

MEAN PERIODIC FUNCTIONS AND ORDINARY DIFFERENTIAL EQUATIONS.

by

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Except where it is otherwise indicated, this thesis is my own work.

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INTRODUCTION

The term, mean periodic function, was coined by Delsarte [1] in 1935 to denote a complex-valued continuous function, f , of a real variable t that satisfies an integral equation of the form

$$\int f(t-r)k(r)dr = 0 \quad \text{for all } t$$

where k is a not identically zero continuous function with compact support.

This name is suggested by the fact that f is a periodic function of period τ whose average is zero if and only if $\int_0^\tau f(t-r)dr = 0$ for all t .

A more complete theory was presented by L. Schwartz [1] in 1947. For any topological vector space, E , of complex-valued functions of a real variable, the function $f \in E$ is defined to be mean periodic in E if the linear combinations of the translates is not dense in E . For $E = \mathcal{C}(R)$, the space of all complex-valued continuous functions defined on the real line and equipped with the topology of convergence uniform on all compact subsets of R , Schwartz obtains properties similar to those of Delsarte and shows that the general and intrinsic definition is equivalent to the following;

A necessary and sufficient condition that $f \in \mathcal{C}(R)$ be mean periodic is that there exists a non-zero measure, μ , with compact support for which

$$\int f(t-r)d\mu(r) = 0 \quad \text{for all } t.$$

The main property of mean periodic functions in $\mathcal{C}(R)$ is that they are limits of linear combinations of exponential monomials. The means Schwartz used to prove this are summarised in the Appendix while the exposition in Chapter One is based on proofs given by Kahane [1], [2], that

make use of the Carleman transform. Both approaches to the theory of mean periodic functions in $\mathcal{C}(\mathbb{R})$ rely on the theory of analytic functions of one complex variable and both extensively use the Fourier transform of a measure. In this thesis, $\hat{\mu}$ is defined as $\hat{\mu} : z \rightarrow \int e^{-zt} d\mu(t)$ instead of $\int e^{-2\pi izt} d\mu(t)$ (Schwartz) or $\int e^{-izt} d\mu(t)$ (Kahane); consequently, statements like " $\hat{\mu}$ is bounded on the real axis" as found in Schwartz and Kahane will read here as " $\hat{\mu}$ is bounded on the imaginary axis".

The scope of this thesis is restricted to mean periodic functions in $\mathcal{C}(\mathbb{R})$ and to the mean periodic functions on a half line introduced by Koosis [3]. Properties of both classes of functions that do not appear in the literature are amongst those included in Chapters 2 and 4. In the case of $\mathcal{C}(\mathbb{R})$ functions, some of these properties characterize mean periodicity as something quite different from almost periodicity.

The relationship of mean periodic functions in $\mathcal{C}(\mathbb{R})$ to ordinary differential equations is considered. It would seem that the connection is practical, only in the case of linear differential equations with constant coefficients.

Finally, mean periodic functions in both $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(\mathbb{R}_+)$ are useful in discussing linear differential-difference equations with constant coefficients.

Chapter One

MEAN PERIODIC FUNCTIONS IN $\mathcal{C}(R)$.§1.1. The Space $\mathcal{C}(R)$.

We denote by $\mathcal{C}(R)$ the complex vector space of all complex valued continuous functions defined on the real line (R) and equipped with the topology of convergence uniform on all subsets of R . This topology may be defined by the seminorms

$$p_k(f) = \sup\{|f(t)| : -k \leq t \leq k\}.$$

As the sets $\{f : f \in \mathcal{C}(R) \text{ and } p_k(f) < \varepsilon\}$, formed when ε ranges over all positive numbers and k ranges over the set N of positive integers, are convex and form a base of neighbourhoods at the origin, $\mathcal{C}(R)$ is locally convex. This topology for $\mathcal{C}(R)$ is Hausdorff, since $p_k(f) = 0$ for each $k \in N \Rightarrow f = 0$.

A bounded invariant metric defining the given topology is

$$\rho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(f-g)}{1+p_k(f-g)}$$

with $\rho(f_n, 0) \rightarrow 0 \Leftrightarrow p_k(f_n) \rightarrow 0$ as $n \rightarrow \infty$ for each $k \in N$.

$$\Leftrightarrow f_n \rightarrow 0 \text{ in } \mathcal{C}(R) \text{ as } n \rightarrow \infty.$$

$$\Leftrightarrow f_n \text{ converges "locally uniformly" to zero as } n \rightarrow \infty.$$

$\mathcal{C}(R)$ is a complete space, for if $\{f_n\}$ is a Cauchy sequence of elements of $\mathcal{C}(R)$, i.e. $\rho(f_m, f_n) \rightarrow 0$ as $m, n \rightarrow \infty$, then there exists an $f \in \mathcal{C}(R)$ such that $\rho(f, f_n) \rightarrow 0$ as $n \rightarrow \infty$. This follows since for each $t \in R$, $\{f_n(t)\}$ is a Cauchy sequence of complex numbers and so

convergent to $f(t)$, say. Then $\{f_n\}$ converges to f locally uniformly and so f is continuous.

Thus, $\mathcal{C}(R)$ is a Fréchet space. As well, it is separable, the countable set of all polynomials having rational coefficients being dense in $\mathcal{C}(R)$.

To see this, take an $f \in \mathcal{C}(R)$, an $\epsilon > 0$ and select $\ell \in \mathbb{N}$ so that

$$\sum_{k=\ell+1}^{\infty} 2^{-k} < \epsilon/2.$$

For the compact interval $[-\ell, \ell]$, by use of Weierstrass's approximation theorem, a polynomial q having rational coefficients can be found for which

$$|f(t) - q(t)| < \epsilon/2\ell \quad \text{for } -\ell \leq t \leq \ell.$$

Since $p_k \leq p_{k+1}$, $2^{-k} < 1$ and $p_k/(1+p_k) < \min(1, p_k)$ for each $k \in \mathbb{N}$, it follows that

$$\sum_{k=1}^{\ell} 2^{-k} \frac{p_k(f-q)}{1+p_k(f-q)} < \ell p_{\ell}(f-q) < \epsilon/2,$$

and so $\rho(f, q) < \epsilon$.

The support of a function f is the closure of the set $\{t : f(t) \neq 0\}$. The set of all complex valued continuous functions having compact supports is denoted by $\mathcal{C}_c(R)$.

§1.2 The Dual Space of $\mathcal{C}(R)$.

The term, "measure", will be used in this thesis to denote a continuous linear functional on the complex topological linear space $\mathcal{C}(R)$. The set of all such measures is denoted by M_C . The relevant properties of these measures are outlined here; for full details of their theory involving integration theory and Radon measures on $\mathcal{C}(R)$, see Edwards [1], Chapter 4.

Any such measure, μ , assigns to each $f \in \mathcal{C}(R)$ a complex number $\mu(f)$ in such a way that:-

$$\mu(af) = a\mu(f) \quad \text{for every complex number } a,$$

$$\mu(f+g) = \mu(f) + \mu(g) \quad \text{for } f, g \in \mathcal{C}(R), \text{ and}$$

$$\mu(f_n) \rightarrow \mu(f) \quad \text{when } f_n \rightarrow f \text{ in } \mathcal{C}(R) \text{ as } n \rightarrow \infty.$$

A positive measure, λ , is one for which $\lambda(f) \geq 0$ whenever $f \in \mathcal{C}(R)$ and $f \geq 0$. It is customary and often convenient to write $\mu(f)$ as $\int f d\mu$ or $\int f(t) d\mu(t)$.

For a linear functional, μ , on $\mathcal{C}(R)$ to be continuous, (i.e. to be a measure in the above sense), it is necessary and obviously sufficient that there exist a compact set K and a non negative real number C such that

$$|\mu(f)| \leq C \cdot \sup\{|f(t)| : t \in K\} \quad \text{for each } f \in \mathcal{C}(R) \quad (I).$$

For, if (I) were false, there would exist for each $n \in \mathbb{N}$ an $f_n \in \mathcal{C}(R)$ such that $|\mu(f_n)| > np_n(f_n)$.

$$\text{Put } g_n = \frac{f_n}{np_n(f_n)} \quad \text{so that } |\mu(g_n)| > 1 \text{ and } p_m(g_n) \leq 1/n \text{ for } n \geq m.$$

Then, for each $m \in \mathbb{N}$, $p_m(g_n) \rightarrow 0$ as $n \rightarrow \infty$, so that $g_n \rightarrow 0$ in $\mathcal{C}(R)$.

Since $|\mu(g_n)| > 1$, μ would be discontinuous.

For a given measure μ , there exists a smallest compact set K for which (I) holds for a suitable $C = C_K$, and K is called the support of μ . If a continuous function, f , is zero on K , then $\mu(f)$ is zero. The smallest closed interval containing K , i.e. the closed convex envelope of K , will be called the segment of support of μ .

It can be shown that, for any measure μ with compact support K ,

there exists a complex valued function u defined on R that is of bounded variation and constant on each component interval of $R \setminus K$ for which

$$\mu(f) = \int_K f(t) du(t) \quad \text{for each } f \in \mathcal{C}(R).$$

Conversely, any such function u will define through the above formula a measure μ .

If v is a complex-valued Lebesgue integrable function of compact support, then a measure μ is defined by

$$\mu(f) = \int f(t)v(t)dt \quad \text{for each } f \in \mathcal{C}(R).$$

Such a measure is said to have density v relative to Lebesgue measure and, when there is no possibility of confusion, we use μ to denote both the function and the measure. However, the supports of v and μ may not agree when the support of a function is defined as in §1.1.

The norm of the measure μ with compact support is defined to be

$$\|\mu\| = \sup\{|\mu(f)| : f \in \mathcal{C}(R) \text{ and } |f(t)| \leq 1 \text{ for all } t \in K\}$$

We will occasionally have use for distributions. Let $\mathcal{C}^n(R)$ denote the complex vector space of all complex-valued functions on R that have continuous n^{th} order derivatives, equipped with the topology defined by the semi norms $p_k^{(n)}(f) = \sup\{|D^p f(t)| : -k \leq t \leq k, p = 0, 1, \dots, n\}$ for $k \in \mathbb{N}$ and $f \in \mathcal{C}^n(R)$.

Any continuous linear functional on $\mathcal{C}^n(R)$ is (i.e., an extension by continuity of) a distribution of compact support and the least such non-negative integer, n , for which the linear functional is continuous is called the order of the distribution. Thus, a measure is a distribution of

order zero. The distributional derivative, DT , of a distribution T of order n is defined by

$$DT(f) = -T(f') \quad \text{for each } f \in \mathcal{C}_C^\infty(\mathbb{R})$$

and is a distribution of order $n + 1$.

§1.3 The Fourier-Laplace Transform.

The Fourier-Laplace transform of a measure μ is defined here as

$$M(z) = \hat{\mu}(z) = \int e^{-zt} d\mu(t) = \mu(e_{-z}) = \mu(\check{e}_z) = \check{\mu}(e_z)$$

where

$$e_z : t \rightarrow e^{zt}, \quad \check{f} : t \rightarrow f(-t) \quad \text{and} \quad \check{\mu} \text{ is the measure for which}$$

$$\check{\mu}(g) = \mu(\check{g}) \quad \text{for every } g \in \mathcal{C}(\mathbb{R}).$$

This transform proves to be characterized by certain properties.

(i) M is an entire function. To see this, set

$$f_h : t \rightarrow [\exp(-t(z+h)) - \exp(-zt)]/h + t \exp(-tz).$$

Now $p_n(f_h) \rightarrow 0$ as $h \rightarrow 0$ for each $n \in \mathbb{N}$ and so $\mu(f_h) \rightarrow 0$ as $h \rightarrow 0$.

Hence M is complex-differentiable at each point and so analytic in the complex plane.

(ii) M is of exponential type and bounded on the imaginary axis. For if μ has support $K \subset [-L, L]$ and $z = x + iy$, $|M(z)| \leq \|\mu\| \exp(L|x|)$ and so M is of order one and type not exceeding L ; also $|M(iy)| \leq \|\mu\|$.

(iii) From (i) and (ii), the zeros of M have no finite limit point; in fact, by Hadamard's factorization theorem, if the zeros of $M(z)$ are $a_n \neq 0$ of orders p_n respectively ($n = 1, 2, \dots$), and if $M(z)$

has a zero of order k at the origin ($k = 0$ if $M(0) \neq 0$), then

$$M(z) = Az^k e^{cz} \prod_{n=1}^{\infty} (1 - z/a_n)^{p_n} \exp(zp_n/a_n)$$

where c is imaginary and $\sum p_n/|a_n|^2 < \infty$.

(iv) From (ii), the entire function M satisfies

$$\lim_{|z| \rightarrow \infty} \frac{\log |M(z)|}{|z|} \leq L \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\max(\log |M(iy)|, 0)}{1+y^2} dy < \infty$$

so by Levinson [1], pages 25-28,

$$\lim_{R \rightarrow \infty} \int_1^R \log |M(iy)M(-iy)| \frac{1}{y^2} dy \quad \text{exists and is finite.} \quad \text{Then}$$

(a) the set of zeros of M has a finite density $\leq L$, i.e.,

$\lim_{n \rightarrow \infty} n(r)/r$ exists and does not exceed L , where $n(r)$ is the number of zeros of M , counted according to their order, inside the circle $|z| = r$.

(b) Let the non vanishing zeros of M be $a_n = r_n \exp(i\theta_n)$. Then $\sum p_n |\cos \theta_n|/r_n < \infty$, showing that outside any sector containing the imaginary axis, $\sum p_n/|a_n| < \infty$.

For a distribution, T , of order n , the Fourier-Laplace transform will still be an entire function of exponential type. Instead of $\hat{T}$ being bounded on the imaginary axis, $\hat{T}(iy) = O(|y|^n)$.

When a measure, μ , has density u relative to Lebesgue measure and $u \in \mathcal{C}_c^m(\mathbb{R})$, $u(iy) = O(|y|^{-m})$.

For the conditions in which an entire function of exponential type is the Fourier-Laplace transform of a measure of distribution, the Paley-Wiener theorem is applicable. We quote it in two of its derived forms.

(i) If $M(z)$ is an entire function of exponential type that satisfies the conditions

$$M(iy) = O(|y|^{-2}), \quad M(x) = O(e^{\beta x}), \quad x > 0 \quad \text{and} \quad M(x) = O(e^{\alpha x}), \quad x < 0,$$

then M is the Fourier-Laplace transform of a measure whose support is contained in $[\alpha, \beta]$. (Kahane [2], page 14).

(ii) A necessary and sufficient condition that a function be the Fourier-Laplace transform of a distribution with a compact support is that it be an entire function of exponential type that is of polynomial order on the imaginary axis (Edwards, [1], page 390).

Finally, we note that $\hat{\mu} = 0 \Rightarrow \mu = 0$.

§1.4 Convolution Products

The convolution product of a measure $\mu \in M_C$ and a function $f \in \mathcal{C}(R)$ is defined as

$$\mu * f : t \rightarrow \int_K f(t-r) d\mu(r) = \mu(T_t f)$$

where K is the support of μ and the translation operator, T_t takes f into $T_t f : r \rightarrow f(r-t)$. This product is a continuous function, for

$$|\mu * f(r) - \mu * f(s)| \leq \|\mu\| \sup_{t \in K} |f(t-r) - f(t-s)|.$$

The function, $(\mu, f) \rightarrow \mu * f$ is a bilinear map from $M_C \times \mathcal{C}(R)$ into $\mathcal{C}(R)$.

If $f_n \rightarrow 0$ in $\mathcal{C}(R)$ as $n \rightarrow \infty$, then $\mu * f_n \rightarrow 0$ in $\mathcal{C}(R)$. For let ϵ be any positive number, L be any compact subset of R and let μ have support K . An $m \in N$ can be chosen such that when $n > m$,

$$\sup\{|f_n(t-r)| : t \in L, r \in K\} < \epsilon / \|\mu\|$$

whence $|\mu * f_n(t)| < \epsilon$ for $t \in L$.

If $\alpha \in \mathbb{R}$, the α -translate of a measure μ is defined by

$$T_\alpha \mu(f) = \mu(T_{-\alpha} f) \quad \text{for each } f \in \mathcal{C}(\mathbb{R}).$$

The convolution product of two measures μ and $\lambda \in M_C$ is a measure defined by

$$\mu * \lambda(f) = \mu_x [\lambda_y \{f(x+y)\}] \quad \text{for each } f \in \mathcal{C}(\mathbb{R})$$

and the operation is associative and commutative. If λ and μ have segments of supports $[\alpha, \beta]$ and $[\gamma, \delta]$, then $\mu * \lambda$ has support contained in the interval $[\alpha + \gamma, \beta + \delta]$. Thus M_C is a ring w.r.t. $+$, $*$. Also, $(\mu * \lambda)^\wedge = \hat{\mu} \cdot \hat{\lambda}$.

By use of $\mathcal{C}_C^\infty(\mathbb{R})$ functions, any measure or distribution can be regularized. If $\rho \in \mathcal{C}_C^\infty(\mathbb{R})$ and T is a distribution of compact support, then $T * \rho \in \mathcal{C}_C^\infty(\mathbb{R})$.

We will make use of the Dirac measures, either $\delta = \delta_0$ placed at the origin or δ_α placed at any point, α , on the real line. δ_α has support $\{\alpha\}$ and for each $f \in \mathcal{C}(\mathbb{R})$, we have

$$\delta(f) = f(0)$$

$$\delta_\alpha(f) = f(\alpha)$$

$$\delta * f = f$$

$$\delta_\alpha * f = T_\alpha f$$

$$\delta_\alpha * \mu = T_\alpha \mu$$

$$\delta_\alpha * \delta_\beta = \delta_{\alpha + \beta}$$

with $D\delta * f = f'$ when f is absolutely continuous.

The following formulae are valid for distributions as well as measures;

$$T_\alpha(\mu * \lambda) = T_\alpha \mu * \lambda = \mu * T_\alpha \lambda$$

$$T_\alpha(\mu * f) = T_\alpha \mu * f = \mu * T_\alpha f$$

$$D(\mu * \lambda) = D\mu * \lambda = \mu * D\lambda \quad \text{and}$$

$$D(\mu * f) = D\mu * f = \mu * f' \quad \text{when } f \text{ is absolutely continuous.}$$

§1.5 Definition and Basic Properties of Mean Periodic Functions.

Definition. A function $f \in \mathcal{C}(R)$ is mean periodic if there exists a non zero measure, μ , with compact support, satisfying $\mu * f = 0$.

It is evident that any continuous periodic function, f , is mean periodic, for if $\mu = \delta - \delta_\alpha$, where α is the period of f , the $\mu \neq 0$ and $\mu * f = f - T_\alpha f = 0$.

We see that no non zero integrable function, f , can be mean periodic. For if $\mu * f = 0$, then $\hat{\mu} \cdot \hat{f} = 0$. As $\hat{f}$ is continuous, $\hat{f} \neq 0$ and $\hat{\mu}$ is analytic, $\hat{\mu}$ must be identically zero so there exists no non zero measure of compact support satisfying $\mu * f = 0$. This also shows that $\mathcal{C}_C(R)$ functions can not be mean periodic.

Definition. For $f \in \mathcal{C}(R)$, U_f is the linear subspace of $\mathcal{C}(R)$ generated by f and its translates. V_f is the closure of U_f w.r.t. the topology of locally uniform convergence.

Prop. 1.1 $g \in V_f$, $\mu \in M_C$ and $\mu * f = 0 \Rightarrow \mu * g = 0$.

Proof. For a given $\mu \in M_C$, $S_\mu = \{g : g \in \mathcal{C}(R) \text{ and } \mu * g = 0\}$ is a closed translation invariant linear subspace of $\mathcal{C}(R)$. If $\mu * f = 0$, $V_f \in S_\mu$ and so $\mu * g = 0$.

Corollary $\mu * f = 0 \Leftrightarrow \mu * V_f = \{0\}$; and if f is mean periodic and $g \in V_f$, then g is mean periodic.

Theorem 1.2 A necessary and sufficient condition that $f \in \mathcal{C}(R)$ be mean periodic is that $V_f \neq \mathcal{C}(R)$.

Proof. The condition is necessary for if $V_f = \mathcal{C}(R)$ and $\mu \in M_C$ with $\mu * f = 0$, then by Prop. 1.1,

$$\mu(g) = \mu * \check{g}(0) = 0 \quad \text{for all } g \in \mathcal{C}(R).$$

Thus μ is zero and so f is not mean periodic.

Conversely, if the closed linear subspace V_f is different from $\mathcal{C}(R)$, and if g belongs to $\mathcal{C}(R)$ but not V_f , the Hahn-Banach theorem (Edwards, [1], page 119) guarantees the existence of a measure $\mu \in M_C$ such that $\mu(g) \neq 0$ and $\mu(V_f) = 0$. Then

$$\check{\mu} * f(t) = \int f(t-r) d\check{\mu}(r) = \mu(T_t f)$$

and as V_f is translation invariant, $\check{\mu} * f = 0$. Also $\mu \neq 0$ and so f is mean periodic.

Prop. 1.3 (i) If U_f is finite dimensional, then $U_f = V_f$.

(ii) $g \in V_f \Rightarrow V_g \subset V_f$.

(iii) $f \in \mathcal{C}^1(R) \Rightarrow f' \in V_f$.

(i) is due to the fact that any finite dimensional subspace of a Fréchet space is closed. (Edwards, [1], page 65).

(ii). Let $g \in V_f$. Then any finite linear combination of translates of g belong to V_f so that $U_g \subset V_f$. As V_f is closed, $V_g \subset V_f$.

(iii) The function $g_h = (T_{-h}f - f)/h \in U_f$. When $f \in \mathcal{C}^1(R)$, the real and imaginary parts of f are continuously differentiable so by use of the mean value theorem, $g_h \rightarrow f'$ locally uniformly as $h \rightarrow 0$ and so $f' \in V_f$.

We now give further examples of mean periodic functions including polynomials and exponentials $e_a : t \rightarrow e^{at}$ where a is any complex number.

Definition. An exponential monomial is a function $u_n e_a : t \rightarrow t^n e^{at}$ where n is a non-negative integer and a is any complex number.

An exponential polynomial is a finite linear combination of exponential monomials.

A typical exponential polynomial is

$$\sum_{k=1}^n \sum_{q=0}^{p_k} A(k,q) u_q e_{a_k} \quad (A(k,p_k) \neq 0, a_k \text{'s distinct})$$

which generates a translation invariant subspace of dimension $\sum_{k=1}^n (p_k+1)$.

Such a subspace is closed and distinct from $\mathcal{C}(R)$ so that any exponential polynomial is mean periodic.

Other examples of functions that are not mean periodic include $g : t \rightarrow \exp(t^2)$ and $h : t \rightarrow \sum_{n=1}^{\infty} a_n \exp(i\alpha_n t)$ where $\sum |a_n| < \infty$ and $\{\alpha_n\}$ is any sequence of real numbers with a finite limit point. For $g' = 2ug$, $g'' = 2g + 4u_2g$ etc. and so V_g contains the products of g with polynomials. As the polynomials are dense in $\mathcal{C}(R)$, $V_g = \mathcal{C}(R)$ and so g is not mean periodic.

If μ is any measure of compact support satisfying $\mu * h = 0$, it can be shown that $\hat{\mu}(i\alpha_n) = 0$ for each $n \in \mathbb{N}$ (cf. Prop. 2.5) so that $\hat{\mu} = 0$ as $\hat{\mu}$ is an entire function. Then μ is zero so that h is not mean periodic although h is almost periodic.

Prop. 1.4 For any $f \in \mathcal{C}(R)$, the set

$$V'_f = \{\mu : \mu \in M_c \text{ and } \mu * f = 0\}$$

is a weakly closed ideal in the ring M_C . Also $V'_f \neq \{0\} \Leftrightarrow f$ is mean periodic.

Proof. If μ and $\lambda \in V'_f$ and $v \in M_C$, then

$$(\lambda + \mu) * f = 0 \quad \text{and} \quad (v * \mu) * f = v * (\mu * f) = 0.$$

If μ_α is a directed family of elements of V'_f that converge weakly to μ , then $\bigvee \mu_\alpha$ converges weakly to μ and so,

$$\mu * f(t) = \bigvee \mu(T_t f) = \lim_\alpha \bigvee \mu_\alpha(T_t f) = 0$$

for each $t \in R$. Thus $\mu \in V'_f$ showing that V'_f is closed in the weak topology of M_C .

Prop. 1.5. Let V be a closed translation invariant linear subspace of $\mathcal{C}(R)$.

If β is any fixed real number and if $\mu * g(\beta) = 0$ for every $\mu \in M_C$ that satisfies $\mu * V = \{0\}$, then $g \in V$.

For $\bigvee \mu(V) = \{\bigvee \mu(h) : h \in V\}$ and $\bigvee \mu(h) = \int h(r) d\bigvee \mu(r) = \mu * h(0)$ so that $\bigvee \mu(V) = \{0\}$. Also $\bigvee \mu(T_\beta g) = \int g(r - \beta) d\bigvee \mu(r) = \mu * g(\beta) = 0$ so the given condition may be rephrased: for each $\mu \in M_C$ such that $\bigvee \mu(V) = \{0\}$, it follows that $\bigvee \mu(T_\beta g) = 0$.

The approximation principle (Edwards [1], page 129) is a consequence of the Hahn-Banach theorem and may be stated: If S is a closed linear subspace of $\mathcal{C}(R)$, a necessary and sufficient condition that $f \in S$ is that each μ annihilating S also annihilates f .

Hence $T_\beta g \in V$ and as V is translation invariant, $g \in V$.

Prop. 1.6 A necessary and sufficient condition that $u_n e_a \in V_f$ is that for each μ such that $\mu * f = 0$, it follows that $M^{(k)}(a) = 0$ for $k = 0, 1, 2, \dots, n$ where $M = \hat{\mu}$.

Proof. We note that $M^{(k)}(z) = \int_{\mu} (u_k e_z)^{(k)}$ and

$$\mu * u_n e_a(t) = \int (t-r)^n e^{a(t-r)} d\mu(r)$$

$$= \sum_{k=0}^n n C_k e^{at} t^{n-k} \int (-r)^k e^{-ar} d\mu(r)$$

$$= \sum_{k=0}^n n C_k e^{at} t^{n-k} \mu(u_k e_a) \quad \text{or}$$

$$\mu * u_n e_a = \sum_{k=0}^n n C_k u_{n-k} e_a M^{(k)}(a).$$

If $u_n e_a \in V_f$, it is necessary that $M^{(k)}(a) = 0$ for $k = 0, 1, \dots, n$ when $\mu * f = 0$. For then $\mu * u_n e_a = 0$ and as the functions $e_a, u e_a, \dots, u_n e_a$ are linearly independent, $M^{(k)}(a) = 0$ for $k = 0, 1, \dots, n$.

The converse follows from the above formula and Prop. 1.5.

§1.6 The Carleman Transform and Spectrum.

We now introduce the Carleman transform of a mean periodic function.

This transform was originally defined by Carleman for functions $f : f(t) = O(e^{\alpha|t|})$ as $F^+(z) = \int_{-\infty}^0 f(t) e^{-zt} dt$ (defined and analytic for $z > \alpha$ where $z = x + iy$).

$$F^-(z) = - \int_0^{\infty} f(t) e^{-zt} dt \quad (\text{defined and analytic for } z < -\alpha)$$

A different definition is used for mean periodic functions. This will later be shown to agree with the above when the function is of exponential growth

Let f be a given mean periodic function, and let $\mu \in M_C$, $\mu \neq 0$ and $\mu * f = 0$.

Set $f^-(t) = f(t)$ if $t < 0$ and $= 0$ if $t \geq 0$ and $g = \mu * f^-$. Write also $f^+ = f - f^-$ so that $\mu * f^+ = -g$.

When μ has segment of support $[\alpha, \beta]$ and $\mu * f = 0$,

$$\mu * f^-(t) = \int_{\alpha}^{\beta} f^-(t-r) d\mu(r) = 0 \quad \text{when } t \geq \beta.$$

Also $\mu * f^+(t) = 0$ when $t \leq \alpha$ so g has support contained in $[\alpha, \beta]$ and thus $\hat{g}(z) = \int g(t) e^{-zt} dt$ is an entire function of exponential type. Then $\hat{g}/\hat{\mu}$ is a meromorphic function for it is the quotient of two entire functions.

More over, $\hat{g}/\hat{\mu}$ is independent of the non-zero measure μ chosen so that $\mu * f = 0$. For if another non-zero measure, $\lambda \in M_C$ is chosen so that $\lambda * f = 0$ and if $\lambda * f^- = h$, then

$$\mu * h = \mu * (\lambda * f^-) = \lambda * (\mu * f^-) = \lambda * g$$

so that $\hat{\mu} \cdot \hat{h} = \hat{\lambda} \cdot \hat{g}$ and when $\hat{\mu}(z) \hat{\lambda}(z) \neq 0$,

$$\hat{g}(z)/\hat{\mu}(z) = \hat{h}(z)/\hat{\lambda}(z).$$

If the entire functions $\hat{\mu}$, $\hat{\lambda}$, $\hat{g}$ and $\hat{h}$ have zeros at the point a of orders m , n , p and q respectively, (understanding $\hat{\mu}(a) \neq 0 \Leftrightarrow m = 0$, etc.) then from $\hat{\mu} \cdot \hat{h} = \hat{\lambda} \cdot \hat{g}$, $m + q = n + p$. Thus, if $\hat{g}/\hat{\mu}$ has a pole of order $p - m (> 0)$, then so does $\hat{h}/\hat{\lambda}$.

Definition. Let f be a mean periodic function with $\mu \in M_C$, $\mu \neq 0$, $\mu * f = 0$ and let $g = \mu * f^-$. Then the Carleman transform of f is defined as

$$F(f) = \hat{g}/\hat{\mu} \quad \text{or} \quad T(f, z) = \hat{g}(z)/\hat{\mu}(z).$$

Prop. 1.7 When f is mean periodic and $f(t) = O(e^{\alpha|t|})$, the two definitions agree in the sense that

$$T(f, z) = F^+(z) \quad \text{if } x > \alpha \quad \text{and} \quad = F^-(z) \quad \text{if } x < -\alpha.$$

Proof. Let $\mu \neq 0$, $\mu * f = 0$ and $\mu * f^- = g$. For a fixed $x > \alpha$, let e_{-x}^μ be the measure with density e_{-x} relative to μ so that

$$e_{-x}^\mu(h) = \mu(e_{-x}h) \quad \text{for each } h \in \mathcal{C}(\mathbb{R})$$

and

$$e_{-x}^\mu * e_{-x} f^-(t) = \int f^-(t-r) e^{-x(t-r)} e^{-xr} d\mu(r)$$

or

$$e_{-x}^\mu * e_{-x} f = e_{-x} g.$$

Now $(e_{-x}^\mu)^\wedge(iy) = \int e^{-iyt} e^{-xt} d\mu(t) = \hat{\mu}(z)$,

$$(e_{-x} g)^\wedge(iy) = \hat{g}(z) \quad \text{and so}$$

$$\hat{\mu}(z) \int f^-(t) e^{-xt} e^{-iyt} dt = \hat{g}(z) \quad \text{or}$$

$$\hat{\mu}(z) F^+(z) = \hat{\mu}(z) T(f, z).$$

Hence, for $z = x + iy$ and $x > \alpha$, $F^+(z) = T(f, z)$.

Similarly, when $x < -\alpha$, $F^-(z) = T(f, z)$ so $T(f, z)$ is analytic for $|x| > \alpha$.

Definition The spectral set, $[\Lambda_f]$, of a mean periodic function f is the set of all poles of the Carleman transform.

The spectrum, Λ_f , of a mean periodic function f is the set of pairs (a, p) where $a \in [\Lambda_f]$ and p is the order of the pole of the transform $T(f, z)$ at $z = a$.

Note that the spectral set of a mean periodic function f is contained in the set of zeros of $\hat{\mu}$ whenever $\mu \in M_C$, $\mu \neq 0$ and $\mu * f = 0$.

Prop. 1.8 Let (a) f be mean periodic and $\mu \in M_C$, $\mu \neq 0$ and $\mu * f = 0$
 (b) μ have segment of support $[\alpha, \beta]$ and $\hat{\mu}(a) = 0$.
 (c) $\mu * f^- = g$ and $\hat{g}(a) = 0$. (This occurs when $a \notin [\Lambda_f]$.)

Then the function $\lambda : t \rightarrow \int_{\alpha}^t e^{a(t-r)} d\mu(r)$ is such that

(i) λ has support contained in $[\alpha, \beta]$.

(ii) $\hat{\lambda}(z) = \hat{\mu}(z)/(z-a)$.

(iii) $\lambda * f = 0$.

Proof. (i) holds since $\lambda(\beta) = e^{a\beta} \hat{\mu}(a) = 0$ and μ has support in $[\alpha, \beta]$.

(ii) For any $x \in \mathcal{C}_C^\infty(\mathbb{R})$

$$e_{-a} \lambda(x') = \int_{t=\alpha}^{\beta} x'(t) \int_{r=\alpha}^t e^{-ar} d\mu(r) dt$$

$$= \int_{r=\alpha}^{\beta} e^{-ar} d\mu(r) \int_{t=r}^{\beta} x'(t) dt$$

$$= \int_{\alpha}^{\beta} e^{-ar} (x(\beta) - x(r)) d\mu(r)$$

$$= x(\beta) \hat{\mu}(a) - e_{-a} \mu(x)$$

$$= 0 - e_{-a} \mu(x)$$

showing $e_{-a} \mu = D e_a \lambda$. As

$$D e_{-a} \lambda = e_{-a}' \lambda + e_{-a} D \lambda = e_{-a} \mu,$$

$$-a \lambda + D \lambda = \mu,$$

$$-a \lambda(e_{-z}) - \lambda(e_{-z}') = \mu(e_{-z}) \text{ so}$$

$$\lambda(z) = \hat{\mu}(z)/(z-a) \quad -a\hat{\lambda}(z) + z\hat{\lambda}(z) = \hat{\mu}(z)$$

and this is equation (ii).

(iii) Let $\lambda * f = k$. Then $e_{-a} \lambda * e_{-a} f = e_{-a} k$, $D(e_{-a} k) = D(e_{-a} \lambda) * e_{-a} f = e_{-a} \mu * e_{-a} f = e_{-a} \mu * f = 0$ showing $k = ce_a$ where c is constant. Now $\lambda * f(\beta) = \int_{\alpha}^{\beta} f(\beta-t) dt \int_{\alpha}^t e^{a(t-r)} d\mu(r)$.

Set $r = r$, $s = \beta + r - t$ to get, using $g = -\mu * f^+$

$$\lambda * f(\beta) = - \int_{s=\alpha}^{\beta} ds \int_{r=\alpha}^s f(s-r) e^{a(\beta-s)} d\mu(r)$$

$$= -e^{a\beta} \int_{s=\alpha}^{\beta} e^{-as} ds \int_R f^+(s-r) d\mu(r)$$

$$= e^{a\beta} \int_R g(s) e^{-as} ds$$

$$= e^{a\beta} \hat{g}(a) \quad \text{so that}$$

$$\lambda * f = \hat{g}(a) e_a. \quad \text{As } \hat{g}(a) = 0, \quad \lambda * f = 0.$$

Prop. 1.9 If f is mean periodic, $u_n e_a \in V_f \iff (a, m) \in \Lambda_f$ for

some $m > n$.

Proof. If $(a, m) \in \Lambda_f$ and $m > n$, $T(f, z)$ has at $z = a$ a pole of order exceeding n . Then, for each $\mu \in M_C$ such that $\mu \neq 0$ and $\mu * f = 0$, $\hat{\mu}(z)$ has a zero of order at least $n+1$ at $z = a$ so by Prop. 1.6, $u_n e_a \in V_f$.

Conversely, if $u_n e_a \in V_f$, for any μ such that $\mu \neq 0$ and $\mu * f = 0$, $\hat{\mu}(z)$ has a zero of order at least $n+1$ at $z = a$. In the case when $\hat{g}(a) \neq 0$, there is nothing more to prove. However, if $\hat{g}(a) = 0$, by Prop. 1.8, there exists a measure λ such that $\lambda \neq 0$, $\lambda * f = 0$ and

$$\hat{\lambda}(z) = \hat{\mu}(z)/(z-a).$$

Now let $h = \lambda * f^-$ so that h has compact support and $\mu * h = \lambda * g$,

$$\hat{h}(z)\hat{\mu}(z) = \hat{g}(z)\hat{\mu}(z)/(z-a)$$

and as all functions are analytic and $\hat{\mu} \neq 0$, $\hat{h}(z) = \hat{g}(z)/(z-a)$.

If $h(a) = 0$, the order of the zero of $h(z)$ at $z = 0$ is one less than that of $g(z)$ at $z = a$. We need only continue this process a finite number of times until there is a $v \in M_C$ such that $v \neq 0$, $v * f = 0$, $v * f^- = k$ and $\hat{k}(a) \neq 0$. Then $\hat{k}(z)/\hat{v}(z)$ has a pole of order at least $n + 1$ so that $(a, m) \in \Lambda_f$ for some $m > n$.

§1.7 An approximation theorem

The following proposition was used by Kahane to show that a mean periodic function is the limit of a certain sequence of exponential polynomials. The proof given here is due to Koosis [1].

Prop 1.10 If f is mean periodic and its spectral set is empty, then $f = 0$.

Proof. If $\mu \in M_C$, $\mu \neq 0$ and $\mu * f = 0$, we can, by regularization and translation if necessary, assume that it has a density ψ relative to Lebesgue measure where $\psi \in C^2(\mathbb{R})$ and has support contained in $[0, L]$ for some $L > 0$. As in §1.3,

$$\hat{\psi} : z \rightarrow A e^{cz} z^k \prod_{n=1}^{\infty} (1 - z/a_n)^{p_n} \exp(z p_n / a_n)$$

so that

$$\hat{\psi}' : z \rightarrow c\psi(z) + k\psi(z)/z + \sum_{n=1}^{\infty} \hat{\psi}(z) p_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right) \quad (I)$$

It does not affect the proof as a whole to suppose that $\psi(0) = 0$,

for if not, $k = 0$. Since $\hat{\mu}(a_n) = 0$ and the spectral set of f is empty, $\hat{g}(a_n)$ must vanish. So Proposition 1.8 applies to show that the functions

$$\psi_0 : t \rightarrow \int_0^t \psi(r)dr \quad \text{and} \quad \psi_n : t \rightarrow \int_0^t \exp(a_n(t-r))\psi(r)dr$$

have support in $[0, L]$, have Fourier-Laplace transforms $\hat{\psi}(z)/z$ and $\hat{\psi}(z)/(z-a_n)$ respectively, and $\psi_0 * f = 0$, $\psi_n * f = 0$.

Now consider the series

$$\phi = c\psi + k\psi_0 + \sum_{n=1}^{\infty} \xi_n \quad (\text{II})$$

where $\xi_n = p_n\psi_n + p_n\psi/a_n$. By twice integrating by parts using $\psi(0) = \psi'(0) = \psi(L) = \psi'(L) = 0$,

$$\psi_n(t) = \frac{1}{a_n} [-\psi(t) + \int_0^t \psi'(r)\exp(a_n(t-r))dr]$$

and

$$\xi_n(t) = p_n \left[\int_0^t \psi''(r) \{ \exp(a_n(t-r)) - 1 \} dr \right] / a_n^2.$$

The last integral is bounded with respect to $t \in \mathbb{R}$ and $n \in \mathbb{N}$. To see this, note that ψ'' has support contained in $[0, L]$, put $|\psi''| < M$, $a_n = \alpha + i\beta$ and

$$\eta : t \rightarrow \int_0^t \psi''(r)\exp(a_n(t-r))dr.$$

As $\hat{\psi}(a_n) = 0$, $\eta(L) = 0$ and so η has support contained in $[0, L]$. When $\alpha \leq 1$, $|\eta| < M \int_0^t e^{\alpha(t-r)}dr < LMe^L$. When $\alpha \geq 1$, put $\theta = e_{-i\beta}\psi''$ and

$$\rho : t \rightarrow \int_0^t e^{\alpha(t-r)}\theta(r)dr \quad \text{with} \quad \eta = e_{i\beta}\rho.$$

Then θ and ρ have supports contained in $[0, L]$, $|\theta| < M$, $\rho' = \alpha\rho + \theta$

and so the real and imaginary parts of ρ have extreme values when the real and imaginary parts of $\alpha\rho + \theta$ are zero. Hence $|\rho| < 2M|\alpha|$ and so $|\eta| < 2M$ when $\alpha \geq 1$.

This shows that $|\xi_n| \leq Ap_n/|a_n|^2$ where A is a constant and as $\sum p_n/|a_n|^2 < \infty$, the series (II) is uniformly convergent. Since all of its terms have supports in $[0, L]$, the Fourier-Laplace transform of (II) is uniformly convergent on all compact subsets of the plane and equal to the series (I). Then $\hat{\psi}' = \hat{\phi}$ and as $\hat{\psi}' = (\hat{u}\hat{\psi})$ and $\phi, u\psi$ are continuous integrable functions, $\phi = u\psi$.

Since (II) is uniformly convergent and all its terms have supports in $[0, L]$, $u\psi * f = c\psi * f + k\psi_0 * f + \sum_{n=1}^{\infty} \xi_n * f$
 $= 0 + \sum p_n(\psi_n * f + \psi * f/a_n)$
 $= 0.$

Hence $h\psi * f = 0$ for any polynomial h . Let γ be a point such that $\psi(\gamma) \neq 0$ and $\{h_k\}$ be a sequence of polynomials that are non-negative on $[0, L]$, converge to zero on $[0, L]$ outside any neighbourhood of the point γ and are such that $\int_0^L h_k(r)dr = 1/\psi(\gamma)$. Then the sequence $h_k\psi$ converges weakly in M_C to δ_γ as $k \rightarrow \infty$ and so

$$T_\gamma f = \delta_\gamma * f = 0 \text{ or } f = 0.$$

Theorem 1.11. A mean periodic function, f , is the limit in $\mathcal{C}(R)$ of a sequence of exponential polynomials belonging to V_f .

Proof. Let V_0 denote the closed translation invariant linear subspace of $\mathcal{C}(R)$ spanned by the exponential monomials in V_f . It is required to show that $f \in V_0$.

Let μ be a non-zero measure satisfying $\mu * f = 0$. If λ is a non-zero measure and $h = \lambda * f$, then h is mean periodic for $\mu * h = \lambda * \mu * f = 0$. When λ has segment of support $[\alpha, \beta]$,

$$\lambda * f^-(t) = \lambda * f(t) \quad \text{if } t \geq \beta \quad \text{and} \quad = 0 \quad \text{if } t \leq \alpha,$$

so that $h^- = \lambda * f^- + k$ when k has compact support. Thus, $\hat{k}$ is an entire function and if $F(h)$ and $F(f)$ are the Carleman transforms of h and f , from

$$\mu * h^- = \lambda * \mu * f^- + \mu * k,$$

we have

$$\hat{\mu} \cdot F(h) = \hat{\mu} \cdot \hat{\lambda} \cdot F(f) + \hat{\mu} \cdot \hat{k}$$

whence $F(h) = \hat{\lambda} \cdot F(f) + \hat{k}$ as $\hat{\mu} \neq 0$.

Now, if λ is chosen so that $\lambda * V_0 = \{0\}$, then, for each exponential monomial $u_n e_a \in V_f$ (with $(a, n+1) \in \Lambda_f$, the spectrum of f), it follows that $\hat{\lambda}^{(k)}(a) = 0$ for $k = 0, 1, \dots, n$. From the definition of Λ_f as the set of poles of $F(f)$, counted according to their multiplicity, $\hat{\lambda}(f)$ is an entire function. Thus, $F(h)$ is an entire function, the spectral set of h is empty, and so, by Prop. 1.10, $h = 0$.

As $\lambda * f = 0$ whenever $\lambda * V_0 = \{0\}$, Prop. 1.5 shows that $f \in V_0$.

Corollary (i) If V_0 is defined as above, then $V_0 = V_f$.

(ii) A necessary and sufficient condition that a function $f \in \mathcal{C}(R)$, be a mean periodic is that it be the locally uniform limit of a sequence of exponential polynomials, each of which is orthogonal to a fixed non-zero measure with compact support.

A refinement of the above approximation theorem was proved by Schwartz [1] (Théorème 11, page 902) who showed that, if f is a mean periodic function in $\mathcal{C}(R)$ with a formal series

$$f \sim \sum_k \sum_{q=0}^{p_k-1} A(k,q) u_q e_{a_k} \quad (\Lambda_f = (a_k, p_k)),$$

then this series is characteristic of f and is Abel summable in $\mathcal{C}(R)$ to it when the terms are suitably grouped. Thus, the function f is uniquely determined when the coefficients, $A(k,q)$, $q = 0, 1, \dots, p_k-1$, $k \in \mathbb{N}$ are given and the exponential monomials belonging to V_f form a free system. However, in the remainder of this thesis, we will only have use for the simpler approximation theorem proved above.

§1.8 The Mean Period.

For a closed interval, I , let $\mathcal{C}(I)$ denote the Banach space of all continuous complex-valued functions on I with the norm $\|f\| = \sup\{|f(t)| : t \in I\}$. Let $B_f(I)$ be the closed linear subspace of $\mathcal{C}(I)$ generated by the restrictions to I of the exponential monomials belonging to V_f . Whether or not $B_f(I)$ is total in $\mathcal{C}(I)$ is a translation invariant property and depends on the length, ℓ , of I and the "mean period" of f .

Definition. The mean period, L_f , of a mean periodic function f is the infimum of the lengths of the segments of supports of the measures $\mu \in M_C$ such that $\mu \neq 0$ and $\mu * f = 0$.

Prop. 1.12. (i) mean periodic functions with the same spectrum have the same mean period.

$$(ii) \quad L_f = \inf\{\ell : B_f(I) \neq \emptyset(I)\} = \sup\{\ell : B_f(I) = \emptyset(I)\}$$

where ℓ is the length of the closed interval, I .

(iii) If f is zero on an interval of length exceeding its mean period, then $f = 0$.

$$(iv) \quad \Lambda_f = \{(a_n, 1) : n = 1, 2, \dots\} \text{ and } \sum 1/|a_n| < \infty \Rightarrow L_f = 0.$$

(v) If f is a continuous periodic function of period τ and if f has no non-zero Fourier coefficients, then $L_f = \tau$.

Proof. (i) is a consequence of Theorem 1.11, corollary (ii).

(ii) If $B_f(I) \neq \emptyset(I)$, by use of the Hahn-Banach theorem, there exists a non-zero measure μ with support contained in I such that $\mu(B_f(I)) = \{0\}$. Then $\mu * V_f = \{0\}$ so $L_f < \ell$, the length of I .

If $L_f < \ell$, there exists a $\mu \in M_C$ such that $\mu * f = 0$, μ has support in I , $\mu \neq 0$ and $\mu(B_f(I)) = 0$. Thus $B_f(I) \neq B(I)$.

(iii) If f is zero in $[\alpha, \beta]$, then $T_{-\alpha}f$ is zero on $[0, \beta - \alpha]$.

If $L_f < \beta - \alpha$, there exists a measure with support $[0, \gamma]$ where $L_f < \gamma < \beta - \alpha$. Then $\mu * (T_{-\alpha}f)^-$, having support in $[0, \gamma]$ is zero and so the Carleman transform of f is zero. Thus, f is zero.

(iv) For proof, see Kahane [2], page 34.

(v) As f is defined, V_f contains the set of exponentials $\{e_{2\pi i n/\tau} : n \in \mathbb{Z}\}$. Hence $B_f(I) = \emptyset(I)$ if and only if the length of the interval I does not exceed τ , and so, by section (ii) of this proposition, f has mean period τ .

Chapter Two

In this chapter, further properties of mean periodic functions in $\mathcal{C}(R)$ are outlined. Let MP denote the set of all mean periodic functions in $\mathcal{C}(R)$ and MP_0 the set of all mean periodic functions with mean period zero. Let MQ denote the set of all exponential polynomials.

Prop. 2.1 MQ , MP_0 and MP are linear subspaces of $\mathcal{C}(R)$ and each is dense in $\mathcal{C}(R)$. Also

$$MQ \subset MP_0 \subset MP \subset \mathcal{C}(R)$$

and the inclusion is proper in each case.

Proof. For if f and g are mean periodic with $\mu * f = 0$ and $\lambda * g = 0$, where μ and λ are nonzero elements of M_C , then $\mu * \lambda$ is a nonzero element of M_C and

$$\mu * \lambda * (af + bg) = a\lambda * (\mu * f) + b\mu * (\lambda * g) = 0,$$

showing that MP is a linear subspace of $\mathcal{C}(R)$.

If, as well, f and g have mean period zero, by definition, for any $\varepsilon > 0$, there will exist measures μ and λ whose supports are contained in intervals of length $\varepsilon/2$. Then $\lambda * \mu$ is a measure whose support is contained in an interval of length ε , $\mu * \lambda * (af + bg) = 0$ and by the definition of mean period and the fact that ε is an arbitrary positive number, $af + bg$ has mean period zero whence MP_0 is a linear subspace of MP .

It is obvious, on recalling the definition of an exponential polynomial as a finite linear combination of exponential monomials, that MQ is a linear subspace of MP . That any exponential polynomial has mean period zero, follows from Prop. 1.12; or, more directly we show that any exponential monomial $u_n e_a : t \rightarrow t^n e^{at}$ has mean period zero as follows.

The Legendre polynomials, $\phi_0 = 1$ and

$$\phi_n : t \rightarrow \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n \quad \text{for } n = 1, 2, \dots$$

form an orthogonal set on $[-1, 1]$. Thus, if $m \neq n$ and $\delta > 0$,

$$\int_{-\delta}^{\delta} \phi_n(t/\delta) \phi_m(t/\delta) dt = 0, \quad \text{and so, if } P_n \text{ is a polynomial of degree } n < m,$$

$$\int_{-\delta}^{\delta} P_n(t) \phi_m(t/\delta) dt = 0.$$

Hence

$$\int_{-\delta}^{\delta} (x-t)^n \phi_m(t/\delta) dt = 0$$

for $m > n$ and all $x \in \mathbb{R}$. If μ denotes the measure defined for each $f \in \mathcal{C}(\mathbb{R})$ by

$$\mu(f) = \int_{-\delta}^{\delta} f(t) \phi_m(t/\delta) e^{at} dt,$$

then $\mu * (u_n e_a) = 0$ (when $m > n$). As δ is an arbitrary positive number and μ has support $[-\delta, \delta]$, $u_n e_a$ has mean period zero and so $MQ \subset MP_0$.

Let $f = \sum_{n \in \mathbb{Z}} c_n e_{in}$ where $\sum |c_n| < \infty$ and $c_n \neq 0 \iff n = \pm m^2$ for some $m \in \mathbb{N}$. Also, let $g = \sum_{n \in \mathbb{Z}} d_n e_{in}$ where $\sum |d_n| < \infty$ and $d_n \neq 0$ for all $n \in \mathbb{Z}$. Then f and g are continuous periodic functions and so are mean periodic. By Prop. 1.12, f has mean period zero and g has mean period 2π . Since f is not an exponential polynomial, and since

there exist non mean-periodic functions, the inclusions

$MQ \subset MP_0 \subset MP \subset \mathcal{C}(R)$ are all proper.

The next proposition gives properties of mean periodicity.

Prop. 2.2 For $f \in \mathcal{C}(R)$,

- (i) $f \in MQ \Leftrightarrow e_a f \in MQ \Leftrightarrow \int f \in MQ \Leftrightarrow f' \in MQ$;
- (ii) $f \in MP \Leftrightarrow e_a f \in MP$;
- (iii) $f \in MP \Leftrightarrow \int f \in MP$;
- (iv) $f \in MP$ and $f' \in \mathcal{C}(R) \Leftrightarrow f' \in MP$;
- (v) each of $e_a f$, $\int f$, f' (when f is continuously differentiable) has the same mean period as f .

Proof. The first statement is easily verified using elementary integral and differential calculus. For the second, since $f = e_{-a} e_a f$, it suffices to show that $e_a f \in MP$ when $f \in MP$. In this case, there exists a non zero measure $\mu \in M_C$ such that $\mu * f = 0$. Then $e_a \mu * e_a f = e_a (\mu * f) = 0$, where $e_a \mu$ is the measure with density e_a relative to μ , and so $e_a f \in MP$. To show that $e_a f$ has the same mean period as f , let L be the mean period of f so that, for any $\varepsilon > 0$, there exists a measure μ whose support is contained in an interval of length $L + \varepsilon$ and $\mu * f = 0$. As μe_a has the same support as μ , the mean period of $e_a f$ does not exceed $L + \varepsilon$ for any $\varepsilon > 0$ so that the mean period of $e_a f$ does not exceed L . Since $f = e_{-a} e_a f$, the mean period of $e_a f$ is not less than L and so is equal to L .

If $f \in MP$ and $g : t \rightarrow \int_a^t f(r) dr$ ($a \in R$), then g is continuously

differentiable and we may write $D(\mu * g) = \mu * Dg = \mu * f = 0$ showing that $\mu * g$ is constant. As μ is nonzero, $\hat{\mu}$ has at least one non-zero derivative at the origin. If n is the smallest integer for which $D^{(n)} \hat{\mu}(0)$ is non-zero, then $\mu * u_n$ is constant and hence $\mu * (g - cu_n) = 0$ for some constant c . As $cu_n \in MP$, $\mu \in M_C$ and $\mu \neq 0$, $g \in MP$.

When f is mean periodic and continuously differentiable, $f' \in V_f$ (Prop. 1.3(iii)), V_f is a proper subset of $\mathcal{C}(R)$ and so f' is mean periodic. Alternatively, if $\mu * f = 0$, then $\mu * Df = 0$; and, since $f' \in \mathcal{C}(R)$, the distributional derivative Df is equal to the pointwise derivative f' and thus $f' \in MP$.

These two considerations taken together give statements (iii) - (iv).

Turning to the rest of (v); f is assumed to have mean period L so for any $\varepsilon > 0$, there exists a non zero measure μ with support contained in an interval of length $L + \varepsilon$ and $\mu * g = d$ say, a constant.

Now a measure λ can be selected with support contained in an interval of length ε such that $\lambda * d = 0$. Then $(\lambda * \mu) * g = 0$ and as $\lambda * \mu$ has support contained in an interval of $L + 2\varepsilon$, it follows that the mean period of g does not exceed $L + 2\varepsilon$ whence the mean period of $g \leq L$.

Since $f \in V_g$, the mean period of $g \geq L$ showing that $g = \int f$ has mean period L . This also shows that if $f' \in MP$, the mean period of f' is equal to L .

Remark: We note that statements (ii), (iii) and (iv) can be proved using the fact that any mean periodic function f is the locally uniform limit

of a sequence of exponential polynomials f_n , each of which satisfies $\mu * f_n = 0$ for some non-zero $\mu \in M_c$. Then $e_a f_n \rightarrow e_a f$ as $n \rightarrow \infty$, locally uniformly, and as $e_a \mu * e_a f_n = 0$ for each n , $e_a f$ is mean periodic. For the integral and derivative (when it is continuous) of a mean periodic function being mean periodic, see Kahane [1], page 47.

In the following two propositions, more properties of mean periodic functions are given. These properties are not shared by almost periodic functions and they characterize the spaces MQ , MP_0 and MQ even further.

The truncated convolution product of two continuous functions, f and g , is a continuous function defined by

$$f \circledast g : t \rightarrow \int_0^t f(t-r)g(r)dr \quad \text{for all } t \in \mathbb{R}.$$

For details of this product which is associative and commutative, see Erdélyi. With the operations $+$ and $\circledast$, $\mathcal{C}(\mathbb{R})$ is an algebra. This algebra has no units or identity elements w.r.t. $\circledast$, but has divisors of zero; e.g. if u^+ and u^- are the restrictions of the identity function $u : t \rightarrow t$ to the non-negative and negative real axis respectively, then $u^+ \neq 0$, $u^- \neq 0$ and $u^+ \circledast u^- = 0$.

Prop. 2.3 The truncated convolution product of two mean periodic functions is mean periodic.

Proof. This is equivalent to showing $f \circledast f$ is mean periodic when f is, for MP is a linear space and

$$(f+g) \circledast (f+g) = f \circledast f + 2f \circledast g + g \circledast g.$$

As in Chapter 1, let

$$f^-(t) = f(t) \quad \text{if } t \leq 0, = 0 \quad \text{if } t > 0$$

and

$$f^+ = f - f^-.$$

Let $\mu \in M_c$, $\mu \neq 0$ and $\mu * f = 0$. Set $\mu * f^- = v$ so that

$$\mu * f^+ = \mu * f - \mu * f^- = v \quad \text{and recall}$$

that v has compact support. Now

$$\begin{aligned} f^+ \circledast f^+ : t \rightarrow \int_{-\infty}^{\infty} f^+(t-r)f^+(r)dr &= \int_0^t f(t-r)f(r)dr \quad \text{if } t \geq 0 \\ &= 0 \quad \text{if } t \leq 0 \end{aligned}$$

$$\text{i.e.} \quad f^+ * f^+ = (f \circledast f)^+$$

$$\text{Also} \quad f^- * f^- : t \rightarrow \int_{-\infty}^{\infty} f^-(t-r)f^-(r)dr = \begin{cases} \int_t^0 f(t-r)f(r)dr & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 0 \end{cases}$$

$$\text{i.e.} \quad f^- * f^- = -(f \circledast f)^-.$$

$$\text{Thus } f \circledast f = (f \circledast f)^+ + (f \circledast f)^- = f^+ * f^+ - f^- * f^-$$

$$\text{and so } \mu * \mu * (f \circledast f) = \mu * \mu * f^+ * f^+ - \mu * \mu * f^- * f^-$$

$$= \mu * f^+ * \mu * f^+ - \mu * f^- * \mu * f^-$$

$$= (-v) * (-v) - v * v$$

$$= 0.$$

As $\mu * \mu$ is a non-zero measure with compact support, $f \circledast f$ is mean periodic.

This proposition may also be derived by using Theorem 1.11 and a proof is sketched here. As usual u_n and e_a denote the functions $u_n : t \rightarrow t^n$ and $e_a : t \rightarrow e^{at}$.

Let f and g be two mean periodic functions with $\mu * f = 0$ and $\lambda * g = 0$ where $\mu, \lambda \neq 0, \in M_c$. Let g_m be a sequence of exponential polynomials that converge, locally uniformly to g and such that $\lambda * g_m = 0$ with $g_m \in V_g$ for each $m \in \mathbb{N}$. Now, for any complex number a and any non negative integer n , $f \circledast u_n e_a = e_a h$ where

$$h : t \rightarrow \int_0^t (t-r)^n e^{-ar} f(r) dr$$

so that $\mu * (f \circledast u_n e_a) = \mu * e_a h = e_a (h * e_{-a} \mu)$. As

$$D^{n+1}(h * e_{-a} \mu) = e_{-a} \mu * D^{n+1} h$$

$$= n! e_{-a} \mu * e_{-a} f$$

$$= n! e_{-a} (\mu * f) = 0,$$

$h * e_{-a} \mu = P_n$, a polynomial of degree $\leq n$ and $\mu * (f \circledast u_n e_a) = e_a P_n$.

Hence if

$$g_m = \sum_{k=1}^{\infty} \sum_{q \leq p_k} A(m, k, q) u_q e_{a_k}$$

then

$$\mu * (f \circledast g_m) = \sum_{k=1}^{\infty} \sum_{q \leq p_k} B(m, k, q) u_q e_{a_k}$$

and so $\lambda * \mu * (f \circledast g_m) = 0$. As $g_m \rightarrow g$ in $\mathcal{C}(R)$, $f \circledast g_m \rightarrow f \circledast g$

$\mathcal{C}(R)$ as $m \rightarrow \infty$ so that $\lambda * \mu * (f \circledast g) = 0$. As $\lambda * \mu$ is a non zero measure with compact support, $f \circledast g$ is mean periodic.

Corollary. Each of MQ , MP_0 and MP is a subalgebra $(+, \circledast)$ of $\mathcal{C}(R)$.

Proof. It can be shown that

$$u_m e_a \circledast u_n e_a = A_{m,n} u_{m+n+1} e_a$$

and

$$u_m e_a \circledast u_n e_b = P_m e_a + Q_n e_b$$

where P_m and Q_n are polynomials of degree m and n and $A_{m,n}$ is a number. Hence the truncated convolution product of two exponential polynomials is an exponential polynomial. This means MQ is closed under $\circledast$. In view of Prop. 2.1, this shows that MQ is a subalgebra of $\mathcal{C}(R)$.

It has been shown that each of MP_0 and MP is a linear subspace of $\mathcal{C}(R)$. That MP is a subalgebra follows from the preceding proposition. If f and $g \in MP_0$, an argument similar to that used in Prop. 2.1 shows that $f \circledast g$ has mean period zero. Hence MP_0 is a subalgebra of $\mathcal{C}(R)$.

Prop. 2.4 If f is mean periodic and g is an exponential polynomial, then fg is mean periodic.

Proof. By use of propositions 2.1 and 2.2, it is only necessary to show that $u_n f$ is mean periodic when f is. Thus the proposition is proved by showing $uf : t \rightarrow tf(t)$ is mean periodic when f is.

Let $\mu \in M_c$, $\mu \neq 0$ and $\mu * f = 0$. Then

$$\mu * uf : t \rightarrow \int (t-r)f(t-r)d\mu(r)$$

so

$$\mu * uf = u(\mu * f) - \lambda * f = -\lambda * f$$

where $\lambda \in M_c$ is defined by $\lambda(g) = \mu(ug)$ for each $g \in \mathcal{C}(R)$, so

$\mu * \mu * uf = -\mu * \lambda * f = -\lambda * \mu * f = 0$. Since $\mu * \mu$ is a non-zero measure of compact support, uf is mean periodic.

Corollary When g is an exponential polynomial

$$(i) \quad f \in MQ \Rightarrow fg \in MQ$$

$$(ii) \quad f \in MP_0 \Rightarrow fg \in MP_0.$$

Proof The first statement follows directly from the definition of an exponential polynomial. For the second, Prop. 2.2 shows that it suffices to show that uf has mean period zero when f has that property. If f has mean period zero, then for any $\varepsilon > 0$, there exists a $\mu \neq 0$, $\mu \in M_c$ with support contained in an interval of length ε satisfying $\mu * f = 0$. Then, as in the proof of Prop. 2.4, $\mu * \mu * uf = 0$. Since $\mu * \mu$ is a non zero measure with support contained in an interval of arbitrarily small length, uf has mean period zero.

Prop. 2.5 The product of two mean periodic functions need not be mean periodic.

Proof Let f and g be two continuous periodic functions, both with no non-zero Fourier coefficients. Let f have period $2\pi/\alpha$ and g have period $2\pi/\beta$. When α/β is rational, fg is a continuous periodic function with period equal to the L.C.M. of α and β . When α/β is irrational, we show that fg is not mean periodic.

Let μ be any measure of compact support K satisfying $\mu * fg = 0$.

Then the expression

As $\lim_{T \rightarrow \infty} E = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\gamma t} \mu * fg(t) dt$ is zero.

$$\text{Set } \sigma_N^f = \sum_{|p| \leq N} a(p, N) e^{ip\alpha}$$

$$\text{and } \sigma_N^g = \sum_{|q| \leq N} b(q, N) e^{iq\beta}$$

where

$$a(p, N) = (1 - \frac{|p|}{N+1}) \hat{f}(p), \quad b(q, N) = (1 - \frac{|q|}{N+1}) \hat{g}(q)$$

with

$$a(p, N) \rightarrow \hat{f}(p), \quad b(q, N) \rightarrow \hat{g}(q) \quad \text{as } N \rightarrow \infty.$$

Then $\sigma_N^f \rightarrow f$, $\sigma_N^g \rightarrow g$, $h_N = \sigma_N^f \sigma_N^g \rightarrow fg$ and $\mu * h_N \rightarrow \mu * fg$ uniformly as $N \rightarrow \infty$ so that

$$E = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\gamma t} \lim_{N \rightarrow \infty} \mu * h_N(t) dt$$

$$= \lim_{T \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\gamma t} \int_K h_N(t-r) d\mu(r) dt$$

$$= \lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} \int_K e^{i\gamma r} d\mu(r) \frac{1}{T} \int_{-r}^{T-r} e^{i\gamma s} h_N(s) ds \quad (s = t-r)$$

$$= \lim_{N \rightarrow \infty} \sum_{|p| \leq N} \sum_{|q| \leq N} a(p, N) b(q, N) \lim_{T \rightarrow \infty} \int_K e^{i\gamma r} H(T, \theta, r) d\mu(r)$$

where

$$\theta = \gamma + p\alpha + q\beta$$

and

$$H(T, \theta, r) = \frac{1}{T} \int_{-r}^{T-r} e^{i\theta s} ds = \begin{cases} 1 & \text{if } \theta = 0 \\ e^{-i\theta r} (e^{i\theta T} - 1) / i\theta T & \text{if } \theta \neq 0. \end{cases}$$

As $\lim_{T \rightarrow \infty} H(T, \theta, r) = 1$ if $\theta = 0$, 0 if $\theta \neq 0$, and the limit exists uniformly in r , if $\gamma = -\alpha m - \beta n$ where m and n are any fixed integers, the expression E is equal to

$$\begin{aligned} & \lim_{N \rightarrow \infty} \sum_{|p| \leq N} \sum_{|q| \leq N} a(p, N) b(q, N) \int_K e^{i\gamma r} \lim_{T \rightarrow \infty} H(T, \theta, r) d\mu(r) \\ &= \lim_{N \rightarrow \infty} \sum \sum a(p, N) b(q, N) \delta_{mp} \delta_{nq} \int_K e^{i\gamma r} d\mu(r) \\ &= \hat{f}(m) \hat{g}(n) \hat{\mu}(-i\gamma) \end{aligned}$$

showing that $\hat{\mu}(-i\gamma) = 0$ when $\gamma = -\alpha m - \beta n$ for $m, n \in \mathbb{Z}$.

Using the fact that, for any irrational number ξ there exists an infinity of integers p, q satisfying $|\frac{p}{q} - \xi| < k/q^2$, it can be shown that the set of points $\{\alpha m + \beta n \mid m, n \in \mathbb{Z}\}$ is dense on the real line. Thus, if $\mu * (fg) = 0$, then $\hat{\mu}$ vanishes on a set that is dense on the imaginary axis. As the Fourier-Laplace transform of a measure with compact support is an entire function, it follows that $\hat{\mu}$ is zero and so μ is zero.

Hence fg is not a mean periodic function.

Note. Since the product of two almost periodic functions is almost periodic, another difference between almost periodicity and mean periodicity is exhibited.

Chapter Three

DIFFERENTIAL AND DIFFERENTIAL DIFFERENCE EQUATIONS.

We begin this chapter by considering a simple differential equation and its connection with mean periodic functions in $\mathcal{C}(R)$.

Prop. 3.1. Let $f \in \mathcal{C}(R)$ and a be a constant. The equation,

$$\frac{dx}{dt} + ax = f(t) \quad (I)$$

has a mean periodic forcing term, f , if any solution is mean periodic.

Conversely, if f is mean periodic, then all solutions are mean periodic.

The solution and the forcing term have the same mean period.

Proof. (I) may be written as $T*x = f$ where T is a distribution of order one with support at $\{0\}$ and equal to $D\delta + a\delta$. If a solution, x , is mean periodic, and if the non-zero measure $\lambda \in M_C$ satisfies $\lambda*x = 0$, then

$$\lambda*f = \lambda * (T*x) = T * (\lambda*x) = 0,$$

showing f is mean periodic.

If f is mean periodic, and if the non-zero measure $\mu \in M_C$ satisfies $\mu*f = 0$, then $\mu*T*x = 0$. Let ρ be any non-zero $\mathcal{C}_C^\infty(R)$ function and $\psi = \rho*\mu*T$. Then $\psi \neq 0$, $\psi \in \mathcal{C}_C^\infty(R)$ and $\psi*x = 0$, so that x is mean periodic.

For any $g \in \mathcal{C}(R)$, L_g denotes the mean period of g (with $0 \leq L_g < \infty$ for $g \in MP$ and $L_g = \infty$ if $g \notin MP$). Since $\lambda \in M_C, \lambda \neq 0$ and $\lambda*x = 0 \Rightarrow \lambda*f = 0$, it follows that $L_x \geq L_f$. If ϵ is any positive number, a non-zero measure, μ , can be chosen with support in an interval of

length $L_f + \varepsilon$ and satisfying $\mu * f = 0$. Also ρ can be chosen with support in an interval of length ε and as T has point support, $\psi = \rho * \mu * T$ has support contained on an interval of length $L_f + 2\varepsilon$. Since $\psi * x = 0$, $L_x \leq L_f + 2\varepsilon$; and since ε is an arbitrary positive number, $L_x \leq L_f$ and so $L_x = L_f$.

Remark. An alternative way of proving this proposition is outlined.

The solution to (I) with $x(\alpha) = c$ is

$$x(t) = ce^{-a(t-\alpha)} + \int_{\alpha}^t e^{a(r-t)} f(r) dr.$$

When f is mean periodic, the repeated use of Prop. 2.2 shows that x is mean periodic and $L_x = L_f$. As any solution x of (I) is continuously differentiable, $x' \in V_x$, so that $f \in V_x$; hence $x \in MP \Rightarrow f \in MP$.

A generalization of (I) is

$$\frac{dx}{dt} + p(t)x(t) = f(t) \quad (\text{II})$$

and we give examples that show a difficulty in attempting to extend Prop. 3.1 to (II) when p is not constant.

The condition $p \in MP$ does not ensure that $x \in MP$ when $f \in MP$.

For if $f = 0$ and p is the identity function, then $x(t) = A \exp(-t^2)$

which is not mean periodic. Nor does periodicity of p suffice to ensure

that $x \in MP$ when $f \in MP$. The solution to (II) with $x(0) = c$ and p

being a continuous periodic function of period τ is

$$x(t) = \exp(-q(t)-bt) \left\{ c + \int_0^t \exp(q(s)+bs) f(s) ds \right\}$$

where $b = \frac{1}{\tau} \int_0^\tau p(r)dr$, and $q(t) = \int_0^t p(r)dr - bt$ has period τ . However, from Prop. 2.5, $f(s)\exp(q(s))$ need not be mean periodic when q is periodic and f is mean periodic. When $f(s)\exp(q(s))$ is not mean periodic, by Prop. 2.2, $\int_0^t \exp(q(s)+bs)f(s)ds$ is not mean periodic, and so the solution will not be mean periodic.

The mean periodicity of f is not necessary for a solution of (II) to be mean periodic. To see this, choose two periodic functions, p and g , so that g is continuously differentiable and pg is not mean periodic. Then $g' \in MP$ and $f = g' + pg \notin MP$. As the differential equation (II) with $x(0) = g(0)$ has a unique solution, $x = g$ and $x \in MP$ but $f \notin MP$.

Prop. 3.1 can be extended to systems of linear differential equations with constant coefficients. We say a vector $\underline{x}$, whose components $x_1, x_2, \dots, x_n$ are $\mathcal{C}(R)$ functions, is mean periodic if each component is mean periodic. The mean period of a mean periodic vector is the infimum of the lengths of the segments of supports of the non-zero measures, μ , such that $\mu * x_j = 0$ for $j = 1, 2, \dots, n$. Thus $\underline{x}$ has mean period zero if and only if $x_1, x_2, \dots, x_n$ have mean period zero.

Prop. 3.2. For the system

$$\frac{d\underline{x}}{dt} + A\underline{x} = \underline{b}(t), \quad \underline{x}(\alpha) = \underline{c} \quad (\text{III})$$

where $\underline{b}$ is an n -vector-valued function with continuous component functions and A is a constant $n \times n$ matrix, a necessary and sufficient condition that $\underline{x}$ be mean periodic is that $\underline{b}$ be mean periodic, in which case, $\underline{x}$ and $\underline{b}$ have the same mean period.

Proof. Let $\underline{x}$ be mean periodic and satisfy $\lambda * \underline{x} = \underline{0}$, where $\lambda \in M_C$ is non-zero. As $\underline{x}$ is continuously differentiable, $\lambda * \underline{x}' = \underline{0}$; also, $\lambda * A\underline{x} = A\lambda * \underline{x} = \underline{0}$. Thus $\lambda * \underline{b} = \underline{0}$ so that $\underline{b}$ is mean periodic and the mean period of $\underline{x}$ is not less than the mean period of $\underline{b}$.

If $\underline{b}$ is mean periodic and satisfies $\mu * \underline{b} = \underline{0}$, when $\mu \in M_C$ is non-zero, set $\underline{y} = \mu * \underline{x}$, so that

$$\underline{y}' + A\underline{y} = \mu * \underline{x}' + A\mu * \underline{x} = \mu * (\underline{x}' + A\underline{x}) = \mu * \underline{b} = \underline{0}.$$

Then $\underline{y}(t) = e^{-At} \underline{d}$, for some constant vector $\underline{d}$. By use of the Jordan canonical form, (see Coppel, [1], page 46), the elements of the matrix e^{-At} are exponential polynomials with exponents equal to the characteristic values of A . Thus, the components of $\underline{y}$ are exponential polynomials so that $\underline{y}$ is mean periodic with mean period zero. Consequently, a non-zero measure, $\nu \in M_C$ can be found with arbitrarily small support satisfying $\nu * \underline{y} = \underline{0}$. Then $\nu * \mu * \underline{x} = \underline{0}$, showing that $\underline{x}$ is mean periodic with mean period at most equal to the mean period of $\underline{b}$. Thus the mean period of $\underline{x}$ is equal to the mean period of $\underline{b}$.

Remark. Another way of showing that $\underline{y}(t) = e^{-At} \underline{d}$ is mean periodic with mean period zero is as follows. Because of finite dimensionality, there exists an integer $m > 0$ such that $A^{m+1} = \sum_{j=0}^m c_j A^j$ for constants $c_0, c_1, \dots, c_m$, and so

$$e^{At} = \sum_{j=0}^{\infty} A^j t^j / j! = \sum_{j=0}^m f_j(t) A^j$$

where $f_0, f_1, \dots, f_m$ are continuous. For any $\delta > 0$, a non-zero measure ν can be chosen with support contained in $[-\delta, \delta]$ and such that $\nu(f_j) = 0$ ($j = 0, 1, \dots, m$). Then,

$$v * y(t) = [e^{-At} \int e^{Ar} dv(r)] \underline{d} = [e^{-At} \sum_{j=0}^m A^j v(f_j)] \underline{d} = \underline{0}$$

and so y is mean periodic with mean period zero.

We now consider the linear differential-difference equation

$$\sum_{k=1}^m a_k u'(t - \alpha_k) + \sum_{\ell=1}^n b_\ell u(t - \beta_\ell) = f(t), \quad (\text{IV})$$

seeking solutions that are valid on the real axis. [When boundary conditions are imposed, these may restrict the validity of the solution to a half line. This situation is better suited to functions mean periodic on a half line and a special case of (IV) will be discussed in Chapter 5.]

Prop. 3.3. Let $f \in \mathcal{D}(R)$. Also, let $a_1, \dots, a_m, b_1, \dots, b_n$ be non-zero complex numbers, and $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ be real numbers. If the differential-difference equation, (IV) has a mean periodic solution, u , then f is mean periodic. Conversely, if f is mean periodic, then all solutions valid on R are mean periodic.

Proof. The first statement is easily verified. For any solution, u , of (IV) that is valid on R is continuously differentiable. If u is mean periodic, then each translate of u or of u' is mean periodic. As f is a finite linear combination of such translates, it follows that f is mean periodic.

Conversely, let T denote the distribution

$$\sum_{k=1}^m a_k D \delta_{\alpha_k} + \sum_{\ell=1}^n b_\ell \delta_{\beta_\ell}$$

so that (IV) may be written as $T * u = f$. Since f is mean periodic, there

exists a non-zero measure μ of compact support such that $\mu * f = 0$, and so $\mu * T * u = 0$. Let ρ be any non-zero $\mathcal{C}_C^\infty(\mathbb{R})$ function and $\psi = \rho * \mu * T$; then $\psi \neq 0$, $\psi \in \mathcal{C}_C^\infty(\mathbb{R})$, and $\psi * u = 0$. Thus u is mean periodic.

Note. The converse portion of the preceding proposition may also be obtained in a longer but more constructive manner that is briefly outlined here for the simpler equation

$$\frac{du}{dt} + au(t-\omega) = f(t) \quad (V)$$

where $a \in \mathbb{C}$, $a \neq 0$, $\omega \in \mathbb{R}$ and $\omega > 0$.

The characteristic equation of (V) is defined as $h(z) \equiv z + ae^{-z\omega} = 0$, and has at least one root. For if not, $h(z)$ is of the form $e^{g(z)}$ where $g(z)$ is an entire function. As $h(z)$ is of order one, $e^{g(z)} = Ae^{dz}$ where A and d are constants. Thus $z + ae^{-z\omega} = Ae^{dz}$ which is contradictory, and so there exists a complex number, c , such that $h(c) = 0$. If $u = e_c v$, by integration one finds that

$$v(t) + c \int_0^\omega v(t-r)dr = v(0) + c \int_0^\omega v(-r)dr + \int_0^t e^{-cr} f(r)dr = F(t), \text{ say.}$$

If λ denotes the measure, defined for each $g \in \mathcal{C}(\mathbb{R})$ by

$$\lambda(g) = g(0) + c \int_0^\omega g(r)dr,$$

the last equation may be written as $\lambda * v = F$. If also, f is mean periodic, then F is mean periodic and as λ is a non-zero measure of compact support, v is mean periodic. Since $u = e_{-c} v$, u is mean periodic.

Chapter Four

MEAN PERIODIC FUNCTIONS IN $\mathcal{C}(R_+)$

We denote by $\mathcal{C}(R_+)$ the complex vector space of all continuous complex valued functions defined on the non negative real axis (R_+) equipped with the topology of uniform convergence on all compact subsets of R_+ . The concept of mean periodicity in $\mathcal{C}(R_+)$ was introduced by Koosis [1], [2], who defined a function f belonging to $\mathcal{C}(R_+)$ to be mean periodic if W_f , the closed linear subspace generated by the non-negative translates of f , $(T_{-r}f : t \rightarrow f(t+r), r \geq 0)$, is different from $\mathcal{C}(R_+)$. He showed, using the Hahn-Banach theorem that this condition is equivalent to:

f is mean periodic in $\mathcal{C}(R_+)$ if there exists a non-zero measure μ such that $\mu * f = g$ where both μ and g have compact supports in $(-\infty, 0]$.

For $f \in \mathcal{C}(R_+)$, the mean period L_f is defined as the supremum of the numbers $\ell > 0$ such that any continuous function in $(0, \ell]$ can be uniformly approximated by finite linear combinations of the non-negative translates of f . Thus,

f is mean periodic in $\mathcal{C}(R_+)$ $\Leftrightarrow L_f < \infty$.

Also, L_f is equal to the infimum of the numbers $\ell \geq 0$ so that there exists a non-zero measure μ having support in $[-\ell, 0]$ for which $\mu * f$ is zero on $[0, \infty)$.

As in the case of functions mean periodic in $\mathcal{C}(R)$, the Carleman transform $(\hat{g}/\hat{\mu}; \hat{\mu}: z \rightarrow \mu(e_{-z}))$ is defined and unique. This is a meromorphic function and its poles determine the spectrum of f . The spectrum corresponds to the exponential monomials contained in W_f . With respect to approximation of f by exponential polynomials, the following theorem holds.

Let f be mean periodic in $\mathcal{C}(R_+)$ with mean period L_f . Then there exists a number, A_f , such that $0 \leq A_f \leq L_f$ and f is locally uniformly approximable on (A_f, ∞) by exponential polynomials belonging to W_f .

The proofs of these statements are to be found in Koosis, [3].

When convenient, we regard functions belonging to $\mathcal{C}(R_+)$ as being zero on the negative real axis. If $f \in \mathcal{C}(R)$, as in §1.6,

$$f^+(t) = f(t) \text{ if } t \geq 0 \text{ and } = 0 \text{ if } t < 0,$$

so that $f^+ \in \mathcal{C}(R_+)$. $\mathcal{C}(R_+)$ with the operations of $(+, \otimes)$, $(\otimes, \text{the truncated convolution product defined in Chapter 2}),$ is a commutative algebra without a unit. Moreover, it has no divisors of zero, i.e. if $f, g \in \mathcal{C}(R_+)$ and $f \otimes g = 0$, then $f = 0$ or $g = 0$. (See Erdélyi, page 17.)

We let $MP(+)$ denote the set of functions mean periodic in $\mathcal{C}(R_+)$. One difference between MP and $MP(+)$ is as follows: a non-zero integrable function in $\mathcal{C}(R)$ cannot be mean periodic (§1.5) but there exist non-zero integrable functions in $MP(+)$, (see Koosis, [2]). The spectrum of a non-zero integrable $MP(+)$ function lies in the half plane to the left of the imaginary axis.

Prop. 4.1. $f \in MP \Rightarrow f^+ \in MP(+)$.

Proof. Since $f \in MP$, there exists a $\mu \in M_C$ such that $\mu \neq 0$ and $\mu * f = 0$. If the segment of support of μ is $[\alpha, \beta]$, then the measure

$\lambda = T_{-\beta}\mu = \delta_{-\beta} * \mu$ has segment of support $[-(\beta-\alpha), 0]$, and so, when $t \geq 0$

$$\lambda * f^+(t) = \int f^+(t-r) d\lambda(r) = \int f(t-r) d\lambda(r) = \int f(t-r+\beta) d\mu(r) = 0.$$

When $t \leq -(\beta-\alpha)$, $\lambda * f^+(t) = 0$. Thus $\lambda * f^+$ has support in $[-(\beta-\alpha), 0]$ and so $f^+ \in MP(+)$.

Corollary. When $f \in MP$, $(T_{\gamma}f)^+ \in MP(+)$ for any real γ and the mean period of $(T_{\gamma}f)^+$ in $MP(+)$ does not exceed the mean period of f in MP .

Prop. 4.2. Let $f \in MP(+)$. For $\beta > 0$,

(i) $(T_{-\beta}f)^+ \in MP(+)$, and

(ii) $T_{\beta}f$ may be arbitrarily extended to a function, $u \in \mathcal{C}(R_+)$, and $u \in MP(+)$.

Proof. Let $\mu * f = g$ where both μ and g have supports contained in $[-\alpha, 0]$.

(i) If $t \geq 0$, $\mu * (T_{-\beta}f)^+(t) = \int_{-\alpha}^0 f(t-r+\beta) d\mu(r) = 0$, and if $t \leq -\alpha$, $\mu * (T_{-\beta}f)^+(t) = 0$ showing that $(T_{-\beta}f)^+ \in MP(+)$.

(ii) If $u(t) = f(t-\beta)$ for $t \geq \beta$ and $u \in \mathcal{C}(R_+)$, and if

$\lambda = T_{-\beta}\mu$ with λ having support in $[-\alpha-\beta, -\beta]$, then, for $t \geq 0$

$$\begin{aligned} \lambda * u(t) &= \int_{-\alpha-\beta}^{-\beta} u(t-r) d\lambda(r) \\ &= \int_{-\alpha-\beta}^{-\beta} f(t-r-\beta) d\lambda(r) \\ &= \int_{-\alpha}^0 f(t-r) d\mu(r) = 0. \end{aligned}$$

Also, $\lambda * u(t) = 0$ for $t \leq -\alpha-\beta$ so that $u \in MP(+)$.

The following five propositions assert properties of mean periodic functions in $\mathcal{C}(R_+)$ that are analogous to the properties of mean periodic functions in $\mathcal{C}(R)$ described in Chapter Two. $MP_0(+)$ denotes the set of $MP(+)$ functions with mean period zero and $MQ(+)$ denotes the set of MQ functions restricted to R_+ .

(See 11.3.)

Prop. 4.3. $MQ(+)$, $MP_0(+)$ and $MP(+)$ are linear subspaces of $\mathcal{C}(R_+)$ and each is dense in $\mathcal{C}(R_+)$. Also

$$MQ(+) \subset MP_0(+) \subset MP(+) \subset \mathcal{C}(R_+)$$

and the inclusion is proper in each case.

Proof. The proofs for the first sentence resemble those of Prop. 2.1 and, except in the case of $MP(+)$, are omitted.

Let $f_1, f_2 \in MP(+)$, with μ_1 and μ_2 being non-zero measures such that $\mu_1 * f_1 = g_1$ and $\mu_2 * f_2 = g_2$ where μ_1, μ_2, g_1 and g_2 all have compact supports in $(-\infty, 0]$. If $\lambda = \mu_1 * \mu_2$, then

$$\lambda * (af_1 + bf_2) = a\mu_2 * g_1 + b\mu_1 * g_2 = h, \text{ say.}$$

As λ is a non-zero measure and as λ and h have compact supports in $(-\infty, 0]$, $af_1 + bf_2 \in MP(+)$.

For the second sentence, by Propositions 2.1 and 4.1, $MQ(+) \subset MP_0(+)$ and the inclusion is proper. Also, by definition, $MP_0(+) \subset MP(+)$ and this inclusion is seen to be proper by taking a continuous periodic function f of period τ which has no non-zero Fourier coefficients and showing that f^+ has mean period τ in $MP(+)$. Thus, if $\mu * f^+(t) = 0$

for $t \geq 0$, then $\mu * f = 0$ and so, by use of Prop. 1.12(iv), the length of the segment of support of $\mu \geq \tau$ and f^+ has mean period $\geq \tau$; also, we may take $\mu = \delta - \delta_{-\tau}$ so that f^+ has mean period $\leq \tau$.

Finally, the function $\exp(t^2)$ restricted to R_+ is not mean periodic in $\mathcal{C}(R_+)$ for the same reasons as $\exp(t^2)$ is not mean periodic in $\mathcal{C}(R)$. (See §1.5.)

Prop. 4.4. For $f \in \mathcal{C}(R_+)$,

- (i) $f \in MQ(+) \Leftrightarrow e_a f \in MQ(+) \Leftrightarrow \int f \in MQ(+) \Leftrightarrow f' \in MQ(+)$
- (ii) $f \in MP(+) \Leftrightarrow e_a f \in MP(+)$
- (iii) $f \in MP \Leftrightarrow \int f \in MP(+)$
- (iv) $f \in MP(+) \text{ and } f' \in \mathcal{C}(R_+) \Leftrightarrow f' \in MP(+)$
- (v) Each of $e_a f$, $\int f$, and f' (when it is continuous on R_+) has the same mean period as f .

Proof. We show that $f \in MP(+) \Rightarrow \int f \in MP(+)$ and that the mean period of $\int f$ does not exceed the mean period of f . The other proofs have more resemblance to their counterparts in Prop 2.2 and are omitted.

Let $\mu * f = g$ where μ is a non-zero measure and μ and g have compact supports in $[-\alpha, 0]$ for some $\alpha > 0$.

Set $F(t) = \int_{-\infty}^t f(r) dr$ where γ is any real number. As $f(t) = 0$ for $t < 0$, by convention, F is constant on $(-\infty, 0]$. Also, $F \in \mathcal{C}(R)$. Write also

$$G(t) = \int_{-\alpha}^t g(r) dr,$$

so that G is constant on $[0, \infty)$ and zero on $(-\infty, -\alpha]$. Then

$$D(\mu * F) = \mu * F' = \mu * f = g \quad \text{and}$$

$$\mu * F - G = c, \quad \text{a constant.}$$

Choose $\beta > 0$ and let λ be the measure defined by

$$\lambda(h) = \int_{-\beta}^0 (2r+\beta)h(r)dr \quad \text{for each } h \in \mathcal{C}(R).$$

Then $\lambda * G(t) = 0$ for $t \geq 0$ and $t \leq -\alpha - \beta$ and $\lambda * c = 0$, so that

$$\lambda * \mu * F = \lambda * G + \lambda * c = K, \quad \text{say.} \quad \text{As } \lambda * \mu \text{ is a non-zero measure and } \lambda * \mu$$

and K have supports contained in $[-\alpha - \beta, 0]$, $F \in MP(+)$. Also, β can

be made arbitrarily small so that $L_F \leq L_f$.

Prop. 4.5. The (truncated) convolution product of two mean periodic functions in $\mathcal{C}(R_+)$ is mean periodic in $\mathcal{C}(R_+)$.

Proof. We only need show that $f \in MP(+) \Rightarrow f \circledast f \in MP(+)$. As f is zero on $(-\infty, 0)$, $f^+ = f$ and

$$f * f(t) = \int_{-\infty}^{\infty} f(t-r)f(r)dr = \begin{cases} \int_0^t f(t-r)f(r)dr & \text{if } t \geq 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

$$\text{or } f * f = (f \circledast f)^+ = f^+ * f^+ = f \circledast f.$$

For $\mu * f = g$ with $\mu \neq 0$ and μ and g having supports contained in $[-\alpha, 0]$ for some $\alpha > 0$,

$$\mu * \mu * (f * f) = \mu * f * \mu * f = g * g$$

and as $\mu * \mu \neq 0$, $\mu * \mu$ and $g * g$ have compact supports in $(-2\alpha, 0]$, $f * f \in MP(+)$.

Note. As in Prop. 2.3, there is an alternative way of showing the above proposition and this is briefly outlined here. If μ is a non-zero measure with $\mu * f = g$ where μ and g have supports in $[-\alpha, 0]$ with $\alpha > 0$, then $\alpha >$ the mean period of f . Let

$$h(t) = f(t) \quad \text{if } t \geq \alpha \quad \text{and} \quad = 0 \quad \text{if } t < \alpha.$$

Let $k = h - h$ so that k has support in $[0, \alpha]$. As h can be locally uniformly approximated by exponential polynomials belonging to W_f , it can be shown that

$$\mu * \mu * h * h \text{ has support contained in } [0, 2\alpha],$$

$$\mu * \mu * h * k \text{ has support contained in } [-\alpha, 2\alpha], \text{ and}$$

$$\mu * \mu * k * k \text{ has support contained in } [-2\alpha, 2\alpha]$$

whence $f * f = f \circledast f \in MP(+)$.

Corollary. Each of $MP_0(+)$ and $MP(+)$ are subalgebras $(+, \circledast)$ of $\mathcal{C}(R_+)$.

Proof. For $MP_0(+)$, it follows that $f \circledast f$ has mean period zero when f does. As

$$2f_1 \circledast f_2 = (f_1 + f_2) \circledast (f_1 + f_2) - f_1 \circledast f_1 - f_2 \circledast f_2$$

and $MP_0(+)$ is a linear space, $f_1 \circledast f_2 \in MP_0(+)$ when f_1, f_2 do so that $MP_0(+)$ is a subalgebra of $C(R_+)$.

Prop. 4.6. $f \in MP(+)$ and $q \in MQ(+)$ $\Rightarrow fq \in MP(+)$.

Proof. As in Prop. 2.4, it suffices to show that $uf \in MP(+)$ when $f \in MP(+)$.

As usual, u denotes the identity function $u : t \rightarrow t$, and f is zero on $(-\infty, 0)$. Suppose $\mu * f = g$ with μ and g having supports contained in $[-\alpha, 0]$ and $\mu \neq 0$. Then

$$\begin{aligned}\mu * uf &= u(\mu * f) = (u\mu) * f \quad \text{and} \\ \mu * \mu * uf &= \mu * (ug) - (u\mu) * \mu * f \\ &= \mu * (ug) - (u\mu) * g \\ &= h, \quad \text{say.}\end{aligned}$$

Since h and $\mu * \mu$ have supports contained in $[-2\alpha, 0]$ and $\mu * \mu$ is a non-zero measure, $uf \in MP(+)$.

Corollary. $f \in MP_0(+)$ and $q \in MQ(+)$ $\Rightarrow fq \in MP_0(+)$.

Prop. 4.7. The product of two functions means periodic in $\mathcal{C}(R_+)$ need not be mean periodic in $\mathcal{C}(R_+)$.

The proof is based on the calculations in Prop. 2.5. Let E, f, g, f_N, g_N and h_N be defined as in Prop. 2.5, so that f^+ and g^+ are mean periodic in $\mathcal{C}(R_+)$. Suppose that μ has compact support in $(-\infty, 0]$ and

$$\mu * (fg)^+(t) = 0 \quad \text{for } t \geq 0.$$

The expression, E , is then zero. As $h_N^+ \rightarrow fg^+$ uniformly as $N \rightarrow \infty$

and

$$\int_{-r}^{T-r} e^{i\gamma s} h_N^+(s) ds = \int_{-r}^{T-r} e^{i\gamma s} h_N(s) ds$$

DIFFERENTIAL-DIFFERENCE EQUATIONS WITH BOUNDARY CONDITIONS.

(with $s \geq -r \geq 0$ for $r \leq 0$), the calculations in Prop. 2.5 are still valid and so $\hat{\mu}(i\gamma) = 0$ when $\gamma = \alpha m + \beta n$ for any $m, n \in \mathbb{Z}$. Hence, if α/β is irrational, μ must be zero and so fg is not mean periodic in $\mathcal{C}(R_+)$.

$$u'(t) + \sum_{k=0}^n a_k u(t-a_k) = f(t) \quad (I)$$

when subject to a boundary condition that forces the uniqueness of the solution. The mean periodicity of its solution is discussed and these results are compared with the properties of a particular differential-difference equation treated in Bellman and Cooke, [4].

Theorem 3.1. For the equation (I) with

$$(i) \quad f \in \mathcal{C}(R_+),$$

$$(ii) \quad 0 = a_0 < a_1 < \dots < a_n, \quad n \geq 1 \text{ and } a_0 a_n \neq 0,$$

$$(iii) \quad g \text{ is continuous on } [0, a_n],$$

there exists a unique function, u , which is continuous on R_+ , satisfies (I) on (a_n, ∞) and coincides on $[0, a_n]$ with g . Moreover, u is continuously differentiable on (a_n, ∞) .

Proof. Write $a = a_1$, $\beta = a_n$ and let

$$v(t) = f(t) - \sum_{k=1}^n a_k u(t-a_k) \quad (t \geq 0) \quad (II)$$

so that

$$\frac{dv}{dt} + a_0 v(t) = f(t) \quad (t \geq 0) \quad (III)$$

If $t \in [t, \infty)$, $0 \leq t - a_k \leq \beta$ for $k = 1, 2, \dots, n$. So, since

Chapter Five

DIFFERENTIAL-DIFFERENCE EQUATIONS WITH BOUNDARY CONDITIONS.

In this short chapter, we give properties of the differential-difference equation

$$u'(t) + \sum_{k=0}^n a_k u(t-\omega_k) = f(t) \quad (I)$$

when subject to a boundary condition that forces the uniqueness of the solution. The mean periodicity of its solution is discussed and these results are compared with the properties of a particular differential-difference equation treated in Bellman and Cooke, [1].

Theorem 5.1. For the equation (I) with

$$(i) \quad f \in \mathcal{C}(R_+),$$

$$(ii) \quad 0 = \omega_0 < \omega_1 < \dots < \omega_n, \quad n \geq 1 \quad \text{and} \quad a_0 a_n \neq 0,$$

$$(iii) \quad g \text{ is continuous on } [0, \omega_n],$$

there exists a unique function, u , which is continuous on R_+ , satisfies (I) on (ω_n, ∞) and coincides on $[0, \omega_n]$ with g . Moreover, g is continuously differentiable on (ω_n, ∞) .

Proof. Write $\alpha = \omega_1$, $\beta = \omega_n$ and let

$$v(t) = f(t) - \sum_{k=1}^n a_k u(t-\omega_k) \quad (t \geq \beta)$$

so that

$$\frac{du}{dt} + a_0 u(t) = v(t) \quad (t \geq \beta) \quad (II)$$

If $t \in [\beta, \beta+\alpha]$, $0 \leq t - \omega_k \leq \beta$ for $k = 1, 2, \dots, n$. So, since

$u(t) = g(t)$ for $t \in [0, \beta]$, $v(t)$ is known for $t \in [\beta, \beta + \alpha]$. Since $u(\beta) = g(\beta)$, the differential equation (II) has a unique solution on $[\beta, \beta + \alpha]$.

Now assume that u is uniquely determined on $[0, \beta + l\alpha]$ where $l \in \mathbb{N}$, coincides with g on $[0, \beta]$ and satisfies (I) on $(\beta, \beta + l\alpha]$. If $t \in [\beta + l\alpha, \beta + l\alpha + \alpha]$, then $l\alpha \leq t - \omega_k \leq \beta + l\alpha$ for $k = 1, 2, \dots, n$ and so $v(t)$ is known. Since $u(\beta + l\alpha)$ is known, (II) has a unique solution on $[\beta + l\alpha, \beta + l\alpha + \alpha]$, and so, u is uniquely determined on $[0, \beta + l\alpha + \alpha]$, coincides with g on $[0, \beta]$ and satisfies (I) on $[\beta, \beta + l\alpha + \alpha]$. Thus, by induction, we obtain a unique function u that is continuous on $\mathbb{R}_+$, satisfies (I) on (β, ∞) and coincides on $[0, \beta]$ with g . As u and $f \in \mathcal{C}(\mathbb{R}_+)$ and as u satisfies (I) on (β, ∞) , u is continuously differentiable on (β, ∞) .

Theorem 5.2. Granted the hypotheses and notation of Theorem 5.1, a necessary and sufficient condition that u be mean periodic in $\mathcal{C}(\mathbb{R}_+)$ is that f be mean periodic in $\mathcal{C}(\mathbb{R}_+)$.

Proof. There is no restriction in assuming that both u and f are defined on $\mathbb{R}$ and zero on $(-\infty, 0)$. The equation, (I), may be written as

$$u'(t + \beta) + \sum_{k=0}^n a_k u(t + \beta - \omega_k) = f(t + \beta) \quad \text{for } t > 0,$$

or

$$T * u = (T_{-\beta} f)^+ + h$$

where T is the distribution, $D\delta_{-\beta} + \sum_{k=0}^n a_k \delta_{\omega_k - \beta}$ (of order one and support $[-\beta, 0]$) and h is a function with support contained in $[-\beta, 0]$.

If $u \in MP(+)$, there exists a non-zero measure, such that $\lambda * u = v$, having compact supports in $[-\infty, 0]$. Then

$$\lambda * (T_{-\beta} f)^+ = \lambda * (T * u) - \lambda * h = T * v - \lambda * h = \theta, \text{ say.}$$

Since θ has compact support in $(-\infty, 0]$, $(T_{-\beta} f)^+ \in MP(+)$ and so, by Prop. 4.2, $f \in MP(+)$.

Now suppose that $f \in MP(+)$ and $\mu * f = g$ where μ is a non-zero measure, μ and g having compact supports in $(-\infty, 0]$. Then $\mu * (T_{-\beta} f)^+(t) = 0$ for $t \geq 0$ (see Prop. 4.2), and so $\mu * T * u = \phi$ say, where ϕ has compact support in $(-\infty, 0]$. If $\rho \in \mathcal{C}_C^\infty(\mathbb{R})$ and has its support in $(-\infty, 0]$, then so does $\psi = \mu * T * \rho$. Then $\psi * u = \phi * \rho$ and, as $\psi \neq 0$, with ψ and $\phi * \rho$ having compact supports in $(-\infty, 0]$, $u \in MP(+)$.

Corollary. With hypotheses as in Theorem 5.1,

(i) When f is mean periodic, there exists an $\alpha > 0$ such that u is locally uniformly approximable by exponential polynomials on (α, ∞) .

(ii) When $f = 0$, u has mean period not exceeding β . The solution, u , is locally uniformly approximable on (β, ∞) by finite linear combinations of exponential monomials, $u_n e_a$, where $\hat{T}(z) = z + \sum_{k=0}^n a_k \exp(-z\omega_k)$ has a zero at $z = a$ of order exceeding n .

Proof. (i) Since $f \in MP(+)$, $u \in MP(+)$ and so, when $\alpha \geq L_u$, the mean period of u , the statement follows from a theorem quoted in

(ii) That $L_u \leq \beta$ follows from the formula $\rho * T * u = 0$ on $(0, \infty)$, when $f = 0$; and the fact that ρ can be chosen with support contained in an interval of arbitrarily small length; and the fact that T has support $[-\beta, 0]$. Since ρ is an arbitrary $\mathcal{C}_C^\infty(\mathbb{R})$ function with support contained in $(-\infty, 0]$, the spectral set of u is independent of ρ and is thus contained in the set of zeros of $\hat{T}$.

We now compare these results with those stated in Chapter 4 of Bellman and Cooke [1] for

$$a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t) \quad (\omega > 0, a_0 b_1 \neq 0) \quad (\text{III})$$

with characteristic equation

$$h(z) \equiv a_0 z + b_0 + b_1 \exp(-\omega z) = 0.$$

It is shown (loc. cit., page 109) that if g is continuously differentiable on $[0, \omega]$ and u coincides with g on $[0, \omega]$ and if u satisfies (III) on (ω, ∞) , then $u(t) = \sum_{r=1}^{\infty} p_r(t) \exp(s_r t)$ where s_r is a zero of $h(z)$ and p_r is a polynomial of degree less than the order of $h(z)$ at $z = r$. It is also shown that this series is locally uniformly convergent on (ω, ∞) .

If $T = a_0 D \delta_{-\omega} + b_0 \delta_{-\omega} + b_1 \delta$, then $\hat{T}(z) = e^{-z\omega} h(z)$ and so, all of the above follows from our Theorem 5.2, Corollary (ii). Bellman and Cooke give conditions in which the series may converge on $(0, \infty)$; also, they give conditions in which the series may be uniformly convergent. However, none of this follows from the properties stated here that concern mean periodic functions.

Appendix

Also, they remark (loc. cit., page 110) that it is possible to obtain expansions of the solutions of inhomogeneous equations such as

$$a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t), \quad \omega > 0$$

Theorem 5.2 gives a sufficient condition for these solutions to be locally uniformly approximable by exponential polynomials; namely that f is mean periodic in $\mathcal{C}(R_+)$.

satisfying $\mu^2 = 0$.

For the exponential polynomial, $h = \sum_{k=1}^n p_k e^{\lambda_k t}$, where p_k is a polynomial of degree $p_k - 1$, V_h is completely characterized by the set of pairs of numbers $\lambda_k = \tau_k / 2\pi i$, p_k . For any mean periodic function, f , with $V_f \neq \mathcal{C}(R)$ the spectral set, $[A_f]$ is defined as $\{(\lambda_k, p_k) : e^{2\pi i \lambda_k} = V_f\}$. The spectrum, A_f , is the set of pairs (λ_k, p_k) : $\lambda_k \in [A_f]$ and $e^{2\pi i \lambda_k} = V_f$ for $j \leq p_k - 1$ but not $j = p_k$. It is to be noted that $[A_f]$ is contained in the set of zeros of μ .

The main theorem, that any mean periodic function is the limit of linear combinations of exponential monomials belonging to V_f , is stated in 1/; and from 15 to 115, Schwartz leads up to the proof of this theorem in the following steps.

(1) The set of exponential monomials in $V_f \neq \mathcal{C}(R)$ is free. This is shown by constructing measures $\mu_{k,j}$ so that

$$\int t^j \exp(2\pi i \lambda_k t) d\mu_{k,j}(t) = \delta_{k,j}^{V,P} = \begin{cases} = 1 & \text{if } \lambda_k = \lambda \text{ and } j = 1 \\ = 0 & \text{otherwise} \end{cases}$$

for $(\lambda_k, p_k) \in A_f$ and $j \leq p_k - 1$.

(2) V_0 is defined as the closed translation invariant linear subspace generated by the exponential monomials belonging to V_f . Any $g \in V_0$ can be expressed as a formal sum

Appendix

A. Schwartz's Theory of Mean Periodic Functions in $\mathcal{C}(\mathbb{R})$.

These remarks supplement those given in the Introduction and Chapter One and they briefly outline the steps taken in Schwartz [1] to obtain the main theorem. Throughout this appendix, f denotes a mean periodic function in $\mathcal{C}(\mathbb{R})$ and μ denotes a non-zero measure with compact support satisfying $\mu * f = 0$.

For the exponential polynomial, $h = \sum_{k=1}^n P_k e_{r_k}$, where P_k is a polynomial of degree $p_k - 1$, V_h is completely characterized by the set of pairs of numbers $\lambda_k = r_k / 2\pi i$, p_k . For any mean periodic function, f , with $V_f \neq \mathcal{C}(\mathbb{R})$ the spectral set, $[\Lambda_f]$ is defined as $\{\lambda_k : e_{2\pi i \lambda_k} \in V_f\}$. The spectrum, Λ_f , is the set of pairs $(\lambda_k, p_k) : \lambda_k \in [\Lambda_f] \text{ and } u_j e_{2\pi i \lambda_k} \in V_f \text{ for } j \leq p_k - 1 \text{ but not } j = p_k$. It is to be noted that $[\Lambda_f]$ is contained in the set of zeros of $\hat{\mu}$.

The main theorem, that any mean periodic function is the limit of linear combinations of exponential monomials belonging to V_f , is stated in §7; and from §8 to §15, Schwartz leads up to the proof of this theorem in the following steps.

(1) The set of exponential monomials in $V_f \neq \mathcal{C}(\mathbb{R})$ is free. This is shown by constructing measures $\mu_{k,j}$ so that

$$\int t^\ell \exp(2\pi i \lambda_\nu t) d\mu_{k,j}^\nu(t) = \delta_{k,j}^{\nu,p} \begin{cases} = 1 & \text{if } \nu = k \text{ and } \ell = j \\ = 0 & \text{otherwise} \end{cases}$$

for $(\lambda_\nu, p_\nu) \in \Lambda_f$ and $\ell \leq p_\nu - 1$.

(2) V_0 is defined as the closed translation invariant linear subspace generated by the exponential monomials belonging to V_f . Any $g \in V_0$ can be expressed as a formal sum

$$g \sim \sum_{k,j} d_{k,j} u_j e^{2\pi i \lambda_k}$$

where $(\lambda_k, p_k) \in \Lambda_f$, $0 \leq j \leq p_{k-1}$ and $d_{k,j} = \int g(t) d\mu_{k,j}^\vee(t)$.

(3) For $g = f$, there exists a similar formal developement. However, the terms $\int f(t) d\mu_{k,j}^\vee(t)$ will, in general, depend on the measure μ . Even if all these terms are zero, there is no necessity for f to be identically zero.

(4) The formula, $\mu_{k,0}^* g = p_k e^{2\pi i \lambda_k}$, is valid for $g \in V_0$ and also for $g = f$; the symbols being defined as above.

(5) The formal developement of f is unique.

(6) If the spectrum of f is void, then $f = 0$.

(7) Finally, it is shown that there exists a series of terms of suitably grouped exponential polynomials that is Abel summable in $\mathcal{C}(R)$ to f .

B. Finite Dimensional Subspaces of $\mathcal{C}(R)$.

The notation here is that used in Chapter One. Propositions B.1 and B.2 are based on arguments appearing in Edwards [2], 2.2.1 and Problem 11.4 respectively.

Prop. B.1. U_f is finite dimensional $\Rightarrow f$ is an exponential polynomial.

Proof. The trivial case of $U_f = \{0\}$ is excluded; as is the case when U_f is one dimensional with $f = e_a$ for some complex number a . Thus, we suppose that U_f is of dimension $n \geq 2$ with a basis $f_1, f_2, \dots, f_n$ so that $f = \sum_{j=1}^n d_j f_j$ for suitably chosen $d_j \in \mathbb{C}$.

Let $\underline{f}$ denote the n -vector with components $f_1, f_2, \dots, f_n$. As U_f is translation invariant, for each real α , $T_\alpha \underline{f} = A(\alpha) \underline{f}$ where $A(\alpha)$ is an $n \times n$ matrix. By use of the fact that the components of $\underline{f}$ are linearly independent, $A(0) = I$, the unit matrix, $A(\alpha)A(\beta) = A(\alpha+\beta)$ and $A(\alpha)$ is continuous (with $(A(\alpha) - A(\beta))\underline{f} = T_\alpha \underline{f} - T_\beta \underline{f} \rightarrow \underline{0}$ as $\alpha \rightarrow \beta$). Thus, $A(\alpha) \int_0^\gamma A(\beta) d\beta = \int_0^\gamma A(\alpha+\beta) d\beta = \int_\alpha^{\alpha+\gamma} A(r) dr$. As $A(0) = I$ and A is continuous, we may choose and fix γ so that $B = \int_0^\gamma A(\beta) d\beta$ is non-singular. Then $A(\alpha) = B^{-1} \int_\alpha^{\alpha+\gamma} A(r) dr$ showing that A has continuously differentiable elements, and so,

$$\frac{d\underline{f}}{dt} = \lim_{\alpha \rightarrow 0} (T_{-\alpha} \underline{f} - \underline{f})/\alpha = \lim_{\alpha \rightarrow 0} (A(-\alpha) - A(0))\underline{f}/\alpha = C\underline{f}$$

where $C = -A'(0)$ is constant. Since the solutions of this differential equation are vectors with components that are exponential polynomials and as $\underline{f} = \sum_{j=1}^n d_j \underline{f}_j$, $\underline{f}$ is an exponential polynomial.

Prop. B.2. U_f is closed $\Rightarrow U_f$ is finite dimensional.

Proof. For each $n \in \mathbb{N}$, the set S_n equal to

$$\{g: g \in U_f \text{ and } g = \sum_{k=1}^n a_k T_{\alpha_k} f \text{ where } |a_k| \leq n, |\alpha_k| \leq n \text{ for } k = 1, 2, \dots, n\}$$

is bounded for $p_m(g) \leq n p_{n+m}(f)$ for each $m \in \mathbb{N}$ where p_m is the seminorm defined in §1.1. S_n is also equicontinuous since

$$|g(r) - g(s)| \leq \sum_{k=1}^n |a_k| |T_{\alpha_k} f(r) - T_{\alpha_k} f(s)|$$

and $|a_k|, |\alpha_k|$ are bounded for $k = 1, 2, \dots, n$. Hence, S_n is compact by Ascoli's theorem and therefore closed.

Now $U_f = \bigcup_{n=1}^{\infty} S_n$ and U_f is a closed subset of a complete metric space so by Baire's category theorem, there exists a S_n that is not nowhere dense. As this S_n is closed, it must then have a non-void interior whence U_f contains a compact neighbourhood of zero. Thus, by Reisz's theorem, (see Edwards [1], page 65) U_f is finite dimensional.

Combining the above propositions with the fact that if f is an exponential polynomial, then U_f is finite dimensional and so closed, we have

f is an exponential polynomial $\Leftrightarrow U_f$ is closed
 $\Leftrightarrow U_f$ is finite dimensional.

C. Almost Periodic Functions and Mean Periodic Functions.

Almost periodic functions have been mentioned in this thesis and we state some well known properties of these functions.

A continuous complex valued function f on $\mathbb{R}$ is said to be (uniformly) almost periodic if, for each $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that each interval of length T contains a point α for which

$$|f(t+\alpha) - f(t)| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

This definition is equivalent to the following

(i) Any sequence $\{\alpha_n\}$ of real numbers contains a subsequence $\{\alpha_{n_k}\}$ such that the subsequence $\{f(t+\alpha_{n_k})\}$ is uniformly convergent for all $t \in \mathbb{R}$.

(ii) f is the uniform limit of a sequence of generalized trigonometric polynomials $\sum a_n e_{i\alpha_n}$ where $\alpha_n \in \mathbb{R}$.

Such functions are uniformly continuous and bounded. The set of all almost periodic functions form a Complex Banach space with the norm

$\|f\| = \sup\{|f(t)| : t \in \mathbb{R}\}$. When f is almost periodic, $e_{i\alpha} f$ is almost periodic when α is real, f' is almost periodic when it is uniformly continuous and $\int f$ is almost periodic when it is bounded. When f and g are almost periodic, fg is almost periodic but in general $f \otimes g$ is not bounded and so not almost periodic.

An example of an almost periodic function that is not mean periodic has been given in §1.5.. The exponential function is not bounded on $\mathbb{R}$ and provides an example of a mean periodic function that is not almost periodic.

It has been shown by Kahane [1] that a bounded uniformly continuous mean periodic function is almost periodic.

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