}$ is the embedding from section 4.2.

Remark 4.26. By virtue of imposing the constraints and boundary conditions to make our theory 'physical', we have reduced our selection of fields in Γ_P to only those pairs that satisfy these conditions. In particular, we consider the subset of Γ_P

$$\begin{aligned} \mathcal{X} := \{(A, E) \in \Gamma_P \mid\ & \text{Horizon BC, Gauss, Diffeomorphism,} \\ & \text{Hamiltonian constraints hold}\}. \end{aligned} \tag{4.24}$$

In the literature, $\mathcal{X}$ is modelled as a Fréchet submanifold of Γ_P [ABK00],[Thi07].

Notation 4.27. From now on, we will refer to the space $\mathcal{X}_P$ as the *phase space*, rather than Γ_P.

In this section, we have described the classical theory of isolated horizons in the connection framework of general relativity. Why did we consider these specific horizons rather than the standard horizons H of black hole regions? By definition, isolated horizons were defined locally in terms of null 3-surfaces Δ, allowing us to impose boundary conditions on a single slice $\Delta \cap \Sigma$ without reference to the global spacetime. This makes it more plausible that our Hilbert space in the quantum theory will be countable, allowing for a reasonable picture of a true quantum theory. In the following chapters, we outline the mathematics and framework of the quantisation procedure of isolated horizons.

Chapter 5

Loop Quantum Gravity

In the previous chapter, we recast our classical theory in terms of the spin connection A and the densitised dreibein field E. The next step is to pass to a quantum theory in terms of these objects. This is the purpose of the Loop Quantum Gravity (LQG), a mathematically rigorous quantum theory of gravity. The two key objects of this theory are holonomies (of connection forms) and electric fluxes (Lie algebra-valued vector fields), which comprise the so-called *Holonomy-Flux algebra*. A diffeomorphism-invariant representation of a related *-algebra on a Hilbert space will underpin the quantum theory, and in the end provide us with a microscopic description of the Bekenstein-Hawking entropy. Moreover, we will see this is a so-called *background independent theory* - one which does not fix a spacetime metric and works to quantise the spacetime structure itself. This is in contrast to other theories, such as quantum field theory. We will first describe the general setting in this chapter, before moving to define entropy in physical spacetime.

Let us outline the general procedure. Let $P \to M$ be a principal G-bundle over some C^m–differentiable manifold M of dimension D and $\mathcal{A}$ the space of local connection one-forms. Then the cotangent bundle $T^*\mathcal{A}$ is a symplectic Banach manifold carrying the canonical Poisson brackets. One may, of course, obtain the same canonical structure from an action functional and the usual Hamiltonian reduction (as is typically done in quantisation). Still, in this thesis, we will not use action functionals directly. Instead, we will adopt a deformation canonical quantisation approach: the holonomy–flux variables provide a natural Poisson algebra which we promote to a quantum *-algebra. These gauge-covariant variables are well suited to background independence (since everything here is explicitly metric independent). The constraints then generate the group of bundle automorphisms homotopic to the identity (a semi-direct product of Yang–Mills gauge transformations with $\mathrm{Diff}(\Sigma)$), whose implementation in the quantum theory is crucial for applications such as black-hole entropy.

To this end, let us begin with a general description of the Holonomy-Flux algebra. Let $\mathcal{C}$ denote the set of continuous, oriented, piecewise semianalytic compactly supported curves embedded into M.

The phase space $\Gamma_P = T^*\mathcal{A}$ consists of pairs (A, E) of $\mathfrak{g}$ valued connection one-forms and $\mathfrak{g}^*$ valued vector densities E. One can observe that this is an extension of our specific case of an $SU(2)$ bundle as described above. Given a chart U_α with local coordinates $(x^1, ..., x^D)$, we have that the one-forms and vector densities may be expressed componentwise as

$$A(x) = A^i_a(x)dx^a \otimes \tau_a \;\; ; \;\; E(x) = E^a_i(x)\partial_a \otimes \tau^i, \tag{5.1}$$

where $\{\tau_1, ..., \tau_R\}$ denotes a basis for the Lie algebra $\mathfrak{g}$.

With the given symplectic form Ω_P, we obtain a single non-vanishing Poisson bracket:

$$\{A^i_a(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta(x - y). \tag{5.2}$$

Observe that this bracket is distributional, which is a significant obstruction to quantisation, leading to ill-defined operators, singularities in the domain and worse, the breaking of diffeomorphism invariance. Hence, we must smear these with appropriate test functions (as is typical in functional analysis) to obtain well-defined operators or states. Since A is a one-form, the canonical choice of functional in LQG [Thi07] is that of integrals of 1-dimensional submanifolds of M (specifically curves $c \in \mathcal{C}$, the space of continuous, oriented, semianalytic, compactly supported curves on M), in the sense that there is a natural pairing between k-forms and k-dimensional submanifolds, due to Poincaré duality. In particular, we are interested in a *holonomies* of A:

Definition 5.1. Suppose $U \subset M$ be a chart and let $\mathcal{A} := \{A \mid A \in \Omega^1(U, \mathfrak{g})\}$ be the space of smooth local connections one-forms on P (note that the dependence on P here is implicit). A **holonomy** along a curve $c \in \mathcal{C}$ is a map $h_c : \mathcal{A} \to G$ given by

$$h_c(A) := \mathcal{P} \exp\left(\int_c A\right) = 1_G + \sum_{n=1}^{\infty} \int_0^1 dt_1 ... \int_{t_{n-1}}^1 dt_n A(c(t_1))...A(c(t_n)), \tag{5.3}$$

where $\mathcal{P}$ denotes the path ordering and 1_G is the identity on G.

Note that it may be shown [Thi07] Definition 5.1 is equivalent to the usual definition of holonomies on principal bundles.

On the other hand, E is a vector density, and for the Poisson bracket to be non-distributional, we require E to be smeared along $D - 1$ dimensions. Again, the canonical approach [Thi07] here is to consider the Hodge dual $(\star E)$, which is a pseudo $D - 1$ form *. In coordinates, we write

$$(\star E) = \frac{1}{(D-1)!}\epsilon_{aa_1...a_D} E^a_i dx^{a_1} \wedge ... \wedge dx^{a_D} \otimes \tau_i. \tag{5.4}$$

Using the de Rham theorem (which describes a natural pairing with integrating k forms with k-dimensional submanifolds), we arrive at a distribution:

*A pseudo n-form τ is a twisted differential form, in the sense that over a vector space V, τ assigns, for every orientation o of V a n-form τ_o such that a reversal of orientation corresponds to taking the negative of τ_o, i.e. $\tau_{-o} = -\tau_o$. A familiar example of such a form is the standard volume form on $\mathbb{R}^n$, as a change of orientation of basis vectors induces a sign flip due to the antisymmetry of the wedge product.

Definition 5.2. Let $S \subset M$ be a codimension 1 submanifold of M, such that the normal bundle NS is equipped with an orientation, and f a G-invariant, semianalytic compactly supported vector field over the subbundle $\pi^{-1}(S)$. That is, $f \in \mathfrak{X}(\pi^{-1}(S), \mathrm{ad}P)$ The **electric flux** associated to E is the distribution

$$E(S,f) := \int_S (\star E)^i f_i. \tag{5.5}$$

In the literature, S is known as a *face* of M and f a *smearing vector field.* This is since it defines a distribution when paired with $\star E$. Note also that we have f, in a local trivialisation, which will be Lie algebra valued.

We have now defined two basic objects: The holonomy h_c and the electric flux $E(S,f)$. The quantisation procedure requires the construction of a quantum algebra $\mathfrak{Q}$, an abstract $*$-algebra, with operators associated to h_c and $E(S,f)$ together with relations that reflect composition of holonomies/fluxes as well as an appropriate conjugation relation between the operators. Additionally, we also require an appropriate space of functions to act on the configuration space (i.e. functions from $\mathcal{A} \to \mathbb{C}$) to obtain information about the dynamics of the space, as in traditional quantum mechanics. We begin by describing these spaces and relations.

5.1 The Holonomy-Flux Algebra

In this section, we construct the *Holonomy-Flux algebra*, a classical algebra equipped with Poisson brackets inherited from the symplectic structure of the classical phase. There are several ways to construct the holonomy flux algebra with varying methods (see e.g. [Thi07] [Kam11]). We will follow the approach outlined in [LOST06], supplemented by ideas from Chapters 6-7 of [Thi07] and Sections 1-2 of [ABK00].

Let $P \to M$ be a principal G-bundle with atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in\mathcal{I}}$ for M and fibres F for P. We will restrict our attention to the real semianalytic category of smooth curves.[†].

Definition 5.3. An **edge** $e(t)$ on M is an equivalence class of curves $[c(t)] \in \mathcal{C}$, $t \in [0,1]$, where two curves c, c' are said to be equivalent if they share start and endpoints, and holonomies along both curves agree, i.e. given $A \in \mathcal{A}$

$$h_c(A) = h_{c'}(A). \tag{5.6}$$

In particular, e[‡] is an oriented, embedded 1-dimensional C^0 submanifold of M with a two-point boundary, denoted $e_s = e(0)$ and $e_f = e(1)$, given by a finite union of semianalytic edges. Denote the space of all edges as $\mathcal{E}$

We also define edge composition as $\cdot : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ by $e \cdot e' := \overline{e \cup e' \setminus (e \cap e')}$ (subject to obvious reparameterisation). The overline denotes the completion, and $(e \cdot e')_s = e_s$ and

[†]This is the required category to work in to allow for the construction of the quantum *-algebra [Thi07].
[‡]often, we suppress the t-dependence when referring to edges.

$(e \cdot e')_f = e'_f$. Observe that this composition is only defined when $e_f = e'_s$. We define an 'edge inverse' as $e^{-1}(t) = e(1-t)$ to be the traversal along an edge in the opposite direction.

Remark 5.4. It follows $(\mathcal{E}, \cdot)$ forms a groupoid.

Theorem 5.5. *Let $e \in \mathcal{E}$ with starting point $e_s = x$ and endpoint $e_f = y$. Then $A \in \mathcal{A}$ defines a G-equivariant diffeomorphism between fibres:*

$$A_e : \pi^{-1}(x) \to \pi^{-1}(y). \tag{5.7}$$

Proof. (sketch): The edge $e(t)$, parameterised by $t \in [0,1]$ between two points in M lifts to a horizontal curve $\tilde{e}(t) \in P$ such that $\pi(\tilde{e}(t)) = e(t)$. Define $A_e := \tilde{e}(1)$, the endpoint of the lifted curve. Since $e(0) \in \pi^{-1}(x)$ and $e(1) \in \pi^{-1}(y)$, it suffices to show that A_e is G-equivariant. Indeed, we have that there exists a $p \in \pi^{-1}(x)$ such that $\tilde{e}_p(0) = p$, and A_e acts on p by $A_e(p) = \tilde{e}_p(1)$. By G-equivariance of the lift, we have $\tilde{e}_{p \cdot g}(t) = \tilde{e}_p(t) \cdot g$, and hence

$$A_e(p \cdot g) = \tilde{e}_p(1) \cdot g = A_e(p) \cdot g. \tag{5.8}$$

Hence, A_e is a G-equivariant map. Injectivity and surjectivity follow from this, and therefore A_e is a diffeomorphism. □

Denote the collection of all such fibre diffeomorphisms $\pi^{-1}(x) \to \pi^{-1}(y)$ for a given edge e as $\mathcal{A}_e$. Now, each fibre can be realised as a G-torsor (a topological space for which G acts freely and transitively). Choose two basepoints $p_x, p_y \in \pi^{-1}(x), \pi^{-1}(y)$ respectively. Then there exists a bijection [LOST06]:

$$\varphi : \mathcal{A}_e \to G. \tag{5.9}$$

where $A_e(p_x) = R_{\varphi(\mathcal{A}_e)}\, p_y$, the right action. In particular, this map recovers a group element g by $\varphi(A) = g$, if and only if $A_e(p_x) = p_y \cdot g$. This map is completely determined up to left and right action of g, corresponding to changing basepoints. Hence, $\mathcal{A}_e$ inherits the topological structure of G. This is significant, since $\mathcal{A}_e$ also inherits the Haar measure on G, a key ingredient for passage to quantum theory.

We have constructed a space which is topologically the same as G. This gives us a good space to define functions on (which will become our quantum states).

Definition 5.6. A finite collection of edges $\{e_1, ..., e_n\}$ is **independent** if they only intersect each other at the start and endpoints of each edge. An **oriented, finite semianalytic graph** $\gamma \subset M$ is an independent set of edges $\{e_1, ..., e_n\}$, so that $\gamma := \cup_{i=1}^n e_i([a,b])$ where $[a,b]$ is some reparameterisation of the interval $[0,1]$ for each i so that γ can be traversed in unit time.

Denote the set of edges of γ as $E(\gamma)$, and the set of vertices $V(\gamma) := \{e_s, e_f |\ e \in E(\gamma)\}$. Consider an embedded graph $\gamma \subset M$, for each edge $e_k \in E(\gamma)$, we have a collection of bundle isomorphisms $\mathcal{A}_{e_k}$, each of which has the structure of G. Unfortunately [LOST06], there is no natural way of equipping it with a measure. So, for us to define a candidate for a state function in the quantum theory, we are naturally led to consider cylindrical functions:

Definition 5.7. A function $\Psi : \mathcal{A} \to \mathbb{C}$ is **cylindrical** if there exists a finite set of edges $E = \{e_1, ..., e_n\}$ and $\psi \in C^\infty \left(\prod_{k=1}^n \mathcal{A}_{e_k}\right)$ so that

$$\Psi(A) = \psi(A_{e_1}, ..., A_{e_n}), \tag{5.10}$$

for all $A \in \mathcal{A}$. In this case, we say Ψ is *compatible* with E and ψ

Since γ defines a finite set of edges on M, we are led to consider cylindrical functions over γ.

Proposition 5.8. *Denote the algebra of cylindrical functions over γ as Cyl^∞. Then Cyl^∞ is a subalgebra of $\mathrm{Fun}(\mathcal{A}, \mathbb{C})$, with a natural involution and norm:*

$$\Psi^* := \bar{\Psi} \quad ; \quad \|\Psi\| = \sup_{A \in \mathcal{A}} |\Psi|, \tag{5.11}$$

where $\bar{\Psi}$ denotes usual complex conjugation.

This is straightforward to show. Considering the completion $\overline{\mathrm{Cyl}^\infty}$ with respect to the norm, the triple $(\overline{\mathrm{Cyl}^\infty}, \|.\|, *)$ becomes a unital commutative C*-algebra[§]. With this, the space of connections $\mathcal{A}$ is our classical configuration space, and cylindrical functions on this will play the role of 'position functions'. To transition into the quantum theory, we aim to promote the cylindrical functions to 'position wavefunctions' on the quantum configuration space. To do this, it turns out that we must embed $\mathcal{A}$ into the Gel'fand spectrum $\mathrm{Spec}(\overline{\mathrm{Cyl}^\infty})$, which will give us our quantum configuration space of functions, which will form the foundations of the quantum algebra.

Remark 5.9. This is not the only method of constructing the quantum configuration space. For instance, the quatum configuration space $\overline{\mathcal{A}}$ may be identified with the projective limit X_∞ of the spaces $X_l = \mathrm{Hom}(l(\gamma), G)$, and therefore $\overline{\mathcal{A}} = \mathrm{Hom}(\mathcal{E}, G)$, from which one realises $\overline{\mathcal{A}}$ as the spectrum of a cylindrical algebra. For more details on the construction of $\overline{\mathcal{A}}$ using projective techniques, we direct the reader to [Vel04].

In light of this, we define the following:

Definition 5.10. A **generalised connection** $\bar{A}$ on P is a bundle isomorphism

$$\bar{A}_e : \pi^{-1}(e_s) \to \pi^{-1}(e_f), \tag{5.12}$$

such that it is a homomorphism on $\mathcal{E}$. That is, whenever composition is defined:

$$\bar{A}(e \cdot e') = \bar{A}(e) \cdot \bar{A}(e') \;\; ; \;\; \bar{A}(e^{-1}) = \bar{A}^{-1}(e). \tag{5.13}$$

Denote the collection of all generalised connections as $\overline{\mathcal{A}}$ [¶]

[§] In particular, this is a result of taking the completion with respect to the sup norm.
[¶] It follows $\bar{A}$ is a groupoid morphism on $\mathcal{E}$.

It turns out [LOST06] that we can extend every cylindrical function in Cyl^∞ to $\mathcal{A}$:

$$\Psi(\bar{A}) = \psi(\bar{A}_{e_1}, ..., \bar{A}_{e_n}), \tag{5.14}$$

for a set of edges of a graph $\gamma = \cup_{i=1}^n e_i([a, b])$ compatible with Ψ.

Recalling that each $\mathcal{A}_e$ is in bijection with G - a compact Lie group, it can be shown [Thi07],[Vel04] that $\overline{\mathcal{A}}$ can be realised as the projective limit

$$\overline{\mathcal{A}} = \varprojlim_{\gamma} G^{|E(\gamma)|}. \tag{5.15}$$

A theorem by Tychonov [Thi07] states that the projective limit of compact spaces is compact, which guarantees that $\overline{\mathcal{A}}$ is compact. This is a crucial observation for the constructions that follow.

Lemma 5.11. *The map $\overline{\mathrm{Cyl}^\infty} \to \mathbb{C}$; $\Psi \mapsto \Psi(\bar{A})$ is continuous and is a C*-algebra homomorphism.*

Proof. Continuity follows from continuity of the cylindrical functions with respect to the norm. The map $f : \overline{\mathrm{Cyl}^\infty} \to \mathbb{C}$ is a homomorphism, since

$$\begin{aligned} f(\Psi\Phi) = \Psi\Phi(\bar{A}) &= \psi\phi(\bar{A}_{e_1}, ..., \bar{A}_{e_n}) \\ &= \psi(\bar{A}_{e_1}, ..., \bar{A}_{e_n})\phi(\bar{A}_{e_1}, ..., \bar{A}_{e_n}) \\ &= \Psi(\bar{A})\Phi(\bar{A}) = f(\Psi)f(\Phi), \end{aligned} \tag{5.16}$$

since multiplication is defined pointwise. It is a *-homomorphism, since

$$f(\Psi^*) = f(\bar{\Psi}) = \bar{\Psi}(\bar{A}) = f^*(\Psi). \tag{5.17}$$

Since both $\overline{\mathrm{Cyl}^\infty}$ and $\mathbb{C}$ are C*-algebras, f defines a C*-algebra homomorphism. □

Corollary 5.12. *The space of generalised connections $\overline{\mathcal{A}}$ is a subset of the Gel'fand spectrum.*

Conversely, every element of $\mathrm{Spec}(\overline{\mathrm{Cyl}^\infty})$ may be identified with a generalised connection [AL93], which gives an identification both ways. This leads us to the following definition:

Definition 5.13. The **quantum configuration space** of quantum gravity is the space $\overline{\mathcal{A}} \cong \mathrm{Spec}(\overline{\mathrm{Cyl}^\infty})$. This is known as the *Ashtekar–Isham configuration space.*

Similarly to how we generalised the space of connections $\mathcal{A}$, we wish to do the same for the phase space $T^*\mathcal{A}$, i.e. we want to define a 'generalised phase space' of tangent vectors to $\overline{\mathrm{Cyl}^\infty}$. In particular, we wish to define vector fields corresponding to derivations on the quantum configurations.

To begin with, let $S \subset M$ be a face. Denote the subbundle $P_S = \pi^{-1}(S)$ and let f be a smearing vector field on P_S. We can consider the flow associated with f

$$\exp(tf) : P_S \to P_S. \tag{5.18}$$

In particular, over a local trivialisation of the bundle, $\exp(tf)$ corresponds to an element of G (by definition of exp). In particular, it preserves the fibres of P_S, and hence defines an automorphism on each fibre $\pi^{-1}(x) \subset P_S$. Since it preserves the fibres, it defines a one-parameter subgroup of G-actions on the space of generalised connections. Therefore, it will give us a means to discuss infinitesimal 'variations' on the quantum configuration space. To define this, note the following:

Proposition 5.14. *Given a face S, there exists an open neighbourhood $U \subset M$ containing S such that it partitions U into regions corresponding to the positive and negative orientations of the normal bundle of S. Specifically:*

$$U \setminus S = U^- \cup U^+,$$

where $U^\mp$ denotes the negative and positive regions, respectively. Figure 5.1 portrays the two-dimensional case.

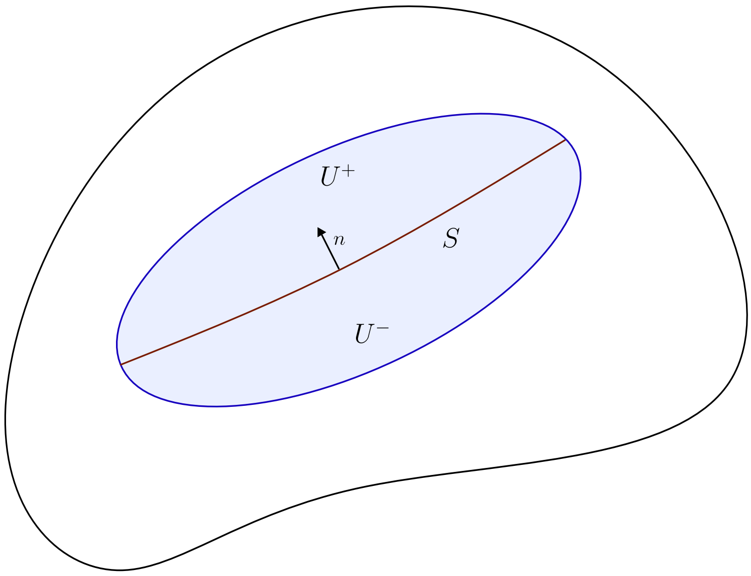


Figure 5.1: A partition of a face on a 2-dimensional manifold. Observe the positive and negative regions defined by a vector n in the normal bundle.

Definition 5.15. Let $e \in \mathcal{E}$ with $e_s = x$. The **generalised flow** of a smearing vector field f on a connection $\bar{A} \in \overline{\mathcal{A}}$ is defined as

$$\theta^{(t)}(\bar{A}_e) := \begin{cases} \bar{A}_e \exp(tf)|_{\pi^{-1}(x)} & \text{if } e \cap S = \{x\} \text{ and } e \setminus x \subset U^+, \\ \exp(tf)|_{\pi^{-1}(x)} \bar{A}_e & \text{if } e \cap S = \{x\} \text{ and } e \setminus x \subset U^-, \\ \bar{A}_e & \text{else.} \end{cases} \tag{5.19}$$

In particular, this preserves $\overline{\mathrm{Cyl}^\infty}$, since the fibres are invariant under G-action. From the t-dependence of the generalised flow, we can make sense of directional derivatives of $\theta^{(t)}$, and therefore by pulling back $\overline{\mathrm{Cyl}^\infty}$, we obtain a definition for derivations on $\overline{\mathrm{Cyl}^\infty}$:

Definition 5.16. A derivation $X_{S,f}$, known as the **generalised vector field** on $\overline{\mathrm{Cyl}^\infty}$ is the derivative

$$X_{S,f}\Psi := \left.\frac{d}{dt}\right|_{t=0} \theta^{*(t)}(\bar{A})\Psi = \left.\frac{d}{dt}\right|_{t=0} \Psi(\theta^{(t)}(\bar{A})). \tag{5.20}$$

Remark 5.17. If we fix a local trivialisation, it turns out this is actually the Hamiltonian vector field associated to the Poisson bracket $X_{S,f}\Psi := \{\Psi, E_{S,f}\}$. Note that the Poisson bracket on $\overline{\mathrm{Cyl}^\infty}$ preserves cylindrical functions, as cylindrical functions only depend on a finite number of edges on an embedded graph γ, the Poisson bracket only depends on a finite number of points (in particular those in $S \cap \gamma$). Hence, they are also well-defined. This indicates that $X_{S,f}$ really is an element of the tangent bundle of $\overline{\mathcal{A}}$

Denote the space of all generalised vector fields as $\mathfrak{X}(\overline{\mathrm{Cyl}^\infty})$. We now have a space of functions Cyl^∞ and generalised vector fields $\mathfrak{X}(\overline{\mathrm{Cyl}^\infty})$ on the quantum configuration space, giving us an analogous 'phase space'. In the spirit of canonical quantisation, these constitute the classical (Poisson) algebra of the canonical theory of general relativity. By equipping this algebra with an involution, we obtain a *-algebra, achieving this subsection's goal of defining the Holonomy-Flux algebra:

Definition 5.18. Let $\mathfrak{X}(\overline{\mathrm{Cyl}^\infty})^{\mathbb{C}}$ denote the complexification of $\mathfrak{X}(\overline{\mathrm{Cyl}^\infty})$. The **Ashtekar–Corichi–Zapata (ACZ) holonomy-flux algebra** is a *-Lie algebra $(\mathfrak{P}, [\cdot,\cdot]_{HF}, *)$, where $\mathfrak{P} := \mathrm{Cyl}^\infty \times \mathfrak{X}(\overline{\mathrm{Cyl}^\infty})^{\mathbb{C}}$, with brackets defined by

$$[(\Psi, Y), (\Psi', Y')]_{HF} := -(Y(\Psi') - Y'(\Psi), [Y, Y']), \tag{5.21}$$

and involution $*$ defined by complex conjugation.

In the next section, we will examine how $\mathfrak{P}$ is used as the foundation of the quantum *-algebra.

5.2 The Quantum *-Algebra

As stated at the beginning of the chapter, we follow the deformation quantisation method of [LOST06] while providing proofs for some of the introduced ideas.

Consider a pair $p = (\Psi, Y) \in \mathfrak{P}$. To construct the quantum algebra, consider the space of formal linear combinations of finite sequences of pairs:

$$\mathfrak{A} = \{(p_1, ..., p_n) \mid p_i \in \mathfrak{P}\} \cong \mathfrak{P} \times ... \times \mathfrak{P}. \tag{5.22}$$

We equip this space with an associative product (defined by concatenation of sequences) and an (antilinear) involution $*$:

$$(p_1, ..., p_n)(q_1, ..., q_n) := (p_1, ..., p_n, q_1, ..., q_n)$$
$$(p_1, ..., p_n)^* := (\bar{p}_n, , ..., \bar{p}_1)$$

Remark 5.19. Observe that $\mathfrak{A}$ is indeed a *-algebra; however, it is not a C*-algebra since we have not defined a norm on it. In particular, upon passing into a suitable representation over a Hilbert space of the quantum *-algebra, we will necessarily have to construct a suitable norm for unbounded operators on Hilbert spaces.

Consider the ideal

$$\begin{aligned}\mathfrak{I} := \langle(\lambda a) - \lambda(a),\ (a+b) - (a) - (b),\\ (a,b) - (b,a) - i[a,b]_{HF},\ (\Psi, a) - (\Psi a)\rangle,\end{aligned} \tag{5.23}$$

for $\lambda \in \mathbb{C},\ a, b \in \mathfrak{P}$ and $\Psi \in \overline{\mathrm{Cyl}^{\infty}}$. This leads us to the quantum *-algebra:

Definition 5.20. The **quantum ACZ holonomy-flux algebra** is the unital *-algebra $(\mathfrak{Q}, *)$ given by

$$\mathfrak{Q} := \mathfrak{A}/\mathfrak{I}. \tag{5.24}$$

The algebra $\mathfrak{Q}$ consists of finite sequences of pairs of cylindrical functions in Cyl^{∞} and complex vector fields on $\overline{\mathrm{Cyl}^{\infty}}$. Physically, this describes a quantum system that could exist in many possible states, and hence we encode all that information in the sequences. By quotienting out by the ideal $\mathfrak{I}$, we obtain the desired linearity, commutator relations and module structure for the quantum theory.

Lemma 5.21. *The map*

$$\mathfrak{P} \to \mathfrak{Q}, \tag{5.25}$$

is an embedding.

This map is non-trivial, and is described in Equation (31) of [LOST06]. Observe that we can represent $\Psi \in \mathrm{Cyl}^{\infty}$ and $Y \in \mathfrak{X}(\overline{\mathrm{Cyl}^{\infty}})$ as the one-element sequences $(\Psi, 0)$ and $(0, Y)$ in $\mathfrak{P}$, respectively. Under the map in (5.25), we denote the images

$$(\Psi, 0) \mapsto \hat{\Psi}\ \ ;\ \ (0, Y) \mapsto \hat{Y}. \tag{5.26}$$

In particular, these generate the whole algebra $\mathfrak{Q}$. Since Y is nothing more than the complexified version of $X_{S,f}$ we have, since $X_{S,f}$ acts as a derivation, that every element of $\mathfrak{Q}$ is a linear combination of products

$$\hat{\Psi},\ \hat{\Psi}_1 \hat{X}_{S_{11}, f_{11}}, \ldots,\ \hat{\Psi}_n \hat{X}_{S_{n,1}, f_{n,1}} \ldots \hat{X}_{S_{n,n}, f_{n,n}}. \tag{5.27}$$

We have now constructed the quantum *-algebra associated with the phase space. The next step in formulating the quantum theory is to represent our quantum algebra on a suitable Hilbert space to describe our states. In particular, we require a representation of the quantum algebra on a Hilbert space. We can achieve this using the *Gel'fand–Naimark–Segal (GNS) construction* of states over a *-algebra, which allows us to construct representations of the quantum algebra that are invariant under the automorphism group of our classical theory, which is necessary to ensure the consistency of the theory.

We begin by defining what we mean by a state on $\mathfrak{Q}$:

Definition 5.22. A **state** on a unital *-algebra $\mathfrak{Q}$ is a linear functional $\omega : \mathfrak{Q} \to \mathbb{C}$ such that, for all $\lambda \in \mathbb{C}$ and $a, b \in \mathfrak{Q}$, the following hold:

1. $\omega(a^*) = \overline{\omega(a)}$
2. $\omega(a^* a) \geq 0$
3. $\omega(1) = 1$

Definition 5.23. Let ω be a state on $\mathfrak{Q}$. The **GNS representation** is a triple $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ where $\mathcal{H}_\omega$ is a Hilbert space, $\pi_\omega : \mathfrak{Q} \to \mathcal{H}_\omega$ a representation and ξ_ω is a cyclic vector on $\mathcal{H}_\omega$ (loosely, a vector for which $\pi_\omega(\mathfrak{Q})\xi_\omega$ spans $\mathcal{H}_\omega$). We call ξ_ω the *vacuum state.*

From here, we would like to know what our Hilbert space is, and it turns out it is very closely related to the quantum algebra itself. We will show this, as presented in [LOST06]. In particular, consider the left ideal of $\mathfrak{Q}$

$$\mathfrak{J} := \langle a \mid \omega(a^* a) = 0 \rangle. \tag{5.28}$$

By quotienting out by this ideal, we obtain the equivalence class

$$[\mathfrak{Q}] = \mathfrak{Q}/\mathfrak{J}. \tag{5.29}$$

Proposition 5.24. $[\mathfrak{Q}]$ *is equipped with an inner product, defined by*

$$([a], [b]) := \omega(a^* b). \tag{5.30}$$

It is a straightforward computation to verify that (5.30) indeed defines an inner product ‖. This now allows us to define our Hilbert space:

Definition 5.25. The Hilbert space of quantum states is the completion of $[\mathfrak{Q}]$ under the norm induced by (5.30). That is,

$$\mathcal{H}_\omega := \overline{[\mathfrak{Q}]}, \tag{5.31}$$

with norm $\|a\|_\omega := \sqrt{([a], [a])}$.

Heuristically, our states are just equivalence classes of the quantum algebra itself, under an appropriate representation. Furthermore, operators on these states are defined by the action of representations of the quantum algebra. More specifically, $\pi_\omega(a)$, $a \in \mathfrak{Q}$ is an operator on $\mathcal{H}_\omega$, defined by

$$\pi_\omega(a)[b] = [ab]. \tag{5.32}$$

We have now constructed the quantum *-algebra $\mathfrak{Q}$ of canonical loop quantum gravity in full generality. In the following section, we will use the machinery here to describe the Hilbert space of states that we will use in the next chapter.

‖ with the caveat that it is antilinear in the first argument, as is the convention in most of physics.

5.3 The Ashtekar–Lewandowski Hilbert Space

Describing our Hilbert space of states comes down to a specific application of the GNS construction with a particular choice of functional ω. In particular, given $P \xrightarrow{\pi} M$, we are after states which are invariant under automorphisms of P (such as gauge transformations). This is of utmost importance, since every bundle automorphism $\tilde{\phi} \in \mathrm{Aut}(P)$ determines a unique diffeomorphism $\phi \in \mathrm{Diff}(M)$ of the underlying manifold. In particular, we need $\pi \circ \tilde{\phi} = \phi \circ \pi$.

So classical constraints such as the diffeomorphisms and Gauss constraints (describing symmetries of the classical theory) are encoded in bundle automorphisms. Therefore, we require the quantum states to also obey such symmetries for consistency.

With this in mind, we begin with the following lemma (Lemma 3.2 of [LOST06]):

Lemma 5.26. *Let $\omega : \mathrm{Cyl}^\infty \to \mathbb{C}$ be a state on the *-algebra $(\mathrm{Cyl}, *)$ with involution given by complex conjugation. Then for any $\Psi \in \mathrm{Cyl}^\infty$*

$$|\omega(\Psi)| \leq 2\|\Psi\|, \tag{5.33}$$

with norm as defined in Proposition 5.8. Hence, ω is continuous with respect to $\|\cdot\|$ and determines a unique extension to the completion $\overline{\mathrm{Cyl}^\infty}$, a C^*-algebra.

Definition 5.27. The **adjoint action** of a bundle automorphism $\tilde{\phi} \in \mathrm{Aut}(P)$ on Cyl^∞ is given by the pullback

$$(\mathrm{Ad}^*_{\tilde{\phi}})(\Psi)(A) = (\mathrm{Ad}^*_{\tilde{\phi}}\psi)(A_{\phi^{-1}(e_1)}, ..., A_{\phi^{-1}(e_n)}). \tag{5.34}$$

Along with this definition, we will utilise the previous lemma in conjunction with the following non-trivial result (Lemma 4.3 of [LOST06]):

Lemma 5.28. *Let ω be a state invariant under $\mathrm{Aut}(P)$. That is, given $\tilde{\phi} \in \mathrm{Aut}(P)$, the action $\alpha_{\tilde{\phi}} : \mathfrak{Q} \to \mathfrak{Q}$ on $\mathfrak{Q}$ satisfies*

$$\omega(a) = \omega(\alpha_{\tilde{\phi}} a) \;\; \forall a \in \mathfrak{Q}, \tag{5.35}$$

where $\alpha_{\tilde{\phi}}(\hat{\Psi}, \hat{X}_{S,f}) := (\widehat{\mathrm{Ad}^*_{\tilde{\phi}^{-1}}\Psi}, \hat{X}_{\phi(S),\tilde{\phi}_* f})$. Then for every flux vector field $X_{S,f}$ in the corresponding GNS representation of this state

$$[\hat{X}_{S,f}] = 0. \tag{5.36}$$

Furthermore, it turns out that only one such invariant state exists on $\mathfrak{Q}$ [LOST06]. Therefore, this state effectively 'kills' the flux component. Physically speaking, it develops a zero vacuum expectation value in this state. Hence, any vector fields $\hat{Y} \in \mathfrak{Q}$ no longer contribute to the state and ω restricts to the subalgebra $\widehat{\mathrm{Cyl}^\infty} \to \mathbb{C}$ consisting of the cylindrical functions $\hat{\Psi}$. In particular, Lemma 5.28 applies and we obtain that ω restricts to a commutative

C*-algebra $\widehat{\overline{\mathrm{Cyl}^\infty}}$ and is continuous with respect to the norm $\|\cdot\|$ on $\overline{\mathrm{Cyl}^\infty}$. The Riesz–Markov–Kakutani representation theorem then applies, and we obtain that there exists a unique Borel measure μ such that

$$\omega(\hat{\Psi}) = \int_{\overline{\mathcal{A}}} \Psi d\mu. \tag{5.37}$$

More concretely [Vel04], given a graph $\gamma \subset M$, a result in [Vel04] gives:

$$\omega(\hat{\Psi}) = \int_{G^{|E(\gamma)|}} (\varrho^{-1})^* \psi \; d\mu_H^1 \otimes ... \otimes d\mu_H^{|E(\gamma)|}, \tag{5.38}$$

where each $d\mu_H^i$ is the Haar measure on the i^{th} G in the product:

Definition 5.29. The **Ashtekar–Lewandowski Hilbert space** of canonical LQG is the Hilbert space $\mathcal{H}_\omega := L^2(\overline{\mathcal{A}}, \mu)$ where ω is the GNS state and μ is the unique Borel measure, both defined above. The latter is known as the *Ashtekar–Lewandowski measure.*

We emphasise that the uniqueness of μ stems from enforcing gauge and diffeomorphism invariance of the principal bundle, providing us with a unique measure that is compatible with the Gauss and diffeomorphism constraints present in LQG [LOST06].

Furthermore, this Hilbert space admits an L^2 inner product, derived from the GNS state. Given $[\hat{\Psi}], [\hat{\Phi}] \in \widehat{\overline{\mathrm{Cyl}^\infty}}$, we have

$$\langle [\hat{\Psi}], [\hat{\Phi}] \rangle = \omega(\Psi^* \Phi) = \int_{\overline{\mathcal{A}}} \overline{\Psi} \Phi d\mu. \tag{5.39}$$

Finally, we remark that $L^2(\overline{\mathcal{A}}, \mu)$ is infinite-dimensional; however, the Borel measure μ obtained from the Riesz–Markov–Kakutani theorem guarantees the support of any $f \in \overline{\mathcal{A}}$ is compact, so functions over this space make sense.

Throughout this chapter, we have seen that we can construct an algebra from the holonomies of connections on our manifold and electric fluxes. We promoted this to a *-algebra by utilising tools from operator theory to obtain an algebra of operators acting on an appropriate Hilbert space of states. By choosing appropriate representations, we obtained the Ashtekar–Lewandowski Hilbert space, which consists of square-integrable functions on the space of connections and is precisely the Hilbert space of states for quantum gravity. We are now ready to quantise the isolated horizon described in Chapter 4 and observe how this leads us to a microscopic description of the Bekenstein-Hawking entropy.

Chapter 6

Quantisation of Isolated Horizons

This chapter is, in a sense, the culmination of all our previous work undertaken in this thesis. We will provide an overview of the procedure of quantising the isolated horizons introduced in Chapter 4, utilising the Ashtekar–Lewandowski Hilbert space of Chapter 5 as the space of quantum states. We will also provide a brief outline of the overall picture, recalling features of our classical theory and introducing some additional structure. In particular, we will decompose and describe the total Hilbert space of the local spacetime with the isolated horizon into states lying completely on (the *surface*, and the complement (the *volume*) of the horizon surface. We will conclude with the imposition of quantised variants of the horizon boundary condition and quantum constraints seen in Chapter 4, leading us to our final quantum theory for the horizon. In particular, imposing the boundary conditions leads to the two Hilbert spaces having different structures. Most of this chapter follows Sections 2-5 of [ABK00] and serves as a preliminary setup for the primary focus of the second section of this thesis: the entropy counting in Chapter 7.

6.1 The Setup

Let us describe the playing field and set up some preliminaries. Recall our classical framework from Chapter 4. We have our spacetime $\mathcal{M} \cong \mathbb{R} \times \Sigma$, $P \to \Sigma$ an $SU(2)$-principal bundle and our configuration variables $(A, E) \in \mathcal{X}$. From our work in Chapter 5, we have a space of generalised connections $\overline{\mathcal{A}}$ constructed from our connection one-forms A. Our states will correspond to cylindrical functions on $\overline{\mathcal{A}}$, so our quantum theory will be built from the Ashtekar-Lewandowski Hilbert space of generalised connections $\mathcal{H} = L^2(\overline{\mathcal{A}}, \mu)$ from Chapter 5. This will be precisely the Hilbert space of states on the isolated horizon. By the nature of its construction, we have that $\overline{\mathcal{A}}$ splits as follows [ABK00][Thi07]:

Proposition 6.1. *The space of generalised connections decomposes as*

$$\overline{\mathcal{A}} = \overline{\mathcal{A}}_V \times \overline{\mathcal{A}}_S, \tag{6.1}$$

where $\overline{\mathcal{A}}_V$ and $\overline{\mathcal{A}}_S$ denote the spaces of generalised connections over the volume (connections *on the spacetime outside Σ) and the surface, respectively.*

Furthermore, the measure on $\mathcal{H}$ also splits to a product measure $\mu = \mu_V \otimes \mu_S$. This stems from the fact that holonomies in the volume are independent from those on the surface [ABK00]. Standard results from measure theory then tell us the Hilbert space naturally splits as the product

$$\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S. \tag{6.2}$$

The goal of the rest of this chapter is to understand and construct each of the constituent Hilbert spaces $\mathcal{H}_V$ and $\mathcal{H}_S$ (the *volume* and *horizon* Hilbert spaces, respectively).

First, let us describe a useful operator on the space of graphs on Σ. Consider a graph $\gamma \subset \Sigma$ which has edges $e_1, ..., e_n$. Then the space of bundle isomorphisms of the graph $\overline{\mathcal{A}}_\gamma = \prod_{i=1}^n \overline{\mathcal{A}}_{e_i}$ (as defined in Chapter 5) is a product of n copies of $SU(2)$ since the bundle is trivial. In particular, we have that $\overline{\mathcal{A}}_\gamma \cong SU(2)^{|E(\gamma)|}$. This allows us to define $SU(2)$-valued vector fields in terms of the cylindrical functions in $\overline{\mathcal{A}}_\gamma$. In particular, given the bijection $\varphi : \overline{\mathcal{A}}_e \to SU(2)$ as defined in Chapter 5, we extend this to the entire graph

$$\begin{aligned} \overline{\varphi} : \overline{\mathcal{A}}_\gamma &\longrightarrow SU(2)^{|E(\gamma)|} \\ \bar{A} &\mapsto (h_{e_1}(\bar{A}), ..., h_{e_n}(\bar{A})). \end{aligned} \tag{6.3}$$

Hence, our cylindrical functions $\Psi \in \overline{\mathrm{Cyl}^\infty}$ can be written as $\Psi(\bar{A}) = \psi \circ \bar{\varphi}(\bar{A})$.

Let us now define left (right) invariant vector fields on $\overline{\mathcal{A}}_\gamma$ as follows. Suppose $\{X_j\}$ is a basis for $\mathfrak{su}(2)$. Then:

$$(L_j\Psi)(e_k) := \left.\frac{d}{dt}\right|_{t=0} \psi(h_{e_1}(\bar{A}), ..., h_{e_k}(\bar{A}) \cdot e^{tX_j}, ..., h_{e_n}(\bar{A})) \tag{6.4}$$

$$(R_j\Psi)(e_k) := \left.\frac{d}{dt}\right|_{t=0} \psi(h_{e_1}(\bar{A}), ..., e^{tX_j} \cdot h_{e_k}(\bar{A}), ..., h_{e_n}(\bar{A})), \tag{6.5}$$

where each $h_{e_k}(\bar{A})$ now acts as an element of $SU(2)$ thanks to the bundle isomorphism $\bar{\varphi}$.

This allows us to define the following classical operator:

Definition 6.2. Let $\gamma \subset \Sigma$ be a graph with edges $e \in E(\gamma)$ and vertices $v \in V(\gamma)$. Define the following operator on the sets of edges and vertices

$$J_k(e, v) := \begin{cases} iL_k(e) & \text{if } e_s = v \text{ (i.e. } e \text{ is outgoing wrt. } v) \\ iR_k(e) & \text{if } e_f = v \text{ (i.e. } e \text{ is incoming wrt. } v). \end{cases} \tag{6.6}$$

The **angular momentum operator** associated to a vertex v is

$$J_k(v) := \sum_{e \in E(\gamma)} J_k(e, v). \tag{6.7}$$

As the name suggests, these have the interpretation of being 'angular momentum' operators based at a vertex v, since they generate $SU(2)$ rotations at that point [ABK00].

Finally, we must implement quantum versions of the Hamiltonian, diffeomorphism, and Gauss constraints, as well as implement a quantum version of the boundary condition (4.23) from Chapter 4.

Let us revisit the horizon boundary condition from (4.23). By restricting our bundle $P \to S \cong S^2$, we will show that this condition determines a smooth nowhere-vanishing section $r : S \to \mathrm{ad}P|_S \cong \mathfrak{su}(2)$. Note that the tangent space at $p \in S$ is isomorphic to that of $T_pS^2 = \{v \in \mathbb{R}^3 \mid v \perp p\}$. Let v_1^a, v_2^a be a basis for this space. Since the canonical momentum $\Theta^i_{ab} \propto E^i_a \propto e^i_a$, it follows that the vectors defined by the triad action $e^i_\alpha = e^i_a v^a_\alpha$, $(\alpha = 1, 2)$ define a subspace of the internal space $\mathrm{Span}\{e^i_1, e^i_2\} \subset W \cong \mathbb{R}^3$ fibrewise. Therefore, the wedge sum defines (up to normalisation) a unit vector on S

$$r_i = \epsilon_{ijk} e^j_1 \wedge e^k_2. \tag{6.8}$$

Since the conjugate momentum (a 2-form) is proportional to this sum, it may be expressed in terms of r_i on the horizon. Since the horizon boundary condition gives us $F^i_{ab} \propto \Theta^i_{ab}$, we obtain a refined boundary condition:

$$\varphi^* F_{ab} = -\frac{2\pi\beta}{A_S} \varphi^* \Theta^i_{ab} r_i. \tag{6.9}$$

From this construction, we see r defines a global section of $\mathrm{ad}P|_S$, and furthermore $\varphi^* F_{ab}$ takes values in the 1-dimensional span $\mathrm{span}\{r\} \subset \mathfrak{su}(2)$. With this, it can be shown [ABK00] that r_i must be fixed by $U(1) \subset SU(2)$ action at every point in S, so for the horizon boundary condition to be satisfied, P must reduce to a $U(1)$ bundle $Q \to S$.

The overall effect of this is that the pullback $\varphi^* A$ is determined by a $U(1)$ connection form $W \in \Omega^1(Q, \mathfrak{u}(1) \cong \mathbb{R})$. The local connection form has components

$$W_a := -\frac{1}{\sqrt{2}} \varphi^* \Gamma^i_a r_i, \tag{6.10}$$

from which it follows our curvature $F = dW$ since $U(1)$ is abelian.

Corollary 6.3. *The reduction of $SU(2)$ to $U(1)$ means that the symplectic form Ω_P in (4.20) picks up an additional term to give us:*

$$\begin{aligned}\Omega_{grav}(\delta(A,\Theta,W),\delta(A',\Theta',W')) = \frac{1}{8\pi G}\bigg(&\int_\Sigma \mathrm{Tr}[\delta A \wedge \delta\Theta' - \delta A' \wedge \delta\Theta'] \\ &+ \frac{A_S}{\beta\pi}\int_S \delta W \wedge \delta W'\bigg),\end{aligned} \tag{6.11}$$

where the second terms actually coincides with $U(1)$-Chern–Simons theory at level $k = \frac{A_S}{4\pi\beta\ell_P^2}$ [ABK00].

From here, we are left to describe the structure of this Hilbert space. In what follows, we will explain the Hilbert space of states in the external space outside of the horizon (*Volume Hilbert space*) and the states lying on the intersection $\Sigma \cap \Delta = S$ (*Surface Hilbert space*). This decomposes $\mathcal{H}$ into two different subspaces as in (6.2). We begin by describing the former.

6.2 The Volume Hilbert Spaces

From our description of $\mathcal{H}$ and the decomposition in Proposition 6.1, it follows that our Hilbert space of states will be $L^2(\overline{\mathcal{A}}_V, \mu_V)$. From measure theory, we know this is an orthogonal subspace of $\mathcal{H}$, and hence these states will be independent of those on the surface. On $L^2(\overline{\mathcal{A}}_V, \mu_V)$ we can define a basis as follows: [Bae96][ABK00]:

Definition 6.4. Denote an element of $L^2(\overline{\mathcal{A}}_V, \mu_V)$ by the triples $\psi = (\gamma, \rho, \nu)$, where

- $\gamma \subset \Sigma$ is a graph in the volume
- $\rho : E(\gamma) \to \Lambda$, defined by $e \mapsto \rho_e$, is a map assigning edges to nontrivial irreducible unitary spin j-representations of $SU(2)$ (denoted by Λ), where $j \in \frac{1}{2}\mathbb{Z}^{\geq 0}$
- $\nu : V(\gamma) \to \bigotimes_{e \in E(\gamma)} \rho_e$, defined by $v \mapsto \nu(v) = \bigotimes_{e_f = v} \rho_{e_f}$, i.e. the tensor product of representations ρ_e that label edges incident to v, defined so that ν_v is orthonormal.

If ν is invariant under the action of $SU(2)$, we call ψ an **open spin network**. An open spin network has the general expression:

$$\psi(A) = \prod_{e \in E(\gamma)} \rho_e(A_e)^{i_e}_{j_e} \prod_{v \in V(\gamma)} \nu(v)^{\dots j_e \dots}_{\dots i_e \dots} \ . \tag{6.12}$$

It can be shown that the collection $\{\psi = (\gamma, \rho, \nu) \mid \gamma \in \mathcal{E}, e \in E(\gamma), v \in V(\gamma)\}$ defines an orthonormal basis for $L^2(\overline{\mathcal{A}}_V, \mu_V)$. Hence, spin networks are the natural basis to describe states on the volume. They can physically be interpreted as 'generating' the geometry of spacetime. As described in [RS95], each vertex may be thought of as depositing a quantum of 'volume' into the spacetime. By considering only those connections in $\overline{\mathcal{A}}_V$ that are invariant under gauge transformations on the volume, we obtain the following:

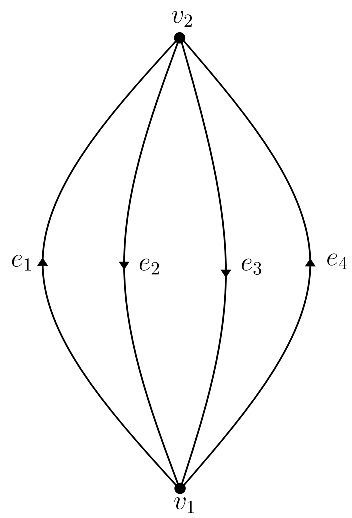
Definition 6.5. The **volume Hilbert space** is the space $\mathcal{H}_V := L^2(\overline{\mathcal{A}}_V / \mathcal{G}_V, \mu_V)$, where $\overline{\mathcal{A}}_V / \mathcal{G}_V$ denotes the subset of gauge-invariant generalised connections, where $\mathcal{G}_V$ is the group of gauge transformations acting on the volume. An orthonormal basis for $\mathcal{H}_V$ consists of spin networks $\psi = (\gamma, \rho, \nu)$, where holonomies along edges of γ are gauge invariant.

By construction, the Ashtekar-Lewandowski measure μ on the entire space $\overline{\mathcal{A}}$ is by construction gauge invariant, so it passes through the quotient by $\mathcal{G}_V$.

Example 6.6. Consider the graph defined in the figure below:

Then this corresponds to the spin network state

$$\psi(A) = \rho_{e_1}(A_{e_1})^i_{i'} \rho_{e_2}(A_{e_2})^j_{j'} \rho_{e_3}(A_{e_3})^k_{k'} \rho_{e_4}(A_{e_4})^l_{l'} \nu(v_1)^{i'l'}_{jk} \nu(v_2)^{j'k'}_{il} . \tag{6.13}$$



For spin 1/2-representations of $SU(2)$, each edge is labelled by either $\pm 1/2$. In this case, each element $\rho_{e_i}(A_{e_i})$ may be represented by a matrix

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \tag{6.14}$$

and each $\nu(v)$ is a linear combination $\nu(v) = \alpha \epsilon_{A_2}^{A_1} \epsilon_{A_3}^{A_4} + \beta \epsilon_{A_3 A_2}^{A_1, A_4}, \quad \alpha, \beta \in \mathbb{C}$ of the antisymmetric tensors with matrix representation $\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With appropriate normalisation, one can obtain a concrete matrix representation of the spin network.

Finally, let us introduce an important operator on $\mathcal{H}$:

Definition 6.7. Let $\Psi \in \overline{\mathrm{Cyl}^\infty}$ and let $\gamma \subset \Sigma$ and let $T \subset \Sigma$ be an oriented 2-surface. Suppose each edge $e \in E(\gamma)$ lies above T, in the sense that $\dot{e} > 0$ on $e \cap T$ and each e intersects T at most once. By subdividing the graph as necessary, we can consider each intersection $e \cap T$ as another vertex of γ. Then the **area operator** on $\overline{\mathrm{Cyl}^\infty}$ is defined as

$$\hat{A}_T(\Psi) := 16\pi\beta^2\ell_P^2 \sum_{v \in V(\gamma)} \sqrt{J_i(v) J_j(v) \kappa^{ij}}\ \Psi, \tag{6.15}$$

recalling the angular momentum operators J_i from (6.2) and κ^{ij} is the usual Cartan–Killing form for $\mathfrak{su}(2)$.

Physically, one can interpret the cylindrical functions as carrying information about the quantised geometry by way of the holonomies, and each intersection of Ψ with the surface T deposits a quantised area unit into T. By examining how much of Ψ permeates T, we arrive at an expression for the area.

Note that our notion of 'above' and 'below' T are defined once a spatial vector in its normal bundle is chosen to represent its orientation. Next, we only considered the special case where each e lies above T, since this is sufficient for isolated horizons. In general, e

can intersect S from above or below, or even lie within S. The general case is dealt with in [Abh96].

Since $\hat{A}_T$ gives us the area of T, it must represent a self-adjoint (or at least essentially self-adjoint) operator, as enforced by the postulates of quantum mechanics. Indeed, we have the following [Abh96]:

Lemma 6.8. *$\hat{A}_T$ is essentially self-adjoint. That is, the closure of $\hat{A}_T$ is self-adjoint.*

To illustrate the significance of this operator, let us return to $\mathcal{H}_V$. Note that for any spin network ψ, we can consider the set of vertices lying in entirely S, denoted by $\mathcal{P} = \{p_1, ..., p_n\}$. We call this the *set of punctures* of S and denote $\mathcal{H}_V^{\mathcal{P}}$ (note if ψ lies purely in the volume, $\mathcal{P} = \emptyset$). Furthermore, for each edge e with endpoint p_i, we can label p_i with the spin label j_i carried by e. This leads us to [ABK00]:

Lemma 6.9. *Let $j = (j_1, ..., j_n)$ be a list of spin labels corresponding to the set of punctures $\mathcal{P}$. Then there exists a subspace $\mathcal{H}_V^{\mathcal{P},j} \subset \mathcal{H}_V$ for which the operator $J(p_i) \cdot J(p_i) = \kappa_{jk} J(p_i)^j J(p_i)^k$ satisfies*

$$J(p_i) \cdot J(p_i)\psi = j_i(j_i + 1)\psi. \tag{6.16}$$

for $\psi \in \mathcal{H}_V^{\mathcal{P},j}$, at p_i and 0 for every $p \neq p_i$.

We will not give a full proof for this; however, the existence of such a subspace relies on the fact that the $J(p_i)$'s correspond to generators of $\mathfrak{su}(2)$. This allows one to take advantage of the Clebsch–Gordon decomposition of irreps of edges incident at p_i (which gives us the $j_i(j_i + 1)$ factor). Then, since the angular momentum operators act independently on each puncture, this gives us the desired orthogonality of the subspaces. Techniques in spectral theory and essential self-adjointness allow us to extend these facts to the volume Hilbert space.

Corollary 6.10. *This lemma implies that $\mathcal{H}_V^{\mathcal{P},j}$ are mutually orthogonal, and since this holds for every spin network, we have an orthogonal decomposition of $\mathcal{H}_V$ as*

$$\mathcal{H}_V = \bigoplus_{\mathcal{P}} \bigoplus_{j} \mathcal{H}_V^{\mathcal{P},j}. \tag{6.17}$$

This is extremely significant, for this decomposition diagonalises the area operator $\hat{A}_S$ on the horizons. In particular, for any $\psi \in \mathcal{H}_V^{\mathcal{P},j}$, ψ is an eigenvector of $\hat{A}_S$ satisfying:

$$\hat{A}_S\psi = 16\pi\beta^2\ell_P^2 \sum_i \sqrt{j_i(j_i + 1)}\psi. \tag{6.18}$$

In summary, we have that the volume Hilbert space $\mathcal{H}_V$ consists of open spin networks (which can be represented as graphs γ with edges carrying labels corresponding to spin representations of $SU(2)$), which can be decomposed into orthogonal eigenspaces of the area operator on the surface S.

This completes one half of the picture. We will now describe the surface Hilbert space.

6.3 The Surface Hilbert Space

Our quantisation process of the surface S follows closely the procedure in Section 4.3 of [ABK00]. Recall from our setup that over the surface S, P reduces to a $U(1)$ subbundle $Q \to S$. Our goal is to describe the space of bundle connections $W \in \Omega^1(U, \mathfrak{u}(1) \cong \mathbb{R})$, and then quantise this to obtain our surface Hilbert space.

Consider a path $\eta : [0,1] \to S$ and denote the space of all paths on S by $P(S)$. Then we have a space of generalised connections on Q consisting of bundle isomorphisms $\overline{W}_\eta : \pi^{-1}(\eta(0)) \to \pi^{-1}(\eta(1))$ such that $\overline{W}_\eta$ is a homomorphism on $P(S)$. In particular, this is an assignment of η to a holonomy $\overline{W}_\eta$. Recall from the previous section that graphs in the volume can intersect S, giving us a set of punctures $\mathcal{P} = \{p_1, ..., p_n\}$. From this, we can define a special type of generalised connection on Q:

Definition 6.11. Let $\overline{W}$ be a generalised connection on Q and let $\mathcal{P}$ be as above. Then $\overline{W}$ is **flat except at the punctures (FEP)** if $\overline{W}$ assigns the same holonomies as some other connection $W_0 \in \Omega^1(Q, \mathbb{R})$ which satisfies:

1. W_0 is flat on $S \setminus \mathcal{P}$

2. Over a fixed trivialisation of Q over a neighbourhood $U_i \ni p_i$ diffeomorphic to $\mathbb{R}^2$ with local coordinates (x, y), W_0 satisfies

$$W_0 = \omega_1 + c\frac{(x-a)dy + (y-b)dx}{(x-a)^2 + (y-b)^2}, \tag{6.19}$$

over $U_i \setminus \{p_i\}$, where $c \in \mathbb{R}$, $\omega_1 \in \Omega^1(U_i \setminus \{p_i\}, \mathbb{R})$ and $p_i = (a, b)$.

The imposition of condition 1 is fairly obvious. However, condition 2 requires some comment. This states that a nontrivial holonomy $\overline{W}_\eta$ cannot be smoothly extended over the puncture without becoming distributional. In particular, since the fundamental group $\pi_1(U_i \setminus \{p_i\}) \cong \mathbb{Z}$, the holonomy around arbitrarily small loops remains nontrivial, so a smooth connection cannot exist in a neighbourhood of p_i. Therefore, there is no globally defined connection $\overline{W}$ which is smooth at p_i. It can be shown [Thi07] that, despite the singularity at the punctures, $\overline{W}$ remains well-defined and satisfies the homomorphism conditions for generalised connections.

Next, define $\overline{\mathcal{A}^\mathcal{P}}$ to be the space of all generalised $U(1)$ that are FEP, and consider the gauge group $\mathcal{G}^\mathcal{P}$ on Q fixing each p_i. Furthermore, let $\mathcal{D}^\mathcal{P}$ be the group of orientation-preserving diffeomorphisms fixing a ray in $T_{p_i}S$ for every i. Then the semi-direct product $\mathcal{G}^\mathcal{P} \rtimes \mathcal{D}^\mathcal{P}$ is a subgroup of $\mathrm{Aut}(Q)$[ABK00][Thi07] and hence acts on $\overline{\mathcal{A}^\mathcal{P}}$. By quotienting by this subgroup, we obtain the following:

Definition 6.12. The space of **physical generalised connections** on Q is a symplectic manifold given by the quotient $\mathcal{X}^\mathcal{P} := \overline{\mathcal{A}^\mathcal{P}}/(\mathcal{G}^\mathcal{P} \rtimes \mathcal{D}^\mathcal{P})$, consisting of generalised connections

$\overline{W} \in \Omega^1(Q, \mathbb{R})$ invariant under $\mathcal{G}^{\mathcal{P}} \rtimes \mathcal{D}^{\mathcal{P}}$, with symplectic structure given by

$$\omega(\delta W, \delta W') := \frac{k}{2\pi} \int_{S^2} \mathrm{Tr}[\delta W \wedge \delta W'], \tag{6.20}$$

where k is a parameter known as the *level*.

Recall that in Corollary 6.3 we have that $k = \frac{A_S}{4\pi\beta\ell_P^2}$.

Remark 6.13. In the literature, this is precisely the symplectic form on the phase space of a $U(1)$-Chern–Simons theory (a particular type of topological quantum field theory). This is significant, as we will see that the quantum theory on the horizon will be directly related to the well-documented quantum Chern-Simons theory, ultimately forcing k to be integral.

Furthermore, $\mathcal{X}^{\mathcal{P}}$ is the space that we will model the surface Hilbert space over. This relies on a key property of $\mathcal{X}^{\mathcal{P}}$'s topological structure.

Theorem 6.14. *$\mathcal{X}^{\mathcal{P}} \cong T^{2(n-1)}$, where $T^{2(n-1)}$ denotes the $2(n-1)$-dimensional torus.*

This is Theorem 1 of [ABK00], and for a complete proof we direct the reader here. We can identify

$$\mathcal{X}^{\mathcal{P}} \cong \mathbb{C}^{n-1}/\Lambda, \tag{6.21}$$

where $\Lambda = (2\pi\mathbb{Z})^{2(n-1)}$ is an integer lattice. In particular, we have local coordinates on $\mathcal{X}^{\mathcal{P}}$ given by $z_j = x_j + iy_j$. This allows us to obtain an explicit form of the symplectic structure on $\mathcal{X}^{\mathcal{P}}$:

Proposition 6.15. *The symplectic structure in Definition 6.12 can be expressed locally as*

$$\omega = \frac{k}{2\pi} \sum_{j=1}^{n-1} dx_j \wedge dy_j. \tag{6.22}$$

The proof of this is shown explicitly in the proof of Theorem 1 of [ABK00]. Furthermore, we have that under the identification in (6.21), $\mathcal{X}^{\mathcal{P}}$ is a Kähler manifold with Kähler structure $h = \frac{k}{2\pi} \sum_{j=1}^{n-1} dz_j \otimes d\bar{z}_j$.

Proposition 6.16. *The Kähler structure h is compatible with the symplectic form ω on $\mathcal{X}^{\mathcal{P}}$.*

The proof of this is a straightforward verification that $\Im(h) = \omega$, which can be seen by substituting $z_j = x_j + iy_j$ into the definition of h and simplifying.

The fact that $(\mathcal{X}^{\mathcal{P}}, h)$ is a Kähler manifold gives us the correct space to perform *geometric quantisation* over - in contrast to the operator theoretic approach in Chapter 5. This is largely due to the boundary conditions we impose on the horizon, and serves as an equivalent yet simpler approach. In essence, this is a geometric method for quantising a classical theory. The general framework for geometric quantisation involves constructing a holomorphic line bundle over a symplectic manifold whose curvature is related to the symplectic form. From this, the Hilbert space is constructed from holomorphic sections of this line bundle.

The topic of geometric quantisation would warrant a thesis of its own, and we will not delve into the intricacies of such a construction; rather, we will give an overview of the procedure as presented in Section 4.3.2 of [ABK00]*.

Since $(\mathcal{X}^{\mathcal{P}}, h)$ is a Kähler manifold, it can also be viewed as a symplectic manifold with symplectic structure given by $\Im(h)$. From here, the authors of [ABK00] construct a holomorphic line bundle $L \to \mathcal{X}^{\mathcal{P}}$. As is the case for standard vector bundles, one can equip L with a connection ∇.

For us to quantise in this way, standard results from geometric quantisation [Woo92] require that the following lemma holds [ABK00]:

Lemma 6.17. *The line bundle $(L, \nabla) \to (\mathcal{X}^{\mathcal{P}}, h)$ with connection curvature $F_\nabla = i\omega$ exists if and only if $k \in \mathbb{Z}$*†

This condition arises from how operators are constructed in geometric quantisation. For more information (and proof), we direct the reader to Appendix A. With this lemma, we define the surface Hilbert space as follows:

Definition 6.18. Given a fixed set $\mathcal{P} = \{p_1, ..., p_n\}$ of punctures on S, the **surface Hilbert space** $\mathcal{H}_S^{\mathcal{P}}$ consists of holomorphic sections of $\Gamma(L)$ with a basis given by

$$\{\psi_{\mathcal{P},a}\}_{a \in \frac{\mathbb{Z}^{n-1}}{k}} \ : \ \psi_{\mathcal{P},a}(z) = \sum_{v \in \Lambda} \exp\left(\frac{k}{2\pi}(iv \cdot z - \frac{1}{2}|v|^2) + ia \cdot (z + iv)\right), \tag{6.23}$$

where a is an n-tuple, with $a_i \in \{1, ..., k\}$, satisfying $\sum_{i=1}^n a_i = 0 \mod k$, and $a_n := -\sum_{i=1}^{n-1} a_i$. Furthermore, this space has an inner product defined by

$$\langle f, g \rangle := \int_{[0,2\pi]^{2(n-1)}} e^{-\frac{k}{2\pi}|y|^2} \bar{f}(z) g(z) d^{n-1}x d^{n-1}y. \tag{6.24}$$

We refer the reader to [ABK00] for a complete construction, and give a brief discussion of this construction here. From the definition of $\mathcal{X}^{\mathcal{P}}$, a holomorphic section of $L \to \mathcal{X}^{\mathcal{P}}$ should correspond to a section of $\mathbb{C}^{n-1}$ which is invariant under the group action of the lattice Λ. Specifically, this corresponds to holomorphic sections invariant under a representation $R : \Lambda \to \mathrm{End}(H\Gamma(L))$. Such sections are known in the literature as *theta functions*, which we have denoted as $\psi_{\mathcal{P},a}$ in the above definition, and can be shown to form a basis for $\mathcal{H}_S^{\mathcal{P}}$ [Mum07]. Furthermore, since a_n is linearly dependent on every other a_i, one can observe that the dimension $\dim \mathcal{H}_S^{\mathcal{P}} = k^{n-1}$.

Next, let us discuss the meaning of the a's, since they are more than just labels for the basis states. In particular, they are tied to gauge transformations at the punctures. Fixing a trivialisation on S, consider the group of gauge transformations $\mathcal{G}(Q) = \{g \mid g : S \to U(1)\}$ of Q. Then we have the following:

*We also direct the reader to [Woo92] for an overview of geometric quantisation as a concept.

†This is known in the literature as the Weil integrality condition, as described in Section 8.3 of [Woo92].

Proposition 6.19. *Consider $\mathcal{G}^{\mathcal{P}}$ as prior. Then*

$$\mathcal{G}(Q)/\mathcal{G}^{\mathcal{P}} \cong U(1)^n. \tag{6.25}$$

Proof. Given $g \in \mathcal{G}(Q)$, these are locally $U(1)$ transformations defined globally on S. Hence, there is a surjective map to $U(1)^n$ given by evaluation of g on $\mathcal{P}$

$$\Upsilon : \mathcal{G}(Q) \to U(1)^n \quad , \quad g \mapsto (g(p_1), ..., g(p_n)). \tag{6.26}$$

Since $\mathcal{G}^{\mathcal{P}}$ was defined as the subgroup of $\mathcal{G}(Q)$ that fixes every puncture, it follows $\ker \Upsilon = \mathcal{G}^{\mathcal{P}}$. An application of the first isomorphism theorem then gives us the desired result. □

In particular, $U(1)^n$ can be represented by acting on $\mathcal{X}^{\mathcal{P}}$ via rotation. I.e. $U(1)^n$-action on $\mathcal{X}^{\mathcal{P}}$ is given by

$$\begin{aligned} T_g : U(1)^n \times \mathcal{X}^{\mathcal{P}} &\longrightarrow \mathcal{X}^{\mathcal{P}} \\ ((e^{i\theta_1}, ..., e^{i\theta_n}), (x_1, y_1, ..., x_{n-1}, y_{n-1})) &\mapsto (e^{i(\theta_n - \theta_1)} x_1, y_1, ..., e^{i(\theta_n - \theta_{n-1})} x_{n-1}, y_{n-1}), \end{aligned} \tag{6.27}$$

where $g \in U(1)^n$ is represented by the n-tuple of rotations $(e^{i\theta_1}, ..., e^{i\theta_n})$. From this, it is of interest in the quantum case to see if (and how) this operator extends to the Hilbert space $\mathcal{H}_S^{\mathcal{P}}$. Indeed, we have the following lemma [ABK00]:

Lemma 6.20. *There exists a unique lift $\tilde{T}_g$ of T_g to a holomorphic map on the line bundle $L \to \mathcal{X}^{\mathcal{P}}$ iff $g \in \mathbb{Z}^n/k \subset U(1)^n$, i.e. is a kth root of unity. Furthermore, $\tilde{T}_g$ defines a unitary operator on $\mathcal{H}_S^{\mathcal{P}}$.*

Building on this further, this lift is uniquely associated with the action of the representation R defined previously, i.e. we associate $\tilde{T}_g \sim R(g)$. The action of this operator on theta functions is given by

$$R(g_1, ..., g_n)\psi_{\mathcal{P},a} = g_1^{a_1} ... g_n^{a_n} \psi_{\mathcal{P},a}. \tag{6.28}$$

This allows us to construct a holonomy operator $\hat{h}$ on $\mathcal{H}_S^{\mathcal{P}}$. Consider a loop η_j around some puncture p_j. Then define the holonomy operator around η_j:

$$\hat{h}_j := R(1, ..., 1, e^{\frac{2\pi i}{k}}, 1, ..., 1). \tag{6.29}$$

From (6.28), this acts on the theta functions as

$$\hat{h}_j \psi_{\mathcal{P},a} = e^{\frac{2\pi i a_j}{k}} \psi_{\mathcal{P},a}. \tag{6.30}$$

So we can see that the eigenvectors of this operator correspond exactly to the basis states of $\mathcal{H}_S^{\mathcal{P}}$, hence the a_i's label the eigenvalues of $\hat{h}_j$, in turn giving us a way to quantise the singularities arising from the punctures. Furthermore, the condition $\sum_{i=1}^n a_i = 0 \mod k$ tells us the product of holonomies of loops around each puncture must be equivalent to the holonomy of a loop enclosing every puncture, which is contractible in $S \setminus \mathcal{P}$. This gives us a realisation of the FEP condition for our $U(1)$ connection.

Finally, note $\mathcal{H}_S^{\mathcal{P}}$ was constructed on some fixed set of punctures $\mathcal{P}$. To describe the total surface Hilbert space, we must allow $\mathcal{P}$ to vary over S. Through projective techniques (for an overview of projective techniques in this context, we refer the reader to [AL95]), one can obtain the surface Hilbert space as the direct limit of fixed-puncture Hilbert spaces [ABK00]:

$$\mathcal{H}_S := \varinjlim_{\mathcal{P}} \mathcal{H}_S^{\mathcal{P}}, \tag{6.31}$$

where here $\mathcal{P}$ ranges over all finite subsets of S.

This concludes the construction of the surface Hilbert space. Next, we will see how this fits together with the volume Hilbert space to give us the total Hilbert space of our spacetime.

6.4 The Kinematic Hilbert Space

Our work has granted us a description of the total Hilbert space, yielding explicitly what $\mathcal{H}_V$ and $\mathcal{H}_S$ are in (6.2). However, this space is, in a sense, 'too large', since it corresponds to potentially unphysical states (those not satisfying correct boundary conditions). Hence, we are interested specifically in the subset $\mathcal{H}$ corresponding to physical states, known as the *kinematical Hilbert space* in literature [ABK00]. This is done by enforcing the quantum version of the boundary condition in (4.23). In this section, we describe how this is implemented and how this yields the kinematical Hilbert space.

Let us first begin with defining another operator, this time on the volume:

Definition 6.21. Consider a codimension 2 surface $T \subset \Sigma$ and graph γ such that each edge lies above T (as in Definition 6.7). The **flux operator** $\hat{\Theta} : \mathcal{H}_V \to \mathcal{H}_V$ is an essentially self-adjoint operator defined explicitly as

$$\hat{\Theta}_{T,f}[\psi_V] = 8\pi\ell_P^2 \sum_{v \in V(\gamma)\cap T} f^i(v) J_i(v)[\psi_V]. \tag{6.32}$$

for an $\mathfrak{su}(2)$-valued function $f : T \to \mathfrak{su}(2)$.

Physically, this can be seen as determining 'how much' of ψ_V permeates any given 2-surface in Σ.

From our discussion in Section 6.1, we saw that the curvature is related directly to the $U(1)$-connection by $F = dW$. This implies that the quantised variant of this operator $\hat{F}$ acts on $\mathcal{H}_S$. From above, we have $\varphi^*\hat{\Theta} \cdot r$ acts on $\mathcal{H}_V$. It follows from the description of the Gauss constraint in Section 4.4 that our quantum states should remain invariant under $SU(2)$ gauge transformations. Actually, they should be further invariant under $U(1) \subset SU(2)$ due to the requirement in Section 6.1 that the section r remains fixed. This leads us to the following:

Definition 6.22. The **quantum boundary condition** on $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ is the requirement of states $\psi_V \otimes \psi_S \in \mathcal{H}$ remain invariant under $U(1)$ gauge symmetries. Explicitly, we require states to satisfy:

$$\left(1 \otimes \exp(i\hat{F})\right) \psi_V \otimes \psi_S = \left(\exp\left(-i\frac{2\pi\beta}{A_0}\varphi^*\hat{\Theta}\cdot r\right) \otimes 1\right) \psi_V \otimes \psi_S. \tag{6.33}$$

The reason we take exponentials of these operators is to ensure unitarity on $\mathcal{H}$, as well as some other subtleties stemming from Chern-Simons theory [ABK00]. With this, we are left to find precisely which states in $\mathcal{H}$ are invariant under $U(1)$ gauge symmetries. We know from (6.28) how $U(1)$ acts on theta functions, so we direct our focus to $\mathcal{H}_V$. In particular, since (6.22) is a condition for states on the horizon surface, we need to describe the action of gauge transformations on the surface.

Let $\mathcal{G}(Q) \subset \mathcal{G}_S$ denote the group of gauge transformations on $P|_S$ that also fix the section $r : S \to \mathfrak{su}(2)$ as prior. Our discussion in the previous section demonstrates that $\mathcal{G}_S$ has a unitary representation on $\mathcal{H}_V$, specifically at the punctures where the spin networks terminate. In particular, for a finite set $\mathcal{P} = \{p_1, ..., p_n\}$ of punctures, we can consider a subgroup $\mathcal{G}(Q)' \subset \mathcal{G}(Q)$ that is the identity everywhere expect at points of $\mathcal{P}$ where it is a $U(1)$ transformation. This reduces our possibly infinite-dimensional gauge group $\mathcal{G}_S$ to a finite-dimensional subgroup with non-trivial action on $\mathcal{P}$. From Section 6.1, it follows that $J(p) \cdot r$ will generate $U(1)$ transformations on the surface. Therefore, by labelling each p_i by some $m_i \in \frac{1}{2}\mathbb{Z}$ (corresponding to the spin label carried by the edges), we obtain a decomposition of $\mathcal{H}_V$ into subspaces $\mathcal{H}_V^{\mathcal{P},m}$ of eigenvectors of $J(p) \cdot r$, for a half-integer list $m = (m_1, ..., m_n) \in \frac{1}{2}\mathbb{Z}^n$. Specifically we define

$$\mathcal{H}_V^{\mathcal{P},m} := \{\psi_V \in \mathcal{H}_V \mid (J(p)\cdot r)\psi_V = \delta^2(p, p_i)m_i\psi_V\}, \tag{6.34}$$

where δ^2 defines the 2-dimensional Dirac delta distribution on S. This is completely analogous to the statement and result of Lemma 6.9 and Corollary 6.10 [Thi07], and gives us a deocmposition of $\mathcal{H}_V$ into eigenspaces.

Lemma 6.23. *The action of $J(p) \cdot r$ diagonalises $\mathcal{H}_V$ into orthogonal subspaces*

$$\mathcal{H}_V = \bigoplus_{\mathcal{P}} \bigoplus_m \mathcal{H}_V^{\mathcal{P},m}. \tag{6.35}$$

Furthermore, this diagonalises the action of $\mathcal{G}(Q)'$. Given $g \in \mathcal{G}(Q)'$, $\psi \in \mathcal{H}_V^{\mathcal{P},m}$, the gauge action satisfies

$$g\psi = g(p_1)^{2m_1}...g(p_n)^{2m_n}\psi. \tag{6.36}$$

Remark 6.24. Since the value of each m_i is tied to the value of the spin label of each edge intersecting p_i, we conclude m_i can take values in $\{-j_i, -j_i + 1, ..., j_i\}$.

As mentioned in [ABK00], the factor of 2 in the exponent is conventional since each m_i is a half-integer. With this lemma, we can finally see which states of $\mathcal{H}$ satisfy the quantum boundary condition: We can now see that the action of the holonomy operator $\hat{h}_j = \exp(i\hat{F_j})$ on $\psi_{\mathcal{P},a} \in \mathcal{H}_S^{\mathcal{P},a}$ at a puncture p_i has eigenvalue $e^{\frac{2\pi i a_j}{k}}$. We also see that by definition of

$\exp(\varphi^* \hat{\Theta} \cdot r)$, the action on $\psi_V \in \mathcal{H}_V^{\mathcal{P},m}$ at p_j has eigenvalue $8\pi\ell_P^2 m_j$. Then if $\Psi_V \otimes \Psi_S \in \mathcal{H}$ is such that Ψ_V, Ψ_S are eigenstates of $\exp(i\hat{F})$, $\exp(\varphi^* \hat{\Theta} \cdot r)$ respectively, then the quantum boundary condition shows us at a puncture p_j:

$$\begin{aligned} \left(1 \otimes \exp(i\hat{F})\right) \Psi_V \otimes \Psi_S &= \left(\exp\left(-i\frac{2\pi\beta}{A_0}\varphi^*\hat{\Theta} \cdot r\right) \otimes 1\right) \Psi_V \otimes \Psi_S \\ \implies \left(1 \otimes \exp\left(\frac{2\pi i a_j}{k}\right)\right) \Psi_V \otimes \Psi_S &= \left(\exp\left(-i\frac{2\pi\beta}{A_0}(8\pi\ell_P^2 m_j)\right) \otimes 1\right) \Psi_V \otimes \Psi_S. \end{aligned} \tag{6.37}$$

Setting $k = \frac{A_0}{4\pi\beta\ell_P^2}$ as in Section 6.3, we get

$$\left(1 \otimes \exp\left(i\frac{8\pi^2\beta\ell_P^2 a_j}{A_0}\right)\right) \Psi_V \otimes \Psi_S = \left(\exp\left(-i\frac{16\pi^2\beta\ell_P^2 m_j}{A_0}\right) \otimes 1\right) \Psi_V \otimes \Psi_S. \tag{6.38}$$

To obtain the identity operator on both sides (corresponding to $U(1)$ gauge invariance), we must have that $2m_j \equiv -a_j \mod k$. The states in the product $\mathcal{H}_V^{\mathcal{P},m} \otimes \mathcal{H}_S^{\mathcal{P},a}$ satisfying this condition define a subspace of the product Hilbert space, and so this corresponds precisely to our physical states:

Definition 6.25. The **kinematical Hilbert space** is the subspace of all states in $\mathcal{H}$ satisfying the quantum boundary condition, specifically, it is the subspace defined by

$$\mathcal{H}^{Kin} := \bigoplus_{\mathcal{P},m,a \,:\, 2m \equiv -a \mod k} \mathcal{H}_V^{\mathcal{P},m} \otimes \mathcal{H}_S^{\mathcal{P},a}. \tag{6.39}$$

This shows us that our states of interest in the quantum theory impose a direct relationship between m and a, and further constrain the possible spin values each edge can take.

6.5 Quantum Constraints and the Physical Hilbert Space

In this section, we describe the final component of the horizon quantum theory, which is the implementation of the classical constraints as described in Chapter 4. We will focus on the resulting Hilbert space, rather than the explicit implementation, which can be found in detail in [ABK00].

We have already imposed the Gauss constraint by requiring that our physical states on the surface be invariant under $U(1)$ gauge transformations. In particular, since the Gauss constraint is defined as generating $SU(2)$ gauge transformations, in our case, this reduces to $U(1)$ transformations to preserve the section r. By implementing this, we recovered $\mathcal{H}^{Kin} \subset \mathcal{H}$.

The diffeomorphism constraint is implemented by constructing states that are invariant under spatial diffeomorphisms. For a complete explicit construction, we refer the reader to [ABK00]. In short, this is implemented through gauge fixing techniques. This is particularly important on the surface, as any two states related by diffeomorphisms are physically equivalent. This implies that any two sets of punctures $\mathcal{P}, \mathcal{P}'$ are equivalent up to permutation, no matter the spin labels given to their constituent elements. However, this becomes problematic when one attempts to perform entropy counting since this may lead to double-counting. Gauge fixing amends this by fixing an order on $\mathcal{P} = \{p_1, ..., p_n\}$ such that the associated spin labels satisfy $j_1 \leq ... \leq j_n$, and restricting to diffeomorphisms that preserve this ordering. This reduces our kinematical Hilbert space further to:

$$\mathcal{H}^{\text{Diff}} := \bigoplus_{n,m,a\,:\,2m\equiv -a \mod k} \mathcal{H}_V^{n,m} \otimes \mathcal{H}_S^{n,a}, \tag{6.40}$$

where the superscript n in each of the constituents corresponds to a fixed n-puncture set, each $m_i \in \{-j_i, -j_i+1, ..., j_i\}$ and $j_i \leq j_{i+1}$ for all $i = 1, 2, ..., n-1$.

Finally, we implement the quantum Hamiltonian constraint. On the classical level, we required the lapse function N to vanish at both $\mathscr{I}^+$ and the horizon Δ. What this implies is that the quantum operator associated with the lapse must leave surface states invariant and only act on the volume states. In particular, implementation of the Hamiltonian constraint means obtaining a quantum operator $\hat{H}[N]$ associated to $H[N]$ from Definition 4.21, where $N \to 0$ at $\mathscr{I}^+$ and Δ. Then states satisfying the Hamiltonian constraint are precisely the subspace $\ker \hat{H}[N] =: \tilde{\mathcal{H}}_V^{n,m} \subset \mathcal{H}_V^{n,m}$ (recalling our description of constraints from Section 4.4). This brings us to our final Hilbert space, corresponding to our physically allowable states:

Definition 6.26. The **physical Hilbert space** of our spacetime is the sum

$$\mathcal{H}^{\text{Phys}} := \bigoplus_{n,m,a\,:\,2m\equiv -a \mod k} \tilde{\mathcal{H}}_V^{n,m} \otimes \mathcal{H}_S^{n,a}, \tag{6.41}$$

for $n \in \mathbb{N}$, m, j are defined as in (6.40) and $a = (a_1, ..., a_n) \in \mathbb{Z}^n/k$ such that $\sum_{i=1}^n a_i = 0$.

It follows from this that $\mathcal{H}^{\text{Phys}} \subset \mathcal{H}_V^{\text{Phys}} \otimes \mathcal{H}_S^{\text{Phys}}$, where we define $\mathcal{H}_V^{\text{Phys}} := \bigoplus_{n,m} \tilde{\mathcal{H}}_V^{n,m}$ and $\mathcal{H}_S^{\text{Phys}} := \bigoplus_{n,a} \mathcal{H}_S^{n,a}$ [ABK00].

To summarise, we began with a total Hilbert space for the local spacetime $\mathcal{H} = L^2(\mathcal{A}, \mu)$, which was certainly infinite-dimensional and would be 'too large' actually to describe physical states. Through implementation of the horizon boundary condition and constraints into the quantum description, we have obtained a much smaller subspace of $\mathcal{H}^{\text{Phys}} \subset \mathcal{H}$ which describes the true physical states of our theory.

Remark 6.27. In the above description, we have not given an explicit form of the Hamiltonian constraint operator $\hat{H}[N]$. Indeed, an explicit representation of the Hamiltonian constraint remains an open question in LQG research. In particular, attempts to construct an explicit representation lead to issues with incorporating graph dependence, closure of the constraint algebra and recovery of the classical constraint in the classical limit. Some progress has been made in relation to obtaining an explicit representation (see [Thi96]). However, critics argue that this construction fails to reproduce results from classical GR. Due to a lack of a universally accepted description of the Hamiltonian constraint, we keep its description abstract and retain only its structural consequences (in particular, those relating to the lapse).

Not only have we described the Hilbert space of physical states, but we have also isolated the degrees of freedom of the volume and surface sectors, a key construction that we will utilise in the next chapter.

Chapter 7

The Microscopic Entropy

We now have all the required tools and assumptions to carry out the entropy calculation of the horizon. By examining states in the physical Hilbert space derived in Section 6.5, we find that by counting certain states corresponding to a particular Hilbert space (the so-called *black hole sector*), we will see that these all give a discrete contribution to the area of the isolated horizon in the quantum theory. From our discussion in Chapter 3, these lead to the emergence of a microscopic description of the Bekenstein–Hawking entropy.

In Section 7.1, following the work undertaken in [ABK00] and [A. 98], we will begin with an outline of the general strategy, specifying precisely which states we will be counting, as well as the physical interpretation of what we will be doing. In particular, we will define the statistical mechanical entropy of the black hole utilising density matrices. We will then work to obtain lower and upper bounds for the entropy, employing some number theoretic techniques as presented in [ABK00]. We will further elaborate on the arguments presented in [ABK00] regarding the lower bound, and we will also provide original proofs concerning the convergence of the partition function.

In Section 7.2, we discuss the result of the counting argument outlined in Section 7.1 and its connection with other areas of theoretical and mathematical physics. We also heuristically propose some directions for future research; in particular, we outline an alternative counting method based on Skein modules and the homology theory that arises from it.

7.1 Entropy Counting

To describe the entropy, we will be utilising the *von Neumann entropy* of the quantum system that our Hilbert space $\mathcal{H}$ describes. This is the most natural description for entropy in this setting, since the von Neumann entropy is inherently related to the probability amplitudes of wavefunctions, which in turn may be utilised in a description of the Shannon entropy of a system (see Definition 3.4). To this end, we recall the following definition from quantum mechanics:

Definition 7.1. Let $\mathcal{H}$ be a Hilbert space. A **density operator** is a positive semidefinite self-adjoint operator $\hat{\rho} : \mathcal{H} \to \mathcal{H}$ such that $\mathrm{Tr}(\hat{\rho}) = 1$.

In particular, the density operator on a Hilbert space, especially in the context of quantum mechanics, allows us to recover the probability of obtaining specific measurements of a system. This allows us to deal with more complex quantum systems, such as when two quantum systems become entangled (which involves tensor products of wavefunctions from both systems).

In our case, we have a density operator $\hat{\rho}$ on $\mathcal{H}^{\text{Phys}} \subset \mathcal{H}_V^{\text{Phys}} \otimes \mathcal{H}_S^{\text{Phys}}$. We will see that by taking an appropriate projection to a certain subspace, we obtain a so-called *reduced density operator* that makes an appearance in the statistical mechanical entropy of the horizon. Firstly, however, we must identify which states would actually contribute to the entropy.

Fix an isolated horizon S_0 with area $A_0 = A(S_0)$ and consider a collection of horizons that fall within some small $\delta > 0$ tolerance of it. This is physically motivated, as it is equivalent to constructing a microcanonical ensemble, which is the collection of all states in a mechanical system that describe a macroscopic variable (energy, temperature, entropy, etc.) whose expectation lies within a certain tolerance of the variable's magnitude. Furthermore, since area eigenvalues are discrete, projecting onto states with an exact area eigenvalue may not agree with the macroscopic description (even if it did, the resulting space of eigenvectors may be empty, and hence not representative of the physical situation). With this, we consider the set of horizons

$$\mathcal{S} = \{S \subset \mathcal{M} \mid S \text{ is an isolated horizon and } A(S) \in [A_0 - \delta, A_0 + \delta],\ \delta > 0\}. \tag{7.1}$$

This now motivates the construction of a subspace of $\mathcal{H}^{\text{Phys}}$ of states which lie on some fixed horizon $S \in \mathcal{S}$:

Definition 7.2. Given a horizon $S \in \mathcal{S}$, the **black hole sector** is a Hilbert space defined as the subspace $\mathcal{H}^{bh} \subset \mathcal{H}^{\text{Phys}}$ as follows:

$$\mathcal{H}^{bh} := \{\psi \in \mathcal{H}^{Phys} \mid \hat{A}_S\psi = \lambda\psi \text{ and } \lambda \in [A_0 - \delta, A_0 + \delta]\ \}, \tag{7.2}$$

i.e. the span of eigenvectors of $\hat{A}_S$ who's eigenvalue lies within the tolerance region described in (7.1).

From Definition 6.26, it follows every $\psi \in \mathcal{H}^{\text{Phys}}$ can be written as

$$\psi = \sum_i \psi_V^i \otimes \psi_S^i \qquad \psi_V^i \in \mathcal{H}_V^{\text{Phys}}, \psi_S^i \in \mathcal{H}_S^{\text{Phys}}. \tag{7.3}$$

However, note ψ_S^i can be *any* linear combination of theta functions, not necessarily area eigenvectors. Hence, we only care about those which appear in the decomposition of $\psi \in \mathcal{H}^{bh}$. Indeed, these form a subspace of $\mathcal{H}_S^{\text{Phys}}$ [ABK00]:

Proposition 7.3. *There exists a subspace $\mathcal{H}_S^{bh} \subset \mathcal{H}_S^{Phys}$ such that any state $\psi \in \mathcal{H}^{bh}$ is of the in (7.3) where $\psi_S^i \in \mathcal{H}_S^{Phys}$.*

Specifically, we are picking out the smallest subspace $\mathcal{H}_S^{bh} \subset \mathcal{H}_S^{\text{Phys}}$ for which $\psi_S^i \in \mathcal{H}_S^{\text{Phys}}$

appear in the decomposition of some $\psi \in \mathcal{H}^{bh}$. Explicitly, we can define this subspace using orthogonal projections:

Let $P_{\mathcal{H}^{bh}}$ denote orthogonal projection onto $\mathcal{H}^{bh} \subset \mathcal{H}^{\text{Phys}}$. Define the **reduced density operator** by projection onto this subspace and taking a partial trace over the volume subspace: $\hat{\rho}_{bh} := \text{Tr}_V P_{\mathcal{H}^{bh}}$, Then define the subspace

$$\mathcal{H}_S^{bh} := \text{supp}(\hat{\rho}_{bh}). \tag{7.4}$$

With this, we can define the von Neumann entropy associated with the quantum states lying on the horizon S. Hence, we have the following definition:

Definition 7.4. The **entropy of an isolated horizon** is the von Neumann entropy associated to the density operator ρ_{bh} on $\mathcal{H}_S^{bh}$ is given by

$$S_{bh} = -\text{Tr}(\hat{\rho}_{bh} \ln \hat{\rho}_{bh}). \tag{7.5}$$

We can see from this definition that it resembles the Shannon entropy used in Bekenstein's original derivation. This is an indicator that we are proceeding in the correct direction.

From thermodynamics, we also know that a system in equilibrium has its entropy maximised. From Chapter 4, we know isolated horizons represent a type of black hole region at equilibrium, and so the associated entropy must be at a maximum. This will correspond to the distribution of states maximising (7.5), which is obtained when every state is equally likely. Specifically, this corresponds to a distribution for which the expectation value of any states is $\frac{1}{\dim \mathcal{H}_S^{bh}}$. In this case, the reduced density operator becomes

$$\hat{\rho}_{bh} = \frac{1}{N_{bh}} \hat{1}_{\mathcal{H}_S^{bh}}, \tag{7.6}$$

where we follow [ABK00] and write $N_{bh} := \dim \mathcal{H}_S^{bh}$. With this, (7.5) becomes

$$\begin{aligned} S_{bh} &= -\text{Tr}\left(\frac{1}{N_{bh}} \hat{1}_{\mathcal{H}_S^{bh}} \ln\left(\frac{1}{N_{bh}} \hat{1}_{\mathcal{H}_S^{bh}}\right)\right) \\ &= -\text{Tr}\left(\frac{1}{N_{bh}} \hat{1}_{\mathcal{H}_S^{bh}} \ln\left(\frac{1}{N_{bh}}\right)\right) \\ &= \ln(N_{bh}). \end{aligned} \tag{7.7}$$

by properties of the matrix logarithm. Accordingly, obtaining the entropy is equivalent to finding the dimension $\mathcal{H}_S^{bh}$. Since the space is finite-dimensional, we will only have a finite number of basis states to count. With this, we have the following theorem:

Theorem 7.5. *Let $S \in \mathcal{S}$ be an isolated horizon of area A_0. Then the von Neumann entropy of this horizon has the form*

$$S_{bh} = \frac{\ln 2}{4\pi\beta\ell_P^2\sqrt{3}} A_0 + o(A_0). \tag{7.8}$$

Notation 7.6. The notation $o(n)$ is used to describe a quantity that decays to 0 when divided by n as $n \to \infty$. In what follows, it will represent high-order corrections to the entropy formula one obtains in the quantum case, which vanish in the classical limit.

In particular, this value is based on N_{bh}, so the rest of this chapter will be dedicated to counting the basis states of this space by determining two upper and lower bounds for this number, which will be the key to proving Theorem 7.5.

We begin the proof with the lower bound.

7.1.1 The Lower Bound

To streamline the counting argument, we give a few preliminary definitions.

Definition 7.7. Let $j = (j_1, ..., j_n) \in \frac{1}{2}\mathbb{Z}^n$ be a list of half-integers corresponding to the spins labelling the edges of a spin network. We will say this list is **permissible** if it satisfies the following conditions:

1. $j_1 \leq ... \leq j_n$,
2. $A_0 - \delta \leq A_j \leq A_0 + \delta$,

where A_j is the eigenvalue associated to the area operator when acting upon a spin network state with spin j, i.e. $A_j = 8\pi\beta\ell_P^2 \sum_i \sqrt{j_i(j_i + 1}$.

Definition 7.8. Let $(m_1, ..., m_n)$ be a list of half-integers corresponding to the spins of the vertices of a spin network. Then this list is **permissible** with respect to a list $j \in \frac{1}{2}\mathbb{Z}^n$ if we have $m_i \in \{-j_i, -j_i + 1, ..., j_i\}$ for all i.

Remark 7.9. We can extend this permissibility definition to $a = (a_1, ..., a_n) \in \mathbb{Z}^n/k$. We define this list to be **permissible** if $a_i = -2m_i \mod k$ for some permissible list m and $\sum_i a_i = 0 \mod k$.

The purpose of these definitions is to give us an appropriate way to order the basis elements of $\mathcal{H}_S^{bh}$. More specifically, such a construction encodes all the states that satisfy the constraint on the area we set for the collection $\mathcal{S}$. One can also interpret this as describing which states can physically appear when considering horizons with areas lying in the interval defined in 7.1. This interpretation will enable us to decompose the surface Hilbert into subspaces corresponding to states indexed by the permissible lists.

One final assumption we require is the following [ABK00]:

Proposition 7.10. *Let j, m be permissible lists. Then there exists $\psi_V \in \mathcal{H}_V^{\mathcal{P},j} \cap \mathcal{H}_V^{\mathcal{P},m}$ such that the Hamiltonian constraint annihilates it, i.e. $\hat{H}\psi_V = 0$.*

With this, since $\psi_V \in \ker \hat{H} \cap \mathcal{H}_V^{\mathcal{P},j} \cap \mathcal{H}_V^{\mathcal{P},m}$, any physical state $\psi_V \otimes \psi_S$ in $\mathcal{H}^{bh}$ is annihilated by $\hat{H}$ for all $\psi_S \in \mathcal{H}_S^{bh}$. Furthermore, in the physical Hilbert space, every permissible list a satisfies $2m \equiv -a \mod k$ and every surface subspace $\mathcal{H}_S^{n,a}$ is one-dimensional, spanned by

ψ_S^a, so each tensor product $\psi_V \otimes \psi_S^a$ is a distinct state. Therefore, $N_{bh} = \dim \mathcal{H}_S^{bh}$ is exactly the number of permissible a-lists. In the case $\mathcal{H}_S^{bh}$ does not contain one of these subspaces, then N_{bh} will be less than the number of permissible a-lists. Still, in either case, it is finite *.

Hence, our strategy further streamlines the process of counting the number of permissible lists a.

With these final preliminary considerations, let us start by obtaining a lower bound on $\dim \mathcal{H}_S^{bh}$. Consider a list $j = (j_1, .., j_N)$ of half integers. The minimal possible contribution to the area occurs when each $j_i = \frac{1}{2}$, since $j = 0$ will give zero overall contribution to the macroscopic area, so we ignore this case.

In the present case, we get the associated area eigenvalue.

$$A_j = 8\pi\beta\ell_P^2 \sum_i \sqrt{\frac{1}{2}\left(\frac{3}{2}\right)} = 4\pi\beta\ell_P^2\sqrt{3}N. \tag{7.9}$$

Now we show that this list is permissible. The first condition is clearly satisfied. For the second condition, the area prescribed by this j must be within some tolerance δ of the fixed area A_0. Since this choice of j corresponds to the smallest collection of spins contributing to the area, this must enforce a condition on δ. Indeed, if we set $\delta > \frac{2}{N}A_j = 8\pi\beta\ell_P^2\sqrt{3}$, there exists an even N which guarantees $A_j \in [A_0 - \delta, A_0 + \delta]$. Hence, j is a permissible list. By Definitions 7.7, 7.8, we can obtain 2^N permissible lists m, since each $m_i = \pm\frac{1}{2}$ by definition.

Now the number of permissible lists a is obtained from m subject to $a_i = -2m_i \mod k$ and $\sum_i a_i = 0 \mod k$. We can determine this by counting how many lists m are such that:

$$\sum_{i=1}^{N} m_i = 0. \tag{7.10}$$

Proposition 7.11. *The number of lists m satisfying (7.10) is given by*

$$\binom{N}{N/2}. \tag{7.11}$$

Proof. Let $(m_1, ..., m_N)$ be a list with each $m_i = \pm\frac{1}{2}$. Since N is even, the condition as listed in (7.10) tells us we must have precisely $N/2$ pairs of elements with opposite signs, since they will cancel each other in the sum. Since we have N possible entries in the list and $N/2$ choices of elements satisfying the condition, it follows that the number of lists satisfying the condition is exactly $\binom{N}{N/2}$. □

Corollary 7.12. *The number of permissible lists a is given by $\binom{N}{N/2}$. Furthermore, these lists are distinct, given the non-trivial case of $k > 2$.*

*It is also remarked in [ABK00] that even if Proposition 7.10 fails, the dimension of $\mathcal{H}_S^{bh}$ will be smaller and again finite.

From this, we now have a lower bound on $\dim \mathcal{H}_S^{bh} = N_{bh}$:

$$N_{bh} \geq \binom{N}{N/2}. \tag{7.12}$$

In physical cases, we consider this when N is large. In this case, we can apply the Stirling approximation

$$\binom{N}{N/2} \approx \frac{2^{N+\frac{1}{2}}}{\sqrt{\pi N}}. \tag{7.13}$$

Hence

$$\begin{aligned} \ln\left(\frac{2^{N+\frac{1}{2}}}{\sqrt{\pi N}}\right) &\leq \ln N_{bh} = S_{bh} \\ \implies S_{bh} &\geq N\ln(2) - o(N). \end{aligned} \tag{7.14}$$

Since j is a permissible list, we also have from our equation for A_j that

$$\begin{aligned} A_0 - \delta \leq 4\pi\beta\ell_P^2\sqrt{3}N &\leq A_0 + \delta, \\ \implies A_0 \leq 4\pi\beta\ell_P^2\sqrt{3}N &\leq A_0 + 2\delta. \end{aligned} \tag{7.15}$$

From this, it follows that the entropy satisfies

$$S_{bh} \geq \frac{\ln(2)}{4\pi\beta\ell_P^2\sqrt{3}}A_0 - o(A_0). \tag{7.16}$$

This is our lower bound. In the next section, we compute the upper bound.

7.1.2 The Upper Bound

The upper bound for S_{bh} is a more involved computation as compared to the lower bound. In the latter, there was only one permissible list we had to consider, namely the list which defined the minimal possible area contributions from eigenstates of $\hat{A}_S$. However, we must now consider *all* permissible lists and their associated eigenvalues, including those stemming from higher spin labels as well as the multiplicities of states in the j, m, a subspace decompositions of $\mathcal{H}_S^{bh}$. In essence, we must show that the number of such lists can be controlled by some large finite value. To deal with this, we will utilise the partition function, which describes the entire microscopic ensemble of states, and use this to control N_{bh}.

To begin, we recognise that N_{bh} is at least finite. Moreover, consider *every* possible permissible list $j \in \frac{1}{2}\mathbb{Z}^n$ with n elements. This allows us to determine the number of permissible lists m stemming from this (and by Remark 7.9, the number of permissible a). In particular, the number of permissible lists m $M(j)$ associated to j is given by [ABK00]

$$M(j) := \prod_{i=1}^{n}(2j_i + 1). \tag{7.17}$$

By summing over all possible lists j, we obtain a crude estimate for N_{bh} as

$$N_{bh} \leq \sum_{j \in \frac{1}{2}\mathbb{Z}^n} M(j) = \sum_{j \in \frac{1}{2}\mathbb{Z}^n} \prod_{i=1}^{n} (2j_i + 1). \tag{7.18}$$

To refine this, we must examine the growth rate of the density of states in $\mathcal{H}_S^{bh}$ over a given patch of area on the horizon. By examining the rate of growth of these states as the area increases, we obtain a bound on N_{bh}, since the state density is directly related to (7.17):

Definition 7.13. The **spectral density** of states on $S \in \mathcal{S}$ is a distribution $g \in \mathcal{D}'((0, \infty))$ defined by the Dirac delta distribution as

$$g(A) := \sum_{j} M(j)\delta(A - A_j). \tag{7.19}$$

Heuristically, one can observe that upon integration over some patch $[A_1, A_2]$ on S one can recover the area of that patch. Furthermore, it follows from (7.18) that

$$N_{bh} \leq \int_{A_0 - \delta}^{A_0 + \delta} g(A) dA. \tag{7.20}$$

This equation now tells us that the dimension N_{bh} is bounded above by the density of states on the horizon. Physically, one would not expect the density of states to grow to infinity. Hence, if we can bound the growth of the spectral density as the area goes to infinity, we obtain a bound for the dimension that works for all horizons in $\mathcal{S}$. As presented in [ABK00], we are led to consider the Laplace transform of (7.19) because, in statistical mechanics, this provides the partition function associated with the states. We define the case relevant to our discussion below:

Definition 7.14. Let $S \in \mathcal{S}$ be an isolated horizon and let g be the spectral density of states on S. The **partition function** associated with g is defined as the Laplace transform:

$$\mathcal{Z}(s) := \mathcal{L}\{g\}(s) = \int_0^\infty g(A) e^{-sA} dA, \tag{7.21}$$

for some complex variable $s \in \mathbb{C}$.

Since this integral is taken up to infinity, we obtain a description of the growth of the spectral density, as well as the behaviour of the statistical properties described by $\mathcal{H}_S^{bh}$ physically. In particular, one can observe that this integral will converge if s is significantly large, since the exponential term decays faster than g for large s. This will, in turn, give us a holomorphic function $\mathcal{Z}(s)$ of s. Next, we examine the holomorphic domain of $\mathcal{Z}(s)$.

Lemma 7.15. *The partition function is holomorphic on the region* $\Re(s) > s_0$ *where* s_0 *is the* *abscissa of convergence*

$$s_0 := \inf \left\{ \sigma \in \mathbb{R} \mid \sum_{j \in \frac{1}{2}\mathbb{Z}^n} M(j) e^{-\sigma A_j} < \infty \right\}. \tag{7.22}$$

Proof. By properties of the Dirac delta distribution, we can express the partition function as

$$\int_0^\infty g(A) e^{-sA} dA = \int_0^\infty \sum_j M(j) e^{-sA} \delta(A - A_j) dA \tag{7.23}$$

$$= \sum_j M(j) e^{-sA_j}. \tag{7.24}$$

This is a Dirichlet series, so it has an abscissa of convergence as defined by s_0 in the statement of the lemma. Next, let us choose and fix $s \in \mathbb{C}$ with the property that $\Re(s) > s_0$. Then there exists an $\epsilon > 0$ such that $\Re(s) \geq s_0 + \epsilon$. In particular, we have

$$\begin{aligned} \left| \sum_j M(j) e^{-sA_j} \right| &\leq \sum_j \left| M(j) e^{-sA_j} \right| \\ &\leq \sum_j M(j) e^{-\Re(s) A_j} \\ &\leq \sum_j M(j) e^{-(s_0+\epsilon) A_j} < \infty. \end{aligned} \tag{7.25}$$

Hence, the sum converges absolutely for a fixed $s \in \mathbb{C}$. Next, consider the half plane $H_{s_0} := \{s \in \mathbb{C} \mid \Re(s) > s_0\}$ and let $K \subset H_{s_0}$ be any compact subset. By compactness, there exists a minimum $m \in \mathbb{R}$ such that $m > s_0$. Then if we set $\epsilon = m - s_0$, we obtain that $\Re(s) \geq s_0 + \epsilon$ for any $s \in K$. In particular, we have that $|M(j)e^{-sA_j}| \leq M(j)e^{-(s_0+\epsilon)A_j}$, which is bounded for all j. An application of the Weierstrass M-test shows that the sum

$$\sum_j |M(j) e^{-sA_j}| \tag{7.26}$$

converges uniformly and absolutely on K. Therefore, since $M(j)e^{-sA_j}$ is an entire function, the sum $\mathcal{Z}(s) = \sum_j M(j)e^{-sA_j}$ is a sum of holomorphic functions on K. There exists a covering of H_{s_0} by compact sets, so therefore $\mathcal{Z}(s)$ extends to a holomorphic function on all of H_{s_0}. Hence, $\mathcal{Z}(s)$ is holomorphic on the domain $\Re(s) > s_0$. □

We are now left to determine exactly what this constant s_0 is. Let us revisit the Dirichlet series in (7.24). By determining this constant, we will also show another result:

Lemma 7.16. *$\mathcal{Z}(s)$ is meromorphic on the half-plane $\Re(s) > 0$. Furthermore, the abscissa of convergence for $\mathcal{Z}(s)$ occurs at*

$$s_0 = \frac{\ln(2)}{4\pi\beta\ell_P^2\sqrt{3}}. \tag{7.27}$$

Proof. Consider the Dirichlet series as in (7.24). By expanding each term under the sum out we obtain:

$$\mathcal{Z}(s) = \sum_{n=1}^{\infty} \sum_{j \in \frac{1}{2}\mathbb{Z}^n} \exp\left[-s\left(8\pi\beta\ell_P^2 \sum_{i=1}^{n} \sqrt{j_i(j_i+1)}\right)\right] \prod_{i=1}^{n}(2j_i+1). \tag{7.28}$$

Note that $M(j)$ counts the number of permissible lists m based on the ordered permissible lists $j_1 \leq ... \leq j_n$. We can rewrite the expression in (7.17) so that, instead of considering ordered permissible lists, we obtain $M(j)$ by grouping each j_i based on the number of times an edge carrying that spin punctures the horizon. More specifically, given a fixed permissible list j, let $n_l := |\{i \in \mathbb{N} \mid j_i = l\}|$. From this, we can see that $\sum_l n_l = n$. By changing how we count the number of permissible lists, we obtain

$$M(j) = \prod_{l=\frac{1}{2}}^{\infty}(2l+1)^{n_l}. \tag{7.29}$$

Using this fact, we can rewrite the partition function as

$$\begin{aligned}\mathcal{Z}(s) &= \sum_{n_l=0}^{\infty} \prod_{l \geq \frac{1}{2}} \left((2l+1)\exp\left[-s\left(8\pi\beta\ell_P^2\sqrt{l(l+1)}\right)\right]\right)^{n_l} \\ &= \prod_{l \geq \frac{1}{2}} \sum_{n_l=0}^{\infty} \left((2l+1)\exp\left[-s\left(8\pi\beta\ell_P^2\sqrt{l(l+1)}\right)\right]\right)^{n_l}.\end{aligned} \tag{7.30}$$

Note that we were able to switch the sum and product since all the n_l's are independent of each other for every l (indeed, this can also be seen by expanding (7.30) for a finite sum/product). We see that the right-hand side has a geometric series under the product. If we assume we are in the half-plane $\Re(s) > s_0$ where $\mathcal{Z}(s)$ is holomorphic, then this series converges to

$$\mathcal{Z}(s) = \prod_{l \geq \frac{1}{2}} \frac{1}{1-(2l+1))\exp\left[-s\left(8\pi\beta\ell_P^2\sqrt{l(l+1)}\right)\right]}. \tag{7.31}$$

In particular, for the series in (7.30) to converge, we require

$$1 > (2l+1))\exp\left[-s\left(8\pi\beta\ell_P^2\sqrt{l(l+1)}\right)+2\pi i n\right], \quad n \in \mathbb{Z} \tag{7.32}$$

$$\implies s > \frac{\ln(2l+1)+2\pi i n}{8\pi\beta\ell_P^2\sqrt{l(l+1)}}. \tag{7.33}$$

From (7.33), we can see that if this is an equality, $\mathcal{Z}(s)$ has simple poles at each of these points, since the denominator in (7.31) vanishes. Since each of these poles is simple, we see that outside of the poles, $\mathcal{Z}(s)$ is holomorphic. Hence, it is meromorphic on the region $\Re(s) > 0$.

Now by inspection, we can see that the terms in the product (7.31) rapidly decay down to 1 as $l \to \infty$. In particular, this means that for a fixed s the region $\Re(s) > \frac{\ln(2l+1)}{8\pi\beta\ell_P^2\sqrt{l(l+1)}}$ does not contain all of the region $\Re(s') > \frac{\ln(2(l+1)+1)}{8\pi\beta\ell_P^2\sqrt{(l+1)(l+2)}}$ for some other s, since this is a decreasing function in l. Therefore, the largest pole must occur at $l = \frac{1}{2}$:

$$\Re(s) > \frac{\ln(2)}{4\pi\beta\ell_P^2\sqrt{3}} =: s_0. \tag{7.34}$$

This is therefore precisely the abscissa of convergence for $\mathcal{Z}(s)$. $\square$

This brings us a step closer to refining the bound on N_{bh}. We know that N_{bh} is bounded above by the integral of the spectral density, as in (7.20). To write this in terms of the partition function, suppose that $s > 0$. Then we have that

$$\begin{aligned}\int_{A_0-\delta}^{A_0+\delta} g(A)dA &\leq \int_0^{A_0+\delta} g(A)dA \\ &\leq \int_0^{A_0+\delta} e^{s(A_0+\delta-A)}g(A)dA \\ &\leq e^{s(A_0+\delta)}\mathcal{Z}(s),\end{aligned} \tag{7.35}$$

where in lines 2 and 3 we utilise the fact that the exponential $e^{A_0+\delta-A} \geq 1$ and $s > 0$. Hence, we get a bound for N_{bh} in terms of the partition function

$$N_{bh} \leq e^{s(A_0+\delta)}\mathcal{Z}(s). \tag{7.36}$$

For $\Re(s) > 0$, $\mathcal{Z}(s)$ has simple pole at $s = s_0$. So what we have on the right-hand side is a product of two factors: one that grows exponentially with s and another that rapidly decreases as s grows but blows up at $s = s_0$. So we are left to minimise this product with respect to s to optimise this bound. This is the content of the following lemma:

Lemma 7.17. *$N_{bh} \leq \frac{C}{(s-s_0)}e^{s(A_0+\delta)}$ when $s \in (s_0, s_0+\epsilon)$ for some $\epsilon > 0$ and C some non-zero constant.*

Proof. Since the largest pole occurs at $l = \frac{1}{2}$, this has the largest contribution to the product in (7.31) when s is close to s_0. Since each term rapidly approaches unity as $l \to \infty$, it is of interest to isolate the $l = \frac{1}{2}$ term from the rest. Hence, we write

$$\mathcal{Z}(s) = \frac{1}{1-2e^{-sA_{1/2}}}\mathcal{Y}(s), \tag{7.37}$$

where we define $\mathcal{Y}(s) := \prod_{l\geq 1}\frac{1}{1-(2l+1)e^{-sA_l}}$ and $A_l = 8\pi\beta\ell_P^2\sqrt{l(l+1)}$ the area eigenvalue for spin l. In particular, we have $A_{1/2} = 4\pi\beta\ell_2^2\sqrt{3}$.

We note that $\mathcal{Y}(s)$ is holomorphic over any ball $B_\epsilon(s_0)$, since we do not have $l = \frac{1}{2}$ contributions for the remainder of the product. Furthermore, it is also finite at $s_0 = \frac{\ln(2)}{4\pi\beta\ell_P^2\sqrt{3}}$.

Therefore, there exists $M < \infty$ bounding $\mathcal{Y}(s)$ over the ball:

$$\sup_{s \in B_\epsilon(s_0)} |\mathcal{Y}(s)| \leq M. \tag{7.38}$$

We now turn our attention to the reciprocal $1 - 2e^{-sA_{1/2}}$. Let us consider the difference $x = s - s_0$ from the pole. Then we can rewrite this term as

$$1 - 2e^{-sA_{1/2}} = 1 - 2e^{-s_0 A_{1/2}} e^{-xA_{1/2}} = 1 - e^{-xA_{1/2}}, \tag{7.39}$$

since $2e^{-s_0 A_{1/2}} = 1$ by definition of s_0 and $A_{1/2}$. Over the interval $(0, x)$, $1 - 2e^{-sA_{1/2}}$ is entire, so by the mean value theorem, there exists $c \in (0, x)$ such that

$$\frac{1 - e^{-xA_{1/2}}}{x} = A_{1/2} c e^{-cA_{1/2}} \implies 1 - e^{-xA_{1/2}} = A_{1/2} c x e^{-cA_{1/2}}. \tag{7.40}$$

Moreover, if $|s - s_0| < \epsilon$ from above, then it follows

$$|1 - e^{-xA_{1/2}}| \geq |A_{1/2} c e^{-\epsilon A_{1/2}} x| = K|s - s_0|, \tag{7.41}$$

where we use the definition of x and $K := A_{1/2} c e^{-\epsilon A_{1/2}} > 0$ is a constant. Therefore, the partition function satisfies

$$|\mathcal{Z}(s)| = \left| \frac{\mathcal{Y}(s)}{1 - 2e^{-sA_{1/2}}} \right| \leq \frac{M}{K|s - s_0|}. \tag{7.42}$$

As both $M, K > 0$, we can set $C = \frac{M}{K}$, which gives us that

$$|\mathcal{Z}(s)| \leq \frac{C}{|s - s_0|} \implies N_{bh} \leq \frac{C}{(s - s_0)} e^{s(A_0 + \delta)}, \tag{7.43}$$

after combining with (7.36) for $s \in (s_0, s_0 + \epsilon)$, proving the lemma. □

By taking the logarithm of both sides of this equation, we obtain an upper bound for the entropy:

$$S_{bh} \leq \ln\left(\frac{C}{s - s_0} e^{s(A_0 + \delta)} \right) = s(A_0 + \delta) + \ln\left(\frac{C}{s - s_0} \right). \tag{7.44}$$

We now see that the right-hand side depends on s. This works in our favour, since we can now further optimise the bound by finding what value of s minimises the right-hand side. In particular, let us use $x = s - s_0$ as prior. Then we can rewrite (7.44) as

$$S_{bh} \leq (x + s_0)(A_0 + \delta) + \ln\left(\frac{C}{x} \right). \tag{7.45}$$

Using simple calculus, one finds the right-hand side is minimised when $x = (s_0 + \delta)^{-1}$ (one can check this is indeed a minimum of the right-hand side of (7.44)). Substituting this back in, we find

$$S_{bh} \leq 1 + s_0 A_0 + s_0 \delta + \ln C(s_0 + \delta). \tag{7.46}$$

Now we observe that terms on the right-hand side, bar $s_0 A_0$ are all $o(A_0)$, which means that the entropy is bounded above by

$$S_{bh} \leq s_0 A_0 + o(A_0) = \frac{\ln 2}{4\pi\beta\ell_P^2\sqrt{3}} A_0 + o(A_0). \tag{7.47}$$

Now note that the terms under the $o(A_0)$ umbrella have δ dependence. But recall from Subsection 7.1.1 that there always exists a permissible list of j's for which $A_j \in [A_0 - \delta, A_0 + \delta]$ as long as $\delta > 8\pi\beta\ell_P^2\sqrt{3}$. Choosing this, we can combine the lower bound in (7.16) with (7.47) to obtain the equality

$$S_{bh} = \frac{\ln 2}{4\pi\beta\ell_P^2\sqrt{3}} A_0 + o(A_0). \tag{7.48}$$

This is precisely (up to higher order area terms) the black hole entropy formula of quantum gravity. However, we still need to check whether this agrees with the classical description in (3.10). Indeed, by fixing the Barbero–Immirzi parameter at $\beta = \beta_0 = \frac{\ln 2}{\sqrt{3}}$, we recover asymptotically (upon restoring units)

$$S_{bh} = \frac{A_0}{4\pi\ell_P^2} = \frac{A_0}{4\pi\hbar}\frac{k_B c^3}{G}, \tag{7.49}$$

the classical Bekenstein–Hawking entropy as in (3.10). This proves Theorem 7.5, and establishes a quantum description of Bekenstein's classical formula. In effect, this completes the main goal of this thesis. In the following, we discuss this result in the broader picture of mathematical physics.

7.2 Discussion and Outlook

By quantising the horizon states and recognising that black holes exhibit von Neumann entropy (similar to the realisation of the need to utilise Shannon entropy that struck Bekenstein in the classical case), we have rigorously obtained a microscopic description of black hole entropy that asymptotically agrees with the classical Bekenstein–Hawking radiation. Furthermore, this derivation was completely background independent and did not rely on perturbative techniques. This is quite remarkable.

However significant this result is, there are clearly some questions left unanswered, particularly those involving the Barbero–Immirzi parameter β. As we have touched on in Chapter 4, the Barbero–Immirzi parameter remains relatively ambiguous and arises in the prelude of the quantisation process. The fixing that we performed by setting $\beta = \beta_0$ is akin to matching this theory with empirical results (particularly the Bekenstein–Hawking entropy) and does not arise naturally in the derivation, at least as it currently stands in the literature. Moreover, alternative counting methods or examining horizons with more exotic topologies

[PBC21] result in different choices of β_0 to recover the classical theory, implying that β indeed parameterises a family of quantum gravity theories which emerge based on the topological spaces of interest.

Hence, no one universal theory can account for all cases. There have been developments in exploring alternative formulations and counting methods of the microscopic entropy (for instance, the very recent review in [ANT25]); however, it is evident that further work is still required to obtain the true universal microscopic origin of the Bekenstein–Hawking entropy. Despite this, this calculation serves as a stepping stone in the path to the unification of general relativity with quantum mechanics.

7.2.1 Connections with QFT and Entanglement Entropy

Building on our discussion above, one can directly compare our results to those in other fundamental theories of our universe. In particular, we have shown that the microscopic origin of black hole entropy can be described by a Chern–Simons topological quantum field theory (TQFT) on the horizon surface, with spin network states puncturing the surface being a realisation of Wilson loops - gauge invariant operators representing observables in Chern–Simons TQFTs.

Furthermore, our procedure of tracing out the volume Hilbert space to reduce our density matrix to the black hole sector ρ_{bh} is precisely the same logic utilised in QFT calculations involving entanglement entropy of states [Sol11]. In particular, since our theory is inherently non-perturbative, we avoid having to perform any renormalisation scheme to avoid UV divergences, thanks to the inherent discreteness of the quantum geometry of the horizon - in contrast to regularisation schemes implemented in most standard QFT calculations.

As a result, the black hole entropy of LQG realises the entanglement entropy of horizon microstates while avoiding the UV divergences present in standard QFT. Our tradeoff here is still, as we've seen, the seemingly arbitrary choice of the Immirzi parameter, which is not present in QFT calculations.

7.2.2 Future Directions

The discovery of black hole entropy in 1973 led to multiple attempts to describe a microscopic origin, apart from the one we presented. In particular, the seminal paper by Andrew Strominger and Cumrun Vafa derives a quantum entropy formula for non-extremal black holes in string theory [SV96], and entanglement entropy was first proposed as a source for black hole entropy [Sor14]. As a result, there is still more that may emerge from this theory.

Of course, one may attempt to continue refining the correction terms for entropy, particularly by employing well-established techniques from QFT to this specific gauge theory. This, however, begs the question as to whether it is possible to obtain a description of the entropy without any higher-order correction terms in the first place? To this end, we propose an alternative direction in a heuristic manner.

Consider the surface Hilbert space $\mathcal{H}_S$. We saw that this is the space of states for a quantum $U(1)$-Chern–Simons theory. It has also been established that such quantum theories may be described by so-called *Skein modules* - a module associated to a 3-manifold M by considering links (considered as immersed submanifolds of M modulo appropriate Skein relations). In particular, these can be used to model quantised Wilson loops in Chern–Simons theory [BJ25] - and the horizon surface provides the correct playing ground to utilise this connection.

Currently, however, the surface S is a 2-manifold, so to implement this construction, we consider a ball D such that $\partial D = S$. This also allows the spin-network edges puncturing S to be regarded as lines in D terminating at S. This hence endows the horizon with a richer mathematical structure based on the Skein module. Furthermore, specific Skein modules allow one to utilise tools from homology (such as the Khovanov/Kauffmann bracket Skein module).

With this, one could lift the state space $\mathcal{H}_S$ to some appropriate (possibly graded) chain complex. Subsequently, one could construct and interpret gradings as corresponding to various physical quantities (i.e. area, spin, charge, etc.) such that the overall Euler characteristic describes the partition function, giving us a homological perspective on the entropy. This could potentially yield new topological invariants and allow one to probe higher-order correction terms. However, at the time of writing, there does not seem to be any publications on this connection, which warrants further investigation.

Nevertheless, approaches like this extend the ideas of LQG (and other areas of mathematical physics) into rigorous mathematical frameworks, bringing us closer to a more complete description of our universe and our existence as a whole.

Appendix A

The Condition for Lemma 6.17

The purpose of this appendix is to shed light on the requirement that our connection curvature must be a multiple of the symplectic form ω. Let us frame this as a lemma.

Lemma A.1. *The curvature of the connection satisfies $F_\nabla = i\omega$.*

To show this, we need some basic facts from geometric quantisation [Woo92].

Given the Kähler manifold $(\mathcal{X}^{\mathcal{P}}, h)$ and $f \in C^\infty(\mathcal{X}^{\mathcal{P}})$, we can construct a *quantum operator* associated to f as

$$\hat{P}(f) := i\nabla_{X_f} + f, \tag{A.1}$$

where X_f is the Hamiltonian vector field associated with f. Geometric quantisation requires that these operators satisfy

$$[\hat{P}(f), \hat{P}(g)] = i\hat{P}(\{f, g\}). \tag{A.2}$$

Loosely, this can be thought of as 'Lie brackets of quantum operators must agree with Poisson brackets of classical functions'. The above equation yields the condition for the curvature in Lemma A.1.

Proof. (Lemma A.1):

Let $f, g \in C^\infty(\mathcal{X}^{\mathcal{P}})$ and let X_f, X_g be their associated Hamiltonian vector fields. We know the connection ∇ along these vector fields satisfies

$$[\nabla_{X_f}, \nabla_{X_g}] = \nabla_{[X_f, X_g]} + F_\nabla(X_f, X_g). \tag{A.3}$$

Since Hamiltonian vector fields satisfy $[X_f, X_g] = -X_{\{f,g\}}$ where $\{.,.\}$ is the usual Poisson bracket associated to ω, the above equation becomes

$$[\nabla_{X_f}, \nabla_{X_g}] = -\nabla_{X_{\{f,g\}}} + F_\nabla(X_f, X_g). \tag{A.4}$$

By combining Equations A.1,A.2, we obtain

$$\begin{aligned}[\hat{P}(f),\hat{P}(g)] &= [i\nabla_{X_f}+f, i\nabla_{X_g}+g]\\ &= -[\nabla_{X_f}-if, \nabla_{X_g}-ig]\\ &= -[\nabla_{X_f},\nabla_{X_g}]+i[\nabla_{X_f},g]+i[f,\nabla_{X_g}]-i[f,g].\end{aligned} \tag{A.5}$$

Note that $[f, g] = 0$, since f, g correspond to scalar pointwise multiplication on sections. Furthermore, we have the bracket

$$\begin{aligned}[\nabla_{X_f},g]s &= \nabla_{X_f}(gs)-g\nabla_{X_f}s\\ &= X_f(g(s))+g\nabla_{X_f}s-g\nabla_{X_f}s = X_f(g(s)),\end{aligned} \tag{A.6}$$

for some section of the line bundle $s \in \Gamma(L)$. This is similar to the other term of this form. Furthermore, $X_f(g) = \mathcal{L}_{X_f}g$, so using this we obtain

$$\begin{aligned}[\hat{P}(f),\hat{P}(g)] &= -[\nabla_{X_f},\nabla_{X_g}]+i\mathcal{L}_{X_f}(g)-i\mathcal{L}_{X_g}(f)\\ &= \nabla_{X_{\{f,g\}}}+2i\{f,g\}-F_\nabla(X_f,X_g).\end{aligned} \tag{A.7}$$

Using Equation A.4 and that the Poisson bracket satisfies $\{f, g\} = \mathcal{L}_{X_f}(g)$. For this expression to yield the right-hand side of A.2, we must necessarily have $F_\nabla = i\omega$, proving the lemma. Indeed, we verify

$$\begin{aligned}[\hat{P}(f),\hat{P}(g)] &= \nabla_{X_{\{f,g\}}}+i\{f,g\}\\ &= i(-\nabla_{X_{\{f,g\}}}+\{f,g\}) = i\hat{P}(\{f,g\}).\end{aligned} \tag{A.8}$$

as required. □

Note that this is the standard requirement for the prequantum line bundle and does not hold for general line bundles. For instance, the bracket $[f, g]$ may not vanish if the concatenation of f, g with some section represents some non-trivial operation, rather than just plain pointwise multiplication.

Appendix B

Basic Statistical Mechanics

This appendix provides the minimal notions from statistical mechanics required for this thesis. We do not follow any particular reference; however, we refer the reader to [Sch21] for a full treatment of the subject.

B.1 Thermodynamics and the First Law

Definition B.1. A **thermodynamic system** is a physical system described by a set of variables known as *macroscopic variables* characterising the physical properties of the system. A thermodynamic system is **isolated** if the macroscopic variables cannot interact with its surroundings.

Furthermore, we say a macroscopic variable is *extensive* if it is scales with the volume of the system (some examples include internal energy U, volume V, particle number N) and *intensive* if they are not extensive (such as temperature T, pressure P).

Given a thermodynamic system, the internal energy U is determined by the heat supplied to the system Q (from an external source, say) and the work W done on the system by its surroundings. In variational form, we have

$$\delta U = \delta Q - \delta W. \tag{B.1}$$

Definition B.2. Let U be the internal energy of a thermodynamic system. The **first law of thermodynamics** states that the differential energy satisfies:

$$dU = \delta Q - \sum_j X_j dY_j, \tag{B.2}$$

where X_j are generalised intensive variables and Y_j corresponding extensive variables.

B.2 Statistical Mechanics

The purpose of statistical mechanics is to link observable macroscopic variables with statistical distributions of configurations (ensembles) of discrete quantum states known as **microstates**. For example, in a physical quantum system of particles, a microstate is the value of a particle's wavefunction.

In contrast, one can define a **macrostate** as a category of microstates with the same macroscopic properties. A physical example is the energy associated with a 2-state paramagnet (a material for which constituent particles align in parallel to an externally applied magnetic field).

Definition B.3. The **multiplicity** Ω of a macrostate is the number of microstates in that macrostate.

In many physical cases, we assume that the probability of every microstate is equally likely (following a uniform distribution). This is sometimes known as *thermal randomisation.*

Definition B.4. Given an isolated system in a macrostate of multiplicity Ω, the **entropy** is defined as

$$S = k_B \ln \Omega. \tag{B.3}$$

In particular, isolated systems are determined by a combination of intensive and extensive macroscopic variables (U, V, P, T, N, etc.). By fixing these, the system is now in a single macrostate Ω (implicitly a function of the macroscopic variables). Taking partial derivatives of the entropy allows us to extract certain thermodynamic quantities.

Example B.5. Given an isolated system with macroscopic variables (U, V, N) and entropy S, the temperature of this system is determined by $\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{V,N \text{ fixed}}$.

For physical calculations, the multiplicity is often very large, so physicists utilise Stirling's approximation to calculate quantities such as entropy. This approximation is given by

$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left(1 + O\left(\frac{1}{N}\right)\right) \tag{B.4}$$

Which means asymptotically

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N. \tag{B.5}$$

The entropy of a system connects with Equation B.2:

Definition B.6. A **reversible process** is a change/evolution of a thermodynamic system that can always be brought back to its initial state. Specifically, this is a process for which $Q = W$.

Lemma B.7. *For reversible processes, there exists an integrating factor (the temperature T) for which*

$$\delta Q_{rev} = TdS. \tag{B.6}$$

Corollary B.8. *For reversible processes, the first law of thermodynamics can be expressed as*

$$dU = TdS - \sum_j X_j dY_j. \tag{B.7}$$

We remark that each of the differentials in B.7 is exact.

Appendix C

Gauge Theory

This appendix serves as a supplement to the discussion on principal bundles. Given a principal G-bundle P, we know each fibre is invariant under G-action. It is then natural to ask about G-invariance when moving between any two fibres, specifically what type of transformations preserve this G-invariance. Such transformations are known as *gauge transformations*, and are prevalent in multiple areas of theoretical physics (typically in quantum field theory and string theory), and encode information about symmetries of the physical theory. Section C.1 provides a formal treatment of the subject and follows [Ham17], while Section C.2 applies these constructions to graphs, following [Bae96].

C.1 Gauge Tranformations

Definition C.1. Let $P \xrightarrow{\pi} M$ be a principal G-bundle. A **gauge transformation** is a bundle automorphism of P $\varphi : P \to P$ such that, for all $p \in P$, $g \in G$

1. φ preserves the fibres: $\pi \circ \varphi = \pi$

2. φ is G-equivariant: $\varphi(p \cdot g) = \varphi(p) \cdot g$.

The gauge transformations of P form a group, which is denoted as $\mathcal{G}(P)$ (sometimes just $\mathcal{G}$ if the bundle is clear from context) or $\mathrm{Aut}(P)$.

More concretely, the group of gauge transformations is characterised by sections of the adjoint bundle over P $\mathcal{G} = \Gamma(\mathrm{Ad}(P))$, since it can be shown that by changing basepoints in M, the associated fibres are related to each other via the adjoint action of G. It follows that there is a correspondence between gauge transformations and G-valued bundle maps:

Proposition C.2. *Define the group* * *of G-equivariant maps*

$$C^\infty(P, G)^G := \{f : P \to G \mid f(p \cdot g) = \mathrm{Ad}_{g^{-1}} f(p)\}. \tag{C.1}$$

*One can check this is a group under pointwise multiplication.

Then the map

$$\begin{aligned} \mathcal{G}(P) &\to C^\infty(P,G)^G \\ \varphi &\mapsto f_\varphi, \end{aligned} \tag{C.2}$$

is an isomorphism, where the map f_φ is defined as the unique group element that satisfies $\varphi(p) = p \cdot f_\varphi(p)$ with inverse $f_\varphi^{-1}(p) = p \cdot f(p)$.

Corollary C.3. *If G is abelian, we get an even stronger statement: The map*

$$\begin{aligned} C^\infty(M,G) &\to C^\infty(P,G)^G \\ h &\mapsto h \circ \pi \end{aligned}, \tag{C.3}$$

is an isomorphism.

Example C.4. Consider an $SU(2)$-principal bundle over Minkowski spacetime $\mathbb{R}^{3,1}$. This is a prototypical example in gauge theories involving the Standard Model. Since $\mathbb{R}^{3,1}$ is globally flat, every principal bundle is trivial. Hence, we have the group of gauge transformations

$$\mathcal{G}(P) \cong C^\infty(\mathbb{R}^{3,1}, SU(2)). \tag{C.4}$$

Hence, we have a global section s. From the isomorphism, we have for any $x \in \mathbb{R}^{3,1}$, $s \mapsto s \circ \pi$ will transform as

$$s(\pi(x) \cdot g) = g^{-1} s(x) g, \tag{C.5}$$

for $g \in SU(2)$. In particular, connection one forms $A \in \Omega^1(\mathbb{R}^{3,1}, \mathfrak{su}(2))$ transform as

$$A \mapsto g^{-1} A g + g^{-1} dg. \tag{C.6}$$

Noting that since our bundle is trivial, the Maurer-Cartan form $\omega_G = d$ everywhere.

Unfortunately, determining gauge groups for arbitrary principal bundles is often not as straightforward as posed in the above example, since we usually do not have a globally trivial bundle. To that end, it is of interest to examine gauge transformations locally over some trivialising subset $U_\alpha \subset M$ (typically a chart), similar to how we treated local connection one forms.

Definition C.5. Let $P \xrightarrow{\pi} M$ be a principal G-bundle and let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an open cover of M. a **local gauge transformation** is a map

$$g_\alpha : U_\alpha \to G, \tag{C.7}$$

defined by the restriction $\varphi_\alpha(p) = p \cdot g_\alpha(\pi(p))$ where $\varphi_\alpha \in \mathcal{G}(P|_{U_\alpha})$

Proposition C.6. *If G is abelian, then the group of G-equivariant maps*

$$C^\infty(P|_{U_\alpha}, G)^G := \{f : P|_{U_\alpha} \to G \mid f(p \cdot g) = \mathrm{Ad}_{g^{-1}} f(p)\},$$

is isomorphic to $C^\infty(U_\alpha, G)$.

Remark C.7. In physics, the group of local gauge transformations is typically what is known as the 'gauge group'. So, rather than regarding gauge transformations as automorphisms of the bundle upstairs, it is generally more intuitive and valuable in the context of physics to think of Lie group-valued maps from the base manifold. Indeed, in particle physics, both treatments are often equivalent since the base manifold is typically taken to be Minkowski spacetime.

C.2 Gauge Theory on Graphs

Extending this notion somewhat, an example of these constructions manifests in gauge theory over graphs. This is especially prevalent in Loop Quantum Gravity. The following treatment closely follows that given in [Bae96].

Definition C.8. By a **finite, directed graph** ϕ, we mean a collection of edges $E(\phi)$ and vertices $V(\phi)$ and functions

$$s : E(\phi) \to V(\phi) \qquad f : E(\phi) \to V(\phi). \tag{C.8}$$

We call $s(e), f(e)$ the **starting point** and **endpoint** of some edge $e \in E$.

Notation C.9. We will often abbreviate $s(e)$ and $f(e)$ as e_s, e_f respectively, since this makes it more apparent that these correspond to the two vertices of an edge e. Note we will also suppress the argument of $E(\phi), V(\phi)$ to E, V when the graph ϕ is fixed or clear from context.

Equipping V with the discrete topology, let $P \to V$ be a principal G-bundle for some compact Lie group G. It is not hard to see that such bundles are always trivialisable [Bae96]. A choice of trivialisation identifies $\Gamma(\mathrm{Ad}P) \cong \mathrm{Fun}(V, G) \cong G^{|V|}$. With this, we define the gauge group as:

$$\mathcal{G} = \Gamma(\mathrm{Ad}P) \cong G^{|V|}. \tag{C.9}$$

Now consider an edge $e \in E(\phi)$, and let $\mathcal{A}_e$ denote the space of all G-equivariant maps $A_e : P_{s(e)} \to P_{f(e)}$, i.e., maps satisfying $A_e(xg) = A_e(x) \cdot g$ for all $g \in G$, $x \in P_{s(e)}$. With this, we can define the *space of connections on* ϕ as the product

$$\mathcal{A} = \prod_{e \in E} \mathcal{A}_e. \tag{C.10}$$

Regarding Chapter 5, this provides us with a slightly different viewpoint of the space of connections as defined in Definition 5.1. The arguments of Section 5.1 also allow one to identify $\mathcal{A}$ with $G^{|E|}$ after a choice of trivialisation. So for $g \in \mathcal{G}$, write g_v to denote the value of g at v. Then g_v is a fibre automorphism of P_v, and acts on $\mathcal{A}$ by conjugation

$$g \cdot A_e = g_{e_f} A g_{e_s}^{-1}. \tag{C.11}$$

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