

STOCHASTIC PROBLEMS IN TRANSPORT THEORY

by

Peter John Brockwell

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P. J. Brockwell

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ABSTRACT

The application of the theory of discontinuous Markov processes and stochastic population processes to problems in the field of transport theory is investigated.

In Chapter 1 it is shown how a scattering process (in which there may, at each collision, be a constant probability that the scattered particle is absorbed and lost to the system) can be characterized by transition probabilities $P_k(S; \theta_1, \dots, \theta_k | \omega; s)$ of a transition $\omega \rightarrow S$ in distance s with exactly k scatterings, the angles of scattering $\theta_1, \dots, \theta_k$ (in order of occurrence) being such that $\theta_1 \in \Theta_1, \dots, \theta_k \in \Theta_k$. Such a characterization is of importance when there is a change of state of the particle at each collision and the corresponding transition probabilities depend on the angle of deflection.

For one-dimensional processes the problems simplify considerably owing to the fact that the angular deflections, θ_1 , can take only two values, 0 or π . A detailed investigation of one-dimensional problems in which the probability of a collision in a small element of path-length δs is $\lambda \delta s + o(\delta s)$ is made in Chapter 2. In order to apply the theory to scattering in three dimensions an approximation is made in which scattered particles are assumed to move in one of a set of 30 directions in space. This model is considered in detail in Chapters 6 and 7.

Chapter 3 deals with a particular one-dimensional problem which arises in the analysis of bubble chamber tracks and is concerned with the stochastic population process generated by the distances

travelled between successive collisions in a fixed length of track. In Chapter 5 we consider one-dimensional processes in which the distances travelled by a particle between successive collisions have either a gamma distribution or a distribution which is a linear combination of negative exponential distributions. Processes of the latter kind can be used to improve the approximation (used by Fermi and others) to the projection on a fixed axis of the three-dimensional motion of a particle which is scattered isotropically.

Chapter 4 is concerned with multiplicative scattering processes in one dimension. New results are derived for a model which has previously been considered by Bellman, Kalaba and Wing and a more complete analysis is given of a model in which, at each collision, a random number of additional particle pairs is produced.

CHAPTER 1

PROBLEMS TO BE CONSIDERED

1. - Introduction.

The processes by which particles such as neutrons or photons are transmitted and reflected by scattering media are essentially probabilistic in nature so that the theory of stochastic processes is a natural tool for their analysis. This has been recognized for some time; analyses from a statistical viewpoint have been given by Sobolev [40,41], Ueno [43,44], Pal [35], and others. However, the techniques of stochastic process theory have by no means been fully exploited, and it is the purpose of this thesis to develop a theory which, by characterizing the history of a single particle in a very detailed manner, will lead to direct solutions of many of the problems of transport theory. The new formulation is based on the theory of discontinuous Markov processes as given by Moyal [31].

The early development of the subject was stimulated by the interest of astronomers in the transfer of radiation through stellar atmospheres; however, with the advent of nuclear reactors, a great deal of attention has since been devoted to the passage of neutrons and gamma radiation through matter. The equations of transfer are essentially the same for all these problems and many solutions can be translated from one field to another. Radiative transfer theory has also been applied by a number of authors to investigate the radiation from novae and the optical properties of the earth's atmosphere and the sea. Extensive references to the applications are given in the books by Chandrasekhar [17], Kourganoff [27], and Sobolev [40].

For astrophysical and geophysical problems, the greatest interest lies in scattering by a medium which is either

- (1) stratified in plane parallel layers,
- or
- (2) spherically symmetric.

Such systems are also of interest in reactor physics. We shall restrict attention to problems of type (1), observing that in the diffusion approximation (see Tait [42], p. 56) there is a close connection between solutions for plane parallel systems and spherically symmetric systems. Thus we shall be concerned exclusively with the scattering of particles by a plane slab $\{x: a \leq x \leq b\}$ where a and b may be infinite.

A general stochastic formulation of the problem will be given in Sections 2, 3, and 4. The notation is somewhat different from that used in the classical approach to transport theory, which is thoroughly discussed in the books by Chandrasekhar [17], Kourganoff [27], and Sobolev [40]. When numerical solutions are obtained, they will, where possible, be compared with solutions already obtained by classical means. Of course, many of our results, in particular those involving the number of collisions and the angular deflections suffered by a particle, are unobtainable from the classical formulation of the problem. Among the problems considered will be those of determining

- (1) the probability that a particle from the source at $x(a \leq x \leq b)$ emerges from a particular face of the slab;
- (2) the distribution of the number of collisions before emergence;
- (3) the state distribution of emergent particles; and
- (4) the angular distribution of emergent particles.

Exact solutions of these problems are rare. The usual approach is to consider the Boltzmann integro-differential equation for $N(\underline{x}, \underline{y}, v, t)$ where $N(\underline{x}, \underline{y}, v, t) d\underline{x} d\underline{y} dv$ is the expected number of particles in an element $d\underline{x} d\underline{y} dv$ of phase space at time t . An explicit solution in the time-independent case was obtained by Wiener and Hopf [49] for the famous Milne problem of isotropic scattering in a half-space. Chandrasekhar, extending the invariance principles of Ambarzumian [1], found exact solutions for a more general class of problems in terms of his H-, X-, and Y-functions, defined as solutions of certain integral equations (see [14], [15]).

For most physical problems in transport theory, some kind of approximation is necessary. A series of papers by Chandrasekhar [10,...,13] developed a method of approximation which replaced the integral in the Boltzmann equation by a sum using Gaussian quadrature. This idea was first put forward by Wick [48]. A second kind of approximation, in which the solution of Boltzmann's equation was expanded in spherical harmonics and the series truncated, was also used by Chandrasekhar [9] and later by Marshak [30], Mark [29], and others, in neutron transport theory. The close relationship between the Gaussian quadrature and spherical harmonic methods was demonstrated by Wang and Guth [45]. Further developments of the spherical harmonic method have been made by Yvon [53], and Wilson and Sen [50]. The method of successive collisions, in which the particle density is replaced by the density of once, twice,..., and n times scattered particles, has been found useful for scattering by thin slabs (see Peebles and Plesset [36]). Monte Carlo methods have also been used extensively; these are particularly useful

for problems with complicated geometry. The method of Carlson [8], which is well suited to large-scale computers, again reduces the integral in the Boltzmann equation to a sum by introducing a finite set of permissible neutron velocities and by assuming a particular functional form for the variation of neutron flux with direction.

Accounts of the various methods at present available for dealing with problems in neutron transport theory are given in the books by Davison [19], Weinberg and Wigner [46], Tait [42], and Woods [52]. The diffusion of gamma radiation is considered in detail in the book by Leipunskii, Novozhilov, and Sakharov [28], and the article of Fano, Spencer, and Berger [20]. Extensive numerical calculations of the reflection functions for radiation scattered isotropically by slabs of finite thickness have been carried out by Bellman, Kalaba, and Prestrud [6]. There appear, however, to be no results available at present pertaining to the distribution of the number of collisions made by a photon before emergence. This question is studied in detail later in this thesis.

The fundamental equations for a general scattering process will be derived in Sect. 4. It will be noticed that this new formulation in terms of probabilities and generating functionals contains a great deal more information about the history of individual particles than does the usual formulation in terms of the function $N(\underline{x}, \underline{\mu}, v, t)$ (or to use the terminology of radiative transfer, in terms of the radiation field intensity $I_v(\underline{x}, \underline{\mu}, t)$). In particular, information is available regarding the distribution of the number of collisions suffered by a particle and also the angular deflections resulting from these collisions.

This more detailed information leads to a new and direct solution of problems in which there is a change of state of the particle at each collision, the corresponding transition probabilities depending on the angle of deflection. One such example occurs when low energy gamma radiation undergoes multiple Compton scattering. This problem is examined in Chapter 7 where a new analysis is given which eliminates certain objections to the previous treatments of the problem by Chandrasekhar [16], and O'Rourke [34].

Several chapters are devoted to one-dimensional processes. Complete and explicit solutions are obtained for the one-dimensional problem in Chapter 2. Some of these results (the probabilities of emergence at boundaries without regard to the number of collisions experienced before emergence) were obtained by Fermi [21] who used this process to approximate the projection on the x-axis of a three-dimensional scattering process in the slab $\{x: a \leq x \leq b\}$. All the other results of Chapter 2, except where specifically mentioned, are believed to be new. The projected motion is of considerable interest because it can be used to find certain quantities such as the probabilities of emergence and the distribution of the number of collisions before emergence for the corresponding three-dimensional process. It is of interest to observe that two of the earliest workers in the field of radiative transfer, Schuster [38], and Schwarzschild [39], used a method which is equivalent to the one-dimensional method of Fermi. Effectively they assumed that the projections on the x-axis of the distances travelled by a photon between collisions are independent and have a negative exponential distribution with mean equal to half the mean free path in three dimensions.

Fermi made the same assumptions except for the value of the mean projected interval between collisions which he chose as $1/\sqrt{3}$ times the mean free path in three dimensions. A far better approximation to the exact distribution of projected intervals than either Fermi's or Schuster's can be found using a linear combination of negative exponential distributions. Improved results using this new one-dimensional approximation are derived in Chapter 5.

Chapter 3 deals with a special case of the one-dimensional process of Chapter 2 which is of interest in the analysis of bubble chamber experiments. Aspects of the process are examined which make possible the estimation of the velocity of a particle by counting the number of distances between bubbles which exceed a fixed value, d .

Chapter 4 treats some aspects of multiplicative transport processes, including some new results for a process which has previously been studied by Bellman, Kalaba, and Wing [4].

Chapters 6 and 7 deal with the application of the ideas of Sect. 4 to three-dimensional processes with scattering and absorption in which the angular, spatial, and state distributions of the scattered particles are required. Special attention is given to the multiple Compton scattering of low energy gamma radiation.

2. - Scattering processes.

Consider a particle whose state ω is specified by its position vector, $\underline{x}$, its direction of motion, $\underline{\mu}$, and a further state variable, v , which will usually be referred to as velocity (although it could equally well, as in the case of photon scattering, refer to frequency or

wavelength); ω stands for the set of variables (x, μ, v) . Further internal state variables such as rotations or vibrations could be included in ω by an obvious generalization of the methods of this thesis.

A scattering process is a stochastic process characterized by the properties that:

- (a) Only the position of the scattered particle changes between collisions which the particle makes with the scattering centers in the medium.
- (b) The probability that a particle makes a collision in any element δs of path-length is given by $\lambda(\omega)\delta s + o(\delta s)$, where $\lambda(\omega)$ depends only on the instantaneous state, ω , of the particle.
- (c) The change of state of the particle at a collision is instantaneous.
- (d) The probability (given a collision) of a transition $(v, \mu) \rightarrow V \times L$ (where V is a set of velocities and L is a set of unit vectors) is given by a transition distribution $\Phi(V \times L | v, \mu)$ dependent only on the velocity, v , and direction of motion, μ , immediately prior to the collision and independent of the previous values of the state variables. For many of the problems of scattering theory, the transition probabilities Φ take the form

$$(2.1) \quad \Phi(du \times d\gamma | v, \mu) = \phi_\theta(du | v) \psi_v(d\gamma | \mu) ,$$

where the probability $\psi_v(d\gamma | \mu)$ depends only on the initial velocity, v , and the angle, θ , between μ and γ while the

transition probabilities $\phi_\theta(du|v)$ for the velocities depend only on u , v , and θ . It will generally be assumed in what follows that $\bar{\Phi}$ is of the form (2.1) and furthermore that the dependence of $\psi_v(dv|\mu)$ on v can be neglected.

- (e) The change of position of the particle between collisions is deterministic: if a particle initially at $\underline{x}$ with direction of motion $\underline{\mu}$ suffers no collisions in a length of path s , its position will then be $\underline{x} + \underline{\mu}s$.

With these assumptions it is clear that a scattering process is a time-homogeneous discontinuous Markov process of the type whose theory is given in Moyal [31]. The independent variable of the process may be taken as either the path-length, s , or the time, t ; the connection between these two formulations following from the relation $ds = v dt$. Thus if we use $\lambda(\omega)$ to denote the collision rate per unit path-length, then the collision rate per unit time is $v\lambda(\omega)$.

In general, it will be assumed (unless otherwise specified) that the collision rate λ is a function of $\underline{x}$ only. The absence of dependence on v corresponds to the one velocity group theory of neutron transport processes (see Davison [19]); by postulating a finite set of permissible velocities as in multi-group neutron transport theory, this limitation could be overcome to some extent. In many cases, however, the assumption that λ is independent of v is a reasonable one (see, for example, Chapter 7).

3. - Note on absorption and multiplication.

In the most general transport processes a collision may have several possible outcomes. The particle may be absorbed and lost to the system, or may, as in neutron multiplication processes, give rise to several new particles of the same type.

Except where specifically indicated, this thesis will be concerned with pure scattering processes, sometimes known as conservative processes, in which the probability, p_s , that a particle is scattered when it makes a collision is unity, i.e., the integral over all directions $\underline{y}$,

$$(3.1) \quad p_s = \int_{\Omega} \phi_v(d\underline{y}|\underline{y}) = 1 \quad .$$

The generalization to processes in which, at each collision, absorption takes place with probability,

$$(3.2) \quad p_a = 1 - p_s \quad ,$$

is very simple and will be indicated in the relevant parts of the text.

Some attention will be given to multiplicative processes in one dimension and to the question of criticality for some simple models. This will be discussed in Chapter 4.

4. - Characterization of a scattering process.

Considered as a Markov process in the state variable $\underline{\omega}$, a scattering process may be characterized by the transition probability function $P(S|\underline{\omega};s)$, i.e., the probability of a transition $\underline{\omega} \rightarrow S$ in path-length s , satisfying the Chapman-Kolmogorov relation

$$(4.1) \quad P(S|\underline{\omega};s_1+s_2) = \int_{\mathcal{S}} P(S|\underline{\eta};s_1)P(d\underline{\eta}|\underline{\omega};s_2) \quad , \quad s_1, s_2 \geq 0 \quad .$$

However, we shall seek a more detailed specification provided by the probabilities $P_k(S; \theta_1, \theta_2, \dots, \theta_k | \omega; s)$ of a transition $\omega \rightarrow S$ in s with exactly k scatterings, the angles of scattering $\theta_1, \theta_2, \dots, \theta_k$ (in order of occurrence) being such that $\theta_1 \in \Theta_1, \dots, \theta_k \in \Theta_k$. In terms of these functions the Chapman-Kolmogorov relation becomes

$$(4.2) \quad P_k(S; \theta_1, \dots, \theta_k | \omega; s_1 + s_2) = \sum_{j=0}^k \int_{\mathcal{J}} P_{k-j}(S; \theta_{j+1}, \dots, \theta_k | \eta; s_1) P_j(d\eta; \theta_1, \dots, \theta_j | \omega; s_2)$$

where $\mathcal{J}$ denotes the set of all possible states ω . We also introduce the sequence of probability distributions $Q_k(S; \theta_1, \dots, \theta_k | \omega; s)$ where Q_k is the probability that the k^{th} collision occurs in $[0, s)$, the angular deflections being such that $\theta_1 \in \Theta_1, \dots, \theta_k \in \Theta_k$, and the state of the particle immediately after the k^{th} collision belonging to the set S .

The relation between the probabilities P_k and Q_k is

$$(4.3) \quad P_k(S; \theta_1, \dots, \theta_k | \omega; s) = \int_{\tau=0}^s \int_{\eta \in \mathcal{J}} P_0(S | \eta; s - \tau) Q_k(d\eta; \theta_1, \dots, \theta_k | \omega; d\tau),$$

where conventionally

$$(4.4) \quad Q_0(S | \omega; s) = \delta(S | \omega) \in(s) = I(S | \omega; s);$$

in equation (4.4) $\delta(S | \omega)$ denotes the characteristic function of the set S , i.e., $\delta(S | \omega) = 1$ if $\omega \in S$, 0 otherwise; $\in(s)$ is the unit step function, i.e., $\in(s) = 1$ if $s > 0$, 0 otherwise. The probabilities Q_k are themselves connected by the recurrence relations

$$(4.5) \quad Q_k(S; \theta_1, \dots, \theta_k | \omega; s) = \int_{\tau=0}^s \int_{\eta \in J} Q_1(S; \theta_k | \eta; s-\tau) Q_{k-1}(d\eta; \theta_1, \dots, \theta_{k-1} | \omega; d\tau) .$$

Thus, given the functions P_0 and Q_1 , the relations (4.3) and (4.5) determine all the probabilities P_k and Q_k .

For any given S and ω , the angular deflections $\theta_1, \dots, \theta_k$ which the particle experiences in travelling a distance s , constitute a finite univariate population process as defined by Moyal [32]. Corresponding to k angular deflections $\theta_1, \dots, \theta_k$ we have k individuals with states $\theta_1, \dots, \theta_k$, each belonging to the individual state space $[0, \pi]$. The generating functional for this population process, i.e., the expectation $\mathcal{E}\xi(\theta_1), \dots, \xi(\theta_k)$, is given by $G(S; \xi(\cdot) | \omega; s) P(S | \omega; s)^{-1}$, where

$$(4.6) \quad G(S; \xi(\cdot) | \omega; s) = P_0(S | \omega; s) + \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_k) P_k(S; d\theta_1, \dots, d\theta_k | \omega; s) ,$$

it is convenient also to introduce the functional

$$(4.7) \quad H(S; \xi(\cdot) | \omega; s) = \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_k) Q_k(S; d\theta_1, \dots, d\theta_k | \omega; s) ,$$

where $\xi(\theta)$ is a complex-valued measurable function of θ such that $\sup_{0 \leq \theta \leq \pi} |\xi(\theta)| \leq 1$. The transition probability $P(S | \omega; s)$ is obtained by setting

$$\xi(\theta) = 1, \quad 0 \leq \theta \leq \pi ,$$

in G .

The concept of population process will again be utilized in Chapter 3 when we consider a one-dimensional scattering process in which distances between successive collisions take the place of angular deflections as the individuals of the population. This will be discussed more fully in Chapter 3.

It follows from equation (4.3) that

$$(4.8) \quad G = P_0 * (I+H) = P_0 + P_0 * H ,$$

where the symbol $*$ denotes the type of composition appearing in the right-hand side of (4.3). If we let

$$H_1 = \int_{\theta_1=0}^{\pi} \xi(\theta_1) Q_1(S; d\theta_1 | \omega; s) ,$$

defining the sequence $\{H_n\}$ by the iteration relation

$$(4.9) \quad H_{n+1} = H_n * H_1 ,$$

and the sequence $\{G_n\}$ by

$$(4.10) \quad G_n = P_0 * H_n ,$$

then it is easily seen to follow from (4.3) and (4.5) that

$$(4.11) \quad H_n(S; \xi(\cdot) | \omega; s) = \int_{\theta_n=0}^{\pi} \dots \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_n) Q_n(S; d\theta_1, \dots, d\theta_n | \omega; s) ,$$

and

$$(4.12) \quad G_n(S; \xi(\cdot) | \omega; s) = \int_{\theta_n=0}^{\pi} \int_{\theta_1=0}^{\pi} \xi(\theta_1) \dots \xi(\theta_n) P_n(S; d\theta_1, \dots, d\theta_n | \omega; s) ,$$

and hence (setting $G_0 = P_0$) that

$$(4.13) \quad H = \sum_{n=1}^{\infty} H_n \quad \text{and} \quad G = \sum_{n=0}^{\infty} G_n .$$

It is now easily seen that G and H must satisfy the fundamental integral equations

$$(4.14) \quad G = P_0 + G * H_1 ,$$

$$(4.15) \quad H = H_1 + H * H_1 .$$

The major part of this thesis will be concerned with finding solutions of the fundamental equations (4.14) and (4.15).

The theory of discontinuous Markov processes in Moyal [31] is easily generalized to show that the expressions (4.6) and (4.7) for G and H are respectively solutions of the integral equations (4.14) and (4.15). These solutions are unique under certain not very stringent conditions, which are fulfilled in the present case if the collision rate $\lambda(\omega)$ is a bounded function of ω (this will be the case throughout this thesis). All transition probabilities obtained from G satisfy the relevant Chapman-Kolmogorov relation.

The foregoing theory applies to general scattering processes which are homogeneous in path length (or time). It remains applicable under the imposition of a variety of boundary conditions, such as absorbing or reflecting boundaries in space. The effect of such boundary conditions is incorporated in the integral equations (4.14) and (4.15) via the definition of the basic distributions P_0 and Q_1 and of the *-operation. It then follows that the distributions P_k and Q_k obtained by *-iterations from P_0 and Q_1 , and hence the generating functionals G and H, automatically satisfy the boundary conditions.

5. - The backward Kolmogorov equations.

We now specialize the theory to processes which have the following properties:

- (1) The collision rate $\lambda(\omega)$ per unit path length depends only on the position of the particle and is written as $\lambda(\underline{x})$.
- (2) Given a collision, the probability of a transition $\underline{\mu} \rightarrow d\underline{y}$ (where $d\underline{y}$ denotes an element of solid angle surrounding the unit vector $\underline{y}$) is given by $\phi(d\underline{y}|\underline{\mu})$.
- (3) If $\psi_\theta(V|\underline{v})$ is the probability of a velocity transition $\underline{v} \rightarrow V$ conditional upon an angular deflection θ , then the function ψ_θ satisfies the commutativity condition

$$\int_0^\infty \psi_{\theta_2}(V|u) \psi_{\theta_1}(du|\underline{v}) = \int_0^\infty \psi_{\theta_1}(V|u) \psi_{\theta_2}(du|\underline{v}) ,$$

or, symbolically

$$\psi_{\theta_2} \circ \psi_{\theta_1} = \psi_{\theta_1} \circ \psi_{\theta_2} .$$

We introduce the notation $\psi_\theta^{(n)}$ for the sequence of distributions defined by the iteration relation $\psi_\theta^{(n)} = \psi_\theta \circ \psi_\theta^{(n-1)}$.

With these postulates the theory simplifies considerably. Postulate (3) implies that the (absolute) velocity distribution conditional on deflections $\theta_1, \dots, \theta_k$ is $\psi_{\theta_1} \circ \psi_{\theta_2} \circ \dots \circ \psi_{\theta_k}$ independently of the order in which the deflections occur. In addition the assumptions (1) and (2) imply that the process is still Markovian, and the theory of Section (4) applies if we omit $\underline{v}$ and take the state of the particle to be $\omega = (\underline{\mu}, \underline{x})$. Let P, Q, G, H , etc., be the various functions defined in Section 4 in terms of $\omega = (\underline{\mu}, \underline{x})$, and let P', Q', G', H' , etc., be the

corresponding functions defined in terms of $\omega' = (v, \underline{\mu}, \underline{x})$. Then

$$(5.1) \quad G'(VxS; \xi(\cdot) | \omega'; s) = P_0(S | \omega'; s) \delta(V | v) \\ + \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \cdots \int_{\theta_1=0}^{\pi} (\psi_{\theta_1} \circ \dots \circ \psi_{\theta_k}) \xi(\theta_1) \dots \xi(\theta_k) P_k(S; d\theta_1, \dots, d\theta_k | \omega'; s),$$

$$(5.2) \quad H'(VxS; \xi(\cdot) | \omega'; s) = \\ \sum_{k=1}^{\infty} \int_{\theta_k=0}^{\pi} \cdots \int_{\theta_1=0}^{\pi} (\psi_{\theta_1} \circ \dots \circ \psi_{\theta_k}) \xi(\theta_1) \dots \xi(\theta_k) Q_k(S; d\theta_1, \dots, d\theta_k | \omega'; s).$$

Thus we can develop the theory initially in terms of P , Q , G , H , etc.

(i.e., in terms of $\underline{\mu}$ and $\underline{x}$ only), introducing the dependence on v later.

Consider now the unrestricted process (i.e., the process with no boundaries). The probability of no collisions in s is $\exp\left[-\int_0^s \lambda(\underline{x} + \underline{\mu}\tau) d\tau\right]$; with no collisions the position of the particle initially at $\underline{x}$ becomes $\underline{x} + \underline{\mu}s$. Hence

$$(5.3) \quad P_0(X, L | \underline{x}, \underline{\mu}; s) = \delta(X | \underline{x} + \underline{\mu}s) \delta(L | \emptyset) \exp\left[-\int_0^s \lambda(\underline{x} + \underline{\mu}\tau) d\tau\right],$$

$$(5.4) \quad H_1(X, L; \xi(\cdot) | \underline{x}, \underline{\mu}; s) = \\ \int_L \xi(\theta) \phi(d\underline{v} | \underline{\mu}) \int_0^s \delta(X | \underline{x} + \underline{\mu}\tau) \exp\left[-\int_0^{\tau} \lambda(\underline{x} + \underline{\mu}\sigma) d\sigma\right] \lambda(\underline{x} + \underline{\mu}\tau) d\tau,$$

where θ is the angle between the vectors $\underline{\mu}$ and $\underline{v}$ and L is a subset of the space of all $\underline{v}$.

Substituting (5.3) and (5.4) in (4.15) we obtain the "backward" equations for G and H , namely

$$(5.5) \quad \left[\underline{\mu} \cdot \nabla - \frac{\partial}{\partial s} - \lambda(\underline{x}) \right] G(\underline{x}, \underline{\mu}; s) = -\lambda(\underline{x}) \int_{\Omega} \xi(\theta) G(\underline{x}, \underline{v}; s) \phi(d\underline{v} | \underline{\mu}),$$

$$(5.6) \quad \left[\mu \cdot \nabla - \frac{\partial}{\partial s} - \lambda(x) \right] H(x, \mu; s) = \\ -\lambda(x) \int_{\Omega} \xi(\theta) \left[H(x, \nu; s) + \delta(x|x) \delta(L|\nu) \right] \phi(d\nu|\mu) ,$$

where the arguments X, L of G and H have been suppressed and Ω denotes the space of all ν .

Consider now processes whose motion is restricted by the imposition of boundary conditions such as absorbing or reflecting barriers. For small s , the generating functionals G and H for the restricted process will differ only by terms of order $o(s)$ from the corresponding functionals for the unrestricted process. Hence G and H satisfy the same "backward" equations (5.5) and (5.6). Thus, contrary to what is the case for purely discontinuous Markov processes (cf. Moyal [31]), the "backward" equations do not have a unique solution and are not equivalent to the basic integral equations of the process. However, solutions of (5.5) and (5.6) which satisfy the required boundary conditions will also satisfy the integral equations appropriate to the process with these boundary conditions and therefore give the unique solution of the problem.

CHAPTER 2

EXACT SOLUTIONS OF ONE-DIMENSIONAL SCATTERING PROBLEMS

1. - Introduction.

One-dimensional processes of the type described in Chapter 1 have been used in radiative transfer theory by Schuster [38] and Schwarzschild [39], and in neutron transport theory by Fermi [21] and Placzek [37]. A series of papers by Bellman, Kalaba, and Wing (see [4] and the references given there) has been devoted to the application of invariance principles to a variety of problems in neutron transport theory and in particular to one-dimensional fission processes. We shall consider fission processes in Chapter 4.

A different type of one-dimensional scattering process in which particles travel a fixed distance $v\tau$ between collisions, and for which the direction of motion is reversed at a collision with probability p and maintained with probability q , has been investigated by Goldstein [24]. He determined the characteristic function of the displacement $v\tau$ of a particle after n collisions, and by letting $\tau \rightarrow 0$, $n \rightarrow \infty$ in such a way that $n\tau = t$, $v\tau = y$, found that the limiting distribution of particles on $(-\infty, \infty)$ obeyed the telegraph equation. Generalizations of Goldstein's model have been investigated by Gupta [25] and Gillis [23], and a further generalization will be derived in Chapter 5, where it is used to improve on Fermi's analysis of the scattering of thermal neutrons by paraffin.

Mycielski and Paszowski [33] studied the behaviour of particles on a line, which move in such a way that, for any segment of measure x , there are probabilities $p_1(x)$, $q_1(x)$, and $r_1(x)$ of reflection,

transmission, and absorption for leftward moving particles, and corresponding probabilities $p_r(x)$, $q_r(x)$, and $r_r(x)$ for particles moving to the right. They showed by invariance arguments that these functions must be of certain restricted types. We shall see later in this chapter (Sect. 6) that our results are consistent with theirs.

We consider a particle which is restricted to move along the x -axis in the manner described in Chapter 1, Sect. 2. In the one-dimensional case, the direction of motion μ clearly has only two possible values which we shall denote by a scalar, μ , having the values $+1$ for motion to the right and -1 for motion to the left. It will be assumed that:

- (1) The collision rate λ per unit path-length is constant.
- (2) Given the occurrence of a collision, the probability of a reversal is a constant, p , and that of a continuation is $q = 1 - p$: thus if we denote by $\phi(v|\mu)$ the probability of a change of direction $\mu \rightarrow v$ at a collision, then $\phi(-\mu|\mu) = p$ and $\phi(\mu|\mu) = q$.
- (3) If $\psi_o(V|v)$ and $\psi_\pi(V|v)$ are the probabilities of a transition $v \rightarrow V$ at a collision conditional respectively on a continuation (deflection through 0) or reversal (deflection through π), then ψ_o and ψ_π satisfy the commutativity condition of Chapter 1, Sect. 5, i.e.,

$$\int_0^\infty \psi_\pi(V|u) \psi_o(du|v) = \int_0^\infty \psi_o(V|u) \psi_\pi(du|v),$$

or symbolically $\psi_\pi \circ \psi_o = \psi_o \circ \psi_\pi$. We introduce the notation $\psi_\pi^{(n)}$, $\psi_o^{(n)}$ for the sequences of distribution defined

respectively by the iteration relations $\psi_{\pi}^{(n)} = \psi_{\pi}^{(n-1)} \circ \psi_{\pi}$
and $\psi_0^{(n)} = \psi_0^{(n-1)} \circ \psi_0$.

With these assumptions we shall show that it is possible to obtain explicit solutions of the problems connected with one-dimensional scattering. The extension of these results to processes with constant non-zero absorption probability per collision is relatively simple, and is considered in Sect. 9.

2. - Integral equations for the one-dimensional process.

The fundamental integral equations of Chapter 1, Sect. 4, are simplified in one dimension since there are only two possible angular deflections, 0 and π , corresponding to continuations and reversals respectively. The generating functionals G and H of Chapter 1 are therefore replaced by the generating functions

$$(2.1) \quad G(S; z, w | \omega; s) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk}(S | \omega; s),$$

$$(2.2) \quad H(S; z, w | \omega; s) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k Q_{rk}(S | \omega; s) - I(S | \omega; s),$$

where $P_{rk}(S | \omega; s)$ is the probability of a transition $\omega \rightarrow S$ in distance s involving exactly r reversals and k continuations; similarly $Q_{rk}(S | \omega; s)$ is the probability that the $(r+k)^{\text{th}}$ collision with r reversals and k continuations occurs in s leaving the particle immediately after the collision in a state belonging to the set S . G_n and H_n are now clearly given by

$$(2.3) \quad G_n = \sum_{j=0}^n z^{n-j} w^j P_{n-j,j},$$

$$(2.4) \quad H_n = \sum_{j=0}^n z^{n-j} w^j Q_{n-j,j} .$$

As in Chapter 1, Sect. 4, the probabilities P_{rk} and Q_{rk} are related by the equation

$$(2.5) \quad P_{rk}(S|\omega;s) = \int_{\tau=0}^s \int_{\eta \in \mathcal{J}} P_{00}(S|\eta;s-\tau) Q_{rk}(d\eta|\omega;d\tau) ,$$

$r,k = 0,1,2,\dots,$

where, as before, $\mathcal{J}$ is the set of all states and

$$Q_{00}(S|\omega;s) = \delta(S|\omega) \in (s) = I(S|\omega;s) .$$

Denoting the type of composition in the right-hand side of (2.5)

symbolically as $P_{00} * Q_{rk}$, it follows that

$$(2.6) \quad G = P_0 * (I+H) = P_0 + P_0 * H ,$$

$$(2.7) \quad H_{n+1} = H_n * H_1 ,$$

and

$$(2.8) \quad G_n = P_0 * H_n .$$

With this definition of the $*$ -operation, the fundamental integral equations (4.14) and (4.15) of Chapter 1 remain the same, i.e.,

$$(2.9) \quad G = P_0 + G * H_1 ,$$

$$(2.10) \quad H = H_1 + H * H_1 .$$

3. - The "backward" equations.

Under the postulates 1), 2), and 3) of Sect. 1, the theory simplifies as already indicated in Chapter 1, Sect. 5. First the (absolute)

velocity distribution conditional on exactly r reversals and k continuations is $\psi_{\pi}^{(r)} \circ \psi_o^{(k)}$ independently of their order. It also follows that the process is still Markovian and the theory of Sect. 2 still applies if we omit v and take the state of the particle to be $\omega = (\mu, x)$. Let P_{rk}, Q_{rk}, G, H , etc., be the various functions defined in Sect. 2 in terms of $\omega = (\mu, x)$, and let P'_{rk}, Q'_{rk}, G', H' , be the corresponding functions in terms of $\omega = (v, \mu, x)$. Then P'_{rk}, Q'_{rk} , split into products of independent distributions:

$$(3.1) \quad P'_{rk} = P_{rk} \times (\psi_{\pi}^{(r)} \circ \psi_o^{(k)}), \quad Q'_{rk} = Q_{rk} \times (\psi_{\pi}^{(r)} \circ \psi_o^{(k)}),$$

and hence

$$(3.2) \quad \begin{aligned} G' &= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk} \times (\psi_{\pi}^{(r)} \circ \psi_o^{(k)}), \\ H' &= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k Q_{rk} \times (\psi_{\pi}^{(r)} \circ \psi_o^{(k)}) - I. \end{aligned}$$

Thus we can develop the theory initially in terms of μ and x only and introduce the dependence on v later. Secondly, we can omit the final direction v from $P_{rk}(X, v | x, \mu; s), Q_{rk}(X, v | x, \mu; s)$ etc., where X denotes a (measurable) subset of the real line, because $v = \mu$ if r is even and $v = -\mu$ if r is odd (i.e., P_{2rk}, Q_{2rk} vanish if $v = -\mu$ and $P_{2r+1,k}, Q_{2r+1,k}$ vanish if $v = \mu$). It will be convenient to express the initial direction μ as a superscript: thus, $P_{rk}^{(\mu)}, Q_{rk}^{(\mu)}, G^{(\mu)}, H^{(\mu)}$, etc.

Consider now the unrestricted process. With constant λ , the probability of no collisions in s is $\exp[-\lambda s]$; with no collisions, the position of the particle initially at x becomes $x + \mu s$. Hence

$$(3.3) \quad \left\{ \begin{array}{l} P_0^{(\mu)}(X|x;s) = \delta(X|x+\mu s) \exp[-\lambda s], \\ H_1^{(\mu)}(X|x;s;z,w) = (pz+qw) \int_0^s \delta(X|x+\mu\tau) \exp[-\lambda\tau] \lambda d\tau = \\ (pz+qw) \int_0^s P_0^{(\mu)}(X|x;\tau) \lambda d\tau, \end{array} \right.$$

and therefore

$$\frac{\partial H_1^{(\mu)}}{\partial s} = \lambda(pz+qw)P_0^{(\mu)}.$$

This last relation together with (2.7) implies that

$$\partial H_{n+1}^{(\mu)} / \partial s = \lambda(pz+qw)G_n^{(\mu)}, \text{ and hence that}$$

$$(3.4) \quad \frac{\partial H^{(\mu)}}{\partial s} = \lambda(pz+qw)G^{(\mu)}.$$

Thus, one need only calculate one of the generating functions G or H .

It turns out that H is the more important for our purpose, particularly in the investigation of limiting distributions as $s \rightarrow \infty$; for this reason we concentrate on H .

With only two possible directions of motion, the backward equation for H [Chapter 1, eq. (5.6)] simplifies immediately to the equation

$$(3.5) \quad \left[\frac{\partial}{\partial s} - \mu \frac{\partial}{\partial x} + \lambda(1-qw) \right] H^{(\mu)} = \lambda pz H^{(-\mu)} + \lambda(pz+qw)\delta(X|x),$$

with initial conditions which from (3.3) are $[H^{(\mu)}]_{s=0} = 0, \mu = \pm 1$.

Introducing the Laplace transform

$$\overline{H^{(\mu)}} = \int_0^\infty \exp[-\sigma s] H^{(\mu)} ds,$$

eq. (3.5) becomes

$$(3.6) \quad \left[\sigma + \lambda(1-qw) - \mu \frac{\partial}{\partial x} \right] \overline{H^{(\mu)}} = \lambda p z \overline{H^{(-\mu)}} + \frac{\lambda}{\sigma} (pz+qw) \delta(X|x).$$

From (3.6) we obtain the 2nd order equation

$$(3.7) \quad \left(\frac{\partial^2}{\partial x^2} - \theta^2 \right) \overline{H^{(\mu)}} = - \frac{\lambda}{\sigma} (pz+qw) [\sigma + \lambda(1-qw+pz)] \delta(X|x) ,$$

where

$$(3.8) \quad \theta^2 = [\sigma + \lambda(1-qw+pz)][\sigma + \lambda(1-qw-pz)] ,$$

which is the transform of a nonhomogeneous telegraphist's equation with leakage; an equation of the same type was obtained by Goldstein [24].

Note that because of (3.4), $G^{(\mu)}$ satisfies the homogeneous part of (3.5), and hence the homogeneous telegraphist's equation

$$(3.9) \quad \left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial x^2} + 2\lambda(1-qw) \frac{\partial}{\partial s} + \lambda^2(1-qw+pz)(1-qw-pz) \right] G^{(\mu)} = 0 .$$

It also follows from (3.4) that

$$(3.10) \quad \lambda(pz+qw) \overline{G^{(\mu)}} = \lambda(pz+qw) \int_0^\infty \exp[-\sigma s] G^{(\mu)} ds = \int_0^\infty \exp[-\sigma s] \frac{\partial H^{(\mu)}}{\partial s} ds = \sigma \overline{H^{(\mu)}} .$$

If the motion is restricted by the imposition of boundary conditions such as absorbing or reflecting barriers, for example if there are two such barriers at a_{-1}, a_1 ($a_{-1} < a_1$) then for an initial position $x \in [a_{-1}, a_1]$, one can see that for small s the generating function H appropriate to this process will differ only by terms of order $o(s)$ from the one appropriate to the unrestricted process. Hence H will satisfy the same "backward" equation (3.5). The solution of (3.5) which

satisfies the required boundary conditions will also satisfy the integral equation appropriate to the process with these boundary conditions, and is hence the unique solution of the problem.

The general solution of (3.6) or (3.7) is

$$(3.11) \quad \overline{H^{(\mu)}} = \begin{cases} A_{11}^{(\mu)} \exp[\theta x] + A_{12}^{(\mu)} \exp[-\theta x] , & x \notin X, \\ A_{21}^{(\mu)} \exp[\theta x] + A_{22}^{(\mu)} \exp[-\theta x] + \frac{\lambda}{\sigma} (pz+qw) \cdot [\sigma + \lambda(1-qw-pz)]^{-1} , & x \in X, \end{cases}$$

where the coefficients $A_{ij}^{(\mu)}$ are related via eq. (3.6) by

$$(3.12) \quad \lambda p z A_{ij}^{(-1)} = [\sigma + \lambda(1-qw) + (-1)^j \theta] A_{ij}^{(1)} , \quad i, j = 1, 2,$$

but are otherwise arbitrary functions of σ , z , and w . The solution satisfying required boundary conditions is obtained by appropriately choosing these functions.

4. - The unrestricted process.

In the unrestricted case, it is convenient to express the solutions in terms of cumulative distributions of displacement: i.e., we set $X = (-\infty, y](*)$ in H . The appropriate boundary conditions for $\overline{H^{(\mu)}}$ are here that $\overline{H^{(\mu)}}$ is a bounded function of x as $x \rightarrow \pm \infty$; hence

$$(4.1) \quad \overline{H^{(\mu)}} = \begin{cases} A_{12}^{(\mu)} \exp[-\theta x] , & x > y, \\ A_{21}^{(\mu)} \exp[\theta x] + \frac{\lambda}{\sigma} (pz+qw) \phi^{-1} , & x \leq y, \end{cases}$$

(*) $[a, b]$ denotes a closed interval, $(a, b]$ an interval open to the left and closed to the right.

where

$$(4.2) \quad \phi = \sigma + \lambda(1-qw-pz) \quad .$$

The two further conditions needed to determine $A_{12}^{(\mu)}$, $A_{21}^{(\mu)}$ are obtained by requiring $\overline{H^{(\mu)}}$ to be continuous at $x = y$: i.e., we set

$$(4.3) \quad [\overline{H^{(\mu)}}]_{x=y-} = [\overline{H^{(\mu)}}]_{x=y+} \quad , \quad \mu = 1, -1.$$

Substituting (4.1) in (4.3) and using (3.12), we find that

$$(4.4) \quad \overline{H^{(1)}} = \begin{cases} -\frac{\lambda(pz+qw)}{2\sigma} \exp[\theta(y-x)](\theta^{-1}-\phi^{-1}) \quad , & x > y, \\ -\frac{\lambda(pz+qw)}{2\sigma} \exp[\theta(x-y)](\theta^{-1}+\phi^{-1}) \\ \quad + \frac{\lambda}{\sigma} (pz+qw)\phi^{-1} \quad , & x \leq y; \end{cases}$$

we obtain $\overline{H^{(-1)}}$ from (4.4) by using (3.6) and $\overline{G^{(\mu)}}$, $\mu = \mp 1$, by using (3.10). Thus (4.4) yields a complete and explicit solution (in Laplace transform) of the one-dimensional scattering problem in the unrestricted case.

We see from (4.4) and (3.6) that $\overline{H^{(\mu)}}$ is a continuous function of $\xi = y-x$, and that its density, say $h^{(\mu)}(\xi) = \partial \overline{H^{(\mu)}} / \partial \xi$ is continuous apart from a finite saltus at $\xi = 0$; introducing the Fourier-Stieltjes transform

$$\overline{\phi^{(\mu)}}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] \overline{H^{(\mu)}}(d\xi) = \int_0^{\infty} [\exp[i\alpha\xi] h^{(\mu)}(\xi) + \exp[-i\alpha\xi] h^{(\mu)}(-\xi)] d\xi \quad ,$$

it is easily seen to follow from (4.4) that

$$(4.5) \quad \overline{\phi^{(1)}}(\alpha) = \frac{\lambda}{\sigma} (pz+qw) \left[\frac{i\alpha}{\theta^2+\alpha^2} + \frac{\sigma + \lambda(1-qw+pz)}{\theta^2+\alpha^2} \right] \quad .$$

Clearly, $\overline{\phi^{(\mu)}}(\alpha)$ is the Laplace transform of the Fourier-Stieltjes transform

$$\phi^{(\mu)}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] H^{(\mu)}(d\xi) .$$

Inverting (4.5) we find that

$$(4.6) \quad \phi^{(1)}(\alpha) = \frac{\lambda(pz+qw)[i\alpha+\lambda(1-qw+pz)]}{\lambda^2(1-qw)^2+\eta^2} \\ \cdot \left[1 - \exp[-\lambda(1-qw)s] \cos ns - \frac{\lambda(1-qw)}{\eta} \exp[-\lambda(1-qw)s] \sin ns \right] \\ + \frac{\lambda(pz+qw)}{\eta} \exp[-\lambda(1-qw)s] \sin ns,$$

where $\eta^2 = \alpha^2 - (\lambda zp)^2$.

We now turn our attention to the limiting distributions as $s \rightarrow \infty$: for the unrestricted process, it is easy to see that $G^{(\mu)} \rightarrow 0$; on the other hand, $\lim_{s \rightarrow \infty} Q_{r,n-r}$ will yield the probability distribution of the position at which the n -th collision with r reversals occurs, and hence $K^{(\mu)} = \lim_{s \rightarrow \infty} H^{(\mu)}$ will yield the generating function for these distributions. From these distributions we can find in the way outlined above the joint distributions of the n -th collision position together with the absolute velocity and direction of motion resulting from this collision. Since $K^{(\mu)} = \lim_{\sigma \rightarrow 0} \overline{\sigma H^{(\mu)}}$, we find from (4.4) that

$$(4.7) \quad K^{(1)} = -\frac{1}{2}\lambda(pz+qw) \exp[-\gamma|y-x|][\gamma^{-1} + \operatorname{sgn}(y-x)\lambda^{-1}(1-qw-pz)]$$

where

$$(4.8) \quad \gamma^2 = \lambda^2(1-qw-pz)(1-qw+pz) .$$

Clearly, the Fourier-Stieltjes transform

$$\psi^{(\mu)}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] K^{(\mu)}(d\xi) = \lim_{s \rightarrow \infty} \phi^{(\mu)}(\alpha) ,$$

and hence

$$(4.9) \quad \psi^{(1)}(\alpha) = \lambda(pz+qw)[i\alpha+\lambda(1-qw+pz)][\lambda^2(1-qw)^2+\eta^2]^{-1} .$$

Equation (4.9) is a special case of a more general result which is derived in Chapter 5, Sect. 2.

Placzek [37] considers the unrestricted process in which $p = q = \frac{1}{2}$ and derives an expression for $\psi_0(y,u)$, where $\psi_0(y,u)dydu$ is the number of neutrons per unit time scattered into the position-lethargy element $dy \times du$. (Lethargy is defined as $\log E_0/E$ where E is the energy and E_0 the initial energy of the particle.) He assumes that the lethargy increases by a fixed amount, c , per collision and that there is an isotropic source at the origin emitting neutrons of lethargy 0 at the rate of one per second.

$\psi_0(y,u)$ in this case can be deduced from eq. (4.4) as follows: assuming neutrons are emitted at the origin (i.e., $x = 0$) with equal probabilities of moving initially to the right or left, it follows that the generating function of the probability density $f_n(y)$ for the point at which a neutron suffers its n -th collision is given by

$$(4.10) \quad \sum_{n=1}^{\infty} f_n(y)z^n = \lim_{s \rightarrow \infty} \frac{1}{2} \left[\frac{\partial H^{(1)}}{\partial y} + \frac{\partial H^{(-1)}}{\partial y} \right] = \lim_{\sigma \rightarrow 0} \frac{\sigma}{2} \left[\frac{\partial \overline{H}^{(1)}}{\partial y} + \frac{\partial \overline{H}^{(-1)}}{\partial y} \right] ,$$

i.e.,

$$(4.11) \quad \sum_{n=1}^{\infty} f_n(y)z^n = \frac{z}{2\sqrt{1-z}} \exp[-\sqrt{1-z}|y|]$$

(taking the mean free path as the unit of length).

Since the increase in lethargy per collision is constant for each collision, it follows that

$$(4.12) \quad \psi_0(y, u) = \delta(u - nc) f_n(y), \quad n = 1, 2, \dots$$

From eq. (4.11) $f_n(y)$ can be expressed as a contour integral,

$$(4.13) \quad f_n(y) = \int_C \frac{1}{2z^n \sqrt{1-z}} \exp[-\sqrt{1-z}|y|] dz, \quad n = 1, 2, \dots,$$

where C is the contour illustrated in Fig. 1.

Changing the variable of integration to w , where $z = 1 + w^2$, and using eq. (4.12), we obtain the result

$$(4.14) \quad \psi_0(y, u) = \delta(u - nc) \frac{1}{\pi} \int_0^\infty \frac{\cos yw}{(1+w^2)^n} dw$$

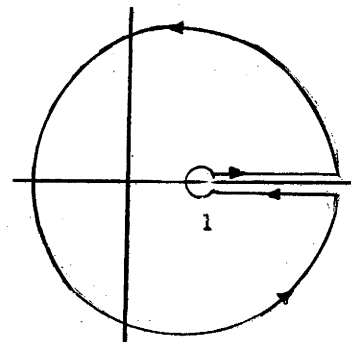


Fig. 1

which was originally obtained by Placzek. For a discussion of the behaviour of the function $\psi_0(y, u)$ we refer the reader to [37].

5. - The process with absorbing boundaries.

Consider now the one-dimensional scattering process with absorbing barriers at a_{-1}, a_1 ($a_{-1} < a_1$). As mentioned in Chapter 1, Sect. 4, these boundary conditions have to be incorporated in the expressions for the basic probability distributions P_0, Q_{10}, Q_{01} , and the *-operation. In the present case, all distributions of interest may be derived from the generating function H restricted by the condition that all collisions prior to the last one occur in the interval $[a_{-1}, a_1]$. This condition is satisfied if we choose $H_1^{(\mu)}$ as in (3.3) and restrict integration with respect to x in the *-operation to the interval

$[a_{-1}, a_1]$. As mentioned in Sect. 3, $\overline{H^{(\mu)}}$ will then satisfy the "backward" eq. (3.6) for initial positions $x \in [a_{-1}, a_1]$. Take $X = (a_{-1}, y]$, where $a_{-1} < y \leq a_1$; then the boundary conditions that must be imposed on the general solution (3.11) are

$$(5.1) \quad \overline{H^{(\mu)}}_{x=a_{-1}} = 0; \quad \overline{H^{(\mu)}}_{x=y+} = \overline{H^{(\mu)}}_{x=y-}; \quad \mu = 1, -1.$$

It is convenient in this case to rewrite (3.11) in the form

$$(5.2) \quad \overline{H^{(\mu)}} = \begin{cases} B_{11}^{(\mu)} \sinh \theta(a_1 - x) + B_{12}^{(\mu)} \cosh \theta(a_1 - x), & a_{-1} < y \leq x \leq a_1, \\ B_{21}^{(\mu)} \sinh \theta(y - x) + B_{22}^{(\mu)} \cosh \theta(y - x) \\ \quad + \frac{\lambda}{\sigma} (pz + qw) \phi^{-1}, & a_{-1} \leq x \leq y \leq a_1, \end{cases}$$

where ϕ was defined in (4.2). Substituting (5.2) in (5.1) and using (3.6) to derive the relations between $B_{ij}^{(1)}$ and $B_{ij}^{(-1)}$, we find the coefficients $B_{ij}^{(\mu)}$ and hence

$$(5.3) \quad \overline{H^{(1)}} = \begin{cases} \frac{\lambda(pz + qw)}{\sigma \sinh \theta \ell} \frac{\sinh \theta(a_1 - x)}{\sinh \theta(a_1 - y)} [\phi^{-1} \sinh \theta \ell - \alpha \sinh \theta(a_1 - y) \\ \quad - \beta \sinh \theta(y - a_{-1})], & a_{-1} < y \leq x \leq a_1, \\ \frac{\lambda(pz + qw)}{\sigma \sinh \theta \ell} [\phi^{-1} \sinh \theta \ell - \alpha \sinh \theta(a_1 - x) \\ \quad - \beta \sinh \theta(x - a_{-1})], & a_{-1} \leq x \leq y \leq a_1, \end{cases}$$

where

$$(5.4) \quad \begin{cases} \ell = a_1 - a_{-1}; & \alpha = \lambda p z \phi^{-1} [\theta + \sigma + \lambda(1 - qw)]^{-1}, \\ \beta = \theta^{-1} \sinh \theta(a_1 - y) - \phi^{-1} \cosh \theta(a_1 - y). \end{cases}$$

Take $X = (a_1, y]$, where $a_1 < y < \infty$; in this case the boundary conditions are

$$(5.5) \quad \overline{[H^{(1)}]}_{x=a_1} = \frac{\lambda(pz+qw)}{\sigma(\sigma+\lambda)} [1 - \exp[-(\sigma+\lambda)(y-a_1)]] , \quad \overline{[H^{(-1)}]}_{x=a_{-1}} = 0.$$

The first of these conditions results from the observation that if initially $\mu = 1$, $x = a_1$ and if $X = (a_1, y]$ with $y > a_1$, then the first collision occurs outside $[a_{-1}, a_1]$; hence $H^{(1)} = H_1^{(1)}$, and therefore

$$\overline{H^{(1)}} = \overline{H_1^{(1)}} = (pz+qw) \int_0^\infty \exp[-\sigma s] ds \int_0^s \delta[(a_1, y] | a_1 + \tau] \exp[-\lambda \tau] \lambda d\tau =$$

$$\frac{\lambda}{\sigma} (pz+qw) \int_0^{y-a_1} \exp[-(\sigma+\lambda)s] ds .$$

Substituting the first expression of (3.11) in (5.5) and using (3.12) we find $A_{1i}^{(\mu)}$, $i = 1, 2$, and hence

$$(5.6) \quad \overline{H^{(1)}} = \frac{\lambda(pz+qw)}{\sigma(\lambda+\sigma)} \frac{[\sigma+\lambda(1-qw)] \cosh \theta(x-a_{-1}) + \theta \sinh \theta(x-a_{-1})}{[\sigma+\lambda(1-qw)] \cosh \theta l + \theta \sinh \theta l}$$

$$\cdot (1 - \exp[-(\sigma+\lambda)(y-a_1)]) .$$

The expression for $\overline{H^{(\mu)}}$ when $X = [y, a_{-1})$ is obtained by symmetry from (5.6):

$$(5.7) \quad \overline{H^{(\mu)}}[[y, a_{-1}) | x; \sigma, a_1, a_{-1}] = \overline{H^{(-\mu)}}[(-a_{-1}, -y] | -x; \sigma, -a_{-1}, -a_1] .$$

The expression for $\overline{H^{(\mu)}}$ when $X = (-\infty, y]$ is a linear combination of the solutions (5.3), (5.6), and (5.7).

The generating function for the limiting distribution as $s \rightarrow \infty$

$$(5.8) \quad K^{(\mu)} = \lim_{s \rightarrow \infty} H^{(\mu)} = \lim_{\sigma \rightarrow 0} \overline{\sigma H^{(\mu)}}$$

follows directly from (5.3), (5.6) and (5.7) and yields the probability distribution of the position at which the n -th collision occurs with the previous $n-1$ all taking place in $[a_{-1}, a_1]$ (see the remarks following eq. (4.6)). If we write $K^{(\mu)} = \sum_{n=1}^{\infty} K_n^{(\mu)}$, where $K_n^{(\mu)} = \lim_{s \rightarrow \infty} H_n^{(\mu)}$, then the iteration relation (2.7) takes a simple form:

$$(5.9) \quad K_{n+1}^{(\mu)}(X|x; z, w) = \int_0^{|x-a_{\mu}|} \{p_z K_n^{(-\mu)}(X|x + \mu s; z, w) + q_w K_n^{(\mu)}(X|x + \mu s; z, w)\} \exp[-\lambda s] \lambda ds;$$

the use of these iterations to calculate an explicit expression for $K^{(\mu)}$ is not prohibitively complicated in this case.

6. - First passage distributions.

In the present Section, we wish to evaluate the probability of a first passage at the barriers $a_{\pm 1}$, together with the velocity and direction of motion distribution at this passage. We can do this using the method outlined at the beginning of Sect. 3 if we know the sequence of probabilities $P_{rk}^{(\mu)}(a_v|x;s)$ that a particle initially at x with direction μ effect a first passage at a_v in a length of path $\leq s$ after suffering exactly r reversals and k continuations in $[a_{-1}, a_1]$. Writing as before P_0 for P_{00} , we can see that $P_0^{(\mu)}(a_v|x;s)$ is the sum of two terms, namely, the probabilities that in path length s the particle (initially at x with direction μ) effect a first passage at a_v with no collisions, and that it suffer its first collision in path length $\leq s$ but after having passed through a_v ; thus

$$(6.1) \quad P_0^{(\mu)}(a_v | x; s) = \delta[(a_v, v^\infty) | x + \mu s] \exp[-\lambda s] + \int_0^s \delta[(a_v, v^\infty) | x + \mu \tau] \exp[-\lambda \tau] \lambda d\tau = \left(\frac{1}{\lambda} \frac{\partial}{\partial s} + 1 \right) Q_1^{(\mu)}[(a_v, v^\infty), s | x] .$$

Hence, taking Laplace transforms,

$$(6.2) \quad \overline{P_0^{(\mu)}}(a_v | x; \sigma) = \left(1 + \frac{\sigma}{\lambda} \right) \overline{Q_1^{(\mu)}}[(a_v, v^\infty) | x; \sigma] = \\ = \left(1 + \frac{\sigma}{\lambda} \right) (pz + qw)^{-1} \overline{H_1^{(\mu)}}[(a_v, v^\infty) | x; \sigma, z, w] .$$

In accordance with the theory developed in Sect. 2 and eq. (2.7), we see that the generating function

$$(6.3) \quad G^{(\mu)} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk}^{(\mu)} = P_0^{(\mu)} * [I + H^{(\mu)}] = \\ = (pz + qw)^{-1} \left(1 + \frac{1}{\lambda} \frac{\partial}{\partial s} \right) H^{(\mu)} ,$$

where $H^{(\mu)}$ is the generating function obtained in Sect. 5. Taking Laplace transforms, we find that

$$(6.4) \quad \overline{G^{(\mu)}}(a_v | x; \sigma, z, w) = (pz + qw)^{-1} \left(1 + \frac{\sigma}{\lambda} \right) \overline{H^{(\mu)}}[(a_v, v^\infty) | x; \sigma, z, w] .$$

In this type of problem, one is mainly interested in the limiting first passage distributions as $s \rightarrow \infty$. Let $M^{(\mu)} = \lim_{s \rightarrow \infty} G^{(\mu)} = \lim_{\sigma \rightarrow 0} \sigma \overline{G^{(\mu)}}$; then clearly

$$(6.5) \quad M^{(\mu)}(a_v | x; z, w) = (pz + qw)^{-1} K^{(\mu)}[(a_v, v^\infty) | x; z, w] ,$$

where $K^{(\mu)}$ is the limiting generating function defined in (5.8).

Explicitly,

$$(6.6) \quad \begin{cases} M^{(1)}(a_1|x;z,w) = \frac{\lambda(1-qw) \sinh \gamma(x-a_{-1}) + \gamma \cosh \gamma(x-a_{-1})}{\lambda(1-qw) \sinh \gamma \ell + \gamma \cosh \gamma \ell} \\ M^{(-1)}(a_1|x;z,w) = \frac{\lambda p z \sinh \gamma(x-a_{-1})}{\lambda(1-qw) \sinh \gamma \ell + \gamma \cosh \gamma \ell} \end{cases}$$

where $\ell = a_1 - a_{-1}$ and γ was defined in (4.8). The expressions for $M^{(\mu)}(a_{-1})$ may be obtained by symmetry from (6.6), as in (5.7).

Letting z and w tend to 1 in (6.6), we obtain the probability $M^{(\mu)}(a_1|x)$ that the particle first emerges from $[a_{-1}, a_1]$ at a_1 (regardless of the number of collisions which occur before emergence). Thus

$$(6.7) \quad \begin{cases} M^{(1)}(a_1|x) = [1 + (x-a_{-1})\lambda p][1 + (a_1-a_{-1})\lambda p]^{-1}, \\ M^{(-1)}(a_1|x) = (x-a_{-1})\lambda p[1 + (a_1-a_{-1})\lambda p]^{-1}. \end{cases}$$

We notice that these are consistent with the general results found by Mycielski and Paszkowski [33] using invariance arguments.

Under postulate (3) of Sect. 1, i.e., that the velocity transition probabilities satisfy the commutativity condition $\psi_\pi \circ \psi_o = \psi_o \circ \psi_\pi$, we can use eq. (3.2) together with the first passage distributions given by (6.6) to determine the distribution of the velocity with which the particle first emerges from the interval $[a_{-1}, a_1]$.

The probability of emerging first at a_v (after any number of previous collisions) is given by $M^{(\mu)}(a_v|x)$. Hence, using (3.2), the distribution of emergent velocity conditional on the initial state (u, μ, x) and first emergence at a_v is given by

$$(6.8) \quad F_{a_v}(V|u, \mu, x) = [M^{(\mu)}(a_v|x)]^{-1} \sum_{i+j \geq 0} P_{ij}^{(\mu)}(a_v|x; \infty) \psi_\pi^{(i)} \circ \psi_o^{(j)}.$$

In principle eq. (6.8) gives the required distribution of emergent velocity. For functions ψ_π , ψ_o of certain forms the equation simplifies considerably, leading to more manageable solutions. We shall now give two such examples.

Example 1.

$$\psi_\pi(dv|u) = \frac{1}{u} f_p\left(\frac{v}{u}\right) dv, \quad 0 \leq v \leq u,$$

$$\psi_o(dv|u) = \frac{1}{u} f_q\left(\frac{v}{u}\right) dv, \quad 0 \leq v \leq u.$$

In this case it is convenient to introduce the Mellin transforms

$$\bar{\psi}_\pi(\alpha) = \int_{v=0}^u f_p\left(\frac{v}{u}\right) \left(\frac{v}{u}\right)^{\alpha-1} \frac{dv}{u},$$

$$\bar{\psi}_o(\alpha) = \int_{v=0}^u f_q\left(\frac{v}{u}\right) \left(\frac{v}{u}\right)^{\alpha-1} \frac{dv}{u},$$

$$\mathcal{F}_{a_v}(\alpha|u, \mu, x) = \int_{v=0}^u \left(\frac{v}{u}\right)^{\alpha-1} F_{a_v}(dv|u, \mu, x).$$

Then, taking Mellin transforms in eq. (6.8) we have

$$(6.9) \quad \mathcal{F}_{a_v}(\alpha|u, \mu, x) = [M^{(\mu)}(a_v|x)]^{-1} M^{(\mu)}(a_v|x; \bar{\psi}_\pi(\alpha), \bar{\psi}_o(\alpha)),$$

where the generating functions $M^{(\mu)}(a_v|x; z, w)$ are obtained from eq. (6.6).

The moments of the distribution of emergent velocity are easily found from eq. (6.9); for example, the n -th moment is given by

$$(6.10) \quad \mathcal{G}_{a_v}(v^n|u, \mu, x) = u^n [M^{(\mu)}(a_v|x)]^{-1} M^{(\mu)}(a_v|x; \bar{\psi}_\pi(n+1), \bar{\psi}_o(n+1)).$$

Example 2.

$$\psi_{\pi}(dv|u) = \delta(v-c_1 u) dv, \quad 0 \leq c_1 \leq 1,$$

$$\psi_0(dv|u) = \delta(v-c_2 u) dv, \quad 0 \leq c_2 \leq 1.$$

This is the case in which, as a result of a collision, the logarithmic energy of the particle decreases by a fixed amount depending only on the angle of deflection. The emergent distribution, as distinct from the previous example, is now discrete. If the particle, starting with velocity u , makes i collisions with reversals and j collisions without reversals, then the resulting velocity is given by

$$v = u c_1^i c_2^j.$$

Taking Mellin transforms as in Example 1, again defining

$$\mathcal{F}_{a_v}(\alpha|u, \mu, x) = \int_{v=0}^u \left(\frac{v}{u}\right)^{\alpha-1} F_{a_v}(dv|u, \mu, x).$$

we find that

$$(6.11) \quad \mathcal{F}_{a_v}(\alpha|u, \mu, x) = [M^{(\mu)}(a_v|x)]^{-1} M^{(\mu)}(a_v|x; c_1^{\alpha-1}, c_2^{\alpha-1}).$$

7. - Higher-order passage distributions and further boundary conditions.

The results of Sect. 6 can be generalized to give passage distributions for any prescribed sequence of passages through points on $(-\infty, \infty)$. Let us take two points a_1, a_{-1} ($-\infty < a_{-1} < a_1 < \infty$). These divide the real line into three regions $(-\infty, a_{-1})$, $[a_{-1}, a_1]$, (a_1, ∞) which we shall denote by L_{-1}, L_0, L_1 , respectively. An arbitrary sequence of passages through the points a_1, a_{-1} can then be specified by a sequence of

numbers $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ where $\gamma_j = -1, 0$ or 1 . This sequence indicates that the particle starts in L_{γ_0} , makes a first passage from L_{γ_0} to L_{γ_1} , thence to L_{γ_2} , thence to L_{γ_3} and eventually to L_{γ_n} . Clearly $\{\gamma_0, \dots, \gamma_n\}$ must be a nearest neighbour sequence, i.e., $|\gamma_j - \gamma_{j-1}| = 1$, $j \geq 1$. For example, the sequence $\{0, 1, 0, 1, 0, -1, 0\}$ is of the required type.

We introduce the following generalization of the function $M^{(\mu)}(a_\nu | x; z, w)$ defined by equation (6.5). Let $P_{rk}^{\gamma_0 \gamma_1 \dots \gamma_n}(x, \mu)$ be the probability that a particle which starts at x with direction μ ($-\infty < x < \infty$, $\mu = \pm 1$) makes the sequence of first passages corresponding to the sequence $\{\gamma_0, \dots, \gamma_n\}$ and experiences exactly r reversals and k continuations before making the last of these from $L_{\gamma_{n-1}}$ to L_{γ_n} . Then we define

$$M_{\gamma_0 \dots \gamma_n}^{(\mu)}(x; z, w) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} P_{rk}^{\gamma_0 \dots \gamma_n}(x, \mu) z^r w^k, \quad |z| \leq 1, |w| \leq 1.$$

Clearly the passage $L_{\gamma_{n-1}} \rightarrow L_{\gamma_n}$ must occur through $a_{\gamma_{n-1} + \gamma_n}$. Furthermore

$$(7.1) \begin{cases} M_{0,1}^{(\mu)}(x; z, w) = M^{(\mu)}(a_1 | x; z, w), \\ M_{0,-1}^{(\mu)}(x; z, w) = M^{(\mu)}(a_{-1} | x; z, w), \end{cases}$$

where the expressions on the right-hand side are found directly from eqs. (6.6). The functions $M_{\pm 1,0}^{(\mu)}(x; z, w)$ can also be found from (6.6) by letting $a_1 \rightarrow \infty$ or $a_{-1} \rightarrow -\infty$ as required. Thus for example, letting a_{-1} tend to $-\infty$ in the first of eqs. (6.6) and then replacing a_1 by

a_{-1} , we find that

$$(7.2) \quad M_{-1,0}^{(1)}(x; z, w) = \exp[-\gamma(a_{-1}-x)] .$$

The higher order passage probabilities are then determined recursively from the relation

$$(7.3) \quad M_{\gamma_0 \dots \gamma_n}^{(\mu)}(x; z, w) = M_{\gamma_{n-1} \gamma_n}^{(\gamma_{n-1} - \gamma_{n-2})}(a_{\gamma_{n-1} + \gamma_{n-2}}; z, w) M_{\gamma_0 \dots \gamma_{n-1}}^{(\mu)}(x; z, w) .$$

Suppose now that the particle moves in the interval $(-\infty, a_{-1}]$ and that there is a reflecting barrier at a_{-1} , i.e., when the particle strikes the barrier at a_{-1} its direction of motion is reversed (reflection is not counted as a collision) and the process continues. The argument we are about to use extends in an obvious way to the problem of scattering with two reflecting barriers. Let X be a subinterval of $(-\infty, a_{-1}]$ and let $R_{rk}^{(\mu)}(X|x)$ be the probability, conditional on initial position and direction (x, μ) , that the particle makes its $(r+k)^{\text{th}}$ collision (r of these being reversals and k being continuations) in the interval X . Then the generating function,

$$R^{(\mu)}(X|x) = \sum_{r+k \geq 1} R_{rk}^{(\mu)}(X|x) z^r w^k, \text{ is given by}$$

$$(7.4) \quad R^{(\mu)}(X|x) = K_{\infty}^{(\mu)}(X|x) + \sum_{n=0}^{\infty} M_{\infty}^{(\mu)}(a_{-1}|x; z, w)$$

$$K_{\infty}^{(-1)}(X|a_{-1}) \left[M_{\infty}^{(-1)}(a_{-1}|a_{-1}; z, w) \right]^n ,$$

where $K_{\infty}^{(\mu)}(X|x)$ is the analogue for the half-line $(-\infty, a_{-1})$ of the

generating function defined by eq. (5.8) for the interval $[a_{-1}, a_1]$.

Thus

$$(7.5) \quad K_{\infty}^{(\mu)}(X|x) = \lim_{\sigma \rightarrow 0} \overline{H^{(\mu)}}(X|x; \sigma, a_{-1}, -\infty),$$

where $\overline{H^{(\mu)}}$ is defined by eqs. (5.6) and (5.7). Similarly $M_{\infty}^{(\mu)}(a_{-1}|x; z, w)$ is obtained from (6.6) upon letting a_{-1} tend to $-\infty$ and then replacing a_1 by a_{-1} . The n^{th} term on the right-hand side of (7.4) corresponds to the particle being reflected $(n-1)$ times at the barrier before making its $(r+k)^{\text{th}}$ collision in X . Equation (7.4) can be written more simply as

$$(7.6) \quad R^{(\mu)}(X|x) = M_{\infty}^{(\mu)}(a_{-1}|x; z, w) K_{\infty}^{(-1)}(X|a_{-1}) \left[1 - M_{\infty}^{(-1)}(a_{-1}|a_{-1}; z, w) \right]^{-1} + K_{\infty}^{(\mu)}(X|x).$$

We observe finally that the same argument can also be applied to scattering problems on a line which is made up of a number of adjacent homogeneous segments. At each passage through an interface the particle, instead of being reflected, passes into a different segment in which it may have a different mean free path.

8. - The steady-state distribution of particles in a finite interval.

For an interval of length ℓ we define the following probabilities:

p_ℓ = Prob {a particle incident at one end emerges for the first time from the opposite end};

p'_ℓ = Prob {a particle incident at one end emerges for the first time from the same end}.

From eq. (6.7) we have immediately that

$$(8.1) \quad p_\ell = [1 + \lambda p \ell]^{-1}, \quad p'_\ell = \lambda p [1 + \lambda p \ell]^{-1}.$$

Suppose that N particles per second are incident upon the left-hand end of the interval $[a, b]$ and suppose that the system is in a steady state. Then the expected number of particles per second passing to the right through any point x in the interval for the k -th time is given by

$$(8.2) \quad \mathcal{E}_{n_{kR}} = N p_{x-a} [p'_{b-x} p'_{x-a}]^k.$$

Hence the total expected number of particles per second passing to the right through x ,

$$(8.3) \quad \mathcal{E}_{n_R} = \sum_{k=1}^{\infty} \mathcal{E}_{n_{kR}} = N p_{x-a} [1 - p'_{b-x} p'_{x-a}]^{-1}.$$

Similarly the expected number of particles per second passing to the left through x ,

$$(8.4) \quad \mathcal{E}_{n_L} = N p_{x-a} p'_{b-x} [1 - p'_{b-x} p'_{x-a}]^{-1}.$$

Substituting for p_ℓ , p'_ℓ from eq. (8.1) gives

$$(8.5) \quad \begin{cases} \xi_{n_L} = N(b-x)\lambda p[1 + (b-a)\lambda p]^{-1} \\ \xi_{n_R} = N(1 + (b-x)\lambda p)(1 + (b-a)\lambda p)^{-1} \end{cases}$$

9. - The effect of absorption.

Consider now the effect of introducing a third possible outcome at each collision; namely, that the particle is absorbed. The probabilities of reversal and continuation will still be denoted by p and q respectively; however, $(p+q)$ is now less than 1. The probability of absorption, given a collision, will be denoted by r where $r = 1-p-q$.

Once a particle is absorbed, its motion ceases completely. To the set of all possible states $\mathcal{J}$ in the pure scattering process, we must therefore adjoin the family of states $A(\omega)$, a particle being in state $A(\omega)$ if it has been absorbed and if its state immediately before absorption was ω where $\omega \in \mathcal{J}$. The enlarged state space we shall denote by $\mathcal{J}^*$.

The transition probabilities $P_{rk}(S|\omega;s)$ and $Q_{rk}(S|\omega;s)$ are then defined exactly as in Sect. 2; it should be observed that $Q_{rk}(S^*|\omega;s)$ is zero for all sets S^* exterior to $\mathcal{J}$ since it is impossible for the particle to be absorbed at the $(r+k)^{\text{th}}$ collision if r of the collisions were reversals and the remaining k were continuations. We observe also that $\left[1 - \sum_{r=0}^n \int_{\mathcal{J}} Q_{r,n-r}(d\eta|\omega;\infty)\right]$ is the probability that a particle initially in state ω is absorbed before making n collisions.

The fundamental equations (2.9) and (2.10) are unchanged by the possibility of absorption at a collision. In fact, the only results

which are no longer valid are those in whose derivation the fact that $p+q = 1$ has been used. These are equations (4.11) to (4.14) which can easily be modified if $p+q < 1$ (the density function $f_n(y)$ will then no longer be normalized to unity) and equations (6.7) which become

$$(9.1) \quad \left\{ \begin{array}{l} M^{(1)}(a_1|x) = \frac{(1-q) \sinh(x-a_{-1})\lambda\sqrt{rs} + \sqrt{rs} \cosh\lambda(x-a_{-1})\sqrt{rs}}{(1-q) \sinh\lambda t \sqrt{rs} + \sqrt{rs} \cosh\lambda t \sqrt{rs}}, \\ M^{(-1)}(a_1|x) = \frac{p \sinh(x-a_{-1})\lambda\sqrt{rs}}{(1-q) \sinh\lambda t \sqrt{rs} + \sqrt{rs} \cosh\lambda t \sqrt{rs}}, \end{array} \right.$$

where $t = a_1 - a_{-1}$, $r = 1-p-q$, and $s = 1-q+p$.

Equation (8.1) which was obtained directly from (6.7) must also be changed and the results of Sect. 8 modified accordingly.

It will sometimes be of interest to know the probabilities

$P_{mk}(S|\omega;s)$ and $Q_{mk}(S|\omega;s)$ conditional upon no absorption in the first $(m+k)$ collisions. These conditional probabilities are simply

$P_{mk}(1-r)^{-m-k}$ and $Q_{mk}(1-r)^{-m-k}$; the corresponding generating functions are therefore obtained by replacing z by $z/(1-r)$ and w by $w/(1-r)$ in the generating functions G and H respectively.

CHAPTER 3

THE APPLICATION OF A SIMPLE ONE-DIMENSIONAL PROCESS IN BUBBLE CHAMBER EXPERIMENTS

1. - Introduction.

The material of this chapter is concerned with the process of Chapter 2 in the special case where there is no absorption or reversal, i.e., $p = r = 0$ and $q = 1$. Nevertheless, this process is of considerable interest in physics partly because it occurs in the formation of bubble chamber tracks; the theory of such processes can be used to estimate the velocity of particles passing through the chamber.

Consider a particle which moves along the half-line $[0, \infty)$ in such a way that it has probability $\lambda \delta u + o(\delta u)$ of forming a bubble in any small element $(u, u + \delta u)$ of its track (λ is constant). Following bubble formation, the particle continues moving in the same direction and generates further bubbles in the same way. The particle is observed only in the line segment $0 \leq u \leq x$, and successive bubbles are formed at $X_1, X_2, \dots, X_n$ ($0 \leq X_1 \leq X_2 \leq \dots \leq X_n \leq x$) where $X_1, \dots, X_n$ are random variables, and the number, n , of bubbles in $[0, x]$ is also random (having a Poisson distribution). The distances

$$Y_1 = X_1,$$

$$Y_i = X_i - X_{i-1}, \quad 2 \leq i \leq n,$$

constitute a finite univariate population process as defined by Moyal [32], the individuals being the distances Y_i , each with state space $[0, x]$.

These distances Y_i between successive bubbles are used to estimate the mean free path, λ^{-1} , and hence the velocity of the particle whose

track is observed. Owing to the non-zero diameter, d , of the bubbles, only those distances Y_i which are greater than the bubble diameter are observed. These are then used to estimate the mean free path by equating the total number of observed distances to the expected number. Such a procedure requires knowledge of the distribution of the number of distances Y_i which are greater than the bubble diameter d .

It is the purpose of this chapter to give a complete description of the population process generated by the Y_i . We shall determine the probability generating functional for the process and use this to find explicit expressions for the k^{th} order factorial moment densities and the distribution of the number of distances Y_i such that $Y_i > d$.

We shall assume throughout that the mean free path λ^{-1} for bubble formation is unity. This implies no loss of generality, being equivalent to measuring all distances in units of λ^{-1} .

2. The probability generating functional and factorial moment densities.

The process may be described as a population process in the following way: the state space of each individual is $[0, x]$ and corresponding to n bubbles with successive mutual distances $Y_1, \dots, Y_n$ we have n individuals with states $Y_1, \dots, Y_n$ in $[0, x]$; the process may then be characterized by the integral-valued random function N on a class of subsets of $[0, x]$, where $N(A|Y_1, \dots, Y_n)$ is the number of individuals in the subset A corresponding to a total of n individuals with states $Y_1, \dots, Y_n$; i.e., $N(A|Y_1, \dots, Y_n) = \sum_{i=1}^n \delta(A|Y_i)$ where $\delta(A|Y) = 1$ if $Y \in A$ and $\delta(A|Y) = 0$ if $Y \notin A$.

We define the integral with respect to N of any measurable function θ on the individual state space $\mathcal{X}(=[0,x])$ to be the random variable

$$\begin{aligned} \int_{\mathcal{X}} \theta(u) N(du|Y_1, \dots, Y_n) &= \sum_{i=1}^n \int_{\mathcal{X}} \theta(u) \delta(du|Y_i) \\ &= \sum_{i=1}^n \theta(Y_i). \end{aligned}$$

The probability generating functional for the process is defined, for any bounded measurable function ξ on $\mathcal{X}$ as the expectation

$$\begin{aligned} G[\xi] &= \mathcal{E} \xi(Y_1) \xi(Y_2), \dots, \xi(Y_n) \\ &= \mathcal{E} \exp \int_{\mathcal{X}} \log \xi(u) N(du|Y_1, \dots, Y_n). \end{aligned}$$

Consideration of the point at which the first bubble in $[0,x]$ occurs leads to the following equation for G :

$$(2.1) \quad G[\xi, x] = e^{-x} + \int_0^x \xi(y) G[\xi, x-y] e^{-y} dy.$$

Taking Laplace transforms with respect to x and defining

$$\begin{aligned} H[\xi, s] &= \int_0^\infty e^{-su} G[\xi, u] du, \\ \phi(s) &= \int_0^\infty e^{-su} \xi(u) du, \end{aligned}$$

we find from equation (2.1) that

$$(2.2) \quad H[\xi, s] = (s+1)^{-1} [1 - \phi(s+1)]^{-1}.$$

It has been assumed here that $\xi(u)$ is defined independently of x for all non-negative values of u and in such a way that $\phi(s)$ exists for some s with positive real part. These restrictions are not severe; since for a given x any bounded measurable function ξ on $[0, x]$ can be extended to a function on $[0, \infty)$ having the required properties.

The Laplace transform $H[\xi, s]$ of equation (2.2) can be inverted to give the solution

$$\begin{aligned}
 (2.3) \quad G[\xi, x] &= \sum_{n=0}^{\infty} [f * g^{(n)}](x) \\
 &= e^{-x} \left[1 + \sum_{n=1}^{\infty} \int_{u_n=0}^x \int_{u_{n-1}=0}^{u_n} \cdots \int_{u_1=0}^{u_2} \xi(u_n - u_{n-1}) \cdots \right. \\
 &\quad \left. \xi(u_2 - u_1) \xi(u_1) du_1 \cdots du_n \right],
 \end{aligned}$$

where

$$f(x) = e^{-x},$$

$$g(x) = e^{-x} \xi(x),$$

$$[\phi * \psi](x) = \int_0^x \phi(x-y) \psi(y) dy,$$

$$g^{(n)}(x) = [g * g^{(n-1)}](x), \quad n = 1, 2, \dots,$$

and

$$g^{(0)}(x) = \delta(x), \text{ the Dirac delta function.}$$

The solution (2.3) clearly converges for all functions ξ which are bounded and measurable on $\mathcal{X}$.

The k^{th} order factorial moment distribution $M_{(k)}(A_1, \dots, A_k; x)$ is defined for any family of measurable disjoint subsets $A_1, \dots, A_k$ of the individual state space $\mathcal{X}$ by

$$(2.4) \quad M_{(k)}(A_1, \dots, A_k; x) = \mathcal{E} N(A_1) N(A_2) \dots N(A_k)$$

It is shown by Moyal [32] that, provided the factorial moment distributions exist, they are related to the probability generating functional by

$$(2.5) \quad \int_{\mathcal{X}^k} \xi_1(u_1) \dots \xi_k(u_k) M_{(k)}(du_1, du_2, \dots, du_k; x) = \lim_{n \rightarrow 1} \delta_{\xi_1 \dots \xi_k}^k G[n, x],$$

where $\delta_{\xi_1 \dots \xi_k}^k G[n]$ is the k^{th} order variation of G defined by

$$\delta_{\xi_1 \dots \xi_k}^k G[n] = \left\{ \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} G \left[n + \sum_{i=1}^k \lambda_i \xi_i \right] \right\}_{\lambda_1 = \dots = \lambda_k = 0},$$

where $\sup_{x \in \mathcal{X}} |\eta(x)| < 1$ and $\lambda_1, \dots, \lambda_k$ are complex variables.

The corresponding factorial moment densities are defined by

$$(2.6) \quad m_{(k)}(u_1, \dots, u_k; x) du_1 \dots du_k = M_{(k)}(du_1, \dots, du_k; x).$$

Taking Laplace transforms of equation (2.5) with respect to x , choosing

$$\xi_i(u) = \begin{cases} \mu \exp[-\mu(x-u)] & , \quad x \geq u_i, \\ 0 & , \quad x < u_i, \end{cases}$$

$$\eta(u) = c < 1,$$

and letting $c \rightarrow 1$, $\mu \rightarrow \infty$ we find that the Laplace transform of

$m_{(k)}(u_1, \dots, u_k; x)$ is given by

$$(2.7) \quad \int_0^\infty m_{(k)}(u_1, \dots, u_k; x) e^{-sx} dx = \frac{k!}{s} \left(1 + \frac{1}{s}\right)^k \exp\left[-(s+1) \sum_{i=1}^k u_i\right].$$

Hence

$$(2.8) \quad m_{(k)}(u_1, \dots, u_k; x) = (k!)^2 \exp\left\{-\left[\sum_{i=1}^k u_i\right]\right\} \sum_{i=0}^k \frac{\left(x - \sum_{i=1}^k u_i\right)^i}{(k-i)!(i!)^2}.$$

3. - The number of distances exceeding the bubble diameter, d.

Denote by N the number of individuals Y such that $Y > d$ and let p_k be the probability that $N = k$, $k = 1, 2, \dots$. Then the probability generating function

$$(3.1) \quad g(z, s) = \sum_{k=0}^{\infty} p_k z^k, \quad |z| \leq 1,$$

is obtained from the generating functional $G[\xi, x]$ upon defining $\xi(u)$ as

$$\xi(u) = \begin{cases} z & , \quad u > d, \\ 1 & , \quad u \leq d. \end{cases}$$

From equation (2.2) we find that the Laplace transform of $g(z, x)$ with respect to x is given by

$$(3.2) \quad h(z, s) = \left[s + e^{-(s+1)d} - ze^{-(s+1)d} \right]^{-1}.$$

This can be inverted to give

$$(3.3) \quad g(z, x) = \sum_{0 \leq n \leq x/d} \frac{(z-1)^n}{n!} (x-nd)^n e^{-nd},$$

and hence

$$(3.4) \quad p_k = \frac{1}{k!} \left[\frac{\partial^k g(z, x)}{\partial z^k} \right]_{z=0} = \sum_{k \leq n \leq x/d} \frac{(-)^{n-k} e^{-nd}}{k! (n-k)!} (x-nd)^n .$$

The factorial moments of N are obtained immediately from equation (3.3) using the relation

$$\mathcal{E}N(N-1)\dots(N-k+1) = \left[\frac{\partial^k g(z, s)}{\partial z^k} \right]_{z=1} .$$

$$\text{Thus} \quad \mathcal{E}N(N-1)\dots(N-k+1) = \begin{cases} (x-kd)^k e^{-kd} , & 1 \leq k \leq x/d, \\ 0 & , \quad k > x/d. \end{cases}$$

The first two moments agree well with a Monte Carlo computation which was performed by Dr. L. G. Hyman of the Argonne National Laboratory High Energy Physics Division, who raised this problem in connection with its bubble chamber applications.

CHAPTER 4

MULTIPLICATIVE TRANSPORT PROCESSES IN ONE DIMENSION

1. - Introduction.

The following simple model for the multiplication of neutrons in a one-dimensional rod has been studied by Bellman, Kalaba, and Wing [3,4,51] in order to clarify the concepts of criticality and critical size. It is assumed, as in the scattering model of Chapter 2, that a neutron travels with constant direction between collisions, the probability of a collision in any small interval δs of path-length being $\lambda \delta s + o(\delta s)$. As a result of a collision, the parent neutron is replaced by two neutrons which leave the point of collision in opposite directions and continue to move and multiply in the same manner as the parent neutron. Bellman, Kalaba, and Wing, using the principle of invariant imbedding (in which the process is considered as a function of the length of the rod), determine the expected number of transmitted and reflected neutrons generated over all time by a single incident neutron. They use as a criterion for criticality the condition that these expected values become infinite, in this way showing that the critical length of the rod is given by $\pi/2\lambda$. They also derive equations for the probabilities

(1.1) $p_n(t)$ = the probability that a single neutron incident on a rod of thickness t eventually gives rise to a total of n reflected neutrons,

and for the corresponding generating function,

$$(1.2) \quad u(t,z) = \sum_{n=0}^{\infty} p_n(t) z^n, \quad ,$$

and make the following conjectures regarding the behaviour of the probabilities $p_n(t)$;

$$(1.3) \quad p_n(t) < p_{n-1}(t) \quad , \quad \begin{matrix} n = 1, 2, \dots, \\ t > 0, \end{matrix}$$

$$(1.4) \quad p_n(t) \text{ is unimodal and has a maximum at } t = y_n, \text{ where}$$

$$\frac{\pi}{2\lambda} > y_n > y_{n-1} > 0$$

$$\text{and } \lim_{n \rightarrow \infty} y_n = \frac{\pi}{2\lambda} = \text{critical length.}$$

In this chapter we shall consider a modified model in which a random number, v , of neutron pairs emerges from a collision (v neutrons moving in each direction) in addition to the parent neutron which continues to move with its original direction of motion. We show that for this model explicit solutions can be obtained for the probabilities $p_n(t)$ and in particular we show that there is a certain length of rod, t_c , for which $\sum_{n=0}^{\infty} p_n(t) < 1$ when $t > t_c$. This critical length, t_c , is the same as is obtained by applying the condition that the mean number of particles produced become infinite. However, the property that $\sum_{n=0}^{\infty} p_n(t) < 1$ for lengths greater than t_c is clearly a stronger property of criticality than is implied by the behaviour of the mean number of emergent particles. Following the detailed analysis of the model just described, we shall return to the model of Bellman, Kalaba, and Wing, and show that it also exhibits the property of criticality in this stronger sense.

In the particular case when the random variable v is equal to 1 with probability 1, i.e., exactly two additional particles are produced at each collision, the conjectures (1.3) and (1.4) of Bellman, Kalaba, and Wing are established.

2. - The distribution of the number of emergent particles.

Consider a particle which originates at a point x in the rod $-a \leq x \leq a$. Its initial direction of motion is μ ($\mu = 1$ for motion to the right and -1 for motion to the left). It moves in the manner described in the introduction, i.e., the probability of a collision in a small element of path-length δs is $\lambda \delta s + o(\delta s)$, and at each collision it gives rise to v new particles (in addition to itself) moving in the same direction and v new particles moving in the opposite direction; v is a random variable taking the values $0, 1, 2, \dots$, with associated probabilities $\pi_0, \pi_1, \pi_2, \dots$, respectively. Without loss of generality we shall assume throughout that the collision rate λ is equal to 1.

We introduce the probabilities $P_n(\mu, x)$ that a single particle originating at x with direction μ ultimately gives rise to a total of n particles emerging from the right-hand end of the rod. The corresponding generating function is defined by

$$(2.1) \quad G(\mu, x) = \sum_{n=0}^{\infty} P_n(\mu, x) z^n, \quad |z| \leq 1.$$

Consideration of the point at which the first collision occurs gives the following integral equation for $G(\mu, x)$:

$$(2.2) \quad G(\mu, x) = z^{\delta_{1,\mu}} e^{\mu(x-a)} + \mu \int_{-x}^{\mu a} G(\mu, y) \sum_{n=0}^{\infty} \pi_n G(\mu, y)^n G(-\mu, y)^n e^{\mu(x-y)} dy.$$

Differentiating this equation with respect to x , we find that $G(\mu, x)$ must satisfy the differential equation

$$(2.3) \quad \frac{d}{dx} G(\mu, x) = \mu G(\mu, x) \left[1 - \sum_{n=0}^{\infty} \pi_n G(\mu, x)^n G(-\mu, x)^n \right]$$

with boundary conditions

$$(2.4) \quad G(\mu, \mu a) = z^{\delta_{1, \mu}}.$$

From equation (2.3) it follows that $G(1, x)G(-1, x)$ is independent of x , i.e.,

$$(2.5) \quad G(1, x)G(-1, x) = K(z),$$

where K is a function of z with the properties

$$(2.6) \quad \begin{cases} |K(z)| \leq 1, & |z| \leq 1, \\ |K(z)| < 1, & |z| < 1. \end{cases}$$

Substituting $K(z)$ for $G(1, x)G(-1, x)$ in equation (2.3) and using the boundary conditions (2.4) we find that

$$(2.7) \quad G(\mu, x) = z^{\delta_{1, \mu}} \exp \left[(\mu x - a) \left(1 - \sum_{n=0}^{\infty} \pi_n K(z)^n \right) \right],$$

where $K(z)$ is a root of the equation

$$(2.8) \quad \theta = z \exp \left[-2a \left(1 - \sum_{n=0}^{\infty} \pi_n \theta^n \right) \right],$$

having the properties (2.6).

By Lagrange's Theorem (Whittaker and Watson [54]), if $|z| < 1$ equation (2.8) has only one root θ such that $|\theta| < 1$. From conditions (2.6) this must therefore be the required root $K(z)$ appearing in equation (2.7). Making use of Lagrange's expansion, an explicit solution will now be derived for $G(\mu, x)$.

We introduce functions $f(\theta)$ and $\phi(\theta)$, in which z , x , and a are regarded as fixed, defined by the following equations:

$$(2.9) \quad f(\theta) = z^{\delta_{1, \mu}} \exp \left[(\mu x - a) \left(1 - \sum_{n=0}^{\infty} \pi_n \theta^n \right) \right],$$

$$(2.10) \quad \phi(\theta) = \exp \left[-2a(1 - \sum_{n=0}^{\infty} \pi_n \theta^n) \right].$$

For $|z| < 1$ the generating function $G(\mu, x)$ is obtained from $f(\theta)$ upon replacing θ by the root $K(z)$ of (2.8) in the interior of the unit circle.

By Lagrange's expansion therefore

$$(2.11) \quad G(\mu, x) = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \left\{ \frac{d^{n-1}}{d\theta^{n-1}} \left[\phi(\theta)^n \frac{d}{d\theta} f(\theta) \right] \right\}_{\theta=0}.$$

From equation (2.11) it can be shown directly that the coefficient of z^n in $G(1, x)$, i.e., $P_n(1, x)$, is given by

$$(2.12) \quad \begin{cases} P_1(1, x) = \exp[(x-a)(1-\pi_0)] \\ P_n(1, x) = \frac{a-x}{n-1} \exp[(1-\pi_0)(x-2na+a)] \sum_{j=0}^{n-2} (n-j-1) \gamma_j^{(n-1)} \pi_{n-j-1}, \end{cases}$$

where

$$(2.13) \quad \begin{cases} \gamma_0^{(n)} = 1, \\ \gamma_j^{(n)} = \sum_{m=1}^j \frac{(2na+a-x)^m}{m!} \sum_{i_1+\dots+i_m=j} \pi_{i_1} \pi_{i_2} \dots \pi_{i_m}, \end{cases} \quad \begin{matrix} n \geq 2, \\ j \geq 1, \end{matrix}$$

the inner summation being over all sets of positive integers

$(i_1, i_2, \dots, i_m)$ such that $i_1 + i_2 + \dots + i_m = j$.

The probabilities $P_n(-1, x)$ can be found immediately from (2.12)

using the relation

$$(2.14) \quad P_n(-1, x) = P_{n+1}(1, -x), \quad n=0, 1, 2, \dots,$$

which is a direct consequence of equation (2.7).

The reflection probabilities $p_n(t)$ defined by (1.1) are obtained from $P_n(-1, x)$ on setting $a = x = t/2$. Thus

$$(2.15) \quad \left\{ \begin{array}{l} p_n(t) = \frac{t}{n} \exp[-(1-\pi_0)(n+1)t] * \\ \left\{ n\pi_n + \sum_{j=1}^{n-1} (n-j)\pi_{n-j} \sum_{m=1}^j \frac{[(n+1)t]^m}{m!} \sum_{i_1+\dots+i_m=j} \pi_{i_1} \dots \pi_{i_m} \right\}, \\ n \geq 2, \\ p_0(t) = \exp[-t(1-\pi_0)] \quad , \quad p_1(t) = t\pi_1 \exp[-2t(1-\pi_0)] . \end{array} \right.$$

In the particular case when $v = 1$ with probability 1, i.e., if

$$(2.16) \quad \pi_n = \delta_{1,n} \quad , \quad n = 0, 1, 2, \dots,$$

then $p_n(t)$ takes the very simple form,

$$(2.17) \quad p_n(t) = \frac{(n+1)^{n-1}}{n!} t^n \exp[-(n+1)t] \quad , \quad n = 0, 1, 2, \dots$$

The maximum with respect to t of this function occurs when $t = n/n+1$ and it will be shown in Sect. 3 that the critical length for this particular model is $t_c = 1$. This proves conjecture (1.4). The proof of conjecture (1.3) follows trivially from equation (2.17).

3. - Criticality in the model of Sect. 2.

The results of Sect. 2 may be summarized as follows: for $|z| < 1$ the generating functions $G(\mu, x)$ are defined by

$$(3.1) \quad G(\mu, x) = z^{\delta_{1,\mu}} \exp[(\mu x - a)(1 - \sum_{n=0}^{\infty} \pi_n K(z)^n)] \quad ,$$

where $K(z)$ is the unique root of the equation

$$(3.2) \quad \theta = z \exp[-2a(1 - \sum_{n=0}^{\infty} \pi_n \theta^n)] \quad ,$$

in the interior of the unit circle, $|\theta| = 1$. We shall now examine the limiting behaviour of $G(\mu, x)$ as z tends to 1 and show that

$$(3.3) \quad \lim_{z \rightarrow 1} G(\mu, x) < 1 \quad \text{if } 2a > 1/\bar{v}, x \neq \mu a,$$

$$(3.4) \quad \lim_{z \rightarrow 1} G(\mu, x) = 1 \quad \text{otherwise,}$$

where $\bar{v} = \sum_{n=0}^{\infty} n\pi_n$ is the mean number of additional particle pairs emitted per collision. Thus there is a certain "critical" length of rod, $t_c = 1/\bar{v}$, beyond which there is a non-zero probability $(1 - \lim_{z \rightarrow 1} G(\mu, x))$ that a single initial particle at x with direction μ will eventually produce an infinite number of descendants. The critical length is usually derived (see Bellman, Kalaba, and Wing [3]) by considering the equation for the mean number of particles produced (derived on the assumption that $\lim_{z \rightarrow 1} G(\mu, x) = 1$) and taking as the critical length that length for which the solution first becomes infinite. As we shall see, this procedure leads to the same value, $1/\bar{v}$, of t_c . The fact that the mean number of particles produced by a single ancestor is infinite for rod-lengths greater than t_c is clearly implied by (3.3) but not conversely.

To prove (3.3) and (3.4) we first observe that if $x = \mu a$ then $G(\mu, x) = z^{\delta_1, \mu}$ so that $\lim_{z \rightarrow 1} G(\mu, x) = 1$ for all values of a . It will be assumed in what follows that this trivial case is excluded, i.e., $x \neq \mu a$.

Now if $2a > 1/\bar{v}$, then by Lagrange's theorem, equation (3.2) has a unique root $\theta = \psi(z)$ such that $|\psi(z)| < 1$ for $|z| \leq 1$. By the uniqueness of $K(z)$ and $\psi(z)$ we must have

$$(3.5) \quad \psi(z) = K(z) \quad \text{for all } z \text{ such that } |z| < 1.$$

Hence $\lim_{z \rightarrow 1} K(z) = \lim_{z \rightarrow 1} \psi(z)$ so that

$$(3.6) \quad \left| \lim_{z \rightarrow 1} K(z) \right| = \left| \lim_{z \rightarrow 1} \psi(z) \right| < 1.$$

Using this last result, it follows from equation (3.1) that

$$\lim_{z \rightarrow 1} G(\mu, x) < 1 \quad \text{if } 2a > 1/\sqrt{v}.$$

This completes the proof of (3.3).

To prove (3.4) we observe from (2.5) that $K(1)$ must be real and less than or equal to 1; $K(1)$ must also satisfy the equation

$$(3.7) \quad K(1) = \exp[-2a(1 - \sum_{n=0}^{\infty} \pi_n K(1)^n)].$$

However, if $2a \leq 1/\sqrt{v}$ the only real root of this equation which is less than or equal to 1 is $K(1) = 1$. Substituting in equation (3.1) gives

$$\lim_{z \rightarrow 1} G(\mu, x) = 1 \quad \text{if } 2a \leq 1/\sqrt{v},$$

thus completing the proof of the result (3.4).

Now let us consider the behaviour of the mean number, $M(\mu, x)$, of particles ultimately emerging from the right-hand end of the rod conditional upon a single particle originating at x with direction μ , and conditional upon the total number of particles emerging from the right-hand end of the rod being finite. In terms of the generating function $G(\mu, x)$ we have

$$(3.8) \quad M(\mu, x) = \left\{ [G(\mu, x)]^{-1} \frac{\partial}{\partial z} G(\mu, x) \right\}_{z=1}.$$

Hence

$$(3.9) \quad M(\mu, x) = \delta_{1, \mu} + \frac{(a-x) \sum_n n \pi_n \gamma^n}{1-2a \sum_n n \pi_n \gamma^n},$$

where $\gamma = \lim_{z \rightarrow 1} K(z)$. Setting $\gamma = 1$ in (3.9) we get

$$(3.10) \quad M(\mu, x) = \delta_{1, \mu} + \frac{(a-x)\bar{v}}{1-2a\bar{v}},$$

which is the solution of the equation for the mean

$$(3.11) \quad \frac{d}{dx} M(\mu, x) = -\mu \bar{v} [M(-\mu, x) + M(\mu, x)],$$

with boundary conditions

$$(3.12) \quad M(\mu, a\mu) = \delta_{1, \mu}.$$

The reason for the discrepancy between (3.9) and (3.10) is that equation (3.11) is obtained by differentiating (2.3) with respect to z and setting $z = 1$ and $G(\mu, x) = 1$. It therefore ceases to be valid in the supercritical case when $\lim_{z \rightarrow 1} G(\mu, x) < 1$; this is indicated by the fact that when $2a > 1/\bar{v}$, $M(\mu, x)$ as given by (3.10) becomes negative (for further discussion see also Moyal [55]).

In the model of Bellman, Kalaba, and Wing [3] only two particles emerge from each collision (one moving in each direction) and the equation corresponding to (3.11) for the mean number of emergent particles (based on the assumption that $\lim_{z \rightarrow 1} G(\mu, x) = 1$) is

$$(3.13) \quad \frac{d}{dx} M(\mu, x) = -\mu M(-\mu, x),$$

with the boundary conditions

$$(3.14) \quad M(\mu, a\mu) = \delta_{1,\mu} \quad .$$

The solution of equations (3.13) and (3.14) can be written

$$(3.15) \quad M(\mu, x) = \delta_{1,\mu} \frac{\cos(a+x)}{\cos 2a} + \delta_{-1,\mu} \frac{\sin(a+x)}{\cos 2a} \quad .$$

Bellman, Kalaba, and Wing took the critical length for their model to be the smallest rod-length for which the mean number of particles produced by a single ancestor (calculated on the assumption that $\lim_{z \rightarrow 1} G(\mu, x) = 1$) becomes infinite. Hence, from (3.15) this criterion gives a critical length $t_c = \pi/2$. Applying the same argument to the model considered previously, we find from (3.10) that the critical length is $t_c = 1/\sqrt{v}$. Thus we arrive at the same value as was previously derived by considering the behaviour of $\lim_{z \rightarrow 1} G(\mu, x)$.

4. - Criticality in the model of Bellman, Kalaba, and Wing.

It has been shown that for the model of Sect. 2 there is a critical length $t_c = 1/\sqrt{v}$ such that

$$(4.1) \quad \lim_{z \rightarrow 1} G(\mu, x) < 1 \quad \text{if } 2a > t_c, \quad x \neq \mu a,$$

$$(4.2) \quad \lim_{z \rightarrow 1} G(\mu, x) = 1 \quad \text{otherwise.}$$

The same kind of behaviour will now be established for the model of Bellman, Kalaba, and Wing in which, at each collision, the incident particle is lost and two new particles emerge (one in each direction) from the point of collision.

Instead of considering, as in Sect. 2, the probability $P_n(\mu, x)$ that a single particle originating at x ($-a \leq x \leq a$) with direction μ ultimately

generates n particles emerging from the right-hand end of the rod, it is more convenient now to consider the probability $Q_n(\mu, x)$ that the total number of particles ultimately emerging from the rod is n . The corresponding generating function is defined by

$$H(\mu, x) = \sum_{n=0}^{\infty} Q_n(\mu, x) z^n, \quad |z| \leq 1.$$

Clearly $\sum_{n=0}^{\infty} P_n(\mu, x) = \sum_{n=0}^{\infty} Q_n(\mu, x) = \lim_{z \rightarrow 1} H(\mu, x)$ so that it will be sufficient to prove (4.1) and (4.2) with G replaced by H . In fact, we shall prove the following:

$$(4.3) \quad \lim_{z \rightarrow 1} H(\mu, x) < 1 \quad \text{if } 2a > \pi/2, \quad x \neq \mu a,$$

$$(4.4) \quad \lim_{z \rightarrow 1} H(\mu, x) = 1 \quad \text{otherwise.}$$

The critical length t_c is thus $\pi/2$ as already found by Bellman, Kalaba, and Wing from their study of the mean number of particles produced by a single ancestor (see Sect. 3).

Proceeding as in Sect. 2, we find that the equations for the generating functions $H(\mu, x)$ are

$$(4.5) \quad \frac{d}{dx} H(\mu, x) = \mu H(\mu, x) [1 - H(-\mu, x)],$$

$$(4.6) \quad H(\mu, \mu a) = z.$$

Introducing new functions $u(x, z)$ and $v(x, z)$ defined by

$$(4.7) \quad u(x, z) = \frac{1}{2} [H(1, x) + H(-1, x)],$$

$$(4.8) \quad v(x, z) = \frac{1}{2} [H(1, x) - H(-1, x)],$$

we find from (4.5) that

$$(4.9) \quad \frac{du}{dx} = v \quad ,$$

$$(4.10) \quad \frac{dv}{dx} = u - u^2 + v^2 \quad .$$

Inspection of equation (4.5) shows that

$$(4.11) \quad \log H(1,x) - H(1,x) + \log H(-1,x) - H(-1,x) = \chi(z)$$

where $\chi(z)$ is independent of x .

Suppose that $H(1,0) = \alpha(z)$. Then by symmetry $H(-1,0)$ is also equal to $\alpha(z)$. Since a particle which starts at $x = a$ moving to the right can have no collisions, we must have $H(1,a) = z$. We denote by $\beta(z)$ the corresponding function $H(-1,a)$. For real values of z less than or equal to 1, equation (4.5) shows that

$$(4.12) \quad \beta(z) \leq H(-1,x) \leq \alpha(z) \leq H(1,x) \leq z \quad \text{for all } x \geq 0.$$

Thus in order to prove (4.3), it will be sufficient (using (4.5) and (4.12)) to show that

$$(4.13) \quad \lim_{z \rightarrow 1} \alpha(z) < 1 \quad \text{for } 2a > \pi/2.$$

To prove (4.4) it will be sufficient (using (4.5) and (4.12)) to show that

$$(4.14) \quad \lim_{z \rightarrow 1} \alpha(z) = 1 \quad \text{for } 2a \leq \pi/2 \quad .$$

Substituting the values of $H(1,x)$ and $H(-1,x)$ at $x = 0$ and $x = a$ in equation (4.11) we find that

$$(4.15) \quad 2[\log \alpha(z) - \alpha(z)] = \log[z\beta(z)] - \beta(z) - z = \chi(z) \quad .$$

Rewriting equation (4.11) in terms of $u(x,z)$ and $v(x,z)$ and using the relation between $\alpha(z)$ and $\chi(z)$ gives

$$(4.16) \quad u^2 - v^2 = \alpha^2 e^{-2(\alpha-u)} \quad ,$$

and hence from (4.9)

$$(4.17) \quad \frac{du}{dx} = \sqrt{u^2 - \alpha^2 e^{-2(\alpha-u)}} \quad ,$$

the positive value of the square root being appropriate if z is real, positive, and less than or equal to 1, and if $x \geq 0$.

Integrating equation (4.17) between the limits $x = 0$ and $x = a$ leads to the relationship

$$(4.18) \quad a = \int_{\alpha(z)}^{\frac{1}{2}[z+\beta(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2 e^{-2(y-\alpha)}}} \quad \text{where } z\beta e^{-(z+\beta)} = \alpha^2 e^{-2\alpha} \quad .$$

We shall now use equation (4.18) to deduce the limiting behaviour of $\alpha(z)$ as z tends to 1.

Suppose that z is real and positive and $z \leq 1$ so that $\alpha(z)$ and $\beta(z)$ are both real and positive and $\alpha \leq 1$, $\beta \leq 1$. Define $\gamma(z)$ to be the unique real root of the equation

$$\gamma e^{-(1+\gamma)} = \alpha^2 e^{-2\alpha}$$

such that $\gamma \leq 1$ for $z \leq 1$. It then follows from the relations

$$\beta z e^{-(z+\beta)} = \alpha^2 e^{-2\alpha} \quad \text{and} \quad \beta(z) \leq z$$

that $z+\beta(z) \leq 1 + \gamma(z)$.

We wish to show that $\lim_{z \rightarrow 1} \alpha(z) < 1$ if $2a > \pi/2$ and that $\lim_{z \rightarrow 1} \alpha(z) = 1$ if $2a \leq \pi/2$. Suppose first that $\lim_{z \rightarrow 1} \alpha(z) = 1$. Then from equation (4.18)

$$\begin{aligned} a &= \lim_{z \rightarrow 1} \int_{\alpha(z)}^{\frac{1}{2}[z+\beta(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2} e^{2(y-\alpha)}} \\ &\leq \lim_{z \rightarrow 1} \int_{\alpha(z)}^{\frac{1}{2}[1+\gamma(z)]} \frac{dy}{\sqrt{y^2 - \alpha^2} e^{2(y-\alpha)}} \\ &= \lim_{\delta \rightarrow 0} \int_{1-\delta}^{1-\delta/\sqrt{2}} \frac{dy}{\sqrt{y^2 - (1-\delta)^2} e^{2(y-1+\delta)}} \\ &= \lim_{\delta \rightarrow 0} \int_{\delta/\sqrt{2}}^{\delta} \frac{dv}{\sqrt{v^2 - \delta^2}} = \frac{\pi}{4} \end{aligned}$$

Thus if $\lim_{z \rightarrow 1} \alpha(z) = 1$ then $a \leq \pi/4$, i.e., if $2a > \pi/2$, then $\lim_{z \rightarrow 1} \alpha(z) < 1$. Conversely, if $\lim_{z \rightarrow 1} \alpha(z) = \alpha(1) < 1$, then

$$a = \int_{\alpha(1)}^{\frac{1}{2}[1+\beta(1)]} \frac{dy}{\sqrt{y^2 - \alpha(1)^2} e^{2[y-\alpha(1)]}} > \frac{\pi}{4}$$

and hence $\lim_{z \rightarrow 1} \alpha(z) = 1$ if $2a \leq \pi/2$. This completes the proof of the properties (4.1) and (4.2) for the model of Bellman, Kalaba, and Wing.

CHAPTER 5

GENERALIZED ONE-DIMENSIONAL SCATTERING PROCESSES

WITH APPLICATIONS IN THREE DIMENSIONS

1. - Introduction.

Consider a particle which moves along the x-axis starting from a point x in the interval $[a_{-1}, a_1]$. It continues to move with constant direction until it makes a collision, as a result of which its direction of motion is reversed with probability p and unchanged with probability q . The probability that the particle is absorbed at any collision is $r = 1-p-q$. The distances travelled between successive collisions are assumed to be independently distributed with density function $f(x)$ and characteristic function $\phi(\alpha) = \int_0^{\infty} f(x) \exp(i\alpha x) dx$.

Results for the special case where the distances between successive collisions have a negative exponential distribution, i.e., $f(x) = \lambda \exp(-\lambda x)$, $\phi(\alpha) = \lambda(\lambda - i\alpha)^{-1}$, have already been derived in Chapter 2. It is the purpose of the present chapter to extend these results to include more general distributions for the distances travelled between collisions. In Sect. 2 we determine the characteristic function (for arbitrary $\phi(\alpha)$ and no barriers) of the displacement of the particle at its n -th collision, given that it makes at least n collisions before being absorbed. In Sects. 4 and 5 we generalize the results of Chapter 2 for scattering processes with absorbing boundaries at a_{-1} and a_1 to processes where the distances between collisions have probability density functions belonging to one of the following two classes:

$$(I) \quad f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x),$$

$$(II) \quad f(x) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i x), \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1.$$

As an application of these generalized results we shall discuss the method used by Fermi [21] to study the scattering of thermal neutrons in a three-dimensional plane slab $\{x: a_{-1} \leq x \leq a_1\}$ of paraffin. Fermi supposed that the motion of the neutrons is as follows: between collisions each neutron moves in a straight line and in any element δs of its path it has probability $\lambda \delta s + o(\delta s)$ of making a collision, where λ^{-1} is its (constant) mean free path; given that it makes a collision, the neutron is absorbed with probability r and scattered isotropically with probability $1-r$. Fermi obtained approximate results for this process by supposing that the projection of the motion on the x -axis is a one-dimensional scattering process of the type studied in Chapter 2 with collision rate $\lambda\sqrt{3}$ per unit path length (as opposed to λ for the three-dimensional motion) and probabilities $r, \frac{1-r}{2}, \frac{1-r}{2}$ of absorption, reversal, and continuation respectively.

This is an approximate procedure because, as Fermi points out, the projections of the distances travelled between collisions do not in fact have a negative exponential distribution as do the distances themselves.

For a more precise study of the motion of the projection of the particle on the x -axis, we shall determine in Sect. 6 the true characteristic function, $\phi(\alpha)$, of the projections of the distances travelled between collisions. In the absence of barriers we can then use the results of Sect. 2 to derive the exact characteristic function of the x -displacement at the n -th collision. When there are barriers present

(at $x = a_{-1}$ and $x = a_1$) we shall use the generalized results of Sect. 5 to improve on Fermi's results for the scattering of particles by a slab of finite thickness.

Fermi's method and the method to be described both apply only to isotropic scattering. For anisotropic scattering, the projected displacements between successive collisions are no longer independent.

2. - The general one-dimensional process with no boundaries.

Consider the following generalization of the process studied in Chapter 2. A particle starting from the origin moves in steps X_i , $i=1,2,\dots$, along the x-axis where $\{|X_i|\}$ is a sequence of identically and independently distributed random variables with characteristic function $\phi(\alpha)$. At the end of each step the particle suffers a collision, the probabilities of reversal, continuation and absorption at each collision being p , q and r respectively. For the moment we assume that $r=0$ (we show at the end of this Section how to adapt the results when $r>0$) and determine the characteristic function $\sigma_n^{(\mu)}(\alpha)$ of the displacement at the n -th collision, given that the initial direction of motion was μ ($\mu=+1$ for motion to the right and -1 for motion to the left); thus

$$(2.1) \quad \sigma_n^{(\mu)}(\alpha) = \mathcal{E}\left\{\exp\left[i\alpha \sum_{j=1}^n X_j\right] \mid \mu\right\}.$$

Given that m of the first n steps $X_1, \dots, X_n$ have the sign of the initial direction μ , we have

$$(2.2) \quad \sigma_n^{(\mu)}(\alpha) = [\phi(\mu\alpha)]^m [\phi(-\mu\alpha)]^{n-m}.$$

Hence, defining

$$(2.3) \quad P_{mn}^{(\mu)} = \text{Prob} \{m \text{ of the first } n \text{ steps have the sign of the initial direction, } \mu\},$$

(observing that these probabilities are independent of μ) we have

$$(2.4) \quad \sigma_n^{(\mu)}(\alpha) = \sum_{k=1}^n P_{kn}^{(\mu)} [\phi(\mu\alpha)]^k [\phi(-\mu\alpha)]^{n-k},$$

i.e.,

$$(2.5) \quad \sigma_n^{(\mu)}(\alpha) = [\phi(-\mu\alpha)]^n P_n^{(\mu)} ([\phi(\mu\alpha)][\phi(-\mu\alpha)]^{-1}), \quad n \geq 1,$$

where $P_n^{(\mu)}(z) = \sum_{k=0}^n P_{kn}^{(\mu)} z^k$ and $P_{on}^{(\mu)}$ is defined for convenience as $P_{on}^{-1} \delta_{o,n}$.

Consideration of the collision at which the first reversal occurs yields the recurrence relation

$$(2.6) \quad P_{mn}^{(\mu)} = \sum_{k=1}^m (1-p)^{k-1} p P_{(n-m)(n-k)}^{(\mu)}, \quad 1 \leq m \leq n,$$

from which it follows that if $n \geq 1$

$$(2.7) \quad P_n^{(\mu)}(z) = z^n \sum_{m=1}^n \sum_{k=1}^m p(1-p)^{k-1} z^{m-n} P_{(n-m)(n-k)}^{(\mu)} \\ = z^n \sum_{k=1}^n p(1-p)^{k-1} P_{n-k}^{(\mu)}(z^{-1}), \quad n \geq 1.$$

This equation can be written in the more convenient form

$$(2.8) \quad P_{n+1}^{(\mu)}(z) - z(1-p) P_n^{(\mu)}(z) = pz^{n+1} P_n^{(\mu)}(z^{-1}),$$

together with a second equation which is obtained from (2.8) on replacing z by z^{-1} , namely

$$(2.9) \quad P_{n+1}^{(\mu)}(z^{-1}) - z^{-1}(1-p)P_n^{(\mu)}(z^{-1}) = pz^{-n-1}P_n^{(\mu)}(z) \quad .$$

Elimination of $P_n^{(\mu)}(z^{-1})$ from (2.8) and (2.9) leads to a second-order difference equation

$$(2.10) \quad P_{n+2}^{(\mu)}(z) - (1-p)(1+z)P_{n+1}^{(\mu)}(z) + z(1-2p)P_n^{(\mu)}(z) = 0 \quad ,$$

and introducing the generating function

$$(2.11) \quad P^{(\mu)}(z, w) = \sum_{n=1}^{\infty} P_n^{(\mu)}(z) w^n \quad , \quad |w| < 1 \quad , \quad |z| \leq 1 \quad ,$$

we find from (2.10) that

$$(2.12) \quad P^{(\mu)}(z, w) = wz \frac{1 - w(1-2p)}{1 - (1-p)(1+z)w + (1-2p)zw^2} \quad .$$

Hence, from equation (2.5), we obtain for the generating function of $\sigma_n^{(\mu)}(\alpha)$,

$$(2.13) \quad \sum_{n=1}^{\infty} \sigma_n^{(\mu)}(\alpha) t^n = t\phi(\alpha) \frac{1 - (1-2p)t \phi(-\mu\alpha)}{1 - (1-p)[\phi(\mu\alpha) + \phi(-\mu\alpha)]t + (1-2p)\phi(\mu\alpha)\phi(-\mu\alpha)t^2} \quad .$$

Notice that if we replace $\phi(\alpha)$ by $\lambda[\lambda - i\alpha]^{-1}$ then $\sigma_n^{(\mu)}(\alpha)$ becomes the characteristic function of the displacement at the n-th collision for the model considered in Chapter 2. The same generating function could be obtained from equation (4.9), Chapter 2, by taking $z=w=t$ and $x=0$.

If there is a positive probability ($r>0$) of absorption at each collision then we define $\tau_n^{(\mu)}(\alpha)$ to be the characteristic function of

the displacement at the n -th collision, given that the particle makes at least n collisions before being absorbed. Then the generating function $\sum_{n=1}^{\infty} \tau_n^{(\mu)}(\alpha) t^n$ is obtained directly from (2.13) upon replacing p by $p^* = p(1-r)^{-1}$.

3. - The one-dimensional first passage problem.

Suppose that the initial coordinate of the particle in the interval $[a_{-1}, a_1]$ is x and that its initial direction of motion is μ ; $\mu = +1$ for motion to the right and -1 for motion to the left. For distances between successive collisions having probability density function $f(x)$ and for probabilities p, q, r of reversal, continuation and absorption at a collision we wish to find the probabilities $P_{rk}^{(\mu)}(a_{\pm 1}|x)$ of first emergence at $a_{\pm 1}$ after exactly r reversals and k continuations.

The solution when $f(x) = \lambda \exp(-\lambda x)$ is given in Chapter 2 as a generating function,

$$(3.1) \quad M^{(\mu)}(a_v|x; z, w) \equiv \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_v|x) z^r w^k.$$

From the basic equation (4.14) of Chapter 1 or by a simple direct argument it can be deduced that for arbitrary $f(x)$ the generating functions $M^{(\mu)}(a_v|x)$ (omitting explicit reference to z and w) satisfy

$$(3.2) \quad M^{(\mu)}(a_v|x) = \mu \delta_{v\mu} \int_{a_\mu}^{\mu\infty} f(\mu y - \mu x) dy \\ + \mu \int_x^{a_\mu} f(\mu y - \mu x) [pzM^{(-\mu)}(a_v|y) + qwM^{(\mu)}(a_v|y)] dy.$$

$\delta_{v\mu}$ here denotes Kronecker's delta, i.e., $\delta_{v\mu} = \begin{cases} 1 & , \quad v = \mu \\ 0 & , \quad v \neq \mu \end{cases}$.

A large number of distributions encountered in practice can be closely approximated by a gamma distribution with suitably chosen parameters or by a linear combination of negative exponential distributions. If $f(x)$ belongs to either of these two families, it is possible to solve equation (3.2). In Sect. 4 we solve (3.2) when $f(x)$ is of the form

$$(3.3) \quad f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x), \quad 0 \leq x < \infty,$$

and in Sect. 5 we do the same when $f(x)$ is of the form

$$(3.4) \quad f(x) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i x), \quad 0 \leq x < \infty, \\ 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1.$$

Both families include the negative exponential distribution as a special case.

Although further generalizations of the results of Chapter 2 can be derived in a similar manner, we shall restrict ourselves to the determination of $M^{(\mu)}(a_v | x)$ which is sufficient for the discussion of Fermi's model in Sect. 6.

4. - First passage probabilities when the distances between collisions have a gamma distribution.

When $f(x)$ is given by equation (3.3) the integral equation (3.2) takes the form

$$\begin{aligned}
 (4.1) \quad M^{(\mu)}(a_\nu | x) = & \mu \delta_{\mu\nu} \int_{a_\mu}^{\infty} \frac{\lambda^n \mu^{n-1} (y-x)^{n-1}}{(n-1)!} \exp\{-\lambda\mu(y-x)\} dy \\
 & + \mu \int_x^{a_\mu} \frac{\lambda^n \mu^{n-1} (y-x)^{n-1}}{(n-1)!} \exp\{-\lambda\mu(y-x)\} [pzM^{(-\mu)}(a_\nu | y) \\
 & + qwM^{(\mu)}(a_\nu | y)] dy .
 \end{aligned}$$

Using the symbol D to denote the operator $\frac{\partial}{\partial x}$, this equation can be rewritten as

$$(4.2) \quad [(\lambda - \mu D)^n - qw\lambda^n] M^{(\mu)} - \lambda^n pzM^{(-\mu)} = 0, \quad \mu = \pm 1,$$

with boundary conditions

$$(4.3) \quad M^{(\mu)}(a_\nu | a_\mu) = \delta_{\nu\mu},$$

$$(4.4) \quad (\lambda - \mu D)^i M^{(\mu)}(a_\nu | x) = \delta_{\nu\mu} \lambda^i \text{ at } x = a_\mu, \quad i=1, \dots, n-1.$$

The general solution of the equations (4.2) is

$$(4.5) \quad M^{(\mu)}(a_\nu | x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)}(z, w) \exp(\theta_i x),$$

where $\theta_i, i=1, \dots, 2n$, are the roots (assumed distinct) of the equation

$$(4.6) \quad [(\lambda - \theta)^n - qw\lambda^n] [(\lambda + \theta)^n - qw\lambda^n] + \lambda^{2n} p^2 z^2 = 0.$$

The terms in the solution (4.5) corresponding to a k -tuple root θ of (4.6)

are of course of the form $[A_{\nu 1}^{(\mu)} + A_{\nu 2}^{(\mu)} x + \dots + A_{\nu k}^{(\mu)} x^{k-1}] \cdot \exp(\theta x)$.

Equation (4.2) can be used to express $A_{vi}^{(-1)}$ in terms of $A_{vi}^{(1)}$, $i=1, \dots, 2n$, and the $2n$ boundary conditions (4.3), (4.4) then determine the coefficients $A_{vi}^{(\mu)}(z, w)$ completely.

Although this solves the first passage problem in principle it will generally be difficult, especially for a gamma distribution with large parameter n , to obtain an explicit solution in terms of z and w . However the probabilities of emergence at either boundary,

$$(4.7) \quad P^{(\mu)}(a_v | x) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_v | x) = [M^{(\mu)}(a_v | x)]_{z=w=1},$$

are easily found by setting $z=w=1$ in equations (4.2), (4.3), (4.4), and solving numerically. Thus $P^{(\mu)}(a_v | x)$ satisfies

$$(4.8) \quad [(\lambda - \mu D)^n - q\lambda^n]P^{(\mu)} - p\lambda^n P^{(-\mu)} = 0,$$

with boundary conditions

$$(4.9) \quad P^{(\mu)}(a_v | a_\mu) = \delta_{v\mu},$$

$$(4.10) \quad (\lambda - \mu D)^i P^{(\mu)}(a_v | x) = \delta_{v\mu} \lambda^i \text{ at } x=a_\mu, \quad i=1, \dots, n-1.$$

The solution of these equations is given, for $p+q < 1$, by

$$(4.11) \quad P^{(\mu)}(a_v | x) = \sum_{i=1}^{2n} A_{vi}^{(\mu)} \exp(\theta_i x),$$

where θ_i , $i=1, \dots, 2n$ are the roots of (4.6) with $z=w=1$. If $p+q = 1$ then $\theta = 0$ is a double root of (4.6) and the solution is of the form

$$(4.12) \quad P^{(\mu)}(a_v | x) = A_{v1}^{(\mu)} + A_{v2}^{(\mu)} x + \sum_{i=3}^{2n} A_{vi}^{(\mu)} \exp \theta_i x.$$

The equations for the coefficients $A_{vi}^{(\mu)}$ are obtained as above and can be solved numerically.

The moments of the joint distribution of the number of reversals and continuations before emergence can be obtained by finding the appropriate partial derivatives of (4.2), (4.3) and (4.4) with respect to z and w , setting $z=w=1$ and solving numerically. For example the mean number of reversals before emergence, conditional on emergence at a_v , is given by

$$(4.13) \quad m = [P^{(\mu)}(a_v|x)]^{-1} \left[\frac{\partial}{\partial z} M^{(\mu)}(a_v|x) \right]_{z=w=1}.$$

5. - Linear combinations of negative exponential distributions.

In this section we determine the generating functions $M^{(\mu)}(a_v|x)$ for density functions $f(x)$ of the form

$$(5.1) \quad f(x) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i x), \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1.$$

For such a function $f(x)$ equation (3.2) becomes

$$(5.2) \quad M^{(\mu)}(A_v|x) = \mu \delta_{\mu v} \int_{a_\mu}^{\mu\infty} \sum_{i=1}^n p_i \lambda_i \exp\{\lambda_i \mu(x-y)\} dy \\ + \mu \int_x^{a_\mu} \sum_{i=1}^n p_i \lambda_i \exp\{\lambda_i \mu(x-y)\} [pzM^{(-\mu)}(a_v|y) + qwM^{(\mu)}(a_v|y)] dy.$$

To solve (5.1) we first determine the constants $c_1, c_2, \dots, c_n$ satisfying the equations

$$(5.3) \quad c_n \lambda_i^n + c_{n-1} \lambda_i^{n-1} + \dots + c_1 \lambda_i + 1 = 0, \quad i=1, 2, \dots, n.$$

With this set of constants c_k we deduce from (5.2) that $M^{(\mu)}(a_v|x)$ satisfies the differential equation

$$(5.4) \quad [c_n(\mu D)^n + c_{n-1}(\mu D)^{n-1} + \dots + c_1 \mu D + 1 + \phi(\mu D)q\omega] M^{(\mu)} + \phi(\mu D)pzM^{(-\mu)} = 0,$$

where D again denotes the operator $\frac{\partial}{\partial x}$ and

$$\phi(\theta) = c_1 \sum_1^n p_i \lambda_i + c_2 \left[\sum_1^n p_i \lambda_i^2 + \sum_1^n p_i \lambda_i \theta \right] + \dots + c_n \left[\sum_1^n p_i \lambda_i^n + \dots + \sum_1^n p_i \lambda_i \theta^{n-1} \right].$$

The boundary conditions to be satisfied by $M^{(\mu)}(a_v|x)$ are

$$(5.5) \quad M^{(\mu)}(a_v|a_\mu) = \delta_{v\mu},$$

$$(5.6) \quad (\mu D)^j M^{(\mu)}(a_v|x) = \delta_{v\mu} \sum_{i=1}^n \lambda_i^j p_i - \sum_{r=1}^j \left\{ (\mu D)^{j-r} [q\omega M^{(\mu)}(a_v|x) + pzM^{(-\mu)}(a_v|x)] \cdot \sum_{i=1}^n \lambda_i^r p_i \right\} \text{ at } x = a_\mu, \quad j=1, \dots, n-1.$$

The general solution of equation (5.4) is given by

$$(5.7) \quad M^{(\mu)}(a_v|x) = \sum_{i=1}^{2n} A_{vi}^{(\mu)}(z, w) \exp(\theta_i x),$$

where $\theta_1, \dots, \theta_{2n}$ are the roots (assumed distinct) of the equation

$$(5.8) \quad [c_n \theta^n + \dots + c_1 \theta + 1 + q\omega \phi(\theta)][c_n (-\theta)^n + \dots + c_1 (-\theta) + 1 + q\omega \phi(-\theta)] - p^2 z^2 \phi(\theta) \phi(-\theta) = 0.$$

Multiple roots of equation (5.8) are treated in the same way as in

Sect. 4. From equation (5.4), $A_{vi}^{(-1)}$ can be expressed in terms of $A_{vi}^{(1)}$

for each i and the $2n$ boundary conditions (5.5), (5.6) then determine the coefficients $A_{\nu i}^{(\mu)}$ completely.

As in Sect. 4 an explicit solution for $M^{(\mu)}(a_{\nu}|x)$ in terms of z and w will in general be extremely complicated; however the probabilities of emergence, $P^{(\mu)}(a_{\nu}|x)$, and the moments of the joint distribution of the numbers of reversals and continuations before emergence can easily be found numerically as before.

An illustration of the use of equation (5.4) and its solution will be given in Sect. 7 where we discuss Fermi's method for determining the probabilities of reflection and transmission of thermal neutrons by a slab of paraffin.

6. - Three-dimensional scattering with no boundaries.

We now consider Fermi's model for three-dimensional scattering which was described in the introduction. As indicated in Sect. 1, Fermi's approximation for the projected motion of the particle on the X -axis supposes that the projected displacements between successive collisions are independently distributed with probability density function

$$(6.1) \quad f_1(x) = \sqrt{3} \lambda \exp(-\sqrt{3}\lambda x), \quad 0 \leq x < \infty,$$

where λ^{-1} is the mean free path of the particle in three dimensions.

The characteristic function corresponding to $f_1(x)$ is

$$(6.2) \quad \phi_1(\alpha) = \int_0^{\infty} \exp(i\alpha x) f_1(x) dx = \sqrt{3} \lambda (\sqrt{3}\lambda - i\alpha)^{-1}.$$

The exact density function $f(x)$ for the projected displacements can in fact be found as follows. Suppose that the particle has just suffered a collision and moves off in a direction whose angle with the positive X-axis has cosine μ , $-1 \leq \mu \leq 1$. The probability that the projection on the X-axis of the distance travelled to the next collision lies in $(x, x+\delta x)$ is $\frac{\lambda}{|\mu|} \exp \left\{ -\frac{\lambda x}{|\mu|} \right\} \delta x + o(\delta x)$. Since scattering is assumed to be isotropic $|\mu|$ is a random variable uniformly distributed on $[0,1]$. Hence

$$(6.3) \quad f(x) = \lambda \int_0^1 \theta^{-1} \exp \left(-\frac{\lambda x}{\theta} \right) d\theta, \quad 0 \leq x < \infty,$$

and the corresponding characteristic function is

$$(6.4) \quad \phi(\alpha) = \int_0^\infty \exp(i\alpha x) f(x) dx = -\frac{\lambda}{i\alpha} \log \left(1 - \frac{i\alpha}{\lambda} \right).$$

Expanding (6.2) and (6.4) in powers of $i\alpha$, we see that the exact distribution and Fermi's approximating distribution have the same mean square but their means and all the other moments about the origin are different. In fact for isotropic scattering, as we shall see later, it is the mean square which determines the asymptotic behaviour of the particle after a large number of collisions.

When there are no boundaries the exact characteristic function $\tau_n^{(\mu)}(\alpha)$ for the x-displacement at the n-th collision, given that the particle experiences at least n collisions before being absorbed, can be found immediately from equation (2.13). Since we are assuming isotropic scattering we replace p in equation (2.13) by $p^* = \frac{1}{2}$. For an isotropic source of particles at $x = 0$ the appropriate characteristic

function is $\tau_n(\alpha) = \frac{1}{2} [\tau_n^{(1)}(\alpha) + \tau_n^{(-1)}(\alpha)]$ which is found to be (expanding (2.13) as a power series in t)

$$(6.5) \quad \tau_n(\alpha) = \left[\frac{\lambda}{2i\alpha} \log \frac{\lambda+i\alpha}{\lambda-i\alpha} \right]^n .$$

The corresponding Fermi approximation is obtained from (2.13) by using, instead of the exact characteristic function $\phi(\alpha)$ given by (6.4), the corresponding approximation $\phi_1(\alpha)$ given by (6.2). This leads to the approximation of $\tau_n(\alpha)$ by the function

$$(6.6) \quad \tau_n^*(\alpha) = \left[\frac{3\lambda^2}{3\lambda^2 + \alpha^2} \right]^n .$$

The distributions corresponding to $\tau_n(\alpha)$ and $\tau_n^*(\alpha)$ are compared most conveniently by considering their moments. Thus if S_n is a random variable with characteristic function $\tau_n(\alpha)$ and S_n^* is a random variable with characteristic function $\tau_n^*(\alpha)$, then

$$\mathcal{E}\{S_n^{2k+1}\} = 0, \quad \mathcal{E}\{S_n^{*2k+1}\} = 0, \quad k=0,1,2,\dots,$$

$$\mathcal{E}\{S_n^2\} = \frac{2n}{3} \lambda^{-2}, \quad \mathcal{E}\{S_n^{*2}\} = \frac{2n}{3} \lambda^{-2},$$

$$\mathcal{E}\{S_n^4\} = \left[\frac{4n^2}{3} + \frac{52n}{15} \right] \lambda^{-4}, \quad \mathcal{E}\{S_n^{*4}\} = \left[\frac{4n^2}{3} + \frac{4n}{3} \right] \lambda^{-4},$$

and for large n the moments differ only by terms which are $O(n^{k-1})$, i.e.,

$$\mathcal{E}\{S_n^{2k}\} = \frac{2k!}{k!} \left(\frac{n}{3\lambda^2} \right)^k + O(n^{k-1}), \quad \mathcal{E}\{S_n^{*2k}\} = \frac{2k!}{k!} \left(\frac{n}{3\lambda^2} \right)^k + O(n^{k-1}),$$

$k=1,2,\dots$

From equations (5.9) and (5.10) we see that the displacements S_n and S_n^* are both distributed as sums of independent random variables. We can therefore apply the central limit theorem to deduce that as $n \rightarrow \infty$, $n^{-1/2} S_n$ and $n^{-1/2} S_n^*$ are asymptotically normally distributed. In fact the asymptotic distributions are identical, both having mean zero and variance $\frac{2}{3} \lambda^{-2}$. This suggests that Fermi's approximation is a good one when the number of collisions is large. The parameter $\lambda \sqrt{3}$ of his approximating negative exponential distribution is in fact determined uniquely by the condition that $S_n^*/\sqrt{n}$ and $S_n/\sqrt{n}$ have the same asymptotic behaviour.

7. - Boundary value problems in three dimensions.

Let us consider again the three-dimensional model described in the introduction; this time however with the planes $x = a_{-1}$ and $x = a_1$ as absorbing boundaries. The particle starts at a point with x-coordinate $x \in [a_{-1}, a_1]$ and continues to move as described in the introduction until it is either absorbed at a collision or crosses one of the boundaries, after which it is lost to the system. The problem is to determine the probabilities that it eventually crosses one or another of the boundaries, and to investigate the distribution of the number of collisions made before doing so.

As in Sect. 6 we consider the projection of the motion on the x-axis; this reduces the problem to the one-dimensional first passage problem of Sect. 3, in which the density function $f(x)$ is given by equation (6.3), i.e.,

$$(7.1) \quad f(x) = \lambda \int_0^1 \theta^{-1} \exp(-\lambda x \theta^{-1}) d\theta, \quad 0 \leq x < \infty.$$

The solution of the basic equations (3.2) for this particular density function is very difficult; the approach used by Fermi was to replace $f(x)$ by an approximating function

$$(7.2) \quad f_1(x) = \sqrt{3} \lambda \exp(-\sqrt{3}\lambda x) \quad ,$$

in which case the equations (3.2) are relatively simple to solve.

It is possible to improve on Fermi's procedure using the results of Sect. 5. We seek an improved approximating function

$$(7.3) \quad f_n(x) = \sum_{i=1}^n \lambda_i p_i \exp(-\lambda_i x) \quad ,$$

for the exact density function $f(x)$, choosing the p_i and λ_i by fitting moments about the origin of $f_n(x)$ to those of $f(x)$. Taking $n=2$ we can fit the first three moments of $f_2(x)$ to those of $f(x)$ by choosing

$$(7.4) \quad f_2(x) = \frac{1}{2} \lambda_1 \exp(-\lambda_1 x) + \frac{1}{2} \lambda_2 \exp(-\lambda_2 x) \quad , \quad 0 \leq x < \infty \quad ,$$

where $\lambda_1 = \lambda(3 - \sqrt{3})$ and $\lambda_2 = \lambda(3 + \sqrt{3})$.

Table 1 shows for comparison the first five moments of the density functions $f(x)$, $f_1(x)$ and $f_2(x)$ about the origin; Fig. 2 shows the graphs of the density functions themselves. Better approximations could be derived by taking $n > 2$, but even $f_2(x)$ gives substantially better approximations than $f_1(x)$.

Table 1

k	k^{th} moment about the origin		
	$f(x)$	$f_1(x)$	$f_2(x)$
1	$\frac{1}{2} \lambda^{-1}$	$\frac{1}{3} \sqrt{3} \lambda^{-1}$	$\frac{1}{2} \lambda^{-1}$
2	$\frac{2}{3} \lambda^{-2}$	$\frac{2}{3} \lambda^{-2}$	$\frac{2}{3} \lambda^{-2}$
3	$\frac{3}{2} \lambda^{-3}$	$\frac{2}{3} \sqrt{3} \lambda^{-3}$	$\frac{3}{2} \lambda^{-3}$
4	$\frac{24}{5} \lambda^{-4}$	$\frac{8}{3} \lambda^{-4}$	$\frac{14}{3} \lambda^{-4}$
5	$20 \lambda^{-5}$	$\frac{40}{9} \sqrt{3} \lambda^{-5}$	$\frac{55}{3} \lambda^{-5}$

The generating functions $M^{(\mu)}(a_v | x; z, w)$ for the projected motion on the x-axis are given by equations (3.2) which become, upon replacing the density function $f(x)$ by the approximating function $f_2(x)$, equations of the form (5.2) with $n=2$, $p_1=p_2=1/2$, $\lambda_1=3-\sqrt{3}$ and $\lambda_2=3+\sqrt{3}$. (The mean free path for the three-dimensional motion, i.e., λ^{-1} , has here been used as the unit of length.)

As already shown in Sect. 5 the solution for $M^{(\mu)}(a_v | x; z, w)$ is of the form

$$(7.5) \quad M^{(\mu)}(a_v | x) = \sum_{i=1}^4 A_{vi}^{(\mu)}(z, w) \exp(\theta_i x) \quad ,$$

where the functions $\theta_i(z, w)$ and $A_{vi}^{(\mu)}(z, w)$ are to be determined from equations (5.3), (5.4), (5.5) and (5.6).

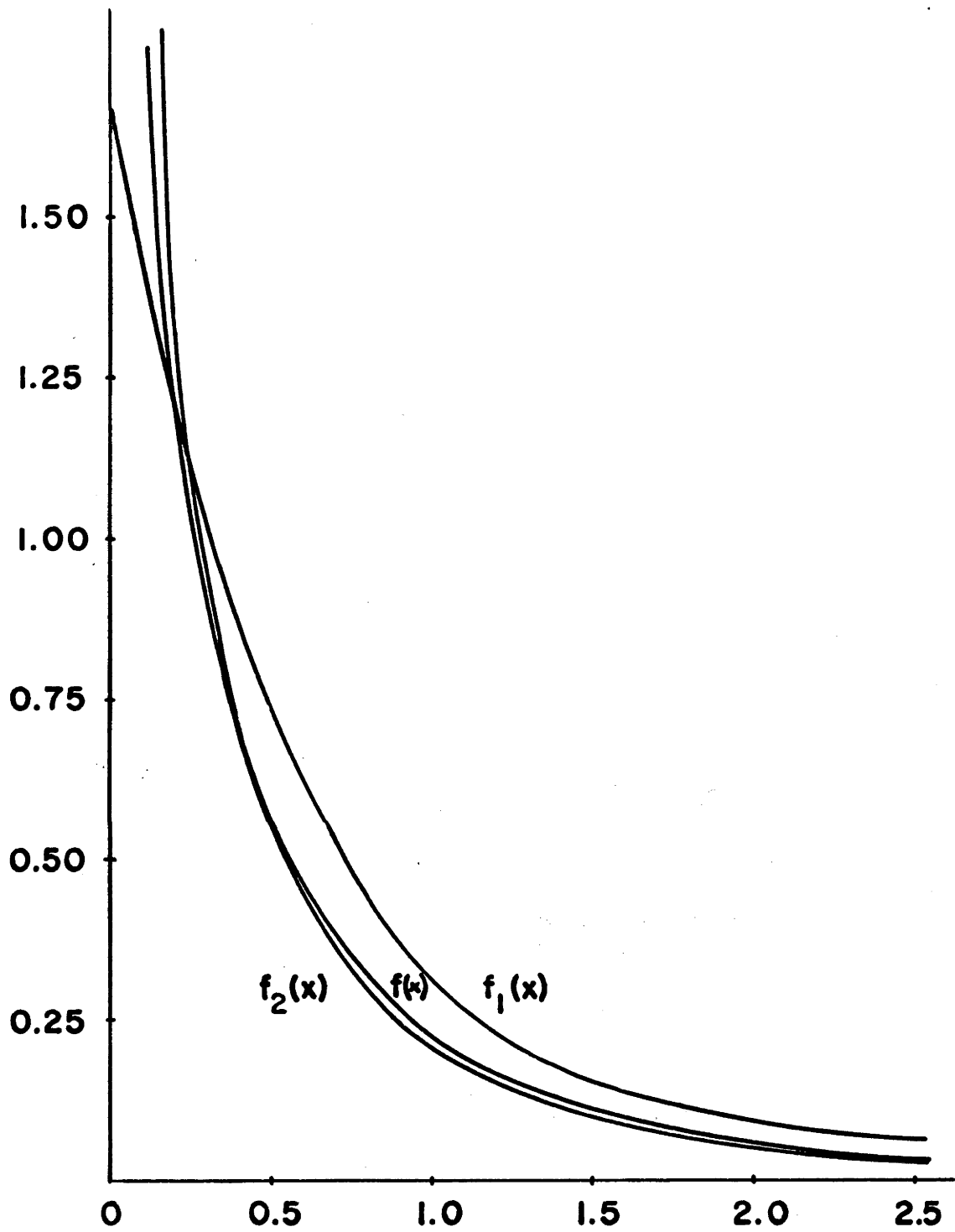


Figure 2

Substituting the above numerical values for $\lambda_1, \lambda_2, p_1$ and p_2 in (5.3) and solving for c_1 and c_2 , we find

$$(7.6) \quad c_1 = -1, \quad c_2 = \frac{1}{6}.$$

$\theta_1, \dots, \theta_4$ are therefore the roots of the equation

$$(7.7) \quad \theta^4 - 3\theta^2[8(1-qw)+3(q^2w^2-p^2z^2)] + 36[(1-qw)^2-p^2z^2] = 0.$$

Since scattering is assumed to be isotropic $p = q = \frac{1-r}{2}$ where r is the probability of absorption at a collision.

The coefficients $A_{vi}^{(1)}, A_{vi}^{(-1)}$ are related by equation (5.4), which in this case takes the form

$$(7.8) \quad [D^2 - D(6-3qw) + 6(1-qw)]M^{(1)} - 3pz[2-D]M^{(-1)} = 0.$$

Thus

$$(7.9) \quad A_{vi}^{(-1)} = \frac{\theta_i^2 - (6-3qw)\theta_i + 6(1-qw)}{3pz(2-\theta_i)} A_{vi}^{(1)} = k_i(z, w) A_{vi}^{(1)}, \quad i=1, \dots, 4.$$

The coefficients $A_{vi}^{(1)}$ are found from the boundary conditions (5.5),

(5.6) which can now be written

$$(7.10) \quad \left\{ \begin{array}{l} \sum_{i=1}^4 A_{vi}^{(1)} \exp(\theta_i a_1) = \delta_{1v} \\ \sum_{i=1}^4 k_i A_{vi}^{(1)} \exp(\theta_i a_{-1}) = \delta_{-1v} \\ \sum_{i=1}^4 (\theta_i + 3qw + 3pz k_i) A_{vi}^{(1)} \exp(\theta_i a_1) = 3\delta_{1v} \\ \sum_{i=1}^4 (-k_i \theta_i + 3qwk_i + 3pz) A_{vi}^{(1)} \exp(\theta_i a_{-1}) = 3\delta_{-1v} \end{array} \right. .$$

Equations (7.8), (7.9) and (7.10) provide a complete solution to the first passage problem in generating function form. The probabilities of emergence at the two boundaries regardless of the number of collisions before emergence, i.e., $P^{(\mu)}(a_v|x) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_v|x)$, are obtained when z and w are replaced throughout by unity. Moments of the distribution of the number of reversals and continuations before emergence are obtained by differentiating partially with respect to z and w and setting $z=w=1$.

8. - The albedo for a semi-infinite slab.

To illustrate the application of these results we now consider a stream of particles incident on the face $x = 0$ of the half-space $\{x: 0 \leq x < \infty\}$. We shall use (7.5), (7.9) and the appropriate boundary conditions to determine the albedo, i.e., the proportion of incident particles which ultimately emerge again from the half-space. The boundary conditions at a_1 will now be replaced by the condition that $P^{(\mu)}(a_v|x)$ is bounded as $x \rightarrow \infty$.

Denote by $p(x)$ the probability that a particle from an isotropic source with coordinate $x(x > 0)$ emerges from the half-space without being absorbed, i.e.,

$$p(x) = \frac{1}{2} [P^{(1)}(0|x) + P^{(-1)}(0|x)] .$$

Then the probability that a particle incident on the half-space at an angle θ to the positive x -axis ultimately emerges again is given by

$$(8.1) \quad \beta(\theta) = (1-r) \int_0^{\infty} \exp(-x \sec \theta) p(x) \sec \theta \, dx.$$

If there is a stream of particles incident on the half-space in such a way that the angle of incidence θ has probability density function $\phi(\theta)$, $0 \leq \theta \leq \pi/2$, then the albedo is given by

$$(8.2) \quad \beta = \int_0^{\pi/2} \beta(\theta) \phi(\theta) d\theta .$$

In the particular case $\phi(\theta) = 2 \cos\theta \sin\theta$, i.e., for incident particles obeying the cosine law, Wick [47] obtained an exact asymptotic expression for β when r is small, namely,

$$(8.3) \quad \beta = 1 - \frac{4}{\sqrt{3}} r^{1/2} + \dots ;$$

Fermi [21], using the approximation $f_1(x)$ to derive $p(x)$, evaluated the integrals (8.1) and (8.2) to find

$$(8.4) \quad \beta = 1 - \left(1 + \frac{2}{\sqrt{3}}\right) r^{1/2} + \dots .$$

From the approximation $f_2(x)$, proceeding as in Sect. 7, we find that

$$(8.5) \quad p(x) = A \exp(\theta_1 x) + B \exp(\theta_2 x) ,$$

where

$$\theta_1 = -\sqrt{6(1+r+\sqrt{1+r+r^2})} ,$$

$$\theta_2 = -\sqrt{6(1+r-\sqrt{1+r+r^2})} ,$$

$$A = -\frac{1}{2}(1+k_1) \frac{k_2(2\theta_2+3+3r)-3+3r}{2k_1k_2(\theta_1-\theta_2)+3(1-r)(k_1-k_2)} ,$$

$$B = \frac{1}{2}(1+k_2) \frac{k_1(2\theta_1+3+3r)-3+3r}{2k_1k_2(\theta_1-\theta_2)+3(1-r)(k_1-k_2)} ;$$

k_1 and k_2 are defined by equation (7.9) with $z=w=1$.

From (8.1) we obtain

$$(8.6) \quad \beta(\theta) = (1-r) [A(1-\theta_1 \cos \theta)^{-1} + B(1-\theta_2 \cos \theta)^{-1}] ,$$

and in the particular case $\phi(\theta) = 2 \cos \theta \sin \theta$ the integral (8.2) can be evaluated explicitly to give

$$(8.7) \quad \beta = -2(1-r)[A\theta_1^{-2}(\theta_1 + \log[1-\theta_1]) + B\theta_2^{-2}(\theta_2 + \log[1-\theta_2])] ,$$

i.e.,

$$(8.8) \quad \beta = 1 - \left[\frac{2\sqrt{3}-3}{12} \log (1+2\sqrt{3}) + \frac{13}{6}\sqrt{3} - \frac{3}{2} \right] r^{1/2} + \dots .$$

Comparing the coefficients of $r^{1/2}$ in (8.3), (8.4) and (8.8), we see that Fermi's approximation gives 2.1547, the improved approximation gives 2.3106 and the exact value is 2.3094. This gives an idea of the accuracy of the approximation. The same method of course can be applied to determine the albedo and transmission probability for slabs of finite thickness.

9. - Transmission of particles through slabs of finite thickness.

The following table gives the probabilities of transmission through a slab of thickness t mean free paths and the mean number of collisions experienced by transmitted particles when the particles are incident isotropically on one face of the slab. The results using both Fermi's and the improved approximation are given. In Chapter 7 these results will be compared with those obtained using a new method of solution which also takes into account directions of motion in space and the angular deflections experienced by scattered particles. The tabulated

results refer to the case of zero absorption, i.e., $r=0$; however corresponding results could easily be obtained for any non-zero value of r .

Table 2

Thickness t	Probability of transmission		Mean number of collisions	
	Fermi's app.	Improved app.	Fermi's app.	Improved app.
0.5	.69783	.62083	0.6016	0.9184
1.0	.53590	.47929	1.5000	1.9973
2.0	.36603	.33743	4.0981	4.8266
4.0	.22401	.21294	12.360	13.543
8.0	.12613	.12255	46.856	43.142

When there is no absorption (i.e., $r=0$) the application of Fermi's method (and the improved approximation) to the scattering of particles in the interior of a thick slab can be justified asymptotically using Wald's identity (Bartlett [2], p. 17). Suppose there is an isotropic source of particles at $x=0$ and that scattering is isotropic with no absorption, i.e., for the projected motion on the X-axis $r=0$, $p=q=\frac{1}{2}$. Let N be a random variable denoting the number of collisions made by the particle before emerging from $[-a, b]$, $-a \leq 0 \leq b$. P_a and P_b will denote the probabilities of first departures from $-a$ and b respectively.

The Fermi approximations to P_a , P_b and $E N$ (the expected value of N) can be deduced from equation (6.6) of Chapter 2. They are

$$(9.1) \quad P_a = \frac{\sqrt{3}b+1}{\sqrt{3}(a+b)+2}, \quad P_b = \frac{\sqrt{3}a+1}{\sqrt{3}(a+b)+2},$$

$$(9.2) \quad \mathcal{E}N = \frac{\sqrt{3}(\ell^3 - a^3 - b^3) - 3(2\ell^2 - a^2 - b^2)}{2\sqrt{3}\ell + 4}, \text{ where } \ell = a+b.$$

Now as we have already seen in the unrestricted case, the x-displacement at the n-th collision is distributed as a sum of n independent random variables, each with mean zero and variance $\frac{2}{3}$. Using Wald's identity (Bartlett, p. 20), we can deduce from this that for a and b large, P_a , P_b and $\mathcal{E}N$ are given asymptotically by

$$(9.3) \quad P_a = \frac{b}{a+b}, \quad P_b = \frac{a}{a+b},$$

$$(9.4) \quad \mathcal{E}N = \frac{3}{2} ab.$$

Examination of (9.3) and (9.4) shows that Fermi's approximations are in fact asymptotic to these values. The approximations we derived earlier also have this property.

We remark finally that for scattering by very thick slabs with $r > 0$ the penetration of particles is largely determined by the tail of the exact density function $f(x)$ and not by its second moment about the origin.

CHAPTER 6

A THREE-DIMENSIONAL SCATTERING MODEL

1. - Introduction.

In Chapter 5 we developed a simple method for determining the probabilities of transmission and reflection, and the distribution of the number of collisions experienced by a particle which undergoes scattering in a three-dimensional plane slab. Unfortunately, this procedure is limited to problems with isotropic scattering and clearly gives no indication of the direction of motion of emergent particles or the angular deflections which they experience as a result of collisions in the scattering material.

For a more general treatment of scattering processes we return to the formulation of Chapter 1 in terms of the generating functionals $G(S; \xi(\cdot) | \omega; s)$ and $H(S; \xi(\cdot) | \omega; s)$. To solve the basic equations (4.14) and (4.15) of Chapter 1 in closed form would be an extremely difficult, if not impossible, task. However, by choosing a certain finite set of directions in which the particle is restricted to move, the backward Kolmogorov equations (5.5) and (5.6) of Chapter 1 can be reduced as in the one-dimensional case to a finite set of ordinary linear differential equations.

It is desirable to choose a set of directions of motion with the following property: if $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$ are unit vectors in the permissible directions of motion, then the set of scalar products $\{\tilde{a}_i \cdot \tilde{a}_1, \tilde{a}_i \cdot \tilde{a}_2, \dots, \tilde{a}_i \cdot \tilde{a}_n\}$ is the same for each i ($i=1, \dots, n$). This means that for any direction of motion prior to a collision, the set of all possible angular deflections resulting from the collision is the

same independently of the initial direction of motion; that is to say, we can define a single set of n possible angular deflections $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ which take an arbitrary initial direction of motion into each member of the set; we also define a set of transition probabilities $\{p_1, p_2, \dots, p_m\}$ corresponding to the distinct angular deflections $\{\theta_1, \theta_2, \dots, \theta_m\}$ which constitute a subset of $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$.

Thus the probabilities p_i , $i=1, \dots, m$, replace the transition function $\phi(dy|\mu)$ of Chapter 1, Sect. 5. Similarly, the generating functionals G and H are replaced by generating functions

$$(1.1) \quad G(S; z_1, \dots, z_m | \omega; s) = \sum_{i_1 + \dots + i_m \geq 0} z_1^{i_1} \dots z_m^{i_m} P^{i_1 \dots i_m}(S | \omega; s),$$

$$(1.2) \quad H(S; z_1, \dots, z_m | \omega; s) = \sum_{i_1 + \dots + i_m \geq 0} z_1^{i_1} \dots z_m^{i_m} Q^{i_1 \dots i_m}(S | \omega; s) - I(S | \omega; s),$$

where $P^{i_1 \dots i_m}(S | \omega; s)$ is the probability of a transition $\omega \rightarrow S$ in distance s with exactly i_1 deflections through angle $\theta_1, \dots, i_m$ deflections through θ_m ; similarly $Q^{i_1 \dots i_m}(S | \omega; s)$ is the probability that the first $(i_1 + \dots + i_m)$ collisions occur in distance s , that i_k of these collisions result in deflections through θ_k , $k=1, \dots, m$, and that the particle is left immediately after the $(i_1 + \dots + i_m)^{th}$ collision in a state belonging to the set S .

In two dimensions the obvious choice for the permissible directions of motion is the set $\mathcal{M}_n$ of directions making angles $2k\pi/n$, $k=0, 1, \dots, n-1$, with the unit vector $\hat{i}$ in the direction of the x -axis.

The transition probabilities p_j , $j=0,1,\dots,n-1$ are the probabilities that the particle is deflected through an angle $2j\pi/n$, given that it makes a collision.

In three dimensions, suitable sets of directions can be chosen by taking the lines joining the center of a regular polyhedron to its vertices, face-centers, or edge-centers. The largest set obtainable in this way is the set of 30 directions corresponding to mid-points of the edges of a regular icosahedron. In fact, each finite subgroup of the three-dimensional real orthogonal group generates a suitable set of directions of motion; consequently sets of at least 120 directions can be constructed (the degenerate sets obtained from rotations about a fixed axis are excluded). For the purpose of developing the theory, we shall consider the sets of 30 directions corresponding to the mid-points of the edges of a regular icosahedron which, as we shall see, leads to good numerical approximations. In Sect. 9 the construction and use of a set of 120 directions will be described. The two-dimensional problem will not be examined in detail since it is of limited interest and since the analysis is exactly parallel to the three-dimensional case.

2. - Geometry of the icosahedron.

The cosines of the angles between the faces of a regular icosahedron are ± 1 , $\pm \frac{1}{3}\sqrt{5}$, and $\pm \frac{1}{3}$ (see for example Cundy and Rollett [18]). From these we can readily determine the coordinates of the face centers. An icosahedron of unit inradius can be oriented (Fig. 3) so that they are given by

$$\pm [1, 0, 0],$$

$$\begin{aligned} & \pm \left[\frac{\sqrt{5}}{3}, \frac{2}{3}, 0 \right], \pm \left[\frac{\sqrt{5}}{3}, -\frac{1}{3}, \frac{\sqrt{3}}{3} \right], \pm \left[\frac{\sqrt{5}}{3}, -\frac{1}{3}, -\frac{\sqrt{3}}{3} \right], \\ & \pm \left[\frac{1}{3}, \frac{\sqrt{5}}{3}, \frac{\sqrt{3}}{3} \right], \pm \left[\frac{1}{3}, \frac{3-\sqrt{5}}{6}, \frac{\sqrt{15}+\sqrt{3}}{6} \right], \pm \left[\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, \frac{\sqrt{15}-\sqrt{3}}{6} \right], \\ & \pm \left[\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, -\frac{\sqrt{15}-\sqrt{3}}{6} \right], \pm \left[\frac{1}{3}, \frac{3-\sqrt{5}}{6}, -\frac{\sqrt{15}+\sqrt{3}}{6} \right], \\ & \pm \left[\frac{1}{3}, \frac{\sqrt{5}}{3}, -\frac{\sqrt{3}}{3} \right]. \end{aligned}$$

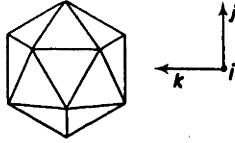


Fig. 3.

From these coordinates it is a matter of elementary algebra to determine the vertices and the mid-points of the edges. It is then found that the cosines of the angles between the lines joining the center of the icosahedron to the mid-points of its edges are

$\mu_1, \mu_2, \mu_3, \dots, \mu_{10}$, where

$$(2.1) \quad \left\{ \begin{array}{ll} \mu_1 = 1 & = -\mu_{10} \\ \mu_2 = \frac{\sqrt{5}+1}{4} & = -\mu_9 \\ \mu_3 = \frac{1}{2} & = -\mu_8 \\ \mu_4 = \frac{\sqrt{5}-1}{4} & = -\mu_7 \\ \mu_5 = 0 & = -\mu_6 \end{array} \right.$$

Throughout the remainder of this chapter we shall assume that the icosahedron is oriented as in Fig. 4 with the 30 possible directions of motion along the lines joining the center of the icosahedron to the mid-points of its edges.

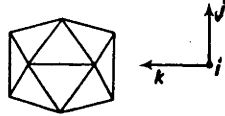


Fig. 4.

In the notation of Sect. 1 the number of directions, n , is equal to 30 and the unit vectors $\{a_1, \dots, a_{30}\}$ are the unit vectors in the directions just indicated. The set of angular deflections $\{\epsilon_1, \dots, \epsilon_{30}\}$ contains 9 distinct angles $\theta_1, \theta_2, \dots, \theta_9$ where

$$(2.2) \quad \left\{ \begin{array}{ll} \cos \theta_1 = \mu_1 & , \quad \cos \theta_9 = \mu_{10} = -\mu_1 & , \\ \cos \theta_2 = \mu_2 & , \quad \cos \theta_8 = \mu_9 = -\mu_2 & , \\ \cos \theta_3 = \mu_3 & , \quad \cos \theta_7 = \mu_8 = -\mu_3 & , \\ \cos \theta_4 = \mu_4 & , \quad \cos \theta_6 = \mu_7 = -\mu_4 & , \\ \cos \theta_5 = \mu_5 & , \end{array} \right.$$

and the angles $\theta_2, \theta_3, \dots, \theta_8$ each occur four times in the set $\{\epsilon_1, \dots, \epsilon_{30}\}$. Assuming that the probability of a deflection through angle θ depends only on θ , we need therefore define only 9 distinct transition probabilities $\{p_1, {}^4p_2, \dots, {}^4p_8, p_9\}$ corresponding to angular deflections $\{\theta_1, \theta_2, \dots, \theta_9\}$. Collisions which result in deflections through $\theta_1, \dots, \theta_9$ are said to be of types 1, ..., 9 respectively.

We can achieve a further simplification by grouping the directions of motion into classes as follows. The number α , $\alpha=1, \dots, 10$, will be used to denote the set of all directions of motion whose cosine with the positive x-axis is μ_α . The definition must be modified slightly for the directions 5 and 6 perpendicular to the x-axis. Direction 5 consists (Fig. 4) of the two unit vectors $\pm \hat{k}$ and direction 6 consists of the unit vectors $\pm \hat{j}$. This splitting of the directions perpendicular to the x-axis is necessary for the process, when defined in terms of the state variables (x, α) to be Markovian.

In practice, of course, the particles can move in any direction in space, the probabilities $\phi(d\tilde{y}|\tilde{\mu})$ of a change of direction $\tilde{\mu} \rightarrow \tilde{y}$ at a collision being given by

$$(2.3) \quad \phi(d\tilde{y}|\tilde{\mu}) = \sigma(\theta)d\tilde{y} \quad ,$$

where θ is the angle between $\tilde{\mu}$ and $\tilde{y}$, and $d\tilde{y}$ is an element of solid angle surrounding the unit vector $\tilde{y}$. The first step in any physical problem is, therefore, to determine the probabilities $p_1, \dots, p_9$ corresponding to the continuous function $\sigma(\theta)$. These probabilities are chosen so that

$$(2.4) \quad p_i = k\sigma(\theta_i) \quad , \quad i=1, \dots, 9,$$

the constant k being uniquely determined by the normalization condition

$$(2.5) \quad p_1 + 4 \sum_{i=2}^8 p_i + p_9 = 1-r \quad ,$$

where r is the probability of absorption at a collision.

3. - The general solution.

We shall be concerned primarily with first passage problems for a scattered particle when it emerges from the slab $\{x: 0 \leq x \leq t\}$. Once the particle leaves the slab it continues to move in a straight line so that it never returns. In the notation of Chapter 1, Sect. 4, we seek the functional $G(S; \xi(\cdot) | \omega; \infty)$ in which the initial state ω is represented by an x -coordinate ($0 \leq x \leq t$) and a direction $\beta (\beta=1, \dots, 10)$, and the set S is the Cartesian product $X^C \times \Omega$ where X^C is the exterior of the slab and Ω is a set of directions. Since there are only a finite number of directions of motion, it will suffice to find G when S is of the form $X^C \times \{\alpha\}$, $\alpha=1, \dots, 10$. As already explained in Sect. 1, the functional G reduces to a multivariate generating function when the set of directions of motion is finite.

It is convenient to introduce the notation,

$P_{\alpha\beta}^{i_1 \dots i_9}(x) = \text{Prob} \{ \text{the first passage of the particle from the slab occurs after } i_j \text{ collisions of type } j, j=1, \dots, 9, \text{ and the direction of emergence is } \alpha, \text{ given that the initial position was } x \text{ and the initial direction of motion was } \beta \}$,

$$P_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x).$$

The functional $G(X^C \times \{\alpha\}; \xi(\cdot) | x, \beta; \infty)$ then reduces to the generating function

$$(3.1) \quad G_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) z_1^{i_1} \dots z_9^{i_9}.$$

We also define the column vectors $\tilde{P}_\alpha^{i_1 \dots i_9}(x)$, $\tilde{G}_\alpha(x)$, and $\tilde{P}_\alpha(x)$, whose i -th components ($i=1, \dots, 10$) are $\tilde{P}_{\alpha i}^{i_1 \dots i_9}(x)$, $\tilde{G}_{\alpha i}(x)$, and $\tilde{P}_{\alpha i}(x)$ respectively.

The backward Kolmogorov equation (eq. (5.5) of Chapter 1) can now be written

$$(3.2) \quad A_\theta(z_1, \dots, z_9) \tilde{G}_\alpha(x) = 0,$$

where $A_\theta(z_1, \dots, z_9)$ is the 10×10 matrix

$$\begin{bmatrix} \mu_1 \theta + (1) - 1 & (2222) & (3333) & (4444) & (55) & (55) & (6666) & (7777) & (8888) & (9) \\ (2) & \mu_2 \theta + (1234) - 1 & (2345) & (2356) & (46) & (37) & (4578) & (5678) & (6789) & (8) \\ (3) & (2345) & \mu_3 \theta + (1267) - 1 & (2457) & (28) & (46) & (3568) & (3489) & (5678) & (7) \\ (4) & (2356) & (2457) & \mu_4 \theta + (1368) - 1 & (37) & (28) & (2479) & (3568) & (4578) & (6) \\ (5) & (4466) & (2288) & (3377) & (19) - 1 & (55) & (3377) & (2288) & (4466) & (5) \\ (5) & (3377) & (4466) & (2288) & (55) & (19) - 1 & (2288) & (4466) & (3377) & (5) \\ (6) & (4578) & (3568) & (2479) & (37) & (28) & -\mu_4 \theta + (1368) - 1 & (2457) & (2356) & (4) \\ (7) & (5678) & (3489) & (3568) & (28) & (46) & (2457) & -\mu_3 \theta + (1267) - 1 & (2345) & (3) \\ (8) & (6789) & (5678) & (4578) & (46) & (37) & (2356) & (2345) & -\mu_2 \theta + (1234) - 1 & (2) \\ (9) & (8888) & (7777) & (6666) & (55) & (55) & (4444) & (3333) & (2222) & -\mu_1 \theta + (1) - 1 \end{bmatrix}$$

The symbol $(i_1 \dots i_n)$ is here used to denote the expression

$(p_{i_1} z_{i_1} + \dots + p_{i_n} z_{i_n})$ and θ denotes the operator $\lambda^{-1} \partial / \partial x$.

The boundary conditions for equation (3.2) are

$$(3.3) \quad \begin{cases} G_{\alpha\beta}(0) = \delta_{\alpha\beta}, & \beta = 7, 8, 9, 10, \\ G_{\alpha\beta}(t) = \delta_{\alpha\beta}, & \beta = 1, 2, 3, 4. \end{cases}$$

If the slab is semi-infinite, i.e., if $t = \infty$, the second of these is

replaced by the condition that $G_{\alpha\beta}(x)$ be a bounded function of x as $x \rightarrow \infty$.

In solving equation (3.2) it will be assumed that the mean free path λ^{-1} , is unity. If λ is any function of x the problem can be reduced to the case of unit λ by the simple coordinate transformation

$$(3.4) \quad x' = \int_0^x \lambda(u) du.$$

It is apparent from equation (3.2) that $G_{\alpha 5}(x)$ and $G_{\alpha 6}(x)$ can be expressed as linear combinations of $G_{\alpha 1}(x), \dots, G_{\alpha 4}(x), G_{\alpha 7}(x), \dots, G_{\alpha 10}(x)$; equation (3.2) can therefore be rewritten in the form

$$(3.5) \quad \frac{\partial}{\partial x} \underline{g}_{\alpha}(x) = R(z_1, \dots, z_9) \underline{g}_{\alpha}(x) \quad ,$$

where $R(z_1, \dots, z_9)$ is an easily determined 8×8 matrix and $\underline{g}_{\alpha}(x)$ is the column vector whose transpose is

$$(3.6) \quad \underline{g}_{\alpha}^t(x) = [G_{\alpha 1}(x) \dots G_{\alpha 4}(x) \ G_{\alpha 7}(x) \dots G_{\alpha 10}(x)] \ .$$

The solution of equation (3.5) is

$$(3.7) \quad \underline{g}_{\alpha}(x) = \exp(xR) \underline{g}_{\alpha}(0) \quad ,$$

where $\underline{g}_{\alpha}(0)$ is uniquely determined by the boundary conditions. In principle this result completely solves the scattering problem for the icosahedral model. The evaluation of the components of the matrix $\exp(xR)$ is extremely difficult for arbitrary $z_1, \dots, z_9$; however, the angular distribution, which is obtained when all the z_i are set equal to 1, can be determined numerically as will now be shown. The determination of the moments of the distribution of the number of collisions of each type experienced by the particle before emergence will be considered in Sect. 4.

The emergent angular distribution of particles for fixed initial position x and direction β is given by $P_{\alpha\beta}(x)$, $\alpha=1,\dots,10$, which is obtained from $G_{\alpha\beta}(x)$ when $z_1,\dots,z_9$ are each replaced by unity. From equation (3.7), defining $\tilde{p}_\alpha(x) = [g_\alpha(x)]_{z_1=\dots=z_9=1}$, we see immediately that the solution for $\tilde{p}_\alpha(x)$ is

$$(3.8) \quad \tilde{p}_\alpha(x) = \exp[xR(1,\dots,1)] \tilde{p}_\alpha(0) ,$$

where $\tilde{p}_\alpha(0)$ is uniquely determined by the boundary conditions

$$(3.9) \quad \begin{cases} P_{\alpha\beta}(0) = \delta_{\alpha\beta} & , \beta = 7,8,9,10 , \\ P_{\alpha\beta}(t) = \delta_{\alpha\beta} & , \beta = 1,2,3,4 . \end{cases}$$

Explicit numerical determinations of the probabilities $P_{\alpha\beta}(x)$ are most conveniently carried out by reducing $R(1,\dots,1)$ to canonical form, i.e., by finding a matrix T such that $T^{-1}R T = \Lambda$, where Λ is the canonical form of the matrix R . Then

$$(3.10) \quad \exp[xR(1,\dots,1)] = T \exp(x\Lambda)T^{-1}.$$

Details of the numerical calculations are given in Appendix 1. It is worth noticing that the solution for isotropic scattering is very much simplified owing to the fact that the eigenvectors of the matrix R , and hence the components of the matrix T , take a simple form. Thus the eigenvalues γ_i of R are the roots of the equation

$$(3.11) \quad \theta^8[13+2r]-\theta^6[156+99r]+\theta^4[448+752r]-\theta^2[320+1600r]+960r = 0 ,$$

where r is the probability of absorption at a collision; the right eigenvector of R corresponding to γ_i is simply

$$(3.12) \quad \tilde{e}_i = \begin{bmatrix} (1-\mu_1\gamma_i)^{-1} \\ \vdots \\ (1-\mu_4\gamma_i)^{-1} \\ (1-\mu_7\gamma_i)^{-1} \\ \vdots \\ (1-\mu_{10}\gamma_i)^{-1} \end{bmatrix}.$$

4. - Moments of the distribution of the number of collisions.

Let N_i , $i=1, \dots, 9$, be the (random) number of collisions of type i experienced by a particle which originates at x with direction β and emerges from the slab $\{x: 0 \leq x \leq t\}$ in direction α . We seek the moments of the distribution of the random variable

$$(4.1) \quad N = \sum_{i=1}^9 a_i N_i, \quad ,$$

where $a_1, \dots, a_9$ is a set of arbitrary real constants. In particular we shall consider the determination of the first two moments $\mathcal{E}N$ and $\mathcal{E}N^2$, third and higher order moments being found in the same way.

In terms of the probabilities $P_{\alpha\beta}^{i_1 \dots i_9}(x)$, the mean $\mathcal{E}N$ can be written

$$(4.2) \quad \mathcal{E}N = [P_{\alpha\beta}(x)]^{-1} M_{\alpha\beta}(x), \quad ,$$

where

$$M_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} (a_1 i_1 + \dots + a_9 i_9) P_{\alpha\beta}^{i_1 \dots i_9}(x), \quad ,$$

i.e.,

$$(4.3) \quad M_{\alpha\beta}(x) = \left\{ \sum_{i=1}^9 a_i \frac{\partial}{\partial z_i} G_{\alpha\beta}(x) \right\}_{z_1 = \dots = z_9 = 1}.$$

In the notation of Sect. 3 we require to determine the vectors

$$\underline{M}_{\alpha}(x) = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} \underline{G}_{\alpha}(x) \right\}_{z_1=\dots=z_9=1}$$

From equation (3.2) the equation for $\underline{M}_{\alpha}(x)$ is

$$(4.4) \quad A_{\theta}(1, \dots, 1) \underline{M}_{\alpha}(x) + B \underline{P}_{\alpha}(x) = 0,$$

where $\underline{P}_{\alpha}(x)$ is found as described in Sect. 3 and B is the matrix

$$(4.5) \quad B = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} A_{\theta}(z_1, \dots, z_9) \right\}_{z_1=\dots=z_9=1}.$$

It suffices to determine the vectors

$$\underline{m}_{\alpha}(x) = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} \underline{g}_{\alpha}(x) \right\}_{z_1=\dots=z_9=1},$$

since the components $M_{\alpha 5}(x)$ and $M_{\alpha 6}(x)$ of $\underline{M}_{\alpha}(x)$ can be expressed, using (4.4), as linear combinations of the components of $\underline{m}_{\alpha}(x)$. Equation (4.4) can thus be rewritten

$$(4.6) \quad \frac{\partial}{\partial x} \underline{m}_{\alpha}(x) = R(1, \dots, 1) \underline{m}_{\alpha}(x) + \underline{V}_{\alpha}(x),$$

where $R(z_1, \dots, z_9)$ is the same matrix appearing in equation (3.5) and

$$\underline{V}_{\alpha}(x) = \left\{ \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} R(z_1, \dots, z_9) \underline{p}_{\alpha}(x) \right\}_{z_1=\dots=z_9=1}.$$

The solution of equation (4.6) is given by

$$(4.7) \quad \underline{m}_{\alpha}(x) = \exp[xR(1, \dots, 1)] \underline{m}_{\alpha}(0) + \int_0^x \exp[(x-y)R(1, \dots, 1)] \underline{V}_{\alpha}(y) dy,$$

where $\underline{m}_{\alpha}(0)$ is uniquely determined by the boundary conditions

$$(4.8) \quad \begin{cases} M_{\alpha\beta}(t) = 0, & \beta = 1, 2, 3, 4, \\ M_{\alpha\beta}(0) = 0, & \beta = 7, 8, 9, 10. \end{cases}$$

The second moment $\mathcal{E}N^2$ is found in a similar manner. Thus

$$\mathcal{E}N^2 = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} (a_1 i_1 + \dots + a_9 i_9)^2 P_{\alpha\beta}^{i_1 \dots i_9}(x),$$

i.e.,

$$(4.9) \quad \mathcal{E}N^2 = [P_{\alpha\beta}(x)]^{-1} \left[S_{\alpha\beta}(x) + \left\{ \sum_{k=1}^9 a_k^2 \frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right\}_{z_1 = \dots = z_9 = 1} \right],$$

where

$$(4.10) \quad S_{\alpha\beta}(x) = \left\{ \sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} G_{\alpha\beta}(x) \right\}_{z_1 = \dots = z_9 = 1}.$$

Defining, as before, the vector $\mathcal{S}_\alpha(x)$ and the corresponding eight-component vector $\mathcal{s}_\alpha(x)$ we find from (3.2) that the equation for $\mathcal{S}_\alpha(x)$ is

$$(4.11) \quad A_\theta(1, \dots, 1) \mathcal{S}_\alpha(x) + 2B \mathcal{M}_\alpha(x) = 0,$$

which can be rewritten

$$(4.12) \quad \frac{\partial}{\partial x} \mathcal{s}_\alpha(x) = R(1, \dots, 1) \mathcal{s}_\alpha(x) + \mathcal{W}_\alpha(x),$$

where

$$\begin{aligned} \mathcal{W}_\alpha(x) = & \left\{ \sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} R(z_1, \dots, z_9) p_\alpha(x) \right. \\ & \left. + 2 \sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} R(z_1, \dots, z_9) \mathcal{m}_\alpha(x) \right\}_{z_1 = \dots = z_9 = 1}. \end{aligned}$$

The solution of (4.12) is given by

$$(4.13) \quad \mathcal{s}_\alpha(x) = \exp[xR(1, \dots, 1)] \mathcal{s}_\alpha(0) + \int_0^x \exp[(x-y)R(1, \dots, 1)] \mathcal{W}_\alpha(y) dy,$$

where $s_{\alpha}(0)$ is uniquely determined by the boundary conditions

$$(4.14) \quad \begin{cases} s_{\alpha\beta}(0) = 0, & \beta = 7, 8, 9, 10, \\ s_{\alpha\beta}(t) = 0, & \beta = 1, 2, 3, 4. \end{cases}$$

The only term in equation (4.9) not yet accounted for is the sum

$$\left\{ \sum_{k=1}^9 a_k^2 \frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right\}_{z_1=\dots=z_9=1},$$

which is simply $M_{\alpha\beta}(x)$ with a new set of constants $a_1^2, \dots, a_9^2$ replacing the constants $a_1, \dots, a_9$. (If each a_i is either 1 or 0, as will often be the case, then $a_i^2 = a_i$ and no additional computation is required.)

The determination of each of the moments of N requires the solution of a two-point boundary value problem. However, it is sufficient to solve the problem for a small number of slab thicknesses owing to an iteration relation which holds for the generating functions $G_{\alpha\beta}(0)$.

This will be considered in Sect. 5.

5. - An iteration relation for the generating functions $G_{\alpha\beta}(0)$.

Consider a particle which is incident on the left face of the slab $\{x: 0 \leq x \leq t\}$ in direction β ($\beta=1, \dots, 6$). If $t > 0$, the possible directions of emergence, α , are 1, 2, 3, 4, in which case the particle is transmitted, or 7, 8, 9, 10, in which case the particle is reflected.

We define the two 4×6 matrices of generating functions

$$(5.1) \quad G_T^*(t) = [G_{ij}(0)] \quad , \quad i=1, \dots, 4, \quad j=1, \dots, 6,$$

$$(5.2) \quad G_R^*(t) = [G_{(11-i)j}(0)], \quad i=1, \dots, 4, \quad j=1, \dots, 6,$$

and the analogous 4×4 matrices $G_T(t)$ and $G_R(t)$ obtained by deleting the

last two columns of $G_T^*(t)$ and $G_R^*(t)$ respectively. Clearly the matrices $G_T^*(t)$, $G_T(t)$ refer to transmitted particles and $G_R^*(t)$, $G_R(t)$ to reflected particles. The following theorem relates the transmission and reflection matrices for slab thickness (t_1+t_2) to the corresponding matrices for thicknesses t_1 and t_2 . Given the matrices G_T^* and G_R^* for a set of slab thicknesses $t_1, \dots, t_n$, the result of the theorem can be used to generate G_T^* and G_R^* for any linear combination of the thicknesses $t_1, \dots, t_n$, with positive integral coefficients. Corollaries 1, 2 and 3 show how the results can also be applied to the matrices

$$(5.3) \quad P_{R(T)}^* = [G_{R(T)}^*]_{z_1=\dots=z_9=1},$$

$$(5.4) \quad M_{R(T)}^* = \left[\sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} G_{R(T)}^* \right]_{z_1=\dots=z_9=1},$$

$$(5.5) \quad S_{R(T)}^* = \left[\sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} G_{R(T)}^* \right]_{z_1=\dots=z_9=1},$$

and thus to the determination of the moments of the distribution of the random variable $N = \sum_{i=1}^9 a_i N_i$ defined in Sect. 4. Matrices $P_{R(T)}$, $M_{R(T)}$ and $S_{R(T)}$ are defined from $P_{R(T)}^*$, $M_{R(T)}^*$ and $S_{R(T)}^*$ as before, by deleting the fifth and sixth columns.

Theorem.

If $G_T^*(t)$, $G_T(t)$, $G_R^*(t)$ and $G_R(t)$ are the transmission and reflection matrices defined above, then

$$(5.6) \quad G_T^*(t_1+t_2) = G_T(t_2) [I - G_R(t_1)G_R(t_2)]^{-1} G_T^*(t_1),$$

$$(5.7) \quad G_R^*(t_1+t_2) = G_R^*(t_1) + G_T(t_1)G_R(t_2) [I - G_R(t_1)G_R(t_2)]^{-1} G_T^*(t_1).$$

Proof.

Let $\pi_{\alpha\beta}^{i_1 \dots i_9}$ be the probability of emergence in direction α from a slab of thickness t_1 after i_k collisions of type k , $k=1, \dots, 9$, conditional on incident direction β . Let $\psi_{\alpha\beta}^{i_1 \dots i_9}$ be the corresponding probability for a slab of thickness t_2 , and let $P_{\alpha\beta}^{i_1 \dots i_9}$ be the corresponding probability for the slab of thickness (t_1+t_2) obtained by placing the two smaller slabs together.

Then the transmission probabilities $P_{\alpha\beta}^{i_1 \dots i_9}$, $\alpha=1, \dots, 4$, $\beta=1, \dots, 6$, are given by

$$(5.8) \quad P_{\alpha\beta}^{i_1 \dots i_9} = \sum_{n=0}^{\infty} \sum_{m_j^{(1)} + \dots + m_j^{(2n+2)} = i_j} \sum_{\gamma_1=1}^4 \dots \sum_{\gamma_{2k}=7}^{10} \sum_{\gamma_{2k+1}=1}^4 \dots \sum_{\gamma_{2n+1}=1}^4 \psi_{\alpha\gamma_{2n+1}}^{m_1^{(2n+2)} \dots m_9^{(2n+2)}} \dots \pi_{\gamma_1\beta}^{m_1^{(1)} \dots m_9^{(1)}},$$

where the k -th factor on the right-hand side ($2 \leq k \leq 2n+1$) is

$$d_k = \begin{cases} \psi_{\gamma_{2n+3-k}\gamma_{2n+2-k}}^{m_1^{(2n+3-k)} \dots m_9^{(2n+3-k)}} & \text{if } k \text{ is odd,} \\ \pi_{\gamma_{2n+3-k}\gamma_{2n+2-k}}^{m_1^{(2n+3-k)} \dots m_9^{(2n+3-k)}} & \text{if } k \text{ is even.} \end{cases}$$

The n -th term in the outer summation arises from particles which make $(n+1)$ passages through the interface between the two slabs before emerging from the extreme right-hand face. The second summation is over all non-negative sets of integers $m_j^{(h)}$, $j=1, \dots, 9$, $h=1, \dots, 2n+2$, such that

$$\sum_{h=1}^{2n+2} m_j^{(h)} = i_j, \quad j=1, \dots, 9.$$

Multiplying equation (5.8) by $z_1^{i_1} \dots z_9^{i_9}$ and summing over all non-negative sets of integers $i_1, \dots, i_9$ we find immediately that

$$(5.9) \quad G_T^*(t_1+t_2) = G_T(t_2) \sum_{n=0}^{\infty} [G_R(t_1)G_R(t_2)]^n G_T^*(t_1).$$

Since the sum of the moduli of the elements in any column of the matrix $G_R(t_1)G_R(t_2)$ is strictly less than 1, it follows (cf. Frechet[22], p. 105) that the spectral radius of $G_R(t_1)G_R(t_2)$ is also less than 1. The summation in eq. (5.9) can therefore be replaced by $[I - G_R(t_1)G_R(t_2)]^{-1}$, thus establishing the result (5.6) for $G_T^*(t_1+t_2)$.

Equation (5.7) for the reflection matrix $G_R^*(t_1+t_2)$ is established in an exactly parallel manner.

Corollary 1.

Setting $z_1 = \dots = z_9 = 1$ in equations (5.6) and (5.7) we find immediately that

$$(5.10) \quad P_T^*(t_1+t_2) = P_T(t_2) [I - P_R(t_1)P_R(t_2)]^{-1} P_T^*(t_1),$$

$$(5.11) \quad P_R^*(t_1+t_2) = P_R^*(t_1) + P_T(t_1)P_R(t_2) [I - P_R(t_1)P_R(t_2)]^{-1} P_T^*(t_1).$$

Corollary 2.

Defining $M_T^*(t)$ and $M_R^*(t)$ as in equation (5.4) it follows, on differentiating (5.6) and (5.7) with respect to z_k , $k=1, \dots, 9$, that

$$(5.12) \quad M_T^*(t_1+t_2) = M_T(t_2)V^{-1} P_T^*(t_1) + P_T(t_2) V^{-1}WV^{-1} P_T^*(t_1) \\ + P_T(t_2) V^{-1}M_T^*(t_1) ,$$

$$(5.13) \quad M_R^*(t_1+t_2) = M_R^*(t_1) + XV^{-1} P_T^*(t_1) + P_T(t_1)P_R(t_2) V^{-1}WV^{-1} P_T^*(t_1) \\ + P_T(t_1)P_R(t_2) V^{-1} M_T^*(t_1) ,$$

where $V = I - P_R(t_1)P_R(t_2),$

$$W = M_R(t_1)P_R(t_2) + P_R(t_1)M_R(t_2),$$

and $X = M_T(t_1)P_R(t_2) + P_T(t_1)M_R(t_2) .$

Corollary 3.

Differentiating equations (5.6) and (5.7) twice in order to form the derivatives appearing in equation (5.5) for S_T^* and S_R^* , we find that

$$(5.14) \quad S_T^*(t_1+t_2) = S_T(t_2) V^{-1} P_T^*(t_1) + P_T(t_2) V^{-1} S_T^*(t_1) \\ + P_T(t_2) V^{-1}ZV^{-1} P_T^*(t_1) + 2 M_T(t_2) V^{-1}WV^{-1} P_T^*(t_1) \\ + 2 P_T(t_2) V^{-1}WV^{-1} M_T^*(t_1) + 2 M_T(t_2) V^{-1} M_T^*(t_1) \\ + 2 P_T(t_2) V^{-1}WV^{-1}WV^{-1} P_T^*(t_1) ,$$

$$(5.15) \quad S_R^*(t_1+t_2) = S_R^*(t_1) + YV^{-1} P_T^*(t_1) + P_T(t_1)P_R(t_2) V^{-1} S_T^*(t_1) \\ + 2 XV^{-1} M_T^*(t_1) + 2 P_T(t_1)P_R(t_2) V^{-1}WV^{-1}WV^{-1} P_T^*(t_1) \\ + 2 P_T(t_1)P_R(t_2) V^{-1}WV^{-1} M_T^*(t_1) + 2 XV^{-1}WV^{-1} P_T^*(t_1) \\ + P_T(t_1)P_R(t_2) V^{-1}ZV^{-1} P_T^*(t_1) ,$$

where $Y = S_T(t_1)P_R(t_2) + 2 M_T(t_1)M_R(t_2) + P_T(t_1)S_R(t_2)$,

and $Z = S_R(t_1)P_R(t_2) + 2 M_R(t_1)M_R(t_2) + P_R(t_1)S_R(t_2)$.

The application of this theorem and its corollaries is very simple, involving as it does only elementary operations on 4×4 and 4×6 matrices.

6. - The distribution of emergent velocity.

Consider again a slab $[0, t]$ and a particle that originates at a point x inside the slab moving in direction β . We wish to find the distribution of the particle's emergent velocity given that it emerges from the slab in direction α . In the notation of the preceding sections $G_{\alpha\beta}(x)[P_{\alpha\beta}(x)]^{-1}$ is the generating function for the joint distribution, conditional on emergence in direction α , of the number of collisions of each type before emergence.

Let $\psi_i(dv|u)$ be the transition probability for the absolute velocities u and v before and after a collision of type i . If we assume, as in Chapter 1. Sect. 5, that

$$(6.1) \quad \int_w \psi_i(dv|w)\psi_j(dw|u) = \int_w \psi_j(dv|w)\psi_i(dw|u) , \quad i, j=1, 2, \dots, 9 ,$$

then the probability $F_{\alpha\beta}(V|u, x)$ that the emergent velocity belongs to the set V , conditional on initial state (β, u, x) and emergent direction α , is given by $G_{\alpha\beta}(x)[P_{\alpha\beta}(x)]^{-1}$ with terms such as $z_1^{i_1} \dots z_9^{i_9}$ replaced by $\psi_1^{(i_1)} \circ \psi_2^{(i_2)} \circ \dots \circ \psi_9^{(i_9)}$. The functions $\psi^{(n)}$ and $\psi \circ \phi$ are defined as follows:

$$(6.2) \quad \begin{cases} \psi^{(n)} = \int \psi(V|w) \psi^{(n-1)}(dw|u) , & n=1,2,\dots , \\ \psi^{(0)} = \delta(V|u) = \begin{cases} 1, & u \in V , \\ 0, & u \notin V, \end{cases} \\ \psi \circ \phi = \int \psi(V|w) \phi(dw|u) . \end{cases}$$

Further simplification occurs if we assume particular forms for $\psi_i(dv|u)$ as in the following examples

Example 1.

$$\psi_i(dv|u) = \begin{cases} \frac{1}{u} f_i\left(\frac{v}{u}\right) dv , & 0 \leq v \leq u \\ 0 , & \text{otherwise.} \end{cases}$$

It is convenient in this case to introduce the Mellin transforms

$$(6.3) \quad \begin{cases} \Psi_i(\omega) = \int_{v=0}^u f_i\left(\frac{v}{u}\right) \left(\frac{v}{u}\right)^{\omega-1} \frac{dv}{u} , \\ \mathcal{F}_{\alpha\beta}(\omega|u,x) = \int_{v=0}^u \left(\frac{v}{u}\right)^{\omega-1} F_{\alpha\beta}(dv|u,x) , \end{cases}$$

from which, using the preceding remarks, we find

$$(6.4) \quad \mathcal{F}_{\alpha\beta}(\omega|u,x) = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1+\dots+i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) \Psi_1(\omega)^{i_1} \dots \Psi_9(\omega)^{i_9} .$$

Clearly the right side of (6.4) is just $[P_{\alpha\beta}(x)]^{-1} G_{\alpha\beta}(x)$ with z_k replaced by $\Psi_k(\omega)$, $k=1,\dots,9$.

If u is the initial velocity of a particle and v is its emergent velocity, then we denote by $\mathcal{E}_{\alpha\beta}[(v/u)^n|u,x]$ the expectation of the random variable $(v/u)^n$ conditional upon initial state (β, u, x) and emergent direction α . To determine $\mathcal{E}_{\alpha\beta}[(v/u)^n|u,x]$ we put $\omega=n+1$ in $\mathcal{F}_{\alpha\beta}(\omega|u,x)$. $P_{\alpha\beta}(x)$ is determined as in Sect. 3 and $G_{\alpha\beta}(x)$ is determined from the equation

$$(6.5) \quad A_{\theta}(\psi_1(n+1), \dots, \psi_9(n+1)) G_{\alpha}(x) = 0,$$

with boundary conditions $G_{\alpha\beta}(0) = \delta_{\alpha\beta}$, $\beta = 7, 8, 9, 10$, $G_{\alpha\beta}(t) = \delta_{\alpha\beta}$, $\beta = 1, 2, 3, 4$.

Example 2.

$$\psi_i(dv|u) = \delta(v - c_i u) dv, \quad 0 \leq c_i \leq 1, i=1, \dots, 9.$$

This model corresponds to a physical situation in which the logarithmic energy of the particle decreases by a fixed amount for each particular angular deflection. In this case the distribution of emergent velocity is discrete since a particle which starts with velocity u and makes i_1 collisions of type 1, i_2 of type 2, ..., and i_9 of type 9 has a resultant velocity

$$v = u c_1^{i_1} \dots c_9^{i_9}.$$

If we again take Mellin transforms as in Example 1 we find that

$$(6.6) \quad \mathcal{F}_{\alpha\beta}(\omega|u,x) = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) c_1^{(\omega-1)i_1} \dots c_9^{(\omega-1)i_9},$$

which is simply $[P_{\alpha\beta}(x)]^{-1} G_{\alpha\beta}(x)$ with z_k replaced by $c_k^{\omega-1}$, $k=1, \dots, 9$.

To find moments of the distribution, say $\xi_{\alpha\beta}[(v/u)^n | u, x]$, we put $\omega=n+1$ as before, determining $P_{\alpha\beta}(x)$ as in Sect. 3 and $G_{\alpha\beta}(x)$ from the equation

$$(6.7) \quad A_\theta(c_1^n, \dots, c_9^n) G_\alpha(x) = 0, \quad \alpha=1, \dots, 10,$$

with boundary conditions $G_{\alpha\beta}(0) = \delta_{\alpha\beta}$, $\beta=7, 8, 9, 10$, $G_{\alpha\beta}(t) = \delta_{\alpha\beta}$, $\beta=1, 2, 3, 4$.

7. - The steady-state distribution of particles in a finite slab.

Suppose now that there is a steady stream of particles incident on one face of the slab $\{x: 0 \leq x \leq t\}$ and that the time-independent solution for the mean number of particles per second crossing the plane $x=\xi$ in a given direction is required.

The incident particles can be represented by a column vector $\underline{n} = [n_i]$ where n_i is the number of particles incident per second in direction i , $i=1, \dots, 6$. The particles crossing the plane $x=\xi$ can be represented by two 4-component vectors, $\underline{n}^L$, the vector of particles crossing $x=\xi$ in directions 1, 2, 3, 4, and $\underline{n}^R$, the vector of particles crossing $x=\xi$ in directions 10, 9, 8, 7. In terms of the matrices $P_T^*(t)$, $P_R^*(t)$, $P_T(t)$, $P_R(t)$ defined by equation (5.3), the mean number of particles per second crossing the plane $x=\xi$ from the left for the $(k+1)$ th time is

$$(7.1) \quad \underline{n}_k^L = [P_R(\xi) P_R(t-\xi)]^k P_T^*(\xi) \underline{n}, \quad k=0, 1, 2, \dots$$

Hence the total number of particles per second crossing the plane $x=\xi$ from the left is given by

$$(7.2) \quad \tilde{n}^L = \sum_{k=0}^{\infty} \tilde{n}_k^L = [I - P_R(\xi) P_R(t-\xi)]^{-1} P_T^*(\xi) \tilde{n}.$$

Similarly the total number of particles per second crossing the plane $x = \xi$ from the right is given by

$$(7.3) \quad \tilde{n}^R = \sum_{k=0}^{\infty} \tilde{n}_k^R = [I - P_R(t-\xi) P_R(\xi)]^{-1} P_R(t-\xi) P_T^*(\xi) \tilde{n}.$$

We remark that the argument of this section can also be used when there is a distribution of sources in the interior of the scattering medium.

8. - The effect of absorption.

Consider now the modified process in which there is at each collision a constant probability of absorption, $r = 1 - p_1 - p_9 - \sum_{j=2}^8 p_j$. The Kolmogorov eqs. (3.2) for the generating functions $G_{\alpha\beta}(x)$ remain the same and so also do the boundary conditions. Thus the behaviour of the particles when there is absorption (i.e., $r > 0$) is determined by exactly the same equations, the only difference being that the probabilities $p_1, p_2, \dots, p_8, p_9$ will no longer add to 1. All the results of this chapter therefore apply without modification to the process with absorption. The numerical calculations are in fact simplified when there is a non-zero probability of absorption owing to the fact that the matrix $R(1, \dots, 1)$ (see equation (3.5)) no longer has zero as a double eigenvalue.

9. - A model which uses 120 directions of motion.

As already indicated in Sect. 1, it is possible to choose a set of 120 unit vectors $\{a_1, \dots, a_{120}\}$ in three dimensions having the property that the set of scalar products $\{a_i \cdot a_1, \dots, a_i \cdot a_{120}\}$ is the same for each i , $i=1, \dots, 120$. This can be done as follows: each triangular face of the icosahedron is subdivided by its altitudes into 6 congruent triangles. If we consider the set of 120 triangles formed by subdividing each of the 20 faces in this way and consider the unit vectors along the lines joining the centre of the icosahedron to a representative point on each small triangle (for example the centroid), we arrive at a set of 120 unit vectors a_i having the above property.

A numerical study using the set of 120 directions obtained using the centroids of the small triangles has been carried out by Brockwell and Landwehr[7]. The icosahedron is oriented as in Fig. 4, Sect. 2, so that the directions of motion fall into classes 1, ..., 30 where class i consists of the 4 directions of motion making an angle with the positive x -axis whose cosine is μ_i . The numerical values of μ_i are as follows:

$$\mu_1 = 0.971400229 = -\mu_{30}$$

$$\mu_2 = 0.906819049 = -\mu_{29}$$

$$\mu_3 = 0.794101899 = -\mu_{28}$$

$$\mu_4 = 0.777656688 = -\mu_{27}$$

$$\mu_5 = 0.689607354 = -\mu_{26}$$

$$\mu_6 = 0.664939538 = -\mu_{25}$$

$$\mu_7 = 0.619944324 = -\mu_{24}$$

$$\mu_8 = 0.495863814 = -\mu_{23}$$

$$\mu_9 = 0.351455904 = -\mu_{22}$$

$$\mu_{10} = 0.313483534 = -\mu_{21}$$

$$\mu_{11} = 0.286874724 = -\mu_{20}$$

$$\mu_{12} = 0.281792875 = -\mu_{19}$$

$$\mu_{13} = 0.208989089 = -\mu_{18}$$

$$\mu_{14} = 0.112717150 = -\mu_{17}$$

$$\mu_{15} = 0.104494545 = -\mu_{16}$$

The set of angular deflections $\{\varepsilon_1, \dots, \varepsilon_{120}\}$ for this model contains 76 distinct angles $\{\theta_1, \dots, \theta_{76}\}$ (see Ref. [7]). The analogue of the generating function $G_{\alpha\beta}(x)$ of Sect. 3 is therefore of the form

$$(9.1) \quad G'_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_{76} \geq 0} P_{\alpha\beta}^{i_1 \dots i_{76}}(x) z_1^{i_1} \dots z_{76}^{i_{76}}, \quad \alpha=1, \dots, 30, \\ \beta=1, \dots, 30,$$

and the Kolmogorov equation (3.2) is replaced by

$$(9.2) \quad A'_\theta(z_1, \dots, z_{76}) G'_\alpha(x) = 0,$$

with boundary conditions

$$(9.3) \quad G'_{\alpha\beta}(0) = \delta_{\alpha\beta}, \quad \beta=16, \dots, 30, \\ G'_{\alpha\beta}(t) = \delta_{\alpha\beta}, \quad \beta=1, \dots, 15.$$

The matrix $A'_\theta(z_1, \dots, z_{76})$ is a 30 x 30 matrix as compared with the 10 x 10 matrix $A_\theta(z_1, \dots, z_9)$ of Sect. 3. Equation (9.2) can be rewritten in the more convenient form

$$(9.4) \quad \frac{\partial}{\partial x} \tilde{G}'_{\alpha}(x) = R'(z_1, \dots, z_{76}) \tilde{G}'_{\alpha}(x) \quad ,$$

and the analysis then proceeds exactly as for the case of 30 directions already described. (Since there are no directions of motion perpendicular to the x-axis, there is no analogue of the vector $\tilde{g}_{\alpha}(x)$ introduced in Sect. 3.)

Numerical results derived for this model are given in detail in Ref. [7]. For the purpose of this thesis we shall consider only the model which employs 30 directions. In Chapter 7 some of the results will be compared with the corresponding results obtained from the 120-direction approximation.

CHAPTER 7

PHYSICAL APPLICATIONS OF THE THREE-DIMENSIONAL MODEL

1. - Introduction.

In this chapter we shall examine a number of physical applications of the three-dimensional model developed in Chapter 6. The earliest problems to receive attention from astronomers were concerned with scattering by a three-dimensional half-space (this being the idealization of a stellar atmosphere). The Milne problem is concerned with the determination of the angular distribution of emergent radiation from a half-space when the source of radiation is a very large distance below the surface. For isotropic scattering the exact solution of this problem was obtained by Wiener and Hopf [49] in closed form and is given in the book by Hopf [26]. Explicit solutions of the problems of transport theory, however, are very rare. Various approximate methods of analysis have been devised; for example the spherical harmonics method, in which the distribution of particles is expanded in terms of Legendre polynomials and the series truncated (see for example Chandrasekhar [9]), and the Gaussian quadrature method devised by Wick [48] and extensively developed by Chandrasekhar [10,...,13], in which the integral in the transport equation is replaced by an approximating sum. Wang and Guth [45] formulated the spherical harmonic method in such a way as to demonstrate its close relationship with the Gaussian quadrature method. Chandrasekhar [14,15], making use of invariance principles introduced by Ambarzumian [1], has found exact solutions to many of the problems of radiative transfer by expressing the solutions in terms of his X-, Y- and H-functions, defined as solutions of certain integral equations.

In Sect. 2 we apply the methods of Chapter 6 to the Milne problem with isotropic scattering law and compare the results with the known exact solutions as given by Chandrasekhar [17], p. 135, and with the results of the standard approximation techniques. Scattering by slabs of finite thickness is also considered and the matrices $P_R^*(T)$, $M_R^*(T)$ and $S_R^*(T)$ of Chapter 6 are tabulated for several slab thicknesses t in the special case where the coefficients a_i in the linear combination $N = \sum_{i=1}^9 a_i N_i$ are all 1. From these matrices it is possible to deduce the mean and variance of the total number of collisions experienced by an incident particle before emerging from the slab. As an illustration (and also for the sake of comparison with the one-dimensional method of Chapter 5) we determine the probability of transmission and the mean and variance of the number of collisions experienced by transmitted particles when the particles are isotropically incident on one face of the slab.

Analogous results are derived in Sect. 3 when the particles are scattered according to Rayleigh's law, i.e., the probability, given a collision, of being scattered through an angle θ into the element of solid angle $d\Omega$ is given by

$$(1.1) \quad \sigma(\theta)d\Omega = \frac{3}{16\pi} (1 + \cos^2\theta) d\Omega .$$

In Sect. 4 we examine the multiple Compton scattering of low-energy gamma radiation applying the theory of Chapter 6 to determine the spectral distribution of gamma radiation which is transmitted through a slab of scattering material. This problem has previously been studied

by Chandrasekhar [16] and O'Rourke [34] using Gaussian quadrature; however, as Chandrasekhar points out, their results, although derived formally for a monochromatic source, are of physical significance only when integrated with respect to a relatively smooth spectral intensity of incident radiation. Another undesirable feature of their results is the existence of a non-zero (though small) probability of a negative wave-length increment which for Compton scattering is physically impossible. The form of the solution derived in Sect. 4 does not exhibit these features. Instead of the isotropic scattering law used in [16] and [34] we use the more accurate Rayleigh law (1.1). As in the previous treatments of the problem the effects of polarization and photo-electric absorption have been neglected; however, cases in which there is a constant probability of absorption at each collision could be treated in exactly the same way (cf. Chapter 6, Sect. 8).

2. - Isotropic scattering in a plane slab.

We shall be concerned first with the Milne problem, i.e., with isotropic scattering in the half-space $[x: 0 \leq x < \infty]$. In particular, we consider a particle which starts from a point with coordinate x ($0 \leq x < \infty$) moving initially in direction β . The distribution of emergent direction of such a particle is given (in the icosahedral approximation of Chapter 6) by the probabilities $P_{\alpha\beta}(x)$, $\alpha=6,7,8,9,10$. The Milne problem is of course concerned with particles which can move in any direction in space and which are scattered according to a continuous scattering law. The first step therefore is to set up a discrete scattering function to replace the continuous isotropic scattering law,

i.e., to determine the transition probabilities $[p_1, {}^4p_2, \dots, {}^4p_8, p_9]$ defined in Chapter 6, Sect. 2. Now for isotropic scattering the probability $\sigma(\theta)d\Omega$ of being scattered through an angle θ into the element $d\Omega$ of solid angle is defined by the equation

$$(2.1) \quad \sigma(\theta) = (4\pi)^{-1}, \quad 0 \leq \theta \leq \pi.$$

The probabilities p_i , $i=1, \dots, 9$ are determined by requiring them to be proportional to $\sigma(\theta_i)$, $i=1, \dots, 9$, where θ_i is the angular deflection corresponding to a deflection of type i . This condition, together with the condition that for pure scattering

$$(2.2) \quad p_1 + p_9 + 4 \sum_{i=2}^8 p_i = 1,$$

uniquely determines the probabilities p_i . Clearly for isotropic scattering

$$(2.3) \quad p_i = \frac{1}{30}, \quad i=1, \dots, 9.$$

The probabilities $P_{\alpha\beta}(x)$ can now be found as described in Chapter 6, Sect. 3. Since $P_{\alpha\beta}(x)$ is bounded as $x \rightarrow \infty$, the solution (3.8) of Chapter 6 must be of the form

$$(2.4) \quad \underset{\sim}{p}_{\alpha}(x) = a_{\alpha}^{(1)} \exp(\gamma_1 x) \underset{\sim}{e}_1 + a_{\alpha}^{(2)} \exp(\gamma_2 x) \underset{\sim}{e}_2 + a_{\alpha}^{(3)} \exp(\gamma_3 x) \underset{\sim}{e}_3 + a_{\alpha}^{(4)} \exp(\gamma_4 x) \underset{\sim}{e}_4$$

where γ_i , $i=1, \dots, 4$ are the non-positive roots of the characteristic equation (3.11) of Chapter 6 and the eigenvectors $\underset{\sim}{e}_i$, $i=1, \dots, 4$ are defined by equation (3.12). The constants $a_{\alpha}^{(1)}, \dots, a_{\alpha}^{(4)}$ are uniquely

determined for each α by the boundary conditions

$$(2.5) \quad P_{\alpha\beta}(0) = \delta_{\alpha\beta}, \quad \beta=7,8,9,10.$$

Solving for $\gamma_1, \dots, \gamma_4$ and $a_{\alpha}^{(1)}, \dots, a_{\alpha}^{(4)}$ we find that (for zero absorption, i.e., $r=0$)

$$\left\{ \begin{array}{l} \gamma_1 = -2.852422, \\ \gamma_2 = -1.665073, \\ \gamma_3 = -1.044615, \\ \gamma_4 = 0 \end{array} \right.,$$

and

$$\left\{ \begin{array}{l} a_7^{(1)} = .099600, a_7^{(2)} = .01861, a_7^{(3)} = .001664, a_7^{(4)} = .11906, \\ a_8^{(1)} = -.054192, a_8^{(2)} = .10555, a_8^{(3)} = .004615, a_8^{(4)} = .23289, \\ a_9^{(1)} = -.036417, a_9^{(2)} = -.10500, a_9^{(3)} = .029344, a_9^{(4)} = .48106, \\ a_{10}^{(1)} = -.008992, a_{10}^{(2)} = -.01916, a_{10}^{(3)} = .035621, a_{10}^{(4)} = .16790. \end{array} \right.$$

We see at once from these results that for a particle which starts from a point several mean free paths within the scattering material the emergent angular distribution is very close to the emergent angular distribution for a particle which starts at infinity. Probabilities of the various emergent directions in this limiting case are independent of the initial direction of motion of the particle and are given by

$$(2.6) \quad p_{\alpha}^{(\infty)} = a_{\alpha}^{(4)} e_{\alpha},$$

i.e.,

$$(2.7) \quad \begin{bmatrix} P_{7\beta}(\infty) \\ P_{8\beta}(\infty) \\ P_{9\beta}(\infty) \\ P_{10\beta}(\infty) \end{bmatrix} = \begin{bmatrix} .11906 \\ .23289 \\ .48016 \\ .16790 \end{bmatrix} \quad \text{for all } \beta \quad .$$

In order to relate the discrete angular distribution obtained in this way to the continuous distribution which would be obtained if the discrete set of transition probabilities $\{p_1, p_2, \dots, p_8, p_9\}$ were replaced by a uniform distribution over the whole of the unit sphere we proceed as follows. Since the probabilities $P_{7\beta}$, $P_{8\beta}$, $P_{9\beta}$ and $P_{10\beta}$ of equation (2.7) correspond to areas $8\pi/15$, $8\pi/15$, $8\pi/15$ and $2\pi/15$ respectively of the unit sphere we can estimate the probability density per unit solid angle of the emergent angular distribution (in directions whose angles with the positive x-axis have cosines μ_6 , μ_7 , μ_8 , μ_9 and μ_{10}) by

$$\begin{cases} \pi(\mu_6) = 0 \\ \pi(\mu_\alpha) = 15 P_{\alpha\beta}/8\pi, \quad \alpha=7,8,9, \\ \pi(\mu_{10}) = 15 P_{10\beta}/2\pi \end{cases} .$$

Hence if $f(\mu)$ is the corresponding probability density normalized so

that $\int_{-1}^0 f(\mu) d\mu = 1$, then $f(\mu_6), \dots, f(\mu_{10})$ are estimated by

$$\begin{cases} f(\mu_6) = 0, \\ f(\mu_\alpha) = 3.75 P_\alpha (P_7 + P_8 + P_9 + P_{10})^{-1}, \quad \alpha=7,8,9, \\ f(\mu_{10}) = 15 P_{10} (P_7 + P_8 + P_9 + P_{10})^{-1} \end{cases} .$$

The results are tabulated below together with the exact solution and the approximations given by Wang and Guth [45] and Chandrasekhar [9,17].

Table 3.

μ	Exact solution	Icosahedral approximation	Gaussian quadrature approximation $n=4$	Wang and Guth's S.H. method $n=3$ r odd	Chandrasekhar's S.H. method $n=3$
1	2.5182	2.5185	2.5096	2.5128	2.4837
0.8090	1.7997	1.8006	1.7922	1.7948	1.7702
0.5000	0.8716	0.8733	0.8660	0.8680	0.8520
0.3090	0.4441	0.4465	0.4384	0.4417	0.4324
0	0	0	0	0	0

As can be seen from these figures the icosahedral approximation is very good compared with the other standard approximations. Densities at values of μ other than those given could be obtained by interpolation.

For scattering by slabs of finite thickness, solutions are obtained exactly as described in Chapter 6, Sect. 3. The necessary calculations were carried out using a computer program which is described in the Appendix. Numerical tabulations of the matrices $P_R^*(T)$, $M_R^*(T)$ and $S_R^*(T)$ are given in the Appendix for various values of the slab thickness, t . The matrices $M_R^*(T)$ and $S_R^*(T)$ have been tabulated only for one particular set of coefficients $a_1, \dots, a_9$, namely that in which $a_i=1$, $i=1, \dots, 9$. When used in equations (4.2) and (4.9) they therefore give the mean and mean square of the total

number of collisions experienced by a particle incident on the slab in direction β and emergent in direction α . An application in which the coefficients a_i are not all unity is discussed in connection with the multiple scattering of low energy gamma radiation in Sect. 4.

For the sake of numerical comparison with the results of Chapter 5 and with the higher order approximation using 120 directions instead of 30, the transmission probability and the mean and variance of the number of collisions experienced by transmitted particles are given in Table 4 for particles which are incident isotropically on slabs of various thicknesses. In Table 5 the transmission probabilities are compared with those obtained in Chapter 5 using both Fermi's method and its generalization and also with the results of Brockwell and Landwehr [7] for the model with 120 directions. A similar comparison for the mean number of collisions experienced by transmitted particles is made in Table 6.

Table 4

Thickness, t	Transmission probability	Mean number of collisions	Variance of number of collisions
0.5	.63011	0.8712	0.7167
1.0	.48638	1.9398	2.8628
2.0	.33958	4.8106	14.486
4.0	.21366	13.550	85.519
8.0	.12290	43.121	631.65

Transmission of particles isotropically incident on a slab of thickness t (mean free paths).

Table 5.

Thickness, t	0.5	1.0	2.0	4.0	8.0
Fermi's approximation	.69783	.53590	.36603	.22401	.12613
Method of Chapter 5	.62083	.47929	.33743	.21294	.12255
30 direction approx.	.63011	.48638	.33958	.21366	.12290
120 direction approx.	.62940	.48512	.33904	.21331	.12266

Comparison of transmission probabilities.

Table 6.

Thickness, t	0.5	1.0	2.0	4.0	8.0
Fermi's approximation	0.6016	1.5000	4.0981	12.360	46.856
Method of Chapter 5	0.9184	1.9973	4.8266	13.543	43.142
30 direction approx.	0.8712	1.9398	4.8106	13.550	43.121
120 direction approx.	0.8733	1.9394	4.7901	13.512	43.064

Comparison of mean numbers of collisions.

3. - Rayleigh scattering.

A major feature of the approach to scattering problems described in Chapter 6 is the fact that the method of solution applies unchanged to a very general class of scattering laws. In this section we consider the same problems as in Sect. 2 except that the isotropic scattering law is replaced by Rayleigh's law, i.e., the probability, given a collision that the particle is scattered through an angle θ into the element of solid angle $d\Omega$ is given by $\sigma(\theta)d\Omega$ where

$$(3.1) \quad \sigma(\theta) = \frac{3}{16\pi} (1 + \cos^2\theta) .$$

Using the argument of Sect. 2 to determine the discrete approximation to the continuous scattering function $\sigma(\theta)$ we find that

$$(3.2) \quad \left\{ \begin{array}{l} p_1 = p_9 = \frac{1}{20} \\ p_2 = p_8 = \frac{1}{320}(11 + \sqrt{5}) \\ p_3 = p_7 = \frac{1}{32} \\ p_4 = p_6 = \frac{1}{320}(11 - \sqrt{5}) \\ p_5 = \frac{1}{40} \end{array} \right.$$

When these probabilities are substituted in the fundamental equation (3.2) of Chapter 6 the solution proceeds in exactly the same way as for isotropic scattering, the only difference being that the determination of the eigenvalues and eigenvectors of the matrix $R(1, \dots, 1)$ requires the use of a computer.

Table 7 shows the result of applying the icosahedral approximation to the Milne problem with Rayleigh scattering, i.e., to the determination of the angular distribution $(f(\mu))$ of emergent photons originating at a large distance below the surface of a half-space. The exact solution as given by Chandrasekhar [17] is compared with the icosahedral approximation, Chandrasekhar's Gaussian quadrature approximation [11] and the approximation of Wilson and Sen [50].

Table 7.

μ	Exact solution	Icosahedral approximation	Chandrasekhar's Gaussian quadrature approximation		Wilson and Sen $n=1$
			$n=2$	$n=3$	
1	2.5292	2.5287	2.4962	2.5152	2.5376
0.8090	1.8036	1.8042	1.7748	1.7910	1.8181
0.5000	0.8689	0.8706	0.8488	0.8593	0.8812
0.3090	0.4404	0.4430	0.4273	0.4334	0.4468
0	0	0	0	0	0

The numerical results for Rayleigh scattering by slabs of finite thickness are again tabulated in the Appendix as the transmission and reflection matrices, P_T^* , P_R^* , M_T^* , M_R^* , S_T^* , S_R^* , the latter four being calculated for the particular linear combination in which $a_i=1$, $i=1, \dots, 9$. These matrices were used to derive the transmission probabilities and the mean and variance of the number of collisions experienced by transmitted particles which are incident isotropically on one face of the slab. A comparison is made in Table 8 of results obtained using 30 directions of motion with the corresponding results obtained using 120 directions. We observe that Fermi's method and the improved method of Chapter 5 cannot be applied to this problem since they are restricted to the case of isotropic scattering. Figure 5 shows the angular distribution of particles ($f(\mu)$) transmitted through slabs of various thicknesses when the particles are again incident isotropically.

4. The multiple Compton scattering of low energy gamma radiation.

The multiple Compton scattering of low energy ($E \ll 0.5$ Mev) photons in slabs of finite thickness has been studied analytically by Chandrasekhar [16] and O'Rourke [34] using the Gaussian quadrature approximation to Boltzmann's equation. Provided the momenta of the scattering electrons and the energy of the scattered photons are small the problem is somewhat simplified owing to the fact that the Klein-Nishina differential cross-section (i.e., the cross-section for scattering through a given angle per unit solid angle) is then effectively energy independent and is given, in Thomson units per electron, by:

$$(4.1) \quad \sigma(\theta) = \frac{3}{16\pi} (1 + \cos^2\theta) ,$$

where θ is the angle of deflection resulting from a collision and

$$1 \text{ Thomson unit} = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 = 0.665 \text{ barns.}$$

The mean free path d of the photons is related to $\sigma(\theta)$ by the equation

$$(4.2) \quad d^{-1} = N \int \sigma(\theta) d\Omega$$

where N is the number of electrons per unit volume. Thus d is independent of the photon energy and, by the transformation (3.4) of Chapter 6, may be taken as unity. Assuming the momenta of the scattering electrons are sufficiently small for Compton's relation to be applicable, then the wavelength of each scattered photon increases

according to the equation:

$$(4.3) \quad \lambda' = \lambda + 1 - \cos\theta, \quad ,$$

where λ and λ' are the wavelengths in Compton units (1 Compton unit = $h/mc = 0.02426 \text{ \AA}$) before and after a collision and θ is the angle of scattering.

For scattering by plane slabs of finite thickness both Chandrasekhar and O'Rourke derive the spectral distribution of forward scattered radiation integrated over all forward directions of emergence. They consider only the isotropic approximation to (4.1), i.e., $\sigma(\theta) = 1/4\pi$. In this section we shall make use of the more accurate cross-section (4.1) (which is in fact the Rayleigh scattering law considered in Sect. 3). The angular distribution of photons transmitted by a finite slab has already been determined in Sect. 3. We shall now show how to derive the spectral distribution of the emergent radiation for each angle of emergence.

Chandrasekhar's method for the Compton scattering problem depends on the replacement of the radiation field intensity $I(x, \mu, \lambda')$ by the first two terms of a power series in $(\lambda' - \lambda)$, i.e.,

$$I(x, \mu, \lambda') = I(x, \mu, \lambda) + (\lambda' - \lambda) \frac{\partial I}{\partial \lambda}(x, \mu, \lambda) \quad .$$

The resulting approximation to the Boltzmann equation is then solved by Gaussian quadrature. As a consequence of this power series expansion Chandrasekhar's results (and also those of O'Rourke) are of physical significance only when integrated with respect to a relatively smooth spectral distribution of incident radiation. The method

to be used does not suffer from this limitation. Assuming a monochromatic source of radiation of wavelength λ_0 we are led quite naturally to determine the distribution function of the emergent wavelength λ as the sum of two components, an absolutely continuous component due to the scattered radiation and a simple discontinuity at λ_0 whose magnitude is equal to the probability π_0 that an incident photon passes through the slab without being scattered.

Like Chandrasekhar and O'Rourke we shall neglect photoelectric absorption and polarization; however if we assume a constant probability of absorption at any collision the problem can be solved in exactly the same way as explained in Chapter 6, Sect. 8.

The discrete approximation to the continuous cross-section (4.1) has already been derived in Sect. 3 and is given by equations (3.2). In addition we must introduce the Compton increments δ_i ($i=1, \dots, 9$), δ_i being the increase in wavelength of a photon resulting from a collision of type i . If θ_i is the angle of deflection resulting from a collision of type i then

$$(4.4) \quad \delta_i = 1 - \cos\theta_i \quad .$$

Thus

$$(4.5) \quad \left\{ \begin{array}{ll} \delta_1 = 0 , & \delta_6 = \frac{1}{4}(3 + \sqrt{5}) , \\ \delta_2 = \frac{1}{4}(3 - \sqrt{5}) , & \delta_7 = \frac{3}{2} , \\ \delta_3 = \frac{1}{2} , & \delta_8 = \frac{1}{4}(5 + \sqrt{5}) , \\ \delta_4 = \frac{1}{4}(5 - \sqrt{5}) , & \delta_9 = 2 . \\ \delta_5 = 1 , & \end{array} \right.$$

We now consider a photon which originates at a point x in the slab $[x: 0 \leq x \leq t]$ with initial direction β and initial wavelength λ_0 . Let $\overline{\Phi}_{\alpha\beta}(\lambda|x)$ denote the distribution of its increase in wavelength due to multiple scattering when it emerges from the slab, given that it emerges with direction α . The Mellin-Stieltjes transform

$$(4.6) \quad \overline{\Phi}_{\alpha\beta}(s|x) = \int_0^\infty \lambda^{s-1} d\overline{\Phi}_{\alpha\beta}(\lambda|x)$$

is clearly given by

$$(4.7) \quad \overline{\Phi}_{\alpha\beta}(s|x) = [P_{\alpha\beta}(x)]^{-1} K_{\alpha\beta}(s, x)$$

where

$$(4.8) \quad K_{\alpha\beta}(s, x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) (i_1 \delta_1 + \dots + i_9 \delta_9)^{s-1}$$

and $\delta_1, \dots, \delta_9$ are defined by eq. (4.5). If n is a positive integer then $\overline{\Phi}_{\alpha\beta}(n+1|x)$ is the n -th moment about the origin of the wavelength increment λ . These moments can be determined exactly as described in Chapter 6, Sect. 4.

For radiation incident isotropically on the face $x = 0$ of the slab, the direction of motion of an incident photon is a random variable β which takes the values 1, 2, 3, 4, 5, 6 with associated probabilities $r_1 = \frac{1}{15}$, $r_2 = \frac{4}{15}$, $r_3 = \frac{4}{15}$, $r_4 = \frac{4}{15}$ and $r_5 = r_6 = \frac{1}{15}$. The mean and mean square of the increase in wavelength of the transmitted radiation are given by

$$(4.9) \quad \mathcal{L}(\lambda) = \frac{1}{\pi} \left[\sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_\beta K_{\alpha\beta}(2, 0) \right]$$

$$(4.10) \quad \mathcal{E}(\lambda^2) = \frac{1}{\pi} \left[\sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} K_{\alpha\beta}(3,0) \right]$$

where $\pi = \sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} P_{\alpha\beta}(0)$ is the probability that an incident photon is transmitted through the slab and the functions $K_{\alpha\beta}(2,x)$ and $K_{\alpha\beta}(3,x)$ are found as described in Chapter 6, Sect. 4.

Numerical determinations of π , $\mathcal{E}(\lambda)$, $\mathcal{E}(\lambda^2)$ for the slab thicknesses $t = 2/3$, $4/3$, and 2 leads to the following results:

Table 9.

t	2/3	4/3	2
π	.57154	.42321	.33850
$\mathcal{E}(\lambda)$	1.0794	2.6124	4.6021
$\mathcal{E}(\lambda^2)$	4.1550	16.898	44.233

We shall use these moments together with certain exact parameters which will now be determined to estimate the transmitted spectral intensity for the continuous model when particles are incident isotropically on one face of a slab of thickness t .

It will be assumed that the incident radiation is monochromatic so that the spectral intensity (rate of incidence of photons per unit wavelength interval) is given by

$$(4.11) \quad I(\lambda) = \delta(\lambda - \lambda_0) ,$$

where $\delta(x)$ is the Dirac delta function and λ_0 is the incident wavelength.

If $\eta(\lambda)$ is the spectral intensity corresponding to a transmitted wavelength $(\lambda + \lambda_0)$, i.e., if the rate of transmission of

photons with wavelength in the small interval $(\lambda_0 + \lambda, \lambda_0 + \lambda + \delta\lambda)$ is $\eta(\lambda)\delta\lambda + o(\delta\lambda)$, then $\eta(\lambda)$ is clearly independent of λ_0 and can be written:

$$(4.12) \quad \eta(\lambda) = \pi_0 \delta(\lambda) + \gamma(\lambda) \quad , \quad 0 \leq \lambda < \infty \quad ,$$

where π_0 is the probability that an incident photon is transmitted through the slab with no collisions and $\gamma(\lambda)$ is the intensity of the transmitted radiation which is scattered at least once in its passage through the slab. It is worth noticing at this point that the spectral intensities found by Chandrasekhar and O'Rourke (Fig. 7) are not of the form (4.12); in fact they differ in two respects, first in the complete absence of a delta function component and secondly in the existence of a non-zero probability of a negative wavelength increment which for Compton scattering is physically impossible.

Making use of the moments derived above from the icosahedral approximation it is possible, as we shall now show, to fit a transmitted spectral distribution of the correct form as given by eq. (4.12).

We first observe that the values π_0 and $\gamma(0)$ appearing in (4.12) can be determined exactly as follows. If μ is the cosine of the angle made by an incident photon with the positive x-axis, then for isotropically incident radiation μ is a random variable uniformly distributed on $[0,1]$. The probability π_0 is therefore given by:

$$(4.13) \quad \pi_0 = \int_0^1 \exp(-t\mu^{-1}) d\mu \quad .$$

For a single collision the probability density $f(\lambda)$ corresponding to an increase in wavelength λ is readily found from equations (4.1) and (4.3) to be

$$(4.14) \quad f(\lambda) = \frac{3}{8} (2 - 2\lambda + \lambda^2), \quad 0 \leq \lambda \leq 2.$$

The only contribution to the intensity $\gamma(0)$ comes from photons which are scattered exactly once. Hence

$$(4.15) \quad \gamma(0) = \frac{3}{4} \int_0^1 t\mu^{-1} \exp(-t\mu^{-1}) d\mu,$$

For any given slab thickness, π_0 and $\gamma(0)$ are easily found from equations (4.14) and (4.15) using tables of the exponential integral.

Thus for thicknesses $t = 2/3, 4/3$ and 2 they are given by the following table:

Table 10.

t	2/3	4/3	2
π_0	.2478	.09204	.03753
$\gamma(0)$	.1992	.1287	.07335

From these exact values of π_0 , $\gamma(0)$ and from the icosahedral approximations to π , $\mathcal{E}(\lambda)$ and $\mathcal{E}(\lambda^2)$ we now fit a spectral intensity function having the required form (4.12). Expressing $\mathcal{E}(\lambda)$ and $\mathcal{E}(\lambda^2)$ in terms of the transmitted spectral intensity $\eta(\lambda)$ we have

$$(4.16) \quad \mathcal{E}(\lambda) = \frac{1}{\pi} \int_0^\infty \lambda \eta(\lambda) d\lambda = \frac{1}{\pi} \int_0^\infty \lambda \gamma(\lambda) d\lambda,$$

$$(4.17) \quad \mathcal{E}(\lambda^2) = \frac{1}{\pi} \int_0^\infty \lambda^2 \eta(\lambda) d\lambda = \frac{1}{\pi} \int_0^\infty \lambda^2 \gamma(\lambda) d\lambda.$$

There are two further conditions to be satisfied by $\gamma(\lambda)$, namely

$$(4.18) \quad \int_0^{\infty} \gamma(\lambda) d\lambda = \pi - \pi_0$$

and

$$(4.19) \quad \gamma(0) = \frac{3}{4} \int_0^1 t\mu^{-1} \exp(-t\mu^{-1}) d\mu \quad .$$

Using the numerically determined values of π_0 , $\gamma(0)$, π , $\xi(\lambda)$ and $\xi(\lambda^2)$ the parameters a , b , α , β were obtained by fitting a curve of the form

$$(4.20) \quad \gamma^*(\lambda) = a \exp(-\alpha\lambda) + b\lambda \exp(-\beta\lambda)$$

to the scattered component $\gamma(\lambda)$ of the transmitted spectral intensity. The parameters a , α , b and β are given, for each slab thickness, in the table below. Figure 6 shows the graphs of the fitted functions $\gamma^*(\lambda)$.

t	2/3	4/3	2
a	0.1992	0.1287	0.07335
α	1.2931	1.1269	0.8790
b	0.07886	0.04053	0.01924
β	0.6817	0.4322	0.2974

We remark finally that the assumption of an isotropically incident stream of radiation is not necessary for the analysis. Any incident angular distribution which can be reasonably well approximated by a distribution on the set of 30 directions in space which we have used can be dealt with in the same way. Furthermore, although we have here considered only the overall emergent spectral

distribution for transmitted radiation the same method can be used to determine the spectral distribution for particular directions of emergence.

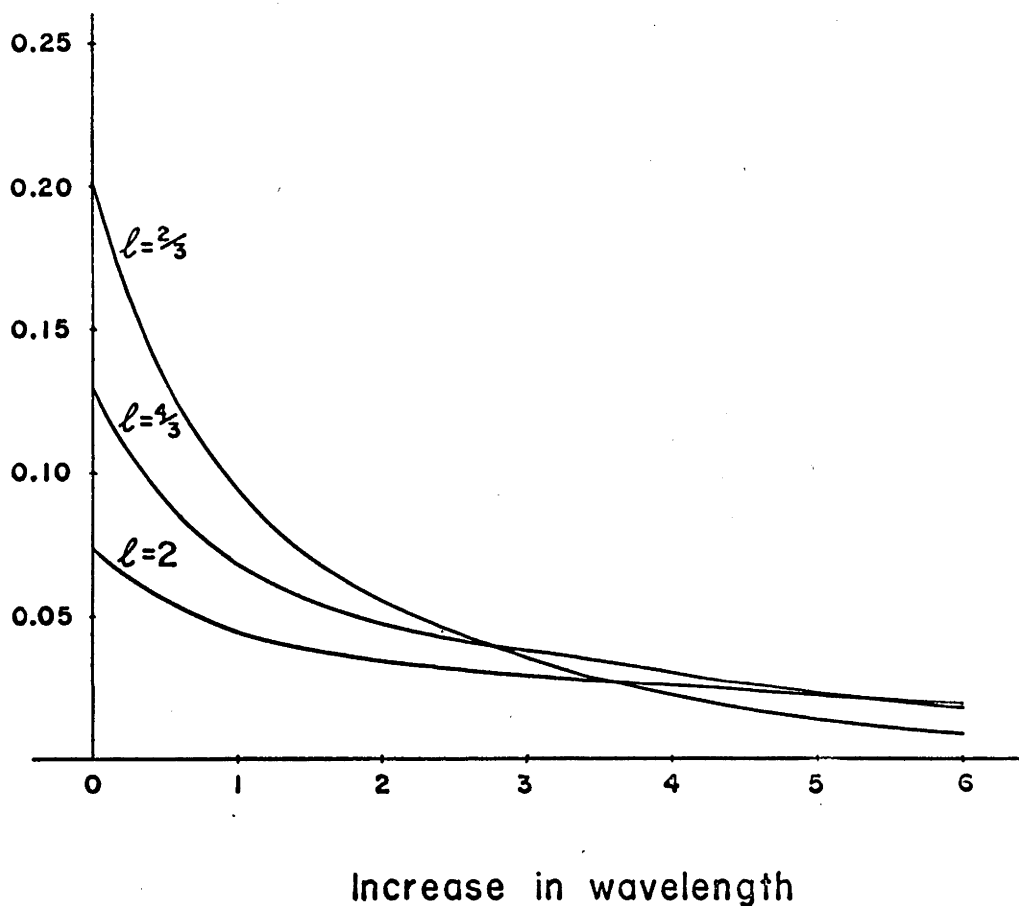


Fig. 6

The spectral intensity of scattered radiation transmitted through a slab of thickness l . In addition the intensity of the transmitted radiation will have a component $\pi_0 \delta(\lambda)$ due to unscattered photons. The wavelength increase is measured in Compton units.

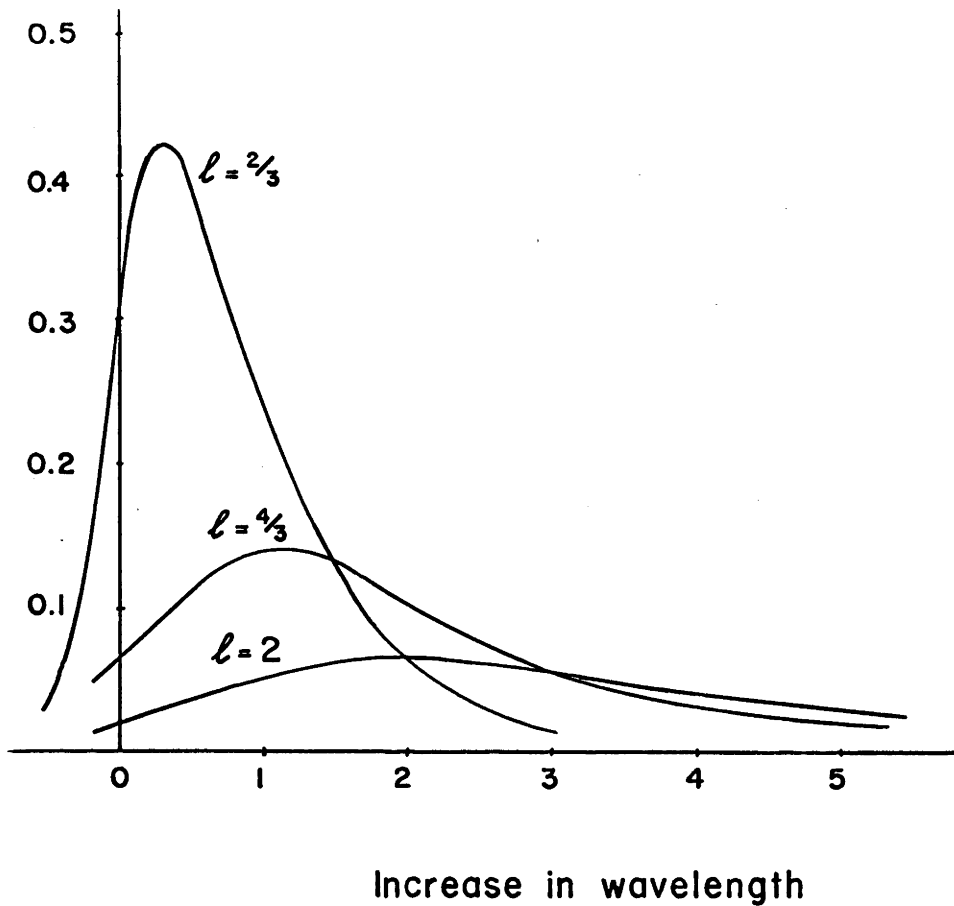


Fig. 7.

The results of Chandrasekhar and O'Rourke showing the spectral intensity of emergent radiation measured in units of the integrated incident intensity. The wavelength increments are again measured in Compton units.

APPENDIX

NUMERICAL CALCULATIONS

1. - Introduction.

In this appendix we shall first outline the method used to solve equations (3.2), (4.4) and (4.11) of Chapter 6 for $P_{\alpha\beta}(x)$, $M_{\alpha\beta}(x)$ and $S_{\alpha\beta}(x)$ respectively. The results are printed out by a computer program as the matrices P_R^* , P_T^* , M_R^* , M_T^* , S_R^* and S_T^* defined in Chapter 6 by equations (5.3), (5.4) and (5.5). A brief description is given in Sect. 3 of a second program which uses the matrices for thickness t to generate the corresponding matrices for thicknesses $2^n t$, $n=1,2,\dots$. In Sect. 4 the matrices are then tabulated for a number of slab thicknesses and several scattering cross-sections. For all the tabulated results the coefficients a_i in the linear combination, $N = \sum a_i N_i$, are taken as unity.

2. - Program Transport.

Program Transport makes use of the backward Kolmogorov equation

$$A_\theta(z_1, \dots, z_9) G_\alpha(x) = 0,$$

with boundary conditions

$$\begin{cases} G_{\alpha\beta}(0) = \delta_{\alpha\beta}, & \beta = 7, 8, 9, 10, \\ G_{\alpha\beta}(t) = \delta_{\alpha\beta}, & \beta = 1, 2, 3, 4, \end{cases}$$

to solve for

$$P_{\alpha\beta}(x) = [G_{\alpha\beta}(x)]_{z_1=\dots=z_9=1},$$

$$M_{\alpha\beta}(x) = \left[\sum_{k=1}^9 a_k \frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right]_{z_1=\dots=z_9=1},$$

$$\text{and } S_{\alpha\beta}(x) = \left[\sum_{j=1}^9 \sum_{k=1}^9 a_j a_k \frac{\partial^2}{\partial z_j \partial z_k} G_{\alpha\beta}(x) \right]_{z_1=\dots=z_9=1}.$$

As already described in Chapter 6, Sects. 3 and 4, this requires the solution of three two-point boundary value problems.

The input data consists of the discrete approximation $\{p_1, \dots, p_9\}$ to the scattering function, $\sigma(\theta)$, and the coefficients $\{a_1, \dots, a_9\}$ of the particular linear combination under consideration. The probability of absorption at each collision is given by

$$r = 1 - p_1 - p_9 - \sum_{i=2}^8 p_i.$$

The first step in the program is to determine the matrices B and $R(1, \dots, 1)$ (denoted for simplicity by R), which have already been defined in Chapter 6, Sects. 3 and 4. The eigenvalues $\lambda_1, \dots, \lambda_8$ of R (assumed to be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \leq \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8$) and the corresponding eigenvectors $\underline{e}_1, \dots, \underline{e}_8$ are then determined. We observe that the eigenvalues have the property $\lambda_i = -\lambda_{9-i}$, $i=1, \dots, 4$, and that $\lambda_4 = \lambda_5 = 0$ if there is zero probability of absorption at each collision. Let $T = [\underline{t}_1 \underline{t}_2 \dots \underline{t}_8]$ be a matrix such that $T^{-1}RT$ is in Jordan canonical form. If all the eigenvalues of R are distinct then such a matrix is obtained by choosing $\underline{t}_i = \underline{e}_i$, $i=1, \dots, 8$; if, on the other hand, the probability of absorption, r , is zero then $\lambda_4 = \lambda_5 = 0$ and we obtain a suitable matrix T by choosing $\underline{t}_5$ such that $R\underline{t}_5 = \underline{e}_4$ and the remaining $\underline{t}_i$ such that $\underline{t}_i = \underline{e}_i$, $i \neq 5$.

If $\Lambda = T^{-1}RT$ then $\exp[xR] = T \exp[x\Lambda]T^{-1}$. The evaluation of $\exp[x\Lambda]$ is trivial. Thus if the eigenvalues of R are all distinct then $\exp[x\Lambda]$ is a diagonal matrix whose i -th diagonal element is $\exp[\lambda_i x]$; if $r = 0$, implying that $\lambda_4 = \lambda_5 = 0$ then $\exp[x\Lambda]$ is the same as before except for an additional element x immediately above the 5th diagonal element.

From equation (3.8), Chapter 6, $\tilde{p}_\alpha(x)$ can be written in one of the forms:

$$\begin{aligned}\tilde{p}_\alpha(x) &= \sum_{k=1}^8 \tilde{p}_\alpha^{(k)} \exp[\lambda_k x] && \text{if } r \neq 0, \\ \tilde{p}_\alpha(x) &= \sum_{\substack{k=1 \\ k \neq 5}}^8 \tilde{p}_\alpha^{(k)} \exp[\lambda_k x] + x \tilde{p}_\alpha^{(5)} && \text{if } r = 0,\end{aligned}$$

where $\tilde{p}_\alpha^{(k)} = c_{\alpha k} t_k$, $k \neq 4, 5$, and

$$\tilde{p}_\alpha^{(4)} = c_{\alpha 4} t_4, \quad \tilde{p}_\alpha^{(5)} = c_{\alpha 5} t_5 \quad \text{if } r \neq 0,$$

$$\tilde{p}_\alpha^{(4)} = c_{\alpha 4} t_4 + c_{\alpha 5} t_5, \quad \tilde{p}_\alpha^{(5)} = c_{\alpha 5} t_4 \quad \text{if } r = 0.$$

The unknown constants $c_{\alpha k}$, $k=1, \dots, 8$, are determined by the boundary conditions (3.9) of Chapter 6.

The 10-component vectors $\tilde{P}_\alpha(x)$ corresponding to the 8-component vectors $\tilde{p}_\alpha(x)$ are easily determined using eq. (3.2), Chapter 6, to find the additional components $\tilde{P}_{\alpha 5}(x)$ and $\tilde{P}_{\alpha 6}(x)$. Proceeding in this way we can express the vectors $\tilde{P}_\alpha(x)$ as follows:

$$\begin{aligned} \tilde{P}_\alpha(x) &= \sum_{k=1}^8 \tilde{P}_\alpha^{(k)} \exp[\lambda_k x] & \text{if } r \neq 0, \\ \tilde{P}_\alpha(x) &= \sum_{\substack{k=1 \\ k \neq 5}}^8 \tilde{P}_\alpha^{(k)} \exp[\lambda_k x] + x \tilde{P}_\alpha^{(5)} & \text{if } r = 0. \end{aligned}$$

The next step is to form the vectors $\tilde{B}P_\alpha^{(k)}$, $k=1, \dots, 8$, and thence (from eq. (4.4), Chapter 6) to determine the 8-component vector $\tilde{V}_\alpha(x)$ of eq. (4.6), Chapter 6, in the form

$$\begin{aligned} \tilde{V}_\alpha(x) &= \sum_{k=1}^8 \tilde{V}_\alpha^{(k)} \exp[\lambda_k x] & \text{if } r \neq 0, \\ \tilde{V}_\alpha(x) &= \sum_{\substack{k=1 \\ k \neq 5}}^8 \tilde{V}_\alpha^{(k)} \exp[\lambda_k x] + x \tilde{V}_\alpha^{(5)} & \text{if } r = 0. \end{aligned}$$

The integral, $\int_0^t \exp[-yR] \tilde{V}_\alpha(y) dy$, is evaluated analytically by expanding the integrand as a sum of the form

$$\begin{aligned} \tilde{I}_\alpha(y) &= \sum_{i=1}^8 \sum_{j=1}^8 \exp[(\lambda_j - \lambda_i)y] \tilde{I}_\alpha^{(ij)} & \text{if } r \neq 0, \\ \tilde{I}_\alpha(y) &= \sum_{\substack{i=1 \\ i \neq 5}}^8 \sum_{\substack{j=1 \\ j \neq 5}}^8 \exp[(\lambda_j - \lambda_i)y] \tilde{I}_\alpha^{(ij)} \\ &+ \sum_{i=1}^8 y [\exp(-\lambda_i y) \tilde{I}_\alpha^{(i5)} - \exp(\lambda_i y) \tilde{I}_\alpha^{(5i)}] - y^2 \tilde{I}_\alpha^{(55)} & \text{if } r = 0. \end{aligned}$$

The boundary conditions (4.8) of Chapter 6 define four components of each of the vectors $\tilde{m}_\alpha(t)$ and $\tilde{m}_\alpha(0)$. The remaining eight unknown

components are determined from the equation

$$\exp[-tR] \underline{m}_\alpha(t) = \underline{m}_\alpha(0) + \int_0^t \exp[-yR] \underline{V}_\alpha(y) dy ,$$

which is a direct consequence of (4.7), Chapter 6.

Having found $\underline{m}_\alpha(0)$ it is now possible to expand $\underline{m}_\alpha(x)$ as follows:

$$\begin{aligned} \underline{m}_\alpha(x) &= \sum_{k=1}^8 \underline{m}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{k=1}^8 \underline{m}_\alpha^{(k+8)} x \exp[\lambda_k x] && \text{if } r \neq 0, \\ \underline{m}_\alpha(x) &= \sum_{\substack{k=1 \\ k \neq 5}}^8 \underline{m}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{\substack{k=1 \\ k \neq 4,5}}^8 \underline{m}_\alpha^{(k+8)} x \exp[\lambda_k x] \\ &\quad + \underline{m}_\alpha^{(5)} x + \underline{m}_\alpha^{(12)} x^2 + \underline{m}_\alpha^{(13)} x^3 && \text{if } r = 0 . \end{aligned}$$

Equation (4.4), Chapter 6, is then used to determine $\underline{M}_{\alpha 5}(x)$ and

$\underline{M}_{\alpha 6}(x)$ thus giving the 10-component vectors $\underline{M}_\alpha^{(k)}$ such that

$$\begin{aligned} \underline{M}_\alpha(x) &= \sum_{k=1}^8 \underline{M}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{k=1}^8 \underline{M}_\alpha^{(k+8)} x \exp[\lambda_k x] && \text{if } r \neq 0, \\ \underline{M}_\alpha(x) &= \sum_{\substack{k=1 \\ k \neq 5}}^8 \underline{M}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{\substack{k=1 \\ k \neq 4,5}}^8 \underline{M}_\alpha^{(k+8)} x \exp[\lambda_k x] \\ &\quad + \underline{M}_\alpha^{(5)} x + \underline{M}_\alpha^{(12)} x^2 + \underline{M}_\alpha^{(13)} x^3 && \text{if } r = 0 . \end{aligned}$$

The determination of the vectors $\underline{S}_\alpha(x)$ proceeds in much the same way as the determination of $\underline{M}_\alpha(x)$. Thus we begin (see Chapter 6, eq. (4.11)) by forming the vectors $2B \underline{M}_\alpha^{(k)}$, $k=1, \dots, 16$, and thence (from eq. (4.11), Chapter 6) we derive the 8-component vector $\underline{W}_\alpha(x)$ of eq. (4.12), Chapter 6, in the form

$$\underline{W}_\alpha(x) = \sum_{k=1}^8 \underline{W}_\alpha^{(k)} \exp[\lambda_k x] + \sum_{k=1}^8 \underline{W}_\alpha^{(k+8)} x \exp[\lambda_k x] \quad \text{if } r \neq 0 ,$$

$$W_{\alpha}(x) = \sum_{\substack{k=1 \\ k \neq 5}}^8 W_{\alpha}^{(k)} \exp[\lambda_k x] + \sum_{\substack{k=1 \\ k \neq 4,5}}^8 W_{\alpha}^{(k+8)} x \exp[\lambda_k x] \\ + W_{\alpha}^{(5)} x + W_{\alpha}^{(12)} x^2 + W_{\alpha}^{(13)} x^3 \quad \text{if } r = 0.$$

The integral, $\int_0^t \exp[-yR] W_{\alpha}(y) dy$, is again evaluated analytically by expanding the integrand as a sum of the form

$$I_{\alpha}(y) = \sum_{i=1}^8 \left\{ \sum_{j=1}^8 \exp[(\lambda_j - \lambda_i)y] I_{\alpha}^{(ij)} + \sum_{j=1}^8 y \exp[(\lambda_j - \lambda_i)y] I_{\alpha}^{(i,j+8)} \right\} \quad \text{if } r \neq 0,$$

$$I_{\alpha}(y) = \sum_{\substack{i=1 \\ i \neq 5}}^8 \left\{ \sum_{\substack{j=1 \\ j \neq 5}}^8 \exp[(\lambda_j - \lambda_i)y] I_{\alpha}^{(ij)} + \sum_{\substack{j=1 \\ j \neq 4,5}}^8 y \exp[(\lambda_j - \lambda_i)y] I_{\alpha}^{(i,j+8)} \right. \\ \left. + y \exp(-\lambda_i y) I_{\alpha}^{(i5)} + y^2 \exp(-\lambda_i y) I_{\alpha}^{(i12)} + y^3 \exp(-\lambda_i y) I_{\alpha}^{(i13)} \right\} - \sum_{\substack{j=1 \\ j \neq 5}}^8 y \exp(\lambda_j y) I_{\alpha}^{(5j)} \\ - \sum_{\substack{j=1 \\ j \neq 4,5}}^8 y^2 \exp(\lambda_j y) I_{\alpha}^{(5,j+8)} \\ - y^2 I_{\alpha}^{(55)} - y^3 I_{\alpha}^{(5,12)} - y^4 I_{\alpha}^{(5,13)} \quad \text{if } r = 0.$$

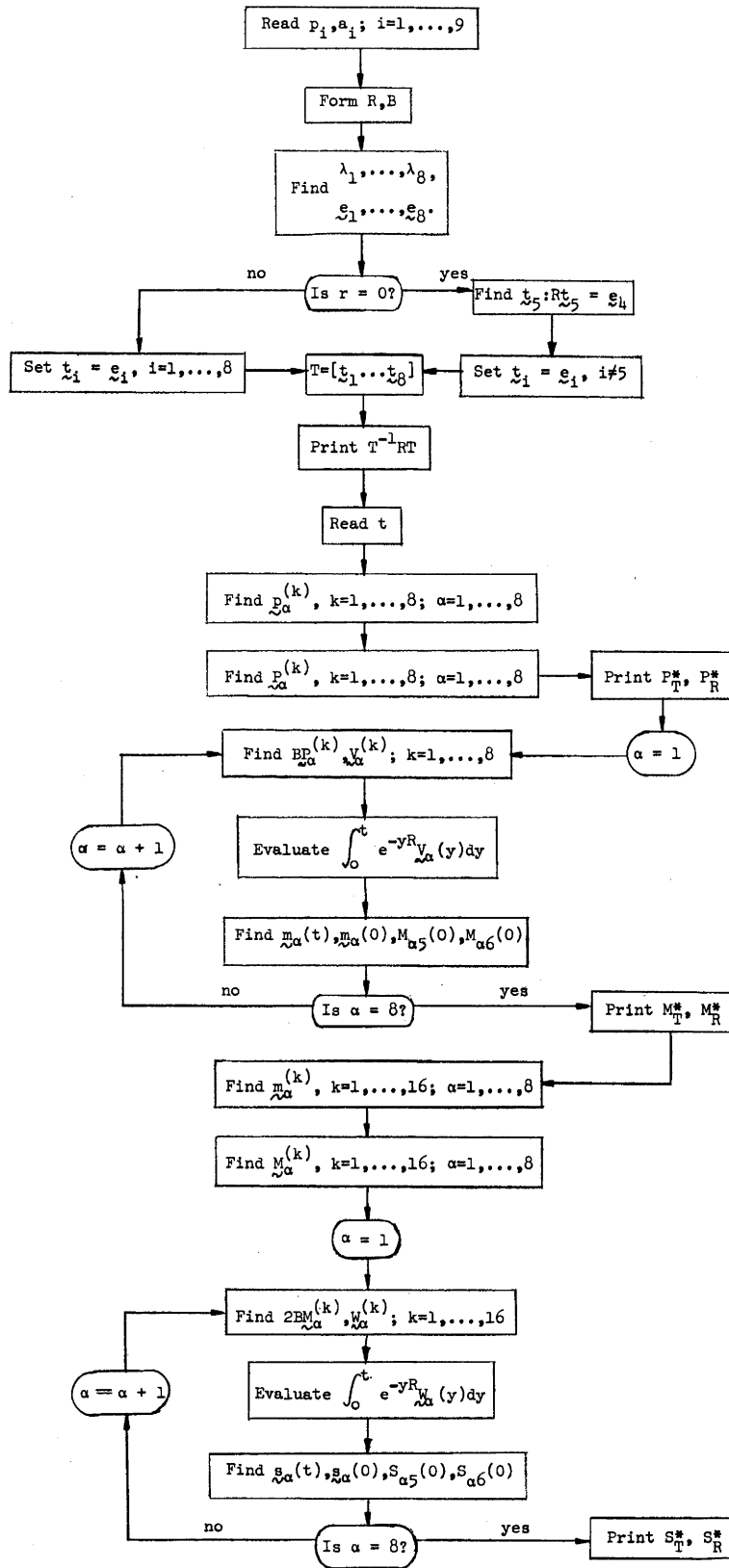
The boundary conditions (4.14) of Chapter 6 define four components of each of the vectors $s_{\alpha}(t)$ and $s_{\alpha}(0)$. The remaining eight unknown components are determined from the equation

$$\exp[-tR] \tilde{s}_\alpha(t) = \tilde{s}_\alpha(0) + \int_0^t \exp[-yR] \tilde{W}_\alpha(y) dy ,$$

which is a direct consequence of eq. (4.13), Chapter 6.

It would be possible now to proceed in the same way as before, obtaining an expansion for $\tilde{s}_\alpha(x)$ analogous to the one above for $\tilde{m}_\alpha(x)$, and proceeding to find third and higher order moments of $N = \sum_{i=1}^9 a_i N_i$. However the program, as written, stops at the determination of $\tilde{S}_\alpha(0)$ and $\tilde{S}_\alpha(t)$ which are easily found from $\tilde{s}_\alpha(0)$, $\tilde{s}_\alpha(t)$ and equation (4.11) of Chapter 6.

The output of Program Transport consists of the matrices P_T^* , P_R^* , M_T^* , M_R^* , S_T^* and S_R^* defined in Chapter 6. A block diagram of the program is given on the next page.



3. - Program Iterate.

The evaluation of the integrals $\int_0^t \exp[-yR] V_\alpha(y) dy$ and $\int_0^t \exp[-yR] W_\alpha(y) dy$ becomes numerically unstable for large values of t since sums of large positive and negative numbers are involved. This difficulty can be overcome by using the method of iteration described in Chapter 6, Sect. 5, to determine the matrices $P_{T(R)}^*$, $M_{T(R)}^*$ and $S_{T(R)}^*$ for thicker slabs.

Program Iterate uses as input data the six matrices $P_{T(R)}^*(t)$, $M_{T(R)}^*(t)$ and $S_{T(R)}^*(t)$; these are then used to generate the corresponding matrices for thicknesses $2t, 4t, \dots, 2^n t$ by a direct application of equations (5.10), ..., (5.15) of Chapter 6. Each new matrix element is found as a sum of positive terms so that the question of instability does not arise. The only errors are rounding errors and those introduced in the inversion of the 4×4 matrix $(I - P_R^2)$ (see Sect. 5, Chapter 6).

In Sect. 4 transmission and reflection matrices are tabulated for isotropic and Rayleigh scattering laws with absorption probabilities 0 and 0.05. Program Transport was used to derive the results for thickness $t = 0.5$ and Program Iterate was then used to generate the corresponding results for thicknesses 1, 2, 4, 8 and 16. Only the results for $t = 0.5$, $t = 2$ and $t = 16$ are tabulated.

4. - Numerical results.

We recall from Chapter 6 that the 4×6 matrices G_T^* and G_R^* are defined as follows: denoting the (i,j) th element of G_T^* by $\{G_T^*\}_{ij}$ we have

$$\{G_T^*\}_{ij} = G_{ij}(0)$$

$$\{G_R^*\}_{ij} = G_{(11-i)j}(0) .$$

The matrices $P_{R(T)}^*$, $M_{R(T)}^*$ and $S_{R(T)}^*$ are defined in terms of $G_{R(T)}^*$ by equations (5.3), (5.4) and (5.5) of Chapter 6.

Consider a particle incident on the face $x = 0$ of the slab $\{x : 0 \leq x \leq t\}$ with direction j , $j=1, \dots, 6$, and suppose that it makes a random number, N_k , of collisions of type k ($k=1, \dots, 9$) before emerging from the slab. Let N denote the random variable, $N = \sum_{k=1}^9 a_k N_k$. Then the (i,j) th element of P_T^* , $\{P_T^*\}_{ij}$, denotes the probability that the particle emerges from the slab with direction i , and $\{P_R^*\}_{ij}$ denotes the probability that it emerges with direction $(11-i)$. Similarly $\{M_T^*\}_{ij} / \{P_T^*\}_{ij}$ is the expectation, $\mathcal{E}N$, conditional on emergent direction i , and $\{M_R^*\}_{ij} / \{P_R^*\}_{ij}$ is the expectation, $\mathcal{E}N$, conditional on emergent direction $(11-i)$. In the same way $[\{S_T^*\}_{ij} + \{\bar{M}_T^*\}_{ij}] / \{P_T^*\}_{ij}$ and $[\{S_R^*\}_{ij} + \{\bar{M}_R^*\}_{ij}] / \{P_R^*\}_{ij}$ are the expectations, $\mathcal{E}N^2$, conditional on emergent directions i and $(11-i)$ respectively; the matrix $\bar{M}_{T(R)}^*$ is defined as the matrix $M_{T(R)}^*$ in which the set of constants $\{a_1, \dots, a_9\}$ is replaced by a new set $\{a_1^2, \dots, a_9^2\}$.

The tabulated results which follow were all obtained with $a_1 = a_2 = \dots = a_9 = 1$. Isotropic and Rayleigh cross-sections were used with absorption probabilities 0 and 0.05. Although numerical results were obtained for thicknesses $t = 0.5, 1, 2, 4, 8$ and 16, only the results for $t = 0.5, 2$ and 16 are listed.

In addition to the results tabulated below, the matrices $P_R^*(T)$, $M_R^*(T)$, $\overline{M_R^*}(T)$ and $S_R^*(T)$ have been calculated using the Rayleigh cross-section with no absorption, thicknesses $t = 2/3, 4/3$ and 2 and $a_1, \dots, a_9$ as defined by equation (4.5), Chapter 7. These results were used to derive the results in Chapter 7, Sect. 4.

I. - Isotropic scattering with no absorption.

$$p_1 = p_2 = \dots = p_9 = 1/30,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 0.5$.

P_T^*

.624971202	.021530545	.029192636	.036365861	.037435668	.037435668
.069674307	.620288379	.109934548	.136409600	.136007862	.136007862
.058385273	.067943288	.459027078	.111668018	.100497647	.100497647
.044950676	.052103831	.069014631	.281150850	.063616459	.063616459

P_R^*

.018972398	.022299095	.030880567	.039741847	.053486863	.053486863
.072161387	.084879214	.117827773	.152196999	.211222205	.211222205
.061761133	.072821568	.101862115	.133100531	.203573000	.203573000
.049123625	.058134080	.082260652	.109366294	.194160297	.194160297

M_T^*

.033597669	.039265675	.053399305	.066809008	.069021679	.069021679
.127066392	.148423161	.201502601	.251436676	.254336616	.254336616
.106798611	.124535456	.168149797	.208046242	.197102076	.197102076
.082580476	.096040264	.128579648	.157002910	.135290982	.135290982

M_R^*

.034261009	.040224207	.055504325	.071018387	.088686714	.088686714
.130168268	.152905408	.211346083	.271120970	.346474010	.346474010
.111008650	.130619063	.181510550	.234766775	.323299392	.323299392
.087783554	.103558996	.145093846	.190038472	.294898374	.294898374

S_T^*

.055124232	.064519237	.088143475	.111001193	.115574636	.115574636
.208788637	.244333843	.333628046	.419816398	.434601039	.434601039
.176286951	.206193473	.281092772	.352833147	.358962704	.358962704
.137205020	.160355595	.218062877	.272686762	.270849898	.270849898

S_R^*

.055452441	.064993465	.089184546	.113081093	.124479998	.124479998
.210323270	.246551232	.338495902	.429541772	.476299508	.476299508
.178369091	.209201972	.287697534	.366029539	.415910339	.415910339
.139775918	.164070357	.226218696	.288985008	.342352184	.342352184

$$p_1 = p_2 = \dots = p_9 = 1/30,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 2.0$.

P^*_T

.182533895	.048218653	.046484408	.041583132	.027987641	.027987641
.156038837	.242881246	.150367030	.132394947	.086869682	.086869682
.092968816	.092931936	.102830694	.071495969	.044692431	.044692431
.051399578	.050570370	.044186939	.037773969	.022366334	.022366334

P^*_R

.061232207	.066011295	.073627113	.076820537	.074117999	.074117999
.213617037	.231500226	.261906509	.277561741	.280672729	.280672729
.147254225	.161867124	.190372710	.210707824	.246117110	.246117110
.094955406	.106019151	.130224597	.151661882	.217176074	.217176074

M^*_T

.229110606	.237260342	.236605735	.217747438	.147507458	.147507458
.767790594	.793301634	.786429935	.719621467	.483511277	.483511277
.473211471	.486040430	.474586170	.428367545	.283460038	.283460038
.269150636	.274870942	.264745702	.236461389	.155555877	.155555877

M^*_R

.256283092	.271655027	.288715219	.284478997	.225354617	.225354617
.879094132	.934218640	1.00011737	.993761247	.808140967	.808140967
.577430438	.618106529	.675611118	.688209531	.611102018	.611102018
.351635379	.379583031	.425336881	.447140810	.456736942	.456736942

S^*_T

1.69650142	1.76948614	1.79644856	1.67904727	1.14798532	1.14798532
5.72617742	5.96892124	6.05028631	5.64620638	3.85332034	3.85332034
3.59289713	3.73928259	3.77477410	3.50915594	2.38506939	2.38506939
2.07541657	2.15665893	2.16877764	2.00939255	1.36177330	1.36177330

S^*_R

1.74830214	1.83490326	1.89458789	1.80224121	1.26728572	1.26728572
5.93787167	6.23628139	6.45150786	6.15017696	4.34376273	4.34376273
3.78917579	3.98725114	4.14738962	3.97845239	2.85156319	2.85156319
2.22769265	2.34915856	2.45881880	2.37667599	1.74384455	1.74384455

$$p_1 = p_2 = \dots = p_9 = 1/30,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P*
T

.016184136	.014301916	.011223908	.009284038	.005977830	.005977830
.046281973	.040899556	.032097275	.026549779	.017094939	.017094939
.022447817	.019837207	.015567904	.012877243	.008291432	.008291432
.011475702	.010141113	.007958574	.006583063	.004238719	.004238719

P*
R

.124772705	.123412562	.119116976	.114259316	.098151082	.098151082
.399371441	.399410458	.395023363	.387100745	.350981143	.350981143
.238233952	.244137865	.255612214	.264386412	.280568459	.280568459
.141232281	.147859328	.163399789	.178959408	.234696399	.234696399

M*
T

2.38504214	2.12226806	1.67872073	1.39254241	0.89763469	0.89763469
6.86780372	6.11084845	4.83344071	4.00938718	2.58443682	2.58443682
3.35744147	2.98723065	2.36263779	1.95978865	1.26326203	1.26326203
1.72127709	1.53144963	1.21121600	1.00468583	0.64760940	0.64760940

M*
R

3.93394810	3.58363928	2.94340261	2.49552779	1.65264459	1.65264459
11.5969003	10.5888147	8.73999966	7.44004973	4.96929966	4.96929966
5.88680523	5.40161686	4.51024442	3.88142587	2.67219260	2.67219260
3.08464200	2.84184613	2.39885312	2.09034099	1.51729596	1.51729596

S*
T

503.700965	449.390360	356.592509	296.150771	190.989806	190.989806
1454.25775	1297.42335	1029.47895	854.974962	551.377205	551.377205
713.185020	636.252984	504.836185	419.257444	270.379604	270.379604
366.062486	326.571377	259.115351	215.189642	138.775777	138.775777

S*
R

565.954882	506.958724	404.443276	336.662711	217.364744	217.364744
1640.55289	1469.80370	1172.92980	976.517008	630.563617	630.563617
808.886554	724.910482	578.790112	482.023041	311.356868	311.356868
416.137997	372.996308	297.906623	248.159777	160.359416	160.359416

II. - Isotropic scattering with absorption probability, $r = 0.05$.

$$p_1 = p_2 = \dots = p_9 = 19/600,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 0.5$.

P*
T

.623357516	.019644740	.026628514	.033158689	.034123322	.034123322
.063571715	.613160627	.100260024	.124341809	.123812613	.123812613
.053257027	.061964102	.450957072	.101689248	.091073045	.091073045
.040986394	.047494345	.062847411	.273627944	.057176410	.057176410

P*
R

.017325951	.020365949	.028212478	.036326775	.049202268	.049202268
.065905595	.077530087	.107667084	.139157001	.194471610	.194471610
.056424956	.066541917	.093132226	.121802027	.187908899	.187908899
.044902363	.053153245	.075277793	.100211734	.179828236	.179828236

M*
T

.029451676	.034416054	.046786486	.058503848	.060403168	.060403168
.111372691	.130071846	.176504384	.220089552	.222195131	.222195131
.093572972	.109085708	.147171328	.181868775	.171219991	.171219991
.072314733	.084066728	.112401084	.136966180	.116451926	.116451926

M*
R

.030066545	.035304547	.048737709	.062405770	.078669227	.078669227
.114247913	.134226582	.185628701	.238336128	.307779052	.307779052
.097475418	.114724847	.159556059	.206637917	.288448861	.288448861
.077137774	.091036300	.127709256	.167589946	.264740567	.264740567

S*
T

.044112951	.051628341	.070520076	.088784842	.092375744	.092375744
.167072820	.195502300	.266891704	.335730360	.347119955	.347119955
.141040152	.164948144	.224783677	.281999348	.286083333	.286083333
.109744100	.128237586	.174285182	.217751047	.215184289	.215184289

S*
R

.044398400	.052040786	.071425523	.090593829	.100138342	.100138342
.168407520	.197430809	.271125415	.344188989	.383468149	.383468149
.142851046	.167564722	.230528079	.293477007	.335727918	.335727918
.111980130	.131468495	.181378765	.231926937	.277529121	.277529121

$$p_1 = p_2 = \dots = p_9 = 19/600,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 2.0$.

P*
T

.172873753	.038227604	.036552885	.032468919	.021823581	.021823581
.123707126	.209528084	.117436387	.102371636	.066755686	.066755686
.073105770	.072579679	.083082127	.053772515	.033026752	.033026752
.040133792	.039102485	.033233242	.028062499	.016016808	.016016808

P*
R

.050275177	.054378479	.061207100	.064516645	.064204194	.064204194
.175972454	.191419877	.218769784	.234432421	.244913655	.244913655
.122414200	.135207162	.161017179	.180552067	.218627627	.218627627
.079746959	.089545217	.111587315	.131854809	.196225630	.196225630

M*
T

.154667978	.159735492	.158205007	.144720661	.097700550	.097700550
.516914912	.532286385	.523118762	.474943083	.317139810	.317139810
.316410015	.323305174	.311455822	.277660380	.181625698	.181625698
.178884575	.181412115	.171603552	.150826830	.097902521	.097902521

M*
R

.178190478	.189516607	.203366620	.202662013	.166354148	.166354148
.613288624	.654329707	.708367552	.713063490	.603737583	.603737583
.406733241	.437795224	.485935880	.503723878	.472120395	.472120395
.250504024	.272366017	.311318477	.334689988	.366968349	.366968349

S*
T

0.92855470	0.96734687	0.97905800	0.91242562	0.62207256	0.62207256
3.13040022	3.25836440	3.29032902	3.05962894	2.08047911	2.08047911
1.95811601	2.03353517	2.04143172	1.88781583	1.27613440	1.27613440
1.12782010	1.16867426	1.16673434	1.07372716	0.72287358	0.72287358

S*
R

0.96907587	1.01852542	1.05587430	1.00894882	0.71627266	0.71627266
3.29601750	3.46755718	3.60442404	3.45457712	2.46799901	2.46799901
2.11174861	2.22765656	2.33332197	2.25592270	1.64579969	1.64579969
1.24712932	1.31953104	1.39423690	1.36233890	1.02742527	1.02742527

$$p_1 = p_2 = \dots = p_9 = 19/600,$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P*
T

.000352489	.000294581	.000217640	.000176298	.000112643	.000112643
.000953282	.000796840	.000588695	.000476868	.000304685	.000304685
.000435279	.000363833	.000268797	.000217737	.000139119	.000139119
.000217916	.000182147	.000134569	.000109006	.000069648	.000069648

P*
R

.068348175	.070535578	.073480093	.074306334	.070402839	.070402839
.228257927	.238218095	.254347267	.262801275	.262871689	.262871689
.146960186	.157195256	.177743842	.193886231	.227066734	.227066734
.091847680	.100381155	.119828281	.138425033	.200384209	.200384209

M*
T

.016690976	.014290396	.010813682	.008828099	.005655648	.005655648
.046244693	.039571355	.029927590	.024428056	.015648687	.015648687
.021627365	.018496268	.013981129	.011409979	.007308839	.007308839
.010912131	.009330687	.007051755	.005754610	.003686135	.003686135

M*
R

.449191468	.436655775	.401691425	.366586494	.271662790	.271662790
1.41304778	1.38381535	1.29338155	1.19623701	.914030340	.914030340
.803382851	.799353759	.775167154	.742177527	.625140068	.625140068
.453125827	.456921879	.458690937	.456062003	.444822905	.444822905

S*
T

.954305460	.829061352	.637416624	.523235959	.335881048	.335881048
2.68289891	2.32956135	1.79004227	1.46910484	.942995655	.942995655
1.27483325	1.10630696	.849571438	.697106729	.447428178	.447428178
.646755216	.561148114	.430835652	.353492686	.226878520	.226878520

S*
R

7.39116311	6.88876570	5.83650318	5.00792254	3.30089043	3.30089043
22.2925142	20.8385857	17.7516760	15.2857691	10.1136002	10.1136002
11.6730064	10.9711391	9.44786135	8.20072810	5.48582918	5.48582918
6.19013270	5.83864423	5.06832869	4.43053729	3.00875903	3.00875903

III. - Rayleigh scattering with no absorption.

$$p_1 = p_9 = 1/20, p_2 = p_8 = (11+\sqrt{5})/320, p_3 = p_7 = 1/32,$$

$$p_4 = p_6 = (11-\sqrt{5})/320, p_5 = 1/40.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 0.5$.

P*
T

.628839256	.023313157	.027204354	.031966379	.031504321	.031643801
.075442962	.623433936	.104846582	.132072485	.122962859	.132556464
.054408709	.064798752	.466708908	.110531046	.115773526	.094070843
.039512618	.050447201	.068311943	.290203564	.064176301	.073333858

P*
R

.023019048	.024192050	.028751844	.034776891	.043928168	.044110552
.078287118	.088274313	.112239761	.147061788	.187055815	.204412807
.057503689	.069367987	.110792733	.131293547	.242703358	.184058654
.042986601	.056172605	.081143874	.122094299	.191895653	.235813021

M*
T

.035290514	.039219186	.049882204	.061261607	.061901956	.062359886
.126915952	.147786304	.195256791	.249281046	.242791171	.254417984
.099764409	.120675334	.179444645	.213446857	.219694112	.197341164
.075723511	.095216887	.131917412	.174024434	.141907055	.152742600

M*
R

.036090400	.040242835	.051783488	.064817286	.077572637	.078131901
.130228550	.152393169	.204506915	.268233412	.323974012	.344110413
.103566975	.126392225	.194313639	.240133695	.374424135	.313539202
.080118572	.102456046	.148410785	.212145018	.304241373	.352436565

S*
T

.047472230	.058423158	.083683403	.108472520	.112767757	.114082862
.189061310	.233270538	.334238356	.434065765	.446871097	.453760025
.167366806	.206570665	.297749970	.383051197	.390398091	.391178280
.134079408	.165798369	.236738659	.306321941	.299650004	.304991678

S*
R

.047698798	.058790694	.084582949	.110369368	.120924668	.122396592
.190250683	.235210600	.338967155	.444146661	.489873693	.498516106
.169165897	.209493223	.305113978	.398163823	.459713977	.457404570
.136424041	.169648929	.246078775	.326502044	.386791765	.397979061

$$p_1 = p_9 = 1/20, p_2 = p_8 = (11 + \sqrt{5})/320, p_3 = p_7 = 1/32,$$

$$p_4 = p_6 = (11 - \sqrt{5})/320, p_5 = 1/40.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 2.0$.

P_T^*

.185347454	.049103081	.045264344	.039932841	.026453279	.026288698
.158900909	.244122232	.148462489	.131392281	.084405453	.085234421
.090528688	.091754865	.105000536	.071751614	.045382007	.043750845
.049359706	.050187386	.044344936	.038125194	.022148287	.021978473

P_R^*

.066675461	.068156044	.070506066	.070884496	.064303610	.064281503
.220557592	.235153902	.254943468	.272167572	.256305107	.272980699
.141012133	.157563728	.201741448	.209918413	.285758245	.226544019
.087618056	.103958762	.129736714	.165827586	.215244012	.258941341

M_T^*

.223034918	.232219466	.232833294	.214933432	.144482909	.143537509
.751477978	.786151675	.787479427	.725100007	.482530631	.480725528
.465666590	.486689053	.485552021	.438044734	.288523014	.284801641
.265672333	.276963558	.270726534	.242647034	.157871042	.156598109

M_R^*

.250866389	.266130003	.281510488	.276306547	.212728592	.212233589
.861214779	.925818070	.995468551	.998424088	.787752357	.804298659
.563020978	.615233401	.701196613	.706810369	.668732288	.605242448
.341533676	.381364067	.436832831	.479451630	.470719725	.517629469

S_T^*

1.62326857	1.72664329	1.78754495	1.68337027	1.14307281	1.13600273
5.58753505	5.93957313	6.13752412	5.76893289	3.90819546	3.88373601
3.57508992	3.79319853	3.90186348	3.64976238	2.46134442	2.44444221
2.08076008	2.20353629	2.25567720	2.10140622	1.41206184	1.40232263

S_R^*

1.66787254	1.78615051	1.88144801	1.80490481	1.25979775	1.25399306
5.78010448	6.19705269	6.54426017	6.29713149	4.41806084	4.40161895
3.76289603	4.04457523	4.30217360	4.16835967	2.98235185	2.96346318
2.23098504	2.40529020	2.57618795	2.52308095	1.84932619	1.85104840

$$p_1 = p_9 = 1/20, p_2 = p_8 = (11 + \sqrt{5})/320, p_3 = p_7 = 1/32,$$

$$p_4 = p_6 = (11 - \sqrt{5})/320, p_5 = 1/40.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P*
T

.016315134	.014387962	.011234061	.009247615	.005878870	.005845583
.046560424	.041060834	.032060108	.026391124	.016777297	.016682301
.022468122	.019814236	.015470865	.012735251	.008096021	.008050180
.011430680	.010080512	.007870818	.006479071	.004118859	.004095537

P*
R

.130573011	.125769089	.115991869	.108142569	.087932043	.087756838
.406997323	.403529316	.387984911	.381107701	.325379384	.341603419
.231983739	.239787862	.266758038	.263139785	.319499983	.260063383
.133671567	.145570188	.162629331	.192756883	.232317544	.275902758

M*
T

2.40007734	2.13277352	1.67988752	1.38720499	0.88306728	0.87793563
6.90180011	6.13277273	4.83018590	3.98854123	2.53900091	2.52424915
3.35977506	2.98521907	2.35098978	1.94128531	1.23575720	1.22857893
1.71467967	1.52348719	1.19978031	0.99068640	0.63063518	0.62697223

M*
R

3.94029950	3.58804096	2.93679090	2.47885062	1.61741491	1.60832342
11.6111445	10.6079656	8.73628520	7.41916674	4.88252363	4.87395997
5.87358182	5.39932121	4.52595981	3.87871663	2.69140948	2.61548952
3.06402789	2.83386954	2.39717871	2.10494661	1.50720793	1.54774627

S*
T

506.961234	451.811653	357.112750	295.276509	188.074101	186.970578
1462.09322	1303.00103	1029.85760	851.520198	542.366085	539.184069
714.225504	636.487001	503.041095	415.924381	264.916461	263.362400
364.981839	325.251775	257.055404	212.537275	135.371911	134.577823

S*
R

569.113015	509.441140	405.048085	335.759180	214.138260	212.867032
1648.58616	1476.05939	1174.01404	973.382419	620.894459	617.211892
810.096174	725.580585	577.472126	478.974896	305.655392	303.835537
415.021173	371.799001	296.022766	245.610967	156.809575	155.889062

IV. - Rayleigh scattering with absorption probability, $r = 0.05$.

$$p_1 = p_9 = 19/400, p_2 = p_8 = 19(11 + \sqrt{5})/6400, p_3 = p_7 = 19/640,$$

$$p_4 = p_6 = 19(11 - \sqrt{5})/6400, p_5 = 19/800.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 0.5$.

P*
T

.627131693	.021422285	.024810609	.029033370	.028544426	.028662577
.069323971	.616324426	.095484568	.120128917	.111358904	.120379468
.049621219	.059012709	.458093755	.100317878	.105256502	.084672193
.035887219	.045885164	.061999858	.281869476	.057439367	.066061604

P*
R

.021271771	.022250448	.026264145	.031668437	.040194806	.040350988
.072003964	.080936853	.102421072	.134183034	.171445785	.187806427
.052528290	.063299703	.101443212	.119764671	.224535412	.168931810
.039144341	.051253358	.074018637	.111879070	.177149538	.218670863

M*
T

.031404727	.034648553	.043651484	.053356448	.053774072	.054150921
.112125072	.129979190	.170571996	.217441446	.210717630	.221453424
.087302968	.105419291	.157178846	.185683862	.191307754	.170049327
.065952197	.083055242	.114758937	.151659544	.121485839	.131538235

M*
R

.032154070	.035603909	.045415829	.056646053	.068281112	.068746958
.115216670	.134265399	.179139517	.234976949	.285837679	.304575426
.090831659	.110714311	.170961510	.210333053	.335069210	.277352730
.070018373	.089753208	.129992976	.186934676	.271643259	.316912389

S*
T

.037900257	.046609124	.066709719	.086418972	.089803552	.090832157
.150830293	.186024415	.266356864	.345793903	.355513233	.361081681
.133419439	.164617596	.237302605	.304869665	.310104242	.310294644
.106819724	.132081518	.188419815	.243672009	.236891781	.241255685

S*
R

.038096731	.046927734	.067489356	.088062677	.096888425	.098051801
.151861337	.187705987	.270454307	.354530468	.392854874	.399963021
.134978712	.167149954	.243688380	.317961055	.370384720	.367765258
.108851454	.135418589	.196510739	.261164054	.312558998	.322095374

$$p_1 = p_9 = 19/400, p_2 = p_8 = 19(11 + \sqrt{5})/6400, p_3 = p_7 = 19/640,$$

$$p_4 = p_6 = 19(11 - \sqrt{5})/6400, p_5 = 19/800.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 2.0$.

P_T^*

.175910654	.039315458	.035508463	.030960421	.020432986	.020308163
.127227495	.211083656	.115558506	.101211691	.064389615	.065282868
.071016927	.071419085	.084828796	.053683101	.033536991	.032073780
.038269185	.038659426	.033177981	.028195110	.015731471	.015614910

P_R^*

.055900165	.056743695	.058428054	.058988058	.055010305	.055006941
.183626456	.195438345	.212123979	.228949483	.221636712	.237468695
.116856109	.131099829	.171263982	.179029054	.255525624	.199464989
.072913250	.087450921	.110646040	.144559727	.193708298	.235063768

M_T^*

.151764230	.156704068	.155132998	.142064546	.095124702	.094490623
.507105019	.527187616	.521550628	.476268463	.314628439	.313826210
.310265998	.322336016	.317635437	.282147819	.183958929	.181063451
.175601437	.181918365	.174376942	.153633229	.098480153	.097637096

M_R^*

.176235603	.186293365	.197232809	.195004401	.154785787	.154523771
.602857995	.648510155	.701192481	.712605849	.581986195	.598256317
.394465620	.433360788	.504822390	.514569675	.522003202	.462438899
.241038697	.272191214	.318021549	.359949904	.376209511	.420612655

S_T^*

0.88403769	0.93891202	0.96874114	0.90944016	0.61593975	0.61201829
3.03838314	3.22415145	3.31770943	3.10624321	2.09665032	2.08311696
1.93748229	2.05045720	2.09636261	1.94898065	1.30698416	1.29732328
1.12412987	1.18647933	1.20453628	1.11337867	0.74280430	0.73732286

S_R^*

0.91877040	0.98522975	1.04183495	1.00410826	0.70759005	0.70465258
3.18827045	3.42454084	3.63432406	3.51787876	2.49734128	2.49033049
2.08366991	2.24613580	2.40838166	2.35345647	1.71853798	1.70646500
1.24114608	1.34371012	1.45451609	1.44310118	1.08980695	1.09403818

$$p_1 = p_9 = 19/400, p_2 = p_8 = 19(11 + \sqrt{5})/6400, p_3 = p_7 = 19/640,$$

$$p_4 = p_6 = 19(11 - \sqrt{5})/6400, p_5 = 19/800.$$

$$a_1 = a_2 = \dots = a_9 = 1.$$

Slab thickness, $t = 16.0$.

P^*_T

.000364025	.000302034	.000220603	.000177450	.000111722	.000111236
.000977401	.000811136	.000592430	.000476540	.000300028	.000298723
.000441206	.000366142	.000267420	.000215108	.000135431	.000134842
.000219340	.000182022	.000132944	.000106938	.000067328	.000067035

P^*_R

.074298286	.073069097	.070728837	.068734869	.061103446	.061059065
.236456565	.242519798	.247675353	.257094356	.239221454	.254932306
.141457674	.153071786	.187891326	.192179327	.263739323	.207620769
.084960979	.098201306	.118773356	.150990571	.197725456	.239052906

M^*_T

.017033474	.014511792	.010885487	.008832227	.005580197	.005552731
.046961146	.039982124	.029971162	.024312724	.015359457	.015284070
.021770974	.018523197	.013876101	.011253979	.007109052	.007074258
.010917233	.009286634	.006955342	.005640634	.003563049	.003545626

M^*_R

.447105739	.433409438	.395185088	.357885005	.258187190	.257224783
1.40254241	1.37832101	1.28535178	1.19277874	.886672955	.900865141
.790370182	.794391090	.793015350	.750956521	.671868200	.611275434
.442370198	.455600938	.464116654	.479640966	.452055755	.495939247

S^*_T

.967187137	.837232720	.639149655	.521736488	.330465115	.328723986
2.70934202	2.34381957	1.78805387	1.45924066	.924184736	.919327834
1.27829933	1.10507807	.842413462	.687324210	.435259144	.432978035
.644901772	.557380338	.424789723	.346555779	.219454255	.218305224

S^*_R

7.32483786	6.84869281	5.81257717	4.98042893	3.24780942	3.22827590
22.1628356	20.7996755	17.7718756	15.2973974	10.0211894	9.96482783
11.6251545	10.9836232	9.51086570	8.26544693	5.49283749	5.45589242
6.15614877	5.84308588	5.10832713	4.48232416	3.02945340	3.02088738

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Exact Solutions of One-Dimensional Scattering Problems.

P. J. BROCKWELL and J. E. MOYAL

Australian National University - Canberra

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Summary. — A one-dimensional model for particle scattering which has been applied by Fermi to neutron transport theory is examined using the theory of discontinuous Markov processes. The process is characterized by the transition probabilities $P_{rk}(S|\omega; s)$ that a transition from a state ω to a state belonging to the set S occurs in path length s accompanied by r collisions with reversals and k collisions without reversals. Assuming constant mean free path the behaviour of the particle, whose state is specified by a position co-ordinate x , a direction of motion μ and an absolute velocity v , is determined completely and explicitly under a variety of boundary conditions.

1. — Introduction.

We consider a particle whose state ω is specified by its position vector x , its direction of motion μ (μ is a unit vector) and its absolute velocity v : thus ω stands for the set of variables (x, μ, v) ; the results we obtain can be generalized in an obvious way to the case where ω includes further internal state variables of the particle, such as rotations or vibrations. A scattering process is a stochastic process characterized by the fact that only the position of the scattered particle changes between collisions of the particle with the scattering centres of the medium, the collision rate $\lambda(\omega)$ depends only on the instantaneous state ω of the particle, the change of state of the particle at each collision is instantaneous and the probability given that a collision has occurred at x of a transition $(v, \mu) \rightarrow V \times L$ (where V is a set of velocities and L a set of directions of motion) is given by a distribution $\varphi(V \times L|v, \mu; x)$ conditional only on the velocity v and direction of motion μ immediately prior to the collision and independent of previous values of the state variables. The change of position

between collisions is deterministic: if the particle initially at x with direction of motion μ suffers no collisions in a length of path s , its position will be $x + \mu s$. With these assumptions, it is clear that a scattering process is a time-homogeneous *discontinuous Markov process* of the type whose theory is given in MOYAL ⁽¹⁾. The independent variable of the process may be taken as either the *path length* s or the *time* t , the connection between these two formulations follows from the relation $ds = v dt$; thus if we use λ to denote the collision rate per unit path-length s (i.e., λ is the inverse of the mean free path), then the collision rate per unit time is λv .

If the scattering medium is *homogeneous*, then λ is independent of x and μ , φ is independent of x and depends on the *change* in μ . If furthermore the collision rate depends only on the *density* of the scatterer, then the inverse mean free path λ is a constant. We shall make these two assumptions in the present paper, which is devoted to the study of one-dimensional scattering processes, where the position x is a point on the line, and μ has only two possible values: $+1$ for motion to the right, -1 for motion to the left. Thus we assume that:

1) The collision rate per unit path-length λ is constant.

2) Given a collision, the probability of a *reversal* of direction is a constant p ($0 \leq p \leq 1$) and that of a *continuation* is hence $q = 1 - p$: thus if we denote by $\varphi(v|\mu)$ the probability of a transition $\mu \rightarrow v$ in a collision, then $\varphi(-\mu|\mu) = p$ and $\varphi(\mu|\mu) = q$.

Because of these assumptions, it is convenient to choose the path-length s as independent variable. Let $\psi_p(V|v)$, $\psi_q(V|v)$ be the probabilities of a transition $v \rightarrow V$ in a collision conditional respectively on a reversal or a continuation (thus $\varphi(V \times \{\mu\} | v, \mu) = q\psi_q(V|v)$ and $\varphi(V \times \{-\mu\} | v, \mu) = p\psi_p(V|v)$). We shall make the further assumption that ψ_p and ψ_q *commute* in the following sense:

$$3) \int_0^\infty \psi_p(V|u) \psi_q(du|v) = \int_0^\infty \psi_q(V|u) \psi_p(du|v), \text{ or symbolically, } \psi_p \circ \psi_q = \psi_q \circ \psi_p.$$

We introduce the notation $\psi_p^{(n)}$, $\psi_q^{(n)}$ for the sequences of distributions defined respectively by the iteration relations $\psi_p^{(n)} = \psi_p^{(n-1)} \circ \psi_p$ and $\psi_q^{(n)} = \psi_q^{(n-1)} \circ \psi_q$.

With these assumptions, we shall show that it is possible to obtain explicit solutions of the problems connected with one-dimensional scattering. An extension of the methods developed here to more general scattering processes will form the subject of a separate publication.

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One-dimensional scattering problems have been studied by a number of authors. GOLDSTEIN ⁽²⁾ considered particles which travel a fixed distance $v\tau$ between collisions and for which the direction of motion is reversed at a collision with probability p and maintained with probability q . He determined the characteristic function for the displacement $v\tau$ of a particle after n collisions and letting $\tau \rightarrow 0$, $n \rightarrow \infty$ in such a way that $n\tau = t$, $v\tau = y$, found that the limiting distribution of particles on $(-\infty, \infty)$ obeyed the telegraph equation. Generalizations of Goldstein's model have been investigated by GUPTA ⁽³⁾ and GILLIS ⁽⁴⁾. MYCIELSKI and PASZKOWSKI ⁽⁵⁾ studied the case of a particle moving on a line in such a way that for any segment of measure x there are probabilities $p(x)$, $q(x)$ and $r(x)$ of transmission, absorption and reflection for leftward moving particles and corresponding probabilities $P(x)$, $Q(x)$ and $R(x)$ for particles moving to the right. They found that these functions must be of certain restricted types. A series of papers by BELLMAN, KALABA and WING (including ⁽⁶⁾, ⁽⁷⁾, ⁽⁸⁾ and ⁽⁹⁾) deals with problems in neutron transport theory using the principle of invariant imbedding in which the process is considered as a function of the boundaries. The emphasis in these papers is on fission problems and the methods used are fundamentally different from ours which are based on the theory of discontinuous Markov processes. The unrestricted model which we shall consider in Sect. 4 was first used in the study of neutron transport theory by FERMI ⁽¹⁰⁾ and later by PLACZEK ⁽¹¹⁾.

2. - Equations for the transition probabilities.

Considered as a Markov process in the state variables ω , a scattering process is completely characterized by the transition probability function $P(S|\omega; s)$ (the probability of a transition $\omega \rightarrow S$ in path-length s) satisfying the Chapman-Kolmogorov relation

$$(2.1) \quad P(S|\omega; s_1 + s_2) = \int_{\mathcal{V}} P(S|\eta; s_1) P(d\eta|\omega; s_2) \quad (s_1, s_2 \geq 0),$$

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⁽⁹⁾ R. BELLMAN, R. KALABA and G. M. WING: *Journ. Math. Mech.*, **8**, 575 (1959).

⁽¹⁰⁾ E. FERMI: *Ric. Scient.*, **7** (2), 13 (1936).

⁽¹¹⁾ G. PLACZEK: Manhattan Project Report, A-25.

or, symbolically, $P(s_1 + s_2) = P(s_1) \circ P(s_2)$. However, we shall require a more detailed specification provided by the transition probabilities $P_{rk}(S|\omega; s)$, probabilities of a transition $\omega \rightarrow S$ in s involving exactly r reversals and k continuations. These functions characterize a Markov process in the state variables and numbers of reversals and continuations, the Chapman-Kolmogorov relation taking the form

$$(2.2) \quad P_{rk}(S|\omega; s_1 + s_2) = \sum_{i=0}^r \sum_{j=0}^k \int_{\Omega} P_{r-i, k-j}(S|\eta; s_1) P_{ij}(\mathrm{d}\eta|\omega; s_2) \quad (s_1, s_2 \geq 0).$$

We also introduce the sequence of probability distributions $Q_{rk}(S, s|\omega)$, the probability that the $(r+k)$ -th collision with r reversals of a particle initially in state ω occurs in s leaving the particle (immediately after the collision) in a state belonging to the set S . The relation between the P_{rk} and the Q_{rk} is

$$(2.3) \quad P_{rk}(S|\omega; s) = \int_0^s \int_{\Omega} P_{00}(S|\eta; s - \tau) Q_{rk}(\mathrm{d}\eta, \mathrm{d}\tau|\omega) \quad (r, k = 0, 1, \dots),$$

where conventionally $Q_{00}(S, s|\omega) = \delta(S|\omega) \varepsilon(s) = I(S, s|\omega)$; here $\delta(S|\omega)$ is the characteristic function of S ($\delta = 1$ if $\omega \in S$ and $\delta = 0$ if $\omega \notin S$) and $\varepsilon(s)$ is the unit step function ($\varepsilon(s) = 0$ if $s \leq 0$, $\varepsilon(s) = 1$ if $s > 0$).

We express symbolically the type of composition in the right-hand side of (2.3) as follows: $P_{rk} = P_{00} * Q_{rk}$. The Q_{rk} form a double Markov chain governed by the iteration relation

$$(2.4) \quad Q_{rk} = Q_{r-1, k} * Q_{10} + Q_{r, k-1} * Q_{01} \quad (r, k = 1, 2, \dots).$$

Thus given P_{00} , Q_{01} and Q_{10} , the relations (2.3) and (2.4) determine all the P_{rk} and Q_{rk} . We also introduce the probabilities $P_n = \sum_{i=0}^n P_{n-i, i}$ and $Q_n = \sum_{i=0}^n Q_{n-i, i}$, with $P_0 = P_{00}$ [where of course n is the number of collisions, irrespective of whether they are reversals or continuations]. We now define the generating function

$$(2.5) \quad G(S|\omega; s, z, w) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk}(S|\omega; s),$$

$$(2.6) \quad H(S, s|\omega; z, w) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k Q_{rk}(S, s|\omega) - I(S, s|\omega).$$

where z, w are complex variables of modulus ≤ 1 .

The transition probability $P(S|\omega; s)$ is obtained by setting $z = w = 1$ in G . It follows from (2.3) that

$$(2.7) \quad G = P_0 * (I + H) = P_0 + P_0 * H.$$

If we let $H_1 = zQ_{10} + wQ_{01}$, defining the sequence $\{H_n\}$ by the iteration relation

$$(2.8) \quad H_{n+1} = H_n * H_1 \quad (n=1, 2, \dots)$$

and the sequence $\{G_n\}$ by $G_n = P_0 * H_n$, then it is easily seen to follow from (2.3) and (2.4) that

$$(2.9) \quad H_n = \sum_{j=0}^n z^{n-j} w^j Q_{n-j,j}, \quad G_n = \sum_{j=0}^n z^{n-j} w^j P_{n-j,j} \quad (n=1, 2, \dots),$$

and hence that (setting $G_0 = P_0$)

$$(2.10) \quad H = \sum_{n=1}^{\infty} H_n, \quad G = \sum_{n=0}^{\infty} G_n.$$

It is now easy to see that G and H must satisfy the fundamental integral equations:

$$(2.11) \quad G = P_0 + G * H_1,$$

$$(2.12) \quad H = H_1 + H * H_1.$$

The theory of discontinuous Markov processes in MOYAL⁽¹⁾ is easily generalized to show that the expressions (2.5) and (2.6) for G and H are respectively solutions of the integral eqs. (2.11) and (2.12). These solutions are unique under certain not very stringent conditions, which are fulfilled in the present case if the collision rate λ is a bounded function (and hence *a fortiori* if λ is constant). All transition probabilities obtained from G satisfy the relevant Chapman-Kolmogorov relation.

The foregoing theory applies to general one-dimensional scattering processes (homogeneous in path-length or time). It remains applicable under the imposition of a variety of boundary conditions, such as absorbing or reflecting boundaries in space. The effect of such boundary conditions is incorporated in the integral eqs. (2.11), (2.12) via the definition of the basic distributions P_0 , Q_{10} and Q_{01} and of the $*$ -operation. It then follows that the distributions P_{rk} , Q_{rk} , obtained by $*$ -iterations from P_0 , Q_{10} and Q_{01} , and hence the generating functions G and H , automatically satisfy the boundary conditions.

3. - The « backward » equations.

We now specialize the theory to one-dimensional processes satisfying postulates 1), 2) and 3) of Sect. 1. Because of these postulates, the theory simplifies considerably. Firstly, the (absolute) velocity distribution conditional on exactly r reversals and k continuations is $\psi_p^{(r)} \circ \psi_q^{(k)}$ independently of their order. This implies that the process is still Markovian and the theory of Sect. 2 still applies if we omit v and take the state of the particle to be $\omega = (\mu, x)$. Let P_{rk} , Q_{rk} , G , H etc., be the various functions defined in Sect. 2 in terms of $\omega = (\mu, x)$, and let P'_{rk} , Q'_{rk} , G' , H' etc., the corresponding functions in terms of $\omega = (v, \mu, x)$. Then P'_{rk} , Q'_{rk} split into products of independent distributions:

$$(3.1) \quad P'_{rk} = P_{rk} \times (\psi_p^{(r)} \circ \psi_q^{(k)}), \quad Q'_{rk} = Q_{rk} \times (\psi_p^{(r)} \circ \psi_q^{(k)}),$$

and hence

$$(3.2) \quad G' = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk} \times (\psi_p^{(r)} \circ \psi_q^{(k)}), \quad H' = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k Q_{rk} \times (\psi_p^{(r)} \circ \psi_q^{(k)}) - I.$$

Thus we can develop the theory initially in terms of μ and x only and introduce the dependence on v later. Secondly, we can omit the final direction v from $P_{rk}(X, v|x, \mu; s)$, $Q_{rk}(X, v, s|x, \mu)$ etc., where X denotes a (measurable) subset of the real line, because $v = \mu$ if r is even and $v = -\mu$ if r is odd (i.e., $P_{2r,k}$, $Q_{2r,k}$ vanish if $v = -\mu$ and $P_{2r+1,k}$, $Q_{2r+1,k}$ vanish if $v = \mu$). It will be convenient to express the initial direction μ as a superscript: thus, $P_{rk}^{(\mu)}$, $Q_{rk}^{(\mu)}$, $G^{(\mu)}$, $H^{(\mu)}$ etc.

Consider now the unrestricted process. With constant λ , the probability of no collisions in s is $\exp[-\lambda s]$; with no collisions, the position of the particle initially at x becomes $x + \mu s$. Hence

$$(3.3) \quad \left\{ \begin{aligned} P_0^{(\mu)}(X|x, s) &= \delta(X|x + \mu s) \exp[-\lambda s], \\ H_1^{(\mu)}(X, s|x; z, w) &= (pz + qw) \int_0^s \delta(X|x + \mu \tau) \exp[-\lambda \tau] \lambda d\tau = \\ &= (pz + qw) \int_0^s P_0^{(\mu)}(X|x; \tau) \lambda d\tau, \end{aligned} \right.$$

and therefore

$$\frac{\partial H_1^{(\mu)}}{\partial s} = \lambda(pz + qw) P_0^{(\mu)}.$$

This last relation together with (2.8) implies that $\partial H_n^{(\mu)}/\partial s = \lambda(pz + qw)G_n^{(\mu)}$, and hence that

$$(3.4) \quad \frac{\partial H^{(\mu)}}{\partial s} = \lambda(pz + qw)G^{(\mu)}.$$

Thus, one need only calculate one of the generating functions G or H . It turns out that H is the more important for our purpose, particularly in the investigation of limiting distributions as $s \rightarrow \infty$; for this reason we concentrate on H .

Substituting (3.3) in (2.12), we obtain the « backward » equation for H :

$$(3.5) \quad \left[\frac{\partial}{\partial s} - \mu \frac{\partial}{\partial x} + \lambda(1 - qw) \right] H^{(\mu)} = \lambda pz H^{(-\mu)} + \lambda(pz + qw) \delta(X|x),$$

with initial conditions which from (3.3) are $[H^{(\mu)}]_{s=0} = 0$, $\mu = \pm 1$. Introducing the Laplace transform

$$\overline{H^{(\mu)}} = \int_0^\infty \exp[-\sigma s] H^{(\mu)} ds,$$

eq. (3.5) becomes

$$(3.6) \quad \left[\sigma + \lambda(1 - qw) - \mu \frac{\partial}{\partial x} \right] \overline{H^{(\mu)}} = \lambda pz \overline{H^{(-\mu)}} + \frac{\lambda}{\sigma} (pz + qw) \delta(X|x).$$

From (3.6) we obtain the 2nd order equation

$$(3.7) \quad \left(\frac{\partial^2}{\partial x^2} - \theta^2 \right) \overline{H^{(\mu)}} = -\frac{\lambda}{\sigma} (pz + qw) [\sigma + \lambda(1 - qw + pz)] \delta(X|x),$$

where

$$(3.8) \quad \theta^2 = [\sigma + \lambda(1 - qw + pz)][\sigma + \lambda(1 - qw - pz)],$$

which is the transform of a nonhomogeneous telegraphist's equation with leakage; an equation of the same type was obtained by GOLDSTEIN in (2). Note that because of (3.4), $G^{(\mu)}$ satisfies the homogeneous part of (3.5), and hence the homogeneous telegraphist's equation

$$(3.9) \quad \left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial x^2} + 2\lambda(1 - qw) \frac{\partial}{\partial s} + \lambda^2(1 - qw + pz)(1 - qw - pz) \right] G^{(\mu)} = 0.$$

It also follows from (3.4) that

$$(3.10) \quad \lambda(pz + qw) \overline{G^{(\mu)}} = \lambda(pz + qw) \int_0^\infty \exp[-\sigma s] G^{(\mu)} ds = \\ = \int_0^\infty \exp[-\sigma s] \frac{\partial H^{(\mu)}}{\partial s} ds = \sigma \overline{H^{(\mu)}}.$$

Consider now processes whose motion is restricted by the imposition of boundary conditions such as absorbing or reflecting barriers. Suppose we have two such barriers at a_{-1} , a_1 ($a_{-1} < a_1$): then for an initial position $x \in [a_{-1}, a_1]$, one can see that for small s the generating function H appropriate to this process will differ only by terms of order $o(s)$ from the one appropriate to the unrestricted process. Hence H will satisfy the same « backward » equation (3.5). Thus, contrary to what is the case for purely discontinuous Markov processes (cf. MOYAL ⁽¹⁾) the « backward » equation does not have a unique solution and is not equivalent to the basic integral equation for the process. However, a solution of (3.5) which satisfies the required boundary conditions will also satisfy the integral equation appropriate to the process with these boundary conditions, and is hence the unique solution of the problem.

The general solution of (3.6) or (3.7) is

$$(3.11) \quad H^{(\mu)} = \begin{cases} A_{11}^{(\mu)} \exp[\theta x] + A_{12}^{(\mu)} \exp[-\theta x], & x \notin X, \\ A_{21}^{(\mu)} \exp[\theta x] + A_{22}^{(\mu)} \exp[-\theta x] + \frac{\lambda}{\sigma} (pz + qw) \cdot \\ \cdot [\sigma + \lambda(1 - qw - pz)]^{-1}, & x \in X, \end{cases}$$

where the coefficients $A_{ij}^{(\mu)}$ are related via eq. (3.6) by

$$(3.12) \quad \lambda pz A_{ij}^{(-1)} = [\sigma + \lambda(1 - qw) + (-1)^j \theta] A_{ij}^{(w)}, \quad i, j = 1, 2,$$

but are otherwise arbitrary functions of σ , z and w . The solution satisfying required boundary conditions is obtained by appropriately choosing these functions.

4. - The unrestricted process.

In the unrestricted case, it is convenient to express the solutions in terms of cumulative distributions of displacement: *i.e.*, we set $X = (-\infty, y]$ (*) in H .

(*) $[a, b]$ denotes a closed interval, $(a, b]$ an interval open to the left and closed to the right.

The appropriate boundary conditions for $\overline{H^{(\mu)}}$ are here that $\overline{H^{(\mu)}}$ is a bounded function of x as $x \rightarrow \pm \infty$; hence

$$(4.1) \quad \overline{H^{(\mu)}} = \begin{cases} A_{12}^{(\mu)} \exp[-\theta x], & x > y, \\ A_{12}^{(\mu)} \exp[\theta x] + \frac{\lambda}{\sigma} (pz + qw) \varphi^{-1}, & x \leq y, \end{cases}$$

where

$$(4.2) \quad \varphi = \sigma + \lambda(1 - qw - pz).$$

The two further conditions needed to determine $A_{12}^{(\mu)}$, $A_{21}^{(\mu)}$ are obtained by requiring $\overline{H^\mu}$ to be continuous at $x = y$: i.e., we set

$$(4.3) \quad [\overline{H^{(\mu)}}]_{x=y-} = [\overline{H^{(\mu)}}]_{x=y+}, \quad \mu = 1, -1.$$

Substituting (4.1) in (4.3) and using (3.12), we find that

$$(4.4) \quad \overline{H^{(\mu)}} = \begin{cases} -\frac{\lambda(pz + qw)}{2\sigma} \exp[\theta(y - x)](\theta^{-1} - \varphi^{-1}), & x > y, \\ -\frac{\lambda(pz + qw)}{2\sigma} \exp[\theta(x - y)](\theta^{-1} + \varphi^{-1}) + \frac{\lambda}{\sigma} (pz + qw) \varphi^{-1}, & x \leq y; \end{cases}$$

we obtain $\overline{H^{(-1)}}$ from (4.4) by using (3.6) and $\overline{G^{(\mu)}}$, $\mu = \pm 1$ by using (3.10). Thus (4.4) yields a complete and explicit solution (in Laplace transform) of the one-dimensional scattering problem in the unrestricted case.

We see from (4.4) and (3.6) that $\overline{H^{(\mu)}}$ is a continuous function of $\xi = y - x$, and that its density, say $h^{(\mu)}(\xi) = \partial \overline{H^{(\mu)}} / \partial \xi$ is continuous apart from a finite saltus at $\xi = 0$; introducing the Fourier-Stieltjes transform

$$\overline{\varphi^{(\mu)}}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] \overline{H^{(\mu)}}(d\xi) = \int_0^{\infty} [\exp[i\alpha\xi] h^{(\mu)}(\xi) + \exp[-i\alpha\xi] h^{(\mu)}(-\xi)] d\xi,$$

it is easily seen to follow from (4.4) that

$$(4.5) \quad \overline{\varphi^{(1)}}(\alpha) = \frac{\lambda}{\sigma} (pz + qw) \left[\frac{i\alpha}{\theta^2 + \alpha^2} + \frac{\sigma + \lambda(1 - qw + pz)}{\theta^2 + \alpha^2} \right].$$

Clearly, $\overline{\varphi^{(\mu)}}(\alpha)$ is the Laplace transform of the Fourier-Stieltjes transform

$$\varphi^{(\mu)}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] H^{(\mu)}(d\xi).$$

Inverting (4.5) we find that

$$(4.6) \quad \varphi^{(1)}(\alpha) = \frac{\lambda(pz + qw)[i\alpha + \lambda(1 - qw + pz)]}{\lambda^2(1 - qw)^2 + \eta^2} \cdot \left[1 - \exp[-\lambda(1 - qw)s] \cos \eta s - \frac{\lambda(1 - qw)}{\eta} \exp[-\lambda(1 - qw)s] \sin \eta s \right] + \frac{\lambda(pz + qw)}{\eta} \exp[-\lambda(1 - qw)s] \sin \eta s,$$

where $\eta^2 = \alpha^2 - (\lambda zp)^2$.

We now turn our attention to the limiting distributions as $s \rightarrow \infty$: for the unrestricted process, it is easy to see that $G^{(\mu)} \rightarrow 0$; on the other hand, $\lim_{s \rightarrow \infty} Q_{r,n-r}$ will yield the probability distribution of the position at which the n -th collision with r reversals occurs, and hence $K^{(\mu)} = \lim_{s \rightarrow \infty} H^{(\mu)}$ will yield the generating function for these distributions. From these distributions we can find in the way outlined above the joint distributions of the n -th collision position together with the absolute velocity and direction of motion resulting from this collision. Since $K^{(\mu)} = \lim_{s \rightarrow \infty} \sigma H^{(\mu)}$, we find from (4.4) that

$$(4.7) \quad K^{(1)} = -\frac{1}{2}\lambda(pz + qw) \exp[-\gamma|y - x|][\gamma^{-1} + \operatorname{sgn}(y - x)\lambda^{-1}(1 - qw - pz)]$$

where

$$(4.8) \quad \gamma^2 = \lambda^2(1 - qw - pz)(1 - qw + pz).$$

Clearly, the Fourier-Stieltjes transform

$$\psi^{(\mu)}(\alpha) = \int_{-\infty}^{\infty} \exp[i\alpha\xi] K^{(\mu)}(d\xi) = \lim_{s \rightarrow \infty} \varphi^{(\mu)}(\alpha),$$

and hence

$$(4.9) \quad \psi^{(1)}(\alpha) = \lambda(pz + qw)[i\alpha + \lambda(1 - qw + pz)][\lambda^2(1 - qw)^2 + \eta^2]^{-1}.$$

Equation (4.9) is a special case of a more general result which is derived in Appendix I.

PLACZEK ⁽¹¹⁾ considers the unrestricted process in which $p = q = \frac{1}{2}$ and derives an expression for $\psi_0(y, u)$, where $\psi_0(y, u) dy du$ is the number of neutrons per unit times scattered into the position-lethargy element $dy \times du$. (Lethargy is defined as $\log E_0/E$ where E is the energy and E_0 the initial energy of the particle.) He assumes that the lethargy increases by a fixed amount, c , per collision and that there is an isotropic source at the origin emitting neutrons of lethargy 0 at the rate of one per second.

$\psi_0(y, u)$ in this case can be deduced from eq. (4.4) as follows: assuming neutrons are emitted at the origin (i.e., $x = 0$) with equal probabilities of

moving initially to the right or left it follows that the generating function of the probability density $f_n(y)$ for the point at which a neutron suffers its n -th collision is given by

$$(4.10) \quad \sum_{n=1}^{\infty} f_n(y) z^n = \lim_{\sigma \rightarrow \infty} \frac{1}{2} \left[\frac{\partial H^{(1)}}{\partial y} + \frac{\partial H^{(-1)}}{\partial y} \right] = \lim_{\sigma \rightarrow 0} \frac{\sigma}{2} \left[\frac{\partial \bar{H}^{(1)}}{\partial y} + \frac{\partial \bar{H}^{(-1)}}{\partial y} \right],$$

i.e.

$$(4.11) \quad \sum_{n=1}^{\infty} f_n(y) z^n = \frac{z}{2\sqrt{1-z}} \exp[-\sqrt{1-z}|y|]$$

(taking the mean free path as the unit of length).

Since the increase in lethargy per collision is constant for each collision it follows that

$$(4.12) \quad \psi_0(y, u) = \delta(u - nc) f_n(y), \quad n = 1, 2, \dots$$

From eq. (4.11) $f_n(y)$ can be expressed as a contour integral,

$$(4.13) \quad f_n(y) = \int_C \frac{1}{2z^n \sqrt{1-z}} \exp[-\sqrt{1-z}|y|] dz, \quad n = 1, 2, \dots,$$

where C is the contour illustrated in Fig. 1. Changing the variable of integration to w , where $z = 1 + w^2$, and using eq. (4.12) we obtain the result

$$(4.14) \quad \psi_0(y, u) = \delta(u - nc) \frac{1}{\pi} \int_0^{\infty} \frac{\cos yw}{(1 + w^2)^n} dw$$

which was originally obtained by PLAZCEK. For a discussion of the behaviour of the function $\psi_0(y, u)$ we refer the reader to (11).

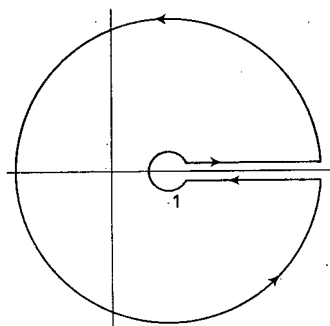


Fig. 1.

5. - The process with absorbing boundaries.

Consider now the one-dimensional scattering process with absorbing barriers at a_{-1} , a_1 ($a_{-1} < a_1$). As mentioned in Sect. 2, these boundary conditions have to be incorporated in the expressions for the basic probability distributions P_0 , Q_{10} , Q_{01} and the $*$ -operation. In the present case, all distributions of interest may be derived from the generating function H restricted by the condition that all collisions prior to the last one occur in the interval $[a_{-1}, a_1]$.

This condition is satisfied if we choose $H_1^{(\mu)}$ as in (3.3) and restrict integration with respect to x in the $*$ -operation to the interval $[a_{-1}, a_1]$. As mentioned in Sect. 3, $\overline{H^{(\mu)}}$ will then satisfy the «backward» eq. (3.6) for initial positions $x \in [a_{-1}, a_1]$. Take $X = (a_{-1}, y]$, where $a_{-1} < y \leq a_1$; then the boundary conditions that must be imposed on the general solution (3.11) are

$$(5.1) \quad [\overline{H^{(\mu)}}]_{x=a_\mu} = 0; \quad [\overline{H^{(\mu)}}]_{x=y+} = [\overline{H^{(\mu)}}]_{x=y-}; \quad \mu = 1, -1.$$

It is convenient in this case to rewrite (3.11) in the form

$$(5.2) \quad H^{(\mu)} = \begin{cases} B_{11}^{(\mu)} \sinh \theta(a_1 - x) + B_{12}^{(\mu)} \cosh \theta(a_1 - x), & a_{-1} < y \leq x \leq a_1, \\ B_{21}^{(\mu)} \sinh \theta(y - x) + B_{22}^{(\mu)} \cosh \theta(y - x) + \frac{\lambda}{\sigma} (pz + qw) \varphi^{-1}, & a_{-1} \leq x \leq y \leq a_1, \end{cases}$$

where φ was defined in (4.2). Substituting (5.2) in (5.1) and using (3.6) to derive the relations between $B_{ij}^{(1)}$ and $B_{ij}^{(-1)}$, we find the coefficients $B_{ij}^{(\mu)}$ and hence

$$(5.3) \quad \overline{H^{(1)}} = \begin{cases} \frac{\lambda(pz + qw) \sinh \theta(a_1 - x)}{\sigma \sinh \theta l} \frac{\sinh \theta(a_1 - y)}{\sinh \theta(a_1 - y)} [\varphi^{-1} \sinh \theta l - \alpha \sinh \theta(a_1 - y) - \\ - \beta \sinh \theta(y - a_{-1})], & a_{-1} < y \leq x \leq a_1, \\ \frac{\lambda(pz + qw)}{\sigma \sinh \theta l} [\varphi^{-1} \sinh \theta l - \alpha \sinh \theta(a_1 - x) - \beta \sinh \theta(x - a_{-1})], & a_{-1} \leq x \leq y \leq a_1, \end{cases}$$

where

$$(5.4) \quad \begin{cases} l = a_1 - a_{-1}; & \alpha = \lambda pz \varphi^{-1} [\theta + \sigma + \lambda(1 - qw)]^{-1}, \\ \beta = \theta^{-1} \sinh \theta(a_1 - y) - \varphi^{-1} \cosh \theta(a_1 - y). \end{cases}$$

Take $X = (a_1, y]$, where $a_1 < y < \infty$; in this case the boundary conditions are

$$(5.5) \quad [\overline{H^{(1)}}]_{x=a_1} = \frac{\lambda(pz + qw)}{\sigma(\sigma + \lambda)} [1 - \exp[-(\sigma + \lambda)(y - a_1)]]; \quad [\overline{H^{(-1)}}]_{x=a_{-1}} = 0.$$

The first of these conditions results from the observation that if initially $\mu = 1$, $x = a_1$ and if $X = (a_1, y]$ with $y > a_1$, then the first collision occurs outside $[a_{-1}, a_1]$; hence $H^{(1)} = H_1^{(1)}$, and therefore

$$\begin{aligned} \overline{H^{(1)}} = \overline{H_1^{(1)}} &= (pz + qw) \int_0^\infty \exp[-\sigma s] ds \int_0^s \delta[(a_1, y] | a_1 + \tau] \exp[-\lambda \tau] \lambda d\tau = \\ &= \frac{\lambda}{\sigma} (pz + qw) \int_0^{y-a_1} \exp[-(\sigma + \lambda)s] ds. \end{aligned}$$

Substituting the first expression of (3.11) in (5.5) and using (3.12) we find $A_{ii}^{(\mu)}$ $i=1, 2$, and hence

$$(5.6) \quad \overline{H^{(1)}} = \frac{\lambda(pz + qw)[\sigma + \lambda(1 - qw)] \cosh \theta(x - a_{-1}) + \theta \sinh \theta(x - a_{-1})}{\sigma(\lambda + \sigma) [\sigma + \lambda(1 - qw)] \cosh \theta l + \theta \sinh \theta l} \cdot (1 - \exp[-(\sigma + \lambda)(y - a_1)]) .$$

The expression for $\overline{H^{(u)}}$ when $X = [y, a_{-1}]$ is obtained by symmetry from (5.6):

$$(5.7) \quad \overline{H^{(u)}}[[y, a_{-1}] | x; \sigma, a_1, a_{-1}] = \overline{H^{(-u)}}((-a_{-1}, -y] | -x; \sigma, -a_1, -a_{-1}) .$$

The expression for $\overline{H^{(u)}}$ when $X = (-\infty, y]$ is a linear combination of the solutions (5.3), (5.6) and (5.7).

The generating function for the limiting distribution as $s \rightarrow \infty$

$$(5.8) \quad K^{(\mu)} = \lim_{s \rightarrow \infty} H^{(\mu)} = \lim_{\sigma \rightarrow 0} \sigma \overline{H^{(\mu)}}$$

follows directly from (5.3), (5.6) and (5.7) and yields the probability distribution of the position at which the n -th collision occurs with the previous $n-1$ all taking place in $[a_{-1}, a_1]$ (see the remarks at the end of Sect. 4). If we write $K^{(\mu)} = \sum_{n=1}^{\infty} K_n^{(\mu)}$, where $K_n^{(\mu)} = \lim_{s \rightarrow \infty} H_n^{(\mu)}$, then the iteration relation (2.8) takes a simple form:

$$(5.9) \quad K_{n+1}^{(\mu)}(X | x; z, w) = \int_0^{|x-a_{\mu}|} \{pz K_n^{(-\mu)}(X | x + \mu s; z, w) + qw K_n^{(\mu)} \cdot (X | x + \mu s; z, w)\} \exp[-\lambda s] \lambda ds ;$$

the use of these iterations to calculate an explicit expression for $K^{(\mu)}$ is not prohibitively complicated in this case.

6. - First passage distributions.

In the present Section, we wish to evaluate the probability of a 1st passage at the barriers $a_{\pm 1}$, together with the velocity and direction of motion distribution at this passage. We can do this using the method outlined at the beginning of Sect. 3 if we know the sequence of probabilities $P_{rk}^{(\mu)}(a_{\pm 1} | x; s)$ that a particle initially at x with direction μ effect a 1st passage at $a_{\pm 1}$ in a length of path $\leq s$ after suffering exactly r reversals and k continuations in $[a_{-1}, a_1]$. Writing as before P_0 for P_{00} , we can see that $P_0^{(\mu)}(a_{\pm 1} | x; s)$ is the sum of two terms, namely, the probabilities that in path length $\leq s$ the par-

ticle (initially at x with direction μ) effect a 1st passage at $a_{\pm 1}$ with no collisions, and that it suffer its 1st collision in path length $\leq s$ but outside the interval $[a_{-1}, a_1]$; thus

$$(6.1) \quad P_0^{(\mu)}(a_{\pm 1}|x; s) = \delta[(a_{\pm 1}, \pm \infty)|x + \mu s] \exp[-\lambda s] + \int_0^s \delta[(a_{\pm 1}, \pm \infty) \cdot |x + \mu \tau] \exp[-\lambda \tau] \lambda d\tau = \left(\frac{1}{\lambda} \frac{\partial}{\partial s} + 1 \right) Q_1^{(\mu)}[(a_{\pm 1}, \pm \infty), s|x].$$

Hence, taking Laplace transforms,

$$(6.2) \quad \overline{P}_0^{(\mu)}(a_{\pm 1}|x; \sigma) = \left(1 + \frac{\sigma}{\lambda} \right) \overline{Q}_1^{(\mu)}[(a_{\pm 1}, \pm \infty)|x; \sigma] = \left(1 + \frac{\sigma}{\lambda} \right) (pz + qw)^{-1} \overline{H}_1^{(\mu)}[(a_{\pm 1}, \pm \infty)|x; \sigma, z, w].$$

In accordance with the theory developed in Sect. 2 and eq. (2.7), we see that the generating function

$$(6.3) \quad G^{(\mu)} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} z^r w^k P_{rk}^{(\mu)} = P_0^{(\mu)} * [I + H^{(\mu)}] = (pz + qw)^{-1} \left(1 + \frac{1}{\lambda} \frac{\partial}{\partial s} \right) H^{(\mu)},$$

where $H^{(\mu)}$ is the generating function obtained in Sect. 5. Taking Laplace transforms, we find that

$$(6.4) \quad \overline{G}^{(\mu)}(a_{\pm 1}|x; \sigma, z, w) = (pz + qw)^{-1} \left(1 + \frac{\sigma}{\lambda} \right) \overline{H}^{(\mu)}[(a_{\pm 1}, \pm \infty)|x; \sigma, z, w].$$

In this type of problem, one is mainly interested in the limiting 1st passage distributions as $s \rightarrow \infty$. Let $M^{(\mu)} = \lim_{s \rightarrow \infty} G^{(\mu)} = \lim_{\sigma \rightarrow 0} \sigma \overline{G}^{(\mu)}$; then clearly

$$(6.5) \quad M^{(\mu)}(a_{\pm 1}|x; z, w) = (pz + qw)^{-1} K^{(\mu)}[(a_{\pm 1}, \pm \infty)|x; z, w],$$

where $K^{(\mu)}$ is the limiting generating function defined in (5.8). Explicitly,

$$(6.6) \quad \begin{cases} M^{(1)}(a_1|x; z, w) = \frac{\lambda(1 - qw) \sinh \gamma(x - a_{-1}) + \gamma \cosh \gamma(x - a_{-1})}{\lambda(1 - qw) \sinh \gamma l + \gamma \cosh \gamma l} \\ M^{(-1)}(a_1|x; z, w) = \frac{\lambda p z \sinh \gamma(x - a_{-1})}{\lambda(1 - qw) \sinh \gamma l + \gamma \cosh \gamma l} \end{cases}$$

where $l = a_1 - a_{-1}$ and γ was defined in (4.8). The expressions for $M^{(\mu)}(a_{-j})$ may be obtained by symmetry from (6.6), as in (5.7).

Letting z and w tend to 1 in (6.6) we obtain the probability $M^{(\mu)}(a_1|x)$ that the particle first emerges from $[a_{-1}, a_1]$ at a_1 (regardless of the number of collisions which occur before emergence). Thus

$$(6.7) \quad \begin{cases} M^{(1)}(a_1|x) = [1 + (x - a_{-1})\lambda p][1 + (a_1 - a_{-1})\lambda p]^{-1}, \\ M^{(-1)}(a_1|x) = (x - a_{-1})\lambda p[1 + (a_1 - a_{-1})\lambda p]^{-1}. \end{cases}$$

We notice that these are consistent with the general results found by MYCIELSKI and PASZKOWSKI ⁽⁵⁾ using invariance arguments.

Under postulate (3) of Sect. 1, *i.e.*, that the velocity transition probabilities satisfy the commutativity condition $\psi_p \circ \psi_q = \psi_q \circ \psi_p$, we can use eq. (3.2) together with the first passage distributions given by (6.6) to determine the distribution of the velocity with which the particle first emerges from the interval $[a_{-1}, a_1]$.

The probability of emerging first at $a_{\pm 1}$ (after any number of previous collisions) is given by $M^{(\mu)}(a_{\pm 1}|x)$. Hence, using (3.2), the distribution of emergent velocity conditional on the initial state (u, μ, x) and first emergence at $a_{\pm 1}$ is given by

$$(6.8) \quad F_{a_{\pm 1}}(V|u, \mu, x) = [M^{(\mu)}(a_{\pm 1}|x)]^{-1} \sum_{i+j \geq 0} P_{ij}^{(\mu)}(a_{\pm 1}|x; \infty) \psi_p^{(i)} \circ \psi_q^{(j)}.$$

In principle eq. (6.8) gives the required distribution of emergent velocity. For functions ψ_p, ψ_q of certain forms the equation simplifies considerably leading to more manageable solutions. We shall now give two such examples.

Example 1.

$$\begin{aligned} \psi_p(dv|u) &= \frac{1}{u} f_p \left(\frac{v}{u} \right) dv, & 0 \leq v \leq u, \\ \psi_q(dv|u) &= \frac{1}{u} f_q \left(\frac{v}{u} \right) dv, & 0 \leq v \leq u. \end{aligned}$$

In this case it is convenient to introduce the Mellin transforms

$$\begin{aligned} \bar{\psi}_p(\alpha) &= \int_{v=0}^u f_p \left(\frac{v}{u} \right) \left(\frac{v}{u} \right)^{\alpha-1} \frac{dv}{u}, \\ \bar{\psi}_q(\alpha) &= \int_{v=0}^u f_q \left(\frac{v}{u} \right) \left(\frac{v}{u} \right)^{\alpha-1} \frac{dv}{u}, \\ \mathcal{F}_{a_{\pm 1}}(\alpha|u, \mu, x) &= \int_{v=0}^u \left(\frac{v}{u} \right)^{\alpha-1} F_{a_{\pm 1}}(dv|u, \mu, x). \end{aligned}$$

Then, taking Mellin transforms in eq. (6.8) we have

$$(6.9) \quad \mathcal{F}_{a\pm 1}(\alpha|u, \mu, x) = [M^{(\mu)}(a_{\pm 1}|x)]^{-1} M^{(\mu)}(a_{\pm 1}|x; \bar{\psi}_p(\alpha), \bar{\psi}_a(\alpha)),$$

where the generating functions $M^{(\mu)}(a_{\pm 1}|x; z, w)$ are obtained from eq. (6.6). The moments of the distribution of emergent velocity are easily found from eq. (6.9); for example the n -th moment is given by

$$(6.10) \quad \mathcal{E}_{a\pm 1}(v^n|u, \mu, x) = u^n [M^{(\mu)}(a_{\pm 1}|x)]^{-1} M^{(\mu)}(a_{\pm 1}|x; \psi_p(n+1), \psi_a(n+1)).$$

$$\begin{aligned} \text{Example 2.} \quad \psi_p(dv|u) &= \delta(v - c_1 u) dv, & 0 \leq c_1 \leq 1, \\ \psi_a(dv|u) &= \delta(v - c_2 u) dv, & 0 \leq c_2 \leq 1. \end{aligned}$$

This is the case in which, as a result of a collision, the logarithmic energy of the particle decreases by a fixed amount depending only on the angle of deflection. The emergent distribution, as distinct from the previous example, is now discrete. If the particle, starting with velocity u , makes i collisions with reversal and j collisions without reversals then the resulting velocity is given by

$$v = u c_1^i c_2^j.$$

Taking Mellin transforms as in Example 1, again defining

$$\mathcal{F}_{a\pm 1}(\alpha|u, \mu, x) = \int_{v=0}^u \left(\frac{v}{u}\right)^{\alpha-1} F_{a\pm 1}(dv|u, \mu, x),$$

we find that

$$(6.11) \quad \mathcal{F}_{a\pm 1}(\alpha|u, \mu, x) = [M^{(\mu)}(a_{\pm 1}|x)]^{-1} M^{(\mu)}(a_{\pm 1}|x; c_1^{\alpha-1}, c_2^{\alpha-1}).$$

7. - Higher-order passage distributions and further boundary conditions.

The results of Sect. 6 can be generalized to give passage distributions for any prescribed sequence of passages through points on $(-\infty, \infty)$. Let us take two points a_1, a_{-1} ($-\infty < a_{-1} < a_1 < \infty$). These divide the real line into three regions $(-\infty, a_{-1}), [a_{-1}, a_1], (a_1, \infty)$ which we shall denote by L_{-1}, L_0, L_1 , respectively. An arbitrary sequence of passages through the points a_1, a_{-1} can then be specified by a sequence of numbers $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ where $\gamma_i = -1, 0$ or 1 . This sequence indicates that the particle starts in L_{γ_0} , makes a first passage from L_{γ_0} to L_{γ_1} , thence to L_{γ_2} , thence to L_{γ_3} and eventually to L_{γ_n} .

Clearly $\{\gamma_0, \dots, \gamma_n\}$ must be a nearest neighbour sequence, *i.e.*, $|\gamma_j - \gamma_{j-1}| = 1$, $j \geq 1$. For example the sequence $\{0, 1, 0, 1, 0, -1, 0\}$ is of the required type.

The generating function for the passage probabilities at the last passage (*i.e.*, from $L_{\gamma_{n-1}}$ to L_{γ_n}) will be denoted by $M_{\gamma_0 \dots \gamma_n}^{(\mu)}(x; z, w)$. We see at once that $M_{\gamma_0 \dots \gamma_n}^{(\mu)}$ must correspond to a passage through $a_{\gamma_{n-1} + \gamma_n}$. We observe also that

$$(7.1) \quad \begin{cases} M_{0,1}^{(\mu)}(x; z, w) = M^{(\mu)}(a_1 | x; z, w), \\ M_{0,-1}^{(\mu)}(x; z, w) = M^{(\mu)}(a_{-1} | x; z, w), \end{cases}$$

where the right-hand side expressions are found using eqs. (6.6). $M_{\pm 1,0}^{(\mu)}(x; z, w)$ can also be found from (6.6) by letting $a_1 \rightarrow \infty$ or $a_{-1} \rightarrow -\infty$ as required. For example from the first of eqs. (6.6) we find that

$$(7.2) \quad M_{-1,0}^{(1)}(x; z, w) = \exp[-\gamma(a_{-1} - x)].$$

The higher-order passage probabilities can then be determined recursively from the relation

$$(7.3) \quad M_{\gamma_0 \dots \gamma_n}^{(\mu)}(x; z, w) = M_{\gamma_{n-1} \gamma_n}^{(\gamma_{n-1} - \gamma_{n-2})}(a_{\gamma_{n-1} + \gamma_n}; z, w) M_{\gamma_0 \dots \gamma_{n-1}}^{(\mu)}(x; z, w).$$

Now suppose that the particle moves in the interval $(-\infty, a_{-1}]$ and that there is a reflecting barrier at a_{-1} , *i.e.*, when the particle strikes the barrier at a_{-1} its direction of motion is reversed (reflection is not counted as a collision) and the process continues. The argument we use can be extended in an obvious way to the problem of scattering with two reflecting barriers.

If $X \subseteq (-\infty, a_{-1}]$ and $R_{ij}^{(\mu)}(X|x)$ is the probability, conditional on initial position and direction (x, μ) that the $(i+j)$ -th collision occurs in X with i of these collisions being reversals, then the generating function

$$(7.4) \quad R^{(\mu)}(X|x; z, w) = \sum_{i+j \geq 1} R_{ij}^{(\mu)}(X|x) z^i w^j$$

is given by

$$(7.5) \quad R^{(\mu)}(X|x; z, w) = K_{-1}^{(\mu)}(X|x; z, w) + \sum_{n=0}^{\infty} M^{(\mu)}(a_{-1} | x; z, w) \cdot K_{-1}^{(-1)}(X | a_{-1}; z, w) [M^{(-1)}(a_{-1} | a_{-1}; z, w)]^n,$$

where $K_{-1}^{(\mu)}(X|x; z, w)$ is the analogue for the interval $(-\infty, a_{-1})$ of the generating function $K^{(\mu)}$ defined by eq. (5.8) for the interval $[a_{-1}, a_1]$. It is found from $K^{(\mu)}$ by letting a_{-1} tend to $-\infty$ and then replacing a_1 by a_{-1} . We notice that the n -th term on the right-hand side of (7.5) corresponds to $(n-1)$

previous reflections at the barrier. Equation (7.5) can be written more simply

$$(7.6) \quad R^{(\mu)}(X|x; z, w) = M^{(\mu)}(a_{-1}|x; z, w) K_{-1}^{(-1)}(X|a_{-1}; z, w) \cdot \\ \cdot [1 - M^{(-1)}(a_{-1}|a_{-1}; z, w)]^{-1} + K_{-1}^{(\mu)}(X|x; z, w).$$

We observe that the same method can also be applied to scattering on a line which is made up of a number of adjacent homogeneous strata. At each passage through an interface the particle, instead of being reflected, passes into a new stratum in which it may have a different mean free path.

8. - The steady-state distribution of particles in a finite interval.

For an interval of length l we define the following probabilities:

$p_i = \text{Prob}\{\text{a particle incident at one end emerges for the first time from the opposite end}\};$

$p'_i = \text{Prob}\{\text{a particle incident at one end emerges for the first time from the same end}\}.$

From eq. (6.7) we have immediately that

$$(8.1) \quad p_i = [1 + \lambda pl]^{-1}, \quad p'_i = \lambda pl[1 + \lambda pl]^{-1}.$$

Suppose that N particles per second are incident upon the left-hand end of the interval $[a, b]$ and suppose that the system is in a steady state. Then the expected number of particles per second passing to the right through any point x in the interval for the k -th time is given by

$$(8.2) \quad \mathcal{E}n_{kR} = N p_{x-a} [p'_{b-x} p'_{x-a}]^{-1}.$$

Hence the total expected number of particles per second passing to the right through x ,

$$(8.3) \quad \mathcal{E}n_R = \sum_{k=1}^{\infty} \mathcal{E}n_{kR} = N p_{x-a} [1 - p'_{b-x} p'_{x-a}]^{-1}.$$

Similarly the expected number of particles per second passing to the left through x ,

$$(8.4) \quad \mathcal{E}n_L = N p_{x-a} p'_{b-x} [1 - p'_{b-x} p'_{x-a}]^{-1}.$$

Substituting for p_i , p'_i from eq. (8.1) gives

$$(8.5) \quad \begin{cases} \mathcal{E}n_L = N(b-x)\lambda p[1+(b-a)\lambda p]^{-1}, \\ \mathcal{E}n_R = N[1+(b-x)\lambda p][1+(b-a)\lambda p]^{-1} \end{cases}$$

APPENDIX I

There is an alternative method in the unrestricted case (Sect. 4) which can be used to determine the Fourier-Stieltjes transform of the distribution of displacement at the n -th collision when the distances travelled by the particle between successive collisions are independently distributed with characteristic function $\varphi(\alpha)$. We have assumed so far that $\varphi(\alpha) = \lambda[\lambda - i\alpha]^{-1}$. The argument to be used is essentially a generalization of that used by GOLDSTEIN ⁽²⁾ to deal with a process of diffusion by discrete movements in which the magnitude of each step (or distance between successive collisions) is fixed.

We suppose that the particle starts at the origin and moves in steps X_i , $i = 1, 2, \dots$, where $\{X_i\}$ is a sequence of identically and independently distributed random variables with characteristic function $\varphi(\alpha)$. The probabilities of reversal and continuation at a collision are assumed as before to be p and q ($= 1 - p$), respectively. We define

$$P_{mn}^{(\mu)} = \text{Prob} \{m \text{ of the first } n \text{ steps are positive, given that the initial direction of motion was } \mu\}$$

and

$$\sigma_n^{(\mu)}(\alpha) = \mathcal{E}\{\exp[i\alpha \sum_{j=1}^n X_j | \mu]\},$$

i.e., $\sigma_n^{(\mu)}(\alpha)$ is the characteristic function of the displacement at the n -th collision conditional on initial direction μ .

Given that m of the steps $X_1, \dots, X_n$ are positive we have

$$\mathcal{E}_m\{\exp[i\alpha \sum_{j=1}^n X_j | \mu]\} = [\varphi(\alpha)]^m [\bar{\varphi}(\alpha)]^{n-m}.$$

Hence

$$(A.1) \quad \sigma_n^{(1)}(\alpha) = \sum_{k=1}^n P_{kn}^{(1)} [\varphi(\alpha)]^k [\bar{\varphi}(\alpha)]^{n-k} = [\bar{\varphi}(\alpha)]^n P_n^{(1)} \left[\frac{\varphi(\alpha)}{\bar{\varphi}(\alpha)} \right], \quad n \geq 1,$$

where $P_n^{(1)}(z) = \sum_{k=0}^n P_{kn}^{(1)} z^k$ and $P_{0n}^{(1)}$ is defined for convenience as $p^{-1}\delta_{0,n}$. We need only determine $\sigma_n^{(1)}(\alpha)$ owing to the fact that $\sigma_n^{(-1)}(\alpha)$ can easily be derived from it by considering the whole process to be reflected in the origin.

Consideration of the collision at which the first reversal occurs yields the recurrence relation

$$(A.2) \quad P_{mn}^{(1)} = \sum_{r=1}^m (1-p)^{r-1} p P_{(n-m)(n-r)}^{(1)}, \quad m \leq n, n \geq 0,$$

from which it follows that

$$\begin{aligned}
 (\text{A.3}) \quad P_n^{(1)}(z) &= z^n \sum_{m=0}^n \sum_{r=1}^m p(1-p)^{r-1} z^{m-n} P_{(n-m)(n-r)}^{(1)} = \\
 &= z^n \sum_{r=1}^n p(1-p)^{r-1} P_{n-r}^{(1)}(z^{-1}), \quad n \geq 1.
 \end{aligned}$$

This equation can be written in the more convenient form

$$(\text{A.4}) \quad P_{n+1}^{(1)}(z) - z(1-p)P_n^{(1)}(z) = pz^{n+1}P_n^{(1)}(z^{-1}),$$

together with a second equation which is obtained from (A.4) on replacing z by z^{-1} , namely

$$(\text{A.5}) \quad P_{n+1}^{(1)}(z^{-1}) - z^{-1}(1-p)P_n^{(1)}(z^{-1}) = pz^{-n-1}P_n^{(1)}(z).$$

Elimination of $P_n(z^{-1})$ from (A.4) and (A.5) leads to a second-order difference equation

$$(\text{A.6}) \quad P_{n+2}^{(1)}(z) - (1-p)(1+z)P_{n+1}^{(1)}(z) + z(1-2p)P_n^{(1)}(z) = 0,$$

and introducing the generating function

$$(\text{A.7}) \quad P^{(1)}(z, w) = \sum_{n=1}^{\infty} P_n^{(1)}(z)w^n, \quad |w| < 1, |z| \leq 1,$$

we find from (A.6) that

$$(\text{A.8}) \quad P^{(1)}(z, w) = wz \frac{1 - w(1-2p)}{1 - (1-p)(1+z)w + (1-2p)zw^2}$$

Hence from eq. (A.1) we obtain for the generating function of $\sigma_n^{(1)}(\alpha)$,

$$(\text{A.9}) \quad \sum_{n=1}^{\infty} \sigma_n^{(1)}(\alpha)t^n = t\varphi(\alpha) \frac{1 - (1-2p)t\bar{\varphi}(\alpha)}{1 - [1-p][\varphi(\alpha) + \bar{\varphi}(\alpha)]t + [1-2p]\varphi(\alpha)\bar{\varphi}(\alpha)t^2}.$$

If in this equation we replace $\varphi(\alpha)$ by $\lambda[\lambda - i\alpha]^{-1}$ then $\sigma_n^{(1)}(\alpha)$ becomes the characteristic function of displacement at the n -th collision for the system considered in Sect. 4. The same generating function can be obtained from eq. (4.9) taking $x=0$ and putting $z=w=t$.

From eq. (A.9) it is a simple matter to derive the moments of the distribution of $\sum_{i=1}^n X_i$. For example if $\mathcal{E}|X_i| = m$, $\mathcal{E}X_i^2 = s^2$ we find that

$$(\text{A.10}) \quad \left\{ \begin{aligned} \mathcal{E} \left\{ \sum_{i=1}^n X_i \mid 1 \right\} &= \frac{m}{2p} [1 - (1-2p)^n], \\ \mathcal{E} \left\{ \left(\sum_{i=1}^n X_i \right)^2 \mid 1 \right\} &= \frac{(2p-1)m^2}{2p^2} [1 - (2-2p)^n] + n \left[s^2 + \frac{1-2p}{p} m^2 \right], \end{aligned} \right.$$

and these results reduce to those obtained by GOLDSTEIN for the case of constant $|X_i|$.

APPENDIX II

Consider now the model studied in Sects. 1-3 with the following modification. Instead of having only two possibilities at each collision, *i.e.*, reversal and continuation, we introduce a nonzero probability of absorption. Thus at each collision we have probabilities p, q, r of reversal, continuation and absorption respectively where $p + q + r = 1$. The only difference between the analysis of this process and the pure scattering process which we have considered so far is that p and q no longer add to 1. The basic equations and the boundary conditions are unchanged. Thus the general solution (3.11) remains valid and so also does the relation (3.4) expressing $G^{(\mu)}$ in terms of $H^{(\mu)}$.

The only results of Sects. 1-8 which must be modified for the process with absorption are those in whose derivation the relation $p + q = 1$ is used. These are the results for isotropic scattering at the end of Sect. 4 which are easily modified and the eqs, (6.7) which for scattering with absorption become

$$(6.7') \quad \begin{cases} M^{(1)}(a_1|x) = \frac{(1-q)\lambda \sinh \lambda(x-a_{-1}) \sqrt{rs} + \lambda \sqrt{rs} \cosh \lambda(x-a_{-1}) \sqrt{rs}}{(1-q)\lambda \sinh \lambda \sqrt{rs} + \lambda \sqrt{rs} \cosh \lambda \sqrt{rs}} \\ M^{(-1)}(a_1|x) = \frac{p\lambda \sinh \lambda(x-a_{-1}) \sqrt{rs}}{(1-q)\lambda \sinh \lambda \sqrt{rs} + \lambda \sqrt{rs} \cosh \lambda \sqrt{rs}}, \quad s = 1 - q + p. \end{cases}$$

From (6.7') one can readily deduce the results of FERMI⁽¹⁰⁾, eq. (37). Equation (8.1) which was derived from (6.7) must also be changed and the results of Sect. 8 modified accordingly.

The result (A.9) of Appendix I remains valid provided p is replaced by $p(1-r)^{-1}$ and $\sigma_n^{(\mu)}(x)$ is interpreted as the characteristic function of the displacement at the n -th collision conditional on the occurrence of at least n collisions before absorption.

RIASSUNTO (*)

Facendo uso della teoria dei processi markoviani discontinui, si esamina un modello unidimensionale dello scattering delle particelle che è stato applicato da Fermi alla teoria del trasporto dei neutroni. Il processo è caratterizzato dalle probabilità di transizione $P_{rk}(S|\omega, s)$ che una transizione dallo stato ω ad uno stato appartenente alla classe S subisce in un percorso di lunghezza s in cui avvengono r collisioni con inversione e k collisioni senza inversione. Supponendo che il cammino libero medio sia costante il comportamento della particella, il cui stato è precisato dalla coordinata di posizione x , dalla direzione di moto μ e dalla velocità assoluta v , è determinato completamente ed esplicitamente per varie condizioni limite.

(*) Traduzione a cura della Redazione.

P. J. BROCKWELL

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Generalized One-Dimensional Scattering Processes with Applications in Three Dimensions.

P. J. BROCKWELL

Australian National University - Canberra

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Summary. — Results which have been obtained by BROCKWELL and MOYAL ⁽¹⁾ for one-dimensional scattering processes, in which the distances between successive collisions have a negative exponential distribution, are generalized to include processes where the distribution is either a γ -distribution or a linear combination of negative exponential distributions. The results are applied to the scattering of particles in three dimensions leading to an improvement on the results of FERMI ⁽²⁾.

1. — Introduction.

Consider a particle which moves along the X -axis starting from a point x in the interval $[a_{-1}, a_1]$. It continues to move with constant velocity until it makes a collision as a result of which its direction of motion is reversed with probability p and unchanged with probability q . The probability that the particle is absorbed at any collision is $r=1-p-q$. The distances travelled between successive collisions are assumed to be independently distributed with probability density function $f(x)$ and characteristic function

$$\varphi(\alpha) = \int_0^{\infty} f(x) \exp[i\alpha x] dx.$$

The characteristic function of the displacement of the particle at its n -th collision, given that it makes at least n collisions before being absorbed, has

⁽¹⁾ P. J. BROCKWELL and J. E. MOYAL: *Nuovo Cimento*, **33**, 776 (1964).

⁽²⁾ E. FERMI: *Ric. Scient.*, **7** (2), 13 (1936).

already been found by BROCKWELL and MOYAL ⁽¹⁾ for arbitrary $\varphi(x)$ and no barriers. However the process with absorbing barriers at a_{-1} and a_1 has been studied only when the distances between successive collisions have a negative exponential distribution, *i.e.*, $f(x) = \lambda \exp[-\lambda x]$, $\varphi(x) = \lambda(\lambda - ix)^{-1}$.

It is the purpose of this paper to generalize the results of ⁽¹⁾ for scattering processes with boundaries at a_{-1} and a_1 to processes where the distances between collisions have probability density function belonging to one of the following two classes:

$$(I) \quad f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp[-\lambda x],$$

$$(II) \quad f(x) = \sum_{i=1}^n p_i \lambda_i \exp[-\lambda_i x], \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1.$$

As an application of these generalized results we shall discuss the method used by FERMI ⁽²⁾ to study the scattering of thermal neutrons in a three-dimensional plane slab $\{x: a_{-1} \leq x \leq a_1\}$ of paraffin. FERMI supposed that the motion of the neutrons is as follows: between collisions each neutron moves in a straight line and in any element δs of its path it has probability $\lambda \delta s + O(\delta s)$ of making a collision where λ^{-1} is its (constant) mean free path; given that it makes a collision, the neutron is absorbed with probability r and scattered isotropically with probability $1-r$. FERMI obtained approximate results for this process by supposing that the projection of the motion on the X -axis is a one-dimensional scattering process in which the projected distances between collisions have a negative exponential distribution with parameter $\sqrt{3}\lambda$ (as opposed to λ for the three-dimensional motion) and probabilities r , $(1-r)/2$, $(1-r)/2$ of absorption, reversal and continuation respectively.

This is an approximate procedure because, as Fermi points out, the projections of the distances travelled between collisions are not in fact negative exponential variates as are the distances themselves.

For a more precise study of the motion of the projection of the particle on the X -axis we shall first determine the true characteristic function, $\varphi(x)$, of the projections of the distances travelled between collisions. In the absence of barriers we can then use the results of BROCKWELL and MOYAL ⁽¹⁾ to derive the exact characteristic function of the displacement at the n -th collision. When there are barriers present (at $x = a_1$ and $x = a_{-1}$) we shall use the generalized results of Sect. 4 to improve on Fermi's results for the scattering of particles by a slab of finite thickness.

Fermi's method and the method to be described both apply only to isotropic scattering. For anisotropic scattering the projected displacements between successive collisions are no longer independent.

2. - Integral equations for the first-passage probabilities.

Suppose that the initial co-ordinate of the particle in the interval $[a_{-1}, a_1]$ is x and that its initial direction of motion is μ ; $\mu = +1$ for motion to the right and -1 for motion to the left. For distances between successive collisions having probability density function $f(x)$ and for probabilities p, q, r of reversal, continuation and absorption at a collision we wish to find the probabilities $P_{rk}^{(\mu)}(a_{\pm 1}|x)$ of first emergence at $a_{\pm 1}$ after exactly r reversal and k continuations.

The solution when $f(x) = \lambda \exp[-\lambda x]$, is given by BROCKWELL and MOYAL⁽¹⁾ as a generating function,

$$(2.1) \quad M^{(\mu)}(a_v|x; z, w) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_v|x) z^r w^k.$$

From the basic eqs. (2.11) and (2.12) of ⁽¹⁾ or by a simple direct argument it can be deduced that for arbitrary $f(x)$ the generating functions $M^{(\mu)}(a_v|x)$ (omitting explicit reference to z and w) satisfy

$$(2.2) \quad M^{(\mu)}(a_v|x) = \mu \delta_{v\mu} \int_{a_\mu}^{\mu\infty} f(\mu y - \mu x) dy + \\ + \mu \int_x^{a_\mu} f(\mu y - \mu x) [pz M^{(-\mu)}(a_v|y) + qw M^{(\mu)}(a_v|y)] dy.$$

$\delta_{\nu\mu}$ here denotes Kronecker's delta, i.e. $\delta_{\nu\mu} = \begin{cases} 1, & \nu = \mu, \\ 0, & \nu \neq \mu. \end{cases}$

A large number of distributions encountered in practice can be closely approximated by a γ -distribution with suitably chosen parameters or by a linear combination of negative exponential distributions. If $f(x)$ belongs to either of these two families it is possible to solve eq. (2.2). In Sect. 3 we solve (2.2) when $f(x)$ is of the form

$$(2.3) \quad f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp[-\lambda x], \quad 0 \leq x < \infty,$$

and in Sect. 4 we do the same when $f(x)$ is of the form

$$(2.4) \quad f(x) = \sum_{i=1}^n p_i \lambda_i \exp[-\lambda_i x], \quad 0 \leq x < \infty, \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1.$$

Both families included the negative exponential distribution as a special case.

Although further generalization of the results of (1) can be derived in a similar manner we shall restrict ourselves to the determination of $M^{(\mu)}(a_\nu|x)$ which is sufficient for the discussion of Fermi's model in Sect. 6.

3. - First-passage probabilities when the distances between collisions have a gamma distribution.

When $f(x)$ is given by eq. (2.3) the integral eq. (2.2) takes the form

$$(3.1) \quad M^{(\mu)}(a_\nu|x) = \mu \delta_{\nu\mu} \int_{a_\mu}^{\infty} \frac{\lambda^n \mu^{n-1} (y-x)^{n-1}}{(n-1)!} \exp[-\lambda\mu(y-x)] dy + \\ + \mu \int_x^{a_\mu} \frac{\lambda^n \mu^{n-1} (y-x)^{n-1}}{(n-1)!} \exp[-\lambda\mu(y-x)] [pz M^{(-\mu)}(a_\nu|y) + qw M^{(\mu)}(a_\nu|y)] dy.$$

Using the symbol D to denote the operator $\partial/\partial x$ this equation can be rewritten as

$$(3.2) \quad [(\lambda - \mu D)^n - qw\lambda^n] M^{(\mu)} - \lambda^n pz M^{(-\mu)} = 0, \quad \mu = \pm 1,$$

with boundary conditions

$$(3.3) \quad M^{(\mu)}(a_\nu|a_\mu) = \delta_{\nu\mu},$$

$$(3.4) \quad (\lambda - \mu D)^i M^{(\mu)}(a_\nu|x) = \delta_{\nu\mu} \lambda^i \quad \text{at } x = a_\mu, \quad i=1, \dots, n-1.$$

The general solution of the eqs. (3.2) is

$$(3.5) \quad M^{(\mu)}(a_\nu|x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)}(z, w) \exp[\theta_i x],$$

where $\theta_i, i=1, \dots, 2n$, are the roots (assumed distinct) of the equation

$$(3.6) \quad [(\lambda - \theta)^n - qw\lambda^n][(\lambda + \theta)^n - qw\lambda^n] + \lambda^{2n} p^2 z^2 = 0.$$

The terms in the solution (3.5) corresponding to a k -ple root θ of (3.6) are of course of the form $[A_{\nu 1}^{(\mu)} + A_{\nu 2}^{(\mu)} x + \dots + A_{\nu k}^{(\mu)} x^{k-1}] \exp[\theta x]$.

Equation (3.2) can be used to express $A_{\nu i}^{(-1)}$ in terms of $A_{\nu i}^{(1)}, i=1, \dots, 2n$,

and the $2n$ boundary conditions (3.3), (3.4) then determine the coefficients $A_{\nu i}^{(\mu)}(z, w)$ completely.

Although this solves the first-passage problem in principle, it will generally be difficult, especially for a γ -distribution with large parameter n , to obtain an explicit solution in terms of z and w . However the probabilities of emergence at either boundary,

$$(3.7) \quad P^{(\mu)}(a_\nu | x) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_\nu | x) = [M^{(\mu)}(a_\nu | x)]_{z=w=1},$$

are easily found by setting $z=w=1$ in eqs. (3.2), (3.3), (3.4) and solving numerically. Thus $P^{(\mu)}(a_\nu | x)$ satisfies

$$(3.8) \quad [(\lambda - \mu D)^n - q\lambda^n]P^{(\mu)} - p\lambda^n P^{(-\mu)} = 0$$

with boundary conditions

$$(3.9) \quad P^{(\mu)}(a_\nu | a_\mu) = \delta_{\nu\mu},$$

$$(3.10) \quad (\lambda - \mu D)^i P^{(\mu)}(a_\nu | x) = \delta_{\nu\mu} \lambda^i \quad \text{at } x = a_\mu, i=1, \dots, n-1.$$

The solution of these equations is given, for $p+q < 1$, by

$$(3.11) \quad P^{(\mu)}(a_\nu | x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)} \exp[\theta_i x],$$

where θ_i , $i=1, \dots, 2n$, are the roots of (3.6) with $z=w=1$. If $p+q=1$ then $\theta=0$ is a double root of (3.6) and the solution is of the form

$$(3.12) \quad P^{(\mu)}(a_\nu | x) = A_{\nu 1}^{(\mu)} + A_{\nu 2}^{(\mu)} x + \sum_{i=3}^{2n} A_{\nu i}^{(\mu)} \exp[\theta_i x].$$

The equations for the coefficients $A_{\nu i}^{(\mu)}$ are obtained as above and can be solved numerically.

The moments of the joint distribution of the number of reversals and continuations before emergence can be obtained by finding the appropriate partial derivatives of (3.2), (3.3) and (3.4) with respect to z and w , setting $z=w=1$ and solving numerically. For example the mean number of reversals before emergence, conditional on emergence at a_ν , is given by

$$(3.13) \quad m = [P^{(\mu)}(a_\nu | x)]^{-1} \left[\frac{\partial}{\partial z} M^{(\mu)}(a_\nu | x) \right]_{z=w=1}.$$

4. - Linear combinations of negative exponential distributions.

In this Section we determine the generating functions $M^{(\mu)}(a_\nu|x)$ for density functions $f(x)$ of the form

$$(4.1) \quad f(x) = \sum_{i=1}^n p_i \lambda_i \exp[-\lambda_i x], \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^n p_i = 1.$$

For such a function $f(x)$ eq. (2.2) becomes

$$(4.2) \quad M^{(\mu)}(a_\nu|x) = \mu \delta_{\nu\mu} \int_{a_\mu}^{\mu\infty} \sum_{i=1}^n p_i \lambda_i \exp[\lambda_i \mu(x-y)] dy + \\ + \mu \int_x^{a_\mu} \sum_{i=1}^n p_i \lambda_i \exp[\lambda_i \mu(x-y)] [pz M^{(-\mu)}(a_\nu|y) + qw M^{(\mu)}(a_\nu|y)] dy.$$

To solve (4.2) we first determine the constants $a_1, a_2, \dots, a_n$ satisfying the equations

$$(4.3) \quad a_n \lambda_i^n + a_{n-1} \lambda_i^{n-1} + \dots + a_1 \lambda_i + 1 = 0, \quad i=1, 2, \dots, n.$$

With this set of constants a_k we deduce from (4.2) that $M^{(\mu)}(a_\nu|x)$ satisfies the differential equation

$$(4.4) \quad [a_n(\mu D)^n + a_{n-1}(\mu D)^{n-1} + \dots + a_1 \mu D + 1 + \varphi(\mu D)qw] M^{(\mu)} + \\ + \varphi(\mu D)pz M^{(-\mu)} = 0,$$

where D again denotes the operator $\partial/\partial x$ and

$$\varphi(\theta) = a_1 \sum_{i=1}^n p_i \lambda_i + a_2 \left[\sum_{i=1}^n p_i \lambda_i^2 + \sum_{i=1}^n p_i \lambda_i \theta \right] + \dots + a_n \left[\sum_{i=1}^n p_i \lambda_i^n + \dots + \sum_{i=1}^n p_i \lambda_i \theta^{n-1} \right].$$

The boundary conditions to be satisfied by $M^{(\mu)}(a_\nu|x)$ are

$$(4.5) \quad M^{(\mu)}(a_\nu|a_\mu) = \delta_{\nu\mu},$$

$$(4.6) \quad (\mu D)^j M^{(\mu)}(a_\nu|x) = \delta_{\nu\mu} \sum_{i=1}^n \lambda_i^j p_i - \sum_{r=1}^j \{ (\mu D)^{j-r} [qw M^{(\mu)}(a_\nu|x) + \\ + pz M^{(-\mu)}(a_\nu|x)] \sum_{i=1}^n \lambda_i^r p_i \} \quad \text{at } x = a_\mu, \quad j=1, \dots, n-1.$$

The general solution of eq. (4.2) is given by

$$(4.7) \quad M^{(\mu)}(a_\nu | x) = \sum_{i=1}^{2n} A_{\nu i}^{(\mu)}(z, w) \exp[\theta_i x],$$

where $\theta_1, \dots, \theta_{2n}$ are the roots (assumed distinct) of the equation

$$(4.8) \quad [a_n \theta^n + \dots + a_1 \theta + 1 + qw\varphi(\theta)] \cdot \\ \cdot [a_n(-\theta)^n + \dots + a_1(-\theta) + 1 + qw\varphi(-\theta)] - p^2 z^2 \varphi(\theta) \varphi(-\theta) = 0.$$

Multiple roots of eq. (4.8) are treated in the same way as in Sect. 3. From eq. (4.4) $A_{\nu i}^{(-1)}$ can be expressed in terms of $A_{\nu i}^{(1)}$ for each i and the $2n$ boundary conditions (4.5), (4.6) then determine the coefficients $A_{\nu i}^{(\mu)}$ completely.

As in Sect. 3 an explicit solution for $M^{(\mu)}(a_\nu | x)$ in terms of z and w will in general be extremely complicated; however the probabilities of emergence, $P^{(\mu)}(a_\nu | x)$, and the moments of the joint distribution of the numbers of reversals and continuations before emergence can easily be found numerically as before.

An illustration of the use of eq. (4.4) and its solution will be given in Sect. 6 where we discuss Fermi's method for determining the probabilities of reflection and transmission of thermal neutrons by a slab of paraffin.

5. - Three-dimensional scattering with no boundaries.

We now consider Fermi's model for three-dimensional scattering which was described in the Introduction. As indicated in Sect. 1 Fermi's approximation for the projected motion of the particle on the X -axis supposes that the projected displacements between successive collisions are independently distributed with probability density function

$$(5.1) \quad f^*(x) = \sqrt{3}\lambda \exp[-\sqrt{3}\lambda x], \quad 0 \leq x < \infty,$$

where λ^{-1} is the mean free path of the particle in three dimensions. The characteristic function corresponding to $f^*(x)$ is

$$(5.2) \quad \varphi^*(\alpha) = \int_0^\infty \exp[i\alpha x] f^*(x) dx = \sqrt{3}\lambda (\sqrt{3}\lambda - i\alpha)^{-1}.$$

The exact density function $f(x)$ for the projected displacements can in fact be found as follows. Suppose that the particle has just suffered a collision and

moves off in a direction whose angle with the positive X -axis has cosine μ , $-1 \leq \mu \leq 1$. The probability that the projection on the X -axis of the distance travelled to the next collision lies in $(x, x + \delta x)$ is

$$\frac{\lambda}{|\mu|} \exp \left[-\frac{\lambda x}{|\mu|} \right] \delta x + o(\delta x).$$

Since scattering is assumed to be isotropic, $|\mu|$ is a random variable uniformly distributed on $[0, 1]$. Hence

$$(5.3) \quad f(x) = \lambda \int_0^1 \theta^{-1} \exp \left[-\frac{\lambda x}{\theta} \right] d\theta, \quad 0 \leq x < \infty,$$

and the corresponding characteristic function is

$$(5.4) \quad \varphi(\alpha) = \int_0^\infty \exp[i\alpha x] f(x) dx = -\frac{\lambda}{i\alpha} \log \left(1 - \frac{i\alpha}{\lambda} \right).$$

Expanding (5.2) and (5.4) in powers of $i\alpha$ we see that the exact distribution and Fermi's approximating distribution have the same mean square but their means and all the other moments about the origin are different. In fact for isotropic scattering, as we shall see later, it is the mean square which determines the asymptotic behaviour of the particle after a large number of collisions. However using the distributions considered in Sect. 4 it is possible to find much better approximations to the exact density function (5.4). For example the probability density

$$(5.5) \quad g(x) = \frac{1}{2} \lambda_1 \exp[-\lambda_1 x] + \frac{1}{2} \lambda_2 \exp[-\lambda_2 x], \quad 0 \leq x < \infty,$$

where $\lambda_1 = \lambda(3 - \sqrt{3})$, $\lambda_2 = \lambda(3 + \sqrt{3})$, has exactly the same first three moments about the origin as $f(x)$ as well as having all its higher moments much closer to the true values than those of Fermi's approximating distribution. Even better fits can of course be obtained by allowing linear combinations of more than two negative exponential distributions. We shall use the function $g(x)$ in Sect. 6 to obtain improved solutions for scattering problems with boundaries.

For scattering with no boundaries exact solutions can be derived using the results of BROCKWELL and MOYAL ⁽¹⁾. Denote by $\tau_n^{(\mu)}(\alpha)$ the characteristic function of the x -displacement, S_n , at the n -th collision given that the particle experiences at least n collisions before absorption and that the initial direction

of motion was μ . If p , q , r are the probabilities (for the projected motion) of reversal, continuation and absorption at a collision, then the generating function, $G^{(\mu)}(\alpha; t) \equiv \sum_{n=1}^{\infty} \tau_n^{(\mu)}(\alpha) t^n$, is obtained from eq. (A.9) of (1) on replacing p by $p^* = p(1-r)^{-1}$. For an isotropic source of particles at $x=0$ the appropriate generating function is clearly

$$(5.6) \quad G(\alpha; t) = \frac{1}{2} [G^{(1)}(\alpha; t) + G^{(-1)}(\alpha; t)],$$

i.e., from eq. (A.9) of (1),

$$(5.7) \quad G(\alpha; t) = \frac{t}{2} \frac{\varphi(\alpha) + \overline{\varphi(\alpha)} - 2(1-2p^*)t\varphi(\alpha)\overline{\varphi(\alpha)}}{1 - [1-p^*][\varphi(\alpha) + \overline{\varphi(\alpha)}]t + [1-2p^*]\varphi(\alpha)\overline{\varphi(\alpha)}t^2}.$$

For isotropic scattering $p^* = \frac{1}{2}$ and $\varphi(\alpha)$ is given by eq. (5.4). Hence, substituting in (5.7) we obtain the result

$$(5.8) \quad G(\alpha; t) = \frac{t}{2} \frac{\lambda \log((\lambda + i\alpha)/(\lambda - i\alpha))}{i\alpha - (t\lambda/2) \log((\lambda + i\alpha)/(\lambda - i\alpha))},$$

which generalizes the results of (1) and (3) for one-dimensional scattering processes to three-dimensional processes without boundaries.

It is a simple matter to expand (5.8) in powers of t giving

$$(5.9) \quad \tau_n(\alpha) = \left[\frac{\lambda}{2i\alpha} \log \frac{\lambda + i\alpha}{\lambda - i\alpha} \right]^n.$$

Denoting the x -displacement at the n -th collision obtained from Fermi's approximation by S_n^* and the corresponding characteristic function by $\tau_n^*(\alpha)$ we find, from (5.2) and (5.7), that

$$(5.10) \quad \tau_n^*(\alpha) = \left[\frac{3\lambda^2}{3\lambda^2 + \alpha^2} \right]^n.$$

The distributions corresponding to $\tau_n(\alpha)$ and $\tau_n^*(\alpha)$ are compared most conveniently by considering their moments. Thus

$$\begin{aligned} \mathcal{E}\{S_n^{2k+1}\} &= 0, & \mathcal{E}\{S_n^{*2k+1}\} &= 0, & k &= 0, 1, 2, \dots, \\ \mathcal{E}\{S_n^2\} &= \frac{2n}{3} \lambda^{-2}, & \mathcal{E}\{S_n^{*2}\} &= \frac{2n}{3} \lambda^{-2}, \\ \mathcal{E}\{S_n^4\} &= \left[\frac{4n^2}{3} + \frac{52n}{15} \right] \lambda^{-4}, & \mathcal{E}\{S_n^{*4}\} &= \left[\frac{4n^2}{3} + \frac{4n}{3} \right] \lambda^{-4}, \end{aligned}$$

(3) G. PLACZEK: Manhattan Project Report, A-25.

and for large n the moments differ only by terms which are $O(n^{k-1})$, i.e.,

$$\mathcal{E}\{S_n^{2k}\} = \frac{2k!}{k!} \left(\frac{n}{3\lambda^2}\right)^k + O(n^{k-1}), \quad \mathcal{E}\{S_n^{*2k}\} = \frac{2k!}{k!} \left(\frac{n}{3\lambda^2}\right)^k + O(n^{k-1}), \quad k = 1, 2, \dots$$

From eqs. (5.9) and (5.10) we see that the displacements S_n and S_n^* are both distributed as sums of independent random variables. We can therefore apply the central limit theorem to deduce that as $n \rightarrow \infty$, $n^{-1/2}S_n$ and $n^{-1/2}S_n^*$ are asymptotically normally distributed. In fact the asymptotic distributions are identical, both having mean zero and variance $\frac{2}{3}\lambda^{-2}$. This suggests that Fermi's approximation is a good one when the number of collisions is large. The parameter $\lambda\sqrt{3}$ of his approximating negative exponential distribution is in fact determined uniquely by the condition that $S_n^*/\sqrt{n}$ and $S_n/\sqrt{n}$ have the same asymptotic behaviour.

6. - Scattering by a slab of finite thickness.

We have already mentioned in Sect. 5 that the distribution of the projected displacements between successive collisions in a three-dimensional scattering process can be closely approximated by a linear combination of negative exponential distributions. The following Table indicates the accuracy of the approximation $g(x)$ (defined by (5.5)) as compared with Fermi's approximation $f^*(x)$.

TABLE I. - *Moments of the exact and approximating distributions.* By taking linear combinations of more than two negative exponential distributions it is possible to fit still higher-order moments exactly but the algebra then becomes more complicated.

n	n -th moment about the origin		
	$f(x)$	$f^*(x)$	$g(x)$
1	$(1/2)\lambda^{-1}$	$(1/3)\sqrt{3}\lambda^{-1}$	$(1/2)\lambda^{-1}$
2	$(2/3)\lambda^{-2}$	$(2/3)\lambda^{-2}$	$(2/3)\lambda^{-2}$
3	$(3/2)\lambda^{-3}$	$(2/3)\sqrt{3}\lambda^{-3}$	$(3/2)\lambda^{-3}$
4	$(24/5)\lambda^{-4}$	$(8/3)\lambda^{-4}$	$(14/3)\lambda^{-4}$
5	$20\lambda^{-5}$	$(40/9)\sqrt{3}\lambda^{-5}$	$(55/3)\lambda^{-5}$

Consider now the problem of scattering in the slab $\{x: a_{-1} \leq x \leq a_1\}$. With $f^*(x)$ as an approximation to $f(x)$ FERMI derived the first-passage probabilities $P^{(\mu)}(a_\nu | x)$. Using $g(x)$ as an approximation to $f(x)$ and applying the results of Sect. 4 we shall improve on his results. In fact it is possible to use Sect. 4 to obtain the generating functions $M^{(\mu)}(a_\nu | x; z, w)$ explicitly.

The solution of eqs. (4.3), (4.4), (4.5) for $M^{(\mu)}(a_v|x)$ is given by

$$(6.1) \quad M^{(\mu)}(a_v|x) = \sum_{i=1}^4 A_{vi}^{(\mu)}(z, w) \exp[\theta_i x],$$

where the functions $\theta_i(z, w)$, $A_{vi}^{(\mu)}(z, w)$ are to be found as described in Sect. 4.

Using the particular density function $g(x)$ and taking λ^{-1} as the unit of length we have, in the notation of Sect. 4,

$$\begin{aligned} p_1 &= p_2 = \frac{1}{2}, \\ \lambda_1 &= 3 - \sqrt{3}, \\ \lambda_2 &= 3 + \sqrt{3}. \end{aligned}$$

Substituting in (4.3) and solving for a_1, a_2 we find

$$(6.2) \quad a_1 = -1, \quad a_2 = \frac{1}{6}.$$

$\theta_1, \dots, \theta_4$ are therefore the roots of the equation

$$(6.3) \quad \theta^4 - 3\theta^2[8(1 - qw) + 3(q^2w^2 - p^2z^2)] + 36[(1 - qw)^2 - p^2z^2] = 0.$$

Since scattering is assumed to be isotropic $p = q = (1 - r)/2$, where r is the probability of absorption at a collision.

The coefficients $A_{vi}^{(1)}, A_{vi}^{(-1)}$ are related by eq. (4.4) which in this case takes the form

$$(6.4) \quad [D^2 - D(6 - 3qw) + 6(1 - qw)]M^{(1)} - 3pz[2 - D]M^{(-1)} = 0.$$

Thus

$$(6.5) \quad A_{vi}^{(-1)} = \frac{\theta_i^2 - (6 - 3qw)\theta_i + 6(1 - qw)}{3pz(2 - \theta_i)} A_{vi}^{(1)} = k_i(z, w) A_{vi}^{(1)}, \quad i = 1, \dots, 4.$$

The coefficients $A_{vi}^{(1)}$ are found from the boundary conditions (4.5), (4.6) which can now be written

$$(6.6) \quad \left\{ \begin{aligned} \sum_{i=1}^4 A_{vi}^{(1)} \exp[\theta_i a_1] &= \delta_{1v}, \\ \sum_{i=1}^4 k_i A_{vi}^{(1)} \exp[\theta_i a_{-1}] &= \delta_{-1v}, \\ \sum_{i=1}^4 (\theta_i + 3qw + 3pz k_i) A_{vi}^{(1)} \exp[\theta_i a_1] &= 3\delta_{1v}, \\ \sum_{i=1}^4 (-k_i \theta_i + 3qw k_i + 3pz) A_{vi}^{(1)} \exp[\theta_i a_{-1}] &= 3\delta_{-1v}. \end{aligned} \right.$$

Equations (6.1), (6.5) and (6.6) provide a complete solution to the first-passage problem in generating function form. The probabilities of emergence at the two boundaries regardless of the number of collisions before emergence, *i.e.*, $P^{(\mu)}(a_\nu|x) = \sum_{r+k \geq 0} P_{rk}^{(\mu)}(a_\nu|x)$, are obtained when z and w are replaced throughout by unity. Moments of the distribution of the number of reversals and continuations before emergence are obtained by differentiating partially with respect to z and w and setting $z=w=1$.

To illustrate the application of these results we now consider a stream of particles incident on the face $x=0$ of the half-space $\{x: 0 \leq x < \infty\}$. We shall use (6.1), (6.5) and the appropriate boundary conditions to determine the albedo, *i.e.*, the proportion of incident particles which ultimately emerge again from the half-space. The boundary conditions at a_1 will now be replaced by the condition that $P^{(\mu)}(a_\nu|x)$ is bounded as $x \rightarrow \infty$.

Denote by $p(x)$ the probability that a particle from an isotropic source with co-ordinate x ($x > 0$) emerges from the half-space without being absorbed. Then the probability that a particle incident on the half-space at an angle θ to the positive X -axis ultimately emerges again is clearly given by

$$(6.7) \quad \beta(\theta) = (1-r) \int_0^\infty \exp[-x \sec \theta] p(x) \sec \theta \, dx.$$

If there is a stream of particles incident on the half-space in such a way that the angle of incidence θ has probability density function $\varphi(\theta)$, $0 \leq \theta \leq \pi/2$, then the albedo is given by

$$(6.8) \quad \beta = \int_0^{\pi/2} \beta(\theta) \varphi(\theta) \, d\theta.$$

In the particular case $\varphi(\theta) = 2 \cos \theta \sin \theta$, *i.e.*, for incident particles obeying the cosine law, WICK⁽⁴⁾ obtained an exact asymptotic expression for β when r is small, namely

$$(6.9) \quad \beta = 1 - \frac{4}{\sqrt{3}} r^{\frac{1}{2}} + \dots$$

FERMI⁽²⁾, using the approximation $f^*(x)$ to derive $p(x)$ evaluated the integrals (6.7) and (6.8) to find

$$(6.10) \quad \beta = 1 - \left(1 + \frac{2}{\sqrt{3}}\right) r^{\frac{1}{2}} + \dots$$

(4) G. C. WICK: *Rend. Lincei*, 23, 774 (1936).

From the approximation $g(x)$ as described above we find that

$$(6.11) \quad p(x) = A \exp[\theta_1 x] + B \exp[\theta_2 x],$$

where

$$\begin{aligned} \theta_1 &= -\sqrt{6(1+r+\sqrt{1+r+r^2})}, \\ \theta_2 &= -\sqrt{6(1+r-\sqrt{1+r+r^2})}, \\ A &= -\frac{1}{2}(1+k_1) \frac{k_2(2\theta_2+3+3r)-3+3r}{2k_1k_2(\theta_1-\theta_2)+3(1-r)(k_1-k_2)}, \\ B &= \frac{1}{2}(1+k_2) \frac{k_1(2\theta_1+3+3r)-3+3r}{2k_1k_2(\theta_1-\theta_2)+3(1-r)(k_1-k_2)}; \end{aligned}$$

k_1 and k_2 are defined by eq. (6.5) with $z=w=1$. From (6.7) we obtain

$$(6.12) \quad \beta(\theta) = (1-r)[A(1-\theta_1 \cos \theta)^{-1} + B(1-\theta_2 \cos \theta)^{-1}]$$

and in the particular case $\varphi(\theta) = 2 \cos \theta \sin \theta$ the integral (6.8) can be evaluated explicitly to give

$$(6.13) \quad \beta = -2(1-r)[A\theta_1^{-2}(\theta_1 + \log[1-\theta_1]) + B\theta_2^{-2}(\theta_2 + \log[1-\theta_2])],$$

i.e.,

$$(6.14) \quad \beta = 1 - \left[\frac{2\sqrt{3}-3}{12} \log(1+2\sqrt{3}) + \frac{13}{6}\sqrt{3} - \frac{3}{2} \right] r^{\frac{1}{2}} + \dots$$

Comparing the coefficients of $r^{\frac{1}{2}}$ in (6.9), (6.10) and (6.14) we see that Fermi's approximation gives 2.1547, the improved approximation gives 2.3106 and the exact value is 2.3094. This gives an idea of the accuracy of the approximation. The same method of course can be applied to determine the albedo and transmission probability for slabs of finite thickness.

The following Table gives the probabilities of transmission through a slab

TABLE II. - *Probability of transmission by a slab of thickness a .*

a	$r=0$		$r=\frac{1}{3}$	
	$f^*(x)$	$g(x)$	$f^*(x)$	$g(x)$
0	1.0000	1.0000	1.0000	1.0000
$\frac{1}{2}$	0.6978	0.6208	0.5783	0.4721
1	0.5359	0.4793	0.3472	0.2801
2	0.3660	0.3374	0.1258	0.1125
4	0.2240	0.2129	0.0170	0.0190

of thickness a (with the mean free path again used as the unit of length) obtained using Fermi's approximation $f^*(x)$ and the improved approximation $g(x)$ of eq. (5.5). It is assumed that the particles are incident isotropically on one face of the slab.

When there is no absorption (*i.e.*, $r=0$) the application of Fermi's method (and the improved approximation) to the scattering of particles in the interior of a thick slab can be justified asymptotically using Wald's identity (BARTLETT⁽⁵⁾, p. 17). Suppose there is an isotropic source of particles at $x=0$ and that scattering is isotropic with no absorption, *i.e.*, for the projected motion on the X -axis $r=0$, $p=q=\frac{1}{2}$. Let N be a random variable denoting the number of collisions made by the particle before emerging from $[-a, b]$, $-a \leq 0 \leq b$. P_a and P_b will denote the probabilities of first departures from $-a$ and b , respectively.

The Fermi approximation to P_a , P_b and $\mathcal{E}N$ (the expected value of N) can be deduced from eq. (6.6) of (1). They are

$$(6.15) \quad P_a = \frac{\sqrt{3}b + 1}{\sqrt{3}(a+b) + 2}, \quad P_b = \frac{\sqrt{3}a + 1}{\sqrt{3}(a+b) + 2},$$

$$(6.16) \quad \mathcal{E}N = \frac{\sqrt{3}(l^3 - a^3 - b^3) - 3(2l^2 - a^2 - b^2)}{2\sqrt{3}l + 4}, \quad \text{where} \quad l = a + b.$$

Now as we have already seen in the unrestricted case the x -displacement at the n -th collision is distributed as a sum of n independent random variables, each with mean zero and variance $\frac{2}{3}$. Using Wald's identity (BARTLETT⁽⁵⁾, p. 20) we can deduce from this that for a and b large, P_a , P_b and $\mathcal{E}N$ are given asymptotically by

$$(6.17) \quad P_a = \frac{b}{a+b}, \quad P_b = \frac{a}{a+b},$$

$$(6.18) \quad \mathcal{E}N = \frac{3}{2}ab.$$

Examination of (6.15) and (6.16) shows that Fermi's approximations are in fact asymptotic to these values. The approximations we derived earlier also have this property.

We remark finally that for scattering by very thick slabs with $r > 0$ the penetration of particles is largely determined by the tail of the exact density function $f(x)$ and not by its second moment about the origin.

(5) M. S. BARTLETT: *Stochastic Processes* (London, 1955).

* * *

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RIASSUNTO (*)

Si generalizzano i risultati, ottenuti da BROCKWELL e MOYAL ⁽¹⁾ per processi di scattering unidimensionali, in cui le distanze fra le collisioni successive hanno una distribuzione esponenziale negativa, ad includere processi in cui la distribuzione è una distribuzione γ o una combinazione lineare di distribuzioni esponenziali negative. Si applicano i risultati allo scattering di particelle in tre dimensioni, giungendo ad un miglioramento dei risultati ottenuti da FERMI ⁽²⁾.

(*) *Traduzione a cura della Redazione.*

The Scattering of Particles in an Infinite Plane Slab.

P. J. BROCKWELL

Australian National University - Canberra

(ricevuto il 10 Marzo 1964)

Summary. — A three-dimensional scattering process is studied using the theory of discontinuous Markov processes, extending results previously obtained in one dimension by Brockwell and Moyal. The scattered particles are assumed to move in one of a set of thirty directions in space. Constant mean free path is assumed, scattering is axially symmetric and velocity transition probabilities $\psi_\theta(dv|u)$ are defined for each collision depending on the angular deflection, θ . It is shown how the results obtained for the discrete set of directions can be used to approximate the physically more realistic continuous case; a numerical comparison with exact results is given for the well-known Milne problem with isotropic scattering. For a steady stream of particles incident on one face of a slab $\{x: a \leq x \leq b\}$ it is shown how to determine the time-independent solution for the mean number of particles per second crossing a plane $x=\xi$ in any given direction. The distribution of velocity and direction with which particles emerge from a slab is also determined assuming the probabilities $\psi_\theta(dv|u)$ satisfy a certain commutativity condition. If there is a constant (nonzero) probability of absorption at a collision no essential modification of the analysis is required.

1. — Introduction.

It is the purpose of this paper to extend the results obtained by BROCKWELL and MOYAL⁽¹⁾ for particle scattering in one dimension to scattering in two and three dimensions. We consider a particle which moves through a (two- or three-dimensional) slab of scattering material $\{x: a \leq x \leq b\}$ in the following manner: between successive collisions the particle moves in a straight line with constant velocity. The probability that a collision occurs in any element δs of path-length is given by $\lambda \delta s + o(\delta s)$ where λ^{-1} is the mean free

⁽¹⁾ P. J. BROCKWELL and J. E. MOYAL: *Nuovo Cimento*, **33**, 776 (1964).

path of the particle. Given that a collision occurs there is a transition probability $q(d\mathbf{v}|\boldsymbol{\mu})$ for the unit vectors $\boldsymbol{\mu}, \mathbf{v}$ in the incident and emergent directions of motion, respectively. It will be assumed that scattering is axially symmetric, i.e. $q(d\mathbf{v}|\boldsymbol{\mu})$ is a function of $\mathbf{v} \cdot \boldsymbol{\mu}$ only, and there is also defined a set of transition probabilities $\psi_\theta(dv|u)$ for the velocities u and v before and after a collision, where θ is the angle through which the particle is deflected as a result of the collision.

Such a model arises in a variety of problems, among them the theory of radiative transfer, the multiple scattering of neutrons and the multiple scattering of charged particles. The statistical problem is to determine the spatial, angular and energy distributions of the scattered radiation or particles. The usual approach to this problem (see for example DAVISON⁽²⁾) is to consider the Boltzmann integro-differential equation for $N(\mathbf{x}, \boldsymbol{\mu}, v, t)$ where $N(\mathbf{x}, \boldsymbol{\mu}, v, t) \cdot d\mathbf{x} d\boldsymbol{\mu} dv$ is the expected number of particles at time t in the element $d\mathbf{x} d\boldsymbol{\mu} dv$ of phase-space. It is usually not possible to find an exact solution of this equation in closed form. One exception is the time-independent solution obtained by WIENER and HOPF⁽³⁾ for isotropic scattering in a half-space. This is the classical Milne problem of radiative transfer in stellar atmospheres.

Various approximate methods for solving the Boltzmann equation have been devised. The method of spherical harmonics, in which the distribution is expanded in terms of Legendre polynomials, has been applied to problems with isotropic and anisotropic scattering (see for example CHANDRASEKHAR⁽⁴⁾) and in particular to various problems of neutron transport theory by MARSHAK⁽⁵⁾, MARK⁽⁶⁾ and others. A second method of approximation developed by WICK⁽⁷⁾ uses Gaussian quadrature to replace the integral in the Boltzmann equation by an approximating sum. WANG and GUTH⁽⁸⁾ have formulated the spherical harmonic method in such a way as to demonstrate its close relationship with the Gaussian quadrature method. Good accounts of the methods available for dealing with problems in neutron transport theory are given in the books by DAVISON⁽²⁾ and WEINBERG and WIGNER⁽⁹⁾. A very general probabilistic theory of the processes occurring in nuclear reactors has been given by PÁL⁽¹⁰⁾.

(2) B. DAVISON: *Neutron Transport Theory* (London, 1957).

(3) N. WIENER and E. HOPF: *Berlin Ber. Math. Phys. Klasse*, 696 (1931).

(4) S. CHANDRASEKHAR: *Astrophys. Journ.*, **99**, 180 (1944).

(5) R. E. MARSHAK: *Phys. Rev.*, **71**, 443 (1947).

(6) C. MARK: National Research Council of Canada Report MT-97 (1945).

(7) G. C. WICK: *Zeits. f. Phys.*, **121**, 702 (1943).

(8) M. C. WANG and E. GUTH: *Phys. Rev.*, **84**, 1092 (1951).

(9) A. M. WEINBERG and E. P. WIGNER: *The Physical Theory of Neutron Chain Reactors* (Chicago, 1958).

(10) L. PÁL: *Suppl. Nuovo Cimento*, **7**, 25 (1958).

The method we shall use is a direct generalization of that used by BROCKWELL and MOYAL ⁽¹⁾ in one dimension. We recall that solutions were obtained in that case for the generating functions

$$G(\Omega, \Sigma | \mu, x) = \sum_{i+j \geq 0} P_{ij}(\Omega, \Sigma | \mu, x) z^i w^j,$$

$$H(\Omega, X | \mu, x) = \sum_{i+j \geq 1} Q_{ij}(\Omega, X | \mu, x) z^i w^j,$$

where $P_{ij}(\Omega, \Sigma | \mu, x) = Pr$ (the first passage of the particle through the boundary Σ occurs after exactly i collisions with reversals and j collisions without reversals and the direction of motion at the first passage belongs to the set Ω , given that the initial position of the particle was x and its initial direction μ) and $Q_{ij}(\Omega, X | \mu, x) = Pr$ (the $(i+j)$ -th collision occurs in $\Omega \times X$ and i of the first $(i+j)$ collisions are reversals, given that the initial position of the particle was x and its initial direction of motion μ).

The functions G and H were determined from pairs of linear differential equations with boundary conditions depending on the particular sets Ω , X and Σ in question. It was also shown how these generating functions can be used, assuming the transition probabilities for the velocities before and after a collision satisfy a certain commutativity condition, to determine the distribution of emergent velocity of the particle from a rod. We shall now show how these ideas can be extended to scattering in two and three dimensions.

In one dimension the problem is simplified, first because there are only two possible directions of motion and secondly because the position of the particle is specified by a scalar. In the present paper we are concerned with axially symmetric scattering in a homogeneous infinite slab, $\{x: a \leq x \leq b\}$. We are interested only in the x -co-ordinate of the particle so position is again specified by a scalar. However the direction of motion μ belongs to a continuously infinite set of possible directions so that it is not possible as in the one-dimensional case to write down pairs of linear differential equations for G and H . However if we restrict the particle to move in one of a set of n possible directions corresponding to n points uniformly spaced on the unit circle in two dimensions or the unit sphere in three dimensions then we are again led to a finite system of linear differential equations for the two- and three-dimensional analogues of G and H . As we shall see later such a discrete system can be used to obtain approximate solutions for the continuous case. Together with the set of n possible directions we introduce n transition probabilities for the directions of motion before and after a collision, these probabilities depending only on the angle between the initial and final directions of motion. It will be assumed throughout that the mean free path, λ^{-1} , is independent of the velocity of the particle.

In two dimensions n may be chosen arbitrarily large, the corresponding set $\mathcal{M}_n$ of directions consisting of those making angles $2k\pi/n$, $k = 0, 1, \dots, n-1$, with the unit vector $\mathbf{i}$ in the direction of the x -axis. We also introduce probabilities p_j , $j = 0, 1, \dots, n-1$, that the particle is deflected through an angle $2j\pi/n$, given that it makes a collision. Clearly the set of directions $\mathcal{M}_n$ is closed with respect to these deflections, closed in this sense meaning that if a particle originally moves in a direction belonging to the set $\mathcal{M}_n$ and suffers deflections only through angles $2j\pi/n$ then its direction of motion remains in $\mathcal{M}_n$.

In three dimensions n cannot be chosen arbitrarily large, in fact the maximum number of uniformly spaced points on the unit sphere is thirty corresponding to the mid-points of the edges of a regular icosahedron. We shall examine in detail a three-dimensional system with thirty such equally spaced directions oriented relative to the slab as follows. Consider a regular icosahedron placed so that the line joining its centre to the mid-point of one edge lies along the x -axis. The thirty possible directions of motion are then along the lines joining the centre to the mid-points of all the edges. This is discussed more fully in Sect. 2.

We shall consider only the three-dimensional case in detail since this is of greatest practical interest and since the analysis in two dimensions is exactly parallel. Numerical results are given for the emergent angular distribution of particles originating from a point below the surface of a half-space of scattering material when the scattering is isotropic. As the distance of the source below the surface becomes very large a limiting distribution is approached. From this limiting distribution in the discrete case we derive an approximation to the corresponding distribution for a continuously isotropic scattering function. The approximation is compared with the exact solution of HOPF⁽¹¹⁾ and with the Gaussian quadrature and spherical harmonic approximations described by CHANDRASEKHAR⁽¹²⁾ and WANG and GUTH⁽⁸⁾. The comparison is quite favourable and moreover the method is equally applicable to the case of anisotropic scattering in finite or infinite slabs. Furthermore for certain velocity transition probabilities $\psi_0(dv|u)$ which satisfy the commutativity assumption made in one dimension, it is possible to find expressions for the joint distribution of emergent direction and energy for particles originating at a point in the slab.

It is to be noted that good approximations to the continuous case will be obtained only if the distribution $\varrho(d\mathbf{v}|\mu)$ can be reasonably well approximated by a discrete distribution on the set of thirty directions. If for example the angle of scattering is less than a few degrees with probability 1 then the icosahedral approximation is not appropriate.

(11) E. HOPF: *Mathematical Problems of Radiative Equilibrium* (Cambridge, 1934).

(12) S. CHANDRASEKHAR: *Radiative Transfer* (London, 1950).

2. - Geometry of the icosahedron.

The cosines of the angles between the faces of a regular icosahedron are ± 1 , $\pm \frac{1}{3}\sqrt{5}$ and $\pm \frac{1}{3}$ (see for example CUNDY and ROLLETT⁽¹³⁾). From these we can readily determine the co-ordinates of the face centres. An icosahedron of unit inradius can be oriented (Fig. 1) so that they are given by

$$\pm [1, 0, 0],$$

$$\pm \left[\frac{\sqrt{5}}{3}, \frac{2}{3}, 0 \right], \pm \left[\frac{\sqrt{5}}{3}, \frac{1}{3}, \frac{\sqrt{3}}{3} \right], \pm \left[\frac{\sqrt{5}}{3}, -\frac{1}{3}, -\frac{\sqrt{3}}{3} \right],$$

$$\pm \left[\frac{1}{3}, \frac{\sqrt{5}}{3}, \frac{\sqrt{3}}{3} \right], \pm \left[\frac{1}{3}, \frac{3-\sqrt{5}}{6}, \frac{\sqrt{15}+\sqrt{3}}{6} \right], \pm \left[\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, \frac{\sqrt{15}-\sqrt{3}}{6} \right],$$

$$\pm \left[\frac{1}{3}, -\frac{3+\sqrt{5}}{6}, -\frac{\sqrt{15}-\sqrt{3}}{6} \right], \pm \left[\frac{1}{3}, \frac{3-\sqrt{5}}{6}, -\frac{\sqrt{15}+\sqrt{3}}{6} \right],$$

$$\pm \left[\frac{1}{3}, \frac{\sqrt{5}}{3}, -\frac{\sqrt{3}}{3} \right].$$

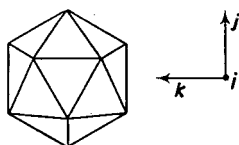


Fig. 1.

From these co-ordinates it is a matter of elementary algebra to determine the vertices and the mid-points of the edges. It is then found that the cosines of the angles between the lines joining the centre of the icosahedron to the mid-points of its edges are $\pm\mu_1$, $\pm\mu_2$, $\pm\mu_3$, $\pm\mu_4$, μ_5 where

$$(2.1) \quad \begin{cases} \mu_1 = 1, \\ \mu_2 = \frac{\sqrt{5}+1}{4}, \\ \mu_3 = \frac{1}{2}, \\ \mu_4 = \frac{\sqrt{5}-1}{4}, \\ \mu_5 = 0. \end{cases}$$

Throughout the remainder of this paper we shall assume that the ico-

(13) H. M. CUNDY and A. P. ROLLETT: *Mathematical Models* (Oxford, 1952).

hedron is oriented as in Fig. 2 with the thirty possible directions of motion along the lines joining the centre of the icosahedron to the mid-points of its edges. The directions of motion will be numbered 0, 1,

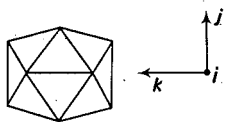


Fig. 2.

2, 3, 4, 5, 6, 7, 8, 9 according as the cosine of the angle made with the unit vector i (along the x -axis) is equal to $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_5, -\mu_4, -\mu_3, -\mu_2, -\mu_1$. We distinguish between the directions 4 (represented in Fig. 2 by the unit vectors $\pm k$) and the direc-

tions 5 (represented by $\pm j$) even though they are both perpendicular to the x -axis. The remaining directions fall into groups of four with the exception of 0 and 9 which occur only once.

3. - The general solution.

As described in the preceding Section we assign numbers 0, 1, ..., 9 to the directions of motion depending upon the cosine of the angle made with the positive x -axis. In the same way we refer to collisions as being of type 1, 2, ..., 9 according as the cosine of the angle of deflection is equal to $\mu_1, \dots, \mu_4, \mu_5, -\mu_4, \dots, -\mu_1$.

We introduce the following definitions:

λ^{-1} = mean free path, *i.e.*, the probability of a collision in any element δs of path length is given by $\lambda \delta s + o(\delta s)$,

$p_1, 4p_2, 4p_3, \dots, 4p_8, p_9$ are the probabilities that a collision is of types 1, 2, 3, ..., 8, 9 respectively.

For pure scattering $p_1 + p_9 + 4 \sum_{i=2}^8 p_i = 1$.

$P^{i_1, \dots, i_9}(\Omega, \Sigma | \beta, x) = Pr$ (the 1-st passage through the surface Σ occurs after i_j collisions of type j , $j = 1, \dots, 9$, and the direction of motion at the 1-st passage belongs to the set Ω , given that the initial position of the particle was x and its initial direction β),

$Q^{i_1, \dots, i_9}(\Omega, X | \beta, x) = Pr$ (the $(i_1 + \dots + i_9)$ -th collision occurs in $\Omega \times X$ and i_j of the first $(i_1 + \dots + i_9)$ collisions are of type j , $j = 1, \dots, 9$, given that the initial position of the particle was x and its initial direction of motion β).

The corresponding generating functions are defined as

$$(3.1) \quad \begin{cases} G(\Omega, \Sigma | \beta, x) = \sum_{i_1 + \dots + i_9 \geq 0} P^{i_1, \dots, i_9}(\Omega, \Sigma | \beta, x) z_1^{i_1} \dots z_9^{i_9}, \\ H(\Omega, X | \beta, x) = \sum_{i_1 + \dots + i_9 \geq 1} Q^{i_1, \dots, i_9}(\Omega, X | \beta, x) z_1^{i_1} \dots z_9^{i_9}. \end{cases}$$

It is convenient to introduce a more concise notation for the case when Ω consists of a single direction, $\{\alpha\}$. We shall drop explicit mention of Σ and X as these can be specified as required. Thus we are led to define

$$(3.2) \quad \left\{ \begin{array}{l} P_{\alpha\beta}^{i_1 \dots i_9}(x) = P^{i_1 \dots i_9}(\{\alpha\}, \Sigma | \beta, x), \quad \alpha, \beta = 0, 1, \dots, 9, \\ G_{\alpha\beta}(x) = G(\{\alpha\}, \Sigma | \beta, x), \\ P_{\alpha\beta}(x) = \sum_{i_1 \dots i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x), \\ P_{\alpha}^{i_1 \dots i_9}(x) = \begin{bmatrix} P_{\alpha 0}^{i_1 \dots i_9}(x) \\ \vdots \\ P_{\alpha 9}^{i_1 \dots i_9}(x) \end{bmatrix}, \quad G_{\alpha}(x) = \begin{bmatrix} G_{\alpha 0}(x) \\ \vdots \\ G_{\alpha 9}(x) \end{bmatrix}, \\ P_{\alpha}(x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha}^{i_1 \dots i_9}(x). \end{array} \right.$$

Analogous notation is used for the corresponding Q 's and H 's.

The backward Kolmogorov equations for the generating functions $G_{\alpha\beta}(x)$ can now be written

$$(3.3) \quad A(z_1, \dots, z_9) G_{\alpha}(x) = 0, \quad \alpha = 0, \dots, 9,$$

where $A(z_1, \dots, z_9)$ is a 10×10 matrix. If I is the 10×10 unit matrix then $A - I$ is equal to

$$\begin{bmatrix} -\mu_1\theta - (1) & -(2222) & -(3333) & -(4444) & -(55) & -(55) & -(6666) & -(7777) & -(8888) & -(9) \\ -(2) & -\mu_2\theta - (1234) & -(2345) & -(2356) & -(46) & -(37) & -(4578) & -(5678) & -(6789) & -(8) \\ -(3) & -(2345) & -\mu_3\theta - (1267) & -(2457) & -(28) & -(46) & -(3568) & -(3489) & -(5678) & -(7) \\ -(4) & -(2356) & -(2457) & -\mu_4\theta - (1368) & -(37) & -(28) & -(2479) & -(3568) & -(4578) & -(6) \\ -(5) & -(4466) & -(2288) & -(3377) & -(19) & -(55) & -(3377) & -(2288) & -(4466) & -(5) \\ -(5) & -(3377) & -(4466) & -(2288) & -(55) & -(19) & -(2288) & -(4466) & -(3377) & -(5) \\ -(6) & -(4578) & -(3568) & -(2479) & -(37) & -(28) & \mu_4\theta - (1368) & -(2457) & -(2356) & -(4) \\ -(7) & -(5678) & -(3489) & -(3568) & -(28) & -(46) & -(2457) & \mu_3\theta - (1267) & -(2345) & -(3) \\ -(8) & -(6789) & -(5678) & -(4578) & -(46) & -(37) & -(2356) & -(2345) & \mu_2\theta - (1234) & -(2) \\ -(9) & -(8888) & -(7777) & -(6666) & -(55) & -(55) & -(4444) & -(3333) & -(2222) & \mu_1\theta - (1) \end{bmatrix}$$

The symbol $(i_1 \dots i_n)$ is used to denote the expression $(p_{i_1} z_{i_1} + \dots + p_{i_n} z_{i_n})$ and θ denotes the operator $(1/\lambda)(\partial/\partial x)$.

The solution of eq. (3.3) is of the form

$$(3.4) \quad G_{\alpha\beta}(x) = \sum_{k=1}^4 [B_{\alpha\beta}^k(z_1, \dots, z_9) \exp[\lambda x \theta_k(z_1, \dots, z_9)] + C_{\alpha\beta}^k(z_1, \dots, z_9) \exp[-\lambda x \theta_k(z_1, \dots, z_9)]],$$

where $\pm \theta_k(z_1, \dots, z_9)$, $k = 1, \dots, 4$, are the roots (assumed distinct) of the determinantal equation

$$(3.5) \quad |A(z_1, \dots, z_9)| = 0$$

regarded as an eighth degree equation in θ . In a given problem the functions $B_{\alpha\beta}^k(z_1, \dots, z_9)$ and $C_{\alpha\beta}^k(z_1, \dots, z_9)$ are determined by the relations between them arising from the original eq. (3.3) and by the boundary conditions.

For a given set of boundary conditions the complete solution for $G_{\alpha\beta}(x)$ as a function of $z_1, \dots, z_9$ is generally too complicated to determine explicitly. However it is a relatively straightforward numerical procedure to determine moments of the corresponding joint distribution. For example suppose we have particles starting with direction β at a point x in the slab $a < x < b$ and we wish to find the expected number $m_{\alpha\beta}^k$ of collisions of type k for the particles emerging from the slab in direction α . In this case we take Σ to consist of the two planes $\{x:x=a\}$, $\{x:x=b\}$ and obtain

$$(3.6) \quad m_{\alpha\beta}^k(x) = \frac{1}{P_{\alpha\beta}(x)} \sum_{i_1+\dots+i_9 \geq 0} i_k P_{\alpha\beta}^{i_1 \dots i_9}(x).$$

Now

$$(3.7) \quad P_{\alpha\beta}(x) = [G_{\alpha\beta}(x)]_{z_1=\dots=z_9=1}$$

and

$$(3.8) \quad \sum_{i_1+\dots+i_9 \geq 0} i_k P_{\alpha\beta}^{i_1 \dots i_9}(x) = \left[\frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right]_{z_1=\dots=z_9=1}.$$

Both these quantities can be determined numerically from the equations

$$(3.9) \quad \begin{cases} A(z_1, \dots, z_9) G_{\alpha}(x) = 0, \\ \left[\frac{\partial}{\partial z_k} A(z_1, \dots, z_9) \right] G_{\alpha}(x) + A(z_1, \dots, z_9) \left[\frac{\partial}{\partial z_k} G_{\alpha}(x) \right] = 0 \end{cases}$$

with $z_1, \dots, z_9$ replaced by 1 throughout. $\partial A / \partial z$ here denotes the matrix $[(\partial / \partial z) a_{ij}]$ where a_{ij} are the elements of A . The appropriate boundary con-

ditions are

$$(3.10) \quad \begin{cases} P_{\alpha\beta}(a) = \delta_{\alpha\beta}, & \beta = 6, 7, 8, 9, \\ P_{\alpha\beta}(b) = \delta_{\alpha\beta}, & \beta = 0, 1, 2, 3, \\ \left[\frac{\partial}{\partial z_k} G_{\alpha\beta}(x) \right]_{z_1=\dots=z_9=1} = 0 & \begin{cases} x = a, & \beta = 6, 7, 8, 9, \\ x = b, & \beta = 0, 1, 2, 3. \end{cases} \end{cases}$$

We shall show in Sect. 4 how, under certain assumptions regarding the transition probability densities $\varphi_\theta(v|u)$ for the velocities before and after a collision, the same kind of argument can be used to find the distribution of emergent velocity at various angles.

4. - Isotropic scattering in a half-space.

In this Section we shall illustrate the application of the foregoing results to the well-known Milne problem of isotropic scattering in a half-space. In particular we shall derive the distribution of emergent angle for particles originating at a point x in the half-space $[0, \infty)$ and use the results for thirty equally spaced directions to derive an approximation for the case of a continuously isotropic scattering function. The exact solution as $x \rightarrow \infty$ is given by HOPF⁽¹¹⁾ and approximate methods have been devised by CHANDRA-SEKHAR⁽¹²⁾, WANG and GUTH⁽⁸⁾ and others. Very close agreement is found between the limit of our solution as $x \rightarrow \infty$ and the corresponding exact solution as obtained by HOPF. A table of numerical results is given to show the comparison.

We consider the half-space $0 \leq x < \infty$ and take for Σ the surface $x = 0$. We shall determine the distribution of emergent angle $P_{\alpha\beta}(x)$, $\alpha = 6, 7, 8, 9$, for a particle which starts at x , $0 \leq x < \infty$, moving in direction β . As we have already seen in Sect. 2 the vectors $P_\alpha(x)$ satisfy the equation

$$(4.1) \quad A(1, \dots, 1) P_\alpha(x) = 0.$$

The roots of the determinantal equation $|A(1, \dots, 1)| = 0$, regarded as an equation in θ are $\pm\theta_1, \pm\theta_2, \pm\theta_3, \pm\theta_4$, where

$$(4.2) \quad \begin{cases} \theta_1 = 0, \\ \theta_2 = 1.044615, \\ \theta_3 = 1.665073, \\ \theta_4 = 2.852422. \end{cases}$$

Since the elements of $\mathbf{P}_\alpha(x)$, being probabilities, are bounded as $x \rightarrow \infty$, the required solution of (4.1) is of the form

$$(4.3) \quad \mathbf{P}_\alpha(x) = \mathbf{A}_\alpha + \mathbf{B}_\alpha \exp[-\lambda\theta_2 x] + \mathbf{C}_\alpha \exp[-\lambda\theta_3 x] + \mathbf{D}_\alpha \exp[-\lambda\theta_4 x],$$

where $\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha, \mathbf{D}_\alpha$ are to be determined from the boundary conditions at $x=0$. These are

$$(4.4) \quad P_{\alpha\beta}(0) = \delta_{\alpha\beta}, \quad \beta = 6, 7, 8, 9.$$

Also from eqs. (4.1) we have the relations

$$(4.5) \quad \left\{ \begin{array}{l} A_{\alpha 0}(1 + \mu_1 \theta_1) = \dots = A_{\alpha 3}(1 + \mu_4 \theta_1) = \\ \qquad \qquad \qquad = A_{\alpha 4} = A_{\alpha 5} = A_{\alpha 6}(1 - \mu_4 \theta_1) = \dots = A_{\alpha 9}(1 - \mu_1 \theta_1), \\ B_{\alpha 0}(1 + \mu_1 \theta_2) = \dots = B_{\alpha 3}(1 + \mu_4 \theta_2) = \\ \qquad \qquad \qquad = B_{\alpha 4} = B_{\alpha 5} = B_{\alpha 6}(1 - \mu_4 \theta_2) = \dots = B_{\alpha 9}(1 - \mu_1 \theta_2), \\ C_{\alpha 0}(1 + \mu_1 \theta_3) = \dots = C_{\alpha 4} = C_{\alpha 5} = \dots = C_{\alpha 9}(1 - \mu_1 \theta_3), \\ D_{\alpha 0}(1 + \mu_1 \theta_4) = \dots = D_{\alpha 4} = D_{\alpha 5} = \dots = D_{\alpha 9}(1 - \mu_1 \theta_4), \end{array} \right.$$

where

$$(4.6) \quad \left\{ \begin{array}{l} \mu_1 = 1, \\ \mu_2 = \frac{\sqrt{5} + 1}{4}, \\ \mu_3 = \frac{1}{2}, \\ \mu_4 = \frac{\sqrt{5} - 1}{4}, \\ \mu_5 = 0. \end{array} \right.$$

These relations, together with the boundary conditions, completely determine the vectors $\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha, \mathbf{D}_\alpha$ leading to the solution

$$(4.7) \quad \mathbf{P}_\alpha(x) = a_\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b_\alpha \exp[-\lambda\theta_2 x] \begin{bmatrix} 1 \\ 1.10813 \\ 1.34310 \\ 1.54567 \\ 2.04462 \\ 2.04462 \\ 3.01924 \\ 4.28020 \\ 13.2005 \\ -45.8280 \end{bmatrix} +$$

$$+ c_x \exp[-\lambda\theta_3 x] \begin{bmatrix} 1 \\ 1.13549 \\ 1.45431 \\ 1.75966 \\ 2.66507 \\ 2.66507 \\ 5.48974 \\ 15.9144 \\ -7.67873 \\ -4.00719 \end{bmatrix} + d_x \exp[-\lambda\theta_4 x] \begin{bmatrix} 1 \\ 1.16470 \\ 1.58783 \\ 2.04758 \\ 3.85242 \\ 3.85242 \\ 32.4954 \\ -9.03877 \\ -2.94605 \\ -2.07967 \end{bmatrix},$$

where

$$(4.8) \quad \begin{cases} a_6 = 0.11906, & a_7 = 0.23289, & a_8 = 0.48016, & a_9 = 0.16790, \\ b_6 = 0.000814, & b_7 = 0.002257, & b_8 = 0.014352, & b_9 = 0.017422, \\ c_6 = 0.006984, & c_7 = 0.039606, & c_8 = -0.039400, & c_9 = -0.007190, \\ d_6 = 0.025854, & d_7 = -0.014067, & d_8 = -0.009453, & d_9 = -0.002334. \end{cases}$$

We see at once from these results that for a particle which originates from a point several mean free paths within the slab of scattering material the distribution of emergent angle is very close to the distribution of emergent angle for a particle which starts at infinity. Probabilities of the various emergent directions in this limiting case are independent of the initial direction of motion of the particle and are given by

$$(4.9) \quad \begin{cases} P_6 = 0.11906, \\ P_7 = 0.23289, \\ P_8 = 0.48016, \\ P_9 = 0.16790. \end{cases}$$

From these results we can derive an approximation to the solution in the physically more realistic continuous case by estimating the probability density $f(\mu)$ corresponding to the direction whose angle with the positive x -axis has cosine μ . $f(\mu)$ is normalized so that $\int_0^1 f(\mu) d\mu = 1$ and estimates are made for the directions 5, 6, 7, 8, 9, i.e. for $\mu = \mu_5, -\mu_4, -\mu_3, -\mu_2, -\mu_1$, by supposing the probabilities P_6, P_7, P_8, P_9 to be distributed over $\frac{2}{15}, \frac{2}{15}, \frac{2}{15}$ and $\frac{1}{15}$ respectively of the unit sphere.

The results are tabulated below together with the exact solution and the approximations given by WANG and GUTH⁽⁸⁾ and CHANDRASEKHAR⁽¹²⁾.

μ	Exact solution	Icosahedral approximation	Gaussian quadrature approximation $n=4$	Wang and Guth's S.H. method $n=3$ r odd	Chandra-sekhar's S.H. method $n=3$
1	2.5184	2.5185	2.5096	2.5128	2.4837
0.8090	1.7998	1.8006	1.7922	1.7948	1.7702
0.5000	0.8716	0.8733	0.8660	0.8680	0.8520
0.3090	0.4441	0.4465	0.4384	0.4417	0.4324
0	0	0	0	0	0

As can be seen from these figures the icosahedral approximation is very good compared with the other standard approximations. Densities at values of μ other than those given could be obtained by interpolation.

Another feature of the icosahedral method is that it provides information about the dependence of the emergent distribution on x which is not available from the other methods.

The icosahedral method can also be used to study (for finite x) the number of collisions of each type before emergence at any given angle.

5. - The stationary distribution of particles in a finite slab.

Suppose now that there is a steady stream of particles incident on one face of the slab $\{x: 0 \leq x \leq b\}$. It will be shown in this Section how the time-independent solution for the mean number of particles per second crossing a plane $x = \xi$ in any given direction can be determined.

Consider a plane slab of scattering material of thickness l and define $P_{\alpha\beta}$ as the probability that a particle incident on the slab in direction β emerges from the slab in direction α . Define a forward transfer matrix, P_i , and a reverse transfer matrix, P'_i , for the slab as follows:

$$(5.1) \quad P_i = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{30} & \cdot & \cdot & P_{33} \end{bmatrix} = \begin{bmatrix} P_{99} & P_{98} & P_{97} & P_{96} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{69} & \cdot & \cdot & P_{66} \end{bmatrix},$$

$$(5.2) \quad P'_i = \begin{bmatrix} P_{90} & P_{91} & P_{92} & P_{93} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{60} & \cdot & \cdot & P_{63} \end{bmatrix} = \begin{bmatrix} P_{09} & P_{08} & P_{07} & P_{06} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_{39} & \cdot & \cdot & P_{36} \end{bmatrix}.$$

Suppose we have a number of particles incident on the slab, say n_0 in direction 0, n_1 in direction 1, n_2 in direction 2 and n_3 in direction 3. These particles can be represented by a vector

$$\mathbf{n} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

Of these incident particles n'_0, n'_1, n'_2, n'_3 will eventually emerge from the opposite face of the slab moving in directions 0, 1, 2, 3 respectively and $n''_0, \dots, n''_3$ will eventually be reflected from the incident face emerging in directions 9, 8, 7, 6 respectively. The transmitted and reflected particles are represented by the vectors

$$(5.3) \quad \mathbf{n}' = P_i \mathbf{n} \quad \text{and} \quad \mathbf{n}'' = P'_i \mathbf{n}.$$

Clearly the forward and reverse transfer matrices for a homogeneous isotropic medium do not depend on the face at which the particles are incident.

Now let us examine the density of particles at points inside a slab $(0, b)$ assuming the system to be in a steady state. Suppose $\mathbf{n}$ particles per second are incident on the left face of the slab. If the system is in equilibrium the mean number of particles per second crossing the plane $x = \xi$ from the left for the $(k + 1)$ th time is

$$(5.4) \quad \mathbf{n}'_k = [P'_\xi P'_{b-\xi}]^{-1} P_\xi \mathbf{n}, \quad k = 0, 1, 2, \dots$$

Hence the total number of particles per second crossing the plane $x = \xi$ from the left is given by

$$(5.5) \quad \sum_{k=0}^{\infty} \mathbf{n}'_k = [I - P'_\xi P'_{b-\xi}]^{-1} P_\xi \mathbf{n},$$

where I is the 4×4 unit matrix. Similarly the total number of particles per second crossing the plane $x = \xi$ from the right is given by

$$(5.6) \quad \sum_{k=0}^{\infty} \mathbf{n}''_k = [I - P'_{b-\xi} P'_\xi]^{-1} P'_{b-\xi} P_\xi \mathbf{n}.$$

The probabilities $P_{\alpha\beta}$ which appear in the transfer matrixes P_i and P'_i are determined by the same method as was used in Sect. 3, *i.e.*, by solving the equation

$$(5.7) \quad A(1, 1, \dots, 1) P_\alpha(x) = 0$$

subject to the boundary conditions

$$(5.8) \quad \begin{cases} P_{\alpha\beta}(0) = \delta_{\alpha\beta}, & \beta = 6, 7, 8, 9, \\ P_{\alpha\beta}(l) = \delta_{\alpha\beta}, & \beta = 0, 1, 2, 3. \end{cases}$$

The transfer matrices given by (5.1), (5.2) are then determined from the relations

$$(5.9) \quad \left\{ \begin{array}{ll} P_{\alpha\beta} = P_{\alpha\beta}(0), & \beta = 0, 1, 2, 3 \\ P_{\alpha\beta} = P_{\alpha\beta}(l), & \beta = 6, 7, 8, 9 \end{array} \right\} \alpha = 0, 1, 2, 3.$$

The solution $P_{\alpha\beta}(x)$ of (5.7) subject to the conditions (5.8) is given (for isotropic scattering) by

$$(5.10) \quad P_{\alpha\beta}(x) = \begin{bmatrix} 1 & (\lambda x + \mu_1) & (1 - \mu_1 \theta_2)^{-1} & (1 + \mu_1 \theta_2)^{-1} & (1 - \mu_1 \theta_3)^{-1} & (1 + \mu_1 \theta_3)^{-1} & (1 - \mu_1 \theta_4)^{-1} & (1 + \mu_1 \theta_4)^{-1} \\ 1 & (\lambda x + \mu_2) & (1 - \mu_2 \theta_2)^{-1} & (1 + \mu_2 \theta_2)^{-1} & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda x + \mu_3) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda x + \mu_4) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda x & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \lambda x & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (\lambda x - \mu_4) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda x - \mu_3) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda x - \mu_2) & (1 + \mu_2 \theta_2)^{-1} & (1 - \mu_2 \theta_2)^{-1} & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda x - \mu_1) & (1 + \mu_1 \theta_2)^{-1} & (1 - \mu_1 \theta_2)^{-1} & (1 + \mu_1 \theta_3)^{-1} & (1 - \mu_1 \theta_3)^{-1} & (1 + \mu_1 \theta_4)^{-1} & (1 - \mu_1 \theta_4)^{-1} \end{bmatrix} \begin{bmatrix} A_\alpha \\ B_\alpha \\ A_\alpha^{(3)} e^{\lambda x \theta_3} \\ A_\alpha^{(-3)} e^{-\lambda x \theta_3} \\ A_\alpha^{(2)} e^{\lambda x \theta_2} \\ A_\alpha^{(-2)} e^{-\lambda x \theta_2} \\ A_\alpha^{(4)} e^{\lambda x \theta_4} \\ A_\alpha^{(-4)} e^{-\lambda x \theta_4} \end{bmatrix}$$

$\mu_1, \dots, \mu_4, \theta_2, \dots, \theta_4$ have the numerical values given in Sect. 4 and $A_\alpha, B_\alpha, A_\alpha^{(2)}, \dots, A_\alpha^{(-4)}$ are found, for any given thickness l , from the equations

$$(5.11) \quad \begin{cases} A_\alpha - \mu_j B_\alpha + \sum_{k=2,3,4} (1 + \mu_j \theta_k)^{-1} A_\alpha^{(k)} + \sum_{k=-2,-3,-4} (1 - \mu_j \theta_k)^{-1} A_\alpha^{(k)} = \delta_{\alpha(10-j)}, & j = 1, 2, 3, 4, \\ A_\alpha + (\lambda l + \mu_j) B_\alpha + \sum_{k=2,3,4} (1 - \mu_j \theta_k)^{-1} A_\alpha^{(k)} \exp[\lambda l \theta_k] + \sum_{k=-2,-3,-4} (1 + \mu_j \theta_k)^{-1} A_\alpha^{(k)} \exp[-\lambda l \theta_k] = \delta_{\alpha(j-1)}, & j = 1, 2, 3, 4. \end{cases}$$

6. - The distribution of emergent velocity.

Consider a slab $[a, b]$ and a particle that originates at a point x inside the slab moving in direction β . We wish to find the distribution of the particle's emergent velocity given that it emerges from the slab in direction α . If as before Σ is taken to consist of the two surfaces $\{x: x = a\}$ and $\{x: x = b\}$ then $G_{\alpha\beta}(x)[P_{\alpha\beta}(x)]^{-1}$ is the generating function for the joint distribution, conditional on emergence in direction α , of the number of collisions of each type before emergence.

Let $\psi_i(dv|u)$ be the transition probability for the absolute velocities u and v before and after a collisions of type i . If we assume, as in the one-dimensional case ⁽¹⁾, that

$$(6.1) \quad \int_w \psi_i(dv|w) \psi_i(dw|u) = \int_w \psi_i(dw|w) \psi_i(dw|u), \quad i, j = 1, 2, \dots, 9,$$

then the probability $F_{\alpha\beta}(V|u, x)$ that the emergent velocity belongs to the set V , conditional on initial state (β, u, x) and emergent direction α , is given by $G_{\alpha\beta}(x)[P_{\alpha\beta}(x)]^{-1}$ with terms such as $z_1^{i_1} \dots z_9^{i_9}$ replaced by $\psi_1^{(i_1)} \circ \psi_2^{(i_2)} \circ \dots \circ \psi_9^{(i_9)}$. The functions $\psi^{(n)}$ and $\psi \circ \varphi$ are defined as follows:

$$(6.2) \quad \begin{cases} \psi^{(n)} = \int \psi(V|w) \psi^{(n-1)}(dw|u), & n = 1, 2, \dots, \\ \psi^{(0)} = \delta(V|u) = \begin{cases} 1, & u \in V, \\ 0, & u \notin V, \end{cases} \\ \psi \circ \varphi = \int \psi(V|w) \varphi(dw|u). \end{cases}$$

Further simplification occurs if we assume particular forms for $\psi_i(dv|u)$ as in the following examples

Example 1

$$\psi_i(dv|u) = \begin{cases} \frac{1}{u} f_i\left(\frac{v}{u}\right) dv, & 0 \leq v \leq u \\ 0, & \text{otherwise.} \end{cases}$$

It is convenient in this case to introduce the Mellin transforms

$$(6.3) \quad \begin{cases} \Psi_i(\omega) = \int_{v=0}^u f_i\left(\frac{v}{u}\right) \left(\frac{v}{u}\right)^{\omega-1} \frac{dv}{u}, \\ \mathcal{F}_{\alpha\beta}(\omega|u, x) = \int_{v=0}^u \left(\frac{v}{u}\right)^{\omega-1} F_{\alpha\beta}(dv|u, x), \end{cases}$$

from which, using the preceding remarks, we find

$$(6.4) \quad \mathcal{F}_{\alpha\beta}(\omega|u, x) = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) \Psi_1(\omega)^{i_1} \dots \Psi_9(\omega)^{i_9}.$$

Clearly the right side of (6.4) is just $[P_{\alpha\beta}(x)]^{-1} G_{\alpha\beta}(x)$ with z_k replaced by $\Psi_k(\omega)$, $k = 1, \dots, 9$.

To find moments of the emergent velocity distribution, for example, $E_{\alpha\beta}((v/u)^n|u, x)$, we put $\omega = n + 1$ in $\mathcal{F}_{\alpha\beta}(\omega|u, x)$. $P_{\alpha\beta}(x)$ is determined as in Sect. 3 and $G_{\alpha\beta}(x)$ is determined from the equation

$$(6.5) \quad A(\Psi_1(n+1), \dots, \Psi_9(n+1))G(x) = 0$$

with boundary conditions $G_{\alpha\beta}(a) = \delta_{\alpha\beta}$, $\beta = 6, 7, 8, 9$, $G_{\alpha\beta}(b) = \delta_{\alpha\beta}$, $\beta = 0, 1, 2, 3$.

Example 2

$$\psi_i(dv|u) = \delta(v - c_i u)dv, \quad 0 \leq c_i \leq 1, \quad i = 1, \dots, 9.$$

This model corresponds to the physical situation in which the logarithmic energy of the particle decreases by a fixed amount for each particular angular deflection. In this case the distribution of emergent velocity is discrete since a particle which starts with velocity u and makes i_1 collisions of type 1, i_2 of type 2, ..., and i_9 of type 9 has a resultant velocity

$$v = u c_1^{i_1} \dots c_9^{i_9}.$$

If we again take Mellin transforms as in example 1 we find that

$$(6.6) \quad \mathcal{F}_{\alpha\beta}(\omega|u, x) = [P_{\alpha\beta}(x)]^{-1} \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1 \dots i_9}(x) c_1^{(\omega-1)i_1} \dots c_9^{(\omega-1)i_9},$$

which is simply $[P_{\alpha\beta}(x)]^{-1} G_{\alpha\beta}(x)$ with z_k replaced by $c_k^{\omega-1}$, $k = 1, \dots, 9$.

To find moments of the distribution, say $E_{\alpha\beta}((v/u)^n|u, x)$, we put $\omega = n + 1$ as before, determining $P_{\alpha\beta}(x)$ as in Sect. 3 and $G_{\alpha\beta}(x)$ from the equation

$$(6.7) \quad A(c_1^n, \dots, c_9^n) G_{\alpha}(x) = 0$$

with boundary conditions $G_{\alpha\beta}(a) = \delta_{\alpha\beta}$, $\beta = 6, 7, 8, 9$, $G_{\alpha\beta}(b) = \delta_{\alpha\beta}$, $\beta = 0, 1, 2, 3$.

7. - The effect of absorption.

Consider now the modified process in which there is at each collision a constant probability of absorption, $r = 1 - p_1 - p_9 - 4 \sum_2^8 p_j$. The Kolmogorov eqs. (3.3) for the generating functions $G_{\alpha\beta}(x)$ remain the same and so also do the boundary conditions. Thus the behaviour of the particles when there is absorption (*i.e.*, $r > 0$) is determined by exactly the same equations, the only

difference being that the probabilities $p_1, 4p_2, \dots, 4p_8, p_9$ will no longer add to 1. All the results of this paper therefore, except for those of Sect. 4 and eqs. (5.10) and (5.11) which were derived for a particular set of probabilities p_i , apply without modification to the process with absorption.

* * *

In conclusion I would like to thank Mr. J. E. MOYAL for a number of very helpful discussions during the course of this work.

RIASSUNTO (*)

Per mezzo della teoria dei processi markoviani discontinui si studia un processo di scattering tridimensionale, estendendo i risultati ottenuti in precedenza per una dimensione da Brockwell e Moyal. Si suppone che le particelle diffuse si muovano in una di un gruppo di trenta direzioni nello spazio. Si suppone che il cammino libero medio sia costante, lo scattering ha simmetria assiale e le probabilità di transizione della velocità $\psi_\theta(dv|u)$ sono definite per ciascuna collisione in dipendenza della deflessione angolare θ . Si dimostra come i risultati ottenuti per il gruppo discreto di direzioni possono essere usati per approssimare il caso continuo fisicamente più realistico; si fa un confronto con i risultati esatti per il ben noto problema di Milne con scattering isotropico. Si mostra come determinare, per un flusso continuo di particelle incidenti su una faccia di una lastra $\{x: a \leq x \leq b\}$, la soluzione indipendente dal tempo per il numero medio di particelle per secondo che attraversano un piano $x=\xi$ in ogni data direzione. Si determina anche la distribuzione della velocità e della direzione con cui le particelle emergono da una lastra, nell'ipotesi che le probabilità $\psi_\theta(dv|u)$ soddisfacciano una certa condizione di commutatività. Se si ha una probabilità di assorbimento costante (non nulla) in una collisione, non è richiesta alcuna sostanziale modificazione dell'analisi.

(*) Traduzione a cura della Redazione.

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The Multiple Compton Scattering of Low Energy Gamma Radiation

By P. J. BROCKWELL

Department of Statistics, Australian National University, Canberra†

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ABSTRACT

The emergent angular and spectral distributions of low energy gamma radiation after multiple Compton scattering by a homogeneous plane slab are determined by a method which is based on the theory of Markov processes and the replacement of the set of all possible directions of motion in space by a finite set of directions corresponding to thirty points uniformly spaced on the unit sphere.

§ 1. INTRODUCTION

THE multiple Compton scattering of low energy ($E \ll 0.5$ mev) photons has been studied analytically by Chandrasekhar (1948) and O'Rourke (1952, 1953) using the Gaussian quadrature and spherical harmonic approximations to Boltzmann's equation. The problem is somewhat simplified at low photon energies because the Klein-Nishina differential cross section (i.e. the cross section for scattering through a given angle per unit solid angle) is then effectively energy-independent and is given, in Thomson units per electron, by:

$$\sigma(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta). \quad (1)$$

where θ is the angle of deflection resulting from a collision and

$$1 \text{ Thomson unit} = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 = 0.665 \text{ barns.}$$

The mean free path d of the photons is related to $\sigma(\theta)$ by the equation:

$$d^{-1} = N \int \sigma(\theta) d\Omega, \quad (2)$$

where N is the number of electrons per unit volume. We shall use d as the unit of length throughout. Given that a collision occurs, the probability that a photon is scattered through an angle θ into the element $d\Omega$ of solid angle is given by

$$\frac{3}{16\pi} (1 + \cos^2 \theta) d\Omega.$$

If we make the assumption that the momenta of the scattering electrons are small so that Compton's relation is applicable, then the wavelength

† Present address: Applied Mathematics Division, Argonne National Laboratory, Argonne Illinois.

Like Chandrasekhar and O'Rourke we shall neglect photoelectric absorption; however if we assume a constant probability of absorption at any collision the problem can be solved in exactly the same way as explained in Brockwell (1964), §7. A further advantage of the method to be described is that it can be used to determine not only the overall emergent spectral intensities but also the intensities in particular directions of emergence.

§ 2. APPLICATION OF THE ICOSAHERAL APPROXIMATION

Using the method of Brockwell (1964) we replace the set of all directions in space by 30 directions corresponding to points uniformly spaced on the unit sphere. For a detailed description of this procedure the reader is referred to the above paper. The directions of motion then fall into classes numbered 1, 2, . . . , 10 such that the angles they make with the positive X -axis have cosines $\mu_1, \mu_2, \dots, \mu_{10}$ respectively, where

$$\left. \begin{aligned} \mu_1 &= 1, & \mu_6 &= \mu_5, \\ \mu_2 &= \frac{\sqrt{5}+1}{4}, & \mu_7 &= -\mu_4, \\ \mu_3 &= \frac{1}{2}, & \mu_8 &= -\mu_3, \\ \mu_4 &= \frac{\sqrt{5}-1}{4}, & \mu_9 &= -\mu_2, \\ \mu_5 &= 0, & \mu_{10} &= -\mu_1. \end{aligned} \right\} \dots \dots \dots (4)$$

The directions perpendicular to the X -axis have been split into two classes (5 and 6) for the purpose of deriving the eqns. (13) below. Combining these directions into a single class would destroy the Markovian nature of the process. In the same way collisions are said to be of types 1, . . . , 9 according as the angle of deflection has cosine $\mu_1, \dots, \mu_4, \mu_5, -\mu_4, \dots, -\mu_1$. The probabilities that collisions are of types 1, . . . , 9 will be denoted by $p_1, 4p_2, \dots, 4p_8, p_9$ respectively. For pure scattering, i.e. no absorption,

$$p_1 + p_9 + 4 \sum_{k=2}^8 p_k = 1. \quad \dots \dots \dots (5)$$

To determine the discrete approximation to the continuous differential cross section:

$$\sigma(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta), \quad \dots \dots \dots (6)$$

it is assumed that the probabilities $p_1, \dots, p_5$ are in the ratio

$$1 + \mu_1^2 : 1 + \mu_2^2 : 1 + \mu_3^2 : 1 + \mu_4^2 : 1 + \mu_5^2.$$

The normalization condition (5) and the symmetry condition, $p_j = p_{10-j}$,

then determine the probabilities p_j uniquely. We find in this way that

$$\left. \begin{aligned} p_1 &= p_9 = \frac{1}{20}, \\ p_2 &= p_8 = \frac{1}{320} (11 + \sqrt{5}), \\ p_3 &= p_7 = \frac{1}{32}, \\ p_4 &= p_6 = \frac{1}{320} (11 - \sqrt{5}), \\ p_5 &= \frac{1}{40}. \end{aligned} \right\} \dots \dots \dots (7)$$

The wavelength increments δ_i for collisions of type i , $i = 1, \dots, 9$ are given by Compton's equation as:

$$\delta_i = 1 - \cos \theta_i, \quad \dots \dots \dots (8)$$

where θ_i is the angle of deflection for a collision of type i . Thus:

$$\left. \begin{aligned} \delta_1 &= 0, \\ \delta_2 &= \frac{1}{4} (3 - \sqrt{5}), \quad \delta_6 = \frac{1}{4} (3 + \sqrt{5}), \\ \delta_3 &= \frac{1}{2}, \quad \delta_7 = \frac{3}{2}, \\ \delta_4 &= \frac{1}{4} (5 - \sqrt{5}), \quad \delta_8 = \frac{1}{4} (5 + \sqrt{5}), \\ \delta_5 &= 1, \quad \delta_9 = 2. \end{aligned} \right\} \dots \dots \dots (9)$$

Consider now a photon which originates at a point with coordinate x in the slab ($\mathbf{x}: a \leq x \leq b$). Supposing the photon to be restricted to move in one of the above set of 30 directions we define the probabilities:

$P_{\alpha\beta}^{i_1, \dots, i_9}(x) = Pr$ (the first departure of the particle from the slab occurs after exactly i_j collisions of type j , $j = 1, \dots, 9$, and the direction of motion at departure is α , given that the initial position of the particle is x and its initial direction of motion β).

The corresponding generating functions are given by:

$$G_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1, \dots, i_9}(x) z_1^{i_1} \dots z_9^{i_9}. \quad \dots \dots (10)$$

It is convenient also to define the column vectors $\mathbf{P}_\alpha^{i_1, \dots, i_9}(x)$, $\mathbf{G}_\alpha(x)$, whose i th components ($i = 1, \dots, 10$) are $P_{\alpha i}^{i_1, \dots, i_9}(x)$ and $G_{\alpha i}(x)$ respectively.

We then write:

$$\mathbf{P}_{\alpha\beta}(x) = \sum_{i_1 + \dots + i_9 \geq 0} \mathbf{P}_{\alpha\beta}^{i_1, \dots, i_9}(x), \quad \dots \dots \dots (11)$$

$$\mathbf{P}_\alpha(x) = \sum_{i_1 + \dots + i_9 \geq 0} \mathbf{P}_\alpha^{i_1, \dots, i_9}(x), \quad \dots \dots \dots (12)$$

observing that $\mathbf{P}_\alpha(x)$ is obtained from $\mathbf{G}_\alpha(x)$ on replacing each of $z_1, \dots, z_9$ by unity.

As described in Brockwell (1964) the generating functions $G_{\alpha\beta}(x)$ are found from the backward Kolmogorov equations:

$$A_\theta(z_1, \dots, z_9) \mathbf{G}_\alpha(x) = \mathbf{0}, \quad \dots \dots \dots (13)$$

with boundary conditions:

$$\left. \begin{aligned} G_{\alpha\beta}(a) &= \delta_{\alpha\beta}, \quad \beta = 7, 8, 9, 10, \\ G_{\alpha\beta}(b) &= \delta_{\alpha\beta}, \quad \beta = 1, 2, 3, 4. \end{aligned} \right\} \dots \dots \dots (14)$$

If I is the 10×10 identity matrix then $A_\theta(z_1, \dots, z_9) - I$ is given by:

$$\begin{bmatrix} -\mu_1\theta-(1) & -(2222) & -(3333) & -(4444) & -(55) & -(55) & -(6666) & -(7777) & -(8888) & -(9) \\ -(2) & -\mu_2\theta-(1234) & -(2345) & -(2356) & -(46) & -(37) & -(4578) & -(5678) & -(6789) & -(8) \\ -(3) & -(2345) & -\mu_3\theta-(1267) & -(2457) & -(28) & -(46) & -(3568) & -(3489) & -(5678) & -(7) \\ -(4) & -(2356) & -(2457) & -\mu_4\theta-(1368) & -(37) & -(28) & -(2479) & -(3568) & -(4578) & -(6) \\ -(5) & -(4466) & -(2288) & -(3377) & -(19) & -(55) & -(3377) & -(2288) & -(4466) & -(5) \\ -(5) & -(3377) & -(4466) & -(2288) & -(55) & -(19) & -(2288) & -(4466) & -(3377) & -(5) \\ -(6) & -(4578) & -(3568) & -(2479) & -(37) & -(28) & \mu_4\theta-(1368) & -(2457) & -(2356) & -(4) \\ -(7) & -(5678) & -(3489) & -(3568) & -(28) & -(46) & -(2457) & \mu_3\theta-(1267) & -(2345) & -(3) \\ -(8) & -(6789)^* & -(5678) & -(4578) & -(46) & -(37) & -(2356) & -(2345) & \mu_2\theta-(1234) & -(2) \\ -(9) & -(8888) & -(7777) & -(6666) & -(55) & -(55) & -(4444) & -(3333) & -(2222) & \mu_1\theta-(1) \end{bmatrix}$$

The symbol $(i_1 \dots i_n)$ is used to denote the expression $(p_{i_1} z_{i_1} + \dots + p_{i_n} z_{i_n})$ and θ denotes the operator $\partial/\partial x$. If b is infinite the second boundary condition is replaced by the condition that $G_{\alpha\beta}(x)$ be a bounded function of x as $x \rightarrow \infty$.

The solution of eqns. (13) and (14) is:

$$G_{\alpha\beta}(x) = \sum_{k=1}^8 A_{\alpha\beta}^{(k)}(z_1, \dots, z_9) \exp(\theta_k x), \quad \dots \quad (15)$$

where $\theta_1, \dots, \theta_8$ are the roots of the determinantal equation:

$$|A_\theta(z_1, \dots, z_9)| = 0, \quad \dots \quad (16)$$

a fourth degree equation in θ^2 . The coefficients $A_{\alpha\beta}^{(k)}(z_1, \dots, z_9)$ are completely determined by eqn. (13) and the boundary conditions but the explicit solution for $G_{\alpha\beta}(x)$ in terms of $z_1, \dots, z_9$ is very cumbersome. For most purposes it suffices to determine the moments of the corresponding joint distribution and this can easily be done numerically.

The emergent angular distribution for given x and β is given by $P_{\alpha\beta}(x)$ which is obtained from $G_{\alpha\beta}(x)$ on letting $z_1, \dots, z_9$ all tend to 1. Alternatively $P_{\alpha\beta}(x)$ can be found directly from eqns. (13) and (14) on setting $z_1 = \dots = z_9 = 1$ throughout. The solution is found to be of the form:

$$P_{\alpha\beta}(x) = A_{\alpha\beta}^{(1)} + A_{\alpha\beta}^{(2)}x + \sum_{k=3}^8 A_{\alpha\beta}^{(k)} \exp(\theta_k x), \quad \dots \quad (17)$$

the first two terms arising from the double root $\theta = 0$ of the equation:

$$|A_\theta(1, \dots, 1)| = 0, \quad \dots \quad (18)$$

and the remaining terms from the other six roots $\theta_3, \dots, \theta_8$. The constants $A_{\alpha\beta}^{(k)}$ are determined uniquely by (13) and the boundary conditions:

$$\left. \begin{aligned} P_{\alpha\beta}(a) &= \delta_{\alpha\beta}, & \beta &= 7, 8, 9, 10, \\ P_{\alpha\beta}(b) &= \delta_{\alpha\beta}, & \beta &= 1, 2, 3, 4. \end{aligned} \right\} \quad \dots \quad (19)$$

Numerical solutions of these equations will be given in § 3.

Now we consider the spectral distribution of the emergent photons. Let $\Phi_{\alpha\beta}(\lambda | x)$ denote the distribution function of the increase in wavelength due to multiple scattering of a photon which starts from x with direction β and emerges from the slab with direction α . The Mellin-Stieltjes transform

$$\Phi_{\alpha\beta}(s | x) = \int_0^\infty \lambda^{s-1} d\Phi_{\alpha\beta}(\lambda | x) \quad \dots \quad (20)$$

is clearly given by:

$$\Phi_{\alpha\beta}(s | x) = [P_{\alpha\beta}(x)]^{-1} K_{\alpha\beta}(s, x), \quad \dots \quad (21)$$

where

$$K_{\alpha\beta}(s, x) = \sum_{i_1 + \dots + i_9 \geq 0} P_{\alpha\beta}^{i_1, \dots, i_9}(x) (i_1 \delta_1 + \dots + i_9 \delta_9)^{s-1}$$

and $\delta_1, \dots, \delta_9$ are defined by eqn. (9). If n is a positive integer then $\Phi_{\alpha\beta}(n+1 | x)$ is the n th moment about the origin of the wavelength

increment λ . These moments can be determined by expressing $K_{\alpha\beta}(n+1, x)$ in terms of derivatives of $G_{\alpha\beta}(x)$ at $z_1 = \dots = z_9 = 1$. Thus, for example:

$$K_{\alpha\beta}(2, x) = \left\{ \sum_{p=1}^9 \delta_p \frac{\partial}{\partial z_p} G_{\alpha\beta}(x) \right\}_{z_1 = \dots = z_9 = 1}$$

and

$$K_{\alpha\beta}(3, x) = \left\{ \sum_{p=1}^9 \sum_{q=1}^9 \delta_p \delta_q \frac{\partial^2}{\partial z_p \partial z_q} G_{\alpha\beta}(x) + \sum_{p=1}^9 \delta_p^2 \frac{\partial^2}{\partial z_p^2} G_{\alpha\beta}(x) \right\}_{z_1 = \dots = z_9 = 1}.$$

It follows from eqn. (13) that $K_{\alpha\beta}(2, x)$ satisfies the equation:

$$A_{\theta}(1, \dots, 1) \mathbf{K}_{\alpha}(2, x) + B \mathbf{P}_{\alpha}(x) = \mathbf{0}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where B is the matrix:

$$B = \left\{ \sum_{p=1}^9 \delta_p \frac{\partial A_{\theta}}{\partial z_p} \right\}_{z_1 = \dots = z_9 = 1}$$

and the boundary conditions for $\mathbf{K}_{\alpha}(2, x)$ are:

$$\left. \begin{aligned} K_{\alpha\beta}(2, b) &= 0, & \beta &= 1, 2, 3, 4, \\ K_{\alpha\beta}(2, a) &= 0, & \beta &= 7, 8, 9, 10, \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Further differentiation of eqn. (13) gives equations for $K_{\alpha\beta}(n, x)$, $n > 2$ which, together with the boundary conditions

$$\left. \begin{aligned} K_{\alpha\beta}(n, b) &= 0, & \beta &= 1, 2, 3, 4, \\ K_{\alpha\beta}(n, a) &= 0, & \beta &= 7, 8, 9, 10, \end{aligned} \right\}$$

completely determine the higher order moments $\Phi_{\alpha\beta}(n | x)$, $n > 2$.

In §4 we shall determine numerically the first two moments of the wavelength increase for radiation transmitted through slabs of various thicknesses. These moments will then be used to estimate the spectral distribution of the transmitted radiation.

§ 3. THE ANGULAR DISTRIBUTION OF EMERGENT RADIATION

As already described in §2, the distribution of emergent direction for a photon which starts from a point x ($a \leq x \leq b$) moving in direction β is found from the equations:

$$A_{\theta}(1, \dots, 1) \mathbf{P}_{\alpha}(x) = 0, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

with boundary conditions (19).

It is convenient to write the solution (17) of these equations in the form:

$$\mathbf{P}_{\alpha}(x) = \mathbf{C}_{\alpha} + \mathbf{B}_{\alpha}^{(1)} + \mathbf{B}_{\alpha}^{(2)}x + \sum_{k=3}^8 \mathbf{B}_{\alpha}^{(k)} \exp(\theta_k x), \quad . \quad . \quad . \quad (25)$$

where $\theta_3, \dots, \theta_8$ ($\theta_3 < \theta_4 < \dots < \theta_8$) are the non-zero roots of (18), $\theta_1 = \theta_2 = 0$ and $\mathbf{C}_{\alpha}$ is the column vector whose i th component is $\mu_i B_{\alpha i}^{(2)}$, $i = 1, \dots, 10$. From eqn. (24) it then follows that the vectors $\mathbf{B}_{\alpha}^{(k)}$ satisfy the relations:

$$A_{\theta_k}(1, \dots, 1) \mathbf{B}_{\alpha}^{(k)} = \mathbf{0}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For each value of k eqn. (26) can be used to determine the constants $\psi_\beta^{(k)}$, such that

$$\psi_\beta^{(k)} = B_{\alpha\beta}^{(k)} [B_{\alpha 5}^{(k)}]^{-1}, \quad \left. \begin{array}{l} \beta = 1, \dots, 10, \\ k = 1, \dots, 8, \end{array} \right\} \dots \dots \dots (27)$$

and thus to express $B_{\alpha\beta}^{(k)}$ in terms of $B_{\alpha 5}^{(k)}$ for every k and β . From the boundary conditions (19) it then follows that the matrix whose i th row is:

$$[B_{15}^{(i)} B_{25}^{(i)} B_{35}^{(i)} B_{45}^{(i)} B_{75}^{(i)} B_{85}^{(i)} B_{95}^{(i)} B_{105}^{(i)}], \quad i = 1, \dots, 8,$$

is the inverse of the matrix whose i th row is:

$$\begin{aligned} & [\psi_i^{(1)} \psi_i^{(2)} \cdot (\mu_i + l) \psi_i^{(3)} \cdot \exp(\theta_3 l) \psi_i^{(4)} \cdot \exp(\theta_4 l) \dots \psi_i^{(8)} \cdot \exp(\theta_8 l)], \quad i = 1, \dots, 4, \\ & [\psi_{i+2}^{(1)} \psi_{i+2}^{(2)} \mu_{i+2} \psi_{i+2}^{(3)} \quad \psi_{i+2}^{(4)} \quad \dots \psi_{i+2}^{(8)}], \quad i = 5, \dots, 8. \end{aligned}$$

We have supposed here that $a = 0$ and $b = l$, i.e. l is the thickness of the slab. Having determined the constants $B_{\alpha 5}^{(i)}$ the probabilities $P_{\alpha\beta}(x)$ are given explicitly by eqns. (25) and (27).

For scattering in the semi-infinite slab $\{x: 0 \leq x < \infty\}$ the eqns. (24) to (27) are still valid but the modified boundary conditions (i.e. that $P_{\alpha\beta}(x)$ be a bounded function of x as $x \rightarrow \infty$) require that

$$\mathbf{C}_\alpha = \mathbf{B}_\alpha^{(2)} = \mathbf{B}_\alpha^{(6)} = \mathbf{B}_\alpha^{(7)} = \mathbf{B}_\alpha^{(8)} = \mathbf{0}. \quad \dots \dots \dots (28)$$

The boundary conditions in this case lead to the result:

$$\begin{bmatrix} B_{75}^{(1)} & B_{85}^{(1)} & B_{95}^{(1)} & B_{105}^{(1)} \\ B_{75}^{(3)} & B_{85}^{(3)} & B_{95}^{(3)} & B_{105}^{(3)} \\ B_{75}^{(4)} & B_{85}^{(4)} & B_{95}^{(4)} & B_{105}^{(4)} \\ B_{75}^{(5)} & B_{85}^{(5)} & B_{95}^{(5)} & B_{105}^{(5)} \end{bmatrix} = \begin{bmatrix} \psi_7^{(1)} & \psi_7^{(3)} & \psi_7^{(4)} & \psi_7^{(5)} \\ \psi_8^{(1)} & \psi_8^{(3)} & \psi_8^{(4)} & \psi_8^{(5)} \\ \psi_9^{(1)} & \psi_9^{(3)} & \psi_9^{(4)} & \psi_9^{(5)} \\ \psi_{10}^{(1)} & \psi_{10}^{(3)} & \psi_{10}^{(4)} & \psi_{10}^{(5)} \end{bmatrix}^{-1}, \quad (29)$$

where the constants $\psi_\beta^{(k)}$ are defined exactly as before. The probabilities $P_{\alpha\beta}(x)$ are again found explicitly from eqns. (25) and (27).

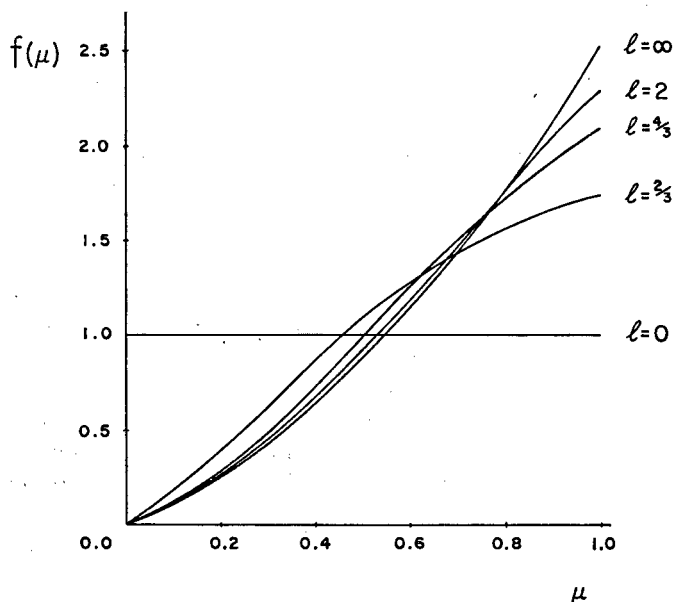
In order to relate the discrete angular distributions obtained in this way to the continuous distributions which would be obtained from the exact differential cross section (1) we proceed as in Brockwell (1954) supposing the discrete probabilities to be distributed over the appropriate regions of the unit sphere. Thus if P_1, P_2, P_3, P_4 are the probabilities of emergence in directions 1, 2, 3, 4 for the discrete model and $f(\mu)$ is the probability density (normalized so that $\int_0^1 f(\mu) d\mu = 1$) of the cosine of the angle between the emergent direction and the X -axis in the continuous model, then $f(\mu_i)$, $i = 1, 2, 3, 4, 5$ is estimated by:

$$\left. \begin{aligned} f(\mu_1) &= \frac{15P_1}{P_1 + P_2 + P_3 + P_4}, \\ f(\mu_i) &= \frac{15P_i}{4(P_1 + P_2 + P_3 + P_4)}, \quad i = 2, 3, 4, \\ f(\mu_5) &= 0. \end{aligned} \right\} \dots \dots \dots (30)$$

Numerical calculations of $f(\mu)$ were carried out for the photons transmitted through a slab of given thickness when the photons are incident isotropically on one face of the slab. The results are shown in fig. 1 for several values of slab thickness.

The proportions of incident photons transmitted through the slab for thicknesses $l = \frac{2}{3}, \frac{4}{3}, 2$ were found to be 0.5715, 0.4232 and 0.3385 respectively.

Fig. 1



The angular distribution of emergent photons transmitted through a slab of thickness l .

The accuracy of this procedure is indicated in the following table which shows the emergent angular distribution ($f(\mu)$) in the special case of photons originating at a large distance below the surface of a half-space. The exact solution as given by Chandrasekhar (1960) is compared with the icosahedral approximation, Chandrasekhar's Gaussian quadrature approximation (1944), and the approximation of Wilson and Sen (1964).

Table 1

μ	Exact solution	Icosahedral approximation	Chandrasekhar's method		Wilson and Sen $n=1$
			$n=2$	$n=3$	
1.0000	2.5292	2.5287	2.4962	2.5152	2.5376
0.8090	1.8036	1.8042	1.7748	1.7910	1.8181
0.5000	0.8689	0.8706	0.8488	0.8593	0.8812
0.3090	0.4404	0.4430	0.4273	0.4334	0.4468
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

§ 4. THE SPECTRAL DISTRIBUTION OF TRANSMITTED RADIATION

We again consider a monochromatic stream of photons incident isotropically on the face $x=0$ of the slab $\{\mathbf{x}: 0 \leq x \leq l\}$. The spectral intensity (rate of incidence of photons per unit wavelength interval) of the incident radiation is given by:

$$I(\lambda) = \delta(\lambda - \lambda_0), \quad (31)$$

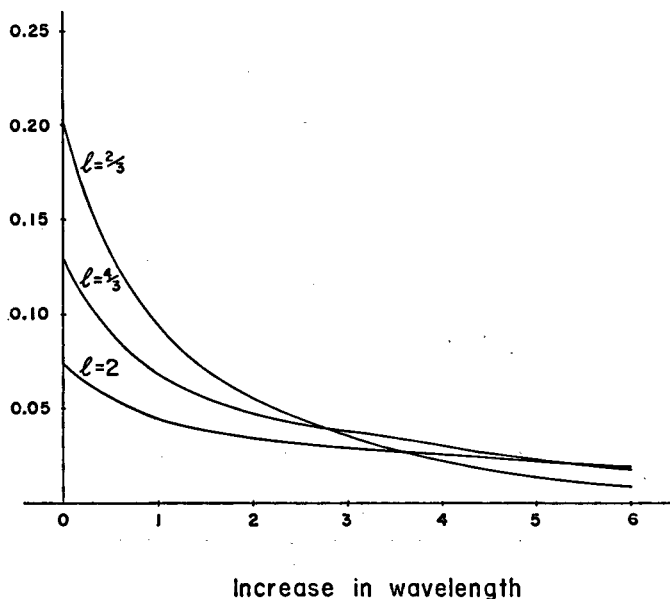
where $\delta(x)$ is the Dirac delta function and λ_0 is the incident wavelength. It is the purpose of this section to examine the spectral intensity of the radiation emerging from the face $x=l$.

Suppose $\eta(\lambda)$ is the spectral intensity corresponding to an emergent wavelength $(\lambda + \lambda_0)$, i.e. the rate of emergence of photons with wavelength in the small interval $(\lambda_0 + \lambda, \lambda_0 + \lambda + \delta\lambda)$ is $\eta(\lambda)\delta\lambda + o(\delta\lambda)$. Then $\eta(\lambda)$ is clearly independent of λ_0 and can be written:

$$\eta(\lambda) = \pi_0 \delta(\lambda) + \gamma(\lambda), \quad 0 \leq \lambda < \infty, \quad (32)$$

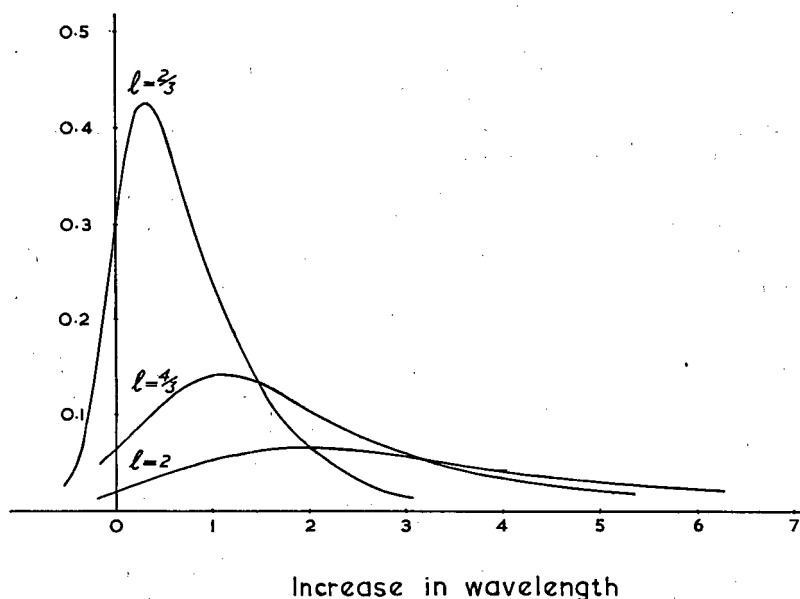
where π_0 is the probability that an incident photon is transmitted through the slab with no collisions and $\gamma(\lambda)$ is the intensity of the emergent radiation which is scattered at least once in its passage through the slab. It is worth noticing at this point that the spectral intensities found by Chandrasekhar and O'Rourke (fig. 3) are not of the form (32); in fact they differ in two respects, first in the complete absence of a delta function component and

Fig. 2



The spectral intensity of scattered radiation transmitted through a slab of thickness l . In addition the intensity of the transmitted radiation will have a component $\pi_0 \delta(\lambda)$ due to unscattered photons. The wavelength increase is measured in Compton units.

Fig. 3



O'Rourke's curves showing the spectral intensity of emergent radiation measured in units of the integrated incident intensity. The wavelength increments are again measured in Compton units.

secondly in the existence of a non-zero probability of a negative wavelength increment which for Compton scattering is physically impossible. As Chandrasekhar points out, his results are physically significant only when integrated with respect to a smooth spectral distribution of incident radiation.

Making use of the icosahedral approximation described in the preceding paragraphs it is possible, as we shall now show, to determine the moments of the distribution of emergent wavelength and to fit an emergent spectral distribution of the correct form as given by eqn. (32). We first evaluate π_0 and $\gamma(0)$ exactly and then use the icosahedral approximation as described in § 2 to determine the first two moments of the increase in wavelength of the transmitted radiation.

If μ is the cosine of the angle made by an incident photon with the positive X -axis, then for isotropically incident radiation μ is a random variable uniformly distributed on $[0, 1]$. The probability π_0 is therefore given by:

$$\pi_0 = \int_0^1 \exp(-l\mu^{-1}) d\mu. \quad \dots \dots \dots (33)$$

For a single collision the probability density $f(\lambda)$ corresponding to an increase in wavelength λ is readily found from eqns. (1) and (3) to be:

$$f(\lambda) = \frac{3}{8}(2 - 2\lambda + \lambda^2), \quad 0 \leq \lambda \leq 2. \quad \dots \dots \dots (34)$$

The only contribution to the intensity $\gamma(0)$ comes from photons which are scattered exactly once. Hence:

$$\gamma(0) = \frac{3}{4} \int_0^1 l\mu^{-1} \exp(-l\mu^{-1}) d\mu. \quad (35)$$

For any given slab thickness, π_0 and $\gamma(0)$ are easily found from eqns. (33) and (35) using tables of the exponential integral.

We now return to the icosahedral model of §2 in order to discuss the moments of the emergent spectral distribution. For radiation incident isotropically on the face $x=0$, the direction of motion of an incident photon is a random variable β which takes the values 1, 2, 3, 4, 5, 6 with associated probabilities $r_1=1/15$, $r_2=4/15$, $r_3=4/15$, $r_4=4/15$ and $r_5=r_6=1/15$. The mean and mean square of the increase in wavelength of the transmitted radiation are given by:

$$\mathcal{E}(\lambda) = \frac{1}{\pi} \left[\sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} K_{\alpha\beta}(2, 0) \right], \quad (36)$$

$$\mathcal{E}(\lambda^2) = \frac{1}{\pi} \left[\sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} K_{\alpha\beta}(3, 0) \right], \quad (37)$$

where $\pi = \sum_{\alpha=1}^4 \sum_{\beta=1}^6 r_{\beta} P_{\alpha\beta}(0)$ is the probability that an incident photon is transmitted through the slab and the functions $K_{\alpha\beta}(2, x)$ and $K_{\alpha\beta}(3, x)$ are found as described in §2.

Expressing $\mathcal{E}(\lambda)$ and $\mathcal{E}(\lambda^2)$ in terms of the spectral intensity $\eta(\lambda)$ we have:

$$\mathcal{E}(\lambda) = \frac{1}{\pi} \int_0^{\infty} \lambda \eta(\lambda) d\lambda = \frac{1}{\pi} \int_0^{\infty} \lambda \gamma(\lambda) d\lambda, \quad (38)$$

$$\mathcal{E}(\lambda^2) = \frac{1}{\pi} \int_0^{\infty} \lambda^2 \eta(\lambda) d\lambda = \frac{1}{\pi} \int_0^{\infty} \lambda^2 \gamma(\lambda) d\lambda. \quad (39)$$

There are two further conditions to be satisfied by $\gamma(\lambda)$, namely:

$$\int_0^{\infty} \gamma(\lambda) d\lambda = \pi - \pi_0 \quad (40)$$

and

$$\gamma(0) = \frac{3}{4} \int_0^1 l\mu^{-1} \exp(-l\mu^{-1}) d\mu. \quad (41)$$

Table 2

l	2/3	4/3	2
π_0	0.2478	0.09204	0.03753
$\gamma(0)$	0.1992	0.1287	0.07335
π	0.5715	0.4232	0.3385
$\mathcal{E}(\lambda)$	1.0794	2.6124	4.6021
$\mathcal{E}(\lambda^2)$	4.1550	16.898	44.233
α	0.1992	0.1287	0.07335
α	1.2931	1.1269	0.8790
b	0.07886	0.04053	0.01924
β	0.6817	0.4322	0.2974

The results of numerical calculations of π_0 , $\gamma(0)$, π , $\mathcal{E}(\lambda)$ and $\mathcal{E}(\lambda^2)$ are shown in table 2 for various slab thicknesses. The parameters a , b , α , β were obtained by fitting a curve of the form:

$$\gamma^*(\lambda) = a \exp(-\alpha\lambda) + b\lambda \exp(-\beta\lambda) \quad . \quad . \quad . \quad (42)$$

to the scattered component $\gamma(\lambda)$ of the transmitted spectral intensity; the form of the second term on the right-hand side of (42) was chosen to give a sufficiently slow rate of decay with increasing λ and the parameters a , α , b and β were determined from the four conditions (38), (39), (40) and (41).

Figure 2 shows the graphs of the fitted functions $\gamma^*(\lambda)$. The probability density $\zeta(\lambda)$ of the increase in wavelength of scattered radiation (normalized so that $\int_0^\infty \zeta(\lambda) d\lambda = 1$) is related to $\gamma(\lambda)$ by the equation:

$$\zeta(\lambda) = (\pi - \pi_0)^{-1} \gamma(\lambda) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and is approximated by $\zeta^*(\lambda) = (\pi - \pi_0)^{-1} \gamma^*(\lambda)$.

We remark finally that the assumption of an isotropically incident stream of radiation is not necessary for the analysis. Any incident angular distribution which can be reasonably well approximated by a distribution on the set of 30 directions in space which we have used can be dealt with in the same way. Furthermore, although we have here considered only the overall emergent spectral distribution for transmitted radiation the same method can be used to determine the spectral distribution for particular directions of emergence.

§ 5. A SERIES SOLUTION FOR THIN SLABS

The fundamental eqns. (13) lead directly to a series solution for the generating functions $G_{\alpha\beta}(x)$ which is useful when dealing with thin slabs. As distinct from the usual thin slab approximations (which are based on the assumption of not more than some small number of collisions) the present method expresses $G_{\alpha\beta}(x)$ as a power series in x which can be truncated when the slab thickness l is sufficiently small.

It is apparent from eqns. (13) that $G_{\alpha 5}(x)$ and $G_{\alpha 6}(x)$ can be expressed as linear combinations of $G_{\alpha 1}(x), \dots, G_{\alpha 4}(x), G_{\alpha 7}(x), \dots, G_{\alpha 10}(x)$; the equations can then be re-written in the form:

$$\frac{\partial}{\partial x} \mathbf{g}_\alpha(x) = R(z_1, \dots, z_9) \mathbf{g}_\alpha(x), \quad . \quad . \quad . \quad . \quad (44)$$

where $R(z_1, \dots, z_9)$, is an easily determined 8×8 matrix and

$$\mathbf{g}_\alpha(x) = \begin{bmatrix} G_{\alpha 1}(x) \\ \vdots \\ G_{\alpha 4}(x) \\ G_{\alpha 7}(x) \\ \vdots \\ G_{\alpha 10}(x) \end{bmatrix}.$$

The solution of this set of differential equations is:

$$\mathbf{g}_\alpha(x) = \exp(xR) \mathbf{g}_\alpha(0), \quad (45)$$

where $\mathbf{g}_\alpha(0)$ is determined by the boundary conditions:

$$G_{\alpha\beta}(0) = \delta_{\alpha\beta}, \quad \beta = 7, 8, 9, 10, \quad (46)$$

$$G_{\alpha\beta}(l) = \delta_{\alpha\beta}, \quad \beta = 1, 2, 3, 4, \quad (47)$$

In principle this result completely solves the scattering problem for the icosahedral approximation, however, the evaluation of the components of the matrix $\exp(xR)$ is extremely difficult for arbitrary $z_1, \dots, z_9$. Now the power series expansion

$$\exp(xR) = \sum_{k=0}^{\infty} \frac{(xR)^k}{k!} \quad (48)$$

is convergent for all x and provided x is sufficiently small the exponential function can be replaced by the first few terms of the series. Hence if the slab thickness l is small (implying that x is small since $0 \leq x \leq l$) an approximation to $\mathbf{g}_\alpha(x)$ is given by:

$$\mathbf{g}_\alpha(x) = \left[\sum_{k=0}^n \frac{(xR)^k}{k!} \right] \mathbf{g}_\alpha(0), \quad 0 \leq x \leq l, \quad (49)$$

and $\mathbf{g}_\alpha(0)$ can be determined from the boundary conditions (46) and (47).

In particular this method reduces the determination of the transmission probabilities $P_{\alpha\beta}(0)$, $\beta = 1, 2, \dots, 6$, to the numerical inversion of a single matrix.

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