LONG RANGE DEPENDENT PROCESSES AND FRACTIONAL BROWNIAN MOTION

by

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DECLARATION

In accordance with the regulations of The Australian National University, I declare that the work described in this thesis was carried out by myself, except where due reference was made and stated in the text.

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To My Parents

Related Publications

The following papers have appeared or been accepted for publication:

HEYDE, C. C. and DAI, W. (1995) On the robustness to small trends of estimation based on the smoothed periodogram. *J. Time Series Anal.*, to appear.

HEYDE, C. C. and DAI, W. (1995) On the robustness of limit theorems. *Bull. Internat. Statist. Inst.*, Vol. 56, Book 2 , pp. 549–555.

ABSTRACT

The aim of this thesis is to investigate long range dependent processes and their applications.

The first chapter of the thesis gives an introduction to long range dependent processes. In particular, we give a brief introduction to fractional Brownian motion which has been proposed as a model for the long range dependence observed in a variety of hydrological and geophysical time series.

In Chapter 2 we study the robustness of asymptotic normality of estimation using smoothed periodogram methodology in circumstances where long range dependence appears. We also study the robustness of some estimators of the Hurst index H which characterizes certain models of long range dependence.

Modern statistical theory is notable for attempts at relaxing the assumption of independent structure of processes under consideration. In Chapter 3 we investigate asymptotic properties, such as the law of large numbers, the central limit theorem and the uniformity of convergence, of sample mean, sample covariance and some quadratic forms involving data from long range dependent processes. Moreover, we develop a method for hypothesis testing of the mean of time series perturbed from stationarity by a deterministic trend.

In Chapters 4 and 5, we are concerned with the applications of long range dependence to mathematical finance. It is of practical and theoretical importance to take into account possible long range dependence in research on the behaviour of stock price movement. We propose and study a stochastic model — fractional Black-Scholes model — a stochastic differential equation driven by fractional Brownian motion. The fractional Black-Scholes model includes the standard Black-Scholes model as a special case and is able to account for long range dependence in financial markets. In Chapter 4, we develop stochastic inte-

gration with respect to fractional Brownian motion which is not a semimartingale. In Chapter 5, we establish the Itô formula for fractional Brownian motion. Then we use Itô's formula to prove the existence and uniqueness of the solution to the fractional Black-Scholes equation.

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Chapter 1

INTRODUCTION TO LONG RANGE DEPENDENCE

1.1 Introduction to Long Range Dependence

A stationary second order process is said to have long range dependence (LRD) if its covariance function at lag k decays, as $k \to \infty$, so slowly that the summation of the covariance over the lags $k = 0, \pm 1, \pm 2, \ldots$ diverges. The slow decay of the covariance function corresponds to a singularity at the origin in the spectrum. To fix ideas, let us consider a second order stationary time series $\{X_t : t = 1, 2, \ldots\}$. We write

$$\gamma_k = \text{cov}\,(X_t, X_{t+k})\ ,\ k = 0, \pm 1, \pm 2, \ldots$$

for the covariance function of $\{X_t\}$ at lag k and

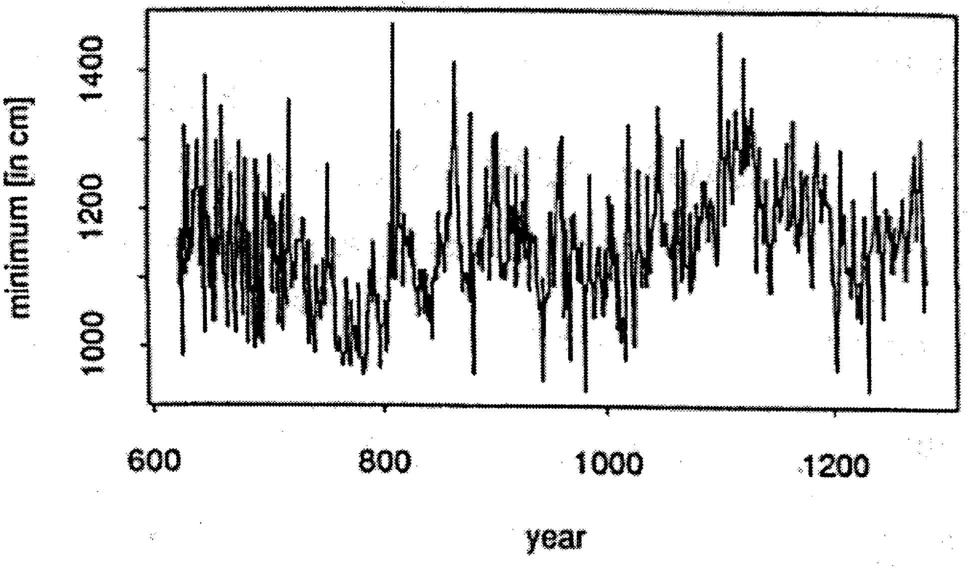
$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-ik\omega}\ ,\ -\pi < \omega < \pi$$

for the spectral density of $\{X_t\}$ (if it exists). Then the dichotomy between short range dependence (SRD) and LRD can be characterized by the criteria given in Table 1.1 (see, for example, Heyde and Gay, 1992).

Table 1.1: SRD and LRD

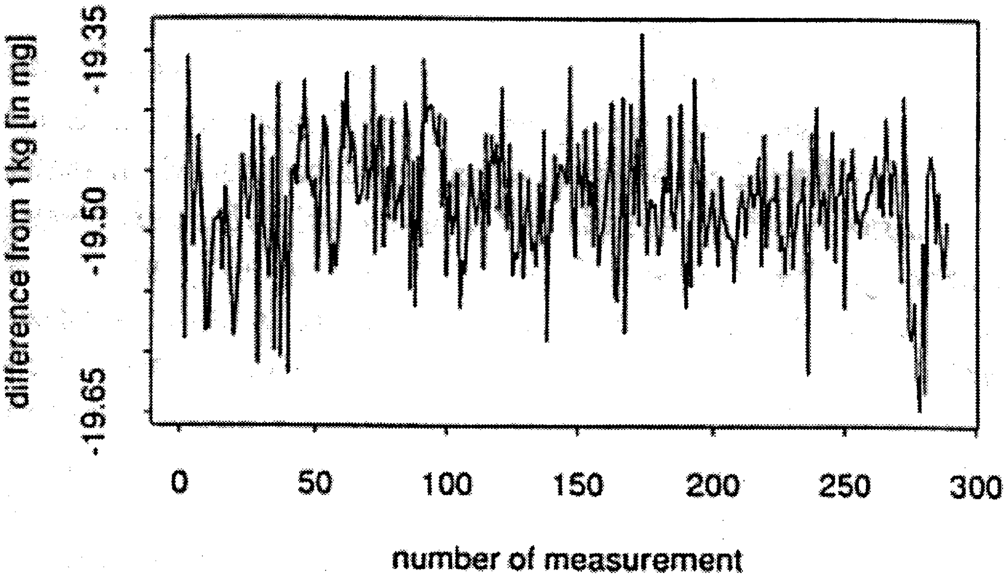
SRD	**LRD**
(i) $\sum \gamma_k$ converges	(i) $\sum \gamma_k$ diverges
(ii) $f(0)$ finite	(ii) $f(\omega)$ singular at $\omega = 0$

Figure 1.1: Nile River minima (from Beran, 1992b, p. 405)



As early as 1895 the astronomer Newcomb discussed the phenomenon of LRD in astronomical data sets and called it "semi-systematic" errors. Pearson (1902) observed slowly decaying correlations in simulated astronomical observations. Due to the vast number of examples from hydrology and geophysics, LRD is recognized by most hydrologists and geophysicists to be the rule rather than the exception. A typical data set where LRD is exhibited is the record of the Nile River minima (Figure 1.1, taken from Beran, 1992b, p. 405). This data played a key role in

Figure 1.2: NBS precision measurements on the 1-kg check standard (from Beran, 1992b, p. 406)



the discovery of LRD in hydrological data by the famous hydrologist Hurst (1951). Nowadays the phenomenon of LRD has been observed in a wide range of areas of statistical applications, such as hydrology (Hurst, 1951), geophysics (Lawrance and Kotegoda, 1977), agriculture (Smith, 1938; Whittle, 1956, 1962), astronomy (Pearson, 1902; Jeffreys, 1939), meteorology (Haslett and Raftery, 1989) and economics (Robinson, 1994c; Peters, 1994). The phenomenon of LRD even occurs in situations where every precaution was taken to prevent dependence between the observations. A typical example of such high-quality data measured under ideal circumstances are the measurements of the 1-kg check standard weight provided by the U.S. National Bureau of Standards (NBS) Washington (Figure 1.2, taken from Beran, 1992b, p. 406). The correlations of NBS precision measurements seem to decay with a rate approximately proportional to $|k|^{-\alpha}$ with $\alpha = 0.8$. For details of the analysis of the data sets of Nile River minima and NBS precision measurements, see, for example, Beran (1992b).

Though most known examples of LRD are time series, LRD is not restricted to this type of data. See, for example, Gay and Heyde (1990) for random fields with LRD. Smith (1938) noticed LRD in a random field setting of agricultural field trials.

In the research of stochastic processes with LRD, two paradigm models are the fractional Gaussian noise and the fractional ARIMA model.

A fractional Gaussian noise (see, for example, Sinai, 1976) is the first order difference of a fractional Brownian motion which was introduced analytically by Mandelbrot and Van Ness (1968). We shall discuss fractional Brownian motion in Section 1.2. A fractional Gaussian noise is a stationary Gaussian process $\{ X_t \}$ with covariance

$$\gamma_k = \operatorname{cov}\,(X_t\,,\, X_{t+k}\,) = \frac{\sigma^2}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right)\,, \tag{1.1}$$

where $\sigma^2 = \operatorname{Var}(X_1)$. The spectral density of $\{ X_t \}$ is of the form

$$f(\omega) = 2\,b\,(H,\sigma^2)\,(1-\cos\omega)\sum_{j=-\infty}^{\infty} |2\,\pi\,j+\omega|^{-2H-1}\,,\ -\pi \le \omega \le \pi\,, \tag{1.2}$$

where $b\,(H,\sigma^2) = (2\pi)^{-1}\sigma^2\sin(\pi H)\,\Gamma(2H+1)$, $H \in (0,1)$. The parameter H is called the Hurst index. When $H = 1/2$, $\{ X_t \}$ is the standard Gaussian noise. When $H \in (1/2,1)$, $f(\omega)$ has a pole of the form $b\,|\omega|^{1-2H}$ at zero and $\sum \gamma_k = \infty$. When $H \in (0,1/2)$, $f(0) = 0$ and $\sum \gamma_k = 0$, and in this case, $\{ X_t \}$ is not long range dependent (LRD).

Fractional ARIMA models were introduced by Granger and Joyeux (1980) and Hosking (1981). (See also Adenstedt, 1974.) Fractional ARIMA models are a natural generalization of standard ARIMA(p,d,q) models proposed by Box and Jenkins (1970). For any $-1/2 < d < 1/2$, a fractional ARIMA process is defined as

$$\Phi(B)\,(1-B)^d = \Theta(B)\varepsilon_t\,, \tag{1.3}$$

where $\{ \varepsilon_t \}$ are independent and identically distributed (i.i.d.) normal random variables with mean zero, B denotes the backshift operator, Φ and Θ are polynomials of finite degree having no roots in or on the unit circle, $\Phi(B)$ denotes the autoregressive part and $\Theta(B)$ the moving average part of the process, and

$$(1-B)^d = \sum_{k=0}^{\infty} \begin{pmatrix} d \\ k \end{pmatrix} (-B)^k$$

is the fractional difference operator. The spectral density of the fractional ARIMA model is given as

$$f(\omega) = \frac{|\Theta(e^{i\omega})|^2}{|\Phi(e^{i\omega})|^2} \left|1 - e^{i\omega}\right|^{-2d}, \quad -\pi \leq \omega \leq \pi \,. \tag{1.4}$$

When $d = 0$, definition (1.3) gives the usual ARIMA model. When $0 < d < 1/2$, LRD appears.

The Hurst index H in (1.1) and the fractional difference index d in (1.3) are connected through the formula $d = H - 1/2$.

In each of the above models, the spectral density has the form

$$f(\omega) \sim L_1(\omega)\, |\omega|^{-\alpha}\,, \quad \text{as } \omega \to 0\,, \tag{1.5}$$

and the covariance function γ_k at lag k behaves as

$$\gamma_k \sim L_2(k)\, |k|^{\alpha-1}\,, \quad \text{as } k \to \infty\,, \tag{1.6}$$

where $0 < \alpha < 1$, $L_1(\omega)$ is a slowly varying function at the origin while $L_2(k)$ slowly varying for $|k| \to \infty$. Here the symbol "$\sim$" means that the ratio of the left- and right-hand sides tends to 1.

In general, a stationary second order process is said to exhibit LRD if its spectral density satisfies (1.5), or, equivalently, if γ_k satisfies (1.6).

The recognition of stationary random processes and random fields subject to LRD is a problem of particular contemporary scientific importance. This is because such processes typically exhibit apparent trends or periodicities which appear persistent but are eventually ephemeral.

Slowly decaying covariances have, if ignored, disastrous effects on classical statistical results of stationary processes. The usual methods of analysis of stationary processes assume exponential decay of the covariance function, which is a typical case of SRD. A large number of limit theorems which, in the classical approach were always studied under the assumption that the underlying random variables were independent, continue to hold under certain SRD structures. However, as dependence becomes stronger, at a certain point, new phenomena arise. For example, models with weak dependence (such as ARMA models and Markov processes) are well known and often used in practice. Already for these models, interval estimates and prediction intervals can differ considerably from the i.i.d. case, though asymptotic rates of convergence remain the same. The asymptotic standard deviation of the sample mean is typically of the form $\sigma/\sqrt{N}$, where N is the sample size and σ a certain constant. When the dependence becomes stronger so that, for example, the covariance is not summable any more, the consequence of the strong dependence for classical tests and confidence intervals can be calamitous. As an example, let us consider a case where the covariance of the data decays as $|k|^{-\alpha}$, $\alpha \in (0,1)$, then the standard deviation of the sample mean of the data decays at a rate of $N^{-\alpha/2}$. If $\alpha = 0.4$, one needs approximately $100,000$ observations to achieve the same precision as from 100 independent observations drawn from a population with the same variance. Another example is that slowly decaying covariance can cause goodness-of-fit tests for a distribution to reject the null hypothesis with probability tending to one with increasing sample size.

Stationary processes or random fields with LRD have drawn considerable interest in statistical research. Although still in the beginning stages, progress in the development of statistical methodology for LRD has been made along several lines, including location and scale estimation, estimation of the Hurst index and fractal dimension, spectral estimation and testing, tests for LRD, goodness of fit tests, prediction and regression. For surveys of general areas of LRD, see, for example, Beran (1992b), Peters (1994) and Robinson (1994c).

Many standard methods and results in classical SRD situations should be investigated under the possibility of LRD. The development of statistical methodology for LRD will certainly be a rewarding task for future research.

1.2 Fractional Brownian Motion

Fractional Brownian motion (FBM) is a generalization of Brownian motion. Brownian motion is a good model if the process under study is unknown and a large number of small contributing influences are involved. However, the increments of Brownian motion are independent. In reality independence is just an extreme case. The restrictive assumption required for Brownian motion becomes a special case. Fractional Brownian motion arises naturally from the examination of the conditions of the validity of limit theorems relating to Brownian motion. Kolmogorov (1940) first introduced FBM to probability. Mandelbrot and Van Ness (1968) gave an analytical definition of FBM. They proposed FBM as a model for the LRD widely observed in hydrological and geophysical time series.

A FBM (denoted as B_H for mathematical convenience) is a Gaussian process with covariance

$$\operatorname{cov}\left(B_H(t) \,,\, B_H(s) \right) = \frac{1}{2} \operatorname{Var}\left(B_H(1) \right) \left(t^{2H} + s^{2H} - |t-s|^{2H} \right)$$

for any $t,\, s \geq 0$. Here $H \in (0\,,\,1)$ is the so called Hurst index. When $H = 1/2$, $B_H(t) = B(t)$ is a Brownian motion. When $0 < H < 1/2$, it has long been thought that B_H is of less interest (Mandelbrot, 1982). Actually this is not so because of the relationship between the Hurst index and the spectral exponent of fractional noises (see, for example, Peters, 1994, Chapter 13). But we will not pursue this matter in this thesis, for when $0 < H < 1/2$, the summation of the covariance of the increment process of B_H over lags $k = 0\,,\, \pm 1\,,\, \pm 2\,,\, \ldots$ is zero, which is not the case for LRD. When $1/2 < H < 1$, the increment process of FBM is LRD, which is the case with which this thesis is concerned.

For any arbitrarily fixed $\tau \geq 0$, let

$$X(t) = B_H(t+\tau) - B_H(t)$$

be the increment process of B_H, then $\{X(t)\}$ is a stationary and self-similar process. In particular, the increment sequence $\{B_H(t+1) - B_H(t) : t = 1, 2, \ldots\}$ is the interesting fractional Gaussian noise (equation (1.1) or (1.2)).

A number of simulations of FBM series are shown in Figures 1.3, 1.4, 1.5 and 1.6 for different values of H. Panel (a)'s of Figures 1.3 – 1.6 show simulated series (fractional Gaussian noises) with $H = 0.25, 0.50, 0.75$ and 0.90 respectively. Here we can see that as H draws closer to 1, the series becomes less erratic and has more consecutive observations with the same signs. Panel (b)'s of Figures 1.3 – 1.6 are plotted as cumulative time series (FBM). Again, as H increases, the cumulative line becomes more smooth and less jagged.

The feature which most distinguishes FBM from Brownian motion is that FBM is no longer a semimartingale, which makes the stochastic analysis of FBM a challenge because the standard Itô calculus is no longer available.

Figure 1.3: Simulated fractional Brownian motion: H=0.25

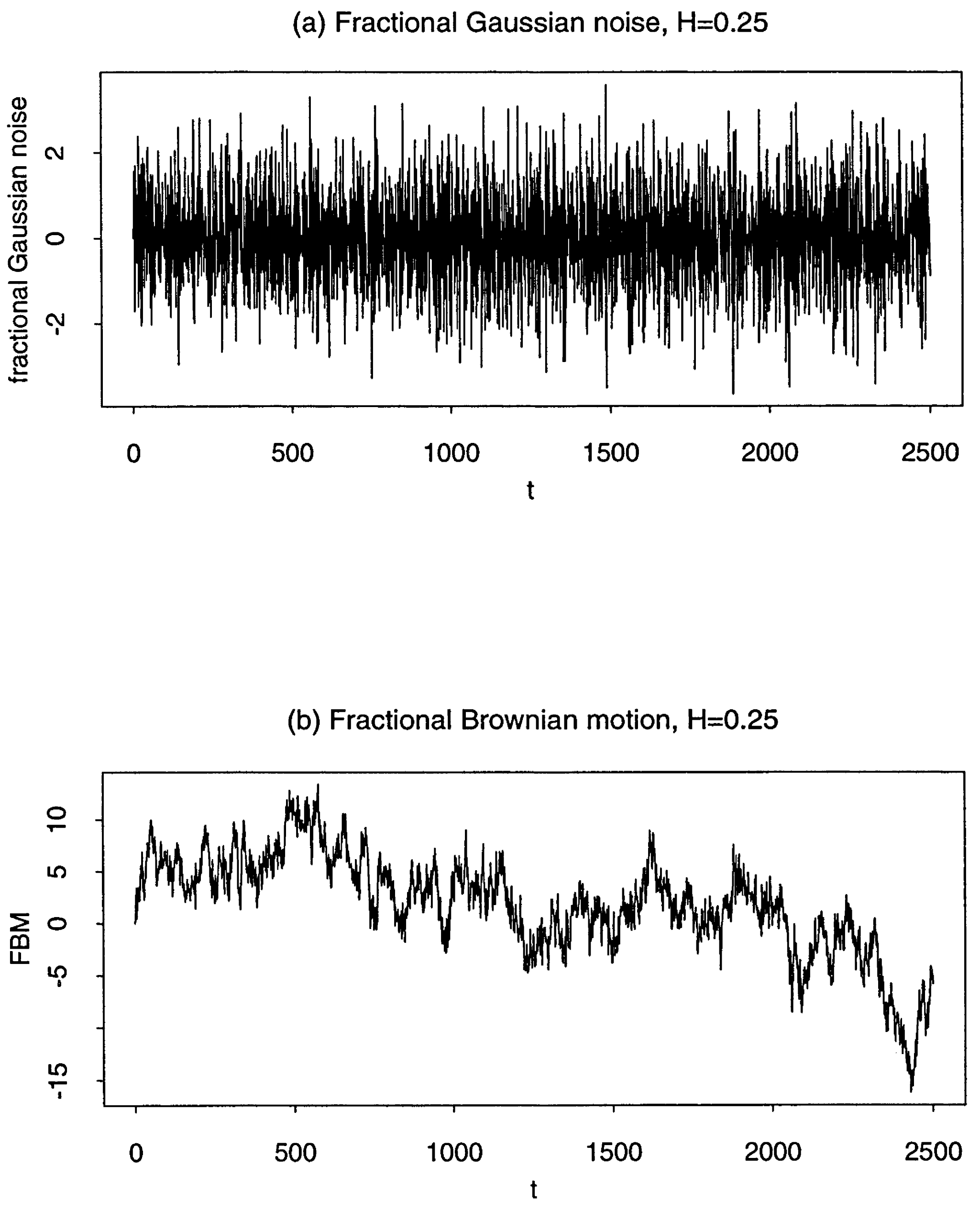


Figure 1.4: Simulated fractional Brownian motion: H=0.50 – Brownian motion

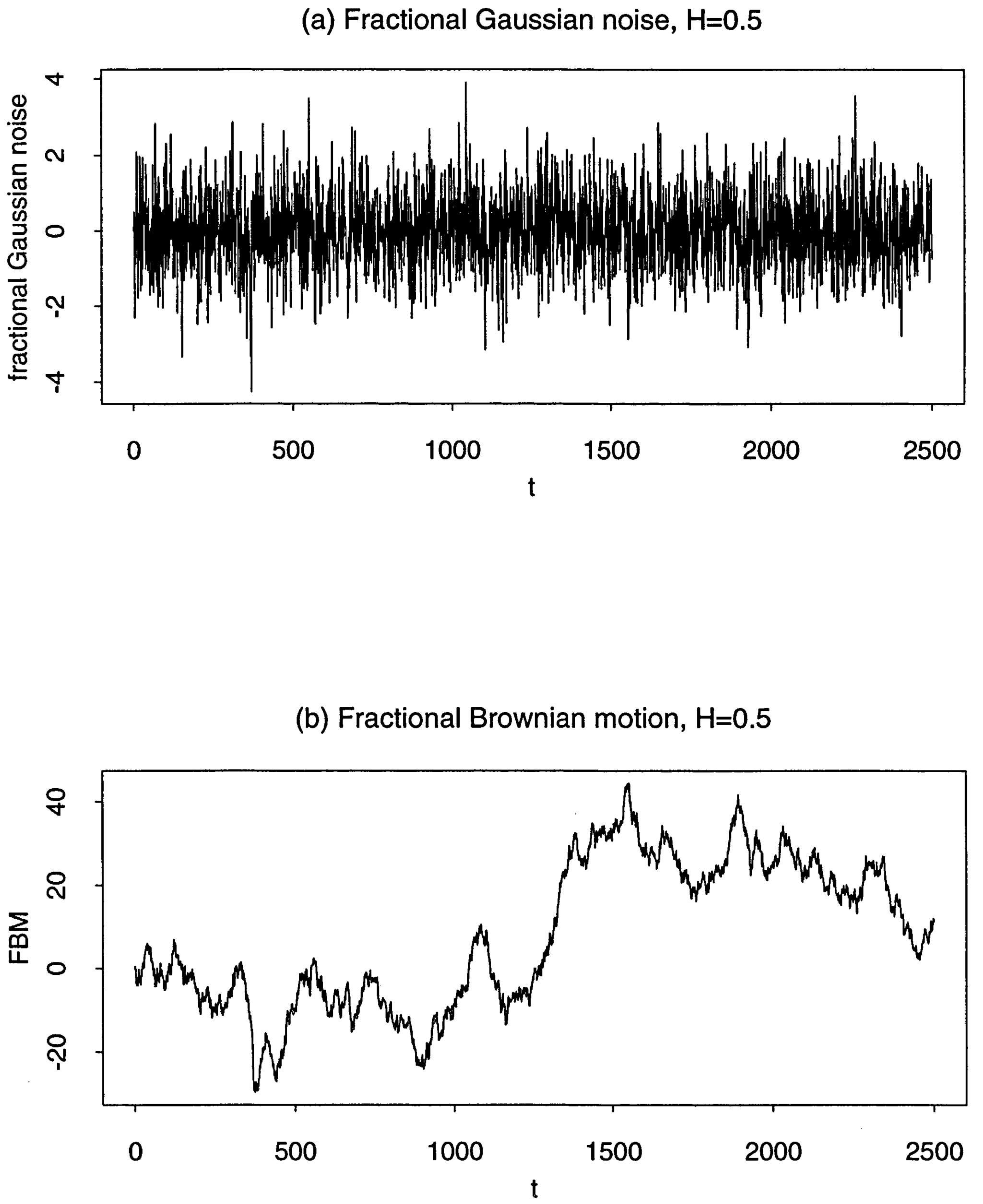
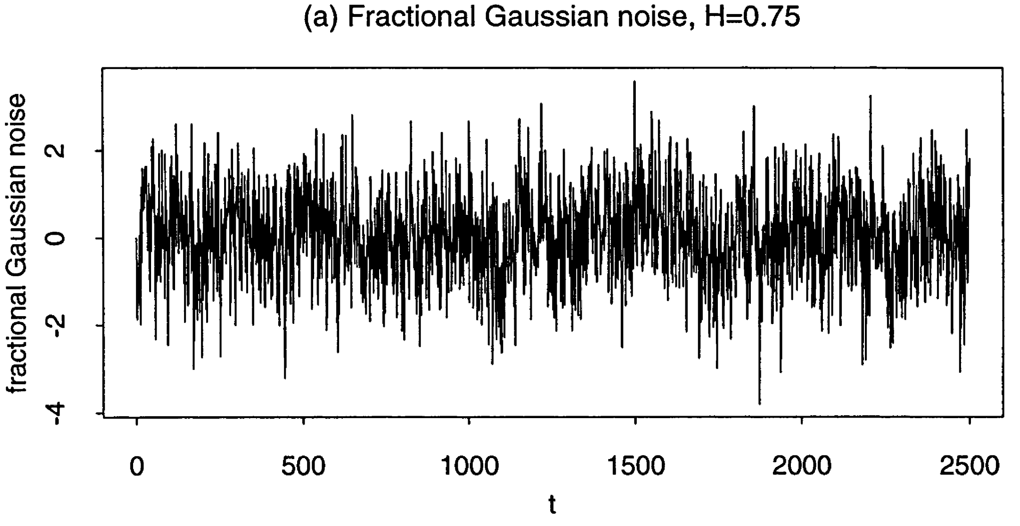


Figure 1.5: Simulated fractional Brownian motion: H=0.75



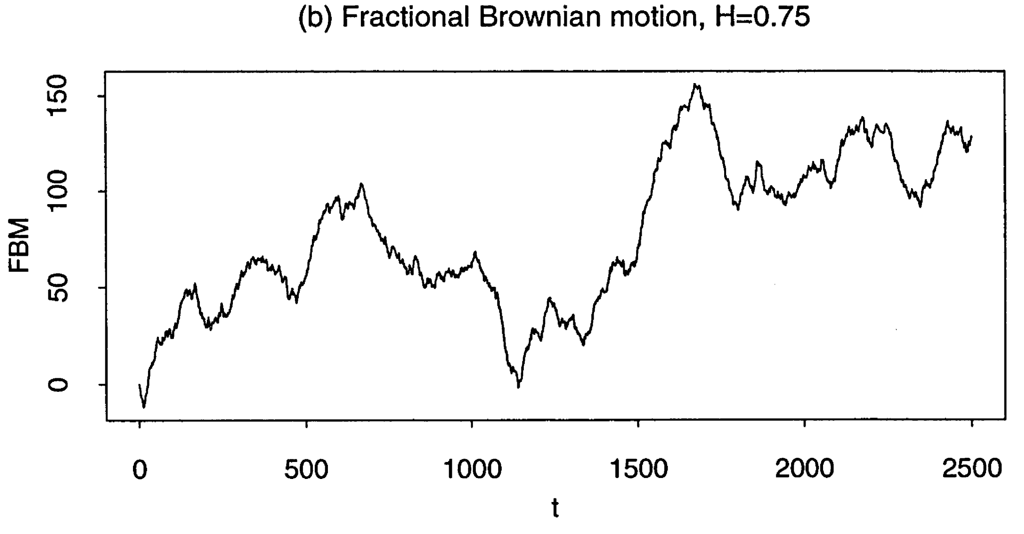
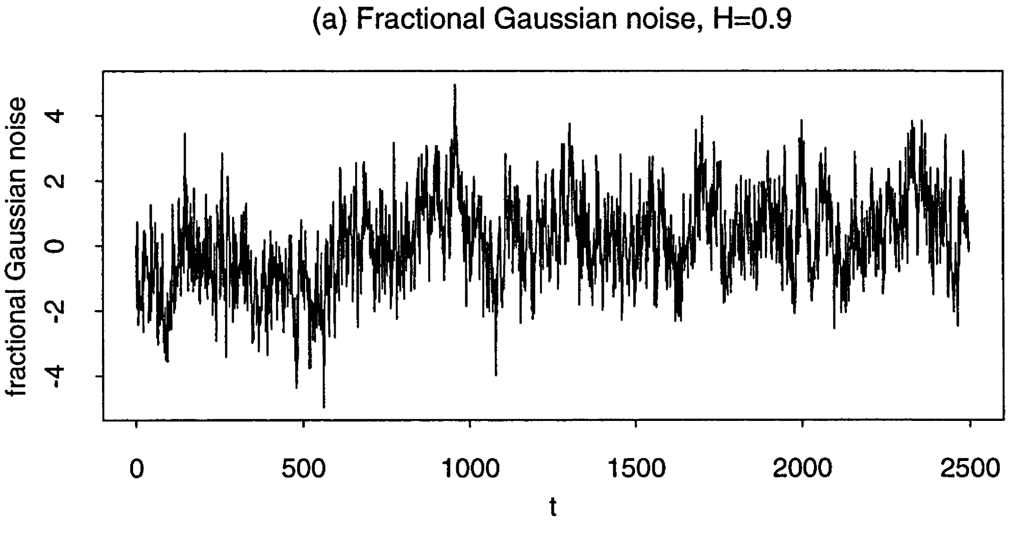
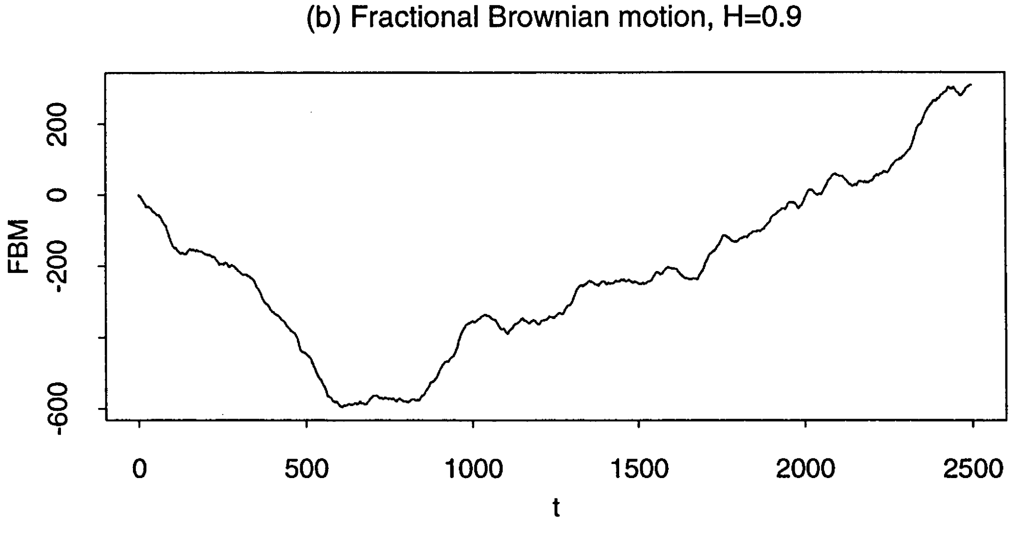


Figure 1.6: Simulated fractional Brownian motion: H=0.90





1.3 Summary of Thesis

The development of a theory of stationary long range dependent processes has become a very active field of research in the last decade. The importance of statistical methods with LRD in virtually all areas of applications has now been demonstrated by numerous examples.

The results in this thesis have been arrived at in the course of the investigation of problems of statistical inference and stochastic analysis of stationary processes in circumstances where LRD has been taken into account.

Many of the key asymptotic normality theorems have not been investigated systematically for their sensitivity to disturbances of the sufficient conditions under which they hold. In Chapter 2, we study the robustness, or sensitivity, of asymptotic normality of estimates based on smoothed periodograms, when there is possible LRD, with respect to small perturbations of the models under consideration. Asymptotic normality is ubiquitous but in some cases the limits may be very vulnerable to departures from standard conditions, such as stationarity or independence. However, in reality, "minor" departures to these conditions can be expected. Thus, what constitutes "minor" and whether such departures matter are issues of concern. It is shown in Chapter 2 that a smoothed periodogram approach to model fitting and parameter estimation is highly robust to the presence of a small trend if the underlying processes are short range dependent (SRD). If the underlying processes are LRD, the robustness properties are still good but deteriorate with increasing Hurst index.

Modern asymptotic theory has been characterized by attempts at relaxing the "independent" part of the assumption of an i.i.d. underlying population. As was noted previously, the independence is not a necessary condition for a great number of limit theorems. However, as dependence structures of the underlying population become stronger new phenomena arise.

The aim of Chapter 3 is to investigate some of the most important limit theorems when the underlying processes are subjected to LRD. The partial sums, the sample covariance functions and, more generally, some quadratic forms are the issues of interest in Chapter 3. Here we assume the processes under consideration are generalized linear processes of the form:

$$Y_t = \sum_{\nu=-\infty}^{\infty} b_{t-\nu}\xi_\nu .$$

The law of large numbers for the partial sums from $\{ Y_t \}$ is given in Chapter 3 in the case where $\{ Y_t \}$ is LRD.

There are theorems (see, for example, Ibragimov and Linnik, 1971) showing that the ordinary central limit theorem (CLT) holds for the partial sums from $\{ Y_t \}$ when the weights $\{ b_\nu \}$ decay to zero fast enough and the innovation process $\{ \xi_\nu \}$ is i.i.d. or a martingale difference sequence. In Chapter 3 we give more general conditions on the weights and the innovation under which the CLT holds for $\{ Y_t \}$ being LRD.

Many statistical inferences are based on quadratic forms of sample data, such as the sample covariance function

$$\hat{\gamma}_T(t) = T^{-1} \sum_{s=1}^{T-t} (Y_s - EY_s)(Y_{t+s} - EY_{t+s}) ,$$

where $Y_1 , Y_2 , \ldots , Y_T$ is a sample of size T. It has been shown that a strong law of large number holds for $\hat{\gamma}_T(t)$. See, for example, Hannan (1970), Hannan and Heyde (1972). It is of great importance for the $\hat{\gamma}_T(t)$'s to converge to their true values uniformly in t. Theorems relating the uniform convergence of $\hat{\gamma}_T(t)$ have been considered in the cases where $\{ Y_t \}$ has SRD. In Chapter 3, we show that under very general conditions the convergence of $\hat{\gamma}_T(t)$ to the true value $\gamma(t)$, uniformly in t, holds for $\{ Y_t \}$ being LRD.

In reality we have cases where the processes concerned are perturbed from stationarity. Often a non-stationary series $\{ X_t \}$ can be distinguished as a series having a trend in the mean:

$$X_t = Y_t + m_t ,$$

where $\{ Y_t \}$ is a stationary series with vanishing mean and $\{ m_t \}$ is a deterministic trend. In Chapter 3, we develop methodology for hypothesis testing of the mean m_t. We solve the problem in the case where $\{ Y_t \}$ may exhibit LRD.

In Chapters 4 and 5 of the thesis we are concerned with the applications of long range dependent processes to financial markets. In mathematical finance, stochastic differential equations driven by Brownian motion are a powerful tool used to describe the movement of stock prices. This is because Brownian motion is a Gaussian process with the martingale property, or principle of fair games of chance, upon which the traditional capital market theory has been based. Brownian motion has independent increments and is therefore, a SRD process. However, in recent years it has become increasingly obvious that LRD time series are widespread in financial data. There does not seem to exist any model that accurately explains the dynamics of the stock price movement in financial mathematics. It is of practical and theoretical importance to take into account LRD in the research of the fluctuating behaviour of financial markets.

The aim of Chapters 4 and 5 is to propose and study a stochastic model — the fractional Black-Scholes model which includes the famous Black-Scholes model as a special case and is able to account for LRD in the stock price movement. A fractional Black-Scholes model is a stochastic differential equation driven by FBM. Because FBM is not a semimartingale, in order to define stochastic differential equations driven by B_H, it is necessary to define the stochastic integral with respect to B_H first. In Chapter 4 we achieve this goal.

As soon as we have defined stochastic integration with respect to B_H, we derive the Itô formula for B_H, which is a very powerful tool for stochastic analysis of B_H.

In the final part of Chapter 5 we prove the existence and uniqueness of the solution of fractional Black-Scholes equation. An unrealized goal is to develop an option pricing formula based on the fractional Black-Scholes model.

Chapter 2

THE ROBUSTNESS TO SMALL TRENDS OF ESTIMATORS WITH POSSIBLE LONG RANGE DEPENDENCE

2.1 Introduction

Asymptotic normality is ubiquitous in probability and statistics to the extent that it has assumed the role of expected behaviour, alternatives to which have almost curiosity value. However, many of the key asymptotic normality theorems have not been subjected to systematic investigation of their sensitivity to perturbations of the sufficient conditions under which they hold. In some cases they may be very vulnerable to departures from standard conditions such as those of stationarity or independence etc. However, "minor" departures can be expected in many applications. For example, many time series observed over long periods may be perturbed from stationarity as a result of substantial events whose effect diminishes over an extended period before eventually disappearing. The examples of such events we have in mind are the stock market crash of October 1987 and its influence on various economic time series or the eruption of Mt. Pinatubo in the Philippines in June 1991 and its effect on some environmental series, such as ozone levels. Thus, what constitutes "minor" and whether such departures matter are issues of concern.

In this chapter we assume that $\{X_t, t = 1, 2, \dots\}$ is a time series of the form

$$X_t = Y_t + h_t \tag{2.1}$$

where $\{Y_t\}$ is a (strictly) stationary time series with mean μ, (in some circumstances, without loss of generality, we assume $\mu = 0$ for convenience), and spectral density $f(\omega, \theta)$, where the parameter $\theta \in \Theta \subseteq \mathbf{R}^p$, $p \geq 1$, Θ being open and the Hurst index H being one of the components of the vector parameter θ. The $\{h_t\}$ in (2.1) is a deterministic trend. We shall be concerned with circumstances where any major trend, such as would be observed by plotting the sample mean against the sample size, has been removed and all that remains are minor effects for which $T^{-1}\sum_{t=1}^{T} h_t \to 0$, as $T \to \infty$.

The process $\{Y_t\}$ is what we believe (or hope) is being observed but in reality the data are in the contaminated form $\{X_t\}$. Standard inferences, however, are performed on the basis of the model for $\{Y_t\}$ and in this chapter we wish to address the issue of whether the conclusions are robust against this kind of departure from the model.

We shall be particularly concerned with this question for models which may be subject to LRD. It is increasingly obvious that long range dependent phenomena are widespread in nature and that the stationary noise in such circumstances is characterized by apparent trends and cycles which may persist over substantial periods. This makes the issue of whether a deterministic trend is or is not present quite difficult to determine and it is clearly important to have methods which are unaffected by small trends if possible.

For time series in which LRD may need to be taken into account, there are approaches to analysis in both the time domain and the frequency domain. The classical time domain approach is via R/S (rescaled adjusted range) analysis which was introduced by Hurst and popularized by Mandelbrot. However, this approach, as described, for example, in Mandelbrot and Taqqu (1979), is rather heuristic and provides no precise consistency results for H and no confidence statement for H. Furthermore, it has been shown by Bhattacharya, Gupta and Waymire (1983) that if, in (2.1), $\{Y_t\}$ is a SRD series and the trend h_t satisfies $h_t = C(M+t)^{-\delta}$, $C > 0$, $0 < \delta < \frac{1}{2}$, then an R/S analysis produces an apparent Hurst index of $H = 1-\delta$. This effect throws into question the considerable work that has been done using R/S analysis on time series in finance (see, for example Peters, 1994), where external events, such as political decisions, not infrequently cause departures from stationarity.

In this chapter we concentrate on the frequency domain methodology. The main focus of this chapter is on the robustness of some statistical procedures that are conducted on the basis that no contamination is believed present. We will discuss the robustness of a class of estimators based on smoothed periodograms in Section 2.2. It is shown that a smoothed periodogram approach to model fitting and parameter estimation is highly robust to the presence of a small trend if the underlying stationary process is SRD. If the underlying process is LRD, the robustness properties are still good but now depend on the Hurst index of the process and deteriorates with increasing Hurst index.

The proof of the main results will appear in Section 2.4.

As an application of the method we used to prove the main result of this chapter, Theorem 2.1, we investigated the robustness of some estimators of the Hurst index H. The relevant results are the contents of Section 2.3. In Section 2.3, we are concerned with the case where the spectral density of $\{Y_t\}$ satisfies

CONDITION A. For some $H \in (0, 1)$, as $\omega \to 0_+$

$$f(\omega, \theta) \sim L_\theta(1/\omega)\,\omega^{-\alpha(\theta)}, \tag{2.2}$$

where $0 < \alpha < 1$ and the Hurst index H is one of the components of the parameter vector θ. Here $L_\theta(\omega)$ is a slowly varying function at infinity. That is, $L_\theta(\omega)$ is a positive, measurable function satisfying, as $\omega \to \infty$, for all $t > 0$,

$$\lim_{\omega \to \infty} \frac{L_\theta(t\omega)}{L_\theta(\omega)} = 1 \ .$$

(See, for example, Feller, 1966, Volume II.)

Condition A asserts that $f(\cdot\,,\,\theta)$ is regularly varying at $\omega = 0_+$ and is unbounded at $\omega = 0$. Apart from being integrable (due to covariance stationarity) $f(\cdot\,,\,\theta)$ is not even required to satisfy any smoothness assumptions and need not to be in L_p for any $p > 1$. (It would be in L_p, $p < 1/(2H-1)$ if $f(\cdot\,,\,\theta)$ were smooth away from $\omega = 0$.)

Two examples of time series exhibiting property (2.2) are the fractional Gaussian noise and fractional ARIMA models given by (1.2) and (1.4).

In Section 2.3 we will discuss two estimators for the Hurst index H, $\hat{H}$ and $\hat{H}_{m,q}$, given by Robinson (1994a, 1994b). We shall show that under some conditions, $\hat{H}$ and $\hat{H}_{m,q}$ are robust against certain small trends.

2.2 Robustness to Small Trends of Estimators Based on Smoothed Periodograms

In this section we are concerned with the robustness of inferences, carried out on a time series, LRD or SRD, contaminated by a small trend, with respect to departures from stationarity. We shall concentrate on a frequency domain approach to the analysis of time series using smoothed periodogram analysis.

Let $\{Y_t\}$ be a stationary time series with unknown parameter θ, where $\theta \in \Theta \subseteq \mathbf{R}^p$, $p \geq 1$, Θ being open and the Hurst index H may be one of the components of θ. A smoothed periodogram of $\{Y_t\}$ is defined as

$$G_{T,Y}(\theta) = \int_{-\pi}^{\pi} g(\omega, \theta) \{I_{T,Y}(\omega) - E\, I_{T,Y}(\omega)\}\, d\omega \ ,$$

where

$$I_{T,Y}(\omega) = \left| \sum_{t=1}^{T} Y_t \exp(-it\omega) \right|^2 \Big/ (2\pi T) \tag{2.3}$$

is the periodogram of $\{Y_t\}$, $g(\omega, \theta)$ is in a suitably restricted class of smoothing functions $\mathcal{G}$. For details of the restriction to $\mathcal{G}$, see, for example, Heyde and Gay (1993).

Smoothed periodogram analysis is considerably more informative than the direct use of the periodogram in discriminating between the SRD with monotonic trends and the LRD, as has been advocated by Künsch (1986). Parameter estimation based on smoothed periodogram analysis has a long history. The idea derives from suggestive forms of Whittle's estimation procedure, which has formed the backbone of asymptotic estimation since its discovery (Whittle, 1951). The general methodology has been developed by many statisticians, including Whittle (1951, 1952, 1953, 1954), Hannan (1970, 1973), Dunsmuir and Hannan (1976), Brillinger (1975), Dzhaparidze (1970, 1971, 1986), Kabaila (1980, 1983), Kulpberger (1985), Fox and Taqqu (1986), Rosenblatt (1985), (see Heyde and Gay, 1993), and amongst others.

Smoothed periodograms provide a class of estimating functions. Heyde and Gay (1993) derived the asymptotic normality of $G_{T,Y}$ without Gaussian assumptions on the underlying processes. Therefore, the asymptotic quasi-likelihood approach can be used to give the estimate of θ and establish relative optimality results. (See, for example, Heyde, 1995, Chapter 5.) The procedure of parameter estimation, for a sample $\{Y_t : t = 1, 2, \ldots, T\}$, is to choose an asymptotic quasi-score estimating function as

$$G^*_{T,Y}(\theta) = \int_{-\pi}^{\pi} -\frac{\partial f^{-1}}{\partial \theta}(\omega, \theta) \{ I_{T,Y}(\omega) - E\, I_{T,Y}(\omega) \}\, d\omega \ , \tag{2.4}$$

where $f(\omega, \theta)$ is the spectral density, and an estimator for θ is obtained by solving the equation

$$G^*_{T,Y}(\theta) = 0 .$$

Strong consistency and asymptotic normality results are available for the estimation of θ. These come from an asymptotic study of the smoothed periodogram form (2.4) obtained by minimizing

$$\int_{-\pi}^{\pi} \{ I_{T,Y}(\omega)/f(\omega, \theta) + \log f(\omega, \theta) \} \, d\omega \tag{2.5}$$

through differentiating under the integral sign in (2.5).

This procedure encompasses many random processes or random fields with either SRD or LRD. A well developed inferential theory is now available for this approach and, as we shall show, it is relatively robust against replacing $\{ Y_t \}$ by $\{ X_t \}$ of (2.1).

In our analysis we shall use a general even smoothing function $g(\omega, \theta)$ rather than the specific form $-(\partial/\partial\theta) f^{-1}(\omega, \theta)$ which is suggested by the Whittle procedure and also comes from an asymptotic quasi-score estimating function approach (see Heyde and Gay, 1989) and hence is associated with minimum size asymptotic confidence zones for θ . The general estimator of θ is obtained as a solution of the estimating equation

$$\int_{-\infty}^{\infty} g(\omega, \theta)(I_{T,Y}(\omega) - f(\omega, \theta)) d\omega = 0 \ .$$

Our results focus on smoothed periodogram convergence results of the form

$$T^{\frac{1}{2}} \int_{-\pi}^{\pi} g(\omega, \theta) \ (I_{T,X}(\omega) - f(\omega, \theta)) \ d\omega \xrightarrow{\mathrm{d}} \mathbf{MVN}(0 \, , \, V) \, , \tag{2.6}$$

where $I_{T,X}(\omega)$ is the periodogram of $\{ X_t \}$, **MVN** denotes the multivariate normal law with mean vector zero and covariance matrix V and $\xrightarrow{\mathrm{d}}$ denotes the convergence in distribution. The usual Taylor expansion methods based on (2.6) lead to estimators of θ which are taken to encapsulate both model dimension and its parameter values. We shall not pursue this matter here, however, as the trends $\{h_t\}$ are not directly involved in this analysis. See, for example, Rosenblatt (1985) for a discussion of the Taylor series expansion asymptotics and Gay and Heyde (1990) for discussion of a flexible family of spectral densities $f(\omega, \theta)$ which could be used in this setting. For example, if we take the minimum asymptotic variance choice of $-(\partial/\partial\theta)f^{-1}(\omega, \theta)$ for our $g(\omega, \theta)$, results from Rosenblatt (1985) such as Lemma 4, p.107 which gives consistency and Theorem 3, p.108 which gives asymptotic normality, continue to hold without change for estimators calculated on the basis of the contaminated periodogram $I_{T,X}(\omega)$ rather that $I_{T,Y}(\omega)$ under the conditions of the following Theorem 2.1

In this section, we show that a trend of the form $|h_t| \leq Ct^{-\beta}$, $t = 1, 2, \ldots$, where $C > 0$ and $\beta > 0$ are constants, does not affect the analysis in the case of a SRD* process $\{Y_t\}$ (where the star denotes absolute convergence of the lagged covariance sum $\sum_{k=0}^{\infty} |\gamma_k| < \infty$, rather than the convergence of $\sum_{k=0}^{\infty} \gamma_k$ of the SRD definition).

In the case of LRD the corresponding robustness properties are still good but not quite so striking. Under some minor regularity conditions we have that a trend of the form $|h_t| \leq Ct^{-\beta}$, $C > 0$, $\beta > 0$, $t = 1, 2, \ldots$ does not affect the analysis provided $\beta > \min(1/4,\ H - 1/2)$. (Note that the SRD* result corresponds to the case $H = \frac{1}{2}$.)

THEOREM 2.1. *Suppose that $\{Y_t\}$ is a stationary mean zero process and $g(\omega)(= g(\omega,\ \theta))$ is a real, even, integrable function for $\omega \in [-\pi, \pi]$. Let $\{X_t\}$ be given by (2.1). Write*

$$g_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\omega)\, e^{-it\omega}\, d\omega\ , \tag{2.7}$$

$$\gamma(t) = E\, Y_s\, Y_{s+t}\ ,$$

$$I_{T,X}(\omega) = \frac{1}{2\pi} \left| \sum_{t=1}^{T} X_t e^{-it\omega} \right|^2 , \tag{2.8}$$

$$I_{T,Y}(\omega) = \frac{1}{2\pi} \left| \sum_{t=1}^{T} Y_t e^{-it\omega} \right|^2 . \tag{2.9}$$

We assume further that g_t satisfy

$$|\, g_t \,| = o(|t|^{-1}) \,, \text{ as } t \to \infty \,. \tag{2.10}$$

If the trend $\{\, h_t \,\}$ is such that

$$|\, h_t \,| \le C_1 t^{-\beta} \;, \; t = 1\,, 2\,, \ldots \tag{2.11}$$

for constant $C_1 > 0$, $\frac{1}{4} < \beta < 1$, then as $T \to \infty$,

$$E \left| T^{\frac{1}{2}} \int_{-\pi}^{\pi} g(\omega) \,(\, I_{T,X}(\omega) - I_{T,Y}(\omega) \,)\; d\omega \right|^2 \longrightarrow 0 \tag{2.12}$$

provided that

(A) $\{\, Y_t \,\}$ is short range dependent, or*

(B) $\{\, Y_t \,\}$ is long range dependent with covariance function

$$|\, \gamma_t \,| \le C_2 \,|\, t \,|^{-\alpha} \;, \; t = 1,\; 2, \ldots \tag{2.13}$$

for $C_2 > 0$, $0 < \alpha < 1$, where $\alpha + 2\beta > 1$ and the Fourier coefficients g_t in (2.7) satisfy

$$|\, g_t \,| \le C_3 \,|\, t \,|^{-\eta} \tag{2.14}$$

for some $\eta > \frac{3}{2} - \alpha > 1$.

Remark on Theorem 2.1. The parameter α in (2.13) is typically related to the Hurst index H through $\alpha = 2(1 - H)$. The discussion of Section 2.1 uses this formulation.

The proof of Theorem 2.1 will be given in Section 2.4.

THEOREM 2.2 *Suppose* $\{X_t\}$, $\{Y_t\}$, $\{h_t\}$ *and* $g(\omega,\theta)$ *are as in Theorem 2.1. Assume further that*

$$T^{\frac{1}{2}} \int_{-\pi}^{\pi} g(\omega,\theta)\,\left(I_{T,Y}(\omega) - f(\omega,\theta)\right)\,d\omega \xrightarrow{\mathrm{d}} \mathbf{MVN}(0,V) \tag{2.15}$$

as $T \to \infty$, *where* $f(\omega,\theta)$ *is the spectral density of* $\{Y_t\}$, *and* $\mathbf{MVN}(0,V)$ *is a multivariate normal vector with mean zero and covariance matrix* V. *Then under the conditions of Theorem 2.1,*

$$T^{\frac{1}{2}} \int_{-\pi}^{\pi} g(\omega,\theta)\,\left(I_{T,X}(\omega) - f(\omega,\theta)\right)\,d\omega \xrightarrow{\mathrm{d}} \mathbf{MVN}(0,V)\,. \tag{2.16}$$

Remark on Theorem 2.2. The convergence result (2.15) holds under broad circumstances of SRD or LRD. For detailed conditions see, for example, Giraitis and Surgailis (1990) and Heyde and Gay (1993). In particular the results apply to fractional Gaussian noise and fractional ARIMA processes in the case of the Whittle estimation procedure (i.e. $g(\omega,\theta) = (\partial/\partial\theta)f^{-1}(\omega,\theta)$). That the condition (2.14) is a consequence of standard LRD assumptions (Fox and Taqqu, 1986, Section 2; Giraitis and Surgailis, 1990, Section 1) can be seen from, for example, the considerations of Lemma 5 of Fox and Taqqu (1986). Typically $\eta = 2 - \alpha$. One sufficient condition for (2.14) is that g is absolutely continuous.

Proof of Theorem 2.2. Note that

$$\begin{aligned} T^{\frac{1}{2}}\int_{-\pi}^{\pi} g(\omega,\theta)\left(I_{T,X}(\omega) - f(\omega,\theta)\right)d\omega &= T^{\frac{1}{2}}\int_{-\pi}^{\pi} g(\omega,\theta)\,\left(I_{T,X}(\omega) - I_{T,Y}(\omega)\right)d\omega \\ &\quad + T^{\frac{1}{2}}\int_{-\pi}^{\pi} g(\omega,\theta)\,\left(I_{T,Y}(\omega) - f(\omega,\theta)\right)d\omega\,. \end{aligned}$$

Therefore Theorem 2.2 follows immediately from Theorem 2.1.

As an application of Theorem 2.1 and Theorem 2.2, we consider a case studied by Fox and Taqqu (1986).

In their 1986 paper, Fox and Taqqu investigated an approximate maximum likelihood procedure to estimate θ. They adapted the approach of Whittle (1953), introduced for weakly dependent random variables. Consider a quadratic form

$$S_{T,Y}^2(\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(\omega,\ \theta))^{-1}\, I_{T,Y}(\omega)d\omega\,. \tag{2.17}$$

We assume the compactness of the closure $\overline{\Theta}$, where Θ is the parameter set of θ. Let $\hat{\theta}_{T,Y}$ be a value of θ which minimizes $S_{T,Y}^2(\theta)$

$$\hat{\theta}_{T,Y} = \arg\ \min\left\{\int_{-\pi}^{\pi} (f(\omega,\ \theta))^{-1}\, I_{T,Y}(\omega)d\omega :\quad \theta \in \Theta\right\}. \tag{2.18}$$

When $\{\, Y_t \,\}$ is a stationary Gaussian sequence, Theorem 1 and Theorem 2 of Fox and Taqqu (1986) show that, with probability 1,

$$\lim_{T\to\infty} \hat{\theta}_{T,Y} = \theta_0$$

and $\hat{\theta}_{T,Y}$ is asymptotically normal with rate of convergence $T^{-1/2}$. Subsequently, Dahlhaus (1989) has proved that $\hat{\theta}_{T,Y}$ is not only asymptotically normal but also asymptotically efficient in the sense of Fisher. Dahlhaus (1989) also derived an exact maximum likelihood estimator and its consistency and asymptotic normality.

Now, if we put

$$\hat{\theta}_{T,X} = \arg\ \min\left\{\int_{-\pi}^{\pi} (\, f\,(\,\omega\,,\,\theta\,)\,)^{-1}\, I_{T,X}(\omega)\, d\omega\ :\ \theta \in \Theta\right\} \tag{2.19}$$

then, by our Theorem 2.1 and Theorem 2.2, we have $\hat{\theta}_{T,X} - \hat{\theta}_{T,Y} \longrightarrow 0$ as $T \to \infty$ and $\sqrt{T}(\hat{\theta}_{T,X} - \theta_0)$ is asymptotically normal. In fact we have the following more general results.

THEOREM 2.3 *Let $\{\, X_t \,\}$, $\{\, Y_t \,\}$ and $\{\, h_t \,\}$ be given by (2.1). The parameter θ of the spectral density $f(\omega\,,\,\theta)$ is in $\Theta \subseteq \mathbf{R}^p$, $p \geq 1$ and the closure $\overline{\Theta}$ is compact. We shall call θ_0 the true value of the parameter. For a smoothing function $g(\omega,\ \theta)$ let*

$$\Theta_0 = \left\{\, \theta \in \overline{\Theta}\ :\ \int_{-\pi}^{\pi} g(\omega\,,\,\theta)\, f(\omega\,,\,\theta_0)\, d\omega\ \ \textit{is not a constant}\ \right\}.$$

Assume $g(\omega\,,\,\theta)$, $\{\,Y_t\,\}$ *and* $\{\,h_t\,\}$ *satisfy the conditions in Theorem 2.2, and that*

$$\lim_{T\to\infty}\int_{-\pi}^{\pi} g(\omega\,,\,\theta)I_{T,Y}(\omega)d\omega = \int_{-\pi}^{\pi} g(\omega\,,\,\theta)f(\omega\,,\,\theta_0)d\omega \tag{2.20}$$

uniformly in $\theta\in\Theta$ *and* $\Theta_0\cap\Theta$ *is not empty. Write*

$$\hat{\theta}_{T,Y}(g) = \arg\min\left\{\int_{-\pi}^{\pi} g(\omega\,,\,\theta)I_{T,Y}(\omega)\,d\omega\;:\;\theta\in\Theta\right\}\,, \tag{2.21}$$

$$\hat{\theta}_{T,X}(g) = \arg\min\left\{\int_{-\pi}^{\pi} g(\omega\,,\,\theta)I_{T,X}(\omega)d\omega\;:\;\theta\in\Theta\right\}\,. \tag{2.22}$$

If

$$\hat{\theta}_{T,Y}(g) \xrightarrow{\mathrm{P}} \theta_0\,, \tag{2.23}$$

then

$$\hat{\theta}_{T,X}(g) \xrightarrow{\mathrm{P}} \theta_0\,. \tag{2.24}$$

Remark on Theorem 2.3. Condition (2.20) will be discussed for the case where $g\,(\,\omega\,,\,\theta\,)\;=\;f^{-1}(\,\omega\,,\,\theta\,)$ in Theorem 3.11 of Chapter 3. Condition (2.23) is a natural requirement for parameter estimation. Again, by the property of smoothed periodograms, the Gaussianity of the underlying process is not necessary here.

We shall give the proof of Theorem 2.3 in Section 2.4.

2.3 Robustness of Estimators of H

In this section, as an application of the method we use for deriving Theorem 2.1, we study the robustness of various estimators for the Hurst index H. We are concerned with the case where the underlying series satisfies

$$|\gamma_k| = |\,\mathrm{cov}(\,Y_t\,,\,Y_{t+k}\,)\,| \sim const\,|k|^{2H-2}\,,\ \text{as}\ |k|\to\infty\,.$$

and (2.2)

$$f(\omega,\ \theta) \sim L_\theta(1/\omega)|\omega|^{-\alpha(\theta)}\,,\ \text{as}\ \omega\to 0_+\,,$$

where $0 < \alpha(\theta) = 2H - 1 < 1$ and $L_\theta(\omega)$ varies slowly at infinity and H is one of the components of the vector parameter θ.

The reason for this is that it has been noticed recently (Robinson, 1994b) that when the parametric form of $f(\omega, \theta)$ is misspecified, some highly desirable properties, such as $T^{1/2}$-consistency and asymptotic normality, of estimators will, in general, fail. In particular, misspecification of $f(\omega, \theta)$ at high frequencies can lead to an inconsistent estimate of H, which characterizes low frequency behaviour (Robinson, 1994a).

To overcome such criticisms attention has recently been given to stationary processes whose spectral densities satisfy (2.2). For some purposes, $L_\theta(\omega)$ in (2.2) can be of unknown form, for others, we require

$$L_\theta(\omega) = GM(\omega) \tag{2.25}$$

where $G > 0$ is one component of θ and $M(\omega)$ is a known function. Because (2.2) and (2.25) make no parametric assumption about f outside a neighbourhood of the origin, the models satisfying (2.2) or (2.25) are called semiparametric models.

Semiparametric estimators of H in (2.2) or (2.25) have been proposed which can be justified as consistent in the absence of full parametric or other global assumptions on $f(\omega, \theta)$. For example Geweke and Porter-Hudak (1983) have defined a semiparametric estimate for H. Robinson (1992) has established desirable asymptotic properties of modified and more efficient versions of this estimate.

Robinson (1994a) suggested an estimator for H, which, unlike the estimators of Geweke and Porter-Hudak (1983), is not defined in closed form, but dominates these estimators in several respects. It is asymptotically more efficient. Much weaker assumptions than Gaussianity are imposed. The estimator had been suggested earlier by Künsch (1987), and can be described as follows:

Consider the objective function (see Künsch, 1987)

$$Q_{T,Y}(G,H) = \frac{1}{m}\sum_{j=1}^{m}\left\{ \log\left(G\,\omega_j^{2H-1}\right) + \frac{\omega_j^{2H-1}}{G} I_{T,Y}(\omega_j\,) \right\}\,, \tag{2.26}$$

where $I_{T,Y}$ is defined by (2.3), G is given by (2.25) and

$$\omega_j = \frac{2\pi j}{T}\,,\; j = 1\,,\,2\,,\,\ldots\,,\,m\,, \tag{2.27}$$

where $m = m(T)$ will be described shortly. The estimator $(\hat{G}_{T,Y}\;,\;\hat{H}_{T,Y})$ is defined by minimizing (2.26)

$$(\hat{G}_{T,Y}\;,\;\hat{H}_{T,Y}) = \arg\min\{\,Q_{T,Y}(G\,,\,H)\;:\;(G\,,\,H)\in\Theta\,\}\;, \tag{2.28}$$

where $\Theta = (\,0\,,\,\infty\,)\times[\,\delta_1\,,\,\delta_2\,]$, δ_1 and δ_2 are numbers chosen such that $0<\delta_1<\delta_2<1$. We can choose δ_1 and δ_2 arbitrarily close to 0 and 1 respectively, or we can choose them to reflect weak knowledge about H_0, for example, $\delta_1 = \frac{1}{2}$ if we are confident that $f(\omega,\;\theta)\not\to 0$ as $\omega\to 0$. We shall show in Theorem 2.4 that $(\hat{G}_{T,Y}\;,\hat{H}_{T,Y})$ is robust against a small trend $\{\,h_t\,\}$ satisfying (2.11) for some $0<\beta<1$.

THEOREM 2.4 *Let $\{\,X_t\,\}$, $\{\,Y_t\,\}$ and $\{\,h_t\,\}$ be given by (2.1) and the spectral density $f\,(\,\omega\,,\,\theta\,)$ satisfy (2.25). Assume further that*

$$|\,h_t\,| \le const\,|\,t\,|^{-\beta}\;,\;\;\beta + H > \frac{3}{2}\,. \tag{2.29}$$

Write

$$Q_{T,X}(G\,,\,H) = \frac{1}{m}\sum_{j=1}^{m}\left\{ \log\left(G\omega_j\,\right) + \frac{\omega_j^{2H-1}}{G} I_{T,X}(\omega_j) \right\}\,, \tag{2.30}$$

$$Q_{T,Y}(G\,,\,H) = \frac{1}{m}\sum_{j=1}^{m}\left\{ \log(G\omega_j) + \frac{\omega_j^{2H-1}}{G} I_{T,Y}(\omega_j) \right\}\,, \tag{2.31}$$

where $m = [\,\log T\,]$, $[\cdot]$ denote integer part. Then,

$$E\left|T^{\frac{1}{2}}\left(Q_{T,X}(G\,,\,H) - Q_{T,Y}(G,\;\,H)\right)\right|^2 \longrightarrow 0\,,\; as\; T\to\infty\,. \tag{2.32}$$

Remark on Theorem 2.4. If we use $m(T) = T^{\tau}$ to replace $m(T) = [\log T]$ in (2.30) and (2.31) then for suitably chosen τ, (2.32) still holds.

The proof of Theorem 2.4 will be given in Section 2.4.

Another statistic to deal with inference on H in the case of (2.2) or (2.25) is based on the discretely averaged periodogram, where the averaging is done over a neighbourhood of the origin which slowly degenerates to zero as the sample size T increases (see Robinson, 1994b). A suitable normalized version of the averaged periodogram is well known to converge in probability to the spectrum at the origin under weak dependence conditions (see, for example, Brillinger, 1975, Chapter 6). The procedure for estimating H from a sample $\{Y_t : \; t = 1, 2, \ldots, T\}$ is as follows:

Write

$$F(\lambda) = \int_0^{\lambda} f(\omega, \theta)\, d\omega \,, \tag{2.33}$$

and

$$\hat{F}_{T,Y}(\lambda) = \frac{2\pi}{T} \sum_{j=1}^{[T\lambda/(2\pi)]} I_{T,Y}(\omega_j) \,. \tag{2.34}$$

Then $\hat{F}_{T,Y}(\lambda)$ is the discretely averaged periodogram and is a special case of the general class of weighted periodogram spectrum estimates. That is, under some conditions,

$$\frac{\hat{F}_{T,Y}(\lambda_m)}{F(\lambda_m)} \xrightarrow{\text{P}} 1 \,, \;\; \text{as } T \to \infty \,,$$

where $\lambda_m = \frac{2\pi m}{T}$, $\xrightarrow{\text{P}}$ denotes the convergence in probability and m satisfies

CONDITION B. $$\frac{1}{m} + \frac{m}{T} \longrightarrow 0 \,, \;\; \text{as } T \to \infty \,.$$

The estimator of H suggested by Robinson (1994b) is

$$\hat{H}_{m,q} = 1 - \frac{\log\{\hat{F}_{T,Y}(q\lambda_m)/\hat{F}_{T,Y}(\lambda_m)\}}{2\log q} \tag{2.35}$$

for any $q > 0$, where m satisfies Condition B. Because $\hat{H}_{m,q} = \hat{H}_{mq,1/q}$, we can restrict q to the interval $(0, 1)$. $\hat{H}_{m,q}$ nearly always lies in the stationary region

$(-\infty, 1)$; it cannot exceed 1 and it equals 1 only if $\hat{F}_{T,Y}(q\lambda_m) = \hat{F}_{T,Y}(\lambda_m)$. $\hat{H}_{m,q}$ is scale and location invariant due to the ratio in the numerator of (2.35) and the invariance of $\hat{F}$ to location. We shall show in Theorem 2.6 that $\hat{H}_{m,q}$ is robust against the small trend $\{h_t\}$.

Theorem 2.6 is based on Robinson's Theorem 1 (Robinson, 1994b). We give Robinson's result and relevant conditions first.

CONDITION A. For some $H \in (0, 1)$, as $\omega \to 0_+$,

$$f(\omega) \sim L(1/\omega)\, \omega^{1-2H} ,$$
$$|\gamma(t)| \sim const\, |t|^{2H-2} .$$

CONDITION B. $$\frac{1}{m} + \frac{m}{T} \longrightarrow 0 , \text{ as } T \to \infty .$$

CONDITION B′. $$m(T) = [\log T] .$$

CONDITION C. Y_t has the representation

$$Y_t = \mu + \sum_{j=0}^{\infty} a_j \varepsilon_{t-j} , \qquad \sum_{j=0}^{\infty} a_j^2 < \infty , \tag{2.36}$$

where

C1. $E(\varepsilon_t \varepsilon_s) = 0 , \ t \neq s$;

C2. $E(\varepsilon_r \varepsilon_s \varepsilon_t \varepsilon_u) = \begin{cases} \sigma^4 , \text{if } r = s > t = u , \\ 0 , \ \text{if } r = s > t > u , \text{ or } r > s = t > u , \text{ or } r > s > t \geq u ; \end{cases}$

C3. there exists a nonnegative random variable ε such that, for all $\eta > 0$ and some $K > 0$,

$$E(\varepsilon^2) < \infty , \quad P(|\varepsilon_t| > \eta) \leq K\, P(\varepsilon > \eta) ;$$

C4. $\frac{1}{T} \sum_{t=1}^{T} E(\varepsilon_t^2 \,|\, \varepsilon_s^2, s < t) \xrightarrow{\text{P}} \sigma^2$, as $T \to \infty$.

Condition C is satisfied if $\{\varepsilon_t\}$ is an i.i.d. sequence with finite variance, or, more generally, if the ε_t forms a stationary square integrable martingale difference sequence.

THEOREM 2.5 *(Robinson, 1994b) Let Conditions A, B and C hold. Then,*

$$\frac{\hat{F}_{T,Y}(\lambda_m)}{F(\lambda_m)} \xrightarrow{P} 1 , \quad as\ T \to \infty . \tag{2.37}$$

Our main result on the robustness of $\hat{H}_{m,q}$ is the following theorem:

THEOREM 2.6 *Let $\{X_t\}$, $\{Y_t\}$ and $\{h_t\}$ be given by (2.1). Assume the spectral density $f(\omega, \theta)$ satisfies (2.25), where $M(\omega)$ is bounded in a neighbourhood of zero, and $\{h_t\}$ satisfies*

$$|h_t| \le const\ |t|^{-\beta} , \quad \beta + H > 1 . \tag{2.38}$$

Write

$$\hat{F}_{T,X}(\lambda) = \frac{2\pi}{T} \sum_{j=1}^{[T\lambda/(2\pi)]} I_{T,X}(\omega_j) \tag{2.39}$$

and $\hat{F}_{T,Y}(\lambda)$ is defined by (2.34). Then, under Conditions A, B′ and C, for any $q \in (0\,,\,1)$ we have

$$\frac{\hat{F}_{T,X}(\lambda_m)}{\hat{F}_{T,Y}(\lambda_m)} \xrightarrow{P} 1 \tag{2.40}$$

and

$$\frac{\hat{F}_{T,X}(q\lambda_m)}{\hat{F}_{T,Y}(q\lambda_m)} \xrightarrow{P} 1 . \tag{2.41}$$

Remark on Theorem 2.6. We can change $m(T)$ in Condition B′ to Condition B″:

CONDITION B″.

$$m = T^{\tau}$$

with

$$\tau = \begin{cases} \begin{cases} > 1 + \frac{2\beta-1}{2H-1} , & H+\beta < 1 \\ > 1 + \frac{H-\beta}{1-2H} , & H+\beta > 1 \end{cases} , & 0 < H < \frac{1}{2} ; \\ \begin{cases} < 1 + \frac{2\beta-1}{2H-1} , & H+\beta < 1 \\ < 1 + \frac{H-\beta}{1-2H} , & H+\beta > 1 \end{cases} , & \frac{1}{2} < H < 1 . \end{cases}$$

H	$H+\beta<1$	$H+\beta>1$
$(0,\ \frac{1}{2})$	$\tau\ >1+\frac{1-2\beta}{1-2H}$	$\tau>1-\frac{\beta-H}{1-2H}$
$(\frac{1}{2},\ 1)$	$\tau<1+\frac{1-2\beta}{1-2H}$	$\tau<1-\frac{\beta-H}{1-2H}$

Condition B$''$ can be put in the following table:

Due to Theorem 2.4 and Theorem 2.6 we get robustness of $(\hat{G}_{T,Y}\ ,\ \hat{H}_{T,Y})$ and $\hat{H}_{m,q}$ as follows:

COROLLARY 2.7 *Assume that $\{X_t\}$, $\{Y_t\}$ and $\{h_t\}$ satisfy the conditions of Theorem 2.4. Let $(\hat{G}_{T,X}\ ,\ \hat{H}_{T,X})$ be defined by (2.28) with $\{X_t\}$ replacing $\{Y_t\}$. If, as $T\to\infty$,*

$$(\hat{G}_{T,Y}\ ,\ \hat{H}_{T,Y}) \xrightarrow{\mathrm{P}} (G_0\ ,\ H_0)\ , \tag{2.42}$$

then,

$$(\hat{G}_{T,X}\ ,\ \hat{H}_{T,X}) \xrightarrow{\mathrm{P}} (G_0\ ,\ H_0)\ . \tag{2.43}$$

For the detailed conditions under which (2.42) holds see Robinson (1994a). We shall give the proof of Corollary 2.7 in Section 2.4.

COROLLARY 2.8 *Assume that $\{X_t\}$, $\{Y_t\}$ and $\{h_t\}$ satisfy the conditions of Theorem 2.6. Then under conditions of A, B$'$ and C, or, A, B$''$ and C we have*

$$\left|\,\hat{H}_{m,q}(X)-\hat{H}_{m,q}(Y)\right| \xrightarrow{\mathrm{P}} 0\ ,\ \textit{as } T\to\infty\ . \tag{2.44}$$

The proof of Corollary 2.8 follows immediately from Theorem 2.6.

2.4 Proofs

2.4.1 Proof of Theorem 2.1

We shall establish Theorem 2.1 by showing that as $N \to \infty$,

$$E\left|T^{\frac{1}{2}} \int_{-\pi}^{\pi} g(\omega,\theta)\left(I_{T,X}(\omega) - I_{T,Y}(\omega)\right) d\omega\right|^2 \longrightarrow 0\,. \tag{2.45}$$

We have

$$I_{T,X}(\omega) - I_{T,Y}(\omega) = \frac{1}{2\pi T}\left\{\left|\sum_{t=1}^{T} X_t e^{-it\omega}\right|^2 - \left|\sum_{t=1}^{T} Y_t e^{-it\omega}\right|^2\right\}$$

$$= \frac{1}{2\pi T}\left\{\left|\sum_{t=1}^{T}(Y_t + h_t)e^{-it\omega}\right|^2 - \left|\sum_{t=1}^{T} Y_t e^{-it\omega}\right|^2\right\}$$

$$= \frac{1}{2\pi T}\sum_{s=1}^{T}\sum_{t=1}^{T}(Y_t h_s + Y_s h_t + h_s h_t)e^{-i(t-s)\omega}\,,$$

and

$$\int_{-\pi}^{\pi} g(\omega,\theta)\left(\, I_{T,X}(\omega) - I_{T,Y}(\omega)\,\right) d\omega = \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}(Y_t h_s + Y_s h_t + h_s h_t) g_{t-s}$$

the g_t's being the Fourier coefficients of $g(\omega,\theta)$.

In order to complete the proof of (2.45), it is sufficient to show that as $N \to \infty$,

$$E\left|T^{-\frac{1}{2}}\sum_{s=1}^{T}\sum_{t=1}^{T} Y_t h_s g_{t-s}\right|^2 \longrightarrow 0\,, \tag{2.46}$$

and

$$T^{-\frac{1}{2}}\sum_{s=1}^{T}\sum_{t=1}^{T} h_s h_t g_{t-s} \longrightarrow 0\,. \tag{2.47}$$

To deal with (2.46) we write

$$\sum_{s=1}^{T}\sum_{t=1}^{T} Y_t h_s g_{t-s} = \sum_{t=1}^{T} Y_t h_t g_0 + \sum_{t=2}^{T}\sum_{s=1}^{t-1} Y_t h_s g_{t-s} + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T} Y_t h_s g_{t-s}\,.$$

Using the inequality $|a+b|^2 \le 2(|a|^2+|b|^2)$, we have

$$E\left(T^{-\frac{1}{2}}\sum_{s=1}^{T}\sum_{t=1}^{T} Y_t h_s g_{t-s}\right)^2 \le \frac{1}{T}\left(2\,g_0^2 S_1 + 4(S_2+S_3)\right) \tag{2.48}$$

where

$$S_1 = E\left(\sum_{t=1}^{T} Y_t h_t\right)^2, \quad S_2 = E\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1} Y_t h_s\, g_{t-s}\right)^2, \quad S_3 = E\left(\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} Y_t h_s\, g_{t-s}\right)^2$$

and (2.46) holds if $T^{-1}S_i \to 0$ as $T \to \infty$, $i = 1, 2, 3$.

Now

$$S_1 = \sum_{s=1}^{T}\sum_{t=1}^{T} E(Y_s Y_t) h_s h_t = \gamma_0 \sum_{t=1}^{T} h_t^2 + 2\sum_{t=2}^{T}\sum_{s=1}^{t-1} \gamma_{t-s} h_s h_t$$

and $T^{-1}\sum_{t=1}^{T} h_t^2 \to 0$ as $T \to \infty$ since $\beta > 0$. In the case of SRD*,

$$T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1} |\,\gamma_{t-s} h_s h_t\,| \le T^{-1}\sum_{t=1}^{T} |\,h_t\,|\,(2\,C_1 \sum_{s=1}^{\infty} |\,\gamma_s\,|) \longrightarrow 0,$$

while if Condition (B) is satisfied,

$$T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1} |\,\gamma_{t-s} h_s h_t\,| \le C_1^2 C_2 T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1} (t-s)^{-\alpha} t^{-\beta} s^{-\beta}$$

and as $T \to \infty$,

$$\sum_{s=1}^{t-1} (t-s)^{-\alpha} s^{-\beta} \sim \int_1^{t-1} (t-s)^{-\alpha} s^{-\beta} ds \sim t^{1-\alpha-\beta} B(1-\alpha, 1-\beta)\ , \tag{2.49}$$

(B denoting the beta function) so that, since $\alpha + 2\beta > 1$,

$$\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{t-1} |\,\gamma_{t-s} h_s h_t\,| = 0$$

also holds and thus $T^{-1}S_1 \to 0$ as $T \to \infty$.

Next, writing

$$c_t = \sum_{s=1}^{t-1} h_s g_{t-s}\ , \tag{2.50}$$

we have

$$S_2 = E\left(\sum_{t=2}^{T} Y_t c_t\right)^2 = \sum_{t=2}^{T}\sum_{s=2}^{T} \gamma_{t-s} c_s c_t$$

and, in the rest of this section, using C to denote a constant which may be different in different equations,

$$|\, c_t \,| \leq \sum_{s=1}^{t-1} |\, h_s\, g_{t-s} \,| \leq C \sum_{s=1}^{t-1} s^{-\beta}(t-s)^{-1} \leq C \sum_{s=1}^{t-1} s^{-\beta}(t-s)^{-\delta} \tag{2.51}$$

for any $0 < \delta < 1$ so chosen that $\beta + \delta > 1$, $2\beta + \alpha + 2\delta > 3$, and, as in (2.49),

$$\sum_{s=1}^{t-1} s^{-\beta}(t-s)^{-\delta} \sim t^{1-\delta-\beta} B(1-\delta, 1-\beta) \tag{2.52}$$

as $t \to \infty$. Thus, if

$$T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} |\, \gamma_{t-s} \,| s^{1-\delta-\beta}\, t^{1-\delta-\beta} \longrightarrow 0 \,, \tag{2.53}$$

we have $T^{-1} S_2 \to 0$ as $T \to \infty$.

In the case of SRD*, (2.53) holds since $\beta + \delta > 1$ and

$$\sum_{s=1}^{T} |\, \gamma_{t-s} \,| s^{1-\delta-\beta} \leq 2 \sum_{s=0}^{\infty} |\, \gamma_s \,| < \infty.$$

On the other hand, if (B) is satisfied we use

$$\sum_{s=1}^{T} \sum_{t=1}^{T} |\, \gamma_{t-s} \,| s^{1-\delta-\beta} t^{1-\delta-\beta} = \gamma_0 \sum_{t=1}^{T} t^{2(1-\delta-\beta)} + 2 \sum_{t=2}^{T} \sum_{s=1}^{t-1} |\, \gamma_{t-s} \,| t^{1-\delta-\beta} s^{1-\delta-\beta}$$

and $T^{-1} \sum_{t=1}^{T} t^{2(1-\delta-\beta)} \to 0$ while

$$\begin{aligned} T^{-1} \sum_{t=2}^{T} \sum_{s=1}^{t-1} |\, \gamma_{t-s} \,| t^{1-\delta-\beta} s^{1-\delta-\beta} &\leq C_2 T^{-1} \sum_{t=2}^{T} t^{1-\delta-\beta} \sum_{s=1}^{t-1} (t-s)^{-\alpha} s^{1-\delta-\beta} \\ &\leq C\, T^{-1} \sum_{t=1}^{T} t^{3-2(\beta+\delta)-\alpha} \longrightarrow 0 \end{aligned}$$

as $T \to \infty$ since $2(\beta + \delta) + \alpha > 3$ and hence (2.53) holds and $T^{-1} S_2 \to 0$ as $T \to \infty$.

Finally, writing

$$d_t = \sum_{s=t+1}^{T} h_s\, g_{t-s} \,, \tag{2.54}$$

we have

$$S_3 = E \left(\sum_{t=1}^{T-1} Y_t d_t \right)^2 = \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \gamma_{t-s}\, d_s\, d_t$$

and

$$|\, d_t \,| \leq \sum_{s=t+1}^{T} |\, h_s \, g_{t-s} \,| \leq C \sum_{s=t+1}^{T} s^{-\beta}(s-t)^{-1} \leq C\, t^{-\beta} \log T \tag{2.55}$$

so that $T^{-1}S_3 \to 0$ provided

$$T^{-1}(\log T)^2 \sum_{t=1}^{T} \sum_{s=1}^{T} |\, \gamma_{t-s} \,| s^{-\beta}\, t^{-\beta} \longrightarrow 0 \tag{2.56}$$

as $T \to \infty$.

In the case of SRD*, (2.56) holds since

$$\sum_{t=1}^{T} |\, \gamma_{t-s} \,| s^{-\beta} \leq 2 \sum_{s=0}^{\infty} |\, \gamma_s \,| < \infty$$

and $\sum_{t=1}^{T} t^{-\beta} \sim (1-\beta)^{-1} T^{1-\beta}$ as $T \to \infty$. Also, if (B) is satisfied we use

$$\begin{aligned} \sum_{t=1}^{T} \sum_{s=1}^{T} |\, \gamma_{t-s} \,| s^{-\beta}\, t^{-\beta} &= \gamma_0 \sum_{t=1}^{T} t^{-2\beta} + 2 \sum_{t=2}^{T} \sum_{s=1}^{t-1} |\, \gamma_{t-s} \,| s^{-\beta} t^{-\beta} \\ \leq\; & C \left\{ T^{1-2\beta} + \sum_{t=1}^{T} t^{-\beta} \sum_{s=1}^{t-1} (t-s)^{-\alpha} s^{-\beta} \right\} \leq C \left\{ T^{1-2\beta} + \sum_{t=1}^{T} t^{1-\alpha-2\beta} \right\} \\ \leq\; & C\, T^{2-\alpha-2\beta} \end{aligned}$$

in view of (2.49) and hence (2.56) holds since $\alpha + 2\beta > 1$. This completes the proof of (2.46).

We now proceed to establish (2.47). We have

$$T^{-\frac{1}{2}} \sum_{s=1}^{T} \sum_{t=1}^{T} h_s\, h_t\, g_{t-s} \equiv T^{-\frac{1}{2}} (g_0 S_4 + S_5 + S_6)$$

where

$$S_4 = \sum_{t=1}^{T} h_t^2 \,, \quad S_5 = \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} h_t\, h_s\, g_{t-s} \,, \quad S_6 = \sum_{t=2}^{T} \sum_{s=1}^{t-1} h_t\, h_s\, g_{t-s}$$

and (2.47) holds if $T^{-\frac{1}{2}} S_j \to 0, \;\; j = 4, 5, 6,$ as $T \to \infty$.

Since $\frac{1}{4} < \beta < 1$, as $T \to \infty$,

$$S_4 \leq C \int_1^T t^{-2\beta} dt = C\, T^{1-2\beta}$$

and $T^{-\frac{1}{2}}S_4 \to 0$. Also, using (2.54) and (2.55),

$$| S_5 | = \left| \sum_{t=1}^{T-1} h_t d_t \right| \leq C \sum_{t=1}^{T-1} | h_t | t^{-\beta} \log T \leq C\, T^{1-2\beta} \log T$$

and $T^{-\frac{1}{2}}S_5 \to 0$. Finally, using (2.50) and (2.52) where δ is now chosen so that $0 < \delta < 1$ and $2\beta + \delta > \frac{3}{2}$,

$$| S_6 | = \left| \sum_{t=2}^{T} h_t c_t \right| \leq C \sum_{t=2}^{T} | h_t | t^{1-\delta-\beta} \leq C\, T^{2-2\beta-\delta}$$

and $T^{-\frac{1}{2}}S_6 \to 0$. This completes the proof of (2.47) and hence of Theorem 2.1.

2.4.2 Proof of Theorem 2.3

If $\hat{\theta}_{T,X}$ does not converge to θ_0 there is a subsequence converging to $\theta_1 \in \overline{\Theta}$, $\theta_1 \neq \theta_0$. Call this subsequence $\hat{\Theta}_M(X)$, where $\{M\}$ is a subsequence of $\{ N = 1 , 2, , \dots , \}$. Then by the definition of $\hat{\Theta}_M(X)$, (2.20) and Theorem 2.2 we have

$$\begin{aligned} \liminf_{M\to\infty} \int_{-\pi}^{\pi} g(\omega,\, \hat{\Theta}_M(X)) I_{M,X}(\omega)\, d\omega &\leq \liminf_{M\to\infty} \int_{-\pi}^{\pi} g(\omega,\, \theta) I_{M,X}(\omega)\, d\omega \\ = \lim_{M\to\infty} \int_{-\pi}^{\pi} g(\omega,\, \theta) I_{M,Y}(\omega)\, d\omega &= \int_{-\pi}^{\pi} g(\omega,\, \theta) f(\omega,\, \theta_0)\, d\omega \qquad (2.57) \end{aligned}$$

for any $\theta \in \Theta$. On the other hand,

$$\begin{aligned} \liminf_{M\to\infty} \int_{-\pi}^{\pi} g(\omega,\, \hat{\Theta}_M(X)) I_{M,X}(\omega)\, d\omega &\geq \liminf_{N\to\infty} \int_{-\pi}^{\pi} g(\omega,\, \hat{\Theta}_N(X)) I_{N,X}(\omega)\, d\omega \\ = \int_{-\pi}^{\pi} g(\omega,\, \theta_1) f(\omega,\, \theta_0)\, d\omega\,. \end{aligned}$$

Since $\Theta_0 \cap \Theta$ is not empty we can choose $\theta_2 \in \Theta_0 \cap \Theta$ such that

$$\int_{-\pi}^{\pi} g(\omega,\, \theta_2) f(\omega,\, \theta_0) d\omega < \int_{-\pi}^{\pi} g(\omega,\, \theta_1) f(\omega,\, \theta_0) d\omega$$

which contradicts (2.57). This proves Theorem 2.3.

2.4.3 Proof of Theorem 2.4

By (2.30) and (2.31) we have

$$
\begin{aligned}
T^{\frac{1}{2}}\left(Q_{T,X}(G,\ H) - Q_{T,Y}(G,\ H)\right) &= T^{\frac{1}{2}}\frac{1}{m}\sum_{j=1}^{m}\frac{\omega_j^{2H-1}}{G}\left(I_{T,X}(\omega_j) - I_{T,Y}(\omega_j)\right) \\
&= \frac{T^{1/2}}{G\,m}\sum_{j=1}^{m}\omega_j^{2H-1}\frac{1}{2\pi\,T}\sum_{t,s=1}^{T}(h_tY_s + h_sY_t + h_th_s)\,e^{-i\,(t-s)\,\omega_j} \\
&= \frac{T^{-1/2}}{2\pi G\,m}\sum_{t,s=1}^{T}(h_tY_s + h_sY_t + h_th_s)\sum_{j=1}^{m}\omega_j^{2H-1}\,e^{-i\,(t-s)\,\omega_j}\ .
\end{aligned}
$$

When $m = [\log T]$ and $\beta + H > 3/2$ we have

$$
\begin{aligned}
& E\left|\frac{1}{T^{1/2}\,m}\sum_{t,s=1}^{T} h_tY_s\sum_{j=1}^{m}\omega_j^{2H-1}\,e^{-i\,(t-s)\,\omega_j}\right|^2 \\
= \ & \frac{1}{T\,m^2}\sum_{t,s=1}^{T} h_th_s\sum_{j,k=1}^{m}\omega_j^{2H-1}\omega_k^{2H-1}\sum_{u,v=1}^{T} EY_u\,Y_v e^{i\,(t-u)\,\omega_j - i\,(s-v)\,\omega_k} \\
\leq \ & \frac{1}{T\,m^2}\sum_{t,s=1}^{T}|\,h_th_s\,|\sum_{j,k=1}^{m}\omega_j^{2H-1}\omega_k^{2H-1}\sum_{u,v=1}^{T}|\,\gamma(u-v)\,| \\
\sim \ & \frac{C}{T\,m^2}T^{2-2\beta}\left(\sum_{j=1}^{m}\left(\frac{j}{T}\right)^{2H-1}\right)^2\sum_{u=1}^{T}(T-u)\,u^{2H-2} \\
\sim \ & C\,m^{4H-2}T^{1-2\beta+2-4H+2H} \\
= \ & C\,(\log T)^{4H-2}T^{3-2H-2\beta} \rightarrow 0\ ,\ \text{as}\ T\rightarrow\infty\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|\frac{1}{T^{1/2}\,m}\sum_{t,s=1}^{T} h_th_s\sum_{j=1}^{m}\omega_j^{2H-1}\,e^{-i\,(t-s)\,\omega_j}\right| \leq \frac{1}{T^{1/2}\,m}\sum_{t,s=1}^{T}|\,h_th_s\,|\sum_{j=1}^{m}\omega_j^{2H-1} \\
\sim \ & \frac{1}{T^{1/2}\,m}T^{2-2\beta}\,\frac{m^{2H}}{T^{2H-1}} = m^{2H-1}T^{5/2-2\beta-2H}\longrightarrow 0\ ,\ \text{as}\ T\rightarrow\infty\ .
\end{aligned}
$$

This finishes the proof of Theorem 2.4.

2.4.4 Proof of Theorem 2.6

We notice first that for any $\lambda \in (0\ ,\ 2\pi)$,

$$\begin{aligned}
\hat{F}_{T,X}(\lambda) &= \frac{1}{T^2}\sum_{j=1}^{[T\lambda/2\pi]}\sum_{t,s=1}^{T} X_t X_s e^{i(t-s)\lambda j} \\
&= \frac{1}{T^2}\sum_{j=1}^{[T\lambda/2\pi]}\sum_{t,s=1}^{T} (Y_t + h_t)(Y_s + h_s)\, e^{i(t-s)\lambda j} \\
&= \frac{1}{T^2}\sum_{j=1}^{[T\lambda/2\pi]}\sum_{t,s=1}^{n} \{Y_t Y_s + h_t Y_s + h_s Y_t + h_t h_s\}\, e^{i(t-s)\lambda_j} \\
&\equiv \hat{F}_{T,\,Y}(\lambda) + I(\lambda) + II(\lambda) + III(\lambda)\,,
\end{aligned}$$

say, in an obvious notation. Thus

$$\frac{\hat{F}_{T,\,X}(\lambda)}{\hat{F}_{T,\,Y}(\lambda)} = 1 + \frac{I(\lambda) + II(\lambda) + III(\lambda)}{\hat{F}_{n,\,Y}(\lambda)} \quad .$$

So in order to establish Theorem 2.6 it is sufficient to show that as $T \to \infty$,

$$\frac{I(\lambda_m)}{\hat{F}_{T,Y}(\lambda_m)}\ ,\quad \frac{II(\lambda_m)}{\hat{F}_{T,Y}(\lambda_m)} \quad \text{and} \quad \frac{III(\lambda_m)}{\hat{F}_{T,Y}(\lambda_m)} \longrightarrow 0\ . \tag{2.58}$$

We consider $I(\lambda_m)/\hat{F}_{T,Y}(\lambda_m)$ first. Since as $T \to \infty$, from Theorem 2.5 (2.37),

$$\frac{\hat{F}_{T,Y}(\lambda_m)}{F(\lambda_m)} \xrightarrow{P} 1\ ,$$

$$\frac{I(\lambda)_m}{\hat{F}_{T,Y}(\lambda_m)} = \frac{I(\lambda_m)}{F(\lambda_m)}\,\frac{F(\lambda_m)}{\hat{F}_{T,Y}(\lambda_m)}\quad .$$

If we can show that

$$\frac{I(\lambda_m)}{F(\lambda_m)} = \frac{1}{T^2 F(\lambda_m)}\sum_{j=1}^{m}\sum_{t,\,s=1}^{T} h_t Y_s e^{i(t-s)\lambda_j} \xrightarrow{P} 0$$

then the first term in (2.58) goes to zero in probability. Under Condition A and (2.25) and we have

$$\begin{aligned}
F(\lambda_m) &= \int_0^{\lambda_m} f(\omega)d\omega \sim C\lambda_m^{2-2H} \\
&\sim\ C\left(\frac{m}{T}\right)^{2-2H} = C\left(\frac{[\log T\,]}{T}\right)^{2-2H}\,,
\end{aligned}$$

$$
\begin{aligned}
I(\lambda_m) &= \frac{1}{T^2}\sum_{j=1}^{m}\sum_{t,\ s=1}^{T} h_t Y_s e^{i(t-s)\lambda_j} \\
&= \frac{1}{T^2}\sum_{s=1}^{T} Y_s \left(\sum_{t=1}^{T}\sum_{j=1}^{m} h_t e^{i(t-s)\lambda_j}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
E\,|\,I(\lambda_m)\,|^2 &= \frac{1}{T^4}\sum_{u,\ v=1}^{T}\gamma(u-v)\left(\sum_{t=1}^{T}\sum_{j=1}^{m} h_t e^{i(t-u)\lambda_j}\right)\left(\sum_{s=1}^{T}\sum_{k=1}^{m} h_s e^{-i(s-v)\lambda_k}\right) \\
&\le \frac{1}{T^4}\left|\sum_{u,\ v=1}^{T}\gamma(u-v)\right|\left(\sum_{t=1}^{T}\sum_{j=1}^{m}|\,h_t\,|m\right)^2 \\
&\sim \frac{C\,m^2}{T^4}T^{2-2\beta}\left|\sum_{u,\ v=1}^{n}\gamma(u-v)\right| \sim \frac{C\,m^2}{T^4}T^{2-2\beta}\sum_{u=1}^{T}(T-u)u^{2H-2} \\
&\sim \frac{C\,m^2}{T^4}T^{2-2\beta}\times T^{2+2H-2} = C\,m^2 T^{2H-2\beta-2}\,.
\end{aligned}
$$

Thus under Condition B$'$,

$$
\begin{aligned}
E\left|\frac{I(\lambda_m)}{F(\lambda_m)}\right|^2 &\sim Cm^2T^{2H-2\beta-2}\left(\frac{m}{T}\right)^{2(2H-2)} \\
&\sim C(\log T)^{4H-2}\,T^{6H-6-2\beta} \longrightarrow 0\,,\ \text{as } T\to\infty\,.
\end{aligned}
$$

So the first term in (2.58) goes to zero in probability. By the same argument, the second term in (2.58) goes to zero in probability as $T\to\infty$. To deal with the third term in (2.58) we notice that

$$
\begin{aligned}
\left|\frac{III(\lambda)}{\hat{F}_{T,\,Y}(\lambda)}\right| &\sim \left|\frac{C}{T^2}\sum_{t,\ s=1}^{T} h_t h_s \sum_{j=1}^{m} e^{i(t-s)\lambda_j}\left(\frac{m}{T}\right)^{2H-2}\right| \\
&\le \frac{C}{T^2}\sum_{s,t=1}^{T}|\,h_s h_t\,|\,M\left(\frac{m}{T}\right)^{2H-2} \\
&= Cm^{2H-1}T^{2-2H-2\beta} \longrightarrow 0\,,\ \text{as } T\to\infty
\end{aligned}
$$

by (2.38) and Condition B$'$. This finishes the proof of Theorem 2.6.

If Conditions A, B$'$ and C of Theorem 2.6 are replaced by Conditions A, B$''$ and C it can be shown by the same argument used here that the results of Theorem 2.6 still hold.

2.4.5 Proof of Corollary 2.7

Clearly we can write

$$\hat{H}_{T,Y} = \arg \min_{H\in\Theta} R_{T,Y}(H) \,,$$

where

$$\begin{aligned} R_{T,Y}(H) &= \log \hat{G}_{T,Y}(H) - (2H-1)\frac{1}{m}\sum_{j=1}^{m} \log \omega_j \\ \hat{G}_{T,Y}(H) &= \frac{1}{m}\sum_{j=1}^{m} \omega_j I_{T,Y}(\omega_j)\,. \end{aligned}$$

Since

$$\hat{G}_{T,X}(H) - \hat{G}_{T,Y}(H) = \frac{1}{m}\sum_{j=1}^{m} \omega_j^{2H-1}\left(I_{T,X}(\omega_j) - I_{T,Y}(\omega_j)\right) \,,$$

$$E\left|\hat{G}_{T,X}(H)-\hat{G}_{T,Y}(H)\right|^2 = E\left|\frac{1}{m}\sum_{j=1}^{m}\omega_j^{2H-1}\left(\frac{1}{2\pi T}\sum_{s,t=1}^{T}(h_tY_s+h_sY_t+h_th_s)e^{i\,(t-s)\,\omega_j}\right)\right|^2 .$$

It is easy to see that, under condition (2.29), as $T\to\infty$,

$$\begin{aligned} &\left|\frac{1}{m\,T}\sum_{j=1}^{m}\omega_j^{2H-1}\sum_{s,t=1}^{T} h_t h_s e^{i\,(t-s)\,\omega_j}\right| \le \frac{1}{m\,T}\sum_{j=1}^{m}\omega_j^{2H-1}\left(\sum_{t=1}^{T}|h_t|\right)^2 \\ &\sim \frac{1}{m\,T}\frac{m^{2H}}{T^{2H-1}}T^{2-2\beta} = C\,T^{2-2H-2\beta} \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} &E\left|\frac{1}{m}\sum_{j=1}^{m}\omega_j^{2H-1}\frac{1}{2\pi\,T}\sum_{s,t=1}^{T} h_t Y_s e^{i\,(t-s)\,\omega_j}\right|^2 \\ &\le \frac{1}{m^2\,T^2}\sum_{j,k=1}^{m}\omega_j^{2H-1}\omega_k^{2H-1}\left(\sum_{t=1}^{T}|h_t|\right)^2\sum_{u,v=1}^{T}|\,\gamma(u-v)\,| \\ &\sim \frac{1}{m^2\,T^2}\,\frac{m^{2H}}{T^{2(2H-1)}}\,T^{2-2\beta}\,T^{2H} = C\,m^{4H-2}T^{2-2H-2\beta} \longrightarrow 0\,. \end{aligned}$$

Hence, $\hat{G}_{T,X}(H) - \hat{G}_{T,Y}(H) \xrightarrow{\mathrm{P}} 0$. Therefore, as $\log x$ is a monotonic and continuous function,

$$\hat{H}_{T,X} - \hat{H}_{T,Y} \xrightarrow{\mathrm{P}} 0\,.$$

Thus by (2.42)

$$(\hat{G}_{T,X}\,,\ \hat{H}_{T,X}) \xrightarrow{\mathrm{P}} (G_0\,,\ H_0)\,.$$

This ends the proof of Corollary 2.7

Chapter 3

SOME ASYMPTOTIC PROPERTIES OF THE SAMPLE MEAN AND SAMPLE COVARIANCE OF LONG RANGE DEPENDENT SERIES

3.1 Introduction

A stationary Gaussian time series is described completely by its mean and covariance function. If the series is not Gaussian these quantities are also of importance although they will generally not determine the process uniquely. In this chapter we study the properties of estimators of these quantities. We are concerned with the limits of the sample mean, sample covariance and the problem of testing the mean of a class of time series which is characterized by the behaviour of its spectral density near the origin of the form

$$f(\omega) \sim L(1/\omega)|\omega|^{1-2H}\ , \text{ as } \omega \to 0\ . \tag{3.1}$$

Here, as in Chapter 2, $L(\omega)$ is a slowly varying function at infinity, while the Hurst index H satisfies $0 < H < 1$. When $1/2 < H < 1$, the series exhibits LRD.

Modern analysis of time series has been characterized by attempts to relax the "independent" part of the i.i.d. assumption. It is well known that a great number of limit theorems, which in the classical approach were always studied under the assumption that the underlying random variables were independent, continue to hold under certain dependence structures. Independence is not necessary. On the other hand, as dependence becomes stronger, at a certain point new phenomena arise. One is then led to ask: how much dependence, qualitatively and quantitatively, is allowed for the classical limit results to hold, and what are the results when dependence becomes stronger. It is the aim of this chapter to extend the limiting theorems for the sample mean (Section 3.2), sample covariance (Section 3.3) and some quadratic forms (Section 3.4) to LRD series.

When the spectral density of a time series satisfies (3.1), we make use of the well known fact that a stationary (in the wide sense) process is a process of moving averages if and only if its spectral distribution is absolutely continuous, (see, for example, Doob, 1953, p. 499), and we assume the series we are interested in is a linear process

$$Y_t = \sum_{\nu=-\infty}^{\infty} b_{t-\nu}\xi_\nu \tag{3.2}$$

where

$$\sum_{j=-\infty}^{\infty} b_j^2 < \infty\ ,\quad E\xi_\nu = 0\ ,\quad E\xi_\nu\xi_\mu = \delta_{\nu,\mu}\ . \tag{3.3}$$

Here $\delta_{\nu,\mu}$ is the Kronecker symbol. The Wold Decomposition Theorem suggests the generality of such a formulation.

If the innovations in non-explosive AR, MA or ARMA models form i.i.d. sequences the asymptotic normality of parametric or non-parametric estimators can be proved. Also, when the innovations in (3.2) are i.i.d. or martingale differences, the series allows the central limit theorem (CLT) to be established with the same rate of convergence as in the case of i.i.d. data, if the weights in (3.2) decay fast enough (see Moran, 1943, with many subsequent citations). In such schemes the innovations need not actually be strictly stationary. So both aspects of the i.i.d. assumption are relaxed. Many alternative general schemes of characterizing serial dependence are available. We will discuss them in more detail in Section 3.2.

In Section 3.2 we are concerned with the laws of large numbers and the CLT for the partial sums of $\{Y_t\}$ satisfying (3.2) and (3.3) in the presence of LRD. The ordinary CLT holds under modest conditions for the partial sums when the summands are weakly dependent. There is weak dependence when the summands satisfy a suitable mixing condition, a martingale type condition, are moving averages with adequately chosen weights or satisfy various special dependence structures. When the weights decay to zero but not fast enough to allow the covariance sequence of the series to be summable the dependence of the series becomes too strong for the usual CLT to hold for general orthogonal innovations. We will give conditions on the weights and the innovations under which the CLT holds for the partial sums.

A great deal of time series analysis is based upon quadratic functions of the data. In particular, many inferential results relate to theorems concerning the sample covariance $\hat{\gamma}_T(t) = T^{-1}\sum_{s=1}^{T-t}(Y_s - EY_s)(Y_{s+t} - EY_{s+t})$, $Y_1, \ldots, Y_T$ being a sample of T consecutive observations on some process $\{Y_t\}$. Much of the statistical literature on stationary time series has been concerned with studying the asymptotic behaviour of $\hat{\gamma}_T(t)$. It is well known that, under certain classical conditions on process $\{Y_t\}$, a strong law of large numbers holds for $\hat{\gamma}_T(t)$ (see, for example, Hannan, 1970, Chapter IV). It has been shown that, using limit theorems for martingales, the scope of the classical inferential theory can be appreciably widened in a natural way (see, for example, Hannan and Heyde, 1972).

It is of great importance for $\hat{\gamma}_T(t)$'s to converge to their true values uniformly in t, because many statistics are concerned with the use of the covariance matrix of the underlying process. Theorems relating the almost sure uniform convergence of the sample covariance function to its true value have been considered in the circumstances of SRD (see, for example, Hannan, 1974; An, Chen and Hannan, 1982). In Section 3.3 we will prove that under very general conditions, the convergence of $\hat{\gamma}_T(t)$ to $\gamma(t)$, uniformly in t, holds for linear processes with LRD.

Often a non-stationary time series $\{X_t\}$ can be distinguished as a series having trend in the mean

$$X_t = Y_t + m_t\,,\ \ t = 0\,,\,1\,,\,\dots\,, \tag{3.4}$$

where $\{Y_t\}$ is a stationary process with mean zero and $\{m_t\}$ is a deterministic trend. Then $\{m_t\}$ is the mean function of $\{X_t\}$. Various parametric and non-parametric tests have been suggested by various authors for testing different hypotheses regarding $\{m_t\}$ ($m_t = m$ for all t, or m_t is a monotonic function of t etc.), see, for example, Grenander and Rosenblatt (1957), Parthasarathy (1961), Sen (1965) and Subba Rao (1968).

In Section 3.4 we are concerned with the testing of mean of a series of form (3.4), where $\{Y_t\}$ satisfies (3.2) and (3.3). The problem of testing the mean function

$$H_0 : \qquad m_t = 0\,,\ \ t = 0\,,\,1\,,\,\dots \tag{3.5}$$

has been formally solved by Grenander and Rosenblatt (1957) in the case when the process $\{Y_t\}$ is Gaussian and a simple hypothesis (3.5) is being tested against a simple alternative. When $\{Y_t\}$ is of the form of (3.2) and (3.3), where $\{\xi_\nu\}$ is an i.i.d. sequence and $\{Y_t\}$ is a short range dependent series, in other words, $\sum_{j=-\infty}^{\infty}|b_j| < \infty$, Parthasarathy (1961) has formally solved the problem of testing the null hypothesis (3.5) against all alternatives which satisfy certain conditions. This will be discussed in more detail in Section 3.4 where we deal with the same testing problem in the more general case where $\{Y_t\}$ may exhibit LRD.

3.2 Asymptotic Distribution of Partial Sums from Linear Processes with Long Range Dependence

In this section we discuss the asymptotic properties of large sample inference for means in certain models in the presence of LRD. The focus is on the asymptotic distribution of the sample mean. Let $\{X_t\}$ be a stationary time series with mean $EX_t = m$ and the spectral density of $\{X_t\}$ exists. Then $\{X_t\}$ can be described as

$$X_t = m + Y_t \tag{3.6}$$

where $\{Y_t\}$ is given by (3.2) and (3.3):

$$Y_t = \sum_{\nu=-\infty}^{\infty} b_{t-\nu}\xi_\nu \,, \quad \sum_{j=-\infty}^{\infty} b_j^2 < \infty \,, \quad E\xi_\nu = 0 \,, \quad E\xi_\nu\xi_\mu = \delta_{\nu,\mu} \,.$$

We assume further that

$$|\gamma(t)| \equiv |E(X_s - m)(X_{s+t} - m)| \sim const\ |t|^{2H-2} \,, \quad \text{as } t \to \infty \,. \tag{3.7}$$

For the observations $\{X_1, \ldots, X_T\}$ let the sample mean of the observations be defined as

$$\hat{m}_T = \frac{1}{T}\sum_{t=1}^{T} X_t \,. \tag{3.8}$$

When $\sum_{j=-\infty}^{\infty} |b_j| < \infty$, therefore $\sum_{t=-\infty}^{\infty} |\gamma(t)| < \infty$, $\hat{m}_T$ usually has very nice properties. For example, under various regularity conditions $\sqrt{T}(\hat{m}_T - m)$ is asymptotically normal with mean zero and variance $\sum \gamma(t)$. When $\sum |b_j| = \infty$, $\{X_t\}$ is LRD, $\hat{m}_T$ is still a good estimator for m. First of all, in spite of the slowly decaying covariance function: $|\gamma(t)| \sim const\ |t|^{2H-2}$, $\hat{m}_T$ does not lose much efficiency compared to the best linear unbiased estimator (BLUE). (To calculate the BLUE one would have to know or estimate all covariances.) An explicit formula for the asymptotic efficiency (given by the ratio of the two asymptotic variances) was derived by Adenstedt (1974)

$$\text{eff}(\hat{m}_T\,,\,\hat{m}_{BLUE}) = \frac{\pi(2H-1)H}{B(3/2-H\,,\,3/2-H)\sin\pi(H-1/2)}\,.$$

Numerically this is above 0.98 for $H \in [1/2, 1]$. (Beran and Künsch, 1985; Samarov and Taqqu, 1988.) For most practical purposes, an efficiency loss of 2% does not matter so that the sample mean is not only much easier to calculate but also a sufficiently accurate estimator of m.

Because of the properties mentioned above, $\hat{m}_T$ is widely used for inference on m. It is of importance to investigate the asymptotic properties of $\hat{m}_T$, particularly the asymptotic normality of $\hat{m}_T$. In Theorems 3.1 and 3.3 we summarize the well known large sample results of $\hat{m}_T$. For completeness we give the proof of these results in this thesis although they have been mentioned in a great deal of literature without proof.

THEOREM 3.1 *For a stationary series $\{X_t\}$ given by (3.6) and (3.7),*

$$E\hat{m}_T = m\,, \tag{3.9}$$

$$Var(\hat{m}_T) \sim const\, T^{2H-2}\,,\ as\ T \to \infty\,. \tag{3.10}$$

Therefore, $\hat{m}_T$ is an unbiased asymptotically consistent estimator for m.

Proof of Theorem 3.1. In the following we use C to denote a generic constant, which may differ in different equations.

Since $\{X_t\}$ is stationary, we immediately have the following equations:

$$E\hat{m}_T = \frac{1}{T}\sum_{t=1}^{T} EX_t = m\,.$$

$$\text{Var}(\hat{m}_T) = E\left(\frac{1}{T}\sum_{t=1}^{T}(X_t - m)\right)^2 = \frac{1}{T^2}\sum_{s,t=1}^{T}\gamma(t-s)$$

$$= \frac{1}{T^2}\sum_{|t|\le T-1}(T-|t|)\gamma(t) \sim \frac{C}{T^2}\sum_{t=1}^{T}(T-t)t^{2H-2} \sim C\,T^{2H-2}\,.$$

This finishes the proof of Theorem 3.1.

From Theorem 3.1 we know that the law of large numbers holds for $\hat{m}_T$, i.e. $\hat{m}_T \xrightarrow{P} m$ as $T \to \infty$. We have further that the strong law of large numbers for $\hat{m}_T$ holds for certain series with LRD. Before we state the strong law of large numbers for $\hat{m}_T$ we need the following

LEMMA 3.2 *Let $\{Y_t\}$ be a stationary series with $EY_t = 0$. Then $\bar{Y}_T \equiv T^{-1}\sum_{t=1}^{T} Y_t$ converges with probability one to zero if there are constants $K > 0$, $\alpha > 0$, such that*

$$Var\bar{Y}_T \leq K\,T^{-\alpha}\ . \tag{3.11}$$

The proof of Lemma 3.2 can be found in Hannan (1970), page 206.

THEOREM 3.3 *Let $\{X_t\}$ be as in Theorem 3.1, where $0 < H < 1$. Then with probability one,*

$$\hat{m}_T \longrightarrow m\ ,\ as\ T \to \infty\ . \tag{3.12}$$

Proof of Theorem 3.3. Theorem 3.3 follows immediately from Theorem 3.1 and Lemma 3.2.

For the CLT for $\hat{m}_T$, without loss of generality we need only to consider the CLT for the partial sums of a random sequence $\{X_t\}$ with mean zero. Many papers have been concerned with the partial sums, after appropriate norming and centering, of a random sequence $\{X_t\}$. For example, Bernstein and Lévy (1935) discussed what was later referred to as the martingale condition

$$E(X_k \mid X_j\ :\ j \leq k-1) = 0 \tag{3.13}$$

in this context. Doob (1953) and many others have developed an extensive set of theoretical results based on martingale conditions. When a condition like (3.13) is not satisfied a strong version of mixing conditions (stronger than those typically employed in ergodic theory) has been introduced as sufficient conditions for asymptotic normality. Kolmogorov and Rozanov (1960) showed that a sufficient condition for a stationary Gaussian process to be strong mixing is that it have an absolutely continuous spectrum with a positive continuous spectral density function.

A variation of this type of result given by Sun (1963, 1965) leads towards the literature on limit theory for functions of strongly dependent Gaussian processes. Sun (1965) assumed

$$X_t = H(u_t) \ , \tag{3.14}$$

where $H(u)$ is a possibly non-linear function such that $EH(u_t) = 0$, $EH(u_t)^2 < \infty$, and $\{ u_t \}$ is a zero mean, unit variance stationary Gaussian process such that $\gamma(j) = E(u_t u_{t+j})$ satisfies the condition

$$\sum_j \gamma(j)^2 < \infty \ , \ \lim_{T\to\infty} \frac{1}{T} \sum_{t=-T+1}^{T-1} (T - |t|)\gamma(t) \text{ exists and is finite.} \tag{3.15}$$

Then $T^{1/2}\hat{m}_T$ is asymptotically normal with mean zero and finite variance. Breuer and Major (1983) replaced (3.15) by the condition

$$\sum_j \gamma(j)^k < \infty \tag{3.16}$$

when $H(u)$ admits the Hermite expansion

$$H(u) = \sum_{j=0}^{\infty} \frac{C_j}{j\,!} H_j(u) \ , \ C_j = EH(u_t)H_j(u_t)$$

where $H_j(u)$ is the jth Hermite polynomial with leading coefficient one and $k = \inf\{j \, : \, C_j \neq 0\}$. Ho and Sun (1987) extended Breuer and Major's (1983) work to the non-instantaneous filter

$$X_t = H(u_{t+t_1} \, , \, \dots \, , \, u_{t+t_d}) \ . \tag{3.17}$$

Such results indicate that even if the underlying process $\{ u_t \}$ has strong autocorrelation, some non-linear function (3.14) or (3.17) may satisfy the CLT with

$T^{-1/2}$ norming. However, linear functions of u_t, or non-linear ones with $k = 1$, do not satisfy conditions (3.15) or (3.16), and moreover, nor do non-linear functions with $k > 1$ if the autocorrelation is strong enough. An early counter example to the CLT was provided by Rosenblatt (1961), in the case $X_t = u_t^2 - 1$ with $\text{cov}(X_1, X_{1+j}) \sim \textit{const}\, |j|^{2H-2}$, where H is the Hurst index and satisfies $3/4 < H < 1$; here $T^{2-2H}\hat{m}_T$ has a non-normal limit distribution.

Taqqu (1975) considerably generalized Rosenblatt's result, and indicated how limiting normality, as well as non-normality, can hold when the usual CLT does not.

A natural generalization of Hermite polynomials to non-normal distributions is provided by the Appell polynomials $A_j(x)$. The Appell polynomials have been used by Giraitis and Surgailis (1986) in developing limit theory not only for non-linear functions of linear processes but also for more general processes, and for multivariate situations. This theory tends towards a generalization of the results described above. Invariance principles for the case of non-stationary as well as stationary X_t generated by linear and mixing processes have been given by Akonom and Gourieroux (1987), Silveira (1991).

An alternative line of research extends the usual CLT with $T^{-1/2}$ norming in case $X_t = Y_t$, where Y_t is given by (3.2) and (3.3), but not necessarily being a Gaussian process. The CLT has been given for the case when $\{\xi_\nu\}$ is an i.i.d. sequence, or a sequence of stationary and non-stationary martingale differences $\{\xi_\nu\}$ with the restriction that

$$\sum \gamma(t) = \sum_k \sum_j b_j b_{j+k} < \infty \,. \tag{3.18}$$

However, strong autocorrelation in Y_t entails a relaxation of (3.18). Ibragimov and Linnik (1971) established the CLT for the generalized linear processes with the restriction that the innovations are i.i.d. sequences. In detail, write

$$S_T = Y_1 + Y_2 + \cdots + Y_T \,, \tag{3.19}$$

$$\sigma_T^2 \quad = \quad \mathrm{Var}(Y_1 + Y_2 + \cdots + Y_T)\,, \tag{3.20}$$

and assume that $\{\,\xi_\nu\,\}$ in (3.2) is an i.i.d sequence, and as $T \to \infty$,

$$\sigma_T^2 = \mathrm{Var}(S_T) = \sum_j (b_{j-1} + \cdots + b_{j-T})^2 \longrightarrow \infty\,, \tag{3.21}$$

where the rate of increase can be arbitrarily slow. Then

$$\sigma_T^{-1} S_T \xrightarrow{\mathrm{d}} N(0\,,\,1)\,. \tag{3.22}$$

Existing extensions of Ibragimov and Linnik's (1971) result have been in several directions. Davydov (1970) and Gorodetskii (1977) established invariance principles in Ibragimov and Linnik's (1971) setting. Surgailis (1981, 1983) investigated a sliding average process

$$X_k = \sum_{j=-\infty}^{k} b_{k-j}\xi_j \ \ , \ \ k = 0\,,\,\pm 1\,,\,\pm 2\,,\,\cdots\,. \tag{3.23}$$

Here $\{\,\xi_\nu\,\}$ is an i.i.d. sequence of random variable with mean zero and variance one and finite moments of any order. For a function $H(u)$ being possibly nonlinear such that $EH(X_k) = 0$, $E(H(X_k)^2) < \infty$, under certain conditions the partial sums of $H(X_k)$ satisfy an invariance principle in that, suitably normalized and centred, the partial sums of $\{\,H(X_k)\,\}$ converge to a self-similar process.

In the following, we are concerned with the asymptotic distribution of $\sigma_T^{-1} S_T$ for a generalized linear process $\{\,Y_t\,\}$ in the case where the innovation $\{\,\xi_\nu\,\}$ is not assumed to be an i.i.d. sequence, and the weights of (3.2) are not summable, so the process $\{\,Y_t\,\}$ may be LRD. The idea of our result follows.

When $\{\,\xi_\nu\,\}$ satisfies $E\xi_\nu\xi_\mu = \delta_{\nu,\mu}$, we have

$$\sigma_T^2 = \sum_{k=-\infty}^{\infty} (b_{k-1} + b_{k-2} + \cdots + b_{k-T})^2 \equiv \sum_{k=-\infty}^{\infty} b_{k,T}^2$$

where

$$b_{k,T} = b_{k-1} + \cdots + b_{k-T}\,. \tag{3.24}$$

Writing

$$a_{k,T} = \frac{b_{k,T}}{\sigma_T} \;, \tag{3.25}$$

and assuming

CONDITION A. *Let a positive constant sequence* $\{\varepsilon_j\}$ *and an integer* $N = N(T)$ *be chosen such that, as* $j \to \infty$, $\varepsilon_j \to 0_+$ *and*

$$\sum_{|k|>N} a_{k,T}^2 < \varepsilon_T \;, \; T = 1 \;, \; 2 \;, \; \ldots$$

leads to the following theorem

THEOREM 3.4 *Let* $\{Y_t\}$ *satisfy (3.2) and (3.3), and* $\{\varepsilon_j\}$ *and* $N = N(T)$ *satisfy Condition A. Assume further that*

1. $\{\xi_\nu\}$ *satisfies*

$$E\xi_\nu^2\xi_\mu^2 = E\xi_\nu^2 E\xi_\mu^2 \;; \tag{3.26}$$

$$\text{for some } \delta > 0 \;, \qquad \max_{-\infty<\nu<\infty} E|\xi_\nu|^{2+\delta} < \infty \;; \tag{3.27}$$

$$E(\xi_\nu \mid \xi_\mu \;:\; \mu < \nu) = 0 \;. \tag{3.28}$$

2. $\{a_{k,T}\}$ *satisfies*

$$\lim_{T\to\infty} \sum_{k=1}^{N(T)} a_{k,T}^2 < \infty \;.$$

3. $\{\sigma_T\}$ *satisfies*

$$\lim_{T\to\infty} \sigma_T^2 = \infty \;.$$

Then, as $T \to \infty$,

$$\sigma_T^{-1}(Y_1 + Y_2 + \cdots + Y_T) \xrightarrow{\mathrm{d}} \mathrm{N}(0 \;, \; \sigma^2) \tag{3.29}$$

for some $\sigma^2 > 0$.

Remarks on Theorem 3.4.

1. Since $\sum_{k=-\infty}^{\infty} a_{k,T}^2 = \sigma_T^{-2} \sum_{k=-\infty}^{\infty} b_{k,T}^2 = \sigma_T^{-2}\sigma_T^2 = 1$, Condition A and Condition 2 of Theorem 3.4 are reasonable.

2. If $\{\xi_\nu\}$ is an i.i.d. sequence and $E|\xi_\nu|^{2+\delta} < \infty$, then (3.26) and (3.28) are fulfilled. Condition (3.28) implies that $E\xi_\nu\xi_\mu = 0$, $\nu \neq \mu$.

3. If

$$|\gamma(t)| = |EY_sY_{s+t}| \sim const\ |t|^{2H-2}\ , \text{ as } t \to \infty,\ 0 < H < 1\ ,$$

then

$$\sigma_T^2 = \sum_{s,t=1}^{T} \gamma(t-s) = O(T^{2H}) \longrightarrow \infty\ , \text{ as } T \to \infty\ ,$$

therefore Condition 3 holds. Thus the conditions of Theorem 3.4 are very general for linear processes. In particular, the result of Theorem 3.4 can be used in the case where the linear processes are LRD.

We will give the proof of Theorem 3.4 in Section 3.5. From Theorem 3.4 we have the following corollaries:

COROLLARY 3.5 *Let $\{Y_t\}$ be a long range dependent stationary series satisfying (3.2)and (3.3), and let the covariance function $\gamma(t)$ of $\{Y_t\}$ satisfy*

$$|\gamma(t)| \sim const\ |t|^{2H-2}\ ,\ as\ t \to \infty\ ,\ \frac{1}{2} < H < 1\ . \tag{3.30}$$

If the innovation $\{\xi_t\}$ of $\{Y_t\}$ satisfies the conditions of Theorem 3.4, then

$$\frac{1}{T^H}(Y_1 + \cdots + Y_T)$$

is asymptotically normally distributed.

COROLLARY 3.6 *Let*

$$X_t = m + Y_t$$

where $\{ Y_t \}$ *is a long range dependent stationary series satisfying the conditions of Corollary 3.5, and write*

$$\hat{m}_T = \frac{1}{T} \sum_{t=1}^{T} X_t \, .$$

Then, as $T \to \infty$*,*

$$\frac{\hat{m}_T - m}{(\, Var(\hat{m}_T))^{1/2}} \xrightarrow{\mathrm{d}} \mathrm{N}(0\,,\,\sigma_1^2)$$

for some $\sigma_1^2 > 0$*.*

3.3 Asymptotics of Sample Covariance of Linear Processes with Long Range Dependence

Given the observations $\{ Y_t \,:\, t = 1\,,2\,,\,\ldots\,,T\}$ one can define the sample covariance function:

$$\hat{\gamma}_T(t) = \begin{cases} \frac{1}{T}\sum_{s=1}^{T-t} (\, Y_s - \mu\,)\,(\, Y_{t+s} - \mu\,) \;, & 0 \le t \le T-1 \;, \\ 0 \;, & T \le t \;, \\ \hat{\gamma}_T(-t) \;, & t < 0 \,, \end{cases} \tag{3.31}$$

if the mean μ of the process is known. If μ is unknown an analogue may be defined by replacing μ with the sample mean: $\hat{\mu}_T = T^{-1}\sum_{s=1}^{T} Y_s$ in (3.31). For a series $\{ Y_t \}$ given by (3.2) and (3.3) there is a substantial literature concerning the asymptotic behaviour of $\hat{\gamma}_T(t)$. Much of this has employed the i.i.d. assumption of the innovations $\{ \xi_\nu \}$ in (3.2) and the convergence of $\sum |b_j|$. (See, for example, Anderson, 1958; Hannan, 1970; Davis and Resnick, 1985a, 1985b, 1986. Note that in Davis and Resnick's papers, ξ_ν's have infinite variance.) It is well known that a strong law of large numbers and a central limit theorem hold for $\hat{\gamma}_T(t)$ under the conditions mentioned above (see, for example, Hannan, 1970, Chapters 4 and 6), or certain naturally widened conditions on the innovations $\{ \xi_\nu \}$, such as the martingale condition, which relaxes the restriction of i.i.d. assumption on $\{ \xi_\nu \}$, or on the process $\{ Y_t \}$, typically an ergodic condition, and the convergence of

$\sum_j b_j^2$, which corresponds to the case of LRD when $\sum_j b_j$ diverges. (See Hannan and Heyde, 1972.) In their 1972 paper, Hannan and Heyde derived a strong law of large numbers for $\hat{\gamma}_T(t)$ and a central limit theorem for the sample autocorrelation $\hat{\rho}_T(t) \equiv \hat{\gamma}_T(t)/\hat{\gamma}_T(0)$ under the conditions that $\{ Y_t \}$ is ergodic and the innovations $\{ \xi_\nu \}$ of $\{ Y_t \}$ are such that, for all ν, the following hold almost surely

$$E(\xi_\nu | \mathcal{F}_{\nu-1}) = 0 , \tag{3.32}$$

$$E(\xi_\nu^2 | \mathcal{F}_{\nu-1}) = 1 . \tag{3.33}$$

Here $\mathcal{F}_\nu$ is the σ-field generated by ξ_μ, $\mu \leq \nu$.

Many statistics are concerned with the covariance matrix $\Sigma_T = (\gamma(j-k))$, $1 \leq j, k \leq T$. Therefore it is of importance to have uniform convergence of $\hat{\gamma}_T(t)$ to $\gamma(t)$. If the process $\{ Y_t \}$ is ergodic it has been shown that

$$\lim_{T\to\infty} \sup_{-\infty<t<\infty} |\hat{\gamma}_T(t) - \gamma(t)| = 0 \text{ , a.s.}$$

as $T \to \infty$. (See, for example, Hannan, 1974.) In this section we shall obtain a strong law of large numbers for $\hat{\gamma}_T(t)$ under different conditions on $\{ \xi_\nu \}$ and we do not assume that $\{ Y_t \}$ is ergodic. In Theorem 3.7 we are concerned with relaxing the i.i.d. assumption of $\{ \xi_\nu \}$ and we are interested in the case where LRD may occur. We shall show that $\hat{\gamma}_T(t)$ converges to $\gamma(t)$, uniformly in t, in the mean square sense and with probability one.

THEOREM 3.7 *Let $\{ Y_t \}$ be given by (3.2) and (3.3), therefore, $\mu = 0$ in (3.31). Assume further that the covariance function $\gamma(t)$ and the fourth moments of $\{ \xi_t \}$ satisfy the following conditions respectively*

$$|\gamma(t)| \sim const\, |t|^{2H-2} \text{ , as } T \to \infty \text{ , where } 0 < H < 1 \text{ ,} \tag{3.34}$$

$$\max_\nu \left\{ E\xi_\nu^4 \right\} < \infty \text{ ,} \tag{3.35}$$

$$E\xi_u\xi_v\xi_s\xi_t = \begin{cases} E\xi_u^4 \text{ ,} & u = v = s = t \text{ ,} \\ E\xi_u^2 E\xi_t^2 \text{ ,} & u = v > s = t \text{ ,} \\ 0 \text{ ,} & u = v > s > t \text{ or } u > v = s > t \text{ or } u > v > s \geq t \text{ .} \end{cases} \tag{3.36}$$

Then

$$E\left|\hat{\gamma}_T(t)-\gamma(t)\right|^2 \longrightarrow 0\ ,\ as\ T\to\infty \tag{3.37}$$

uniformly in t. If, furthermore, the fourth moments of $\{Y_t\}$ *are such that*

$$\max_t E|Y_t|^4 < \infty\,, \tag{3.38}$$

then, as $T\to\infty$*,*

$$\hat{\gamma}_T(t)-\gamma(t) \longrightarrow 0 \text{ with probability one,} \tag{3.39}$$

uniformly in t.

We give the proof of Theorem 3.7 in Section 3.5

By Theorem 3.7 we can often use $\hat{\gamma}_T(t)$ to replace $\gamma(t)$ for any fixed t and T large enough when $\gamma(t)$ is unknown.

As an application of Theorem 3.7 let us consider the statistic:

$$I_T(\omega)=\frac{1}{2\pi T}\sum_{s,t=1}^{T} Y_sY_te^{i(t-s)\omega}=\frac{1}{2\pi}\sum_{t=-T+1}^{T-1}\hat{\gamma}_T(t)e^{-it\omega} \tag{3.40}$$

and form

$$\hat{R}_T(\tau)=\int_{-\pi}^{\pi} I_T(\omega)e^{-i\tau\omega}\phi(\omega)d\omega\ . \tag{3.41}$$

Here $\phi(\omega)$ is a continuous weight function reflecting the relative importance of the various frequencies. The optimal choice of $\phi(\omega)$ will depend on the nature of the spectrum of $\{\,Y_t\,\}$. Write

$$R(\tau)=\int_{-\pi}^{\pi} f(\omega)e^{-i\tau\omega}\phi(\omega)d\omega \tag{3.42}$$

where $f(\omega)$ is the spectral density of $\{\,Y_t\,\}$. Precisely as in Theorem 2 of Hannan (1974), it may be shown that

THEOREM 3.8 *Under the conditions of Theorem 3.7,*

$$\lim_{T\to\infty}\ \sup_{-\infty<\tau<\infty}\left|\hat{R}_T(\tau)-R(\tau)\right|=0 \tag{3.43}$$

with probability one and in probability.

3.4 Testing the Mean of Linear Processes with Long Range Dependence

In this section we are concerned with the problem of testing the mean function of a time series. Consider a time series $\{X_t\}$ of the form

$$X_t = Y_t + m_t \ , t = 1 \ , 2 \ , \dots \ , \tag{3.44}$$

where $\{Y_t\}$ is a stationary process with mean zero and $\{m_t\}$ is the mean function of $\{X_t\}$. As a prelude to any times series analysis it would be prudent to first test the null hypothesis

$$H_0 : \qquad m_t = 0 \ , \quad t = 1 \ , 2 \ , \dots \tag{3.45}$$

against all alternatives.

The problem of testing the mean function has been solved in a formal sense by Grenander and Rosenblatt (1957) in the case when the process $\{Y_t\}$ is Gaussian and a simple hypothesis (3.45) is being tested against a simple alternative. In detail, given the observations of a Gaussian process with a sample size N, a likelihood ratio test will lead to a test statistic $\mathbf{X}'\Sigma^{-1}\mathbf{X}$ where $\mathbf{X}' = (X_1 \ , \dots \ , X_N)$, $\Sigma = (\gamma(j-k))$, $1 \le j, \ k \le N$, and large values of the statistic are significant. Under H_0, $\mathbf{X}'\Sigma^{-1}\mathbf{X}$ has a χ^2_N distribution and it can easily be seen that a test based on this statistic is consistent against all alternatives for which

$$\lim_{N\to\infty} N^{-1/2}\mathbf{m}'\Sigma^{-1}\mathbf{m} = \infty \tag{3.46}$$

where $\mathbf{m}' = (m_1 \ , , \dots \ , m_N)$. (See, for example, Parthasarathy,1961.) Of course we are assuming here that Σ is known.

In his 1961 paper, Parthasarathy formally solved the problem of testing the null hypothesis (3.45) against all the alternatives which satisfy condition (3.46) when the process $\{Y_t\}$ is a linear process

$$Y_t = \sum_{\nu=-\infty}^{\infty} a_{t-\nu}\xi_\nu \ .$$

Here and in the rest of this section we use a_ν to replace b_ν in order for the notation to be consistent with that of Parthasarathy's. The proposed test statistic based on the observations $\{X_1, X_2, \dots, X_N\}$ is defined as:

$$T_N = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} |\sum_{t=1}^{N} X_t e^{it\omega}|^2 / f(\omega) d\omega \ , \tag{3.47}$$

where $f(\omega)$ is the spectral density of $\{Y_t\}$. Parthasarathy (1961) derived the asymptotic normality of $\sqrt{N}(T_N/N - 1)$ under the conditions that $\{\xi_\nu\}$ is an i.i.d. random sequence and $a_\nu = O(\nu^{-\beta})$, as $\nu \to \infty$, with $\beta > 3/2$; therefore $\sum |a_\nu| < \infty$, in other words, $\{Y_t\}$ is SRD. Under the asymptotic normality of $\sqrt{N}(T_N/N - 1)$ the test for the null hypothesis of (3.45) is as follows: reject H_0 if and only if

$$\sigma^{-1}\sqrt{N}\,(T_N/N - 1) > K_\alpha \ , \tag{3.48}$$

where K_α is the α percent point of the standard normal distribution, σ is the variance of the limiting distribution of $\sqrt{N}(T_N/N - 1)$.

Parthasarathy also proved that when $\{\xi_\nu\}$ are normally distributed the test given by the critical region

$$\sigma^{-1}(N)^{\frac{1}{2}}(T_N/N - 1) > K_\alpha \tag{3.49}$$

is consistent against all alternatives $\{m_t : t = 0, 1, 2, \dots\}$ which satisfy condition (3.46).

In this section we first show, in Theorem 3.9, that if $m_t \geq C\,t^{-\alpha}$ for some constant $C > 0$ and $0 < \alpha < 1$, then (3.46) holds if $\alpha < \frac{1}{4}$ in the case of SRD and $\alpha < \frac{3}{4} - H$ in the case of LRD (which is relevant only when $H < \frac{3}{4}$). Then we consider the same testing problem (3.45) for the process $\{Y_t\}$ given by (3.2) and (3.3). In Theorem 3.10 we show the consistency of the test based on (3.49) against all alternatives satisfying a version of condition (3.46), in which we use matrix B_N, defined by (3.50) below, instead of Σ^{-1}.

The main result of this section is Theorem 3.11. We have seen that the test for (3.45) and its consistency are based on the asymptotic normality of $\sqrt{N}(T_N/N - 1)$. In Theorem 3.11 we assume that $\{\xi_\nu\}$ is an i.i.d. random sequence with finite fourth moment, and, instead of the SRD of $\{Y_t\}$, we assume that the spectral density of $\{Y_t\}$ satisfies certain conditions, which allow the presence of both SRD and LRD, therefore relaxing the restrictions given in Parthasarathy's (1961) paper. Under our general conditions we will prove that the CLT holds for T_N.

THEOREM 3.9 *Assume that the mean function $\{m_t\}$ in (3.44) satisfies*

$$m_t > C\,t^{-\beta}\,,\ t = 1\,,\,2\,,\,\ldots$$

for some constant C, and $0 < \beta < 1$. Then, (i) in the case of SRD, (3.46) holds if $\beta < \frac{1}{4}$. (ii) in the case of LRD, where $\gamma(t) \sim const\ t^{2H-2}$, as $|t| \to \infty$, $\frac{1}{2} < H < 1$, (3.46) holds if $\beta < \frac{3}{4} - H$ and $H < \frac{3}{4}$.

The proof of Theorem 3.9 will be given in Section 3.5.

We now introduce some notation. Let $\{Y_t\}$ satisfy (3.2) and (3.3), that is

$$Y_t = \sum_{\nu=-\infty}^{\infty} a_{t-\nu}\xi_\nu\ ,\quad \sum_{j=-\infty}^{\infty} a_j^2 < \infty\ ,\quad E\xi_\nu = 0\,,\quad E\xi_\nu\xi_\mu = \delta_{\nu,\mu}\ .$$

Write

$$\begin{aligned} I_{N,X}(\omega) &= \frac{1}{2\pi N}\left|\sum_{t=1}^{N} X_t e^{it\omega}\right|^2 , & I_{N,Y}(\omega) &= \frac{1}{2\pi N}\left|\sum_{t=1}^{N} Y_t e^{it\omega}\right|^2 , \\ \gamma(t) &= \int_{-\pi}^{\pi} e^{it\omega} f(\omega)d\omega\ , & b(t) &= \int_{-\pi}^{\pi} e^{it\omega} f^{-1}(\omega)d\omega\ , \\ R_N &= (\gamma(t-s))_{t,s=1,N}\ , & B_N &= (b(t-s))_{t,s=1,N}\ . \end{aligned} \tag{3.50}$$

Note that R_N and B_N are N by N Toeplitz matrices with entries $\gamma(t-s)$ and $b(t-s)$ respectively.

Consider the process (3.44), the null hypothesis (3.45) and the test statistic (3.47). We have a consistency result given by:

THEOREM 3.10 *If the limiting distributions of*

$$\sqrt{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{I_{N,Y}(\omega)}{f(\omega)}d\omega - 1\right) \tag{3.51}$$

and

$$\frac{1}{\sqrt{N}}\sum_{t,s=1}^{N} m_t Y_s b(t-s) \tag{3.52}$$

exist as $N \to \infty$, and if

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}[m_1,\cdots,m_N]B_N[m_1,\cdots,m_N]' = \infty \ , \tag{3.53}$$

then the test given by the critical region

$$\sigma^{-1}\sqrt{N}\,(T_N/N - 1) > K_\alpha \ , \tag{3.54}$$

where K_α is the upper α percent point of the limit distribution for (3.51), is consistent against all alternatives $\{\, m_t \ : t = 1\,, \ \ ,\ldots\}$ satisfying condition (3.53).

Remarks on Theorem 3.10.

1. The conditions under which (3.51) holds will be given by Theorem 3.11.

2. Regarding condition (3.53), we notice that B_N is a Toeplitz matrix with the entries uniformly bounded. Therefore if the trend $\{\, m_t \,\}$ is very small, condition (3.53) may not hold. For example, assume

$$m_t \ = \ \begin{cases} |t|^{-\beta}\,, & t \neq 0\,, \\ 0\,, & t = 0\,, \end{cases}$$

where $\frac{3}{4} < \beta < 1$, then,

$$\begin{aligned} \left|\frac{1}{\sqrt{N}}\sum_{t,s=1}^{N} m_t m_s b(t-s)\right| &\leq \frac{const}{\sqrt{N}}\sum_{t,s=1}^{N}\left|m_t m_s\right| \sim \frac{const}{\sqrt{N}}N^{2(1-\beta)} \\ &\sim const\ N^{3/2-\beta} \longrightarrow 0 \end{aligned}$$

as $N \to \infty$. If $1/2 < \beta \leq 3/4$ and $f(\omega)$ is such that

$$m \leq f^{-1}(\omega) \leq M$$

for any real ω , where m and M are two constants, then by the results of Grenander and Szegö (1958, p. 64, (b)) we know that $0 \le \sum m_t m_s b(t-s) \longrightarrow 0$ as $N \to \infty$. Thus for a trend $\{m_t\}$ satisfying $m_t \sim const\,|t|^{-\beta}$ as $t \to \infty$, if $\beta > 3/4$, or if $1/2 < \beta \le 3/4$ and $f^{-1}(\omega)$ is bounded on $\mathbf{R}^1$, condition (3.53) does not hold. If $\beta < 1/2$, $\sum_t m_t^2 = \infty$, in this case, we cannot use the result of Grenander and Szegö (1958) and it is very difficult to estimate the quadratic $\sum_{t,s} m_t m_s b(t-s)$. So for $0 < \beta < 1/2$, Condition (3.53) is unclear for general spectral density $f(\omega)$.

Proof of Theorem 3.10. In order to prove Theorem 3.10, it is sufficient to show that $P\Big(reject\ H_0 \,|\, \{m_t\}\Big) \to 1$ as $N \to \infty$.

Since

$$\begin{aligned}
P\left(reject\ H_0 \,|\, \{m_t\}\right) &= P\left\{\frac{\sqrt{N}}{\sigma}\left(\frac{T_N}{N}-1\right) > K_\alpha \,\middle|\, \{m_t\}\right\} \\
&= P\left\{\frac{\sqrt{N}}{\sigma}\left(\frac{1}{4\pi^2 N}\int_{-\pi}^{\pi}\frac{|\sum_{t=1}^N X_t e^{it\omega}|^2}{f(\omega)}d\omega - 1\right) > K_\alpha \,\middle|\, \{m_t\}\right\},
\end{aligned}$$

and we notice that,

$$\begin{aligned}
|\sum_{t=1}^N X_t e^{it\omega}|^2 &= \sum_{s,t=1}^N X_s X_t e^{i(t-s)\omega} = \sum_{s,t=1}^N (Y_s + m_s)(Y_t + m_t)\, e^{i(t-s)\omega} \\
&= \sum_{s,t=1}^N (Y_s Y_t + m_s Y_t + m_t Y_s + m_s m_t)\, e^{i(t-s)\omega} \\
&= 2\pi N I_{N,Y}(\omega) + \sum_{s,t=1}^N (m_s Y_t + m_t Y_s + m_s m_t)\, e^{i(t-s)\omega},
\end{aligned}$$

then,

$$\begin{aligned}
P\left(reject\ \ H_0 | \{m_t\}\right) = P\Bigg\{ & \frac{\sqrt{N}}{\sigma}\Bigg(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{I_{N,Y}(\omega)}{f(\omega)}d\omega - 1 \\
& + \frac{1}{4\pi^2 N}\sum_{s,t=1}^N m_s Y_t b(t-s) + \frac{1}{4\pi^2 N}\sum_{s,t=1}^N m_t Y_s b(t-s) \\
& + \frac{1}{4\pi^2 N}\sum_{s,t=1}^N m_s m_t b(t-s)\Bigg) > K_\alpha \,\Bigg|\, \{m_t\}\Bigg\} \\
= 1 &
\end{aligned}$$

by (3.51), (3.52) and (3.53), which proves Theorem 3.10.

Now we consider the asymptotic distribution of the test statistic T_N. For the case where $\{ Y_t \}$ in (3.44) is a linear process, Parthasarathy (1961) derived the asymptotic normality of T_N, after suitably norming and centring, under the conditions that $a_\nu = O(\nu^{-\beta})$, where $\beta > 3/2$, the spectral density $f(\omega)$ of $\{ Y_t \}$ satisfies $\int_{-\pi}^{\pi}(f(\omega))^{-1} exp\{ i\, t\, \omega \}\, d\omega = O(t^{-1})$, and $\{ \xi_\nu \}$ is an i.i.d. random sequence. Here, the summability of $\{ a_\nu \}$ was of essential importance for the proof of Parthasarathy's results. When $\sum_\nu |a_\nu| < \infty$, $\{ Y_t \}$ is SRD. In our case we are concerned with the situation where $\{ Y_t \}$ is LRD. In other words, the summability of a_ν's is not necessary. The aim of Theorem 3.11 is to show that the CLT holds for T_N under more general conditions on the moving average coefficients a_ν and the spectral density of $\{ Y_t \}$.

THEOREM 3.11 *Suppose $\{ Y_t \}$ in (3.44) satisfy (3.2) and (3.3) and the spectral density $f(\omega)$ and the covariance function $\gamma(t)$ of $\{ Y_t \}$ satisfy*

$$f(\omega) \sim L(\omega)|\omega|^{1-2H}\,,\ as\ \omega \to 0\ ,\quad \gamma(t) \sim const\ t^{2H-2}\,,\ as\ t \to \infty \tag{3.55}$$

respectively, where $L(\omega)$ is slowly varying at $\omega = 0$, $0 < H < 1$ and the innovations $\{ \xi_\nu \}$ are an i.i.d. random sequence with $E\xi_0^4 < \infty$. Assume further that one of the two conditions (C1), (C2) is fulfilled

(C1) $\frac{1}{2} < H < \frac{3}{4}$,

(C2) *$f^{-1}(\omega)$ is absolutely continuous and that the derivative $Df^{-1}(\omega)$ is equal almost everywhere to an absolutely continuous function.*

Then under the null hypothesis (3.45)

$$\sqrt{N}\left(\frac{T_N}{N} - 1\right) \xrightarrow{\mathrm{d}} N(0, \sigma^2) \tag{3.56}$$

as $N \to \infty$, where $\sigma^2 = E\xi_0^4 + 1$. The proposed test of (3.45) is as follows: Reject

H_0 *if and only if*

$$\sigma^{-1}\sqrt{N}\left(\frac{T_N}{N} - 1\right) > K_\alpha$$

where K_α *is the upper* α *percent point of the normal distribution with mean zero and variance one.*

Remarks on Theorem 3.11.

1. If the moving average coefficients of $\{Y_t\}$ satisfy $a_\nu = O(\nu^{-\beta})$, the Hurst index H and the exponential index β are related by $H = \frac{3}{2} - \beta$. Thus, when $3/4 < \beta < 1$, $1/2 < H < 3/4$. By Theorem 3.11 (under condition C1), the CLT holds for T_N. Together with Parthasarathy's results, we conclude that the asymptotic normality of T_N holds when either $3/4 < \beta < 1$ or $3/2 < \beta$. When $1 < \beta \leq 3/2$, we have $0 < H < 1/2$. Then $\{Y_t\}$ is a SRD process. If further the spectral density $f(\omega)$ satisfies $f(\omega) \sim const\ |\omega|^{1-2H}$, as $\omega \to 0$, or more generally, if the spectral density of a stationary linear process satisfies $f(\omega) \sim L(\omega)|\omega|^{1-2H}$, as $\omega \to 0$, where $L(\omega)$ is a slowly varying function at the origin, then $f^{-1}(\omega) \sim L^{-1}(\omega)|\omega|^{2H-1}$. Therefore it is reasonable to assume that $f^{-1}(\omega)$ is subject to Dirichlet's Conditions, (see, for example, Carslaw, 1930, pp. 226 and 269), thus $\int_{-\pi}^{\pi} f^{-1}(\omega)exp\{\,i\,t\,\omega\,\}\,d\omega = O(t^{-1})$, which is one of the assumptions made by Parthasarathy (1961). If, furthermore, the derivative $Df^{-1}(\omega)$ satisfies condition (C2) of Theorem 3.11, the CLT holds for T_N.

2. In reality, it is not unusual that we have a sequence of observations $\{X_1, X_2, \ldots, X_N\}$ from a stochastic process whose spectral density is unknown. In this case we try to replace T_N in (3.47) by

$$\hat{T}_N = \frac{N}{2\pi}\int_{-\pi}^{\pi} I_{N,X}(\omega)\hat{f}_M^{-1}(\omega)\,d\omega\,,$$

where $I_{N,X}$ is the periodogram of observation $\{X_1, X_2, \ldots, X_N\}$, M is an integer such that as $N \to \infty$,

$$\frac{1}{M} + \frac{M}{N} \longrightarrow 0\,,$$

and $\hat{f}_M(\omega)$ is an estimator of $f(\omega)$, such as a Truncated estimator, Bartlett estimator, Tukey-Hanning estimator or similar other. When $\{X_t\}$ is SRD, the asymptotics of $\hat{f}_M(\omega)$ are well known. See, for example, Hannan (1970, pp. 273 – 281). It may be reasonable to use $\hat{T}_N$ to replace T_N for the hypothesis testing problem (3.45). When $\{X_t\}$ is LRD, we have been investigating the asymptotics of $\hat{f}_M(\omega)$. We hope that

$$\sqrt{N}\left(\frac{\hat{T}_N}{N}-1\right)$$

is asymptotically normally distributed. If this is so, we can certainly use $\hat{T}_N$ to replace T_N for the hypothesis testing problem (3.45)

3.5 Proofs

3.5.1 Proof of Theorem 3.4

Let

$$\zeta_{T,1}=\sum_{|k|>N} a_{k,T}\xi_k \ , \quad \zeta_{T,j}=a_{-N+j-2,T}\xi_{-N+j-2} \ , \quad j=2\ ,\ \dots\ ,\ 2N+2\ , \tag{3.57}$$

where $N=N(T)$ is given by Condition A. Let

$$\begin{aligned} S_{T,k} &= \sum_{j=2}^{2k+2}\zeta_{T,j} \\ &= a_{-N,T}\xi_{-N}+a_{-N+1,T}\xi_{-N+1}+\cdots+a_{-N+2k}\xi_{-N+2k} \end{aligned} \tag{3.58}$$

for $k=0\ ,\ 1\ ,\ \dots\ ,\ N(T)$; then

$$\sigma_T^{-1}S_T=\sum_{k=-\infty}^{\infty}a_{k,T}\xi_k=\zeta_{T,1}+S_{T,N(T)}\ .$$

Theorem 3.4 can be established by showing that, as $T\to\infty$,

$$\zeta_{T,1}\xrightarrow{\mathrm{P}}0\ , \tag{3.59}$$

and

$$S_{T,N(T)} \xrightarrow{\mathrm{d}} \mathrm{N}(0\ ,\ \sigma^2)\ ,\ \text{as } T \to \infty\ . \tag{3.60}$$

To deal with(3.59) we notice that by Condition A,

$$E|\zeta_{T,1}|^2 = E|\sum_{|k|>N} a_{k,T}\xi_k|^2 = \sum_{|k|>N} a_{k,T}^2 < \varepsilon_T \longrightarrow 0$$

as $T \to \infty$. Then

$$\zeta_{T,1} \xrightarrow{\mathrm{P}} 0\ .$$

Before we start to prove (3.60) we need the following lemmas

LEMMA 3.12

$$|a_{k,T}| \le a_T \equiv \frac{1}{\sigma_T}\left\{4\left(\sum_{j=-\infty}^{\infty} b_j^2\right)^{1/2}\sigma_T\ +\ 4\sum_{j=-\infty}^{\infty} b_j^2\right\}^{1/2}\ . \tag{3.61}$$

We shall give the proof of Lemma 3.12 after the proof of Theorem 3.4.

LEMMA 3.13 *Let* $\{S_{ni}\ ,\ \mathcal{F}_{n,i}\ ,\ 1 \le i \le k_n\ ,\ n \ge 1\}$ *be a zero mean square integrable martingale array with differences* X_{ni}*, and let* η^2 *be an a.s finite random variable. Suppose that*

$$\max_i |X_{ni}| \xrightarrow{\mathrm{P}} 0\ , \tag{3.62}$$

$$\sum_i X_{ni}^2 \xrightarrow{\mathrm{P}} \eta^2 \tag{3.63}$$

$$E(\max_i X_{ni}^2) \text{ is bounded in } n \tag{3.64}$$

and the σ*-fields* $\mathcal{F}_{n,i}$ *are nested*

$$\mathcal{F}_{n,i} \subseteq \mathcal{F}_{n+1,i} \text{ for } 1 \le i \le k_n\ ,\ n \ge 1\ . \tag{3.65}$$

Then $S_{nk_n} = \sum_i X_{ni} \xrightarrow{\mathrm{d}} Z$ *(stably), where the random variable* Z *has characteristic function* $E\exp\{-\frac{1}{2}\eta^2 t^2\}$.

Lemma 3.13 is Theorem 3.2 of Hall and Heyde (1980, p. 58).

To deal with (3.60) let

$$\mathcal{F}_{T,l} = \sigma\{\, \xi_{-N(T)}\,,\, \xi_{-N(T)+1}\,,\, \cdots\,,\, \xi_{-N(T)+2l}\,\}\,,\, l = 1\,,\, 2\,,\, \ldots\,,\, N(T)\,.$$

Obviously $\mathcal{F}_{T,l}$ is nested. From (3.57) and (3.58) $ES_{T,k} = 0$. For $k \geq l$, by (3.28)

$$\begin{aligned} E\,(\, S_{T,k} \,|\, \mathcal{F}_{T,l}\,) &= E\,(a_{-N,T}\xi_{-N} + \cdots + a_{-N+2k,T}\xi_{-N+2k} \,|\, \mathcal{F}_{T,l}) \\ &= a_{-N,T}\xi_{-N} + \cdots + a_{-N+2l,T}\xi_{-N+2l} = S_{T,l}\,. \end{aligned}$$

So $\{\, S_{T,k}\,,\, \mathcal{F}_{T,k} \;:\; k = 1\,,\, \ldots\,,\, N(T)\,\}$ is a zero mean square integrable martingale array. Now we show that conditions (3.62) – (3.65) are fulfilled by our martingale array with η^2 in (3.63) being a constant; then (3.60) follows from Lemma 3.13. Put

$$X_{T,j} = S_{T,j} - S_{T,j-1} = a_{-N+2j,T}\xi_{-N+2j} + a_{-N+2j-1,T}\xi_{-N+2j-1}\,.$$

Then by (3.27) and Lemma 3.12, for any $\epsilon > 0$,

$$\begin{aligned} & P\left(\max_{1\leq j\leq N(T)} \{|X_{T,j}|\} \geq \epsilon\right) = P\left(\cup_{j=1}^{N(T)} |X_{T,j}| \geq \epsilon\right) \\ \leq\ & \sum_{j=1}^{N(T)} P(|X_{T,j}| \geq \epsilon) \leq \sum_{j=1}^{N(T)} \frac{E|X_{T,j}|^{2+\delta}}{\epsilon^{2+\delta}} \\ =\ & \frac{1}{\epsilon^{2+\delta}} \sum_{j=1}^{N(T)} E\left|a_{-N+2j,T}\xi_{-N+2j} + a_{-N+2j-1,T}\xi_{-N+2j-1}\right|^{2+\delta} \\ \leq\ & \frac{1}{\epsilon^{2+\delta}} \sum_{j=1}^{N(T)} E\left\{2^{1+\delta}\left(|a_{-N+2j,T}\xi_{-N+2j}|^{2+\delta} + |a_{-N+2j-1,T}\xi_{-N+2j-1}|^{2+\delta}\right)\right\} \\ \leq\ & \frac{2^{1+\delta}}{\epsilon^{2+\delta}} \max_{1\leq j\leq N(T)} \left\{E|\xi_j|^{2+\delta}\right\} \sum_{j=1}^{N(T)} \left(|a_{-N+2j,T}|^{2+\delta} + |a_{-N+2j-1,T}|^{2+\delta}\right) \\ \leq\ & \frac{2^{1+\delta}}{\epsilon^{2+\delta}} \max_{-\infty<j<\infty} \left\{E|\xi_j|^{2+\delta}\right\} a_T^{\delta} \sum_{j=1}^{N(T)} \left(a^2_{-N+2j,T} + a^2_{-N+2j-1,T}\right) \\ \leq\ & \frac{2^{2+\delta}}{\epsilon^{2+\delta}} \max_{-\infty<j<\infty} \left\{E|\xi_j|^{2+\delta}\right\} a_T^{\delta} \longrightarrow 0\,,\ \text{as } T \to \infty \end{aligned}$$

by (3.61) and Condition 3 of Theorem 3.4. So $\{\, X_{T,j}\,\}$ is uniformly negligible.

To deal with (3.63) we notice that, by (3.26), (3.61) and Condition 3 of the theorem,

$$\begin{aligned}
\sum_{j=1}^{N(T)} X_{T,j}^2 &= \sum_j \left(a_{-N+2j,T}\xi_{-N+2j} + a_{-N+2j-1,T}\xi_{-N+2j-1}\right)^2 \\
&= \sum_j \Big(a_{-N+2j,T}^2\xi_{-N+2j}^2 + 2a_{-N+2j,T}\xi_{-N+2j}a_{-N+2j-1,T}\xi_{-N+2j-1} \\
&\qquad\qquad +a_{-N+2j-1,T}^2\xi_{-N+2j-1}^2\Big) .
\end{aligned}$$

Now

$$\begin{aligned}
&E\left|\sum_{j=1}^{N(T)} a_{-N+2j,T}^2(\xi_{-N+2j}^2 - 1)\right|^2 \\
&= \sum_{j,k=1}^{N(T)} a_{-N+2j,T}^2 a_{-N+2k,T}^2 E(\xi_{-N+2j}^2 - 1)\xi_{-N+2k}^2 - 1) \\
&= \sum_{j,k=1}^{N(T)} a_{-N+2j,T}^2 a_{-N+2k,T}^2 E\left(\xi_{-N+2j}^2\xi_{-N+2k}^2 - 1\right) \\
&= \sum_{j=1}^{N(T)} a_{-N+2j,T}^4 E\left(\xi_{-N+2j}^4 - 1\right) \\
&\le \left|E\xi_0^4 - 1\right| a_T^2 \sum_{j=1}^{N(T)} a_{-N+2j,T}^2 \longrightarrow 0 \text{ , as } T \to \infty ,
\end{aligned}$$

and

$$\begin{aligned}
&E\left|\sum_{j=1}^{N(T)} a_{-N+2j,T}\xi_{-N+2j}a_{-N+2j-1,T}\xi_{-N+2j-1}\right|^2 \\
&= \sum_{j,k=1}^{N(T)} a_{-N+2j,T}a_{-N+2j-1,T}a_{-N+2k,T}a_{-N+2k-1,T} \\
&\qquad E\xi_{-N+2j}\xi_{-N+2j-1}\xi_{-N+2k}\xi_{-N+2k-1} \\
&= \sum_{j=1}^{N(T)} a_{-N+2j,T}^2 a_{-N+2j-1,T}^2 E\xi_{-N+2j}^2\xi_{-N+2j-1}^2 \\
&= \left(E\xi_0^2\right)^2 \sum_{j=1}^{N(T)} a_{-N+2j,T}^2 a_{-N+2j-1,T}^2 \\
&\le \left(E\xi_0^2\right)^2 a_T^2 \sum_{j=1}^{N(T)} a_{-N+2j,T}^2 \le \left(E\xi_0^2\right)^2 a_T^2 \longrightarrow 0 ,
\end{aligned}$$

as $T \to \infty$. Hence

$$\sum_{j=1}^{N(T)} X_{T,j}^2 - \sum_{j=1}^{N(T)} a_{-N+2j,T}^2 - \sum_{j=1}^{N(T)} a_{-N+2j-1,T}^2 \xrightarrow{\mathrm{P}} 0 \ , \text{ as } T \to \infty \ .$$

By Condition 2 of the theorem we have

$$\sum_{j=1}^{N(T)} X_{T,j}^2 \xrightarrow{\mathrm{P}} \lim_{T\to\infty} \left\{ \sum_{j=1}^{N(T)} a_{j-1,T}^2 + \sum_{j=1}^{N(T)} a_{j,T}^2 \right\} \equiv \sigma^2 \ .$$

Finally, by (3.61),

$$\begin{aligned}
E\left(\max_{1\le j\le N(T)} X_{T,j}^2\right) &= E\left(\max_{1\le j\le N(T)} \left\{ \left(a_{-N+2j,T}\xi_{-N+2j} + a_{-N+2j-1,T}\xi_{-N+2j-1}\right)^2 \right\}\right) \\
&\le \ 4E \max_{1\le j\le N(T)} \left\{ a_{-N+2j,T}^2 \xi_{-N+2j}^2 + a_{-N+2j-1,T}^2 \xi_{-N+2j-1}^2 \right\} \\
&\le \ 8E \max_{-N(T)\le j\le N(T)} \left\{ a_{j,T}^2 \xi_j^2 \right\} \le 8E\left(\sum_{j=-N(T)}^{N(T)} a_{j,T}^2 \xi_j^2\right) \le 8 \sum_{j=-N(T)}^{N(T)} a_{j,T}^2 \le 8 \ ,
\end{aligned}$$

and therefore (3.64) holds. Thus by Lemma 3.13

$$S_{T,N(T)} \xrightarrow{\mathrm{d}} \mathbf{N}(0\ ,\ \sigma^2)\ ,$$

as $T \to \infty$. Therefore, as $T \to \infty$, we have

$$\sigma_T^{-1} S_T = \zeta_{T,1} + S_{T,N(T)} \xrightarrow{\mathrm{d}} \mathbf{N}(0\ ,\ \sigma^2)\ .$$

This ends the proof of Theorem 3.4

Proof of Lemma 3.12. By the definition of $b_{j,T}$ we have

$$\begin{aligned}
b_{j,T}^2 &= (b_{j-1} + \cdots + b_{j-T})^2 \\
&= \ (b_{j-1} + b_{j-1-1} + \cdots + b_{j-1-(T-1)})^2 \\
&= \ \Big(b_{j-1,T} + (b_{j-1} - b_{j-1-T})\Big)^2 \\
&= \ b_{j-1,T}^2 + 2(b_{j-1} - b_{j-1-T})b_{j-1,T} + (b_{j-1} - b_{j-1-T})^2\ .
\end{aligned}$$

Summing over $j = k-l\,,\ k-l+1\,,\ \ldots\,,\ k$ gives

$$b_{k-l,T}^2 = b_{k-l-1,T}^2 \ + \ 2\sum_{j=k-l}^{k} (\,b_{j-1} - b_{j-1-T}\,)\, b_{j-1,T} \ + \ \sum_{j=k-l}^{k} (\,b_{j-1} - b_{j-1-T}\,)^2$$

$$
\begin{aligned}
&\le b_{k-l-1,T}^2 \;+\; 2\sum_{j=k-l}^{k}(\,b_{j-1} - b_{j-1-T}\,)\,b_{j-1,T} \;+\; 2\sum_{j=k-l}^{k}(\,b_{j-1}^2 \;+\; b_{j-1-T}^2\,)\\
&\le b_{k-l-1,T}^2 \;+\; 2\sum_{j=k-l}^{k}(\,b_{j-1} - b_{j-1-T}\,)\,b_{j-1,T} + 4\sum_{j=-\infty}^{\infty} b_j^2\\
&\le b_{k-l-1,T}^2 \;+\; 2\left\{\sum_{j=k-l}^{k}(\,b_{j-1} - b_{j-1-T}\,)^2\right\}^{1/2}\left\{\sum_{j=k-l}^{k} b_{j-1,T}^2\right\}^{1/2} \;+\; 4\sum_{j=-\infty}^{\infty} b_j^2\\
&\le b_{k-l-1,T}^2 \;+\; 2\left\{4\sum_{j=-\infty}^{\infty} b_j^2\right\}^{1/2}\left\{\sum_{j=-\infty}^{\infty} b_{j-1,T}^2\right\}^{1/2} \;+\; 4\sum_{j=-\infty}^{\infty} b_j^2\\
&= \;\; b_{k-l-1,T}^2 \;+\; 4\left\{\sum_{j=-\infty}^{\infty} b_j^2\right\}^{1/2}\sigma_T \;+\; 4\sum_{j=-\infty}^{\infty} b_j^2\,.
\end{aligned}
$$

So

$$
\left|\frac{b_{k,T}}{\sigma_T}\right|^2 \le \frac{1}{\sigma_T^2}\left\{\; b_{k-l-1,T}^2 \;+\; 4\left(\sum_{j=-\infty}^{\infty} b_j^2\right)^{1/2}\sigma_T \;+\; 4\sum_{j=-\infty}^{\infty} b_j^2\;\right\}
$$

for any l. Let $l \to \infty$, we have

$$
\left|\frac{b_{k,T}}{\sigma_T}\right| \le \frac{1}{\sigma_T}\left\{\;4\left(\sum_{j=-\infty}^{\infty} b_j^2\right)^{1/2}\sigma_T \;+\; 4\sum_{j=-\infty}^{\infty} b_j^2\;\right\}^{1/2}.
$$

This completes the proof of Lemma 3.12

3.5.2 Proof of Theorem 3.7

We prove (3.37) first. Since for $|t| \ge T$,

$$
|\,\hat{\gamma}_T(t) - \gamma(t)\,| = |\,\gamma(t)\,| \sim const\,|t|^{2H-2} \le const\;T^{2H-2} \longrightarrow 0
$$

as $T \to \infty$, we need only to consider the case where $0 \le |t| \le T-1$. Now

$$
\hat{\gamma}_T(t) - \gamma(t) = \frac{1}{T}\sum_{j,k=-\infty}^{\infty} b_j\,b_k\sum_{s=1}^{T-t}\{\,\xi_{s-j}\,\xi_{s+t-k} - \delta_{j,k-t}\,\} \;-\; \frac{1}{T}\sum_{j,k=-\infty}^{\infty} b_j\,b_k\,\delta_{j,k-t}\,.
$$

Since

$$
|\,\gamma(t)\,| = \left|\sum_j b_j\,b_{j+t}\right| \sim const\;|t|^{2H-2}\,,
$$

$$\frac{t}{T}\left|\sum_{j,k=-\infty}^{\infty} b_j\, b_k\, \delta_{j,k-t}\right| = \frac{t}{T}\left|\sum_j b_j\, b_{j+t}\right| = \frac{t}{T}|\,\gamma(t)\,| \sim const\ \frac{|t|\cdot|t|^{2H-2}}{T}$$
$$\leq const\ T^{2H-2} \longrightarrow 0 \tag{3.66}$$

as $T \to \infty$, in order to prove Theorem 3.7 it is sufficient to show that

$$E\left|\frac{1}{T}\sum_{j,k=-\infty}^{\infty} b_j\, b_k \sum_{s=1}^{T-t}\{\xi_{s-j}\,\xi_{s+t-k} - \delta_{j,k-t}\}\right|^2 \longrightarrow 0\ , \text{ as } T \to \infty\ , \tag{3.67}$$

uniformly in t. We have

$$\begin{aligned}
&E\left|\frac{1}{T}\sum_{j,k=-\infty}^{\infty} b_j\, b_k \sum_{s=1}^{T-t}\{\xi_{s-j}\,\xi_{s+t-k} - \delta_{j,k-t}\}\right|^2 \\
&= \frac{1}{T^2}\sum_{j_1,k_1=-\infty}^{\infty}\ \sum_{j_2,k_2=-\infty}^{\infty}\ \sum_{s_1,s_2=1}^{T-t} b_{j_1}\, b_{k_1} b_{j_2} b_{k_2} E\,\{(\xi_{s_1-j_1}\xi_{s_1+t-k_1} - \delta_{j_1,k_1-t}) \times \\
&\qquad\qquad (\xi_{s_2-j_2}\xi_{s_2+t-k_2} - \delta_{j_2,k_2-t})\} \\
&= \frac{1}{T^2}\Bigg\{\sum_{j_1,k_1=-\infty,k_1\neq j_1+t}^{\infty} b_{j_1}\, b_{k_1} \sum_{j_2,k_2=-\infty,k_2\neq j_2+t}^{\infty} b_{j_2}\, b_{k_2} \sum_{s_1,s_2=1}^{T-t} E\xi_{s_1-j_1}\xi_{s_1+t-k_1}\xi_{s_2-j_2}\xi_{s_2+t-k_2} \\
&\qquad + \sum_{j_1=-\infty}^{\infty} b_{j_1}\, b_{j_1+t} \sum_{j_2=-\infty}^{\infty} b_{j_2}\, b_{j_2+t} \sum_{s_1,s_2=1}^{T-t} E\left\{\left(\xi^2_{s_1-j_1}-1\right)\left(\xi^2_{s_2-j_2}-1\right)\right\}\Bigg\} \\
&\equiv \frac{1}{T^2}\{\,I + II\,\}\ .
\end{aligned}$$

We consider II first. By (3.35) and (3.36),

$$\begin{aligned}
II &= \sum_{j_1=-\infty}^{\infty} b_{j_1}\, b_{j_1+t} \sum_{j_2=-\infty}^{\infty} b_{j_2}\, b_{j_2+t} \sum_{s_1,s_2=1}^{T-t} E\left\{\left(\xi^2_{s_1-j_1}-1\right)\left(\xi^2_{s_2-j_2}-1\right)\right\} \\
&= \sum_{j_1=-\infty}^{\infty} b_{j_1}\, b_{j_1+t} \sum_{j_2=-\infty}^{\infty} b_{j_2}\, b_{j_2+t} \sum_{s_1=1,s_2=s_1-j_1+j_2}^{T-t} E\left(\xi^2_{s_1-j_1}-1\right)^2 \\
&\leq const\ T \sum_{j_1=-\infty}^{\infty} b_{j_1}\, b_{j_1+t} \sum_{j_2=-\infty}^{\infty} b_{j_2}\, b_{j_2+t} = const\ T\gamma(t)^2\ .
\end{aligned}$$

Since $\gamma(t)$ is uniformly bounded in t, we have

$$\frac{1}{T^2}II = \frac{const\ \gamma(t)^2}{T} \to 0\ , \text{ as } T \to \infty \tag{3.68}$$

uniformly in t. Now we estimate I as follows. By (3.3) and (3.36),

$$I = \sum_{j_1,k_1=-\infty,k_1\neq j_1+t}^{\infty} b_{j_1}\, b_{k_1} \sum_{j_2,k_2=-\infty,k_2\neq j_2+t}^{\infty} b_{j_2}\, b_{k_2} \sum_{s_1,s_2=1}^{T-t} E\xi_{s_1-j_1}\xi_{s_1+t-k_1}\xi_{s_2-j_2}\xi_{s_2+t-k_2}$$

$$
\begin{aligned}
&= \sum_{j_1,k_1=-\infty,k_1\neq j_1+t}^{\infty} b_{j_1}\, b_{k_1} \sum_{j_2,k_2=-\infty,k_2\neq j_2+t}^{\infty} b_{j_2}\, b_{k_2} \sum_{s_1,s_2=1}^{T-t} \delta_{s_1-j_1,s_2-j_2}\delta_{s_1-k_1,s_2-k_2} \\
&\qquad + \sum_{j_2,k_2=-\infty,k_2\neq j_2+t}^{\infty} b_{j_2}\, b_{k_2} \sum_{s_1,s_2=1}^{T-t} \delta_{s_1-j_1,s_2+t-k_2}\delta_{s_1+t-k_1,s_2-j_2} \\
&\equiv III + IV\ .
\end{aligned}
\tag{3.69}
$$

Now

$$
\begin{aligned}
III &= \sum_{j_1,k_1=-\infty,k_1\neq j_1+t}^{\infty} b_{j_1}\, b_{k_1} \sum_{j_2,k_2=-\infty,k_2\neq j_2+t}^{\infty} b_{j_2}\, b_{k_2} \sum_{s_1,s_2=1}^{T-t} \delta_{s_1-j_1,s_2-j_2}\delta_{s_1-k_1,s_2-k_2} \\
&= \sum_{s_1,s_2=1}^{T-t} \sum_{j_1-\infty}^{\infty} b_{j_1}\, b_{s_2-s_1+j_1} \sum_{k_1=-\infty,k_1\neq j_1+t}^{\infty} b_{k_1}\, b_{s_2-s_1+k_1} \\
&\sim \sum_{s_1,s_2=1}^{T-t} \left(\sum_{j}^{\infty} b_j\, b_{s_2-s_1+j}\right)\left(\sum_{k}^{\infty} b_k\, b_{s_2-s_1+k}\right) = \sum_{s_1,s_2=1}^{T-t} (\,\gamma(s_2-s_1)\,)^2 \\
&= \sum_{|s|<T-t} (\,T-t-|s|\,)\,(\,\gamma(s_2-s_1)\,)^2 \\
&\sim 2\sum_{s=1}^{T-t} (\,T-t-|s|\,)\,(\,\gamma(s_2-s_1)\,)^2 \sim const \sum_{s=1}^{T-t}(\,T-t-|s|\,)s^{4H-4}\ .
\end{aligned}
$$

If $4\,H-3>0$,

$$
\sum_{s=1}^{T-t}(\,T-t-s\,)\,s^{4H-4} \sim const\ (t-t)^{2+4H-3-1}B\,(\,2\,,\,4H-3\,) \leq const\ T^{4H-2}\ ,
$$

where $B(p\,,q)$ is the Beta function. If $4H-3<0$, we can choose $0<\alpha<3$ such that $4H-\alpha>0$ and $4H-\alpha-1<0$, so then we have

$$
\begin{aligned}
\sum_{s=1}^{T-t}(\,T-t-s\,)\,s^{4H-4} &= \sum_{s=1}^{T-t}(\,T-t-s\,)\,s^{4H-\alpha-1}s^{\alpha-3} \\
&\leq \sum_{s=1}^{T-t}(\,T-t-s\,)\,s^{4H-\alpha-1} \sim const\ (\,T-t\,)^{2+4H-\alpha-1} \leq const\ T^{4H-\alpha+1}\ .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{T^2}III &\leq \begin{cases} \frac{const}{T^2}T^{4H-2}, & \frac{3}{4}<H<1\,, \\ \frac{const}{T^2}T^{4H+1-\alpha}, & 0<H\leq\frac{3}{4}\,, \end{cases} \\
&= \begin{cases} const\ T^{4H-4}\,, & \frac{3}{4}<H<1\,, \\ const\ T^{4H-\alpha-1}\,, & 0<H\leq\frac{3}{4}\,, \end{cases}
\end{aligned}
\tag{3.70}
$$

and, as $T \to \infty$, $T^{-2} III \to 0$, uniformly in t. Finally, we evaluate IV in (3.69):

$$
\begin{aligned}
IV &= \sum_{j_2,k_2=-\infty,k_2\neq j_2+t}^{\infty} b_{j_2}\, b_{k_2} \sum_{s_1,s_2=1}^{T-t} \delta_{s_1-j_1,s_2+t-k_2}\delta_{s_1+t-k_1,s_2-j_2} \\
&= \sum_{s_1,s_2=1}^{T-t} \sum_{j=-\infty}^{\infty} b_j\, b_{s_2-s_1+t+j} \sum_{k=-\infty}^{\infty} b_k\, b_{s_1-s_2+t+k} \\
&= \sum_{s_1,s_2=1}^{T-t} \gamma(s_2-s_1+t)\,\gamma(s_1-s_2+t) \\
&= \sum_{|s|<T-t} (\,T-t-|s|\,)\,\gamma(s+t)\,\gamma(-s+t) \\
&\sim \mathit{const} \sum_{s=0,s\neq t}^{T-t} (\,T-t-s\,)\,|s+t|^{2H-2}|s-t|^{2H-2} + T\gamma(2t)\,\gamma(0) \\
&= \mathit{const} \sum_{u=t,u\neq 2t}^{T} (T-u)u^{2H-2}|u-2t|^{2H-2} + T\,\gamma(2t)\,\gamma(0) \\
&\sim \mathit{const}\, T^{2H} + T\,\gamma(2t)\,\gamma(0)\ .
\end{aligned}
$$

Thus, as $T \to \infty$,

$$
\frac{1}{T^2} IV \sim \mathit{const}\, T^{2H-2} + T^{-1}\gamma(2t)\,\gamma(0) \leq \mathit{const}\,(\,T^{2H-2} + T^{-1}\,) \longrightarrow 0\,, \tag{3.71}
$$

uniformly in t.

Now we prove (3.39). From the proof of (3.37) we know that there are constants $K > 0$ and $\alpha > 0$ such that

$$
E\,|\,\hat{\gamma}_T(t) - \gamma(t)\,|^2 < K\,T^{-\alpha}\,;
$$

see (3.66), (3.68), (3.70) and (3.71). We choose $\beta \geq 1$ such that $\beta\,\alpha > 1$. Let $T(M)$ be the smallest integer not smaller that M^β, i.e.

$$
M^\beta \leq T(M) < M^\beta + 1\,. \tag{3.72}
$$

Then,

$$
\sum_{M=1}^{\infty} P\left(\left|\hat{\gamma}_{T(M)}(t) - \gamma(t)\right| \geq \epsilon\right) \leq K\,\epsilon^{-2} \sum_{M=1}^{\infty} M^{-\alpha\,\beta} < \infty\,. \tag{3.73}
$$

Thus, by the Borel-Cantelli lemma, only finitely many of the events occur, whose probabilities are summed on the left hand side of (3.73). Therefore,

$$
\left|\,\hat{\gamma}_{T(M)} - \gamma(t)\,\right| < \epsilon \ \text{ with probability one}
$$

for M large enough, uniformly in t. Since ϵ is arbitrary, $\hat{\gamma}_{T(M)} - \gamma(t)$ converges to zero with probability one, uniformly in t. Now, for any fixed t we write

$$\tilde{Y}_s = Y_s Y_{t+s} .$$

For

$$T(M) \leq T \leq T(M+1) , \tag{3.74}$$

we have

$$\begin{aligned}
&\left| \hat{\gamma}_T(t) - \gamma(t) - \frac{T(M)}{T} \left(\hat{\gamma}_T(t) - \gamma(t) \right) \right| \\
&= \left| \frac{1}{T} \sum_{s=1}^{T-t} \tilde{Y}_s - \gamma(t) \frac{T(M)}{T} \left(\frac{1}{T(M)} \sum_{s=1}^{T(M)-t} \tilde{Y}_s - \gamma(t) \right) \right| \\
&= \left| \frac{1}{T} \sum_{s=T(M)-t+1}^{T-t} \tilde{Y}_s - \gamma(t) \left(1 - \frac{T(M)}{T} \right) \right| \\
&\leq \frac{1}{T(M)} \sum_{s=T(M)-t+1}^{T(M+1)-t} \left| \tilde{Y}_s \right| + \left| \gamma(t) \right| \frac{T - T(M)}{T} \\
&\leq \frac{1}{T(M)} \sum_{s=T(M)-t+1}^{T(M+1)-t} \left| \tilde{Y}_s \right| + \left| \gamma(t) \right| \frac{T(M+1) - T(M)}{T(M)} .
\end{aligned}$$

Then,

$$\begin{aligned}
&E \left\{ \max_{T(M) \leq T \leq T(M+1)} \left| \hat{\gamma}_T(t) - \gamma(t) - \frac{T(M)}{T} \left(\hat{\gamma}_{T(M)}(t) - \gamma(t) \right) \right|^2 \right\} \\
&\leq \frac{2}{T^2(M)} E \left\{ \left(\sum_{s=T(M)-t+1}^{T(M+1)-t} \left| \tilde{Y}_s \right| \right)^2 + \gamma^2(t) \left(T(M+1) - T(M) \right)^2 \right\} .
\end{aligned} \tag{3.75}$$

Since, by (3.38),

$$\begin{aligned}
&E \left| \tilde{Y}_s \right|^2 = E \left| Y_s^2 Y_{s+t}^2 \right| \leq \left(E\, Y_s^4 \right)^{1/2} \left(E\, Y_{s+t}^4 \right)^{1/2} < \infty , \\
&E \left(\sum_{s=T(M)-t+1}^{T(M+1)-t} \left| \tilde{Y}_s \right| \right)^2 \leq const \, \left(T(M+1) - T(M) \right)^2 \text{ , uniformly in } t .
\end{aligned}$$

Therefore, the expression (3.75) in bounded by

$$const \, \frac{\left(T(M+1) - T(M) \right)^2}{T^2(M)} \leq \frac{const}{M^2} \text{ , uniformly in } t ,$$

for

$$\frac{T(M+1)-T(M)}{T(M)} \le \frac{(M+1)^{\beta}+1-M^{\beta}}{M^{\beta}} \le \frac{const}{M}\,.$$

Applying Chebyshev's inequality and the Borel-Cantelli lemma we have that, as $T \to \infty$,

$$\left| \hat{\gamma}_T(t) - \gamma(t) - \frac{T(M)}{T}\,(\,\hat{\gamma}_T(t) - \gamma(t)\,) \right|$$

converges to zero with probability one uniformly in t for T satisfying (3.74). Thus, for $T(M)$ satisfying (3.72),

$$\max_{T(M)\le T\le T(M+1)} \left| \hat{\gamma}_T(t) - \hat{\gamma}_{T(M)}(t) \right|$$

converges to zero with probability one, uniformly in t (because, for $T(M) \le T \le T(M+1)$, $T(M)/T \to 1$). This ends the proof of Theorem 3.7.

3.5.3 Proof of Theorem 3.9

We have

$$\mathbf{m}'\Sigma^{-1}\mathbf{m} = \max_{b_j' s} \frac{(\sum_1^N b_t m_t)^2}{\sum_1^N \sum_1^N b_s\, b_t \gamma_{s-t}} \ge \frac{(\sum_1^N m_t t^{\delta})^2}{\sum_1^N \sum_1^N s^{\delta} t^{\delta} \gamma_{s-t}} \tag{3.76}$$

for some δ yet to be specified, and under the assumptions on $\{\,m_t\,\}$,

$$\left(\sum_1^N m_t\right)^2 \ge \begin{cases} c\,N^{2(1+\delta-\beta)}\,, & \delta > \beta-1\,, \\ c\,(\log N)^2\,, & \delta = \beta - 1\,, \end{cases}$$

for some $c > 0$ and all $N > 1$. In the case of SRD we have convergence of $\sum_1^{\infty} \gamma_i$ and hence, taking $\delta = 0$ in (3.76),

$$N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{i-j} = \sum_{|i|\le N-1} (\,1 - N^{-1}|i|\,)\,\gamma_i \to \sum_{i=-\infty}^{\infty} \gamma_i$$

which is finite. Thus, the right hand side of the inequality (3.76) tends to infinity as $N \to \infty$ if $2(1-\beta) > \frac{3}{2}$, i.e. $\beta < \frac{1}{4}$, which gives (i). In the case of LRD when $\gamma_k \sim C\,k^{2H-2}$, $\frac{1}{2} < H < 1$, as $k \to \infty$ we take $\delta = \beta - 1$. Then

$$\sum_{s=1}^{N} \sum_{t=1}^{N} s^{\beta-1}\, t^{\beta-1}\, \gamma_{t-s} = \gamma_0 \sum_{t=1}^{N} t^{2(\beta-1)} + 2 \sum_{t=2}^{N} \sum_{s=1}^{t-1} \gamma_{t-s}\, s^{\beta-1}\, t^{\beta-1}$$

and

$$\sum_{t=1}^{\infty} t^{2(\beta-1)} < \infty$$

for $\beta \leq \frac{3}{4} - H$, $\frac{1}{2} < H < \frac{3}{4}$. Also,

$$\sum_{s=1}^{t-1} (t-s)^{2H-2} s^{\beta-1} \sim \int_1^{t-1} (t-s)^{2H-2} s^{\beta-1} ds = t^{2H-2+\beta} \int_{t^{-1}}^{1-t^{-1}} (1-u)^{2H-2} u^{\beta-1} du$$
$$\sim t^{2H-2+\beta} B(2H-1, \beta),$$

as $t \to \infty$, B denoting the beta function. Consequently,

$$\sum_{t=2}^{T} \sum_{s=1}^{t-1} \gamma_{t-s} s^{\beta-1} t^{\beta-1} \sim C \sum_{t=2}^{N} t^{2(H+\beta)-3} \sim C T^{2(H+B-1)}$$

as $N \to \infty$ for some $C > 0$ and the right hand side of (3.76) with $\delta = \beta - 1$ tends to infinity as $N \to \infty$ if $2(H+\beta) - \frac{3}{2} < 0$, i.e. $\beta < \frac{3}{4} - H$ when $H < \frac{3}{4}$. This gives (ii) and completes the proof of Theorem 3.9.

3.5.4 Proof of Theorem 3.11

Under the null hypothesis,

$$\begin{aligned}
\sqrt{N}\left(\frac{T_N}{N} - 1\right) &= \sqrt{N}\left(\frac{1}{4\pi^2 N} \int_{-\pi}^{\pi} \frac{|\sum_{t=1}^{N} X_t e^{it\omega}|^2}{f(\omega)} d\omega - 1\right) \\
&= \sqrt{N}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_{N,X}(\omega)}{f(\omega)} d\omega - 1\right) = \sqrt{N}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{I_{N,Y}(\omega)}{f(\omega)} d\omega - 1\right) \\
&= \frac{\sqrt{N}}{2\pi} \int_{-\pi}^{\pi} \Big(I_{N,Y}(\omega) - f(\omega) \Big) f^{-1}(\omega)\, d\omega \\
&= \frac{\sqrt{N}}{2\pi} \int_{-\pi}^{\pi} \Big(I_{N,Y}(\omega) - E I_{N,Y}(\omega) + E I_{N,Y}(\omega) - f(\omega) \Big) f^{-1}(\omega)\, d\omega .
\end{aligned}$$

In order to establish Theorem 3.11 it is sufficient to show that, as $N \to \infty$,

$$\frac{\sqrt{N}}{2\pi} \int_{-\pi}^{\pi} \Big(I_{N,Y}(\omega) - E I_{N,Y}(\omega) \Big) f^{-1}(\omega)\, d\omega \xrightarrow{\text{d}} N(0\ \sigma)$$

and

$$\frac{\sqrt{N}}{2\pi} \int_{-\pi}^{\pi} \Big(E\, I_{N,Y}(\omega) - f(\omega) \Big) f^{-1}(\omega)\, d\omega \longrightarrow 0 .$$

We put this in the following lemmas

LEMMA 3.14 *(Giraitis and Surgailis, 1990, Theorem 2.) Let $\hat{b}(\omega)$ be a real, even, integrable function on $[-\pi, \pi]$ and $b(t) = \int_{-\pi}^{\pi} exp\{i t\omega\}\hat{b}(\omega)\, d\omega$. Let $Q_N = \sum_{s,t=1}^{N} b(t-s)Y_sY_t$, where $\{Y_t\}$ is given by (3.2) and $\{\xi_\nu\}$ in (3.2) is now assumed to be an i.i.d. random sequence with mean zero, variance one and finite fourth moment. Let R_N and B_N be defined by (3.50). If*

$$Tr(R_N B_N)^2/N \longrightarrow (2\pi)^3 \int_{-\pi}^{\pi} \Big(f(\omega)\, \hat{b}(\omega) \Big)^2 d\omega < \infty\ , \qquad (3.77)$$

then the distribution of $(Q_N - EQ_N)/\sqrt{N}$ tends to a normal distribution $N(0\ ,\ \sigma^2)$ with

$$\sigma^2 = 16\,\pi^3 \int_{-\pi}^{\pi} \Big(f(\omega)\, \hat{b}(\omega) \Big)^2 d\omega\ +\ (E\xi_1^4 - 1)\Big(2\pi \int_{-\pi}^{\pi} f(\omega)\, \hat{b}(\omega)\, d\omega \Big)^2 . \qquad (3.78)$$

Remarks on Lemma 3.14. In Theorem 2 of Giraitis and Surgailis (1990), instead of $E\xi_1^4 - 1$ in formula (3.78), the fourth order semiinvariant of ξ_1, $\chi_4 = E\xi_1^4 - 3$, is used. According to our calculation, χ_4 should be replaced by $E\xi_1^4 - 1$. This can be seen by checking the proof of formula (3.2) in their paper.

LEMMA 3.15 *Under the conditions of Theorem 3.11,*

$$\frac{\sqrt{N}}{2\pi} \int_{-\pi}^{\pi} \Big(I_{N,Y}(\omega) - EI_{N,Y}(\omega) \Big)\, f^{-1}(\omega)\, d\omega \xrightarrow{\ \mathrm{d}\ } N\,(\,0\,,\,\sigma^2\,) \qquad (3.79)$$

as $N \to \infty$.

LEMMA 3.16 *Under the conditions of Theorem 3.11,*

$$\sqrt{N} \int_{-\pi}^{\pi} \Big(EI_{N,Y}(\omega) - f(\omega) \Big)\, f^{-1}(\omega)\, d\omega \xrightarrow{\ \mathrm{P}\ } 0 \qquad (3.80)$$

as $N \to \infty$.

Proof of Lemma 3.15. We have

$$\frac{\sqrt{N}}{2\pi} \int_{-\pi}^{\pi} \Big(I_{N,Y}(\omega) - EI_{N,Y} \Big)\, f^{-1}(\omega) d\omega$$

$$= \frac{\sqrt{N}}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi N}\sum_{s,t=1}^{N}\left(Y_t\,Y_s\,e^{i\,(t-s)\,\omega} - E\,Y_t\,Y_s\,e^{i\,(t-s)\,\omega}\right)\,f^{-1}(\omega)\,d\omega\,d\omega$$

$$= \frac{1}{4\pi^2\sqrt{N}}\left(\sum_{s,t=1}^{N}Y_t\,Y_s\int_{-\pi}^{\pi}f^{-1}(\omega)\,e^{i\,(t-s)\,\omega}\,d\omega - E\sum_{s,t=1}^{N}Y_t\,Y_s\int_{-\pi}^{\pi}f^{-1}(\omega)\,e^{i\,(t-s)\,\omega}\,d\omega\right)$$

$$= \frac{1}{4\pi^2\sqrt{N}}\left(\sum_{s,t=1}^{N}Y_t\,Y_s\,b(t-s) - E\sum_{s,t=1}^{N}Y_t\,Y_s\,b\,(t-s)\right)$$

$$\equiv \frac{1}{4\pi^2\sqrt{N}}\,(\,Q_N - EQ_N\,)\,. \tag{3.81}$$

Now we can use the result of Theorem 2 of Giraitis and Surgailis (1990) to establish Lemma 3.15. That is we need to check that formula (3.77) in Lemma 3.14 holds in our case. Indeed,

$$\infty > \int_{-\pi}^{\pi}\left(f(\omega)\,f^{-1}(\omega)\right)^2\,d\omega$$

$$= \int_{-\pi}^{\pi}\left(\left(\frac{1}{2\pi}\sum_{t=-\infty}^{\infty}\gamma(t)\,e^{i\,t\,\omega}\right)\left(\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}b(s)\,e^{i\,s\,\omega}\right)\right)^2\,d\omega$$

$$= \frac{1}{16\pi^4}\int_{-\pi}^{\pi}\left(\left(\lim_{N\to\infty}\sum_{t=-N}^{N}\gamma(t)\,e^{i\,t\,\omega}\right)\left(\lim_{N\to\infty}\sum_{s=-N}^{N}b(s)\,e^{i\,s\,\omega}\right)\right)^2\,d\omega. \tag{3.82}$$

Since $g(\omega)\,\equiv\,1$ is continuous on $[\,-\pi\,,\,\pi\,]$ and $g(-\pi)\,=\,g(\pi)$ so $g(\omega)$ is the summation of a uniformly convergent Fourier series, so (3.82) can be rewritten as

$$\frac{1}{16\pi^4}\lim_{N\to\infty}\int_{-\pi}^{\pi}\left(\sum_{t,s=-N}^{N}\gamma(t)\,b(s)\,e^{i\,(t+s)\,\omega}\right)^2\,d\omega$$

$$= \frac{1}{16\pi^4}\lim_{N\to\infty}\sum_{t,s,u,v=-N}^{N}\gamma(t)\,b(s)\,\gamma(u)\,b(v)\int_{-\pi}^{\pi}e^{i\,(t+s-u-v)\,\omega}\,d\omega$$

$$= \frac{1}{16\pi^4}\lim_{N\to\infty}\sum_{\substack{t,\,s,\,u,\,v\,=\,-N\\ t\,+\,s\,=\,u\,+\,v}}^{N}\gamma(t)\,b(s)\,\gamma(u)\,b(v)\,2\pi$$

$$= \frac{1}{8\pi^3}\lim_{N\to\infty}\sum_{\substack{t,\,s,\,u,\,v\,=\,-N\\ t\,+\,s\,=\,u\,+\,v}}^{N}\gamma(t)\,b(s)\,\gamma(u)\,b(v)$$

$$= \frac{1}{8\pi^3}\lim_{N\to\infty}\frac{1}{N}\sum_{t,s,j,k=1}^{N}\gamma(j-t)\,b(s-j)\,\gamma(k-s)\,b(t-k)$$

$$= \frac{1}{8\pi^3}\lim_{N\to\infty}\frac{1}{N}Tr\left((R_NB_N)^2\right).$$

Then by using Lemma 3.14 to (3.81) we have

$$\frac{1}{\sqrt{N}}\Big(Q_N - EQ_N\Big) \xrightarrow{\mathrm{d}} N(0\,,\,\sigma_1^2)$$

where $\sigma_1^2 = 16\pi^4(E\xi_1^4 + 1)$, so

$$\begin{aligned}
&\frac{\sqrt{N}}{2\pi}\int_{-\pi}^{\pi}\Big(I_{N,Y}(\omega) - EI_{N,Y}(\omega)\Big)\, f^{-1}(\omega)\, d\omega \\
= &\ \frac{1}{4\pi^2\sqrt{N}}\Big(Q_N - EQ_N\Big) \xrightarrow{\mathrm{d}} N\,(\,0\,,\,E\xi_1^4 + 1\,)\,.
\end{aligned}$$

This finishes the proof of Lemma 3.15.

Proof of Lemma 3.16. Now

$$EI_{N,Y}(\omega) = \frac{1}{2\pi}E\left|\sum_{t=1}^{N} Y_t\, e^{-it\omega}\right|^2 = \frac{1}{2\pi N}\sum_{|t|\le N-1}(N - |t|)\,\gamma(t)\, e^{-it\omega}\,,$$

$$f(\omega) = \frac{1}{2\pi}\sum_{t-\infty}^{\infty}\gamma(t)\, e^{-it\omega} = \frac{1}{2\pi}\sum_{|t|\le N-1}\gamma(t)\, e^{-it\omega} + \frac{1}{2\pi}\sum_{|t|\ge N}\gamma(t)\, e^{-it\omega}\,,$$

$$EI_{N,Y}(\omega) - f(\omega) = -\frac{1}{2\pi}\sum_{|t|\le N-1}|t|\,\gamma(t)\, e^{-it\omega} - \frac{1}{2\pi}\sum_{|t|\ge N}\gamma(t)\, e^{-it\omega}\,,$$

$$\begin{aligned}
&\sqrt{N}\int_{-\pi}^{\pi}\Big(EI_{N,Y}(\omega) - f(\omega)\Big)\, f^{-1}(\omega)\, d\omega \\
= &\ -\frac{\sqrt{N}}{2\pi}\int_{-\pi}^{\pi} f^{-1}(\omega)\Big(\sum_{|t|\ge N}\gamma(t)\, e^{-it\omega} + \frac{1}{N}\sum_{|t|\le N-1}|t|\,\gamma(t)\, e^{-it\omega}\Big)\, d\omega \\
= &\ -\frac{\sqrt{N}}{2\pi}\sum_{|t|\ge N}\gamma(t)\, b(t) - \frac{1}{2\pi\sqrt{N}}\sum_{|t|\le N-1}|t|\,\gamma(t)\, b(t)\,.
\end{aligned}$$

Note that under condition (C1) of Theorem 3.11, $|b(t)| \le const\,|t|^{-1}$, as $t \to \infty$. Since $H < \frac{3}{4}$, as $N \to \infty$ we have

$$\begin{aligned}
&\Big|\sqrt{N}\sum_{|t|\ge N}\gamma(t)\, b(t)\Big| \le const\,\sqrt{N}\sum_{t\ge N} t^{2H-2}t^{-1} \sim const\,\sqrt{N}N^{2H-2} \\
&\quad = const\; N^{2(H-\frac{3}{4})} \longrightarrow 0\,, \\
&\Big|\frac{1}{\sqrt{N}}\sum_{|t|\le N-1}|t|\,\gamma(t)\, b(t)\Big| \le \frac{const}{\sqrt{N}}\sum_{0<t<N-1} t^{1+2H-2-1} \sim const\; N^{2(H-\frac{3}{4})} \longrightarrow 0\,.
\end{aligned}$$

Under condition (C2) of Theorem 3.11, $|b(t)| \le const\,|t|^{-2}$, as $t \to \infty$. So as $N \to \infty$,

$$\Big|\sqrt{N}\sum_{t\ge N}\gamma(t)\, b(t)\Big| \le const\,\sqrt{N}\sum_{t\ge N} t^{2H-2-2} \longrightarrow 0\,,$$

and

$$\left| \frac{1}{\sqrt{N}} \sum_{|t| \leq N-1} |t| \, \gamma(t) \, b(t) \right| \leq \frac{const}{\sqrt{N}} \sum_{1 \leq t \leq N} t^{1+2H-2-1} = \frac{c}{\sqrt{N}} \sum_{1 \leq t \leq N} t^{2H-3} \longrightarrow 0 \, .$$

This finishes the proof of Lemma 3.16 and hence of Theorem 3.11.

Chapter 4

FRACTIONAL BROWNIAN MOTION AND ITS STOCHASTIC INTEGRAL

4.1 Introduction to Chapters 4 and 5

In the rest of this thesis we are concerned with the applications of long range dependent processes to other disciplines, such as financial markets. We are interested in taking into account the LRD when modelling stock price movements. One of such models we have in mind is the so called Black-Scholes model, of which more details will be given in Chapter 5.

In mathematical finance, stochastic calculus is a powerful tool for describing the stock price behaviour because of the apparent random fluctuations of the stock price. Stochastic differential equations driven by semimartingales are traditionally used to model the dynamics of stock prices. In particular, those driven by Brownian motion are widely used, because Brownian motion has desirable mathematical characteristics. For example, Brownian motion is a continuous semimartingale with independent increments therefore analytical solutions can often be obtained for systems driven by Brownian motion due to the knowledge of the stochastic analysis of semimartingale. The independent increments of Brownian motion means that Brownian motion is a SRD process.

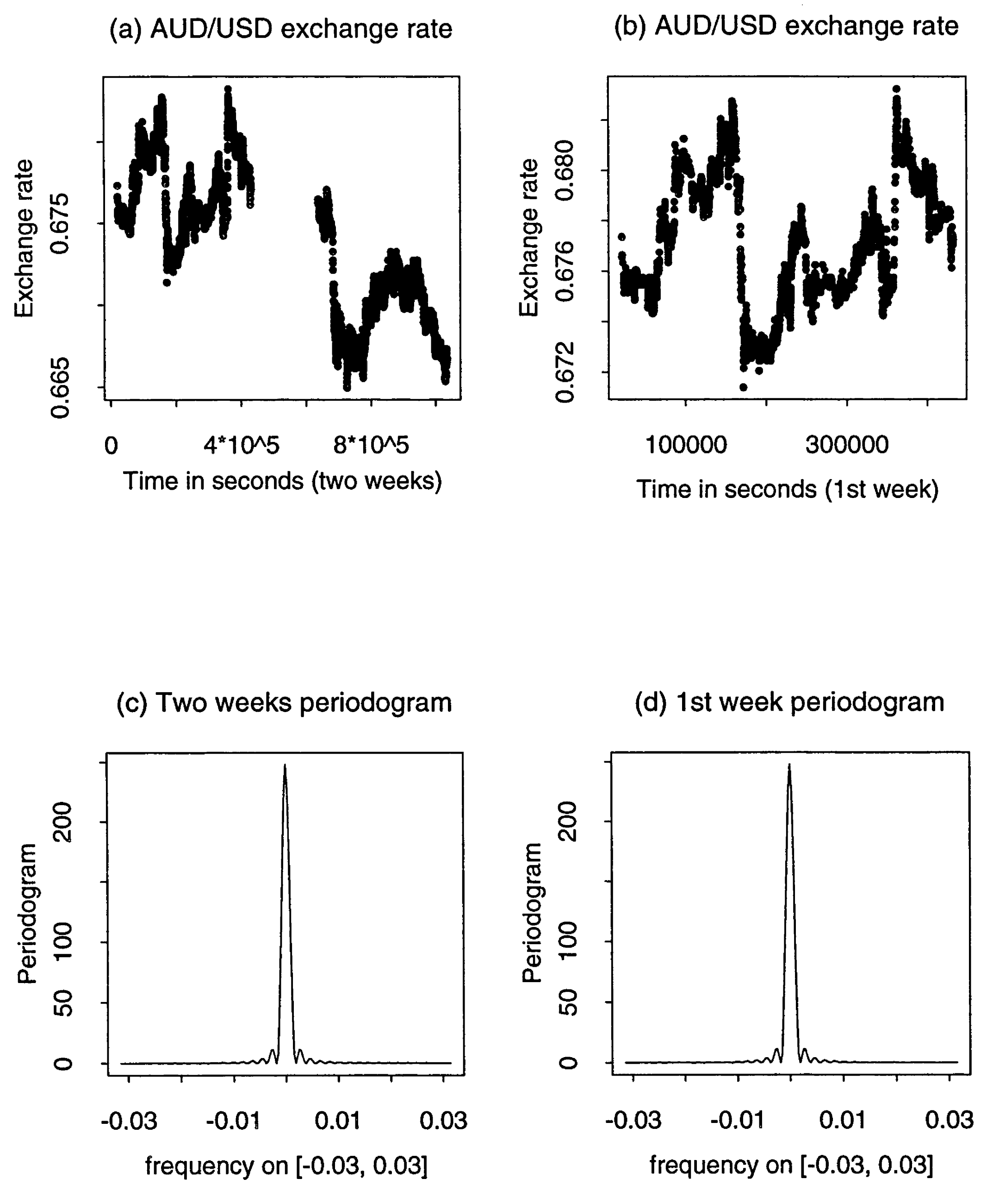
However, there has been a lot of empirical evidence from actual stock price data suggesting stochastic models which are not covered by the setting of stochastic differential equations driven by SRD processes. Various analyses of data from financial markets have provided enough evidence of the existence of LRD in economic data. For example, the periodograms (Figure 4.1, (c) and (d)) of the exchange rate between Australian and US Dollars (Figure 4.1, (a) and (b)) have very sharp peaks near the origin.

The presence of LRD, or the Joseph Effect (see, Mandelbrot, 1971), is well documented in many economic time series. See, for example, Booth and Kaen (1979) for gold prices; Booth, Kaen and Koveos (1982) for foreign exchange rates; Helms, Kaen and Rosenman (1984) for future contracts; Mandelbrot (1971) for asset returns; Fama and French (1988), Greene and Fielitz (1977) and Poterba and Summers (1988) for stock returns. For a formal definition of the returns see, for example, Hull (1993). Here we give a rough description of the stock returns. Write S_T for the stock price at a future time T and S_t are the current time t. Then the stock return between t and T is given by

$$\eta = \frac{1}{T-t} \log \frac{S_T}{S_t} .$$

However until now there seems not to exist any model that accurately explains the dynamics of fluctuating behaviour of financial markets, such as the prices of contingent claims, interest rates and exchange rates. Generally speaking, statistical analysis has been based on assuming the normality of underlying distribution. However, it is well known that a lot of data from financial markets are not normally distributed, most of them have heavy tails, much heavier than the normal. For example, Figure 4.2 (c) and (d) show that the densities of the return of the exchange rates between Australian dollars and US dollars have heavy tails. Figure 4.2 (e) and (f) show that the returns are not normally distributed. (Panel (a) of Figure 4.1 depicts a real data set representing the value of the Australian Dollar versus the US Dollar recorded during 16 – 28 August 1993 by the Bankers

Figure 4.1: Long range dependence of the data from exchange rates



Trust Australia Ltd. Panel (b) of Figure 4.1 is the plot of the raw data of the first week.)

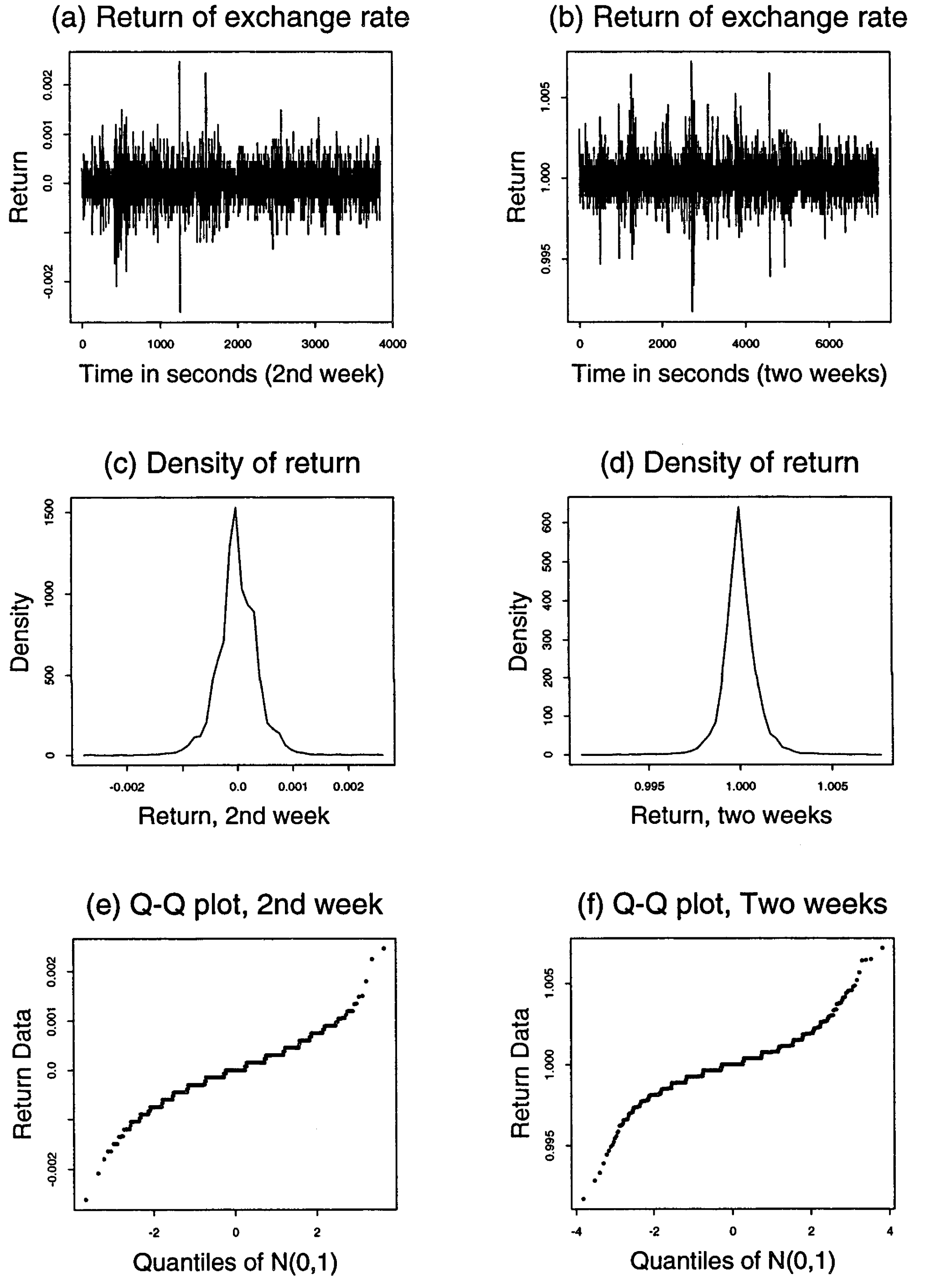
There is also various literature about analysis of stable processes, which are typical processes with fat tailed distributions. See, for example, Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994) and references therein. Some finance data may be well modelled by such processes.

Another reason for the inaccurate modelling of financial markets is that a lot of statistical approaches used to deal with financial data are based on implicit or explicit assumptions of SRD in the structure of the data, which may not be satisfied by most of the cases in reality. On the contrary, LRD appears extensively in financial data. The assumption of SRD turns out to be a special case. In this sense the well known Black-Scholes model is an extreme case for it results in a degenerate autocorrelation function. In other words, the single-period stock prices are uncorrelated, hence, SRD.

It is undisputed (see, for example, Mandelbrot, 1971) that the presence of LRD would have important practical and theoretical implications for many problems in modern financial economics, for example, the pricing of derivative securities using martingale methods. Some authors (see, for example, Greene and Fielitz, 1977; Kunitomo, 1993) have shown that the Black-Scholes model would not be an adequate process for stock price but should be replaced by a model which uses fractional Brownian motion as the driving process for stock returns.

The aim of Chapters 4 and 5 is to propose and study a "revised" Black-Scholes model which includes the ordinary Black-Scholes model as a special case and is able to account for LRD in the stock market price movement. We call the resulting model a fractional Black-Scholes model for it is obtained through replacing Brownian motion as a driving process in the Black-Scholes model by fractional Brownian motion with the Hurst index $H \in [1/2, 1)$.

Figure 4.2: Non-normality of the return of exchange rates



In order to define a stochastic differential equation driven by fractional Brownian motion which is not a semimartingale, it is necessary to define the stochastic integral with respect to fractional Brownian motion first. Chapter 4 is devoted to this task.

Having established stochastic integrals with respect to fractional Brownian motion in Chapter 4, we propose a parametric model — the fractional Black-Scholes model in Chapter 5. There we also study the stochastic calculus with respect to fractional Brownian motion. We derive Itô's formula for fractional Brownian motion, which is a powerful tool for stochastic analysis.

4.2 Introduction to Fractional Brownian Motion and its Stochastic Integrals

Fractional Brownian motion arises naturally from an examination of the conditions of validity of the central limit theorem. As is well known, the theory of the central limit theorem began with the study of sequences of independent random variables and Markov processes. However, the classical conditions required for the validity of the central limit theorem are most likely not satisfied in many cases, a fact that raises difficult and interesting problems of practical and theoretical character. Strong dependence is often found and needs to be taken into account. One task suggested by finding evidence of strong dependence is to single out and study in detail some specific families of random functions that could, in some way, be expected to be typical of what happens in the absence of asymptotic independence. Fractional Brownian motion has been proposed as a model for the LRD postulated to occur in a variety of hydrological and geophysical time series. See, for example, Hurst (1951, 1956) and Mandelbrot and Van Ness (1968).

Since Mandelbrot and Van Ness (1968) gave an analytical definition of fractional Brownian motion, it has been widely accepted for mathematical modelling in science and engineering. In the most widely applied models for stationary LRD processes, attention has recently been given to models in which the spectral densities are proportional to ω^{-r}, $0 < r < 1$, for ω near the origin or, equivalently, that the asymptotic decay of the covariance functions at lag t is proportional to t^{r-1}. Fractional Brownian motion is a standard example for the processes with spectral densities having the properties mentioned above. For a discussion of the spectrum of fractional Brownian motion see, for example, Flandrin (1989). Fractional Brownian motion is also a typical process for parametric models obtained as the increments of self-similar processes. Many results have been established on statistical inference for stationary increments of self-similar processes and other long range dependent processes. See, for example, Vervaat (1987), Beran (1992a), Heyde and Gay (1989, 1993) and Robinson (1994b, 1994c) and relevant references given there. However, the stochastic analytical aspects of fractional Brownian motion has not born rich fruit yet due to the absence of the semimartingale property.

The theory of stochastic analysis was initiated and developed by Itô (1942). This theory was first applied to Kolmogorov's problem (Kolmogorov, 1931) of determining Markov processes. Today Itô's theory is applied not only to Markov processes but also to a large class of stochastic processes where the framework provides us with a powerful tool for description and analysis. The class of stochastic processes to which Itô's theory can be applied is now extended to a class of stochastic processes called semimartingales. Such processes appear to be the most general for which a regular theory of stochastic integral with natural properties such as linearity and having properties of the type of the Lebesgue theorem about dominated convergence for the Lebesgue integrals is available. It is now a well known fact of the stochastic calculus that a "reasonable" stochastic integral is possible only with respect to semimartingale integrators, (see, for example, Dellacherie and Meyer, 1982, Theorem VIII.80), and semimartingales can even be characterised by this feature.

We write $B_H(t)$ for fractional Brownian motion, where $1/2 \leq H < 1$ is the Hurst index. Although $B_H(t)$ is not a semimartingale, this does not imply that integrating with respect to it is out of the question. It just means that the integral must be interpreted with care. In the rest of this chapter, we are concerned with the establishment of stochastic integrals with respect to $B_H(t)$.

The problem of defining stochastic integrals with respect to $B_H(t)$ is being investigated by a number of authors. We understand that Terrin has proposed to define the stochastic integral $\int_0^1 B_H(t)\, dB_H(t)$ by extending the idea of Chan and Wei (1988), but no details are available. When we were preparing our research report on this chapter, we got to know that Lin (1996) has defined stochastic integrals with respect to $B_H(t)$. In Lin's paper, the integrals with respect to $B_H(t)$ are defined for those integrands ϕ's such that ϕ is a deterministic bounded Borel function. In details, if $\phi : \mathbf{R} \to \mathbf{R}$ is a bounded Borel function, then

$$\int_0^t \phi(\, B_H(s)\,)\, dB_H(s) \equiv \int_0^{B_H(t)} \phi(s)\, ds\,, \tag{4.1}$$

$$\int_0^1 \phi(\, s\,)\, dB_H(s) \equiv \text{“lim of Riemann-Stieltjes sum”}\,. \tag{4.2}$$

First, for definition (4.1), the boundedness of ϕ on $(-\infty, \infty)$ is important for the right-hand side of (4.1) being well defined since the range of sample paths of $B_H(t)$ is in $(-\infty, \infty)$. Secondly, definition (4.1) is defined only for integrands being deterministic functions, so the stochastic differential equation

$$dX_t = a\,(t, X_t)\; dt + b\,(t, X_t)\; dB_H(t) \tag{4.3}$$

is not covered by Lin's framework. Neither unbounded but Riemann-Stieltjes integrable deterministic functions are included as integrands in his work. We understand that our Definition 4.9 includes Lin's definition (4.1) as a special case while Lin's definition (4.2) is a special case of our Definition 4.8 where we define integral $\int_0^1 f(t)\,dB_H(t)$ for f being a L_2-integrable function in $[0,1]$. In our framework, we can define stochastic differential equation (4.3) involving stochastic process X_t in the integrals through function $b(t,x)$.

Towards the end of writing this thesis, we had information that Gripenberg and Norros (1994) had done work on establishing stochastic integrals with respect to $B_H(t)$. They have defined the integrals for three kinds of integrands. First, for the integrands of simple stochastic processes:

$$Y_t = \sum_{j=1}^{k} X_j 1_{(T_{j-1},T_j]}(t)\,,$$

where X_j's and T_j's are random variables and 1_D is the indicator function, they define

$$\int_{-\infty}^{\infty} Y_t\,dB_H(t) = \sum X_j\left(B_H(T_j) - B_H(T_{j-1})\right)\,. \tag{4.4}$$

Secondly, if the integrands are stochastic processes $\{Y_t\}$ with bounded variation, they define

$$\int_a^b Y_t\,dB_H(t) = Y_b B_H(b) - Y_a B_H(a) - \int_a^b B_H(t)\,dY_t\,. \tag{4.5}$$

Thirdly, they have commented on defining stochastic integrals with respect to $B_H(t)$ for more general integrands. If the integrands are deterministic then the integrals can be defined by using L_2-limits. And another possible method of defining stochastic integrals with respect to $B_H(t)$ is to use the integral representation (4.6) of $B_H(t)$. But they have given no more details about the latter two cases. No stochastic differential equations driven by $B_H(t)$ are discussed.

The scope of our setting covers much more general type of integrands. Comparing our work with that of Lin (1996) and Gripenberg and Norros (1994), we see that the main advantage of our framework is that we can define general stochastic integrals with respect to B_H, therefore, we can establish Itô's formula with respect to B_H, which is very important for stochastic calculus.

In this chapter we establish stochastic integrals of the form $\int_0^1 X(t)\, dB_H(t)$. When $H = 1/2$, it is well known that $B_H(t) = B(t)$ and integration with respect to Brownian motion has been defined. Thus, in the rest of this chapter, we are concerned with the case where $1/2 < H < 1$. The method we use is different from those mentioned above. In order to explain our idea of constructing stochastic integrals with respect to $B_H(t)$, let us recall the method used in constructing Itô's stochastic integrals. Let $X(t)$ be a measurable and square integrable process. The classical Itô theory deals with the stochastic integral $\int_0^1 X(t)dB(t)$. Although the sample functions of $B(t)$ are not of bounded variation, so that $\int_0^1 X(t)dB(t)$ can not be defined as an ordinary Riemann-Stieltjes integral in the individual sample functions, the integral in question is, however, a generalised Riemann-Stieltjes integral. The procedure for the construction of $\int_0^1 X(t)dB(t)$ is reminiscent of the definition of the Riemann-Stieltjes integral. The key points here are: first, the Gaussian property of $B(t)$, secondly, that the sample functions of $B(t)$ have moduli of continuity $1/2$, that is $E|B(t_1)-B(t_2)| \sim const\, |t_1-t_2|^{1/2}$ as $|t_1-t_2| \to 0$, and finally, the orthogonality of the increments of $B(t)$. (For the concept of moduli of continuity of a function, see, for example, Courant and John, 1953.) Looking in detail at the sample functions of $B_H(t)$, we notice that the sample functions of $B_H(t)$ have moduli of continuity equal to H. That is, $E|B_H(t_1) - B_H(t_2)| \sim const\, |t_1 - t_2|^H$ as $|t_1 - t_2| \to 0$, where $1/2 < H < 1$. We will see this from Corollary 4.1 of Section 4.3.1. Furthermore, $B_H(t)$ is a Gaussian process. These facts are the basis for our construction of the stochastic integral with integrator $dB_H(t)$. The procedure for defining the integral $\int_0^1 X(t)dB_H(t)$ is analogous to that of $\int_0^1 X(t)dB(t)$. We use the fact that the modulus of continuity of the sample functions of $B_H(t)$ is greater than $1/2$ to circumvent the shortcoming of the lack of semimartingale property for $B_H(t)$.

The organization of this chapter is as follows. In Section 4.3 of this chapter we give the definition of B_H and its basic properties. In Section 4.4, we establish the stochastic integral with respect to B_H. In Subsection 4.4.1 we establish the stochastic integral

$$I(f) = \int_0^1 f(t)dB_H$$

for $f \in C(D)$. In Subsection 4.4.2 we define $I(f)$ for $f \in L_2(D)$. The meaning of the notation $C(D)$ and $L_2(D)$ will be given at the beginning of Section 4.4. In Subsection 4.4.3 we define the stochastic integral

$$\int_0^1 \phi(B_H(t))dB_H(t)$$

for those ϕ's satisfying a Lipschitz condition. In Subsection 4.4.4 we define the stochastic integral

$$\int_0^1 X(t,\omega)dB_H(t,\omega)$$

for stochastic processes $X(t,\omega)$ satisfying certain conditions. In each of these subsections, we also discuss the basic properties of the corresponding integrals. In Subsection 4.4.5 we define the stochastic processes

$$\{ I(f)(t) \equiv \int_0^t f(s)dB_H(s,\omega) \ : \ t \geq 0 \} \ ,$$
$$\{ I(\phi)(t) \equiv \int_0^t \phi(B_H(s,\omega)dB_H(s,\omega) \ : \ t \geq 0 \} \ ,$$

and

$$\{ I(X)(t) \equiv \int_0^t X(s,\omega)dB_H(s,\omega) \ : \ \ t \geq 0 \} \ .$$

Then we study some sample properties of these processes. The proofs of our main results appear in Section 4.5.

4.3 Definition and Basic Properties of Fractional Brownian Motion

4.3.1 Definition

There are several definitions of fractional Brownian motion. The following was given by Mandelbrot and Van Ness (1968).

DEFINITION 4.1 *Let H be such that $0 < H < 1$ and let b_0 be an arbitrary real number. Let $B(t,\omega)$ be a Brownian motion. We call the following random function $B_H(t,\omega)$ fractional Brownian motion with parameter H and starting value b_0*

$$B_H(0,\omega) = b_0$$

and

$$B_H(t\,,\omega) - B_H(0\,,\omega) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-s)^{H-1/2} - (-s)^{H-1/2}\right] dB(s\,,\omega) + \int_0^t (t-s)^{H-1/2} dB(s\,,\omega)\right\}\,, \tag{4.6}$$

where we can take the integration in the pointwise sense (as well as the mean square sense) by using the usual methods involving integration by parts.

From Definition 4.1 it follows that B_H is a centred Gaussian process with $B_H(0) = b_0$. The parameter H in Definition 4.1 is the so called Hurst index. When $H = 1/2$, $B_{1/2}(t) = B(t)$ is standard Brownian motion. Now let us consider an increment process $\{\, B_H(\cdot+\tau) - B_H(\cdot)\,\}$ of B_H, where $\tau > 0$ is an arbitrary fixed constant. It can be seen from Corollary 4.1 that the increment process is a stationary and not a long range dependent process when $0 < H \leq 1/2$, but long range dependent if $1/2 < H < 1$. The following corollary summarizes some well known facts about B_H (see, for example, Mandelbrot and Van Ness, 1968).

COROLLARY 4.1 *For any* $T \geq 0$,

$$E[B_H(t)] = b_0 \ , \tag{4.7}$$

$$Var[B_H(1)] = [\Gamma(H+\frac{1}{2})]^{-2}\left\{\int_{-\infty}^{0}\left[(t-s)^{H-1/2}-(-s)^{H-1/2}\right]^2 ds + \frac{1}{2H}\right\}. \tag{4.8}$$

By writing

$$V_H = Var\left(B_H(1)\right) \ , \tag{4.9}$$

we have

$$E\left(B_H(t)\right)^2 = t^{2H}V_H + b_0^2 \ . \tag{4.10}$$

Furthermore, we have the T^H *law for* B_H *as follows: for any* t *and* $\tau \geq 0$,

$$E\left(B_H(t+\tau) - B_H(t)\right)^2 = \tau^{2H}V_H \ . \tag{4.11}$$

For any t *and* $s \geq 0$,

$$cov(\, B_H(t)\, ,\, B_H(s)\,) = \frac{1}{2}V_H\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \ , \tag{4.12}$$

$$E(B_H(t)B_H(s)) = b_0^2 + \frac{1}{2}V_H\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \ . \tag{4.13}$$

The proof of Corollary 4.1 follows from Definition 4.1, the martingale property of Brownian motion and the knowledge of Wiener-Itô integration.

4.3.2 Self-similarity properties

In this subsection, we list some interesting and important results of fractional Brownian motion. More details can be found in, for example, Mandelbrot and Van Ness (1968).

One of the most distinguishing properties of fractional Brownian motion is the so called self-similar property. For a general reference to self-similar processes, see, for example, articles in the Collection of Eberlein and Taqqu (1986).

DEFINITION 4.2 *The increment of a random process* $\{ X(t) : -\infty < t < \infty \}$ *is said to be self-similar with parameter* H $(H \geq 0)$ *if for any* $h > 0$ *and any* t_0

$$\{ X(t + t_0) - X(t_0) \} \stackrel{\mathrm{d}}{=} \{ h^{-H}[X(ht + t_0) - X(t_0)] \} , \tag{4.14}$$

where the notation $\{X(t)\} \stackrel{\mathrm{d}}{=} \{Y(t)\}$ *means that processes* $X(t)$ *and* $Y(t)$ *have the same state space and finite joint distribution functions.*

DEFINITION 4.3 *A process* $X(t)$ *is said to have stationary increments if all finite dimensional vectors of the form*

$$[\, X(t_1 + \tau) - X(t_0 + \tau) \,,\, X(t_2 + \tau) - X(t_1 + \tau) \,,\, \ldots \,,\, X(t_n + \tau) - X(t_{n-1} + \tau) \,]$$

have distributions independent of τ.

THEOREM 4.2 *The increments of* $B_H(t)$ *are stationary and self-similar with parameter* H.

The stationary sequence $B_H(n+1) - B_H(n)$ (often called fractional Gaussian noise) is ergodic by the general result that any stationary Gaussian sequence with continuous spectral measure is ergodic and weakly mixing (see, for example, Cornfeld, Fomin and Sinai, 1982).

Theorem 4.2 provides the motivation for the introduction of fractional Brownian motion. An interesting fact about B_H is that, in some sense, the inverse of Theorem 4.2 holds. Namely

PROPOSITION 4.3 *If* $X(t)$ *has self-similar and stationary increments and is mean square continuous, then* $0 < H < 1$.

The proofs of Theorem 4.2 and Proposition 4.3 can be found in, for example, Mandelbrot and Van Ness (1968).

PROPOSITION 4.4 *If $X(t)$ is a non-constant Gaussian process satisfying the conditions of Proposition 4.3, then it must be a fractional Brownian motion.*

The proof of Proposition 4.4 follows from the fact that a Gaussian process is determined by its mean and covariance functions.

4.3.3 Continuity and differentiation of sample functions

From (4.11) we know that $B_H(t)$ is a mean square continuous process. Furthermore, its sample function is continuous.

PROPOSITION 4.5 *For $0 < H < 1$, $B_H(t)$ has a continuous sample function with probability one, where t is in any compact set.*

In order to prove Proposition 4.5, we need the following

LEMMA 4.6 *(Kolmogorov) Let $\{X(t) : t \in [a, b]\}$ be a separable process such that for some $r > 0$, $\varepsilon > 0$ and $C \geq 0$ and all t, $t + \delta \in [a, b]$ with δ sufficiently small,*

$$E|X(t+\delta) - X(t)|^r \leq C\,|\delta|^{1+\varepsilon}\ ,$$

then $X(t)$ is a.s. sample continuous.

For a proof of Lemma 4.6, see, for example, Loève (1978, Vol. II, p. 185, Corollary).

Proof of Proposition 4.5. If $H > 1/2$, the statement follows immediately from (4.11) and Lemma 4.6. In any case we can choose k such that $0 < k < H$ and let $C(s) = 1$ if $s \leq 0$, $= 0$ if $s > 0$. Then,

$$\Gamma(H + 1/2)^{1/k} E|B_H(t+\tau) - B_H(t)|^{1/k} = \Gamma(H + 1/2)^{1/k} E|B_H(\tau) - B_H(0)|^{1/k}$$

$$= E\left|\int_{-\infty}^{\tau}\left[(\tau-s)^{H-1/2}-C(s)(-s)^{H-1/2}\right]dB(s)\right|^{1/k}$$
$$= |\tau|^{H/k}E\left|\int_{-\infty}^{1}\left[(1-s)^{H-1/2}-C(s)(-s)^{H-1/2}\right]dB(s)\right|^{1/k}.$$

This finishes the proof of Proposition 4.5 by Lemma 4.6.

Like Brownian motion, fractional Brownian motion is not mean square differentiable and it almost surely does not have a differentiable sample function. For simplicity we take $B_H(0) = 0$ from now on.

PROPOSITION 4.7 *$B_H(t)$ is almost surely not differentiable. In fact,*

$$\lim_{t\to t_0}\left|\frac{B_H(t)-B_H(t_0)}{t-t_0}\right| = \infty \tag{4.15}$$

with probability one.

For the proof of Proposition 4.7, see, for example, Mandelbrot and Van Ness (1968).

Proposition 4.7 implies a very important feature of fractional Brownian motion which makes the application of Itô's calculus impossible:

THEOREM 4.8 *When $1/2 < H < 1$, fractional Brownian motion $B_H(t)$ is not a semimartingale.*

For a proof of Theorem 4.8, see, for example, Lin (1996).

Because $B_H(t)$, $1/2 < H < 1$, is not a semimartingale, we spend the rest of this chapter on establishing stochastic integrals with integrator $dB_H(t)$.

4.3.4 Equivalent definitions

To end this section, we give a number of equivalent definitions of fractional Brownian motion. The equivalence of these definitions follows from the fact that a Gaussian process is determined by its mean and covariance functions.

DEFINITION 4.4 *A centred Gaussian process* $\{X(t)\}$ *with* $X_0 = 0$ *is a fractional Brownian motion if*

$$Var(X_t - X_s) = |t - s|^{2H}\, Var(X_1)$$

for any $t,\ s \geq 0$.

DEFINITION 4.5 *A centred Gaussian process* $\{X(t)\}$ *with* $X_0 = 0$ *is a fractional Brownian motion if*

$$Cov(X_s\,,\,X_t) = \frac{1}{2}\, Var(X_1)(t^{2H} + s^{2H} - |t - s|^{2H})$$

for any $t,\ s \geq 0$.

DEFINITION 4.6 *A centred Gaussian process* $\{X(t)\}$ *with* $X_0 = 0$ *is a fractional Brownian motion if*

1. *For any* $t_1,\ t_2,\ s_1,\ s_2$ *and* $h \geq 0$,

$$(X_{t_2} - X_{t_1}\,,\, X_{s_2} - X_{s_1}) \stackrel{\mathrm{d}}{=} (X_{t_2+h} - X_{t_1+h}\,,\, X_{s_2+h} - X_{s_1+h})\ .$$

2. *There is an* $H \in (0\,,\,1)$ *such that*

$$X_{t+\tau} - X_t \stackrel{\mathrm{d}}{=} h^{-H}(X_{t+h\tau} - X_t)$$

for all $t,\ \tau$ *and* $h \geq 0$.

4.4 Definition of Stochastic Integral with respect to Fractional Brownian Motion

The classical stochastic integral $\int_0^1 X(t)dB(t)$ is defined first for integrands which are adapted stochastic processes and whose sample functions are step functions. For more general integrands, satisfying certain conditions, the integrals are obtained through a routine limiting procedure. The validity of the procedure is guaranteed by the orthogonality of the increments of $B(t)$. The orthogonality of a stochastic process means certain continuity of the second moments, or the covariance functions of the process. It has been shown that similar stochastic integrals can be established with integrators from a large class of Gaussian processes with covariance functions satisfying certain continuity and smoothness conditions, which includes Brownian motion as an example. See, for example, Yeh (1975). The framework of Yeh is based on some relatively strong conditions on the covariance function of the Gaussian integrators. In some sense, the second partial derivative of the covariance function must be bounded, which is not true in the case of fractional Brownian motion.

Although fractional Brownian motion does not meet either conditions needed by Itô's setting or Yeh's, we can use the ideas of both to construct the stochastic integrals with respect to fractional Brownian motion.

To begin our task, we introduce some notation first.

Let $D = [0,1] \subset \mathbf{R}$. Let $L_2(D)$ be the collection of real valued Lebesgue measurable functions which are square integrable over D and write $\mathcal{L}_2(D)$ for the Hilbert space of equivalence classes of functions in $L_2(D)$ modulo a.e. equality on D. Let $(\Omega, \mathcal{B}, \mathbf{P})$ be a probability space. We distinguish likewise between $L_2(\Omega)$ and $\mathcal{L}_2(\Omega)$. Let $C(D)$ be the collection of real valued continuous functions on D and $\mathcal{C}(D)$ be the collection of those elements of $\mathcal{L}_2(D)$ each of which has a version in $C(D)$. We shall write $(\cdot\,,\,\cdot)$ and $\|\cdot\|$ for inner product and norm in both $\mathcal{L}_2(D)$ and $\mathcal{L}_2(\Omega)$ since there will be no ambiguity from the context.

Let B_H be a fractional Brownian motion on D with long range dependence, starting from zero. In other words, $B_H(0) = 0$ almost surely, and the Hurst index satisfies $1/2 < H < 1$. Let

$$\Gamma(s\,,\,t) \equiv EB_H(s)B_H(t) = \frac{V_H}{2}(s^{2H} + t^{2H} - |s-t|^{2H}) \tag{4.16}$$

be the covariance function of $B_H(t)$. The basic properties of $\Gamma(s\,,\,t)$ are summarized in the following two lemmas, which are prerequisite for the validity of the definition of the integral $\int_0^1 X(t)dB_H(t)$.

LEMMA 4.9 *For any* $(t\,,\,s) \in D \times D$,

1. Γ *is continuous on* $D \times D$.
2. *Let*
$$\gamma(t) = \lim_{s\uparrow t} \frac{\Gamma(s\,,\,t) - \Gamma(t\,,\,t)}{s-t} - \lim_{s\downarrow t} \frac{\Gamma(s\,,\,t) - \Gamma(t\,,\,t)}{s-t}\,.$$
Then for any $t \in [\,0\,,\,1\,]$, $\gamma(t) \equiv 0$.
3. $\dfrac{\partial^2\Gamma(s\,,t)}{\partial t\partial s}$ *and* $\dfrac{\partial^2\Gamma(s\,,t)}{\partial s^2}$ *exist and are continuous on* $T_1 \cup T_2$, *where* T_1 *and* T_2 *are two open triangles*
$$T_1 = \{\,(s\,,\,t) \in D \times D\,;\ \ s \in (0,t)\,,\ \ t \in (0\,,\,1)\,\}\,,$$
and
$$T_2 = \{\,(s\,,\,t) \in D \times D\,;\ \ s \in (t\,,\,1)\,,\ \ t \in (0\,,\,1)\,\}\,.$$

LEMMA 4.10 *For any partition* β *of* $[0\,,\,1]$

$$\beta\ :\qquad 0 = a_0 < a_1 < \cdots < a_q = 1\ ,$$

let $\Gamma_{k,l} = \Gamma(a_k\ ,\ a_l)$ *and* $\Delta\Gamma_{k,l} = \Gamma_{k,l} - \Gamma_{k,l-1} - \Gamma_{k-1,l} + \Gamma_{k-1,l-1}$. *Then for any* $k \neq l$,

$$\begin{aligned}
&\left|\Delta\Gamma_{k,l} - \frac{\partial^2\Gamma}{\partial y \partial x}(a_{k-1}\ ,\ a_{l-1})(a_k - a_{k-1})(a_l - a_{l-1})\right| \\
&\leq\ C|a_{k-1} - a_{l-1}|^{2H-2-\alpha}(a_k - a_{k-1})(a_l - a_{l-1})\,[\,(a_k - a_{k-1})^\alpha + (a_l - a_{l-1})^\alpha] \\
&\quad +o((a_k - a_{k-1})(a_l - a_{l-1}))\ , \qquad (4.17)
\end{aligned}$$

where C *is a nonnegative constant, and* α *is a positive constant such that* $-1 < 2H - 2 - \alpha < 0$.

The proofs of Lemmas 4.9 and 4.10 and other results obtained in this section are given in Section 4.5.

In the following three subsections we give details of stochastic integrals with respect to B_H step by step. In the last subsection of this section, we define stochastic integral processes with respect to B_H.

4.4.1 Stochastic Integral $\int_0^1 f(s)dB_H(s,\omega)$ for $f \in$ C(D)

In this subsection we give the definition of integral $\int_0^1 f(t)dB_H(t)$ through Theorem 4.11, where $f \in C(D)$. Then we discuss some properties of $\int_0^1 f(t)dB_H(t)$.

THEOREM 4.11 *Let* $\mathcal{P}^n$ *be a partition of* D *given by*

$$\mathcal{P}^n \ :\qquad 0 = a_{n,0} < a_{n,1} < \cdots < a_{n,q(n)} = 1$$

for $n = 1\ ,\ 2\ ,\ \ldots$ *and* $\lim_{n\to\infty} |\mathcal{P}^n| = 0$, *where* $|\mathcal{P}^n|$ *is the maximum of the lengths of the subintervals of partition* $\mathcal{P}^n$. *Corresponding to* $\mathcal{P}^n$ *and a collection of* $q(n)$ *real numbers*

$$\alpha^n = \{\ \alpha_{n,k} \in [\,a_{n,k-1}\ ,\ a_{n,k}\,]\ ,\ k = 1\ ,\ 2\ ,\ \ldots\ ,\ q(n)\ \}\ ,$$

let the Riemann-Stieltjes sum of $f \in C(D)$ with respect to B_H be defined by

$$S(f, \beta^n, \alpha^n)(\omega) = \sum_{k=1}^{q(n)} f(\alpha_{n,k})\{B_H(a_{n,k}, \omega) - B_H(a_{n,k-1}, \omega)\}$$

for $\omega \in \Omega$. Then $\{S(f, \beta^n, \alpha^n), n = 1, 2, \ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. Furthermore the element in $\mathcal{L}_2(\Omega)$ to which this Cauchy sequence converges is determined by f independently of the sequences $\{\beta^n, n = 1, 2, \ldots\}$ and $\{\alpha^n, n = 1, 2, \ldots\}$.

DEFINITION 4.7 *For $f \in C(D)$ we define the stochastic integral*

$$I(f) = \int_0^1 f(t) dB_H(t)$$

to be the element in $\mathcal{L}_2(\Omega)$ to which the sequence $\{S(f, \beta^n, \alpha^n), n = 1, 2, \ldots\}$ in Theorem 4.11 converges.

THEOREM 4.12 *For $f, g \in C(D)$ and $\alpha, \beta \in R^1$ the stochastic integral I satisfies the following*

1. $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$,
2. $E[I(f)] = 0$,
3. $(I(f), I(g)) = \int_{D\times D} f(s)g(t)\frac{\partial^2\Gamma(s, t)}{\partial t \partial s} m_L(d(s,t))$, *where m_L denotes the Lebesgue measure on $\mathbf{R}^2$,*
4. $\|I(f)\|^2 = \int_{D\times D} f(s)f(t)\frac{\partial^2\Gamma}{\partial t\partial s}(s,t) m_L(d(s,t))$,
5. *$I(f)$ is distributed as $N(0, \|I(f)\|^2)$,*
6. *$\{I(f), f \in L_2(D)\}$ is a Gaussian system of random variables.*

4.4.2 Stochastic Integral $\int_0^1 f(s)dB_H(s,\omega)$ for $f \in L_2(D)$

LEMMA 4.13 *If $\{ f_n : n = 1, 2, \dots \} \subset C(D)$ satisfies, as n, $m \to \infty$, $\int_0^1 |f_n(s) - f_m(s)|^2 ds \to 0$, then $\{ I(f_n) : n = 1, 2, \dots \}$ satisfies $E|I(f_n) - I(f_m)|^2 \longrightarrow 0$ as n, $m \to \infty$.*

LEMMA 4.14 *For $f \in L_2(D)$, let $\{ f_n : n = 1, 2, \dots \} \subset C(D)$ be such that $\lim_{n\to\infty} \|f_n - f\| = 0$. Then by Lemma 4.13, $\{ I(f_n) : n = 1, 2, \dots \}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. Furthermore the element in $\mathcal{L}_2(\Omega)$ to which our Cauchy sequence converges is independent of the sequence $\{ f_n : n = 1, 2, \dots \}$.*

DEFINITION 4.8 *For $f \in L_2(D)$, we define the stochastic integral*

$$I(f) = \int_0^1 f(t)dB_H(t)$$

to be the element in $\mathcal{L}_2(\Omega)$ to which the sequence $\{ I(f_n) : n = 1, 2, \dots \}$ in Lemma 4.14 converges .

THEOREM 4.15 *The stochastic integral $\int_0^1 f(t)dB_H(t)$, $f \in L_2(D)$, satisfies (1)—(6) of Theorem 4.12.*

4.4.3 Stochastic Integral $\int_0^1 \phi(B_H(t))dB_H(t)$

THEOREM 4.16 *Let ϕ be a real function on R^1 and satisfy the Lipschitz condition*

$$|\phi(t) - \phi(s)| \le C \ |t - s|^\beta \tag{4.18}$$

almost everywhere on R^1 in Lebesgue measure, where C is a constant independent of t and s, and $\beta > 0$ satisfies $H(1 + \beta) > 1$. For any partition

$$\mathcal{B} : \qquad 0 = a_0 < a_1 < \cdots < a_q = 1 \tag{4.19}$$

and q real numbers

$$\alpha = \{ \alpha_j \in [a_{j-1} , a_j] : j = 1 , 2 , \dots , q \} , \tag{4.20}$$

let the Riemann-Stieltjes sum of $\phi(B_H)$ with respect to B_H be defined by

$$S(\phi , \beta , \alpha) = \sum_{k=1}^{q} \phi(B_H(\alpha_k))\{ B_H(a_k) - B_H(a_{k-1}) \} ,$$

then $S(\phi , \beta , \alpha) \in \mathcal{L}_2(\Omega)$, that is

$$E|S(\phi , \beta , \alpha)|^2 < \infty .$$

Let $\{ \beta^n : n = 1 , 2 , \dots \}$ be a sequence of partitions given by

$$\beta^n \quad : \qquad 0 = a_{n,0} < a_{n,1} < \cdots < a_{n,q(n)} = 1$$

for $n = 1 , 2 , \dots$, and $\lim_{n\to\infty} |\beta^n| = 0$, where $|\beta^n|$ is the maximum of the lengths of the subintervals of partition β^n and

$$\alpha^n = \{ \alpha_{n,k} \in [a_{n,k-1} , a_{n,k}] , k = 1 , 2 , \dots , q(n) \}$$

be the corresponding collections of $q(n)$ real numbers. Then $\{ S (\phi , \beta^n , \alpha^n) : n = 1 , 2 , \dots \}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. Furthermore, the element in $\mathcal{L}_2(\Omega)$ to which this Cauchy sequence converges is independent of the sequences $\{ \beta^n : n = 1 , 2 , \dots \}$ and $\{ \alpha^n : n = 1 , 2 , \dots \}$.

DEFINITION 4.9 *For the fractional Brownian motion B_H, and a deterministic function ϕ satisfying (4.18) we define the stochastic integral*

$$I(\phi(B_H)) = \int_0^1 \phi(B_H(t))dB_H(t)$$

to be the element in $\mathcal{L}_2(\Omega)$ to which the sequence $\{S(\phi , \beta^n , \alpha^n) : n = 1 , 2 , \dots\}$ in Theorem 4.16 converges.

THEOREM 4.17 *For any real numbers a and b, and any real functions ϕ_1, ϕ_2 on R^1 satisfying*

$$|\phi_1(t) - \phi_1(s)| \leq const\, |t-s|^{\beta_1} \quad a.e - m_l$$

and

$$|\phi_2(t) - \phi_2(s)| \leq const\, |t-s|^{\beta_2} \quad a.e - m_l \ ,$$

where a.e.- m_l means almost everywhere on R^1 in Lebesgue measure, while H and β_1, β_2 satisfy: $H(1+\beta_1) > 1$, $H(1+\beta_2) > 1$, the stochastic integral $I(\phi(B_H))$ defined by Definition 4.9 satisfies

$$I(a\phi_1(B_H) + b\phi_2(B_H)) = aI(\phi_1(B_H)) + bI(\phi_2(B_H)) \ . \tag{4.21}$$

If Φ is an absolutely continuous function on R^1 and for almost every $t \in (-\infty, \infty)$, $\Phi' \equiv \phi$ exists and satisfies

$$|\phi(t) - \phi(s)| \leq const\, |t-s|^{\beta} \ a.e. - m_l \tag{4.22}$$

with $H(1+\beta) > 1$, then

$$\Phi(B_H(1)) - \Phi(B_H(0)) = \int_0^1 \phi(B_H(s)) dB_H(s) \ . \tag{4.23}$$

4.4.4 Stochastic Integral $\int_0^1 X(t,\omega) dB_H(t,\omega)$

In this subsection we are concerned with extending stochastic integrals with respect to $B_H(t)$ from deterministic integrands $f(t)$ and some specified stochastic processes $\phi(B_H(t))$ to more general ones. First of all, for a simple process

$$X(t,\omega) = \sum_{j=1}^{k} A_j(\omega) 1_{(T_{j-1},T_j]}(t) \ ,$$

where $A_j(\omega)$ and T_j are random variables, $j = 1, 2, \ldots, k$, and $1_D(t)$ denotes the indicator function, we define the integral as

$$\int_{-\infty}^{\infty} X_s \, dB_H(s) = \sum_{j=1}^{k} A_j \Big(B_H(T_j) - B_H(T_{j-1}) \Big) \ . \tag{4.24}$$

Secondly, if $\{X(t)\}$ is a process with locally bounded variation, it is always possible to define the integral by the formula of integration by parts as

$$\int_0^1 X_s\, dB_H(s) = X_1\, B_H(1) - X_0\, B_H(0) - \int_0^1 B_H(s)\, dX_s\,. \qquad (4.25)$$

Finally, for general stochastic process $\{X(t)\}$, we consider some integrands satisfying the conditions given in the following theorem. We give the definition of the integrals for these integrands in Definition 4.10.

THEOREM 4.18 *Let $\{X(t) \;:\; -\infty < t < \infty\}$ be a real valued stochastic process, such that for any $t \in (-\infty,\, \infty)$ and any real valued random variables $\xi(\omega)$ and $\eta(\omega)$,*

$$E|X(t)|^4 \;<\; \infty\,, \qquad (4.26)$$

$$E|X(\xi(\omega)\,,\,\omega) - X(\eta(\omega)\,,\,\omega)|^4 \;\le\; const\, E|\xi(\omega) - \eta(\omega)|^{4\beta}\,, \qquad (4.27)$$

where $H + \beta > 1$ and the constant in the right hand side of (4.27) is independent of ξ and η. For any partition β of $[0\,,\,1]$ given by (4.19) and q real numbers α given by (4.20), let the Riemann-Stieltjes sum of X with respect to B_H be defined as

$$S(X\,,\,\beta,\,\alpha) \equiv \sum_{k=1}^{q} X(\alpha_k)[\,B_H(a_k) - B_H(a_{k-1})\,]\,.$$

Then $S(X\,,\,\beta,\,\alpha) \in \mathcal{L}_2(\Omega)$. Let $\{\beta^n \,:\, n = 1\,,\,2\,,\ldots\}$ and $\{\alpha^n \,:\, n = 1\,,\,2\,,\ldots\}$ be as given in Theorem 4.16; then $\{S(X\,,\,\beta^n\,,\,\alpha^n) \,:\, n = 1\,,\,2\,,\ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. Furthermore, the element in $\mathcal{L}_2(\Omega)$ to which this Cauchy sequence converges is independent of the sequences $\{\beta^n \,:\, n = 1\,,\,2\,,\ldots\}$ and $\{\alpha^n \,:\, n = 1\,,\,2\,,\ldots\}$.

DEFINITION 4.10 *For the fractional Brownian motion B_H and a stochastic process X satisfying (4.26) and (4.27), we define the stochastic integral*

$$I(X) = \int_0^1 X(s,\omega)dB_H(s,\omega)$$

to be the element in $\mathcal{L}_2(\Omega)$ *to which the sequence* $\{S(X, \beta^n, \alpha^n) : n = 1, 2, \ldots\}$ *in Theorem 4.18 converges.*

THEOREM 4.19 *For any real numbers* a *and* b*, nonnegative real numbers* β_1, β_2 *such that* $H+\beta_1 > 1$ *and* $H+\beta_2 > 1$*, and real valued stochastic processes* $\{X_1(t) : -\infty < t < \infty\}$ *and* $\{X_2(t) : -\infty < t < \infty\}$ *satisfying (4.26) and for any real valued random variables* ξ *and* η*,*

$$E|X_1(\xi) - X_1(\eta)|^4 \leq const\, E|\xi - \eta|^{4\beta_1} , \tag{4.28}$$

$$E|X_2(\xi) - X_2(\eta)|^4 \leq const\, E|\xi - \eta|^{4\beta_2} , \tag{4.29}$$

where the constants in the right hand sides of (4.28) and (4.29) are independent of ξ *and* η*. Then the stochastic integral* $I(X)$ *defined by Definition 4.10 satisfies*

$$I(aX_1 + bX_2) = aI(X_1) + bI(X_2) . \tag{4.30}$$

If the stochastic process $\{X(t, \omega) : -\infty < t < \infty, \omega \in \Omega\}$ *is such that*

$$\frac{\partial X}{\partial t}(t, \omega) \equiv x(t, \omega)$$

exists almost everywhere on R^1 *and* $x(t)$ *satisfies (4.26) and (4.27) then the stochastic integral*

$$\int_0^1 x(B_H(s), \omega)\, dB_H(s) = \int_0^1 \frac{\partial X}{\partial t}(B_H(s), \omega)\, dB_H(s)$$

exists, and

$$X(B_H(1), \omega) - X(B_H(0), \omega) = \int_0^1 \frac{\partial X}{\partial t}(B_H(s), \omega)\, dB_H(s). \tag{4.31}$$

4.4.5 Stochastic Integral Processes $\{I(f)(t) : t \geq 0\}$, $\{I(\phi)(t) : t \geq 0\}$ and $\{I(X)(t) : t \geq 0\}$

It is now easy to extend the definition of integral operator I from $I : \mathcal{L}_2([0, 1]) \to \mathcal{L}_2(\Omega)$ to $I : \mathcal{L}_2([a, b]) \to \mathcal{L}_2(\Omega)$ by the same idea as presented in Section 4.4.2, where a and b are any real numbers. Indeed for any $f \in \mathcal{L}_2([a, b])$, we define $I(f) \equiv \int_a^b f(s)dB_H(s)$, then I is a linear operator from $\mathcal{L}_2([a\ ,\ b])$ to $\mathcal{L}_2(\Omega)$ and Theorem 4.12 still holds. Thus for any $f \in \mathcal{L}_2(R^1)$ we define a stochastic process $\{\, I(f)(t)\ :\ t \geq 0 \,\}$ as

$$I(f)(t) \equiv \int_0^t f(s)dB_H(s)\ . \tag{4.32}$$

By the same argument, we define stochastic processes $\{\, I(\phi)(t)\ :\ t \geq 0 \,\}$ and $\{\, I(X)(t)\ :\ t \geq 0 \,\}$ as

$$I(\phi)(t) \equiv \int_0^t \phi(B_H(s))dB_H(s) \tag{4.33}$$

and

$$I(X)(t) \equiv \int_0^t X(s)dB_H(s) \tag{4.34}$$

respectively so long as ϕ and X satisfy the conditions of Theorem 4.16 and Theorem 4.18 respectively.

THEOREM 4.20 *For any constants $-\infty < a < b < \infty$, the sample functions of $\{\, I(f)(t)\ :\ a \leq t \leq b\}$, $\{\, I(\phi)(t)\ :\ a \leq t \leq b\}$ and $\{\, I(X)(t)\ :\ a \leq t \leq b\}$ are uniformly continuous on $[a\ ,\ b]$ with probability one, hence for almost every $\omega \in \Omega$ the sample functions $\{\ I(f)(t)\ :\ t \geq 0\ \}$, $\{\ I(\phi)(t)\ :\ t \geq 0\ \}$ and $\{I(X)(t)\ :\ t \geq 0\}$ are continuous functions on $[0\ ,\ \infty)$ with probability one.*

THEOREM 4.21 *Let $I(f)(t)$, $I(\phi)(t)$ and $I(X)(t)$ be defined by (4.32), (4.33) and (4.34). Then we have the following*

1. *For any $t \geq 0$,*

$$E\left[I(f)(t)\right]^2 < \infty\ , \tag{4.35}$$

$$E\left[I(\phi)(t)\right]^2 < \infty\ , \tag{4.36}$$

$$E\left[I(X)(t)\right]^2 < \infty\ . \tag{4.37}$$

2. *The deterministic functions $E\left[I(f)(\cdot)\right]^2$, $E\left[I(\phi)(\cdot)\right]^2$ and $E\left[I(X)(\cdot)\right]^2$ are continuous on $[0\ ,\ \infty)$.*

3. For any $0 \leq a < b$,

$$\int_a^b E\left[I(\phi)(t)\right]^2 dt < \infty \ , \tag{4.38}$$

$$\int_a^b E\left[I(X)(t)\right]^2 dt < \infty \ , \tag{4.39}$$

$$\int_a^b E\left[I(f)(t)\right]^2 dt \leq (b-a)\int_{[a,b]\times[a,b]} |f(x)f(y)|\frac{\partial^2\Gamma}{\partial y \partial x}(x,y)dxdy < \infty. \tag{4.40}$$

4.5 Proofs

In this section, we use C to denote a constant, which may differ in different expressions.

4.5.1 Proofs of Lemmas 4.9 and 4.10

Proof of Lemma 4.9.

1 Since H> 0, from (4.16), $\Gamma(s\ ,\ t)$ is continuous on $D \times D$.

2 Let

$$\gamma_1(t) = \lim_{s \uparrow t} \frac{\Gamma(s\ ,\ t) - \Gamma(t\ ,\ t)}{s-t} \quad , \quad \gamma_2(t) = \lim_{s \downarrow t} \frac{\Gamma(s\ ,\ t) - \Gamma(t\ ,\ t)}{s-t} \ ;$$

then, $\gamma_1(t) = V_H H t^{2H-1} = \gamma_2(t)$ and $\gamma(t) = \gamma_1(t) - \gamma_2(t) \equiv 0$ for any t in $[0\ ,\ 1]$.

3

$$\begin{aligned}
\frac{\partial\Gamma(s\ ,\ t)}{\partial s} &= \begin{cases} H\,V_H(s^{2H-1} + (t-s)^{2H-1}) & (s\ ,\ t) \in T_1\ , \\ H\,V_H(s^{2H-1} - (s-t)^{2H-1}) & (s\ ,\ t) \in T_2\ , \end{cases} \\
\frac{\partial^2\Gamma(s\ ,\ t)}{\partial t \partial s} &= (2H-1)H\,V_H|t-s|^{2H-2}\ , (s\ ,\ t) \in T_1 \cup T_2\ , \\
\frac{\partial^2\Gamma(s\ ,\ t)}{\partial s^2} &= (2H-1)H\,V_H(s^{2H-1} - |t-s|^{2H-1})\ ,\ (s\ ,\ t) \in T_1 \cup T_2\ .
\end{aligned}$$

So $\partial^2\Gamma(s\ ,\ t)/\partial t\partial s$ and $\partial^2\Gamma(s\ ,\ t)/\partial s^2$ are continuous on $T_1 \cup T_2$.

Proof of Lemma 4.10. By the definitions of $\Gamma_{k,l}$ and $\Delta\Gamma_{k,l}$ we have

$$\begin{aligned}\Delta\Gamma_{k,l} &= -\frac{V_H}{2}\left\{|a_k - a_l|^{2H} - |a_k - a_{l-1}|^{2H} - |a_{k-1} - a_l|^{2H} + |a_{k-1} - a_{l-1}|^{2H}\right\} \\ &= -\frac{V_H}{2}\left\{g(a_k\,,\, a_l) - g(a_{k-1}\,,\, a_l) - g(a_k\,,\, a_{l-1}) + g(a_{k-1}\,,\, a_{l-1})\right\}, \qquad (4.41)\end{aligned}$$

where $g(x\,,\,y) = |x-y|^{2H}$. If $a_k - a_{k-1} \le a_l - a_{l-1}$, therefore,

$$\begin{aligned}\Delta\Gamma_{k,l} &= -\frac{V_H}{2}\Bigg\{ \frac{\partial g}{\partial x}(a_{k-1}\,,\, a_l)(a_k - a_{k-1}) + \frac{\partial^2 g}{\partial x^2}(a_{k-1}\,,\, a_l)\frac{(a_k - a_{k-1})^2}{2} \\ &\qquad + o((a_k - a_{k-1})^2) - \frac{\partial g}{\partial x}(a_{k-1}\,,\, a_{l-1})(a_k - a_{k-1}) \\ &\qquad - \frac{\partial^2 g}{\partial x^2}(a_{k-1}\,,\, a_{l-1})\frac{(a_k - a_{k-1})^2}{2} + o((a_k - a_{k-1})^2) \Bigg\} \\ &= -\frac{V_H}{2}\Bigg\{ \left[\frac{\partial^2 g}{\partial y \partial x}(a_{k-1}\,,\, a_{l-1})(a_l - a_{l-1}) + o(a_l - a_{l-1})\right](a_k - a_{k-1}) \\ &\qquad + \left[\frac{\partial^3 g}{\partial y \partial x^2}(a_{k-1}, a_{l-1})(a_l - a_{l-1}) + o(a_l - a_{l-1})\right]\frac{(a_k - a_{k-1})^2}{2} \\ &\qquad + o((a_k - a_{k-1})^2) \Bigg\} \\ &= \Bigg\{ \frac{\partial^2 \Gamma}{\partial y \partial x}(a_{k-1}\,,\, a_{l-1})(a_k - a_{k-1})(a_l - a_{l-1}) \\ &\qquad - \frac{V_H}{2}\frac{\partial^3 g}{\partial y \partial x^2}(a_{k-1}\,,\, a_{l-1})(a_l - a_{l-1})\frac{(a_k - a_{k-1})^2}{2} \\ &\qquad + o((a_k - a_{k-1})(a_l - a_{l-1})) \Bigg\}.\end{aligned}$$

(The last equation follows from $(-V_H/2)(\partial^2 g/\partial y \partial x) = \partial^2\Gamma/\partial y \partial x$.) Then

$$\begin{aligned}&\left|\Delta\Gamma_{k,l} - \frac{\partial^2 \Gamma}{\partial y \partial x}(a_{k-1}, a_{l-1})(a_k - a_{k-1})(a_l - a_{l-1})\right| \\ &\le \left|\frac{V_H}{2}\frac{\partial^3 g}{\partial y \partial x^2}(a_{k-1}, a_{l-1})(a_l - a_{l-1})\frac{(a_k - a_{k-1})^2}{2}\right| + o((a_k - a_{k-1})(a_l - a_{l-1})) \\ &= C\,|a_{k-1} - a_{l-1}|^{2H-3}(a_l - a_{l-1})(a_k - a_{k-1})^2 + o((a_k - a_{k-1})(a_l - a_{l-1}))\,.\end{aligned}$$

Since $-1 < 2H - 2 < 0$ we can choose $\alpha > 0$ such that $-1 < 2H - 2 - \alpha < 0$. If $a_k - a_{k-1} \le a_l - a_{l-1}$ and $k < l$ then,

$$\frac{a_k - a_{k-1}}{|a_{l-1} - a_{k-1}|} \le 1$$

and we have

$$
\begin{aligned}
&|a_{k-1}-a_{l-1}|^{2H-3}(a_l-a_{l-1})(a_k-a_{k-1})^2\\
&=|a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})(a_k-a_{k-1})^\alpha\left(\frac{a_k-a_{k-1}}{|a_{k-1}-a_{l-1}|}\right)^{1-\alpha}\\
&\le |a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})(a_k-a_{k-1})^\alpha\\
&\le |a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})\left[(a_k-a_{k-1})^\alpha+(a_l-a_{l-1})^\alpha\right] .
\end{aligned}
$$

If $a_k-a_{k-1}\le a_l-a_{l-1}$ and $k>l$,

$$
\begin{aligned}
&|a_{k-1}-a_{l-1}|^{2H-3}(a_l-a_{l-1})(a_k-a_{k-1})^2\\
&\le |a_{k-1}-a_{l-1}|^{2H-3}(a_l-a_{l-1})^2(a_k-a_{k-1})\\
&=|a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})(a_l-a_{l-1})^\alpha\left(\frac{a_l-a_{l-1}}{|a_{k-1}-a_{l-1}|}\right)^{1-\alpha}\\
&\le |a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})(a_l-a_{l-1})^\alpha\\
&< |a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})\left[(a_k-a_{k-1})^\alpha+(a_l-a_{l-1})^\alpha\right] .
\end{aligned}
$$

Thus when $a_k-a_{k-1}\le a_l-a_{l-1}$ we have

$$
\begin{aligned}
&\left|\Delta\Gamma_{k,l}-\frac{\partial^2\Gamma}{\partial y\partial x}(a_{k-1},a_{l-1})(a_k-a_{k-1})(a_l-a_{l-1})\right|\\
&\quad\le C\;|a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})\left[(a_k-a_{k-1})^\alpha+(a_l-a_{l-1})^\alpha\right]\\
&\qquad+o((a_k-a_{k-1})(a_l-a_{l-1})) .
\end{aligned}
$$

If $a_k-a_{k-1}>a_l-a_{l-1}$, since $g(x\,,\,y)=g(y\,,\,x)$,

$$
\Delta\Gamma_{k,l}=-\frac{V_H}{2}\left\{g(a_l\,,\,a_k)-g(a_l\,,\,a_{k-1})-g(a_{l-1}\,,\,a_k)+g(a_{l-1}\,,\,a_{k-1})\right\} .
$$

Then by the same argument used above and noticing that

$$
\frac{\partial^2 g}{\partial y\partial x}(x,y)=\frac{\partial^2 g}{\partial y\partial x}(y\,,\,x)=\frac{-2}{V_H}\frac{\partial^2\Gamma}{\partial y\partial x}(x\,,\,y)\,,
$$

we have

$$
\begin{aligned}
&\left|\Delta\Gamma_{k,l}-\frac{\partial^2\Gamma}{\partial y\partial x}(a_{k-1}\,,\,a_{l-1})(a_k-a_{k-1})(a_l-a_{l-1})\right|\\
&\quad\le C|a_{k-1}-a_{l-1}|^{2H-2-\alpha}(a_l-a_{l-1})(a_k-a_{k-1})\left[(a_k-a_{k-1})^\alpha+(a_l-a_{l-1})^\alpha\right]\\
&\qquad+o((a_k-a_{k-1})(a_l-a_{l-1})) .
\end{aligned}
$$

This establishes (4.17).

4.5.2 Proof of the results of Section 4.4.1

Proof of Theorem 4.11. We first prove that for any partition

$$\text{ß} \ : \qquad 0 = a_0 < a_1 < \cdots < a_q = 1$$

and q real numbers

$$\alpha = \{ \alpha_j \in [a_{j-1}, a_j] \ : \ j = 1, 2, \dots, q \},$$

$S(f, \text{ß}, \alpha) \in \mathcal{L}_2(\Omega)$. Indeed, since $f \in \mathcal{L}_2(\Omega)$, we can choose a version of f in $L_2(\Omega)$ and still denote this version as f, such that at the finite points $\{ \alpha_j, j = 1, 2, \dots, q\}$ $f(\alpha_j) < \infty$, $j = 1, 2, \dots$ and $max\{|f(\alpha_j)| : j = 1, 2, \dots, q\} \equiv C < \infty$. Then, by Lemma 4.10,

$$E|S(f, \text{ß}, \alpha)|^2 = E\left\{\sum_j f(\alpha_j)[B_H(a_j) - B_H(a_{j-1})]\right\}^2$$
$$= \sum_{j,k} f(\alpha_j)f(\alpha_k)E\{[B_H(a_j) - B_H(a_{j-1})][B_H(a_k) - B_H(a_{k-1})]\}$$
$$= \sum_{j,k} f(\alpha_j)f(\alpha_k)\Delta\Gamma_{j,k} \le C\sum_{j,k}|\Delta\Gamma_{j,k}| \le C\int_{[0,1]\times[0,1]}\left|\frac{\partial^2\Gamma}{\partial y\partial x}(x, y)\right| dxdy < \infty .$$

Then we prove that $\{S(f, \text{ß}^n, \alpha^n) : n = 1, , \dots\}$ is a Cauchy sequence in $\mathcal{L}_2(\omega)$.

For any $\epsilon > 0$, since f is continuous on $[0, 1]$, there is a $\delta > 0$ such that $|f(t') - f(t'')| < \epsilon$ for any t', $t'' \in [0, 1]$ and $|t' - t''| < \delta$. Let m and n be so large that $|\text{ß}^m| < \delta/2$ and $|\text{ß}^n| < \delta/2$. Let

$$\text{ß}^{(n,m)} \ : \qquad 0 = a_0 < a_1 < \cdots < a_{q(n+m)} = 1$$

be the partition of D obtained by superposition of ß^n and ß^m. Then,

$$S(f, \text{ß}^m, \alpha^m) - S(f, \text{ß}^n, \alpha^n)$$
$$= \sum_j f(\alpha_{m,j})[B_H(a_{m,j}) - B_H(a_{m,j-1})] - \sum_k f(\alpha_{n,k})[B_H(a_{n,k-1}) - B_H(a_{n,k-1})]$$
$$= \sum_l C_l [B_H(a_l) - B_H(a_{l-1})] ,$$

where C_l's are of the form $C_l = f(\alpha_{m,u_l}) - f(\alpha_{n,v_l})$ for some $u_l \in \{1, 2, \dots, q(n)\}$ and $v_l \in \{1, 2, \dots, q(m)\}$, and $|C_l| < \epsilon$ for $l = 1, 2, \dots, q(n+m)$. So,

$$
\begin{aligned}
&E|S(f, ß^m, \alpha^m) - S(f, ß^n, \alpha^n)|^2 \\
&= \sum_{k,l=1}^{q(n,m)} C_k C_l \operatorname{Cov}(B_H(a_k) - B_H(a_{k-1}), B_H(a_l) - B_H(a_{l-1})) \\
&= \sum_{k,l=1}^{q(n,m)} C_k C_l \Delta\Gamma_{k,l} = \sum_{k=1}^{q} C_k^2 \Delta\Gamma_{k,k} + \sum_{k \leq l} C_k C_l \Delta\Gamma_{k,l}.
\end{aligned}
$$

Since $\Delta\Gamma_{k,k} = V_H |a_k - a_{k-1}|^{2H}$, we have

$$
\begin{aligned}
&\sum_{k=1}^{q} C_k^2 \Delta\Gamma_{k,k} \leq C \sum_{k=1}^{q} (a_k - a_{k-1})^{2H} = C \sum_{k=1}^{q} (a_k - a_{k-1})(a_k - a_{k-1})^{2H-1} \\
&\leq C \max\{|ß^n|^{2H-1}, |ß^m|^{2H-1}\} \sum_{k=1}^{q} (a_k - a_{k-1}) \\
&= C \max\{|\mathcal{B}^n|^{2H-1}, |\mathcal{B}^m|^{2H-1}\} \longrightarrow 0,
\end{aligned}
$$

as n, $m \to \infty$. By Lemma 4.10,

$$
\begin{aligned}
|\Delta\Gamma_{k,l}| < &\left| \frac{\partial^2 \Gamma}{\partial y \partial x}(a_{k-1}, a_{l-1})(a_k - a_{k-1})(a_l - a_{l-1}) \right| \\
&+ C\, |a_{k-1} - a_{l-1}|^{2H-2-\alpha} \times \\
&\quad \Big\{ (a_k - a_{k-1})(a_l - a_{l-1}) \left[(a_k - a_{k-1})^\alpha + (a_l - a_{l-1})^\alpha \right] \\
&\quad + o((a_k - a_{k-1})(a_l - a_{l-1})) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\left| \sum_{k \leq l} C_k C_l \Delta\Gamma_{k,l} \right| \leq \sum_{k \leq l} |C_k C_l \Delta\Gamma_{k,l}| \leq \epsilon^2 \sum_{k \leq l} |\Delta\Gamma_{k,l}| \\
&\leq \epsilon^2 \Big\{ \sum_{k \leq l} \left| \frac{\partial^2 \Gamma}{\partial y \partial x}(a_{k-1}, a_{l-1})(a_k - a_{k-1})(a_l - a_{l-1}) \right| \\
&\quad + \sum_{k \leq l} C |a_{k-1} - a_{l-1}|^{2H-2-\alpha} (a_k - a_{k-1})(a_l - a_{l-1}) \times \\
&\quad\quad \left[(a_k - a_{k-1})^\alpha + (a_l - a_{l-1})^\alpha \right] \\
&\quad + \sum_{k \leq l} o((a_k - a_{k-1})(a_l - a_{l-1})) \Big\} \leq \epsilon^2 C.
\end{aligned}
$$

Now we have proved that

$$
E|S(f, ß^m, \alpha^m) - S(f, ß^n, \alpha^n)|^2 \longrightarrow 0,
$$

as n, $m \to \infty$. This implies that $\{ S(f, \mathfrak{B}^n, \alpha^n) : n = 1, 2, \ldots \}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$.

To show that the element in $\mathcal{L}_2(\Omega)$ to which this Cauchy sequence converges is independent of the sequences $\{ \mathfrak{B}^n \}$ and $\{ \alpha^n \}$ with $|\mathfrak{B}^n| \to 0$ as n $\to \infty$, let $\{ \mathcal{D}^n \}$ and $\{ \beta^n \}$ be another pair of such sequences. Then since the sequence $|\mathfrak{B}^1|$, $|\mathcal{D}^1|$, $|\mathfrak{B}^2|$, $|\mathcal{D}^2|$, $\ldots$ converges to 0, the sequence

$$S(f, \mathfrak{B}^1, \alpha^1), S(f, \mathcal{D}^1, \beta^1), S(f, \mathfrak{B}^2, \alpha^2), S(f, \mathcal{D}^2, \beta^2), \ldots$$

is a Cauchy sequence in $\mathcal{L}_2(\Omega)$ and the two subsequences $\{ S(f, \mathfrak{B}^n, \alpha^n) : n = 1, 2, \ldots \}$ and $\{ S(f, \mathcal{D}^n, \beta^n) : n = 1, 2, \ldots \}$ converge to the same element in $\mathcal{L}_2(\Omega)$. This finishes the proof of Theorem 4.11.

Proof of Theorem 4.12. Let $\{ \mathfrak{B}^n \}$ and $\{ \alpha^n \}$ be given in Theorem 4.11. Then (1) follows immediately since

$$S(\alpha f + \beta g, \mathfrak{B}^n, \alpha^n) = \alpha S(f, \mathfrak{B}^n, \alpha^n) + \beta S(g, \mathfrak{B}^n, \alpha^n).$$

Also, (2) follows from

$$E[S(f, \mathfrak{B}^n, \alpha^n)] = \sum_k f(\alpha_k) E[B_H(a_k) - B_H(a_{k-1})] = 0$$

and

$$E[I(f)] = \lim_{n \to \infty} E[S(f, \mathfrak{B}^n, \alpha^n)] = 0 .$$

To prove (3), we note that, from the definitions of $I(f)$ and $I(g)$, we have

$$(I(f), I(g)) = E\, I(f)\, I(g) = \lim_{n \to \infty} E S(f, \mathfrak{B}^n, \alpha^n) S(g, \mathfrak{B}^n, \alpha^n) .$$

Let $\mathfrak{B}^n$ be $0 = a_0 < a_1 < \cdots < a_q = 1$ and α^n be such that $\alpha_k = a_k$, $k = 1, 2, \ldots, q$, then

$$S(f, \mathfrak{B}^n, \alpha^n) = \sum_k f(a_k) [B_H(a_k - a_{k-1})] ,$$
$$S(g, \mathfrak{B}^n, \alpha^n) = \sum_k g(a_k) [B_H(a_k - a_{k-1})] .$$

Since $E\,[\,B_H(a_k - a_{k-1})\,]\,[\,B_H(a_l - a_{l-1})\,] = \Delta\Gamma_{k,l}$ it follows that

$$E\,S\,(\,f\,,\,\mathcal{B}^n\,,\,\alpha^n\,)\,S\,(\,g\,,\,\mathcal{B}^n\,,\,\alpha^n\,) = \sum_k f(a_k)g(a_k)\Delta\Gamma_{k,k} + \sum_{k\neq l} f(a_k)g(a_l)\Delta\Gamma_{k,l}\;. \tag{4.42}$$

Since f and g are continuous on $[\,0\,,\,1\,]$, $|f(a_k)\,g(a_l)| < C$ for some constant C and any k and l , and when $n \to \infty$, $\sum_{k=1}^{q} |\Delta\Gamma_{k,k}| \to 0$ as shown in the proof of Theorem 4.11, $\sum_{k=1}^{q} f(a_k)g(a_k)\Delta\Gamma_{k,k} \to 0$. Now let us consider the second term in the right hand side of (4.42):

$$\begin{aligned}
\sum_{k\neq l} f(a_k)\,g(a_l)\Delta\Gamma_{k,l} &= \sum_{k\neq l} f(a_k)g(a_l)\left[\frac{\partial^2\Gamma}{\partial y\partial x}(a_{k-1}\,,\,a_{l-1})(a_k - a_{k-1})(a_l - a_{l-1})\right.\\
&\qquad\qquad \left. +o((a_k - a_{k-1})(a_l - a_{l-1}))\right]\\
&= \int_{T_1\cup T_2} f(x)\,g(y)\frac{\partial^2\Gamma}{\partial y\partial x}(x,y)\,dx\,dy + o(1)\;,
\end{aligned}$$

so (3) follows. Now (4) is a particular case of (3). (6) holds since $\{\,I(f)\,,\ f\,\in\,C(D)\,\}$ is a subspace of the closure of the collection of all linear combinations of members of the Gaussian system in $\mathcal{L}_2(\Omega)$. Then (5) holds in view of (2) and (4).

4.5.3 Proof of the results of Section 4.4.2

Proof of Lemma 4.13. From (1) and (4) of Theorem 4.12 we have

$$\begin{aligned}
&E\,|\,I(f_n) - I(f_m)\,|^2 = E\,|\,I(f_n - f_m)|^2\\
&= \int_{D\times D} (f_n(s) - f_m(s))\,(f_n(t) - f_m(t))\frac{\partial^2\Gamma}{\partial s\partial t}(s\,,\,t)m_L(d(s,t))\\
&= C\int_{D\times D} (f_n(s) - f_m(s))\,(f_n(t) - f_m(t))\,|s-t|^{2H-2}\,m_L(d(s,t))\\
&= C\int_{D\times D}\left[(f_n(s) - f_m(s))|s-t|^{H-1}\right]\times\\
&\qquad\qquad \left[(f_n(t) - f_m(t))|s-t|^{H-1}\,m_L(d(s,t))\right]\;.
\end{aligned} \tag{4.43}$$

Notice that

$$\int_{D\times D} (f_n(s) - f_m(s))^2|s-t|^{2H-2}\,m_L(d(s,t))$$

$$= \lim_{\epsilon_1,\epsilon_2\to 0}\left[\int_{D_{\epsilon_1}}(f_n(s)-f_m(s))^2|s-t|^{2H-2}\,ds\,dt \right.$$
$$\left. + \int_{D_{\epsilon_2}}(f_n(s)-f_m(s))^2\,|s-t|^{2H-2}\,ds\,dt\right] ,$$

where

$$\begin{aligned} D_{\epsilon_1} &= \{(s,t) : 0\le s\le 1-\epsilon_1 , s+\epsilon_1\le t\le 1\} , \\ D_{\epsilon_2} &= \{(s,t) : \epsilon_2\le s\le 1 , 0\le t\le s-\epsilon_2\} , \end{aligned}$$

and

$$\begin{aligned} &\int_{D_{\epsilon_1}}(f_n(s)-f_m(s))^2|s-t|^{2H-2}\,ds\,dt \\ &= \int_0^{1-\epsilon_1}\int_{s+\epsilon_1}^1(f_n(s)-f_m(s))^2|s-t|^{2H-2}\,ds\,dt \\ &= C\int_0^{1-\epsilon_1}(f_n(s)-f_m(s))^2\left[(1-s)^{2H-1}-\epsilon_1^{2H-1}\right]\,ds \\ &\le C\int_0^{1-\epsilon_1}(f_n(s)-f_m(s))^2\,ds \le C\,\|\,f_n-f_m\,\|^2<\infty . \end{aligned}$$

(for $\frac{1}{2}<H<1$, $2H-1>0$, $(1-s)^{2H-1}-\epsilon_1^{2H-1}\le 1$). By a similar argument,

$$\int_{D_{\epsilon_2}}(f_n(s)-f_m(s))^2\,|s-t|^{2H-2}\,ds\,dt \le C\,\|\,f_n-f_m\,\|^2<\infty .$$

Thus $(f_n(s)-f_m(s))|s-t|^{H-1}\in L_2(D\times D)$. From (4.43) we have

$$\begin{aligned} E\,|\,I(f_n)-I(f_m)|^2 &\le C\left\{\int_{D\times D}(f_n(s)-f_m(s))^2\,|s-t|^{2H-2}\,m_L(d(s,t))\right\}^{\frac{1}{2}}\times \\ &\quad \left\{\int_{D\times D}(f_n(t)-f_m(t))^2\,|s-t|^{2H-2}\,m_L(d(s,t))\right\}^{\frac{1}{2}} \\ &= C\int_{D\times D}(f_n(s)-f_m(s))^2|s-t|^{2H-2}m_L(d(s,t)) \\ &= C\lim_{\epsilon_1\,,\,\epsilon_2\to 0}\left\{\int_{D_{\epsilon_1}}(f_n(s)-f_m(s))^2\,|s-t|^{2H-2}\,ds\,dt\right. \\ &\quad \left. + \int_{D_{\epsilon_2}}(f_n(s)-f_m(s))^2\,|s-t|^{2H-2}\,ds\,dt\right\} \\ &\le C\,\|\,f_n-f_m\,\|^2 . \end{aligned}$$

Thus $\{\,I(f_n) : n=1,2,\ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$ so long as $\{\,f_n : n=1,2,\ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(D)$.

Proof of Lemma 4.14. Since $\lim_{n\to\infty} \| f_n - f_m \| = 0$, $\{ f_n : n = 1, 2, \ldots\} \subset C(D)$ is a Cauchy sequence in $\mathcal{L}_2(D)$. By Lemma 4.13, $\{ I(f_n) : n = 1, 2, \ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. So there is an element in $\mathcal{L}_2(\Omega)$ to which $\{ I(f_n) : n = 1, 2, \ldots\}$ converges. To show that this element does not depend on the choice of the sequence $\{ f_n : n = 1, 2, \ldots \} \subset C(D)$ converging to f in $\mathcal{L}_2(D)$, let $\{ g_n : n = 1, 2, \ldots \} \subset C(D)$ be another such sequence. Then $\{ f_1, g_1, f_2, g_2, \ldots \}$ is a sequence from C(D) which converges to f in $\mathcal{L}_2(D)$. Then $\{ I(f_1), I(g_1), I(f_2), I(g_2), \ldots \}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$ and its subsequences $\{ I(f_n) : n = 1, 2, \ldots \}$ and $\{ I(g_n) : n = 1, 2, \ldots \}$ converge to the same element in $\mathcal{L}_2(\Omega)$.

4.5.4 Proof of the results of Section 4.4.3

Proof of Theorem 4.16. In order to prove Theorem 4.16, we need the following lemma

LEMMA 4.22 *For any $r > 0$, s and t we have*

$$E \,|\, B_H(s) - B_H(t) \,|^r \le const \, |t - s|^{r\,H} \, . \tag{4.44}$$

The proof of Lemma 4.22 follows from the Hölder inequality and (4.11).

To deal with Theorem 4.16, we first prove that, for any partition ß and corresponding collection of real numbers α given by (4.19) and (4.20) respectively, $E|S(\phi, ß, \alpha)|^2 < \infty$. To this end, we notice that from (4.18) it follows that

$$|\phi(B_H(t))| \le |\phi(B_H(t)) - \phi(0)| + |\phi(0)| \le C\, |B_H(t)|^\beta + |\phi(0)| \, .$$

Thus,

$$E\,|S\,(\phi, ß, \alpha\,)\,|^2 = E\left\{ \sum_j \phi(\,B_H(\alpha_j)\,)\,[\,B_H(a_j) - B_H(a_{j-1})\,] \right\}^2$$

$$
\begin{aligned}
&= \sum_{j,k} E\,\phi(\,B_H(\alpha_j)\,)\,\phi(\,B_H(\alpha_k)\,)\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,[\,B_H(a_k) - B_H(a_{k-1})\,] \\
&\le C \sum_{j,k} E\,\Big\{\,[\,|B_H(\alpha_j)|^\beta + |\phi(0)|\,]\,[\,|B_H(\alpha_k)|^\beta + |\phi(0)|\,]\times \\
&\qquad\qquad [\,|B_H(a_j) - B_H(a_{j-1})|\,]\,[\,|B_H(a_k) - B_H(a_{k-1})|\,]\,\Big\} \\
&< \infty\ ,
\end{aligned}
$$

for $B_H(t) - B_H(s)$ and $B_H(u)$ are normal random variables for any t, s and u, and $\phi(0)$ can be chosen to be finite by (4.18).

Now we prove that $\{S(\phi\,,\,\text{ß}^n\,,\,\alpha^n)\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. For any partitions ß^n and ß^m and corresponding real numbers α^n and α^m, let

$$
\text{ß}^{(n,m)}\quad :\qquad\qquad 0 = a_0 < a_1 \ldots < a_{q(n,m)} = 1
$$

be the partition obtained by superposition of ß^n and ß^m, and

$$
\alpha^{(n,m)} = \{\,\alpha_l \in [\,a_{l-1}\,,\,a_l\,]\,,\,k = 1\,,\,2\,,\,\ldots\,,\,q(n,m)\,\}
$$

be the corresponding collection of $q(n,m)$ real numbers. Since

$$
\begin{aligned}
S\,(\,\phi\,,\,\text{ß}^n\,,\,\alpha^n\,) &= \sum \phi(\,B_H(\alpha_{n,j})\,)\,[\,B_H(a_{n,j}) - B_H(a_{n,j-1})\,]\ , \\
S\,(\,\phi\,,\,\text{ß}^m\,,\,\alpha^m\,) &= \sum \phi(\,B_H(\alpha_{m,j})\,)\,[\,B_H(a_{m,j}) - B_H(a_{m,j-1})\,]\ ,
\end{aligned}
$$

then

$$
S\,(\,\phi\,,\,\text{ß}^n,\,\alpha^n\,) - S\,(\,\phi\,,\,\text{ß}^m,\,\alpha^m\,) = \sum_l \xi_l\,[\,B_H(a_j) - B_H(a_{j-1})\,]\ ,
$$

where ξ_l's are of the form

$$
\xi_l = \phi(\,B_H(\alpha_{n,u_l})\,) - \phi(\,B_H(\alpha_{m,v_l})\,)
$$

for some $u_l \in \{1\,,\,2\,,\,\ldots\,,\,q(n)\,\}$ and $v_l \in \{1\,,\,2\,,\,\ldots\,,\,q(m)\,\}$. Therefore

$$
\begin{aligned}
&E\,|\,S\,(\,\phi\,,\,\text{ß}^n\,,\,\alpha^n\,) - S\,(\,\phi\,,\,\text{ß}^m\,,\,\alpha^m)|^2 \\
&\quad = \sum_{j,l} E\,\{\,\xi_j\xi_l\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,[\,B_H(a_l) - B_H(a_{l-1})\,]\,\} \\
&\quad = \sum_j E\,\xi_j^2\,[\,B_H(a_j) - B_H(a_{j-1})\,]^2 \\
&\qquad + \sum_{j\neq l} E\,\xi_j\xi_l\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,[\,B_H(a_l) - B_H(a_{l-1})\,]\,. \qquad (4.45)
\end{aligned}
$$

For the first term of (4.45), by (4.18) and (4.44), we have

$$\begin{aligned}
&\sum_j E\left\{\xi_j^2\,[\,B_H(a_j)-B_H(a_{j-1})\,]^2\right\}\\
&\qquad\le \sum_j \left(E\,\xi_j^4\right)^{1/2}\left\{E\,[\,B_H(a_j)-B_H(a_{j-1})\,]^4\right\}\\
&\qquad\le C\sum_j \left(E\,\xi_j^4\right)^{1/2}(\,a_j-a_{j-1}\,)^{2H}\,.
\end{aligned}$$

Since $\xi_j=\phi(\,B_H(\alpha_{n,u_l})\,)-\phi(\,B_H(\alpha_{m,v_l})\,)$, by (4.18) we have

$$E\,\xi_j^4\le C\,E\,|B_H(\alpha_{n,u_l})-B_H(\alpha_{m,v_l})\,|^{4\beta}\le C\,|\alpha_{n,u_l}-\alpha_{m,v_l}|^{4\beta H}\le C\;,$$

so,

$$\begin{aligned}
\sum_j \left(E\,\xi_j^4\right)^{1/2}&(a_j-a_{j-1})^{2H}\le C\max\{\,|\text{ß}^n\,|^{2H-1}\,,\,|\text{ß}^m\,|^{2H-1}\,\}\sum_j(\,a_j-a_{j-1}\,)\\
&=C\max\{\,|\text{ß}^n\,|^{2H-1}\,,\,|\text{ß}^m\,|^{2H-1}\,\}\longrightarrow 0
\end{aligned}$$

as $|\text{ß}^n|$ and $|\text{ß}^m|\to 0$, for $2\,H-1>0$.

Now let us consider the second term in (4.45). By (4.44) we have

$$\begin{aligned}
&\sum_{j\neq l} E\,\xi_j\xi_l\,[\,B_H(a_j)-B_H(a_{j-1})\,]\,[\,B_H(a_l)-B_H(a_{l-1})\,]\\
&\le \sum_{j\neq l}\left(E\,\xi_j^2\,\xi_l^2\right)^{1/2}\{\,E\,(\,[\,B_H(a_j)-B_H(a_{j-1})\,]\,[\,B_H(a_l)-B_H(a_{l-1})\,]\,)\,\}^{1/2}\\
&\le \sum_{j\neq l}(\,E\,\xi_j^4\,)^{1/4}(\,E\,\xi_l^4\,)^{1/4}\left\{E\,[\,B_H(a_j)-B_H(a_{j-1})\,]^4\right\}^{1/4}\times\\
&\qquad\qquad\left\{E\,[\,B_H(a_l)-B_H(a_{l-1})\,]^4\right\}^{1/4}\,. \qquad (4.46)
\end{aligned}$$

Since

$$\begin{aligned}
&E\,\xi_j^4\le C\,|\alpha_{n,u_j}-\alpha_{m,v_j}|^{4\beta H}\;,\\
&(\,E\,\xi_j^4\,)^{1/4}\le C\,|\alpha_{n,u_j}-\alpha_{m,v_j}|^{\beta H}\le C\;|\alpha_{n,u_j}-\alpha_{m,v_j}|^{r}\;,
\end{aligned}$$

where $r=1$, if $\beta\,H\ge 1$; or $r=\beta\,H$, if $\beta\,H<1$. Therefore, the right hand side of (4.46) is

$$\le\;\; C\sum_{j\neq l}|\alpha_{n,u_j}-\alpha_{m,v_j}|^r\,|\alpha_{n,u_l}-\alpha_{m,v_l}|^r\,(\,a_j-a_{j-1}\,)^H\,(\,a_l-a_{l-1}\,)^H$$

$$\begin{aligned}
&\le C \sum_{j \neq l} (a_j - a_{j-1})^r (a_l - a_{l-1})^r (a_j - a_{j-1})^H (a_l - a_{l-1})^H \\
&= \sum_{j \neq l} (a_j - a_{j-1})^{r+H} (a_l - a_{l-1})^{r+H} \\
&\le C \max\{ | ß^n |^{r+H-1} | ß^{mm} |^{r+H-1} \} \longrightarrow 0
\end{aligned}$$

as $n, m \to \infty$, for $r + H = 1 + H > 1$, or $r + H = H(1 + \beta) > 1$. Hence $E | S(\phi, ß^n, \alpha^n) - S(\phi, ß^m, \alpha^m) | \to 0$ as $n, m \to \infty$. In other words, $\{ S(\phi, ß^n, \alpha^n), n = 1, 2, \dots \}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$.

To show that the element in $\mathcal{L}_2(\Omega)$ to which this Cauchy sequence converges is independent of the sequences $\{ ß^n \}$ and $\{ \alpha^n \}$ with $| ß^n | \to 0$ as $n \to \infty$, let $\{ \mathcal{D}^n \}$ and $\{ \beta^n \}$ be another pair of such sequences. Then since the sequence $| ß^1 |, | \mathcal{D}^1 |, | ß^2 |, | \mathcal{D}^2 |, \dots$ converges to 0, the sequence

$$S(\phi, ß^1, \alpha^1), S(\phi, \mathcal{D}^1, \beta^1), S(\phi, ß^2, \alpha^2), S(\phi, \mathcal{D}^2, \beta^2), \dots$$

is a Cauchy sequence in $\mathcal{L}_2(\Omega)$ and the two subsequences $\{ S(\phi, ß^n, \alpha^n) : n = 1, 2, \dots \}$ and $\{ S(\phi, \mathcal{D}^n, \beta^n) : n = 1, 2, \dots \}$ converge to the same element in $\mathcal{L}_2(\Omega)$. This finishes the proof of Theorem 4.16.

Proof of Theorem 4.17. For any partition

$$ß : \qquad 0 = a_0 < a_1 < \cdots < a_q = 1 \tag{4.47}$$

and q real numbers

$$\alpha = \{ \alpha_j \in [a_{j-1}, a_j] : j = 1, 2, \dots, q \},$$

since

$$S(a\phi_1 + b\phi_2, ß, \alpha) = a S(\phi_1, ß, \alpha) + b S(\phi_2, ß, \alpha),$$

(4.21) follows immediately. To deal with (4.23) we choose the partition as that in (4.47) and the q real numbers as follows

$$\alpha = \{ a_{j-1} : j = 1, 2, \dots, q \}.$$

By Theorem 4.16, the Newton-Leibniz Formula and (4.22) we have

$$\begin{aligned}
& E\left|\Phi(B_H(1)) - \Phi(B_H(0)) - \int_0^1 \phi(B_H(s))dB_H(s)\right|^2 \\
= \;& E\,|\,\Phi(B_H(1)) - \Phi(B_H(0)) - \lim_{|ß|\to 0} S(\phi\,,\,ß\,,\,\alpha)\,|^2 \\
= \;& \lim_{|ß|\to 0} E\,|\,\Phi(B_H(1)) - \Phi(B_H(0)) - S(\phi\,,\,ß\,,\,\alpha)\,|^2 \\
= \;& \lim_{|ß|\to 0} E\left|\Phi(B_H(1)) - \Phi(B_H(0)) - \sum_{j=1}^{q} \phi(B_H(a_{j-1}))[B_H(a_j) - B_H(a_{j-1})]\right|^2 \\
= \;& \lim_{|ß|\to 0} E\,|\,\sum_{j=1}^{q}\{\Phi(B_H(a_j)) - \Phi(B_H(a_{j-1})) \\
& \qquad -\phi(B_H(a_{j-1}))\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,\}\,|^2 \\
= \;& \lim_{|ß|\to 0} E\left|\sum_{j=1}^{q}\left\{\int_{B_H(a_{j-1})}^{B_H(a_j)} \phi(s)\,ds - \phi(B_H(a_{j-1}))[B_H(a_j) - B_H(a_{j-1})]\right\}\right|^2 \\
= \;& \lim_{|ß|\to 0} E\left|\sum_{j=1}^{q}\int_{B_H(a_{j-1})}^{B_H(a_j)} [\phi(s) - \phi(B_H(a_{j-1}))]\,ds\right|^2 \\
= \;& \lim_{|ß|\to 0} \sum_{j,k=1}^{q} E\left\{\left[\int_{B_H(a_{j-1})}^{B_H(a_j)} [\phi(s) - \phi(B_H(a_{j-1}))]\,ds\right] \times \right. \\
& \qquad \left.\left[\int_{B_H(a_{k-1})}^{B_H(a_k)} [\phi(s) - \phi(B_H(a_{k-1}))]\,ds\right]\right\} \\
\le \;& \lim_{|ß|\to 0} \sum_{j,k=1}^{q} E\left\{\int_{B_H(a_{j-1})\wedge B_H(a_j)}^{B_H(a_{j-1})\vee B_H(a_j)} |\phi(s) - \phi(B_H(a_{j-1}))|\,ds\, \times \right. \\
& \qquad \left.\int_{B_H(a_{k-1})\wedge B_H(a_k)}^{B_H(a_{k-1})\vee B_H(a_k)} |\phi(s) - \phi(B_H(a_{j-1}))|\,ds\right\} \\
\le \;& \lim_{|ß|\to 0} C \sum_{j,k=1}^{q} E\left\{\int_{B_H(a_{j-1})\wedge B_H(a_j)}^{B_H(a_{j-1})\vee B_H(a_j)} |s - B_H(a_{j-1})|^{\beta}\,ds \times \right. \\
& \qquad \left.\int_{B_H(a_{k-1})\wedge B_H(a_k)}^{B_H(a_{k-1})\vee B_H(a_k)} |s - B_H(a_{k-1})|^{\beta}\,ds\right\} \\
= \;& C \lim_{|ß|\to 0} \sum_{j,k=1}^{q} E\left[\,|B_H(a_j) - B_H(a_{j-1})|^{1+\beta}|B_H(a_k) - B_H(a_{k-1})|^{1+\beta}\right] \\
= \;& 0\,.
\end{aligned} \tag{4.48}$$

Note that (4.48) follows from the same argument as that used for the proof of Theorem 4.16. This ends the proof of Theorem 4.17.

4.5.5 Proof of the results of Section 4.4.4

Proof of Theorem 4.18. In order to prove that $S\,(\,X\,,\,\text{ß}\,,\,\alpha\,) \in \mathcal{L}_2(\Omega)$, it is sufficient to prove that $E\,|\,S\,(\,X\,,\,\text{ß}\,,\,\alpha\,)\,|^2 < \infty$. Indeed by the Schwarz inequality and (4.26) we have

$$
\begin{aligned}
&E\,|\,S\,(\,X\,,\,\text{ß}\,,\,\alpha\,)\,|^2 = E\left\{\sum_j X(\alpha_j)\,[\,B_H(a_j) - B_H(a_{j-1})\,]\right\}^2 \\
&= \sum_{j,k} EX(\alpha_j)X(\alpha_k)[B_H(a_j) - B_H(a_{j-1})][B_H(a_k) - B_H(a_{k-1})] \\
&\leq \sum_{j,k} \left\{E[X(\alpha_j)X(\alpha_k)]^2\right\}^{\frac{1}{2}} \left\{E[B_H(a_j) - B_H(a_{j-1})]^2[B_H(a_k) - B_H(a_{k-1})]^2\right\}^{\frac{1}{2}} \\
&\leq \sum_{j,k} \left\{EX^4(\alpha_j)EX^4(\alpha_k)\right\}^{\frac{1}{4}} \left\{E[B_H(a_j) - B_H(a_{j-1})]^2[B_H(a_k) - B_H(a_{k-1})]^2\right\}^{\frac{1}{2}} \\
&< \infty\,.
\end{aligned}
$$

Next we prove that $\{\,S(X\,,\,\text{ß}^n\,,\,\alpha^n)\ :\ n = 1\,,\,2\,,\,\ldots\,\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. We notice that

$$
\begin{aligned}
&E\,|\,S\,(\,X\,,\,\text{ß}^n\,,\,\alpha^n) - S(X\,,\,\text{ß}^m\,,\,\alpha^m\,)\,|^2 \\
&\quad = E\left\{\sum_j \zeta_j\,[B_H(a_j) - B_H(a_{j-1})]\right\}^2 ,
\end{aligned}
\tag{4.49}
$$

where ζ_j's are of the form $\zeta_j = X\,(\,\alpha_{n,u_j}\,) - X\,(\,\alpha_{m,v_j}\,)$ for some $\alpha_{n,u_j} \in \{\,1\,,\,2\,,\,\ldots\,,\,q(n)\,\}$ and $\alpha_{m,v_j} \in \{\,1\,,\,2\,,\,\ldots\,,\,q(m)\,\}$. Then by (4.27) and the same argument used in the proof of Theorem 4.16 we have that (4.49) equals

$$
\begin{aligned}
&= \sum_{j,k} E\,\{\zeta_j\zeta_k\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,[\,B_H(a_k) - B_H(a_{k-1})]\} \\
&\leq \sum_{j,k}(\,E\zeta_j^4)^{1/4}\,(\,E\zeta_k^4)^{1/4}\left\{\,E\,|\,B_H(a_j) - B_H(a_{j-1})\,|^4\,\right\}^{1/4} \times \\
&\qquad\qquad \left\{\,E\,|\,B_H(a_k) - B_H(a_{k-1})\,|^4\,\right\}^{1/4} \\
&\leq C\sum_{j,k}(\,a_j - a_{j-1}\,)^{H+\beta}\,(\,a_k - a_{k-1}\,)^{H+\beta} \longrightarrow 0\,,
\end{aligned}
$$

as $n,\ m \to \infty$. Therefore, $\{\,S\,(\,X\,,\,\text{ß}^n\,,\,\alpha^n\,)\ :\ n = 1\,,\,2\,,\,\ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$.

It is obvious now that the element in $\mathcal{L}_2(\Omega)$ to which our Cauchy sequence converges is independent of the sequences $\{ß^n : n = 1, 2, \ldots\}$ and $\{\alpha^n : n = 1, 2, \ldots\}$. This finishes the proof of Theorem 4.18.

Proof of Theorem 4.19. We prove the existence of $\int_0^1 x(B_H(s))\, dB_H(s)$ first. We put this in the following lemma

LEMMA 4.23 *Assume $x(t)$ satisfies (4.26) and (4.27). For any partition ß and corresponding q real numbers α given by (4.19) and (4.20), let*

$$S(x(B_H), ß, \alpha) = \sum_j x(B_H(\alpha_j))\,[B_H(a_j) - B_H(a_{j-1}] .$$

Then $S(x(B_H), ß, \alpha) \in \mathcal{L}_2(\Omega)$. Let $\{ß^n : n = 1, 2, \ldots\}$ and $\{\alpha^n : n = 1, 2, \ldots\}$ be as given in Theorem 4.18. Then $\{S(x(B_H), ß^n, \alpha^n) : n = 1, 2, \ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. Furthermore, the element in $\mathcal{L}_2(\Omega)$ to which this Cauchy sequence converges is independent of the sequences $\{ß^n : n = 1, 2, \ldots\}$ and $\{\alpha^n : n = 1, 2, \ldots\}$. So we can define the stochastic integral $\int_0^1 x(B_H(s))\, dB_H(s)$ to be the element in $\mathcal{L}_2(\Omega)$ to which the sequence $\{S(x(B_H), ß^n, \alpha^n) : n = 1, 2, \ldots\}$ converges.

Proof of Lemma 4.23. Since $x(t)$ satisfies (4.26) and(4.27),

$$\begin{aligned} E\,|S(x(B_H), ß, \alpha)|^2 &= E\left|\sum_j x(B_H(\alpha_j))\,[B_H(a_j) - B_H(a_k)]\right|^2 \\ &= E\left|\sum_j [x(B_H(\alpha_j)) - x(B_H(0)) + x(B_H(0))]\,[B_H(a_j) - B_H(a_k)]\right|^2 \\ &< \infty . \end{aligned}$$

Thus $S(x(B_H), ß, \alpha)) \in \mathcal{L}_2(\Omega)$. Now we show that $\{S(x(B_H), ß, \alpha)) : n = 1, 2, \ldots\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$. Using the same notation and argument as in the proof of Theorem 4.18 we have

$$E|\,S(x(B_H)\,,\,ß^n\,,\,\alpha^n))-S(x(B_H)\,,\,ß^m\,,\,\alpha^m))\,|^2 = E\,|\sum_j \zeta_j\,[\,B_H(a_j)-B_H(a_{j-1})\,|^2, \tag{4.50}$$

where ζ_j's are of the form $\zeta_j = x(B_H(\alpha_{n,u_j}) - x(B_H(\alpha_{m,v_j})$ for some $\alpha_{n,u_j} \in \{\,1\,,\,2,,\,\ldots\,,\,q(n)\,\}$ and $\alpha_{m,v_j} \in \{\,1\,,\,2,,\,\ldots\,,\,q(m)\,\}$. Then by (4.26) we have that equation (4.50) equals

$$\begin{aligned}
&= \sum_{j,k} E\,\zeta_j\zeta_k\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,[\,B_H(a_k) - B_H(a_{k-1})\,] \\
&\leq \sum_{j,k} \left(E\zeta_j^4\right)^{1/4} \left(E\zeta_k^4\right)^{1/4} \left(E\,[\,B_H(a_j) - B_H(a_{j-1})\,]^4\right)^{1/4} \times \\
&\qquad\qquad \left(E\,[\,B_H(a_k) - B_H(a_{k-1})\,]^4\right)^{1/4} \\
&\leq C\sum_{j,k}(a_j - a_{j-1})^{H+r}\,(a_k - a_{k-1})^{H+r} \longrightarrow 0\,,
\end{aligned}$$

as $n,\, m \to \infty$, for $H + r > 1$, where r is such chosen that $r = 1$, if $\beta \geq 1$; $r = \beta$, if $\beta < 1$. We have proved that $\{\,S\,(\,x(B_H)\,,\,ß^n\,,\,\alpha^n\,)\,\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega)$, and hence it is easy to see that the rest of Lemma 4.23 is true.

We move now to prove Theorem 4.19. Since for any real numbers a and b, any processes X_1 and X_2 satisfying (4.28) and (4.29), and any partition ß and corresponding α,

$$S(aX_1 + bX_2\,,\,ß\,,\,\alpha) = aS(X_1\,,\,ß\,,\,\alpha) + bS(X_2\,,\,ß\,,\,\alpha)\,,$$

so (4.30) follows immediately.

To deal with (4.31) we notice that

$$\begin{aligned}
&X\,(\,B_H(1)\,,\,\omega\,) - X\,(\,B_H(0)\,,\,\omega\,) = \sum_j\,[\,X\,(\,B_H(a_j)\,,\,\omega\,) - X\,(\,B_H(a_{j-1})\,,\,\omega\,)\,] \\
&= \sum_j[\,x(\alpha_j) - x(\,B_H(a_j)\,) + x(B_H(a_j)\,]\,[\,B_H(a_j) - B_H(a_{j-1})\,] \\
&= \sum_j[\,x(\alpha_j) - x(\,B_H(a_j)\,)\,]\,[\,B_H(a_j) - B_H(a_{j-1})\,] \\
&\qquad + \sum_j x(\,B_H(a_j)\,)\,[\,B_H(a_j) - B_H(a_{j-1})\,]\,,
\end{aligned} \tag{4.51}$$

where $\alpha_j = \alpha_j(\omega)$ is a random variable with values between $B_H(a_{j-1})$ and $B_H(a_j)$, $j = 1, 2, \ldots, q$. Since $x(t)$ satisfies (4.27), by the same argument used in the proof of Theorem 4.18, we have

$$
\begin{aligned}
E & \left\{ \sum_j [\, x(\alpha_j) - x(B_H(a_j))\,] [\, B_H(a_j) - B_H(a_{j-1})\,] \right\}^2 \\
& = \sum_{j,k} E\, [\, x(\alpha_j) - x(B_H(a_j))\,] [\, x(\alpha_k) - x(B_H(a_k))\,] \times \\
& \qquad\qquad [\, B_H(a_j) - B_H(a_{j-1})\,] [\, B_H(a_k) - B_H(a_{k-1})\,] \\
& \le C \sum_{j,k} (a_j - a_{j-1})^{H+r} (a_k - a_{k-1})^{H+r} \longrightarrow 0,
\end{aligned}
$$

as $|\,ß\,| \to 0$, for some r such that $H + r > 1$. Hence from (4.51) we have

$$
\begin{aligned}
X(\, B_H(1), \omega\,) - X(\, B_H(0), \omega\,) &= \lim_{|ß| \to 0} \sum_j x(\, B_H(a_j)\,) [\, B_H(a_j) - B_H(a_{j-1})\,] \\
&= \int_0^1 x(\, B_H(s)\,)\, dB_H(s)\,.
\end{aligned}
$$

This finishes the proof of Theorem 4.19.

4.5.6 Proof of the results of Section 4.4.5

Proof of Theorem 4.20. For any process $\{\, \xi(t) \,:\, -\infty < t < \infty \,\}$ and any numbers $-\infty < a < b < \infty$, if the sample function $\{\, \xi(t) \,:\, a \le t \le b \,\}$ is continuous on $[\, a, b]$ with probability one, then the sample function $\{\, \xi(t) \,:\, -\infty < t < \infty \,\}$ is continuous on R^1 with probability one. So in order to prove Theorem 4.20 it is sufficient to show that the sample functions of $\{\, I(f)(t) \,:\, 0 \le t \le 1 \,\}$, $\{\, I(\phi)(t) \,:\, 0 \le t \le 1 \,\}$ and $\{\, I(X)(t) \,:\, 0 \le t \le 1 \,\}$ are continuous functions on $[0, 1\,]$ with probability one. To deal with this, we need the following lemma

LEMMA 4.24 *Let $\{\, \xi(t) \,:\, t \in [\, 0, T\,]\,\}$ be a separable process satisfying the following conditions: there exists a non-negative monotonically non-decreasing* *function $g(h)$ and a function $q(c, h)$, $h \geq 0$, such that*

$$\mathbf{P}\{\,|\xi(t+h)-\xi(t)| > c\,g(h)\,\} \leq q(c, h) \tag{4.52}$$

and

$$G=\sum_{n=0}^{\infty} g(2^{-n}T) < \infty\ , \quad Q(c)=\sum_{n=0}^{\infty} 2^n\, q(c,\ 2^{-n}T) < \infty\ , \tag{4.53}$$

then $\xi(t)$ is continuous with probability one.

For the proof of Lemma 4.24, see, for example, Gihman and Skorohod (1971, pp. 190–191, Lemma 1 and Theorem 6).

To establish Theorem 4.20, it is now sufficient to check that the conditions (4.52) and (4.53) of Lemma 4.24 hold for $\{\, I(f)(t) \,:\, 0 \leq t \leq 1\,\}$, $\{\, I(\phi)(t) \,:\, 0 \leq t \leq 1\,\}$ and $\{\, I(X)(t) \,:\, 0 \leq t \leq 1\,\}$. Let

$$g(h) = h^{\frac{\alpha}{2}}\ , \quad q(c, h) = C\, h^{2H-\alpha} \tag{4.54}$$

where C is a constant independent of h and α and which may be different in different expressions, α satisfies $2H-\alpha-1>0$. We are going to show that, for such chosen g and q, (4.52) and (4.53) hold for $\{\, I(\phi)(t) \,:\, 0 \leq t \leq 1\,\}$, $\{\, I(X)(t) \,:\, 0 \leq t \leq 1\,\}$ and $\{\, I(f)(t) \,:\, 0 \leq t \leq 1\,\}$. From (4.54), notice that $T=1$, $2H-\alpha-1>0$, we have

$$G=\sum_{n=0}^{\infty} g(2^{-n}\,T) = \sum_{n=0}^{\infty} 2^{-\frac{\alpha}{2}n} < \infty\ ,$$

and

$$Q(c)=\sum_{n=0}^{\infty} 2^n\, q(c, 2^{-n}\,T) = \sum_{n=0}^{\infty} 2^n 2^{-(2H-\alpha)n} = \sum_{n=0}^{\infty} 2^{-(2H-\alpha-1)n} < \infty,$$

so (4.53) holds for g and q given by (4.54). Now let us consider (4.52) for $I(\phi)(t)$. We have

$$\mathbf{P}\left\{|I(\phi)(t+h)-I(\phi)(t)| > h^{\frac{\alpha}{2}}\right\} \leq \frac{1}{h^{\alpha}} E|I(\phi)(t+h)-I(\phi)(t)|^2\ . \tag{4.55}$$

For any t $\in R^1$ let

$$\Phi(t) = \int_0^t \phi(s)ds \ . \tag{4.56}$$

Since ϕ satisfies the conditions of Theorem 4.16, (4.56) is well defined. By Theorem 4.17,

$$\Phi(B_H(t+h)) - \Phi(B_H(t)) = \int_t^{t+h} \phi(B_H(s))dB_H(s) \ .$$

Thus,

$$\begin{aligned} E|I(\phi)(t+h) - I(\phi)(t)|^2 &= E\left|\int_t^{t+h} \phi(B_H(s))dB_H(s)\right|^2 \\ &= E|\Phi(B_H(t+h)) - \Phi(B_H(t))|^2 \\ &= E|\phi(B_H(\theta))[B_H(t+h) - B_H(t)]|^2 \ , \end{aligned} \tag{4.57}$$

where $B_H(\theta)$ is between $B_H(t)$ and $B_H(t+h)$, therefore

$$|B_H(\theta) - B_H(t)| \leq |B_H(t+h) - B_H(t)| \ . \tag{4.58}$$

By condition (4.18) of Theorem 4.16 and (4.58) we have

$$\begin{aligned} & E|\phi(B_H(\theta))[B_H(t+h) - B_H(t)]|^2 \\ \leq \ & E\left\{[|\phi(B_H(\theta)) - \phi(B_H(t))| + |\phi(B_H(t))|]\,|B_H(t+h) - B_H(t)|\right\}^2 \\ \leq \ & E\Big\{\Big[C|B_H(\theta) - B_H(t)|^\beta + |\phi(B_H(t)) - \phi((B_H(0))| + |\phi(B_H(0))|\Big] \times \\ & \qquad |B_H(t+h) - B_H(t)|\Big\}^2 \\ \leq \ & E\Big\{\Big[C|B_H(t+h) - B_H(t)|^\beta + |\phi(B_H(t)) - \phi(B_H(0))| + |\phi(B_H(0))|\Big] \times \\ & \qquad |B_H(t+h) - B_H(t)|\Big\}^2 \\ \leq \ & CE\Big\{\Big[|B_H(t+h) - B_H(t)|^\beta + |B_H(t) - B_H(0)|^\beta + |\phi(0)|\Big] \times \\ & \qquad |B_H(t+h) - B_H(t)|\Big\}^2 \\ \leq \ & C\Big\{E\Big[|B_H(t+h) - B_H(t)|^\beta + |B_H(t) - B_H(0)|^\beta + |\phi(0)|\Big]^4\Big\}^{1/2} \times \\ & \qquad \Big\{E\,[\,B_H(t+h) - B_H(t)\,]^4\Big\}^{1/2} \\ \leq \ & C\left\{h^{4\beta H} + t^{4\beta H} + \phi(0)^{4\beta}\right\}^{1/2} h^{2H} \\ \leq \ & C\,h^{2H} \ . \end{aligned} \tag{4.59}$$

The second last inequality follows from (4.44). From (4.55), (4.57) and (4.59), $\{ I(\phi)(t) : 0 \le t \le 1 \}$ is continuous with probability one.

Next we show that (4.52) holds for $\{I(X)(t) : 0 \le t \le 1\}$. Note that

$$\mathbf{P}\left\{|I(X)(t+h) - I(X)(t)| > h^{\frac{\alpha}{2}}\right\} \le h^{-\alpha} E|I(X)(t+h) - I(X)(t)|^2 . \tag{4.60}$$

Let

$$Y(t , \omega) = \int_0^t X(s , \omega) dB_H(s) .$$

It is obvious that $Y(t)$ is well defined and satisfies the conditions of Theorem 4.19. Using the same argument used above we have

$$\begin{aligned} E|I(X)(t+h) - I(X)(t)|^2 &= E\left|\int_t^{t+h} X(B_H(s)) dB_H(s)\right|^2 \\ = \;& E|Y(B_H(t+h)) - Y(B_H(t))|^2 = E\left|X(B_H(\theta))[B_H(t+h) - B_H(t)]\right|^2 \\ \le \;& Ch^{2H} . \end{aligned} \tag{4.61}$$

From (4.60) and (4.61) we have

$$\mathbf{P}\left\{|I(X)(t+h) - I(X)(t)| > h^{\frac{\alpha}{2}}\right\} \le Ch^{2H} .$$

By Lemma 4.24, $\{I(X)(t) : 0 \le t \le 1\}$ is continuous with probability one. Finally we are going to prove that (4.52) holds for $\{I(f)(t) : 0 \le t \le 1\}$. To this end, we need the following well known fact

LEMMA 4.25 *Let $g(x)$ be a Lebesgue integrable function on $[a , b]$, then for almost every $x_0 \in [a , b]$,*

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0 - h_1}^{x_0 + h_2} |g(x) - g(x_0)| dx = 0 \tag{4.62}$$

where $h_1 \ge 0$, $h_2 \ge 0$ and $h_1 + h_2 = h$.

Let

$$D_\gamma = \{(u , v) : t \le u \le t + h - \gamma , u + \gamma \le v \le t + h\} , \tag{4.63}$$

$$T_h = \{(u , v) : t \le u \le t + h , u < v \le t + h\} .$$

Since

$$f(u)f(v)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)=f(v)f(u)\frac{\partial^2\Gamma}{\partial u\partial v}(v\,,\,u)\,,$$

by Theorem 4.12 (4), we have

$$\begin{aligned} E|I(f)(t+h)-I(f)(t)|^2 &= \int_{[t,t+h]\times[t,t+h]} f(u)f(v)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\,du\,dv \\ &= 2\int_{T_h} f(u)f(v)\frac{\partial^2\Gamma}{\partial u\partial v}(u,v)\,du\,dv\;. \end{aligned} \tag{4.64}$$

It is easy to see that $f^2(u)\partial^2\Gamma/\partial u\partial v(u\,,\,v)$ is Lebesgue integrable on T_h. Then by the Schwarz inequality

$$\begin{aligned} &\int_{T_h} f(u)f(v)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\,du\,dv \\ &= \int_{T_h}\left\{f(u)\left[\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\right]^{\frac{1}{2}}\right\}\left\{f(v)\left[\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\right]^{\frac{1}{2}}\right\}du\,dv \\ &\le \left\{\int_{T_h} f^2(u)\frac{\partial^2\Gamma}{\partial u\partial v}(u,v)\,du\,dv\right\}^{\frac{1}{2}}\left\{\int_{T_h} f^2(v)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\,du\,dv\right\}^{\frac{1}{2}} \\ &= \int_{T_h} f^2(u)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\,du\,dv\;. \end{aligned} \tag{4.65}$$

For h^{2H} there exists a $\gamma>0$ such that

$$\int_{T_h} f^2(u)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\,du\,dv \le \int_{D_\gamma} f^2(u)\frac{\partial^2\Gamma}{\partial u\partial v}(u\,,\,v)\,du\,dv + h^{2H} \tag{4.66}$$

where D_γ is given by (4.63). Then,

$$\begin{aligned} \int_{D_\gamma} f^2(u)\frac{\partial^2\Gamma}{\partial u\partial v}(u,v)du\,dv &= \int_t^{t+h-\gamma} f^2(u)\int_{u+\gamma}^{t+h}(v-u)^{2H-2}dv\,du \\ &= \int_t^{t+h-\gamma} f^2(u)\frac{1}{2H-1}\left[(t+h-u)^{2H-1}-\gamma^{2H-1}\right]du \\ &\le \frac{1}{2H-1}\int_t^{t+h-\gamma} f^2(u)(t+h-u)^{2H-1}du \\ &\le \frac{1}{2H-1}\int_t^{t+h} f^2(u)(t+h-u)^{2H-1}du \\ &\le \frac{h^{2H-1}}{2H-1}\int_t^{t+h} f^2(u)\,du\;. \end{aligned} \tag{4.67}$$

Since $f(u)\in\mathcal{L}_2(R^1)$, we can assume that $f^2(t)<\infty$. Then from Lemma 4.25, we have

$$\int_t^{t+h} f^2(u)\,du \le \int_t^{t+h} |f^2(u)-f^2(t)|+f^2(t)\,du = f^2(t)\,h+o(h)\;. \tag{4.68}$$

Thus from (4.64), (4.65), (4.66), (4.67) and (4.68) we have

$$E|I(f)(t+h) - I(f)(t)|^2 \leq Cf^2(t)h^{2H} , \tag{4.69}$$

where C is a constant independent of t and h. Therefore

$$\begin{aligned} &\mathbf{P}\left\{|I(f)(t+h) - I(f)(t)| > h^{\frac{\alpha}{2}}\right\} \\ \leq\ & h^{-\alpha}E|I(f)(t+h) - I(f)(t)|^2 \leq Cf^2(t)h^{2H} \leq Cf^2(t)h^{2H-\alpha} . \end{aligned}$$

Thus (4.52) holds for $\{I(f)(t) \ : \ 0 \leq t \leq 1\}$. This finishes the proof of Theorem 4.20.

Proof of Theorem 4.21.

(a) From the definitions of $I(f)(t)$, $I(\phi)(t)$ and $I(X)(t)$ we know that (4.35), (4.36) and (4.37) are obvious.

(b) From (4.69) we have

$$\begin{aligned} &\left|E\left[I(f)(t)\right]^2 - E\left[I(f)(s)\right]^2\right| = \left|E\left\{[I(f)(t)]^2 - [I(f)(s)]^2\right\}\right| \\ =\ & |E\left\{[I(f)(t) + I(f)(s)][I(f)(t) - I(f)(s)]\right\}| \\ \leq\ & \left\{E\left[I(f)(t) + I(f)(s)\right]^2\right\}^{\frac{1}{2}} \left\{E\left[I(f)(t) - I(f)(s)\right]^2\right\}^{\frac{1}{2}} \\ \leq\ & \left\{E\left[I(f)(t) + I(f)(s)\right]^2\right\}^{\frac{1}{2}} \cdot C \cdot |t-s|^{2H} \longrightarrow 0 , \end{aligned}$$

as $t - s \to 0$.

From the proof of Theorem 4.20, particularly, from (4.57) and(4.59), we have

$$\begin{aligned} &\left|E\left[I(\phi)(t)\right]^2 - E\left[I(\phi)(s)\right]^2\right| \\ \leq\ & \left\{E\left[I(\phi)(t) + I(\phi)(s)\right]^2\right\}^{\frac{1}{2}} \left\{E\left[I(\phi)(t) - I(\phi)(s)\right]^2\right\}^{\frac{1}{2}} \\ \leq\ & \left\{E\left[I(\phi)(t) + I(\phi)(s)\right]^2\right\}^{\frac{1}{2}} C\,|t-s|^{H} \longrightarrow 0 , \end{aligned}$$

as $|t - s| \to 0$. From (4.62) we have

$$\begin{aligned} &\left|E\left[I(X)(t)\right]^2 - E\left[I(X)(s)\right]^2\right| \leq \left\{E\left[I(X)(t) + I(X)(s)\right]^2\right\}^{\frac{1}{2}} C\,|t-s|^{2H} \\ &\qquad \longrightarrow 0 , \end{aligned}$$

as $|t-s| \to 0$. This establishes (2) of Theorem 4.21.

(c) Now (4.38) and (4.39) are obvious. We need only to show (4.40). For any $a \le t \le b$, let

$$ß_t \quad : \qquad a = a_0^t < a_1^t < \cdots < a_{q(t)}^t = t$$

be a partition of $[a, t]$,

$$\alpha^t = \{ \alpha_j^t \in [a_{j-1}^t, a_j^t] \ : \ j = 1, 2, \ldots, q(t) \}$$

be q(t) real numbers,

$$ß_b \quad : \qquad a = a_0^b < a_1^b < \cdots < a_{q(b)}^b = b$$

be a partition of $[a, b]$, such that $q(t) \le q(b)$ and $a_j^b = a_j^t$ when $1 \le j \le q(t)$. Let

$$\alpha^b = \{ \alpha_j^b \in [a_{j-1}^b, a_j^b] \ : \ j = 1, 2, \ldots, q(b) \}$$

be $q(b)$ real numbers such that $\alpha_j^b = \alpha_j^t$ when $1 \le j \le q(t)$. Let

$$ß \quad : \qquad a = a_0 < a_1 < \cdots < a_q = b$$

be any partition of $[a, b]$, and

$$\alpha = \{ \alpha_j \in [a_{j-1}, a_j] \ : \ j = 1, 2, \ldots, q \}$$

be a corresponding collection of q real numbers. From Lemma 4.10 we have

$$\begin{aligned}
&\int_a^b E\,[I(f)(t)]^2\, dt \\
&= \int_a^b \lim_{|ß_t| \to 0} E \left\{ \sum_{j=1}^{q(t)} f(\alpha_j^t) \left[B_H(a_j^t) - B_H(a_{j-1}^t) \right] \right\}^2 dt \\
&= \int_a^b \lim_{|ß_t| \to 0} \sum_{j,k=1}^{q(t)} f(\alpha_j^t) f(\alpha_k^t) E \left[B_H(a_j^t) - B_H(a_{j-1}^t) \right] \left[B_H(a_k^t) - B_H(a_{k-1}^t) \right] dt \\
&\le \int_a^b \liminf_{|ß_b| \to 0} \sum_{j,k=1}^{q(b)} \left| f(\alpha_j^b) f(\alpha_k^b) E \left[B_H(a_j^b) - B_H(a_{j-1}^b) \right] \left[B_H(a_k^b) - B_H(a_{k-1}^b) \right] \right| dt \\
&\le \int_a^b \liminf_{|ß| \to 0} \sum_{j,k=1}^{q} \left| f(\alpha_j) f(\alpha_k) E \left[B_H(a_j) - B_H(a_{j-1}) \right] \left[B_H(a_k) - B_H(a_{k-1}) \right] \right| dt
\end{aligned}$$

$$
\begin{aligned}
&= (b-a)\liminf_{|ß|\to 0} \sum_{j,k=1}^{q} \Big\{ f(\alpha_j)f(\alpha_k)E\left[B_H(a_j)-B_H(a_{j-1})\right]\times \\
&\qquad\qquad\qquad\qquad [B_H(a_k)-B_H(a_{k-1})]\ \Big\} \\
&= (b-a)\lim_{|ß|\to 0}\Big\{\ \sum_{j,k=1}^{q}\left| f(\alpha_j)f(\alpha_k)\frac{\partial^2\Gamma}{\partial y\partial x}(a_{j-1},a_{k-1})(a_j-a_{j-1})(a_k-a_{k-1})\right| \\
&\qquad\qquad\qquad +o(1)\Big\} \\
&= (b-a)\int_{[a,b]\times[a,b]} |f(x)f(y)|\,\frac{\partial^2\Gamma}{\partial y\partial x}(x,y)\,dx\,dy\ .
\end{aligned}
$$

This ends the proof of (4.40) and hence the proof of Theorem 4.21.

Chapter 5

STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION AND THEIR APPLICATIONS TO ECONOMICS

5.1 Introduction

Stochastic differential equations were developed out of the need to assign meaning to ordinary differential equations involving continuous stochastic processes. They have become a fundamental part of modern probability theory and found vast applications since their establishment. A great deal of literature about stochastic differential equations is now available. See, for example, Ikeda and Watanabe (1981), Karatzas and Shreve (1988) and Protter (1990).

A stochastic differential equation can typically be written symbolically as

$$dX_t = a\,(t\,,\,X_t\,)\,dt + b\,(t\,,\,X_t\,)\,d\xi_t\,, \tag{5.1}$$

where $\{\,\xi_t\,\}$ is a semimartingale, $a(t,x)$ and $b(t,x)$ satisfy certain conditions to ensure the existence of a unique solution of (5.1). The symbolic differential (5.1) is interpreted as the stochastic integral equation

$$X_t(\omega) = X_0(\omega) + \int_0^t a\,(s\,,\,X_s\,)\,dt + \int_0^t b\,(s\,,\,X_s\,)\,d\xi_s\,, \tag{5.2}$$

where $X_0(\omega)$ is a random variable. When a and b are constants and $\xi_t = B(t)$ is a Brownian motion, a unique solution of (5.2) is

$$X_t = X_0 \exp\left\{ \left(a - \frac{1}{2} b^2 \right) t + b\, B(t) \right\} . \tag{5.3}$$

There has been diverse applications of stochastic differential equations in other disciplines. For example, given the apparent random fluctuations of share prices on the stock exchange, it has seemed natural to use stochastic differential equations in models of share price dynamics or, more generally, in models of investment finance. One of the first to do this was Merton(1971, 1973), whose simple model contains the basic ideas that have been used in recent, more sophisticated models. Merton considered an investor who chooses between two different types of investment, one risky and the other safe. The investor must implement a strategy which will maximize some utility function, such as his net wealth or cash flow, while avoiding bankruptcy. At each instant of time the investor must select the fraction f of his wealth that he will put into the risky investment, with the remaining fraction $1 - f$ going into the safe one. If his current consumption rate is $C \geq 0$, then his wealth X_t satisfies the stochastic differential equation

$$dX_t = f\,(b\,X_t\,dt + \beta\,X_t\,dB(t)) + (1 - f)\,a\,X_t\,dt - C\,dt\ , \tag{5.4}$$

where a, b and β are positive constants with $a < b$. Equation (5.4) can be rewritten as the Itô stochastic differential equation

$$dX_t = (\,\{\,(1 - f)\,a + f\,b\,\}\,X_t - C)\,dt + f\,\beta\,X_t\,dB(t)\ . \tag{5.5}$$

In the early 1970s, Black and Scholes (1973) made a major breakthrough by deriving a differential equation that must be satisfied by the price of any derivative security dependent on a non-dividend-paying stock. They used the equation to obtain values for European call and put options on the stock. A European call option with strike price C gives the right to buy the stock at time T at the fixed price C. The resulting payoff is then given by

$$f(X_T) = (X_T - C)^+ = \begin{cases} X_t - C, & X_T > C, \\ 0, & X_T \le C. \end{cases} \tag{5.6}$$

The stock price of any derivative security is a stochastic process. We use S_t to denote the stock price at time t. The expected rate of return per unit time from the stock is μ. The volatility of the stock price is σ. Both μ and σ are assumed to be constants. Let ΔS be the change in the stock price. Then the proportional return provided by the stock in a short period of time Δt is $\Delta S/S$. The Black-Scholes model describes $\Delta S/S$ as

$$\frac{\Delta S}{S} = \mu\,\Delta t + \sigma\,\varepsilon\,\sqrt{\Delta t}\,. \tag{5.7}$$

The term $\mu\,\Delta t$ in (5.7) is the expected value of the return from the stock, while the term $\sigma\,\varepsilon\,\sqrt{\Delta t}$ is the stochastic component of the return, where ε is a normally distributed random variable. A continuous time version of (5.7) is

$$dS_t = \mu\,S_t\,dt + \sigma\,S_t\,dB(t)\;, \tag{5.8}$$

where μ and σ are constants, $B(t)$ is a standard Brownian motion.

Equation (5.8) is the classical Black-Scholes model. In their pathbreaking paper (Black and Scholes, 1973), Black and Sholes succeeded in using equation (5.8) to obtain exact formulas for the price of European call and put options. For the details of the Black and Scholes formulas see, for example, Kloeden and Platen, 1992. Nowadays, the Black-Scholes model has been widely used for studying stock price behaviour. There has been vast literature about Black-Scholes analysis. See, for example, Merton (1973), Cox and Ross (1976), Cox and Rubinstein (1984), Harrison and Kreps (1979), Harrison and Pliska (1981), Karatzas (1989) and Hull (1993).

In the framework of Black-Scholes, the driving process of the dynamic system is a Brownian motion which is a short range dependent process. However, during the last few decades, researchers have noticed the existence of long range dependent phenomena occurring in data from financial markets. For details, see Section 4.1. Nowadays, the presence of LRD (or the Joseph Effect) has been increasingly accepted in mathematical financial literature.

Since there has been various evidence for the existence of LRD in financial data, in this chapter, we use the results obtained in Chapter 4 to take the LRD into account in the Black-Scholes model through the use of $B_H(t)$. In this chapter, a plausible counterpart to the now-classical Black-Scholes model is suggested as

$$dS_t = \mu \, S_t \, dt + \sigma \, S_t \, dB_H(t) \,, \tag{5.9}$$

where $B_H(t)$ is a normalized standard fractional Brownian motion, that is, the Gaussian process with $E\,(\,B_H(t) - B_H(s)\,) = 0$ and $E\,(\,B_H(t) - B_H(s)\,)^2 = |\,t-s\,|^{2\,H}$. We are interested in the case where $1/2 < H < 1$. The interpretation of (5.9) will be given in Section 5.2 where we give a definition of a general stochastic differential equation driven by fractional Brownian motion:

$$dX_t = a\,(\,t\,,\,X_t\,)\,dt + b\,(t\,,\,X_t\,)\,dB_H(t)\,. \tag{5.10}$$

In Section 5.4, we derive a solution of (5.9).

We recall that Itô's formula is a powerful tool for studying stochastic differential equations given by (5.1). To deal with (5.10), we need a corresponding version of Itô's formula in which fractional Brownian motion replaces the semimartingale. We come to this in Section 5.3.

5.2 Definition of Stochastic Differential Equations Driven by Fractional Brownian Motion

In this section, we are concerned with taking long range dependence into account in financial markets. We achieve this by introducing a "fractional" Black-Scholes model

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_H(t)\,. \tag{5.11}$$

Equation (5.11) is a symbolic form. The interpretation of it is given through Definition 5.1 in which we consider general forms of stochastic differential equations driven by fractional Brownian motion.

In the rest of this chapter we suppose that $(\Omega\,,\,\mathcal{F}\,,\,P\,)$ is a complete probability space associated with a standard normalized fractional Brownian motion $B_H(t)$ on a finite interval $[\,0\,,\,T\,]$. We further assume that $1/2 < H < 1$.

DEFINITION 5.1 *Let $a(\,t\,,\,\omega\,)$ and $b(\,t\,,\,\omega\,)$: $[\,0\,,\,T\,]\times\Omega \to \mathbf{R}$ be two stochastic processes. We say that a stochastic process $\{\,X(t)\ :\ t\in[\,0\,,\,T\,]\,\}$ has a stochastic differential with respect to fractional Brownian motion $B_H(t)$*

$$dX_t = a(\,t\,)\,dt + b(\,t\,)\,dB_H(\,t\,)\,, \tag{5.12}$$

if for any $(\,t\,,\,\omega\,)\in[\,0\,,\,T\,]\times\Omega$, the following holds

$$X(\,t\,,\,\omega\,) = X_0(\,\omega\,) + \int_0^t a(\,s\,,\,\omega\,)\,ds + \int_0^t b(\,s\,,\,\omega\,)\,dB_H(\,s\,,\,\omega\,)\,, \tag{5.13}$$

where X_0 is a random variable. The stochastic integral $\int_0^t a(\,s\,,\,\omega\,)\,ds$ is an ordinary Riemann-Stieltjes integral for each $\omega\in\Omega$ while $\int_0^t b(\,s\,,\,\omega\,)\,dB_H(\,s\,,\,\omega\,)$ is defined as in Chapter 4.

Remarks on Definition 5.1. Generally speaking, the integral $\int_0^t a(\,s\,,\,\omega\,)\,ds$ exists under standard conditions on $a(\,s\,,\,\omega\,)$. The integral $\int_0^t b(\,s\,,\,\omega\,)\,dB_H(\,s\,,\,\omega\,)$

exists only under the conditions given in Chapter 4 for defining stochastic integrals with respect to $B_H(t)$. We will discuss equation(5.12) in more detail in Section 5.3.

DEFINITION 5.2 *(fractional Black-Scholes model.) The stochastic differential equation*

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_H(t) \tag{5.14}$$

is called a fractional Black-Scholes model, where μ and σ are constants and the Hurst index satisfies $1/2 \le H < 1$.

Remarks on Definition 5.2. When $H = 1/2$, (5.14) is the well known Black-Scholes model. Since the Black-Scholes model has been studied thoroughly, we concentrate here on the case where $1/2 < H < 1$. We discuss equation(5.14) in Section 5.4.

5.3 Itô's Formula with respect to Fractional Brownian Motion

When we consider stochastic differential equations driven by Brownian motion

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dB(t)\,, \tag{5.15}$$

Itô's formula is a powerful tool for dealing with their calculus. When we are concerned with stochastic differential equations driven by fractional Brownian motion

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dB_H(t)\,, \tag{5.16}$$

we have noticed that a version of Itô's formula plays the same role in dealing with equation(5.16). The aim of this section is the following theorem

THEOREM 5.1 (Itô's formula with respect to fractional Brownian motion) *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $B_H(\tau)$ be a fractional Brownian motion on $[0, T]$ such that $1/2 < H < 1$ and $B_H(0) = 0$ a.e. (therefore $EB_H(\tau) \equiv 0$ for any $\tau \in [0, T]$). Assume stochastic processes $a(\tau, \omega)$, $b(\tau, \omega)$ and $X(\tau, \omega)$ are such that for any $[t_0, t] \subseteq [0, T]$,*

1. *$a(\tau, \omega)$ is Riemann-Stieltjes integrable on $[t_0, t]$ for each $\omega \in \Omega$;*
2. *$\int_{t_0}^t b(\tau)\, dB_H(\tau)$ exists in the sense described in Chapter 4;*
3. *Either of the following holds*

   3.1 for any $0 \le s \le t_1 \le t_2, t_3 \le t_4 \le T$, $\{b(\tau) : 0 \le \tau \le T\}$ and $\{B_H(\tau) : 0 \le \tau \le T\}$ are such that

$$E\left\{\Big(b(t_1) - b(s)\Big)\Big(b(t_3) - b(s)\Big)\Big(B_H(t_2) - B_H(t_1)\Big)\Big(B_H(t_4) - B_H(t_3)\Big)\right\}$$
$$= \tag{5.17}$$
$$E\Big\{\Big(b(t_1) - b(s)\Big)\Big(b(t_3) - b(s)\Big)\Big\}E\Big\{\Big(B_H(t_2) - B_H(t_1)\Big)\Big(B_H(t_4) - B_H(t_3)\Big)\Big\},$$

   or,

   3.2 the second derivative $d^2 b(t)/dt^2$ exists, and for any $0 \le s \le t_1 \le t_2, t_3 \le t_4 \le T$, $\left\{b'(\tau) = db(\tau)/d\tau : s \le \tau \le \max\{t_1, t_3\}\right\}$ and $(B_H(t_1), B_H(t_2), B_H(t_3), B_H(t_4))$ are such that for any random variables ξ and η such that ξ and η are measurable with respect to $\sigma\left\{b'(\tau) : s \le \tau \le \max\{t_1, t_3\}\right\}$ and $E|\xi|^4 < \infty$, $E|\eta|^4 < \infty$, the following holds

$$\begin{aligned} &E\left\{\Big(b'(s)(t_1 - s) + \xi\Big)\Big(b'(s)(t_3 - s) + \eta\Big) \times \right.\\ &\qquad \left.\Big(B_H(t_2) - B_H(t_1)\Big)\Big(B_H(t_4) - B_H(t_3)\Big)\right\} \\ &= E\left\{\Big(b'(s)(t_1 - s) + \xi\Big)\Big(b'(s)(t_3 - s) + \eta\Big)\right\} \times \\ &\qquad E\left\{\Big(B_H(t_2) - B_H(t_1)\Big)\Big(B_H(t_4) - B_H(t_3)\Big)\right\}, \end{aligned} \tag{5.18}$$

   furthermore,

$$\sup_{0 \le t \le T} E\left|\frac{db}{dt}(t, \omega)\right|^4 < \infty\ , \quad \sup_{0 \le t \le T} E\left|\frac{d^2 b}{dt^2}(t, \omega)\right|^4 < \infty\ ; \tag{5.19}$$

4.

$$X_t - X_{t_0} = \int_{t_0}^{t} a(\tau\,,\,\omega)\,d\tau + \int_{t_0}^{t} b(\tau\,,\,\omega)\,dB_H(\tau)\,, \tag{5.20}$$

where the first integral in (5.20) is an ordinary Riemann-Stieltjes integral for each $\omega \in \Omega$, while the second is an Itô integral defined in Chapter 4.

Assume that a two variable function $U\,(\,t\,,\,x\,)\,:\,[\,0\,,\,T\,] \times \mathbf{R} \to \mathbf{R}$ has uniformly continuous partial derivatives $\partial U/\partial t$, $\partial U/\partial x$ and $\partial^2 U/\partial x^2$. Assume further that

$$\sup_{0\le t\le T} E\,|\,U\,(\,t\,,\,X_t\,)\,|^2 < \infty\,, \tag{5.21}$$

$$\sup_{0\le t\le T} E\,\left|\frac{\partial U}{\partial t}\Big(\,t\,,\,X_t\Big)\right|^2 < \infty\,, \tag{5.22}$$

$$\sup_{0\le t\le T} E\,\left|\frac{\partial U}{\partial x}\Big(\,t\,,\,X_t\Big)\right|^2 < \infty\,, \tag{5.23}$$

$$\sup_{0\le t\le T} E\,\left|\frac{\partial^2 U}{\partial x^2}\Big(\,t\,,\,X_t + O_{L_2}(1)\Big)\right|^2 < \infty\,, \tag{5.24}$$

$$\sup_{0\le t\le T} E\,\,|\,a\,(\,t\,)\,|^2 < \infty\,, \tag{5.25}$$

$$\sup_{0\le t\le T} E\,\,|\,b\,(\,t\,)\,|^2 < \infty\,, \tag{5.26}$$

$$E\,\,|\,b\,(\,t\,) - b\,(\,s\,)\,| < const\,|\,t - s\,|^{\beta}\,,\ \beta \ge 0\,, \tag{5.27}$$

where $O_{L_2}(1)$ means a term such that $E\,|\,O_{L_2}(1)\,|^2 < \infty$. Let $Y_t = U\,(\,t\,,\,X_t\,)$. If, for any $0 \le t \le T$,

$$\int_0^t b\,(\,\tau\,,\,\omega\,)\,\frac{\partial U}{\partial x}\Big(\,\tau\,,\,X_\tau\Big)\,dB_H(\tau)$$

exists in the sense described in Chapter 4, then the following holds

$$\begin{aligned} Y_t - Y_{t_0} = &\int_{t_0}^{t} \left\{ \frac{\partial U}{\partial t}\Big(\tau\,,\,X_\tau\Big) + a\,(\,\tau\,,\,\omega\,)\frac{\partial U}{\partial x}\Big(\tau\,,\,X_\tau\Big)\right\}\,d\tau\,+ \\ &\int_{t_0}^{t} b\,(\,\tau\,,\,\omega\,)\frac{\partial U}{\partial x}\Big(\,\tau\,,\,X_\tau\Big)\,dB_H(\tau)\,, \end{aligned} \tag{5.28}$$

or, equivalently,

$$dY_t = \left\{ \frac{\partial U}{\partial t}\Big(t\,,\,X_t\Big) + a\,(\,t\,,\,\omega\,)\frac{\partial U}{\partial x}\Big(t\,,\,X_t\Big)\right\}\,dt + b\,(\,t\,,\,\omega\,)\frac{\partial U}{\partial x}\Big(\,t\,,\,X_t\Big)\,dB_H(t)\,. \tag{5.29}$$

Remarks on Theorem 5.1.

1. Since $E(B_H(t+\Delta) - B_H(t))^2 = |\Delta|^{2H}$, where $2H > 1$, there is no term
$$\frac{1}{2} b^2(\tau, \omega) \frac{\partial^2 U}{\partial x^2}\Big(\tau, X_t\Big)\, d\tau$$
in (5.28), in contrast to that of the usual Itô formula with respect to Brownian motion.

2. The requirements on $a(\tau)$, $b(\tau)$, $X(\tau)$ and $U(\tau, X_\tau)$, such as Conditions 1, 2 and 4 of the theorem, and the moment conditions (5.21) – (5.26) are standard.

3. Conditions 3.1 and 3.2 are important for Itô's formula to be true in the case of fractional Brownian motion. Many stochastic processes can be chosen as $b(\tau)$. For example,
$$b(\tau) = A_1 \tau + A_2 \,,$$
where A_1 and A_2 are two random variable with $E\, A_1^2 < \infty$ and A_1 is independent of $\{\, B_H(\tau)\, \}$.

The proof of Theorem 5.1 will be given in Subsection 5.5.1.

5.4 Applications of Stochastic Calculus of Fractional Brownian Motion

5.4.1 Summary of some other results on stochastic calculus of $B_H(t)$

A number of authors have been interested in the stochastic analysis of $B_H(t)$. For example, Lin (1996) defined the stochastic integral with respect to $B_H(t)$ in the case where the integrands are either deterministic bounded functions or the compositions of deterministic bounded functions and $B_H(t)$. He also investigated stochastic differential equations of the form

$$dX_t = f(t, X_t)\,dt + g(t)\,dB_H(t)\,. \tag{5.30}$$

In this subsection we summarize some of his results.

DEFINITION 5.3 *Let $g(t) : \mathbf{R} \to \mathbf{R}$ be a bounded Borel function. Define*

$$\int_0^t g(B_H(\tau))\,dB_H(\tau) = \int_0^{B_H(t)} g(\tau)\,d\tau\,, \tag{5.31}$$

$$\int_0^{B_H(t)} g(\tau)\,d\tau = \lim_{|\mathcal{P}^n|\to 0} \sum_j g\left(t_{j-1}^n\right)\left(B_H\left(t_j^n\right) - B_H\left(t_{j-1}^n\right)\right)\,, \tag{5.32}$$

where the sequence of partition of $[\,0\,,\,t\,]$ is given as the same as that in Chapter 4.

Remarks on Definition 5.3. As discussed in Section 4.2, Definition 5.31 is a special case of our Definition 4.9, while (5.32) is a special case of Definition 4.8. Lin studied the existence and uniqueness of equation 5.30. He found the following result

THEOREM 5.2 *(Lin, 1996) Let $f\,(\,s\,,\,x\,)$ and $g\,(\,s\,)$ be Borel functions such that*

1. *$g\,:\,[\,0\,,\,\infty\,) \to \mathbf{R}$ is bounded,*
2. *$|\,f\,(\,s\,,\,x\,)\,| \le K\,|\,x\,| + K$,*
3. *$|\,f\,(\,s\,,\,x\,) - f\,(\,s\,,\,y\,)\,| \le K\,|\,x - y\,|$.*

Here K is a positive constant. Then the stochastic differential equation

$$\begin{cases} dX_t = f\,(\,t\,,\,X_t\,)\,dt + g\,(\,s\,)\,dB_H(t) \\ X_0 = A\,(\,\omega\,) \end{cases} \tag{5.33}$$

has a unique solution, whose paths are continuous. Here $A\,(\,\omega\,) \in L_2(\Omega)$.

For the proof of Theorem 5.2, see Lin (1996).

5.4.2 The existence and uniqueness of the solution of the fractional Black-Scholes equation

In this subsection we are interested in solving a stochastic differential equation – the fractional Black-Scholes model defined in Section 5.2. We will use Itô's formula (5.29) and Theorem 5.2 to prove the uniqueness of the solution of the fractional Black-Scholes equation (5.14). In detail, we have the following theorems:

THEOREM 5.3 *The stochastic differential equation*

$$\begin{cases} dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_H(t) \\ S_{t_0} = A\,(\omega\,) \end{cases} \tag{5.34}$$

has a solution

$$S_t = A\, \exp\Big\{\mu\,(t - t_o) + \sigma\,(\, B_H(t) - B_H(t_0)\,)\Big\}\ , \tag{5.35}$$

where $A\,(\,\omega\,)$ is a positive random variable, such that $E\,|A(\omega)\,|^2 < \infty$, μ and σ are constants.

THEOREM 5.4 *The solution of (5.34) is unique.*

Remarks on Theorems 5.3 and 5.4. We derived a solution of (5.34) before we read the work of Lin. Subsequently, we have used his result (Theorem 5.2) to prove the uniqueness of (5.34). The original method we used to prove Theorem 5.3 is given in Subsection 5.5.2. Here we use the result of Lin to show the existence and uniqueness through Theorems 5.3 and 5.4.

Proof of Theorems 5.3 and 5.4. Let us consider a stochastic equation

$$\begin{cases} dX_t = \mu\, dt + \sigma\, dB_H(t)\,, \\ X_{t_o} = \log\, A\,, \end{cases} \tag{5.36}$$

where μ, σ and A are as given in Theorem 5.3. Then it is easy to see that

$$X_t = X_{t_0} + \mu\,(t - t_0) + \sigma\,(B_H(t) - B_H(t_0))$$

is a solution of (5.36) and furthermore, from Theorem 5.2, it is the unique solution. Now let

$$S_t = \exp\{X_t\},$$

then by Itô's formula (5.29) we have

$$\begin{aligned} dS_t &= \mu \exp\{X_t\}\,dt + \sigma \exp\{X_t\}\,dB_H(t) \\ &= \mu\,S_t\,dt + \sigma\,S_t\,dB_H(t), \end{aligned}$$

therefore,

$$S_t = A \exp\Big\{\mu\,(t - t_0) + \sigma\,(B_H(t) - B_H(t_0))\Big\}$$

is the unique solution of equation (5.34).

5.5 Proofs

5.5.1 Proof of Theorem 5.1

In order to prove Theorem 5.1, we need the following lemma, the proof of which will be given after the proof of Theorem 5.1.

LEMMA 5.5 *Assume stochastic processes $a(\tau)$ and $b(\tau)$ satisfy the conditions of Theorem 5.1. Then, for any $t, s \in [0, T]$ such that $|t-s| \to 0$, we have*

$$\begin{aligned} &\int_s^t a(\tau)\,d\tau + \int_s^t b(\tau)\,dB_H(\tau) \\ &\quad = a(s)(t-s) + b(s)\Big(B_H(t) - B_H(s)\Big) + o_{L_2}\Big(|t-s|\Big), \end{aligned} \tag{5.37}$$

where $o_{L_2}(|t-s|)$ means a term such that

$$\left(E\left|o_{L_2}(|t-s|)\right|^2\right)^{1/2} = o(|t-s|).$$

Proof of Theorem 5.1. For any interval $[t_0, t] \subseteq [0, T]$ and any sequence of partitions

$$\mathfrak{B}^{(n)}: \qquad t_0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{q(n)}^{(n)} = t$$

with $|\mathfrak{B}^{(n)}| \to 0$ as $n \to \infty$, write

$$\begin{aligned}
&\Delta t_j^{(n)} = t_{j+1}^{(n)} - t_j^{(n)} \ , \ \Delta X_j^{(n)} = X_{t_{j+1}^{(n)}} - X_{t_j^{(n)}} \ , \\
&\Delta B_{H,j}^{(n)} = B_H\left(t_{j+1}^{(n)}\right) - B_H\left(t_j^{(n)}\right) , \\
&\Delta U_j^{(n)} = U\left(t_{j+1}^{(n)}, X_{t_{j+1}^{(n)}}\right) - U\left(t_j^{(n)}, X_{t_j^{(n)}}\right)
\end{aligned}$$

for $j = 0, 1, \ldots, q(n)-1$, $n = 1, 2, \ldots$. Then we have

$$Y_t - Y_{t_0} = U(t, X_t) - U(t_0, X_{t_0}) = \lim_{n\to\infty} \sum_{j=0}^{q(n)} \Delta U_j^{(n)} . \tag{5.38}$$

From a knowledge of calculus we have

$$\begin{aligned}
\Delta U_j^{(n)} &= U\left(t_{j+1}^{(n)}, X_{t_{j+1}^{(n)}}\right) - U\left(t_j^{(n)}, X_{t_j^{(n)}}\right) \\
&= U\left(t_{j+1}^{(n)}, X_{t_{j+1}^{(n)}}\right) - U\left(t_j^{(n)}, X_{t_{j+1}^{(n)}}\right) \\
&\quad + U\left(t_j^{(n)}, X_{t_{j+1}^{(n)}}\right) - U\left(t_j^{(n)}, X_{t_j^{(n)}}\right) \\
&= \frac{\partial U}{\partial t}\left(t_j^{(n)} + \theta_n \Delta t_j^{(n)}, X_{t_{j+1}^{(n)}}\right)\Delta t_j^{(n)} + \frac{\partial U}{\partial x}\left(t_j^{(n)}, X_{t_j^{(n)}}\right)\Delta X_j^{(n)} \\
&\quad + \frac{\partial^2 U}{\partial x^2}\left(t_j^{(n)}, X_{t_j^{(n)}} + \delta_n \Delta X_j^{(n)}\right)\left(\Delta X_j^{(n)}\right)^2 ,
\end{aligned} \tag{5.39}$$

where $\theta_n = \theta_n(\omega)$ and $\delta_n = \delta_n(\omega)$ are random variables such that $0 \le \theta_n, \delta_n \le 1$ and $\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \delta_n = 0$ in the $L_2(\Omega)$ sense. Since $\partial U/\partial x$ is uniformly continuous and the stochastic process X_t is continuous in the sense of $L_2(\Omega)$ (as well as with probability one, see Theorem 4.20, Chapter 4), we have

$$\lim_{n\to\infty} \sum_{j=0}^{q(n)} \frac{\partial U}{\partial t}\left(t_j^{(n)} + \theta_n \Delta t_j^{(n)}, X_{t_{j+1}^{(n)}}\right)\Delta t_j^{(n)} = \int_{t_0}^{t} \frac{\partial U}{\partial t}\left(\tau, X_\tau\right) d\tau . \tag{5.40}$$

From Lemma 5.5 and (5.20) we have

$$\begin{aligned}
\Delta X_j^{(n)} &= X_{t_{j+1}^{(n)}} - X_{t_j^{(n)}} = \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} a(\tau)\, d\tau + \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} b(\tau)\, dB_H(\tau) \\
&= a\left(t_j^{(n)}\right)\Delta t_j^{(n)} + b\left(t_j^{(n)}\right)\Delta B_{H,j}^{(n)} + o_{L_2}\left(\Delta t_j^{(n)}\right) ,
\end{aligned}$$

where $o_{L_2}(\Delta t_j^{(n)})$ means a term such that

$$\left(E\left|o_{L_2}\left(\Delta t_j^{(n)}\right)\right|^2\right)^{1/2}=o\left(\Delta t_j^{(n)}\right).$$

Therefore, by (5.23),

$$\begin{aligned}
&\sum_{j=0}^{q(n)-1}\frac{\partial U}{\partial x}\left(t_j^{(n)},X_{t_j^{(n)}}\right)\Delta X_j^{(n)}\\
&\quad=\sum_{j=0}^{q(n)-1}\frac{\partial U}{\partial x}\left(t_j^{(n)},X_{t_j^{(n)}}\right)\left\{a\left(t_j^{(n)}\right)\Delta t_j^{(n)}+b\left(t_j^{(n)}\right)\Delta B_{H,j}^{(n)}+o_{L_2}\left(\Delta t_j^{(n)}\right)\right\}\\
&\quad=\sum_{j=0}^{q(n)-1}\frac{\partial U}{\partial x}\left(t_j^{(n)},X_{t_j^{(n)}}\right)\left\{a\left(t_j^{(n)}\right)\Delta t_j^{(n)}+b\left(t_j^{(n)}\right)\Delta B_{H,j}^{(n)}\right\}\\
&\qquad\qquad+\sum_{j=0}^{q(n)-1}o_{L_2}\left(\Delta t_j^{(n)}\right)\\
&\quad=\sum_{j=0}^{q(n)-1}\frac{\partial U}{\partial x}\left(t_j^{(n)},X_{t_j^{(n)}}\right)\left\{a\left(t_j^{(n)}\right)\Delta t_j^{(n)}+b\left(t_j^{(n)}\right)\Delta B_{H,j}^{(n)}\right\}+o_{L_2}(1),
\end{aligned}$$

where $o_{L_2}(1)$ means a term such that $\lim_{n\to\infty}E\,|\,o_{L_2}(1)\,|^2=0$. Hence

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{j=0}^{q(n)-1}\frac{\partial U}{\partial x}\left(t_j^{(n)},X_{t_j^{(n)}}\right)\Delta X_j^{(n)}\\
&\quad=\int_{t_0}^{t}\frac{\partial U}{\partial x}\Big(\tau,X_\tau\Big)\Big\{a(\tau)+b(\tau)\,dB_H(\tau)\Big\}.
\end{aligned}\tag{5.41}$$

From Lemma 5.5, (5.25), (5.26) and equation (4.11) of Chapter 4, we have

$$\begin{aligned}
\left(\Delta X_j^{(n)}\right)^2&=\left(X_{t_{j+1}^{(n)}}-X_{t_j^{(n)}}\right)^2=\left\{\int_{t_j^{(n)}}^{t_{j+1}^{(n)}}a(\tau)\,d\tau+\int_{t_j^{(n)}}^{t_{j+1}^{(n)}}b(\tau)\,dB_H(\tau)\right\}^2\\
&=\left\{a\left(t_j^{(n)}\right)\left(\Delta t_j^{(n)}\right)+b\left(t_j^{(n)}\right)\Delta B_{H,j}^{(n)}+o_{L_2}\left(\Delta t_j^{(n)}\right)\right\}^2\\
&=a^2\left(t_j^{(n)}\right)\left(\Delta t_j^{(n)}\right)^2+b^2\left(t_j^{(n)}\right)\left(\Delta B_{H,j}^{(n)}\right)^2+o_{L_2}\left(\Delta t_j^{(n)}\right)\\
&=o_{L_2}\left(\Delta t_j^{(n)}\right).
\end{aligned}$$

Then, by (5.24),

$$\frac{\partial^2 U}{\partial x^2}\left(t_j^{(n)},X_{t_j^{(n)}}+\delta_n\Delta X_j^{(n)}\right)\left(\Delta X_j^{(n)}\right)^2=o_{L_2}\left(\Delta t_j^{(n)}\right)$$

and hence

$$\lim_{n\to\infty}\sum_{j=0}^{q(n)-1}\frac{\partial^2 U}{\partial x^2}\left(t_j^{(n)},X_{t_j^{(n)}}+\delta_n\Delta X_j^{(n)}\right)\left(\Delta X_j^{(n)}\right)^2=0.\tag{5.42}$$

Now, from (5.38), (5.39), (5.40), (5.41) and (5.42) we have

$$
\begin{aligned}
Y_t - Y_{t_0} &= \lim_{n\to\infty} \sum_{j=0}^{q(n)-1} \Delta U_j^{(n)} \\
&= \int_{t_0}^{t} \frac{\partial U}{\partial t}\Big(\tau\,,\, X_\tau\Big)\, d\tau + \int_{t_0}^{t} \frac{\partial U}{\partial x}\Big(\tau\,,\, X_\tau\Big) \Big\{ a(\tau)\, d\tau + b(\tau)\, dB_H(\tau) \Big\} \\
&= \int_{t_0}^{t} \left\{ \frac{\partial U}{\partial t}\Big(\tau\,,\, X_\tau\Big) + \frac{\partial U}{\partial x}\Big(\tau\,,\, X_\tau\Big)\, a(\tau) \right\} d\tau \\
&\quad + \int_{t_0}^{t} \frac{\partial U}{\partial x}\Big(\tau\,,\, X_\tau\Big)\, b(\tau)\, dB_H(\tau)\,.
\end{aligned}
$$

This finishes the proof of Theorem 5.1. Next we move to establish Lemma 5.5.

Proof of Lemma 5.5. Since $a\,(\,t\,,\,\omega\,)$ is Riemann-Stieltjes integrable, as $|t-s| \to 0$, from Lemma 4.25 of Chapter 4 we have

$$
\int_s^t a(\tau)\, d\tau = a(s)\,(t-s) + o_{L_2}(\,|t-s|\,)\,.
$$

Hence, in order to finish the proof of Lemma 5.5, we need only to show that

$$
\int_s^t b(\tau)\, dB_H(\tau) = b(s)\Big(\, B_H(t) - B_H(s) \Big) + o_{L_2}(\,|t-s|\,)\,. \tag{5.43}
$$

Without loss of generality, we assume $s < t$. Let a sequence of partitions of $[\,s\,,\,t\,]$ be given as

$$
\mathfrak{B}^{(n)}\ : \qquad s = t_0^{(n)} < t_1^{(n)} < \cdots < t_{q(n)}^{(n)} = t\,;
$$

then

$$
\begin{aligned}
&E\left| \int_s^t b(\tau)\, dB_H(\tau) - b(s)\Big(\, B_H(t) - B_H(s) \Big) \right|^2 \\
&\quad = E\left| \int_s^t \Big(b(\tau) - b(s) \Big)\, dB_H(\tau) \right|^2 \\
&\quad = \lim_{n\to\infty} E\left| \sum_{j=1}^{n} \Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big) \right|^2 .
\end{aligned} \tag{5.44}
$$

Now we consider the term on the right hand side of (5.44) without taking the limit yet. We have

$$
E\left| \sum_{j=1}^{n} \Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big) \right|^2
$$

$$= \sum_{j,k=1}^{n} E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(b\left(t_{k-1}^{(n)}\right) - b(s) \Big) \Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big) \times$$
$$\Big(B_H\left(t_k^{(n)}\right) - B_H\left(t_{k-1}^{(n)}\right) \Big)$$
$$= \sum_{j=1}^{n} E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big)^2 \Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big)^2$$
$$+ \sum_{j\neq k} E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(b\left(t_{k-1}^{(n)}\right) - b(s) \Big) \times$$
$$\Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big) \Big(B_H\left(t_k^{(n)}\right) - B_H\left(t_{k-1}^{(n)}\right) \Big)$$
$$\equiv A_n + B_n \text{ , say.} \tag{5.45}$$

If Condition 3.1 of Theorem 5.1 holds then, by (5.27),

$$\begin{aligned}
A_n &= \sum_{j=1}^{n} E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big)^2 \Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big)^2 \\
&= \sum_{j=1}^{n} E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big)^2 E\Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big)^2 \\
&\leq const \sum_{j=1}^{n} \left| t_j^{(n)} - s \right|^{\beta} \left(t_j^{(n)} - t_j^{(n)} \right)^{2H} \\
&\leq const\, |t-s|^{\beta+2H-1} \sum_{j=1}^{n} \left(t_j^{(n)} - t_j^{(n)} \right) \\
&= const\, |t-s|^{\beta+2H} = o(|t-s|)\,.
\end{aligned} \tag{5.46}$$

To deal with B_n in (5.45) under Condition 3.1 of Theorem 5.1, we use the notation $\Gamma_{j,k}$, $\Delta\Gamma_{j,k}$ and α appearing in Lemma 4.10 of Chapter 4. Since $|\partial^2\Gamma/\partial y \partial x|$ is integrable in $\{(x,y) : s \leq x \neq y \leq t\}$, by Lemma 4.10, (5.27) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
B_n &= \sum_{j\neq k} \Big\{ E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(b\left(t_{k-1}^{(n)}\right) - b(s) \Big) \Big\} \times \\
&\qquad \Big\{ E\Big(B_H\left(t_j^{(n)}\right) - B_H\left(t_{j-1}^{(n)}\right) \Big) \Big(B_H\left(t_k^{(n)}\right) - B_H\left(t_{k-1}^{(n)}\right) \Big) \Big\} \\
&= \sum_{j\neq k} \Big\{ E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(b\left(t_{k-1}^{(n)}\right) - b(s) \Big) \Big\} \Delta\Gamma_{j,k} \\
&\leq \sum_{j\neq k} \Big\{ E\Big(b\left(t_{j-1}^{(n)}\right) - b(s) \Big) \Big(b\left(t_{k-1}^{(n)}\right) - b(s) \Big) \Big\} \times \\
&\qquad \Big\{ \left| \frac{\partial^2 \Gamma}{\partial y\, \partial x} \left(t_{j-1}^{(n)}, t_{k-1}^{(n)} \right) \right| \left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right)
\end{aligned}$$

$$
\begin{aligned}
& \quad + C \left| t_{j-1}^{(n)} - t_{k-1}^{(n)} \right|^{2H-2-\alpha} o\left(\left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \\
& \quad + o\left(\left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \Big\} \\
\leq & \; const \sum_{j \neq k} \left| t_{j-1}^{(n)} - s \right|^{\beta/2} \left| t_{k-1}^{(n)} - s \right|^{\beta/2} \times \\
& \quad \left\{ \left| \frac{\partial^2 \Gamma}{\partial y \, \partial x} \left(t_{j-1}^{(n)}, t_{k-1}^{(n)} \right) \right| \left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right. \\
& \quad + C \left| t_{j-1}^{(n)} - t_{k-1}^{(n)} \right|^{2H-2-\alpha} o\left(\left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \\
& \quad + o\left(\left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \Big\} \\
\leq & \; const \, |t-s|^{\beta} \sum_{j \neq k} \left\{ \left| \frac{\partial^2 \Gamma}{\partial y \, \partial x} \left(t_{j-1}^{(n)}, t_{k-1}^{(n)} \right) \right| \left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right. \\
& \quad + C \left| t_{j-1}^{(n)} - t_{k-1}^{(n)} \right|^{2H-2-\alpha} o\left(\left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \\
& \quad + o\left(\left(t_j^{(n)} - t_{j-1}^{(n)} \right) \left(t_k^{(n)} - t_{k-1}^{(n)} \right) \right) \Big\} \\
\longrightarrow & \; const \, |t-s|^{\beta} \iint_{[s,t]^2} \left| \frac{\partial^2 \Gamma}{\partial y \, \partial x} (x, y) \right| dy \, dx \\
= & \; const \, |t-s|^{\beta + 2H} = o(|t-s|) . \qquad (5.47)
\end{aligned}
$$

So, in the case of Condition 3.1, from (5.45), (5.46) and (5.47), Lemma 5.5 holds. Finally, we consider the case of Condition 3.2. Following (5.45) and using the inequalities of Condition 3.2, we have

$$
\begin{aligned}
A_n = & \sum_{j=1}^{n} E \left\{ \left(b'(s) \left(t_{j-1}^{(n)} - s \right) + o_{L_4} \left(t_{j-1}^{(n)} - s \right) \right)^2 \right. \\
& \quad \left. \left(B_H \left(t_j^{(n)} \right) - B_H \left(t_{j-1}^{(n)} \right) \right)^2 \right\} \\
\leq & \left\{ E \left(b'(s) \left(t_{j-1}^{(n)} - s \right) + o_{L_4} \left(t_{j-1}^{(n)} - s \right) \right)^4 \right\}^{1/2} \\
& \quad \left\{ E \left(B_H \left(t_j^{(n)} \right) - B_H \left(t_{j-1}^{(n)} \right) \right)^4 \right\}^{1/2} \\
\leq & \sum_{j=1}^{q(n)} \left\{ const \left(t_{j-1}^{(n)} - s \right)^2 + o\left(\left(t_{j-1}^{(n)} - s \right)^2 \right) \right\} \left(t_j^{(n)} - t_{j-1}^{(n)} \right)^{2H} \\
\leq & \; const \, (t-s)^2 \sum_{j=1}^{q(n)} \left(t_j^{(n)} - t_{j-1}^{(n)} \right)^{2H} = o(t-s) . \qquad (5.48)
\end{aligned}
$$

By the same argument, under Condition 3.2, we have

$$
\begin{aligned}
B_n =& \sum_{j\neq k} E\left\{\left(b'(s)\left(t_{j-1}^{(n)}-s\right)+o_{L_4}\left(t_{j-1}^{(n)}-s\right)\right)\left(b'(s)\left(t_{j-1}^{(n)}-s\right)+o_{L_4}\left(t_{j-1}^{(n)}-s\right)\right)\right.\\
&\qquad \left.\left(B_H\left(t_j^{(n)}\right)-B_H\left(t_{j-1}^{(n)}\right)\right)\left(B_H\left(t_k^{(n)}\right)-B_H\left(t_{k-1}^{(n)}\right)\right)\right\}\\
=& \sum_{j\neq k} E\left\{\left(b'(s)\left(t_{j-1}^{(n)}-s\right)+o_{L_4}\left(t_{j-1}^{(n)}-s\right)\right)\left(b'(s)\left(t_{k-1}^{(n)}-s\right)\right.\right.\\
&\qquad \left.\left.+\,o_{L_4}\left(t_{k-1}^{(n)}-s\right)\right)\right\}\times\\
&\qquad E\left\{\left(B_H\left(t_j^{(n)}\right)-B_H\left(t_{j-1}^{(n)}\right)\right)\left(B_H\left(t_k^{(n)}\right)-B_H\left(t_{k-1}^{(n)}\right)\right)\right\}\\
\leq&\ \ const\,|t-s|^2\sum_{j\neq k}|\Delta\Gamma_{j,k}| = const\,|t-s|^{2+2H} = o(t-s)\,. \qquad (5.49)
\end{aligned}
$$

Thus, in the case of Condition 3.2, from (5.45), (5.48) and (5.49), Lemma 5.5 holds. This completes the proof of Lemma 5.5, and hence Theorem 5.1.

5.5.2 Another proof of Theorem 5.3

In this subsection we give our original proof of Theorem 5.3. In order to do so, we first give Lemma 5.6, whose proof will be given after the proof of Theorem 5.3.

LEMMA 5.6 *For any t and δ, we have*

$$\exp\left\{B_H(t+\delta)-B_H(t)\right\} = 1+(\,B_H(t+\delta)-B_H(t)\,)+o_{L_2}(|\delta|)\,, \qquad (5.50)$$

$$B_H(t+\delta)-B_H(t) = O_{L_2}(|\delta|^H)\,, \qquad (5.51)$$

where $o_{L_2}(|\delta|)$ means a term satisfying

$$\left(E\,|o_{L_2}(|\delta|)|^2\right)^{1/2} = o(|\delta|)\,,$$

while $O_{L_2}(|\delta|^H)$ means

$$\left(E\,|O_{L_2}(|\delta|^H)|^2\right)^{1/2} = O(|\delta|^H)\,.$$

We now establish Theorem 5.3. For any $t \in [\, t_0\,,\, T\,]$ and $\omega \in \Omega$, let

$$S_t(\omega) = A\,(\,\omega\,)\,\exp\Big\{\,\mu\,(t - t_o) + \sigma\,(B_H(t\,,\,\omega) - B_H(t_0\,,\,\omega)\,)\,\Big\}\;;$$

then we have

$$\begin{aligned} S_{t+\Delta} - S_t &= A\,\exp\Big\{\,\mu\,(t - t_o + \Delta) + \sigma\,(B_H(t+\Delta) - B_H(t_0)\,)\Big\} \\ &\quad -A\,\exp\Big\{\,\mu\,(t - t_o) + \sigma\,(B_H(t) - B_H(t_0)\,)\Big\} \\ &= A\,\exp\Big\{\,\mu\,(t - t_o) + \sigma\,(B_H(t) - B_H(t_0)\,)\Big\} \times \\ &\quad \Big(\exp\Big\{\mu\,\Delta + \sigma\,(B_H(t+\Delta) - B_H(t)\,)\Big\} - 1\Big)\,. \end{aligned} \tag{5.52}$$

From Lemma 5.6 we have

$$\begin{aligned} &\exp\Big\{\mu\,\Delta + \sigma\,(B_H(t+\Delta) - B_H(t)\,)\Big\} \\ &\quad = \exp\Big\{\mu\,\Delta\Big\}\exp\Big\{\sigma\,(B_H(t+\Delta) - B_H(t)\,)\Big\} \\ &\quad = \Big(\,1 + \mu\,\Delta + o(\Delta)\Big)\,\Big(1 + \sigma\,(\,B_H(t+\Delta) - B_H(t)\,) + o_{L_2}(|\Delta|)\,\Big) \\ &\quad = 1 + \mu\,\Delta + \sigma\,\Big(\,B_H(t+\Delta) - B_H(t)\,) + o_{L_2}(|\Delta|\Big)\,. \end{aligned} \tag{5.53}$$

Therefore,

$$S_{t+\Delta} - S_t = S_t\Big\{\mu\,\Delta + \sigma\,(\,B_H(t+\Delta) - B_H(t)\,)\Big\} + S_t\,o_{L_2}(|\Delta|)\,.$$

In other words,

$$dS_t = \mu\,S_t\,dt + \sigma\,S_t\,dB_H\;,$$

so S_t is a solution of the fractional Black-Scholes equation.

Proof of Lemma 5.6. By the Taylor expansion we have

$$\begin{aligned} &\exp\Big\{\,B_H(t+\Delta) - B_H(t)\Big\} = 1 + (\,B_H(t+\Delta) - B_H(t)\,) \\ &+\frac{1}{2\,!}\Big(\,B_H(t+\Delta) - B_H(t)\Big)^2 + \cdots + \frac{1}{n\,!}\Big(\,B_H(t+\Delta) - B_H(t)\Big)^n + \cdots\,. \end{aligned} \tag{5.54}$$

For $n \geq 2$, let

$$Y_n = \frac{1}{2!} + \frac{1}{3!}\Big(\,B_H(t+\Delta) - B_H(t)\Big) + \cdots + \frac{1}{n!}\Big(\,B_H(t+\Delta) - B_H(t)\Big)^{n-2}\,.$$

Then, for any integers $n \geq 2$ and $p \geq 1$,

$$\begin{aligned}
\Big(E \mid Y_{n+p} - Y_n \mid^4 \Big)^{\frac{1}{4}} &= \left\{ E \left(\sum_{j=1}^{p} \frac{1}{(n+j)!} \Big(B_H(t+\Delta) - B_H(t) \Big)^{(n+j-2)} \right)^4 \right\}^{\frac{1}{4}} \\
&\leq \sum_{j=1}^{p} \left\{ E \left(\frac{1}{(n+j)!} \Big(B_H(t+\Delta) - B_H(t) \Big)^{(n+j-2)} \right)^4 \right\}^{\frac{1}{4}} \\
&= \sum_{j=1}^{p} \frac{1}{(n+j)!} \left\{ E \Big(B_H(t+\Delta) - B_H(t) \Big)^{4(n+j-2)} \right\}^{\frac{1}{4}} \\
&= \sum_{j=1}^{p} \frac{1}{(n+j)!} \left\{ |\Delta|^{H4(n+j-2)} \Big(4(n+j-2) - 1 \Big)!! \right\}^{\frac{1}{4}} \\
&= \sum_{j=1}^{p} \frac{1}{(n+j)!} |\Delta|^{H(n+j-2)} \left\{ \Big(4(n+j-2) - 1 \Big)!! \right\}^{\frac{1}{4}} ,
\end{aligned}$$

where "!!" means a two step factorial, that is,

$$\begin{aligned}
(2n - 1)!! &= 1 \cdot 3 \cdot 5 \cdots (2n - 1) , \\
(2n)!! &= 2 \cdot 4 \cdot 6 \cdots 2n .
\end{aligned}$$

Let

$$a_n = \frac{|\Delta|^{H(n-2)}}{n!} \left\{ \Big(4(n-2) - 1 \Big)!! \right\}^{\frac{1}{4}} ,$$

and observe that

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \left(\frac{|\Delta|^{H(n+1-2)} \{[4(n+1-2) - 1]!!\}^{\frac{1}{4}}}{(n+1)!} \right) \Big/ \left(\frac{|\Delta|^{H(n-2)} \{[4(n-2) - 1]!!\}^{\frac{1}{4}}}{n!} \right) \\
&= \left(|\Delta|^H \Big\{ (4n-5)!! \Big\}^{\frac{1}{4}} \right) \Big/ \left((n+1) \Big\{ (4n-9)!! \Big\}^{\frac{1}{4}} \right) \\
&= \frac{|\Delta|^H \Big\{ (4n-5)(4n-7) \Big\}^{\frac{1}{4}}}{(n+1)} \longrightarrow 0 ,
\end{aligned}$$

as $n \to \infty$. Thus $\sum_{n=1}^{\infty} a_n < \infty$, and for any $p \geq 1$, as $n \to \infty$, we have

$$\sum_{j=1}^{p} a_{n+j} = \sum_{j=1}^{p} \frac{|\Delta|^{H(n+j-2)}}{(n+j)!} \left\{ \Big(4(n+j-2) - 1 \Big)!! \right\}^{\frac{1}{4}} \longrightarrow 0 .$$

Therefore, for any $p \geq 1$, we have

$$\left\{ E \Big| Y_{n+p} - Y_n \Big|^4 \right\}^{\frac{1}{4}} \longrightarrow 0 .$$

That is $\lim_{n\to\infty} Y_n$ exists in $L_4(\Omega)$ sense. Hence

$$
\begin{aligned}
& E\left|\frac{1}{2!}\Big(B_H(t+\Delta)-B_H(t)\Big)^2+\cdots\frac{1}{n!}\Big(B_H(t+\Delta)-B_H(t)\Big)^n+\cdots\right|^2 \\
&= E\left|\Big(B_H(t+\Delta)-B_H(t)\Big)^4\left\{\frac{1}{2!}+\cdots\frac{1}{n!}\Big(B_H(t+\Delta)-B_H(t)\Big)^{n-2}+\cdots\right\}^2\right| \\
&= E\left|\Big(B_H(t+\Delta)-B_H(t)\Big)^4\left(\lim_{n\to\infty} Y_n\right)^2\right| \\
&\le \left(E\Big(B_H(t+\Delta)-B_H(t)\Big)^8\right)^{1/2}\left(E\left(\lim_{n\to\infty} Y_n\right)^4\right)^{1/2} \\
&= const\left(E\left(\lim_{n\to\infty} Y_n\right)^4\right)^{1/2}|\Delta|^{4H}. \qquad (5.55)
\end{aligned}
$$

Notice that $\lim_{n\to\infty} Y_n \in L_4(\Omega)$ and $1/2 < H < 1$, by (5.54) and (5.55) we see that (5.50) of Lemma 5.6 holds. Equation(5.51) of Lemma 5.6 is obvious. This ends the proof of Lemma 5.6.

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