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# Transmitter Power Estimation for Uncooperative Emitters with the Cayley-Menger Determinant

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*Abstract*— **The problem of estimating the transmission power levels of an uncooperative radio transmitter is examined in this short paper. The Cayley-Menger matrix is shown to provide an underlying geometrical constraint on the possible solutions and is subsequently used in a novel fashion as the basis for an estimator of the transmission power.**

## I. The Received Signal Strength Model

For an electromagnetic transmitter with unity directional gain, wavelength $\lambda$ and transmission power $\mathrm{P}_{\mathrm{TX}}$, the power density of the signal at a distance $r$ is proportional to

$$\mathrm{P}_{\mathrm{RX}} \propto \frac{\mathrm{P}_{\mathrm{TX}}}{(r/\lambda)^2}$$

A more general inverse power law that accounts for the effects of shadowing and environmental conditions is typically required in practice. At a *reference* distance $d_0$ from the transmitter, the signal power is given by $\mathrm{P}_{\mathrm{T}}$. This is the transmission power for the practical purposes of this paper. Then the received signal strength L at a distance $d$ from the transmitter reference $d_0$ is proportional to

$$\mathrm{L} \propto 10^{\zeta/10} \frac{\mathrm{P}_{\mathrm{T}}}{(d/\lambda)^{\xi}}$$

where $\zeta$ is a statistical shadowing component and $\xi$ is the path-loss exponent. The shadowing component $\zeta$ has been shown to obey a zero-mean Gaussian distribution and is caused by local obstacles surrounding the receiver [1], [2], [3]. Generally, it is independent of the transmission distance. Note that it is well known that the received signal strength is highly variable, e.g. $\zeta$ may have a large variance.

In many applications of localization and electronic warfare it is desirable to know the transmission power $\mathrm{P}_{\mathrm{T}}$ of an uncooperative target emitter. For example, in order to employ the common received signal strength model in emitter range-finding (and subsequent range-based localization) it is important to know the emitter's transmission power [4], [5]. That is, localization algorithms that use received signal strength based range-finding require knowledge of the emitter's transmission power. However, they have the advantage of using only relatively cheap sensor technology.

Furthermore, if we can estimate the transmitter power of an active radar then we can essentially upper-bound the radar's operational range. This may also permit more efficient electronic counter-measures and optimal electronic jamming.

The contribution of this paper is a novel algorithm based on geometrical constraints that can be used to estimate the transmission power $\mathrm{P}_{\mathrm{T}}$ of an uncooperative target emitter from the signals received at multiple receivers. An uncooperative emitter can be thought of as any wireless transmitter that does not actively broadcast its transmission power.

Geometric constraint based estimation has been used in a number of papers [6], [7], [8], [9] for localization with noisy measurements. The work in [6] uses the Cayley-Menger determinant for range-only localization. The work in [7], [8], [9] covers bearing-only, time-of-arrival-based and hybrid positioning scenarios using various geometrically motivated constraints. In [10], a geometrically derived constraint is used to estimate the path-loss exponent of a wireless transmission channel based on the received signal strength measurements at a number of receivers.

The rest of this paper is organized as follows. In Section II we introduce the notion of a geometrical constraint on the inter-vertex distances of an Euclidean simplex. Specifically, in Section II we introduce the Cayley-Menger determinant. In Section III we use the Cayley-Menger determinant to derive a closed-form algorithm for estimating the transmission power of a wireless emitter from the received signal strength measurements taken at multiple receivers. In Section IV we illustrate the performance of the algorithm through simulation and in Section V we give a conclusion.

## II. Geometric Constraints

Let us define the Cayley-Menger matrix of a single $n$-tuple of points $p_0, \cdots, p_{n-1}$ in $m$-dimensional space as

$$\mathbf{M}(p_0, \cdots, p_{n-1}) \triangleq$$

$$\begin{bmatrix} 0 & d^2(p_0,p_1) & \cdots & d^2(p_0,p_{n-1}) & 1 \\ d^2(p_1,p_0) & 0 & \cdots & d^2(p_1,p_{n-1}) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d^2(p_{n-1},p_0) & d^2(p_{n-1},p_1) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

where $d^2(p_i, p_j)$, for $i \neq j$ and $i, j \in \{0, \cdots, n-1\}$ is the Euclidean distance squared between the points $p_i$ and $p_j$.

Let $\mathrm{V}(p_0, \cdots, p_{n-1})$ denote the volume of a Euclidean simplex $\{p_0, p_1, \cdots, p_{n-1}\}$, then the following relation can be given in terms of the determinant of the Cayley-Menger matrix [11], [12]

$$\mathrm{V}^2(p_0, p_1, \ldots, p_{n-1}) = \frac{(-1)^n}{2^{n-1}((n-1)!)^2} \det |\mathbf{M}|$$

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where $\det|\mathbf{M}| = \det|\mathbf{M}(p_0, p_1, \cdots, p_{n-1})|$ is the determinant of the Cayley-Menger matrix. A simplex of the $n$-tuple of points is defined here to be the smallest $(n-1)$-dimensional convex hull enclosing the $n$-tuple of points. The following theorem is introduced in a number of places.

**Theorem 1** ([11], [12], [13])**.** *Consider an $n$-tuple of points $p_0, \cdots, p_{n-1}$ in $m$-dimensional space. If $n \geq m+2$, then the Cayley-Menger matrix is singular, namely*

$$\det|\mathbf{M}(p_0, \ldots, p_{n-1})| = 0$$

*where $\mathbf{M}(p_0, \cdots, p_{n-1})$ is the Cayley-Menger matrix.*

*Proof:* [Sketch] The proof is based on understanding the volume relationship given by

$$\mathrm{V}^2(p_0, p_1, \ldots, p_{n-1}) = \frac{(-1)^n}{2^{n-1}((n-1)!)^2} \det|\mathbf{M}| \tag{1}$$

of a Euclidean simplex of the $n$-tuple of points in $m$-dimensional space. When $n \geq m+2$ then $\mathrm{V} = 0$ since there cannot exist a $(n-1)$-dimensional convex manifold with non-zero volume in $(n-k)$-dimensional space where $k = n - m$ and $k \geq 2$. Moreover, it is clear that (1) only vanishes when $\det|\mathbf{M}| = 0$ holds. □

That is, considering an $n$-tuple of points in $m$-dimensional space with $n \geq m+2$, a geometrical constraint exists between the inter-point distances that must be satisfied. In [12] a more technical result is given.

**Theorem 2.** *Consider an $n$-tuple of points $p_0, \cdots, p_{n-1}$ in $m$-dimensional space. If $n \geq m+1$, then the rank of the Cayley-Menger matrix is at most $m+1$.*

Consider a single transmitter denoted by $p_0$ and a number of receivers, $p_1, p_2, \cdots, p_n$, measuring the received signal strength (RSS). We have a set of measurements $L_i$, for $i \in \{1, \cdots, n\}$. Furthermore, assume we know the distances between the receiving stations $d_{ij} = d_{ji}$ for $i \neq j$ and $i, j \in \{1, \cdots, n\}$. This assumption may be particularly valid for cellular phone and electronic warfare systems. Finally, we assume that all receivers are in general position relative to the emitter.

Suppose the measurement $\mathrm{L}_i$ at receiver $i$ is noiseless and given by $\mathrm{L}_i = \frac{\mathrm{P_T}}{(d_{0i}/\lambda)^\xi}$. Then normalizing with respect to the wavelength $\lambda$ it follows that

$$d_{0i}^2 = \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_i^{2/\xi}} \tag{2}$$

We can then derive $n-2$ independent equations (which the geometry tell us must be satisfied) on the $d_{0i}$ in (2) in the form

$$\det\begin{vmatrix} 0 & \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_1^{2/\xi}} & \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_2^{2/\xi}} & \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_k^{2/\xi}} & 1 \\ \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_1^{2/\xi}} & 0 & d_{12}^2 & d_{1k}^2 & 1 \\ \frac{\mathrm{P}_T^{2/\xi}}{\mathrm{L}_2^{2/\xi}} & d_{12}^2 & 0 & d_{2k}^2 & 1 \\ \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_k^{2/\xi}} & d_{1k}^2 & d_{2k}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

where $k = 3, \ldots, n$.

## III. Transmission Power Estimation

Consider the problem of estimating the transmitter power level $\mathrm{P_T}$ of an uncooperative target emitter given only the noisy received signal strength measurements and an $a\ priori$ estimate of $\xi$. We assume that it may be possible to estimate $\xi$ for the general environment in question at some $a\ priori$ time. Indeed, the Cayley-Menger determinant has previously been used, e.g. see [10], to estimate $\xi$. The approach in [10] could easily be incorporated into the proposed algorithm of this paper. We assume $\xi$ is constant.

Assume $n \geq 4$ receivers measure the RSS of a single emitter and the inter-receiver distances are known. The path-loss exponent is known from an $a\ priori$ estimation. Then the transmission power level of the single transmitter can be estimated by finding an *optimal* solution to the following system of equations

$$\det\begin{vmatrix} 0 & \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_1^{2/\xi}} & \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_2^{2/\xi}} & \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_k^{2/\xi}} & 1 \\ \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_1^{2/\xi}} & 0 & d_{12}^2 & d_{1k}^2 & 1 \\ \frac{\mathrm{P}_T^{2/\xi}}{\mathrm{L}_2^{2/\xi}} & d_{12}^2 & 0 & d_{2k}^2 & 1 \\ \frac{\mathrm{P_T}^{2/\xi}}{\mathrm{L}_k^{2/\xi}} & d_{1k}^2 & d_{2k}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0 \tag{3}$$

where $k = 3, \ldots, n$. Letting $x = \mathrm{P_T}^{2/\xi}$ we determine that (3) results in $n-2$ quadratic equations of the form

$$a_j x^2 + b_j x + c_j = 0 \tag{4}$$

where $j \in \{1, \ldots, n-2\}$ such that $j = k-2$. The system of quadratic equations (4) is overdetermined and the equations are independent. In a noiseless environment there will exist a unique and consistent solution. However, the measurement set $\mathrm{L}_i$, for $i \in \{1, \ldots, n\}$ is inconsistent and noisy and thus a consistent solution is unlikely to exist.

A closed-form linear least squares solution is now outlined in order to solve the system of quadratics. The system of quadratics (4) can be given as

$$x^2 + b_j^* x + c_j^* = 0$$

where $b_j^* = b_j/a_j$, $c_j^* = c_j/a_j$, $x = \mathrm{P_T}^{2/\xi}$ and $j \in \{1, \ldots, n-2\}$. Subtracting the quadratic indexed by $j = 1$ from the quadratics indexed by $j = 2, \ldots, n-2$ results in $n-3$ linear equations in the unknown $x = \mathrm{P_T}^{2/\xi}$ given in the form of

$$(b_j^* - b_1^*)x + (c_j^* - c_1^*) = 0$$

where $j \in \{2, \ldots, n-2\}$. Stacking the equations results in the following linear system

$$\begin{bmatrix} (b_2^* - b_1^*) \\ \vdots \\ (b_{n-2}^* - b_1^*) \end{bmatrix} x = \begin{bmatrix} (c_1^* - c_2^*) \\ \vdots \\ (c_1^* - c_{n-2}^*) \end{bmatrix}$$

which can be written succinctly as $\mathbf{b}x = \mathbf{c}$. When $n = 4$, the number of linear equations equals the number of unknowns and the solution is simply given by $x = (b_2^* - b_1^*)^{-1}(c_1^* - c_2^*)$. When $n > 4$, the approach most often taken is to minimize the norm $\|\mathbf{b}x - \mathbf{c}\|^2$. This is a linear least squares problem for which a closed form solution does exist and is given by

$$x = (\mathbf{b}^T\mathbf{b})^{-1}\mathbf{b}^T\mathbf{c} \tag{5}$$

where $(\mathbf{b}^T\mathbf{b})^{-1}\mathbf{b}^T$ is the pseudo-inverse of $\mathbf{b}$. It may be possible to improve the performance of the linear least squares approach by realizing that in general $\mathbf{b}x - \mathbf{c} = \mathbf{e}$ where $\mathbf{e}$ is a so-called artificial equation error. If one were to know the covariance matrix for $\mathbf{e}$ then it would be possible to minimize the more accurate norm $\|\mathbf{b}x - \mathbf{c}\|^2_{\mathbf{W}}$ where the weighting matrix $\mathbf{W}$ is the covariance of $\mathbf{e}$. Unfortunately, $\mathbf{W}$ is not simple to derive and for the application considered in this paper $\mathbf{W}$ depends on the true parameter being estimated.

## IV. Simulation Analysis

A numerical analysis is conducted in this section based on computer simulations. The root mean squared error (RMSE) over 100000 simulation runs is calculated and we plot the RMSE as a percentage of the true value,

$$\%RMSE = \frac{100\%}{True}\sqrt{\frac{1}{M}\sum_{i=1}^{M}(Estimate - True)^2}$$

where $M$ is the number of simulation runs. The target is fixed at $(25, 25)$ units while the $n$ sensors (i.e. receivers) are distributed evenly on a $n$-polygon centered on the target with a radius of approximately 24 units. The sensor indices are randomly assigned among the $n$-polygon nodes at each simulation run and according to a uniform distribution. Such a geometry may be typical in a number of electronic warfare systems where the target emitter's location is known (at least roughly) and the sensors move to surround the target.

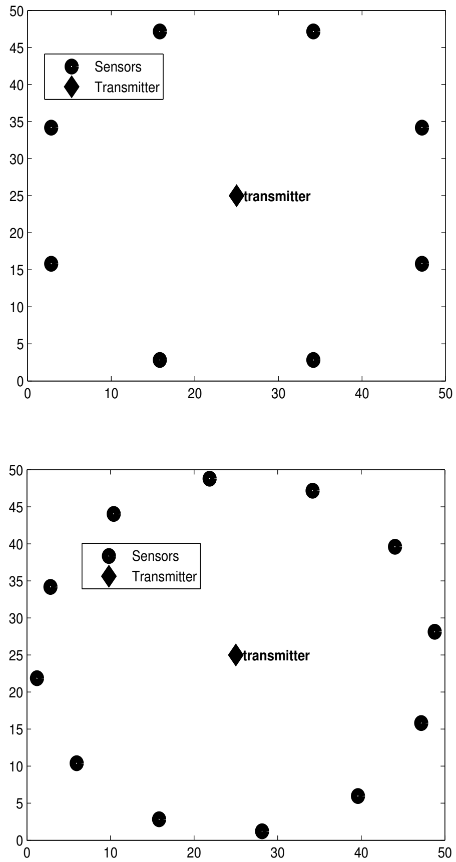
Knowing the emitter's transmission power is paramount in determining it's operational range and in minimum power radio jamming. Moreover, knowledge of an emitter's transmission power is required for RSS-based localization [4].

The typical scene is depicted in Figure 1 (a) for a single target and eight sensors and in Figure 1 (b) for a single target and twelve sensors.

In an electronic warfare scenario it may be quite reasonable to assume a greater number of receivers (sensors) are available for detecting the received signal strength since this parameter does not necessarily require costly sensor technology. Each sensor takes a single measurement with the measurement at the $i^{th}$ receiver given by

$$\mathrm{L}_i = \frac{\mathrm{P_T}}{d_{0i}^{\xi}}v + e$$

where $v = 10^{\zeta/10}$ and $e$ are noise inputs. The shadowing component and additive noise obey $\zeta \sim \mathcal{N}(0, \sigma_\zeta^2)$ and $e \sim \mathcal{N}(0, \sigma_e^2)$. The true power is $\mathrm{P_T} = 1$ for simplicity and the path-loss exponent is varied from $\xi = 2$ to $\xi = 4$.




(a).

(b).

Fig. 1. The transmitter and receiver geometry for (a). eight receivers and (b). twelve receivers. During each run the sensor indexes are randomly distributed among the $n$-polygon nodes according to a uniform distribution.

*1) Simulation Case 1:* In the first simulation example let the noise standard deviations be given by [3] $\sigma_\zeta = 1.6$ dB and $\sigma_e = 5\%$ of the true measurement. The estimated power $\%RMSE$ is plotted in Figure 2 for the cases involving eight to twelve receivers.

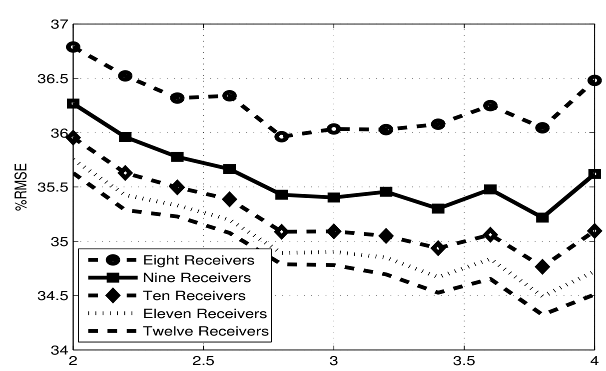



Fig. 2. The power estimate $\%RMSE$ for eight to twelve receivers and $\sigma_\zeta = 1.6$ dB and $\sigma_e = 5\%$ of the true measurement.

*2) Simulation Case 2:* In the second example the deviations are $\sigma_\zeta = 1$ dB and $\sigma_e = 5\%$ of the true measurement. The estimated power $\%RMSE$ is plotted in Figure 3.

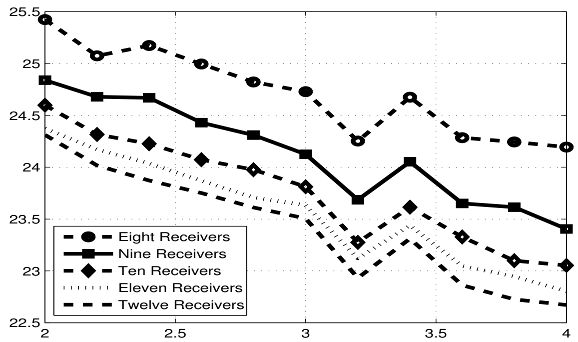



Fig. 3. The power estimate $\%RMSE$ for eight to twelve receivers and $\sigma_\zeta = 1$ dB and $\sigma_e = 5\%$ of the true measurement.

*3) Simulation Case 3:* In the third example the deviations are $\sigma_\zeta = 0$ dB and $\sigma_e = 15\%$ of the true measurement. The estimated power $\%RMSE$ is plotted in Figure 4.

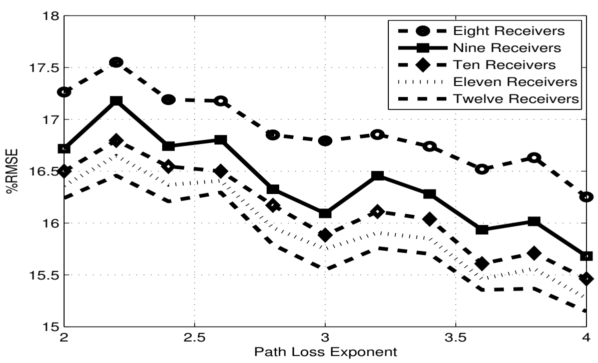



Fig. 4. The power estimate $\%RMSE$ for eight to twelve receivers and $\sigma_\zeta = 0$ dB and $\sigma_e = 15\%$ of the true measurement.

*4) Simulation Discussion:* As expected, the simulation results indicate that the shadowing noise is the dominant factor in determining the estimator's accuracy. For a typical outdoor (suburban/rural) scenario, the standard deviation of the shadowing component may be approximately given by $\sigma_\zeta \approx 1.5$ dB, see [3]. From the first simulation example it can be seen that a reasonable performance is obtained with eight to twelve receivers and performance improvements can be made with $> 12$ receivers. However, it should be noted that RSS-based measurements are extremely difficult to work with when the noise increases (which is commonly seen in practice). This paper does not explore such scenarios in detail since one should expect there is a limit on the measurement noise beyond which it is impractical to consider the use of such measurements for accurate parameter estimation.

We note that, in electronic warfare systems and sensor networks, a large number of receivers can measure the RSS with cheap sensor technology (compared to other technologies). Thus, it may be possible to exploit a large number of sensors in order to reduce the measurement uncertainty.

The estimator's performance looks to improve as the path-loss exponent increases; such a phenomenon has been seen elsewhere when studying RSS-based localization [14], [15].

The geometrically constrained approach has not assumed any knowledge of the noise statistics in its implementation and simply exploits the underlying geometrical relationship between the transmitter and the receivers. Performance improvements may be possible if the estimator bias and noise statistics are examined in detail or numerical optimization algorithms are employed.

Although, the results in this paper were derived only in $\mathbb{R}^2$, it is clearly straightforward to extend the results to systems in $\mathbb{R}^3$. In $\mathbb{R}^3$ we require a minimum of five RSS sensors in order to estimate the emitter's transmission power level.

## V. Conclusion

In this short paper we explored the idea of exploiting an underlying geometrical constraint in order to estimate the transmission power of an uncooperative emitter.

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