Quantum correlations and an intertwining operator on two dimensional hyperbolic space.

Antoine Thanh Gansemer

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Disclaimer

I hereby declare that the work undertaken in this thesis has been conducted by me alone, except where indicated in the text.

Chapters 2, 3, 4 and 5 are the primary new work in this thesis. The work in Chapter 4 appears in the preprint [GH24] and is joint work with Andrew Hassell. A large portion of the content of Chapter 5 comes from unpublished notes of Andrew Hassell and Charlotte Groom and my contributions to this chapter are a sharpening of the estimates to scales h^α for any $\alpha < 1$. This is clarified in the introduction to Chapter 5.

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Abstract

In this thesis, we study an intertwining operator on two dimensional hyperbolic space introduced by Anantharaman and Zelditch in [AZ12]. This operator intertwines two natural flows: one is the geodesic flow on the cotangent bundle of hyperbolic space, which can be thought of as the evolution of a classical observable under classical dynamics. The other flow is the flow on the space of quantum observables (operators) given by the conjugation of a quantum observable with the free Schrödinger propagator on hyperbolic space. This can be thought of as the evolution of a quantum observable under quantum dynamics.

In the first part of the thesis, we modify the construction of the intertwining operator from [AZ12]. The modified intertwining operator is an isometry on the space of square integrable symbols in a particularly natural way without destroying the intertwining property. This allows us to exactly relate two dynamical quantities on the noncompact hyperbolic space, the quantum correlation and the classical correlation, with no error term. Then for the purposes of trying to extend this exact relation to a compact quotient, we explore the structure of the intertwining operator as a Fourier Integral Operator and we describe some features of the canonical relation associated to the intertwining operator.

We will also study the behaviour of a matrix element $\langle \mathrm{Op}(a)\phi_j, \mathrm{Op}(b)\phi_k\rangle$ when we pull the symbol of $\mathrm{Op}(a)$ back via geodesic flow, i.e. we study the quantity $\langle \mathrm{Op}(g_t^* a)\phi_j, \mathrm{Op}(b)\phi_k\rangle$. The trace over the diagonal matrix elements of $\langle \mathrm{Op}(g_t^* a)\phi_j, \mathrm{Op}(b)\phi_k\rangle$ can be considered the "principal part" of the time t quantum correlation between operators $\mathrm{Op}(a)$ and $\mathrm{Op}(b)$. Here ϕ_j are a basis of Laplace eigenfunctions on a compact hyperbolic surface and $\mathrm{Op}(a)$ is given by the Zelditch quantisation. We show these matrix elements decay exponentially as $|t| \to \infty$ under certain assumptions on the symbols. Consequently, the trace also decays in a similar manner under similar assumptions on the symbols. To prove this, we use ideas developed recently on the microlocal analysis of hyperbolic dynamical systems.

Lastly, we provide a chapter exploring similar ideas in the contrasting geometric context of the Euclidean torus. The significance here is that we have the well known Weyl quantisation in this context which satisfies the desirable "exact Egorov" property we hope to develop an analogy of in the hyperbolic case. We apply our analysis in the Euclidean case towards understanding toral quantum ergodicity at small length scales.

Contents

Introduction

0.1 Background and motivation.

The features of Laplace eigenfunctions and their eigenvalues on a Riemannian manifold reflect structures present in the geometry of the space. Inversely, the geometry of the space constrains the possible solutions to the Laplace eigenvalue equation on a manifold. We are particularly interested in the asymptotic features of high frequency eigenfunctions on manifolds which have hyperbolic (and thus chaotic) dynamics.

Our thesis is motivated by a question of small scale quantum ergodicity (or relatedly, the quantum variance decay rate problem) in the context of Laplace eigenfunctions on a compact hyperbolic surface.

Physicists Feingold and Peres [FP86] and Eckhardt et al [EFK+95] suggest that for uniformly hyperbolic Hamiltonians (of which the free Hamiltonian on a compact hyperbolic surface is an example), the variances of quantum observables up to energy E, $V(a,E)$ defined in (1), exhibits a decay of the form

$$V(a,E) := \frac{1}{N(E)} \sum_{E_j \le E} |\langle \mathrm{Op}(a)\phi_j, \phi_j\rangle - \overline{a}|^2 \sim V(a)E^{-1/2}, \qquad E \to \infty \tag{1}$$

where E_j are the eigenvalues and ϕ_j are an associated orthonormal basis of eigenfunctions for the quantised Hamiltonian (such as the Laplace-Beltrami operator when the Hamiltonian is just the kinetic energy term) in $L^2(X)$. Here X is the configuration space, a smooth (and usually compact) Riemannian manifold, $N(E)$ is the number of eigenvalues E_j less than or equal to E, a is a classical observable and $\mathrm{Op}(a)$ the associated quantum observable, $\overline{a}$ is the average of a over phase space and $V(a)$ is the classical variance

$$V(a) := \int_{-\infty}^{\infty} \left(\int_{S^*X} a \circ g_t(x,\hat{\xi}) a(x,\hat{\xi}) dx d\hat{\xi} - \mathrm{vol}(S^*X)|\overline{a}|^2 \right) dt, \tag{2}$$

where g_t is the Hamiltonian flow of the classical chaotic Hamiltonian on phase space

S^*X. The integral is well-defined for hyperbolic dynamical systems which typically have an exponential decay of correlations. Consequently, the integrand in t of (2) decays exponentially as $t \to \pm\infty$ and so (2) is well-defined.

In our thesis, as part of a broader research program, we will explore an approach to this problem via a study of an intertwining operator on hyperbolic space introduced by Anantharaman-Zelditch in [AZ12]. This operator was introduced in [AZ12] as an operator which intertwines two natural flows on hyperbolic space. We will use both $\mathbb{D}$, the Poincaré disk and $\mathbb{H}^2$, the upper half plane as models of hyperbolic space in our thesis.

The first of the two flows is the classical flow which is the geodesic flow on hyperbolic space which acts by pullback on functions on $T^*\mathbb{D}$. It is given by

$$G_t^* a(g, r) := a(g\mathbf{a}_{rt}, r) \tag{3}$$

where we identify $g \in G$ with $S^*\mathbb{D}$ where G is either $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{PSU}(1, 1)$ depending on the model of hyperbolic space we use. We identify $(g, r) \in G \times \mathbb{R}^+$ with $S^*\mathbb{D} \times \mathbb{R}^+ \simeq T^*\mathbb{D} \backslash 0$ through a polar coordinate map in the fibres of $T^*\mathbb{D}$. The right action on g by the particular group element $\mathbf{a}_{rt} \in G$ (which we will define in chapter 1) generates the speed r geodesic flow on $S^*\mathbb{D}$. In particular, this flow is a unitary one-parameter flow on the Hilbert space of symbols $L^2(T^*\mathbb{D})$.

The second flow is the quantum flow on the class of Hilbert-Schmidt operators. Recall that the Hilbert-Schmidt operators on $L^2(\mathbb{D})$ is a Hilbert space of operators under the inner product

$$\langle A, B \rangle = \mathrm{Trace}(A^* B) \tag{4}$$

with the trace taken over $L^2(\mathbb{D})$. On the space of Hilbert-Schmidt operators, the conjugation of an operator by the (free) Schrödinger group $U(t) = e^{it\Delta}$ is interpreted as the evolution of a quantum observable by the quantum flow in the Heisenberg picture of quantum mechanics, it is given by

$$V_t A := e^{-it\Delta} A e^{it\Delta}. \tag{5}$$

The Zelditch quantisation of [Zel86] provides an exact correspondence between symbols and operators in hyperbolic space through the use of the Helgason-Fourier transform, see Section 1.3. Given the Schwartz kernel of an operator on hyperbolic space, $K(z, w)$, $z, w \in \mathbb{D}$, Zelditch associates to this the symbol $a(z, b, r)$ where $z \in \mathbb{D}$, $b \in \mathbb{S}^1$, $r \in \mathbb{R}^+$ through taking the Helgason-Fourier transform of $w \mapsto K(z, w)$. As (z, b) can be identified with $S^*\mathbb{D}$, by taking $b \in \mathbb{S}^1$ as the unique forward endpoint of the geodesic on $\mathbb{D}$ which $(z, \hat{\xi}) \in S^*\mathbb{D}$ uniquely defines, we identify $(z, b, r) \in \mathbb{D} \times \mathbb{S}^1 \times \mathbb{R}^+$ with

$S^*\mathbb{D} \times \mathbb{R}^+$ (and thus with $T^*\mathbb{D}\backslash 0$ through identifying (b, r) with polar coordinates in the fibres of $T^*\mathbb{D}$) and hence one has identified in an isometric way, the Hilbert space of Hilbert-Schmidt operators and the Hilbert space of square integrable symbols $L^2(T^*\mathbb{D})$. Moreoever, Zelditch's construction is equivariant under the left action of G and hence it descends to an identification of operators and symbols on any quotient of hyperbolic space.

A consequence of this is that the flow in (5) induces a corresponding flow on the space of square integrable symbols, which we will also denote (abusively) by V_t, in particular,

$$V_t a = \sigma(e^{-it\Delta}\mathrm{Op}(a)e^{it\Delta}) \tag{6}$$

where σ is the complete symbol map (of Zelditch) and $\mathrm{Op}(a)$ is the Zelditch quantisation of the symbol a. V_t is then also a unitary one-parameter flow on the space $L^2(T^*\mathbb{D})$ of symbols. We intertwine these two unitary one-parameter flows, V_t and G_t^* on the space of symbols $L^2(T^*\mathbb{D})$, following in the footsteps of Anantharaman and Zelditch.

The construction of the intertwining operator by Anantharaman and Zelditch follows one of their earlier papers, [AZ07] which introduces the (diagonal) Patterson-Sullivan distributions and the diagonal Wigner distributions. These are families of distributions which act on symbols on the cosphere bundle S^*X, where $X = \Gamma\backslash\mathbb{D}$ is now a compact hyperbolic surface, the left quotient of $\mathbb{D}$ by a cocompact group $\Gamma \subset G$. They are families indexed by an eigenvalue λ_j of the Laplace-Beltrami operator on X. Crucially, the Patterson-Sullivan distributions are invariant under the geodesic flow G_t^* on symbols over T^*X and the Wigner distributions are invariant under the quantum flow V_t. In [AZ07], they show that these two distributions are asympotically equivalent as $\lambda_j \to \infty$ and are exactly related through an operator which is indexed by λ_j, denoted L_{r_j} in their paper where $\lambda_j = \frac{1}{4} + r_j^2$. (To be slightly more precise, one makes a consistent choice of r_j for each λ_j through the formula $\lambda_j = \frac{1}{4} + r_j^2$ since the diagonal Wigner and Patterson-Sullivan distributions are really indexed by r_j and in fact depend on the choice of r_j).

The sequel to [AZ07], appearing as [AZ12] was able to construct an intertwining operator $\mathcal{L}$ between the two flows G_t^* and V_t. We will make some modifications to their construction of $\mathcal{L}$ in this thesis which will improve its boundedness properties on $L^2(T^*\mathbb{D})$ (in fact it will become an isometry on $L^2(T^*\mathbb{D})$) without destroying its key property of intertwining G_t^* and V_t. We shall use a bolded $\boldsymbol{\mathcal{L}}$ to denote the intertwining operator we construct in this thesis, to avoid confusion and to emphasise the difference with $\mathcal{L}$ as constructed in [AZ12]. On the whole of hyperbolic space, say on the disk model, one replaces the discrete family of Wigner distributions and Patterson-Sullivan distributions in [AZ07] with a continuous family of such distributions which depend on a pair of dual variables

$(b,r),(b',r') \in \mathbb{S}^1 \times \mathbb{R}^+$, where $\mathbb{S}^1 \times \mathbb{R}^+$ is the Helgason-Fourier dual space to $\mathbb{D}$. These are called the Wigner and Patterson-Sullivan transforms respectively and we will denote the versions that we construct in this thesis by $\mathbf{W}$ and $\mathbf{PS}$. Their key function is to diagonalise V_t and G_t^* respectively, such that V_t and G_t^* both become unitarily equivalent to the multiplication operator $e^{it(r^2-r'^2)}$ on $L^2_{S,S}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}, db\,dr\,db'dr')$, where the subscript S,S refers to a hyperbolic scattering matrix condition which we require to extend the Helgason-Fourier dual space from $\mathbb{S}^1 \times \mathbb{R}^+$ to $\mathbb{S}^1 \times \mathbb{R}$. Here db is the normalised Haar measure on $\mathbb{S}^1$. As V_t and G_t^* are unitarily equivalent to the *same* multiplication operator $e^{it(r^2-r'^2)}$ on the *same* Hilbert space, it follows that they are unitarily equivalent to each other and the unitary equivalence is provided by the intertwining operator, $\boldsymbol{\mathcal{L}} = \frac{1}{\sqrt{2}}\mathbf{PS}^{-1} \circ \mathbf{W}$. I.e. $\boldsymbol{\mathcal{L}} \circ V_t = G_t^* \circ \boldsymbol{\mathcal{L}}$. (The $\sqrt{2}$ comes from extending the Helgason-Fourier dual space).

The work of Anantharaman and Zelditch in [AZ12] constructed $\mathcal{L}$ which intertwines G_t^* and V_t on much more mysterious Hilbert spaces of symbols that are slightly ad hoc and which have less clear properties in terms of the growth and regularity of symbols.

Rather remarkably, the intertwining operator $\boldsymbol{\mathcal{L}}$ also has an expression as an oscillatory integral operator (or Fourier integral operator) acting on the symbol. This is derived in Theorem 15 in Chapter 2. We study this expression to try to get some further mapping properties of $\boldsymbol{\mathcal{L}}$ on (possibly variable order) Sobolev spaces.

At this stage, let us try to place the analysis of the intertwining operator into the broader context of a research program which studies small scale quantum ergodicity on a compact hyperbolic surface. In chapter 5, for any $\alpha > 0$, we introduce the quantity which is the h^α-scale quantum variance of the eigenfunction density. Similar to (1), it is

$$V(\varphi_h, h) := h^2 \sum_{j=1}^{\infty} e^{-2h^2\lambda_j^2} |\langle \varphi_h \phi_{h_j}, \phi_{h_j} \rangle - c|^2 \tag{7}$$

where $\lambda_j^2 = h_j^{-2}$ are the eigenvalues and ϕ_{h_j} are an associated orthonormal basis of eigenfunctions in $L^2(M)$, with M a compact Riemannian manifold. We have that $c = \mathrm{vol}(M)^{-1}$ and φ_h is a smooth cutoff to a h^α-radius neighbourhood of some point $x_0 \in M$, with integral equal to 1 uniformly in h. The Gaussian weight, $e^{-2h^2\lambda_j^2}$, effectively cuts off the sum to terms where $\lambda_j \lesssim h^{-1}$.

Using lemmas 33 and 34 in Chapter 5, we can prove that on an arbitrary smooth, compact, Riemannian manifold, the h^α-scale quantum variance of the eigenfunction density on such manifolds M, namely (7), are controlled by the $h \to 0$ asymptotics of the following quantity

$$h^2\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\mathrm{QCor}_h(\varphi_h e^{-h^2\Delta},\varphi_h e^{-h^2\Delta})(t)dt \tag{8}$$

where $T>0$ is arbitrary (and so can depend on h) and

$$\mathrm{QCor}_h(A,B)(t) := \mathrm{Trace}\{U_h(-t)AU_h(t)B\} \tag{9}$$

is the *quantum correlation* at time t for two quantum observables A and B, with the trace taken over $L^2(M)$. Here $U_h(t)=e^{ith\Delta}$ is the h-semiclassical free Schrödinger propagator. In our analysis of small scale quantum ergodicity on the torus, we use the particular geometric structure of the torus to make an analysis of quantity (8). For a general smooth and compact Riemannian manifold M and a fixed $\alpha>0$, if one can show that for each h, there is a choice of $T>0$ such that

$$h^2\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\mathrm{QCor}_h(\varphi_h e^{-h^2\Delta},\varphi_h e^{-h^2\Delta})(t)dt = \frac{1}{8\pi\mathrm{vol}(M)}+o(1) \quad \text{as} \quad h\to 0 \tag{10}$$

then this would prove the h^α-scale quantum variance goes to zero as $h\to 0$ on the manifold M, and hence for any orthonormal basis of $L^2(M)$ eigenfunctions with eigenvalue h_j^{-2}, there exists a density one subsequence that equidistribute at the scale h_j^α as $j\to\infty$.

In fact it is already known via standard theorems in semiclassical analysis (see eg. Egorov's theorem and traces of pseudodifferential operators in [Zwo12]) that on a smooth, compact Riemannian manifold M of dimension d,

$$(2\pi h)^d\mathrm{QCor}_h(A,B)(t) = \mathrm{ClassicalCor}(a,b)(t)+O_t(h) \tag{11}$$

where

$$\mathrm{ClassicalCor}(a,b)(t) = \int_{T^*M}(G_t^*a)(x,\xi)b(x,\xi)dxd\xi \tag{12}$$

is the classical correlation between the h-principal symbols a and b of A and B respectively, which we can assume have differential order $-d-\epsilon$ for any $\epsilon>0$ so that (11) is well-defined. For manifolds with hyperbolic geodesic flow, the $O_t(h)$ error in Egorov's theorem becomes

$$O(he^{\beta|t|}) \tag{13}$$

where $\beta>0$ is the Lyapunov exponent of the flow (the maximal expansion rate of the flow along unstable leaves). To apply this to asymptotic estimates as $h\to 0$, we require the choice of t to be small enough depending on h, smaller than the so called Ehrenfest time i.e

$$|t| \leq \frac{\rho}{\beta} \log h^{-1} \tag{14}$$

for any $\rho < 1/2$. For a system with hyperbolic classical dynamics, one has exponential decay of the classical correlation, but the logarithmic error of Ehrenfest restricts the h-dependent time scales at which we can take $t \to \infty$. *The construction of the intertwining operator is therefore a tool which may allow us to bypass the Ehrenfest bound.* In Chapter 5, in the simpler setting of the torus, we can bypass the Ehrenfest bound with the Weyl quantisation and show that for any orthonormal basis of Laplacian eigenfunctions on $L^2(\mathbb{T}^2)$ with corresponding eigenvalues h_j^{-2}, there is a density one subsequence of which equidistributes at the small scale h_j^α for any $\alpha < 1$ as $h_j \to 0$.

We believe that the intertwining operator could be used to better understand quantity (8) on a compact hyperbolic surface. How this could occur is as follows (we will work non-semiclassically for now). We have, by definition of V_t, that

$$U(-t)\mathrm{Op}(a)U(t) = \mathrm{Op}(V_t a) \tag{15}$$

where Op is the Zelditch quantisation and $U(t) = e^{it\Delta}$ is the Schrödinger propagator. Additionally, on the whole of hyperbolic space $\mathbb{D}$, with Zelditch quantisation, we have exactly

$$\mathrm{Trace}\{\mathrm{Op}(f)\mathrm{Op}(g)^*\} = \int_{\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}^+} f(z,b,r)\overline{g(z,b,r)}e^{\langle z,b\rangle}|c(r)|^2 d\mathrm{vol}(z)dbdr, \tag{16}$$

where $e^{\langle z,b\rangle} = \frac{1-|z|^2}{|z-b|^2}$ is the Poisson kernel and $c(r) = 2^{-(\frac{1}{2}+ir)}\frac{\Gamma(1/2+ir)}{\Gamma(1/2)\Gamma(ir)}$ where $\Gamma(z)$ is the Gamma function. Therefore, for symbols a_1 and a_2 which are square integrable in the norm induced by (16),

$$\mathrm{QCor}(\mathrm{Op}(a_1), \mathrm{Op}(a_2))(t) = \mathrm{Trace}\left\{U(-t)\mathrm{Op}(a_1)U(t)\mathrm{Op}(a_2)^*\right\} \tag{17}$$

is exactly

$$\int_{\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}^+} V_t a_1(z,b,r)\overline{a_2(z,b,r)}e^{\langle z,b\rangle}|c(r)|^2 d\mathrm{vol}(z)dbdr. \tag{18}$$

We show in Theorem 14 in Chapter 2 that $\boldsymbol{\mathcal{L}}$ is an isometry on

$$L^2_S(\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}, e^{\langle z,b\rangle}|c(r)|^2 d\mathrm{vol}(z)dbdr). \tag{19}$$

Here the subscript S is again a hyperbolic scattering matrix condition to extend the Helgason-Fourier dual space from $\mathbb{S}^1 \times \mathbb{R}^+$ to $\mathbb{S}^1 \times \mathbb{R}$. Hence, we can apply this to (18) (after extending the integral to $r \in \mathbb{R}$) to get

$$\int_{\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}} \boldsymbol{\mathcal{L}} V_t a_1(z,b,r)\overline{\boldsymbol{\mathcal{L}} a_2(z,b,r)} e^{\langle z,b\rangle}|c(r)|^2 d\mathrm{vol}(z)dbdr, \tag{20}$$

to which we can then apply the intertwining identity $\boldsymbol{\mathcal{L}} \circ V_t = G_t^* \circ \boldsymbol{\mathcal{L}}$ to get

$$\int_{\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}} G_t^* \boldsymbol{\mathcal{L}} a_1(z,b,r)\overline{\boldsymbol{\mathcal{L}} a_2(z,b,r)} e^{\langle z,b\rangle}|c(r)|^2 d\mathrm{vol}(z)dbdr \tag{21}$$

which has the form of the classical correlation of $\boldsymbol{\mathcal{L}} a_1$ and $\boldsymbol{\mathcal{L}} a_2$ (as one can identify $\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}^+$ with $T^*\mathbb{D}\backslash 0$). In particular, what this shows is that on the whole hyperbolic space $\mathbb{D}$, for square integrable symbols, we have as a corollary of Theorem 14 in Chapter 2 that

$$\mathrm{QCor}(\mathrm{Op}(a_1), \mathrm{Op}(a_2))(t) = \mathrm{ClassicalCor}(\boldsymbol{\mathcal{L}} a_1, \boldsymbol{\mathcal{L}} a_2)(t) \tag{22}$$

with no error term. This requires $\boldsymbol{\mathcal{L}}$ to satisfy both the intertwining relations $\boldsymbol{\mathcal{L}} \circ V_t = G_t^* \circ \boldsymbol{\mathcal{L}}$ *and* for $\boldsymbol{\mathcal{L}}$ to be an isometry on

$$L^2_S(\mathbb{D}\times\mathbb{S}^1\times\mathbb{R}, e^{\langle z,b\rangle}|c(r)|^2 d\mathrm{vol}(z)dbdr) \simeq L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr). \tag{23}$$

One needs an argument to move this line of reasoning over onto a compact quotient of hyperbolic space. There, one could hope to use a decay of correlations argument to analyse the asymptotics of (the compact version of) expression (21). We will showcase one approach we take to try to develop an analogue of (22) on a compact hyperbolic surface. On a compact hyperbolic surface, $\Gamma\backslash\mathbb{D}$ where $\Gamma \subset G$ is a discrete cocompact subgroup, the trace of the product of pseudodifferential operators B^*A over $L^2(\Gamma\backslash\mathbb{D})$ is not just the integral of the symbols $a\bar{b}$ over $T^*(\Gamma\backslash\mathbb{D})$ like it is on hyperbolic space. Instead, as we will derive in Chapter 4, it has the expression, for $w > 1/2$,

$$\mathrm{Trace}\, B^*A = \int_{\Gamma\backslash G\times S_w} a(g,r)\overline{\mathfrak{T}b(g,r)}dgdr \tag{24}$$

where S_w is the strip $\{r \in \mathbb{C} : |\Im(r)| < w\}$ in the complex plane of width w and where $\mathfrak{T}b$ is a distribution with microlocal regularity away from a certain dynamical subbundle of $T^*(\Gamma\backslash G)$. If one could find a function space which includes $\mathfrak{T}b$ on which $\boldsymbol{\mathcal{L}}$ is bounded, one could seek to produce an analogue of (22) on a compact hyperbolic surface. In fact we will not achieve this goal in the thesis, but we will present our analysis with this approach in Chapter 2 and Chapter 3. A different approach to the same question, that we will not have the chance to present in the thesis, is to use the left-G equivariance of $\boldsymbol{\mathcal{L}}$ to study periodic sums over Γ. This is a topic for future study.

Let us mention now some of the broader work in this area. The quantum ergodic theorem of Schnirelman, Zelditch and Colin de Verdière [Shn74, CdV85, Zel87] states that

$$V(\sigma(A),E) := \frac{1}{N(E)} \sum_{E_j \leq E} |\langle A\phi_{E_j}, \phi_{E_j}\rangle - \overline{\sigma(A)}|^2 = o(1) \tag{25}$$

as $E \to \infty$ where $V(\sigma(A),E)$ is the same quantum variance as in (1) when the Riemannian manifold M has an ergodic geodesic flow, here $\sigma(A)$ is the principal symbol of the pseudodifferential operator A. Zelditch in [Zel94] found a quantitative upper bound on this quantum variance decay rate in the context of manifolds with (possibly variable) negative curvature. In this geometry, he found that

$$V(\sigma(A),E) = O((\log E)^{-1}). \tag{26}$$

as $E \to \infty$. This was reproduced by Schubert [Sch05] and extended to systems whose classical dynamics have a certain 'rate of classical ergodicity'. Work from Hezari-Rivière [HR16] and Han [Han15] on the small scale quantum ergodicity problem found that an analogue of statement (25) holds on compact manifolds of negative sectional curvature when you allow symbols to shrink to a point as $E \to \infty$ where the effective radius of a symbol is an power of $(\log E)^{-1}$. On the d-dimensional torus $\mathbb{T}^d$, the same small scale quantum ergodicity problem was considered by Hezari-Rivière in [HR15] on the configuration space of $\mathbb{T}^d$ who found an analogue of (25) holds when one allows a symbol to shrink as $E \to \infty$ with a radius which is of order $E^{-\kappa}$ for an explicit $\kappa(d) > 0$. This same problem on the torus was also considered by Lester-Rudnick [LR17], who found small scale quantum ergodicity holds for balls with shrinking radius

$$E^{-\frac{1}{2(d-1)}+o(1)}, \quad \text{as} \quad E \to \infty. \tag{27}$$

This is the sharp rate (up to the $E^{o(1)}$ term) as shown in their paper. On the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, Luo and Sarnak in [LS95] and a subsequent extension by Jakobson [Jak97], (see also recent work by Luo-Sarnak [LS04] and Sarnak-Woodbury-Zhao [SZW19]) showed the quantum variance of the (Hecke) cusp eigenforms have a decay like

$$V(\sigma(A),E) = O_{A,\epsilon}(E^{-\frac{1}{2}+\epsilon}) \tag{28}$$

for any $\epsilon > 0$, which is sharp up to the E^{ϵ} term.

Work on the intertwining operator on hyperbolic space, as we mentioned, was progressed by Anantharaman and Zelditch in a pair of papers [AZ07] and [AZ12]. The Patterson-Sullivan distributions that were introduced in [AZ07] were generalised to rank one symmetric spaces by Schröder in [Sch10] and Hilgert-Schröder in [HS09]. For higher rank symmetric spaces, the extension of Patterson-Sullivan distributions to this geometry were given by [HHS12].

On hyperbolic space and its quotients, the boundary value of a Laplace eigenfunction was characterised [1] in terms of the Poisson transform by Helgason, see [Hel74] and [Hel81], with further work on the boundary distribution regularity by Otal [Ota98]. This can be reinterpreted, as was done by [DFG15] in the case of compact hyperbolic manifolds, as a relationship between the classical Ruelle resonant states for the geodesic flow and the eigenstates of the Laplacian. This correspondence between classical Ruelle resonant states and quantum resonant states on compact hyperbolic surfaces was extended to convex co-compact hyperbolic surfaces in [GHW18] for quantum resonances $\lambda \notin -\mathbb{N}$.

We will also rely on the work developed in the last 20 years on Ruelle resonances for uniformly hyperbolic flows, although we will take a more microlocal approach following [FS10], [DZ13] and [NZ15]. For a different approach see [Liv04], [BL07].

Our thesis is divided into this present introductory chapter, a preliminary chapter and four main chapters. The preliminary chapter, Chapter 1 will define some of the geometric concepts and notation which we will use in the following chapters. We will assume knowledge of microlocal and semiclassical analysis and not cover this in the preliminary chapter. For more on microlocal and semiclassical analysis, see [Zwo12], [GS94], ([DZ19],Appendix E), [Hör03, Hör07, Hör09], [Hör71], [DH72] for some references.

Chapter 2 begins by constructing an intertwining operator on hyperbolic space. There are some key differences which our intertwining operator has with the intertwining operator in Anantharaman-Zelditch in [AZ12]. We will also need a modified Helgason-Fourier transform compared to what appears in [Hel81] and consequently a modified Zelditch calculus, so the construction of these will appear in this chapter too. Afterwards, we are in a place to construct the Wigner and Patterson-Sullivan transforms **W** and **PS** respectively. There are two main theorems in Chapter 2, the first is Theorem 14 which states that $\boldsymbol{\mathcal{L}}$ is an isometry of $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$ into $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$, and thus a unitary isomorphism onto its range in $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$. Here $G\times\mathbb{R}$ can be identified with $S^*\mathbb{D}\times\mathbb{R}$ and the subscript S refers again to a hyperbolic scattering matrix condition to extend the Helgason-Fourier dual space. That the intertwining operator is an isometry on the space of square integrable symbols is not true with the Anantharman-Zelditch intertwining operator. In their work [AZ12], their intertwining operator is in fact not a bounded operator on the space of square integrable symbols, $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$.

In Chapter 2, we will also prove Theorem 15 which shows that the intertwining operator,

[1] Away from an exceptional set of eigenvalues.

and its inverse can be expressed as oscillatory integral operators. Some proofs in this chapter do follow the arguments made in [AZ07] and [AZ12], with modifications made where necessary.

The next chapter, Chapter 3, continues the study of the intertwining operator as an oscillatory integral operator or a Fourier integral operator. In this chapter we will derive the canonical relation of the phase function associated to the intertwining operator. We shall prove that the modified intertwining operator has a smooth canonical relation and that the projections of the canonical relation onto the left factor and onto the right factor have blowdown singularities on a particular codimension one submanifold. This is presented in the two theorems of this chapter, Theorem 18 and Theorem 22.

The relationship of this chapter to the previous Chapter 2 is that with our construction of the intertwining operator, the amplitude (or symbol) of the intertwining operator vanishes to first order on this particular codimension one submanifold where the left and right projections of the canonical relation have a blowdown singularity. This is in contrast to the Anantharaman-Zelditch construction whose canonical relation has the same blowdown singularity with regards to the left and right projection but whose symbol does not vanish on this submanifold. This can be interpreted as the reason why their operator is not bounded on $L^2(T^*\mathbb{D})$ and the one we construct is bounded (and moreover an isometry).

Motivated by the derivation of (21), we consider in Chapter 4 the long time asymptotics of matrix elements

$$\langle \mathrm{Op}(g_t^* a_1)\phi_j, \mathrm{Op}(a_2)\phi_j\rangle_{L^2(\Gamma\backslash\mathbb{D})} \tag{29}$$

and subsequently of traces of the form,

$$\mathrm{Trace}\{\mathrm{Op}(a_2)^*\mathrm{Op}(g_t^* a_1)\} \tag{30}$$

as $t \to \infty$. Here Op is the Zelditch quantisation, $X = \Gamma\backslash\mathbb{D}$ is a compact hyperbolic surface, $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of Laplace eigenfunctions of $L^2(\Gamma\backslash\mathbb{D})$ where ϕ_j is the j'th eigenfunction (taken by ordering the eigenvalues by magnitude and counting multiplicity) with eigenvalue $\lambda_j = \frac{1}{4} + r_j^2$. The pullback by geodesic flow on S^*X is denoted g_t^*. Although our goal would be to include the intertwining operator in this analysis, Chapter 4 does not involve the intertwining operator at all.

In the main theorem of Chapter 4, Theorem 23, we will show that if $a_1(\cdot, r_j)$ has mean zero over S^*X, then there exists some $\alpha > 0$ such that,

$$\langle \mathrm{Op}(g_t^* a)\phi_j, \mathrm{Op}(a_2)\phi_j\rangle = O_j(e^{-\alpha|t|}), \qquad t \to \pm\infty \tag{31}$$

where $\alpha > 0$ is related to the resonance expansion of the geodesic flow. Moreoever, if $a(\cdot, r_j)$ has mean zero for every r_j and a_1 and a_2 are of sufficiently negative order in the Hörmander symbol class $S^m_{1,0}(T^*X)$, then

$$\mathrm{Trace}\{\mathrm{Op}(a_2)^*\mathrm{Op}(g_t^*a_1)\} = O(e^{-\alpha|t|}), \qquad t \to \pm\infty. \tag{32}$$

The proof is based on the existence of the Helgason boundary values of eigenfunctions, the properties of the Zelditch calculus and a classical decay of correlations estimate for two functions on S^*X, one of which belongs to a Faure-Sjöstrand anisotropic Sobolev space (see eg. [FS10]).

Chapter 5 will consider the significantly different geometry of the (two-dimensional) torus and study the question of small scale equidistribution of toral eigenfunctions. It is in this chapter that we will reduce the study of small scale quantum ergodicity to the study of expression (8) involving the quantum correlation. This reduction holds for all smooth, compact, Riemannian manifolds, not just the torus. We will however use the particular geometry of the torus and the existence of the Weyl quantisation on the torus to analyse this 'averaged quantum correlation'. We will show, for any $\alpha < 1$, that one needs to average the toral quantum correlation over a large time $T \sim h^{-\alpha}$ to prove a small scale quantum ergodicity result on the torus at the scale h^{α}. In particular, our methods show small scale quantum ergodicity down to the Planck scale for the two dimensional torus, this agrees with the work of [LR17]. This chapter provides an example of the sort of analysis one can make to prove polynomial small scale quantum ergodicity when one has an intertwining operator (captured in the Weyl quantisation on the torus) that exactly intertwines the classical and quantum flows and therefore liberates us from the restriction to Ehrenfest time.

Chapter 1

Preliminaries

1.1 Hyperbolic space, metric, volume form, dynamics.

Let us fix the upper half plane, $\mathbb{H}^2$ as the set $\{z = x+iy : x \in \mathbb{R}, y > 0\}$ and equip it with the standard smooth (holomorphic) structure considered as an open subset of $\mathbb{C}$. Further to this, consider the standard hyperbolic metric on $\mathbb{H}^2$, given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This metric induces the following volume form and (positive) Laplace-Beltrami operator respectively:

$$d\mathrm{Vol} = \frac{dxdy}{y^2}, \qquad \Delta_{\mathbb{H}^2} = y^2 \left(D_x^2 + D_y^2\right), \quad D_x = \frac{1}{i}\frac{\partial}{\partial x}, \quad D_y = \frac{1}{i}\frac{\partial}{\partial y}.$$

We can map $\mathbb{H}^2$ bijectively into the unit disk $\mathbb{D} = \{z = x+iy : |z| < 1\}$ using the Cayley transform,

$$z \mapsto \frac{z-i}{z+i}. \tag{1.1}$$

We define this map to be an smooth isometry between $\mathbb{H}^2$ and $\mathbb{D}$, inducing the hyperbolic metric on the unit disk

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \tag{1.2}$$

The map extends smoothly to the boundary of $\mathbb{H}^2$ (the set $\{x + iy \in \mathbb{C} : y = 0\}$), which is mapped smoothly to $\partial\mathbb{D} \setminus \{1\}$. We introduce a point at infinity in the upper half-plane model which maps to $1 \in \partial\mathbb{D}$ to make this a bijection. We will on occasion use B to refer to the boundary of hyperbolic space, this is $\mathbb{S}^1$ in the Poincaré disk model.

Taking a look now at the isometry group of $\mathbb{H}^2$, it is well-known that the fractional linear action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d},$$

is an isometry of $\mathbb{H}^2$. The isometry group of the disk model is $\mathrm{PSU}(1,1)$ which is conjugate to $\mathrm{PSL}(2,\mathbb{R})$ by the Cayley transform. We write G for $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSU}(1,1)$, depending on the model of hyperbolic space we are using (and we will move between the two models freely), and let g or γ denote an element of G. It is easy to check that $\mathrm{PSL}(2,\mathbb{R})$ acts transitively and freely on $S\mathbb{H}^2$, hence we can identify $S\mathbb{H}^2$ with $G = \mathrm{PSL}(2,\mathbb{R})$ under the following identification

$$\mathrm{PSL}(2,\mathbb{R}) \ni g \mapsto (g(i), g'(i)i) \in S\mathbb{H}^2.$$

In particular this identifies the identity element of G with the unit vector (i,i) based at $i \in \mathbb{H}^2$ pointing along the imaginary axis. With this identification of G with $S\mathbb{H}^2$, the geodesic flow starting at a point $h \in G$ after time t is denoted by $g_t(h)$ and is given by the right action, $g_t(h) = h\mathbf{a}_t$, where $\mathbf{a}_t$ is the element of G given by

$$\mathbf{a}_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

The geodesics on $\mathbb{H}^2$ or $\mathbb{D}$ trace out circular arcs orthogonal to the boundary at conformal infinity. This fact is well known in hyperbolic geometry. See, for example, [Hel81] for a proof of this fact.

We define the stable horocyclic flow and the unstable horocyclic flow on G by the right action with the elements $\mathbf{n}_u$ and $\overline{\mathbf{n}_v}$ respectively, where

$$\mathbf{n}_u := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad \overline{\mathbf{n}_v} := \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

We also define the element $\mathbf{k}_\theta$, given by

$$\mathbf{k}_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad \theta \in [0,2\pi), \tag{1.3}$$

whose right action fixes the base point z and rotates by angle θ in the fibre of $S\mathbb{H}^2$. Notice that $\mathbf{k}_{2\pi} = \mathbf{k}_0$ in $\mathrm{PSL}(2,\mathbb{R})$. In the disk model, these flows are given by matrices (which one can easily see by conjugating the matrices in $\mathrm{PSL}(2,\mathbb{R})$ by the Cayley transform)

$$\mathbf{a}_t^{\mathbb{D}} = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}, \qquad \mathbf{n}_u^{\mathbb{D}} = \begin{pmatrix} 1+\frac{iu}{2} & \frac{-iu}{2} \\ \frac{iu}{2} & 1-\frac{iu}{2} \end{pmatrix}, \tag{1.4}$$

and

$$\overline{\mathbf{n}_u}^{\mathbb{D}} = \begin{pmatrix} 1-\frac{iu}{2} & \frac{-iu}{2} \\ \frac{iu}{2} & 1+\frac{iu}{2} \end{pmatrix}, \qquad \mathbf{k}_\theta^{\mathbb{D}} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}. \tag{1.5}$$

We can define global coordinates (z,θ) on $G \simeq S\mathbb{D}$ by taking $(z,\theta=0)$ to be the unit vector at z such that the geodesic emanating from this vector tends to a fixed point, say 1, on $\partial\mathbb{D}$, and so that the action of $\mathbf{k}_\theta$ acts by $(z,\theta') \mapsto (z,\theta'+\theta)$.

The group G has a bi-invariant Haar measure, which we denote dg, given by $d\mathrm{Vol}(z)d\theta$ in (z,θ) coordinates.

The elements $\mathbf{a}_t$ form a subgroup of G denoted A. Similarly the elements $\mathbf{n}_u$, $\overline{\mathbf{n}_v}$ and $\mathbf{k}_\theta$ form subgroups, denoted N, $\overline{N}$ and K respectively. We define the vector fields H, X_+, X_- on $G \simeq S\mathbb{H}^2$ as the infinitesimal generators of the subgroups A, N and $\overline{N}$ respectively; that is, they generate the geodesic flow, the stable horocyclic flow and the unstable horocyclic flow respectively. Let us note that H, X_+ and X_- are left-invariant vector fields on G because the flows they generate act as right actions on G, and hence commute with the left action of G on itself. The Lie brackets of these vector fields satisfy:

$$[H,X_+] = X_+, \quad [H,X_-] = -X_-, \quad [X_+,X_-] = 2H. \tag{1.6}$$

which are easily derived from the matrix expressions for $\mathbf{a}_t, \mathbf{n}_u$, and $\overline{\mathbf{n}_v}$. Let us note that on $\mathrm{PSL}(2,\mathbb{R})$, by taking the derivatives of the one-parameter groups $\mathbf{a}_t, \mathbf{n}_u, \overline{\mathbf{n}_u}$ appearing above, we get that the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is spanned by

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.7}$$

We can visualise these three flows on G using the identification $G \simeq S\mathbb{D}$. Please see Figure 1.1.

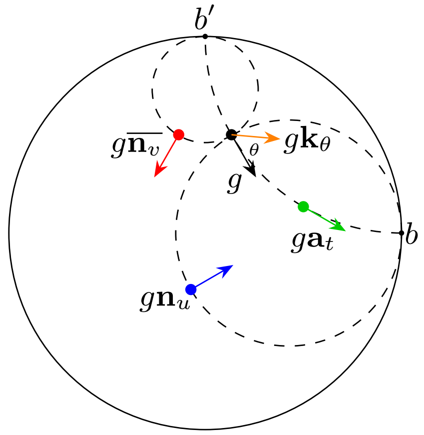
Let us also introduce coordinates $(z,b) \in \mathbb{D} \times \mathbb{S}^1$. Given $(z,v) \in S\mathbb{D}$, there is a unique $b \in \mathbb{S}^1$ such that the geodesic flow applied to (z,v) has forward (conformal) endpoint $b \in \mathbb{S}^1$ as $t \to \infty$. In this way the fibre $S_z\mathbb{D}$ and $\mathbb{S}^1$ are diffeomorphic (in a z-dependent way). Hence we have identifications $G \simeq S\mathbb{D} \simeq \mathbb{D} \times \mathbb{S}^1$.

Fix a point $z \in \mathbb{D}$ and $b \in \mathbb{S}^1$. We define the *horocycle* through z and tangent to b as the limit of a family of curves through z. Explicitly, consider the curve

$$C(z,\beta) := \{z' \in \mathbb{D} : \mathrm{dist}_{\mathbb{D}}(z',\beta) = \mathrm{dist}_{\mathbb{D}}(z,\beta)\}$$

where $\mathrm{dist}_{\mathbb{D}}(z',\beta)$ refers to the distance on $\mathbb{D}$ between points z' and β induced by the hyperbolic metric, (1.2). These are (hyperbolic) circles in the hyperbolic disk with centre point β passing through z. As β tends towards the point $b \in \mathbb{S}^1$ at infinity, say along a geodesic, the curves $C(z,\beta)$ converge smoothly to a curve that we call the horocycle through z tangent to b. In the disk model, a horocycle is a Euclidean circle passing through z and tangent to the boundary, $\mathbb{S}^1$ of the disk at b. See Figure 1.1. For comparison, the corresponding sets in Euclidean space $\mathbb{R}^n$ relative to a point ω on the sphere at infinity are hyperplanes with normal vector ω.

Figure 1.1: By identifying g with a point in $S\mathbb{D}$, the geodesic flow, stable horocyclic flow, unstable horocyclic flow and rotation in the fibres are represented by the right action on g by $\mathbf{a}_t$, $\mathbf{n}_u$, $\overline{\mathbf{n}_v}$ and $\mathbf{k}_\theta$ respectively. The images of the two horocyclic flows trace out horocycles in the base point z, which (in the disk model) are circles tangent to the conformal boundary at the forward and backward endpoints (b and b' respectively) of the geodesic determined by g.



Let us now state the definition of an Anosov flow and present a short direct proof that the geodesic flow on $S\mathbb{H}^2$ is Anosov.

Definition 1 (Anosov flow)**.** A smooth flow φ_t on a smooth manifold M is called *uniformly hyperbolic* or *Anosov* if, for all $x \in M$, there exist a splitting $T_xM = E_0(x) \oplus E_u(x) \oplus E_s(x)$ which is preserved by the flow, i.e. $E_i(\varphi_t(x)) = d\varphi_t(x)(E_i(x))$ for all $x \in M$, $t \in \mathbb{R}$ and $i = 0, u, s$, where $E_0(x)$ is spanned by the vector field which generates φ_t at x, and such that, for any fixed norm on the fibres T_xM, there exists constants $C > 0$, $\lambda > 0$ such that

$$\begin{aligned} |d\varphi_t(x)(v)| &\leq Ce^{-\lambda t}|v|, \quad \text{if} \quad v \in E_s(x), t > 0 \\ |d\varphi_t(x)(v)| &\leq Ce^{-\lambda |t|}|v|, \quad \text{if} \quad v \in E_u(x), t < 0 \end{aligned} \tag{1.8}$$

Consequently, $E_s(x)$ is called the stable subspace of T_xM and $E_u(x)$ is called the unstable subspace of T_xM.

Lemma 1 (geodesic flow on $S\mathbb{H}^2$ is an Anosov flow)**.** *The geodesic flow on $S\mathbb{H}^2$ is an Anosov flow with stable and unstable leaves given by vectors orthogonal to the stable and unstable horocycles respectively.*

Proof. Fix a point $h \in G \simeq S\mathbb{H}^2$ and consider the geodesic flow $g_t(h) = h\mathbf{a}_t$. Consider the two leaves through h given by the unstable and stable horocycle flows respectively, $u \mapsto h\mathbf{n}_u$ and $u \mapsto h\overline{\mathbf{n}_u}$. These leaves are generated by X_+ and X_- respectively and we claim the vector spaces over h spanned by X_+ and X_- are $E_s(h)$ and $E_u(h)$ respectively. Let us fix a left-invariant norm on the fibres of TG by setting $|H| = |X_+| = |X_-| = 1$. We can make the explicit (matrix) computation that $\mathbf{n}_u\mathbf{a}_t = \mathbf{a}_t\mathbf{n}_{ue^{-t}}$ and $\overline{\mathbf{n}_u}\mathbf{a}_t = \mathbf{a}_t\overline{\mathbf{n}_{ue^t}}$ which shows that

$$dg_t(h)X_+ = e^{-t}X_+, \qquad dg_t(h)X_- = e^tX_-.$$

This shows that the line bundles $E_s(h)$ and $E_u(h)$ are preserved by the g_t flow and that X_+ spans the stable sub-bundle and X_- spans the unstable sub-bundle with Lyapunov exponent $\lambda = 1$. Hence the geodesic flow g^t on $G \simeq S\mathbb{H}^2$ is Anosov. □

The splitting of the tangent bundle into stable, unstable and flow directions induces a splitting of the cotangent bundle. These bundles are denoted E_s^*, E_u^* and E_0^* and are defined by

$$\begin{aligned} E_s^* &= (E_s \oplus E_0)^0 \\ E_u^* &= (E_u \oplus E_0)^0 \\ E_0^* &= (E_s \oplus E_u)^0 \end{aligned} \tag{1.9}$$

where the superscript 0 indicates the annihilator space. These spaces can also be defined by the following equations, where h, u, s are the symbols of $-iH, -iX_+, -iX_-$ respectively:

$$\begin{aligned} E_s^* &= \{h = u = 0\} \\ E_u^* &= \{h = s = 0\} \\ E_0^* &= \{s = u = 0\} \end{aligned} \tag{1.10}$$

Then under the bicharacteristic flow of H, that is the geodesic flow lifted to the cotangent bundle of $S\mathbb{H}^2$, we have $\dot{u} = u$ and $\dot{s} = -s$, so the flow is expanding on E_u^* and contracting on E_s^* with respect to the natural left-invariant dual norm on the fibres of $T^*(S\mathbb{H}^2)$ as the 'time' parameter for the bicharacteristic flow increases, which explains the notation for these line bundles. To be precise, the bicharacteristic flow is given as the map $(g_t)_*$:

$T^*(S\mathbb{H}^2) \to T^*(S\mathbb{H}^2)$ defined by

$$(g_t)_* : (x, \xi) \mapsto (g_t(x), (dg_t(x)^T)^{-1}\xi) \tag{1.11}$$

where x is a point on $S\mathbb{H}^2$ and ξ is a point on the cotangent fibre $T_x^*(S\mathbb{H}^2)$. This is the same definition as appears in [FS10], Section 1.

This splitting of the cotangent bundle is important in the definition of anisotropic Sobolev spaces (see Section 4.4 in Chapter 4).

We now introduce a useful identity in hyperbolic geometry,

Lemma 2. *We have the following identity. If we let γ by a hyperbolic isometry in the disk model, then $|\gamma z - \gamma w|^2 = |\gamma'(z)||\gamma'(w)||z - w|^2$ for any $z, w \in \mathbb{D}$ where $|\cdot|$ indicates Euclidean distance.*

Proof. Any such isometry can be written as

$$\gamma z = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}$$

where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$. Then we can explicitly check the claim:

$$\begin{aligned} |\gamma z - \gamma w|^2 &= \left| \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} - \frac{\alpha w + \beta}{\overline{\beta} w + \overline{\alpha}} \right|^2 \\ &= \left| \frac{(\alpha z + \beta)(\overline{\beta} w + \overline{\alpha}) - (\alpha w + \beta)(\overline{\beta} z + \overline{\alpha})}{(\overline{\beta} z + \overline{\alpha})(\overline{\beta} w + \overline{\alpha})} \right|^2 \\ &= \frac{\left||\alpha|^2 z - |\beta|^2 z + |\beta|^2 w - |\alpha|^2 w\right|^2}{|\overline{\beta} z + \overline{\alpha}|^2 |\overline{\beta} w + \overline{\alpha}|^2} \\ &= \frac{|z - w|^2}{|\overline{\beta} z + \overline{\alpha}|^2 |\overline{\beta} w + \overline{\alpha}|^2} = |\gamma'(z)||\gamma'(w)||z - w|^2 \end{aligned} \tag{1.12}$$

The last line follows from the calculation

$$\begin{aligned} |\gamma'(z)| &= \left| \frac{\alpha}{\overline{\beta} z + \overline{\alpha}} - \frac{(\alpha z + \beta)\overline{\beta}}{(\overline{\beta} z + \overline{\alpha})^2} \right| \\ &= \frac{|\alpha(\overline{\beta} z + \overline{\alpha}) - (\alpha z + \beta)\overline{\beta}|}{|\overline{\beta} z + \overline{\alpha}|^2} \\ &= \frac{1}{|\overline{\beta} z + \overline{\alpha}|^2} \end{aligned} \tag{1.13}$$

□

1.2 Harmonic analysis on $\mathbb{D}$.

Definition 2 (Busemann function)**.** Given $z \in \mathbb{D}$ and $b \in \mathbb{S}^1$. The Buseman function of z and b, denoted $\langle z, b\rangle$, is defined to be the signed distance from $0 \in \mathbb{D}$ to the horocycle through z tangent to b, with sign convention that $\langle z, b\rangle$ tends to infinity as $z \to b$ along any geodesic.

The family of geodesics emanating from $b \in \mathbb{S}^1$ are orthogonal to the family of horocycles tangent to b. The identity $\mathbf{n}_u\mathbf{a}_t = \mathbf{a}_t\mathbf{n}_{ue^{-t}}$ has the geometric interpretation that the signed distance between horocycles is well-defined, as any geodesic segment between the two horocycles has the same length t. In particular it is the quantity given by $\langle z, b\rangle - \langle w, b\rangle$ where z is any point on the first horocycle and w is any point on the other horocycle. See Figure 1.2.

The function $e^{\langle z,b\rangle}$ coincides with the classical Poisson kernel on the disk,

$$e^{\langle z,b\rangle} = P(z, b) := \frac{1 - |z|^2}{|z - b|^2}. \tag{1.14}$$

In fact, by direct calculation one can verify that this function is constant on horocycles based at b, it equals 1 when $z = 0$, and the gradient $\nabla_z\langle z, b\rangle$, in the hyperbolic metric, has unit length. Using this fact about the gradient and the harmonicity of $P(z, b)$ (notice that harmonicity coincides for the Euclidean and the hyperbolic metric), it is easy to check that for $\mu \in \mathbb{C}$ and $b \in \partial\mathbb{D}$, the function $z \mapsto e^{\mu\langle z,b\rangle}$ is an eigenfunction of the hyperbolic Laplacian with eigenvalue $\mu(1 - \mu)$. We refer to these powers of the Poisson kernel $e^{\langle z,b\rangle}$ as (hyperbolic) plane waves, as they are analogous to Euclidean plane waves. Helgason used these hyperbolic plane waves to define a non-Euclidean Fourier transform on the hyperbolic disk.

The Busemann function is also useful in linking the two coordinate systems (z, θ) and (z, b) that we have discussed. In fact, for fixed z we have $d\theta = e^{\langle z,b\rangle}db$. The easiest way to derive this is to consider a harmonic function u on the disc with boundary values $f(b)$. Then we have by the mean value theorem

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(b(\theta))\, d\theta = \frac{1}{2\pi}\int_{\mathbb{S}^1} P(z, b)f(b)\, db. \tag{1.15}$$

This implies that the Haar measure in (z, b) coordinates is

$$dg = d\mathrm{Vol}(z)d\theta = e^{\langle z,b\rangle}d\mathrm{Vol}(z)db. \tag{1.16}$$

One of the important identities involving the Busemann function is how it changes under

Figure 1.2: On the left, a horocycle through z and b and a horocycle through w and b on the hyperbolic disk. The so-called Busemann function, $\langle z, b\rangle$, is the distance from the origin o to this horocycle. $\langle z, b\rangle - \langle w, b\rangle$ represents the signed distance between the given horocycles. On the right, a family of horocycles through a single point $b \in B$. Any two horocycles through the same point b are equidistant to each other.
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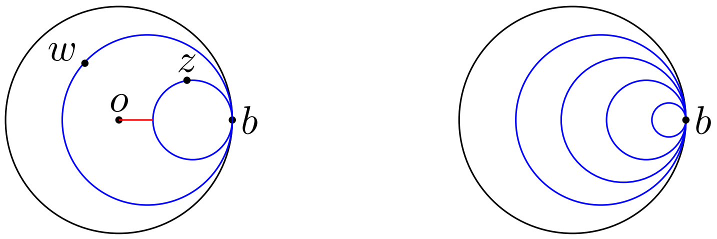

a left action of $\gamma \in G$ (acting on both z and b, that is, using the coordinates (z, b) on G). We note that $\langle z, b\rangle$ is not invariant under γ since it depends on a choice of origin in $\mathbb{D}$. However, a difference $\langle z, b\rangle - \langle w, b\rangle$ is invariant. We thus have

$$\langle \gamma z, \gamma b\rangle - \langle \gamma w, \gamma b\rangle = \langle z, b\rangle - \langle w, b\rangle \text{ for all } \gamma \in G.$$

Setting $w = 0$ and noting that $\langle 0, b\rangle = 0$ by definition, we find that for all $\gamma \in G$, we have

$$\langle \gamma z, \gamma b\rangle = \langle z, b\rangle + \langle \gamma 0, \gamma b\rangle. \tag{1.17}$$

Another useful identity involving the Busemann function is

$$\gamma'(b) = e^{-\langle \gamma(0), \gamma(b)\rangle}, \tag{1.18}$$

which can be derived from (1.14) and the fact that $u \circ \gamma$ is harmonic on the disc if and only if u is harmonic.

1.3 Helgason-Fourier transform and Zelditch pseudodifferential calculus on $\mathbb{D}$

In this section we will recall a version of the Fourier transform for hyperbolic space introduced by Helgason in [Hel74] and how Zelditch uses this to create a pseudodifferential calculus for hyperbolic space in [Zel86]. We remark that in Chapter 2, we shall derive a slightly different Helgason-Fourier transform that we call the *spectral* Helgason-Fourier transform. Hence we will also define a slightly different Zelditch quantisation on hyperbolic space (we may as well call it the *spectral* Zelditch calculus). In this chapter, we shall recall the previous work of Helgason and Zelditch for reference. We recall the following theorem from page 1120 of [AZ12], note that a similar version with different normalisation appears in Theorem 4.2 in [Hel81].

Theorem 3 (Helgason's non-Euclidean Fourier transform)**.** *For any complex-valued function f on $\mathbb{D}$, define the non-Euclidean Fourier transform by*

$$\mathcal{F}f(b,r) := \int_{\mathbb{D}} f(z) e^{(\frac{1}{2}-ir)\langle z,b\rangle} dVol(z).$$

for any $b \in B$ and $r \in \mathbb{C}$ for which this exists. If $f \in C_c^\infty(\mathbb{D})$, then there exists a pointwise inversion formula

$$f(z) = \frac{1}{2}\int_{\mathbb{R}\times B} e^{(\frac{1}{2}+ir)\langle z,b\rangle} \mathcal{F}f(b,r)dp(r)db, \quad dp(r) := \frac{r}{2\pi}\tanh\left(\frac{\pi r}{2}\right) dr. \tag{1.19}$$

This non-Euclidean Fourier transform extends to an isometry of $L^2(\mathbb{D}, dVol(z))$ to $L^2(\mathbb{R}^+ \times B, dp(r)db)$, where $dp(r)$ is as in (1.19).

Zelditch (in [Zel86]) uses Helgason's non-Euclidean Fourier transform to define a left-invariant pseudodifferential quantisation on $\mathbb{D}$. For any operator $A : C^\infty(\mathbb{D}) \mapsto C^\infty(\mathbb{D})$, Zelditch defines the *complete symbol* of A to be the function $a(z,b,r) \in C^\infty(\mathbb{D}\times B\times\mathbb{R}^+)$ by

$$Ae^{(\frac{1}{2}+ir)\langle z,b\rangle} = a(z,b,r)e^{(\frac{1}{2}+ir)\langle z,b\rangle} \tag{1.20}$$

Given a symbol, $a \in C^\infty(\mathbb{D}\times B\times\mathbb{R}^+)$, the definition of the pseudodifferential operator $\mathrm{Op}(a)$ acting on $u \in C_c^\infty(\mathbb{D})$ is consequently,

$$\mathrm{Op}(a)u(z) = \frac{1}{2\pi}\int_{\mathbb{D}_w\times B_b\times\mathbb{R}_r^+} a(z,b,r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}e^{(\frac{1}{2}-ir)\langle w,b\rangle}u(w)dwdbdp(r) \tag{1.21}$$

where $dp(r)$ is the same Plancherel measure as in (1.19). The Fourier inversion formula shows that $Au = \mathrm{Op}(a)u$ for A with complete symbol $a(z,b,r)$ and $u \in C_c^\infty(\mathbb{D})$, hence we have an exact correspondence between symbols $a \in C^\infty(\mathbb{D}\times B\times\mathbb{R}^+)$ and operators $A : C^\infty(\mathbb{D}) \mapsto C^\infty(\mathbb{D})$ on the class of functions $C_c^\infty(\mathbb{D})$.

Let Γ be a subgroup of G. The Zelditch calculus has the important property that a Zelditch quantised pseudodifferential operator, $\mathrm{Op}(a)$, commutes with the left action by Γ, in the sense that it commutes with the operators T_γ, $\gamma \in \Gamma$ acting by $(T_\gamma f)(z) = f(\gamma z)$, if and only if the symbol a is Γ-invariant in the sense

$$a(\gamma z, \gamma b, r) = a(z,b,r) \text{ for all } \gamma \in \Gamma, (z,b) \in G, r \in \mathbb{R}^+. \tag{1.22}$$

Now suppose that Γ is a discrete cocompact subgroup of G, that is, the quotient $\Gamma\backslash\mathbb{D}$ is a compact hyperbolic surface X. The invariance property just described shows that if a is a Γ-invariant symbol, then $\mathrm{Op}(a)$ at least formally maps Γ-invariant functions to Γ-invariant functions, and hence induces an operator on X.

Given an operator on $\mathbb{D}$ with Schwartz kernel $K(z,w)$ that is singular only on the diagonal and which is invariant under the left Γ-action of a discrete cocompact group, i.e $K(z,w) = K(\gamma z, \gamma w)$, if $K(z,\cdot)$ decays fast enough (so that the series below in (1.23) converges absolutely), we can define an induced operator on $\Gamma\backslash\mathbb{D}$ which has Schwartz kernel K^Γ defined by the series

$$K^\Gamma(z,w) = \sum_{\gamma\in\Gamma} K(z,\gamma w). \tag{1.23}$$

The fact that this is the appropriate induced kernel can be checked by the calculation

$$\begin{aligned}\int_X K^\Gamma(z,w) f^\Gamma(w)dw &= \sum_{\gamma\in\Gamma}\int_{\mathbb{D}} K(z,\gamma w) f^\Gamma(w)dw \\ &= \sum_{\gamma\in\Gamma}\int_{\mathbb{D}} K(z,w) f^\Gamma(\gamma^{-1}w)dw = \int_{\mathbb{D}} K(z,w) f(w)dw\end{aligned} \tag{1.24}$$

where f^Γ can be considered as the restriction of the Γ-periodic (automorphic) function f to a fundamental domain X, such that

$$\sum_{\gamma\in\Gamma} f^\Gamma(\gamma^{-1}w) = f(w). \tag{1.25}$$

Rather than consider f^Γ as a sharp cutoff of f to a fundamental domain, we could also define $f^\Gamma(w) := \chi(w)f(w)$ where χ is a smooth function on $\mathbb{D}$ with the property that

$$\sum_{\gamma\in\Gamma} \chi(\gamma^{-1}w) = 1. \tag{1.26}$$

The required decay of $K(z,w)$ away from the diagonal is implied by the symbol $a(z,b,r)$, having an analytic continuation in r to a strip of suitable width, cf. Eguchi-Kowata [EK76]. Section 4.3 of [AZ12] and the paper of [Zel86] goes into more detail regarding the definitition of $\mathrm{Op}(a)$ on a compact hyperbolic surface.

We also finally remark that through the identification of

$$S^*X \times \mathbb{R}^+ \simeq T^*X\backslash 0 \tag{1.27}$$

via polar coordinates in the fibres of T^*X for $X = \Gamma\backslash\mathbb{D}$, we can identify

$$\Gamma\backslash(\mathbb{D} \times B) \times \mathbb{R}^+ \simeq T^*X\backslash 0 \tag{1.28}$$

since $\Gamma\backslash(\mathbb{D} \times B) \simeq \Gamma\backslash G \simeq S^*X$. Consequently, the class of smooth functions on the cotangent bundle, with zero section removed, of X can be identified with the class of smooth functions in coordinates $z \in \mathbb{D}, b \in B, r \in \mathbb{R}^+$ which is left-Γ-invariant in the (z, b) variables. Hence, we can identify certain classes of symbols with symbols in the Zelditch calculus. For example, the positively homogeneous symbols of order m in the fibre, i.e $S^m_{1,0}(T^*X\backslash 0)$, is equivalent via this identification to the left Γ-invariant class of symbols, $a \in C^\infty(\Gamma\backslash G \times \mathbb{R}^+)$, where for any $s \in \mathbb{N}$ and any multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, there exists a constant $C_{s,\alpha} > 0$ such that

$$|(r\partial_r)^s H^{\alpha_1} X_+^{\alpha_2} X_-^{\alpha_3} a(g, r)| < C_{s,\alpha} r^m. \tag{1.29}$$

The work of Zelditch, [Zel86], gives formulae for the composition, adjoint and commutator of pseudodifferential operators in this calculus, as well as deriving an analogue of Egorov's theorem and Friedrichs symmetrization.

Chapter 2

Helgason's hyperbolic Fourier Transform, Zelditch's pseudodifferential calculus and Anantharaman-Zelditch's intertwining operator

2.1 Introduction

In this chapter we will derive a non-Euclidean Fourier transform in two dimensional hyperbolic space and we will also derive some of its properties: an inversion formula and a Plancherel theorem. This will turn out to be similar to the work of Helgason for his Helgason-Fourier transform as described in [Hel81], Theorem 4.2, however our normalisation will differ from Helgason's. We might call the version we derive here the *spectral* Helgason-Fourier transform because of the spectral normalisation.

Consequently, the spectral Helgason-Fourier transform allows us to use a version of Zelditch's quantisation on hyperbolic space as in [Zel86], however it will be again slightly different because we are using the spectral Helgason-Fourier transform which is normalised differently than the Helgason-Fourier transform that Zelditch uses in [Zel86]. So using this spectral version of Zelditch's pseudodifferential calculus on hyperbolic space, we will define the (spectral) Wigner distribution.

We will then construct a Patterson-Sullivan distribution which is similar, but again different, to what appears in the work of Anantharman-Zelditch in [AZ12]. These constructions naturally allow us to state one of the main theorems of this chapter, Theorem 14. The theorem states that the classical flow G_t^* and the quantum flow V_t, which we defined in the introduction of the thesis, are unitarily equivalent to one another between Hilbert spaces of L^2 symbols.

We will use the spectral theory of the hyperbolic Laplace-Beltrami operator on the Poincaré disk to derive the $L^2(\mathbb{D})$ theory of the spectral Helgason-Fourier transform. Doing so will also allow us to introduce the hyperbolic scattering matrix and we shall also explore the role that the hyperbolic scattering matrix plays in the spectral Helgason-Fourier transform, particularly in extending the Helgason-Fourier dual variables from $(b, r) \in \mathbb{S}^1 \times \mathbb{R}^+$ to $(b, r) \in \mathbb{S}^1 \times \mathbb{R}$. This also presents a new point of view on the symmetry properties of the spectral Helgason-Fourier transform and Zelditch's hyperbolic calculus under $r \mapsto -r$.

2.2 Deriving the $L^2(\mathbb{D})$ theory of the Helgason-Fourier transform via the spectral theorem.

In this section we shall show how the spectral theory of the hyperbolic Laplace-Beltrami operator can be used to derive the $L^2(\mathbb{D})$ theory of the Helgason-Fourier transform, we will carefully follow Borthwick's book in certain parts ([Bor16], Chapter 4). Note that Borthwick's book stops just short of connecting the resolution of the spectral measure of the Laplacian to the Helgason-Fourier transform.

The Poincaré disk is the set $\mathbb{D} := \{z = x + iy \in \mathbb{C} : |z| < 1\}$. We follow Borthwick and take $\rho(z) = (1 - |z|^2)/(1 + |z|^2)$ as a boundary defining function for $\mathbb{D}$, meaning ρ vanishes but $d\rho \neq 0$ on the boundary of the Poincaré disk, $\mathbb{S}^1$. We then define a generalised eigenfunction on $\mathbb{D}$ with spectral value $s \in \mathbb{C}$ as the following function of $z \in \mathbb{D}$, $b \in \mathbb{S}^1$,

$$E(s; z, b) = \lim_{z' \to b} \rho(z')^{-s} R_{\mathbb{D}}(s; z, z'), \tag{2.1}$$

where $z' \in \mathbb{D}$ and $R(s; z, z')$ is the resolvent kernel for the operator $\Delta_{\mathbb{D}} - s(1-s)$, with $\Delta_{\mathbb{D}}$ being the hyperbolic Laplace-Beltrami operator on $\mathbb{D}$, see Chapter 1. The resolvent kernel has an explicit form as proved in [Bor16] Section 4.1, with simple poles at $s \in -\mathbb{N}$,

$$R_{\mathbb{D}}(s; z, z') = \frac{\Gamma(s)^2}{4\pi} \sigma^{-s} \mathbf{F}(s, s; 2s; \sigma^{-1}) \tag{2.2}$$

where $\sigma = \cosh^2(d(z, z')/2)$ is a function of the hyperbolic distance between the points z and z' in $\mathbb{D}$, Γ is the Gamma function and $\mathbf{F}$ is the regularised hypergeometric function of Olver,

$$\begin{aligned}\mathbf{F}(a,b;c;w) &:= \frac{F(a,b;c;w)}{\Gamma(c)} \quad \text{where} \\ F(a,b;c;w) &:= 1 + \frac{ab}{1!c}w + \frac{a(a+1)b(b+1)}{2!c(c+1)}w^2 + \dots .\end{aligned} \tag{2.3}$$

$F(a,b;c;w)$ is the Gauss hypergeometric series, converging for $|w| < 1$, see [AS65] for more details on these special functions.

The reason the resolvent kernel has this form is because $R_{\mathbb{D}}(s;z,z')$ is a function of $d(z,z')$ as $\Delta_{\mathbb{D}}$ is an invariant operator under the isometry group of $\mathbb{D}$, $\mathrm{PSU}(1,1)/\mathrm{U}(1)$. Then by changing $\Delta_{\mathbb{D}}$ into hyperbolic polar coordinates, the equation to be solved for $R_{\mathbb{D}}(s;z,z') = f_s(d(z,z'))$ is a second order ordinary differential equation in the radial variable $r = d(z,z') > 0$, which is solvable and has the form of a certain hypergeometric function in r with parameters depending on s. Substituting back $r = d(z,z')$ gets the answer. The details of this derivation are in [Bor16] section 4.1.

Using the explicit formula for the resolvent kernel, Borthwick computes

$$\begin{aligned}E(s;z,b) &= \lim_{z'\to b} \rho(z')^{-s} R_{\mathbb{D}}(s;z,z') \\ &= \frac{2^{-s}}{2s-1}\frac{\Gamma(s)}{\Gamma(1/2)\Gamma(s-1/2)}\left(\frac{1-|z|^2}{|z-b|^2}\right)^s .\end{aligned} \tag{2.4}$$

Substituting $s = 1/2 + ir$, $r \in \mathbb{C}$, we get

$$\begin{aligned}E(1/2+ir;z,b) &= \frac{2^{-(\frac{1}{2}+ir)}}{2ir}\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)}\left(\frac{1-|z|^2}{|z-b|^2}\right)^{\frac{1}{2}+ir} \\ &= \frac{2^{-(\frac{1}{2}+ir)}}{2ir}\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)} e^{\left(\frac{1}{2}+ir\right)\langle z,b\rangle}\end{aligned} \tag{2.5}$$

where we set $e^{\langle z,b\rangle}$ to be $\frac{1-|z|^2}{|z-b|^2}$. One can check that under $s = 1$, the classical Poisson kernel for the disk is recovered,

$$E(1;z,b) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-b|^2}. \tag{2.6}$$

We have the following identity, proved in [Bor16] section 4.3,

$$R_{\mathbb{D}}(s;z,w) - R_{\mathbb{D}}(1-s;z,w) = (1-2s)\int_0^{2\pi} E(s;z,e^{i\theta'})E(1-s;w,e^{i\theta'})d\theta'. \tag{2.7}$$

We can apply this in Stone's formula for the spectral measure of $\Delta_{\mathbb{D}}$ on a domain $\mathcal{D} \subset$

$L^2(\mathbb{D})$. We take the usual domain on which $\Delta_{\mathbb{D}}$ is a self-adjoint operator, the Sobolev space $H^2(\mathbb{D})$. Therefore by Stone's theorem (see eg. [RS80], Chapter 8),

$$\frac{1}{2}(P[a,b]+P(a,b)) = \frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\int_a^b (\Delta-\lambda-i\epsilon)^{-1}-(\Delta-\lambda+i\epsilon)^{-1}d\lambda \tag{2.8}$$

where $P(E)$ is the spectral projection operator onto the part of the spectrum of $\Delta_{\mathbb{D}}$ in the borel set $E\subset\mathbb{R}$.

Fixing a small $\epsilon>0$, we define $s_{+\epsilon}$ to be the solution to $s(1-s)=\lambda+i\epsilon$ with $\Re(s_{+\epsilon})\geq 1/2$ and $s_{-\epsilon}$ to be the solution to $s(1-s)=\lambda-i\epsilon$ with $\Re(s_{-\epsilon})\geq 1/2$, where $\lambda\in\mathbb{R}$. Solving the quadratic equation, we have the solutions $s_{+\epsilon}$ and $s_{-\epsilon}$ which are

$$\begin{aligned} s_{+\epsilon} &= \frac{1}{2}+\sqrt{\frac{1}{4}-(\lambda+i\epsilon)}, \\ s_{-\epsilon} &= \frac{1}{2}+\sqrt{\frac{1}{4}-(\lambda-i\epsilon)}. \end{aligned} \tag{2.9}$$

We let s_+ and s_- to be the limit of $s_{+\epsilon}$ and $s_{-\epsilon}$ as $\epsilon\to 0^+$. So when $\lambda<1/4$, we have $s_+, s_-\in(1/2,1)$. The resolvent kernel, (2.2), is analytic for $s\in(1/2,1)$ which implies that the spectral measure is supported in $\lambda\geq 1/4$ (since applying the spectral theorem to a spectral window supported in $\lambda<1/4$ would give zero because the analyticity of the resolvent kernel means

$$\lim_{\epsilon\to 0^+}((\Delta-\lambda-i\epsilon)^{-1}-(\Delta-\lambda+i\epsilon)^{-1})=0 \tag{2.10}$$

where the limit is taken to be strong convergence). Now for $\lambda\geq 1/4$, we get

$$\begin{aligned} s_+ &= \frac{1}{2}-i\sqrt{\lambda-\frac{1}{4}}, \\ s_- &= \frac{1}{2}+i\sqrt{\lambda-\frac{1}{4}}. \end{aligned} \tag{2.11}$$

Since $\lambda\geq 1/4$, we can write $\sqrt{\lambda-1/4}=r$ for $r>0$, and so we have

$$s_+=\frac{1}{2}-ir, \quad s_-=\frac{1}{2}+ir, \quad \lambda=\frac{1}{4}+r^2, \quad r>0. \tag{2.12}$$

The spectral interval $\lambda\in[1/4,\infty)$, through the formula $\lambda=1/4+r^2$, is in bijection with $r\in[0,\infty)$ and the Lebesgue measure on $[1/4,\infty)_\lambda$ pushes forward to the measure $2rdr$ on the interval $[0,\infty)_r$.

It is also possible, using the explicit formula for the resolvent, to show that the spectral measure has no atoms, and is absolutely continuous with respect to the Lebesgue measure. To note down for the record, using the explicit formula (2.2), one can deduce (as in [Bor16] section 4.2) that

$$R_{\mathbb{D}}(s;z,z') - R_{\mathbb{D}}(1-s;z,z') = \frac{1}{2}\cot(\pi s)P_{s-1}(\cosh d(z,z')) \tag{2.13}$$

where P_ν is a Legendre function of parameter ν (see [AS65], Chapter 8 for details). In particular $P_\nu(z)$ is analytic in $\nu \in \mathbb{C}$, so the spectral measure is absolutely continuous with respect to the Lebesgue measure. Therefore, this work can be put together to express the spectral theorem for the Laplace operator in the following way,

$$P[a,b] = \frac{1}{2\pi i}\int_a^b \left[R_{\mathbb{D}}\left(\frac{1}{2}-ir\right) - R_{\mathbb{D}}\left(\frac{1}{2}+ir\right)\right] 2rdr. \tag{2.14}$$

Here, we can substitute the resolvent identity, (2.7), to get

$$P[a,b]f(z) = \frac{2}{\pi}\int_0^{2\pi}\int_a^b E\left(\frac{1}{2}-ir;z,e^{i\theta}\right)\left(\int_{\mathbb{D}} E\left(\frac{1}{2}+ir;w,e^{i\theta}\right)f(w)dw\right) r^2 dr d\theta. \tag{2.15}$$

Let us choose to distribute a factor of $2ir$ into the generalised eigenfunction, this means that we define a modified generalised eigenfunction, say denoted $e(s;z,b)$, through

$$e\left(\frac{1}{2}+ir;z,e^{i\theta}\right) := 2irE\left(\frac{1}{2}+ir;z,e^{i\theta}\right). \tag{2.16}$$

With this change, the spectral theorem now reads as

$$P[a,b]f(z) = \int_0^{2\pi}\int_a^b e\left(\frac{1}{2}-ir;z,e^{i\theta}\right)\left(\int_{\mathbb{D}} e\left(\frac{1}{2}+ir;w,e^{i\theta}\right)f(w)dw\right) dr\frac{d\theta}{2\pi}. \tag{2.17}$$

This suggests to us that, for a function $f \in C_c^\infty(\mathbb{D})$, we should define

$$\begin{aligned} \boldsymbol{\mathcal{F}} f(\theta,r) &:= \int_{\mathbb{D}} e\left(\frac{1}{2}+ir;w,e^{i\theta}\right)f(w)dw \\ &= \int_{\mathbb{D}} c(r)e^{(\frac{1}{2}+ir)\langle w,b\rangle}f(w)dw \end{aligned} \tag{2.18}$$

where

$$c(r) = 2^{-(\frac{1}{2}+ir)}\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)}, \qquad b = e^{i\theta}, \qquad e^{\langle w,b\rangle} = \frac{1-|w|^2}{|w-b|^2}. \tag{2.19}$$

Note that this is similar to the Helgason-Fourier transform as in [Hel81] Chapter 4, it is a modification of the Helgason-Fourier transform where we carry the $c(r)$ factor with the hyperbolic plane wave $e^{(\frac{1}{2}+ir)\langle z,b\rangle}$. We call $\boldsymbol{\mathcal{F}}$ defined in (2.18) the *spectral* Helgason-Fourier transform owing to the spectral point of view we take.

Continuing on, we read the spectral theorem as saying

$$P[a,b]f(z) = \int_0^{2\pi}\int_a^b e\left(\frac{1}{2}-ir;z,e^{i\theta}\right)\boldsymbol{\mathcal{F}}f(\theta,r)dr\frac{d\theta}{2\pi} \tag{2.20}$$

and since $P[0,R]f$ converges to f in the strong $L^2(\mathbb{D})$ topology as $R\to\infty$, we have that

$$f(z) = \lim_{R\to\infty}\int_0^{2\pi}\int_0^R e\left(\frac{1}{2}-ir;z,e^{i\theta}\right)\boldsymbol{\mathcal{F}}f(\theta,r)dr\frac{d\theta}{2\pi}. \tag{2.21}$$

This suggests that we should define the operator $\boldsymbol{\mathcal{F}}^{-1}$ on functions $g\in C_c^\infty(\mathbb{S}^1\times[0,\infty))$ by

$$\boldsymbol{\mathcal{F}}^{-1}g(z) := \int_0^{2\pi}\int_0^\infty e\left(\frac{1}{2}-ir;z,e^{i\theta}\right)g(e^{i\theta},r)dr\frac{d\theta}{2\pi}. \tag{2.22}$$

So equation (2.21) says that

$$f = \boldsymbol{\mathcal{F}}^{-1}(\boldsymbol{\mathcal{F}}f) \tag{2.23}$$

as an equality between functions in $L^2(\mathbb{D})$, by virtue of the spectral theorem. We shall show below in Theorem 4 that the notation $\boldsymbol{\mathcal{F}}^{-1}$ is justified because it is indeed the inverse operator to the continuous extension of $\boldsymbol{\mathcal{F}}$ to the domain $L^2(\mathbb{D},dz)$.

There are some further questions we can pose on characterising the range of the operators $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{F}}^{-1}$ on different domains for which equation (2.23) still holds. These questions are a little subtle. For example, we haven't specified or characterised when equation (2.21) holds pointwise, only that the limit exists as the convergence of functions in the $L^2(\mathbb{D})$ topology. Nonetheless, we can easily check that for $f\in C_c^\infty(\mathbb{D})$ the convergence of (2.21) holds pointwise too. Helgason provides a proof of the range of his Helgason-Fourier transform acting on the class of functions $C_c^\infty(\mathbb{D})$, in an analogue of the Paley-Wiener theorem in the hyperbolic context, in [Hel81] Theorem 4.2. There are also others who have worked on characterising the image of the Helgason-Fourier transform acting on other classes of functions, for example, the range of the Helgason-Fourier transform on the space of Schwartz functions of L^p type on $\mathbb{D}$ has been characterised by Eguchi-Kowata [EK76]. These are interesting questions that we do not explore further in this thesis.

To finish this section, we note that

$$\begin{aligned}\|P[a,b]f\|^2_{L^2(\mathbb{D},dz)} &= \langle P[a,b]f, f\rangle_{L^2(\mathbb{D},dz)} \\ &= \int_0^{2\pi}\int_a^b \left(\int_{\mathbb{D}} \overline{f(z)} e\left(\frac{1}{2} - ir; z, e^{i\theta}\right) dz\right) \boldsymbol{\mathcal{F}} f(\theta, r) dr \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi}\int_a^b |\boldsymbol{\mathcal{F}} f(\theta,r)|^2 dr \frac{d\theta}{2\pi}\end{aligned} \tag{2.24}$$

If we take $a \to 0$ and $b \to \infty$, we have a version of Plancherel's theorem that

$$\|f\|_{L^2(\mathbb{D},dz)} = \|\boldsymbol{\mathcal{F}} f\|_{L^2(\mathbb{S}^1\times[0,\infty), dr d\theta/2\pi)} \tag{2.25}$$

Therefore $\boldsymbol{\mathcal{F}}$ is an isometry from $L^2(\mathbb{D}, dz)$ to $L^2(\mathbb{S}^1 \times [0,\infty), dr\frac{d\theta}{2\pi})$. In fact, we have more,

Theorem 4. *$\boldsymbol{\mathcal{F}}$ is a unitary isomorphism between $L^2(\mathbb{D}, dz)$ and $L^2(\mathbb{S}^1 \times [0,\infty), dr\frac{d\theta}{2\pi})$.*

We believe that a version of this theorem holds in other models of two dimensional hyperbolic geometry too. While we use the boundary defining function $\rho(z) = (1-|z|^2)/(1+|z|^2)$ particular to $\mathbb{D}$, we suspect that the use of a different boundary defining function to any given model will only change the measure on $B \times [0,\infty)$, where B is the boundary. (Note that in the upper half plane, the function $\rho(x+iy) = y$ is not a good boundary defining function because $d\rho = 0$ at the point at infinity).

The proof of Theorem 4 will require a few lemmas and definitions. We shall first state and prove the required lemmas and then give a proof of Theorem 4 at the end of this subsection. The argument we shall give is similar to the proof Helgason gives in [Hel81], Chapter 4.

First we make a definition.

Definition 3. $r \in \mathbb{C}\backslash i(\frac{1}{2} + \mathbb{Z}_{\geq 0})$ is called *simple* if the mapping of $L^2(\mathbb{S}^1, \frac{d\theta}{2\pi}) \to C^\infty(\mathbb{D})$ given by

$$F(\theta) \mapsto f(z) = \int_0^{2\pi} e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) F(\theta) \frac{d\theta}{2\pi} \tag{2.26}$$

is one-to-one.

Note that we consider $r \notin i(\frac{1}{2} + \mathbb{Z}_{\geq 0})$ to exclude the poles of the Gamma function, which appear in the definition of $e(\frac{1}{2} + ir; z, e^{i\theta})$. The importance of this definition for our purposes is the following,

Lemma 5. *If r is simple, then the set $\{\boldsymbol{\mathcal{F}} f(\cdot, r) : f \in C_c^\infty(\mathbb{D})\}$ is dense in $L^2(\mathbb{S}^1, \frac{d\theta}{2\pi})$.*

Proof. If this were not so, let $F \neq 0$, $F \in L^2(\mathbb{S}^1, \frac{d\theta}{2\pi}) \cap C_c^\infty(\mathbb{S}^1)$ satisfy

$$\int_0^{2\pi} \boldsymbol{\mathcal{F}} f(\theta, r) F(\theta) \frac{d\theta}{2\pi} = 0 \tag{2.27}$$

for all $f \in C_c^\infty(\mathbb{D})$. This implies that

$$\int_{\mathbb{D}} f(z) \left(\int_0^{2\pi} e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) F(\theta) \frac{d\theta}{2\pi} \right) dz = 0 \tag{2.28}$$

for all $f \in C_c^\infty(\mathbb{D})$, which implies by the density of $C_c^\infty(\mathbb{D})$ in $L^2(\mathbb{D}, dz)$, that

$$\int_0^{2\pi} e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) F(\theta) \frac{d\theta}{2\pi} = 0 \tag{2.29}$$

for every $z \in \mathbb{D}$. However, this contradicts the simplicity of r. Therefore, for r simple, the set $\{\boldsymbol{\mathcal{F}} f(\cdot, r) : f \in C_c^\infty(\mathbb{D})\}$ is dense in $L^2(\mathbb{S}^1, \frac{d\theta}{2\pi})$. □

We have the following lemma,

Lemma 6. *Every $r \in \mathbb{C} \backslash i(\frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup i\mathbb{Z}_{\geq 0}$ is simple.*

Proof. Let $r \in \mathbb{C} \backslash i(\frac{1}{2} + \mathbb{Z}_{\geq 0})$. We set $k := -(\frac{1}{2} + ir)$, note that $k \notin \mathbb{Z}_{\geq 0}$ because this is equivalent to $r \notin i(\frac{1}{2} + \mathbb{Z}_{\geq 0})$. Given $F(\theta) \in L^2(\mathbb{S}^1, \frac{d\theta}{2\pi}) \cap C_c^\infty(\mathbb{S}^1)$, we aim to show that for

$$\begin{aligned} f(z) &:= \int_0^{2\pi} e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) F(\theta) \frac{d\theta}{2\pi} \\ &= 2^{-(\frac{1}{2}+ir)} \frac{\Gamma(\frac{1}{2} + ir)}{\Gamma(1/2)\Gamma(ir)} \int_0^{2\pi} \left(\frac{|z - e^{i\theta}|^2}{1 - |z|^2} \right)^k F(\theta) \frac{d\theta}{2\pi}, \end{aligned} \tag{2.30}$$

$f = 0$ implies $F = 0$. We observe that this fails to be true when $r = im$, $m \in \mathbb{Z}_{\geq 0}$, because the prefactor

$$2^{-(\frac{1}{2}+ir)} \frac{\Gamma(\frac{1}{2} + ir)}{\Gamma(1/2)\Gamma(ir)} \tag{2.31}$$

vanishes. So let us now also assume $r \notin i\mathbb{Z}_{\geq 0}$. The strategy of the proof will be to assume $f = 0$ and use the fact that the q-jet of f at $z = 0$ is zero for every $q \in \mathbb{N}$. This fact will allow us to show that the $\pm q$'th Fourier mode of F is zero for every $q \in \mathbb{N}$ and hence $F = 0$. We let $f = 0$. This implies that

$$\int_0^{2\pi} F(\theta) \frac{d\theta}{2\pi} = 0. \tag{2.32}$$

Moreover, as we shall show in Lemma 7, as a consequence of the fact that

$$\left.\frac{\partial^n f}{\partial z^n}\right|_{z=0} = \left.\frac{\partial^n f}{\partial \overline{z}^n}\right|_{z=0} = 0 \tag{2.33}$$

for every $n \in \mathbb{N}$, we get that

$$\int_0^{2\pi} e^{\pm in\theta} F(\theta) \frac{d\theta}{2\pi} = 0 \tag{2.34}$$

for every $n \in \mathbb{N}$, and so $F = 0$. □

We have used the result of the following technical lemma in the proof of Lemma 6.

Lemma 7. *For $k \notin \mathbb{Z}_{\geq 0}$ and*

$$A(z, \overline{z}, \theta) := \left(\frac{|z - e^{i\theta}|^2}{1 - |z|^2}\right)^k \tag{2.35}$$

we have for every $n \in \mathbb{N}$,

$$\left.\frac{\partial^n}{\partial z^n}\right|_{z=0} A(z, \overline{z}, \theta) = k(k-1)\ldots(k-(n+1))(-1)^n e^{-in\theta} \tag{2.36}$$

and

$$\left.\frac{\partial^n}{\partial \overline{z}^n}\right|_{z=0} A(z, \overline{z}, \theta) = k(k-1)\ldots(k-(n+1))(-1)^n e^{in\theta}. \tag{2.37}$$

Proof. Note that

$$A(z, \overline{z}, \theta) = \exp(k \log g(z, \overline{z}, \theta)) \tag{2.38}$$

where

$$g(z, \overline{z}, \theta) := (1 - |z|^2)^{-1}(1 + |z|^2 - ze^{-i\theta} - \overline{z}e^{i\theta}). \tag{2.39}$$

We note that any derivatives in z or $\overline{z}$ of $(1 - |z|^2)^{-1}$ will vanish when evaluated at $z = \overline{z} = 0$. Hence the behaviour of higher order derivatives of g at $z = \overline{z} = 0$ are determined by the derivatives of

$$h(z, \overline{z}, \theta) := 1 + |z|^2 - ze^{-i\theta} - \overline{z}e^{i\theta}. \tag{2.40}$$

We note that

$$\left.\frac{\partial h}{\partial z}\right|_{z=0} = -e^{-i\theta}, \quad \left.\frac{\partial h}{\partial \overline{z}}\right|_{z=0} = -e^{i\theta}. \tag{2.41}$$

From these observations, we deduce that

$$\left.\frac{\partial g}{\partial z}\right|_{z=0} = -e^{-i\theta}, \quad \left.\frac{\partial g}{\partial \overline{z}}\right|_{z=0} = -e^{i\theta},$$

and any higher derivatives of g in z or $\overline{z}$ vanish when evaluated at $z = \overline{z} = 0$. Now let us consider $A(z, \overline{z}, \theta)$. The n'th derivative of A with respect to z equals

$$k(k-1)\dots(k-(n+1))g(z,\overline{z},\theta)^{k-n}\left(\frac{\partial g}{\partial z}\right)^n + P(z,\overline{z},\theta)$$

where $P(z, \overline{z}, \theta)$ contains higher order derivatives in g. When we evaluate this term at $z = \overline{z} = 0$, we get

$$\left.\frac{\partial^n A}{\partial z^n}\right|_{z=0} = k(k-1)\dots(k-(n+1))(-1)^n e^{-in\theta}$$

because P vanishes when evaluated at $z = \overline{z} = 0$. Since $k \notin \mathbb{Z}_{\geq 0}$, the n'th derivative of A with respect to z is a non-zero multiple of $e^{-in\theta}$. Likewise, taking the n'th derivative of A with respect to $\overline{z}$ gives us

$$\left.\frac{\partial^n A}{\partial \overline{z}^n}\right|_{z=0} = k(k-1)\dots(k-(n+1))(-1)^n e^{in\theta}.$$

□

Definition 4. We say that $\psi \in C_c^\infty(\mathbb{D})$ is a *radial function* if for any $k \in K$, we have $\psi(k \cdot z) = \psi(z)$. We let $I_c^\infty(\mathbb{D})$ be the subset of radial functions in $C_c^\infty(\mathbb{D})$.

We will make use of the fact that the spectral Helgason-Fourier transform of radial functions $\psi(z)$ are constant in θ.

Lemma 8. *For $\psi \in I_c^\infty(\mathbb{D})$, we have that $\boldsymbol{\mathcal{F}}\psi(\theta, r)$ is constant in θ.*

Proof. Let $\psi \in I_c^\infty(\mathbb{D})$. We have that

$$\boldsymbol{\mathcal{F}}\psi(\theta', r) = \int_{\mathbb{D}} e\left(\frac{1}{2} + ir; z, k_{\theta'-\theta} \cdot e^{i\theta}\right)\psi(z)dz \tag{2.42}$$

where $k_{\theta'-\theta} \in K = \mathrm{PSO}(2)$. By a change of variables, letting $z = k_{\theta'-\theta} \cdot z'$, we have

$$\boldsymbol{\mathcal{F}}\psi(\theta', r) = \int_{\mathbb{D}} e\left(\frac{1}{2} + ir; k_{\theta'-\theta} \cdot z', k_{\theta'-\theta} \cdot e^{i\theta}\right)\psi(k_{\theta'-\theta} \cdot z')d(k_{\theta'-\theta} \cdot z'). \tag{2.43}$$

Since $d(k \cdot z') = dz'$ for any $k \in K$, and because $\langle k \cdot z', k \cdot e^{i\theta}\rangle = \langle z', e^{i\theta}\rangle + \langle k \cdot o, k \cdot e^{i\theta}\rangle$ and $\langle k \cdot o, k \cdot e^{i\theta}\rangle = \langle o, k \cdot e^{i\theta}\rangle = 0$ for any $k \in K$, we have, using the radial property of ψ,

$$\boldsymbol{\mathcal{F}}\psi(\theta', r) = \boldsymbol{\mathcal{F}}\psi(\theta, r). \tag{2.44}$$

Hence $\boldsymbol{\mathcal{F}}\psi(\theta, r)$ is constant in θ for any radial $\psi \in I_c^\infty(\mathbb{D})$. □

Because of this property for radial $\psi \in I_c^\infty(\mathbb{D})$, we will use the shorthand $\boldsymbol{\mathcal{F}}\psi(r)$ to refer to its spectral Helgason-Fourier transform. We have the following lemma regarding the spectral Helgason-Fourier transform of the convolution of two functions $f, \psi \in C_c^\infty(\mathbb{D})$, one of which is radial. Note that it analogises the same result for the Fourier transform of the convolution of two functions on Euclidean space.

Definition 5. For $f \in C_c^\infty(\mathbb{D})$ and $\psi \in I_c^\infty(\mathbb{D})$, we define the *convolution*

$$f * \psi(z) := \int_G f(g \cdot o)\psi(g^{-1} \cdot z)dg. \tag{2.45}$$

Lemma 9. *For $f \in C_c^\infty(\mathbb{D})$ and $\psi \in I_c^\infty(\mathbb{D})$, we have*

$$\boldsymbol{\mathcal{F}}(f * \psi)(\theta, r) = 2^{(\frac{1}{2}+ir)}\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\boldsymbol{\mathcal{F}}f(\theta, r)\boldsymbol{\mathcal{F}}\psi(r). \tag{2.46}$$

Proof. We have

$$\begin{aligned}\boldsymbol{\mathcal{F}}(f * \psi)(\theta, r) &= \int_{\mathbb{D}}\left(\int_G f(g \cdot o)\psi(g^{-1} \cdot z)dg\right) e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) dz \\ &= \int_G f(g \cdot o)\left(\int_{\mathbb{D}} \psi(g^{-1} \cdot z)e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) dz\right) dg.\end{aligned} \tag{2.47}$$

Changing variables by setting $z = g \cdot z'$ and noting $d(g \cdot z') = dz'$, we reduce the above expression to

$$\int_G f(g \cdot o)\left(\int_{\mathbb{D}} \psi(z')e\left(\frac{1}{2} + ir; g \cdot z', e^{i\theta}\right) dz'\right) dg. \tag{2.48}$$

We note that $\langle g \cdot z', e^{i\theta}\rangle = \langle z', g^{-1} \cdot e^{i\theta}\rangle + \langle g \cdot o, e^{i\theta}\rangle$, hence

$$e\left(\frac{1}{2} + ir; g \cdot z', e^{i\theta}\right) = 2^{(\frac{1}{2}+ir)}\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}e\left(\frac{1}{2} + ir; z', g^{-1} \cdot e^{i\theta}\right) e\left(\frac{1}{2} + ir; g \cdot o, e^{i\theta}\right), \tag{2.49}$$

and thus we have

$$\begin{aligned}\boldsymbol{\mathcal{F}}(f * \psi)(\theta, r) &= 2^{(\frac{1}{2}+ir)}\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\boldsymbol{\mathcal{F}}\psi(r)\left(\int_G f(g \cdot o)e\left(\frac{1}{2} + ir; g \cdot o, e^{i\theta}\right) dg\right) \\ &= 2^{(\frac{1}{2}+ir)}\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\boldsymbol{\mathcal{F}}\psi(r)\boldsymbol{\mathcal{F}}f(\theta, r).\end{aligned} \tag{2.50}$$

□

We make a note that

$$2^{(\frac{1}{2}+ir)}\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\boldsymbol{\mathcal{F}}\psi(r) = \mathcal{F}\psi(r) \tag{2.51}$$

where $\mathcal{F}\psi(r)$ is the usual Helgason-Fourier transform of the radial function $\psi(r)$. We remark that the Helgason-Fourier transform of a radial function is sometimes called the spherical transform in certain literature. So we can rephrase Lemma 9 as stating that

$$\boldsymbol{\mathcal{F}}(f * \psi)(r) = \mathcal{F}\psi(r)\boldsymbol{\mathcal{F}}f(\theta, r) \tag{2.52}$$

for $f \in C_c^\infty(\mathbb{D})$ and $\psi \in I_c^\infty(\mathbb{D})$.

Proof of Theorem 4. The isometry property of the spectral Helgason-Fourier transform, (2.25), implies that $\boldsymbol{\mathcal{F}}$ is injective onto a closed subspace of $L^2(\mathbb{S}^1 \times [0,\infty), dr d\theta/2\pi)$. What remains is to show that $\boldsymbol{\mathcal{F}}$ is surjective.

Let us suppose that $F \in L^2(\mathbb{S}^1 \times [0,\infty), dr d\theta/2\pi) \cap C_c^\infty(\mathbb{S}^1 \times [0,\infty))$ is orthogonal to the range of the spectral Helgason-Fourier transform, i.e that

$$\int_0^{2\pi}\int_0^\infty \boldsymbol{\mathcal{F}}f(\theta, r)F(\theta, r)dr\frac{d\theta}{2\pi} = 0 \tag{2.53}$$

for all $f \in C_c^\infty(\mathbb{D})$. We can replace $\boldsymbol{\mathcal{F}}f(\theta, r)$ with $\boldsymbol{\mathcal{F}}(f * \psi)(\theta, r)$ for any $\psi \in I_c^\infty(\mathbb{D})$ as $f * \psi \in C_c^\infty(\mathbb{D})$. Since $\boldsymbol{\mathcal{F}}(f * \psi)(\theta, r) = \mathcal{F}\psi(r)\boldsymbol{\mathcal{F}}f(\theta, r)$, the effect of this replacement is that we get

$$\int_0^{2\pi}\int_0^\infty \boldsymbol{\mathcal{F}}(f*\psi)(\theta, r)F(\theta, r)dr\frac{d\theta}{2\pi} = \int_0^\infty \mathcal{F}\psi(r)\left(\int_0^{2\pi} \boldsymbol{\mathcal{F}}f(\theta, r)F(\theta, r)\frac{d\theta}{2\pi}\right)dr = 0 \tag{2.54}$$

for all $\psi \in I_c^\infty(\mathbb{D})$ and all $f \in C_c^\infty(\mathbb{D})$. The Helgason-Fourier transform bijects $I_c^\infty(\mathbb{D})$ onto $\mathcal{H}_e(\mathbb{C})$, the space of even entire functions in $r \in \mathbb{C}$ of exponential type, as proved in [Hel81], Theorem 4.7. Therefore by the Stone-Weierstrass theorem, this implies

$$\int_0^{2\pi} \boldsymbol{\mathcal{F}}f(\theta, r)F(\theta, r)\frac{d\theta}{2\pi} = 0 \tag{2.55}$$

for every $r \in [0,\infty)$. Recalling that this set of r is simple and for r simple, the set $\{\boldsymbol{\mathcal{F}}f(\cdot, r) : f \in C_c^\infty(\mathbb{D})\}$ is dense in $L^2(\mathbb{S}^1, \frac{d\theta}{2\pi})$, we therefore have that

$$\int_0^{2\pi} P(\theta)F(\theta, r)\frac{d\theta}{2\pi} = 0 \tag{2.56}$$

for all $P \in L^2(\mathbb{S}^1, \frac{d\theta}{2\pi})$ and for all $r \in [0,\infty)$. We can choose P to be the characteristic function on any closed arc $I \subset \mathbb{S}^1$, so for any such I, we get

$$\int_I F(\theta, r)\frac{d\theta}{2\pi} = 0 \tag{2.57}$$

for all $r \in [0, \infty)$. Then integrating the above function of r over a closed interval $J \subset [0, \infty)$, we get

$$\int_{I \times J} F(\theta, r) \frac{d\theta}{2\pi} dr = 0. \tag{2.58}$$

Since this is true for every rectangle $I \times J \subset \mathbb{S}^1 \times [0, \infty)$, by the Stone-Weierstrass theorem $F = 0$. $\square$

In the sections which follow we will continue with the more concise notation,

$$e(z, b, r) := e\left(\frac{1}{2} + ir; z, e^{i\theta}\right) = 2^{-(\frac{1}{2}+ir)} \frac{\Gamma(\frac{1}{2} + ir)}{\Gamma(1/2)\Gamma(ir)} \left(\frac{1 - |z|^2}{|z - b|^2}\right)^{\frac{1}{2}+ir} \tag{2.59}$$

where $b = e^{i\theta}$.

2.3 The hyperbolic scattering matrix and symbols.

We define the hyperbolic scattering matrix with spectral value $r \in \mathbb{C}$ through its Schwartz kernel as

$$S_{\mathbb{D}}(r; b, b') := 4\pi ir \lim_{z \to b, z' \to b'} (\rho(z)\rho(z'))^{-(\frac{1}{2}+ir)} R_{\mathbb{D}}\left(\frac{1}{2} + ir; z, z'\right), \tag{2.60}$$

for $b, b' \in \mathbb{S}^1$, recalling $R_{\mathbb{D}}(\frac{1}{2} + ir; z, z')$ is the resolvent kernel of the hyperbolic Laplacian, equation (2.2), and

$$\rho(z) = \frac{1 - |z|^2}{1 + |z|^2}$$

is a choice of boundary defining function for $\mathbb{D}$.

Note that

$$\sigma(z, z') := \cosh^2(d(z, z')/2) = \frac{|1 - z\overline{z'}|^2}{(1 - |z|^2)(1 - |z'|^2)} \tag{2.61}$$

and hence

$$(\rho(z)\rho(z')\sigma(z, z'))^{-(\frac{1}{2}+ir)} = ((1 + |z|^2)(1 + |z'|^2))^{(\frac{1}{2}+ir)} \frac{1}{|1 - z\overline{z'}|^{1+2ir}} \tag{2.62}$$

which implies

$$\lim_{z \to b, z' \to b'} (\rho(z)\rho(z')\sigma(z, z'))^{-(\frac{1}{2}+ir)} = 2^{1+2ir} \frac{1}{|b - b'|^{1+2ir}}. \tag{2.63}$$

Since the resolvent kernel (2.2) is $\sigma^{-(\frac{1}{2}+ir)}$ multiplied by a convergent power series in σ^{-1}, as we take the limit with $z \to b$, $z' \to b'$ with $b \neq b'$, the distance between z and z' tends to infinity and so $\sigma^{-1} \to 0$. Therefore, in limit (2.60) defining the hyperbolic scattering kernel, we obtain

$$4\pi ir A(r) \lim_{z\to b, z'\to b'} (\rho(z)\rho(z')\sigma(z,z'))^{-(\frac{1}{2}+ir)} = 4\pi ir A(r) 2^{1+2ir} \frac{1}{|b-b'|^{1+2ir}}. \tag{2.64}$$

where $A(r)$ is the constant term in (2.2), the power series expansion of $R_{\mathbb{D}}(s;z,z')$ in terms of σ^{-1}. In particular,

$$A(r) = \frac{\Gamma(\frac{1}{2}+ir)^2}{4\pi\Gamma(1+2ir)}.$$

We can use the Legendre duplication formula for the Gamma function[1] to express this prefactor differently:

$$\begin{aligned} A(r) = \frac{\Gamma(\frac{1}{2}+ir)^2}{4\pi\Gamma\left(2\left(\frac{1}{2}+ir\right)\right)} &= \frac{\Gamma(\frac{1}{2}+ir)^2}{4\pi\pi^{-\frac{1}{2}}2^{2ir}\Gamma\left(\frac{1}{2}+ir\right)\Gamma(1+ir)} \\ &= \frac{\Gamma(\frac{1}{2}+ir)}{4\pi^{\frac{1}{2}}2^{2ir}ir\Gamma(ir)}. \end{aligned} \tag{2.65}$$

Hence, from (2.63) and (2.65), the scattering matrix kernel in the disk model with the particular choice of boundary defining function $\rho = \frac{1-|z|^2}{1+|z|^2}$ is

$$S_{\mathbb{D}}(r;b,b') = 2\pi \frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)} \frac{1}{|b-b'|^{1+2ir}}. \tag{2.66}$$

Let us make a remark as to how these scattering matrix kernels act as distributions. It is clear that for $\Im(r) > 0$ and $ir \notin \frac{1}{2} - \mathbb{N}$, $S_{\mathbb{D}}(r;b,b')$ is a well-defined integral kernel because the singularity at $b = b'$ is integrable, however as $\Im(r) \to 0^+$ we no longer retain this integrability. Nonetheless, we can make sense of the action of $S_{\mathbb{D}}(r)$ via an analytic continuation argument. Fixing $f \in C_c^\infty(B)$ and $\Im(r) > 0$, we can take a Taylor expansion of f around b, so $f(b') = f(b) + R_b(b')$ where $R_b(b')$ vanishes to first order at $b' = b$, hence

$$\int_{\mathbb{S}^1} \frac{f(b')}{|b-b'|^{1+2ir}} db' = \int_{\mathbb{S}^1} \frac{f(b)}{|b-b'|^{1+2ir}} db' + \int_{\mathbb{S}^1} \frac{R_b(b')}{|b-b'|^{1+2ir}} db'. \tag{2.67}$$

We can compute the first term explicitly, We take stereographic coordinates for $b' \in \mathbb{S}^1$ centred around b (so b is the 'point at infinity' in stereographic coordinates). Without

[1] Recall the duplication formula states that $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\Gamma(1/2)\Gamma(2z)$ for $z \notin -\mathbb{N}/2$.

loss of generality, we can change coordinates so that $b = 1$, then we have

$$b' = \mathbf{n}_u(-1) = \frac{u^2-1}{u^2+1} + i\frac{2u}{u^2+1}, \qquad u \in \mathbb{R}. \tag{2.68}$$

In these coordinates we calculate

$$|b-b'|^{-(1+2ir)} = \frac{(1+u^2)^{\frac{1}{2}+ir}}{2^{1+2ir}} \tag{2.69}$$

and

$$db' = \frac{1}{\pi}\frac{du}{1+u^2}, \tag{2.70}$$

hence

$$\int_B \frac{1}{|b-b'|^{1+2ir}} db' = \frac{2^{-(1+2ir)}}{\pi} \int_{\mathbb{R}} (1+u^2)^{-(\frac{1}{2}-ir)} du. \tag{2.71}$$

We change coordinates again with $t = (1+u^2)^{-1}$ which gives

$$\frac{2^{-(1+2ir)}}{\pi} \int_{\mathbb{R}} (1+u^2)^{-(\frac{1}{2}-ir)} du = \frac{2^{-(1+2ir)}}{\pi} \int_0^1 t^{-1-ir}(1-t)^{-\frac{1}{2}} dt. \tag{2.72}$$

We recall the Beta function is defined as

$$B(z_1, z_2) := \int_0^1 t^{-1+z_1}(1-t)^{-1+z_2} dt \tag{2.73}$$

which is well defined for $\Re(z_1), \Re(z_2) > 0$. Moreover, the following identity holds[2],

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}, \tag{2.74}$$

where $\Gamma(z)$ is the Gamma function. Consequently, following on from equations (2.71) and (2.72), this implies that

$$\int_B \frac{1}{|b-b'|^{1+2ir}} db' = \frac{2^{-(1+2ir)}}{\pi} \frac{\Gamma(-ir)\Gamma(1/2)}{\Gamma(\frac{1}{2}-ir)}. \tag{2.75}$$

We thus have for $\Im(r) > 0$,

$$\begin{aligned}\int_B S(r;b,b')f(b')db' &= 2\pi\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)} \int_B \frac{f(b')}{|b-b'|^{1+2ir}} db' \\ &= 2^{-2ir}\frac{\Gamma(\frac{1}{2}+ir)\Gamma(-ir)}{\Gamma(\frac{1}{2}-ir)\Gamma(ir)} f(b) + 2\pi\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)} \int_B \frac{R_b(b')}{|b-b'|^{1+2ir}} db'.\end{aligned} \tag{2.76}$$

[2]See https://en.wikipedia.org/wiki/Beta_function for a simple derivation of this.

Note that the linear functional (acting on f) on the right hand side of equation (2.76) is well defined for $\Im(r) > -1/2$ and $ir \notin \frac{1}{2} - \mathbb{N}$ because $R_b(b')$ has one order of vanishing at $b' = b$.

Therefore (2.76) is the unique analytic (in r) extension of the action of $S_{\mathbb{D}}(r; b, b')$ from $\Im(r) > 0$ to $\Im(r) > -1/2$, avoiding an exceptional discrete set. We can also observe from (2.76) that $\mathscr{S}_{\mathbb{D}}(0; b, b') = -\delta_b(b')$ because

$$\begin{aligned} &\lim_{r\to 0} 2^{-2ir} \frac{\Gamma(\frac{1}{2} + ir)\Gamma(-ir)}{\Gamma(\frac{1}{2} - ir)\Gamma(ir)} = -1, \quad \text{and} \\ &\lim_{r\to 0} 2\pi \frac{\Gamma(\frac{1}{2} + ir)}{\Gamma(1/2)\Gamma(ir)} = 0. \end{aligned} \tag{2.77}$$

One can continue to define the action of the hyperbolic scattering matrix $S(r)$ to all $r \in \mathbb{C}$, avoiding a discrete set via this analytic continuation argument by taking higher order Taylor expansions of f about b.

2.3.1 Scattering as the matrix map from the Fourier basis of non-Euclidean plane waves with $r > 0$ to $r < 0$.

The reason for us introducing the hyperbolic scattering matrix is that this is the matrix kernel which maps us from the Fourier basis of non-Euclidean plane waves with $r > 0$ to the Fourier basis of non-Euclidean plane waves with $r < 0$.

Lemma 10. *We claim that*

$$e(z, b, -r) = \int_B S_{\mathbb{D}}(-r; b, b') e(z, b', r) db', \qquad \textit{for all} \quad r \in \mathbb{R}, b \in \mathbb{S}^1, z \in \mathbb{D}. \tag{2.78}$$

Proof. We will first study

$$\int_B |b - b'|^{-1+2ir} \left(\frac{1 - |z|^2}{|z - b'|^2} \right)^{\frac{1}{2}+ir} db \tag{2.79}$$

for $z \in \mathbb{D}, b \in \mathbb{S}^1$ and $r \in \mathbb{R}$ fixed and show that it equals

$$\left(\frac{1 - |z|^2}{|z - b|^2} \right)^{\frac{1}{2}-ir} \tag{2.80}$$

up to multiplication by a complex function of r. We will then reintroduce the prefactor functions of r for $e(z,b',r), S(-r;b,b')$ and $e(z,b,-r)$ to prove the lemma. Let us define

$$F(z,b,r) := (1-|z|^2)^{\frac{1}{2}+ir}\int_{\mathbb{S}^1}|z-b'|^{-1-2ir}|b-b'|^{-1+2ir}db'. \tag{2.81}$$

We can assume $\Im(r) < 0$ so the integrand is integrable and then argue by analytic continuation in r like before. We observe that $F(e^{i\theta}z, e^{i\theta}b, r) = F(z,b,r)$ for every $\theta \in \mathbb{R}$, hence without loss of generality, we can set $b = 1$. The claim is then that

$$F(z,1,r) = \left(\frac{1-|z|^2}{|z-1|^2}\right)^{\frac{1}{2}-ir}. \tag{2.82}$$

It is helpful to express z and b' in a different coordinate system, say the horocyclic-geodesic coordinates based at $b = 1$. So we let $b' = \mathbf{n}_v(-1)$ like before, where $v \in \mathbb{R}$ parameterises $b' \in \mathbb{S}^1$ and we let $z = \mathbf{a}_t\mathbf{n}_u 0$ where $t, u \in \mathbb{R}$ parameterise $z \in \mathbb{D}$. Note that in these coordinates, we have

$$\begin{aligned} b' &= \mathbf{n}_v(-1) = \frac{u^2-1}{u^2+1} + i\frac{2u}{u^2+1}, \quad \text{and} \\ z &= \mathbf{a}_t\mathbf{n}_u 0 = 1 - \frac{2}{1+e^t+iue^t}. \end{aligned} \tag{2.83}$$

Then we can calculate that

$$\left(\frac{1-|z|^2}{|z-1|^2}\right)^{\frac{1}{2}-ir} = e^{(\frac{1}{2}-ir)t} \tag{2.84}$$

in these coordinates. One can also verify the explicit calculation that

$$|z-b'|^2 = |\mathbf{a}_t\mathbf{n}_u 0 - \mathbf{n}_v(-1)|^2 = \frac{4e^t(e^t + (e^{t/2}u - e^{-t/2}v)^2)}{(1+v^2)((1+e^t)^2 + (ue^t)^2)}. \tag{2.85}$$

Note that for $z = \mathbf{a}_t\mathbf{n}_u 0$, we calculate

$$1-|z|^2 = \frac{4e^t}{(1+e^t)^2+(ue^t)^2} \tag{2.86}$$

and for $b' = \mathbf{n}_v(-1)$, we calculate

$$|b'-1|^2 = |\mathbf{n}_v(-1)-1|^2 = \frac{4}{1+v^2}, \qquad db' = \frac{1}{\pi}\frac{dv}{1+v^2}. \tag{2.87}$$

Therefore $F(z,1,r)$ in (2.81) equals

$$\frac{1}{\pi}4^{-\frac{1}{2}+ir}\int_{\mathbb{R}}(e^t+(e^{t/2}u-e^{-t/2}v)^2)^{-\frac{1}{2}-ir}(1+v^2)^{\frac{1}{2}+ir}(1+v^2)^{\frac{1}{2}-ir}\frac{dv}{1+v^2} \tag{2.88}$$

which reduces to

$$\frac{1}{\pi}4^{-\frac{1}{2}+ir}\int_{\mathbb{R}}(e^t+(e^{t/2}u-e^{-t/2}v)^2)^{-\frac{1}{2}-ir}dv. \tag{2.89}$$

Note that we can take a factor of $e^{(-\frac{1}{2}-ir)t}$ from the integrand to reduce the integral to

$$\frac{1}{\pi}4^{-\frac{1}{2}+ir}e^{(-\frac{1}{2}-ir)t}\int_{\mathbb{R}}(1+(u-e^{-t}v)^2)^{-\frac{1}{2}-ir}dv. \tag{2.90}$$

We then change variables letting $w=u-e^{-t}v$, note that $dv=e^t dw$, to further reduce to

$$\frac{1}{\pi}4^{-\frac{1}{2}+ir}e^{(\frac{1}{2}-ir)t}\int_{\mathbb{R}}(1+w^2)^{-\frac{1}{2}-ir}dw. \tag{2.91}$$

We have, by a change of variables $t=(1+w^2)^{-1}$, that

$$\int_{\mathbb{R}}(1+w^2)^{-(\frac{1}{2}+ir)}dw=\int_0^1 t^{-1+ir}(1-t)^{-\frac{1}{2}}dt=\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}. \tag{2.92}$$

Hence we conclude that

$$F(z,b,r)=\frac{2^{-(1-2ir)}}{\pi}\frac{\Gamma(1/2)\Gamma(ir)}{\Gamma(\frac{1}{2}+ir)}\left(\frac{1-|z|^2}{|z-b|^2}\right)^{\frac{1}{2}-ir}. \tag{2.93}$$

We now reintroduce the prefactor functions of r for $e(z,b',r)$ and $S(-r;b,b')$, recalling that

$$\begin{aligned}
e(z,b',r)&=2^{-(\frac{1}{2}+ir)}\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(1/2)\Gamma(ir)}\left(\frac{1-|z|^2}{|z-b'|^2}\right)^{\frac{1}{2}+ir}, \quad \text{and}\\
S_{\mathbb{D}}(-r;b,b')&=2\pi\frac{\Gamma(\frac{1}{2}-ir)}{\Gamma(1/2)\Gamma(-ir)}\frac{1}{|b-b'|^{1-2ir}},
\end{aligned} \tag{2.94}$$

we get that

$$\begin{aligned}
\int_B S_{\mathbb{D}}(-r;b,b')e(z,b',r)db&=2\cdot 2^{-(\frac{1}{2}+ir)}\frac{\Gamma(\frac{1}{2}+ir)\Gamma(\frac{1}{2}-ir)}{\Gamma(ir)\Gamma(-ir)}F(z,b,r)\\
&=2^{-(\frac{1}{2}-ir)}\frac{\Gamma(\frac{1}{2}-ir)}{\Gamma(1/2)\Gamma(-ir)}\left(\frac{1-|z|^2}{|z-b|^2}\right)^{\frac{1}{2}-ir}\\
&=e(z,b,-r).
\end{aligned} \tag{2.95}$$

□

This identity implies a symmetry for the non-Euclidean Helgason-Fourier transform, in particular, as

$$\boldsymbol{\mathcal{F}} f(b,-r) := \int_{\mathbb{D}} e(z,b,-r) f(z) dz, \tag{2.96}$$

if we then use identity (2.78), we have

$$\begin{aligned} \boldsymbol{\mathcal{F}} f(b,-r) &= \int_{B\times\mathbb{D}} S_{\mathbb{D}}(-r;b,b') e(z,b',r) f(z) db' dz \\ &= \int_B S_{\mathbb{D}}(-r;b,b') \boldsymbol{\mathcal{F}} f(b',r) db'. \end{aligned} \tag{2.97}$$

Hence,

$$\begin{aligned} \int_B \boldsymbol{\mathcal{F}} f(b,-r) e(z,b,r) db &= \int_B \int_B S_{\mathbb{D}}(-r;b,b') \boldsymbol{\mathcal{F}} f(b',r) e(z,b,r) db' db \\ &= \int_B \boldsymbol{\mathcal{F}} f(b',r) e(z,b',-r) db'. \end{aligned} \tag{2.98}$$

Equation (2.98) (or rather a very similar version) appears in Helgason's book in ([Hel81], Theorem 4.2 part(ii)) as a symmetry property of the Helgason-Fourier transform. We have shown that the symmetry property (2.98) of the spectral Helgason-Fourier transform is a consequence of the relationship of $e(z,b,r)$ with the hyperbolic scattering matrix. We believe that this symmetry property of the spectral Helgason-Fourier transform is equivalent to the symmetry of the hyperbolic plane waves, $e(z,b,r)$, under the action of the hyperbolic scattering matrix (in that one can derive the scattering relation (2.78) from symmetry (2.98)), but we don't prove this in the thesis.

The hyperbolic scattering matrix also has consequences for a (spectral) Zelditch quantisation, suppose we have a linear operator $A : C^\infty(\mathbb{D}) \mapsto C^\infty(\mathbb{D})$, Then A has a well-defined complete Zelditch symbol that we will denote by a which is a complex valued function on $\mathbb{D} \times B \times \mathbb{R}$ defined via

$$Ae(z,b,r) = a(z,b,r) e(z,b,r). \tag{2.99}$$

Note that the difference with [Zel86] is that $e(z,b,r)$ is not just $e^{(\frac{1}{2}+ir)\langle z,b\rangle}$. Since

$Ae(z,b,r) = A[e(\cdot,b,r)]$, using equation (2.78), we have

$$\begin{aligned} Ae(z,b,r) = A[e(\cdot,b,r)](z) &= A\left[\int_B S(r;b,b')e(\cdot,b',-r)db'\right](z) \\ &= \int_B S(r;b,b')A[e(\cdot,b',-r)](z)db' \\ &= \int_B S(r;b,b')Ae(z,b',-r)db'. \end{aligned} \tag{2.100}$$

Hence the complete Zelditch symbol of A satisfies the symmetry

$$a(z,b,r)e(z,b,r) = \int_B S(r;b,b')a(z,b',-r)e(z,b',-r)db' \tag{2.101}$$

which, when multiplied by $e(w,b,-r)$ and integrated in b, implies

$$\int_B a(z,b,r)e(z,b,r)e(w,b,-r)db = \int_B a(z,b,-r)e(z,b,-r)e(w,b,r)db \tag{2.102}$$

for all $z,w \in \mathbb{D}$ and $r \in \mathbb{R}$. Therefore the symmetry that Zelditch symbols satisfy under the transformation $(z,b,r) \mapsto (z,b,-r)$ as stated in [AZ12], equation 4.3 is an implication of the symmetry condition in (2.101) which is a consequence of the action of the hyperbolic scattering matrix on the non-Euclidean plane waves.

2.3.2 Contrasting the hyperbolic scattering matrix with the Euclidean scattering matrix, and comments on the Weyl symmetry of a symbol.

Let us make a brief remark in this section about the differences between the scattering symmetry and the Weyl symmetry on symbols. The Weyl symmetry on symbols is that

$$a(z,b,r) = a(z,b',-r) \tag{2.103}$$

as a pointwise equality for all $z \in \mathbb{D}$, $b \in \mathbb{S}^1$ and $r \in \mathbb{R}\backslash\{0\}$, where b' is related to b as the backwards endpoint of the geodesic through z with forward endpoint b. If one identifies $(z,b) \sim g \in G$, then this symmetry is equivalent to

$$a(g,r) = a(gw,-r) \tag{2.104}$$

for all $g \in G$, $r \in \mathbb{R}\backslash\{0\}$, where $w = \mathbf{k}_\pi \in G$ is the Weyl element of G, recalling $\mathbf{k}_\theta$ as defined in the preliminary chapter 1. Note that if we identify $G \simeq S\mathbb{D}$ in the standard way, then the Weyl map $g \mapsto gw$ corresponds to the isometry on $\mathbb{D}$ which is an inversion symmetry on the tangent space $T_z\mathbb{D}$, where z is the base point of $(z,v) \sim g$. The scattering symmetry for Zelditch symbols, (2.101), is quite a different symmetry to the Weyl symmetry and neither is reducible to the other. One can observe this difference simply from the fact that the scattering matrix kernel, $S(r;b,b')$ is supported (as a distribution) over all $(b,b') \in B \times B$ instead of the codimension one submanifold where (b,b') are related by the antipode map through $z \in \mathbb{D}$ (indeed in the hyperbolic scattering matrix there is no $z \in \mathbb{D}$ to speak of!). It seems that [HHS12] have also noticed this incongruency between the Weyl symmetry and the scattering symmetry on symbols in higher rank symmetric spaces.

This is quite different in contrast to the case in Euclidean $\mathbb{R}^n$ space. In this different geometric context, if one denotes a symbol in Euclidean space by $a(z,\omega,r)$, using polar coordinates in the cotangent fibres of z (so $z \in \mathbb{R}^n$, $(\omega, r) \in \mathbb{S}^{n-1} \times \mathbb{R}^+ \simeq \mathbb{R}^n \backslash \{0\}$). Then the Euclidean Weyl symmetry is the pointwise equality

$$a(z,\omega,r) = a(z,-\omega,-r) \tag{2.105}$$

for all $z \in \mathbb{R}^n$, $\omega \in \mathbb{S}^{n-1}$ and $r \in \mathbb{R}\backslash\{0\}$. Here one can make the same geometric interpretation of the Weyl symmetry in the Euclidean case as in the hyperbolic case, in that $-\omega$ is the backwards endpoint on the sphere at infinity of the geodesic through z with forward endpoint ω. Regarding the scattering symmetry, since one has the following identity

$$\exp(irz\cdot\omega) = \exp(-irz\cdot(-\omega)), \tag{2.106}$$

the matrix kernel that relates the Fourier basis of plane waves with $r > 0$ to the Fourier basis of plane waves with $r < 0$ is simply the delta kernel corresponding to the antipode map, i.e. we have the identity (for all z, r and ω),

$$\exp(irz\cdot\omega) = \int_{\mathbb{S}^{n-1}} \exp(-irz\cdot\omega')\delta(\omega+\omega')d\omega'. \tag{2.107}$$

For the standard quantisation on $\mathbb{R}^n$, which relies on the Fourier basis of plane waves $\exp(irz\cdot\omega)$, the Euclidean scattering symmetry is identical to symmetry (2.101), with each object reinterpreted in the Euclidean framework. In other words, the scattering symmetry on Euclidean symbols is

$$a(z,\omega,r)\exp(irz\cdot\omega) = \int_{\mathbb{S}^{n-1}} a(z,\omega',-r)\exp(-irz\cdot\omega')\delta(\omega+\omega')d\omega' \tag{2.108}$$

which, upon evaluation of the delta kernel, is reducible to the Weyl symmetry

$$a(z,\omega,r) = a(z,-\omega,-r). \tag{2.109}$$

So while in Euclidean space, the scattering symmetry and the Weyl symmetry on standard (left-quantised) symbols are identical, this is not true in hyperbolic space with Zelditch symbols.

2.4 Hyperbolic Wigner distributions

Let A be an operator with spectral Zelditch symbol $a(z,b,r)$. Associated to a pair of Helgason-Fourier dual parameters, (b,r) and (b',r') in $\mathbb{S}^1\times\mathbb{C}$, we can define the following distribution acting on the spectral Zelditch symbol of A, called the Wigner distribution,

$$\mathbf{W}a(b,r;b',r') := \int_{\mathbb{D}} a(z,b,r)e(z,b,r)\overline{e(z,b',-r')}dz = \langle Ae(\cdot,b,r), e(\cdot,b',-r')\rangle_{L^2(\mathbb{D})}. \tag{2.110}$$

This is a well-defined operator on symbols $a\in C^\infty(\mathbb{D}_z\times\mathbb{S}^1_b\times\mathbb{R}^+_r)$ which are compactly supported on $\mathbb{D}_z$ for example. One can extend the Wigner action with formula (2.110) to functions $a\in C^\infty(\mathbb{D}_z\times\mathbb{S}^1_b\times\mathbb{R})$, compactly supported in $\mathbb{D}$, which are not necessarily Zelditch symbols of pseudodifferential operators (i.e those functions $a(z,b,r)$ which don't satisfy the scattering symmetry property (2.101)). By observing that $\mathbf{W}a(b,r;b',r')$ is the non-Euclidean Fourier transform of the function $z\mapsto a(z,b,r)e(z,b,r)$, the inversion formula (2.21) states that

$$a(z,b,r)e(z,b,r) = \int_B\int_0^\infty \mathbf{W}a(b,r;b',r')e(z,b',-r')dr'db'. \tag{2.111}$$

and hence we can recover $a(z,b,r)$ from the Wigner action on a.

We remark that, strictly speaking, our Wigner distribution is slightly different from the one defined by Anantharaman and Zelditch [AZ07] and [AZ12] because the hyperbolic plane waves $e(z,b,r)$ are not just $e^{(\frac{1}{2}+ir)\langle z,b\rangle}$ but also carry with them the complex factor $c(r)$.

2.4.1 Wigner transform as a unitary map.

Let us define the Wigner transform on the domain $a\in C_c^\infty(G\times\mathbb{R}^+)$. If we recall the hyperbolic version of Plancherel's theorem for the spectral Helgason-Fourier transform

(2.25) that

$$\|f\|_{L^2(\mathbb{D},dz)} = \|\boldsymbol{\mathcal{F}} f\|_{L^2(B\times\mathbb{R}^+,dbdr)},$$

then recognising that

$$\mathbf{W}a(b,r,b',r') = \boldsymbol{\mathcal{F}}(a(\cdot,b,r)e(\cdot,b,r))(b',-r'),$$

applying Plancherel's theorem in this case is the statement that

$$\int_{B\times\mathbb{R}^+} |\mathbf{W}a(b,r,b',r')|^2 db'dr' = \int_{\mathbb{D}} |a(z,b,r)e(z,b,r)|^2 dz. \tag{2.112}$$

Note that

$$\int_{\mathbb{D}} |a(z,b,r)e(z,b,r)|^2 dz = \int_{\mathbb{D}} |a(z,b,r)|^2 |c(r)|^2 e^{\langle z,b\rangle} dz. \tag{2.113}$$

We also remark that it is not too difficult to compute

$$|c(r)|^2 = \frac{r}{2\pi}\tanh(\pi r) \tag{2.114}$$

using the Euler reflection formula for the Gamma function and the property $z\Gamma(z) = \Gamma(z+1)$ of the Gamma function.

If we then integrate equation (2.112) in variables b and r, we deduce that

$$\int_{B\times\mathbb{R}^+\times B\times\mathbb{R}^+} |\mathbf{W}a(b,r,b',r')|^2 dbdrdb'dr' = \int_{\mathbb{D}\times B\times\mathbb{R}^+} |a(z,b,r)|^2 |c(r)|^2 e^{\langle z,b\rangle} dzdbdr. \tag{2.115}$$

Under the identification $g \simeq (z,b)$, we know that $e^{\langle z,b\rangle}dzdb = dg$ and hence we can interpret (2.115) as the fact that the Wigner transform of a symbol $a(z,b,r)$ extends to an isometry of $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)$ to $L^2(B\times\mathbb{R}^+\times B\times\mathbb{R}^+, dbdrdb'dr')$. For a symbol $a(z,b,r)$, we let $\tilde{a}(z,b,r) := a(z,b,r)c(r)$. This isometrically identifies $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)$ with $L^2(G\times\mathbb{R}^+, dgdr)$. Since equation (2.112) holds for all $b\in B$ and $r\in\mathbb{R}$, it is also clear that equation (2.115) holds if we integrate over $r\in\mathbb{R}$, rather than just the positive part, $r\in\mathbb{R}^+$. Note that the Wigner transform satisfies the condition

$$\mathbf{W}a(b,r,b',r') = \int_B S(r';b',p)\mathbf{W}a(b,r,p,-r)dp \tag{2.116}$$

because the Wigner transform is the hyperbolic Fourier transform of $z\mapsto a(z,b,r)e(z,b,r)$ and the hyperbolic Fourier transform satisfies a symmetry with the scattering matrix, i.e. (2.98). Moreover, if we assume $a(z,b,r)$ is the Zelditch symbol of a pseudodifferential operator and thus satisfies the symmetry (2.101), the Wigner transform satisfies

$$\begin{aligned}\mathbf{W}a(b,r,b',r') &= \int_{\mathbb{D}} a(z,b,r)e(z,b,r)e(z,b',r')dz \\ &= \int_{\mathbb{D}}\int_{B} S_{\mathbb{D}}(r;b,p)a(z,p,-r)e(z,p,-r)e(z,b',r')dpdz \\ &= \int_{B} S_{\mathbb{D}}(r;b,p)\mathbf{W}a(p,-r,b',r')dp\end{aligned} \tag{2.117}$$

which is the analogous symmetry to (2.116) on the factor of left factor of (b,r) in the Wigner transform.

Furthermore, note that equation (2.112) also holds if we integrate over $r' \in \mathbb{R}$, so long as we introduce a factor of $1/2$ on the side with the Wigner distribution to note that we have 'twice' the information necessary to reconstruct the symbol (since we are using the Helgason-Fourier transform where the $r > 0$ and $r < 0$ non-Euclidean plane waves are related via the hyperbolic scattering matrix, which is known to be a unitary operator on $L^2(B, db)$). Therefore we can say that

$$\frac{1}{2}\|\mathbf{W}a\|^2_{L^2_S(B\times\mathbb{R}\times B\times\mathbb{R})} = \|\mathbf{W}a\|^2_{L^2(B\times\mathbb{R}\times B\times\mathbb{R}^+)} = \|a\|^2_{L^2(G\times\mathbb{R},dg|c(r)|^2dr)} = \|\tilde{a}\|_{L^2(G\times\mathbb{R},dgdr)} \tag{2.118}$$

where the subscript S refers to the subspace of $L^2(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$ which satisfies the symmetry (2.116) on the (b',r') factors to the right. To be precise about the $1/2$, we have a bijection

$$L^2_S(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr') \to L^2(B\times\mathbb{R}\times B\times\mathbb{R}^+, dbdrdb'dr') \tag{2.119}$$

which is given by the restriction map to $\mathbb{R}^+$ on the rightmost factor (the r' factor) whose inverse map is the extension map to $r' \in \mathbb{R}$ through the scattering symmetry (2.116). As a consequence, $\frac{1}{\sqrt{2}}\mathbf{W}$ is a unitary isomorphism between

$$L^2(\mathbb{D}\times B\times\mathbb{R}, dg|c(r)|^2dr) \to L^2_S(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr') \tag{2.120}$$

and also between

$$L^2_S(\mathbb{D}\times B\times\mathbb{R}, dg|c(r)|^2dr) \to L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr'). \tag{2.121}$$

where $L^2_S(\mathbb{D}\times B\times\mathbb{R}, dg|c(r)|^2dr)$ is the subspace of symbols (which satisfy the scattering symmetry (2.101)) and $L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$ is the subspace of $L^2(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$ which satisfies both scattering symmetries (2.117) and (2.116).

We summarise these observations into the following theorem.

Theorem 11. *The Wigner transform $a \mapsto \mathbf{W}a$ defined in (2.110) with inversion formula (2.111) extends to a unitary isomorphism of*

$$L^2(\mathbb{D} \times B \times \mathbb{R}^+, dg|c(r)|^2 dr) \longrightarrow L^2(B \times \mathbb{R}^+ \times B \times \mathbb{R}^+, dbdrdb'dr'). \tag{2.122}$$

It also extends to a unitary isomorphism of

$$L^2(\mathbb{D} \times B \times \mathbb{R}, dg|c(r)|^2 dr) \longrightarrow L^2(B \times \mathbb{R} \times B \times \mathbb{R}^+, dbdrdb'dr'). \tag{2.123}$$

The operator $\frac{1}{\sqrt{2}}\mathbf{W}$ extends to a unitary isomorphism of

$$L^2(\mathbb{D} \times B \times \mathbb{R}, dg|c(r)|^2 dr) \longrightarrow L^2_S(B \times \mathbb{R} \times B \times \mathbb{R}, dbdrdb'dr') \tag{2.124}$$

and

$$L^2_S(\mathbb{D} \times B \times \mathbb{R}, dg|c(r)|^2 dr) \longrightarrow L^2_{S,S}(B \times \mathbb{R} \times B \times \mathbb{R}, dbdrdb'dr'). \tag{2.125}$$

2.5 Patterson-Sullivan distributions

2.5.1 Defining the Patterson-Sullivan distributions

In this section we define a distribution called the Patterson-Sullivan distribution, associated to a pair of Helgason-Fourier dual parameters (b, r) and (b', r') with $b \neq b'$. We follow Anantharaman-Zelditch in [AZ12], but we make a subtle modification by introducing a complex factor of r and r' to the distribution.

Firstly, we will remark that one can parameterise $G \simeq S\mathbb{D}$ by coordinates $(b, b', t) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ where $b \neq b'$. Each $(z, v) \in S\mathbb{D}$ determines a unique geodesic that has a forward endpoint $b \in \mathbb{S}^1$ and backward endpoint $b' \in \mathbb{S}^1$ with $b \neq b'$. Moreover, a pair of points $(b, b') \in \mathbb{S}^1 \times \mathbb{S}^1$ with $b \neq b'$ uniquely defines the oriented geodesic with endpoints b and b', say denoted $\gamma_{b,b'}$. We let $z_{b,b'} \in \mathbb{D}$ to be the unique point on $\gamma_{b,b'}$ which is closest to the origin 0. Then for any $t \in \mathbb{R}$, we let z to be the point which is on $\gamma_{b,b'}$ with an oriented distance t from $z_{b,b'}$. Finally, we let $v \in S_z\mathbb{D}$ be the unit tangent vector which is tangent to $\gamma_{b,b'}$ at z, pointing towards b. Hence $(b, b', t) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ with $b \neq b'$ parameterises $S\mathbb{D}$. By dualising with the metric, this also parameterises $S^*\mathbb{D}$. Therefore, for a symbol $a(z, b, r)$, we can also use $a(b, b', t, r)$ to express a in coordinates (b, b', t).

We define

$$\mathbf{PS}a(b,r,b',r') := \gamma\left(\frac{r+r'}{2}\right)\frac{1}{|b-b'|^{1+2i(\frac{r+r'}{2})}}\int_{\mathbb{R}} a\left(b,b',t,\frac{r+r'}{2}\right)e^{i(r-r')t}dt. \tag{2.126}$$

where

$$\gamma\left(\frac{r+r'}{2}\right) = 2^{\frac{1}{2}+i(\frac{r+r'}{2})}c\left(\frac{r+r'}{2}\right) = \frac{\Gamma\left(\frac{1}{2}+i\left(\frac{r+r'}{2}\right)\right)}{\Gamma(1/2)\Gamma\left(i\left(\frac{r+r'}{2}\right)\right)}. \tag{2.127}$$

This is a well-defined operator on functions $a \in C^\infty((\mathbb{S}^1_b \times \mathbb{S}^1_{b'})\backslash\Delta \times \mathbb{R}_t \times \mathbb{R})$ which are absolutely integrable in t for example. Here $(\mathbb{S}^1\times\mathbb{S}^1)\backslash\Delta$ is the set $\{(b,b') \in \mathbb{S}^1\times\mathbb{S}^1 : b \neq b'\}$. We note that the integral in the Patterson-Sullivan distribution is a (Euclidean) Fourier transform of the symbol a in the t variable into the variable $\nu = r - r'$, i.e. along the geodesic (parameterised by arclength) with end points b' and b. So we can equivalently write the Patterson-Sullivan transform as

$$\mathbf{PS}a(b,r,b',r') = \frac{\gamma(\frac{r+r'}{2})}{|b-b'|^{1+i(r+r')}}\hat{a}\left(b,b',-(r-r'),\frac{r+r'}{2}\right) \tag{2.128}$$

where $\hat{a}$ is the Fourier transform of a in the third argument. Setting $\nu = r - r'$ and $R = \frac{r+r'}{2}$, we get

$$\hat{a}(b,b',-\nu,R)c(R) = \mathbf{PS}a\left(b,R+\frac{\nu}{2},b',R-\frac{\nu}{2}\right)2^{-(\frac{1}{2}+iR)}|b-b'|^{1+2iR} \tag{2.129}$$

since $\gamma(R) = 2^{\frac{1}{2}+iR}c(R)$. Therefore one can recover $\tilde{a}(b,b',t,R) := a(b,b',t,R)c(R)$ from its Patterson-Sullivan transform by taking the inverse Fourier transform of the right side in ν.

A subtle point arises here, to invert $\mathbf{PS}$ and recover the value of $\tilde{a}(b,b',t,R)$, $R > 0$ (i.e. the value of the symbol at a particular point) from its Patterson-Sullivan transform, we need the data of $\mathbf{PS}a(b,r,b',r')$ at *all* $(r,r') \in \mathbb{R}^2$ not merely on the set $\{(r,r') : r + r' > 0\}$. This is clear from (2.129) where, for every fixed $R > 0$, one requires the Fourier data $\hat{a}(b,b',-\nu,R)$ at all $\nu \in \mathbb{R}$ to recover $a(b,b',t,R)$ by the Euclidean Fourier inversion formula. This amounts to needing to define $\mathbf{PS}a(b,r,b',r')$ at negative arguments of r and r' (i.e on the extended Helgason-Fourier dual space). We extend the Patterson-Sullivan transform as a function of $r, r' \in \mathbb{R}$ by assuming our symbol $a(b,b',t,r) \simeq a(g,r)$ is extended to be a function of $r \in \mathbb{R}$. Given this, we can then define the operator (which will turn out to be $\mathbf{PS}^{-1}$)

$$P : C^\infty_c(B\times\mathbb{R}\times B\times\mathbb{R}) \to L^2(G\times\mathbb{R}, dg|c(r)|^2dr) \tag{2.130}$$

by

$$Pf(b,b',t,R) = c(R)^{-1}e^{-(\frac{1}{2}+iR)}|b-b'|^{1+2iR}\frac{1}{2\pi}\int_{\mathbb{R}} f\left(b, R+\frac{\nu}{2}, b', R-\frac{\nu}{2}\right)e^{-i\nu t}d\nu. \quad (2.131)$$

Then by the Euclidean Fourier inversion theorem and Plancherel's theorem for the Euclidean Fourier transform, P can be continuously extended to act on the domain $L^2(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$ and P is a right and left inverse to $\mathbf{PS}$ on this domain.

2.5.2 Patterson-Sullivan distribution as a unitary map.

The Plancherel formula for the Euclidean Fourier transform implies that for $a \in C_c^\infty(G\times\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} |a(b,b',t,R)\gamma(R)|^2 dt &= \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{a}(b,b',\nu,R)\gamma(R)|^2 d\nu \\ &= \frac{1}{2\pi}|b-b'|^2\int_{\mathbb{R}}\left|\mathbf{PS}a\left(b,R+\frac{\nu}{2},b',R-\frac{\nu}{2}\right)\right|^2 d\nu.\end{aligned} \quad (2.132)$$

Hence, integrating over b and b', we have

$$\begin{aligned}2\pi\int_{B\times B\times\mathbb{R}} |a(b,b',t,R)\gamma(R)|^2 dt\frac{dbdb'}{|b-b'|^2} = \\ \int_{B\times B\times\mathbb{R}}\left|\mathbf{PS}a\left(b,R+\frac{\nu}{2},b',R-\frac{\nu}{2}\right)\right|^2 d\nu dbdb'.\end{aligned} \quad (2.133)$$

Let us recall a lemma,

Lemma 12. *Under the identification $g \simeq (b,b',t)$,*

$$dg = 4\pi\frac{db\otimes db'}{|b-b'|^2}\otimes dt.$$

A proof of Lemma 12 can be found in [AZ12], Lemma 3.1. Hence if we identify $g \simeq (b,b',t)$, then (2.133) is equivalent under this identification to

$$\frac{1}{2}\int_G |a(g,R)\gamma(R)|^2 dg = \int_{B\times B\times\mathbb{R}}\left|\mathbf{PS}a\left(b,R+\frac{\nu}{2},b',R-\frac{\nu}{2}\right)\right|^2 dbdb'd\nu. \quad (2.134)$$

Noting that

$$|\gamma(R)|^2 = 2|c(R)|^2, \quad (2.135)$$

if we integrate (2.134) over $R \in \mathbb{R}^+$, we get

$$\int_{G\times\mathbb{R}^+} |a(g,R)c(R)|^2 dg dR = \int_{B\times B\times\mathbb{R}\times\mathbb{R}^+} \left|\mathbf{PS}a\left(b, R+\frac{\nu}{2}, b', R-\frac{\nu}{2}\right)\right|^2 db db' d\nu dR. \tag{2.136}$$

We change variables in the right hand side integral from (R,ν) back to (r,r') where

$$r = R + \frac{\nu}{2}, \qquad r' = R - \frac{\nu}{2}.$$

We find that $drdr' = d\nu dR$ and that $\{(R,\nu) : R > 0, \nu \in \mathbb{R}\}$ gets mapped bijectively to $\{(r,r') : r,r' \in \mathbb{R}, r+r' > 0\}$, and hence (2.136) is equivalent to

$$\int_{G\times\mathbb{R}^+} |a(g,R)c(R)|^2 dg dR = \int_{B\times B\times\{r+r'>0\}} |\mathbf{PS}a(b,r,b',r')|^2 db db' dr dr'. \tag{2.137}$$

Therefore, $\mathbf{PS}$ acting on the domain $C_c^\infty(G\times\mathbb{R})$ satisfies the isometry

$$\|a\|_{L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)} = \|\mathbf{PS}a\|_{L^2(B\times B\times\{r+r'>0\}, dbdrdb'dr')}. \tag{2.138}$$

We can extend the definition of $\mathbf{PS}$ by continuity to all of $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)$. The surjectivity of $\mathbf{PS}$ follows by the existence of the inverse (Euclidean) Fourier transform.

We summarise these observations as a theorem.

Theorem 13. *We have that the Patterson-Sullivan transform $a \mapsto \mathbf{PS}a$ defined in (2.126) extends to a unitary isomorphism of*

$$L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr) \longrightarrow L^2(B\times B\times\{r+r'>0\}, dbdb'drdr'). \tag{2.139}$$

It also extends to a unitary isomorphism

$$L^2(G\times\mathbb{R}, dg|c(r)|^2dr) \longrightarrow L^2(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr'). \tag{2.140}$$

Note that we do not require that the scattering symmetry is satisfied here, on any of the L^2 spaces.

2.6 Intertwining the classical and quantum flows on the hyperbolic disk.

Recall in the introduction that we introduced two flows (one-parameter families of operators, which are unitary on certain Hilbert spaces), the classical flow and the quantum flow respectively, denoted G_t^* and V_t, acting on symbols which are defined over $T^*\mathbb{D}\backslash 0$. We identified $T^*\mathbb{D}\backslash 0$ with $S^*\mathbb{D}\times\mathbb{R}^+$, $G\times\mathbb{R}^+$, $\mathbb{D}\times B\times\mathbb{R}^+$ and so on. G_t^* and V_t are defined as

$$(G_t^*a)(g,r) := a(g\mathbf{a}_{rt},r), \qquad (V_ta)(z,b,r) := \sigma(e^{-it\Delta/2}Ae^{it\Delta/2}) \tag{2.141}$$

where A has complete Zelditch symbol a.

V_t is a one-parameter family of operators acting unitarily on $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)$, a Hilbert space of symbols isometrically isomorphic to the Hilbert space of Hilbert-Schmidt operators on $\mathbb{D}$. G_t^* is a one-parameter family of operators acting unitarily on $L^2(G\times\mathbb{R}^+, dg\otimes m(r))$ where $m(r)$ can be any measure on $\mathbb{R}^+$, in particular it is true for $m(r) = |c(r)|^2dr$. One can check directly from the definitions of the Wigner distributions $\mathbf{W}$ in Section 2.4 and Patterson-Sullivan distributions $\mathbf{PS}$ in Section 2.5 that

$$\begin{aligned}\mathbf{PS}(G_t^*a)(b,r,b',r') &= e^{it(r^2-r'^2)/2}\mathbf{PS}a(b,r,b',r'),\\ \mathbf{W}(V_ta)(b,r,b',r') &= e^{it(r^2-r'^2)/2}\mathbf{W}a(b,r,b',r').\end{aligned} \tag{2.142}$$

In particular, the modifications we make to $\mathbf{PS}$ and $\mathbf{W}$ (compared with [AZ12]) do not change this important property of the distributions.

By Theorems 11 and 13 regarding the Wigner and Patterson-Sullivan distributions, we can see that $\mathbf{PS}$ diagonalises the unitary group of operators G_t^* and that $\mathbf{W}$ diagonalises the unitary group of operators V_t, both acting unitarily on the Hilbert $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)$. In particular we have

$$G_t^* = \mathbf{PS}^{-1}e^{it(r^2-r'^2)/2}\mathbf{PS}, \qquad V_t = \mathbf{W}^{-1}e^{it(r^2-r'^2)/2}\mathbf{W}. \tag{2.143}$$

on the Hilbert space $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)$.

We note that G_t^* is unitarily equivalent to the multiplication operator $e^{it(r^2-r'^2)}$ on $L^2(B\times B\times\{r+r'>0\}, dbdb'drdr')$ whereas V_t is unitarily equivalent to the multiplication operator $e^{it(r^2-r'^2)}$ on $L^2(B\times B\times\mathbb{R}^+\times\mathbb{R}^+, dbdb'drdr')$. As we want to intertwine G_t^* and V_t, the fact that these two L^2 spaces are slightly different is a slight obstruction to this goal and it means we need to consider the extensions of $\mathbf{PS}$ and $\mathbf{W}$ on $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$ and $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$ respectively, and we also need to define V_t and G_t^* on these spaces. Note that $t\mapsto e^{-it\Delta/2}Ae^{it\Delta/2}$ is a unitary one-parameter flow on $\mathrm{HS}(\mathbb{D})$, the space of Hilbert-Schmidt operators on $\mathbb{D}$, and since $\mathrm{HS}(\mathbb{D})\simeq L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$, we define V_t as the induced flow on $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$. This also follows from $L^2(G\times\mathbb{R}^+, dg|c(r)|^2dr)\simeq L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$. Likewise G_t^* can be extended as a unitary one-parameter flow on $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$ by defining $G_t^*a(g,r) := a(g\mathbf{a}_{rt},r)$. One can directly check that $\mathbf{W}(V_t a) = e^{it(r^2-r'^2)}\mathbf{W}a$ and $\mathbf{PS}(G_t^* a) = e^{it(r^2-r'^2)/2}\mathbf{PS}a$ holds under their extensions to $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$ and $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$ respectively.

These constructions and Theorem 11 and Theorem 13 provide us with the tools to state the first main theorem of this thesis,

Theorem 14. *Consider $L^2_S(G\times\mathbb{R})$ and $L^2(G\times\mathbb{R})$ with the measure $dg|c(r)|^2dr$ and $L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R})$ with the measure $dbdrdb'dr'$, then the following commutative diagram holds,*

$$\begin{tikzcd} L^2_S(G\times\mathbb{R}) \arrow[r, "\frac{1}{\sqrt{2}}\mathbf{W}"] \arrow[d, "V_t"'] & L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}) \arrow[r, "\mathbf{PS}^{-1}"] \arrow[d, "e^{it(r^2-r'^2)}"] & L^2(G\times\mathbb{R}) \arrow[d, "G_t^*"] \\ L^2_S(G\times\mathbb{R}) \arrow[r, "\frac{1}{\sqrt{2}}\mathbf{W}"'] & L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}) \arrow[r, "\mathbf{PS}^{-1}"'] & L^2(G\times\mathbb{R}) \end{tikzcd} \tag{2.144}$$

In particular $\boldsymbol{\mathcal{L}} := (1/\sqrt{2})\mathbf{PS}^{-1}\circ\mathbf{W}$ is an unitary isomorphism of $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$ into its range in $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$ and we have, for every $t\in\mathbb{R}$,

$$\boldsymbol{\mathcal{L}}\circ V_t = G_t^*\circ\boldsymbol{\mathcal{L}} \tag{2.145}$$

where $\boldsymbol{\mathcal{L}}\circ V_t$ and $G_t^*\circ\boldsymbol{\mathcal{L}}$ are isometries of $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$ into $L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$.

Let us make some remarks regarding Theorem 14. This theorem states that $(1/\sqrt{2})\mathbf{W}$ and $\mathbf{PS}$ diagonalise V_t and G_t^* respectively. Both V_t and G_t^* are unitarily equivalent to the multiplication operator $e^{it(r^2-r'^2)}$ on the same Hilbert space $L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$ and *hence unitarily equivalent to each other*. The intertwining operator $\boldsymbol{\mathcal{L}} = (1/\sqrt{2})\mathbf{PS}^{-1}\circ\mathbf{W}$ is the unitary 'change of basis' which intertwines the two flows.

We remark that we do not yet have a characterisation of the image of $\mathbf{PS}^{-1}$ on $L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$, as Theorem 13 on the Patterson-Sullivan transform only extends the Patterson-Sullivan transform to a unitary isomorphism between

$$L^2(G\times\mathbb{R}, dg|c(r)|^2dr) \tag{2.146}$$

and

$$L^2(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr') \tag{2.147}$$

without any subscripts S involving the hyperbolic scattering matrix. It is only this unknown which currently prevents us from presenting an analogue of the commutative diagram where every arrow is a unitary isomorphism. This is a topic for future research. There is a curious observation one can make regarding the Patterson-Sullivan transform and the hyperbolic scattering matrix, particularly, $\mathbf{PS}$ appears to contain the hyperbolic scattering matrix kernel at spectral value $\frac{r+r'}{2}$ in parameters (b,b'), in other words,

$$\mathbf{PS}a(b,r,b',r') = \frac{1}{2\pi} S_{\mathbb{D}}\left(\frac{r+r'}{2};b,b'\right)\int_{\mathbb{R}} a\left(b,b',t,\frac{r+r'}{2}\right)e^{i(r-r')t}dt \tag{2.148}$$

where $S_{\mathbb{D}}$ is the hyperbolic scattering matrix. We suspect this observation may be useful in characterising the image of $\mathbf{PS}^{-1}$ on $L^2_{S,S}(B\times\mathbb{R}\times B\times\mathbb{R}, dbdrdb'dr')$.

2.6.1 Constructing the intertwining operator and its inverse as oscillatory integrals.

Following on, we shall analyse the operator $\boldsymbol{\mathcal{L}} := (1/\sqrt{2})\mathbf{PS}^{-1}\circ\mathbf{W}$ and in this section we will prove that $\boldsymbol{\mathcal{L}}$ (and its inverse) have expressions as oscillatory integral operators.

Theorem 15.

$$\boldsymbol{\mathcal{L}} = (1/\sqrt{2})\mathbf{PS}^{-1}\circ\mathbf{W} : L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr) \to L^2(G\times\mathbb{R}, dg|c(r)|^2dr) \tag{2.149}$$

has the following oscillatory integral expression

$$\boldsymbol{\mathcal{L}}a(g,R)c(R) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3} C(R,r)a(g\mathbf{a}_\tau\mathbf{n}_u, r)c(r)e^{2i(r-R)\tau}(1+u^2)^{-(\frac{1}{2}+i(2R-r))}d\tau du dr \tag{2.150}$$

where

$$C(R,r) = 2^{\frac{1}{2}+iR}c(2R-r). \tag{2.151}$$

Moreover,

$$\boldsymbol{\mathcal{L}}^{-1} = \sqrt{2}\mathbf{W}^{-1}\circ\mathbf{PS} : \mathcal{D}\to L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr), \tag{2.152}$$

where $\mathcal{D}\subset L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$ is the range of $\boldsymbol{\mathcal{L}}$ on the domain $L^2_S(G\times\mathbb{R}, dg|c(r)|^2dr)$, has the following oscillatory integral expression

$$\boldsymbol{\mathcal{L}}^{-1}a(g,r)c(r) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3} 2^{-(\frac{1}{2}+i(\frac{r+r'}{2}))}(1+u^2)^{-\frac{1}{2}+ir'}a\left(g\mathbf{n}_u\mathbf{a}_\tau, \frac{r+r'}{2}\right)c\left(\frac{r+r'}{2}\right) \times e^{i(r-r')\tau}c(-r')d\tau dr' du. \tag{2.153}$$

Proof. We follow [AZ12] closely to derive the oscillatory integral operator, making the modifications needed that come with our changes to the $\mathbf{W}$ and $\mathbf{PS}$ transform. The key idea to constructing the intertwining operator is to take the Wigner transform and change coordinates so that the integral in z is done with respect to geodesic-horocyclic parameters defined with respect to b. So we let $z = g_{b,b'}\mathbf{a}_t\mathbf{n}_u$ where $g_{b,b'}$ is the unique group element which maps the unit length rightward pointing tangent vector over the basepoint 0 (pointing towards 1) to the unit tangent vector v over the basepoint $z_{b,b'}$, where $z_{b,b'}$ lies on the geodesic between b and b' at the point closest to 0 and v is tangent to this geodesic at $z_{b,b'}$ pointed to orient towards b. Recalling an identity[3] that

$$e^{\langle z_{b,b'},b\rangle} = e^{\langle z_{b,b'},b'\rangle} = \frac{2}{|b-b'|} \tag{2.154}$$

it follows that $e^{\langle z_{b,b'}\mathbf{a}_t\mathbf{n}_u,b\rangle} = \frac{2}{|b-b'|}e^t$ and so

$$e^{(\frac{1}{2}+ir)\langle z_{b,b'}\mathbf{a}_t\mathbf{n}_u,b\rangle} = \left(\frac{2}{|b-b'|}\right)^{1/2+ir} e^{(1/2+ir)t}. \tag{2.155}$$

To compute $e^{\langle z_{b,b'}\mathbf{a}_t\mathbf{n}_u,b'\rangle}$, we note the invariance of this quantity under hyperbolic isometries, so it is sufficient to compute it in the particular case that $b=\infty$ and $b'=0$ in the upper half plane, in which case $z_{b,b'}=i$ and $z_{b,b'}\mathbf{a}_t\mathbf{n}_u$ is the point $x+iy=e^tu+ie^t$. It is easy to see that

$$dz = \frac{dxdy}{y^2} = \frac{d(e^tu)d(e^t)}{e^{2t}} = dtdu. \tag{2.156}$$

Then $\langle z_{b,b'}\mathbf{a}_t\mathbf{n}_u, b'\rangle$ is the signed hyperbolic distance from the horocycle through e^tu+ie^t based at zero, to the point i. It is easy to see by symmetry that the closest point on this horocycle to i is the point where it meets the y axis. Viewing the horocycle as a Euclidean circle, its centre is at βi where $\beta^2 = x^2+(\beta-y)^2$. This we solve to get

$$\beta = \frac{x^2+y^2}{2y} = \frac{e^t(1+u^2)}{2}. \tag{2.157}$$

The horocycle therefore crosses the y axis at $2\beta i$, and the signed hyperbolic distance to i is

$$-\log 2\beta = -t - \log(1+u^2) \tag{2.158}$$

and so it follows that

$$e^{\langle z_{b,b'}\mathbf{a}_t\mathbf{n}_u,b'\rangle} = \frac{2}{|b-b'|}e^{-t}(1+u^2)^{-1}. \tag{2.159}$$

Therefore, referring back to the Wigner transform, (2.110), changing into these coordi-

[3] See a proof in [AZ12], Lemma 2.1

nates, we have

$$\begin{aligned}\mathbf{W}a(b,r,b',r') &= \int_{\mathbb{D}} a(z,b,r)e(z,b,r)e(z,b',r')d\mathrm{Vol}(z)\\ &= \int_{\mathbb{D}} a(z,b,r)c(r)c(r')e^{(\frac{1}{2}+ir)\langle z,b\rangle}e^{(\frac{1}{2}+ir')\langle z,b'\rangle}d\mathrm{Vol}(z)\\ = \left(\frac{2}{|b-b'|}\right)^{1+i(r+r')} & c(r)c(r')\int_{\mathbb{R}^2} a(g_{b,b'}\mathbf{a}_\tau\mathbf{n}_u\cdot 0,b,r)e^{i(r-r')\tau}(1+u^2)^{-(\frac{1}{2}+ir')}d\tau du.\end{aligned} \tag{2.160}$$

Letting $r = R+\nu/2$ and $r' = R-\nu/2$, we get

$$\begin{aligned}\mathbf{W}a\left(b,R+\frac{\nu}{2},b',R-\frac{\nu}{2}\right) &= \left(\frac{2}{|b-b'|}\right)^{1+2iR} c\left(R+\frac{\nu}{2}\right)c\left(R-\frac{\nu}{2}\right)\\ &\times\int_{\mathbb{R}^2} a(g_{b,b'}\mathbf{a}_\tau\mathbf{n}_u\cdot 0,b,R+\nu/2)e^{i\nu\tau}(1+u^2)^{-(\frac{1}{2}+i(R-\nu/2))}d\tau du.\end{aligned} \tag{2.161}$$

We then substitute $(1/\sqrt{2})\mathbf{W}a(b,R+\nu/2,b',R-\nu/2)$ into the inversion formula for the Patterson-Sullivan transform, (2.129), which gives us a symbol. Notice that the powers of of $|b-b'|$ cancel out and we get

$$\begin{aligned}\boldsymbol{\mathcal{L}}a(g_{b,b'}\mathbf{a}_t,R)\gamma(R) &= \frac{1}{2^{3/2}\pi}\int_{\mathbb{R}}\mathbf{W}a\left(b,R+\frac{\nu}{2},b',R-\frac{\nu}{2}\right)|b-b'|^{1+2iR}e^{-it\nu}d\nu\\ &= \frac{2^{2iR}}{\sqrt{2}\pi}\int_{\mathbb{R}^3}F(R,\nu)a(g_{b,b'}\mathbf{a}_\tau\mathbf{n}_u\cdot 0,b,R+\nu/2)e^{i\nu(\tau-t)}(1+u^2)^{-(\frac{1}{2}+i(R-\nu/2))}d\tau du d\nu.\end{aligned} \tag{2.162}$$

with

$$F(R,\nu) = c\left(R+\frac{\nu}{2}\right)c\left(R-\frac{\nu}{2}\right). \tag{2.163}$$

Writing $g = g_{b,b'}\mathbf{a}_t$ and changing variables by letting $\tau - t = \tau'$ (we will suppress the prime notation) and $r = R+\nu/2$, and recalling $\gamma(R) = 2^{\frac{1}{2}+iR}c(R)$, gives

$$\boldsymbol{\mathcal{L}}a(g,R)c(R) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3}C(R,r)a(g\mathbf{a}_\tau\mathbf{n}_u,r)c(r)e^{i2(r-R)\tau}(1+u^2)^{-1/2-i(2R-r)}d\tau du dr. \tag{2.164}$$

where $C(R,r) = 2^{\frac{1}{2}+iR}c(2R-r)$.

To compute the formula for $\boldsymbol{\mathcal{L}}^{-1}$, we note that by the Fourier inversion formula (2.21), we have

$$\boldsymbol{\mathcal{L}}^{-1}a(z,b,r)e(z,b,r) = \sqrt{2}\int_{\mathbb{S}^1}\int_0^\infty \mathbf{PS}a(b,r;b',r')e(z,b',-r')dr'db', \tag{2.165}$$

which, after substitution is

$$\mathcal{L}^{-1}a(z,b,r)e(z,b,r) = \sqrt{2}\int_{\mathbb{S}^1}\int_0^\infty\int_{\mathbb{R}}\gamma\left(\frac{r+r'}{2}\right)\frac{1}{|b-b'|^{1+i(r+r')}} \times a\left(b,b',t,\frac{r+r'}{2}\right)e^{i(r-r')t}e(z,b',-r')dtdr'db'. \quad (2.166)$$

As in Anantharaman-Zelditch [AZ12], let us fix $z=0, b=1$ to get

$$\mathcal{L}^{-1}a(0,1,r)c(r) = \int_{\mathbb{S}^1}\int_0^\infty\int_{\mathbb{R}}\frac{\sqrt{2}\gamma(\frac{r+r'}{2})}{|1-b'|^{1+i(r+r')}}a\left(1,b',t,\frac{r+r'}{2}\right)e^{i(r-r')t}c(-r')dtdr'db'. \quad (2.167)$$

As b is fixed to 1, the point $(1,b',t)$ on hyperbolic space is equivalent to $g=\mathbf{n}_u\mathbf{a}_\tau$ for some u and τ functions of b' and t. We can work out these functions with some identities, as in Section 2.4 of [AZ12], and we come to the fact that for $g=\mathbf{n}_u\mathbf{a}_\tau$, the (b,b',t) coordinates under the identification $g\simeq(b,b',t)$ described in Section 2.3, is

$$b' = \frac{u^2+1}{u^2-1}+i\frac{2u}{u^2+1}, \qquad t=\tau-\frac{\log(1+u^2)}{2}, \qquad b=1. \quad (2.168)$$

By a simple computation,

$$|1-b'|^2 = \frac{4}{1+u^2}, \qquad db' = \frac{1}{\pi}\frac{du}{1+u^2}, \quad (2.169)$$

hence with this change of coordinates, (2.167) becomes

$$\mathcal{L}^{-1}a(0,1,r)c(r) = \sqrt{2}\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}}\frac{\gamma(\frac{r+r'}{2})}{\pi}2^{-(1+i(r+r'))}(1+u^2)^{-\frac{1}{2}+ir'} \times a\left(\mathbf{n}_u\mathbf{a}_\tau,\frac{r+r'}{2}\right)e^{i(r-r')\tau}c(-r')d\tau dr'du. \quad (2.170)$$

Let us comment that we can take the integral in r' over all $\mathbb{R}$, so long as we multiply by $1/2$. The reason for this is, in the spectral Helgason-Fourier inversion formula, the Helgason-Fourier data of a function $f(z)$ for $r>0$ and $r<0$ are both sufficient to reconstruct the function, hence

$$\int_{\mathbb{R}}\int_{\mathbb{S}^1}\mathcal{F}f(b,r)e(z,b,r)dbdr = 2f(z).$$

This extends to the inversion formula for the Wigner transform too.

Since $\boldsymbol{\mathcal{L}}^{-1}$ is left-G equivariant, we have

$$\boldsymbol{\mathcal{L}}^{-1}a(g,r)c(r) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3}\gamma\left(\frac{r+r'}{2}\right)2^{-(1+i(r+r'))}(1+u^2)^{-\frac{1}{2}+ir'} \times a\left(g\mathbf{n}_u\mathbf{a}_\tau, \frac{r+r'}{2}\right)e^{i(r-r')\tau}c(-r')d\tau dr' du. \quad (2.171)$$

which is equivalent to

$$\boldsymbol{\mathcal{L}}^{-1}a(g,r)c(r) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3}2^{-(\frac{1}{2}+i(\frac{r+r'}{2}))}(1+u^2)^{-\frac{1}{2}+ir'}a\left(g\mathbf{n}_u\mathbf{a}_\tau, \frac{r+r'}{2}\right)c\left(\frac{r+r'}{2}\right) \times e^{i(r-r')\tau}c(-r')d\tau dr' du. \quad (2.172)$$

□

Chapter 3

Canonical relation of the intertwining operator

3.1 Introduction

The conclusion of Chapter 2 found that $\boldsymbol{\mathcal{L}}$ was an operator such that $\boldsymbol{\mathcal{L}} \circ V_t = G_t^* \circ \boldsymbol{\mathcal{L}}$ and that $\boldsymbol{\mathcal{L}}$ was an isometry from $L^2_S(G \times \mathbb{R}, dg|c(r)|^2 dr)$ into $L^2(G \times \mathbb{R}, dg|c(r)|^2 dr)$. As a consequence of this, on $\mathbb{D}$ we have the identity that

$$\mathrm{QCor}(A, B)(t) = \mathrm{ClassicalCor}(\boldsymbol{\mathcal{L}}a, \boldsymbol{\mathcal{L}}b)(t) \tag{3.1}$$

where a and b are the Zelditch symbols of A and B respectively. However, to extend this identity onto a compact hyperbolic surface $X = \Gamma\backslash\mathbb{D}$ (where $\Gamma \subset \mathrm{PSU}(1,1)$ is a discrete cocompact group) requires some more work because unlike on $\mathbb{D}$, the trace of operators B^*A is not just the integral of $a\bar{b}$ over the cotangent bundle $T^*(\Gamma\backslash\mathbb{D})$.

There are two possible ways to do this. The first way is to utilse the left-G equivariance of $\boldsymbol{\mathcal{L}}$ to study periodic sums over Γ. This is a rather interesting approach, however we will not have the chance to present a study following this approach in this thesis. A different approach is to use the fact that on a compact hyperbolic surface, we have, for $w > 1/2$,

$$\mathrm{Trace}\, B^*A = \int_{\Gamma\backslash G \times S_w} a(g, r)\overline{\mathfrak{T}b(g, r)}dgdr \tag{3.2}$$

where S_w is the strip $\{r \in \mathbb{C} : |\Im(r)| < w\}$ in the complex plane of width w and where $\mathfrak{T}b$ is a distribution on $G \times S_w$ which has microlocal regularity away from the dynamical sub-bundle $E_s^*(\Gamma\backslash G)$ and supported in r at a countable number of shells r_j corresponding to the eigenvalues $\lambda_j = 1/4 + r_j^2$. We will derive this identity in Lemma 27 and Corollary 28 of Chapter 4. We mention briefly that it arrives by considering the boundary values of eigenfunctions ϕ_j on a compact hyperbolic surface (by viewing it as a periodic eigenfunction on $\mathbb{D}$).

From Theorem 15 in Chapter 2, we have that the intertwining operator $\boldsymbol{\mathcal{L}}$ has the following expression

$$\boldsymbol{\mathcal{L}}a(g,R)c(R) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3} C(R,r)a(g\mathbf{a}_t\mathbf{n}_v,r)c(r)e^{2i(r-R)t}(1+v^2)^{-(\frac{1}{2}+i(2R-r))}drdtdv \quad (3.3)$$

where $C(R,r) = 2^{\frac{1}{2}+iR}c(2R-r)$. If we consider the operator $\widetilde{\boldsymbol{\mathcal{L}}}$ which comes from $\boldsymbol{\mathcal{L}}$ by identifying the functions $a(g,r) \in L^2(G\times\mathbb{R}, dg|c(r)|^2dr)$ in the domain and codomain of $\boldsymbol{\mathcal{L}}$ with the functions $\tilde{a}(g,r) := a(g,r)c(r) \in L^2(G\times\mathbb{R}, dgdr)$, then this defines

$$\widetilde{\boldsymbol{\mathcal{L}}}\tilde{a}(g,R) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}^3} C(R,r)\tilde{a}(g\mathbf{a}_t\mathbf{n}_v,r)e^{2i(r-R)t}(1+v^2)^{-(\frac{1}{2}+i(2R-r))}drdtdv \quad (3.4)$$

for $C(R,r) = 2^{\frac{1}{2}+iR}c(2R-r)$.

Since in Theorem 15 of Chapter 2, we are able to derive an oscillatory integral expression for $\boldsymbol{\mathcal{L}}$ (and hence for $\widetilde{\boldsymbol{\mathcal{L}}}$), we can use this to try to extend the action of $\widetilde{\boldsymbol{\mathcal{L}}}$ on distributions over $G\times S_w$ with such regularity properties as $\mathfrak{T}b$. This is the goal of this chapter. We will derive the canonical relation associated to the phase function of the oscillatory integral representation of the intertwining operator, then we will prove some results and make some comments regarding the structure of this canonical relation. Unfortunately we will be unable to find function spaces on which $\widetilde{\boldsymbol{\mathcal{L}}}$ is bounded and that contain singular functions like $\mathfrak{T}b$. One reason as to why we have difficulties with this approach is that the canonical relation of $\widetilde{\boldsymbol{\mathcal{L}}}$ is not conic (or asymptotically conic) towards 'frequency infinity'. It seems that there is no FIO theory in the literature that applies to $\widetilde{\boldsymbol{\mathcal{L}}}$.

The main theorems we shall prove in this chapter are Theorem 18 and Theorem 22. Theorem 18 provides a reasonably explicit expression for the canonical relation associated to $\widetilde{\boldsymbol{\mathcal{L}}}$, denoted $\mathrm{CR}_{\widetilde{\boldsymbol{\mathcal{L}}}}$, and Theorem 22 states that the projection of $\mathrm{CR}_{\widetilde{\boldsymbol{\mathcal{L}}}}$ onto the left and right factors $T^*(G\times\mathbb{R})$ are immersions except on a codimension one submanifold where the projections have a singularity of blowdown type, which we define in Definition 6.

We recall that an oscillatory integral operator is an operator of the form

$$Tf(x) = \int_{Y\times\mathbb{R}^N} e^{i\phi(x,y,\theta)}\beta(x,y,\theta)f(y)dyd\theta \quad (3.5)$$

where the phase function $\phi(x,y,\theta) \in C^\infty(X\times Y\times\mathbb{R}^N)$ is real and $\beta \in C_c^\infty(X\times Y\times\mathbb{R}^N)$ is the symbol (or 'amplitude' is perhaps a better term in our case). Here X and Y are arbitrary smooth manifolds. For any (real, smooth) phase function $\phi(x,y,\theta)$, we can define the associated canonical relation as

$$\mathrm{CR}_\phi = \{(x, d_x\phi; y, -d_y\phi) : d_\theta\phi = 0\}. \tag{3.6}$$

It is well known that this is a smooth Lagrangian submanifold of $T^*X \times T^*Y$ where $T^*X \times T^*Y$ is equipped with the symplectic form $\omega_X - \omega_Y$ where ω_X and ω_Y are the canonical symplectic forms on T^*X and T^*Y respectively. In this definition, we do not assume that CR_ϕ is necessarily a *conic* Lagrangian submanifold, where the descriptor of 'conic' means that CR_ϕ is invariant with respect to the dilation action in the cotangent fibres of $T^*(X \times Y)$. The case of a conic canonical relation would correspond to the phase function $\phi(x, y, \theta)$ being homogeneous of degree one in θ.

We have that $\widetilde{\boldsymbol{\mathcal{L}}}$ is an oscillatory integral operator because one can express the pullback action of the symbol $a(g, r)$ by the map $M : (g, t, v) \mapsto g\mathbf{a}_t\mathbf{n}_v$ as an oscillatory integral itself. Let us clarify this point slightly. Given a smooth map $\chi : \mathbb{R}^m \mapsto \mathbb{R}^n$ and a function $f : \mathbb{R}^n \mapsto \mathbb{R}$, one has the expression

$$\chi^* f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(\chi(x)-y)\cdot\theta} f(y) dy d\theta \tag{3.7}$$

from the Fourier inversion formula on $\mathbb{R}^n$. Essentially, this is the fact that one can express the delta distribution itself as an oscillatory integral distribution

$$\delta(\chi(x) - y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\chi(x)-y)\cdot\theta} d\theta. \tag{3.8}$$

So for the phase function $\phi(x, y, \theta) = (\chi(x) - y) \cdot \theta$ we can compute the corresponding canonical relation associated to ϕ in the usual way as in (3.6), we get

$$\mathrm{CR}_\phi = \{(x, d\chi(x)^T\theta; \chi(x), \theta) : x \in \mathbb{R}^n, \theta \in \mathbb{R}^n\}. \tag{3.9}$$

where the T-superscript denotes the transpose. Therefore, expanding the pullback action in terms of a phase function, we can express the intertwining operator in equation (3.3) as an oscillatory integral operator in local coordinates. In fact, one has global coordinates by the Iwasawa or KAN decomposition of the group G. Namely, one has $G = KAN$ where K, A and N are the one-dimensional compact subgroup, abelian subgroup and unipotent subgroup respectively. Thus if we let θ, t and u to be the global coordinate which parameterises K, A and N respectively, one can parameterise G globally via $(\theta, t, u) \in \mathbb{R}^3$

where the coordinate map is

$$\begin{aligned} K \times A \times N &\to KAN \\ (\theta, t, u) &\mapsto k_\theta a_t n_u. \end{aligned} \tag{3.10}$$

So, it is possible to express the pullback operation by the map $M : (g, t, v) \mapsto g\mathbf{a}_t\mathbf{n}_u$ as a global oscillatory integral in these global coordinates. One could then derive the associated canonical relation of the global phase function of the intertwining operator via equation (3.6).

However, there is one slight note we should make regarding this approach. The map $K \times A \times N \to KAN$ is a diffeomorphism of manifolds but is not an isomorphism of groups. Suppose we apply this approach to study (the pullback by) the map $g \mapsto g\mathbf{a}_t\mathbf{n}_u$ for a fixed t and u or even the geodesic flow $g \mapsto g\mathbf{a}_t$ for a fixed t. If we derive the canonical relation associated to the phase function of the pullback operator of the map $g \mapsto g\mathbf{a}_t\mathbf{n}_u$ by taking the diffeomorphism of G onto $K \times A \times N$ and taking global coordinates on the latter manifold, we end up with a canonical relation which is a Lagrangian submanifold of $T^*(K \times A \times N) \times T^*(K \times A \times N)$. To bring this back to a Lagrangian submanifold of $T^*G \times T^*G$, we would need to pull this canonical relation back by pulling back $T^*(K \times A \times N)$ to T^*G with the (inverse of) the lift of the map (3.10) to the cotangent bundles. The lift of a diffeomorpism to the cotangent bundle (of the domain and target manifolds) is a symplectomorphism and thus the pullback of the canonical relation in this way will be a Lagrangian submanifold of $T^*G \times T^*G$.

The approach described in the paragraph above is entirely valid, however we will take a different approach to deriving the canonical relation which uses more directly the Lie algebraic structure associated to G, in particular, by respecting the group structure, the canonical relation we describe automatically descends to any $\Gamma\backslash G$, the left quotient of G by a discrete subgroup Γ. We will compute the canonical relation of the pullback by

$$M : (g, t, v) \mapsto g\mathbf{a}_t\mathbf{n}_v \tag{3.11}$$

which is

$$\mathrm{CR}_M = \{((g, t, v), dM|^T_{(g,t,v)}\alpha; g\mathbf{a}_t\mathbf{n}_v, \alpha) : (g, t, v) \in G \times \mathbb{R} \times \mathbb{R}, \alpha \in T^*_{g\mathbf{a}_t\mathbf{n}_v}G\} \tag{3.12}$$

which is just expression (3.9) applied to the map M. Note that (3.9) makes sense on a manifold (it is well-defined independently of the choice of coordinates), and it is here that we can calculate dM^T directly using the Lie algebraic structure associated to G.

We will then compose CR_M with the canonical relation of the phase function associated to the oscillatory integral operator

$$Tf(g,R) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^3} f(g,t,v,r)C(R,r)e^{2i(r-R)t}(1+v^2)^{-(\frac{1}{2}+i(2R-r))}drdtdv \tag{3.13}$$

which is just the intertwining operator $\widetilde{\boldsymbol{\mathcal{L}}}$ in (3.4) with $\tilde{a}(g\mathbf{a}_t\mathbf{n}_v, r)$ replaced with an arbitrary $f(g,t,v,r)$. We'll call the canonical relation CR_T. Then we observe that we have

$$\mathrm{CR}_{\widetilde{\boldsymbol{\mathcal{L}}}} = \mathrm{CR}_T \circ \mathrm{CR}_M\,. \tag{3.14}$$

3.2 Examples of canonical relations for certain other operators of interest.

In this section, we will present two examples of canonical relations associated to two other operators, the Euclidean intertwining operator and the pullback by geodesic flow on $G = \mathrm{PSL}(2,\mathbb{R})$. These canonical relations each illuminate some aspect of the canonical relation of the intertwining operator $\widetilde{\boldsymbol{\mathcal{L}}}$.

The canonical relation of the pullback by geodesic flow will introduce how we utilise the Lie algebraic structure of G and the Euclidean intertwining operator will give us an example of a canonical relation which has similar asymptotically conic properties as the canonical relation of $\widetilde{\boldsymbol{\mathcal{L}}}$.

3.2.1 The Euclidean intertwining operator.

We consider the operator acting unitarily on $L^2(\mathbb{R}^{2n}_{x,\xi})$ defined by $e^{iD_x \cdot D_\xi/2}$, where $D_x = \frac{1}{i}\frac{\partial}{\partial x}$ and $D_\xi = \frac{1}{i}\frac{\partial}{\partial \xi}$. It has the effect of changing quantisation from the standard quantisation of symbols $a(x,\xi)$ on $\mathbb{R}^{2n}$ to the Weyl quantisation of symbols over $\mathbb{R}^{2n}$ (see [Zwo12], Theorem 4.13). Because the Weyl quantisation satisfies an exact Egorov theorem on $\mathbb{R}^n$ and on its compact quotients (we will prove this in Chapter 5), we might call the operator $e^{iD_xD_\xi/2}$ the 'Euclidean intertwining operator'.

The operator $e^{iD_x \cdot D_\xi/2}$ acts on a function $a(x,\xi)$ as

$$e^{iD_x\cdot D_\xi/2}a(x,\xi) = \frac{1}{(2\pi)^{4n}} \int_{\mathbb{R}^{4n}} e^{iX\cdot\theta/2}e^{iX\cdot(x-y)}e^{i\theta\cdot(\xi-\eta)}a(y,\eta)dyd\eta dXd\theta. \tag{3.15}$$

Since the phase function is

$$\phi(x,\xi,y,\eta,X,\theta) = \frac{X\cdot\theta}{2} + X\cdot(x-y) + \theta\cdot(\xi-\eta), \tag{3.16}$$

we can write the canonical relation as

$$\mathrm{CR}_{e^{iD_xD_\xi/2}} = \Big\{\Big((x,\xi;Xdx+\theta d\xi),(y,\eta;Xdy+\theta d\eta)\Big) : \theta/2+x-y=0, X/2+\xi-\eta=0\Big\}. \tag{3.17}$$

Substituting in the relations $\theta/2+x-y=0$ and $X/2+\xi-\eta=0$ for x and for ξ, one gets the canonical relation

$$\Big\{\Big((y-\theta/2,\eta-X/2;Xdy+\theta d\eta),(y,\eta;Xdy+\theta d\eta)\Big) : y,\eta,X,\theta\in\mathbb{R}^n\Big\}. \tag{3.18}$$

We can directly observe that this canonical relation is a smooth Lagrangian submanifold of $T^*\mathbb{R}^{2n}\times T^*\mathbb{R}^{2n}$ and the projection of $\mathrm{CR}_{e^{iD_xD_\xi/2}}$ onto the left and right factors are diffeomorphisms of the canonical relation onto $T^*\mathbb{R}^{2n}$.

We can make a comment regarding asymptotically conic structures in this canonical relation. For example, the fact that $e^{iD_xD_\xi/2}$ acts invariantly on the standard Hörmander symbol classes $S^m_{\rho,\delta}(\mathbb{R}^{2n})$ can be seen as a consequence of the fact that $\mathrm{CR}_{e^{iD_xD_\xi/2}}$ is asymptotically conic in the base variables y and η at a finite X and θ and the asymptotic map at spatial infinity is the identity map. Moreoever, we also notice that the conic action in X and θ is more complicated than first appears because of the appearance of $y-\theta/2$ and $\eta-X/2$. Namely, because of these factors there is not a conic structure in the fibres of the canonical relation. (As such $e^{iD_x\cdot D_\xi/2}$ doesn't have a clear action on symbols with C^∞ singularities). Something similar seems to appear in the canonical relation for $\widetilde{\boldsymbol{\mathcal{L}}}$.

3.2.2 Pullback by the geodesic flow on the upper half plane.

For this example, we will compute the canonical relation of the operator which pulls a function on $G=\mathrm{PSL}(2,\mathbb{R})\simeq S\mathbb{H}^2$ back by the geodesic flow at time t on $S\mathbb{H}^2$. Recall that under the identification $S\mathbb{H}^2\simeq\mathrm{PSL}(2,\mathbb{R})$, the time t geodesic flow from a point $(z,v)\simeq g$ is given by the right action on g by $\mathbf{a}_t$ as defined in Chapter 1. The pullback operator is then $(g_t)^*f(g)=f(g\mathbf{a}_t)$.

Let us fix some notation. We take a frame of basis vectors of the tangent space to G at the identity given by (H,X_+,X_-), where H,X_+, and X_- are elements of the Lie algebra of G that generate the geodesic flow, the stable horocyclic flow and unstable horocyclic flow respectively as in Chapter 1. This frame extends (under the left G-action) to a framed basis about every fibre of TG. Then we let $(\alpha_0,\alpha_u,\alpha_s)$ be the dual frame to

(H, X_+, X_-), so that $(\alpha_0, \alpha_u, \alpha_s)$ extends (again by the left G-action) to a framed basis about every fibre of T^*G. In particular, α_0 spans E_0^*, α_u spans E_u^* and α_s spans E_s^* where E_0^*, E_u^* and E_s^* are the dynamical sub-bundles of T^*G as described in the preliminary Chapter 1. Then for $\mu_0, \mu_u, \mu_s \in \mathbb{R}$, we denote a point in T^*G with fibre coordinates (μ_0, μ_u, μ_s) with regards to the frame $(\alpha_0, \alpha_u, \alpha_s)$, i.e. $(g; \mu_0, \mu_u, \mu_s)$ represents the point $(g; \mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s) \in T^*G$.

Note that for $g_t : g \mapsto g\mathbf{a}_t$, we have calculated in Chapter 1 that $(g_t)_* = dg_t$ acts on the frame (H, X_+, X_-) in the following way:

$$\begin{aligned} (g_t)_*H &= H \\ (g_t)_*X_+ &= e^{-t}X_+ \\ (g_t)_*X_- &= e^{t}X_-. \end{aligned} \tag{3.19}$$

Consequently, the pullback $(g_t)^*$ acts (via the transpose of dg_t) on the frame $(\alpha_0, \alpha_u, \alpha_s)$ as:

$$\begin{aligned} (g_t)^*\alpha_0 &= \alpha_0 \\ (g_t)^*\alpha_u &= e^{t}\alpha_u \\ (g_t)^*\alpha_s &= e^{-t}\alpha_s. \end{aligned} \tag{3.20}$$

Hence this gives us the canonical relation of $(g_t)^*$ as

$$\mathrm{CR}_{(g_t)^*} = \Big\{ \Big((g\mathbf{a}_t; \mu_0, e^{t}\mu_u, e^{-t}\mu_s), (g; \mu_0, \mu_u, \mu_s) \Big) : (g; \mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s) \in T^*G \Big\}. \tag{3.21}$$

We note that $\mathrm{CR}_{(g_t)^*}$ is equivariant under the left G-action, so the canonical relation of $(g_t)^*$ acting on a hyperbolic surface is simply the left quotient of $\mathrm{CR}_{(g_t)^*}$ by the subgroup $\Gamma \subset G$ generating the hyperbolic surface.

We also remark that this canonical relation has a conic structure in the fibre variables and so one can use the geometry of $\mathrm{CR}_{(g_t)^*}$ to inform an analysis of the propagation of C^∞ singularities for the map $(g_t)^*$. Such an analysis of the propagation of singularities of $(g_t)^*$ has been done in Faure, Sjöstrand [FS10] and also in Dyatlov, Zworski [DZ13] for example (both in the wider context of a contact Anosov flow).

3.3 Canonical relation for the intertwining operator.

In this section we will apply the argument sketched in the introduction of this chapter to derive the canonical relation of the intertwining operator.

We recall from expression (3.4) that

$$\widetilde{\mathcal{L}}\tilde{a}(g,R) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} C(R,r)\tilde{a}(g\mathbf{a}_t\mathbf{n}_v,r)e^{2i(r-R)t}(1+v^2)^{-(\frac{1}{2}+i(2R-r))}drdtdv \tag{3.22}$$

where $C(R,r) = 2^{\frac{1}{2}+iR}c(2R-r)$. We will write $C(R,r)$ in polar form as

$$C(R,r) = |C(R,r)|e^{i\theta(r,R)} = \left(\frac{2R-r}{\pi}\tanh(\pi(2R-r))\right)^{1/2} e^{i\theta(r,R)}. \tag{3.23}$$

Note that $|c(r)|^2 = \frac{r}{2\pi}\tanh(\pi r)$. We will write the intertwining operator in the following way,

$$\widetilde{\mathcal{L}}\tilde{a}(g,R) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3}\tilde{a}(g\mathbf{a}_t\mathbf{n}_v,r)e^{i\phi}|C(R,r)|drdt\frac{dv}{\sqrt{1+v^2}} \tag{3.24}$$

with

$$\phi = 2(r-R)t + (r-2R)\log(1+v^2) + \theta(R,r). \tag{3.25}$$

We will let (g,r) be coordinates for the base manifold $G\times\mathbb{R}$ and in the fibres of $T^*(G\times\mathbb{R})$, we will fix a basis $(\alpha_0,\alpha_u,\alpha_s,dr)$, where $\alpha_0,\alpha_u,\alpha_s$ span the dynamically relevant subbundles of T^*G as described in example 3.2.2. The coordinates in this basis will be (μ_0,μ_u,μ_s,k). If we let M be the following map,

$$\begin{aligned} M : G\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} &\to G\times\mathbb{R} \\ (g,t,v,r) &\mapsto (g\mathbf{a}_t\mathbf{n}_v,r), \end{aligned} \tag{3.26}$$

we can write

$$\widetilde{\mathcal{L}}\tilde{a}(g,R) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} M^*\tilde{a}(g,t,v,r)e^{i\varphi}|C(R,r)|drdt\frac{dv}{\sqrt{1+v^2}}. \tag{3.27}$$

Note that, as described in the introduction to this chapter, to determine the canonical relation of the associated phase function to $\mathcal{L}$, we will first compute the canonical relation of M^* and compose this with the canonical relation of the oscillatory integral operator

$$Tf(g,R) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} f(g,t,v,r)e^{i\phi}|C(R,r)|drdt\frac{dv}{\sqrt{1+v^2}} \tag{3.28}$$

where ϕ is (3.25).

3.3.1 The canonical relation of the map M^*.

In this section we study the smooth map $M : (g, t, v, r) \mapsto (g\mathbf{a}_t\mathbf{n}_v, r)$. Since M acts separately as the identity on the r-coordinate, we will suppress the r-coordinate from the calculations below for a cleaner notation.

We will first determine the pushforward of M acting on the basis vectors $(H, X_+, X_-, \partial_t, \partial_v)$ of $T(G \times \mathbb{R}^2_{t,v})$ to the basis vectors (H, X_+, X_-) of TG.

Note that M pushes forward the curve $s \mapsto (g\mathbf{a}_s, t, v)$ on $G\times\mathbb{R}^2_{t,v}$ to the curve $s \mapsto g\mathbf{a}_s\mathbf{a}_t\mathbf{n}_v$ on G. Since the tangent vector H at g is the derivative of the curve $s \mapsto g\mathbf{a}_s$ at $s = 0$, and the tangent vectors X_+ and X_- at g are likewise derivatives at $s = 0$ of the curves $s \mapsto g\mathbf{n}_s$ and $s \mapsto g\overline{\mathbf{n}_s}$ respectively, we have, using the relations described in the preliminary chapter,

$$\begin{aligned} M_*H &= \left.\frac{d}{ds}\right|_{s=0} g\mathbf{a}_s\mathbf{a}_t\mathbf{n}_v \\ &= \left.\frac{d}{ds}\right|_{s=0} (g\mathbf{a}_t\mathbf{n}_v)\mathbf{n}_{v(e^s-1)}\mathbf{a}_s \\ &= H + vX_+ \end{aligned} \tag{3.29}$$

where we interpret $H+vX_+$ as belonging to the tangent space to G at $g\mathbf{a}_t\mathbf{n}_v = M(g, t, v)$. Note that the right action on $g\mathbf{a}_t\mathbf{n}_v$ by $\mathbf{n}_{v(e^s-1)}\mathbf{a}_s$ has first order contact at $s = 0$ with the tangent vector $H+vX_+$ (recalling that X_+ is the infinitesimal generator of the right action by $\mathbf{n}_s$). The calculation for the pushforward of X_+ is very similar, we compute

$$\begin{aligned} M_*X_+ &= \left.\frac{d}{ds}\right|_{s=0} g\mathbf{n}_s\mathbf{a}_t\mathbf{n}_v \\ &= \left.\frac{d}{ds}\right|_{s=0} (g\mathbf{a}_t\mathbf{n}_v)\mathbf{n}_{e^{-t}s} \\ &= e^{-t}X_+. \end{aligned} \tag{3.30}$$

And finally the pushforward of X_- by M is given by the calculation

$$\begin{aligned} M_*X_- &= \left.\frac{d}{ds}\right|_{s=0} g\overline{\mathbf{n}_s}\mathbf{a}_t\mathbf{n}_v \\ &= \left.\frac{d}{ds}\right|_{s=0} (g\mathbf{a}_t\mathbf{n}_v)(\mathbf{n}_{-v}\mathbf{a}_{-t}\overline{\mathbf{n}_s}\mathbf{a}_t\mathbf{n}_v). \end{aligned} \tag{3.31}$$

To complete the calculation in this case and express the result in the (H, X_+, X_-) basis, note that we have the right action on $g\mathbf{a}_t\mathbf{n}_v$ by the group element $\mathbf{n}_{-v}\mathbf{a}_{-t}\overline{\mathbf{n}_s}\mathbf{a}_t\mathbf{n}_v$. To take the derivative we can express this right action in terms of a right action of matrices of $\mathrm{PSL}(2,\mathbb{R})$ which we can then take the derivative of in s, finally we interpret the result in terms of the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$. The result of taking the derivative in s of the matrix product $\mathbf{n}_{-v}\mathbf{a}_{-t}\overline{\mathbf{n}_s}\mathbf{a}_t\mathbf{n}_v$ is the following,

$$\left.\frac{d}{ds}\right|_{s=0} \mathbf{n}_{-v}\mathbf{a}_{-t}\overline{\mathbf{n}_s}\mathbf{a}_t\mathbf{n}_v = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}. \tag{3.32}$$

Evaluating this matrix product returns the single matrix,

$$\left.\frac{d}{ds}\right|_{s=0} \mathbf{n}_{-v}\mathbf{a}_{-t}\overline{\mathbf{n}_s}\mathbf{a}_t\mathbf{n}_v = \begin{pmatrix} -e^t v & -e^t v^2 \\ e^t & e^t v \end{pmatrix} \tag{3.33}$$

which, referencing the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$ in Chapter 1, is equivalent to

$$e^t(-2vH - v^2 X_+ + X_-). \tag{3.34}$$

We can compute $M_*\partial_t$ and $M_*\partial_v$ in similar ways. In particular,

$$\begin{aligned} M_*\partial_t &= \left.\frac{d}{ds}\right|_{s=0} g\mathbf{a}_{t+s}\mathbf{n}_v \\ &= \left.\frac{d}{ds}\right|_{s=0} g\mathbf{a}_t\mathbf{n}_v\mathbf{n}_{v(e^s-1)}\mathbf{a}_s \\ &= H + vX_+ \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} M_*\partial_v &= \left.\frac{d}{ds}\right|_{s=0} g\mathbf{a}_t\mathbf{n}_{v+s} \\ &= \left.\frac{d}{ds}\right|_{s=0} g\mathbf{a}_t\mathbf{n}_v\mathbf{n}_s \\ &= X_+. \end{aligned} \tag{3.36}$$

Therefore we have proved the following,

Lemma 16. *For the map*

$$\begin{aligned} M : G \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} &\to G \times \mathbb{R} \\ (g, t, v, r) &\mapsto (g\mathbf{a}_t\mathbf{n}_v, r), \end{aligned} \tag{3.37}$$

M_* expressed as a matrix acting on the framed basis $(H, X_+, X_-, \partial_t, \partial_u, \partial_r)$ to the framed

basis $(H, X_+, X_-, \partial_r)$ *is,*

$$M_* = \begin{pmatrix} 1 & 0 & -2e^t v & 1 & 0 & 0 \\ v & e^{-t} & -e^t v^2 & v & 1 & 0 \\ 0 & 0 & e^t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.38}$$

Moreover the pullback map M^* *acting on the dual framed basis,*

$$M^* : (\alpha_0, \alpha_u, \alpha_s, dr) \to (\alpha_0, \alpha_u, \alpha_s, dt, dv, dr) \tag{3.39}$$

is

$$M^* = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ -2e^t v & -e^t v^2 & e^t & 0 \\ 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.40}$$

As a consequence of this lemma, we have

Corollary 17. *The canonical relation of* M^* *is*

$$\Big\{ \big((g, t, v, r; M^*(\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s + kdr)), (g\mathbf{a}_t\mathbf{n}_v, r; \mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s + kdr)\big) : \\ (t, v) \in \mathbb{R}^2, (g\mathbf{a}_t\mathbf{n}_v, r; \mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s + kdr) \in T^*(G \times \mathbb{R})\Big\}. \tag{3.41}$$

where explicitly,

$$M^*(\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s + kdr) = (\mu_0 + v\mu_u)\alpha_0 + e^{-t}\mu_u\alpha_u \\ + e^t(-2v\mu_0 - v^2\mu_u + \mu_s)\alpha_s + (\mu_0 + v\mu_u)dt + \mu_u dv + kdr. \tag{3.42}$$

3.3.2 The canonical relation of the intertwining operator.

We will seek to compose the canonical relation of M with the canonical relation associated to the phase function of the oscillatory integral operator

$$Tf(g, R) = \frac{1}{\sqrt{2}\pi} \int_{\mathbb{R}^3 \times G} f(g', t, v, r)\delta_g(g')C(R, r)e^{2i(r-R)t}(1 + v^2)^{-(\frac{1}{2}+i(2R-r))} drdtdvdg'. \tag{3.43}$$

Since g and g' appear only in the phase of $\delta_g(g')$ which has the diagonal canonical relation (in $T^*G \times T^*G$), we only need to look at the phase function

$$\phi = 2(r-R)t + (r-2R)\log(1+v^2) + \theta(R,r) \tag{3.44}$$

to compute the canonical relation of the phase function associated to T. Doing this calculation, one gets

$$\begin{aligned}\mathrm{CR}_T = \Big\{ &((g,R;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s + d_R\phi), (g,t,v,r;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s - d_{(t,v,r)}\phi)) : \\ &\qquad (R,t,v,r) \in \mathbb{R}^4, (g;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s) \in T^*G \Big\}\end{aligned} \tag{3.45}$$

which is equivalent to

$$\begin{aligned}\Big\{ \Big(&(g,R;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s - (2t + 2\log(1+v^2) - \partial_R\theta)dR), \\ &(g,t,v,r;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s - 2(r-R)dt - (r-2R)\frac{2v}{1+v^2}dv \\ &- (2t + \log(1+v^2) + \partial_r\theta)dr) \Big) : (R,t,v,r) \in \mathbb{R}^4, (g;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s) \in T^*G \Big\}.\end{aligned} \tag{3.46}$$

Composing CR_T in (3.46) with CR_{M^*} in (3.41) we get,
Theorem 18.

$$\begin{aligned}\mathrm{CR}_{\widetilde{\mathcal{L}}} = \Big\{ &((\widehat{g},R;\widehat{\mu_0}\alpha_0 + \widehat{\mu_u}\alpha_u + \widehat{\mu_s}\alpha_s + \widehat{k}dR), (g,r;\mu_0\alpha_0 + \mu_u\alpha_u + \mu_s\alpha_s + kdr)) \Big\} \\ &\subset T^*(G\times\mathbb{R}) \times T^*(G\times\mathbb{R})\end{aligned} \tag{3.47}$$

under the following set of relations,

$$\begin{aligned} g &= \widehat{g}\mathbf{a}_t\mathbf{n}_v \\ \widehat{\mu_0} &= \mu_0 + v\mu_u \\ \widehat{\mu_u} &= e^{-t}\mu_u \\ \widehat{\mu_s} &= e^t(-2v\mu_0 - v^2\mu_u + \mu_s) \\ \widehat{k} &= -2t - 2\log(1+v^2) + \partial_R\theta \\ \mu_0 + v\mu_u &= -2(r-R) \\ \mu_u &= -(r-2R)\frac{2v}{1+v^2} \\ k &= -2t - \log(1+v^2) - \partial_r\theta \end{aligned} \tag{3.48}$$

with the following being free variables:

$$g \in G = \mathrm{PSL}(2, \mathbb{R}), \quad r \in \mathbb{R}, \quad \mu_s \in \mathbb{R}, \quad v \in \mathbb{R}, \quad t \in \mathbb{R}, \quad R \in \mathbb{R}. \tag{3.49}$$

3.4 Comments on the structure of the canonical relation of the intertwining operator.

In this section, we will explore the structure of the canonical relation of the intertwining operator in a bit more detail, namely, we will study the smooth structure of the canonical relation and the projection of the canonical relation onto the left and right factors. We will also end with a discussion on the homogeneous structures present (or not present) in the canonical relation and relate some of the ideas in this work within the context of the broader literature surrounding canonical relations and mapping properties of Fourier integral operators associated to a Lagrangian submanifold.

Proposition 19. *The canonical relation of $\widetilde{\boldsymbol{\mathcal{L}}}$ is a smoothly immersed submanifold of $T^*(G \times \mathbb{R}) \times T^*(G \times \mathbb{R})$.*

Proof. We observe that the canonical relation has 8 free variables which (globally) parameterise it, g, r, μ_s, v, t, R. In particular the points in the canonical relation are a smooth function of these 8 variables and so to check that $\mathrm{CR}_{\widetilde{\boldsymbol{\mathcal{L}}}}$ is a submanifold of $T^*(G \times \mathbb{R}) \times T^*(G \times \mathbb{R})$, we will check that the parameterising variables are smooth, invertible functions of the points in $T^*(G \times \mathbb{R}) \times T^*(G \times \mathbb{R})$.

Namely, let $p \in \mathrm{CR}_{\boldsymbol{\mathcal{L}}}$ with $i(p) = ((g, r; \mu_0, \mu_u, \mu_s, k), (\widehat{g}, R, \widehat{\mu_0}, \widehat{\mu_u}, \widehat{\mu_s}, \widehat{k}))$ where i is the inclusion map. Then we need to show that the (global) variables which parameterise p, i.e g, r, μ_s, v, t, R, are smooth, invertible functions of $i(p) \in T^*(G \times \mathbb{R}) \times T^*(G \times \mathbb{R})$. Note that g, r, μ_s and R can be read off directly from $i(p)$ and we only need to solve for t and v as a smooth function of $i(p)$.

To do this, note from the equation $\widehat{k} - 2k = 2t + \partial_R\theta - \partial_r\theta$ we get $t = \widehat{k}/2 - k - \partial_R\theta + 2\partial_r\theta$ which solves for t. To solve for v, note that $\widehat{g} = g\mathbf{n}_{-v}\mathbf{a}_{-t}$, and so we get $g^{-1}\widehat{g}\mathbf{a}_t = \mathbf{n}_{-v}$, which solves for v. □

3.4.1 The left projection.

Let us now study the projection of the canonical relation onto the left coordinates. As in the previous section, we will use (g, r, μ_s, v, t, R) as coordinates that parameterise $\mathrm{CR}_{\boldsymbol{\mathcal{L}}}$. First let us give a definition,

Definition 6. Given a smooth map $f : X \mapsto Y$ between manifolds X and Y and a natural number r, $1 \leq r \leq \min(\dim X, \dim Y)$, we can define the *singularity set of f of order r* as $S := \{x \in X : \operatorname{rank}(df(x)) = \min(\dim X, \dim Y) - r\}$, i.e the points $x \in X$ where $df(x)$ drops rank by r. We say f has a *blowdown singularity of order r* on S, the singularity set of order r, if S is a submanifold of X and if for all $x \in S$,

$$\ker(df(x)) \subset T_x S. \tag{3.50}$$

Note, these are also sometimes called fibered-fold singularities in certain contexts. An example of a blowdown singularity of order 1 is the map

$$f : \mathbb{R}^2 \mapsto \mathbb{R}^2, \quad f(x,y) := (x, xy) \tag{3.51}$$

on the submanifold $x = 0$. More generally, for any $1 \leq k < N$, the map

$$f_k : \mathbb{R}^N \mapsto \mathbb{R}^N, \quad f_k(x_1, x_2, \dots, x_N) := (y_1, y_2, \dots, y_N) \tag{3.52}$$

where $y_i = x_i$ for all $1 \leq i \leq k$ and $y_j = x_1 x_j$ for all $k+1 \leq j \leq N$ is an example of a map which has a blowdown singularity of order $N - k$ on the submanifold $x_1 = 0$.

We now prove

Proposition 20. *The left projection,*

$$\pi_{left} : \mathrm{CR}_{\widetilde{\mathcal{L}}} \ni (g, r, \mu_s, v, t, R) \to (g, r; \mu_0, \mu_u, \mu_s, k) \in T^*(G \times \mathbb{R})$$

of the canonical relation of the intertwining operator is an immersion everywhere except when $r - 2R = 0$, where $d\pi_{left}$ drops rank by 1. In addition, the singularity of π_{left} on $r - 2R = 0$ is a blowdown singularity.

Proof. It is clear that the (identity) map from (g, r, μ_s) to (g, r, μ_s) is immersive. The map from t to k is immersive too because

$$k = -2t - \log(1 + v^2) - \partial_r \theta(r, R) \tag{3.53}$$

and in particular $\partial k / \partial t = -2$, so we need only check that the map from (v, R) to (μ_0, μ_u) is an immersion. We have from (3.48) that

$$\begin{aligned} \mu_0 &= \frac{2v^2}{1+v^2}(r - 2R) - 2(r - R), \\ \mu_u &= -(r - 2R)\frac{2v}{1+v^2}, \end{aligned} \tag{3.54}$$

where the first equation arrives from substituting the second equation of (3.54) into the equation $\mu_0 + v\mu_u = -2(r-R)$ which appears in (3.48).

We then compute the partial derivatives,

$$\begin{aligned} \frac{\partial \mu_0}{\partial v} &= \frac{4v(r-2R)}{(1+v^2)^2}, \qquad \frac{\partial \mu_u}{\partial v} = -\frac{(2-2v^2)(r-2R)}{(1+v^2)^2}, \\ \frac{\partial \mu_0}{\partial R} &= \frac{2-2v^2}{1+v^2}, \qquad \frac{\partial \mu_u}{\partial R} = \frac{4v}{1+v^2}. \end{aligned} \tag{3.55}$$

So the determinant of the Jacobian of the projection from (v, R) to (μ_0, μ_u) is

$$\frac{16v^2}{(1+v^2)^3}(r-2R) + \frac{(2-2v^2)^2}{(1+v^2)^3}(r-2R) \tag{3.56}$$

which is zero only when $r - 2R = 0$. On this submanifold where $r - 2R = 0$, we see that

$$\frac{\partial \mu_0}{\partial v} = 0 \quad \text{and} \quad \frac{\partial \mu_u}{\partial v} = 0 \tag{3.57}$$

whereas $\partial_R \mu_0$ and $\partial_R \mu_u$ are never both zero on this submanifold. In particular, $\ker(d\pi_{\text{left}})$ is everywhere tangential to the hypersurface defined via $r - 2R = 0$. This is therefore a blowdown singularity. □

3.4.2 The right projection.

Proposition 21. *The right projection,*

$$\pi_{right} : \mathrm{CR}_{\tilde{\mathcal{L}}} \ni (g, r, \mu_s, v, t, R) \to (\widehat{g}, R; \widehat{\mu_0}, \widehat{\mu_u}, \widehat{\mu_s}, \widehat{k}) \in T^*(G \times \mathbb{R}) \tag{3.58}$$

of the canonical relation of the intertwining operator is an immersion everywhere, except when $r - 2R = 0$, where the differential $d\pi_{right}$ drops rank by 1. In addition, the singularity of π_{right} on $r - 2R = 0$ is a blowdown singularity.

Proof. We recall from (3.48),

$$\begin{aligned} \widehat{g} &= g\mathbf{n}_{-v}\mathbf{a}_{-t} \\ \widehat{\mu_0} &= \mu_0 + v\mu_u \\ \widehat{\mu_u} &= e^{-t}\mu_u \\ \widehat{\mu_s} &= e^t(-2v\mu_0 - v^2\mu_u + \mu_s) \\ \widehat{k} &= -2t - 2\log(1+v^2) + \partial_R\theta(r, R). \end{aligned}$$

If we substitute

$$\mu_0 = \frac{2v^2}{1+v^2}(r-2R) - 2(r-R),$$
$$\mu_u = -(r-2R)\frac{2v}{1+v^2}.$$

into the equations for $\widehat{\mu_0}$ and $\widehat{\mu_u}$, we get that

$$\begin{aligned} \widehat{\mu_0} &= -2(r-R), \\ \widehat{\mu_u} &= -e^{-t}(r-2R)\frac{2v}{1+v^2}. \end{aligned} \tag{3.59}$$

We know that the map from g to $\widehat{g}$ is immersive irrespective of v and t, the identity map from R to R is also obviously an immersion, as is the map from μ_s to $\widehat{\mu_s}$ for every t. We also see that the map from r to $\widehat{\mu_0}$ is an immersion. So to check where $d\pi_{\text{right}}$ drops rank, we only need to study the map from (v,t) to $(\widehat{\mu_u}, \widehat{k})$. We write out the partial derivatives,

$$\begin{aligned} \frac{\partial\widehat{\mu_u}}{\partial v} &= -\frac{e^{-t}(r-2R)(2-2v^2)}{(1+v^2)^2}, \qquad & \frac{\partial\widehat{k}}{\partial v} &= -\frac{4v}{1+v^2} \\ \frac{\partial\widehat{\mu_u}}{\partial t} &= -\widehat{\mu_u} = -\frac{e^{-t}(r-2R)2v}{1+v^2}, \qquad & \frac{\partial\widehat{k}}{\partial t} &= -2 \end{aligned} \tag{3.60}$$

So the determinant of $d\pi_{\text{right}}$ from (v,t) to $(\widehat{\mu_u}, \widehat{k})$ is

$$\frac{2e^{-t}(r-2R)(2-2v^2)}{(1+v^2)^2} + \frac{8v^2e^{-t}(r-2R)}{1+v^2} \tag{3.61}$$

which can only be zero when $r-2R=0$.

If we restrict π_{right} to the submanifold $r-2R=0$, we see that the kernel of $d\pi_{\text{right}}$ is a one dimensional subspace of the (v,t) tangent space which is again tangential to the singularity surface. Hence π_{right} has a blowdown singularity on the hypersurface defined by $r-2R=0$. □

To conclude this section, we have proved

Theorem 22. *The associated phase function of the intertwining operator $\widetilde{\mathcal{L}}$ has a smooth, Lagrangian canonical relation,*

$$\mathrm{CR}_{\widetilde{\mathcal{L}}} \subset T^*(G\times\mathbb{R}_R)\times T^*(G\times\mathbb{R}_r) \tag{3.62}$$

and the projections of the canonical relation onto the left factor and the right factor are immersions everywhere except for a singular submanifold defined by $r-2R=0$ where the *left and right projections have a blow down singularity.*

We remark that although we have blowdown singularities in the left and right projection maps of the canonical relation on the singular submanifold, the factor of $|C(R, r)|$ which appears in the 'symbol' (or 'amplitude') of the intertwining operator, see (3.23), happens to vanish to first order at $r - 2R = 0$. So the intertwining operator has the structure of an oscillatory integral operator whose symbol vanishes to first order on the singular submanifold. We note that one can carry out the same calculation with the intertwining operator $\mathcal{L}$ in the work of Anantharaman-Zelditch, [AZ12], to find that the associated canonical relation of $\mathcal{L}$ in their work also has a blowdown singularity appearing in the left and right projection maps on the submanifold defined in coordinates by $r - 2R = 0$. However, the lack of the $C(r, R)$ factor in the intertwining operator $\mathcal{L}$ in [AZ12] can be interpreted as the reason why their operator $\mathcal{L}$ isn't bounded on L^2 symbols.

In fact, we studied $\mathcal{L}$ of [AZ12] noticing the canonical relation had such a structure and (prior to realising the connection with Chapter 2) started to consult the work of those who have analysed the mapping properties of oscillatory or Fourier integral operators with singularities on their projection maps. For example, the work of Melrose-Taylor [MT85] (although they treat the case when both projections have folding singularities rather than blowdowns), show a loss of 1/6'th a derivative compared to standard L^2 boundedness of a zero'th order Fourier Integral Operator associated to a canonical transformation as proved in [Hör71]. In this vein, we might also mention the work of [Fel05] which studies the composition of operators which have blowdown/fold singularities.

Lastly, let us comment that from Theorem 18, relations (3.48), we notice that $\mathrm{CR}_{\widetilde{\boldsymbol{\mathcal{L}}}}$ has a conic structure with regards to the *scattering wavefront set* (i.e. at $r \to \infty$). Precisely, we notice from the relations in (3.48) that the canonical relation of $\boldsymbol{\mathcal{L}}$ is conic with regards to the scaling action in a certain set of variables,

$$(r, \mu_0, \mu_u, \mu_s, R, \widehat{\mu_0}, \widehat{\mu_u}, \widehat{\mu_s}) \to (\lambda r, \lambda\mu_0, \lambda\mu_u, \lambda\mu_s, \lambda R, \lambda\widehat{\mu_0}, \lambda\widehat{\mu_u}, \lambda\widehat{\mu_s}) \tag{3.63}$$

for any $\lambda \in \mathbb{R}$. Hence, it is possible to explore the boundedness of $\widetilde{\boldsymbol{\mathcal{L}}}$ on symbols which have nontrivial points in their scattering wavefront set and the question of the construction of a variable order function space on which $\tilde{\boldsymbol{\mathcal{L}}}$ is bounded is an intersting topic for further study. However, this observation doesn't suffice for the study of $\widetilde{\boldsymbol{\mathcal{L}}}$ on singular functions like $\mathfrak{T}b(g, r)$ which have C^∞ singularities. In particular we would want a conic structure (or asymptotically conic structure) with regards to the frequency set of variables $(\mu_0, \mu_u, \mu_s, k, \widehat{\mu_0}, \widehat{\mu_u}, \widehat{\mu_s}, \widehat{k})$. A similar behaviour appears in the canonical relation of the Euclidean intertwining operator in example 3.2.1 which also has a well-defined structure at scattering infinity but not frequency infinity.

Let us recall that the reason for studying the canonical relation of $\widetilde{\mathcal{L}}$ is because it is well known that the geometric properties of the canonical relation of an oscillatory integral operator (or Fourier integral operator) determine the analytic properties of the operator, such as boundedness on various Sobolev spaces. However, we have not been able to find a known calculus of Fourier integral operators which the intertwining operator $\widetilde{\mathcal{L}}$ lies in primarily because of the lack of clear conic structure of the canonical relation, this has frustrated our efforts at obtaining boundedness results for the intertwining operator on other function spaces (such as microlocally weighted Sobolev spaces, or anisotropic Sobolev spaces, such as those we use in Chapter 4) other than L^2.

Chapter 4

Geodesic flow and decay of traces on hyperbolic surfaces

4.1 Introduction

In this chapter we consider traces and matrix elements of time-varying families of pseudodifferential operators defined on a compact hyperbolic surface which have the symbol pulled back by the geodesic flow. Note that, as we mentioned in the introduction, the main result in this chapter does not involve the intertwining operator $\mathcal{L}$ at all. The work in this chapter appears in a arXiv preprint located at https://doi.org/10.48550/arXiv.2402.00230 and is published jointly with Andrew Hassell.

The motivation for the work in this chapter comes in two parts. One is that the intertwining operator intertwines the classical flow with the quantum flow on hyperbolic space and two, in the study of small-scale quantum ergodicity, as we will show in Chapter 5, we are led to the study of time-varying families of traces of the form

$$\mathrm{Trace}\{A_h e^{ith\Delta} e^{-h^2\Delta} A_h e^{-ith\Delta} e^{-h^2\Delta}\}, \tag{4.1}$$

where h is a semiclassical parameter, representing inverse frequency, and A_h is a multiplication operator by a function with support in a ball around a fixed point with a radius tending to zero as $h \to 0$. Here the factors of $e^{-h^2\Delta}$ are a soft frequency cutoff to ensure that the composition is trace class.

More generally, we consider traces of the form

$$h^2\, \mathrm{Trace}\{B_h e^{iht\Delta} A_h e^{-ith\Delta}\}, \tag{4.2}$$

where A_h and B_h are pseudodifferential operators of differential order $-\infty$ and semiclassical order 0 on a compact hyperbolic surface. The factor of h^2 in front is to cancel the growth rate of the trace of an operator of semiclassical order zero as $h \to 0$, which is

$\sim h^{-2}$ in two dimensions. One can ask about the asymptotic property of such a trace as $h \to 0$. In particular, it would be useful to understand under what conditions (4.2) tends to zero as $h \to 0$, and if so, what is the optimal h-dependent choice of t.

We now consider the following *purely formal* calculation.

Notice that on the whole of hyperbolic space, the trace of B^*A is the integral of $a\bar{b}$ over the cotangent bundle, but this is not the case on a hyperbolic surface $X := \Gamma\backslash\mathbb{D}$ where $\Gamma \subset G$ is a discrete cocompact group. Instead, it takes the form, for $w > 1/2$

$$\text{Trace}\, B^*A = \int_{\Gamma\backslash G \times S_w} a(g,r)\overline{\mathfrak{T}b(g,r)}\, dg\, dr \tag{4.3}$$

where $\mathfrak{T}$ is a linear operator mapping symbols to distributions, such that $\mathfrak{T}b$ has support on the countable number of energy shells of radius r_j, where the spectrum of Δ_X is $\{1/4 + r_j^2\}$. S_w is the strip in the complex plane $\{r \in \mathbb{C} : |\Im(r)| < w|\}$ of width w. This identity is derived in Lemma 27 and Corollary 28.

V_t is the action induced on the Zelditch symbol by conjugation with the Schrodinger propagator $e^{it\Delta}$ at time t. Thus the symbol of the operator $e^{iht\Delta}Ae^{-ith\Delta}$ from (4.1) is $V^{th}a$. So we find that

$$\text{Trace}\{B^*e^{ith\Delta}Ae^{-ith\Delta}\} = \int_{\Gamma\backslash G \times \mathbb{R}} (V_{th}a)(g,r)\overline{(\mathfrak{T}b)(g,r)}\, dg\, dr. \tag{4.4}$$

Under some regularity assumption on the intertwining operator $\boldsymbol{\mathcal{L}}$, one would have (and this is where the argument becomes formal)

$$\text{Trace}\{B^*e^{ith\Delta}Ae^{-ith\Delta}\} = \int_{\Gamma\backslash G \times \mathbb{R}} (\boldsymbol{\mathcal{L}}V_{th}a)(g,r)\overline{\boldsymbol{\mathcal{L}}\big(\mathfrak{T}b(g,r)\big)}\, dg\, dr. \tag{4.5}$$

We now apply the intertwining property of $\boldsymbol{\mathcal{L}}$, namely

$$\boldsymbol{\mathcal{L}}V_t = G_t^*\boldsymbol{\mathcal{L}}, \tag{4.6}$$

where G_t is the geodesic flow scaled by the speed factor r on the energy shell of radius r (this is the bicharacteristic flow for the operator $e^{it\Delta/2}$). Thus we obtain

$$\text{Trace}\{B^*e^{iht\Delta/2}Ae^{-ith\Delta/2}\} = \int_{\Gamma\backslash G \times \mathbb{R}} (G_{th}^*\boldsymbol{\mathcal{L}}a)(g,r)\overline{\boldsymbol{\mathcal{L}}\big(\mathfrak{T}b(g,r)\big)}\, dg\, dr. \tag{4.7}$$

Now comparing this identity with (4.3), we see that this is analogous to studying the trace of $B^*A(t)$ where B is held fixed and the symbol of A evolves according to geodesic flow. This is exactly the question we study here: when B is held fixed and the symbol of A

evolves according to geodesic flow, under what additional conditions can we deduce that the trace of $B^*A(t)$ decays, and what is the rate of decay? Our main result, Theorem 23, is an answer to this question.

4.2 Statement of the result

Connecting with the discussion in the introduction to this chapter, while our long-term interest is in the family $A_1^\Gamma(t) = e^{-it\Delta_X/2}A_1^\Gamma e^{it\Delta_X/2}$, the 'quantum evolution of A_1^Γ' in the Heisenberg picture, here, motivated by the RHS of (4.7), we consider the family $\mathrm{Op}(a_1 \circ g_t)$ where g_t is the geodesic flow on $S^*\mathbb{D}$ (notice that these symbols are left-Γ-invariant for all t). Our main result of this chapter is

Theorem 23. *Let $\Gamma \subset G$ be a discrete cocompact group and let $X = \Gamma\backslash\mathbb{D}$ be the compact hyperbolic surface induced by Γ.*

We let $\lambda_j = 1/4 + r_j^2$ be the j'th eigenvalue of the Laplace-Beltrami operator on X and ϕ_j to be the corresponding eigenfunction.

Let a_1, a_2 be symbols in the Hörmander class $S^\alpha_{1,0}(T^*X\backslash 0)$ which are left-invariant by Γ, and, when considered as a symbol in the Zelditch calculus in variables $(g, r) \in \Gamma\backslash G \times \mathbb{R}^+$, have an analytic continuation in r to a strip of width greater than $1/2$.

Let $A_1^\Gamma(t)$ and A_2^Γ be the operators on $L^2(X)$ induced by $A_1(t) = Op(a_1 \circ g_t)$ and $A_2 = Op(a_2)$ using Zelditch quantization. Here g_t is the classical geodesic flow on $S^*\mathbb{H}^2$, or algebraically, it is the pullback by the right action of the subgroup A on the group G.

- *Then there exists $\alpha > 0$, $N > 0$ such that for all $j \in \mathbb{N}$, there exists a distribution $\mathfrak{T}_j a_2 \in D'(\Gamma\backslash G)$ such that for all $s \geq 1$,*

$$\begin{aligned}\langle A_1^\Gamma(t)\phi_j, A_2^\Gamma\phi_j\rangle = \left(\int_{\Gamma\backslash G} a_1(g, r_j)dg\right)\left(\int_{\Gamma\backslash G} \mathfrak{T}_j a_2(g)dg\right) \\ + O_\alpha\left(e^{-\alpha|t|}\|a_1(\cdot, r_j)\|_{H^N}\|\mathfrak{T}_j a_2\|_{\mathcal{H}^{-s}}\right).\end{aligned} \tag{4.8}$$

  where $H^N(\Gamma\backslash G)$ is the Sobolev space of order N over $\Gamma\backslash G$ and $\mathcal{H}^{-s}(\Gamma\backslash G)$ is the anisotropic Sobolev space of Faure-Sjöstrand, [FS10] of order $-s$, likewise over $\Gamma\backslash G$.

- *We further suppose that $a_1(\cdot, r_j)$ has integral zero over S^*X for each j and that a_1, a_2 are symbols in the Hörmander class $S^\alpha_{1,0}(T^*X)$ of order $\alpha \leq -6$ and 0 respectively.*

  Then $(A_2^\Gamma)^*A_1^\Gamma(t)$ is trace class for each t, and

$$\mathrm{Trace}(A_2^\Gamma)^*A_1^\Gamma(t) \tag{4.9}$$

decays exponentially as $t \to \pm\infty$*.*

Remark. We note that there are interesting examples of symbols which satisfy all the conditions required. For example, a Γ-invariant symbol $a(z,b,r) = \phi(z,b)f(r)$ where $\int_{\Gamma\backslash G} \phi = 0$ and f is a smooth symbol in r which decays on $\mathbb{R}^+$ at a polynomial rate with an analytic extension to a strip will satisfy all the conditions of Theorem 23.

We also note that more general symbols could be considered, and -6 is not sharp, but this is sufficient for our interests, since our applications will only require symbols of order $-\infty$.

Furthermore, the assumption that a_1 and a_2 are both left-invariant by Γ, and have analytic continuations in r to a strip of width strictly greater than $1/2$, is so that their Zelditch quantizations $\mathrm{Op}(a_i)$ are well-defined operators A_1^Γ and A_2^Γ on $L^2(X)$. Note that for the result in this chapter, we don't need to consider the spectral Zelditch quantisation we introduce in Chapter 2, the usual Zelditch quantisation of [Zel86] suffices.

Lastly, we do not require that the symbols a_1 and a_2 satisfy the scattering matrix symmetry condition as explored in Section 2.3. We make the choice to consider the symbols a_1 and a_2, defined over $T^*X\backslash 0$, as functions on $\Gamma\backslash G \times \mathbb{R}^+$, through the identification of $T^*X\backslash 0 \simeq \Gamma\backslash G \times \mathbb{R}^+$ with an analytic continuation to a half strip in r of width greater than $1/2$. In more generality, when a symbol $a(z,b,r)$ satisfies the scattering matrix symmetry condition, it ensures that the quantisation of a, $\mathrm{Op}(a)$, acts on a function $u \in C_c^\infty(\mathbb{D})$ in a manner which is well-defined regardless of the choice of Fourier basis, $\{e(z,b,r) : r > 0\}$ or $\{e(z,b,r) : r < 0\}$. However, we don't lose anything by making a choice of the preferred Fourier basis and since the scattering matrix symmetry condition is not preserved under the geodesic flow, we are forced to make such a choice.

4.3 Helgason boundary values of eigenfunctions.

We state a theorem of Helgason which will be useful to us.

Theorem 24 (Boundary distribution of Laplacian eigenfunctions, Theorem 4.3 in [Hel81])**.** *For any* $r \in \mathbb{C}$*, if* φ_r *is a (smooth) function on* $\mathbb{D}$ *which satisfies* $\Delta\varphi_r = (1/4 + r^2)\varphi_r$ *and grows at most exponentially in the hyperbolic distance,* $d_{\mathbb{D}}(0,z)$*, i.e* $|\varphi_r(z)| \leq Ce^{cd_{\mathbb{D}}(0,z)}$ *for every* $z \in \mathbb{D}$ *with* $C, c > 0$ *some arbitrary constants, then there exists a distribution* $T \in \mathcal{D}'(B)$ *such that*

$$\varphi_r(z) = \int_B e^{(\frac{1}{2}+ir)\langle z,b\rangle} T(b)\, db. \tag{4.10}$$

Moreover the distribution T *is unique if* $1/2 + ir \neq 0, -1, -2, \ldots$*. Hence in this case, we* *may label T as T_{φ_r} since T_{φ_r} is uniquely determined from φ_r and vice versa.*

A corollary to Helgason's theorem, Theorem 24, is information about how the distributions $T \in \mathcal{D}'(B)$ vary under a discrete group (also called a Fuschian group) of cocompact isometries $\Gamma \subset G$ when T is the boundary distribution of an eigenfunction on the compact hyperbolic surface $\Gamma\backslash\mathbb{D}$.

Corollary 25. *Let $\varphi_r \in C^\infty(\Gamma\backslash\mathbb{D})$ satisfy $\Delta\varphi = (1/4 + r^2)\varphi$ pointwise in $\mathbb{D}$. Then, for $1/2 + ir \neq 0, -1, -2, \ldots$, there exists a unique boundary distribution, $T_{\varphi_r} \in \mathcal{D}'(B)$, such that for any $\gamma \in \Gamma$,*

$$T(\gamma b) = e^{(\frac{1}{2}-ir)\langle\gamma 0,\gamma b\rangle}T(b).$$

In particular,

$$e^{(-\frac{1}{2}+ir)\langle\gamma z,\gamma b\rangle}T(\gamma b) = e^{(-\frac{1}{2}+ir)\langle z,b\rangle}T(b)$$

for every $\gamma \in \Gamma$, (using identity (1.17)), and so

$$e^{(-\frac{1}{2}+ir)\langle z,b\rangle}T(b) \in D'(\Gamma\backslash G),$$

that is, it is invariant by Γ and therefore forms a well-defined distribution over $\Gamma\backslash G$.

Let us remark that in this article we take T to be a distributional function, differing from the convention in [AZ12] where it is taken to be a distributional density. This changes the form of the transformation under $\gamma \in G$ by the Jacobian factor $\gamma'(b) = e^{\langle\gamma 0,\gamma b\rangle}$ where $\langle\gamma 0, \gamma b\rangle$ is the Busemann bracket.

Let us say a little more on this, recall that for a diffeomorphism, σ of B that $T(d\sigma b) \in \mathcal{D}'(B)$ is defined by

$$\int_B f(b)T(d\sigma b) = \int_B f(\sigma^{-1}b)T(db). \tag{4.11}$$

In particular, this distinguishes $T(db)$ as a distribution density rather than a distribution on B (or distributional function on B).

Recall that if we have a function g which is integrable in B under the measure db, this defines the distribution

$$f \mapsto \int_B f(b)g(b)db.$$

If $g(b)$ is pulled back under a diffeomorphism σ, then the distributional function $g(\sigma b)$ is defined by

$$\int_B f(b)g(\sigma b)db = \int_B f(\sigma^{-1}b)g(b)|\det d\sigma^{-1}(b)|db.$$

Note how this differs from equation (4.11) defining the action of $T(d\sigma b)$. This is because g is a distributional function whereas $T(db)$ a distribution density. The action of $T(d\sigma b)$

corresponds analogously to pulling back the distribution density $g(b)db$ under σ to get $g(\sigma b)d(\sigma b)$.

4.4 Faure-Sjöstrand anisotropic Sobolev spaces

Since the geodesic flow on S^*X is a contact Anosov flow, the theory of anisotropic Sobolev spaces for contact Anosov flows introduced by Faure-Sjöstrand, [FS10], will be useful to us.

We need a method to talk about the regularity of distributions appearing on S^*X. For this purpose, we will use the standard theory of microlocal (or semiclassical) analysis on C^∞ compact manifolds, say as detailed in [Zwo12]. Note that this is a different pseudodifferential calculus than the Zelditch calculus; we use the Zelditch calculus for the exact correspondence it brings between operators on X and symbols on $T^*X\backslash 0 \simeq S^*X \times \mathbb{R}^+$.

We recall Theorem 4 in Nonenmacher-Zworski, [NZ15], which will suffice for our purposes. It reads, in our context as:

Theorem 26 (Theorem 4 in [NZ15])**.** *Let H be the generator of geodesic flow on S^*X. Consider $P = -iH$ as the self-adjoint operator on $L^2(S^*X, d\nu)$ with domain $\mathcal{D}(P) = \{u \in L^2(S^*X) : Pu \in L^2(S^*X)\}$ where ν is the Liouville measure on S^*X. The minimal asymptotic unstable expansion rate for the geodesic flow,*

$$\lambda_0 := \liminf_{t\to\infty} \frac{1}{t} \inf_{x\in S^*X} \log(\det dg_t|_{E^u(x)})$$

equals one because the geodesic flow has constant unit Lyapunov exponent, as shown in Lemma 1. For any $s > 0$, there exists a Hilbert space $\mathcal{H}^{s\mathcal{G}}(S^*X) := e^{-s\mathcal{G}}L^2(S^*X)$ such that

$$C^\infty(S^*X) \subset \mathcal{H}^{s\mathcal{G}}(S^*X) \subset \mathcal{D}'(S^*X)$$

and the operator family $(P - z) : \mathcal{D}^{s\mathcal{G}} \mapsto \mathcal{H}^{s\mathcal{G}}$ is meromorphic in the half plane $\Im z > -s$, admitting finitely many Pollicott-Ruelle resonances (poles of the resolvent $(P - z)^{-1}$) in the strip $\Im s > -1/2 + \epsilon$, for any $\epsilon > 0$, including a simple pole at $z = 0$ where the residue is a rank one projection operator onto the eigenspace of constants on S^*X. The resolvent estimate $||(P - z)^{-1}||_{\mathcal{H}^{s\mathcal{G}}\mapsto\mathcal{H}^{s\mathcal{G}}} \leq C\langle z\rangle^N$ holds for $\Im z > 1/2$, $\Re z > C$ for some constants $C, N > 0$.

We mention here that $\mathcal{G}$ is the Weyl quantisation of an escape function defined on $T^*(S^*X)$ constructed in [NZ15]. Consequently, $e^{-s\mathcal{G}}$ is a variable order pseudodifferential operator. The Hilbert spaces $\mathcal{H}^{sG}$ are similar to those appearing initially in Faure-Sjöstrand, [FS10].

In particular, we can describe the Sobolev regularity at certain regions of fibre infinity for functions in the space $\mathcal{H}^{s\mathcal{G}}(S^*X)$. Letting A_1 be a zeroth order pseudodifferential operator elliptic in an asymptotically conical neighbourhood of the line bundle E_s^* over S^*X. We can say that there exists some $C > 0$ such that

$$C^{-1}\|A_1 f\|_{H^s} \leq \|A_1 f\|_{\mathcal{H}^{s\mathcal{G}}} \leq C\|A_1 f\|_{H^s}$$

where H^s is the standard Sobolev space over S^*X of order s. A similar equality of norms hold if H^s is replaced by H^{-s} and A_1 is instead elliptic of order zero in an asymptotically conic neighbourhood of E_u^*. We colloquially say that the spaces $\mathcal{H}^{s\mathcal{G}}$ are microlocally of order H^s near E_s^* and of order H^{-s} near E_u^*. In our proof below, where $\mathcal{G}$ is fixed, we will simply use $\mathcal{H}^s$ to refer to this scale of anisotropic Sobolev spaces.

We also mention that there is a relationship between the location of the poles of the resolvent of P and the eigenvalues of the Laplacian for all compact hyperbolic manifolds as proved by Dyatlov-Faure-Guillarmou in [DFG15]. Additionally, the resonant states of P (the image of the residue of the resolvent at the poles) are precisely those distributions appearing in Corollary 25 for those r_j corresponding to a Laplace eigenvalue (away from an exceptional set).

4.5 The Proof

Using Theorem 24, we can find an exact formula which relates the quantity $\langle A_1^\Gamma \phi_j, A_2^\Gamma \phi_l\rangle_{L^2(X)}$ to a certain bilinear form involving the Zelditch symbols a_1, a_2 of A_1^Γ and A_2^Γ.

Lemma 27. *Let A_1^Γ and A_2^Γ be two pseudodifferential operators on a compact hyperbolic surface $X = \Gamma\backslash\mathbb{D}$ as in Theorem 23, with Γ-invariant Zelditch symbols a_1 and a_2 respectively. Let ϕ_j, ϕ_k be a pair of Laplace eigenfunctions on $\Gamma\backslash\mathbb{D}$ with eigenvalues $1/4 + r_j^2$ and $1/4 + r_k^2$ respectively. We take r_j, r_k to be in $[0, \infty) \cup i[-1/2, 0]$. Then*

$$\langle A_1^\Gamma \phi_j, A_2^\Gamma \phi_l\rangle_{L^2(X)} = \int_{\Gamma\backslash G} a_1(g, r_j) E_j(g) \left(\int_K \overline{a_2(gk_\theta, r_l) E_l(gk_\theta)} d\theta \right) dg, \tag{4.12}$$

where $E_j(g)$ is the Γ-invariant distribution on G given by

$$E_j(g) = e^{(-1/2+ir_j)\langle z,b\rangle} T_j(b). \tag{4.13}$$

Here E_j is the Γ-invariant distribution mentioned in Corollary 25. We use coordinates (z, b) on G as described above, and $T_j(b)$ is the boundary distribution function for ϕ_j, *making the choice of r_j as above. We also remind the reader that the Haar measure dg is $e^{\langle z,b\rangle} d\,Vol(z)db$ (see (1.16)), accounting for $-1/2$ (instead of $+1/2$) in the exponent in (4.13). We can interpret E_j as the jth Ruelle-Pollicott resonant state on $\Gamma\backslash G$ in the first band; see [DFG15, Section 2].*

Proof. We begin with Helgason's theorem, Theorem 24, on the integral representation for Laplace eigenfunctions. Since ϕ_j, ϕ_l are Laplace eigenfunctions on $\Gamma\backslash\mathbb{D}$ with eigenvalues $1/4 + r_j^2$ and $1/4 + r_l^2$ respectively, we can consider them as left Γ-invariant (or left Γ-periodic) functions on $\mathbb{D}$, satisfying $\Delta_{\mathbb{D}}\phi_j = (1/4 + r_j^2)\phi_j$ and $\Delta_{\mathbb{D}}\phi_l = (1/4 + r_l^2)\phi_l$ respectively. Since $\Gamma\backslash\mathbb{D}$ is compact, they are bounded functions on $\mathbb{D}$ and hence trivially satisfy the exponential growth bound of Helgason's theorem 24. Our choice of r_j, r_l ensures that $1/2 + ir_j$ and $1/2 + ir_l$ don't belong to the exceptional set $0, -1, -2, \ldots$, so we have unique boundary distributions $T_{\phi_j}(b)$ and $T_{\phi_l}(b)$ such that

$$\phi_j(z) = \int_B e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} T_{\phi_j}(b)db, \qquad \phi_l(z) = \int_B e^{(\frac{1}{2}+ir_l)\langle z,b\rangle} T_{\phi_l}(b)db$$

for all $z \in \mathbb{D}$. We then recall that the pseudodifferential operators $A_1 = \mathrm{Op}(a_1)$ and $A_2 = \mathrm{Op}(a_2)$ defined by Zelditch quantisation act on the eigenfunctions in the following way:

$$\begin{aligned} A_1\phi_j(z) &= \int_B a_1(z,b,r_j) e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} T_{\phi_j}(b)db \\ A_2\phi_l(z) &= \int_B a_2(z,b,r_l) e^{(\frac{1}{2}+ir_l)\langle z,b\rangle} T_{\phi_l}(b)db. \end{aligned}$$

In fact, for real r_j, r_l this follows directly from (1.20). On the other hand, both sides of (1.20) have an analytic continuation to the strip of radius $1/2$, so the equality persists for $|\Im r| \leq 1/2$. Therefore, we can express $\langle A_1\phi_j, A_2\phi_l\rangle_{L^2(\Gamma\backslash\mathbb{D},dz)}$, for any j and l, as

$$\int_{\Gamma\backslash\mathbb{D}} \left(\int_B a_1(z,b,r_j) e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} T_{\phi_j}(b)\,db\right) \left(\int_B \overline{a_2(z,b',r_l) e^{(\frac{1}{2}+ir_l)\langle z,b'\rangle} T_{\phi_l}(b')\,db'}\right) d\mathrm{Vol}(z). \tag{4.14}$$

Now writing $g = (z,b)$, we have $(z,b') = gk_\theta$ for some $\theta = \theta(z,b,b')$, since the right action of K rotates in the fibres of the cosphere bundle $S^*\mathbb{H}^2$ keeping the basepoint z fixed. In view of equality (1.16), $dg = e^{\langle z,b\rangle} d\mathrm{Vol}(z)db = d\mathrm{Vol}(z)dzd\theta$, we have $db = e^{\langle z,b\rangle}d\theta$ when z is held fixed. We can therefore express (4.14) as

$$\int_{\Gamma\backslash G} a_1(g,r_j)E_j(g) \left(\int_K \overline{a_2(gk_\theta,r_l)E_l(gk_\theta)}d\theta\right) dg, \tag{4.15}$$

which verifies (4.12). □

We now verify the claim (4.3) in the introduction.

Corollary 28. *Let $w \geq 1/2$ and let S_w be the strip $\{r \in \mathbb{C} \mid |\Im r| \leq w\}$ in the complex plane of width w. Let a_1, a_2 be symbols as in Theorem 23. Then the trace $A_2^* A_1$ is given by the sesquilinear pairing*

$$\langle a_1, \mathfrak{T} a_2 \rangle \tag{4.16}$$

of a_1 against a distribution $\mathfrak{T} a_2$ where $\mathfrak{T}$ maps symbols to distributions on $\Gamma \backslash G \times S_w$, given by

$$\mathfrak{T} a_2(g, r) = \sum_j \mathfrak{T}_j a_2(g) \delta_{r_j}(r), \quad \mathfrak{T}_j a_2(g) = \overline{E_j(g)} \int_K a_2(gk_\theta, r_j) E_j(gk_\theta)\, d\theta. \tag{4.17}$$

The pairing $\langle a_1, \mathfrak{T} a_2 \rangle$ is well-defined for $a_1 \in S_{1,0}^{-4}((\Gamma \backslash G \times S_w))$.

We recall that $a_1 \in S_{1,0}^{-4}(\Gamma \backslash G \times S_w)$ means that $a_1 \in C^\infty(\Gamma \backslash G \times S_w)$ and satisfies the following symbol estimates, for any $s \in \mathbb{N}$ and any multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$,

$$|(r\partial_r)^s H^{\alpha_1} X_+^{\alpha_2} X_-^{\alpha_3} a(g, r)| \leq C_{s,\alpha} (1 + |r|)^{-4}. \tag{4.18}$$

Here $C_{s,\alpha} > 0$ is a constant which is uniform over $\Gamma \backslash G \times S_w$. In particular, each symbol estimate holds uniformly in the strip S_w. We also remark that ∂_r is the derivative with respect to the complex variable r.

Proof. The formula follows immediately from Lemma 27 and the standard formula for the trace

$$\text{Trace } A_2^* A_1 = \sum_{j=0}^{\infty} \langle A_1 \phi_j, A_2 \phi_j \rangle_{L^2(X)} \tag{4.19}$$

using the fact that the ϕ_j are an orthonormal basis for $L^2(X)$. We need to check that the sum is well-defined as a distribution. This follows from the estimates below which verify that the pairing is well-defined for $a_1 \in S_{1,0}^{-4}((\Gamma \backslash G \times S_w))$. We use rather crude estimates for this, which accounts for the loss in the order of a_1. We are unconcerned by this as our intended application is to operators of order $-\infty$, as in (4.1).

Using work of Otal, [Ota98], we find that T_{ϕ_j} is the derivative of a function $F_{\phi_j}(b)$ that is Hölder continuous of order $1/2$. The Hölder $1/2$-norm of F is bounded in [Ota98] by $C(\|\phi_j\|_\infty + \|\nabla \phi_j\|_\infty)$, where C is independent of j. Using standard estimates of eigenfunctions on compact manifolds, for example as proved in [Hör68], gives us an estimate of $C\langle r_j \rangle^{3/2}$ for the Hölder $1/2$-norm of F and *a fortiori* for its L^2 norm. Therefore T_{ϕ_j} itself is in $H^{-1}(\mathbb{S}^1)$ with a norm estimate $C\langle r_j \rangle^{3/2}$, with C uniform in j. It follows that

E_j, as a function of (z, b), coordinates which makes sense locally on $\Gamma\backslash G$, is L^∞ in z with values in $H^{-1}(\mathbb{S}^1_b)$, with a norm estimate $C\langle r_j\rangle^{3/2}$.

We now consider the integral over K in the definition (4.17) of $\mathfrak{T}a_2$. Recall that this is an expression for $A_2\phi_j$. Since A_2 is a pseudodifferential operator of order 0, this is $L^2(X)$ uniformly in r (with constant depending on a finite number of seminorms of the symbol a_2, but not on j. Viewed as a function on $\Gamma\backslash G$, it is uniformly L^2 in z, and constant in b. The product in (4.17) is therefore L^2 in z with values in H^{-1} in b, locally, and therefore H^{-1} locally, satisfying

$$\|\mathfrak{T}_j a_2\|_{H^{-1}(\Gamma\backslash G)} \leq C\langle r_j\rangle^{3/2}, \quad \text{with } C \text{ independent in } j. \tag{4.20}$$

As for a_1, as a symbol of order -4 it is in $H^k(\Gamma\backslash G)$ for each fixed r, with norm bounded by $\langle r\rangle^{-4}$. The pairing of a_1 with the jth summand of (4.17) is therefore $O(\langle r_j\rangle^{-5/2})$, and using Weyl asymptotics, we see that this is summable in j. This verifies the claim that the pairing $\langle \mathfrak{T}a_2, a_1\rangle$ is well-defined for symbols a_1 of order -4.

□

We can interpret (4.16), at least formally, as a sum over j of the classical correlation of the function $a_1(\cdot, r_j)$, with the distribution $\mathfrak{T}_j a_2$. Next we check that $\mathfrak{T}_j a_2$ has the required regularity for which we can apply some results on decay of correlations.

Lemma 29. *The distribution $\mathfrak{T}_j a_2$ in (4.17) is in the anisotropic Sobolev space $\mathcal{H}^{-s}(\Gamma\backslash G)$ described in Section 4.4 for every $s \geq 1$, and the norm of $\mathfrak{T}_j a_2$ in the space $\mathcal{H}^{-s}(\Gamma\backslash G)$ is bounded by $C\langle r_j\rangle^{7/2}$.*

Proof. We have already seen, in the proof of Corollary 28, that $\mathfrak{T}_j a_2$ is in the space $H^{-1}(\Gamma\backslash G)$ with norm bounded by $C\langle r_j\rangle^{3/2}$. To obtain the strengthened conclusion that $\mathfrak{T}_j a_2$ in the space $\mathcal{H}^{-s}(\Gamma\backslash G)$, we use the fact that E_j satisfies equations

$$\begin{aligned} X_+ E_j &= 0 \\ (H + \frac{1}{2} - ir_j)E_j &= 0. \end{aligned} \tag{4.21}$$

It follows that $\mathfrak{T}_j a_2 = (A_2\phi_j)E_j$ satisfies the equations

$$\begin{aligned} X_+^2 \mathfrak{T}_j a_2 &= E_j(X_+^2(A_2\phi_j)) \\ (H + \frac{1}{2} - ir_j)^2 \mathfrak{T}_j a_2 &= E_j((H + \frac{1}{2} - ir_j)^2(A_2\phi_j)). \end{aligned} \tag{4.22}$$

To compute the norm of $\mathfrak{T}_j a_2$ in $\mathcal{H}^{-1}(\Gamma\backslash G)$, we choose an elliptic pseudodifferential operator B of variable order s, equal to the variable order of the Sobolev space $\mathcal{H}^{-1}(\Gamma\backslash G)$, together with an elliptic invertible operator R of order -1; then an equivalent norm is

$$\|B\mathfrak{T}_j a_2\|_{L^2} + \|R\mathfrak{T}_j a_2\|_{L^2}.$$

Here we note that s is defined on $T^*(S^*X) = T^*(\Gamma\backslash G)$, takes values in $[-1,1]$, is equal to 1 in a conic neighbourhood of the line bundle E_u^* and -1 in a conic neighbourhood of E_s^*, and is monotone increasing with respect to geodesic flow in direction $t \to \infty$.

It is immediate that $R\mathfrak{T}_j a_2$ is in L^2 with a norm bound of $C\langle r_j\rangle^{3/2}$, using (4.20). To analyze the other term, we observe that, except at the bundle E_s^*, one or the other of the operators in (4.22) is elliptic. Using microlocal parametrices for these operators on their respective elliptic sets, we may write

$$B = B_1 X_+^2 + B_2(H + \frac{1}{2} - ir_j)^2 + B_3, \tag{4.23}$$

where B_1 and B_2 have variable order $\mathsf{s}-2$, and B_3 has order -1 (we may assume without loss of generality that B_3 is microsupported in a small conic neighbourhood of E_s^*, where B has order -1). Then, $B_i \in \Psi^{-1}(S^*X)$ for $i = 1,2,3$. Using (4.22) we find that $B\mathfrak{T}_j a_2$ is indeed in L^2. (The reason for using the squares of these vector fields in (4.22) was so that we could reduce the orders of B_1 and B_2 by two relative to B, so that they become operators of order -1.) Moreover, the right hand side of (4.22) is in $H^{-1}(S^*X)$, with norm bounded by $C\langle r_j\rangle^{7/2}$, using the same argument as in the previous proof — the extra powers of r_j arise from up to two derivatives applied to $A_2\phi_j$ as well as the term r_j^2 in $(H + \frac{1}{2} - ir_j)^2$. Thus the expressions (4.23) and (4.22) together show that the norm of $B\mathfrak{T}_j a_2$ in L^2 is bounded by $\langle r_j\rangle^{7/2}$. □

Next we prove a lemma which shows exponential decay of the correlation of two functions $f_1 \in C^\infty(S^*X)$ and $f_2 \in \mathcal{H}^{-s}(S^*X)$ as $t \to -\infty$. The argument is relatively standard, but we provide the details for completeness, following the proof of [NZ15, Corollary 5] rather closely. Note that we use the function space $\mathcal{H}^{-s}$ with $s \geq 1$ since the distributions E_j are in this space. We will apply Theorem 26 to the the vector field $-H$ (rather than H) which generates the flow $t \mapsto g_{-t}$ for which all the estimates in Theorem 26 hold with $\mathcal{H}^{-s}$ in place of $\mathcal{H}^s$ and $-H$ in place of H. The reason the estimates still hold under this replacement is simply that the flow g_{-t} reverses the role of the bundles E_u^*, E_s^*. This will mean that the correlations will decay as $t \to -\infty$ rather than $t \to \infty$.

Lemma 30. *Let $f_1 \in C^\infty(S^*X)$ and $f_2 \in \mathcal{H}^{-s}(S^*X)$ be such that one of them has mean zero. Assume that $1 \leq s \leq 2$, and let $0 < \alpha < 1/2$, $N \in \mathbb{N}, \epsilon > 0$ be such that the* *resolvent is holomorphic for $-\alpha < \Im\lambda < 0$ and there is a polynomial bound on the growth of the resolvent of $P = iH$ on the anisotropic space $\mathcal{H}^{-s}$ in the region $\Im\lambda > -\alpha, |\lambda| > \epsilon$ as in Theorem 26:*

$$\|(P-\lambda)^{-1}\|_{\mathcal{H}^{-s}\to\mathcal{H}^{-s}} \le C\langle\lambda\rangle^N, \quad \Im\lambda > -\alpha, |\lambda| > \epsilon. \tag{4.24}$$

Then there is a constant C_α such that for all $t > 0$,

$$\int_{\Gamma\backslash G} f_1(ga_{-t})\overline{f_2(g)}dg \le C_\alpha e^{-\alpha t}\|f_1\|_{H^{N+4}}\|f_2\|_{\mathcal{H}^{-s}}. \tag{4.25}$$

Remark. Before we prove this lemma, we remark that Faure-Tsujii have stronger results. From their works [FT23] and [FT24], it seems from their proof of the band structure of the Pollicott-Ruelle spectrum, that they get a *uniform* estimate of the resolvent in the strip $\Im\lambda \ge -\alpha$ (away from the pole at $\lambda = 0$) on certain anisotropic Sobolev spaces $\widetilde{\mathcal{H}}^s$ that are slightly different from those considered here. If this is so then an abstract result in semigroup theory, the Gearhart-Prüss-Greiner theorem, see page 302 in [EN01], shows that the semigroup is exponentially decaying, that is, that

$$\|e^{tH}f\|_{\widetilde{\mathcal{H}}^s} \le Ce^{-\alpha t}\|f\|_{\widetilde{\mathcal{H}}^s}$$

for $t > 0$. This implies an improvement of the result claimed in the lemma, that the correlation is bounded by

$$\langle e^{tH}f_1, f_2\rangle \le \|e^{tH}f_1\|_{\widetilde{\mathcal{H}}^s}\|f_2\|_{\widetilde{\mathcal{H}}^{-s}} \le Ce^{-\alpha t}\|f_1\|_{\widetilde{\mathcal{H}}^s}\|f_2\|_{\widetilde{\mathcal{H}}^{-s}}$$

The proof we present of Lemma 30 below is a cruder proof sufficient for our purposes where we only require that a spectral gap for the resolvent $(iH-\lambda)^{-1}$ exists with a polynomial growth bound on the resolvent asymptotically in the spectral gap.

Proof. Observe that, by the density of $L^2 \cap \mathcal{H}^{-s}$ in $\mathcal{H}^{-s}$, it is sufficient to prove the estimate for $f_2 \in L^2 \cap \mathcal{H}^{-s}$. Assuming this, we write the correlation using L^2-spectral theory for the self-adjoint operator $P = iH$ on $L^2(S^*X)$ with domain $D(P) = \{f \in L^2 \mid Pu \in L^2\}$. We will also pedantically write P_s for iH on the space $\mathcal{H}^{-s}$ with domain $D(P_s) = \{f \in \mathcal{H}^{-s} \mid Pu \in \mathcal{H}^{-s}\}$.

Applying L^2-spectral theory for the self-adjoint operator P and the functions $f_1, f_2 \in L^2$, we have

$$\langle e^{itP}f_1, f_2\rangle_{L^2(\Gamma\backslash G)} = \int_{\mathbb{R}} e^{it\lambda}d\langle f_1, E(-\infty,\lambda)f_2\rangle \tag{4.26}$$

where $E(I)$ is the spectral projector for P onto the set $I \subset \mathbb{R}$. To gain a decay factor in this integral, we exploit the fact that f_1 is smooth. We let $\overline{f_1} = (P+i)^{N+2} f_1$, which is C^∞, and has mean zero if f_1 does, since $\langle P^j f_1, 1\rangle = \langle f_1, P^j 1\rangle = 0$. We can express the correlation as

$$\langle e^{itP} f_1, f_2\rangle_{L^2(\Gamma\backslash G)} = \int_{\mathbb{R}} \frac{e^{it\lambda}}{(\lambda+i)^{(N+2)}} d\langle \overline{f_1}, E(-\infty,\lambda) f_2\rangle. \tag{4.27}$$

Next we use Stone's formula to express the spectral measure in terms of the resolvent. We can write (4.27) as

$$\lim_{\epsilon\searrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{it\lambda}}{(\lambda+i)^{(N+2)}} \left\langle \overline{f_1}, (P-\lambda-i\epsilon)^{-1} f_2 \right\rangle d\lambda \\ - \lim_{\epsilon\searrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{it\lambda}}{(\lambda+i)^{(N+2)}} \left\langle \overline{f_1}, (P-\lambda+i\epsilon)^{-1} f_2 \right\rangle d\lambda. \tag{4.28}$$

The second term is zero, as we see by shifting the contour of integration to $\Im\lambda = c$, $c < 0$ and sending $c \to -\infty$. For this we simply need the estimate $\|(P-\lambda)^{-1}\|_{L^2\to L^2} \le |\Im\lambda|^{-1}$.

To deal with the first term, where we cannot shift the contour to $\Im\lambda < 0$ because of the spectrum of P along the real line, we pass to the operator P_s, as we may since $f_2 \in \mathcal{H}^{-s}$. We thus have

$$\langle e^{itP} f_1, f_2\rangle_{L^2(\Gamma\backslash G)} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{it\lambda}}{(\lambda+i)^{(N+2)}} \left\langle \overline{f_1}, (P_s-\lambda+i\epsilon)^{-1} f_2 \right\rangle d\lambda. \tag{4.29}$$

The resolvent of P_s has a meromorphic continuation to the half-space $\Im\lambda > -1/2$, with a pole at the origin, [DFG15]. The pole is simple with residue being the projection on to constants. Since either $\overline{f_1}$ or f_2 has mean zero, the inner product in (4.29) is holomorphic across zero so the contour can be pushed down to $\Im\lambda = -\alpha$, using the decay factor $(\lambda+i)^{-(N+2)}$ which overcomes the $\langle\lambda\rangle^N$ growth of the resolvent. Doing so picks up a decay factor $e^{-\alpha t}$ and we obtain the required estimate by estimating

$$\left|\left\langle \overline{f_1}, (P_s-\lambda-i\alpha)^{-1} f_2 \right\rangle\right| \le C\langle\lambda\rangle^N \|\overline{f_1}\|_{\mathcal{H}^s} \|f_2\|_{\mathcal{H}^{-s}} \le C\langle\lambda\rangle^N \|f_1\|_{H^{N+4}} \|f_2\|_{\mathcal{H}^{-s}},$$

and integrating in λ. The last inequality follows as H^2 embeds into $\mathcal{H}^s$ since $1 \le s \le 2$. □

Now we are in a position to prove the main theorem.

Proof of Theorem 23. A pseudodifferential operator on a compact manifold is trace class provided it has order strictly less than minus the dimension of the manifold. Since we have assumed that A_1 has order ≤ -6 and A_2 has order zero, the evolved operator $A_1(t)$ also has order ≤ -6, so $A_2^* A_1(t)$ has order ≤ -6 and is trace class.

We first prove the exponential decay of the trace in the limit $t \to -\infty$.

By Corollary 28, the trace of $A_2^* A_1(t)$ is given by a sum

$$\sum_j \langle g_t^* a_1(\cdot, r_j), \mathfrak{T}_j a_2 \rangle$$

of distributions $\mathfrak{T}_j a_2$ depending on $a_2(\cdot, r_j)$ applied to $g_t^* a_1(\cdot, r_j)$. We first consider an individual term in this sum. By Lemma 29, the distribution $\mathfrak{T}_j a_2$ is in the anisotropic Sobolev space $\mathcal{H}^{-s}$ for $s \geq 1$, with norm in this space $O(\langle r_j \rangle^{7/2})$. Provided that a_1 is a symbol of order -6 or below, the norm of $a_1(\cdot, r_j)$ in any standard Sobolev space H^m is $O(\langle r_j \rangle^{-6})$, from the symbol estimates, showing that each term in the sum is $O(\langle r_j \rangle^{-5/2})$. Using Weyl asymptotics we see that r_j is bounded above and below by a multiple of $\sqrt{j}$ as $j \to \infty$, hence the sum is absolutely convergent. Moreover, according to Lemma 30, each term in the sum is bounded by

$$C_\alpha e^{-\alpha t} \| a_1(\cdot, r_j) \|_{H^{N+4}} \| \mathfrak{T}_j a_2 \|_{\mathcal{H}^{-s}}$$

for some $0 < \alpha < 1/2$. Removing the exponentially decaying factor in time, we can sum the series, leading to the conclusion that the trace decays exponentially in time.

We next briefly discuss the case $t \to \infty$. Observe first that there is nothing in the statement of the theorem to suggest that one direction of time is favoured over another. Examining the proof, we see that in considering 'plane waves' of the form $e^{(1/2+ir)\langle z,b \rangle}$, we made a choice to use plane waves that are constant on *forward* horocycles. One can equally well consider plane waves that are constant on *backward* horocycles; these are just the previous plane waves composed with the inversion map, which is the group element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R}),$$

that is, the element of the subgroup K that rotates by π in the fibres. If we do that, then we find that the corresponding Ruelle-Pollicott resonances $\hat{E}_j$, as in (4.13), are invariant under the generator X_- of unstable horocycle flow rather than the generator X_+ of stable horocycle flow as is the case for E_j. Correspondingly, the $\hat{E}_j$ are in the opposite anisotropic space, that is regular at the stable line bundle E_s^* and rough at the unstable bundle E_u^*. This leads to exponential decay in as $t \to \infty$. We omit the details.

□

Chapter 5

Quantum correlations and small scale quantum ergodicity on the torus

5.1 Introduction

This chapter will consider a significantly different geometry than the other chapters, namely the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2 \backslash 2\pi\mathbb{Z}^2$.[1] We write this chapter because it provides an interesting example of a way to use the exact Egorov theorem of Weyl quantisation on $\mathbb{R}^d$ (effectively an example of the existence of an intertwining operator on Euclidean space) to probe questions in quantum ergodicity. We hope that a similar relationship may be established on hyperbolic surfaces for us to exploit. Moreoever, the method of proof is somewhat interesting, particularly, it reduces a quantum variance estimate to estimating the quantum correlation averaged over long times, this reduction holds on any smooth, compact, Riemannian manifold so it opens the way for an exploration in more general geometries. And let us just note that the long times we require are $T \sim h^{-\alpha}$ to prove h^α-scale quantum ergodicity where $\alpha < 1$ in $\mathbb{T}^2$ and more generally $\alpha < 1/(d-1)$ in $\mathbb{T}^d$, these time scales are well beyond Ehrenfest time.

The geometric structure of $\mathbb{R}^d$ and $\mathbb{Z}^d$ will provide a lot of the tools to make the argument work in this setting which are not directly transferrable to the hyperbolic surface setting.

A large portion of the ideas in this chapter come from unpublished notes of Andrew Hassell and Charlotte Groom. In particular, the notes set up the framework and sketch an argument for small-scale toral quantum ergodicity up to scales h^α for any $\alpha < 1/2$ in two dimensions and more generally $\alpha < 1/d$ in d-dimensions.

[1] While we only detail the analysis for the two dimensional torus here, our arguments can easily be extended to higher dimensionsal toruses.

Our contribution is to sharpen the argument to $\alpha < 1$ in two dimensions and more generally to $\alpha < 1/(d-1)$ in d-dimensions. We will also present the argument in terms of a 'quantum correlation' and remark on the place where the argument holds for more general smooth, compact Riemannian manifolds. This chapter will set up and present the framework for the argument and prove the main result of this chapter, Theorem 32.

5.2 Weyl quantisation and quantum variances on the torus.

Let us fix the torus as the set $\mathbb{T}^2 = \mathbb{R}^2 \backslash 2\pi\mathbb{Z}^2$ with the standard metric. The fundamental domain is the square with side lengths 2π. A smooth function on the torus, $f \in C^\infty(\mathbb{T}^2)$ has a convergent Fourier series expansion

$$f(x) = \frac{1}{(2\pi)^2} \sum_{\xi \in \mathbb{Z}^2} \hat{f}(\xi) e^{ix\cdot\xi}, \qquad x \in \mathbb{T}^2 \tag{5.1}$$

where

$$\hat{f}(\xi) := \int_{\mathbb{T}^2} f(x) e^{-ix\cdot\xi} dx. \tag{5.2}$$

We can define a pseudodifferential operator on functions on the torus in the following way. Note that the theory we espouse here is similar, but not identical to, the theory of microlocal analysis on the torus via Fourier series in Ruzhansky and Turunen [RT10]. Given a symbol $a(x, \xi) \in C^\infty(T^*\mathbb{T}^2)$ which is periodic in the x variable with period 2π, we want to construct from this an operator which is pseudodifferential and which has a Schwartz kernel that is periodic in the left and right variables. One possibility is to define the operator

$$\mathrm{Op}(a) : \frac{1}{(2\pi)^2} \sum_{\xi \in \mathbb{Z}^2} \hat{f}(\xi) e^{ix\cdot\xi} \mapsto \frac{1}{(2\pi)^2} \sum_{\xi \in \mathbb{Z}^2} \hat{f}(\xi) a(x, \xi) e^{ix\cdot\xi}. \tag{5.3}$$

Then this operator maps periodic functions to periodic functions and equivalently, has a Schwartz kernel that is separately periodic in the left and right variables. To check this directly, notice that

$$\mathrm{Op}(a) f(x) = \frac{1}{(2\pi)^2} \sum_{\xi \in \mathbb{Z}^2} \hat{f}(\xi) a(x, \xi) e^{ix\cdot\xi} = \frac{1}{(2\pi)^2} \sum_{\xi \in \mathbb{Z}^2} \int_{\mathbb{T}^2} a(x, \xi) f(y) e^{-iy\cdot\xi} e^{ix\cdot\xi} dy. \tag{5.4}$$

And hence the Schwartz kernel for $\mathrm{Op}(a)$ is

$$K_a(x,y) = \frac{1}{(2\pi)^2}\sum_{\xi\in\mathbb{Z}^2} a(x,\xi)e^{i(x-y)\cdot\xi}. \tag{5.5}$$

We observe that $K_a(x,y)$ is separately periodic in its left factor and its right factor (in x and in y respectively) with respect to the $2\pi\mathbb{Z}^2$ lattice.

Defining the Weyl quantisation of a symbol $a(x,\xi)$ on the torus $\mathbb{R}^2\backslash 2\pi\mathbb{Z}^2$ is slightly subtle because if we have a symbol $a(x,\xi)$ which is periodic in the x variable under the lattice $2\pi\mathbb{Z}^2$, then it isn't necessarily the case that $a((x+y)/2,\xi)$ is separately periodic in the x and y variables under $2\pi\mathbb{Z}^2$ lattice. To manage this, first consider the one dimensional torus $\mathbb{R}\backslash 2\pi\mathbb{Z}$ where we can write a symbol which is periodic in the x variable with respect to $2\pi\mathbb{Z}$ as

$$a(x,\xi) = \frac{a(x,\xi)+a(x+\pi,\xi)}{2} + \frac{a(x,\xi)-a(x+\pi,\xi)}{2}. \tag{5.6}$$

The first term is periodic with period π and the second term is antiperiodic with period π (it equals its negative when translated by π). We can break up any function on $\mathbb{R}$ which is periodic with period 2π into the sum of a function periodic with period π (which we shall call the even part) and a function antiperiodic with period π (which we shall call the odd part). A symbol $a(x,\xi)$ which is periodic in x with respect to the lattice $2\pi\mathbb{Z}^2$ can thus be broken up into the sum of four functions

$$a(\cdot,\xi) = a_{ee}(\cdot,\xi) + a_{eo}(\cdot,\xi) + a_{oe}(\cdot,\xi) + a_{oo}(\cdot,\xi) \tag{5.7}$$

where the subscripts *ee*, *eo*, *oe*, *oo* refers to a symbol which is even/even, even/odd, odd/even and odd/odd respectively under the translation on the left $2\pi\mathbb{Z}$ factor and right $2\pi\mathbb{Z}$ factor of $2\pi\mathbb{Z}^2$ respectively, precisely with respect to the translations

$$\begin{aligned}\mathbb{R}^2 \ni x = (x_1,x_2) &\mapsto (x_1+k,x_2), \quad k\in 2\pi\mathbb{Z},\\ \mathbb{R}^2 \ni x = (x_1,x_2) &\mapsto (x_1,x_2+k), \quad k\in 2\pi\mathbb{Z}.\end{aligned} \tag{5.8}$$

So given a symbol $a(x,\xi)$ which is periodic under the lattice $2\pi\mathbb{Z}^2$, we consider the Schwartz kernel defined by

$$K_a(x,y) := K_{a_{ee}}(x,y) + K_{a_{eo}}(x,y) + K_{a_{oe}}(x,y) + K_{a_{oo}}(x,y), \tag{5.9}$$

where we have

$$\begin{aligned}
K_{a_{ee}}(x,y) &= \frac{1}{(2\pi)^2}\sum_{\xi\in\mathbb{Z}^2} a_{ee}\left(\frac{x+y}{2},\xi\right)e^{i(x-y)\cdot\xi},\\
K_{a_{eo}}(x,y) &= \frac{1}{(2\pi)^2}\sum_{\xi\in\mathbb{Z}^2+(0,\frac{1}{2})} a_{oe}\left(\frac{x+y}{2},\xi\right)e^{i(x-y)\cdot\xi},\\
K_{a_{oe}}(x,y) &= \frac{1}{(2\pi)^2}\sum_{\xi\in\mathbb{Z}^2+(\frac{1}{2},0)} a_{eo}\left(\frac{x+y}{2},\xi\right)e^{i(x-y)\cdot\xi},\\
K_{a_{oo}}(x,y) &= \frac{1}{(2\pi)^2}\sum_{\xi\in\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})} a_{oo}\left(\frac{x+y}{2},\xi\right)e^{i(x-y)\cdot\xi}.
\end{aligned} \tag{5.10}$$

Notice that we have translated the dual lattice by a factor of $1/2$ in each $2\pi\mathbb{Z}$-sublattice of $2\pi\mathbb{Z}^2$ on which the symbol is odd. The effect of this is that each of the Schwartz kernels above are now separately periodic in x and y under the $2\pi\mathbb{Z}^2$ lattice and hence the operator defined by the Schwartz kernel $K_a(x,y)$ is a well-defined pseudodifferential operator acting on functions defined on $\mathbb{T}^2$.[2] To define a h-semiclassical Weyl quantisation on the torus, analogous to $\mathbb{R}^n$, we'll define it via the Schwartz kernel

$$K_a^h(x,y) := \frac{1}{(2\pi)^2}\sum_{I\in\{e,o\}^2}\sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I} a_I\left(\frac{x+y}{2},h\xi\right)e^{i(x-y)\cdot\xi}, \tag{5.11}$$

where we will identify *even* with 0 and *odd* with 1, so that for $I\in\{e,o\}^2$, $\xi\in\mathbb{Z}^2+\frac{1}{2}I$ corresponds to the appropriate translated dual lattice. We remark that for a symbol $a(x,\xi)$, when the quantisation of $a(x,\xi)$ (Weyl or standard) acts on the torus, only a discrete set of frequencies of the symbol are relevant, however as $h\to 0$, all frequencies of $a(x,\xi)$ become relevant.

We can check that an exact Egorov theorem holds for the operator defined by $K_a^h(x,y)$.

Lemma 31 (Exact Egorov theorem on the torus.). *For a symbol $a(x,\xi)\in C^\infty(\mathbb{R}^2)$ which is periodic under the $2\pi\mathbb{Z}^2$ lattice, we have*

$$e^{-ith\Delta}Op_h^w(a)e^{ith\Delta} = Op_h^w(G_t^*a) \tag{5.12}$$

where $Op_h^w(a)$ is defined by the Schwartz kernel in equation (5.11)

Proof. It sufficies to prove that the infinitesimal generators of the two following one pa-

[2] We remark that this easily generalises to $\mathbb{T}^d$, in that case, we define the Schwartz kernel with regards to the symbol a_I where $I\in\{e,o\}^d$ and the entire Schwartz kernel is the sum $\sum_{I\in\{e,o\}^d} K_{a_I}$.

rameter groups agree:

$$\begin{aligned} t &\mapsto \sigma(e^{-ith\Delta}\mathrm{Op}_h^w(a)e^{ith\Delta}), \\ t &\mapsto G_t^* a. \end{aligned} \tag{5.13}$$

To analyse the first one parameter group, note that the generator (on the operator) is $[ih\Delta, \mathrm{Op}_h^w(a)]$. The Schwartz kernel of this commutator is

$$[ih\Delta, \mathrm{Op}_h^w(a)](x,y) = ih\Delta_x K_a^h(x,y) - ih\Delta_y K_a^h(x,y). \tag{5.14}$$

Observe that

$$ih\Delta_x K_a^h(x,y) - ih\Delta_y K_a^h(x,y) = h \sum_{\substack{i\in\{e,o\},\\ j\in\{e,o\}}} i\Delta_x K_{a_{ij}}^h(x,y) - i\Delta_y K_{a_{ij}}^h(x,y). \tag{5.15}$$

Since

$$i\Delta_x \left[a_{ij}\left(\frac{x+y}{2}, h\xi\right) e^{i(x-y)\cdot\xi} \right] = (\Delta_x + |\xi|^2 + 2\xi\cdot\partial_x) a_{ij}\left(\frac{x+y}{2}, h\xi\right) e^{i(x-y)\cdot\xi} \tag{5.16}$$

and

$$i\Delta_y \left[a_{ij}\left(\frac{x+y}{2}, h\xi\right) e^{i(x-y)\cdot\xi} \right] = (\Delta_x + |\xi|^2 - 2\xi\cdot\partial_x) a_{ij}\left(\frac{x+y}{2}, h\xi\right) e^{i(x-y)\cdot\xi}, \tag{5.17}$$

we have that

$$h(i\Delta_x K_{a_{ij}}(x,y) - i\Delta_y K_{a_{ij}}(x,y)) \tag{5.18}$$

is the sum over ξ in a suitably translated dual lattice of the quantity

$$(2\xi\cdot\partial_x a_{ij})\left(\frac{x+y}{2}, h\xi\right) e^{i(x-y)\cdot\xi}. \tag{5.19}$$

Summing over i and j, we deduce that

$$[ih\Delta, \mathrm{Op}_h^w(a)] = \mathrm{Op}_h^w(2\xi\cdot\partial_x a), \tag{5.20}$$

where we note that the vector field $2\xi\cdot\partial_x$ is precisely the generator of geodesic flow on $T^*\mathbb{T}^2$, it is the Hamiltonian vector field of the Hamiltonian $H(x,\xi) = |\xi|^2$. $\square$

5.3 Small scale quantum ergodicity.

We consider the eigenvalue problem $\Delta\phi_{\lambda_j} = \lambda_j^2\phi_{\lambda_j}$ on the torus $\mathbb{T}^2 = \mathbb{R}^2\backslash 2\pi\mathbb{Z}^2$ for the Laplace-Beltrami operator. We let $h = 1/\lambda$ to be the effective wavelength scale of an eigenfunction of eigenvalue λ^2. Let us fix an arbitrary $\varphi \in C_c^\infty(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2}\varphi = 1$ and, for some $0 < \alpha < 1$, define the localisation function $\varphi_h(x) = h^{-2\alpha}\varphi\left(\frac{x-x_0}{h^\alpha}\right)$ which integrates to 1 for every h and which localises to a h^α-radius neighbourhood of $x_0 \in \mathbb{T}^2$. For any orthonormal eigenfunction basis of $L^2(\mathbb{T}^2)$, we want to explore the h^α-scale features of eigenfunctions ϕ with eigenvalue h^{-2} as $h \to 0$. Particularly, using the small scale cutoff φ_h, whether there is an equidistribution of the L^2 mass of eigenfunctions in the h^α-ball as $h \to 0$.

We will consider the quantity

$$V(\varphi_{\lambda^{-1}}, \lambda^{-1}) := \frac{1}{N(\lambda)}\sum_{\lambda_j\leq\lambda}\left(\langle\varphi_{\lambda^{-1}}\phi_j, \phi_j\rangle - \frac{1}{\mathrm{vol}(\mathbb{T}^2)}\right)^2 \tag{5.21}$$

which is the quantum variance up to eigenvalue λ of the eigenfunction densities $|\phi_j|^2dx$. $N(\lambda)$ is the number of eigenfunctions ϕ_j such that the eigenvalue λ_j is at most λ, hence $V(\varphi_{\lambda^{-1}}, \lambda^{-1})$ is clearly the average of the squared deviations of $\langle\varphi_{\lambda^{-1}}\phi_j, \phi_j\rangle$ from $\mathrm{vol}(\mathbb{T}^2)^{-1}$ - which is why it is called a (quantum) variance. We are interested in the asymptotic behaviour of $V(\varphi_{\lambda^{-1}}, \lambda^{-1})$ as $\lambda \to \infty$.

In our analysis we will consider the asymptotically equivalent quantity

$$V(\varphi_h, h) = h^2\sum_{j=1}^{\infty} e^{-2h^2\lambda_j^2}\left(\langle\varphi_h\phi_j, \phi_j\rangle - c\right)^2 \tag{5.22}$$

and study the asymptotics as $h \to 0$, here $c = \mathrm{vol}(\mathbb{T}^2)^{-1} = (4\pi^2)^{-1}$. The weighting factor $e^{-2h^2\lambda_j^2}$ effectively localises to those terms with $\lambda_j \lesssim h^{-1}$. This is then averaged over the number of eigenfunctions with $\lambda_j \lesssim h^{-1}$ which is to first order in h asymptotic to h^{-2} as $h \to 0$ by Weyl's law (see [Zwo12], Chapter 15.3 for a reference to Weyl's law). If (5.22) goes to zero as $h \to 0$, at a fixed scale $\alpha > 0$ and uniformly in $x_0 \in \mathbb{T}^2$, then this proves that the toral eigenfunctions of eigenvalue h_j^{-2} equidistribute everywhere in space at the (small) scale h_j^α with the possible exception of a density zero subsequence of eigenfunctions as $h \to 0$.

The problem of small scale quantum ergodicity on the torus has been considered by Hezari-Rivière in [HR15] and Lester-Rudnick in [LR17]. For small scale quantum ergodicity on negatively curved manifolds there is work by Han in [Han15] and Hezari-Rivière in [HR16]. We mention that in the work of Lester and Rudnick they find that on the d-dimensional torus, small scale equidistribution occurs, for a density one subsequence, at the scale $h^{1/(d-1)-o(1)}$ as $h \to 0$ and they show that this scale is optimal (up to the $o(1)$ term). In particular, when $d = 2$, equidistribution holds up to the Planck scale.

In this chapter, we shall prove

Theorem 32. *For φ_h a small scale cut off of radius h^α about some point $x_0 \in \mathbb{T}^2$, normalised to have integral one, the associated h^α-scale quantum variance $V(\varphi_h, h)$ in (5.22) on the two-dimensional torus goes to zero for any $\alpha < 1$.*

We remark that the methods we present in this chapter are able to show an analogous result for the dimension d torus for any $\alpha < 1/(d-1)$.

There are a few things to note regarding this theorem, we note that it achieves the sharp bound of [LR17]. The approach of the proof is to study the quantum correlation on the torus, which we can reduce to a classical correlation using the exact Egorov property of Weyl quantisation. We can then use the special geometry of the torus (the Poisson summation formula for example) to relate the problem to a lattice intersection problem. The lattice problem bears resemblances to the arguments of [LR17], however the set up towards this argument resembles closer the style of argument of [HR15]. We find this connection between dynamical quantities (classical correlations) and lattice point estimates interesting. We note that [LR17] prove their result by studying the asymptotics of the averaged l^1 discrepancy while the variance, $V(\varphi_h, h)$, we study is the averaged l^2 discrepancy. This implies that Theorem 1.1 in [LR17] (proved in Section 2.2 as a consequence of the asymptotics of the l^1 discrepancy) regarding the small scale equidistribution of a density one subsequence of eigenfunctions is also proved true by our argument (note that the density one subsequence condition can not be improved, as is shown in [LR17] following a personal communication with Bourgain.)

Moreover, as lemmas 33 and 34 in the proof hold more generally on a smooth, compact Riemannian manifold, the method of proof reduces the question of the $h \to 0$ asymptotic of the quantum variance to a question of estimating a quantum correlation averaged over long times, namely the expression

$$h^2 \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) \mathrm{QCor}_h(\varphi_h e^{-h^2\Delta}, \varphi_h e^{-h^2\Delta})(t)dt \tag{5.23}$$

where T can be chosen arbitrarily. In particular, it is this reduction for which we have been trying to apply the intertwining operator to on hyperbolic surfaces. We use the exact Egorov theorem of the Weyl quantisation on the torus to enable us to go to long enough times, of order $h^{-\alpha}$ to make an estimate of (5.23) which enables us to prove small scale toral quantum ergodicity at polynomial scales h^α.

5.4 The Proof.

Lemma 33. *We start by showing that $V(\varphi_h, h)$ in (5.22) is bounded above by*

$$h^2 \,\mathrm{Trace}\left(c^2 e^{-2h^2\Delta} - 2ce^{-2h^2\Delta}\varphi_h + \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)U_h(-t)(\varphi_h e^{-h^2\Delta})U_h(t)(\varphi_h e^{-h^2\Delta})dt\right) \tag{5.24}$$

where $c = \mathrm{vol}(\mathbb{T}^2)^{-1}$, $T \in \mathbb{R}$ is arbitrary and $U_h(t) = e^{ith\Delta}$, the h-semiclassical free Schrödinger propagator.

Proof. Despite the long expression, it arises reasonably simply from expanding the square in (5.22) and using the Schrödinger symmetry, which is that (5.22) remains fixed when ϕ_j is replaced by $U_h(t)\phi_j$ for any $t \in \mathbb{R}$.

To start, note that

$$h^2 \sum_{j=1}^{\infty} e^{-2h^2\lambda_j^2}\left(\langle \varphi_h \phi_j, \phi_j\rangle - c\right)^2 =$$

$$h^2 \sum_{j=1}^{\infty}\left(\langle \varphi_h e^{-\frac{h^2\Delta}{2}}\phi_j, e^{-\frac{h^2\Delta}{2}}\phi_j\rangle - c\langle e^{-h^2\Delta}\phi_j, \phi_j\rangle\right)^2. \tag{5.25}$$

We can replace ϕ_j with $U_h(t)\phi_j = e^{-ith\Delta}\phi_j = e^{-ith\lambda_j^2}\phi_j$ for any $t \in \mathbb{R}$ and preserve quantity (5.25), hence expression (5.25) is equivalent to

$$h^2 \sum_{j=1}^{\infty}\left\langle\left(\frac{1}{T}\int_{\mathbb{R}} \chi_T(t)\left[U_h(-t)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}U_h(t) - ce^{-h^2\Delta}\right]dt\right)\phi_j, \phi_j\right\rangle^2 \tag{5.26}$$

where χ_T is any Lebesgue integrable function on $\mathbb{R}$ such that $\int \chi_T = T$, where $T \in \mathbb{R}$ is arbitrary, and thus can depend on h. We note that the operator

$$\frac{1}{T}\int_{\mathbb{R}} \chi_T(t)\left[U_h(-t)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}U_h(t) - ce^{-h^2\Delta}\right]dt \tag{5.27}$$

is self-adjoint on $L^2(\mathbb{T}^2)$ and hence, with an application of the Cauchy-Schwarz inequality, we can bound (5.26) by

$$h^2 \sum_{j=1}^{\infty}\left\langle\left(\frac{1}{T}\int_{\mathbb{R}} \chi_T(t)\left[U_h(-t)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}U_h(t) - ce^{-h^2\Delta}\right]dt\right)^2\phi_j, \phi_j\right\rangle \tag{5.28}$$

which we can realise as

$$h^2\text{Trace}\left\{\left(\frac{1}{T}\int_{\mathbb{R}}\chi_T(t)\left[U_h(-t)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}U_h(t)-ce^{-h^2\Delta}\right]dt\right)^2\right\} \tag{5.29}$$

with the trace taken over $L^2(\mathbb{T}^2)$. If we expand the squared operator in the trace, use the cyclicity of the trace and the fact the the Schrödinger propagator commutes with the heat operator where necessary, we reduce expression (5.29) to

$$\begin{aligned} &h^2\text{Trace}\left\{\frac{1}{T^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_T(t)\chi_T(s)U_h(s-t)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}U_h(t-s)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}dtds\right\} \\ &\quad + h^2\text{Trace}\left\{-\frac{2}{T}\int_{\mathbb{R}}\chi_T(t)U_h(-t)e^{-\frac{h^2\Delta}{2}}\varphi_h e^{-\frac{h^2\Delta}{2}}U_h(t)dt\cdot ce^{-h^2\Delta}+c^2e^{-2h^2\Delta}\right\}. \end{aligned} \tag{5.30}$$

We can reduce the second trace term in (5.30) to

$$h^2\text{Trace}\left\{c^2e^{-2h^2\Delta}-2c\varphi_h e^{-2h^2\Delta}\right\} \tag{5.31}$$

using the cyclicity of the trace and the commuting of the heat operator and Schrödinger propagator. With a change of variables, the first trace term in (5.30) can be reduced to

$$h^2\text{Trace}\left\{\frac{1}{T^2}\int_{\mathbb{R}}(\chi_T*\iota\chi_T)(t)U_h(-t)(\varphi_h e^{-h^2\Delta})U_h(t)(\varphi_h e^{-h^2\Delta})dt\right\}. \tag{5.32}$$

where $\iota : x \mapsto -x$ is the inversion on $\mathbb{R}$. To fix an example of χ_T, suppose we choose $\chi_T = 1_{[-T/2,T/2]}$. Then in this case, (5.32) equals

$$h^2\text{Trace}\left\{\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)U_h(-t)(\varphi_h e^{-h^2\Delta})U_h(t)(\varphi_h e^{-h^2\Delta})\right\} \tag{5.33}$$

□

Now we prove,

Lemma 34.

$$h^2\,\text{Trace}\{c^2e^{-2h^2\Delta}-2c\varphi_h e^{-2h^2\Delta}\} = -\frac{c}{8\pi}+o(h) \tag{5.34}$$

as $h \to 0^+$ *where* $c = \text{vol}(\mathbb{T}^2)^{-1}$.

Proof. Recall that the heat kernel on a compact Riemannian manifold of dimension n has the following asymptotic expansion in t as $t \to 0^+$ (see eg. [MP49]),

$$e^{-t\Delta}(x,x) = \frac{1}{(4\pi t)^{n/2}} + O(t^{-\frac{n}{2}+1}), \quad t \to 0^+. \tag{5.35}$$

Hence on a two dimensional Riemannian manifold,

$$e^{-2h^2\Delta}(x,x) = \frac{1}{8\pi h^2} + O(1), \quad h \to 0^+. \tag{5.36}$$

Therefore,

$$\begin{aligned} h^2\mathrm{Trace}\{c^2 e^{-2h^2\Delta} - 2c\varphi_h e^{-2h^2\Delta}\} &= h^2\left(\frac{c^2}{8\pi h^2}\left(\int_{\mathbb{T}^2} 1\right) - 2c\frac{1}{8\pi h^2}\left(\int_{\mathbb{T}^2}\varphi_h\right)\right) + o(h) \\ &= \frac{c}{8\pi} - 2\frac{c}{8\pi} + o(h) \\ &= -\frac{c}{8\pi} + o(h). \end{aligned} \tag{5.37}$$

□

We note that Lemmas 33 and 34 are also true on any smooth, compact, Riemannian manifold M. One must simply replace the constant $c = \mathrm{vol}(\mathbb{T}^2)^{-1}$ with $\mathrm{vol}(M)^{-1}$. Lemmas 33 and 34, relate the study of $V(\varphi_h, h)$ on any smooth, compact, Riemannian manifold M to an estimate on the third term of the trace formula, expression

$$h^2\mathrm{Trace}\left\{\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)U_h(-t)(\varphi_h e^{-h^2\Delta})U_h(t)(\varphi_h e^{-h^2\Delta})\right\} \tag{5.38}$$

which we can interpret as

$$h^2\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\mathrm{QCor}_h(\varphi_h e^{-h^2\Delta}, \varphi_h e^{-h^2\Delta})(t)dt \tag{5.39}$$

where we define $\mathrm{QCor}_h(A,B)(t)$ as the ‘Quantum Correlation’ of two quantum observables A and B at time t,

$$\mathrm{QCor}_h(A,B)(t) := \mathrm{Trace}\{U_h(-t)AU_h(t)B\}. \tag{5.40}$$

In the arguments that follow, we will make an analysis of (5.39) specialised to the torus and use its special geometry. However, we make a note that we think quantity (5.39) is sufficiently interesting to deserve its own study on more general geometries. The case of a general hyperbolic surface, and the intertwining operator, has been the focus of the previous chapters.

For the torus we have another lemma,

Lemma 35. *Given two symbols $a(x,\xi), b(x,\xi) \in C^\infty(\mathbb{R}^4)$ which are periodic in x under the $2\pi\mathbb{Z}^2$ lattice, and which decay sufficiently in ξ as $|\xi| \to \infty$ such that $Op^w(a)$ and $Op^w(b)$ are trace class operators on $L^2(\mathbb{T}^2)$ (for example $|a(x,\xi)| \lesssim |\xi|^{-2-\epsilon}$ is sufficient for any $\epsilon > 0$), we can express the trace of the composition $Op^w(a)Op^w(b)$ as the following bilinear form of the symbols a and b;*

$$Trace\{Op^w(a)Op^w(b)\} = \frac{1}{(2\pi)^2} \sum_{I\in\{e,o\}^2} \sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I} \int_{\mathbb{T}^2} a_I(w,\xi) b_I(w,\xi) dw, \tag{5.41}$$

where the trace is taken over $L^2(\mathbb{T}^2)$ and we identify e with the number 0 and o with the number 1 so that the notation $\xi \in \mathbb{Z}^2 + \frac{1}{2}I$ refers to the appropriately translated dual lattice for each term.

Proof. Recall that $\mathrm{Op}^w(a)$ and $\mathrm{Op}^w(b)$ have Schwartz kernels

$$\begin{aligned} K_a(x,y) &= \sum_{I\in\{e,o\}^2} K_{a_I}(x,y), \\ K_b(x,y) &= \sum_{J\in\{e,o\}^2} K_{b_J}(x,y) \end{aligned} \tag{5.42}$$

as described in equations (5.9) and (5.10). We have the standard formula for the trace of the composition of operators in terms of the Schwartz kernels of the operators (when the kernels are sufficiently regular)

$$\mathrm{Trace}\{\mathrm{Op}^w(a)\mathrm{Op}^w(b)\} = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_a(x,y)K_b(y,x)dydx, \tag{5.43}$$

which we can expand out using (5.42) and (5.10) to get

$$\begin{aligned} &\sum_{I,J\in\{e,o\}^2} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_{a_I}(x,y)K_{b_J}(y,x)dydx \\ &= \frac{1}{(2\pi)^4} \sum_{I,J\in\{e,o\}^2} \sum_{\substack{\xi\in\mathbb{Z}^2+\frac{1}{2}I \\ \eta\in\mathbb{Z}^2+\frac{1}{2}J}} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} a_I\left(\frac{x+y}{2},\xi\right) b_J\left(\frac{x+y}{2},\eta\right) e^{i(x-y)\cdot(\xi-\eta)} dxdy. \end{aligned} \tag{5.44}$$

We can change variables letting $w = (x+y)/2$ and $z = x - y$ to give

$$\frac{1}{(2\pi)^4} \sum_{I,J\in\{e,o\}^2} \sum_{\substack{\xi\in\mathbb{Z}^2+\frac{1}{2}I \\ \eta\in\mathbb{Z}^2+\frac{1}{2}J}} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} a_I(w,\xi) b_J(w,\eta) e^{iz\cdot(\xi-\eta)} dz dw \tag{5.45}$$

We note that if $I \neq J$, then the integral in w of $a_I(w,\xi)b_J(w,\eta)$ vanishes because the product $a_I(w,\xi)b_J(w,\eta)$ will be antiperiodic with period π in one of the coordinate directions of $w \in \mathbb{T}^2 = \mathbb{R}^2\backslash 2\pi\mathbb{Z}^2$. Similarly, the integral over $z \in \mathbb{T}^2$ of $e^{iz\cdot(\xi-\eta)}$ is zero unless $\xi = \eta$, when it equals $\mathrm{vol}(\mathbb{T}^2) = (2\pi)^2$, so therefore we conclude that

$$\mathrm{Trace}\{\mathrm{Op}^w(a)\mathrm{Op}^w(b)\} = \frac{1}{(2\pi)^2} \sum_{I\in\{e,o\}^2} \sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I} \int_{\mathbb{T}^2} a_I(w,\xi) b_I(w,\xi) dw. \tag{5.46}$$

□

Remark. We remark that the operator $\varphi_h e^{-h^2\Delta}$ is the h-Weyl quantisation of the symbol $\varphi_h(x)e^{-|\xi|^2}$ up to an error which is $O(h^{1-\alpha})$, which is of lower order in h for any $\alpha < 1$. Moreover, the lower order symbols are of the type $\psi_h(x)p(\xi)e^{-|\xi|^2}$ where p is a polynomial in ξ and $\psi_h(x)$ are smooth cutoffs to a h^α-radius ball with uniformly bounded integral. Our arguments below are robust even if we multiply $e^{-|\xi|^2}$ by a polynomial in ξ. Hence we will use $\mathrm{Op}_h^w(\varphi_h(x)e^{-|\xi|^2})$ in place of $\varphi_h e^{-h^2\Delta}$ in the arguments below.

Lemma 36. *For*

$$A_h = Op_h^w(\varphi_h(x)e^{-|\xi|^2}) \tag{5.47}$$

we have

$$h^2\, Trace\{U_h(-t)A_h U_h(t) A_h\} = \frac{1}{(2\pi)^4}\frac{\pi}{2} \sum_{I\in\{e,o\}^2} \sum_{\substack{m\in\mathbb{Z}^2 \\ k\in 2\pi\mathbb{Z}^2}} a_{I,m}^2 e^{-\frac{1}{2}|k/2h - tm|^2} e^{ik\cdot\frac{1}{2}I}. \tag{5.48}$$

Proof. We note that the exact Egorov theorem on the torus (Lemma 31) implies that

$$U_h(-t)\mathrm{Op}_h^w(a(x,\xi))U_h(t) = \mathrm{Op}_h^w(a(x+2\xi t,\xi)), \tag{5.49}$$

and hence

$$\mathrm{Trace}\{U_h(-t)A_h U_h(t) A_h\} = \mathrm{Trace}\{\mathrm{Op}_h^w(a(x+2\xi t,\xi))\mathrm{Op}_h^w(a(x,\xi))\}. \tag{5.50}$$

Lemma 35 informs us that equation (5.50) is equivalent to

$$\frac{1}{(2\pi)^2}\sum_{I\in\{e,o\}^2}\sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I}\int_{\mathbb{T}^2} a_I(x+2h\xi t,h\xi)a_I(x,h\xi)dx. \tag{5.51}$$

for $a(x,\xi)=\varphi_h(x)e^{-|\xi|^2}$. We can expand a_I in a Fourier series in x since it is periodic with respect to the $2\pi\mathbb{Z}^2$ lattice. We have

$$a_I(x,\xi)=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}^2} a_{I,m}e^{im\cdot x}e^{-|\xi|^2} \tag{5.52}$$

with

$$a_{I,m}=\int_{\mathbb{T}^2}\varphi_{h,I}(x)e^{-im\cdot x}dx. \tag{5.53}$$

Hence

$$a_I(x+2h\xi t,h\xi)=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}^2} a_{I,m}e^{im\cdot x}e^{im\cdot 2h\xi t}e^{-h^2|\xi|^2}, \tag{5.54}$$

and so expression (5.51) is equivalent to

$$\begin{aligned}&\frac{1}{(2\pi)^6}\sum_{I\in\{e,o\}^2}\sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I}\sum_{\substack{m\in\mathbb{Z}^2,\\ n\in\mathbb{Z}^2}}\int_{\mathbb{T}^2} a_{I,m}a_{I,n}e^{i(n-m)\cdot x}e^{-2h^2|\xi|^2}e^{im\cdot 2h\xi t}dx\\ &=\frac{1}{(2\pi)^4}\sum_{I\in\{e,o\}^2}\sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I}\sum_{m\in\mathbb{Z}^2} a_{I,m}^2e^{-2h^2|\xi|^2}e^{im\cdot 2h\xi t}\end{aligned} \tag{5.55}$$

since the integral in x vanishes unless $n=m$, when it equals $\mathrm{vol}(\mathbb{T}^2)=(2\pi)^2$.

We can look to apply the Poisson summation formula (see eg. [Hör03], Chapter 7.2) to the sum over ξ. We have

$$\sum_{\xi\in\mathbb{Z}^2+\frac{1}{2}I} e^{-2h^2|\xi|^2}e^{im\cdot 2h\xi t}=\sum_{\xi\in\mathbb{Z}^2} e^{-2h^2|\xi+\frac{1}{2}I|^2}e^{im\cdot 2h(\xi+\frac{1}{2}I)t} \tag{5.56}$$

and we can calculate the Fourier transform of the exponential function in $\xi\in\mathbb{R}^2$ as

$$\begin{aligned}\mathcal{F}(e^{-2h^2|\xi+\frac{1}{2}I|^2}e^{im\cdot 2h(\xi+\frac{1}{2}I)t})(k) &:= \int_{\mathbb{R}^2} e^{-2h^2|\xi+\frac{1}{2}I|^2}e^{im\cdot 2h(\xi+\frac{1}{2}I)t}e^{-ik\cdot\xi}d\xi\\ &= e^{ik\cdot\frac{1}{2}I}\int_{\mathbb{R}^2} e^{-2h^2|\xi|^2}e^{im\cdot 2h\xi t}e^{-ik\cdot\xi}d\xi\\ &= e^{ik\cdot\frac{1}{2}I}e^{-\frac{1}{2}|k/2h-mt|^2}\int_{\mathbb{R}^2} e^{-2h^2\left|\xi-i\frac{(2mht-k)}{4h^2}\right|^2}d\xi\\ &= \left(\frac{\pi}{2h^2}\right)e^{ik\cdot\frac{1}{2}I}e^{-\frac{1}{2}|k/2h-mt|^2}.\end{aligned} \tag{5.57}$$

So the Poisson summation formula applies to give us

$$\sum_{\xi\in\mathbb{Z}^2} e^{-2h^2|\xi+\frac{1}{2}I|^2} e^{im\cdot 2h(\xi+\frac{1}{2}I)t} = \left(\frac{\pi}{2h^2}\right) \sum_{k\in 2\pi\mathbb{Z}^2} e^{ik\cdot\frac{1}{2}I} e^{-\frac{1}{2}|k/2h-tm|^2} \tag{5.58}$$

and so

$$h^2\mathrm{Trace}\{U_h(-t)A_hU_h(t)A_h\} = \frac{1}{(2\pi)^4}\frac{\pi}{2} \sum_{I\in\{e,o\}^2} \sum_{\substack{m\in\mathbb{Z}^2\\ k\in 2\pi\mathbb{Z}^2}} a_{I,m}^2 e^{-\frac{1}{2}|k/2h-tm|^2} e^{ik\cdot\frac{1}{2}I}. \tag{5.59}$$

$\square$

Note that for $k = 2\pi(p,q) \in 2\pi\mathbb{Z}$, the term $e^{ik\cdot\frac{1}{2}I}$ is either $1, e^{i\pi p}, e^{i\pi q}$ or $e^{i\pi(p+q)}$ for $I = (e,e),(e,o),(o,e),(o,o)$ respectively. In particular they are plus or minus one depending on I, p and q.

Therefore we have that,

$$\begin{aligned} &h^2\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\mathrm{Trace}\{U_h(-t)A_hU_h(t)A_h\}dt \\ &\qquad = \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\frac{1}{(2\pi)^2}\frac{1}{8\pi}\sum_{\substack{I\in\{e,o\}^2\\ m\in\mathbb{Z}^2\\ k\in 2\pi\mathbb{Z}^2}} |a_{I,m}|^2 e^{-\frac{1}{2}|k/2h-tm|^2} e^{ik\cdot\frac{1}{2}I}dt. \end{aligned} \tag{5.60}$$

Now note that since $a_{I,m}$ is the m'th Fourier series coefficient of $\varphi_{h,I}$ where $\varphi_{h,I}$ is the component of φ_h which has parity I, for any $I \neq ee$, we have that the zeroth Fourier series coefficient,

$$a_{I,0} = \int_{\mathbb{T}^2} \varphi_{h,I}(x)dx = 0 \tag{5.61}$$

because $\varphi_{h,I}$ is odd (antiperiodic with period π) in at least one coordinate direction of $\mathbb{T}^2$. Only the even/even parity has a non-zero $m = 0$ Fourier coefficient which is

$$a_{ee,0} = \int_{\mathbb{T}^2} \varphi_{h,ee}(x)dx = \int_{\mathbb{T}^2} \varphi_h(x)dx = 1. \tag{5.62}$$

Consequently, note that when m is $(0,0)$ in the sum appearing in (5.60), we get

$$\frac{1}{(2\pi)^2}\frac{1}{8\pi}\sum_{k\in 2\pi\mathbb{Z}^2} |a_{ee,0}|^2 e^{-\frac{1}{2}|k/2h|^2}. \tag{5.63}$$

There is a nontrivial contribution at $k = (0,0)$ which is exactly

$$\frac{1}{(2\pi)^2}\frac{1}{8\pi}|a_{ee,0}|^2 = \frac{c}{8\pi} \tag{5.64}$$

where $c = \mathrm{vol}(\mathbb{T}^2)^{-1}$. This is exactly the negative of the constant term in Lemma 34. The non-zero k terms when $m = 0$ go to zero as $h \to 0$, namely we can estimate

$$\frac{1}{(2\pi)^2}\frac{1}{8\pi}\sum_{\substack{k\in 2\pi\mathbb{Z}^2\\ k\neq(0,0)}} e^{-\frac{1}{2}|k/2h|^2} \lesssim e^{-c/h^2} = O(h^\infty). \tag{5.65}$$

Note that $f = O(h^\infty)$ means $f = O(h^N)$ for every $N > 0$. Therefore, connecting back to Lemmas 33 and 34, a small scale quantum ergodicity result on the torus at scale h^α comes from a proof that, there exists some choice of T for each h such that

$$\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\frac{1}{(2\pi)^2}\frac{1}{8\pi}\sum_{\substack{I\in\{e,o\}^2\\ k\in 2\pi\mathbb{Z}^2\\ m\in\mathbb{Z}^2\\ m\neq(0,0)}}|a_{I,m}|^2 e^{-\frac{1}{2}|k/2h-tm|^2}e^{ik\cdot\frac{1}{2}I}dt \to 0 \quad \text{as} \quad h\to 0. \tag{5.66}$$

We can make some simplifications of (5.66) to study a related h-asymptotic quantity. In particular, we can drop the constant $(2\pi)^{-2}(8\pi)^{-1}$ as this doesn't affect the $h \to 0$ asymptotics. Similarly, we can integrate over any interval about $t = 0$ of a size which is proportional to T and we can rescale h by a constant to absorb various constant factors. None of these changes will change the $h \to 0$ asymptotics of (5.66). Moreoever, since $e^{ik\cdot\frac{1}{2}I}$ is ± 1 depending on k and I, and $|a_{I,m}| \leq |a_m|$, the $h \to 0$ asymptotics of the sum in (5.66) are bounded by the same sum where we forget the partitioning into odd and even terms.

Consequently, we will aim to show (for any $\alpha < 1$) that there exists some $T > 0$ for each h such that the quantity

$$\frac{1}{T}\int_0^T \sum_{\substack{m\neq(0,0)\\ k\in\mathbb{Z}^2, m\in\mathbb{Z}^2}} |a_m|^2 e^{-|k/h-tm|^2}dt \to 0 \quad \text{as} \quad h\to 0. \tag{5.67}$$

This will suffice to prove (5.66).

Note that since $\varphi_h(x) = h^{-2\alpha}\varphi(x/h^\alpha)$, one has

$$\begin{aligned} a_m = \widehat{\varphi_h}(m) = \int_{\mathbb{R}^2} \varphi_h(x) e^{-im\cdot x} dx = h^{-2\alpha} \int_{\mathbb{R}^2} \varphi(x/h^\alpha) e^{-im\cdot x} dx \\ = \int_{\mathbb{R}^2} \varphi(x) e^{-im\cdot h^\alpha x} dx = \widehat{\varphi}(h^\alpha m). \end{aligned} \tag{5.68}$$

Since φ is a smooth, compactly supported function on $\mathbb{T}^2$, $\widehat{\varphi}$ is a Schwartz function and decays rapidly, so for any $N > 0$ there exists a constant $C > 0$ such that

$$|a_m| = |\widehat{\varphi_h}(m)| = |\widehat{\varphi}(h^\alpha m)| \le C\langle h^\alpha m\rangle^{-N} \tag{5.69}$$

for all $m \in \mathbb{R}^2$. In particular $|a_m|$ decays rapidly as $m \to \infty$. For each fixed m and t, the sum over k is bounded uniformly in m and t, namely there exists $C > 0$ such that

$$|a_m|^2 \sum_{k\in\mathbb{Z}^2} e^{-|k/h - tm|^2} \le C|a_m|^2. \tag{5.70}$$

For any $\epsilon > 0$, we can split the sum in m in (5.67) into $|m| \ge h^{-\alpha-\epsilon}$ and $|m| \le h^{-\alpha-\epsilon}$. Using (5.70), the infinite sum over $|m| \ge h^{-\alpha-\epsilon}$ of $|a_m|^2$ is summable because of the rapid decay of a_m in m and moreoever is $O(h^\infty)$ because of (5.69). So we can focus on the terms in (5.67) with $|m| \le h^{-\alpha-\epsilon}$. Moreover, we have the standard $L^1 - L^\infty$ bound whereby

$$|\widehat{\varphi_h}(m)| \le \int_{\mathbb{R}^2} |\varphi(x)| dx = 1 \tag{5.71}$$

so it is sufficient to prove the result by reducing our analysis to the study of

$$\sum_{\substack{k\in\mathbb{Z}^2 \\ m\in\mathbb{Z}^2 \\ |m|\le h^{-\alpha-\epsilon} \\ m\neq(0,0)}} \frac{1}{T}\int_0^T e^{-|k/h-tm|^2} dt. \tag{5.72}$$

Now the analysis of (5.72) can be split up into the terms where $k = (0,0)$ (and $m \neq (0,0)$) and where $k \neq (0,0)$.

5.4.1 Case 1: $k = (0,0)$ and $m \neq (0,0)$.

When $|k| = 0$ and $|m| \neq 0$, then we are estimating the quantity

$$\sum_{\substack{|m|\le h^{-\alpha-\epsilon} \\ m\neq(0,0)}} \frac{1}{T}\int_0^T e^{-|m|^2 t^2} dt. \tag{5.73}$$

Note that, from a change of variables,

$$\frac{1}{T}\int_0^T e^{-|m|^2t^2}dt \leq \frac{\sqrt{\pi}}{T|m|}, \tag{5.74}$$

So we have,

$$\sum_{\substack{|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}} \frac{1}{T}\int_0^T e^{-|m|^2t^2}dt \lesssim \frac{1}{T}\sum_{\substack{|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}} \frac{1}{|m|} \lesssim \frac{h^{-\alpha-\epsilon}}{T}. \tag{5.75}$$

So if we choose $T = h^{-\alpha-2\epsilon}$, we have that the quantity (5.73) is bounded by h^{ϵ} which goes to zero as $h \to 0$.

5.4.2 Case 2: $k \neq (0,0)$ and $m \neq (0,0)$.

In this case we are estimating the quantity

$$\sum_{\substack{m\in\mathbb{Z}^2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\sum_{\substack{k\in\mathbb{Z}^2\\|k|\neq 0}} \frac{1}{T}\int_0^T e^{-|k/h-tm|^2}dt. \tag{5.76}$$

If we let $k=(p,q)\in\mathbb{Z}^2$ and $m=(m_1,m_2)\in\mathbb{Z}^2$ we can calculate

$$\int_0^T e^{-|k/h-tm|^2}dt = \int_0^T e^{-(p/h-tm_1)^2}e^{-(q/h-tm_2)^2}dt. \tag{5.77}$$

We will find it convenient to introduce variables

$$\rho = \frac{(m_1,m_2)\cdot(p,q)}{|m|}, \quad \text{and} \quad \delta = \frac{(m_2,-m_1)\cdot(p,q)}{|m|}. \tag{5.78}$$

Note that we can interpret ρ as the projection of k onto m and δ as the projection of k onto $m^{\perp}$. Let us change variables in (5.77) by letting

$$t' = t - \frac{1}{h}\left(\frac{m_1p+m_2q}{|m|^2}\right) = t - \frac{1}{h}\frac{\rho}{|m|} \tag{5.79}$$

(this completes the square in the integrand of (5.77)) to get

$$\int_0^T e^{-|k/h-tm|^2}dt = \frac{1}{|m|}e^{-\delta^2/h^2}\int_{\rho h^{-1}-|m|T}^{\rho h^{-1}} e^{-t^2}dt. \tag{5.80}$$

Note that when k and m are collinear points, we have $\delta = 0$, so we notice the largest contribution to the sum in (5.76) occurs for points where k and m are collinear. We can also notice that for m fixed, the map in (5.78) from $(p,q)\in\mathbb{Z}^2$ to (ρ,δ) is a bijection of $\mathbb{Z}^2$ onto $\frac{1}{|m|}\gcd(m_1,m_2)\mathbb{Z}^2$. Hence we can rewrite the sum (5.76) as

$$\sum_{\substack{m\in\mathbb{Z}^2\\ |m|\le h^{-\alpha-\epsilon}\\ |m|\neq 0}}\ \sum_{\substack{(\rho,\delta)\in\frac{\gcd(m_1,m_2)}{|m|}\mathbb{Z}^2\\ (\rho,\delta)\neq(0,0)}} \frac{1}{|m|T}e^{-\delta^2/h^2}\int_{\rho h^{-1}-|m|T}^{\rho h^{-1}} e^{-t^2}dt. \tag{5.81}$$

The benefit of doing this is that (for m fixed) the sum over ρ and δ splits into the product of the sum in ρ and the sum in δ separately. We can split this case further into where $\delta\neq 0$ and where $\delta=0$.

5.4.3 Case 2.1: $\delta\neq 0$.

When $\delta\neq 0$, we have the rapid decay of $e^{-\delta^2/h^2}$ to assist us. Note that

$$\sum_{\substack{\delta\in\frac{\gcd(m_1,m_2)}{|m|}\mathbb{Z}\\ \delta\neq 0}} e^{-\delta^2/h^2}\lesssim e^{-\frac{\gcd(m_1,m_2)^2}{h^2|m|^2}}\le e^{-\frac{1}{h^2|m|^2}}, \tag{5.82}$$

where

$$e^{-\frac{1}{h^2|m|^2}}=O(h^\infty) \tag{5.83}$$

uniformly for $|m|\le h^{-\alpha-\epsilon}$ so long as $\alpha+\epsilon<1$. Because of this strong decay, we can be quite crude with the other sums, the sum

$$\sum_{\rho\in\frac{\gcd(m_1,m_2)}{|m|}\mathbb{Z}}\int_{\rho h^{-1}-|m|T}^{\rho h^{-1}} e^{-t^2}dt \tag{5.84}$$

can be split up into a piece where $\rho>h^{1-\epsilon}|m|T$. For such ρ, we have

$$\rho h^{-1}-|m|T>(h^{-\epsilon}-1)|m|T\gtrsim h^{-\epsilon}|m|T>0 \tag{5.85}$$

and therefore

$$\int_{\rho h^{-1}-|m|T}^{\rho h^{-1}} e^{-t^2}dt\lesssim e^{-(\rho h^{-1}-|m|T)^2} \tag{5.86}$$

and so

$$\sum_{\substack{\rho\in\frac{\gcd(m_1,m_2)}{|m|}\mathbb{Z}\\ \rho>h^{1-\epsilon}|m|T}} e^{-(\rho h^{-1}-|m|T)^2}\lesssim e^{-(h^{-\epsilon}|m|T)^2}=O(h^\infty) \tag{5.87}$$

since $|m| \neq 0$ and $T \neq 0$. As $|m| \leq h^{-\alpha-\epsilon}$ and we chose $T = h^{-\alpha-2\epsilon}$ where $\alpha + \epsilon < 1$, the terms with $|\rho| \leq h^{1-\epsilon}|m|T$ are contained within $|\rho| \leq h^{-5}$. We can bound

$$\sum_{\substack{\rho \in \frac{\gcd(m_1,m_2)}{|m|}\mathbb{Z} \\ |\rho| \leq h^{-5}}} \int_{\rho h^{-1}-|m|T}^{\rho h^{-1}} e^{-t^2} dt \lesssim \sum_{\substack{\rho \in \frac{\gcd(m_1,m_2)}{|m|}\mathbb{Z} \\ |\rho| \leq h^{-5}}} 1 \lesssim h^{-5}|m|. \tag{5.88}$$

The sum (5.84) over $\rho < -h^{-5}$ is easily seen to be $O(h^{\infty})$ uniformly in $|m| \leq h^{-\alpha-\epsilon}$ where $\alpha + \epsilon < 1$.

Thus the sum of (5.81) with $\delta \neq 0$ is

$$\frac{1}{T} \sum_{\substack{m \in \mathbb{Z}^2 \\ |m| \leq h^{-\alpha-\epsilon} \\ |m| \neq 0}} \frac{A_1 A_2}{|m|} \tag{5.89}$$

where A_1 is the sum (5.82) over all $\delta \neq 0$ and A_2 is the sum (5.84) over all ρ. As $A_1 = O(h^{\infty})$ uniformly in m, A_2 is bounded above by $h^{-5}|m|$ and $T = h^{-\alpha-\epsilon}$, it is clear that (5.89) is $O(h^{\infty})$ for $\alpha + \epsilon < 1$.

5.4.4 Case 2.2: $\delta = 0$.

This is the case where k is collinear to m, in which case $|\rho| = |k|$. We have to study the sum

$$\sum_{\substack{m \in \mathbb{Z}^2 \\ |m| \leq h^{-\alpha-\epsilon} \\ |m| \neq 0}} \sum_{\substack{k \in \mathbb{Z}^2 \\ |k| \neq 0 \\ k \text{ collinear to } m}} \frac{1}{|m|T} \int_{|k|h^{-1}-|m|T}^{|k|h^{-1}} e^{-t^2} dt. \tag{5.90}$$

For any $\epsilon > 0$, we can split the k-sum up into a piece where $|k| > h^{1-\epsilon}|m|T$, where $|k|h^{-1} - |m|T \gtrsim h^{-\epsilon}$. For such k,

$$\int_{|k|h^{-1}-|m|T}^{|k|h^{-1}} e^{-t^2} dt \lesssim e^{-\left(\frac{|k|}{h}-|m|T\right)^2} \tag{5.91}$$

and the sum

$$\sum_{\substack{k \in \mathbb{Z}^2 \\ |k| \neq 0 \\ k \text{ collinear to } m \\ |k| > h^{1-\epsilon}|m|T}} e^{-\left(\frac{|k|}{h}-|m|T\right)^2} \lesssim e^{-\frac{1}{h^{\epsilon}}} = O(h^{\infty}). \tag{5.92}$$

Hence the piece of the sum in (5.90) which sums over $|k| > h^{1-\epsilon}|m|T$ is $O(h^{\infty})$ and we are left with the piece

$$\begin{aligned}&\sum_{\substack{m\in\mathbb{Z}^2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\sum_{\substack{k\in\mathbb{Z}^2\\|k|\neq 0\\k \text{ collinear to } m\\|k|\leq h^{1-\epsilon}|m|T}}\frac{1}{|m|T}\int_{|k|h^{-1}-|m|T}^{|k|h^{-1}}e^{-t^2}dt\\ \lesssim&\sum_{\substack{m\in\mathbb{Z}^2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\sum_{\substack{k\in\mathbb{Z}^2\\|k|\neq 0\\k \text{ collinear to } m\\|k|\leq h^{1-\epsilon}|m|T}}\frac{1}{|m|T}\end{aligned} \tag{5.93}$$

Note that if m_1 and m_2 are coprime (where $m = (m_1, m_2) \in \mathbb{Z}^2$), then the number of lattice points k which are collinear to m can be bounded above by a constant multiple of

$$\frac{\text{Radius of the } k\text{-set}}{|m|} = \frac{h^{1-\epsilon}|m|T}{|m|} = h^{1-\epsilon}T. \tag{5.94}$$

When m_1 and m_2 are not coprime, we define the lattice point $m_{\text{primitive}} = m/\gcd(m_1, m_2)$, then the number of lattice points k which are collinear to m can be bounded above by a constant multiple of

$$\frac{\text{Radius of the } k\text{-set}}{|m_{\text{primitive}}|} = \frac{h^{1-\epsilon}|m|T\gcd(m_1,m_2)}{|m|} = h^{1-\epsilon}T\gcd(m_1,m_2). \tag{5.95}$$

Thus, we get

$$\sum_{\substack{m\in\mathbb{Z}^2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\sum_{\substack{k\in\mathbb{Z}^2\\|k|\neq 0\\k \text{ collinear to } m\\|k|\leq h^{1-\epsilon}|m|T}}\frac{1}{|m|T}\lesssim h^{1-\epsilon}\sum_{\substack{m\in\mathbb{Z}^2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\frac{\gcd(m_1,m_2)}{|m|}. \tag{5.96}$$

To estimate this sum, let us partition the sum over the value of $\gcd(m_1, m_2)$. We have that

$$h^{1-\epsilon}\sum_{\substack{m\in\mathbb{Z}^2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\frac{\gcd(m_1,m_2)}{|m|}\leq h^{1-\epsilon}\sum_{q=1}^{h^{-\alpha-\epsilon}}\sum_{\substack{m\in\mathbb{Z}^2\\q|m_1,q|m_2\\|m|\leq h^{-\alpha-\epsilon}\\|m|\neq 0}}\frac{q}{(m_1^2+m_2^2)^{1/2}}. \tag{5.97}$$

We can then change the sum over $|m| \leq h^{-\alpha-\epsilon}$ where $q|m_1$ and $q|m_2$ into a sum over

$n = (n_1, n_2) \in \mathbb{Z}^2$ where $|n| \leq h^{-\alpha-\epsilon}/q$. Doing so, we have that

$$h^{1-\epsilon} \sum_{q=1}^{h^{-\alpha-\epsilon}} q \sum_{\substack{m \in \mathbb{Z}^2 \\ q|m_1, q|m_2 \\ |m| \leq h^{-\alpha-\epsilon} \\ |m| \neq 0}} \frac{1}{(m_1^2 + m_2^2)^{1/2}} = h^{1-\epsilon} \sum_{q=1}^{h^{-\alpha-\epsilon}} q \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq h^{-\alpha-\epsilon}/q \\ |n| \neq 0}} \frac{1}{((qn_1)^2 + (qn_2)^2)^{1/2}} \tag{5.98}$$

We can take a factor of q^{-1} out from the $1/|qn|$ term, hence we arrive at the sum,

$$h^{1-\epsilon} \sum_{q=1}^{h^{-\alpha-\epsilon}} \sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq h^{-\alpha-\epsilon}/q \\ |n| \neq 0}} \frac{1}{|n|} \lesssim h^{1-\epsilon} \sum_{q=1}^{h^{-\alpha-\epsilon}} \frac{h^{-\alpha-\epsilon}}{q} \lesssim h^{1-\alpha-2\epsilon} \log(h^{-\alpha-\epsilon}) \tag{5.99}$$

where we are using the fact that there exists a constant $C > 0$, independent of $X \geq 1$, such that

$$\sum_{\substack{n \in \mathbb{Z}^2 \\ |n| \leq X \\ |n| \neq 0}} \frac{1}{|n|} \leq CX. \tag{5.100}$$

We observe that $h^{1-\alpha-2\epsilon} \log(h^{-\alpha-\epsilon})$ goes to zero as $h \to 0$ for any $\alpha < 1$ as one can choose $\epsilon > 0$ sufficiently small. This proves Theorem 32, that the two-dimensional torus achieves small scale quantum ergodicity (on a density one subsequence) at scales h^α for every $\alpha < 1$.

We briefly conclude by remarking that our arguments in this chapter can be generalised to higher dimensional toruses in a reasonably straightforward fashion. We note that in dimension d, the key estimate is to bound

$$h^{1-\epsilon} \sum_{\substack{m \in \mathbb{Z}^d \\ |m| \leq h^{-\alpha-\epsilon} \\ |m| \neq 0}} \frac{\gcd(m_1, m_2, \ldots, m_d)}{|m|}, \tag{5.101}$$

which can be calculated in a similar fashion to the two-dimensional case by partitioning on the value of $\gcd(m_1, m_2, \ldots, m_d)$. Doing so, we get

$$h^{1-\epsilon} \sum_{q=1}^{h^{-\alpha-\epsilon}} \sum_{\substack{n \in \mathbb{Z}^d \\ |n| \leq h^{-\alpha-\epsilon}/q \\ |n| \neq 0}} \frac{1}{|n|} \lesssim h^{1-\epsilon} \sum_{q=1}^{h^{-\alpha-\epsilon}} \left(\frac{h^{-\alpha-\epsilon}}{q} \right)^{d-1} \lesssim h^{1-\alpha(d-1)-k\epsilon} \tag{5.102}$$

for any $k > d$, where we are using the fact that there exists a $C > 0$ such that

$$\sum_{\substack{n \in \mathbb{Z}^d \\ |n| \leq X \\ |n| \neq 0}} \frac{1}{|n|} \leq C X^{d-1} \tag{5.103}$$

holds for all $X \geq 1$. Since $h^{1-\alpha(d-1)-k\epsilon}$ goes to zero as $h \to 0$ for any $\alpha < 1/(d-1)$ (as $\epsilon > 0$ can be chosen sufficiently small), an analogue of this argument for toruses of dimension d proves that small scale quantum ergodicity on the dimension-d torus occurs at scales h^α for all $\alpha < 1/(d-1)$.

Chapter 6

Conclusion

6.1 Concluding remarks.

We started by exploring the intertwining operator of Anantharaman-Zelditch in [AZ12]. In Chapter 2, we used a spectral Helgason-Fourier transform, derived from the $L^2(\mathbb{D})$ spectral theory of the Laplace-Beltrami operator on $\mathbb{D}$, to modify the Zelditch quantisation of [Zel86]. This also allowed us to provide a new perspective on the symmetry properties of the Helgason-Fourier transform and Zelditch's calculus through the hyperbolic scattering matrix. Additionally, the modifications allowed us to construct modifed Wigner distributions and with a modifed Patterson-Sullivan distribution, we constructed the intertwining operator $\boldsymbol{\mathcal{L}}$ in Chapter 2. We proved two main theorems in that chapter. The first theorem, Theorem 14 stated that $\boldsymbol{\mathcal{L}}$ is a unitary isomorphism of the space of L^2 symbols, $L^2_S(G \times \mathbb{R}, dg|c(r)|^2dr)$ into its range in $L^2(G \times \mathbb{R}, dg|c(r)|^2dr)$, and that $\boldsymbol{\mathcal{L}}$ intertwines the two natural flows, the classical flow G_t^* and the quantum flow V_t that we have been keeping in mind the entire thesis. The second main theorem, Theorem 15 of Chapter 2 provided an oscillatory integral representation of $\boldsymbol{\mathcal{L}}$. Theorem 14 of Chapter 2 allows us to prove an exact correspondence between the quantum correlation and the classical correlation on hyperbolic space.

In seeking to extend these ideas onto a compact quotient, in Chapter 3, we studied the oscillatory integral formulation of $\boldsymbol{\mathcal{L}}$. In the main theorems of this chapter, Theorem 18 and Theorem 22, we were able to derive an expression for the canonical relation of the phase function associated to $\boldsymbol{\mathcal{L}}$ and to show that the left and right projections of the canonical relation have blowdown singularities on a codimension one submanifold, on which the amplitude of $\boldsymbol{\mathcal{L}}$ vanishes to first order. We constrasted the vanishing of the amplitude of $\boldsymbol{\mathcal{L}}$ on the singular submanifold to the amplitude of $\mathcal{L}$ in [AZ12] which has a phase function that generates a canonical relation with the same blowdown singularities, yet with an amplitude that does not vanish on this singular submanifold.

While we were unable to find a way to use this information to extend the exact equivalence of the quantum correlation and classical correlation on $\mathbb{D}$ to a compact quotient $\Gamma\backslash\mathbb{D}$, we were motivated by it to study the matrix elements and traces of operators like $A_1(t)A_2$ where the symbol of A_1 was pulled back by the geodesic flow at time t. In this chapter, Chapter 4, we showed in Theorem 23 that, under reasonably natural assumptions on the Zelditch symbols of A_1 and A_2, the trace of $A_1(t)A_2$ decays exponentially as $t \to \pm\infty$. We proved this by expressing the matrix elements as a classical correlation of two functions on $S^*(\Gamma\backslash\mathbb{D})$, one of which is singular along a dynamical subbundle, and proving that a decay of correlations estimate holds between the two functions.

Our final chapter, Chapter 5 considered the geometry of the torus, and the quantum variance question in this context. We are able to prove that the small scale quantum variance question on any smooth, compact, Riemannian manifold can be related to a question of estimating the long time averages of the quantum correlation. Approaching the problem from this perspective, we used the exact Egorov's theorem associated to Weyl quantisation on the torus to allow us to study the quantum correlation for long times, of order $T \sim h^{-\alpha}$ for a small scale quantum ergodicity result of order h^{α}. We then analysed the resulting quantum correlation using the special geometry of the torus, reducing the problem via the Poisson summation formula to a kind of lattice intersection problem. Analysing the resulting problem, we are able to prove Theorem 32 which confirms small scale quantum ergodicity on the two dimensional torus holds (for density one subsequences) at scales h^{α} for every $\alpha < 1$. We indicate that our argument can be generalised to the dimension d torus where one can prove the same small scale quantum ergodicity holds at scales h^{α} for every $\alpha < 1/(d-1)$.

We conclude by noting that there is a clear research direction suggested by the work in this thesis. We thank the reader for their patience and attention.

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