

Chapter 1

Introduction

The relationship between curvature and topology has traditionally been one of the most popular and highly developed topics in Riemannian geometry. In this area, a central issue of concern is that of determining global topological structures from local metric properties. Of particular interest to us the so-called pinching problem and related sphere theorems in geometry. We begin with a brief overview of this problem, from Hopf's inspiration to the latest developments in Hamilton's Ricci flow.

1.1 Manifolds with Constant Sectional Curvature

One of the earliest insights into the relationship between curvature and topology is the problem of classifying complete Riemannian manifolds with constant sectional curvature, referred to as *space forms*. In the late 1920s Heinz Hopf studied the global properties of such space forms and proved, in his PhD dissertation [Hop25] (see also [Hop26]), the following:

Theorem 1.1 (Uniqueness of Constant Curvature Metrics). *Let M be a complete, simply-connected, n -dimensional Riemannian manifold with constant sectional curvature. Then M is isometric to either $\mathbb{R}^n$, S^n or $\mathbb{H}^n$.*

Furthermore, if the manifold is compact then the space forms are compact quotients of the either $\mathbb{R}^n$, S^n or $\mathbb{H}^n$. Placing this result on solid ground was one of Hopf's tasks during the 1930s, however the classification is still incomplete (the categorisation of hyperbolic space quotients has been extremely problematic).

Given these developments, curiosity permits one to ask if a similar result would hold under a relaxation of the curvature hypothesis. In other words, assuming a compact manifold has a sectional curvature 'varying not too much' (we will later say the manifold is 'pinched'), can one deduce that the underlying manifold is topologically (one would even hope differentiably) identical to one of the above space forms? After rescaling the metric there are three

cases: the pinching problem around $\kappa_0 = +1, 0, -1$. Therefore if the sectional curvature K satisfies $|K - \kappa_0| \leq \varepsilon$, the question becomes one of finding an optimal $\varepsilon > 0$ in which the manifold is identical (in some sense) to a particular space form.

For our purposes, the question of interest is that of positive pinching around $\kappa_0 = 1$; our subsequent discussion will focus entirely on this case. The problem has enjoyed a great deal of interest over the years due to its historical importance both in triggering new results and as motivation for creating new mathematical tools.

1.2 The Topological Sphere Theorem

The only simply connected manifold of constant positive sectional curvature is, by the above theorem, the sphere. A heuristic sense of continuity leads one to hope that if the sectional curvature of a manifold is close to a positive constant, then the underlying manifold will still be a sphere. Hopf himself repeatedly put forward this problem, in particular when Harry Rauch (an analyst and expert in Riemannian surfaces) visited him in Zürich throughout 1948–1949. Rauch was so enthusiastic about Hopf’s pinching that, back at the Institute for Advanced Study in Princeton, he finally managed to prove Hopf’s conjecture with a pinching constant of roughly $\delta \approx 3/4$. Specifically Rauch [Rau51] proved:

Theorem 1.2 (Rauch, 1951). *Let (M^n, g) be a complete Riemannian manifold with $n \geq 2$. If the sectional curvature $K(p, \Pi)$ (where $p \in M$ and Π is a 2-dimensional plane through p) satisfies*

$$\delta k \leq K(p, \Pi) \leq k - \varepsilon$$

for some constant $k > 0$, some $\varepsilon > 0$, all $p \in M$; and all planes Π , where $\delta \approx 3/4$ is the root of the equation $\sin \pi \sqrt{\delta} = \sqrt{\delta}/2$. Then the simply connected covering space of M is homeomorphic to the n -dimensional sphere S^n . In particular if M is simply connected, then M is homeomorphic to S^n .

Rauch’s paper is seminal in two respects. First of all, he was able to control the metric on both sides. Secondly, to get a global result, he made a subtle geometric study in which he proved that (under the pinching assumption) one can build a covering of the manifold from the sphere.

Thereafter Klingenberg [Kli59] sharpened the constant – in the even dimensional case – to the solution of $\sin \pi \sqrt{\delta} = \sqrt{\delta}$ (i.e. $\delta \approx 0.54$). Crucially, the precise notion of an injectivity radius was introduced to the pinching problem. Using this, Berger [Ber60a] improved Klingenberg’s result, under the even dimension assumption, with $\delta = 1/4$. Finally, Klingenberg [Kli61] extended the

injectivity radius lower estimate to odd dimensions. This proved the sphere theorem in its entirety. The resulting *quarter pinched sphere theorem* is stated as follows:

Theorem 1.3 (Rauch–Klingenberg–Berger Sphere Theorem). *If a simply connected, complete Riemannian manifold has sectional curvature K satisfying*

$$\frac{1}{4} < K \leq 1,$$

*then it is homeomorphic to a sphere.*¹

The theorem’s pinching constant is optimal since the conclusion is false if the inequality is no longer strict. The standard counterexample is complex projective space with the Fubini-Study metric (sectional curvatures of this metric take on values between $1/4$ and 1 inclusive). Other counterexamples may be found among the rank one symmetric spaces (see e.g. [Hel62]).

As a matter of convention, we say a manifold is *strictly* δ -pinched in the *global sense* if $0 < \delta < K \leq 1$ and *weakly* δ -pinched in the *global sense* if $0 < \delta \leq K \leq 1$. In which case the sphere theorem can be reformulated as: ‘Any complete, simply-connected, strictly $1/4$ -pinched Riemannian manifold is homeomorphic to the sphere.’

1.2.1 Remarks on the Classical Proof

The proof from the 1960s consisted in arguing that the manifold can be covered with only two topological balls (e.g. see [AM97, Sect. 1]). The proof relies heavily on classical comparison techniques.

Idea of Proof. Choose points p and q in such a way that the distance $d(p, q)$ is maximal, that is $d(p, q) = \text{diam}(M)$. The key property of such a pair of points is that for any unit tangent vector $v \in T_q M$, there exists a minimising geodesic γ from q to p making an acute angle with v . From this one can apply Toponogov’s triangle comparison theorem [Top58] (see also [CE08, Chap. 2]) and Klingenberg’s injectivity radius estimates to show that

$$M = B(p, r) \cup B(q, r)$$

¹ In the literature some quote the sectional curvature as taking values in the interval $(1, 4]$. This is only a matter of scaling. For instance, if the metric is scaled by a factor λ then the sectional curvatures are scaled by $1/\lambda$. Thus one can adjust the maximum principal curvature to 1 , say. What is important about the condition is that the ratio of minimum to maximum curvature remains greater than $1/4$.

for $\pi/2\sqrt{\delta} < r < \pi$ and $1/4 < \delta \leq K \leq 1$. In other words, under the hypothesis of the sphere theorem, we can write M as a union of two balls. One then concludes, by classical topological arguments, that M is *homeomorphic* to the sphere. $\square$

1.2.2 Manifolds with Positive Isotropic Curvature

One cannot overlook the contributions made by Micallef and Moore [MM88] in generalising the classical Rauch–Klingenberg–Berger sphere theorem. By introducing the so-called positive isotropic curvature condition to the problem together with harmonic map theory, they managed to prove the following:

Theorem 1.4 (Micallef and Moore, 1988). *Let M be a compact simply connected n -dimensional Riemannian manifold which has positive curvature on totally isotropic 2-planes, where $n \geq 4$. Then M is homeomorphic to a sphere.*

This is achieved as follows: Firstly, for a n -dimensional Riemannian manifold M with positive curvature on totally isotropic two-planes one can show that any nonconstant conformal harmonic map $f: S^2 \rightarrow M$ has index at least $\frac{n-3}{2}$. The proof uses classical results of Grothendieck on the decomposition of holomorphic bundles over S^2 . Secondly, one can show: If M is a compact Riemannian manifold such that $\pi_k(M) \neq 0$, where $k \geq 2$, then there exists a nonconstant harmonic 2-sphere in M of index $\leq k - 2$. This fact is a modification of the Sacks–Uhlenbeck theory of minimal 2-spheres in Riemannian manifolds. From this, the above theorem easily follows by using duality with these two results, and using the higher-dimensional Poincaré conjecture, which was proved for $n \geq 5$ by Smale and for $n = 4$ by Freedman.

A refined version of the classical sphere theorem now follows by replacing the global pinching hypotheses on the curvature by a pointwise one. In particular we say that a manifold M is *strictly* δ -pinched in the *pointwise sense* if $0 < \delta K(\Pi_1) < K(\Pi_2)$ for all points $p \in M$ and all 2-planes $\Pi_1, \Pi_2 \subset T_p M$. Furthermore, we say M is *weakly* δ -pinched in the *pointwise sense* if $0 \leq \delta K(\Pi_1) \leq K(\Pi_2)$ for all point $p \in M$ and all 2-planes $\Pi_1, \Pi_2 \subset T_p M$. In which case it follows from Berger’s Lemma (see Sect. 2.7.7) that any manifold which is strictly $1/4$ -pinched in the pointwise sense has positive isotropic curvature (and so is homeomorphic to S^n).

1.2.3 A Question of Optimality

Given the above stated Rauch–Klingenberg–Berger sphere theorem, a natural question to ask is: *Can the homeomorphic condition be replaced by a diffeomorphic one?* In answering this, there are some dramatic provisos.

The biggest concerns the method used in the above classical proof – as the argument presents M as a union of two balls glued along their common boundary so that M is homeomorphic to S^n . However, Milnor [Mil56] famously showed that there are ‘exotic’ structures on S^7 , namely:

Theorem 1.5 (Milnor, 1956). *There are 7-manifolds that are homeomorphic to, but not diffeomorphic to, the 7-sphere.²*

Moreover, some exotic spheres are precisely obtained by gluing two half spheres along their equator with a weird identification. Hence the above classical proof cannot do better since it does not allow one to obtain a diffeomorphism between M and S^n in general. As a result, it remained an open question as to whether or not the sphere theorem’s conclusion was optimal in this sense.

To add to matters, it has also been a long-standing open problem as to whether there actually are any concrete examples of exotic spheres with positive curvature. This has now apparently been resolved by Petersen and Wilhelm [PW08] who show there is a metric on the Gromoll–Meyer Sphere with positive sectional curvature.

1.3 The Differentiable Sphere Theorem

There have been many attempts at proving the differentiable version, most of which have been under sub-optimal pinching assumptions. The first attempt was by Gromoll [Gro66] with a pinching constant $\delta = \delta(n)$ that depended on the dimension n and converged to 1 as n goes to infinity. The result for δ independent of n was obtained by Sugimoto, Shiohama, and Karcher [SSK71] with $\delta = 0.87$. The pinching constant was subsequently improved by Ruh [Ruh71, Ruh73] with $\delta = 0.80$ and by Grove, Karcher and Ruh [GKR74] with $\delta = 0.76$. Furthermore, Ruh [Ruh82] proved the differentiable sphere theorem under a pointwise pinching condition with a pinching constant converging to 1 as $n \rightarrow \infty$.

1.3.1 The Ricci Flow

In 1982 Hamilton introduced fundamental new ideas to the differentiable pinching problem. His seminal work [Ham82b] studied the evolution of a heat-type geometric evolution equation:

$$\frac{\partial}{\partial t}g(t) = -2\operatorname{Ric}(g(t)) \quad g(0) = g_0,$$

²In fact Milnor and Kervaire [KM63] showed that S^7 has exactly 28 non-diffeomorphic smooth structures.

herein referred to as the Ricci flow. This intrinsic geometric flow has, over the years, served to be an invaluable tool in obtaining global results within the differentiable category. The first of which is following result due to Hamilton [Ham82b].

Theorem 1.6 (Hamilton, 1982). *Suppose M is a simply-connected compact Riemannian 3-manifold with strictly positive Ricci curvature. Then M is diffeomorphic to S^3 .*

Following this, Hamilton [Ham86] developed powerful techniques to analyse the global behaviour of the Ricci flow. This enabled him to extend his previous result to 4-manifolds by showing:

Theorem 1.7 (Hamilton, 1986). *A compact 4-manifold M with a positive curvature operator is diffeomorphic to the sphere S^4 or the real projective space $\mathbb{R}P^4$.*

Note that we say a manifold has a positive curvature operator if the eigenvalues of R are all positive, or alternatively $R(\phi, \phi) > 0$ for all 2-forms $\phi \neq 0$ (when considering the curvature operator as a self-adjoint operator on two-forms).

Following this, Chen [Che91] show that the conclusion of Hamilton's holds under the weaker assumption that the manifold has a 2-positive curvature operator. Specifically, he proved that: *If M is a compact 4-manifold with a 2-positive curvature operator, then M is diffeomorphic to either S^4 or $\mathbb{R}P^4$.* Moreover, Chen showed that the 2-positive curvature condition is implied by pointwise 1/4-pinching. In which case he managed to show:

Theorem 1.8 (Chen, 1991). *If M^4 is a compact pointwise 1/4-pinched 4-manifold, then M is either diffeomorphic to the sphere S^4 or the real projective space $\mathbb{R}P^4$, or isometric to the complex projective space $\mathbb{C}P^2$.*

Note that we say a manifold has a 2-positive curvature operator if the sum of the first two eigenvalues of R are positive.³ It is easy to see that when $\dim M = 3$, the condition of positive Ricci and 2-positive curvature operator are equivalent (in fact, 2-positive implies positive Ricci in n -dimensions).

However despite this progress it still remained an open problem for over a decade as to whether or not these results could be extended for manifolds M with $\dim M \geq 4$.

³ Equivalently one could also say that $R(\phi, \phi) + R(\psi, \psi) > 0$ for all 2-forms ϕ and ψ satisfying $|\phi|^2 = |\psi|^2$ and $\langle \phi, \psi \rangle = 0$.

1.3.2 Ricci Flow in Higher Dimensions

Huisken [Hui85] was one of the first to study the Ricci flow on manifolds of dimension $n \geq 4$. His analysis focuses on decomposing the curvature tensor into $R_{ijkl} = U_{ijkl} + V_{ijkl} + W_{ijkl}$, where U_{ijkl} denotes the part of the curvature tensor associated with the scalar curvature, V_{ijkl} is the part of the curvature associated with the trace-free Ricci curvature, and W_{ijkl} denotes the Weyl tensor. By following the techniques outlined in [Ham82b], Huisken showed that if the scalar-curvature-free part of the sectional curvature tensor is small compared to the scalar curvature, then the same evolution equation for the curvature yields the same result – a deformation to a constant positive sectional curvature metric.

Theorem 1.9 (Huisken, 1985). *Let $n \geq 4$. If the curvature tensor of a smooth compact n -dimensional Riemannian manifold of positive scalar curvature satisfies*

$$|W|^2 + |V|^2 < \delta_n |U|^2$$

with $\delta_4 = \frac{1}{5}$, $\delta_5 = \frac{1}{10}$, and $\delta_n = \frac{2}{(n-2)(n+1)}$ for $n \geq 6$, then the Ricci flow has a solution $g(t)$ for all times $0 \leq t < \infty$ and $g(t)$ converges to a smooth metric of constant positive curvature in the C^∞ -topology as $t \rightarrow \infty$.

There are also similar results by Nishikawa [Nis86] and Margerin [Mar86] with more recent sharp results by Margerin [Mar94a, Mar94b] as well.

In 2006, Böhm and Wilking [BW08] managed to generalise Chen’s work in four dimensions by constructed a new family of cones (in the space of algebraic curvature operators) which is invariant under Ricci flow. This allowed them to prove:

Theorem 1.10 (Böhm and Wilking, 2006). *On a compact manifold M^n the Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.*

Their paper [BW08] overcomes some major technical problems, particular those related to controlling the Weyl part of the curvature.

Soon thereafter, Brendle and Schoen [BS09a] finally managed to prove:

Theorem 1.11 (Brendle and Schoen, 2007). *Let M be a compact Riemannian manifold of dimension $n \geq 4$ with sectional curvature strictly $1/4$ -pinched in the pointwise sense. Then M admits a metric of constant curvature and therefore is diffeomorphic to a spherical space form.*

The key novel step in the proof is to show that nonnegative isotropic curvature is invariant under the Ricci flow. This was also independently proved by Nguyen [Ngu08, Ngu10]. Brendle and Schoen obtain the abovementioned result by working with the nonnegative isotropic curvature condition on

$M \times \mathbb{R}^2$, which is implied by pointwise $1/4$ -pinching. We will examine this proof by Brendle and Schoen [BS09a] in Chap. 14. In Chaps. 12 and 13 we will look at the methods of Böhm and Wilking [BW08] in detail with the specific aim to proving the differentiable pointwise $1/4$ -pinching sphere theorem for $n \geq 4$.

Finally, in Chap. 15 we survey the following result of Brendle [Bre08] which generalises the earlier results of [Ham82b, Hui85, Che91, BW08, BS09a].

Theorem 1.12 (Brendle, 2008). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ such that $M \times \mathbb{R}$ has positive isotropic curvature. Then there is a unique maximal solution $g(t)$, $t \in [0, T)$ to the Ricci flow with initial metric g_0 which converges to a metric of constant curvature. In particular, M is diffeomorphic to a spherical space form.*

1.4 Where to Next?

There is a big problem crying out for an application of Ricci flow:

*If (M, g_0) is a simply connected compact Riemannian manifold with positive curvature on isotropic 2-planes, is M diffeomorphic to a spherical space form?*⁴

The difficulties are manifold: First, it is very difficult to bring in the condition of simple-connectedness (a global condition) to the analysis of the Ricci flow (which is locally defined). If the assumption is dropped, then there are many more manifolds which can appear and which have positive isotropic curvature, notably the product $S^{n-1} \times S^1$, for which the universal cover is not compact (and is not a sphere!). Second, singularities can happen in the Ricci flow of such metrics where the curvature operators do not approach those of constant sectional curvature (in the above example, the S^{n-1} factor shrinks while the S^1 factor does not).

Hamilton [Ham97] used a surgery argument to analyse the situation in four dimensions. His method is to show that blow-ups at singularities look close to either a shrinking cylinder or a shrinking sphere. He stops the Ricci flow just before a singularity, and cuts all the cylindrical ‘necks’, pasting in smooth caps on the ‘stumps’. Then he continues the Ricci flow again.⁵ In this way he proves that the original manifold must have been a connected sum of spheres with $S^3 \times S^1$ (or quotients of these).

⁴ Recall that Micallef and Moore [MM88] proved that such manifolds are homeomorphic to S^n .

⁵ This argument was an inspiration for Perel’man’s later use of Ricci flow to prove the Poincaré and Geometrisation conjectures. A correction to Hamilton’s argument was later published by Zhu and Chen [CZ06].

The question is: Can the algebraic methods of Böhm and Wilking [BW08] be used – with the preservation of positive isotropic curvature as presented in Chap. 14 – to prove a similar result for manifolds with positive isotropic curvature in higher dimensions? Doing so would in the best scenario prove the following conjecture of Schoen:

Conjecture 1.13 (Schoen, [CTZ08, Sch07]). *For $n \geq 4$, let M be an n -dimensional compact Riemannian manifold with positive isotropic curvature. Then a finite cover of M is diffeomorphic to S^n , $S^{n-1} \times S^1$ or a connected sum of these. In particular, the fundamental group of M is virtually free.*