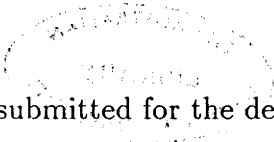


**SURFACE WAVE PROPAGATION
IN
LATERALLY VARYING MEDIA**

by

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ABSTRACT

It is generally acknowledged that the earth exhibits a dominantly radial variation in material properties on which a less pronounced, lateral variation is superimposed. This lateral variation reflects a complex rheology and contains important information regarding the nature of major dynamical processes within the earth. Seismic waves are, without doubt, the most sensitive means currently at our disposal for delineating both large and small scale earth structure, and consequently afford the best prospects for enhancing our knowledge of the forces which shape the planet. However, a comprehensive understanding of the effects of earth structure on the passage of seismic waves is a prerequisite to this objective. This thesis is concerned with the theoretical description of surface wave propagation in several classes of laterally varying, stratified media which model the outer layers of the earth.

The first study deals with the propagation of the regional phase Lg , usually interpreted as a sum of higher mode surface waves, in a crustal waveguide of variable thickness. A ray-based method is introduced which can be used to measure, in a semiquantitative manner, various aspects of scattering from changes in topography of the earth's surface and the Mohorovičić discontinuity. The method is applied to models of central Asia and is used to explain the observed behaviour of Lg in this structurally complex region.

The remainder of the thesis represents a unified treatment of surface wave propagation in different forms of lateral heterogeneity based on representation theorems of classical elastodynamics and wavefield expansions in series of surface wave basis functions. In the first of three parts, the T-matrix formulation of scattering from a single obstacle is extended to surface waves using previously unrecognized orthogonality relations which involve integral properties of the surface wave eigenfunctions. The T-matrix depends only on the size and shape of the obstacle, is valid in both the near- and far-fields, and provides a complete account of the scattering response to a given harmonic wave. Numerical examples are presented to demonstrate the influence of obstacle size and geometry on the scattered wavefield.

The translation properties of the surface wave basis functions are exploited in the second part to derive a composite surface wave T-matrix for an arbitrary multiple obstacle configuration and which accounts for the entire hierarchy of multiple scattering interactions. The importance of multiple scattering in the context of regional phases such as *Lg* is examined by way of several numerical examples.

In the final part the T-matrix description is recast in a reflection/transmission formalism which is developed to describe scattering from obstacles exhibiting a degree of quasi-concentric multilayering about a vertical axis. This description, although highly idealized from a geophysical perspective, leads to a theory for surface wave propagation in media exhibiting a continuous variation in physical properties. The theory relies on the expansion of the Green's function for a stratified reference medium in terms of the surface wave basis functions and is shown to be simply the 3-D extension of coupled mode techniques used to model surface wave propagation in waveguides exhibiting strictly 2-D lateral heterogeneity. Results from numerical experiments are presented to examine the effect of near-source heterogeneity on the wavefield observed at large distances.

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Prior to submitting formal application to undertake PhD study at the Research School of Earth Sciences I expressed to my eventual supervisor, Dr B. L. N. Kennett, my wish to work on a theoretical problem in whole earth seismology. My aims were to acquaint myself with an area of seismology in which I was interested but largely ignorant, and to develop some experience and perhaps proficiency with methods of mathematical and computational physics which, I believe, are indispensable tools for a good seismologist, whether theoretician *or* observationalist. I owe a great deal to Brian for creating an environment in which I was able to fully realize my objectives. The problems he suggested I investigate were challenging, interesting and topical. He provided me with sound guidance and encouragement, and gave unsparingly of his time whenever I wished to discuss difficulties related or unrelated to research. I am also extremely grateful to Brian for his efforts in establishing a research assistantship which allowed me to look after certain fiscal responsibilities I was obliged to assume as an overseas student.

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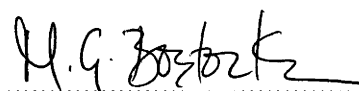
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PREFACE

This thesis is based on work conducted in the Research School of Earth Sciences at The Australian National University between August 1989 and June 1991 while I was a full time research student. The work presented here is believed to be original except where explicit mention is made in the text to the work of others, and has not been submitted for any degree or qualification at any other institution. Much of the material contained in this thesis has, however, been published or is submitted for publication. The correspondence of published papers or manuscripts with chapters and appendices is noted below.



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CHAPTER 1

GENERAL INTRODUCTION

Our understanding of the nature of earth structure has relied heavily upon the development of seismology over the last 100 years. Much of the pioneering work in seismology was devoted to the delineation of major discontinuities within the earth, notably the boundaries of the crust, mantle, inner and outer cores; and subsequently finer and less-pronounced discontinuities and gradients within these principal zones. This work led naturally to the development of seismological tables (e.g. Jeffreys & Bullen, 1940; Herrin, 1968; Dziewonski & Anderson, 1981; Kennett & Engdhal, 1991) which define radially symmetric earth models and incorporate traveltime and, in recent models, also the periods of free oscillation of the normal modes of the earth. The success of these models in predicting travel times and free oscillation periods is testimony to a dominantly radial variation in physical properties within the earth. In recent years, however, numerous studies have indicated that a less-pronounced, but nonetheless significant lateral component of heterogeneity also exists at many scales throughout the earth comprising deviations of a few percent. The delineation of lateral heterogeneity at all depths and scales has become a major objective of earth science since it can provide an indication of the processes which have controlled the planet's evolution and which continue to shape it. Since seismological observations offer what are undeniably the best opportunities for resolving detailed earth structure, a sound theoretical understanding of the effects of lateral heterogeneity on seismic wave propagation is fundamental to this objective. This thesis concerns the propagation of seismic surface waves in the outermost layers of the earth. In order to set the stage for this work we will present a brief overview of the nature of heterogeneity within the earth as revealed by seismological studies with particular reference to the crust and upper mantle. This is followed by a review of the techniques employed in recent studies to describe surface wave propagation in a laterally heterogeneous crust and upper mantle which provides an indication of those areas where our understanding is still incomplete. In the final section of this chapter we outline the organization of the thesis and the approach we have adopted to further the understanding of surface wave propagation in laterally heterogeneous media.

1.1 LATERAL HETEROGENEITY WITHIN THE EARTH

Lateral heterogeneity appears to exist throughout the earth and is manifest in a variety of forms. Within the earth's core, for example, studies of P phases (Poupinet *et al.*, 1983) and the splitting of the earth's free oscillations (Giardini *et al.*, 1987) indicate a strong component of asphericity which has led to the suggestion that the inner core may be anisotropic with an axis of symmetry coincident with earth's axis of rotation. Heterogeneity in the earth's deep interior also exists as perturbations in the topography of the core mantle boundary; in 1972 Cleary and Haddon interpreted precursors to the PKP phase as scattered waves from the undulations in the depth of this discontinuity. More recently, variations in relief of up to 20 km have been mapped on the core-mantle boundary by inversions of P phase traveltimes residuals (Creager & Jordan, 1986; Morelli & Dziewonski, 1987) and bear important implications regarding chemical properties and dynamics in the outer core and lower mantle.

Body wave traveltimes have been used at shallower levels to investigate velocity heterogeneity at a range of scales in the lower mantle and overlying transition zone (420 - 670 km). Long-wavelength structure revealed through P - and S -wave velocity perturbations of up to two percent bears little apparent relation to tectonic structure at the surface (Dziewonski, 1984; Hager *et al.*, 1985), and is difficult to explain through simple steady-state models of mantle convection. Smaller-scale features are also evident through this zone, the most notable, perhaps, being the penetration of subducted oceanic slabs extends to depths of at least 1000 km as suggested by Creager & Jordan (1984).

The research described in this thesis is ultimately directed towards an improved understanding of the effects of lateral heterogeneity in the crust and upper mantle on the propagation of surface waves. Consequently, our concern will lie primarily with the character of heterogeneity through this region. The indication of studies at a full range of scales is that deviations from radially symmetric models of earth structure are, in fact, most pronounced within the upper 400 km of the earth. To date several investigations have made use of the vast quantities of data recently

available through long-period digital seismic networks such as the Global Digital Seismograph Network (GDSN) and the International Deployment of Accelerometers (IDA) to invert for global velocity structure. Nakanishi & Anderson (1984); Nataf *et al.*, 1984; Tanimoto & Anderson (1984, 1985) have used phase velocity measurements of fundamental mode surface waves from a data set comprising 13 earthquakes to expand velocity distributions in terms of spherical harmonics to order 6. Woodhouse and Dziewonski (1984) employed a larger data base (53 earthquakes for a total of 870 paths) and adopted a wave fitting procedure to model mantle waves and iteratively invert for velocity using a spherical harmonic expansion to order 8. The results from the individual studies are remarkably consistent and include several important observations, most notably the expression of surface tectonic structure well into the mantle. Older, more stable cratonic regions (e.g. Australian, Baltic, Canadian Shields) are underlain by high shear velocity roots while areas of young crust, in particular the mid-ocean ridge systems, are characterized by negative upper mantle velocity anomalies. These features are evident to depths of approximately 350 km, although the absolute magnitude of the anomalies appears to decrease from $\pm 8\%$ at 50 km depth to levels of $\pm 1\%$ below 650 km. The resolution of global studies is restricted to longer-wavelength features (~ 2000 km) owing to limited azimuthal path coverage and the use of long-period waves. At shorter periods, the approximation that waves travel great circle paths is no longer valid and the effects of multi-pathing must be considered.

Attempts to map more detailed structure beneath continents have also involved observations of long-period body waves and fundamental-mode surface waves, although at higher frequencies than global studies. Comparative analyses of observed and synthetic seismograms (Burdick, 1981; Grand & Helmberger, 1984) indicate that heterogeneity is strongest in the uppermost mantle and that upper mantle structures in cratonic and tectonic regimes differ considerably (differences in velocity of up to 10% above 170 km depth). The correlation of large-scale velocity structure well into the mantle with major structural features at the surface is also a feature common to tomographic inversion studies of traveltimes (Romanowicz, 1980;

Grand, 1987) and surface wave (Snieder, 1988b,c) data. Tomographic studies have placed tighter constraints on the extent and location of upper mantle heterogeneity. Romanowicz inverted teleseismic *P*-wave arrival times from western and northern Europe and mapped large scale variations in velocity which corresponded closely to major tectonic regimes such as the Alps and the Baltic shield. However the models failed to account for a significant fraction of the standard deviations in the data, a result attributed to substantial scattering from complicated short wavelength structure. Snieder employed linear scattering theory to invert Rayleigh waveforms over central and eastern Europe. The reconstructed velocity models implied horizontal length scales of heterogeneity comparable to or less than the wavelengths of the surface waves (200 km). The distribution of velocity anomalies correlated well with much of the known regional structure although the actual magnitude of the postulated variations is probably too large for the application of linear scattering theory. In regions with denser and more uniform data distributions, more detailed and confident delineation of structure is possible (Humphreys *et al.*, 1984; Spakman *et al.*, 1988). Spakman *et al.* employed a data set of some 500,000 *P*-wave delay times from earthquakes in the eastern Mediterranean and were able to image, quite conclusively, slab penetration at the Hellenic subduction zone to depths of 200+ km beneath the Aegean Sea. Unfortunately, the wider application of detailed inversion studies to other regions of interest is restricted by the non-uniform and frequently sparse distributions of earthquake sources which severely limit horizontal resolution.

The delineation of structure in the crust and upper mantle at still smaller scales requires the analysis of higher frequency seismic waves (> 0.1 Hz). However the effects of scattering from complex structure, which are smoothed out at lower frequencies, renders the interpretation of short-period observations a task which is at best formidable. One approach, which obviates some of the inherent problems is that adopted by Kennett (1987) and Kennett & Bowman (1990). These authors considered upper mantle heterogeneity in the Australian Shield using a stochastic description rather than the identification of specific structure. They recognized

that while upper mantle velocity-depth profiles from long-period body wave studies generally showed close agreement, similar models from short-period studies were far more complex and exhibited greater variation due to the effects of detailed structure. In order to describe both short- and long-period observations, they developed a class of models describing lateral heterogeneity as departures from a stratified reference. The models were devised to accommodate the range of velocity profiles derived from short-period studies and observations of high-frequency, regional surface wave propagation. Reasonable concordance was achieved by horizontal velocity perturbations up to $\pm 2\%$ on a scale of 300 - 500 km with a vertical scale which increased from 100 km at 200 km depth to 200 km at 900 km. Kennett & Nolet (1990) and Kennett (1991) have employed similar concepts to investigate lateral velocity gradients in the upper mantle. A representative model was constructed using the long wavelength structure beneath western Europe as inferred by Nolet (1990) with superimposed small scale perturbations ($\pm 1\%$ with scale lengths of 300 km) to satisfy short-period observations. At depths shallower than 210 km this class of model indicates that lateral gradients may approach $0.001 \text{ km s}^{-1} \text{ km}^{-1}$ and exceed the corresponding vertical gradients.

We may conclude that prominent structural features evident at the earth's surface are characterized by velocity anomalies which may extend significant distances (500+ km) into the mantle. Superimposed on these large-scale variations are short-wavelength perturbations in the lateral velocity distribution. The effects of this composite lateral heterogeneity on seismic wave propagation are most pronounced at regional scales within the crust and upper mantle. Thus in the case of surface waves conditions exist for both multi-pathing (Levshin & Berteussen, 1979) in smoothly varying heterogeneity and complex scattering from abrupt and discontinuous variations in earth structure (Snieder, 1988c). The character of lateral heterogeneity within the earth and, indirectly, the manner in which it affects surface wave propagation must to large degree reflect the processes which govern global dynamics.

1.2 PREVIOUS STUDIES ON SURFACE WAVES IN HETEROGENEOUS MEDIA

Given that the character of lateral heterogeneity within the earth in general, and the crust and upper mantle in particular, is highly dependent upon the scale of investigation it is not surprising that various classes of technique have emerged to describe surface wave propagation in specific environments. In media where the variations in structure are slight and occur over large distances with respect to the dominant wavelengths of the surface waves (e.g. > 50 s propagation at continental scales) a ray based treatment is most useful. The development of a ray theory for surface waves analogous to the geometric ray theory used in body wave studies has been considered by several authors (Bretherton, 1967; Julian, 1970; Woodhouse, 1974). Woodhouse employed the elastic wave Lagrangian and the propagator matrix formalism for a stratified medium to derive equations governing slow variations in amplitude, frequency, and wavenumber of the surface wave train in media with gradual lateral variations. The solution of these equations by the method of characteristics yields ray tracing equations which have been employed in numerous subsequent studies. The application of ray theory is useful in slowly varying media where processes such as mode coupling and reflection are insignificant but where simplifications such as the geometrical optics ('great circle') approximation are no longer warranted. Indeed, it has been shown that surface wave raytracing is, in fact, essential in the study of short-period waves at global scales (Sobel & von Seggern, 1978) and in the analysis of free-oscillations using higher orbit, long-period waves (Schwartz & Lay, 1987). Woodhouse & Wong (1984) were able to explain amplitude anomalies in mantle waves travelling multiple orbits as due to the focussing and defocussing of ray bundles along propagations paths. Yomogida & Aki (1985) derived a method for synthesizing surface wave waveforms in smoothly varying lateral heterogeneity using Gaussian beams. The method was applied (1987) in conjunction with the Born approximation to invert for Rayleigh wave phase velocities at periods of 30-80 s in the Pacific Ocean, using both amplitude and phase anomaly data. The resolution of this class of approach was shown to represent a

significant improvement on more conventional great circle data inversions which employ phase data alone.

In media where lateral heterogeneity exists at scales comparable to or less than the wavelengths of the surface waves some account must be made for the effects of mode coupling and scattering from abrupt or discontinuous structure. For mild heterogeneity this is conveniently accomplished using the Born approximation or, equivalently, first order perturbations schemes. Snieder (1988d) has shown that ray geometrical effects such as raybending and focussing can be identified within the first order Born approximation for the surface wave displacement field. This description also allows for the interaction of the surface wavefield (including modal coupling) with heterogeneity which exhibits abrupt velocity contrasts provided the actual magnitudes are small. Hence the method is very useful in practical terms and permits the inversion of surface wave data where mild lateral variations in earth properties occur at all scales (see Snieder, 1988b,c). A disadvantage however is that the formulation is only valid at large distances and fails to account for near-field effects. Tanimoto (1990) has also applied employed linear perturbation theory and derived a solution for surface waves in mild lateral heterogeneity which assumes a potential representation for Love and Rayleigh waves that satisfies the scalar Helmholtz equation. By employing the Green's function representation theorem he derives integral formulae describing the displacement field perturbation in terms of material perturbations and kernels involving the surface wave potentials. The advantage of this approach is that the kernels may be used to illustrate graphically how the waves sample the heterogeneous medium differently at various frequencies. Although effects such as mode coupling can in principle be accommodated, computational considerations have limited application of the method to periods greater than 20 s where coupling can be ignored.

If lateral variations exceed a few percent in absolute magnitude descriptions of the surface wavefield based on linear scattering theory, although perhaps qualitatively indicative, are no longer quantitatively reliable. The treatment of surface wave propagation across a contact between two individually laterally homogeneous

media, as for example a continental margin, has been extensively studied by several authors (see Malischewsky, 1987). Most of these studies employ an expansion of the stress and displacement fields in terms of modal eigenfunctions to minimize the stress or energy flux mismatch across the contact. Refinements to the approach of this problem include the work of Kazi (1976) which incorporates body wave contributions through the use of the Schwinger-Levine variational principle, and that of Its & Yanovskaya (1985) who studied 3-D reflection and transmission of surface waves incident at oblique angles to weakly curved boundaries between two differing vertical structures. In media where lateral heterogeneity exists as perturbations on a reference structure surface wave propagation is effectively treated using the coupled mode approach of Kennett (1984a). As in the vertical contact problem the stress and displacement are expanded as a series of weighted eigenfunctions (in this case for the reference structure), however the modal coefficients now evolve as the wave proceeds through the heterogeneity. The problem may be cast in terms of transmission and reflection matrices which facilitates the numerical solution by transforming a two-point boundary value problem into an initial value problem. Unfortunately, the formulation is strictly valid for 2-D structures only and fails to incorporate 3-D effects such as coupling between Love and Rayleigh waves.

The major gaps remaining in our understanding of how surface waves propagate in laterally heterogeneous media arise in situations where contrasts are large and linear scattering theory is no longer valid. As mentioned, theories do exist to describe propagation in particular environments, such as subvertical contacts between quarterspaces and media exhibiting strictly 2-D variations in material properties, however consideration of more generally varying 3-D structure is a problem which remains to large degree unsolved. It is of course possible to treat this problem using brute force numerical techniques such as finite element or finite difference methods, however this is computationally infeasible for many practical applications. In addition we seek to gain physical insight into operative mechanisms such as mode coupling, wavetype conversion and near-field scattering, which is not easily afforded by these methods.

1.3 THESIS SCOPE AND ORGANIZATION

It is apparent that the magnitude of the earth's lateral heterogeneity and the variety of forms which it may assume are greatest in the crust and upper mantle. This heterogeneity will exert a significant influence on the character of seismic wave propagation in the outermost layers of the earth especially at higher frequencies, and by identifying this influence we obtain evidence that is fundamental to addressing one of the major objectives of global geophysics: the delineation of earth structure. However, the fulfillment of this objective also requires a comprehensive understanding of elastic wave interaction with lateral heterogeneity. In many regions, the crust and upper mantle behave as a waveguide and a representation of the total wavefield in terms of its surface wave components will incorporate most of the major features observed on regional seismograms. This representation has the advantage that it essentially reduces the dimensionality of the problem by one order and allows us to concentrate on wave propagation in two dimensions parallel to the earth's surface. Although surface waves in a wide variety of laterally heterogeneous environments are effectively described using various methods developed in recent years, many gaps remain in our knowledge of propagation through media exhibiting general variations in 3-D. The objective of this thesis is to narrow some of these gaps.

In chapter 2 we introduce a simple ray-based scheme for describing the propagation of the regional phase *Lg* which relies on an interpretation of crustally guided waves in terms of multiply reflected, constructively interfering shear waves. One of the merits of the method is that it provides an indication of how surface waves interact with variations in the topography of major discontinuities (in this case the free surface and the Moho), a process which is inherently difficult to model using descriptions based on modal eigenfunction expansions. Although it is only semi-quantitative the approach has the advantage over more traditional ray tracing approaches (*cf.* Woodhouse, 1974) that it provides a measure of such complicated processes as modal coupling and wavetype conversion. Simulations of crustal structure in central Asia and California are used to compare the predicted patterns of

surface wave propagation from raytracing with well-documented observations of the *Lg* phase in these structurally complex regions.

The development of an exact theory of surface wave scattering from a discrete obstacle is the focus of chapter 3. The derivation relies on Betti's theorem of elastodynamics and requires the expansion of the incident and scattered fields in terms of two different sets of surface wave basis functions. The basis functions incorporate the Love and Rayleigh wave eigenfunctions and vector cylindrical harmonics to describe the horizontal dependence. The coefficients of the incident and scattered wavefield expansions are then related through a surface wave T-matrix (*cf.* Waterman, 1969) which completely describes the scattering response of an obstacle at a given frequency and depends only on the obstacle's geometrical configuration. In contrast to previous descriptions based on the Born approximation (Snieder, 1986a), the T-matrix formulation remains valid for scatterers exhibiting large velocity contrasts and extending over broad regions; and can be used to examine the behaviour of the scattered wave in the near-field. The method is applied to investigate the effects of scatterer dimension and geometry on the nature of modal coupling and wavetype (Love-Rayleigh) conversion.

In chapter 4 the T-matrix formulation is extended to accommodate scattering from two or more scatterers including the effects of multiple scattering. The development relies on translation operators for the vector cylindrical harmonics which effectively describe propagation of scattered fields between scatterers. The resulting composite T-matrix can be written in a form which involves the individual T-matrices of all scatterers and which clearly indicates the physical processes contributing to the total scattered field. The formulation is applied to several simple dual-scatterer configurations to examine the implications of multiple scattering interactions to the propagation of regional phases such as *Lg* in the earth's crust and upper mantle.

We exploit certain properties of the surface wave basis functions in chapter 5 to investigate propagation and scattering in laterally heterogeneous media which exhibit some degree of cylindrical stratification. In the first instance we consider

media consisting of discrete enveloping cylindrical layers and find that wave propagation is conveniently described using a reflection/transmission matrix formalism analogous to that in the theory of elastic waves in horizontally stratified media. This is somewhat of an academic exercise since few terrestrial environments are accurately modelled in this way. However it leads us to consider the form of solution when lateral heterogeneity exists as 3-D perturbations on a stratified reference medium, a more realistic model in a geophysical context. We employ perturbation theory and invariant embedding to illustrate how surface wave propagation in such media can be described in terms of reflection and transmission properties. This is, as we demonstrate, simply the extension of Kennett's (1984a) coupled mode technique to three dimensions. This theory is applied to a suite of simple models to investigate the effects of near-source heterogeneity on the character of surface waves observed in the far-field.

CHAPTER 2

THE EFFECT OF THREE-DIMENSIONAL STRUCTURE

ON L_g PROPAGATION PATTERNS

2.1 INTRODUCTION

The *Lg* phase propagates in the earth's crust and dominates seismograms at regional distances. It is characterized by frequencies between 1 and 5 Hz and is often of extended duration. Although the wavetrain does not exhibit a clear onset, it builds to an amplitude maximum at a group velocity close to 3.5 km s^{-1} and may, in regions of sedimentary cover, carry significant energy to group velocities of 2.8 km s^{-1} . Recently considerable attention has focussed on the signature of the *Lg* wavetrain in deducing crustal morphology and as an aid to nuclear discrimination; in both instances an understanding of the manner in which *Lg* interacts with zones of crustal heterogeneity is a primary objective.

The sensitivity of *Lg* to major variation in crustal structure has been widely noted; early studies revealed that as little as 100 km of intervening oceanic crust is sufficient to eliminate *Lg* propagation across ocean basins (Ewing *et al.*, 1957). The character of *Lg* propagation across certain continental areas where the crust is known to be complex, most notably central Asia, has been used to constrain structure and provide insight into the operative tectonic processes (Ruzaikan *et al.*, 1977; Kadinsky-Cade *et al.*, 1981; Ni & Barazangi, 1983). These studies have involved the interpretation of short-period seismograms to qualitatively characterize broad geographical zones in terms of their amenability to *Lg* propagation. More quantitative analysis is possible with increased raypath coverage; Kennett *et al.* (1985) inverted a comprehensive set of *Lg* attenuation data for paths in NW Europe to delineate crustal heterogeneity in the North Sea region.

Two different descriptions, which are equivalent in the case of a stratified medium, can be made for the physical nature of *Lg* in a heterogeneous crust: the phase can be considered as a sum of higher mode surface waves whose energy is mostly confined to the crust, or alternatively, as the result of constructive interference between *S*-waves multiply reflected between the free surface and the crust-mantle boundary. For heterogeneous zones, two methods have evolved to model the propagation of *Lg* based on these different views of the propagation process.

The coupled mode technique (Kennett, 1984a) is based on the modal interpretation and solves the boundary value problem exactly; however in zones of exaggerated heterogeneity and variable layer thickness it can become computationally intractable. The method of ray diagrams is a simpler means of assessing the propagation of Lg in a semi-quantitative manner and is not limited by the same circumstances. It involves tracing a set of rays at fixed slowness through a crustal layer of variable thickness, and the disruption in the pattern of the ray system (or equivalently the reflection points) is used as a measure of modal coupling and scattering. Kennett (1986) compared the visually descriptive ray diagrams with coupled mode solutions for simple 2-D earth models and established the validity of the ray based technique.

Our objective in this study is to extend Kennett's (1986) approach to 3-D heterogeneity. After describing a basis for the interpretation of ray diagrams we examine two simple earth models and investigate the effects of oblique incidence on Lg propagation. We then shift our focus to a central Asia to compare our predictions of Lg propagation with observations from previous studies.

2.2 THE RAY DIAGRAM METHOD

Kennett (1986) provided a basis for the interpretation of ray diagrams by comparing them with coupled mode solutions for a suite of 2-D earth models and noting some simple and consistent relationships. The coupled mode technique describes Lg as a sum of modal eigenfunctions, associated with a laterally homogeneous reference model, and weighted by modal coefficients which vary with position. In 2-D these coefficients satisfy a system of non-linear differential equations which can be solved exactly in principle. In practice, however, the method becomes computationally intractable in zones of exaggerated heterogeneity (greater than 5% velocity perturbation or 2 km in boundary perturbation for 1 Hz waves) as consideration of intermode coupling among an increasing number of modes becomes necessary. Maupin (1988) has made use of coupled local modes to give a more direct treatment of variable boundaries; however changes in position of interfaces require extensive

recomputation of modal eigenfunctions. Ray diagrams, in contrast, are most useful where the shape of the crustal waveguide varies and become more difficult to interpret if a non-uniform velocity structure is imposed. The analysis of Kennett (1986) included models amenable to both methods, and demonstrated the close correspondence between the coupled mode and ray diagram results.

The correspondence between the two methods is most easily explained by noting that for a given frequency the Lg wave train will comprise a finite number of modes. Each mode is characterized by a specific phase velocity which can be associated with an S -wave ray system at a particular angle to the vertical. This suggests that the geometrical regularity of a system of rays launched in some systematic fashion may be used as an indication of the coherence and sustained amplitude of a given mode in addition to a measure of the S -wave constructive interference condition.

At fixed frequency, the angle of S -wave propagation to the vertical increases as the phase velocity decreases. For a given angle to the vertical, determined by the choice of incident mode, the behaviour after interaction with heterogeneity can be conveniently characterized by the spread in propagation angles. Increased angles to the vertical correspond to conversion to lower order modes with the change in angle proportional to the spread in intermode coupling. Similarly steeper propagation corresponds to conversion to higher order modes. In general waves incident with higher phase velocities tend to show a larger variation in ray angle after transmission through the heterogeneity due to increased multiple reflections in the complex zone. In those cases where caustics develop in the ray diagrams the equivalent coupled mode solutions show considerable variation in the amplitudes of the modal coefficients with position. The crustal models presented in the following analysis exhibit 3-D heterogeneity. It is no longer possible to construct ray patterns from a system of plane waves in cross section as in the 2-D case. Rather, we have resorted to plan views representing surface/Moho topography via contours and consider a single point source emitting rays at a given angle to the vertical (modal slowness) through a range of azimuths. Regularity in plotted surface/Moho reflection points

and the horizontal projection of raypaths provide a measure of the constructive interference condition in a manner analogous to the pattern of 2-D ray systems considered by Kennett (1986). An added feature of the 3-D analysis is that we are able to examine the way in which the wavefronts tilt as they encounter features oblique to the line of propagation. This 'ray tilt' will lead to a rotation of the polarization of the *S*-waves between the vertical and horizontal planes. As a result energy having an explosive source lying purely in a vertical plane (i.e. represented solely by higher mode Rayleigh waves), may after encounter with the heterogeneity have some component in the horizontal plane, which would be interpreted as conversion to higher mode Love waves. Zones of the model where the ray patterns exhibit a consistent change in tilt can be expected to produce significant conversion.

We have investigated *Lg* propagation in two simple models of crustal structure and in reconstructions of the crustal structure in central Asia based on world topography data, using ray diagrams. All of these models are characterized by simple variation in surface and basement (Moho) topography; physical properties remain constant throughout the crust.

Although a more sophisticated analysis which incorporates crustal stratification might be implemented with slight modification, this has not been done for several reasons. Ray diagrams are at best semiquantitative and are meant to provide a simple means of assessing *Lg* propagation through crustal structure. Moreover, although the shape of the crustal waveguide is the dominant factor influencing *Lg*, other factors such as small scale heterogeneity in crustal velocities cannot be entirely ignored. In addition, when we construct a crustal model from just the surface topography we are forced to make a number of assumptions which may not be well founded, and so a very complex model is not warranted.

Shear velocities of 3.5 km s^{-1} and 4.6 km s^{-1} were chosen to characterize the crust and upper mantle. For a given phase velocity rays were traced away from a point source with multiple reflections at the surface and the Moho. The character of the propagation patterns are indicated by plotting a plan section of the rays and marking the reflection points at the Moho. The magnitude of changes in ray tilt

are illustrated on separate ray diagrams by 'ticks' located perpendicular to the ray at surface and Moho reflection points. This display enables the regular character of the *Lg* wavefront in the homogeneous regions to be compared with the disrupted pattern after passage through the heterogeneity.

2.3 SIMPLE CRUSTAL STRUCTURES

In this section we consider two structures for which the surface and Moho topography are represented as piecewise smooth surfaces and the zones of variation are of limited spatial extent. The ray patterns resulting from these simple configurations are intended to demonstrate the effects of simple changes in topography and aid in the interpretation of more complicated ray diagrams for central Asia presented in the following section.

2.3.1 Crustal Thickening

The first case is of a linear mountain chain and a section across the model along the profile A-B is shown. Ray diagrams were examined for the full range of phase velocities associated with *Lg* propagation (3.5 to 4.6 km s⁻¹) and several general features were observed to characterize the wavefield emergent from the structure. In figure 2.1 we show the ray diagrams for phase velocities of 3.7 km s⁻¹ and 4.3 km s⁻¹ which illustrate the major features of the wavefield. Perpendicular to the axis of the range we note a central corridor marked by a continuous and regular arrangement of bottom reflection points and little variation in ray density with azimuth (even though the angle to the vertical is increased). This corridor is flanked by zones of alternating high and low ray density as the angle between the axis and the wavefront increases.

For particular combinations of source location and phase velocity, some zones may exhibit regularity in both ray density and reflection pattern, and, therefore represent windows through which a significant fraction of the initial *Lg* wave energy can pass. As ray incidence becomes increasingly oblique to the mountain range axis, rays become trapped within the range and few rays penetrate the structure to the crust outside. Thus we might expect linear zones of crustal thickening to act as

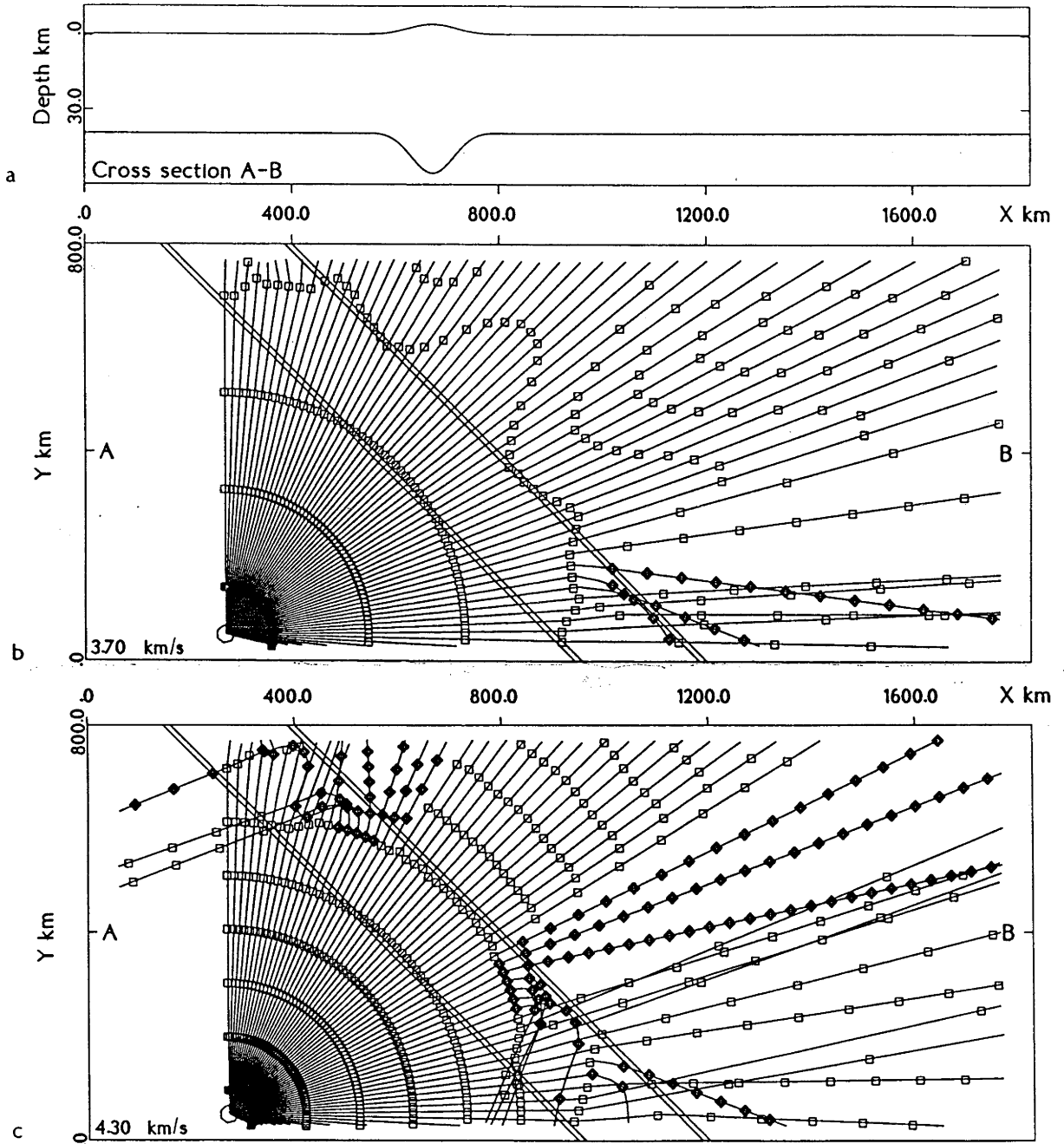


Figure 2.1. Reflection patterns for a linear zone of crustal thickening, simulating a mountain range. A vertical cross section through A-B is shown in a), while b) and c) show ray diagrams generated at phase velocities of 3.7 and 4.3 km s⁻¹. Moho reflection points are plotted as open squares; filled diamonds indicate leakage out of the crustal waveguide.

lateral waveguides, especially for earthquakes in close proximity.

Rays which impinge on the surface at angles to the surface normal less than the critical angle defined by Snell's law will undergo transmission into the upper mantle and are distinguished by filled diamonds marking the Moho reflection points. Kennett (1986) notes that this situation represents a coupling to modes constituting the S_n phase. As observed in the equivalent 2-D model (Kennett, 1986), refraction after emergence from the mountain range occurs most prominently near the maximum phase velocity for Lg (above 4.4 km s^{-1}). The distribution of these emergent refracted rays is dependent on source location and displays little azimuthal preference. In figure 2.2 we show the polarization of the S rays for the same phase velocities as figure 2.1 by means of 'ticks' which show the projection onto a horizontal plane of waves which left the source with energy confined to the vertical plane. Figure 2.2 reveals that changes in ray tilt develop in the zones flanking the central corridor where topographic gradients become increasingly transverse to the direction of propagation. Thus we expect some transfer of energy between vertical and horizontal planes which would be interpreted as a conversion between Love and Rayleigh waves. This feature introduces further complexity to what is already seen to be a very complicated wavefield.

2.3.2 Crustal thinning

The second case represents a region of crustal thinning simulating a zone of localized crustal extension and is designed to illustrate the effect of local transverse gradients in topography. In cross section it is characterized by flanks rising from 30 km to 20 km depth to a plateau at the Moho; its configuration in the horizontal plane is elliptical. Kennett (1986) noted in an analogous 2-D model that constriction of the crustal waveguide had a more severe effect on transmission of Lg than increase in thickness. This is also apparent in the 3-D model (see figure 2.3) where we again display ray patterns for phase velocities of 3.7 and 4.3 km s^{-1} . The effect of the crustal thinning is to steer rays away from the angular window defined by the lateral boundaries of the structure and the source position. This effect, due to

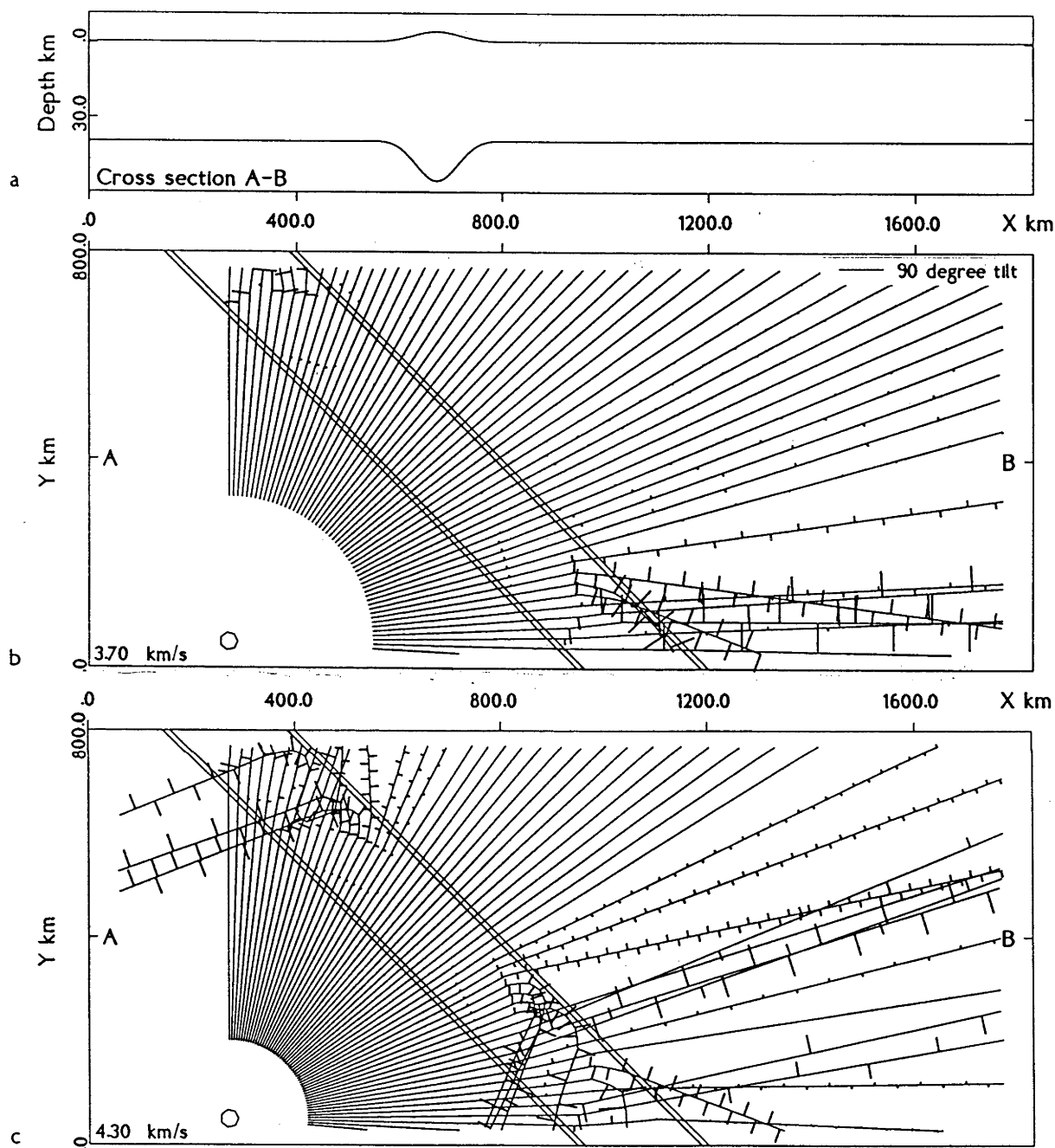


Figure 2.2. Tilt patterns for the same configuration as figure 2.1. The effects of transverse gradients in topography are evidenced through the development of ray tilt at angles oblique to the axis of the zone of thickening. The magnitude of a tick corresponding to a 90° change in tilt is indicated.

the presence of strong gradients in topography oblique to the ray path, gives rise to caustics in the ray pattern, and is particularly noticeable at higher phase velocities as in figure 2.3c. The constriction of the waveguide can also lead to waves traveling across the top of the structure being refracted into the mantle, and hence the low density of emergent rays is complemented by an additional energy loss through conversion to S_n .

In figure 2.4, we show the ray tilt representation of the S -wave polarisation for this crustal thinning model. The three dimensional character of the crustal heterogeneity leads to polarization changes for a larger proportion of the rays than for the linear mountain chain of figure 2.2. At the higher phase velocities the polarization rotation is particularly strong at the lower margin of the heterogeneity where the gradients in Moho topography are nearly transverse to the ray path.

2.4 CENTRAL ASIA

The propagation of Lg is of considerable interest in determining the nature of the earth's crust in central Asia. Extreme variations in topographic relief over the region shown in figure 2.5 testify to a complex and heterogeneous waveguide. To the north of the Indian shield (average elevation 200 m), the Himalayas rise above 6000 m and form the southern perimeter of the Tibetan Plateau, a remarkable physiographic feature covering some $2.5 \times 10^6 \text{ km}^2$ and characterized by an average elevation of about 5000 m. The Tarim Basin separates the Tibetan Plateau to the northwest from a second major mountain range, the Tien Shan.

The pattern of Lg propagation varies significantly over the region (see Figure 2.5; Ruzaike *et al.*, 1977; Ni & Barazangi, 1983). Specifically, Lg propagates very efficiently across the more tectonically stable areas: the Indian Shield, Tarim Basin and Eurasian Platform. The phase is also present for paths along the Himalayas and Tien Shan (although less clear); however it is generally weak or absent for paths crossing the Tibetan Plateau. These observations bear important implications for the interpretation of the tectonic evolution of the region. Additional interest in the nature of Lg propagation across central Asia is prompted by the presence of

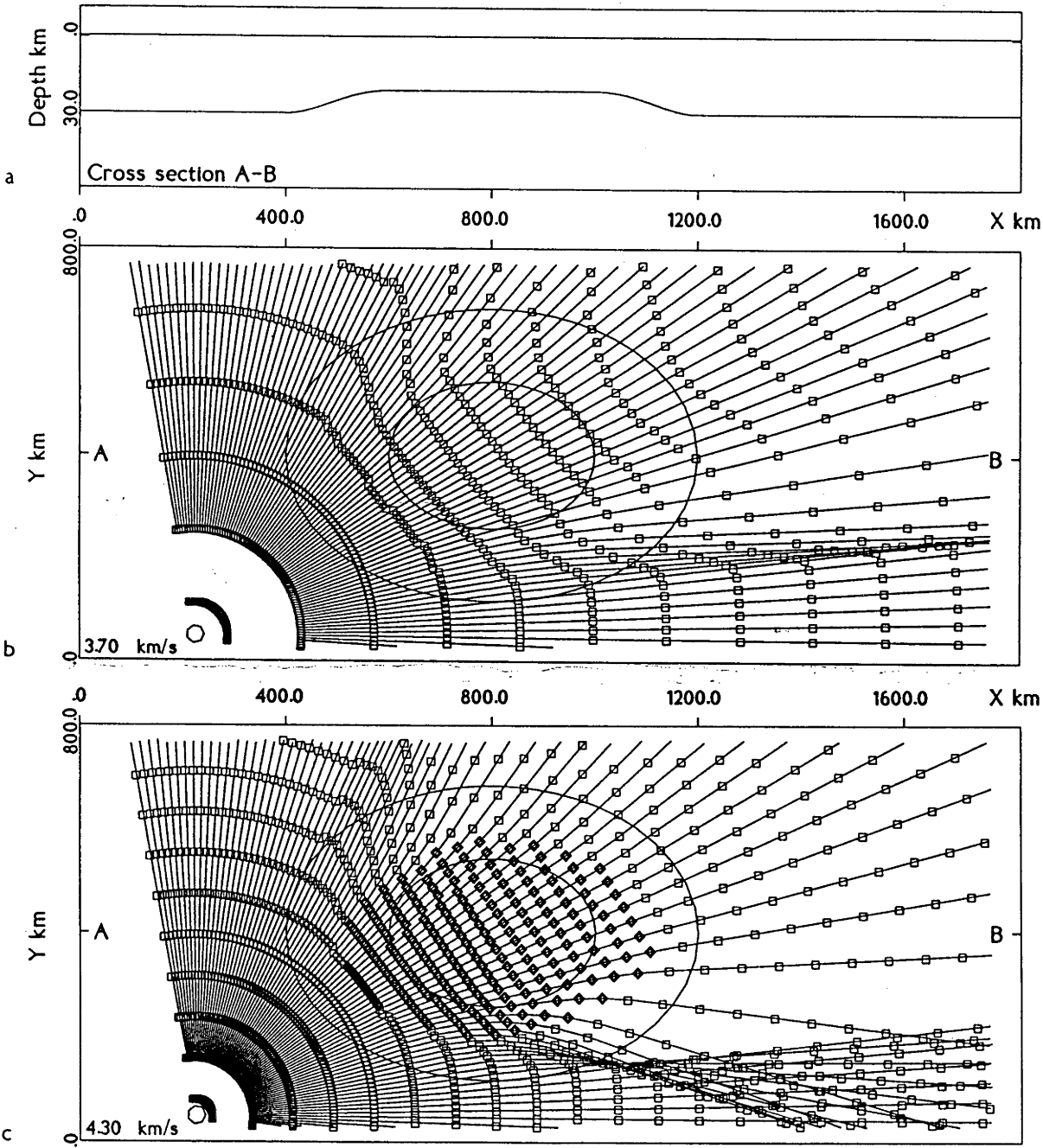


Figure 2.3. Reflection patterns for rays impinging on an elliptical zone of crustal thinning.

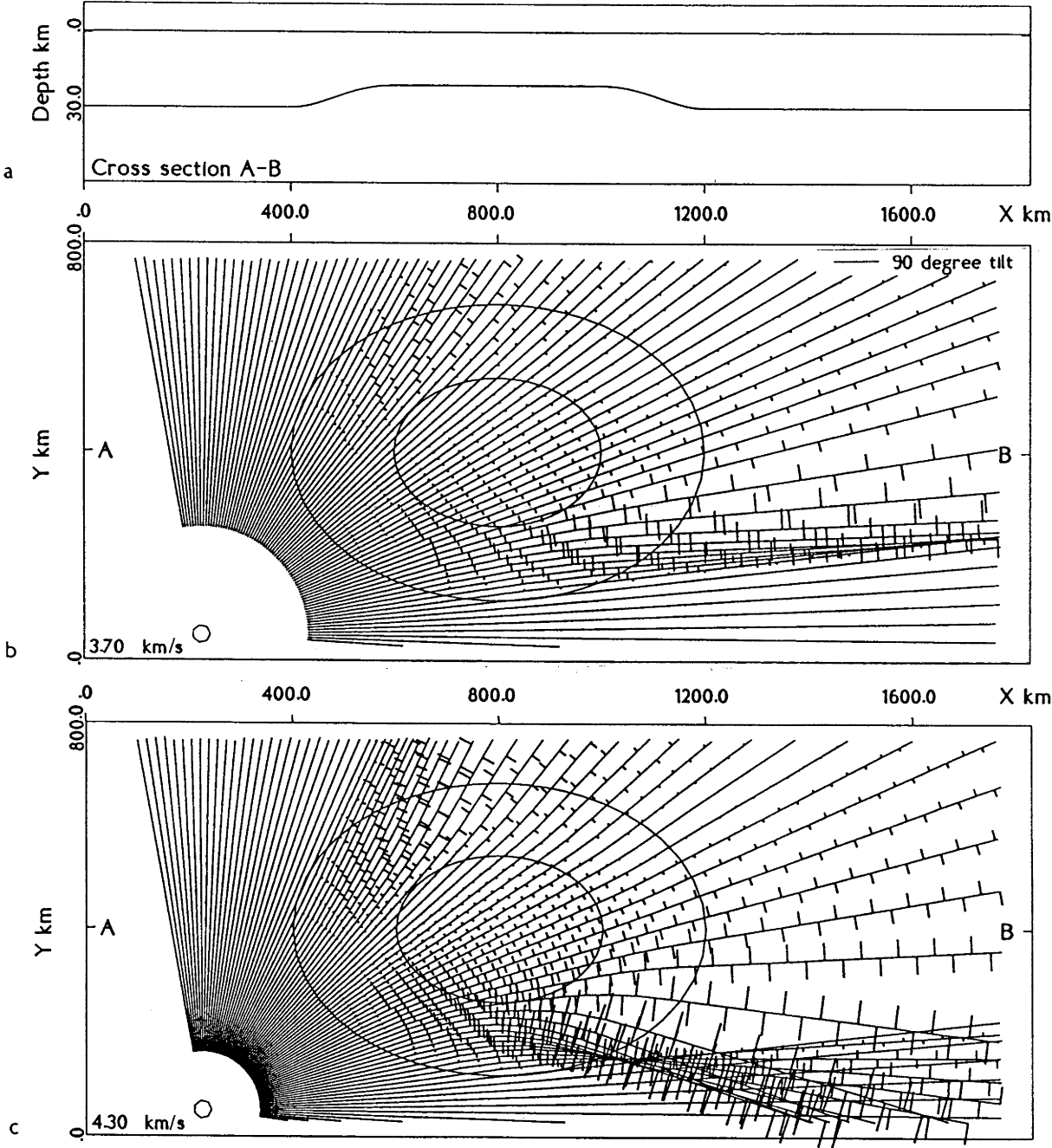


Figure 2.4. Tilt patterns for the same configuration as figure 2.3.

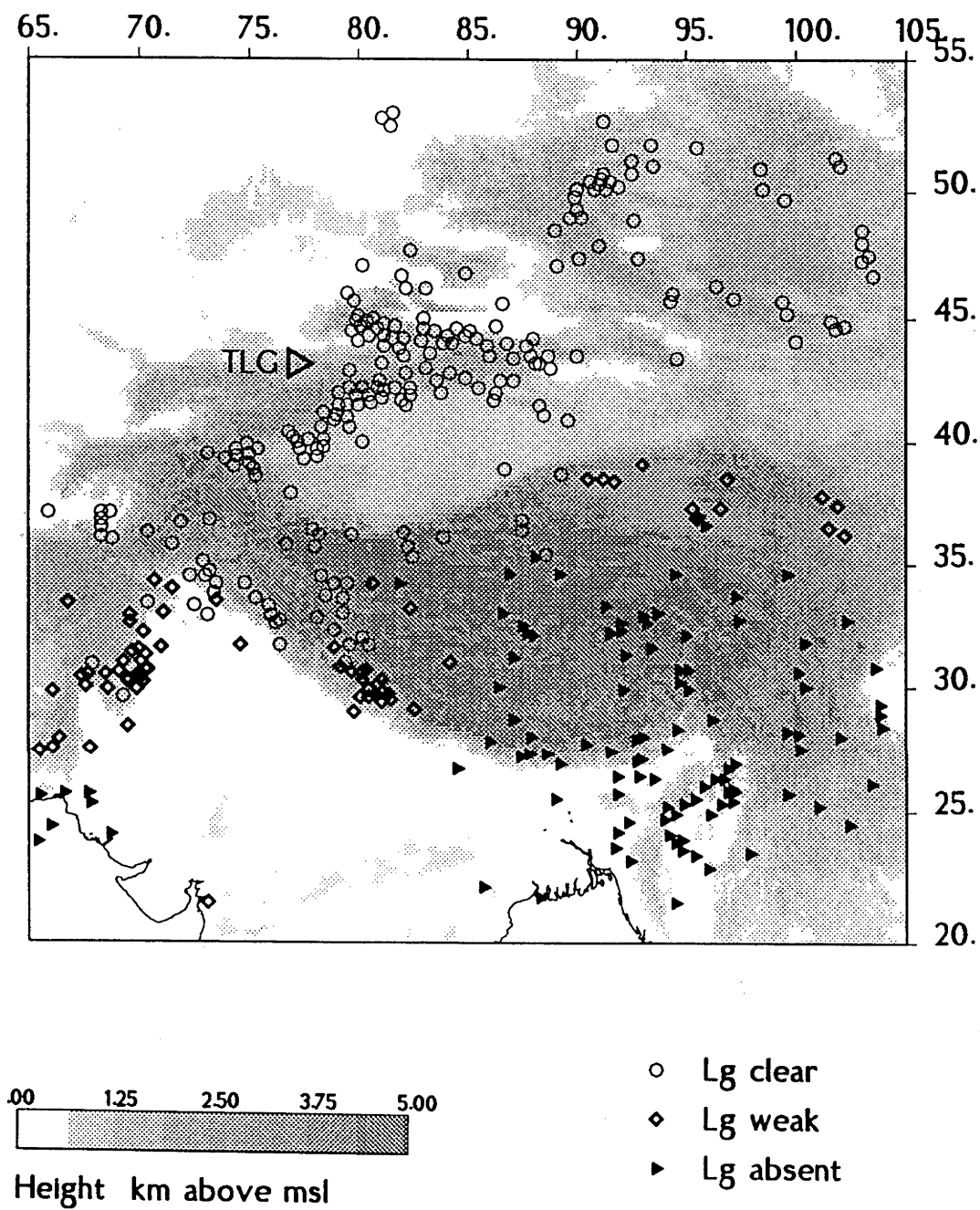


Figure 2.5. Central Asia. a) Topography and epicentral data from Ruzaiikan *et al.* (1977) illustrating characteristics of *Lg* propagation to Talgar (TLG). Data are plotted as circles (*Lg* clear), diamonds (*Lg* weak), and triangles (*Lg* absent).

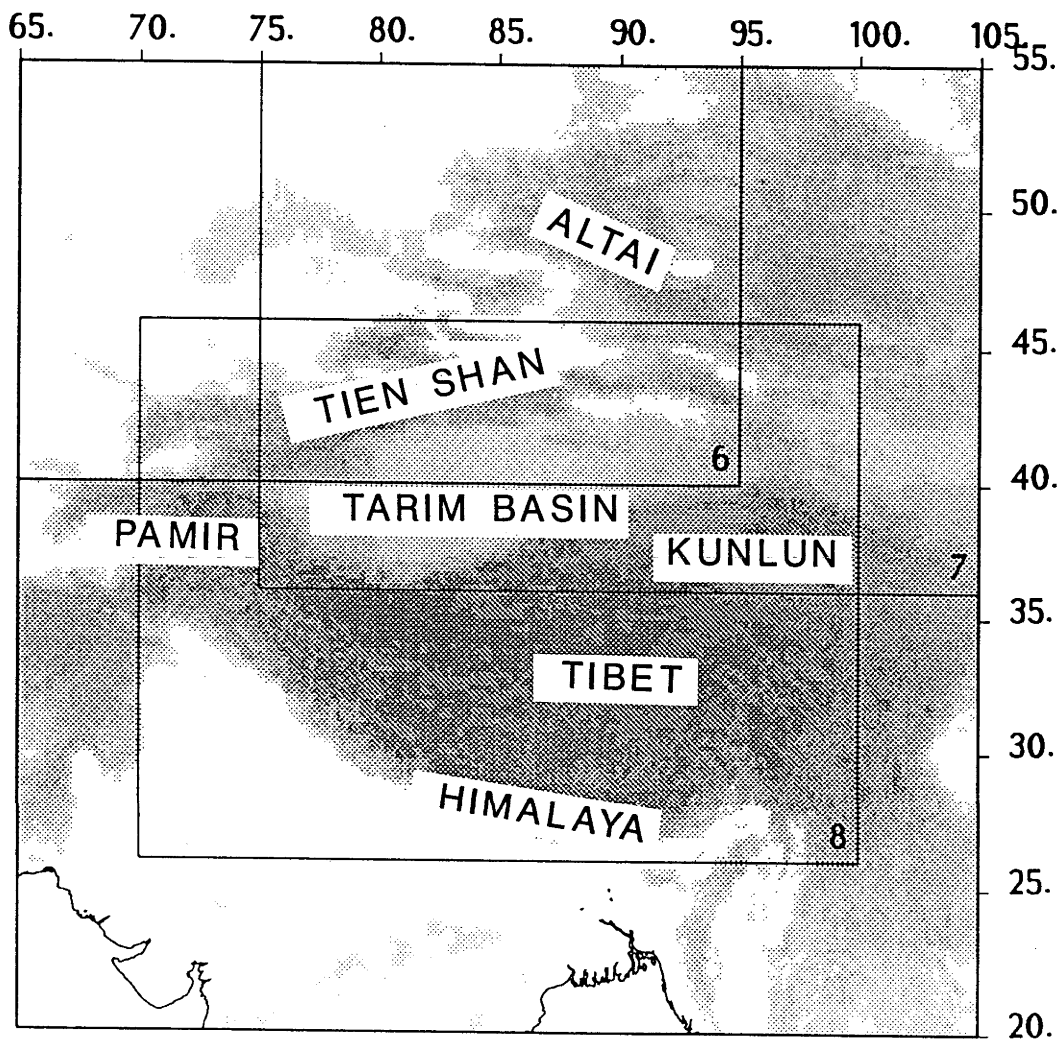


Figure 2.5. Central Asia. b) Reference map identifying major topographic features in a) and relative positions of maps in figures 6, 7 and 8. Note that there is some systematic distortion in this and the following figures due to conversion from spherical to rectangular coordinates.

both Soviet and Chinese nuclear test sites. The prominence of *Lg* on regional seismograms makes the phase valuable for the detection of small events, but the sensitivity of *Lg* to the variation in crustal structure across the region reduces its utility for discrimination and a better knowledge of the effects of heterogeneity on propagation is necessary.

Surface topography for the following models of the crustal structure was constructed by smoothing (25 point average) digital elevation data supplied at 5' intervals from a world topography data base (ETOPO5 compiled by the National Geophysical Data Center, Boulder, Colorado). The depth to the Moho was determined by further smoothing (81 point average) of the surface topography and assuming 90% isostatic compensation occurs within the crust. This assumption, although simplistic, is thought to be reasonable within the limitations of the ray method for a uniform crust.

The choice of a suitable phase velocity in representing *Lg* propagation in the real earth by ray diagrams deserves some discussion. In typical crustal situations the higher modes constituting *Lg* propagate at a range of phase velocities. Within this range certain modes within loosely defined phase velocity windows will dominate at different group velocities. The onset of *Lg* on short-period seismograms is determined by modes at phase velocities of 3.4 to 3.6 km s⁻¹ (i.e. crustal shear velocities). The amplitude maximum which follows is usually characterized by slightly greater phase velocities between 3.7 and 4.0 km s⁻¹ whereas typical values for the late-arriving Airy phases reach 4.3 to 4.5 km s⁻¹. These latter phases play an important role in determining the character of regional seismograms especially in zones of heterogeneity where conversion to *Sn* is significant. An additional factor to consider is the effect of velocity variation within the crust. Although the shape of the crustal wave guide is a major influence on *Lg* propagation, a positive crustal velocity gradient in depth will have some effect on the configuration of the ray systems, flattening them and extending the distances between reflection points. In our simple single layer crustal model this is equivalent to a decrease in phase velocity. By examining the horizontal distance between reflections for single layer

and multilayered crustal models based on equivalent total traveltime, we note that the actual magnitude of the decrease is only significant at very low phase velocities (i.e. within a few percent of the crustal shear velocity). Therefore in modelling effects on the early arrivals it may be appropriate to compensate by reducing the phase velocity parameter for crustal models of uniform velocity. As the overall character of the *Lg* wavetrain is dependent on modal contributions at a full range of phase velocities we have attempted to select model values which both illustrate the general disruption in *Lg* propagation due to changes in the shape of the crustal wave guide, and are pertinent to observations made in previous studies.

Although the major topographic features in the area exist at a larger lateral scale than those in the two simple structural models, the ray patterns can nonetheless be quite sensitive to source position. Attempts have been made in the following analysis to restrict interpretation to those features which remain stable over small changes in source location.

2.4.1 Semipalitinsk

For our first study, we examine *Lg* propagation in the vicinity of the Soviet nuclear test sites located in eastern Kazakhstan, approximately 400 km northeast of Lake Balkhash, near Semipalitinsk. Seismic stations at Novosibirsk (NSB) and Talgar (TLG) are shown as triangles to the north and southeast of the source at Semipalitinsk in figure 2.6. The rather moderate and consistent topography over much of this region permits an assessment of our modelling procedure in what is presumably a relatively uniform portion of the crustal waveguide.

The ray diagrams shown in figure 2.6 are generated at a phase velocity of 4.3 km s^{-1} since anomalous propagation patterns are most pronounced at velocities above 4.0 km s^{-1} . The character of the patterns can be discussed in terms of three broad geographic zones. To the north, elevation decreases very gradually, consequently the ray patterns are very regular and experience a slight increase in phase velocity with distance from the source. For initial phase velocities above 4.45 km s^{-1} , this results in energy loss through conversion to *Sn*, however *Lg* transmission through

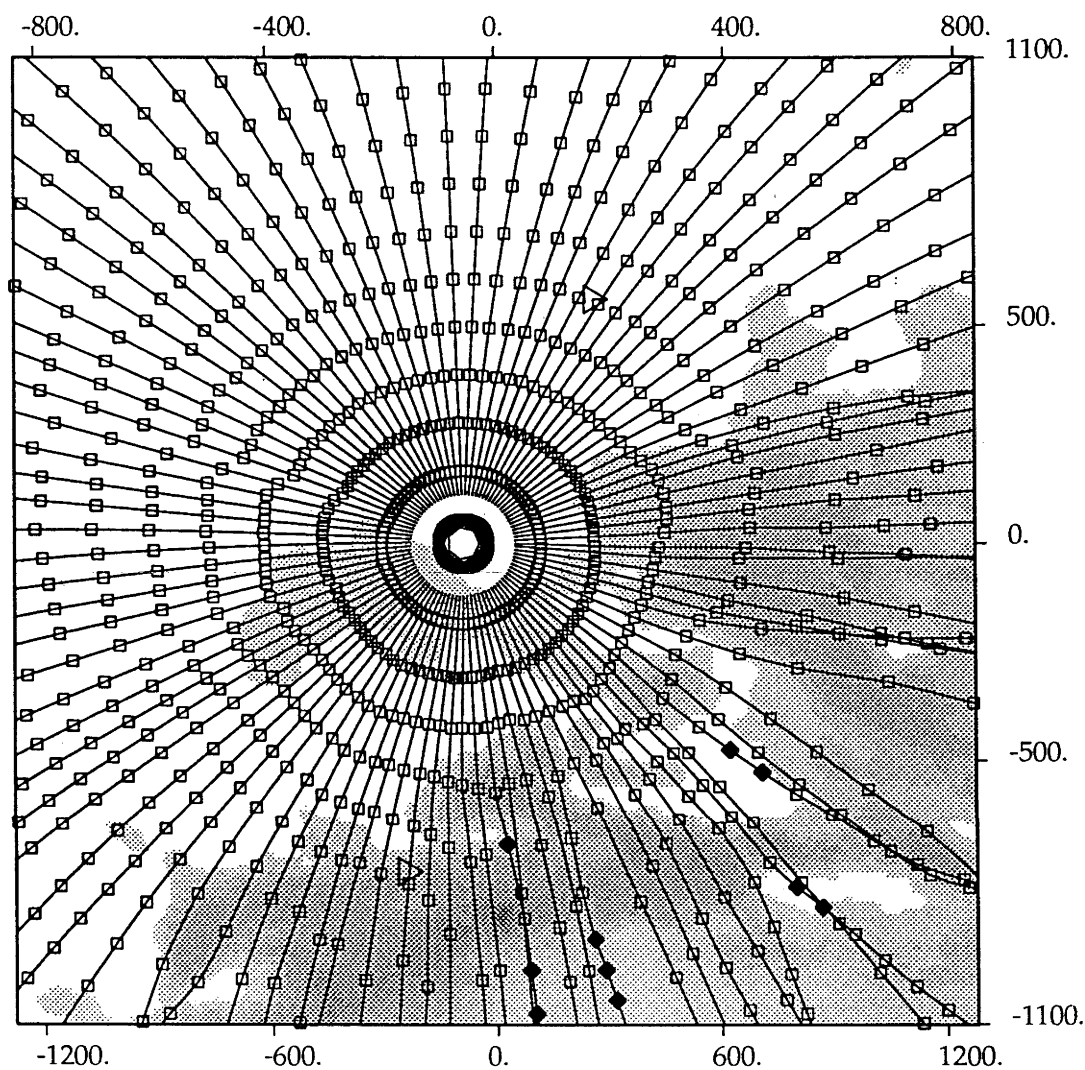


Figure 2.6. Semipalitinsk. a) Reflection pattern for a source at the Soviet nuclear test site near Semipalitinsk at a phase velocity of 4.3 km s^{-1} . Rays in the north-western quadrant represent energy propagating toward western Europe and show very regular patterns indicative of efficient *Lg* propagation.

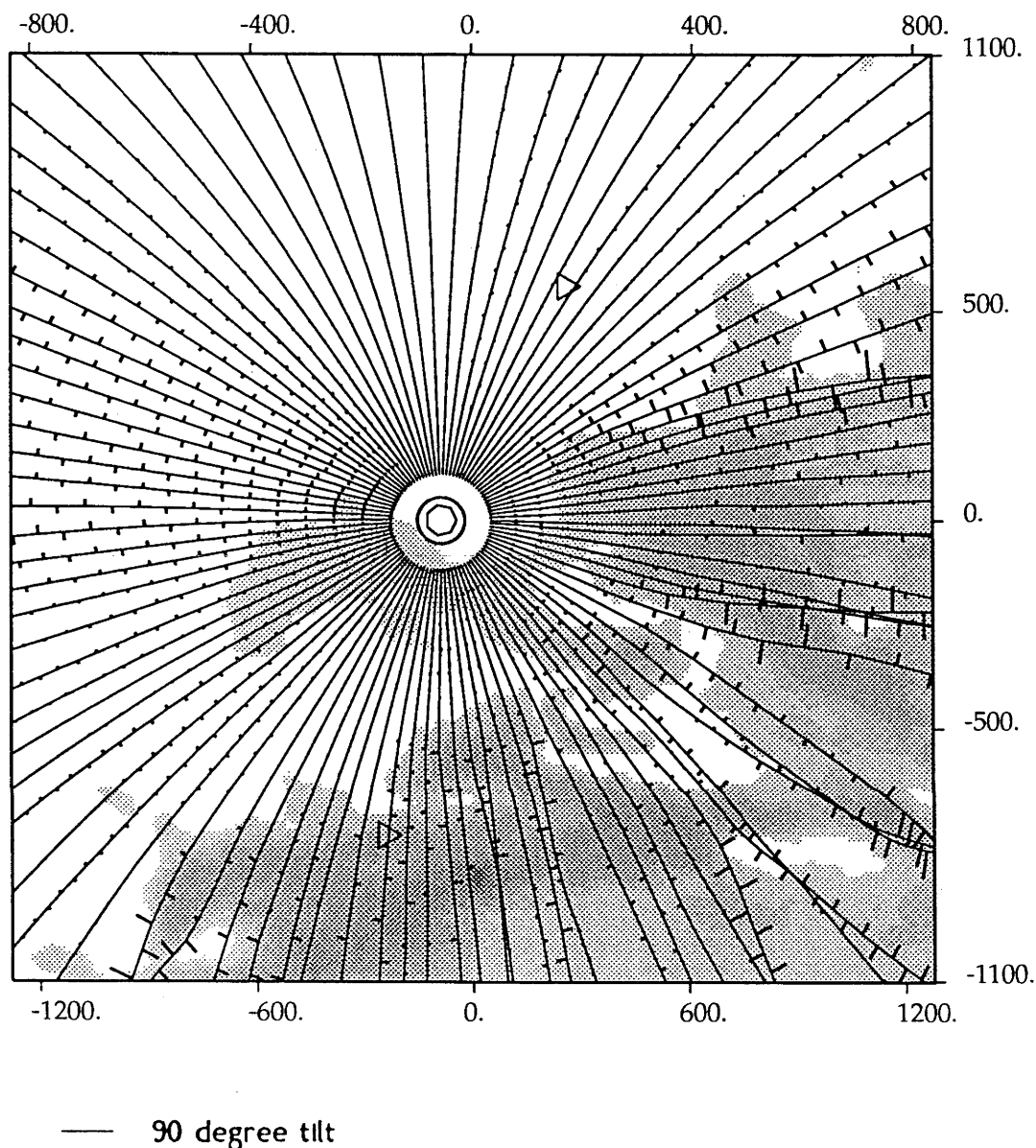


Figure 2.6. Semipalitinsk. b) Tilt pattern for a source at the Soviet nuclear test site near Semipalitinsk at a phase velocity of 4.3 km s^{-1} . The slight but regular, north-south gradient in topography causes rays over the western half of the map to exhibit significant tilt, indicating a transfer of energy between horizontal and vertical planes. The magnitude of a tick corresponding to a 90° change in tilt is indicated.

this zone for lower phase velocities is likely to be extremely efficient. It is worth noting that Lg recorded at NORSAR would propagate through this region. The general topography in Kazakhstan to the west of the test sites remains relatively constant although there are minor local fluctuations. Ray patterns in this zone are slightly less coherent but still suggest efficient Lg transmission. The Tien Shan and Altai chains, mountain ranges rising over 5000 m, border the region to the south and east. Changes in relief are extreme and ray patterns are severely distorted especially at higher phase velocities. The character of Lg recorded at stations along these paths beyond the mountainous regions will undoubtedly be very complex; there is likely to be considerable variability with position and significant attenuation due to conversion to Sn and scattering.

In the corresponding ray tilt diagram (figure 2.6b), as expected, rays passing through mountainous areas, where elevation gradients transverse to direction of propagation are likely to be most pronounced, show the greatest degree of tilt. Also of interest and perhaps of greater significance to the interpretation of source parameters from regional seismograms are the tilt patterns to the north and west. The slight but relatively uniform north-south gradient in elevation over these areas causes a very regular and obvious tilt pattern in eastern Kazakhstan; whereas north of Semipalitinsk, where rays lie subparallel with the gradient, virtually no tilt is evident. This implies that Lg recorded at stations within these two distinct windows might differ markedly in the relative energy distribution among transverse, radial and vertical components. An explosive source at Semipalitinsk generating shear energy primarily on vertical and radial components might nonetheless give rise to Lg in south and central Europe with a dominant transverse component, while Lg recorded further north, at NORSAR for example, would retain much of its Rayleigh-type energy.

2.4.2 Eastern Tien Shan

We now turn our attention to a more complicated region comprising the Tarim Basin, the Altai chain, much of the Tien Shan and the northern fringe of the

Tibetan Plateau. Ruzaikan *et al.* (1977) noted that earthquakes in the eastern Tien Shan radiate clear *Lg* phases to station NSB whereas at TLG the phase is often very weak.

Figure 7a displays the reflection patterns at phase velocities of 3.7 and 4.3 km s⁻¹ for a source in this area of interest. The lower velocity diagram reveals a more consistent and uniform ray pattern at NSB than TLG even though the former station is some 500 km further from the source. This observation would tend to support the explanation of Ruzaikan *et al.* that the difference between signal at TLG and NSB results from different path lengths within the complicated structure of the Tien Shan. The higher phase velocity used to generate the diagram in figure 7b has produced a more severely distorted pattern due to an increased number of multiple reflections. Several specific points are worthy of note. Although the ray density and patterns are still somewhat consistent at NSB, rays in the vicinity of TLG are funnelled into two arms of the Tien Shan in a manner similar to that described for the simple crustal thickening model. Rays in the northern subsidiary arm rapidly lose energy through transmission to the mantle whereas those in the southern arm remain trapped. This funnelling behaviour first becomes apparent at a phase velocity of 3.9 km s⁻¹ and suggests that the main arm may act as a lateral wave guide, trapping and guiding energy within a zone of relatively narrow width (100-200 km). The observations of Ruzaikan *et al.* would tend to suggest that the station at TLG may lie just outside the borders of such a guide.

Ni & Barazangi (1983) note that *Lg* propagation in the Himalayas is less efficient across strike than along strike. This and studies in other mountainous regions (e.g. along the Andes [Chinn *et al.*, 1980]) suggest that mountain chains may play an important role in influencing patterns of *Lg* propagation by focussing and guiding energy along quasi-linear zones of increased crustal thickness.

2.4.3 Kunlun

Lg propagation across the Tibetan Plateau has been the focus of many studies into the nature of the crust near the continental collision margin. Several authors

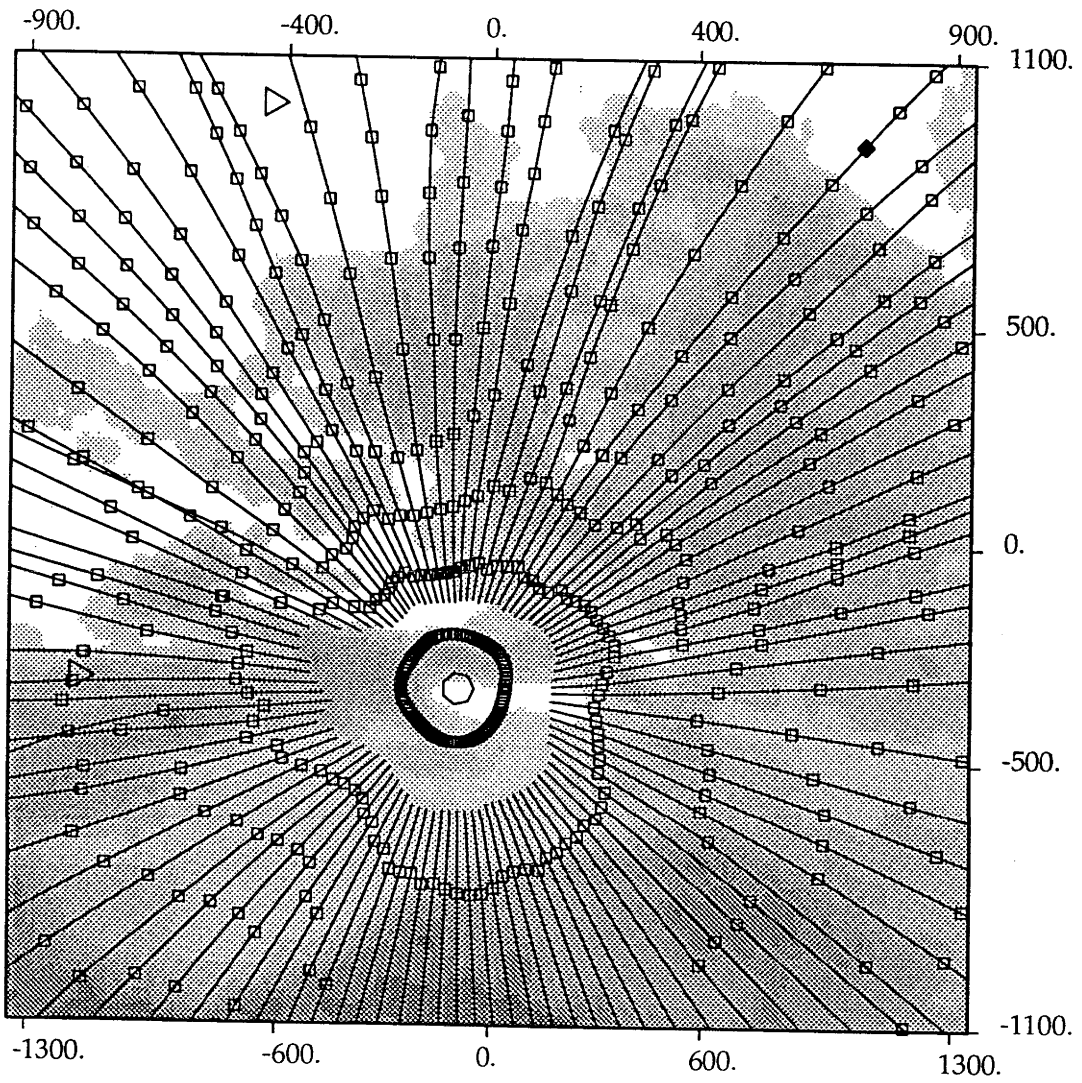


Figure 7. Tien Shan. a) Reflection pattern for a source in the eastern Tien Shan at a phase velocity of 3.7 km s^{-1} . Note distortion in reflection pattern eastwards toward Talgar.

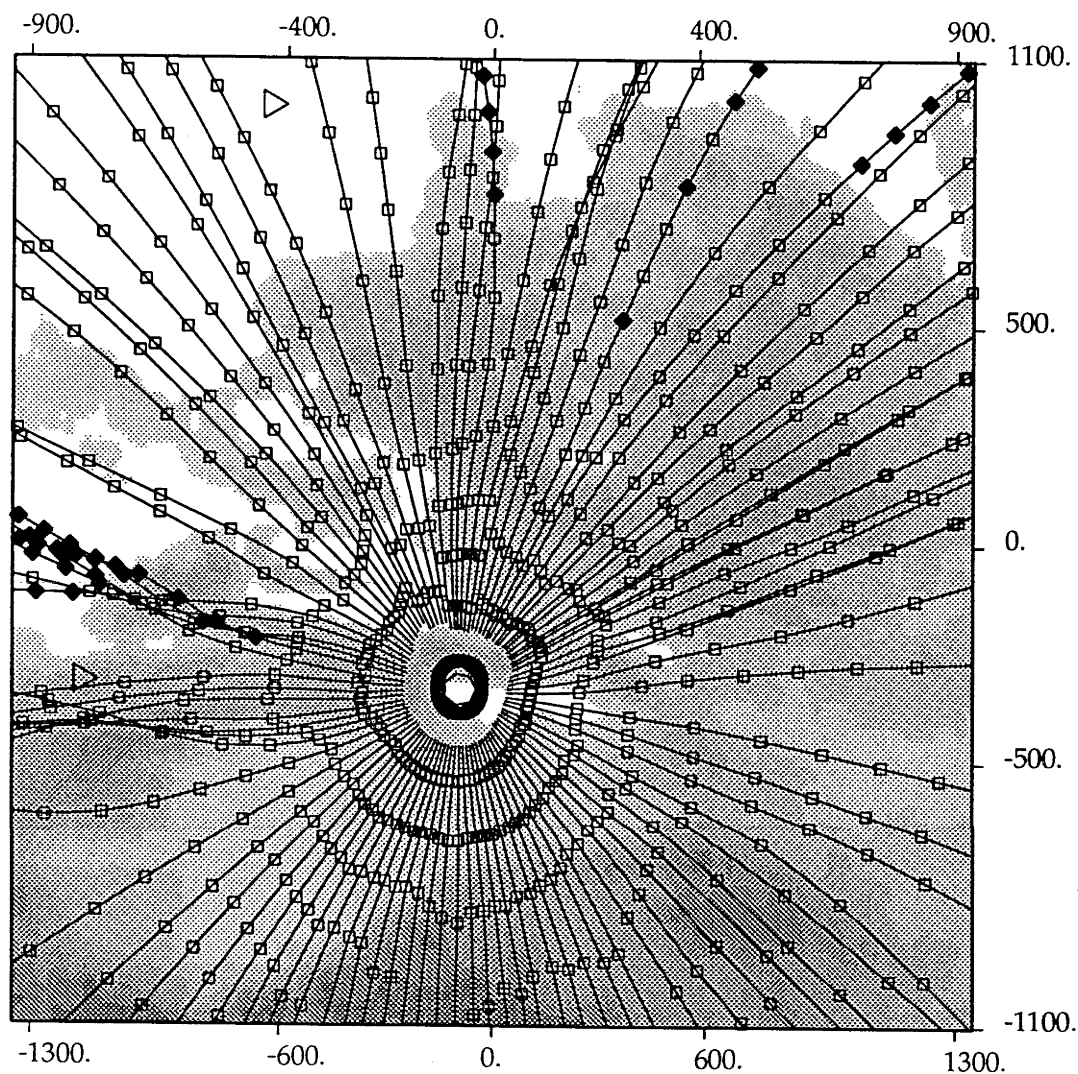


Figure 7. Tien Shan. b) Reflection pattern for a source in the eastern Tien Shan at a phase velocity of 4.3 km s^{-1} . Energy is guided into two arms of the Tien Shan near Talgar, a feature which first becomes apparent at a phase velocity of 3.9 km s^{-1} .

have noted the absence of *Lg* for paths crossing the plateau and the peculiar pattern of propagation for sources within and on its boundaries. In the latter case there appear to be well defined geographic boundaries to effective propagation, notably the 36° line of latitude to the north (Ruzaikan *et al.*, 1977) and the Indus Tsangpo suture to the south (Ni & Barazangi, 1983). *Lg* is rarely recorded from sources within these boundaries.

Ray diagrams at phase velocities of 3.7 and 4.3 km s⁻¹ for a source just above the northern boundary in the Kunlun are shown in figure 8. At higher phase velocities (figure 8b), a large proportion of rays are confined to the plateau by lateral reflection along its borders, and those which do escape have undergone so many vertical reflections at subcritical angles as to be of no significance. In contrast, diagrams at lower phase velocities maintain regular ray patterns through most of the area. Certainly, patterns are fairly regular across the Tarim Basin where propagation is observed to be reasonably efficient (Ruzaikan *et al.*, 1977). Ray systems emerging from the Himalayas are also fairly coherent but lose energy through transmission to the mantle. Our assumptions regarding the structure of the Moho are no longer valid in this region (see Molnar, 1988), where the crust is quite certainly not isostatically compensated; however a rapid transition in crustal thickness does exist and should complicate *Lg* propagation in a similar fashion. Efficient propagation across the Tarim Basin for this particular source location ceases above a phase velocity of approximately 4.0 km s⁻¹ where the effects of source position, with phase velocity, relative to the crustal transition zone (discussed by Kennett [1986]) become manifest. The ray diagram approach agrees with suggestions of previous authors (Ruzaikan *et al.*, 1977) that the absence of *Lg* along paths crossing the Tibetan Plateau is due to changes in the crustal waveguide, notably constriction along its margins. Discriminating between natural and artificial sources from locations within Tibet using *Lg* is rendered exceedingly difficult, if not impossible, by this propagation effect.

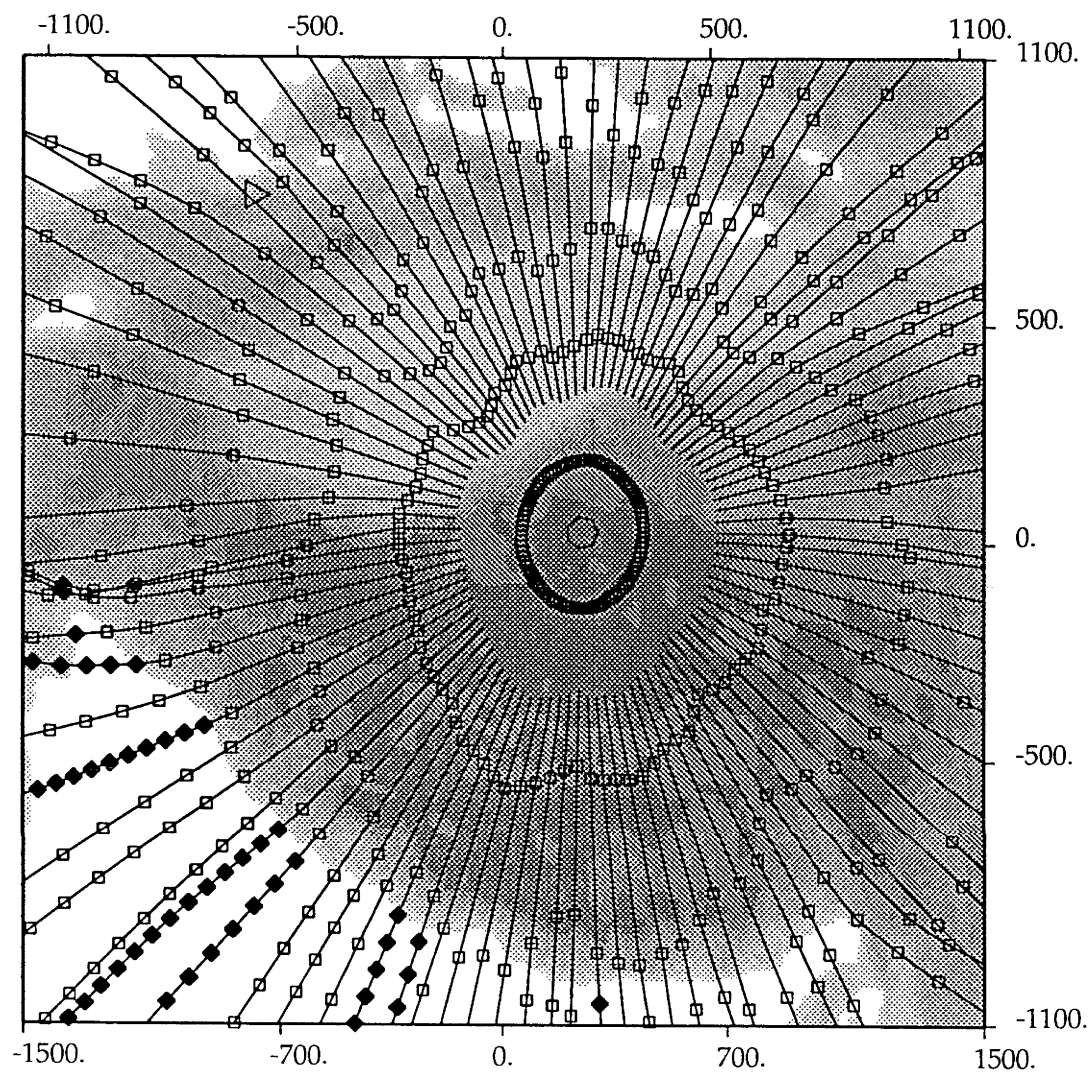


Figure 8. Kunlun. a) Reflection pattern for a source in the Kunlun, northern Tibet at a phase velocity of 3.7 km s^{-1} . Rays south of the Himalayas are characterized by increased phase velocities and widespread transmission into the mantle due to constriction of the crustal wave guide.

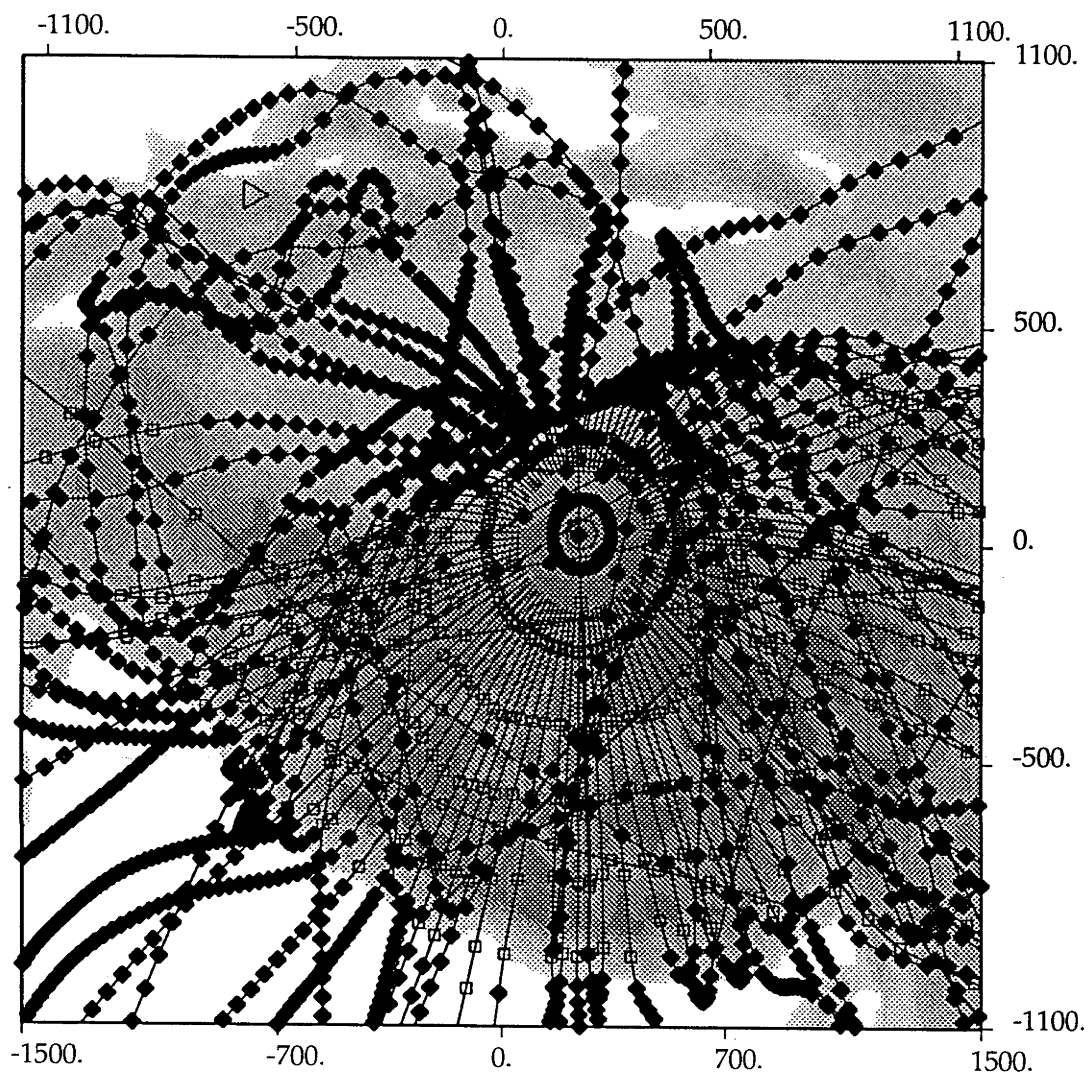


Figure 8. Kunlun. b) Reflection pattern for a source in the Kunlun, northern Tibet at a phase velocity of 4.3 km s^{-1} . The reflection pattern is highly distorted; both transmission into the mantle and reflection along the margins of Tibet contribute to very little coherent energy actually exiting through the crustal waveguide.

2.5 DISCUSSION

We have seen how the simple device of ray diagrams can be extended to three dimensions and used to provide insight into the nature of the interaction of the *Lg* phase with heterogeneity. The interpretation of ray diagrams for crude constructions of the crust in central Asia, based on isostatic compensation, agree with observations from previous studies and indicate that much of the general character of *Lg* propagation can be attributed to changes in the shape of the crustal waveguide. Zones of regular reflection patterns, such as those in Kazakhstan for sources at Semipalitinsk, characterize a waveguide of uniform thickness and generally correspond to zones of efficient *Lg* propagation. Rapid decreases in crustal thickness, as for example the margins of the Tibetan Plateau, form boundaries which severely restrict *Lg* propagation especially at higher phase velocities. Where such changes occur over narrow, quasi-linear zones of crustal thickening (mountain ranges such as the Tien Shan, or the Himalayas), the possibility exists for the development of lateral wave guides. The three dimensional ray representation also affords a simple means of assessing the transfer of energy between horizontal and vertical planes through examination of the change in ray tilt with propagation. Over areas with transverse gradients in surface and Moho topography oblique to the direction of propagation, we expect there to be significant energy transfer corresponding to an interconversion between Love and Rayleigh modes. This feature may be of some value in separating propagation and source effects in the problem of nuclear discrimination where energy at the explosive source is predominantly *P-SV*.

By incorporating detailed information on gravity anomalies and crustal structure where these are available it may be possible to further improve the accuracy of the technique. However, caution should be exercised not to over extend the limits of its usefulness; internal crustal structure undoubtedly plays a significant and complicated role in the propagation of *Lg* as is made evident in its effect on dispersion. The value of the ray approach lies in assessing the main effects of crustal shape on *Lg*.

CHAPTER 3

SURFACE WAVE SCATTERING FROM 3-D OBSTACLES

3.1 INTRODUCTION

The earth exhibits a dominantly radial variation in physical properties, a fact evidenced by the success of one-dimensional radial models in predicting body wave travel times and periods of the earth's free oscillations. Consequently, the seismic response at regional distances in the outer 400 km, where the effects of the earth's sphericity are minimal, can in many instances be accurately modelled by considering propagation in laterally homogeneous, horizontally stratified media. As a specific example, the general character of the surface wave train (e.g. the predominant *Lg* phase) is well described as a sum of independently propagating modes whose individual contributions are determined by a source excitation function (Knopoff *et al.*, 1973; Kennett, 1985). Notwithstanding, it has become increasingly obvious in recent years that most regions of the earth exhibit significant lateral heterogeneity and in consequence we need to develop techniques to determine the character of seismic wave propagation in appropriate heterogeneity models. The indication of a wide range of studies (see e.g. Lay, 1987) is that deviations from uniformly stratified earth models are most pronounced within the crust and upper mantle comprising the outer 400 km of the earth. It is also apparent that this heterogeneity can take on a variety of forms, dependent to some degree on wavelength, from random perturbation through to discrete, isolated scattering bodies.

A number of techniques have emerged to describe the interaction of seismic surface waves with various classes of heterogeneity. Woodhouse (1974) has shown that for a medium with very gradual lateral variation in physical properties with respect to wavelength, a given mode will propagate independently and evolve according to the local environment. In three dimensions local modes can be tracked using a surface wave ray theory developed in the same work. It has since been shown that surface wave ray tracing is essential in the study of short-period waves at global scales (Sobel & von Seggern, 1978) and in the analysis of free oscillations using higher orbit, long period waves (Schwartz & Lay, 1987). In media where physical properties vary more rapidly, the coupling of energy to adjacent modes must be considered. Surface wave propagation across structures exhibiting two-dimensional

heterogeneity of this class is effectively treated using the coupled mode approach of Kennett (1984a) in which the displacement and traction fields are represented by the modal eigenfunctions for a plane stratified reference structure. The character of the waveform as it advances through the perturbed medium is dictated by the evolution of weighting coefficients which satisfy a system of ordinary differential equations. The method provides a useful means of determining the nature of changes to the waveform propagating in this class of structure but is strictly valid only for two dimensional geometries and fails to account for out-of-plane scattering effects. Three-dimensional scattering of surface waves by discrete bodies has been investigated by several authors using first-order perturbation theory (Hudson, 1967; Kennett, 1973; Snieder, 1986a; 1986b). In particular, Snieder has developed a formalism for surface wave scattering employing the far-field surface wave Green's function. The method is linear in deviations from the reference structure, a property used to advantage by Snieder (1988b,c) to invert for lateral heterogeneity across Europe but unfortunately places significant restrictions on the classes of scatterer that can be considered. Snieder (1988a) has since developed a nonlinear treatment of surface wave scattering at large distances from the heterogeneity but this has yet to be extended to the near-field.

The focus of this work is the presentation of a theory which, in principle, may be used to investigate surface wave scattering from obstacles exhibiting arbitrarily large contrasts in physical properties and is valid in both the near- and far-fields. The theory extends the work of Waterman, for acoustic (1969), electromagnetic (1971) and elastic body waves (1976), to surface waves in a plane stratified medium by employing truncated modal expansions (Kennett, 1984a; Maupin & Kennett, 1987) in the construction of basis functions. We begin the development in section 3.2 by formulating the surface wave scattering problem and selecting appropriate basis function expansions for the incident and scattered components of the displacement field based on physical arguments. In section 3.3 we demonstrate via Betti's identity that these basis functions are orthogonal with respect to a specific closed surface integral. This result is exploited in section 3.4 to derive a surface wave

T-matrix which relates the known expansion coefficients of the incident wavefield to those of the scattered field. The assumption of lossless media and time reversal invariance impose restrictions on the structure of this matrix, described in section 3.5, which considerably simplify numerical implementation of the theory. In the remaining sections we examine several simple models at a variety of scales to investigate the relationship between the geometry and dimension of the scatterer and various aspects of surface wave scattering (i.e. wavetype conversion, modal coupling, far-field radiation etc.)

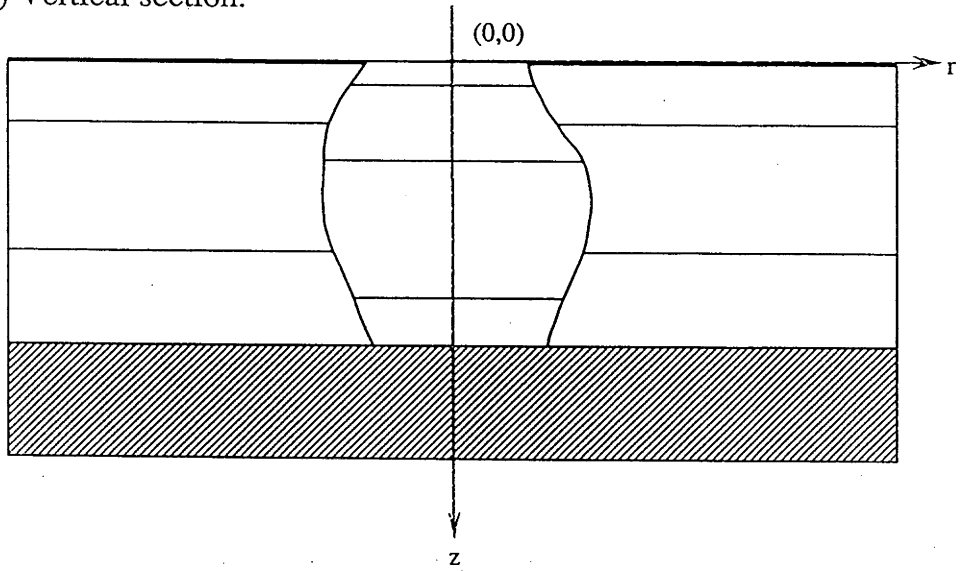
3.2 A FORMALISM FOR SURFACE WAVE BASIS FUNCTIONS

The derivation of the surface wave scattering matrix may proceed along one of two lines; the original analysis for acoustic waves (Waterman, 1969) employed Huyghen's principle and appropriate expansions of the free-space Green's function, a strategy which has been widely adopted in subsequent extensions to the general technique for electromagnetic and elastic waves, and multiple scattering. Although this approach provides insight into the original derivation and development, we will pursue a second, equivalent route which employs Betti's identity (*cf.* Pao, 1978) and which alleviates the need to invoke analytic continuation arguments and complicated triadic manipulations.

Our first concern is to formulate the surface wave scattering problem and define the physical quantities of interest. Although the theory applies to discrete obstacles of arbitrary geometry and constitution, we will simplify matters slightly and consider a scattering body which exhibits variation in the vertical direction alone, and is characterized by a smooth but otherwise arbitrary horizontal cross-section, $R(\theta, z)$, which may vary as it extends from the surface to infinite depth. The obstacle is contained within an otherwise horizontally stratified half-space and a cylindrical coordinate system is defined such that the origin and the z -axis lie wholly within the scatterer as in figure 3.1.

We wish to break the total surface wave displacement field $\mathbf{u}^t$ into its incident, scattered and refracted components, $\mathbf{u}^i$, $\mathbf{u}^s$, $\mathbf{u}^o$, respectively. These are defined

a) Vertical section.



b) Plan view.

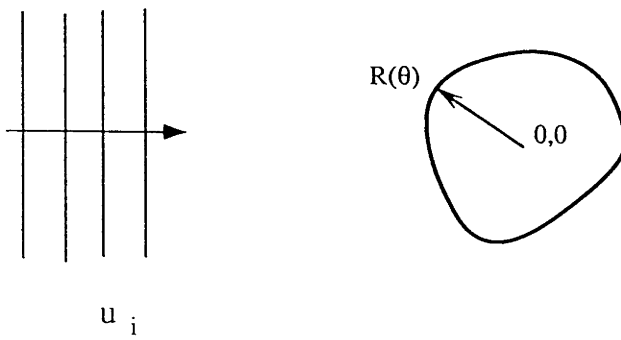


Figure 3.1 Illustration of the problem geometry for surface wave scattering from a discrete obstacle embedded in a laterally homogeneous half-space; a) vertical section, and b) plan view.

such that

$$\begin{aligned} \mathbf{u}^t &= \mathbf{u}^i + \mathbf{u}^s, & r > R(\theta, z); \\ &= \mathbf{u}^o, & r < R(\theta, z). \end{aligned} \quad (3.1)$$

and are to be expanded in sets of basis functions of the form

$$\Psi = \mathbf{W}(z)\mathbf{Y}(r, \theta)e^{-i\omega t}, \quad (3.2)$$

which solve the homogeneous (i.e. no forcing) elastic wave equation in cylindrical coordinates. The depth dependence, $\mathbf{W}(z)$, is composed of the Love and Rayleigh wave eigenfunctions for the appropriate horizontally stratified medium, and $\mathbf{Y}(r, \theta)$ are vector cylindrical harmonics which describe the horizontal behaviour of the fields. The harmonic time dependence $e^{-i\omega t}$ is common to all parts of the wavefield and will henceforth be suppressed.

In seismological applications involving laterally homogeneous stratified media [cf. Kennett (1983)], $\mathbf{Y}(r, \theta)$ is constructed using Bessel functions $J_m(k_n r)$ since the boundary conditions prescribe a solution which is, in general, regular in the vicinity of $r = 0$. We require this class of behaviour of the incident field $\mathbf{u}^i$ since it originates at some distance outside the scatterer and would be finite and continuous in the absence of the heterogeneity. Explicitly then, the incident field will be expanded in basis functions of the form

$$(3.\hat{\psi}^l)_n^m = \begin{cases} W_n(k_n, \omega, z)\hat{\mathbf{T}}_n^m(r, \theta), & l = 1; \\ U_n(k_n, \omega, z)\hat{\mathbf{R}}_n^m(r, \theta) + V_n(k_n, \omega, z)\hat{\mathbf{S}}_n^m(r, \theta), & l = 2 \end{cases} \quad (3.3)$$

in order to represent the modal wavefield (note that throughout this section we will denote quantities corresponding to this regular basis function set by an overhead caret). The functions U_n , V_n , and W_n are the displacement eigenfunctions for the modes which describe the depth dependence and are readily determined for both Love ($l = 1$) and Rayleigh ($l = 2$) waves in plane-stratified earth models. The vector cylindrical harmonics $\hat{\mathbf{R}}_n^m$, $\hat{\mathbf{S}}_n^m$, and $\hat{\mathbf{T}}_n^m$ (Takeuchi and Saito, 1972) describe the radial and azimuthal dependences,

$$\hat{\mathbf{R}}_n^m = \hat{\mathbf{z}}\hat{Y}_n^m, \quad \hat{\mathbf{S}}_n^m = k_n^{-1}\nabla_1\hat{Y}_n^m, \quad \hat{\mathbf{T}}_n^m = -\hat{\mathbf{z}} \times \hat{\mathbf{S}}_n^m, \quad (3.4)$$

where the horizontal standing wave function $\hat{Y}_n^m$ is defined as

$$\hat{Y}_n^m = \epsilon_m J_m(k_n r) \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}. \quad (3.5)$$

Indices n and m refer to modal and azimuthal orders, respectively, $\nabla_1 (= \hat{\mathbf{r}}[\partial/\partial r] + \hat{\theta}[r^{-1}\partial/\partial\theta])$ is the horizontal gradient operator, and the Neumann factor ϵ_m is equal to $1/\sqrt{2}$ for $m = 0$ and 1 otherwise. We have chosen to consider real-valued trigonometric functions under a common index m to describe azimuthal dependence rather than the more conventional complex exponential $e^{im\theta}$ representation, in order to facilitate subsequent analysis.

We note that the scattered field $\mathbf{u}^s$ is defined in the region exterior to the object only; hence the boundary conditions impose no restrictions on its behaviour at the origin. It is therefore appropriate to expand $\mathbf{u}^s$ using some linear combination of the Bessel, $J_m(k_n r)$, and Neumann, $N_m(k_n r)$, functions in the construction of the vector cylindrical harmonics. Given the formulation of the problem, we expect the scattered displacement field to consist of outgoing waves, and so the Hankel function of the first kind $H_m^{(1)}(k_n r)$ ($= J_m(k_n r) + iN_m(k_n r)$), is physically appropriate. A point of significance in later development is that, unlike the regular basis set in (3.3) for the incident wave, the basis functions constructed using $H_m^{(1)}$ will be singular at the origin on account of the imaginary component N_m . The scattered field is thus assembled by replacing $\hat{Y}_n^m$ in (3.3), (3.4) and (3.5) by the outgoing horizontal wavefunction Y_n^m which we define as

$$Y_n^m = \epsilon_m H_m^{(1)}(k_n r) \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}. \quad (3.6)$$

The explicit form of the singular basis set is then

$$(3.\psi^l)_n^m = \begin{cases} W_n(k_n, \omega, z) \mathbf{T}_n^m(r, \theta), & l = 1; \\ U_n(k_n, \omega, z) \mathbf{R}_n^m(r, \theta) + V_n(k_n, \omega, z) \mathbf{S}_n^m(r, \theta), & l = 2. \end{cases} \quad (3.7)$$

From a theoretical standpoint the approach we are about to describe places no restriction on the basis functions used to expand the transmitted field $\mathbf{u}^o$. In general $\mathbf{u}^o$ will be regular and continuous at the origin, and for computational considerations it is obviously desirable to expand $\mathbf{u}^o$ in a basis set that will accurately represent any physical field over the surface of the obstacle in as few terms

as possible. Since we have prescribed a scatterer exhibiting internal stratification in the problem formulation, an appropriate set of basis functions, $(\hat{\phi}^l)_n^m$ will have the same form as that in (3.3) with k_n replaced by κ_n , where κ_n is the horizontal wavenumber associated with the n th order mode for a laterally homogeneous medium with the same vertical velocity profile as the scattering obstacle.

Each of the incident, scattered and refracted wavefields is represented as a sum over modes and angular orders. Over the horizontal coordinates we have a complete orthogonal expansion and the modal basis sets may be considered complete by taking the deepest layers in the stratified media to behave as perfect reflectors. Then, at any given frequency, if the modal dimension N of the function space is chosen sufficiently large, P and S body wave phases in both the half-space and the scatterer are synthesized through modal interference (*cf.* Harvey [1981]; Malischewsky, 1987). Finally, we adopt the eigenfunction normalization given in Appendix A which ensures that each individual, outgoing basis function carries an equivalent (unit) energy across any surface enclosing the z -axis.

To summarize, the three component displacement fields can be written (as illustrated for the incident field) in the form

$$\mathbf{u}^i = \sum_n \left\{ \sum_m (a^1)_n^m [W_n \hat{\mathbf{T}}_n^m] + \sum_m (a^2)_n^m [U_n \hat{\mathbf{R}}_n^m + V_n \hat{\mathbf{S}}_n^m] \right\}, \quad (3.8)$$

where the decomposition in terms of individual modal contributions n , from Love ($l = 1$), and Rayleigh ($l = 2$) wave components is clearly evident. It will simplify the ensuing analysis considerably if we compress this notation by abbreviating the triple summation over l, m, n under a single summation σ . Then we may write

$$\mathbf{u}^i = \sum_l \sum_m \sum_n (a^l)_n^m (\hat{\psi}^l)_n^m = \sum_\sigma a^\sigma \hat{\psi}^\sigma, \quad (3.9a)$$

with similar expressions for the internal and scattered fields:

$$\mathbf{u}^o = \sum_\sigma b^\sigma \hat{\phi}^\sigma, \quad (3.9b)$$

$$\mathbf{u}^s = \sum_\sigma c^\sigma \psi^\sigma. \quad (3.9c)$$

3.3 ORTHOGONALITY RELATIONS

Consider any two displacement fields, $\mathbf{u}$ and $\mathbf{v}$, both characterized by a harmonic time dependence $e^{-i\omega t}$, over a closed surface S which contains no energy sources. Betti's identity asserts that the two fields and their associated tractions ($\mathbf{t}(\mathbf{u})$, $\mathbf{t}(\mathbf{v})$) over S satisfy

$$\int_S dS [\mathbf{t}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{t}(\mathbf{v}) \cdot \mathbf{u}] = 0, \quad (3.10)$$

(cf. Aki & Richards, (1980), eqns 2.35 and 7.94)). We will apply this relation to surface waves propagating in a horizontally stratified halfspace, and will use it to derive orthogonality relations for the basis functions presented in the previous section.

Let $\mathbf{u} = (\hat{\psi}^l)_n^m = \hat{\psi}^\sigma$, $\mathbf{v} = (\hat{\psi}^r)_q^p = \hat{\psi}^\nu$, and S be any surface enclosing the z -axis. Since both $\hat{\psi}^\sigma$ and $\hat{\psi}^\nu$ (and their respective traction fields) are regular throughout S , we have for all $\sigma(l, m, n)$ and $\nu(r, p, q)$ that

$$\int_S dS [\mathbf{t}(\hat{\psi}^\sigma) \cdot \hat{\psi}^\nu - \mathbf{t}(\hat{\psi}^\nu) \cdot \hat{\psi}^\sigma] = 0. \quad (3.11)$$

Now let us define $\mathbf{u}$ and $\mathbf{v}$ to be two functions from our outgoing basis set, ψ^σ and ψ^ν . Note in this instance that the imaginary components of $\mathbf{u}$ and $\mathbf{v}$ are singular at the origin and hence can be thought of as energy sources. Thus, in order to apply (3.10), we consider S to be a closed toroidal surface formed by subtracting two closed cylindrical surfaces S_1, S_2 of radii r_1, r_2 ($r_2 > r_1$) which extend to infinite depth and encompass the z -axis. Since both fields are regular within this surface we may split the integral in (3.10) into separate components over the inner radial surface $S_{r=r_1}$, the outer radial surface $S_{r=r_2}$, the top surface $S_{z=0}$ and the bottom surface $S_{z \rightarrow \infty}$ and write,

$$\begin{aligned} & \int_{S_2-S_1} dS [\mathbf{t}(\psi^\sigma) \cdot \psi^\nu - \mathbf{t}(\psi^\nu) \cdot \psi^\sigma] \\ &= \left\{ \int_{S_{r=r_2}} dS - \int_{S_{r=r_1}} dS + \int_{S_{z=0}} dS + \int_{S_{z \rightarrow \infty}} dS \right\} [\mathbf{t}(\psi^\sigma) \cdot \psi^\nu - \mathbf{t}(\psi^\nu) \cdot \psi^\sigma]. \end{aligned} \quad (3.12)$$

A negative sign is applied to the integral over $S_{r=r_1}$ since we have chosen the outward normal to be positive in the positive radial direction. We note that both

traction fields vanish at the free surface and that displacements tend to 0 as z approaches ∞ ; hence the component integrals over $S_{z=0}$ and $S_{z\rightarrow\infty}$ do not contribute, and we can equate the two remaining components. It can be demonstrated using the modal orthogonality relations presented in Appendix A and the orthogonality of sines and cosines, that both integrals must vanish identically for any choice of σ, ν ($l, m, n; r, p, q$); hence we have

$$\int_S dS [\mathbf{t}(\psi^\sigma) \cdot \psi^\nu - \mathbf{t}(\psi^\nu) \cdot \psi^\sigma] = 0. \quad (3.13)$$

for any surface S enclosing the z -axis.

Alternatively we set $\mathbf{u} = \psi^\sigma$, $\mathbf{v} = \hat{\psi}^\nu$, and apply Betti's identity over the same toroidal surface, $S_2 - S_1$. Once again, contributions from the top and bottom surfaces vanish and we are left with

$$\int_{S_2} dS [\mathbf{t}(\psi^\sigma) \cdot \hat{\psi}^\nu - \mathbf{t}(\hat{\psi}^\nu) \cdot \psi^\sigma] = \int_{S_1} dS [\mathbf{t}(\psi^\sigma) \cdot \hat{\psi}^\nu - \mathbf{t}(\hat{\psi}^\nu) \cdot \psi^\sigma] \quad (3.14)$$

but in this case, the integrals vanish only for $\sigma \neq \nu$, and by employing the definition of the Wronskian of $J_m(k_n r)$ and $H_m^{(1)}(k_n r)$, we see that

$$\int_S dS [\mathbf{t}(\psi^\sigma) \cdot \hat{\psi}^\nu - \mathbf{t}(\hat{\psi}^\nu) \cdot \psi^\sigma] = \frac{2i}{\omega} \delta_{mp} \delta_{qn} \delta_{lr}, \quad (3.15)$$

where δ_{ij} is the Kronecker delta and the modal eigenfunction normalization given in Appendix A has been adopted. This result is due to the imaginary component of $H_m^{(1)}$ which represents an effective energy source for the outgoing basis set. By noting that $J_m = (H_m^{(1)} + H_m^{(2)})/2$, we see that the integral in (3.15) is directly proportional to the total energy flux of an arbitrary outgoing function ψ^σ as shown in Appendix A. It should be re-emphasized that because we are considering propagation in loss-less media conservation of energy applies and the surface of integration in equations (3.11), (3.13), and (3.15) may be replaced by any a surface encompassing the z -axis. This observation will allow us to formulate scattering and transmission matrices for surface waves as described in the following section.

3.4 THE SURFACE WAVE T-MATRIX

We wish now to exploit these orthogonality relations to gain information on the coefficients a^σ , b^σ , and c^σ which describe the incident, transmitted and scattered

surface wave displacement fields in the problem outlined in section 3.2. Recall that we are considering a discrete obstacle extending from the surface to infinite depth, embedded within a stratified halfspace, and enclosing the z -axis of a cylindrical coordinate reference frame. The displacement field immediately outside the obstacle surface, $R(\theta, z)$, can be written as the sum of its incident and scattered parts (i.e. $\mathbf{u}^i + \mathbf{u}^s$) which are, in turn, expanded in terms of our regular and outgoing basis function sets as in (3.9). By inserting $\mathbf{u} = \mathbf{u}^i + \mathbf{u}^s$ and $\mathbf{v} = \hat{\psi}^\nu$ into (3.10), applying the orthogonality relations in the preceding section, and setting S to be the surface of the obstacle approached from the outside, we find

$$\begin{aligned} \int_S dS \left\{ \mathbf{t} \left(\sum_{\sigma} [a^{\sigma} \hat{\psi}^{\sigma} + c^{\sigma} \psi^{\sigma}] \right) \cdot \hat{\psi}^{\nu} - \mathbf{t}(\hat{\psi}^{\nu}) \cdot \sum_{\sigma} [a^{\sigma} \hat{\psi}^{\sigma} + c^{\sigma} \psi^{\sigma}] \right\} \\ = \sum_{\sigma} \left\{ a^{\sigma} \int_{R(\theta, z)} dS [\mathbf{t}(\hat{\psi}^{\sigma}) \cdot \hat{\psi}^{\nu} - \mathbf{t}(\hat{\psi}^{\nu}) \cdot \hat{\psi}^{\sigma}] \right. \\ \left. + c^{\sigma} \int_{R(\theta, z)} dS [\mathbf{t}(\psi^{\sigma}) \cdot \hat{\psi}^{\nu} - \mathbf{t}(\hat{\psi}^{\nu}) \cdot \psi^{\sigma}] \right\} \\ = \frac{2i}{\omega} c^{\nu}. \end{aligned} \quad (3.16)$$

(when performing closed surface integrations over this class of obstacle we need only consider the vertical/subvertical boundaries at finite depth (i.e. $R(\theta, z)$) because as we have already noted traction vanishes at the free surface and displacement tends to zero as z approaches infinity). Since the obstacle and the surrounding halfspace are defined to be in welded contact, we can equate the total displacement and traction field on either side of the obstacle surface (i.e. $\mathbf{u}^i + \mathbf{u}^s = \mathbf{u}^o$ on $R(\theta, z)$) and rewrite (3.16) as

$$\begin{aligned} c^{\nu} &= \frac{\omega}{2i} \int_{R(\theta, z)} dS \left[\mathbf{t} \left(\sum_{\sigma} b^{\sigma} \hat{\phi}^{\sigma} \right) \cdot \hat{\psi}^{\nu} - \mathbf{t}(\hat{\psi}^{\nu}) \cdot \sum_{\sigma} b^{\sigma} \hat{\phi}^{\sigma} \right] \\ &= \frac{\omega}{2i} \sum_{\sigma} \left\{ \int_{R(\theta, z)} dS [\mathbf{t}(\hat{\phi}^{\sigma}) \cdot \hat{\psi}^{\nu} - \mathbf{t}(\hat{\psi}^{\nu}) \cdot \hat{\phi}^{\sigma}] \right\} b^{\sigma} \\ &= \sum_{\sigma} \hat{Q}^{\nu\sigma} b^{\sigma}. \end{aligned} \quad (3.17)$$

If we further expand $\hat{Q}^{\nu\sigma}$, we may write (3.17) as

$$(c^r)_q^p = \frac{\omega}{2i} \sum_l \sum_n \left\{ \sum_m \int_{R(\theta, z)} dS [\mathbf{t}((\hat{\phi}^l)_n^m) \cdot (\hat{\psi}^r)_q^p - \mathbf{t}((\hat{\psi}^r)_q^p) \cdot (\hat{\phi}^l)_n^m] (b^l)_n^m \right\}. \quad (3.18)$$

This result is an expression of Huyghen's principle since it clearly indicates that each modal component q of wavetype r in the scattered wavefield can be seen to

arise from an effective distribution of sources, represented by surface integrals of the (unknown) interior basis functions, over $R(\theta, z)$. Furthermore, we may arrive at a similar expression for the coefficients of incident wavefield a^ν by replacing $\hat{\psi}^\nu$ in (3.16) by ψ^ν ,

$$a^\nu = -Q^{\nu\sigma} b^\sigma, \quad (3.19)$$

where

$$Q^{\nu\sigma} = \frac{\omega}{2i} \left\{ \int_{R(\theta, z)} dS \left[\mathbf{t}(\hat{\phi}^\sigma) \cdot \psi^\nu - \mathbf{t}(\psi^\nu) \cdot \hat{\phi}^\sigma \right] \right\}. \quad (3.20)$$

This provides a corollary to Huygen's principle as expressed in (3.18) by affirming that the traction and displacement components contributing to the effective source field are not, in fact, independent but are related through the known incident wave series coefficients. (Note that in different applications the relationship in (3.19) might be exploited directly to describe the wavefield within the obstacle.) By representing the coefficient vectors $(a^\sigma, b^\sigma, c^\sigma)$ and the tensors $Q^{\nu\sigma}, \hat{Q}^{\nu\sigma}$ as matrix quantities (i.e. $\mathbf{a}, \mathbf{b}, \mathbf{c}; \mathbf{Q}, \hat{\mathbf{Q}}$) we may exploit the simplicity of matrix notation and relate the unknown scattering coefficients to those of the incident wave series as

$$\mathbf{c} = -\hat{\mathbf{Q}}(\mathbf{Q}^{-1})\mathbf{a} \quad (3.21)$$

$$= \mathbf{T}\mathbf{a},$$

where we will define $\mathbf{T}$ as the surface wave T-matrix (not to be confused with the cylindrical harmonic in (3.7)). To gain some appreciation of what this matrix represents, let us partition the coefficient vectors such that Love waves ($l = 1$) are represented by the first $(N \times M)$ elements and Rayleigh waves ($l = 2$) by the remaining $(N \times M)$ elements. We will further subdivide these partitions in terms of N individual modal groups such that, for example, the first M elements describe the wavefield associated with the fundamental Love mode. If we examine the structure of (3.21) in this context, we note that

$$\begin{pmatrix} \mathbf{c}_L \\ \mathbf{c}_R \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{L \rightarrow L} & \mathbf{T}_{R \rightarrow L} \\ \mathbf{T}_{L \rightarrow R} & \mathbf{T}_{R \rightarrow R} \end{pmatrix} \begin{pmatrix} \mathbf{a}_L \\ \mathbf{a}_R \end{pmatrix}, \quad (3.22)$$

where we have designated the wavetype partitions in the column vectors $\mathbf{c}$ and $\mathbf{a}$ by an L (Love) and an R (Rayleigh). The corresponding partitions in $\mathbf{T}$ therefore describe the interaction among wavetypes between the scattered and incident

wavefields. Within each of these 4 wavetype partitions exist $(N \times N)$ ‘modal’ submatrices which define the energy transfer among the various modes in a similar fashion. Thus we see that the scattering matrix $\mathbf{T}$ can be viewed as describing the modal coupling between wavetypes in the scattered field and those in the incident field.

3.5 STRUCTURE OF THE T-MATRIX

In the previous section we presented the derivation of a scattering matrix theory for seismic surface waves. We wish now to examine restrictions on the matrix structure imposed by temporal reciprocity and conservation of energy, which will prove useful in the numerical implementation of the theory in following sections. We follow Waterman (1969) and begin with a reorganization of the total displacement field outside the scatterer in terms of incoming and outgoing surface wave basis functions $\psi^{\sigma*}, \psi^{\sigma}$. The basis set $\psi^{\sigma*}$ is constructed as ψ^{σ} with the Hankel function of the second kind, $H_m^{(2)}$, substituted for $H_m^{(1)}$ in (3.7) (this is equivalent to taking the complex conjugate of ψ^{σ} without operating on the implicit harmonic time dependence). The reorganization is merely a means of simplifying the analysis by posing the problem in a more symmetrical, though less physical, form. We note that $\hat{\psi}^{\sigma} = 1/2(\psi^{\sigma} + \psi^{\sigma*})$ and substitute this relation into our original expression for the external displacement field in (3.1):

$$\begin{aligned}
 \mathbf{u}^t &= \mathbf{u}^i + \mathbf{u}^s \\
 &= \mathbf{a}^T \hat{\boldsymbol{\psi}} + \mathbf{c}^T \boldsymbol{\psi} \\
 &= 1/2[\mathbf{a}^T \boldsymbol{\psi}^* + (\mathbf{a}^T + 2\mathbf{c}^T)\boldsymbol{\psi}] \\
 &= 1/2[\mathbf{a}^T \boldsymbol{\psi}^* + \mathbf{g}^T \boldsymbol{\psi}].
 \end{aligned} \tag{3.23}$$

Here $\mathbf{T}$ denotes transpose and we have dispensed with tensor notation to work with matrix quantities: we consider the wave series coefficients and individual basis functions as column vectors (e.g. $a^{\sigma} = \mathbf{a}$, $\psi^{\sigma} = \boldsymbol{\psi}$). Since we have defined the coefficients of the incoming field as $\mathbf{g} = \mathbf{a} + 2\mathbf{c}$, it becomes apparent that we can relate $\mathbf{g}$ and $\mathbf{a}$ through

$$\mathbf{g} = \mathbf{S}\mathbf{a}. \tag{3.24}$$

where $\mathbf{S}$ equals $(\mathbf{I} + 2\mathbf{T})$.

To consider the effect of energy conservation, we revert temporarily to indicial notation and define the energy flux Σ_i for a given displacement field u_i as

$$\Sigma_i = -\tau_{ij} \frac{\partial u_j}{\partial t}, \quad (3.25)$$

where τ_{ij} is the associated stress tensor, and the summation convention is followed for repeated indices. Since we are dealing with the energy of fields with harmonic time dependence, we will consider the energy flux due to the real component of displacement averaged over a single period:

$$\frac{1}{T} \int_0^T dt \Sigma_i = \frac{i\omega}{4} [\tau_{ij}^* u_j - \tau_{ij} u_j^*]. \quad (3.26)$$

The integrated energy flux s associated with total exterior displacement field $\mathbf{u}^t$ through a closed surface S surrounding the scatterer (or, equivalently, the total power leaving this volume) is found by substituting the first equation in (3.23) into the left hand side of (3.26) and integrating, which after some algebra yields

$$\begin{aligned} s_i(\mathbf{u}^t) = \frac{i\omega}{16} \sum_{\sigma} \sum_{\nu} a^{\sigma*} a^{\nu} \int_S dS [t(\psi^{\sigma}) \cdot \psi^{\nu*} - t(\psi^{\nu*}) \cdot \psi^{\sigma}] \\ + a^{\sigma*} g^{\nu} \int_S dS [t(\psi^{\sigma}) \cdot \psi^{\nu} - t(\psi^{\nu}) \cdot \psi^{\sigma}] \\ + g^{\sigma*} a^{\nu} \int_S dS [t(\psi^{\sigma*}) \cdot \psi^{\nu*} - t(\psi^{\nu*}) \cdot \psi^{\sigma*}] \\ + g^{\sigma*} g^{\nu} \int_S dS [t(\psi^{\sigma*}) \cdot \psi^{\nu} - t(\psi^{\nu}) \cdot \psi^{\sigma*}]. \end{aligned} \quad (3.27)$$

The integrals may be simplified using the orthogonality relations introduced in section 3.3 (see Appendix A and recall that $H_m^{(1)*}(k_n r) = H_m^{(2)}(k_n r) = J_m(k_n r) - iN_m(k_n r)$) so that

$$s_i(\mathbf{u}^t) = -\frac{1}{4} \sum_{\sigma} (a^{\sigma*} a^{\sigma} - g^{\sigma*} g^{\sigma}). \quad (3.28)$$

Since the volume enclosed by S contains no energy sources, we must have that the net energy through the surface vanishes, which in matrix notation, implies

$$\begin{aligned} \mathbf{a}^{*T} \mathbf{a} &= \mathbf{g}^{*T} \mathbf{g} \\ &= \mathbf{a}^{*T} \mathbf{S}^{*T} \mathbf{S} \mathbf{a}, \end{aligned} \quad (3.29)$$

from the definition of $\mathbf{g}$. We conclude, then, that the matrix $\mathbf{S}$ is unitary, or

$$\mathbf{S}^{*T} \mathbf{S} = \mathbf{I} \quad (3.30)$$

(note that this result is dependent upon each basis function carrying an equivalent quantity of energy through a closed surface).

Temporal reciprocity requires that upon time-reversal $\mathbf{u}^t$ still satisfies the wave equation and the same boundary conditions (recall that we have required the scatterer and embedding medium to be nondissipative). We can impose this reciprocity by taking the complex conjugate of the expression in (3.23) (while choosing not to operate on the suppressed harmonic time dependence)

$$\mathbf{u}^{t*} = 1/2 [\mathbf{g}^{*T} \boldsymbol{\psi}^* + \mathbf{a}^{*T} \boldsymbol{\psi}], \quad (3.31)$$

and applying the relation in (3.24) with $\mathbf{a}^*$ and $\mathbf{g}^*$ as our two coefficient vectors

$$\begin{aligned} \mathbf{a}^* &= \mathbf{S} \mathbf{g}^* \\ &= \mathbf{S} \mathbf{S}^* \mathbf{a}. \end{aligned} \quad (3.32)$$

This requires that

$$\mathbf{S} \mathbf{S}^* = \mathbf{I}, \quad (3.33)$$

and comparison with (3.30) indicates then that $\mathbf{S}$ must be symmetric. Consequently, we find that $\mathbf{T}$ must also be symmetric and, furthermore, must satisfy

$$\mathbf{T}^* \mathbf{T} = -\text{Re}(\mathbf{T}), \quad (3.34)$$

If we consider a single component of the basis set in (3.9a) (e.g. $\hat{\psi}^\sigma$) as our incident wave, then (3.34) relates the total energy in the scattered wave to the real component of amplitude of the corresponding unconverted, scattered basis component ψ^σ . This relationship which obviously depends on the conservation of energy is an expression of the optical theorem, first derived for surface waves by Snieder (1988a).

Waterman (1969) proposed a method of inverting for $\mathbf{Q}$ which exploits these constraints on the structure of $\mathbf{T}$, and preserves symmetry and the relation in (3.34) through truncation. In comparison studies (Waterman (1979)) with results using standard matrix inversion schemes, the method invariably achieved convergence at lower truncation levels, and is, consequently, the scheme we have chosen to employ in the following numerical analysis.

3.6 NUMERICAL EXAMPLES

We turn our attention now to the computational development of the theory presented in the preceding sections. Our aim is to examine the general nature of mode coupling and wavetype conversion through an analysis of three geometrically simple models at a range of scales. The problem geometry is shown in figure 3.2 and consists of a two layer, low-velocity plug embedded within a three-layer, stratified half-space with the configuration of the plug boundary differing in each of the three models. The anomaly velocity contrast decreases from 10% in the first layer to 7.5% in the second (specific details regarding physical properties are given in Table 3.1). Contrasts of this magnitude are not likely to be reliably treated using perturbation methods such as the Born approximation especially for large scatterers. The first two models comprise circular and elliptical cylindrical plugs with horizontal cross-sections that remain constant in depth, while the third model is a circular plug which is tapered through the first two layers. The individual layer boundaries in both the half space and the plug are constrained to continue uninterrupted across the plug surface, a restriction which will aid in achieving an accurate representation of the external wavefield on the scattering surface by a computationally tractable number of internal eigenfunctions (see Maupin & Kennett, 1987). We will consider the first 12 Love and Rayleigh modes at 1.0 Hz in our computations. This ensemble incorporates all but the final real Rayleigh mode (vs. complex leaky modes) for the stratified half-space and a large majority of the modes corresponding to the vertical velocity profile of the scatterer, and should permit a reasonable description of propagation/scattering processes in all but the highest-order modes (Kennett, 1984a). Although the geometric simplicity inherent in our model selection may not render it immediately relevant to anomalies occurring in the real earth, the focus of this preliminary study is directed more towards understanding the individual effects of obstacle dimension and variation in vertical and horizontal cross-sections on surface wave scattering, an objective to which our models would appear well suited.

The character of the associated eigenfunctions, shown in figures 3.3 and 3.4,

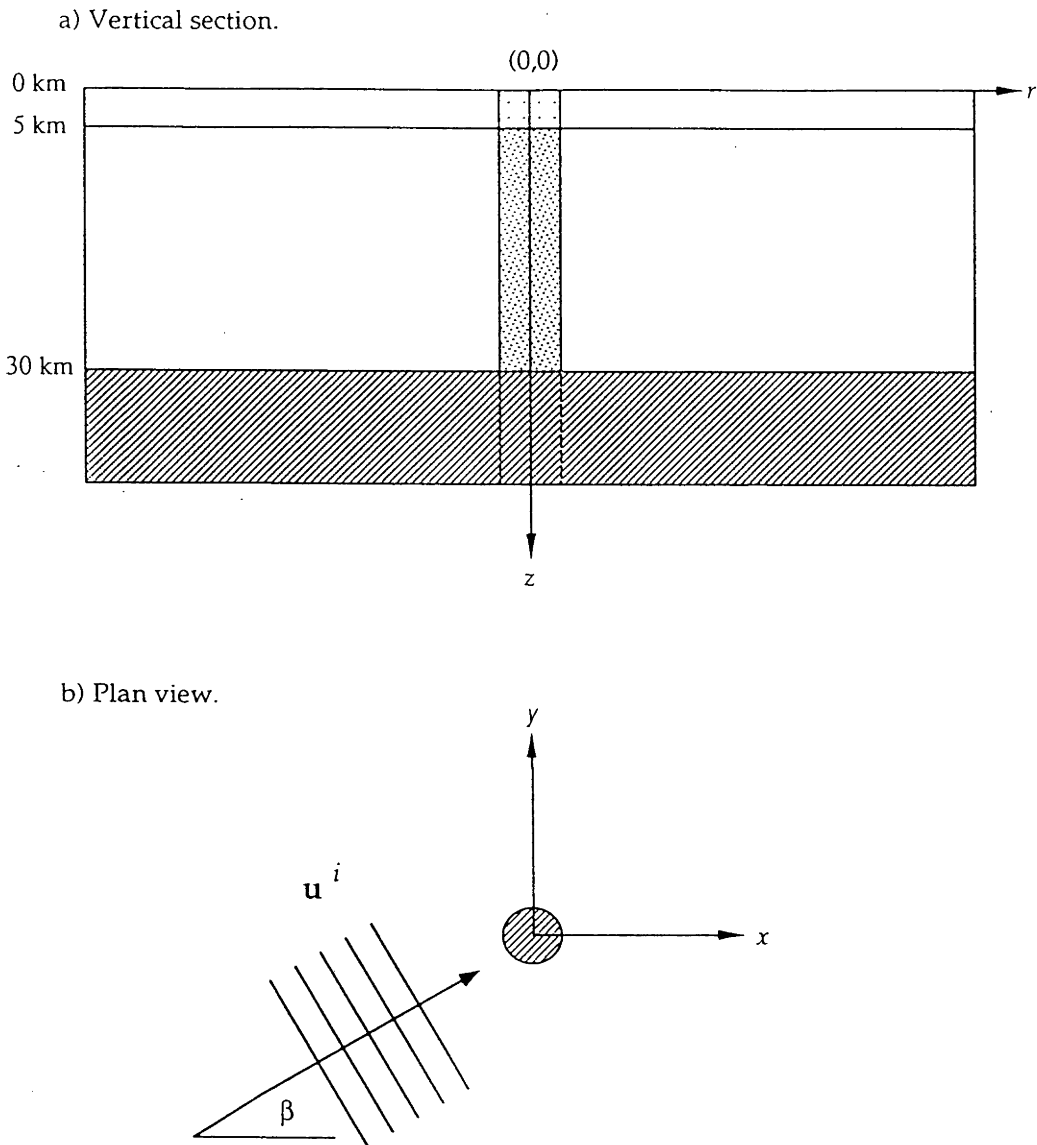


Figure 3.2 Vertical a) and horizontal b) cross-sections for the scatterer described in Section 3.7 comprising a cylindrical plug embedded in a three-layer stratified medium. The obstacles considered in Sections 3.8 and 3.9 share the same vertical velocity profile but differ in the configuration of their horizontal and vertical boundaries.

Table 1. Physical properties of the half-space and scatterer.

	Layer	Thickness (km)	P-wave Velocity (km s ⁻¹)	S-wave Velocity (km s ⁻¹)	Density (kg m ⁻³)
Half-space	1	5.0	5.0	3.0	2600
	2	25.0	6.0	3.5	2800
	3	∞	8.1	4.6	3300
Scatterer	1	5.0	4.5	2.7	2600
	2	25.0	5.55	3.24	2800
	3	∞	8.1	4.6	3300

Table 2. Percentage energy distribution in eigenfunctions 1-12 as a function of layer in the half-space and scatterer.

Modal Order	Layer	Half-space		Scatterer	
		Love	Rayleigh	Love	Rayleigh
1	1	99.2	100.0	99.5	100.0
	2	0.8	0.0	0.5	0.0
	3	0.0	0.0	0.0	0.0
2	1	87.8	85.5	93.6	92.9
	2	12.2	14.5	6.4	7.1
	3	0.0	0.0	0.0	0.0
3	1	0.5	6.0	26.8	38.3
	2	99.5	94.0	73.2	61.7
	3	0.0	0.0	0.0	0.0
4	1	2.1	8.5	6.8	0.8
	2	97.8	91.5	93.2	99.2
	3	0.1	0.0	0.0	0.0
5	1	4.8	8.1	7.7	2.1
	2	95.0	91.8	92.3	97.8
	3	0.2	0.1	0.0	0.1
6	1	8.2	7.4	7.7	3.3
	2	91.5	92.4	92.2	96.6
	3	0.3	0.2	0.1	0.1
7	1	10.9	6.9	7.6	4.5
	2	88.6	92.6	92.2	95.3
	3	0.5	0.5	0.2	0.2
8	1	11.9	6.9	7.7	6.2
	2	87.3	92.2	92.0	93.5
	3	0.8	0.9	0.3	0.3
9	1	11.9	7.4	8.1	9.0
	2	86.7	90.9	91.4	90.5
	3	1.4	1.7	0.5	0.5
10	1	11.8	8.8	8.8	13.2
	2	85.8	87.5	90.4	85.9
	3	2.4	3.7	0.8	0.9
11	1	11.9	10.7	9.7	16.3
	2	83.5	79.3	89.0	82.2
	3	4.6	10.0	1.3	1.5
12	1	11.7	11.6	10.4	15.3
	2	77.6	63.5	87.4	81.4
	3	10.7	24.9	2.2	3.3

deserves some discussion as it will be important in interpreting our results. In this and the following sections we will abbreviate the fundamental Love, Rayleigh and succeeding modes as $L1$, $R1$, $L2$, $R2$... etc. We note from Table 3.2 that the low-order modes ($L1, L2, R1, R2$) in the half-space propagate primarily within the top, low-velocity layer. Energy in the 7 succeeding Rayleigh modes is very similarly allocated with approximately 6-8% in the first layer, 90-92% in the second layer and the remainder in the form of evanescent waves in the underlying half-space. Corresponding Love modes, in contrast, display a systematic redistribution of energy from the middle to top layers with modal order; the proportion of energy in the top and middle layers is 0.5% and 99.5% for $L3$ and shifts gradually to 12% and 87% for $L9$. Love and Rayleigh modes 10, 11, 12 exhibit a redistribution of energy in the middle layer to the top and bottom layers. This is most pronounced in $R12$ which is characterized by an energy distribution of 12% in the top layer, 63% in the middle layer and 25% in the half-space.

The eigenfunctions characterizing the vertical velocity structure of the scatterer are similar to those in the halfspace but differ significantly in several aspects. Firstly, the third Love and Rayleigh modes still retain considerable energy in the topmost layer (27% and 38%, respectively), and thus represent a more gradual transition between the first and second Love and Rayleigh modes which propagate in the top layer and higher-order modes travelling with a majority of energy in the middle layer. The energy in internal modes 4 through 9 is again restricted largely to the middle layer; in this case however it is the Love modes which are more nearly similar in energy distribution and Rayleigh modes which experience a gradual transfer of energy from the middle to top layers. This trend continues for Love and Rayleigh modes 10, 11, 12, with the very little energy entering the bottom layer. This is a point of some importance since none of the internal eigenfunctions employed in our wavefield representation are characterized by the same degree of energy in the underlying half-space as, say, $R12$. In fact, we must look to the highest-order modes (13, 14, 15) for this property in the internal eigenfunctions, and since these are not represented we expect that our results for the highest-order

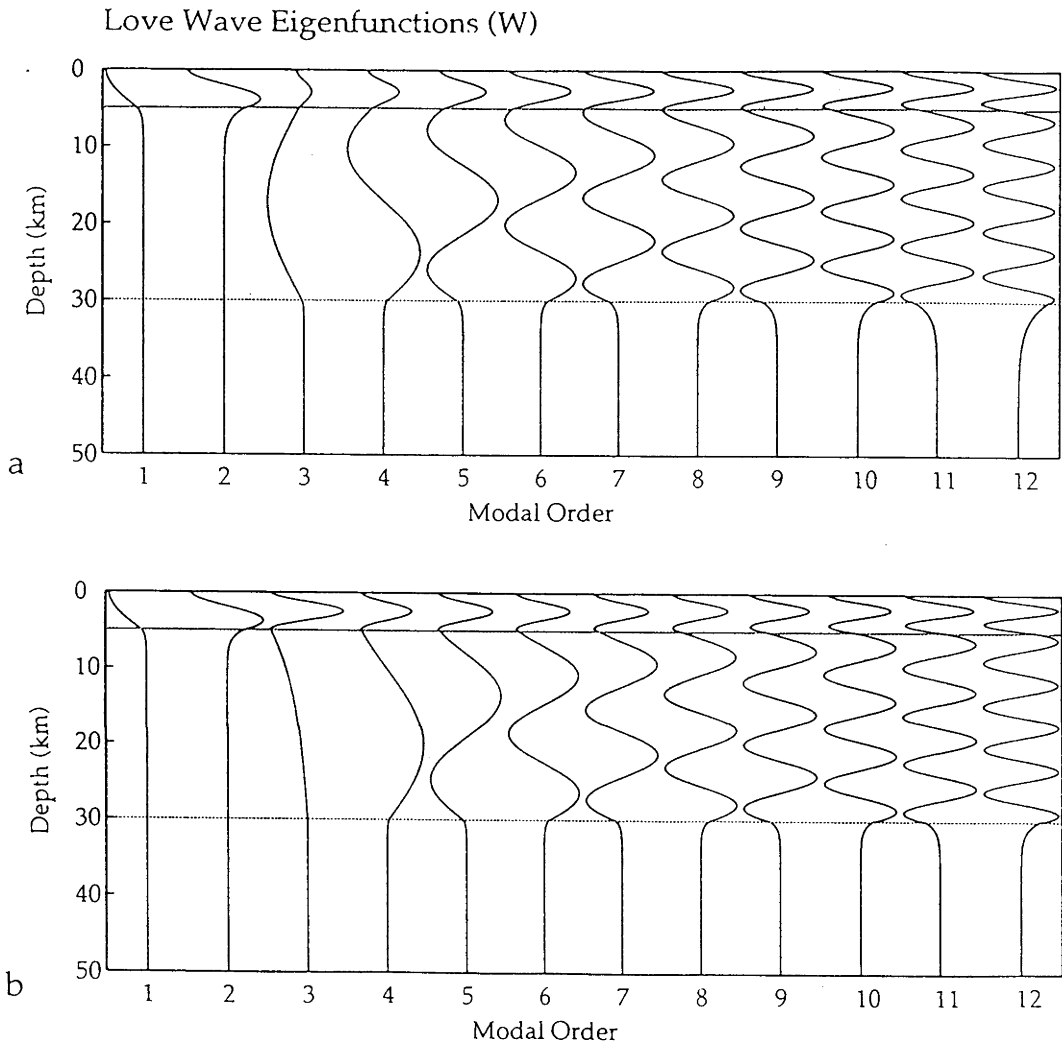


Figure 3.3 Love (W) wave displacement eigenfunctions at 1.0 Hz for a) the half-space and b) the obstacle. Compare with the energy distribution outlined in Table 2.

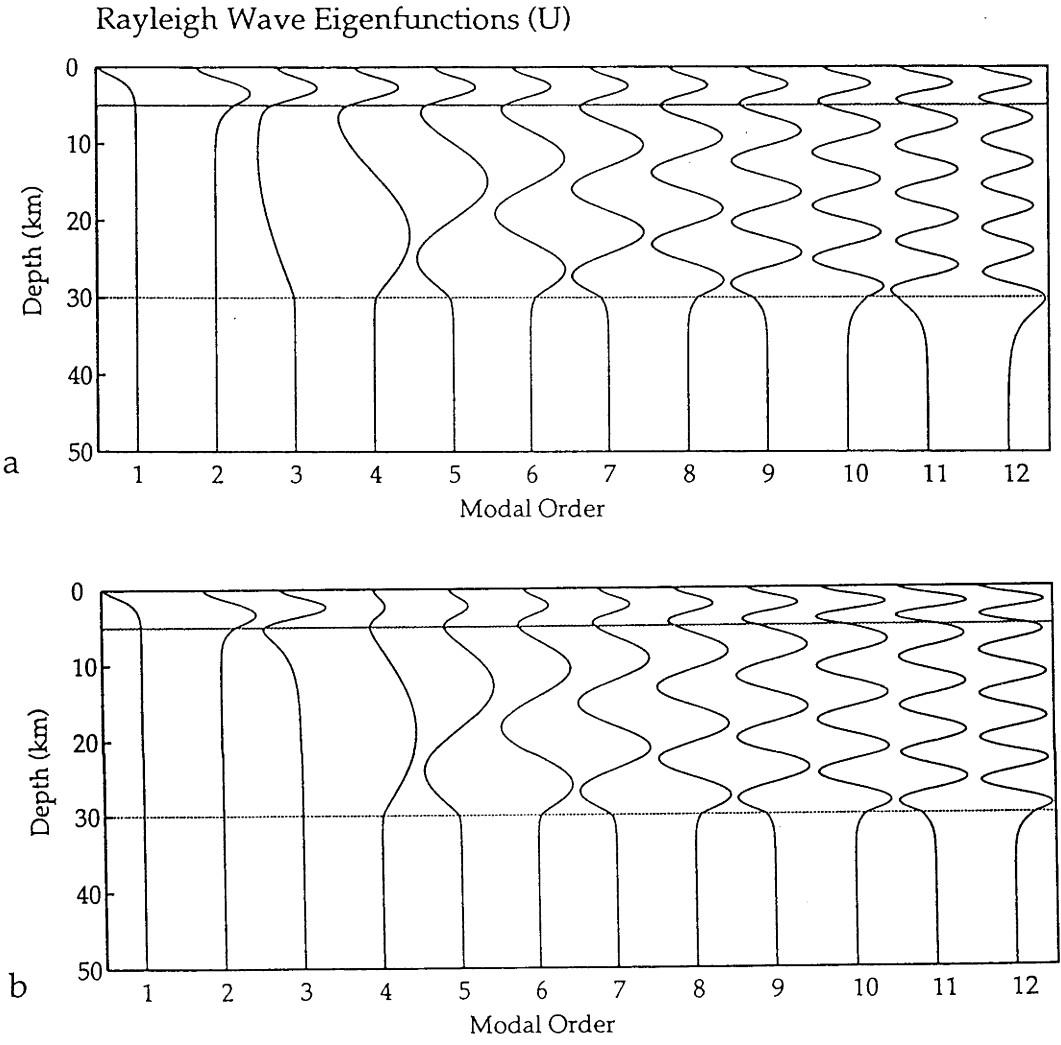


Figure 3.4 Vertical Rayleigh (U) wave displacement eigenfunctions at 1.0 Hz for a) the half-space and b) the obstacle. Note the similarity with the Love wave eigenfunctions in figure 3.3 and compare with the energy distribution outlined in Table 2.

modes may be somewhat inaccurate. Consequently, we restrict our interpretation in the following sections to the first 10 Love and Rayleigh modes alone.

The choice of a suitable truncation level for azimuthal order m will depend upon the horizontal extent of the scatterer; there should be sufficient terms that our incident single mode plane wave expansion is accurate over the entire surface of the scatterer. Related studies (Weaver and Pao, 1979) involving wave scattering from various bodies with smooth surfaces and using standard inversion schemes, indicate that good results may be achieved using values of $m \geq 2ka$, where k is the wavenumber and a is the maximum radius of the scatterer. As mentioned in the previous section however, fewer terms are generally necessary when using the inversion scheme advocated by Waterman (1979) and employed here. Therefore, in the following sections m is chosen such that the higher-order azimuthal terms of the scattered modes no longer contribute significantly to the total amplitude; that is, the truncated scattered wave expansion has begun to converge. This condition will be dictated by the fundamental Rayleigh mode for different earth models since it is characterized by the wavenumber of greatest magnitude. The parameter ka is a convenient means of quantifying the interaction of waves at given frequency with obstacles of varying dimension and we will employ it in this capacity. For practical reasons, associated either directly or indirectly with truncation in azimuthal order m , the method we describe is most useful for $ka \leq 10$ so that we may describe scattering phenomena over a wide range of different classes of interaction (see Aki and Richards, 1980; pp. 748-752).

In all the examples presented below we will consider a single, plane wave mode incident on the plug. The expansion of the incident wave reduces to the problem of expanding the scalar plane wave function $e^{i\mathbf{k}\cdot\mathbf{r}}$ in terms of the regular horizontal wave functions $\hat{Y}_n^m(k_n r, \theta)$ (cf. (3.5)) or

$$e^{i\mathbf{k}_n \cdot \mathbf{r}} = \sum_m a_m \epsilon_m J_m(k_n r) \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}, \quad (3.35)$$

where $|\mathbf{k}_n| = k_n$. The expansion coefficients a_m can be solved for using Fourier

analysis and are given in many textbooks (e.g. Morse and Feshbach, 1953) as

$$a_m = \epsilon_m i^m \begin{pmatrix} \cos m\beta \\ \sin m\beta \end{pmatrix}, \quad (3.36)$$

where β is the angle of the wavevector $\mathbf{k}_n$ with respect to the x -axis. By performing the operations implicit in (3.3), we observe that the coefficients a_m for a given Love or Rayleigh mode are just those employed in the analogous scalar problem.

Finally, we define several terms used in the following sections explicitly to avoid ambiguity and excessive repetition. The term ‘wavetype’ denotes the polarization of the surface wave, that is, as either Love or Rayleigh. ‘Converted’ and ‘unconverted’ are terms employed with regard to the modal order and wavetype of the incident wave; converted modes are all those with the exception of that represented in the incident wave, likewise the the wavetype of the incident wave is the unconverted wavetype. ‘Spectrum’ refers to the scattered energy distribution as a function of modal order.

3.7 CIRCULAR CYLINDER

In our first example, we examine scattering from circular cylinders of varying radii over a range of ka ; specifically the energy distribution among scattered modes and wavytypes, and the behaviour of the scattered far-field amplitudes with azimuth. Since the horizontal cross-sections of this and the following model remain constant in depth, the z -component of the integral in (3.20) can be calculated separately, and need only be computed once for a whole suite of models of differing radii. We can take advantage of sinusoidal orthogonality since radius is constant and independent of azimuth, and equate all elements of $\mathbf{Q}$ comprising different azimuthal orders ($m \neq p$) and dissimilar parity odd/even, even/odd parity interactions ($\cos m\theta \times \sin m\theta$) to zero. This results in a $\mathbf{Q}$ matrix which is very sparse and characterized by a generally multi-diagonal structure within each of the $2N \times 2N$ modal partitions. As a consequence of these simplifications, the circular cylindrical plug represents the simplest of possible scattering surfaces from a computational point of view, and in terms of interpretation allows us to concentrate on the effect

of varying scatterer dimension alone, without considering the effects of irregular boundaries in the horizontal or vertical profiles.

3.7.1 Energy spectra

The analysis begins with what is essentially a point scatterer, a circular cylinder of radius 0.04km corresponding to $ka \sim 0.1$ for the incident fundamental Rayleigh mode. We first examine the energy coupling into scattered modes for a given incident mode. Figure 3.5 shows energy coupling for incident $L2$, $R1$, $L3$, $R3$, $L6$ and $R6$. The logarithm of scattered energy is plotted normalized to that in the unconverted scattered mode. Rather than analyzing each diagram individually we will attempt, in this and the following sections, to isolate some general features which exemplify the scattered wavefield for various incident fields.

As we might expect, the majority of the scattered energy is carried in the unconverted scattered mode and is generally at least an order of magnitude (more for lower order incident modes) greater than that in the next largest contributor. For incident Love modes we find that this contribution comes from a Rayleigh wave, usually at a modal order one greater than the incident mode. The exception to this rule is $L2$ which couples strongly to the entire Rayleigh spectrum, in particular, $R1$. The scattered Rayleigh spectrum for modes of order greater than the incident Love wave tends to display a characteristic alternating pattern with odd-numbered modes (e.g. 5,7,9) low for odd-numbered incident Love modes, and high for Love modes of even order. We also note that the average energy level across the Rayleigh spectrum, in general, significantly exceeds that in the converted Love modes. In addition, the scattered Love spectrum for incident Love outside the unconverted wave, exhibits less variation than the Rayleigh spectrum, the exception once more being $L2$ which is always more strongly coupled than adjacent modes.

Scattered energy from incident Rayleigh modes displays a slightly different pattern. As before the largest contribution arises in the unconverted mode and generally decreases systematically through adjacent Rayleigh modes. The next largest contributor may be Love or Rayleigh; lower-order incident waves ($R1 - R5$)

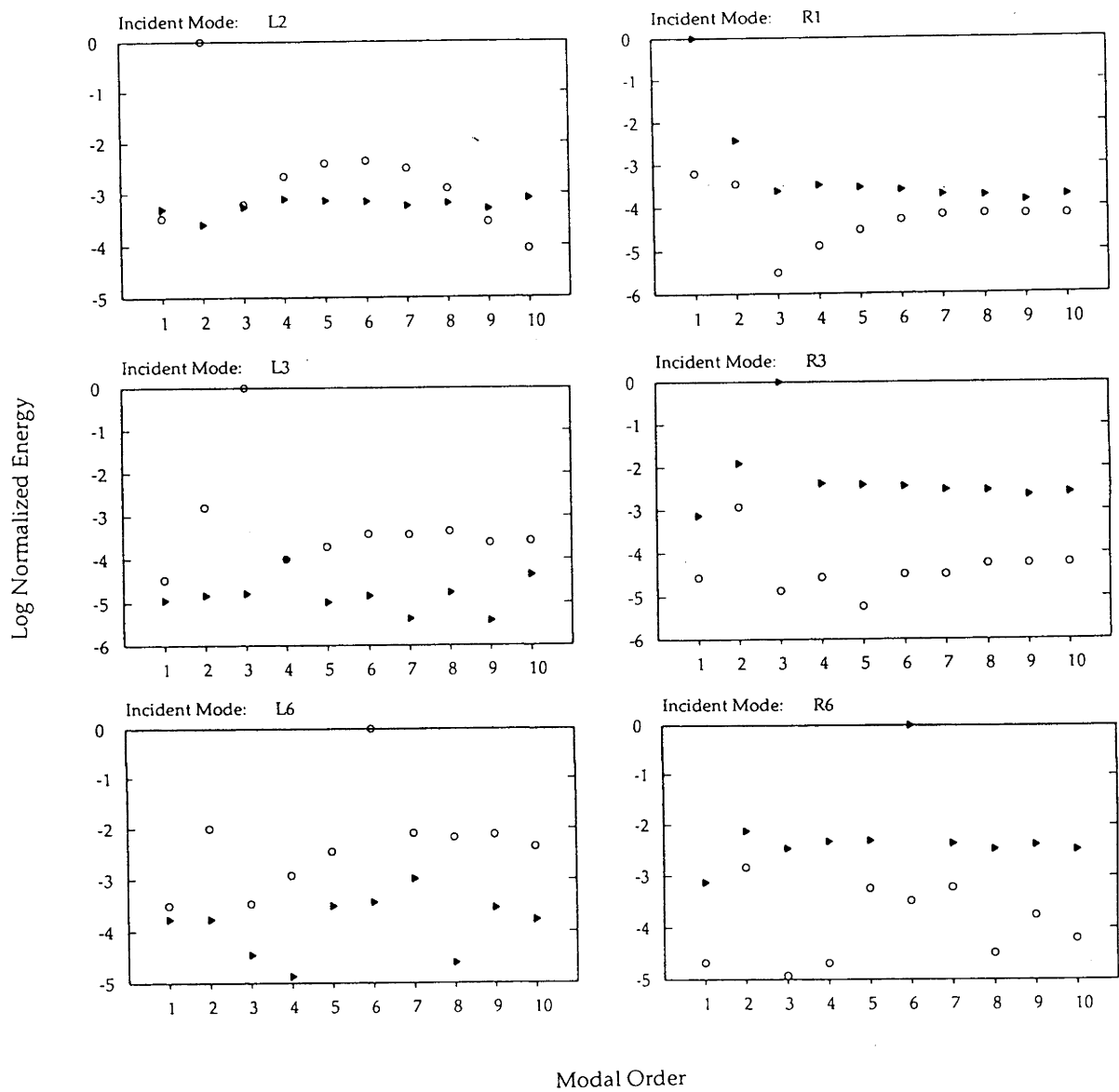


Figure 3.5 Scattered energy spectra from a circular cylinder of radius of $r=0.04\text{km}$ for a selection of incident modes. Circles and triangles denote Love and Rayleigh modes, respectively.

couple strongly to $R2$ whereas a higher-order mode will couple most strongly to its to closest neighbours in the Love spectrum (e.g. incident $R7$ couples strongly into $L6$ and $L8$). A pattern of alternating high and low energy is again evident, this time in the Love spectrum, for modal orders greater than that of the incident wave ($R3$ upwards), although it is less pronounced and tends to converge rather than diverge with modal order. Again scattered modes of odd order (e.g. $L5$, $L7$, $L9$) tend to be high and low for even- and odd-ordered incident Rayleigh modes, respectively.

To interpret these results it is worthwhile recalling the physical character of plane-wave Love and Rayleigh displacements. The displacement vector of a plane incident Love mode oscillates in the horizontal plane perpendicular to the wave vector, whereas plane-wave Rayleigh displacements are confined to the plane formed by the vertical and the wavevector. With this in mind we can make some physically intuitive predictions regarding the principal factors affecting the scattering of surface waves and the nature of their influences. Specifically, we anticipate that the conversion of energy to different modes and wavetypes will be dominated by i) the discordance between internal and external eigenfunctions (i.e. deviations from modal orthogonality resulting from contrasts in material properties), and ii) the geometry of the obstacle surface and its orientation relative to the incident wave. The first of these is responsible for mode coupling in problems where surface waves are normally incident upon vertical boundaries, and where we expect no wavetype conversion. In the case of obstacles whose horizontal cross-sections are independent of depth, such as our circular cylinder, the second (geometric) mechanism will contribute predominantly to wavetype conversion (i.e. Love to Rayleigh, Rayleigh to Love) through the reorientation of displacement vectors after interaction with the obstacle surface at oblique angles of incidence. Where the obstacle surface varies in depth we expect the reorientation of particle displacement to modify the phase velocity of the incident wave, and consequently to contribute to coupling with other modes of similar wavetype. The interplay of all these factors in combination is difficult to visualize and we may expect the associated results to exhibit a complex

behaviour.

The behaviour of $L2$, and to a lesser degree $R2$, for a point scatterer, notably their predominance in the scattered wavefield for higher-order incident modes and the strong coupling to higher-order modes when incident, can be attributed to the discordance between internal and external eigenfunctions. We recall that eigenfunctions for internal modes $L3$ and $R3$ carry considerable energy in both the first and second layers, whilst energy in the corresponding modes of the half-space is confined in large part to the second layer. The internal eigenfunctions may thus act as a means of transferring energy in higher-order incident modes (with a dominant portion of energy in the second layer) to the first layer with a preference for external modes $L2$ and $R2$. Likewise energy in incident $L2$ is very effectively distributed into higher-order scattered modes via the same mechanism. The oscillating high-low patterns observed for higher modes in the energy spectra of the alternate incident wavetype is also likely related to eigenfunction discordance. Recall that the higher-order modes, especially $L4 - L10$, $R4 - R10$, in both the plug and halfspace carry most of their energy in the second layer with higher orders characterized by eigenfunctions with higher frequency oscillations. It seems likely then that the redistribution of incident energy amongst these modes is dictated to some degree by regular phase differences between internal and external eigenfunctions which interfere, in alternating fashion, constructively and destructively.

3.7.2 Radiation patterns

Insight into the effect of scatterer's horizontal cross-section on the wave scattering process may be acquired through the examination of radiation polar plots in which the far-field amplitude of the scattered wave is plotted as a function of azimuth θ (i.e. the angle of observation relative to the coordinate origin) for a given incident mode. Figure 3.6 shows radiation plots for a variety of incident and scattered mode combinations, where we have chosen the wave vector of the incident mode to lie in the direction of the x -axis ($\beta = 0$). Owing to the geometry of the scatterer and incident wave, the plots are necessarily symmetric about the

x -axis, and can be qualitatively characterized in terms of wavetype interactions. Incident Love scattered into Love (hereafter $L \rightarrow L$) frequently exhibits a 4-lobed structure indicative of a strong $\cos 2\theta$ component. Incident Love scattered into Rayleigh and the reverse configuration ($L \rightarrow R, R \rightarrow L$) are also characterized by 4-lobed radiation plots, however the lobe axes lie at oblique angles ($\sim 45^\circ$) to the coordinate axes, indicating a strong $\sin 2\theta$ component. In contrast to $L \rightarrow L, R \rightarrow R$ is often predominantly double lobed and may have a more significant DC ($m = 0$) component. We can compare these results with those derived by Snieder (1986a) for a point scatterer of mild velocity contrast using the Born approximation. He notes that $R \rightarrow R$ will include contributions from DC, $\cos\theta$ and $\cos 2\theta$ components; $L \rightarrow L$ will include $\cos\theta$ and $\cos 2\theta$ contributions; and scattered waves from alternate wavetypes ($L \rightarrow R, R \rightarrow L$) will comprise $\sin\theta$ and $\sin 2\theta$ components (note that θ more generally refers to the difference between the wavevector angle β and of the angle of the point of observation relative to an origin within the scatterer). An examination of the appropriate scattered wave coefficients (as well as a qualitative comparison of respective radiation plots) indicate that the results of both studies are in agreement, in particular the coefficients $(c^l)_n^m$ decrease very rapidly for $m \geq 3$ and hence these components need not be considered for point scatterers even where the Born approximation is no longer valid. It should also be noted, however, that some of these observations (e.g. the lack of a DC component in $L \rightarrow R, R \rightarrow L$) arise more directly as a result of symmetry in the shape of the scatterer (i.e. $r(\theta) = r(\theta + \pi)$) and the associated structure of $\mathbf{Q}$, than the spatial extent of the anomaly.

3.7.3 Effect of dimension

The character of the scattered wave changes only slightly with an order of magnitude increase in the plug radius to $r = 0.4\text{km}$. In particular, the relative energies in the various scattered modes appear almost unaffected by the change in obstacle dimension. This can be understood when we consider that the parameter $k_n r$ characterizing the general level of scattering interaction decreases with modal

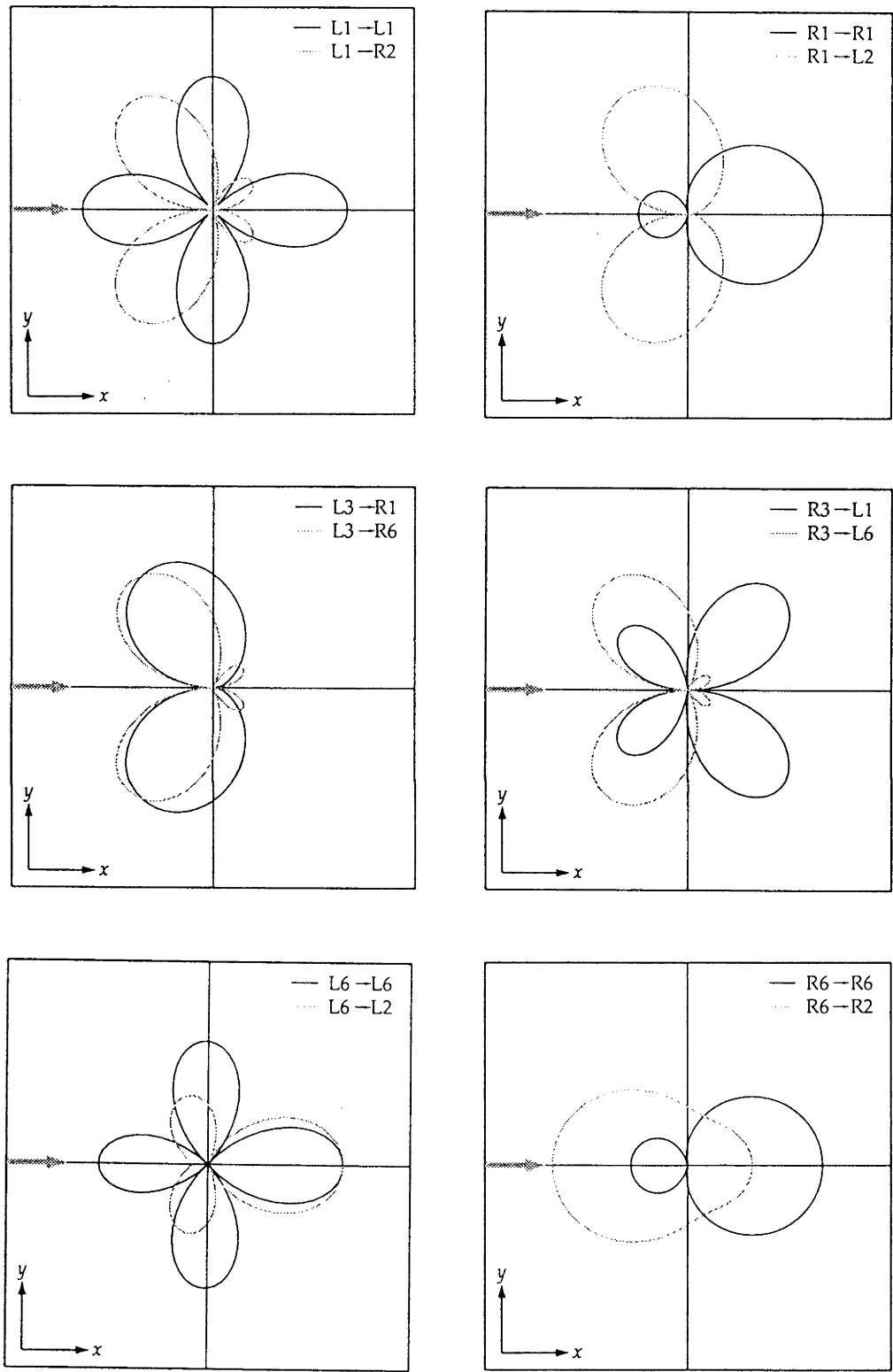


Figure 3.6 Radiation patterns from a circular cylinder of radius $r = 0.04\text{km}$ for a selection of incident and scattered modes with incident wave propagating in the positive x direction.

order (higher-order modes have lower wavenumbers) and remains below 0.8 for all but the lowest-order modes. Consequently, obstacles with a characteristic radius of 0.4km still behave much as point scatterers with a majority of energy residing in azimuthal orders $m \leq 2$. The deviations from point scattering behaviour are slightly more apparent in the radiation plots since these are proportional to the absolute magnitude of each of the coefficients $(c^l)_n^m$ in the expansion rather than the square of their magnitudes. Although the basic structure of the radiation patterns is retained, there is an obvious increase in the relative strength of the forward scattered lobes, especially in the interaction between lower-order modes.

As the radius is increased through to $r = 4.0\text{km}$, these effects become more pronounced, however the basic structure of the converted Love and Rayleigh wave energy spectra remains much the same (compare figures 3.5 and 3.7). The principal difference appears to be an increase in the relative energy of the unconverted scattered mode over other scattered modes, especially those in the spectrum of the converted wavetype. Thus for example, the average energy residing in the Rayleigh spectrum for a given incident Love mode decreases with plug radius while the relative energy distribution within the Rayleigh spectrum is left largely unchanged. This results from the r^{-1} dependence in the interaction between different wavytypes [cf. (3.4)]. A general observation of note is that the energy distribution in the low-order modes ($L1, L2; R1, R2$), which are confined to the topmost layer where the velocity contrast across the plug is greatest, tend to exhibit greater sensitivity to plug radius and may alternate in relative importance as radius is increased especially for low-order, incident modes of the converted wavetype. The enhanced sensitivity and greater variability of these low-order modes with regard to the dimension of the scatterer is a point worth noting. Lower-order modes are characterized by larger wavenumbers and since the dependence on obstacle radius is expressed through the combination $k_n r$, we expect greater sensitivity. In addition, although we expect the average total scattered energy for a given incident mode to increase with the size of the scatterer (as is indeed observed), it is also possible that certain configurations will result in the excitation of resonant modes in the obstacle

leading to fluctuations in scattered energy as a function of dimension. Since the velocity contrast is greatest in the top layer where the low-order modes propagate, they are, consequently, most likely to be strongly affected in this manner.

The effects of increased plug radius are again more obvious in radiation plots shown in figure 3.8 for $r = 4.0\text{km}$ and exhibit a marked contrast with those in figure 3.6. We observe in all cases that the proportion of forward scattered energy has increased at the expense of backscattering with the side lobes less effected. This phenomenon, as Snieder (1986b) observes, is known as the Mie-scattering effect in optics. In addition, the emergence of higher azimuthal terms is evinced either by assymetry about lobe axes or the appearance of additional lobes.

3.8 ELLIPTICAL CYLINDER

We now wish to investigate the effect of changes in the plug radius of curvature on the scattering of surface waves by examining the wavefield scattered from cylinders which exhibit varying degrees of ellipticity in horizontal cross-section. The cross-section remains constant in depth so that, as before, the depth integration may be performed separately. In addition, mirror symmetry about vertical planes through the center of the cylinder may be exploited to further simplify computation. Our original analysis comprised elliptic cylinders with the semi-major axis a (in the x -direction) fixed at 2.0 km while the semi-minor axis b ranged from 2.0 km (circle) through to 0.5 km (aspect ratio of 4). Attempts to run models exhibiting greater aspect ratios resulted in ill-conditioned $\mathbf{Q}$ matrices and, consequently, scattered wave series expansions which did not converge. In theory, this problem can be alleviated by considering a greater number of azimuthal-order terms in our expansions but this is impractical from a numerical point of view on any available computer. A more realistic approach would be to modify the horizontal dependence of the basis functions to better accommodate the nature of the scatterer (*cf.* Weaver & Pao, 1979).

In the following discussion we will concentrate on the field scattered from the cylinder of aspect ratio 4.0 (hereafter E4) since the deviations from the case of

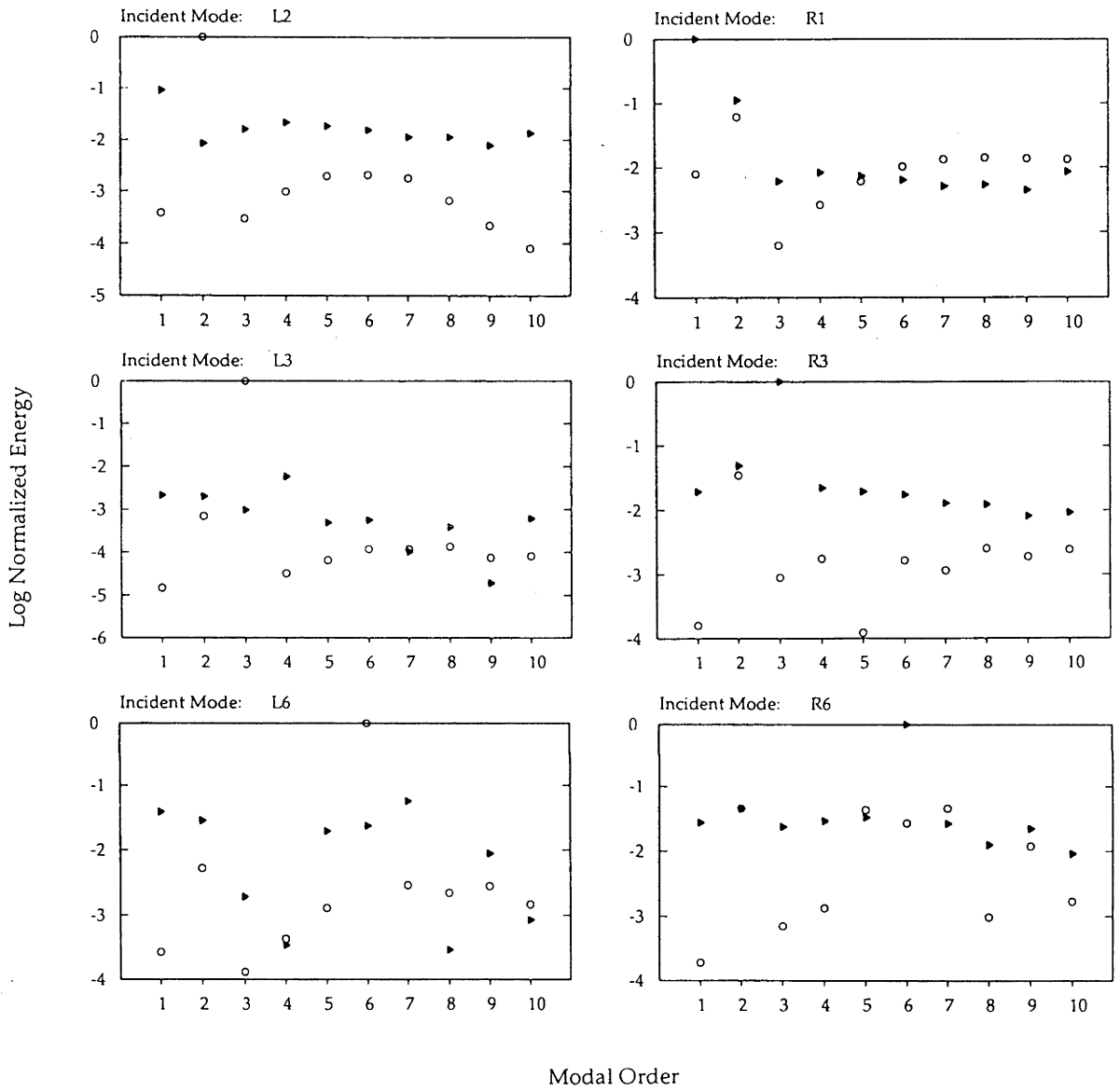


Figure 3.7 Scattered energy spectra from a circular cylinder of radius of $r = 4.0\text{km}$ for the same selection of incident modes as figure 3.5.

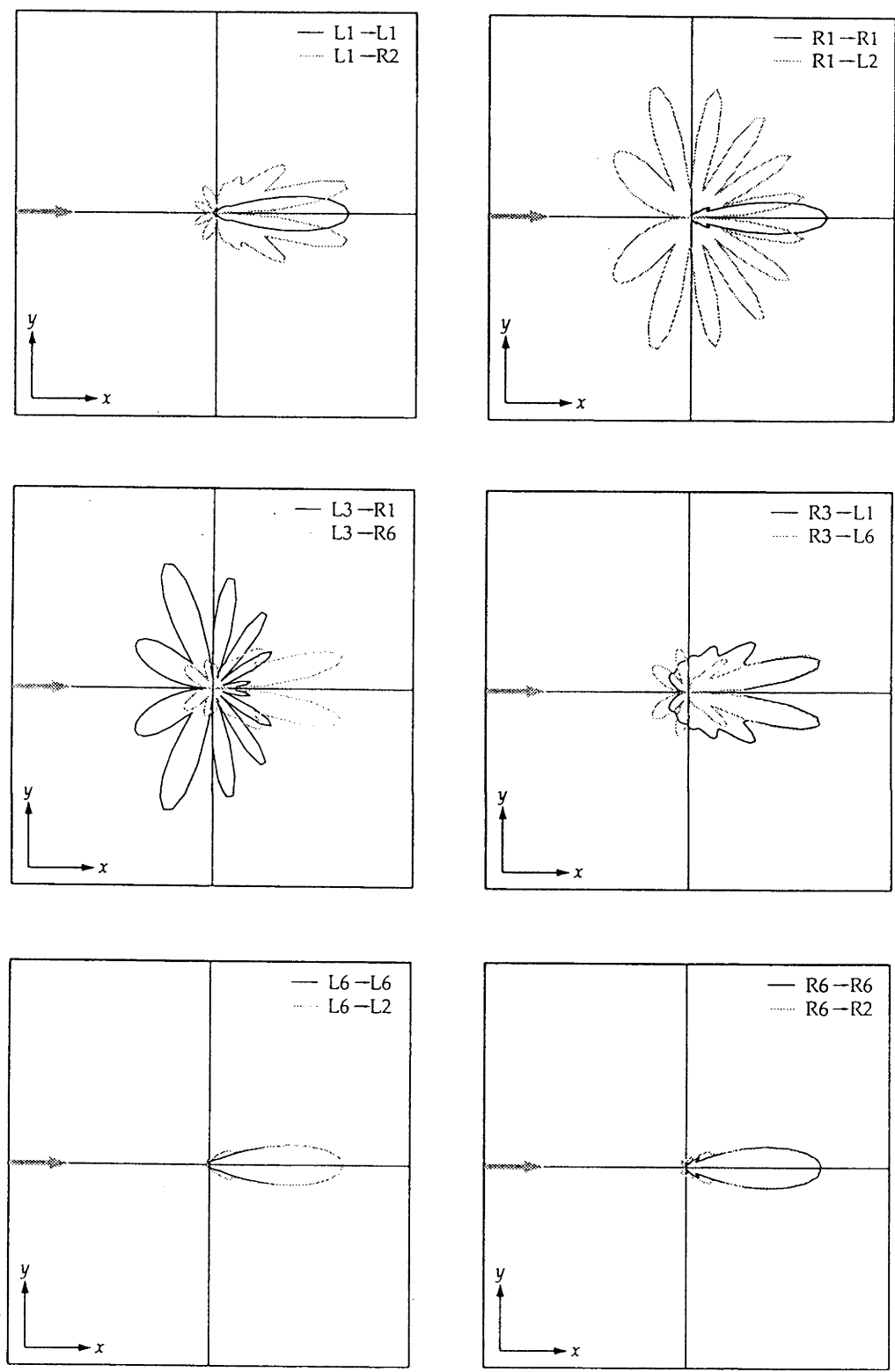


Figure 3.8 Radiation patterns from a circular cylinder of radius $r = 4.0\text{km}$ for the same selection of incident and scattered modes as figure 3.6.

a circular cross-section (radius 2.0 km) vary systematically through the range of ellipticities considered and are most pronounced in this case. In addition, it is appropriate to draw comparisons with results from a circular cylinder of radius 1.0 km (hereafter C1) since both scatterers are characterized by the same cross-sectional area, hence any contrasts can be attributed to geometrical factors alone.

3.8.1 Energy spectra

The general structure of the scattered energy spectra remains remarkably similar for all of the elliptical cylinders considered, and also corresponds very closely to those observed in the case of circular cylinders. The constancy of this feature across the range of models considered to this point re-emphasizes the dominant role played by the eigenfunction discordance in dictating the general character of the scattered wavefield. A slight positive shift (up to one half order in magnitude) is observed in the scattered energy across the entire spectrum of the converted wavetype (relative to the unconverted mode) as the ellipticity is increased; however this appears to be due primarily to the effect noted in the previous section with regard to the cross-sectional area of the scatterer since the energy distribution among the scattered wavetypes for E4 and C1 is more nearly identical. This suggests that the effect of ellipticity (at least over the range of models considered here) on the relative distribution of energy among scattered modes is small with respect to the influence of the modal eigenfunction discordance.

The actual changes in the scattered wavefield become more apparent when we examine the total scattered energy (all modes in both wavetypes) as a function of the angle β of the incident wave, and several interesting patterns emerge. Firstly, the total scattered energy from E4 for all incident Love modes increases from 0 to a maximum near 45° and then decreases steadily once more to 90° ; the actual angle at which the minimum occurs is dependent upon the modal order and may be either 0° or 90° . The ratio of maximum to minimum total scattered energies is constant at approximately 1.8 for all Love modes while the median value is generally very close to the total scattered energy from the model C1. In contrast, the total scattered

energy from model E4 for incident Rayleigh waves increases steadily from 0° to a maximum at 90° and the ratio of maximum to minimum scattered energies is more variable and generally slighter (in the range 1.2-1.7). Another interesting difference is that the total energy scattered from model C1 is always close to the maximum scattered energy from the elliptical model.

3.8.2 Radiation patterns

Some aspects of this behaviour are more readily visualized through examination of the radiation plots shown in figures 3.9, 3.10 and 3.11; and again several obvious trends are evident. As we might expect the unconverted scattered mode exhibits a much larger component of back-scattered energy when impinging on broadside on the scatterer ($\beta = 90^\circ$) than for $\beta = 0^\circ$ (these being the two configurations which must obviously exhibit symmetric radiation patterns). In addition, the forward scattered lobe is considerably narrower at $\beta = 90^\circ$ indicating that significant amplitudes will be observed only over a restricted range of observation azimuths. In comparison, we find that, to large degree, the associated radiation patterns for model C1 exhibit characteristics intermediate to those for the ellipse at $\beta = 0^\circ$ and $\beta = 90^\circ$. At oblique angles the radiation may vary considerably but there is a general tendency for what were side lobes at the angles of symmetry to grow quite significantly at the expense of the 'forward' lobe especially for incident Love modes. These observations hold for more general interactions (i.e. between different modes and wavetypes) and in all cases we note that backscattering becomes increasingly important when the exciting wave is incident the broadside of the scatterer. Indeed for higher-order modes it may exceed the forward scattering component (recall also that higher-order modes are characterized by smaller wavenumbers and hence on the basis of Mie scattering we do not expect forward scattering to be as dominant for a given model as it is for lower-order mode interactions).

3.9 TAPERED CYLINDER

In the final suite of models we are concerned with changes in the horizontal cross-section of the obstacle as a function of depth and their influence on the

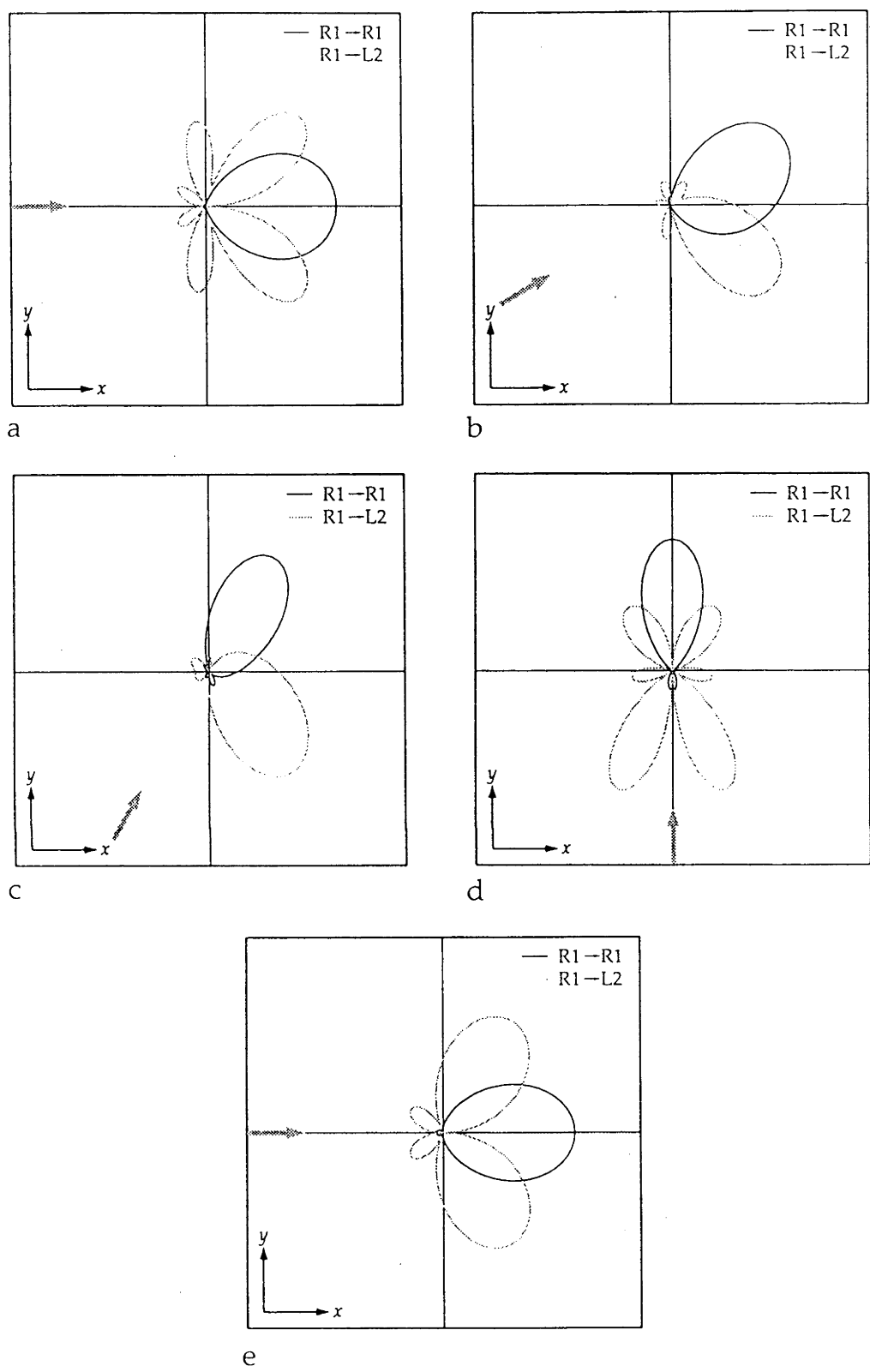


Figure 3.9 Radiation patterns from a scatterer with elliptical cross-section ($a = 2.0\text{km}$, $b = 0.5\text{km}$) (a,b,c,d) and circular cross-section $r = 1.0\text{km}$ (e) as a function of incident angle for incident mode $R1$. The semi-major axis a of the ellipse is parallel to the x -coordinate axis, and the direction of propagation for the incident wave is indicated by a shaded arrow.

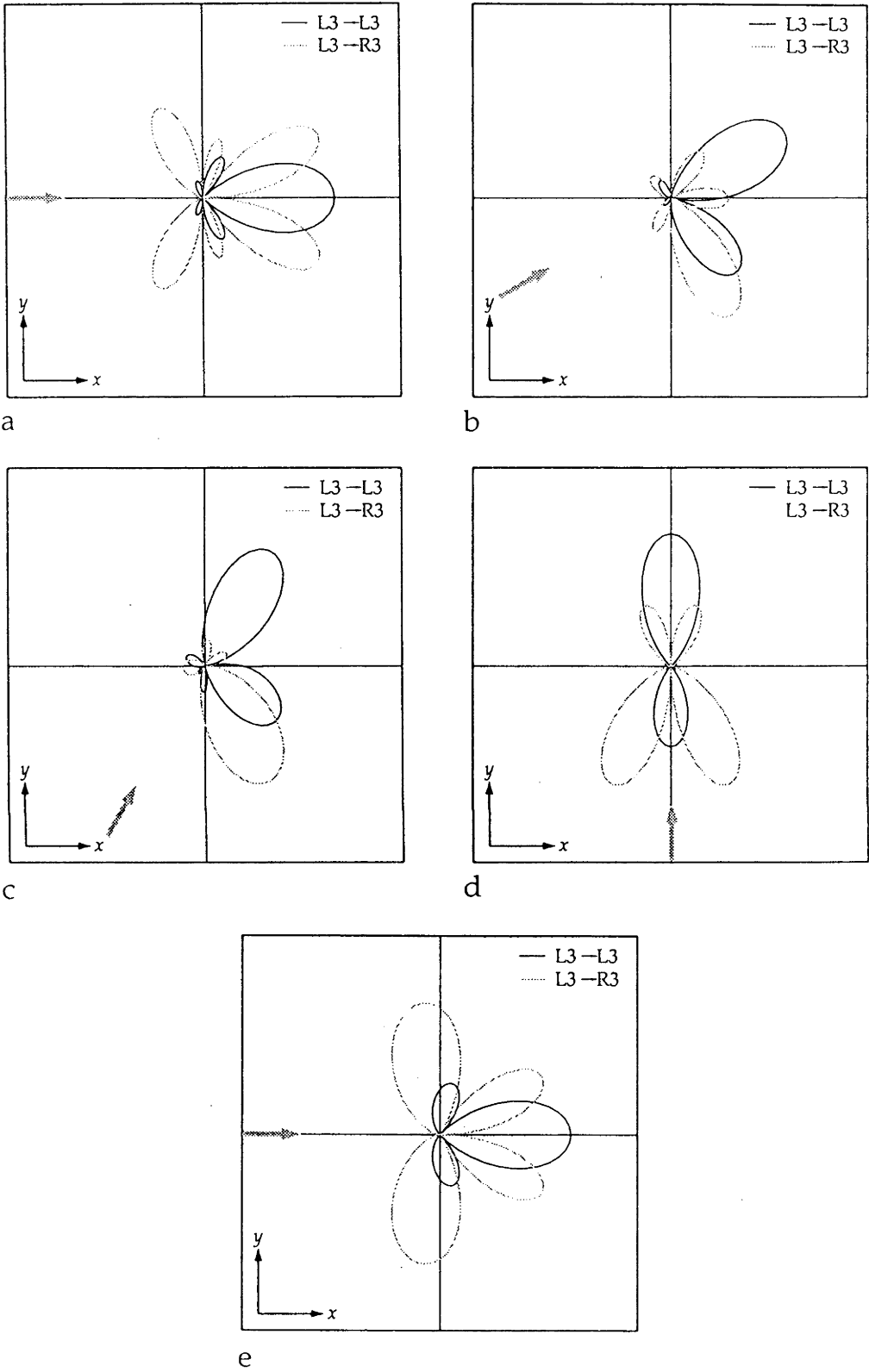


Figure 3.10 As figure 3.9 for incident mode $L3$.

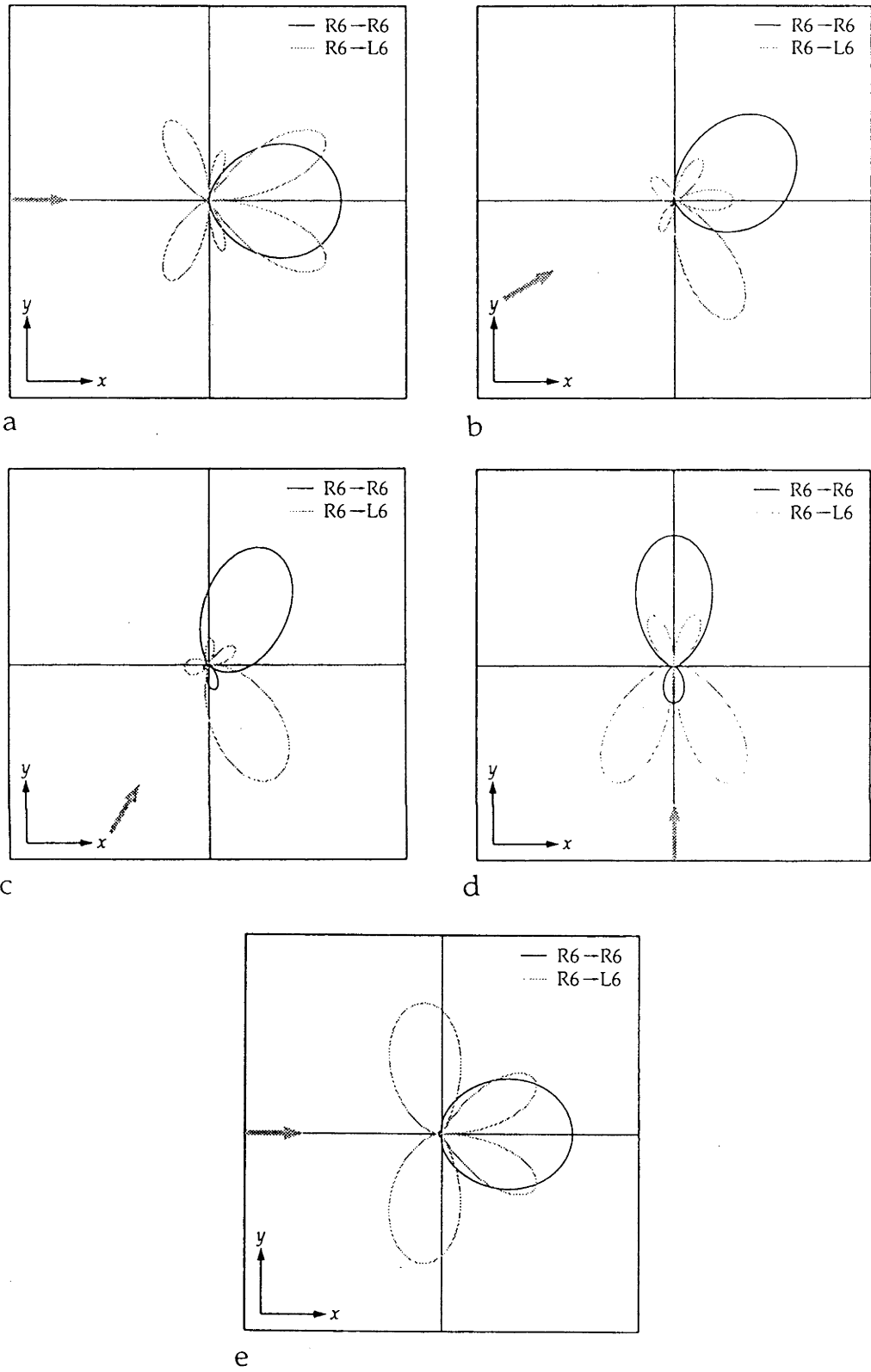


Figure 3.11 As figure 3.9 for incident mode $R6$.

distribution of energy among scattered modes. As alluded to earlier, we expect this class of deviation to have a more significant effect on intermode conversion than those described in the two previous examples. For ease in computation and to avoid unnecessary complication we will consider a plug of circular horizontal cross section whose radius exhibits a cosine depth dependence over one half period from the surface ($r = 2.0\text{km}$) to some depth z_{max} and remains constant at 0.5 km thereafter (see figure 3.12). In the three cases we examine, z_{max} will take the values of 50 km, 20 km and 5 km, which will allow us to observe systematic changes in the behaviour of the scattered field from the limiting case of constant circular cylinder through to a structure exhibiting a rapid variation in radius over a depth interval of a few kilometers. Owing to azimuthal orthogonality we note that the modal partitions in the **Q** and **T** matrices exhibit the same multidiagonal-dominated structure observed for models in section 3.7.

3.9.1 Slight taper

Scattered energy spectra are presented in figure 13a for a tapered cylinder with $z_{max} = 50\text{km}$. This surface does not differ markedly from a circular cylinder of radius $r = 2.0\text{km}$ at depths over which our low order (i.e. $L1, L2, R1, R2$) modes carry the majority of their energy; for example at 5.0 km depth $r = 1.96\text{km}$, and by 10 km depth the radius has not decreased beneath 1.86 km. It is not surprising, then, that the character of the scattered wavefield for low-order incident modes is virtually identical for the two cases since scattering to all modes is determined by the eigenfunction mismatch in the top 10.0 km. The first significant deviations from the constant radius case become evident for incident $L3$ and $R3$ where there is a shift of at least one order of magnitude in $L4$ and $R4$, respectively, and lesser increases for $L2$ and $R2$. Energy in the remaining converted modes is relatively unaffected with the same general levels maintained, although there is a tendency for the alternating patterns observed in the plots of section 3.7 to be smoothed somewhat, especially for modes in the alternate wavetype. Energy spectra for higher-order incident modes ($L4, R4 - L8, R8$) are all affected in the same manner

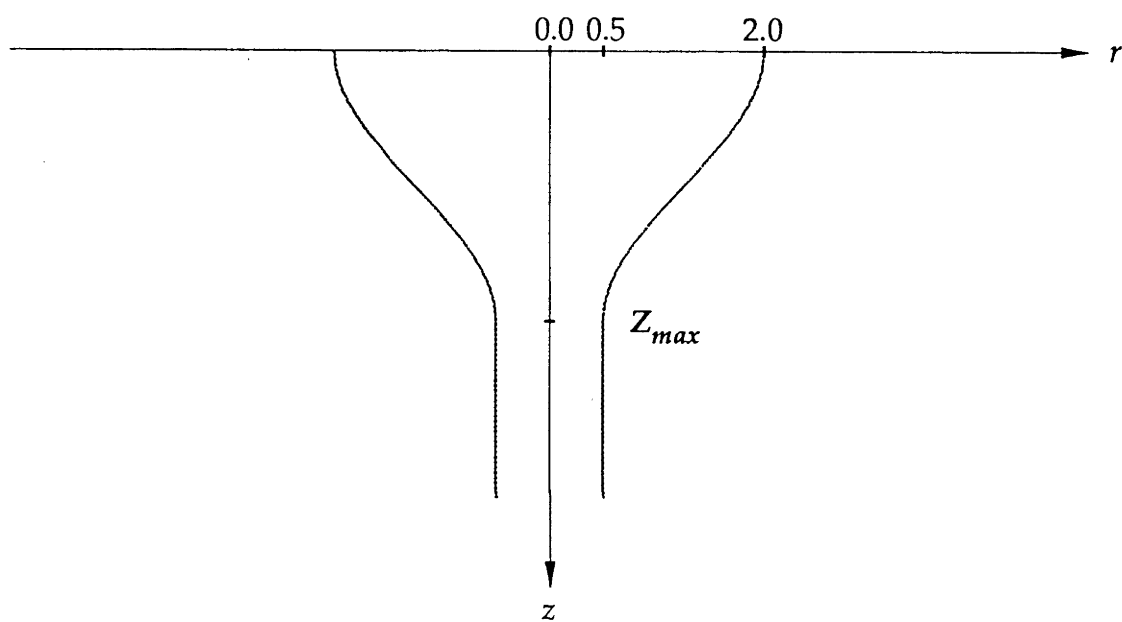


Figure 3.12 Configuration of the tapered cylinder considered in section 3.9. The radius is a half-cosine function of depth of amplitud 0.75 km to depth z_{max} and constant thereafter.

by the variation of the plug surface with depth, the plot for $R6$ in figure 3.13a is representative. In both cases, we find that a large fraction of the energy in the incident mode has been distributed to the two immediately adjacent modes of the same wavetype at the expense of the unconverted mode. Again we note that the general pattern of energy distribution (determined by the eigenfunction discordance) is slightly smoothed but that the basic signature is maintained.

3.9.2 Moderate taper

In the next model z_{max} is chosen to equal 20 km, that is the plug radius is a half-period cosine above 20 km and constant below. In contrast to the previous case the radius has decreased to 1.78 km by the bottom of the first layer and there are some obvious differences in the wavefield scattered from low-order incident modes. We find proportionately larger components of converted $L2$ and $L1$ for incident $L1$ and $L2$, respectively, although in both cases these components still contain less than 1% the energy of the unconverted mode. In the case of incident $L1$ the remaining Love modes are unchanged (aside from $L11, L12$) relative to the unconverted mode whereas, curiously, $R3 - R10$ experience a slight upward shift (a factor of approximately two) with $R1$ and $R2$ essentially unaffected. For incident $L2$ it is the Love spectrum ($L3 - L10$) which has experienced an upward shift of the same order of magnitude and the Rayleigh spectrum which is unaltered relative to the energy in the unconverted mode. No obvious differences exist between the wavefields scattered from the constant radius and tapered cylinders by the incident fundamental mode, because, as we recall, energy in $R1$ is concentrated very close to the surface and is subject to variation above the topmost 4.0 km alone. The principal change in the scattered energy spectra for incident $R2$, in contrast, is a slight positive shift in all converted Rayleigh modes relative to $R2$ with Love modes less markedly affected.

The scattered energy spectra for higher-order incident modes exhibit some interesting differences from both the previous case and the constant radius circular cylinder. As in the previous case incident $L3$ (figure 3.13b) couples strongly to its

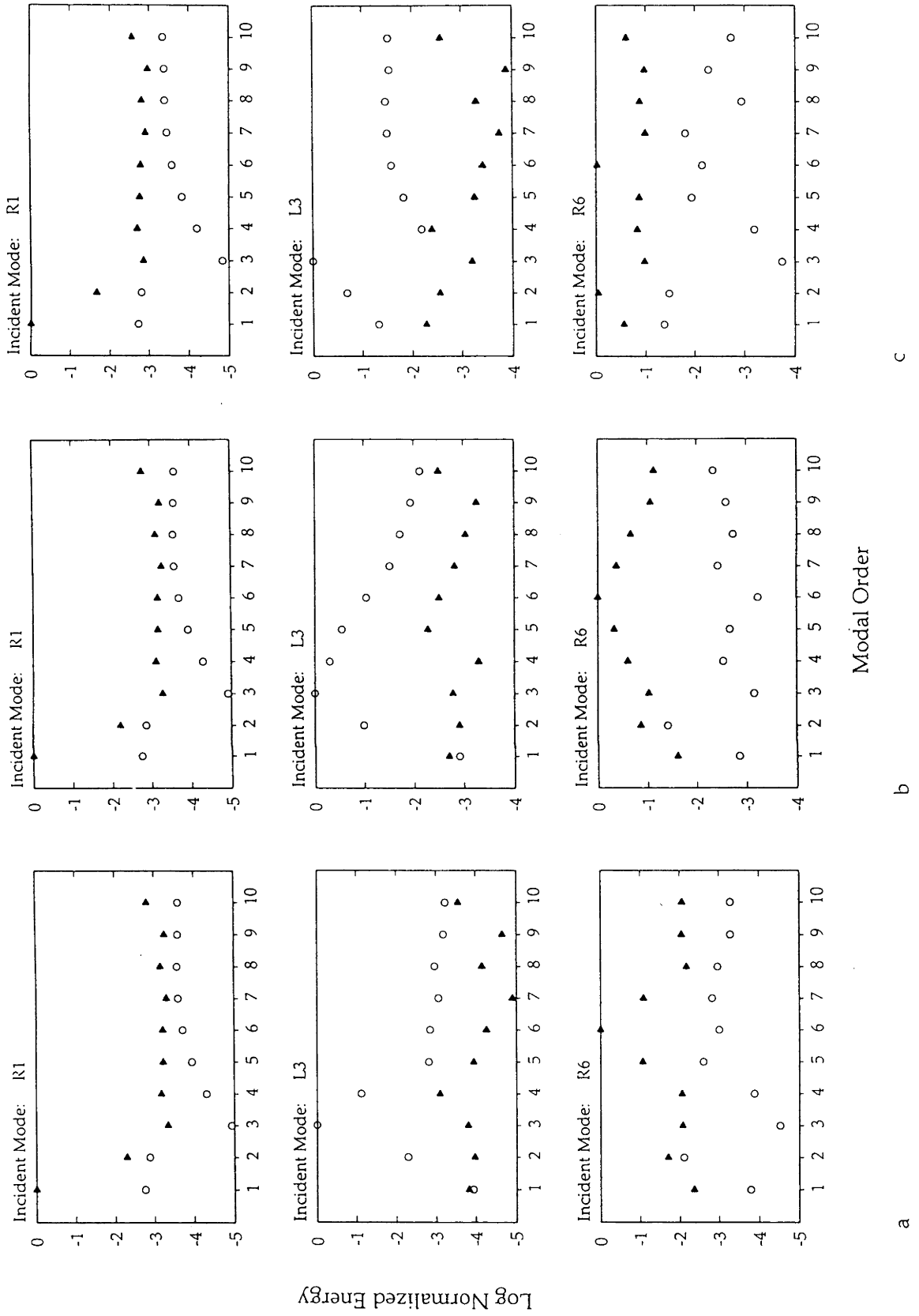


Figure 3.13 Selected scattered energy spectra for a tapered cylinder with the parameter $z_{\max}$ set to a) 50, b) 20 and c) 5 km. Compare with figures 3.5 and 3.7.

adjacent neighbours, $L2$ and $L4$, which are up 0.5 and 3.5 orders respectively from the constant radius model. However the effect is now felt through to more distant modes as well; $L1$ is up one order while modes $L5 - L10$ exhibit a very gradual decrease in energy from 20% the energy in the unconverted mode to less than 1%. The general character of the Rayleigh spectrum for incident $L3$ has not been disrupted to the same degree but is also up by an average of 1 order in magnitude. This basic structure is repeated for all higher-order incident Love modes; that is a broad peak centered on the unconverted mode falls gradually and monotonically with modal order to either side. The one exception to this is scattered $L2$ which, as usual, tends to be more strongly coupled. Scattered Rayleigh modes are up an average of one order of magnitude.

The pattern for higher-order incident Rayleigh modes is similar to Love, and is characterized by a peak which spreads gradually to modes on either flank with lowest energies frequently greater than 1% that in the unconverted wave. The scattered Love wave spectrum, though shifted upwards by approximately a factor of 10 retains more of the character exhibited for the constant radius cylinder than the unconverted wavetype.

3.9.3 Extreme taper

In our final model the portion of the plug exhibiting a depth dependent radius occurs within the first layer, $z_{max} = 5\text{km}$. Hence we can expect to see more pronounced deviations from the constant radius case in our scattered, low-order modes than was apparent for the two previous tapered cylinders. This is confirmed in figure 13c for $L1$, $L2$ and $R2$ whose energy spectra indicate that the effects are again dominant in modes of similar wavetype. Incident $L1$ couples strongly to $L2$ and vice versa; in both cases the scattered mode contains more than 20% of the energy residing in the unconverted mode. Higher-order modes have also increased by a factor of 10 or so, but the general energy distribution is similar to that for the constant radius case, reflecting the fact that for a major fraction of the pertinent depths the radius is constant. Scattered Rayleigh modes from incident low-order

Love modes display little change except that, once more, energy in the higher-order modes has increased relative to that in $R1$ and $R2$. Energy scattered from incident $R1$ deviates only marginally from the constant radius case with an increase in $R2$ of less than one order; the remaining modes in both wavetypes are virtually unchanged from the constant radius case. $R2$ however is now coupled very strongly to $R1$ and indeed the entire Rayleigh spectrum. The principal change in scattered Love modes is a tenfold increase in $L1$ and lesser increases (half an order) in the remaining Love spectrum.

Higher-order mode interactions are again similar in character for Love and Rayleigh waves. Lower-order modes of the same wavetype as the incident wave are strongly coupled, energy in scattered $L2$ and $R2$ are comparable to that in the incident mode. Higher-order converted modes have also experienced an increase in energy (up to 10% that in the unconverted mode) although in the case of incident Love the first few modes preceding the unconverted mode (e.g. $L3, L4, L5$ for incident $L6$) exhibit lower energies. Modal energies in the alternate wavetype are up marginally with little change to general structure, except that energy; in the scattered fundamental modes ($L1, R1$) has increased by up to 3 orders relative to the unconverted mode for some incident Rayleigh waves.

3.10 DISCUSSION

In the first half of this paper we have demonstrated that it is possible to describe surface wave scattering from discrete obstacles, in both the near- and far-field, by using a formulation which exploits the orthogonality of regular and outgoing surface wave basis functions in cylindrical coordinates. Strict validity of this theory requires that our earth models exhibit a perfect reflector at depth in which case the infinite modal set is complete in the sense that all wavefields, including P and S phases, are represented in the modal summation. In the absence of this restriction on our model we are unable to represent the scattered field in its entirety since leaky modes and body wave contributions are not incorporated. Nonetheless we expect surface waves to dominate the scattered field and that a formulation in terms of a finite

set of normal modes alone should provide a reasonably accurate description of the propagation process even in cases where the restrictions are not met. Under these assumptions we have examined several geometrically simple obstacles to investigate the influence of dimension and shape on the scattering process. Although not truly representative of anomalies occurring within the real earth, we expect that some of the more general and qualitative observations may remain applicable in more complicated situations.

The dominant factor influencing the distribution of energy among scattered modes especially for objects characterized by vertical boundaries is the discordance between eigenfunctions in the scatterer and the surroundings. This may be thought of as a departure from orthogonality of the two eigenfunction sets and characterizes the difference in material properties across the obstacle boundary. The effect of this discordance is recognized through the imposition of a definite structure in the scattered energy spectra for any given incident mode which is maintained to varying degrees for a wide range of scales and geometries. The horizontal scale and cross-section of the scatterer play a role of secondary importance in the redistribution of energy among scattered modes and do not alter the relative energy distribution among modes radically. The effect of an increase in dimension over the entire range of ka considered ($0.1 \rightarrow 10.0$ for the fundamental Rayleigh mode) is evidenced in the introduction of considerable complexity into the radiation pattern for a given incident mode through contributions from higher azimuthal orders. In addition there is a significant increase in the ratio of forward to back-scattered energy with the scale of the scatterer, an expression of the Mie scattering phenomenon observed in related disciplines. Finally, although energy spectra retain their general form the proportion of energy scattered into the unconverted mode relative to converted modes generally increases with the scale of the scatterer.

Deviations in the horizontal cross-section from a circular geometry were investigated by considering scatterers exhibiting varying degrees of ellipticity. Here we find that the total scattered energy for higher aspect ratios is highly dependent upon the incident angle of the exciting wave. In addition, the angle at which maximum

energy is scattered differs between the two wavetypes. It occurs at oblique angles ($\sim 45^\circ$) to the principal axes for incident Love modes while Rayleigh modes always scatter maximum total energy when impinging on the broadside of the elliptical cross-section. Furthermore radiation patterns indicate that a larger proportion of energy is backscattered for modes of both wavetypes when incident on the broad side of the ellipse in which case the forward scattered lobe is considerably narrower than for angles of incidence 90° away. In general the behaviour of the scattered wavefield from a circular cylinder of similar area falls between that observed for waves incident upon the ellipse at angles parallel to the two principal axes.

In terms of geometrical factors, boundaries exhibiting departures from a strictly vertical surface appear to have the most dramatic effect on modal coupling, especially in the spectrum of the incident wavetype. For gradual variations in the surface with depth strong coupling is restricted to the 'nearest neighbour' adjacent modes, but for more extreme changes significant coupling will spread across larger portions of the modal spectrum. Curiously, scattered energy in modes of the converted wavetype is not disrupted to the same degree, and although there are minor differences in the relative redistribution of energy between tapered and vertical cylinders the principal difference is a positive energy shift across the entire spectrum.

There is obviously much that can be done in extending the theory presented here to a range of more specific and physically interesting problems. For example certain portions of the earth's crust are known to exhibit lateral heterogeneity of a form which is reasonably modelled as a random distribution of discrete scatterers. It has been shown by researchers in related disciplines (e.g. Varadan, 1979; Boström, 1980) that the T-matrix formalism is ideally suited to treating problems of this type; the extension of these ideas to the surface-wave case will be presented in a subsequent study. Another topic deserving further investigation if the general method is to be applied to situations of greater relevance to anomalies within the earth is the choice of basis functions used to represent the field internal to the scatterer. Since the only restriction on this basis set in the current formulation be

that, upon truncation, it describe the internal field to the same degree of accuracy as those for the incident and scattered fields, it may be feasible given an appropriate geometric model and basis set to examine the effects of features such as topography and non-parallel layer boundaries on surface wave scattering.

Finally there are many parallels between the general T-matrix approach and the reflection/transmission operator formalism employed in problems involving general seismic wave propagation in heterogeneous media (Kennett, 1984b,c). These have already been exploited in particular applications (Boström and Karlsson, 1984; Chen, 1990), and further study appears warranted to investigate this relationship and the possibility of establishing a more unified approach to solving problems of wave propagation in heterogeneous stratified media.

CHAPTER 4

MULTIPLE SCATTERING OF SURFACE WAVES

FROM DISCRETE OBSTACLES

4.1 INTRODUCTION

The structure of the Earth is dominated by a radial variation in physical properties on which a significant component of lateral heterogeneity is superimposed. Results from a variety of studies indicate that this departure from radial stratification is most pronounced in the crust and upper mantle and apparently exists at a wide range of scales (*cf.* Lay, 1987). In certain areas, notably regions which have experienced recent tectonism, the effects of lateral heterogeneity are evidenced by strong scattering of surface waves. This becomes increasingly pronounced at higher frequencies with regional phases such as *Lg* and *Sn* most markedly affected. The scattering of surface waves is manifest in a number of ways, for example through the variability of traveltimes and waveforms across arrays and broad geographic regions (Ruzaikan *et al.*, 1977), azimuth anomalies due to multipathing (Bungum & Capon, 1974), and in the generation of an often prolonged seismic coda (Xie & Mitchell, 1990). Surface wave scattering results in a redistribution of energy both to later portions of the seismogram and across different members of the modal spectrum. Consequently the energy in seismograms at regional ranges is redistributed into a more diffuse signal than would be predicted by calculation for a stratified medium. The actual distinction between this apparent attenuation arising from scattering and intrinsic attenuation due to frictional losses is a primary concern in seismological studies which seek information concerning the physical structure of the earth through the determination of attenuation parameters such as the quality factor Q .

Our interest in the present study is the theoretical description of high frequency surface wave propagation in stratified media containing more than one scatterer. Previous work on the subject is somewhat limited, and has been dominated by two independent approaches, each valid for a specific class of heterogeneity. Coupled mode methods, introduced by Kennett (1984a), treat surface wave propagation in media exhibiting continuous variation in physical properties. The technique is strictly valid for 2-D media only, three dimensional effects such as conversion between Love and Rayleigh waves are not easily accommodated. This restriction renders

the method most useful for investigating the contribution of modal coupling within a given wavetype to the scattering process. For example, by examining stochastic models of the Earth's lateral heterogeneity Kennett (1990) has demonstrated that coupling can account for up to one third of the total attenuation generally observed in regional phase propagation. The second approach involves the application of first-order perturbation theory (i.e. the Born approximation) and has been used to examine more general 3-D scattering through representation theorems which employ a surface wave Green's function. Malin (1980) adopted this approach to model surface wave propagation using the normal modes for a stratified acoustic medium. Wang & Hermann (1988) extended Malin's treatment to elastic layered media, and examined a wide variety of scattering phenomena and their relationship to the character of the surface wave coda. Snieder (1986a, 1988c) has developed a compact formalism for surface wave scattering again employing the Born approximation and the surface wave Green's function for a laterally homogenous medium. This has proved an efficient means of investigating 3-D scattering from weak heterogeneity in a number of applications, for example, to invert for heterogeneity over western Europe. Brandenburg & Snieder (1989) have extended the approach to stochastic earth models to place constraints on attenuation due to scattering. As the authors note, however, their analysis can only be qualitative since the Born approximation violates energy conservation laws and the effects of multiple scattering within the zone of heterogeneity are neglected. Snieder (1988a) has provided valuable insight into the process of attenuation due to scattering by deriving an expression for the surface wave optical theorem which is exact for arbitrarily large heterogeneity contrasts in the far-field. Snieder exploited this information to relate the imaginary component of the forward scattering amplitude to Q in media where multiple scattering can be ignored.

It is apparent that much of the recent work concerning 3-D scattering of surface waves (versus the 2-D coupled mode treatments) in media composed of multiple scatterers has relied heavily on linearized scattering theory and has employed single scattering approximations (as is indeed the case in most body wave studies, see

Wu & Aki, 1989). In the previous chapter, however, we introduced a theory of surface wave scattering from a discrete obstacle, employing a T-matrix formulation (*cf.* Waterman, 1969), which is valid for scatterers exhibiting large contrasts in physical properties with their surroundings. The purpose of this chapter is to extend the T-matrix development to many scatterers and so study the effects of multiple scattering on the total field.

4.2 SURFACE WAVE SCATTERING BY TWO OBSTACLES

4.2.1 *Problem Formulation*

We will begin our theoretical development of surface wave scattering from multiple obstacles by examining the scattered field which arises when a wave is incident upon two scattering bodies. This configuration of two separated scatterers although simple, illustrates the essence of the method and provides a logical framework from which to proceed to the more general case.

We adopt a formulation similar to that used in chapter 3 for the single scatterer problem and will characterize the scatterers in our treatment in the following manner (see figure 4.1). The bounding surface defining the exterior of each obstacle is smooth and extends from the free surface to some lowest layer. This surface may be artificial in the sense that over certain depth intervals there need not exist a contrast in physical properties with the surrounding stratified medium. Although, not strictly required by the theory, we will consider the scatterer interior to be stratified, that is to exhibit variation in the vertical direction alone. In addition it must be possible to define a vertical line extending from the free surface to infinite depth which is wholly confined within the obstacle. Furthermore we assume that the embedding medium and both scatterers share the same bottom layer which behaves as a perfect reflector. This last 'locked mode' restriction ensures (*cf.* Harvey, 1981) that any wavefield within both the embedding medium and the scattering obstacles can be expressed entirely in terms of normal modes since body wave phases are built up through modal interference. Finally, the two scatterers may differ from one another in their shape and internal constitution but must remain completely

distinct so that it is possible to define a circular cylinder enclosing a given obstacle which does not penetrate corresponding circular cylinders for neighbouring obstacles.

4.2.2 *Surface Wave Basis Functions*

Previous authors (e.g. Peterson & Ström, 1973, 1974; Boström, 1980) in treating the multiple scattering problem for the acoustic, electromagnetic and general elastodynamic cases using similar matrix formulations chose to approach the problem by invoking Huyghen's principle with appropriate expansions for the Green's function as a starting point. As in chapter 3 however, we adopt an alternate strategy involving the use of a complete basis function expansion which avoids some of the complications (e.g. triadic manipulations and analytic continuation) inherent in the Green's function approach while yielding essentially the same results. Since the embedding medium and scatterers are both characterized by laterally homogeneous stratification the basis functions in this expansion are separated in their dependence on horizontal and vertical coordinates. As already mentioned, the locked mode restriction allows us to give a complete representation of the vertical dependence of the seismic wavefield with an infinite set of normal modes characterized by their displacement eigenfunctions. The horizontal dependence can be described in a number of ways, and as shown in chapter 3, a vector cylindrical harmonic representation is convenient for scattering problems. If two linearly independent solutions are used in the construction of the harmonics (see below), we have a complete orthogonal expansion in both horizontal and vertical coordinates and can, in principle, represent exactly any wavefield in the given medium. The essential ingredients required to extend the single scatterer formulation to media comprising two or more obstacles are translation operators which allow us to express the vector harmonics for one coordinate frame in those of a different frame. The domain over which some of these operators are valid imposes certain restrictions on the geometry of the scattering obstacles but nonetheless permits the examination of a wide range of configurations.

In applying the T-matrix formulation to scattering from multiple obstacles, we

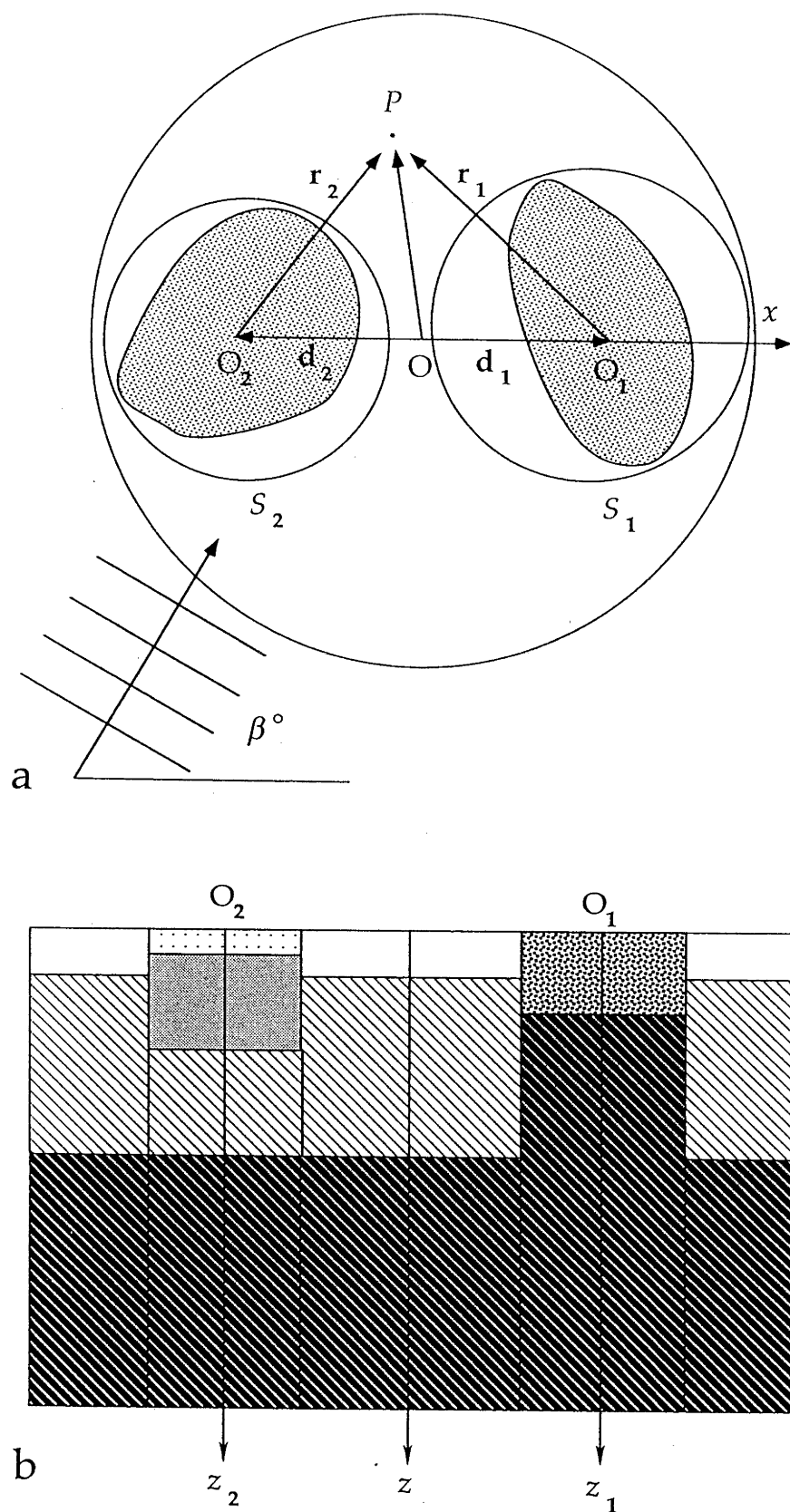


Figure 4.1 Illustration of the problem geometry for surface wave scattering from two obstacles embedded in a laterally-homogeneous half-space; a) plan view, and b) vertical section. Note that obstacle boundaries are not required to be vertical as shown in b).

will choose to arrange the basis functions into two sets as suggested in chapter 3. Explicitly, we define an outgoing set $(\psi^l)_n^m$ as

$$(\psi^l)_n^m = \begin{cases} W_n(k_n, \omega, z) \mathbf{T}_n^m(r, \theta) e^{-i\omega t}, & l = 1; \\ [U_n(k_n, \omega, z) \mathbf{R}_n^m(r, \theta) + V_n(k_n, \omega, z) \mathbf{S}_n^m(r, \theta)] e^{-i\omega t}, & l = 2; \end{cases} \quad (4.1)$$

where the indices l, m, n refer to wavetype, azimuthal order and modal order, respectively. The functions U_n , V_n , and W_n are the displacement eigenfunctions for the modes which describe the depth dependence and are readily determined for both Love ($l = 1$) and Rayleigh ($l = 2$) waves in plane-stratified earth models. The cylindrical harmonics $\mathbf{R}_n^m, \mathbf{S}_n^m, \mathbf{T}_n^m$ are defined as

$$\mathbf{R}_n^m = \hat{\mathbf{z}} Y_n^m, \quad \mathbf{S}_n^m = k_n^{-1} \nabla_1 Y_n^m, \quad \mathbf{T}_n^m = -\hat{\mathbf{z}} \times \mathbf{S}_n^m, \quad (4.2)$$

and are constructed using the outgoing Hankel function $H_m^{(1)}(k_n r)$ in the definition of the horizontal wave function Y_n^m :

$$Y_n^m = \epsilon_m H_m^{(1)}(k_n r) \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}, \quad (4.3)$$

where the Neumann factor ϵ_m is equal to $1/\sqrt{2}$ for $m = 0$ and 1 otherwise. We emphasize that this set of basis functions represents waves propagating outwards from $r = 0$, a fact which becomes obvious when we consider the asymptotic form of $H_m^{(1)}$ as r approaches infinity:

$$H_m^{(1)}(k_n r) \approx \sqrt{\frac{2}{\pi k_n r}} \exp \left[i \left(k_n r - \frac{\pi}{2} (m + 1/2) \right) \right]. \quad (4.4)$$

This type of behaviour is expected of surface waves scattered by obstacles in the embedding stratified medium. We choose therefore to represent the scattered field from a particular obstacle in terms of the basis functions from the set $(\psi^l)_n^m$ with respect to an origin of horizontal coordinates located within that obstacle. Note that the singularity due to the imaginary component of $H_m^{(1)}$ poses no problem to us in this instance since the scattered wave is only strictly defined outside the obstacle's bounding surface. The second set of basis functions, which we designate $(\hat{\psi}^l)_n^m$, is constructed by taking the regular part of $(\psi^l)_n^m$ (an operation we denote by an overhead caret) or, equivalently, substituting J_m for $H_m^{(1)}$ in (4.3), that is

$$(\hat{\psi}^l)_n^m = \begin{cases} W_n(k_n, \omega, z) \hat{\mathbf{T}}_n^m(r, \theta) e^{-i\omega t}, & l = 1; \\ [U_n(k_n, \omega, z) \hat{\mathbf{R}}_n^m(r, \theta) + V_n(k_n, \omega, z) \hat{\mathbf{S}}_n^m(r, \theta)] e^{-i\omega t}, & l = 2; \end{cases} \quad (4.5)$$

This basis set is characterized by standing wave behaviour and is finite-valued at the horizontal coordinate origin (i.e. along the z -axis). Consequently, it is used to expand fields which we expect will be well behaved at a given coordinate origin.

4.2.3 Component Wavefields

The problem configuration is shown in figure 4.1 which illustrates a horizontal plan view of the two scatterers in our embedding medium. The surfaces S_1 and S_2 are circular cylinders which enclose the two scatterers and in which we have located two coordinate origins O_1 and O_2 . A global reference frame is centered at O somewhere between the two obstacles and is associated again with a surface S which contains both scatterers. As in chapter 3 we will consider the total displacement field $\mathbf{u}^t$ within the stratified medium outside S as the sum of an unknown scattered field $\mathbf{u}^s$ and an incident field $\mathbf{u}^i$ which is taken as known and defined to be that which would exist in the absence of any heterogeneity,

$$\mathbf{u}^t = \mathbf{u}^i + \mathbf{u}^s. \quad (4.6)$$

The total scattered field $\mathbf{u}^s$ can in turn be broken into two constituent wavefields $\mathbf{u}^{s1}$ and $\mathbf{u}^{s2}$ associated with the individual fields scattered from either of the two obstacles

$$\mathbf{u}^s = \mathbf{u}^{s1} + \mathbf{u}^{s2}. \quad (4.7)$$

In addition, it is advantageous to introduce 'exciting' fields $\mathbf{u}^{e1}$ and $\mathbf{u}^{e2}$ which represent the total wavefield impinging upon either obstacle and which give rise to $\mathbf{u}^{s1}$ and $\mathbf{u}^{s2}$:

$$\mathbf{u}^{e1} = \mathbf{u}^i + \mathbf{u}^{s2}, \quad (4.8)$$

$$\mathbf{u}^{e2} = \mathbf{u}^i + \mathbf{u}^{s1}. \quad (4.9)$$

Note that in a single scattering approximation the second term in both (4.8) and (4.9) is ignored as a contributing source to the individual scattered fields. Having defined the various component displacement fields of interest, we now wish to

represent these quantities in terms of the basis function expansions in (4.1) and (4.5). The choice of appropriate basis set for a given component of the wavefield will depend on the nature of the field and where it is to be evaluated.

Since the incident field $\mathbf{u}^i$ is taken to originate outside the surface S encompassing both scatterers, it would, in the absence of any heterogeneity, be finite-valued throughout this volume. Hence it is appropriate to expand $\mathbf{u}^i$ in terms of the regular basis set $\hat{\psi}^\sigma$ referred to the global coordinate origin O . We indicate this explicitly by including the position vector $\mathbf{r}$ as an argument (see figure 4.1)

$$\begin{aligned}\mathbf{u}^i &= \sum_l \sum_m \sum_n (a^l)_n^m (\hat{\psi}^l)_n^m(\mathbf{r}) \\ &= \sum_\sigma a^\sigma \hat{\psi}^\sigma(\mathbf{r}).\end{aligned}\tag{4.10}$$

Here, the basis function coefficients for the incident field are denoted by a^σ where we have chosen to abbreviate the triple summation over l, m, n by single summation over the composite index σ (*cf.* chapter 3) for the sake of brevity. In contrast, we expect that outside the surface S the total scattered field $\mathbf{u}^s$ will behave as an outward propagating wavefield; thus we expand $\mathbf{u}^s$ in terms of ψ^l referred O ,

$$\mathbf{u}^s = \sum_\sigma c^\sigma \psi^\sigma(\mathbf{r}).\tag{4.11}$$

Now let us consider the displacement wavefields associated with the individual scatterers explicitly. The two scattered fields $\mathbf{u}^{s1}, \mathbf{u}^{s2}$ are naturally expanded in terms of outgoing basis functions this time referred to the origins O_1 and O_2 , respectively.

$$\mathbf{u}^{s1} = \sum_\sigma c_1^\sigma \psi^\sigma(\mathbf{r}_1),\tag{4.12}$$

$$\mathbf{u}^{s2} = \sum_\sigma c_2^\sigma \psi^\sigma(\mathbf{r}_2).\tag{4.13}$$

Finally, the exciting fields exhibit regular behaviour in the vicinity of the bounding surface of their respective obstacles (e.g. S_1 for $\mathbf{u}^{e1}$) and hence can be written as

$$\mathbf{u}^{e1} = \sum_\sigma b_1^\sigma \hat{\psi}^\sigma(\mathbf{r}_1),\tag{4.14}$$

$$\mathbf{u}^{e2} = \sum_\sigma b_2^\sigma \hat{\psi}^\sigma(\mathbf{r}_2).\tag{4.15}$$

The motivation for introducing these two latter quantities now becomes clear; it was shown in chapter 3 that the coefficients of scattered wave series in the form of (4.12), (4.13) could be related to those of a given exciting field as in (4.14), (4.15) by an infinite set of linear equations which when expressed in algebraic form constitutes a surface wave T-matrix. We will make use of this property shortly, but turn first to an examination of the translation operators for the surface wave basis functions.

4.2.4 Translation Operators

Reconsider for a moment, the basic objective of this section. We wish to derive a composite T-matrix relating the coefficients of the scattered field c^σ to those of the incident field a^σ . Note that our expansions for the corresponding fields are both referred to the origin at O whereas, for example the individual scattered fields from either obstacle are referred to origins at O_1 and O_2 . It is apparent then that to apply the T-matrix formalism we must be able to express basis functions at one coordinate origin to those of a different reference frame. To demonstrate this procedure we will examine the translation properties of the scalar wave functions $\hat{Y}_n^m, Y_n^m$ since the corresponding properties for our vector basis functions follow almost trivially. Consider our two obstacle geometry with the following horizontal quantities defined: $\mathbf{r} = (r, \theta, z)$ the position vector of point P with respect to a reference frame at O , $\mathbf{r}_1 = (r_1, \theta_1, z)$ the same quantity with respect to origin O_1 , and $\mathbf{d}_1 = (d_1, \phi, z)$ the vector separating the two origins such that $\mathbf{r} = \mathbf{r}_1 + \mathbf{d}_1$. To establish the appropriate translation operators for $\hat{Y}_n^m$ we will exploit the following identity:

$$J_m(k_n r) e^{im\theta} = \sum_{p=-\infty}^{\infty} J_{m-p}(k_n d_1) e^{i(m-p)\phi} J_p(k_n r_1) e^{ip\theta_1}. \quad (4.16)$$

where $r = |\mathbf{r}|$ etc. This expression and (19) below are generalizations of Graf's addition theorem, see Erdélyi *et al.*, 1953. We note that the expression in (4.3) incorporates real-valued sinusoids. Thus to pose the relation above in terms of our $\hat{Y}_n^m$ we must separate the complex exponentials in the above expression into sine and cosine quantities on either side of the equality and reduce the sum from

$-\infty \rightarrow \infty$ to $0 \rightarrow \infty$. After some manipulation we can write the result as

$$\hat{Y}_n^m(\mathbf{r}) = \sum_{p=0}^{\infty} A^{mp}(\mathbf{d}_1) \hat{Y}_n^p(\mathbf{r}_1), \quad (4.17)$$

where

$$A^{mp}(\mathbf{d}_1) = \frac{1}{2} \sqrt{\epsilon_m \epsilon_p} \left[J_{m-p}(k_n d_1) \begin{pmatrix} \cos(m-p)\phi & -\sin(m-p)\phi \\ \sin(m-p)\phi & \cos(m-p)\phi \end{pmatrix} \right. \\ \left. + (-1)^p J_{m+p}(k_n d_1) \begin{pmatrix} \cos(m+p)\phi & \sin(m+p)\phi \\ \sin(m+p)\phi & -\cos(m+p)\phi \end{pmatrix} \right]. \quad (4.18)$$

Note that this relationship holds for the wavefunction $\hat{Y}_n^m$ specific to a given surface wave (Love or Rayleigh) mode of index n and involves a summation over azimuthal order p . We can derive the second of the two required translation operators which relates the outgoing horizontal wave function Y_n^m at O_1 to the regular wave functions $\hat{Y}_n^m$ at O_2 using the following identity (here we have posed the relation in a form encountered in the ensuing analysis)

$$H_m^{(1)}(k_n r_1) e^{ip\theta_1} = \sum_{p=-\infty}^{\infty} H_{m-p}^{(1)}(k_n(-d_1 + d_2)) e^{i(m-p)\phi} J_p(k_n r_2) e^{ip\theta_2}, \quad (4.19)$$

where ϕ is now the angle of the vector $-\mathbf{d}_1 + \mathbf{d}_2$. It should be noted that this relation is valid only for $(-\mathbf{d}_1 + \mathbf{d}_2) > r_2$ owing to the singularity in $H_m^{(1)}$. By rearranging this equation in a similar manner we obtain

$$Y_n^m(\mathbf{r}) = \sum_{p=0}^{\infty} B^{mp}(\mathbf{d}_1) \hat{Y}_n^p(\mathbf{r}_1), \quad (4.20)$$

where B^{mp} is computed by substituting the Bessel functions $J_{m\pm p}$ with outgoing Hankel functions $H_{m\pm p}^{(1)}$ in (4.18). The extension of these results to those of our surface wave basis functions is realized by inserting the expressions in (4.17) and (4.20) into the relations given by (4.2). Our task is made easy with the recognition that the differential operators in (4.2) are invariant and cannot depend on the choice of coordinate system. Thus the translation quantities A^{mp}, B^{mp} need only be modified to remain consistent with the conventions we have established thus far. Hence we shall write

$$(\hat{\psi}^l)_n^m(\mathbf{r}) = (\hat{\psi}^l)_n^m(\mathbf{d}_1 + \mathbf{r}_1) = \sum_p (A^l)_n^{mp}(\mathbf{d}_1) (\hat{\psi}^l)_n^p(\mathbf{r}_1), \quad (4.21)$$

and

$$(\psi^l)_n^m(\mathbf{r}_2) = (\psi^l)_n^m(-\mathbf{d}_2 + \mathbf{d}_1 + \mathbf{r}_1) = \sum_p (B^l)_n^{mp}(-\mathbf{d}_2 + \mathbf{d}_1) (\hat{\psi}^l)_n^p(\mathbf{r}_1) \quad (4.22)$$

The requirement that $(-\mathbf{d}_1 + \mathbf{d}_2) > r_2$ for the validity of (4.20) and (4.22) places some restrictions on the spatial relationship of the two obstacles. Since the relation in (4.22) will be required over the surface of the obstacle at S_1 a sufficient (though not necessary) condition for $(-\mathbf{d}_1 + \mathbf{d}_2) > r_2$ to hold is that the two cylinders S_1 and S_2 circumscribing the obstacles do not overlap. Note that we have temporarily expanded the compound index σ to emphasize that the summations exist over azimuthal order (p) only, hence the quantities $(A^l)_n^{mp}, (B^l)_n^{mp}$ are ‘diagonal’ with respect to wavetype and modal indices so that a translation of our basis functions is independent of mode and or wavetype and couples azimuthal orders alone. We note that the modified quantities $(A^l)_n^{mp}, (B^l)_n^{mp}$ will depend on mode and wavetype insofar as we must include the appropriate wavenumber in the calculation of each diagonal submatrix element. To maintain an uncluttered presentation for the remaining analysis we will in general refer to these quantities in terms of the compound indices e.g. $A^{\sigma\nu}, B^{\sigma\nu}$. Finally, it may prove convenient in numerical applications where we are interested the far-field properties of the scattered field to use approximate expressions for the translation operator $A^{\sigma\nu}$. This procedure is detailed in Appendix B.

4.2.5 Dual Scatterer T -Matrix

Now consider the scatterer at O_1 . From our definitions in (4.8), (4.14) and using the relationships (4.21), (4.22) we can write

$$\begin{aligned} \sum_{\sigma} b_1^{\sigma} \hat{\psi}^{\sigma}(\mathbf{r}_1) &= \sum_{\nu} a^{\nu} \hat{\psi}^{\nu}(\mathbf{r}) + \sum_{\tau} c_2^{\tau} \psi^{\tau}(\mathbf{r}_2) \\ &= \sum_{\nu} \sum_{\alpha} a^{\nu} A^{\nu\alpha}(\mathbf{d}_1) \hat{\psi}^{\alpha}(\mathbf{r}_1) + \sum_{\tau} \sum_{\beta} c_2^{\tau} B^{\tau\beta}(-\mathbf{d}_2 + \mathbf{d}_1) \hat{\psi}^{\beta}(\mathbf{r}_1). \end{aligned} \quad (4.23)$$

The orthogonality of the surface wave basis function sets allows us to express b_1^{σ} as

$$b_1^{\sigma} = \sum_{\nu} a^{\nu} A^{\nu\sigma}(\mathbf{d}_1) + \sum_{\tau} c_2^{\tau} B^{\tau\sigma}(-\mathbf{d}_2 + \mathbf{d}_1) \quad (4.24)$$

We can exploit the simplicity of matrix notation and write

$$\mathbf{b}_1^T = \mathbf{a}^T \mathbf{A}(\mathbf{d}_1) + \mathbf{c}_2^T \mathbf{B}(-\mathbf{d}_2 + \mathbf{d}_1), \quad (4.25)$$

so that

$$\mathbf{b}_1 = \mathbf{A}^T(\mathbf{d}_1)\mathbf{a} + \mathbf{B}^T(-\mathbf{d}_2 + \mathbf{d}_1)\mathbf{c}_2, \quad (4.26)$$

where $\mathbf{a}, \mathbf{b}_1$ and $\mathbf{c}_2$ are column vectors; $\mathbf{A}, \mathbf{B}$ are square matrices; and we have denoted transposition by T . Following a similar line of argument for $\mathbf{b}_2$ we must have

$$\mathbf{b}_2 = \mathbf{A}^T(\mathbf{d}_2)\mathbf{a} + \mathbf{B}^T(-\mathbf{d}_1 + \mathbf{d}_2)\mathbf{c}_1, \quad (4.27)$$

As stated previously, $\mathbf{c}_1$ is related to $\mathbf{b}_1$ via the T-matrix for a scatterer at O_1 which we denote $\mathbf{T}_1$ or

$$\mathbf{c}_1 = \mathbf{T}_1\mathbf{b}_1, \quad (4.28)$$

and likewise

$$\mathbf{c}_2 = \mathbf{T}_2\mathbf{b}_2. \quad (4.29)$$

Substituting these relations into (4.26), (4.27) yields

$$\mathbf{c}_1 = \mathbf{T}_1[\mathbf{A}^T(\mathbf{d}_1)\mathbf{a} + \mathbf{B}^T(-\mathbf{d}_2 + \mathbf{d}_1)\mathbf{c}_2], \quad (4.30)$$

$$\mathbf{c}_2 = \mathbf{T}_2[\mathbf{A}^T(\mathbf{d}_2)\mathbf{a} + \mathbf{B}^T(-\mathbf{d}_1 + \mathbf{d}_2)\mathbf{c}_1]; \quad (4.31)$$

where we have two matrix equations with the two scattering coefficient vectors $\mathbf{c}_1, \mathbf{c}_2$ as unknowns which can be solved such that, for $\mathbf{c}_1$ we have

$$\begin{aligned} \mathbf{c}_1 = [\mathbf{I} - \mathbf{T}_1\mathbf{B}^T(\mathbf{d}_1 - \mathbf{d}_2)\mathbf{T}_2\mathbf{B}^T(\mathbf{d}_2 - \mathbf{d}_1)]^{-1} \\ \times \mathbf{T}_1[\mathbf{A}^T(\mathbf{d}_1) + \mathbf{B}^T(\mathbf{d}_1 - \mathbf{d}_2)\mathbf{T}_2\mathbf{A}^T(\mathbf{d}_2)]\mathbf{a}, \end{aligned} \quad (4.32)$$

with a similar relation holding for $\mathbf{c}_2$. We wish to describe the total scattered field $\mathbf{u}^s$ with reference to the global coordinate origin O so we premultiply the expressions for the two individual scattering coefficient vectors $\mathbf{c}_1, \mathbf{c}_2$ by the appropriate translation matrices $-\mathbf{A}^T(-\mathbf{d}_1), -\mathbf{A}^T(-\mathbf{d}_2)$ and incorporate these quantities into equation (4.7) to yield

$$\begin{aligned} \mathbf{c} = \{ \mathbf{A}^T(-\mathbf{d}_1) [\mathbf{I} - \mathbf{T}_1\mathbf{B}^T(\mathbf{d}_1 - \mathbf{d}_2)\mathbf{T}_2\mathbf{B}^T(\mathbf{d}_2 - \mathbf{d}_1)]^{-1} \\ \times \mathbf{T}_1[\mathbf{A}^T(\mathbf{d}_1) + \mathbf{B}^T(\mathbf{d}_1 - \mathbf{d}_2)\mathbf{T}_2\mathbf{A}^T(\mathbf{d}_2)] + \\ \mathbf{A}^T(-\mathbf{d}_2)[\mathbf{I} - \mathbf{T}_2\mathbf{B}^T(\mathbf{d}_2 - \mathbf{d}_1)\mathbf{T}_1\mathbf{B}^T(\mathbf{d}_1 - \mathbf{d}_2)]^{-1} \\ \times \mathbf{T}_2[\mathbf{A}^T(\mathbf{d}_2) + \mathbf{B}^T(\mathbf{d}_2 - \mathbf{d}_1)\mathbf{T}_1\mathbf{A}^T(\mathbf{d}_1)] \} \mathbf{a}, \end{aligned} \quad (4.33)$$

The quantity in curly brackets relates the total scattered and incident fields and is thus simply the T-matrix $\mathbf{T}_{(1,2)}$ for the dual scatterer configuration. By recognizing that $\mathbf{A}^T(-\mathbf{r}) = \mathbf{A}(\mathbf{r})$, $\mathbf{B}^T(-\mathbf{r}) = \mathbf{B}(\mathbf{r})$ and reorganizing the inverse matrix factors slightly we express $\mathbf{T}_{(1,2)}$ as

$$\begin{aligned} \mathbf{T}_{(1,2)} = & \{ \mathbf{A}(\mathbf{d}_1) \mathbf{T}_1 [\mathbf{I} - \mathbf{B}(-\mathbf{d}_1 + \mathbf{d}_2) \mathbf{T}_2 \mathbf{B}(-\mathbf{d}_2 + \mathbf{d}_1) \mathbf{T}_1]^{-1} \\ & \times [\mathbf{I} + \mathbf{B}(-\mathbf{d}_1 + \mathbf{d}_2) \mathbf{T}_2 \mathbf{A}(-\mathbf{d}_2 + \mathbf{d}_1)] \mathbf{A}(-\mathbf{d}_1) \\ & + \mathbf{A}(\mathbf{d}_2) \mathbf{T}_2 [\mathbf{I} - \mathbf{B}(-\mathbf{d}_2 + \mathbf{d}_1) \mathbf{T}_1 \mathbf{B}(-\mathbf{d}_1 + \mathbf{d}_2) \mathbf{T}_2]^{-1} \\ & \times [\mathbf{I} + \mathbf{B}(-\mathbf{d}_2 + \mathbf{d}_1) \mathbf{T}_1 \mathbf{A}(-\mathbf{d}_1 + \mathbf{d}_2)] \mathbf{A}(-\mathbf{d}_2) \} . \end{aligned} \quad (4.34)$$

For the sake of convenience in subsequent analysis and to allow the structure of the equations to be seen more clearly, we will contract the translation operators such that for example $\mathbf{A}_{-1} = \mathbf{A}(-\mathbf{d}_1)$, $\mathbf{B}_{-12} = \mathbf{B}(-\mathbf{d}_1 + \mathbf{d}_2)$ and (4.34) is then written

$$\begin{aligned} \mathbf{T}_{(1,2)} = & \{ \mathbf{A}_1 \mathbf{T}_1 [\mathbf{I} - \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1]^{-1} \times [\mathbf{I} + \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{A}_{-21}] \mathbf{A}_{-1} \\ & + \mathbf{A}_2 \mathbf{T}_2 [\mathbf{I} - \mathbf{B}_{-21} \mathbf{T}_1 \mathbf{B}_{-12} \mathbf{T}_2]^{-1} \times [\mathbf{I} + \mathbf{B}_{-21} \mathbf{T}_1 \mathbf{A}_{-12}] \mathbf{A}_{-2} \} . \end{aligned}$$

This provides an expression for $\mathbf{T}_{(1,2)}$ in the same form as that given for the analogous problem in electromagnetics (Peterson & Ström, 1973), however we have avoided a more complicated derivation involving Green's function manipulations and have proceeded directly from the definition of the T-matrix for an individual obstacle. As mentioned by these authors this form of the dual scatterer T-matrix is advantageous in that it treats the effects of both scatterers in symmetric fashion, and we see that if the magnitude of heterogeneity characterizing one of the two obstacle becomes vanishingly small (i.e. the corresponding T-matrix tends to the zero matrix) then the expression in (4.34) reduces to the T-matrix for a single scatterer with a change of origin. Another point of interest is that we can investigate the physical significance of the various factors by expressing the inverse matrices in (4.34) as a series expansions, that is for example

$$\begin{aligned} [\mathbf{I} - \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1]^{-1} \\ = \mathbf{I} + \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1 + \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1 \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1 + \dots \end{aligned} \quad (4.35)$$

In this form it is clear that the first term in (4.34) represents all the scattering processes which originate from the scatterer at O_1 . For instance terms of the form

$$\mathbf{T}_1 \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1 \mathbf{B}_{-12} \mathbf{T}_2 \mathbf{B}_{-21} \mathbf{T}_1$$

can be seen to represent (reading from left to right) that portion of the incident wave which impinges first on the scatterer at O_1 , undergoes two successive interactions with the scatterer at O_2 and is finally rescattered back into the homogeneous medium from O_1 . We also see then that the matrix factors $\mathbf{B}_{ij}$ are essentially phase factors which, although more complicated in form, play a role analogous to $e^{ik\Delta x}$ for plane waves.

4.3 SURFACE WAVE SCATTERING BY N OBSTACLES

A deterministic description of surface wave scattering from an arbitrary number N obstacles can be approached in a number of ways. The first is naturally to repeat the procedures outlined in the previous section for N scatterers. This course will result in an expression for the scattering coefficient vector characterizing the i th scatterer (i.e. the analogue of (4.30) and (4.31)) of the form

$$\mathbf{c}_i = \mathbf{T}_i[\mathbf{A}_{-i}\mathbf{a} + \sum_{j \neq i}^N \mathbf{B}_{-ij}\mathbf{c}_j], \quad (4.36)$$

which, once solved, leads to the expression of the coefficient vector for the total scattered field as

$$\mathbf{c} = \sum_i \mathbf{A}_i \mathbf{c}_i. \quad (4.37)$$

The complication which we encounter here resides in the solution of (4.36) which represents a system of N matrix equations in N scattering coefficient vectors. The known matrix coefficients are non-commuting operators and (4.36) is thus best solved using iterative techniques which have been given elsewhere (Peterson & Ström, 1973) and will not be repeated here. The advantage of this approach is that the resulting expression for the composite T-matrix, as in (4.33), explicitly identifies the contributions of each scatterer including the complete hierarchy of multiple scattering interactions.

In an alternative approach, we rely on the results from the previous section, where it was demonstrated that a composite T-matrix for two scattering bodies can be constructed in terms of their individual T-matrices in conjunction with a set of translation matrices reflecting the obstacles' spatial relationship. It is natural then

to extend dual scatterer result by considering as one of our scattering entities a configuration of two scatterers for which the composite T-matrix is already known, and applying the formulation outlined in section 2 once more to accomodate a third obstacle. Using this approach we could express the T-matrix for a 3-obstacle configuration as

$$\begin{aligned} \mathbf{T}_{(1,2,3)} = \mathbf{T}_{((1,2),3)} = & \{ \mathbf{T}_{(1,2)} [\mathbf{I} - \mathbf{B}_3 \mathbf{T}_3 \mathbf{B}_{-21} \mathbf{T}_{(1,2)}]^{-1} \times [\mathbf{I} + \mathbf{B}_3 \mathbf{T}_3 \mathbf{A}_{-3}] \\ & + \mathbf{A}_3 \mathbf{T}_3 [\mathbf{I} - \mathbf{B}_{-3} \mathbf{T}_{(1,2)} \mathbf{B}_3 \mathbf{T}_3]^{-1} \times [\mathbf{I} + \mathbf{B}_{-3} \mathbf{T}_3 \mathbf{A}_3] \mathbf{A}_{-3} \}. \end{aligned} \quad (4.38)$$

where we have chosen to refer $\mathbf{T}_{(1,2,3)}$ to the same coordinate origin as $\mathbf{T}_{(1,2)}$. We could however have equally well referred it to the origin at O_3 or any other origin for that matter, either by appropriately modifying the expression in (4.38) or after calculation using the relation in (4.21). In this fashion it is possible, in principle, to construct the T-matrix for an arbitrary configuration recursively, starting with the two closest scatterers and successively adding scatterers or groups of scatterers using the basic form of (4.34), a formulation which is reminiscent of recursion schemes for calculating the transmission-reflection response for a stack of plane, elastic layers. Thus we could represent an N -obstacle T-matrix $\mathbf{T}_{(1,\dots,N)}$ as

$$\mathbf{T}_{(1,\dots,N)} = \mathbf{T}_{((1,\dots,N-1),N)}, \quad (4.39)$$

or equivalently as

$$\mathbf{T}_{(1,\dots,N)} = \mathbf{T}_{((1,\dots,M),(M+1,\dots,N))}. \quad (4.40)$$

Herein lies the principal advantage of the recursive scheme over the method mentioned earlier; by constructing an N -obstacle composite T-matrix out of other T-matrix components we are allowed greater flexibility from a practical point of view in the variety of configurations that can be examined. Note that in principle then, we can calculate the T-matrix for an N -obstacle configuration at a number of frequencies and construct the complete time domain response to a given incident wave via Fourier-Hankel synthesis.

Before proceeding to the numerical implementation of the theory presented in these last two sections and the practical problems involved, a second glance at the

form of equations (4.36) and (4.37) is warranted. By combining both equations we can write the total scattering coefficient vector $\mathbf{c}$ as

$$\mathbf{c} = \sum_i \mathbf{A}_i \mathbf{T}_i [\mathbf{A}_{-i} \mathbf{a} + \sum_{j \neq i}^N \mathbf{B}_{-ij} \mathbf{c}_j], \quad (4.41)$$

By successive iterations this must equal

$$\begin{aligned} \mathbf{c} = & \left[\sum_i^N \mathbf{A}_i \mathbf{T}_i \mathbf{A}_{-i} + \sum_i^N \mathbf{A}_i \mathbf{T}_i \sum_{j \neq i}^N \mathbf{B}_{-ij} \mathbf{A}_j \mathbf{T}_j \mathbf{A}_{-j} \right. \\ & \left. + \sum_i^N \mathbf{A}_i \mathbf{T}_i \sum_{j \neq i}^N \mathbf{B}_{-ij} \mathbf{A}_j \mathbf{T}_j \sum_{k \neq j}^N \mathbf{B}_{-jk} \mathbf{A}_k \mathbf{T}_k \mathbf{A}_{-k} + \dots \right] \mathbf{a}. \end{aligned} \quad (4.42)$$

The terms in square brackets are an expansion in orders of scattering for the composite N -obstacle T-matrix $\mathbf{T}_{(1,\dots,N)}$, for instance the first term represents the effects of single scattering, the second term represents first-order pair interactions and so on. This relation may be useful in assessing the significance of multiple scattering in certain situations. It is interesting to note that this relation is, in fact, equivalent to a more general expression given by Hudson & Knopoff ([1989] equation (40)) of the form:

$$\mathbf{u}^s = \epsilon \sum_i^N \mathbf{S}^i \mathbf{u}^i + \epsilon^2 \sum_i^N \sum_{j \neq i}^N \mathbf{S}^i \mathbf{S}^j \mathbf{u}^i + O(\epsilon^3),$$

where $\mathbf{u}^s$ and $\mathbf{u}^i$ are the total scattered field and incident fields respectively, ϵ is a perturbation parameter and $\mathbf{S}^i$ is a general scattering operator associated with the i th scatterer. Comparison with (4.42) shows explicitly that the scattering operator $\mathbf{S}^i$ must incorporate the effects of propagation of multiply scattered fields between obstacles.

4.4 NUMERICAL EXAMPLES

The theory presented in earlier sections is suited to modelling the propagation of surface waves in environments characterized by a dominantly horizontal stratification and containing discrete scattering obstacles. In the earth this is representative of short-period waves (≤ 1 s) propagating over regional distances ($0 \rightarrow 1000$ km) for which the effects of the earth's sphericity are negligible. The character of crustal and upper mantle heterogeneity and the dominant frequencies of regional

phases are such that the effects of scattering are frequently more pronounced than for waves at longer periods and may become very complex in some regions.

A deterministic treatment of surface wave scattering from multiple scatterers as formulated in the first part of this study can not however provide us with a comprehensive description of the wavefield propagating through a medium such as the earth containing scatterers of variable character and distribution. The limitation is not one of theoretical derivation but rather stems from the numerical aspects of the problem. The method becomes computationally cumbersome when we consider configurations involving more than a few scatterers especially if they are spread over large areas relative to the dominant wavelengths. Hence our objective here is not to model scattering within the earth's crust and upper mantle directly. Rather, we will exploit the T-matrix formulation to investigate several assumptions on which many previous studies on this subject are based by examining a few simple multiple scatterer configurations. There are two principal advantages to our approach in this regard: i) in contrast to single scattering theory, the analysis is not restricted to small scatterers exhibiting mild heterogeneity; and ii) the effects of scattering are accurately described in both the near and far-fields.

This last point is important for it allows us to examine the contributions of first and higher order scattering interactions between multiple scatterers to the behaviour of the total scattered field at a full range of scatterer separations. We can safely surmise that if multiple scattering interactions have a significant effect on the total scattered field from two obstacles at a given separation, then any analysis of surface wave propagation in a heterogeneous earth where the mean distance between scatterers is of similar order must also address this influence. This is particularly relevant in the study of attenuation due to scattering, an area where single scattering has typically been assumed and higher order scattering interactions neglected. Consequently we focus our analysis on the total scattered field from two obstacles as a function of their separation and size, and pay particular attention to the higher order scattering.

The T-matrix for a given obstacle configuration provides a complete description

of the scattering behaviour at a particular frequency and is independent of the form of the incident wave. As a result there exists a wide selection of parameters which might be used to characterize the scattered wavefield. This is especially true in the case of surface waves where the scattering process involves conversion among a large number of modes of two different wavetypes. However, rather than present a comprehensive selection of results for a variety of parameters we will limit ourselves to one or two specific measures of scattering for a particular class of interaction. We will consider a single mode plane wave incident on a two obstacle configuration at angle β to the x -axis which joins the two obstacle centers. In addition we will confine our attention to the far-field power (normalized with respect to the square of the scatterer radius) in the unconverted mode scattered into a 60° window about the angle β . This is commonly considered to be that portion of the scattered power which contributes directly to surface wave attenuation due to scattering (see Brandenburg & Snieder, 1989; Frankel & Clayton, 1986). Although it will not be used to investigate attenuation directly, this parameter is a convenient measure of the scattered field and does provide some indication of how pair interactions might contribute to forward scattering in more general scattering situations.

We will employ an idealized model of the crust and upper mantle as used in chapter 3 to examine surface wave propagation at a frequency of 1 Hz. The model heterogeneity comprises two-layer, low velocity plugs embedded within an otherwise laterally homogeneous three-layered earth. Velocities are constant in each of the layers in both the plugs and the surroundings but increase with layer depth. The velocity contrast across the external boundaries of the scatterers decreases from seven to five percent over the first and second layers respectively, and is meant to represent a maximum typical magnitude of heterogeneity in the earth's crust. For this study we will consider scatterers i) with circular cross sections only as this simplifies some of the analysis and computation and ii) with identical material properties as functions of depth. Neither of these two restrictions is of great importance to the points we wish to address. We have employed the first 12 Love and Rayleigh modes in our computations, an ensemble which should permit a

reasonable description of the scattering process in all but the highest order modes (Kennett, 1984). The number of azimuthal orders m required depends upon the horizontal extent of the scatterers. Thus computations will involve matrices of dimension $N = 24 \times (2m + 1)$ and storage considerations limit our analysis to the examination of obstacles of $ka \leq 10$. This range is quantified through the parameter ka where k is the wavenumber of the surface wave and a is a measure of the average radius of the scatterer.

The specific model parameters are provided in Table 3.1 and for more detailed information on, for example, the characteristics of the eigenfunctions the reader is advised to consult chapter 3. Results will be presented for the fundamental Love and second Rayleigh modes (hereafter $L1$ and $R2$) since these are the first modes which contribute significantly to the Lg phase and unless otherwise stated their behaviour is representative of Love and Rayleigh modes in general (note that the energy in the fundamental Rayleigh mode $R1$ at frequencies as high as 1.0 Hz is confined primarily to surficial sedimentary layers).

4.4.1 Scatterers ($0 < ka \leq 1.0$)

Although the scattered field component of a wave propagating through a bulk medium consisting many point scatterers may differ considerably from that in a medium consisting of scatterers characterized by $ka \approx 1.0$ (*cf.* Aki & Richards, 1980, figure 13.11), the scattered fields from one or two obstacles at these two limits of interaction are qualitatively quite similar in some regards. This similarity is exemplified in the form of their radiation patterns (see figure 4.2) which describe the dependence of the far-field amplitude of a scattered mode as a function of azimuth for a given incident plane wave. These indicate that the same azimuthal orders m dominate for scattering obstacles throughout the range $0 < ka \leq 1.0$. Before proceeding to the scattered field from two-obstacle configurations, it is worthwhile digressing for a moment to review the nature of the scattered field from one obstacle at the low frequency limit of this range of interaction.

In chapter 3 and Snieder (1986a) it is remarked that the far-field radiation patterns from a point scatter for various classes of wavetype conversion are quite

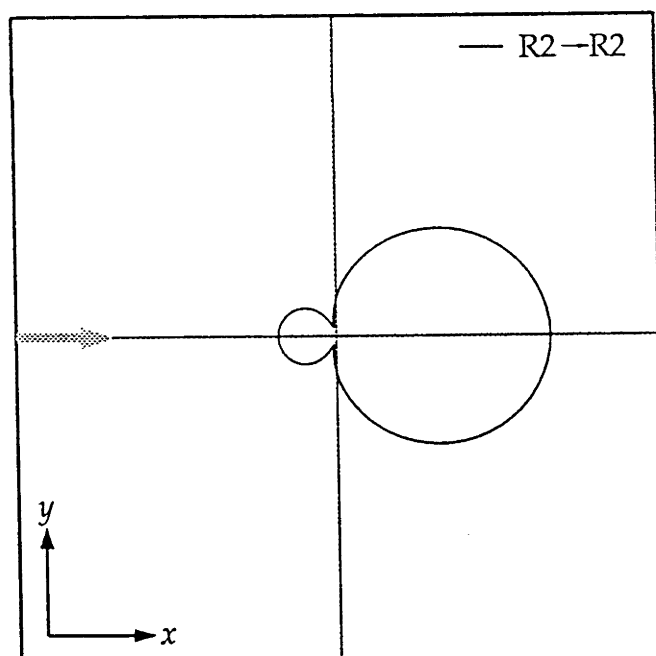
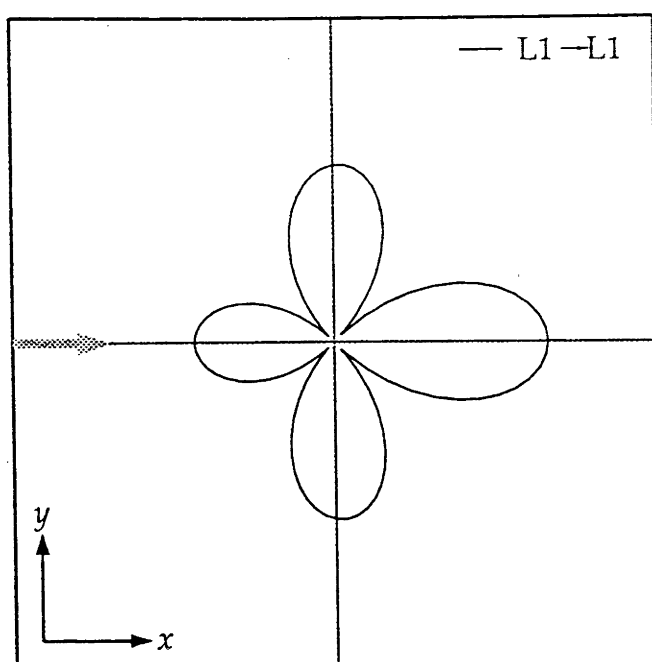


Figure 4.2 Radiation patterns for unconverted $L1$ and $R2$ in the far-field from a scatterer $ka \approx 1.0$. Note the dominance of azimuthal order $m = 2$ for $L1$ and orders $m = 0, 1$ for $R2$.

distinctive. Love to Rayleigh, or $L \rightarrow R$, and $R \rightarrow L$ conversions consist primarily of $\sin\theta$ and $\sin 2\theta$ contributions while $L \rightarrow L$ and $R \rightarrow R$ interactions exhibit DC, $\cos\theta$, and $\cos 2\theta$ components where θ is the angle of observation relative to the direction of propagation of the incident plane wave mode. The association of sine and cosine character with converted and unconverted wavetype interactions respectively, follows in our case from the azimuthal dependence of the surface wave basis functions and the circular symmetry of the scatterer (note that some effect of scatterer geometry is preserved at low frequency interactions; thus for example the scattered field from a very small elliptical scatterer will differ slightly from a circular one). However it is also evident for the models presented in both chapter 3 and Snieder (1986a) that the $L \rightarrow L$ radiation patterns tend to have little energy concentrated in the DC component ($4.m = 0$) and, in general, there is considerably more energy in the $m = 2$ component than those for $R \rightarrow R$ interactions. The reason for this can also be identified in the structure of the surface wave basis functions but now in the radial dependence for small argument ka . It can be shown that the behaviour of point scatterers is largely dictated by the real part of the matrix $\mathbf{Q}$ which we write as $\hat{\mathbf{Q}}$ and the elements of which are defined as (cf. chapter 3)

$$\hat{Q}^{\nu\sigma} = \frac{\omega}{2i} \int_{S^*} dS \left[\mathbf{t}(\hat{\phi}^\sigma) \cdot \hat{\psi}^\nu - \mathbf{t}(\hat{\psi}^\nu) \cdot \hat{\phi}^\sigma \right]. \quad (4.43)$$

Here $\mathbf{t}(\hat{\phi}^\sigma)$ is, for example, the traction of the regular surface wave basis functions for the interior of the scatterer over the surface of the scatterer S^* . The T-matrix $\mathbf{T}$ for a given obstacle is then defined as

$$\mathbf{T} = -\hat{\mathbf{Q}}\mathbf{Q}^{-1}. \quad (4.44)$$

For point scatterers the behaviour of the Bessel function for small arguments requires that the elements of $\hat{\mathbf{Q}}$ are significant only for azimuthal orders $m = 0, 1, 2$ for Rayleigh waves and $m = 1, 2$ for Love waves. The relative magnitudes of these elements are determined by depth integrals of the Love and Rayleigh eigenfunctions U_n, V_n, W_n for the embedding medium and the scatterer. For Rayleigh waves these are

$$m = 0$$

$$\int_0^\infty dz \quad k_n^i k_o^e V_n^i V_o^e [(\lambda^i + \mu^i) - (\lambda^e + \mu^e)] + \frac{1}{2} U_n^i U_o^e (k_o^{e2} \mu^e - k_n^{i2} \mu^i) \\ + (k_n^i \lambda^e \partial_z U_o^e V_n^i - k_o^e \lambda^i \partial_z U_n^i V_o^e) + \frac{1}{2} (k_o^e \mu^e \partial_z V_o^e U_o^e - k_n^i \mu^i \partial_z V_n^i U_o^e), \quad (4.45)$$

$m = 1$

$$\int_0^\infty dz \quad \frac{1}{2} \{ V_n^i V_o^e [k_o^{e2} (\lambda^e + 2\mu^e) - k_n^{i2} (\lambda^i + 2\mu^i)] + k_n^i k_o^e U_n^i U_o^e (\mu^i - \mu^e) \\ + (k_n^i \lambda^i \partial_z U_n^i V_o^e - k_o^e \lambda^e \partial_z U_o^e V_n^i) + (k_o^e \mu^i \partial_z V_n^i U_o^e - k_n^i \mu^e \partial_z V_o^e U_n^i) \}, \quad (4.46)$$

$m = 2$

$$\int_0^\infty dz \quad \frac{1}{2} k_n^i k_o^e V_n^i V_o^e (\mu^i - \mu^e); \quad (4.47)$$

and for Love waves

$m = 1$

$$\int_0^\infty dz \quad \frac{1}{2} W_n^i W_o^e (k_o^{e2} \mu^e - k_n^{i2} \mu^i), \quad (4.48)$$

$m = 2$

$$\int_0^\infty dz \quad \frac{1}{2} k_n^i k_o^e W_n^i W_o^e (\mu^i - \mu^e). \quad (4.49)$$

Here the superscripts i, e indicate that the associated quantities are referred to the scatterer and the embedding medium, respectively, and the relations hold for any two modes o, n . For our simple earth model the relative importance of each of the integrals in (4.45) to (4.49) is such that (4.49) exceeds (4.48), and (4.45), (4.46) exceed (4.47). Indeed it appears that for a majority of typical earth models the behaviour of the depth integrals in (4.45) to (4.49) is qualitatively similar. This can be traced to variational results and will hold in general when the relative perturbations in the shear modulus and density which define the scatterer are approximately equivalent as functions of depth. A consequence of this result is that the surface wave T-matrix for point scatterers preferentially scatters Love waves into $m = 2$ and Rayleigh waves into $m = 0, 1$. This behaviour appears to persist up to $ka \approx 1.0$, beyond which the low order basis functions certainly cease to behave as powers in ka and higher azimuthal orders become increasingly important in the wavefield expansions.

We turn now to examine the scattered field from two-obstacle configurations where each obstacle of radius a is characterized by $ka \approx 1.0$. Consider first the

zeroth order field, that is the scattered field without multiple interactions taken into account, for modes $L1$ and $R2$. The character of the scattered field in general and the forward scattered power in particular, for these two selections, is representative of that for all Love and Rayleigh modes. Further we find that it is primarily the absolute magnitude of the zeroth order scattered power that changes as we look to smaller obstacles at separations of more than a wavelength. In figure 4.3 we consider the variations in the integrated forward scattered power over a window of width 60° as a function of the separation of the two scatterers for three different angles of incidence β : 0° , 45° , and 90° . For all three angles of incidence we find that the envelope of the scattered power decays to a limiting value equivalent to twice the corresponding quantity for a single obstacle. The variation with obstacle separation depends quite noticeably upon the direction of the incident wave and exhibits oscillations of increasing frequency as the angle of incidence β increases from 0° to 90° . The actual differences between Love and Rayleigh waves for a given β are less obvious but still significant. These observations and the general behaviour of the scattered field in the far-field are best understood by employing the asymptotic form of the outgoing surface wave basis functions ψ^σ in terms of complex exponentials (*cf.* (4.4) and Appendix B). It can be shown then that the azimuthal dependence of the scattered amplitude takes the form

$$F_n^l(\theta)\cos[k_nd(\cos\beta - \cos\theta)]$$

Here θ is the angle of observation relative to the x -axis and $F_n^l(\theta)$ is the unconverted far-field radiation pattern for a particular wavetype l and mode n . The differences between the Love and Rayleigh wave behaviour in figure 4.2 are then attributable to the differences in widths of their forward radiation lobes while the dependence on angle of incidence β and separation is clearly dictated by the factor $\cos[k_nd(\cos\beta - \cos\theta)]$ and the fact that we are integrating the scattered energy over a window of finite width. The results for a point scatterer are qualitatively very similar and differ essentially only in absolute magnitude.

Contrasts in the qualitative characteristics of the scattered field for obstacles of $ka < 0.1$ and $ka \approx 1.0$ become more evident when the obstacle separation decreases

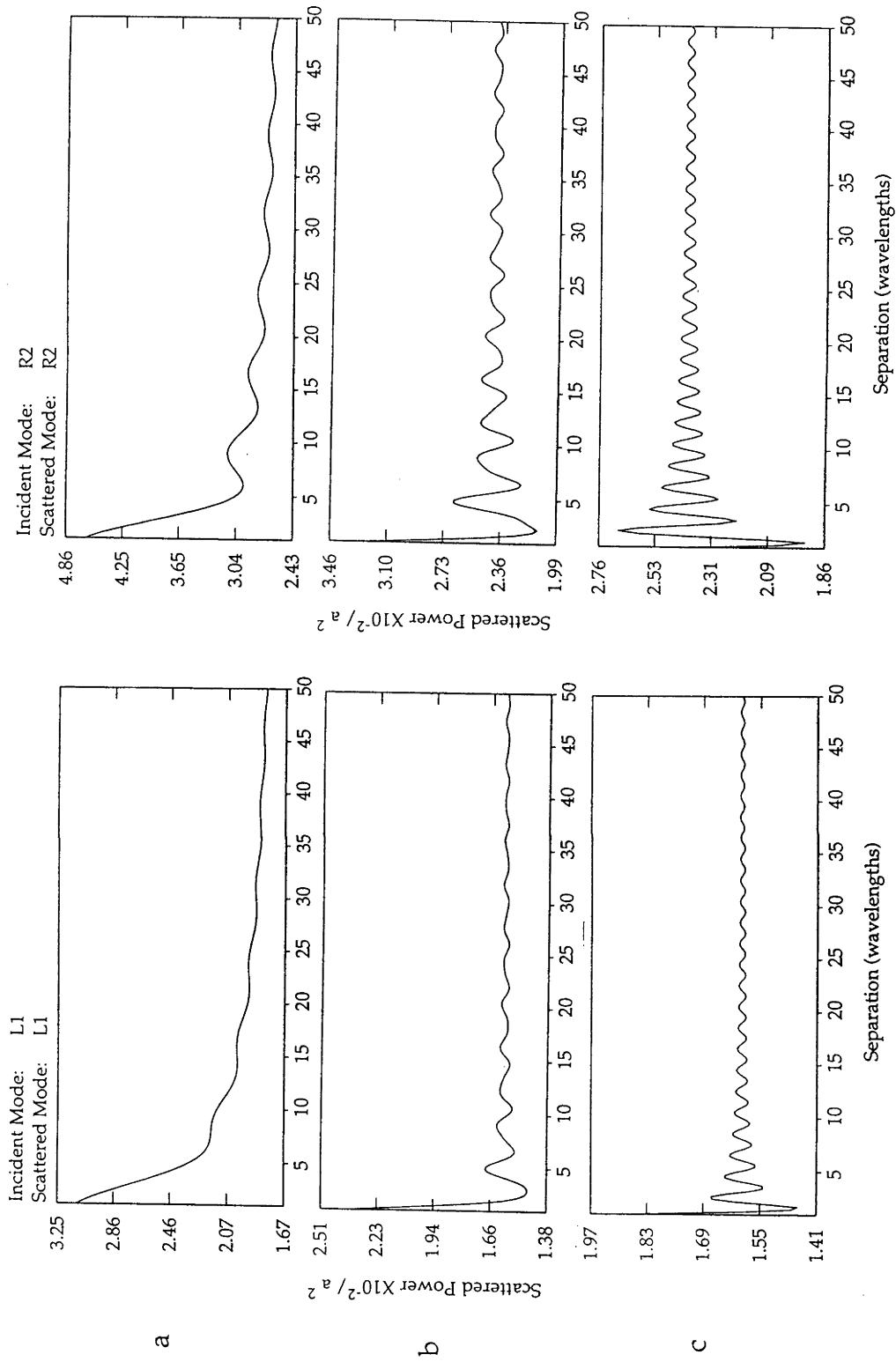


Figure 4.3 Forward scattered power in the zeroth order field from two obstacles $ka \approx 1.0$ as a function of separation for unconverted $L1$ and $R2$. Results are plotted for 3 orientations of incident plane mode a) $\beta = 0^\circ$, b) $\beta = 45^\circ$, and c) $\beta = 90^\circ$.

to under a wavelength, and the effects of multiple scattering interactions are incorporated into the analysis. Consider, for example, an incident, plane mode carrying unit energy across a plane of unit width and infinite depth : the total scattered energy from an object of $ka \approx 1.0$ is down by a factor of 10^{-3} whereas that from an object of $ka \approx 0.1$ is down by 10^{-6} (note that our scatterers share a common depth dependence, thus the volume of the obstacle varies as a^2 and the scattered power from the two sizes of obstacle cannot be ascribed to a simple relation with volume). We expect then that the separations at which multiple interactions become important will be considerably larger for the larger obstacles. This is indeed the case although deviations from the zeroth order field are not dramatic. In figure 4.4 we plot the forward scattered energy from two obstacles of $ka \approx 1.0$ again for modes $L1$ and $R2$ at obstacle separations between 0.3 and 2.0 wavelengths for the modes involved, both with and without multiple interactions taken into account. The effect of multiple scattering is seen as a very low amplitude, low frequency oscillation superimposed on the main lobe of the zeroth order field. At separations greater than one wavelength (or eight separations) the multiple scattering contributes to less than 1% of the forward scattered power and the effects are most noticeable within five separations. In this range we find that for the Love wave the difference between the full treatment and zeroth order power is on the order of 5% whereas for the fundamental Rayleigh mode the deviation is less, approximately 1%. This behaviour does not vary appreciably with the angle of incidence β .

The difference in the magnitude of the multiple scattering contributions to the scattered power for Love and Rayleigh waves noted above exists for all modes and deserves comment. It arises as a result of the difference in the distribution of Love and Rayleigh scattered energies across modal orders m for small scatterers as discussed earlier. The singularities associated with the multipole expansion of our scattered fields (in terms of the outgoing surface wave basis functions) are proportional to order m . A consequence of this is that a scattered Love wave exhibits a 'more' singular behaviour in the 'very' near-field from small scatterers than scattered Rayleigh waves. As the scatterers approach each other to the

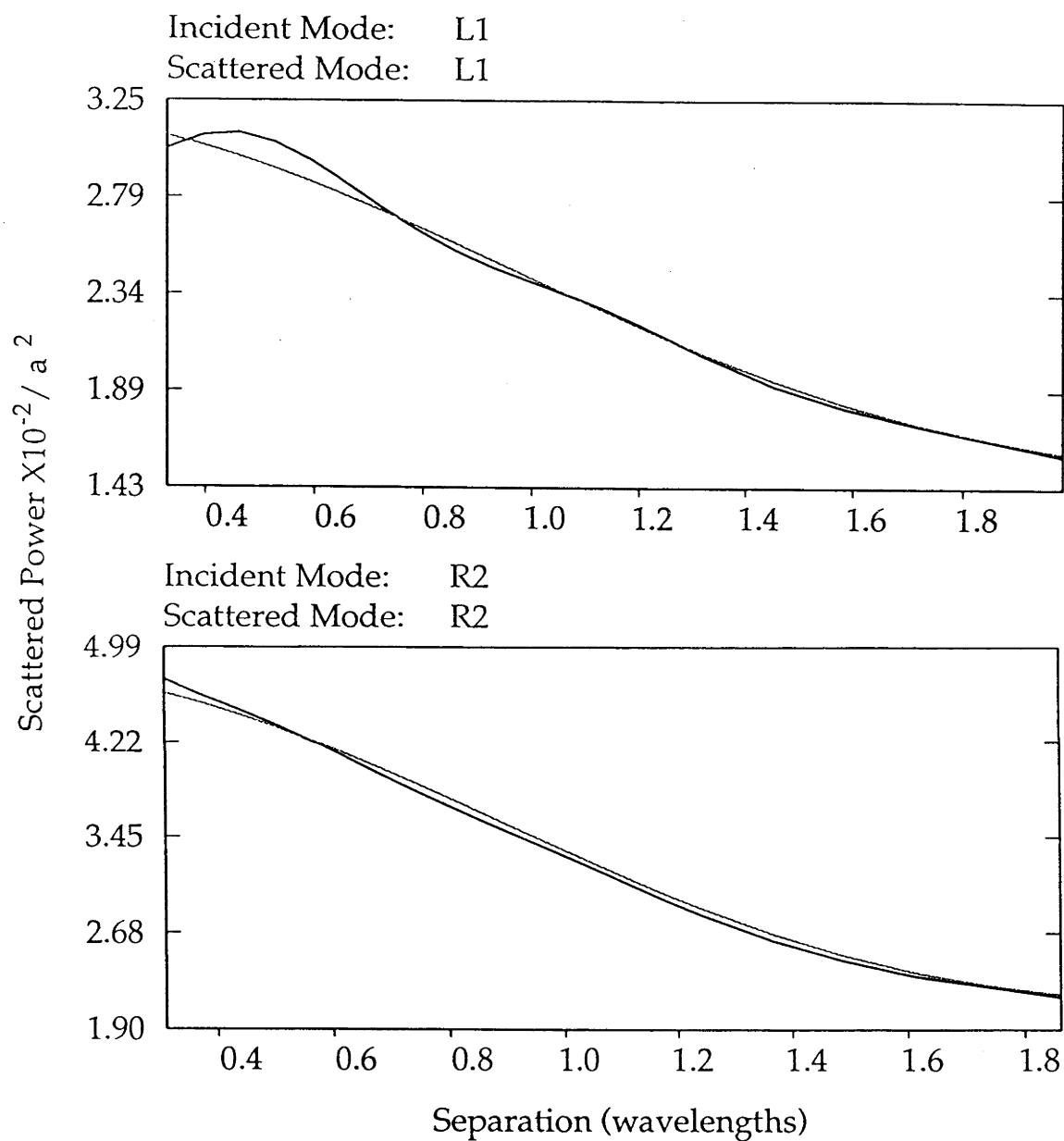


Figure 4.4 Forward scattered power from two obstacles $ka \approx 1.0$ as a function of separation for unconverted $L1$ and $R2$ $\beta = 45^\circ$. The solid and stippled lines represent the total and zeroth order scattered fields, respectively.

point of near-touching (the theoretical limit of our T-matrix formulation) this effect naturally becomes increasingly significant. With smaller scatterers the effect of the singularities is more pronounced since the scatterers can intrude farther into each other's singular regimes before touching. For very small scatterers at close separations the limited number of modes which can be managed in a practical implementation is insufficient to provide an adequately complete description of the wavefield. As a result the behaviour predicted by the truncated T-matrices for point scatterers ($ka \leq 0.1$) will not be quantitatively reliable (see Appendix C for a detailed explanation). Nonetheless this fundamental difference in the nature of the two scattered wavetypes does suggest a possible mechanism by which Love waves might grow preferentially with respect to Rayleigh waves in a medium consisting of dense distributions of point scatterers not necessarily exhibiting large velocity contrasts. We will return to this point later.

It is difficult to draw any simple conclusions regarding the efficiency of Love versus Rayleigh wave multiple scattering interactions for two small obstacles as function of the angle of the incident wave β . The reason for this is that multiple scattering is significant only in the near-field ($4.1.0 < kr < 3.0$ for obstacles of $ka \approx 1.0$). In this regime the Love and Rayleigh horizontal displacement are no longer confined to the azimuthal and radial coordinates, respectively, nor can the radial dependence of each basis function be approximated by a common complex exponential. Hence the radiation pattern (and the location of possible nodes) varies with distance from the scatterer. In addition the near-field components which decay as $1/r$ contribute a $\sin m\theta$ dependence (versus the $\cos m\theta$ dependencies of the remaining components which survive into the far-field) to the radiation pattern for unconverted modes for an incident wave with $\beta = 0^\circ$. The result is that the near-field and far-field radiation patterns may differ substantially. Figure 4.5 shows a plot of the near-field radiation patterns for modes $L1$ and $R2$ from a single obstacle of $ka \approx 1.0$ at a distance equivalent to $kr = 1.6$ and illustrates an appreciable contrast with the far-field radiation patterns in figure 4.2.

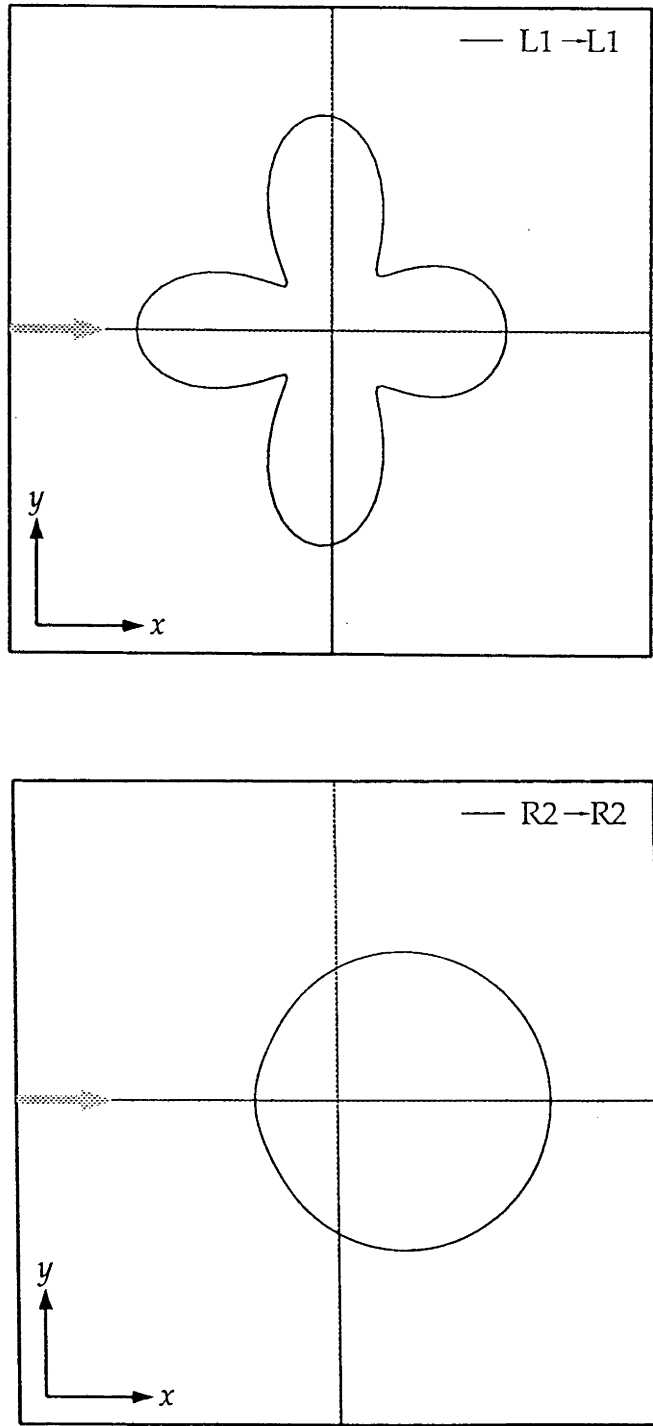


Figure 4.5 Radiation patterns for unconverted $L1$ and $R2$ (horizontal displacement) at a distance $kr = 1.6$ from a scatterer $ka \approx 1.0$. Compare with the corresponding far-field radiation patterns in figure 4.2.

4.4.2 Scatterers ($ka \approx 10.0$)

We proceed now to examine the scattered field as the size of the obstacles increase to the level $ka \approx 10.0$. The field has, from our point of view, become more complicated because of the inclusion of a larger number of angular orders m required for an adequate description of the wavefield. This is apparent in the far-field radiation patterns for a single obstacle which we display for $L1$ and $R2$ in figure 4.6. Note the proportion of energy scattered into the forward direction is considerably larger than for the class of obstacle considered in the previous section. For a plane wave at angle $\beta = 0^\circ$ incident on a scatterer with circular horizontal section this results from a large number of $\cos m\theta$ terms in our expansion which in summation interfere destructively everywhere outside a narrow lobe centered on the forward direction $\theta = 0^\circ$. We expect this will lead to contrasts in the character of the scattered fields of two-obstacle configurations between $ka \approx 10.0$ and $ka \approx 1.0$.

The horizontal extent of the $ka \approx 10.0$ obstacles limit a physically realistic analysis of the scattered field to separations greater than three or four wavelengths for the modes involved since the two obstacles will interpenetrate inside this range. First consider the power in the zeroth order scattered field as a function of separation as shown in figure 4.7. Any qualitative differences with the zeroth order scattered field from two obstacles of $ka \approx 1.0$ (figure 4.3) arises from the radiation patterns $F_n^l(\theta)$ since the dependence on separation distance d is the same in the two cases. For $\beta = 45^\circ, 90^\circ$ the forward scattered power exhibits very little variation about the mean (again equivalent to twice that in the scattered field from one obstacle) even at very small separations (< 10 wavelengths). The forward radiation lobes are sufficiently narrow that within our relatively broad band of integration (60°) there is only a very limited range over which the two lobes overlap and reinforce. The variation in the forward scattered power as a function of separation becomes more pronounced as β approaches 0° since the lobe axes start to coincide along the forward scattering direction.

The total scattered power from an obstacle of $ka \approx 10.0$ exceeds that from an obstacle of $ka \approx 1.0$ by three orders of magnitude although the total volume of

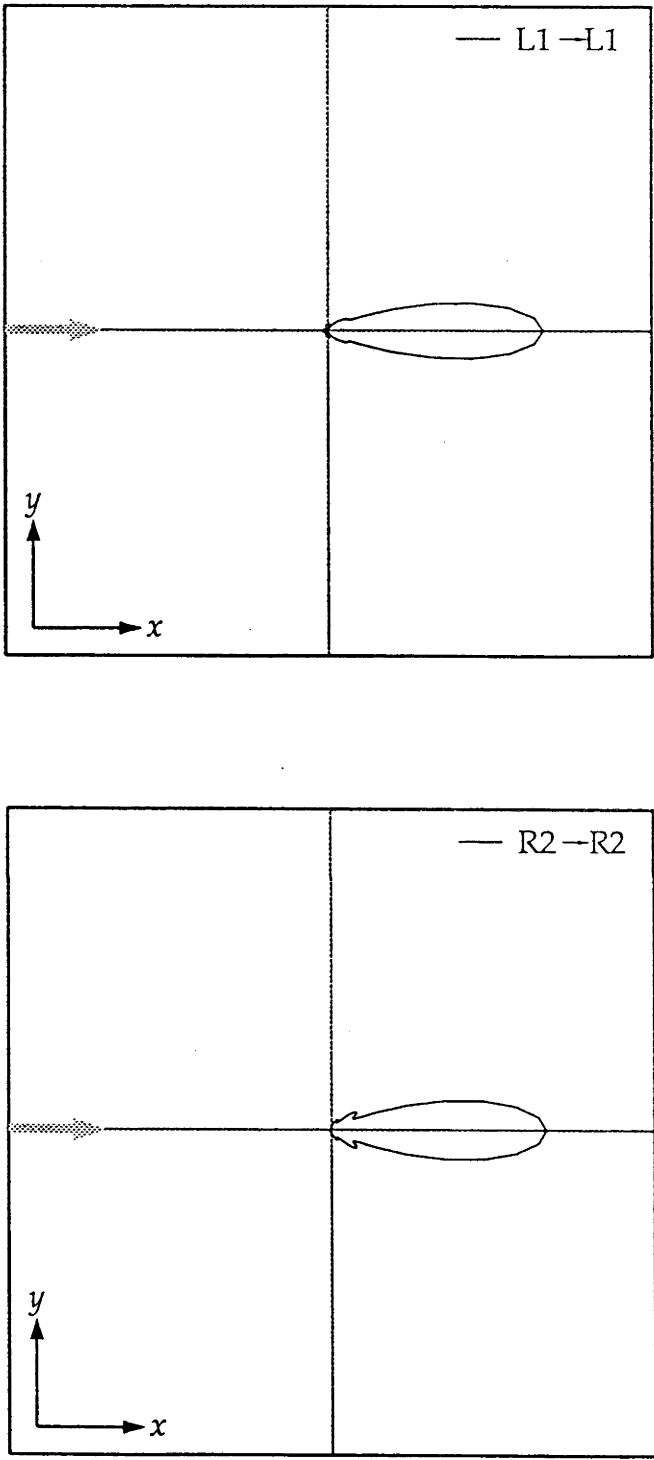


Figure 4.6 Radiation patterns for unconverted $L1$ and $R2$ in the far-field from a scatterer $ka \approx 10.0$. Note in both cases that the majority of energy is scattered into a thin beam about the forward direction.

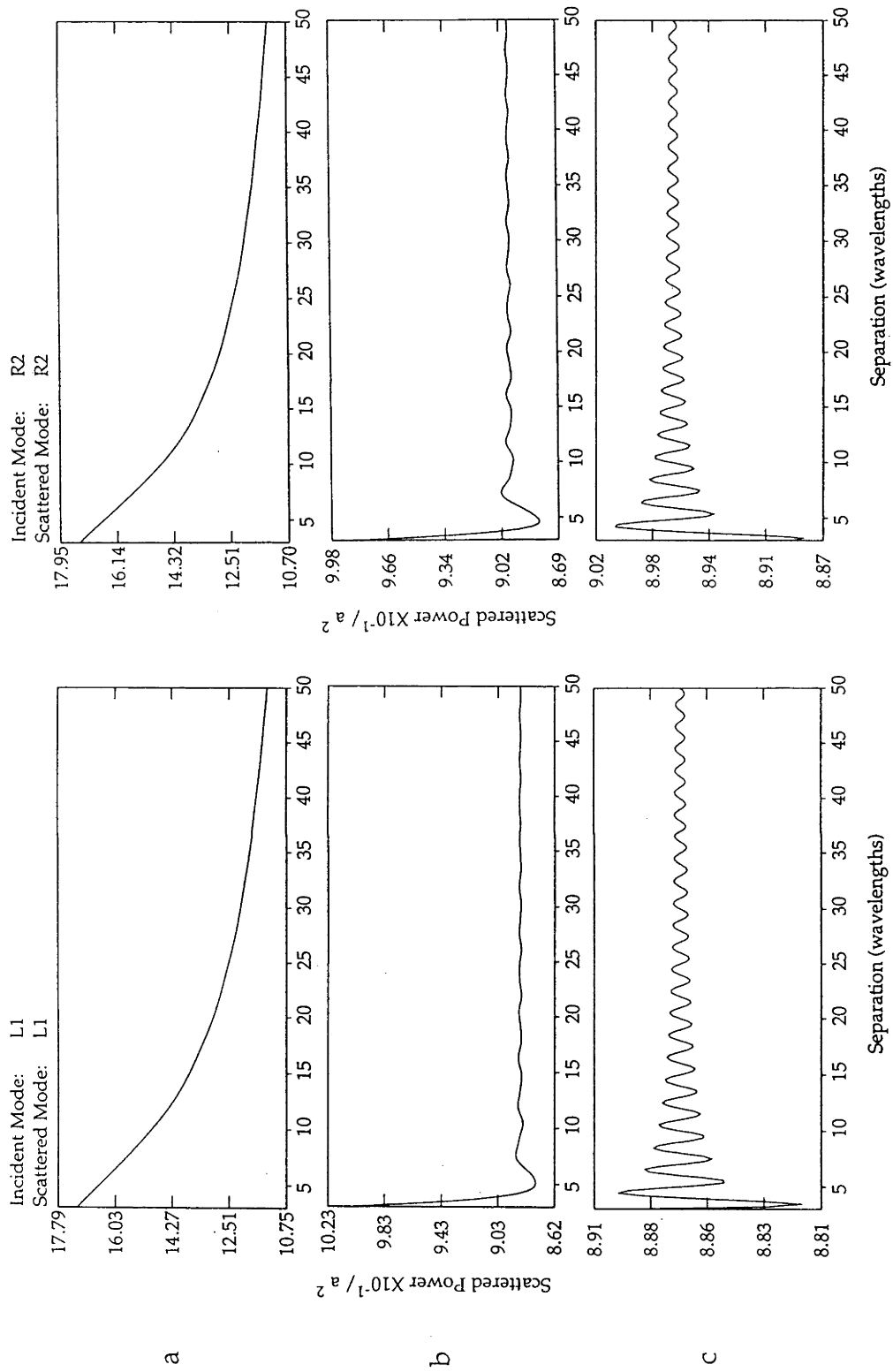


Figure 4.7 Forward scattered power in the zeroth order field from two obstacles $ka \approx 10.0$ as a function of separation for unconverted $L1$ and $R2$. Results are plotted for 3 orientations of incident plane mode a) $\beta = 0^\circ$, b) $\beta = 45^\circ$, and c) $\beta = 90^\circ$.

the scatterer is increased by a factor of 100. Consequently, the effects of multiple scattering will be more pronounced and obvious to greater separations. The zeroth order field will not therefore provide as accurate a description of the total scattered field as was the case for smaller obstacles. The forward scattered power for incident modes $L1$ and $R2$ at $\beta = 45^\circ$ and separations between three and 10 wavelengths is shown in figure 4.8. Again the multiple scattering interactions interfere constructively and destructively with the zeroth order field but it is apparent that the effects are becoming negligible beyond separations of seven or eight wavelengths. At smaller separations the total and zeroth order fields still differ by rather less than 10%. The near-field radiation patterns for a single obstacle (shown in figure 4.9 for $kr = 16.0$) are qualitatively similar to those in the far-field (figure 4.6) insofar as the majority of scattered energy remains confined to a relatively narrow lobe. This suggests that forward scattering from two obstacles at all separations is likely to be most efficient at $\beta = 0^\circ$ and this is indeed observed.

In order to graph the power in both the total and zeroth order scattered fields over the range of separations for which multiple scattering is important at $\beta = 0^\circ$ we need to calculate a large number of T-matrices. A computationally economical alternative which allows us to assess the effect of multiple scattering is to construct the first-order field from one obstacle. That is we consider forward power in that component of the scattered field which results from an initial interaction of the incident wave ($\beta = 0^\circ$) with an obstacle at O_1 , propagation to the second at O_2 , and then rescattering from the second into the embedding medium (see figure 4.1). This component of the scattered field will in fact dominate the multiple interactions since all other multiple scattering involves backscattering from the second obstacle and will presumably, on the basis of radiation patterns, be much reduced in amplitude. Figure 4.10 shows the forward scattered power in this first order scattered field for a large range of separations. We recall that the total forward scattered power from two obstacles at any orientation of the incident plane wave will tend to twice the forward scattered power from a single obstacle as the separation distance increases without bound.

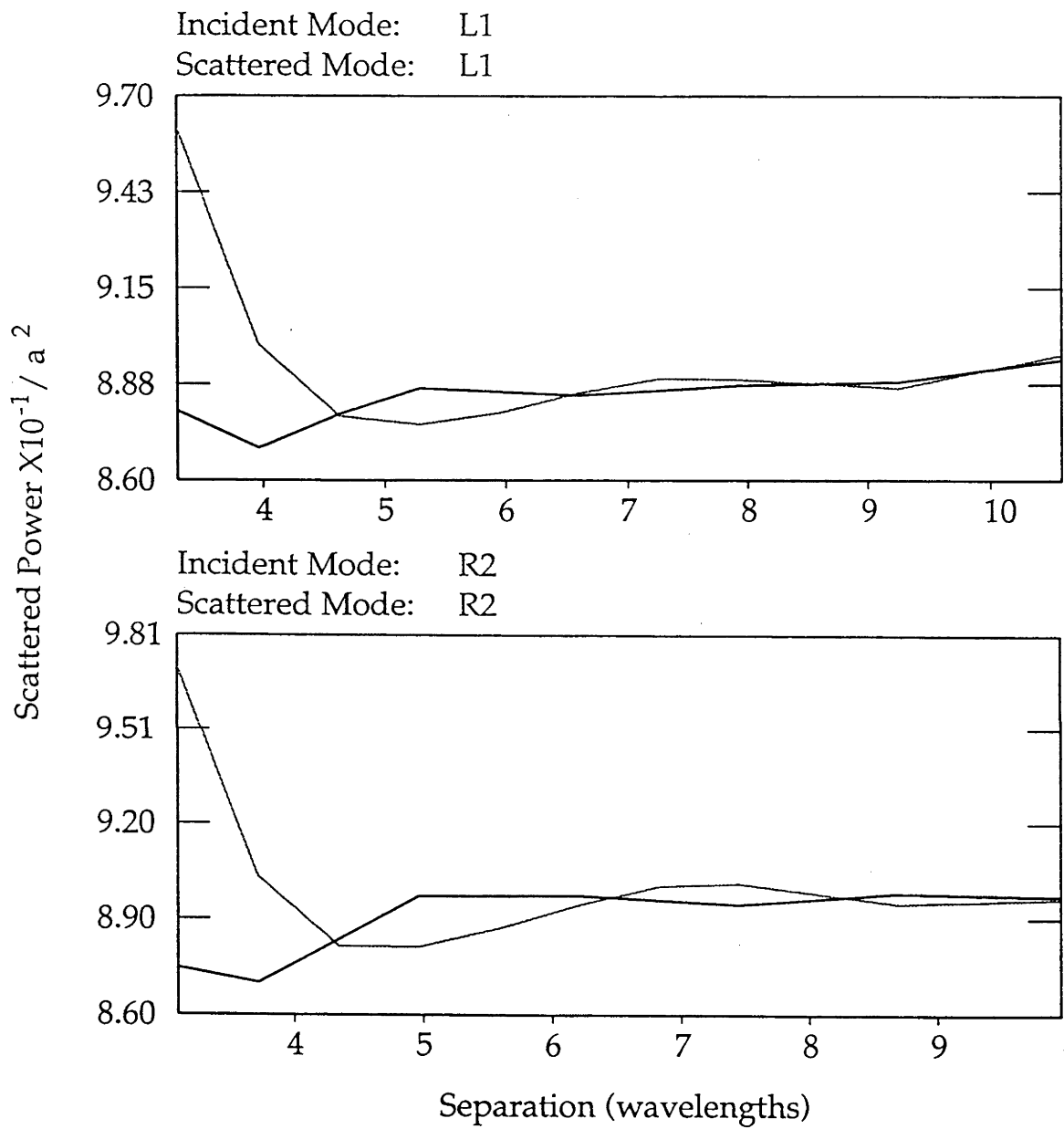


Figure 4.8 Forward scattered power from two obstacles $ka \approx 10.0$ as a function of separation for unconverted $L1$ and $R2$ $\beta = 45^\circ$. The solid and stippled lines represent the total and zeroth order scattered fields, respectively.

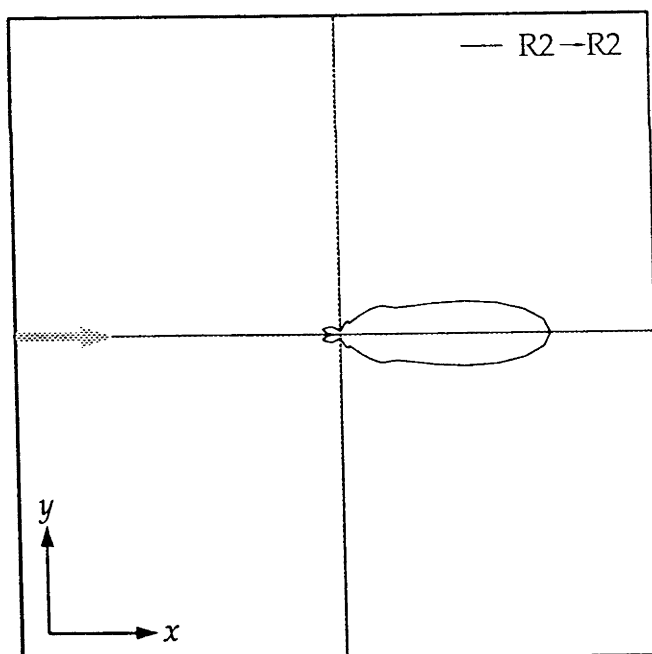
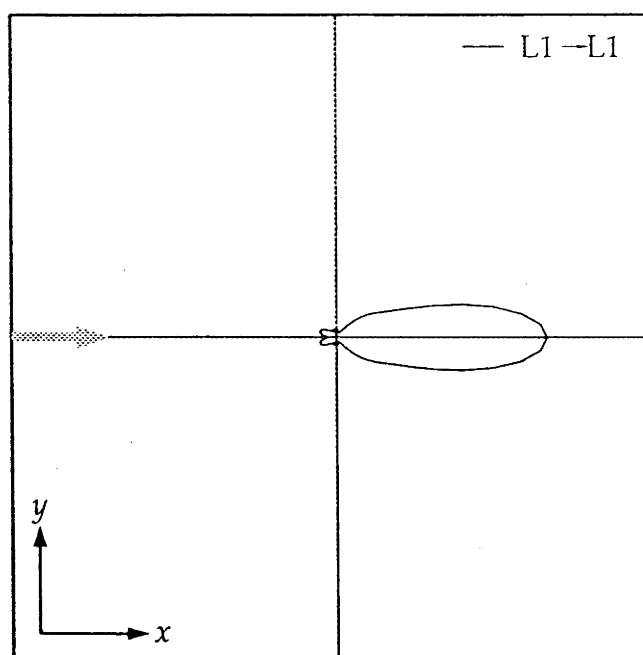


Figure 4.9 Radiation patterns for unconverted $L1$ and $R2$ (horizontal displacement) at a distance $kr = 16.0$ from a scatterer $ka \approx 10.0$. Compare with the corresponding far-field radiation patterns in figure 4.6.

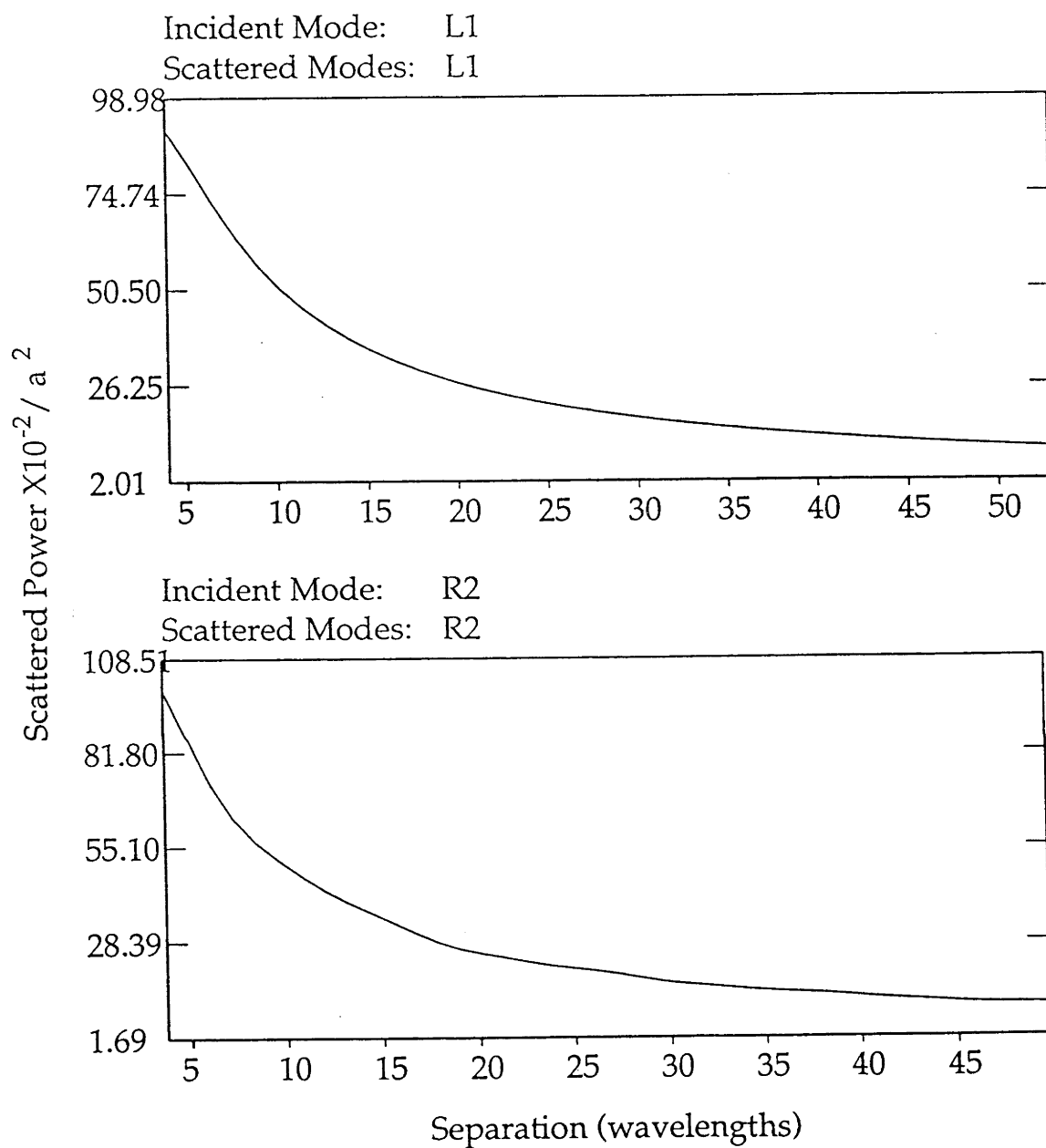


Figure 4.10 Forward scattered power in the first order scattered field off the forward obstacle of a two-obstacle $ka \approx 10$ configuration for unconverted $L1$ and $R2$. Note that the magnitude of this component is comparable to that of the zeroth order approximation of the scattered field (*cf.* figures 4.7 and 4.8).

The forward scattered power from a single obstacle of $ka \approx 10$ for modes $L1$ and $R2$ is on the order of eight to ten times the incident wave intensity, that is the power across a plane of unit width and infinite depth perpendicular to the direction of propagation. This indicates that the power in the first order multiple scattered field is comparable to or exceeds that in the zeroth order field over a large range of separations. If we reconsider the radiation patterns in figures 4.6 and 4.9 it is apparent that a large circular scatterer behaves much as a lens which focusses all scattered energy into a narrow beam centered about the direction of the incident wave vector. This is in fact true of large scatterers of relatively arbitrary shape and is a manifestation of the Mie scattering effect (Born & Wolf, 1959). Thus for two obstacles aligned parallel to the direction of the incident wave the first order multiple field will appear locally to dominate the incident field and will result in a total scattered field which deviates significantly from the zeroth order approximation.

4.5 DISCUSSION

In this chapter we have demonstrated that the surface wave T-matrix formulation presented in chapter 3 can be extended to accomodate the effects of scattering from two or more obstacles. The resulting multiple obstacle T-matrix can be expressed in a form whereby the scattering contributions from each obstacle including multiple scattering are clearly identifiable. In principle the multiple scattering formulation places no restriction on the dimension or magnitude of the heterogeneity contrasts however a deterministic treatment of scattering from many scatterers spread over a large area rapidly becomes computationally intractable. The practical problem arises in attempting to represent the complicated behaviour of the scattered field from a distribution of scatterers in terms of a finite expansion in multipoles at a single origin.

In order to exploit our deterministic formulation as it stands we have elected to investigate the significance of multiple scattering interactions in the earth by considering simply the scattering from two identical obstacles embedded within a

stratified medium. The physical parameters have been tailored to describe wave propagation at 1.0 Hz in an idealized version of the crust and upper mantle so as to model the scattering of regional phases such as *Lg*. The scattering obstacles are defined such that they represent a maximum typical heterogeneity contrast which might occur over broad areas of the earth's crust.

For small scatterers in the range of interaction $ka \leq 1.0$ it appears that multiple scattering becomes important only at separations less than one wavelength. Contributions from multiple scattering to the scattered field from two obstacles at the upper limit of this range may be on the order of a few percent. The range of separations at which multiple scattering interaction between two point scatterers ($ka < 0.1$) must be considered is less still, generally on the order of one tenth of a wavelength for $ka \approx 0.1$. Quantitative estimates of the significance of multiple scattering at this extreme are hindered by inaccuracies arising from assumptions regarding the wavefield representation in our numerical implementation (see Appendix C). However the structure of the surface wave basis functions and the nature of the crust and upper mantle suggest that there may exist fundamental differences in the behaviour of scattered Love and Rayleigh waves in regions where crustal heterogeneity can be modelled by dense distributions of point scatterers. These differences are associated with the representation of the scattered Love and Rayleigh displacement wavefields in terms of their multipole expansions. Love wave energy is preferentially scattered into order $m = 2$ whereas the majority of Rayleigh energy resides in orders $m = 0, 1$. The notion that Love waves from point scatterers are represented by a higher order, dominant multipole suggests a mechanism whereby the scattered wavefield preferentially evolves a Love wave character. In some regions, modelling the crust as a stratified medium containing closely spaced point scatterers ($ka < 0.1$) may not be unrealistic, and does not necessarily require that the heterogeneity occupy a large percentage of the total crustal volume (as for closely spaced scatterers of $ka \geq 1.0$). It is interesting to note in this regard that *Lg* observed at long ranges (900 km) from explosions is known for certain paths to exhibit similar energies on all three components (transverse,

radial and vertical) even though the energy imparted at the source is exclusively of Rayleigh type (Blandford, 1980). Having said this it should also be remarked that any mechanism, regardless of efficiency, which results in a $R \rightarrow L$ energy transfer might be expected to have this effect given a sufficient distance over which to act. This energy transfer would continue until such point as it was balanced by a mechanism for the reverse ($L \rightarrow R$) process.

The importance of multiple scattering will increase with the dimension of the scatterers from $ka = 1.0$. Indeed pair interactions will likely have a very important influence on the scattered field in regions where the dominant correlation distance of the heterogeneity is such that $ka \geq 10.0$ owing to the highly directional character of the radiation pattern and the proportionately greater energies extracted from the incident wave. The predominance of forward scattering from such obstacles can contribute to multiple scattering which is locally comparable in strength to the zeroth order scattered field approximation. In addition, it implies that the observation of multiple scattering effects will depend primarily on scatterers lying within a narrow corridor about an axis joining source and receiver.

CHAPTER 5

REFLECTION/TRANSMISSION OF SURFACE WAVES

IN HETEROGENEOUS MEDIA

5.1 INTRODUCTION

In this chapter we examine how the surface wave basis functions introduced in chapter 3 can be exploited to describe propagation in other forms of heterogeneity. Specifically, we consider propagation in media which, in addition to being stratified in a vertical sense exhibit some degree of cylindrical stratification. We will see that this description, although highly idealized and not immediately applicable to realistic earth models, will lead to a solution for the surface wave field in mild 3-D heterogeneity.

In the first section we review the orthogonality properties of the surface wave basis functions and the manner in which these can be exploited to derive expressions which relate wave series expansions within and outside a region of heterogeneity. The notation used in the previous chapters is re-expressed in a reflection/transmission format to facilitate the ensuing analysis and to illustrate the parallels of the present development with the theory of elastic wave propagation in stratified media (Kennett, 1983). We proceed to analyse the scattering properties of a discrete cylindrical shell of somewhat arbitrary geometry enclosing the z -axis of a particular cylindrical coordinate system. A solution for the scattered wavefield can be posed in a form whereby the physical processes, including internal reverberations, contributing to the total scattering response are clearly identified. The extension of this approach to media consisting of an arbitrary number of enveloping shells using a propagator-matrix development follows trivially and may be employed, for example, to describe scattering from multilayered obstacles.

In the latter half of the chapter we employ perturbation theory to establish the reflection/transmission response of circular cylindrical shell exhibiting mild variations in material properties. Through the use of invariant embedding techniques we derive equations for the reflection and transmission properties of a wave propagating in a continuously varying medium as a function of the radial distance from the source. The correspondence of the present work with the coupled mode description of Kennett (1984a) is demonstrated and our method is shown to be simply a generalization to three dimensions.

5.2 A REFORMULATION

Let us review the T-matrix formulation of chapters 3 and 4 so as to re-express certain quantities in a form which is more amenable to the analysis in subsequent sections of this chapter. Consider a finite, laterally heterogeneous region located within an otherwise laterally homogeneous stratified reference medium as shown in figure 5.1. In the embedding reference medium we find that a representation of the displacement in terms of surface wave basis functions as defined in equations (3.2) and (3.7) of chapter 3 is appropriate and we note that under certain restrictions these functions form a complete orthogonal basis set. Hence they may be used to represent any wavefield which might be expected to occur within the medium. These basis functions can be grouped into two sets $\psi_0^\sigma, \hat{\psi}_0^\sigma$; the first employing an outgoing wave function for its radial dependence, the second a standing wavefunction. The set ψ_0^σ is employed in the expansion of fields which are presumed to have an outgoing character and may exhibit singular behaviour in the vicinity of a particular particular coordinate origin, while the set $\hat{\psi}_0^\sigma$ is suitable for representing fields which are regular throughout a given domain. Now consider the problem of scattering from this discrete, geometrically simple region. We break the wavefield outside the scatterer volume into incident and scattered components. The incident field $\mathbf{u}^i$ which would be finite throughout the volume occupied by the scatterer in the absence of any heterogeneity is expanded as

$$\mathbf{u}^i = \sum_{\sigma} a^{\sigma} \hat{\psi}_0^{\sigma} \quad (5.1)$$

with respect to an origin located within the scatterer. Since the scattered field $\mathbf{u}^s$ is defined outside the scatterer only and exhibits an outgoing character at sufficient distance from the scattering surface, it is expanded as

$$\mathbf{u}^s = \sum_{\sigma} c^{\sigma} \psi_0^{\sigma}. \quad (5.2)$$

In addition we define the wavefunctions $\hat{\psi}_1^{\sigma}, \psi_1^{\sigma}$ to describe the wavefield within the scatterer, and assume that they too form a complete orthogonal, basis set. These functions differ from $\hat{\psi}_0^{\sigma}, \psi_0^{\sigma}$ on account of differences in the material properties

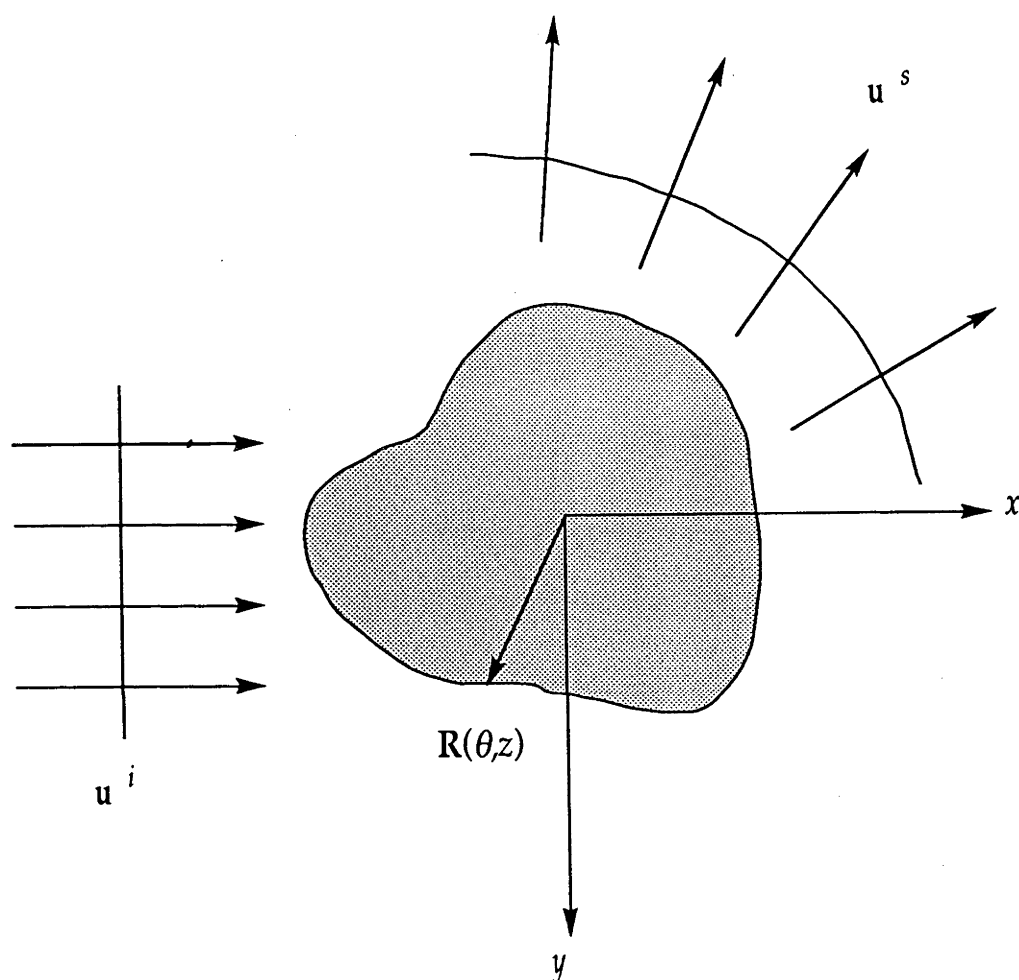


Figure 5.1. The finite scatterer problem: an incident plane surface wave u^i of known properties interacts with a finite heterogeneous region with surface $\mathbf{R}(\theta, z)$ to produce a scattered field u^s . Plan view is shown with z -axis into the page.

defining the scatterer and the outside reference medium. Like the incident field $\mathbf{u}^i$, the internal field $\mathbf{u}^r$ is required to be well behaved throughout the volume occupied by the heterogeneity, and is thus expanded as

$$\mathbf{u}^r = \sum_{\sigma} d^{\sigma} \hat{\psi}_1^{\sigma}. \quad (5.3)$$

The basis set $\psi_0^{\sigma}, \hat{\psi}_0^{\sigma}$ obeys a set of orthogonality relations which are essential to the development of the derivation of the T-matrix:

$$\int_S dS [\mathbf{t}(\psi_0^{\sigma}) \cdot \psi_0^{\nu} - \mathbf{t}(\psi_0^{\nu}) \cdot \psi_0^{\sigma}] = 0, \quad (5.4)$$

$$\int_S dS [\mathbf{t}(\hat{\psi}_0^{\sigma}) \cdot \hat{\psi}_0^{\nu} - \mathbf{t}(\hat{\psi}_0^{\nu}) \cdot \hat{\psi}_0^{\sigma}] = 0, \quad (5.5)$$

$$\int_S dS [\mathbf{t}(\psi_0^{\sigma}) \cdot \hat{\psi}_0^{\nu} - \mathbf{t}(\hat{\psi}_0^{\nu}) \cdot \psi_0^{\sigma}] = \frac{2i}{\omega} \delta_{\sigma\nu}. \quad (5.6)$$

Here S is any surface enclosing the z -axis, $\mathbf{t}(\hat{\psi}_0^{\nu})$ is the traction over this surface associated with, for example, the displacement wavefield $\hat{\psi}_0^{\nu}$, and ω is angular frequency. For the purposes of later developments we will assume that the set $\hat{\psi}_1^{\sigma}, \psi_1^{\sigma}$ independently satisfy the same form of orthogonality given in equations (5.4)-(5.6), so that, for example, a relation analogous to (5.6) is satisfied

$$\int_S dS [\mathbf{t}(\psi_1^{\sigma}) \cdot \hat{\psi}_1^{\nu} - \mathbf{t}(\hat{\psi}_1^{\nu}) \cdot \psi_1^{\sigma}] = \frac{2i}{\omega} \delta_{\sigma\nu}. \quad (5.7)$$

In so doing we implicitly attribute a source and outgoing characteristics for ψ_1^{σ} and require that $\hat{\psi}_1^{\sigma}$ and its derivatives are continuous in the vicinity of the origin. In the event that the heterogeneous region is itself laterally homogeneous ψ_1^{σ} and $\hat{\psi}_1^{\sigma}$ are of course just the normal surface wave basis functions for the horizontally stratified medium with an identical distribution of material properties with depth. The derivation of the T-matrix proceeds by expanding the fields $\mathbf{u}^i$, $\mathbf{u}^s$, $\mathbf{u}^r$ as in (5.1)-(5.3), applying the orthogonality relations (5.4)-(5.6) (for $\psi_0^{\sigma}, \hat{\psi}_0^{\sigma}$), and matching boundary conditions on the scatterer surface to yield matrix equations which independently relate the incident and scattered field coefficients, c^{σ}, a^{σ} , to those of the internal field d^{σ} . The unknown scattered field coefficients are then related to those of the known incident field via the T-matrix by eliminating d^{σ} between the

two equations. Let us define the infinite matrix quantities $\mathbf{Q}_{11}, \mathbf{Q}_{12}, \mathbf{Q}_{21}, \mathbf{Q}_{22}$ with corresponding elements

$$Q_{11}^{\nu\sigma} = \frac{\omega}{2i} \int_S dS [\mathbf{t}(\hat{\psi}_1^\sigma) \cdot \psi_0^\nu - \mathbf{t}(\psi_0^\nu) \cdot \hat{\psi}_1^\sigma], \quad (5.8)$$

$$Q_{12}^{\nu\sigma} = \frac{\omega}{2i} \int_S dS [\mathbf{t}(\psi_1^\sigma) \cdot \psi_0^\nu - \mathbf{t}(\psi_0^\nu) \cdot \psi_1^\sigma], \quad (5.9)$$

$$Q_{21}^{\nu\sigma} = \frac{\omega}{2i} \int_S dS [\mathbf{t}(\hat{\psi}_1^\sigma) \cdot \hat{\psi}_0^\nu - \mathbf{t}(\hat{\psi}_0^\nu) \cdot \hat{\psi}_1^\sigma], \quad (5.10)$$

$$Q_{22}^{\nu\sigma} = \frac{\omega}{2i} \int_S dS [\mathbf{t}(\psi_1^\sigma) \cdot \hat{\psi}_0^\nu - \mathbf{t}(\hat{\psi}_0^\nu) \cdot \psi_1^\sigma], \quad (5.11)$$

where the integrals are evaluated over the surface S of the scatterer. The T-matrix $\mathbf{R}_o$ can then be written as

$$\mathbf{R}_o = -\mathbf{Q}_{21}[\mathbf{Q}_{11}]^{-1}, \quad (5.12)$$

such that the coefficient vectors $\mathbf{c}$ and $\mathbf{a}$, comprising the wave series coefficients c^σ and a^σ , are related by

$$\mathbf{c} = \mathbf{R}_o \mathbf{a}. \quad (5.13)$$

We have broken with the conventions of the two previous chapters and chosen to represent the T-matrix by the symbol $\mathbf{R}_o$. The motivation for this stems from the many parallels in the ensuing analysis with descriptions of the elastic wavefield in plane stratified media (*cf.* Kennett, 1983). It is convenient to view the T-matrix for a zone of heterogeneity simply as a specific implementation of a more general 'reflection' operator which relates some component of the scattered field to the incident field, that is the portion of the total field which would exist in the absence of any lateral heterogeneity. In the same spirit we can define a transmission operator, which in matrix form we denote as $\mathbf{T}_i$, that operates on the incident wavefield to produce the field which propagates within the heterogeneity

$$\mathbf{T}_i = -[\mathbf{Q}_{11}]^{-1}. \quad (5.14)$$

such that

$$\mathbf{d} = \mathbf{T}_i \mathbf{a}. \quad (5.15)$$

By considering an as yet unspecified source located at some depth along the z -axis within the heterogeneous zone, we can define inward reflection and outward transmission matrices as

$$\mathbf{R}_i = -\mathbf{Q}_{12}^T[\mathbf{Q}_{11}^T]^{-1}, \quad (5.16)$$

$$\mathbf{T}_o = -[\mathbf{Q}_{11}^T]^{-1}, \quad (5.17)$$

where T denotes transpose. Thus $\mathbf{R}_i$ relates the coefficients of the outgoing wave series generated by the source to those of a reflected wavefield which results from the presence of the embedding reference medium. Likewise $\mathbf{T}_o$ relates the outgoing wave coefficients within the heterogeneity to those of an outgoing wave series in the reference medium.

5.3 REFLECTION AND TRANSMISSION I

We now consider the scattering properties of a zone of heterogeneity comprising a cylindrical (not necessarily circular) shell which encloses the z -axis as shown in figure 5.2. We allow the heterogeneity to be an arbitrary function of position (r, θ, z) and assume that a complete set of surface wave basis functions, $\hat{\psi}_1^\sigma, \psi_1^\sigma$, satisfying orthogonality relations of the form in (5.4) – (5.6) exists for the wavefield within the shell. The scattering response of the heterogeneity will be characterized in terms of reflection and transmission matrices which, as we show, are essentially generalizations of the T -matrix.

First consider an incident wave $\mathbf{u}^i$ impinging on the heterogeneous shell from the outside. This will give rise to an outgoing scattered wave $\mathbf{u}^s$ and a wavefield $\mathbf{u}^t$ which is transmitted into the embedding medium inside the inner boundary of the shell. According to the anticipated behaviour of these wavefield components we will employ the following expansions

$$\mathbf{u}^i = \sum_{\sigma} a_{\sigma}^{\sigma} \hat{\psi}_0^{\sigma}, \quad (5.18)$$

$$\mathbf{u}^s = \sum_{\sigma} c_{\sigma}^{\sigma} \psi_0^{\sigma}, \quad (5.19)$$

and

$$\mathbf{u}^t = \sum_{\sigma} d_{\sigma}^{\sigma} \hat{\psi}_0^{\sigma}. \quad (5.20)$$

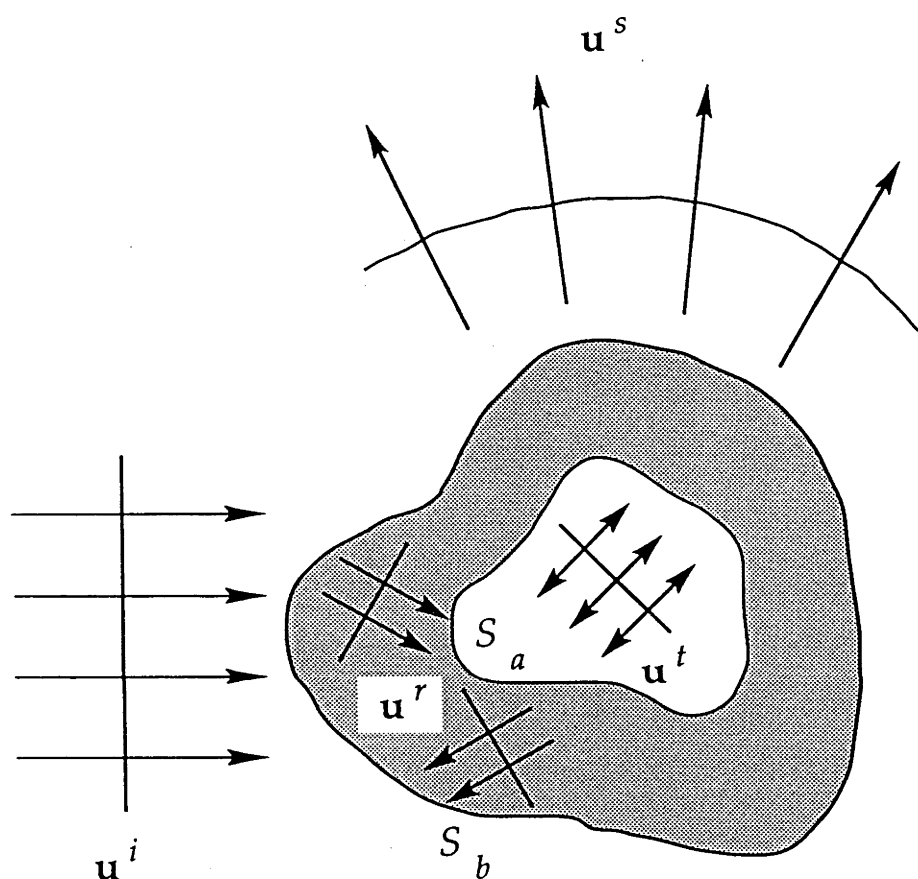


Figure 5.2. Plan view of scattering from a cylindrical shell with arbitrary horizontal section. The inner and outer shell boundaries are defined by the surfaces S_a and S_b respectively. The component fields in the reference medium (u^i , u^s , u^t) and in the heterogeneity (u^r) are expanded in surface wave basis functions according to the boundary conditions.

We must also consider the wavefield $\mathbf{u}^r$ excited within the shell itself and will break this internal field into the following components

$$\mathbf{u}^r = \sum_{\sigma} \left[a_1^{\sigma} \hat{\psi}_1^{\sigma} + c_1^{\sigma} \psi_1^{\sigma} \right]. \quad (5.21)$$

Recall that we must use an expansion in both outgoing and regular basis functions to ensure that the wavefield within the heterogeneity is fully represented. It should be noted that a more symmetrical but equivalent representation of these component fields can be achieved by substituting ingoing wavefunctions for the regular wavefunctions. The ingoing wavefunction for a laterally homogeneous medium incorporates a Hankel function of the second kind to describe the radial dependence rather than the regular Bessel functions. Using this manner of representation the field in the homogeneous region within the shell must be written as

$$\mathbf{u}^t = \sum_{\sigma} \frac{d_{\sigma}^{\sigma}}{2} [\psi_0^{\sigma} + \psi_0^{\sigma*}] \quad (5.22)$$

in order to satisfy the boundary condition of finiteness at the origin. Note also that the ingoing wave function $\psi_0^{\sigma*}$ can be constructed by taking the complex conjugate of the outgoing function ψ_0^{σ} without operating on the implicit time dependence $e^{-i\omega t}$. We will work with representations involving the regular wavefunctions since the development is slightly less complicated and the structure of the equations resembles that for the analogous problem of elastic wave propagation in a radially stratified earth (Illingworth, 1983) more closely. We proceed by employing the orthogonality relations in (5.4)-(5.6) and applying the conditions on the external circular cylindrical surface $S = S_b$. The relationship between the wave series coefficients in the two media at the boundary S_b is then conveniently expressed in a matrix form as

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{c}_0 \end{pmatrix} = \begin{pmatrix} -\mathbf{Q}_{11} & -\mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix}_{S=S_b} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{c}_1 \end{pmatrix}. \quad (5.23)$$

Here the $\mathbf{Q}_{ij}$ partitions are constructed from the tensor elements defined in (5.10)-(5.13) over the cylindrical surface S_b . In a similar manner we may write

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{c}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{22}^T & \mathbf{Q}_{12}^T \\ -\mathbf{Q}_{21}^T & -\mathbf{Q}_{11}^T \end{pmatrix}_{S=S_a} \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{0} \end{pmatrix}, \quad (5.24)$$

relating the field within the shell to that in the embedding medium inside the shell, and where the surface integral quantities are now evaluated over the cylindrical surface S_a . By eliminating the internal field coefficient vector we see that the two matrix quantities behave as wave coefficient propagators which allow us to relate the wavefields in the embedding medium on either side of the shell as

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{c}_0 \end{pmatrix} = \begin{pmatrix} -\mathbf{Q}_{11}|_b \mathbf{Q}_{22}^T|_a + \mathbf{Q}_{12}|_b \mathbf{Q}_{21}^T|_a & -\mathbf{Q}_{11}|_b \mathbf{Q}_{12}^T|_a + \mathbf{Q}_{12}|_b \mathbf{Q}_{11}^T|_a \\ \mathbf{Q}_{21}|_b \mathbf{Q}_{22}^T|_a - \mathbf{Q}_{22}|_b \mathbf{Q}_{21}^T|_a & \mathbf{Q}_{21}|_b \mathbf{Q}_{12}^T|_a - \mathbf{Q}_{22}|_b \mathbf{Q}_{11}^T|_a \end{pmatrix} \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{0} \end{pmatrix} \quad (5.25)$$

Let us now focus on the reflection-transmission properties characterizing the scattering response for the shell. We will define the reflection and transmission matrices $\mathbf{R}_o$, $\mathbf{T}_i$ as those which relate the scattered and transmitted field coefficients to those of an incoming incident field for our heterogeneous shell, that is

$$\mathbf{c}_0 = \mathbf{R}_o \mathbf{a}_0, \quad (5.26)$$

$$\mathbf{d}_0 = \mathbf{T}_i \mathbf{a}_0. \quad (5.27)$$

It follows after some algebra then that

$$\mathbf{R}_o = (\mathbf{Q}_{21}|_b \mathbf{Q}_{22}^T|_a - \mathbf{Q}_{22}|_b \mathbf{Q}_{21}^T|_a)(-\mathbf{Q}_{11}|_b \mathbf{Q}_{22}^T|_a + \mathbf{Q}_{12}|_b \mathbf{Q}_{21}^T|_a)^{-1}, \quad (5.28)$$

$$\mathbf{T}_i = (-\mathbf{Q}_{11}|_b \mathbf{Q}_{22}^T|_a + \mathbf{Q}_{12}|_b \mathbf{Q}_{21}^T|_a)^{-1}. \quad (5.29)$$

At the outset of this section we might just as well have considered an outgoing incident field originating from a given source at the origin and expanded as $\sum_{\sigma} e_{\sigma}^{\sigma} \psi_0^{\sigma}$. In a geophysical context this would correspond to an earthquake whose horizontal location defines the origin of a cylindrical coordinate system (the coefficients e_{σ}^{σ} of the cylindrical harmonic expansion for various different earthquake sources are readily determined, see e.g. Aki & Richards, 1980; section 7.5). In the somewhat contrived event that this earthquake lay at the center of a tubular region of heterogeneity, we would expect a regular scattered field within the volume encompassed by the inner boundary of the annulus and an outgoing field outside the outer boundary. These fields are expanded in the appropriate basis functions ($\hat{\psi}_0^{\sigma}, \psi_0^{\sigma}$, respectively) and following the approach used before we find we can relate the wavefield coefficient on either side of the shell for this situation as

$$\begin{pmatrix} \mathbf{d}_0 \\ \mathbf{e}_0 \end{pmatrix} = \begin{pmatrix} -\mathbf{Q}_{11}|_a \mathbf{Q}_{22}^T|_b + \mathbf{Q}_{12}|_a \mathbf{Q}_{21}^T|_b & -\mathbf{Q}_{11}|_a \mathbf{Q}_{12}^T|_b + \mathbf{Q}_{12}|_a \mathbf{Q}_{11}^T|_b \\ \mathbf{Q}_{21}|_a \mathbf{Q}_{22}^T|_b - \mathbf{Q}_{22}|_a \mathbf{Q}_{21}^T|_b & \mathbf{Q}_{21}|_a \mathbf{Q}_{12}^T|_b - \mathbf{Q}_{22}|_a \mathbf{Q}_{11}^T|_b \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{c}_0 \end{pmatrix} \quad (5.30)$$

Therefore we can define the inward reflection and outward transmission matrices $\mathbf{R}_i$, $\mathbf{T}_o$ for an incident outgoing wave as

$$\mathbf{R}_i = (-\mathbf{Q}_{11}|_a \mathbf{Q}_{12}^T|_b + \mathbf{Q}_{12}|_a \mathbf{Q}_{11}^T|_b)(\mathbf{Q}_{21}|_a \mathbf{Q}_{12}^T|_b - \mathbf{Q}_{22}|_a \mathbf{Q}_{11}^T|_b)^{-1}, \quad (5.31)$$

$$\mathbf{T}_o = (\mathbf{Q}_{21}|_a \mathbf{Q}_{12}^T|_b - \mathbf{Q}_{22}|_a \mathbf{Q}_{11}^T|_b)^{-1}. \quad (5.32)$$

Note that the expression for $\mathbf{R}_o$ is in fact just the T-matrix for a heterogeneous, cylindrical shell. Also observe from (5.29) and (5.32) that

$$\mathbf{T}_o = \mathbf{T}_i^T, \quad (5.33)$$

a property which is also true of matrix implementations of transmission operators in the case of elastic wave propagation and arises from symmetries in the wave equation (*cf.* Kennett, Kerry & Woodhouse, (1978); Thomson, Clarke & Garmany, (1986)). Also observe that by following the example of Kennett (1983) we may recast the propagator matrix in (5.25) in terms of these reflection-transmission quantities:

$$\begin{pmatrix} -\mathbf{Q}_{11}|_b \mathbf{Q}_{22}^T|_a + \mathbf{Q}_{12}|_b \mathbf{Q}_{21}^T|_a & -\mathbf{Q}_{11}|_b \mathbf{Q}_{12}^T|_a + \mathbf{Q}_{12}|_b \mathbf{Q}_{11}^T|_a \\ \mathbf{Q}_{21}|_b \mathbf{Q}_{22}^T|_a - \mathbf{Q}_{22}|_b \mathbf{Q}_{21}^T|_a & \mathbf{Q}_{21}|_b \mathbf{Q}_{12}^T|_a - \mathbf{Q}_{22}|_b \mathbf{Q}_{11}^T|_a \end{pmatrix} = \begin{pmatrix} \mathbf{T}_i^{-1} & -\mathbf{T}_i^{-1} \mathbf{R}_i \\ \mathbf{R}_o \mathbf{T}_i^{-1} & \mathbf{T}_o - \mathbf{R}_o \mathbf{T}_i^{-1} \mathbf{R}_i \end{pmatrix} \quad (5.34)$$

The expressions in (5.28), (5.29), (5.31), (5.32) can be rearranged to gain insight into the physical processes involved in scattering from the heterogeneous shell. For example, after some algebraic manipulation we can write $\mathbf{T}_i$ as

$$\begin{aligned} \mathbf{T}_i &= (\mathbf{I} - \mathbf{Q}_{12}|_b \mathbf{Q}_{21}^T|_a [\mathbf{Q}_{22}^T|_a]^{-1} [\mathbf{Q}_{11}|_b]^{-1})^{-1} \\ &= \mathbf{T}_i|_a (\mathbf{I} - [-\mathbf{Q}_{11}|_b]^{-1} \mathbf{Q}_{12}|_b \mathbf{R}_o|_a)^{-1} \mathbf{T}_i|_b \\ &= \mathbf{T}_i|_a (\mathbf{I} - \mathbf{R}_i|_b \mathbf{R}_o|_a)^{-1} \mathbf{T}_i|_b. \end{aligned} \quad (5.35)$$

Here we have defined $\mathbf{R}_o|_a$ to be the reflection matrix for regular waves at the surface S_a , that is the T-matrix for an obstacle with external boundary S_a embedded within a medium consisting entirely of the heterogeneous material. The quantities $\mathbf{T}_i|_a$, $\mathbf{T}_i|_b$, $\mathbf{R}_i|_b$ are defined in a similar manner. In deriving this expression we have relied on the symmetry of the reflection matrices (see chapter 3 for proof of

symmetry for the T-matrix), a property which is complementary to the relation in (5.33). Thus we have

$$\mathbf{R}_i = \mathbf{R}_i^T. \quad (5.36)$$

Note that all internal reflection interactions can be explicitly identified in the reverberation matrix $(\mathbf{I} - \mathbf{R}_{i|b}\mathbf{R}_{o|a})^{-1}$. We use the formal expansion of an inverse matrix to show that

$$(\mathbf{I} - \mathbf{R}_{i|b}\mathbf{R}_{o|a})^{-1} = \mathbf{I} + \mathbf{R}_{i|b}\mathbf{R}_{o|a} + \mathbf{R}_{i|b}\mathbf{R}_{o|a}\mathbf{R}_{i|b}\mathbf{R}_{o|a} + \dots \quad (5.37)$$

Using this relation terms of the form $\mathbf{T}_{i|a}\mathbf{R}_{i|b}\mathbf{R}_{o|a}\mathbf{T}_{i|b}$ in (5.35) represent waves which been transmitted into the shell, reflected once from the boundary at S_a , once from the boundary at S_b and then transmitted into the interior reference medium (see figure 5.3). By applying the definition of the wave coefficient propagator matrix for a heterogeneous shell and expressing the individual matrix partitions in terms of the reflection and transmission matrices for the specific boundaries (i.e. $\mathbf{R}_{i|b}, \mathbf{R}_{i|a}, \mathbf{T}_{o|b}, \mathbf{T}_{i|a}$ etc.) it can be shown that

$$\mathbf{R}_o = \mathbf{R}_{o|b} + \mathbf{T}_{o|b}\mathbf{R}_{o|a}(\mathbf{I} - \mathbf{R}_{i|b}\mathbf{R}_{o|a})^{-1}\mathbf{T}_{i|b}, \quad (5.38)$$

wherein the effects of reverberation in the scattering process are again clearly evident. In a similar manner we can re-express $\mathbf{T}_o$ and $\mathbf{R}_i$ as

$$\mathbf{R}_i = \mathbf{R}_{i|a} + \mathbf{T}_{i|a}\mathbf{R}_{i|b}(\mathbf{I} - \mathbf{R}_{o|a}\mathbf{R}_{i|b})^{-1}\mathbf{T}_{o|a}, \quad (5.39)$$

and

$$\mathbf{T}_o = \mathbf{T}_{o|a}(\mathbf{I} - \mathbf{R}_{o|a}\mathbf{R}_{i|b})^{-1}\mathbf{T}_{o|a}. \quad (5.40)$$

We emphasize that the expressions for $\mathbf{T}_i$, $\mathbf{R}_o$, $\mathbf{T}_o$, $\mathbf{R}_i$ as given above are exact while we have complete sets of orthogonal basis functions which can be used to describe the field in the heterogeneity and the reference medium. In practice of course it is possible to determine exact expressions for $\hat{\psi}_1^\sigma, \psi_1^\sigma$ only when the heterogeneity is itself laterally homogeneous, in which case $\hat{\psi}_1^\sigma, \psi_1^\sigma$ are just the surface wave basis functions for a laterally homogeneous medium with the same vertical

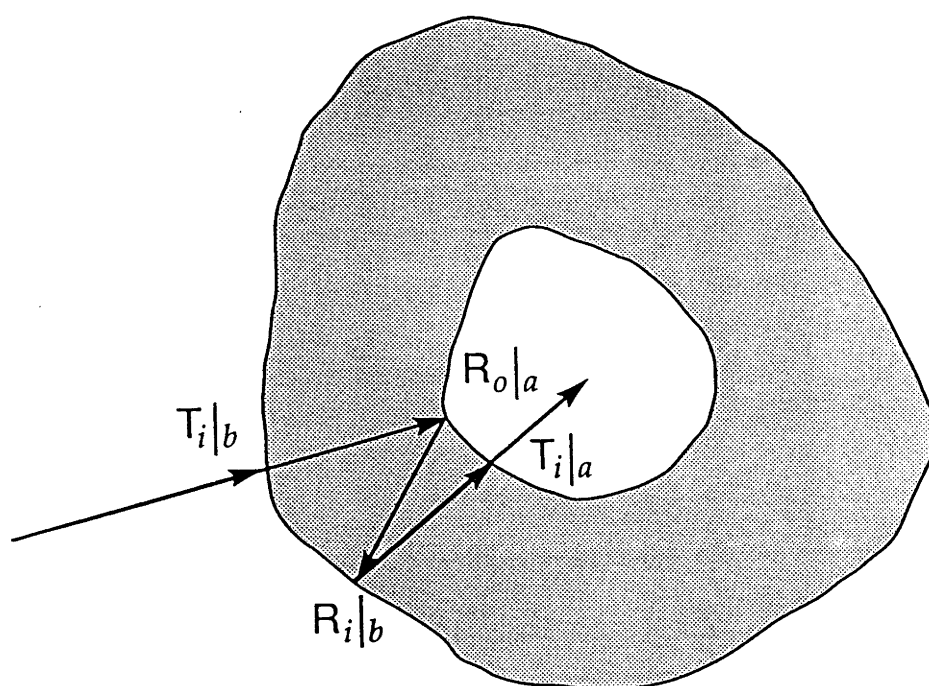


Figure 5.3. Interpretation of the term $\mathbf{T}_{i|a}\mathbf{R}_{i|b}\mathbf{R}_{o|a}\mathbf{T}_{i|b}$ in the formal expansion of the overall, inward transmission matrix $\mathbf{T}_i$ for a heterogeneous shell.

material property distribution. Note that the reflection-transmission formulation we have just presented is applicable both to the propagation of an outgoing wave generated along the z -axis through a tubular zone of heterogeneity and to the scattering of surface waves from a two-layered obstacle for which the materials in the internal layer are the same as those of the embedding medium. It should be obvious now that the methodology might equally well have been applied to a situation where the heterogeneity consisted of a succession of enveloping shells. This observation indicates that, in principle, it is possible to trace the propagation of outgoing waves through a medium consisting of an arbitrary number of enveloping cylinders, or alternatively, to examine the scattering of surface waves from obstacles consisting of many layers. In neither case do we require that the boundaries of the shell be circular, only that they are reasonably smooth so that, for example, the expansion of the scattered field in terms of outgoing waves alone is justified. In the case of a source located at the center of a system of concentric circular shells showing no azimuthal dependence in material properties we would expect coupling of surface wave modes at the shell boundaries but not coupling between azimuthal orders. Furthermore, in this instance Rayleigh and Love waves will decouple and propagate independently in the far-field. By introducing an azimuthal component of heterogeneity either as perturbations in the shell boundaries or in variations in material properties, the energy in individual azimuthal orders among surface wave modes is coupled. This generalization is, in many ways reminiscent of 'super-propagator' approaches to elastic wave propagation in stratified media exhibiting variations in boundary topography where coupling between wavenumbers must be accommodated (see Haines, 1988; Chen, 1990; Kennett, Koketsu & Haines, 1990). In fact it appears that the formulation presented here can be viewed as a specific implementation of the formalism developed by Sabatier (1989) for modelling wave propagation in discontinuous media. He formulates the scattering problem in terms of a general mixed potential-impedance equation which reduces to the Schrödinger equation and the acoustic Helmholtz equation in certain circumstances. The scattering medium comprises a series of successively enclosing domains of variable

properties separated by sphere-like surfaces. Green's function theory is used to construct operators which relate the fields on either side of a surface separating two adjacent domains. The operators are defined by integral equations taken over these surfaces. In addition to the treatment of 'hard' reflectors, Sabatier also treats diffuse scattering from gradients and continuous variations, a topic which we will address in the following sections.

5.4 REFLECTION AND TRANSMISSION II

The formulation presented in the previous section is of limited use in earth science applications. The reason for this is two-fold: i) in practice it is possible to derive exact analytic expressions for surface wave basis functions only when an individual cylindrical shell is laterally homogeneous, and ii) there are very few instances where a heterogeneous region in the earth is accurately modelled as a series of laterally homogeneous, enveloping cylindrical shells. We noted in chapter 1 that evidence from a wide range of seismological studies indicates heterogeneity in the earth can be considered as dominantly radial with a weaker, second order component of lateral heterogeneity which exists as superimposed random fluctuations at a range of scale lengths. The incorporation of Love and Rayleigh wave eigenfunctions in the surface wave basis functions is the essential ingredient that allows us to accommodate the dominating influence of the radial stratification. As we have seen in the last section and the two previous chapters, the effects of variations in the shape of boundaries between two differing, laterally homogeneous media on the scattered field are accommodated by coupling of both azimuthal and modal orders in the basis function expansion. We know from the work of Kennett (1984) that variations in material parameters in a strictly 2-D environment are accounted for by modal coupling; consequently it seems only reasonable that lateral variations in a 3-D cylindrical coordinate reference frame should be described by coupling of modal and azimuthal orders in both wavetypes. This line of reasoning prompts us to investigate the form taken by the reflection and transmission operators when the scattered field is represented as a perturbation on the incident field.

We return to our model of a laterally heterogeneous, cylindrical shell, but now constrain it to be of constant width Δr and so that the internal and external boundary surfaces, S_a and S_b , are perfectly circular with radii r_a and r_b , respectively (see figure 5.4). We write the elasticity tensor c_{ijkl}^1 and density ρ^1 in the heterogeneous zone as

$$c_{ijkl}^1(\mathbf{r}) = c_{ijkl}^0(z) + \Delta c_{ijkl}(r, \theta, z), \quad (5.41)$$

$$\rho^1(\mathbf{r}) = \rho^0(z) + \Delta \rho(r, \theta, z); \quad (5.42)$$

where quantities with superscript 0 are those for the embedding reference medium, quantities prefixed by Δ are the material perturbations. At this point we do not require the perturbations to be continuous functions of radius; that is the material properties on either side of the shell boundary may be discontinuous. In addition we will depart from the notation of the various wavefields in the previous section and adopt one which is better suited to a perturbation approach. That is we choose to represent the total field $\mathbf{u}$ again in terms of a known incident field $\mathbf{u}^0$ and a scattered field $\Delta \mathbf{u}$ which describes the deviation in the wavefield from $\mathbf{u}^0$ as a result of the lateral heterogeneity:

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}^0(\mathbf{r}) + \Delta \mathbf{u}(\mathbf{r}). \quad (5.43)$$

Both $\mathbf{u}$ and $\mathbf{u}^0$ will ultimately arise from a source distribution $\mathbf{F}(\mathbf{r})$ so that they satisfy the following wave equations

$$\left[(c_{ijkl}^0 + \Delta c_{ijkl})(u_{k,l}^0 + \Delta u_{k,l}) \right]_{,j} + (\rho^0 + \Delta \rho) \omega^2 (u_i^0 + \Delta u_i) = -F_i \quad (5.44)$$

and

$$(c_{ijkl}^0 u_{k,l}^0)_{,j} + \rho^0 \omega^2 u_i^0 = -F_i. \quad (5.45)$$

By subtracting the influence of the incident field in the laterally homogeneous reference medium we see that the scattered field $\Delta \mathbf{u}$ satisfies

$$\begin{aligned} (c_{ijkl}^0 \Delta u_{k,l})_{,j} + \rho^0 \omega^2 \Delta u_i &= - \left[(\Delta c_{ijkl} u_{k,l})_{,j} + \Delta \rho \omega^2 u_i \right] \\ &= -f_i \end{aligned} \quad (5.46)$$

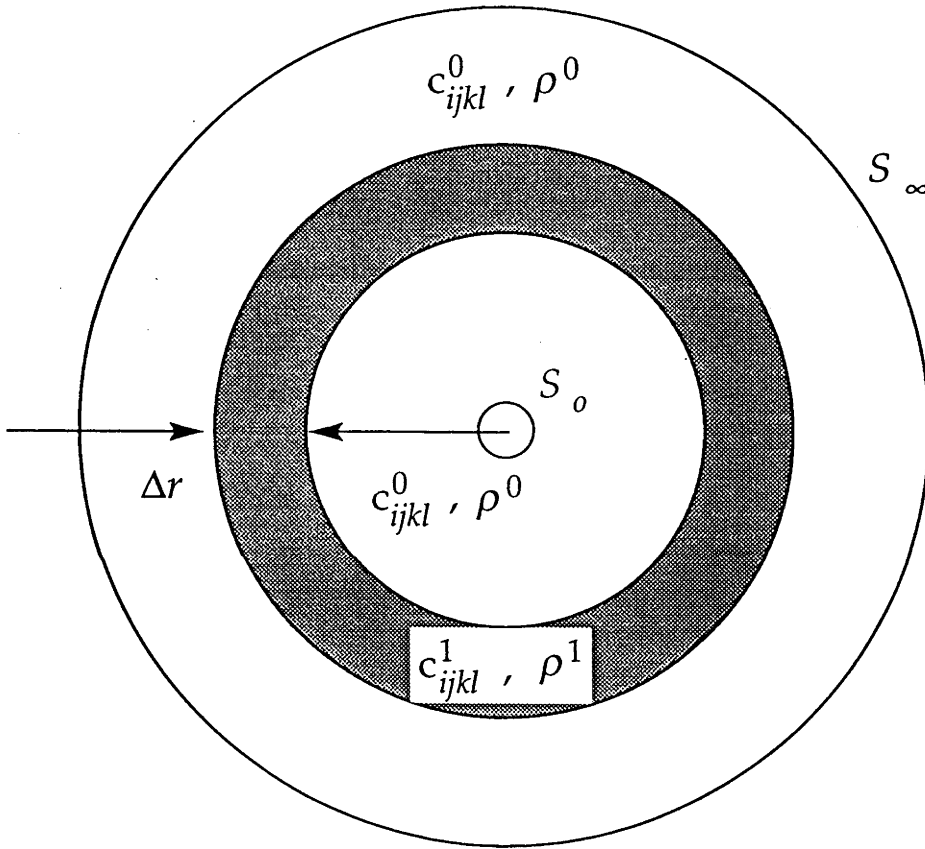


Figure 5.4. Plan view of the heterogeneous shell considered in section 5.4. The inner and outer shell boundaries S_a , S_b are circular, concentric and constant in depth. The material parameters c_{ijkl}^1 and ρ^1 within the shell may vary as a function of position (r, θ, z) but reduce to the corresponding quantities, c_{ijkl}^0 and ρ^0 , of the reference medium at the shell walls.

Thus the scattered field arises as a result of the interaction of the total field with the heterogeneity. We now introduce the surface wave Green's function $G_p(\mathbf{r}|\mathbf{r}')$ for a laterally homogeneous stratified medium which satisfies

$$\left(c_{ijkl}^0 G_{pk,l}(\mathbf{r}|\mathbf{r}')\right)_{,j} + \rho^0 \omega^2 G_{pi}(\mathbf{r}|\mathbf{r}') = -\delta_{ip} \delta(\mathbf{r} - \mathbf{r}'). \quad (5.49)$$

The position vector $\mathbf{r}$ points to the 'observation' coordinates while $\mathbf{r}'$ points to the 'source' coordinates. Thus $G_p(\mathbf{r}|\mathbf{r}')$ represents the response of the medium to a harmonic point force in the p direction located at $\mathbf{r}'$. We follow the conventional approach and express (5.47) in terms of source coordinates $\mathbf{r}'$ and dot with $G_p(\mathbf{r}|\mathbf{r}')$. From this quantity we subtract equation (5.49) dotted with $\Delta u^1(\mathbf{r}')$ to yield

$$\begin{aligned} \Delta u_i(\mathbf{r}') \delta_{ip} \delta(\mathbf{r} - \mathbf{r}') &= f_i(\mathbf{r}') G_{pi}(\mathbf{r}|\mathbf{r}') \\ &+ \left[c_{ijkl}^0 \Delta u_{k,l}(\mathbf{r}')\right]_{,j} G_{pi}(\mathbf{r}|\mathbf{r}') - \left[c_{ijkl}^0 G_{pk,l}(\mathbf{r}|\mathbf{r}') \Delta u_i(\mathbf{r}')\right]_j. \end{aligned} \quad (5.50)$$

We note that the heterogeneity is isolated in the form a circular cylindrical shell V_s centered about the origin. We now integrate over the volume V' shown in figure 5.4 which is bounded by the free surface ($z = 0$), a horizontal surface at infinite depth, and the two external vertical surfaces S_0 and S_∞ . The surface S_0 is of vanishingly small radius whereas the surface S_∞ extends to infinity. This yields

$$\begin{aligned} \Delta u_p(\mathbf{r}) &= \int_{V_s} dV' f_i(\mathbf{r}') G_{pi}(\mathbf{r}|\mathbf{r}') \\ &+ \int_{S_\infty - S_0} dS' \left[c_{ijkl}^0 \Delta u_{k,l}(\mathbf{r}') G_{pi}(\mathbf{r}|\mathbf{r}') - c_{ijkl}^0 G_{pk,l}(\mathbf{r}|\mathbf{r}') \Delta u_i(\mathbf{r}')\right] n_j \end{aligned} \quad (5.51)$$

Here n_j is the outward pointing surface normal and we have employed the tensor divergence theorem together with the symmetries in the isotropic elastic constant c_{ijkl}^0 to reduce the second volume integral to a surface integral. Contributions from the top and bottom surfaces vanish owing to boundary conditions which are implicit in the definition of the surface wave eigenfunctions. Both $G_p(\mathbf{r}|\mathbf{r}')$ and Δu^1 will be outgoing on S_∞ and hence this contribution to the surface integral must vanish as a consequence of the Sommerfeld radiation condition (*cf.* Pao & Mow, 1971). The volume enclosed by S_0 contains no sources for either $G_p(\mathbf{r}|\mathbf{r}')$ or Δu^1 ; hence this integral by Betti's identity must also be zero. A minor complication arises if there are horizontal boundaries in either the reference medium (versus a

continual variation with depth), the heterogeneity or both, across which the material parameters c_{ijkl} , ρ are discontinuous. We must in that case evaluate the surface integral in (5.51) over the individual volumes in which the material properties are continuous. There are no contributions from any of the discrete volumes in the reference medium however we will in general have contributions arising within the region of heterogeneity which can be viewed as applied tractions over the bounding surfaces (*cf.* Herrera & Mal, 1965). It may be shown by applying the boundary conditions that the jump over a given boundary S_n in the quantity $c_{ijkl}^0 \Delta u_{k,l}$ is equivalent to the jump in $-\Delta c_{ijkl} u_{k,l}$ over the same boundary, so that we may write

$$\Delta u_p(\mathbf{r}) = \int_{V_s} dV' \left[(\Delta c_{ijkl} u_{k,l}(\mathbf{r}'))_{,j} + \Delta \rho \omega^2 u_i(\mathbf{r}') \right] G_{pi}(\mathbf{r}|\mathbf{r}') - \sum_n \int_{S_n} dS' \left[\Delta c_{ijkl} u_{k,l} \right]_+^- G_{pi}(\mathbf{r}|\mathbf{r}') n_j \quad (5.52)$$

where the boldface square brackets $\left[\right]_+^-$ signify the difference in value across the discontinuity. We can put this expression into a slightly more symmetric form by following Hudson (1968) and performing the volume integration by parts. The resulting expression for the scattered field is

$$\Delta u_p(\mathbf{r}) = \int_{V_s} dV' \left[-\Delta c_{ijkl} u_{k,l}^0(\mathbf{r}') G_{pi,j}(\mathbf{r}|\mathbf{r}') + \Delta \rho \omega^2 u_i^0(\mathbf{r}') G_{pi}(\mathbf{r}|\mathbf{r}') \right] + \int_{V_s} dV' \left[(-\Delta c_{ijkl} \Delta u_{k,l}(\mathbf{r}')) G_{pi,j}(\mathbf{r}|\mathbf{r}') + \Delta \rho \omega^2 \Delta u_i(\mathbf{r}') G_{pi}(\mathbf{r}|\mathbf{r}') \right], \quad (5.53)$$

and we note that the integration by parts produces a term which exactly cancels the effective traction contributions from discontinuity surfaces in (5.52). This is an important development since it indicates that the expression (5.53) wholly accounts for discontinuities in material parameters throughout the inside of the shell even where these discontinuities are not perfectly horizontal. Note also that we have expanded the total field $\mathbf{u}(\mathbf{r})$ as in (5.43) and assumed that the material perturbations do not affect the boundary conditions (*i.e.* that the material perturbations vanish at the surface and at great depth). At this point we may choose to employ the conventional Born approximation under the assumption that both the contrast in material properties (*i.e.* $\Delta c_{ijkl}, \Delta \rho$) and, consequently, the scattered field are

small (see Hudson & Heritage (1981) for a more precise explanation of the conditions under which this assumption is valid). The total field is then approximated by the incident field $u^0(\mathbf{r})$ and the second term in (5.53) is neglected. However, we can afford to be less restrictive in the magnitude of the material property contrasts by constraining the width of the shell, Δr to be very narrow. It will become obvious shortly that this constraint has no effect on our ultimate objective and indeed will allow us to examine a broader range of heterogeneity. Thus in the event that the shell width is very narrow we may allow $c_{ijkl}(\mathbf{r}, \theta, z), \Delta\rho(\mathbf{r}, \theta, z)$ to be discontinuous across the inner and outer boundaries of the shell. The scattered field contribution to the total field will then be small under the constraint that the volume encompassed by the heterogeneity remains small (*cf.* Hudson, 1977). However, the boundary conditions of continuous displacement and traction across the shell boundaries must be accommodated, and hence following Hudson (1977) we define the tensor $\Delta\tilde{c}_{ijkl}$ where

$$\begin{aligned}\Delta\tilde{c}_{zzzz} &= \Delta\tilde{c}_{\theta\theta\theta\theta} = (\Delta\lambda + 2\Delta\mu) - \Delta\lambda^2/(\lambda + 2\mu) \\ \Delta\tilde{c}_{rrrr} &= (\lambda^0 + 2\mu^0)(\Delta\lambda + 2\Delta\mu)/(\lambda + 2\mu) \\ \Delta\tilde{c}_{zz\theta\theta} &= \Delta\tilde{c}_{\theta\theta zz} = (\lambda^0 + 2\mu^0)\Delta\lambda/(\lambda + 2\mu) \\ \Delta\tilde{c}_{zzrr} &= \Delta\tilde{c}_{rrzz} = \Delta\tilde{c}_{\theta\theta rr} = \Delta\tilde{c}_{rr\theta\theta} = (\lambda^0 + 2\mu^0)\Delta\lambda/(\lambda + 2\mu) \\ \Delta\tilde{c}_{zrrr} &= \Delta\tilde{c}_{rrzz} = \Delta\tilde{c}_{zrrz} = \Delta\tilde{c}_{rrzz} = \mu^0\Delta\mu/\mu \\ \Delta\tilde{c}_{\theta r\theta r} &= \Delta\tilde{c}_{r\theta\theta r} = \Delta\tilde{c}_{\theta rr\theta} = \Delta\tilde{c}_{r\theta r\theta} = \mu^0\Delta\mu/\mu \\ \Delta\tilde{c}_{\theta z\theta z} &= \Delta\tilde{c}_{z\theta\theta z} = \Delta\tilde{c}_{\theta zz\theta} = \Delta\tilde{c}_{z\theta z\theta} = \Delta\mu\end{aligned}\tag{5.54}$$

with all other components of $\Delta\tilde{c}_{ijkl}$ equal to zero. We recognize then that the total field in the reference medium just outside the shell $u^+(\mathbf{r})$ is related to the total field just inside the shell boundary $u^-(\mathbf{r})$ by

$$\Delta c_{ijkl} u_{k,l}^- = \Delta\tilde{c}_{ijkl} u_{k,l}^+.\tag{5.55}$$

Under the assumption that the slab is thin compared with the length scale of the displacement, the total field outside the shell is approximately equal to the incident field ($u^+ \approx u^0$), and so we may approximate the scattered field as

$$\Delta u_p^1(\mathbf{r}) = \int_{V_s} dV' \left[-\Delta\tilde{c}_{ijkl} u_{k,l}^0(\mathbf{r}') G_{pi,j}(\mathbf{r}|\mathbf{r}') + \Delta\rho\omega^2 u_i^0(\mathbf{r}') G_{pi}(\mathbf{r}|\mathbf{r}') \right].\tag{5.56}$$

A superscript ¹ has been appended to our notation of the scattered field Δu^1 to indicate that we are now approximating the total scattered field by the first term in the Born series. To this point we have been unconcerned with the exact representation of the fields u^0 , $G_p(\mathbf{r}|\mathbf{r}')$ and Δu^1 but we will now become more specific. Since the heterogeneity exists as perturbations on a stratified reference medium we will want to represent these wavefields in terms of the surface wave basis functions for the reference medium. Consider in the first instance an incident wavefield u^0 which arises from a source located at the origin within the laterally heterogeneous shell. We can represent this field as

$$u_i^0 = \sum_{\sigma} e^{\sigma} (\psi_0^{\sigma})_i, \quad (5.57)$$

where the coefficients e^{σ} are readily determined for a given earthquake source (see Aki & Richards, 1980; section 7.5). The scattered field Δu^1 is regular within the ring ($r < r_a$) and can be written in terms of the standing wave basis functions

$$\Delta u_i^1 = \sum_{\sigma} d^{\sigma} (\hat{\psi}_0^{\sigma})_i, \quad (5.58)$$

but outside the ring ($r > r_b$) it is outgoing in character and so must be written as

$$\Delta u_i^1 = \sum_{\sigma} c^{\sigma} (\psi_0^{\sigma})_i. \quad (5.59)$$

These representations will be exact in the case that we have a non-lossy waveguide, that is one which exhibits a 'perfect reflector' at some depth which locks all energy into the medium (*cf.* Harvey, 1981; Malischewsky, 1987; Nolet *et al.*, 1989). In this case we can exactly represent the Green's function for a laterally homogeneous stratified medium as a dyadic $\mathcal{G}$ which incorporates the surface wave basis functions such that

$$\mathcal{G} = -\frac{\omega}{2i} \sum_{\sigma} \hat{\psi}_0^{\sigma}(\mathbf{r}_{<}) \psi_0^{\sigma}(\mathbf{r}_{>}) \quad (5.60)$$

This expansion is uniformly convergent for $r \neq r'$ with $r_{>}$, $r_{<}$ the greater and lesser of r , r' , respectively. The expansion of the Green's function in the eigensolutions to this boundary value problem follows from classical Sturm-Liouville theory and a

brief proof is provided in Appendix D. Thus for the scattered field inside the shell where $r < r'$ we can write

$$\begin{aligned}\Delta u_p^1 &= \frac{\omega}{2i} \sum_{\sigma} \sum_{\nu} \left\{ \int_{V_s} dV' \Delta \tilde{c}_{ijkl}(\psi_0^{\sigma})_{k,l}(\mathbf{r}')(\psi_0^{\nu})_{i,j}(\mathbf{r}') \right. \\ &\quad \left. - \Delta \rho \omega^2 (\psi_0^{\sigma})_i(\mathbf{r}')(\psi_0^{\nu})_i(\mathbf{r}') \right\} e^{\sigma}(\hat{\psi}_0^{\sigma})_p(\mathbf{r}) \\ &= \sum_{\eta} d^{\eta}(\hat{\psi}_0^{\eta})_p(\mathbf{r})\end{aligned}\quad (5.61)$$

The orthogonality of the surface wave basis functions permits us to relate the unknown coefficients of the scattered field to those of the known incident field by the matrix relation

$$\mathbf{d} = \tilde{\mathbf{R}}_i \mathbf{e} \quad (5.62)$$

where

$$\tilde{R}_i^{\sigma\nu} = \frac{\omega}{2i} \int_{V_s} dV' \left[\Delta \tilde{c}_{ijkl}(\psi_0^{\sigma})_{k,l}(\mathbf{r}')(\psi_0^{\nu})_{i,j}(\mathbf{r}') - \Delta \rho \omega^2 (\psi_0^{\sigma})_i(\mathbf{r}')(\psi_0^{\nu})_i(\mathbf{r}') \right]. \quad (5.63)$$

Furthermore since we have assumed that the width of the shell Δr is much less than a wavelength, we can approximate this as (omitting the dependence on the coordinates for brevity)

$$\tilde{R}_i^{\sigma\nu} = \frac{\omega}{2i} \int_0^{2\pi} r' d\theta' \int_0^{\infty} dz' \left[\Delta \tilde{c}_{ijkl}(\psi_0^{\sigma})_{k,l}(\psi_0^{\nu})_{i,j} - \Delta \rho \omega^2 (\psi_0^{\sigma})_i(\psi_0^{\nu})_i \right] \Delta r \quad (5.64)$$

Following a similar procedure we can write the scattered field outside the shell as

$$\mathbf{c} = \tilde{\mathbf{T}}_o \mathbf{e}, \quad (5.65)$$

where

$$\tilde{T}_o^{\sigma\nu} = \frac{\omega}{2i} \int_0^{2\pi} r' d\theta' \int_0^{\infty} dz' \left[\Delta \tilde{c}_{ijkl}(\psi_0^{\sigma})_{k,l}(\hat{\psi}_0^{\nu})_{i,j} - \Delta \rho \omega^2 (\psi_0^{\sigma})_i(\hat{\psi}_0^{\nu})_i \right] \Delta r \quad (5.66)$$

Comparison of expression (5.62) with the development in the previous section indicates that $\tilde{\mathbf{R}}_i$ is simply the inward reflection matrix $\mathbf{R}_i$ for a thin shell with the properties defined at the outset of this section. Note however that $\tilde{\mathbf{T}}_i$ is not equivalent to the outward transmission matrix $\mathbf{T}_o$, but rather represents the contribution from scattering, or $\mathbf{T}_o - \mathbf{I}$.

If we specify an incident wave $\mathbf{u}^0$ with a source located somewhere outside the shell, we will, in our usual manner, represent $\mathbf{u}^0$ away from the source area in terms of regular surface wave basis functions

$$u^0 = \sum_{\sigma} a^{\sigma} \hat{\psi}_0^{\sigma}, \quad (5.67)$$

The scattered field however remains defined as in (5.58) and (5.59), and by following similar arguments reflection and transmission matrices $\tilde{\mathbf{R}}_o$ and $\tilde{\mathbf{T}}_i$ can be defined such that

$$\tilde{R}_o^{\sigma\nu} = \frac{\omega}{2i} \int_0^{2\pi} r' d\theta' \int_0^{\infty} dz' \left[\Delta \tilde{c}_{ijkl} (\hat{\psi}_0^{\sigma})_{k,l} (\hat{\psi}_0^{\nu})_{i,j} - \Delta \rho \omega^2 (\hat{\psi}_0^{\sigma})_i (\hat{\psi}_0^{\nu})_i \right] \Delta r \quad (5.68)$$

$$\tilde{T}_i^{\sigma\nu} = \frac{\omega}{2i} \int_0^{2\pi} r' d\theta' \int_0^{\infty} dz' \left[\Delta \tilde{c}_{ijkl} (\hat{\psi}_0^{\sigma})_{k,l} (\hat{\psi}_0^{\nu})_{i,j} - \Delta \rho \omega^2 (\hat{\psi}_0^{\sigma})_i (\hat{\psi}_0^{\nu})_i \right] \Delta r \quad (5.69)$$

We will in general refer to $\tilde{\mathbf{R}}_o, \tilde{\mathbf{T}}_i, \tilde{\mathbf{R}}_i, \tilde{\mathbf{T}}_o$ as differential reflection/transmission matrices since they define the scattering properties of a cylindrical shell of infinitesimal width. Note that the present formulation reveals nothing about the wavefield within the heterogeneous shell, but only of the wavefield within the external reference medium. Also observe that the forms given in (5.64), (5.65), (5.68) and (5.69) automatically satisfy reflection and transmission symmetry relations in the form given by equations (5.33) and (5.36).

5.5 PROPAGATION IN A Laterally Varying Medium

We now wish to apply the results of the previous section, specifically the reflection/transmission properties of a thin heterogeneous shell to the more general problem of wave propagation in media exhibiting a continuous variation in physical properties. An obvious means of proceeding is by the use of invariant embedding techniques which have been employed in reflection/transmission problems for radio waves (Budden, 1955), elastic waves (Tromp & Snieder, 1989), and surface waves (Kennett, 1984a) in media exhibiting heterogeneity in one or two dimensions. The approach then is to replace, for example the real earth which exhibits heterogeneity through its entire volume, by a model which is identical to the earth

over some finite range V but is otherwise laterally homogeneous. By considering a family of similar models with the same general characteristics but different ranges, $V + \Delta V_i$, it is possible to derive differential equations governing the evolution of the reflection/transmission properties as the ranges of the models change.

In order to illustrate this procedure, we follow the general approach of Tromp & Snieder (1989) and consider the situation shown in figure 5.5 where we have a thick shell encircling the origin. This shell exhibits arbitrary heterogeneity throughout its entire volume and we assume that the corresponding reflection/transmission properties are exactly known. We write the corresponding reflection and transmission matrices for this region of heterogeneity as $\mathbf{R}_o$, $\mathbf{R}_i$, $\mathbf{T}_o$, $\mathbf{T}_i$ where as before $\mathbf{R}_o$ relates the coefficients of the expansion of an incoming incident wave to those of the resulting outscattered wave *etc.* Our aim is to determine the effect of welding a thin cylindrical shell of width Δr to the outside of the circular tube, upon the reflection/transmission properties of the heterogeneity as a whole. Recall, for example that the associated inward reflection matrix for the shell is given by

$$\tilde{\mathbf{R}}_i^{\nu\sigma} = \left\{ \frac{\omega}{2i} \int_0^{2\pi} r d\theta' \int_0^\infty dz' \left[\Delta \tilde{c}_{ijkl}(\psi_0^\sigma)_{k,l}(\psi_0^\nu)_{i,j} - \Delta \rho \omega^2 (\psi_0^\sigma)_i (\psi_0^\nu)_i \right] \right\} \Delta r, \quad (5.70)$$

Now consider the reflection of an outgoing wavefield incident upon the heterogeneous tube with the welded outer shell. We can approximate the incoming reflection coefficient of this entire configuration $\mathbf{R}_i(r + \Delta r)$ as the sum of the corresponding coefficient for the tube alone and another term representing the effect of the shell

$$\mathbf{R}_i(r + \Delta r) = \mathbf{R}_i(r) + \mathbf{T}_i(r) \tilde{\mathbf{R}}_i(\Delta r) \mathbf{T}_o(r). \quad (5.71)$$

The second term read from right to left comprises the effect of transmission of the incident wave through the tube to its outer boundary, inward reflection from the shell and subsequent inward transmission through to the inside of the inner tube boundary. Thus we see that higher-order multiple scattering effects have been neglected. Let us rearrange (5.69), and extract the Δr factor such that

$$\frac{\mathbf{R}_i(r + \Delta r) - \mathbf{R}_i(r)}{\Delta r} = \mathbf{T}_i(r) \tilde{\mathbf{R}}_i(\Delta r) \mathbf{T}_o(r). \quad (5.72)$$

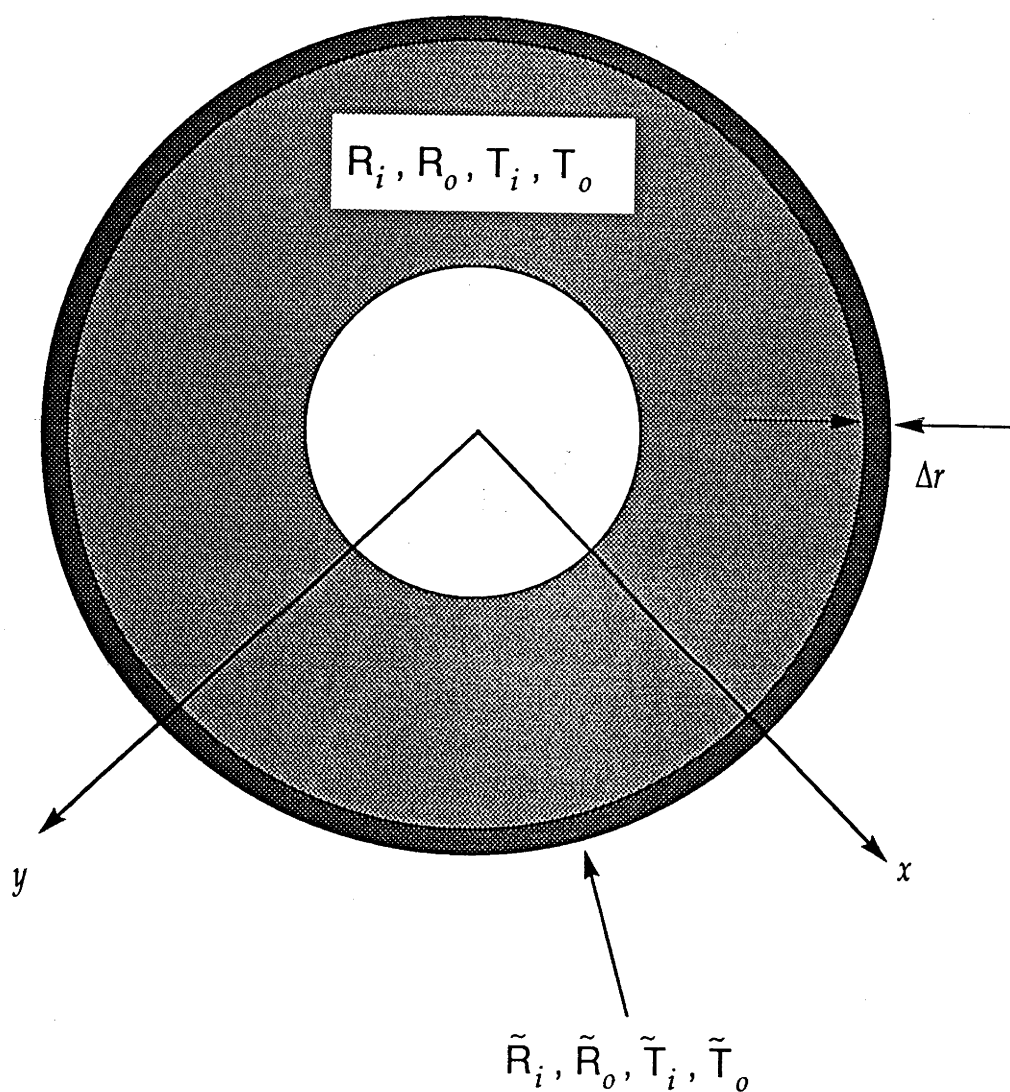


Figure 5.5. The Ricatti equations describing the evolution of the reflection and transmission characteristics of a region of heterogeneity are derived by considering the effect of welding a thin shell (width Δr , properties $\tilde{R}_i, \tilde{R}_o, \tilde{T}_i, \tilde{T}_o$) onto a thick shell with properties R_i, R_o, T_i, T_o .

where it is understood that we have extracted Δr from $\mathbf{R}_i$. As the width of the shell approaches zero we have

$$\frac{d}{dr}\mathbf{R}_i = \mathbf{T}_i\tilde{\mathbf{R}}_i\mathbf{T}_o. \quad (5.73)$$

Similar expressions derived in an analogous manner for the remaining reflection and transmission quantities are

$$\frac{d}{dr}\mathbf{R}_o = \tilde{\mathbf{R}}_o + \tilde{\mathbf{T}}_o\mathbf{R}_o + \mathbf{R}_o\tilde{\mathbf{T}}_i + \mathbf{R}_o\tilde{\mathbf{R}}_i\mathbf{R}_o, \quad (5.74)$$

$$\frac{d}{dr}\mathbf{T}_i = \mathbf{T}_i\tilde{\mathbf{T}}_i + \mathbf{T}_i\tilde{\mathbf{R}}_i\mathbf{R}_o, \quad (5.75)$$

$$\frac{d}{dr}\tilde{\mathbf{T}}_o = \tilde{\mathbf{T}}_o\mathbf{T}_o + \mathbf{R}_o\tilde{\mathbf{R}}_i\mathbf{T}_o. \quad (5.76)$$

Equations (5.73)-(5.76) form a system of $N \times N$ ordinary differential equations of Ricatti type and do, in fact, exactly describe the propagation of surface waves through the laterally heterogeneous medium. This may be demonstrated by recalling that our first-order solution Δu^1 for the field scattered from the thin cylindrical shell is simply the first order term in a Born series which, under conditions established by Hudson & Heritage (1981), will converge to the total scattered field Δu where

$$\Delta u = \sum_n^{\infty} \Delta u^n. \quad (5.77)$$

Higher order terms in the Born series satisfy the recursion relation

$$\Delta u_p^{n+1} = \int_r^{r+\Delta r} dr' \left\{ \int_0^{2\pi} r' d\theta' \int_0^{\infty} dz' \left[\left(\Delta \tilde{c}_{ijkl} \Delta u_{k,l}^n \right)_{,ij} + \Delta \rho \omega^2 \Delta u_i^n \right] G_{pi}(\mathbf{r}|\mathbf{r}') \right\}. \quad (5.78)$$

In this form it is obvious that the higher order terms depend on corresponding powers of Δr and hence must go to zero in the differential limit in, for example, equations (5.72) and (5.73). This may seem surprising at first glance since r is the only independent variable explicitly present in equations (5.73)-(5.76) and we might expect therefore that the Ricatti equations would only account for multiple scattering processes in so far as strictly forward and backward scattering are concerned. However, the surface wave basis functions are indexed in azimuthal order

m as well as wavetype and modal order and this effectively maps the 2-D problem into a 1-D problem (in a similar manner representing the wavefield's depth dependence in terms of the surface wave eigenfunctions allows us to take into account the scattering effects in the vertical dimension). Thus scattering at angles oblique to the direction of propagation are accommodated by a coupling between basis functions of different azimuthal orders.

Equations (5.73)-(5.76) must be supplemented with boundary conditions, and, as in related applications it is appropriate to begin the process at a radius where zero heterogeneity exists, such that $\mathbf{T}_o = \mathbf{I}$, $\mathbf{R}_o = \mathbf{0}$, and integrate the Ricatti equations by successively adding shells of differing material properties until the desired, continuously varying earth model has been constructed. Before proceeding to the next section we should recapitulate several important points. First, the use of the effective elastic modulus tensor $\Delta\tilde{c}_{ijkl}$ in the expressions of the differential reflection and transmission matrices, $\tilde{\mathbf{R}}_i, \tilde{\mathbf{R}}_o, \tilde{\mathbf{T}}_i, \tilde{\mathbf{T}}_o$, is more generally valid than simply employing the direct perturbation in modulus Δc_{ijkl} . It allows us to examine heterogeneity which although continuously varying may, at any given point, deviate considerably from the reference medium. The reason for this is that $\Delta\tilde{c}_{ijkl}$ is derived by applying the continuity of traction across the surface bounding the heterogeneity, and applying the Born approximation under the condition that although the heterogeneity contrast is not necessarily weak, it encompasses an infinitesimal volume. In the limit as the material perturbations become very weak $\Delta\tilde{c}_{ijkl}$ and Δc_{ijkl} are of course equivalent as can be verified from (5.54). The second point of note is that the expressions for $\tilde{\mathbf{R}}_i, \tilde{\mathbf{R}}_o, \tilde{\mathbf{T}}_i, \tilde{\mathbf{T}}_o$ are complete in the sense that the two classes of effective source representation for the scattered field are accommodated in the integral expressions for the differential reflection/transmission matrices: that is i) effective volume forces which accrue from the actual perturbations in material properties from the reference medium, and ii) effective surface tractions applied across levels corresponding to discontinuities in the reference medium, the heterogeneity or both. In addition there is no requirement that the discontinuities in the heterogeneity be horizontal. Using the Ricatti equation approach we can there-

fore, in principle, retrieve the reflection and transmission properties of a band of heterogeneity which includes layers of variable thickness. In practice, however, the resultant scattering effects from this class of heterogeneity are likely to be controlled by interactions over a large number of surface wave modes and thus may be difficult to model numerically.

5.6 THE RELATIONSHIP WITH KENNETT'S COUPLED MODES

One point of interest is the relationship of the present formulation with Kennett's (1984a) coupled mode description of surface wave propagation in 2-D media, and it is worthwhile investigating this matter more closely. The simplest means of establishing the correspondence is to reconsider our formulation in the case of a thin slab of heterogeneous material extending to infinite depth and invariant in one horizontal coordinate, say y . The slab of width Δx is embedded in an otherwise laterally homogeneous stratified reference medium, and is itself horizontally stratified and laterally homogeneous. We employ the arguments presented in equations (5.43)-(5.57) and consider an incident surface wave train, u^0 , propagating in the x -direction. We can expand this wavefield in a sum of surface wave basis functions in a Cartesian coordinate reference frame. These basis functions are defined as in (3.3), (3.7) but now a different form of horizontal wave function is used, specifically we redefine $Y, \hat{Y}$ as

$$Y_n = e^{ik_n x}, \quad (5.79)$$

and

$$\hat{Y}_n = e^{-ik_n x}. \quad (5.80)$$

The geometry of the problem dictates that Love and Rayleigh waves are now decoupled, so that we may consider them independently. By expanding the wavefields Δu^1 and $\mathbf{G}_p(\mathbf{r}|\mathbf{r}')$ using the appropriate surface wave basis functions, expressions for the forward differential reflection matrix elements, for example, are given by

$$\tilde{R}_o^{\sigma\nu} = \frac{\omega}{2i} \int_0^\infty dz' \left[\Delta \tilde{c}_{ijkl} (\hat{\psi}_0^\sigma)_{k,l} (\hat{\psi}_0^\nu)_{i,j} - \Delta \rho \omega^2 (\hat{\psi}_0^\sigma)_i (\hat{\psi}_0^\nu)_i \right] \Delta x, \quad (5.81)$$

An analogous argument leading to the derivation of the Ricatti equations (5.73)-(5.76) is then employed with the appropriate modification of $\Delta\tilde{c}_{ijkl}$ to a 2-D Cartesian reference frame. It may be demonstrated that these 2-D differential reflection/transmission matrices are simply the L_{ij} and K_{ij} of Kennett (1984) (to within a common factor) if Kennett's traction eigenfunctions are recast in terms of displacement eigenfunctions. Our derivation has made no explicit reference as to the nature of the wavefield within the heterogeneity, however the corroboration with Kennett's results suggests that an interpretation in terms of surface wave basis functions weighted by coefficients which vary with radius r is possible. This in turn prompts us to speculate as to whether Maupin's 1988 extension of Kennett's work to incorporate local modes can be made in the present case. This would be useful in the sense that we would no longer be required to consider a wavefield on both sides of the heterogeneity with regard to the same laterally homogeneous reference medium, but we might choose to examine heterogeneity models which exhibited systematic changes from one side to another. In fact one way of accommodating a systematic change in material properties away from the source is through the use of the propagator formalism developed in section 5.3. Consider a situation where we have calculated the reflection and transmission matrices for a shell of heterogeneity embedded within a single stratified reference medium and where we have a prescribed source at $(0,0,h)$. The expansion of the incident wavefield in terms of the outgoing basis functions of the reference medium is known and characterized by the coefficient vector $\mathbf{c}_0$. There will in general be an inward reflected field with corresponding coefficient vector $\mathbf{a}_0$ and an outgoing transmitted field outside the heterogeneous shell described by $\mathbf{c}'_0$ where $\mathbf{c}'_0 = \mathbf{T}_o \mathbf{c}_0$. If we now require the reference medium outside the shell to differ from that within we must calculate the coefficient vector $\mathbf{c}_1$ (referred to the outgoing basis functions of the new reference medium) using a relation of the form

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{c}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{22}^T & \mathbf{Q}_{12}^T \\ -\mathbf{Q}_{21}^T & -\mathbf{Q}_{11}^T \end{pmatrix}_{S=S_+} \begin{pmatrix} \mathbf{T}_i^{-1} & -\mathbf{T}_i^{-1} \mathbf{R}_i \\ \mathbf{R}_o \mathbf{T}_i^{-1} & \mathbf{T}_o - \mathbf{R}_o \mathbf{T}_i^{-1} \mathbf{R}_i \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{c}_0 \end{pmatrix} \quad (5.82)$$

Here the reflection and transmission quantities are those of the heterogeneous shell and quantities such as $\mathbf{Q}_{11}^T$ are defined as in section 5.2 and taken over the circular

cylindrical surface representing the external boundary of the shell S_+ . We could then in turn use the new reflection/transmission quantities (calculated using the type of relation expressed in (5.34)) for this composite configuration as initial conditions to the Ricatti equations in (5.73)-(5.76) and integrate through further models of heterogeneity using now the Green's function for the new reference medium.

5.7 NUMERICAL EXAMPLES

In this section we demonstrate the numerical implementation of the theory developed in previous sections by examining the effect of some rather simplistic models of heterogeneity on the propagation of surface waves. In considering an appropriate application, that is one of some geophysical interest, we note that there are two obvious situations where we find a cylindrical coordinate reference frame convenient for treating surface wave propagation problems. The first, as we have seen in chapters 3 and 4, involves the interaction of an incident plane wave with a finite region of heterogeneity. A cylindrical coordinate geometry is the natural one to adopt in this case since we expect the scattered field to consist of outgoing waves at large distances relative to an origin located within the scatterer. Unfortunately the cylindrical reference frame is less well suited to representing the incident plane wave, and it is this factor which essentially restricts numerical application of the T-matrix formalism to heterogeneous regions with $ka \leq 10.0$ where k is the wavenumber of the fundamental Rayleigh mode and a represents an average radius of the scatterer.

As noted in earlier sections of this chapter, the evolution of the surface wavefield as it propagates outward from a point source at a specified origin is another situation where a representation of the wavefield in a cylindrical coordinate frame is physically appropriate. The specification of the surface wave basis functions in cylindrical coordinates allows us to examine in some detail the character of the near-field, that is the wavefield within several wavelengths of the source. In this section we make use of the perturbation formulation just described to examine the influence of heterogeneity in the general vicinity of the source on the far-field

radiation patterns.

5.71 Numerical Considerations

Our approach will be to build up the outward transmission and reflection matrices, $\mathbf{T}_o$ and $\mathbf{R}_o$, for a thick annular region of heterogeneity using the coupled equations (5.74) and (5.76) by a successive addition of thin shells (each representing a thin cylindrical slice through the total heterogeneity) away from the origin. At first glance numerical implementation of our formulation would appear to be a computationally prohibitive exercise. Equations (5.74) and (5.76) indicate that we require all four of the differential reflection and transmission matrices, $\tilde{\mathbf{R}}_i$, $\tilde{\mathbf{R}}_o$, $\tilde{\mathbf{T}}_i$ and $\tilde{\mathbf{T}}_o$, at each step in the evaluation of the Ricatti equations. Note that the wavefield representations entail an expansion over wavetype (Love/Rayleigh), modal order, azimuthal order, and sinusoidal parity so that a realistic representation of propagation through any heterogeneity model will necessarily require the construction of matrices comprising a very large number of elements. The nature of the individual differential matrix elements is such that the computer coding for their calculation is not easily vectorizable. Furthermore equations (5.74) and (5.76) imply that the evaluation of each differential matrix element at each incremental step in r requires a computationally expensive 2-D integration over a cylindrical surface. We can however minimize the computational overhead by recognizing several key points. First, we need only calculate slightly over half of the elements of the reflection matrix $\tilde{\mathbf{R}}_i$ because of the obvious symmetry inherent in equation (5.70). This of course applies to $\tilde{\mathbf{R}}_o$ as well, however since we must calculate the transmission matrix $\tilde{\mathbf{T}}_o$ ($\tilde{\mathbf{T}}_i$ being the transpose of the $\tilde{\mathbf{T}}_o$) it is simpler to recognize that, with our choice of azimuthal representation (cosines and sines as opposed to complex exponentials), the outward reflection matrix $\tilde{\mathbf{R}}_o$ is directly proportional to the imaginary part of $\tilde{\mathbf{T}}_o$. The fact that the azimuthal dependence of the surface wave basis functions involves simple trigonometric functions allows us to further simplify computations by performing the azimuthal component of the 2-D surface integral analytically. This is accomplished by representing the azimuthal depen-

dence of the heterogeneity in terms of Fourier components, that is, for example

$$\Delta\rho(r, \theta, z) = \sum_p [A^p(r, z)\cos(p\theta) + B^p(r, z)\sin(p\theta)]. \quad (5.79)$$

In that case we can employ simple trigonometric orthogonality relations with respect to the integration of products of sines and cosines over the range $0 \rightarrow 2\pi$ to perform the azimuthal integration. The depth and radial dependencies of the surface wave basis functions are such that both evaluation of the depth integrals in the definition of the differential reflection/transmission matrices as well as the integration of the Ricatti equations must be performed numerically. Hence $\Delta\tilde{c}_{ijkl}$ and $\Delta\rho$ as functions of r and z may be defined in a more or less arbitrary manner.

5.72 Model Specification

In the following sections we will examine the effect of 3 distinct models of heterogeneity on the character of the transmitted wavefield from an explosive source. For each of these models we will consider a reference medium comprising a three-layer half-space with the same material properties as the embedding reference medium employed in chapter 3 for the single scatterer T-matrix problem. The depth dependence of the perturbations in velocity ($\Delta\alpha$, $\Delta\beta$) and density ($\Delta\rho$) for the first and third models is shown in figure 5.6a expressed in percent and is most significant over the top 20 km of the model. The second model exhibits the same form of depth dependence however the depth of onset varies as a function of radius, starting at 0 km depth at the inner radial boundary and moving to 10 km depth at the outer boundary. Note that since the reference medium comprises a stack of uniform layers the absolute values of the perturbation parameters are characterized by jumps at the layer boundaries. The depth dependence is modulated by a radial dependence in the form of a cosine 'bell' (figure 5.6b) so that the perturbations go to zero as the boundaries of the band of heterogeneity are approached. As was noted earlier, the azimuthal variation in material properties is most conveniently described in terms of Fourier components. The first two models we take to exhibit a simple $\cos\theta$ azimuthal dependence, while the third model includes equal contributions from $\cos\theta$, $\cos 2\theta$ and $\cos 3\theta$ (see figure 5.7). The absolute magnitude

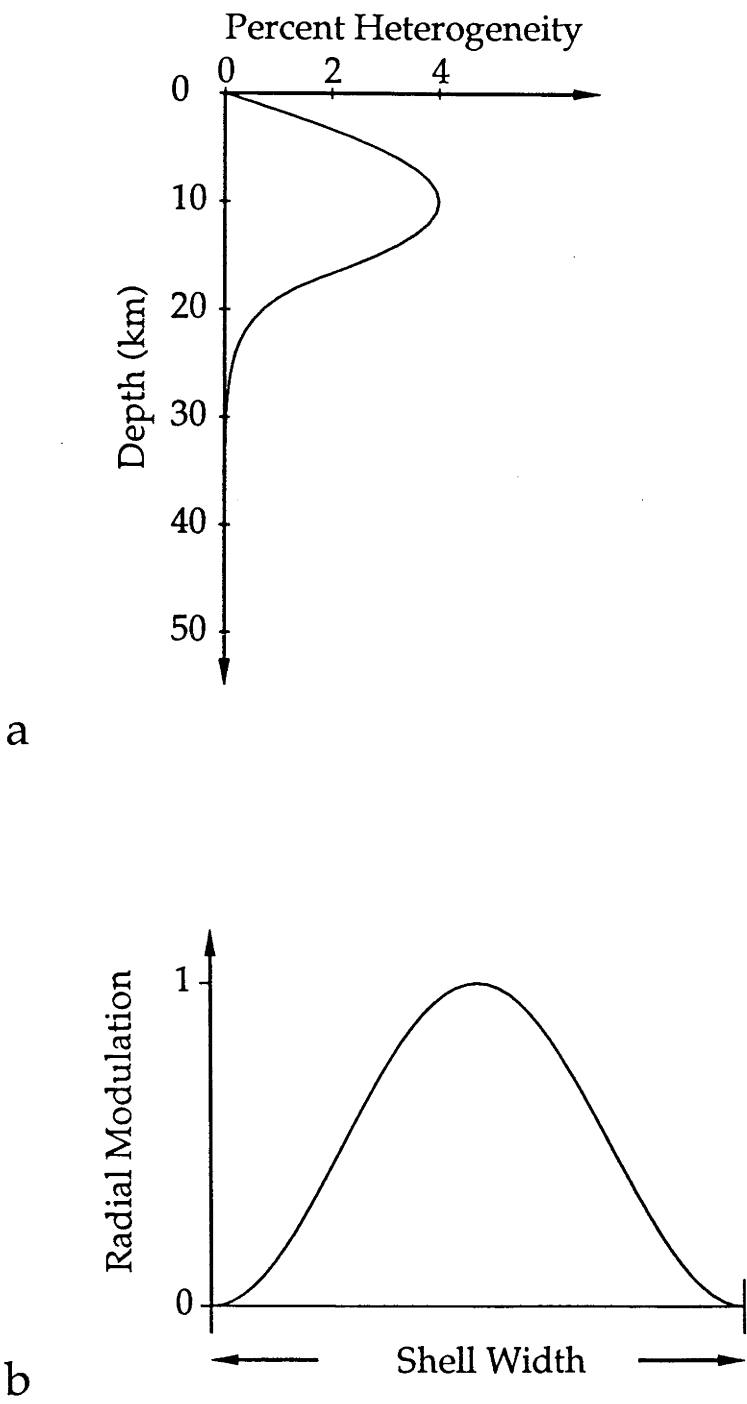


Figure 5.6. Specification of the model heterogeneity. The heterogeneity in velocity and density as a function of depth for models 1 and 2 is shown in a) and is most significant over the top 20 km of the model. The radial dependence consists of a cosine bell over the width of the shell. See the following figure for details regarding the azimuthal description.

of the velocity and density perturbations in all three models is constrained not to exceed $\pm 4\%$ through the heterogeneity (thus the Lamé constants λ , μ vary by up to $\pm 12\%$); this moderate contrast will allow us to work with a computationally tractable number of modes and focus on both the quantitative and qualitative aspects of propagation.

In keeping with earlier work we will consider surface wave propagation at 1.0 Hz so that our model constitutes a simplified representation of the propagation of regional phases such as *Lg* guided within the earth's crust. There is ample evidence to suggest that near-source heterogeneity has a significant influence on the observed character of the seismic wavefield at these frequencies. Several authors (Greenfield, 1971; and Lynnes & Lay, 1989, 1990) have attributed well correlated features in the coda of teleseismic *P*-waveforms to scattering from fundamental and higher surface wave modes in the source region. Furthermore, we find that the recorded waveforms of seismic waves are markedly affected in the reciprocal situation, namely the presence of lateral heterogeneity in the immediate vicinity of receiver sites (*cf.* Key, 1967). In studies at the Nevada Test Site Barker *et al.* (1981) found that the observed character of *Lg* was closely linked to the local geologic structure; receivers at sites on low velocity sediments produced high amplitude, often prolonged wavetrains relative to sites on bedrock. Der *et al.* (1984) examined the intersensor coherence of *Lg* waveforms at several arrays and concluded that the character of *Lg* coda could be best explained if modal coupling due to local structure played a significant role in the scattering process.

It is advantageous for purposes of interpretation to consider an explosive source since the properties of the resultant incident wavefield are invariant in azimuth (i.e. the coefficients of the incident wave series are zero for all $m \neq 0$) and any deviations from this symmetry in the transmitted field must result from scattering. Furthermore, only Rayleigh waves are produced at the source and any Love component in the far-field radiation pattern must also arise from scattering. In fact, as mentioned in previous chapters, observation of a large transverse component of displacement from purely explosive sources is not an uncommon feature on many regional seis-

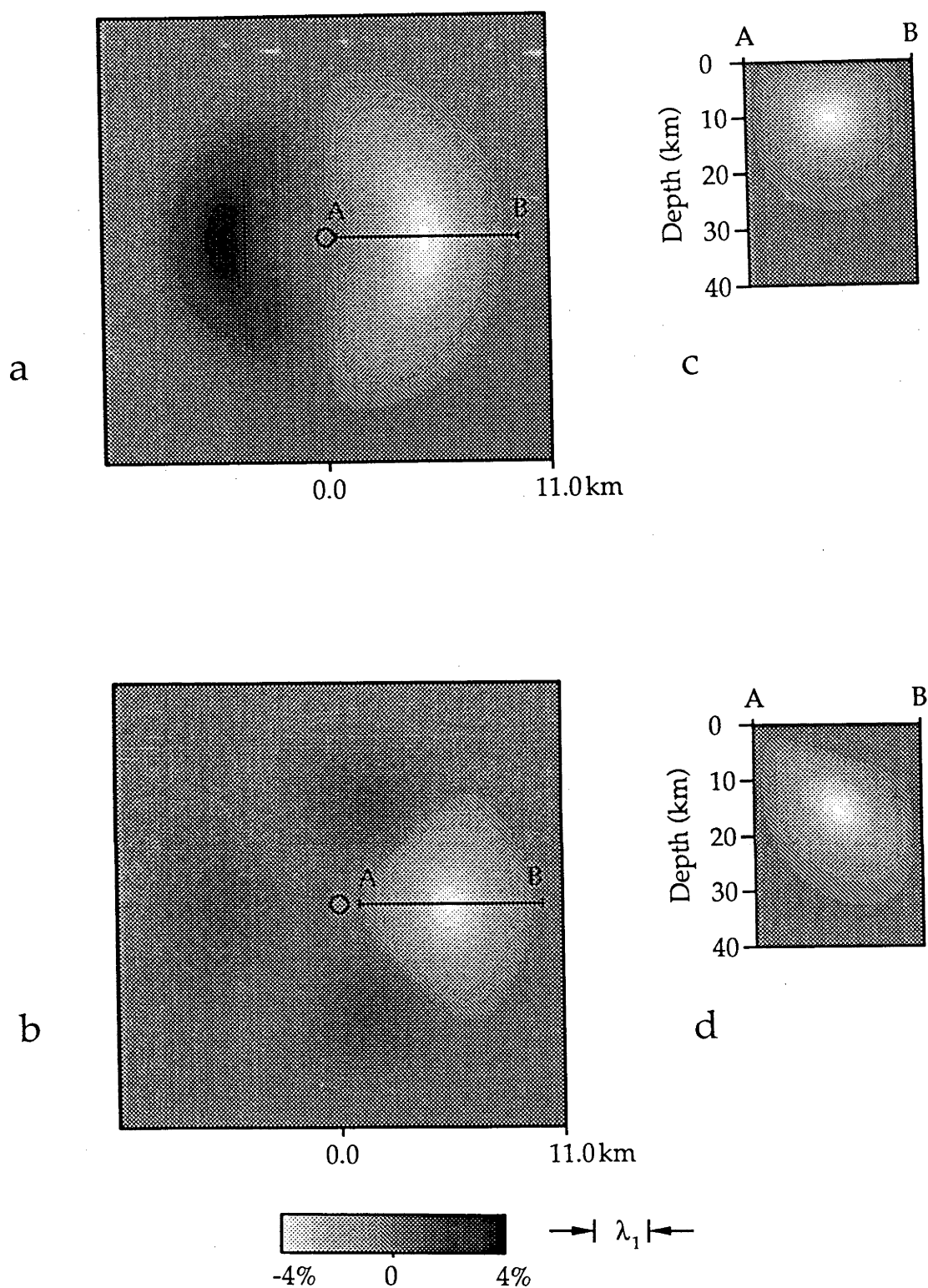


Figure 5.7. Heterogeneity models. Plan view of the heterogeneity in models 1 and 2 is shown in a), the azimuthal variation is specified by a $\cos\theta$ dependence. A corresponding view for model 3 is shown in b) and is characterized by an azimuthal dependence comprising equal contributions from $\cos\theta$, $\cos2\theta$, $\cos3\theta$. A depth slice through the heterogeneity is shown in c) for models 1 and 3 and in d) for model 2. The wavelength λ_1 of the fundamental Rayleigh mode is shown for scale.

mograms. Our formulation should therefore allow us to investigate what role, if any, continuously varying heterogeneity in the near-field plays in producing a significant Love wave component in the scattered far-field. Note that the form of the differential reflection and transmission matrix elements (e.g. (5.64)) and the definition of the surface wave basis functions indicates that wavetype coupling terms are proportional to r^{-2} where r is the distance from the source. Therefore moderate levels of continuously varying heterogeneity are only likely to result in coupling close to the source and should have little effect at greater distances where the displacements of the two wavetypes are essentially separated by coordinate (Love on the transverse component and Rayleigh in the sagittal plane). In theory it is possible to accommodate more rapid fluctuations in material properties to simulate discontinuities having some azimuthal component but at greater distances this requires the introduction of basis functions of increasing order m whose behaviour serves to offset the r^{-2} dependence. However, in such cases the direct T-matrix approach would be both more appropriate and considerably simpler to implement.

In order to keep the descriptions reasonably brief, we will focus on the behaviour of the first, third and sixth Rayleigh modes (hereafter $R1, R3, R6$) as a guide to the general behaviour of the scattered field, except where interactions involving other modes deserve special note. A large fraction of the total energy in these modes is transported over the interval where the model perturbations in velocity and density are significant and so we expect that all scattering effects should be well represented. In addition this selection should provide some indication of differences in character between low-order modes ($R1, R2, L1, L2$) which frequently exhibit their own individual character, and the higher-order modes which comprise the dominant contributions to the Lg wavetrain. As in previous chapters we refer to transmitted energy scattered into modes of an order different to the incident mode as the energy in 'converted' modes.

5.73 Model 1

In our first model we consider a band of heterogeneity characterized by a $\cos\theta$

azimuthal variation in velocity and density perturbations as shown in figure 5.7. The heterogeneity begins approximately one sixth of a wavelength from the source, reaches a maximum contrast at approximately one and one half wavelengths and decreases thereafter to zero at three wavelengths (recall that higher-order surface wave modes are characterized by longer wavelengths). The geometry of the problem and the orthogonality properties of sines and cosines with respect to an integration over $0 \rightarrow 2\pi$ dictates the energy in a basis function of given azimuthal order m interacting with a very thin shell of heterogeneity will be scattered into basis functions of orders $(m+1)$ and $|m-1|$. If we begin with a purely explosive source then the incident wave contains only order $m=0$, as the wavefield advances through the heterogeneity energy is scattered into order $m=1$ and through the process of multiple scattering into higher orders. It becomes clear then that the number of azimuthal orders m which need to be considered in a truncated system is controlled by i) the magnitude of the heterogeneity, ii) the extent of the heterogeneity in a radial direction, and iii) the number of Fourier components p describing the azimuthal distribution of the heterogeneity.

Figure 5.8 shows the partitioning of scattered energy as a function of modal order for incident modes $R1$, $R3$ and $R6$. Note that the values are plotted on a logarithm plot relative to the energy in the incident wave and that the energy plotted for the unconverted mode represents the contribution from scattering plus the incident wave, that is the total energy in the transmitted mode. The relative energy distribution for all three incident modes does not vary significantly over the range of shell widths considered (i.e. that is between approximately one and three wavelengths of the fundamental Rayleigh mode). This indicates that, as in the case of discrete cylinders with vertical walls examined in chapter 3, the distribution of scattered energy is dictated largely by the shapes of the displacement eigenfunctions. In this case $R1$ interacts principally with other low order Rayleigh modes ($R1, R2$) and notably the fundamental Love mode $L1$, whereas scattered energy from $R3$ and $R6$ resides primarily in the unconverted mode and adjacent Rayleigh modes. It is clear that vast majority of incident energy is scattered into

other Rayleigh modes and that Love and Rayleigh waves remain largely decoupled even at these distances. This is due in part to the r^{-2} factor in the coupling factor mentioned earlier which ranges between approximately $1/3$ and $1/400$ over the width of the shell.

As noted in previous chapters it is convenient to examine the directional dependence of scattering field using far-field radiation patterns. When we consider the azimuthal dependence of the surface wave basis functions and our choice of representation it becomes apparent that scattering into the converted wavetype (i.e. $R \rightarrow L$, $L \rightarrow R$) will occur in zones with large gradients in heterogeneity transverse to the general direction of propagation. This observation lends further support to the idea put forward in chapter 2 that the 'tilt' in ray vector caused by transverse gradients in topography provides a measure of interconversion between Love and Rayleigh wavetypes. As we might expect for reasonably mild heterogeneity contrasts, scattered energy resides primarily within basis functions $m = 1$ so that most scattered Rayleigh waves are dominated by a $\cos\theta$ pattern and scattered Love waves by a $\sin\theta$ pattern. There are a few exceptions to this rule, for example $R1 \rightarrow R6$; $R3 \rightarrow L4$; $R6 \rightarrow L5, R1$; the scattered modes for these combinations also have comparable energies in $m = 0$ (Rayleigh only) and $m = 2$ (Rayleigh and Love). If we compare these combinations with their energy spectra in figure 5.7 we note that they are characterized by rather low levels of coupling with the incident mode. In fact the significant contributions of $m = 0$ and $m = 2$ to their radiation patterns are a result of multiple scattering via first order scattering from converted modes (i.e. modes other than the incident mode). In these cases the radiation patterns are somewhat more complex, but owing to the low levels of energy involved, the contributions to the total transmitted field are essentially insignificant.

It is worthwhile examining the relationship between the incident and unconverted scattered modes. We specify the incident mode in terms of basis function coefficients which are entirely real, while the scattered wavefield coefficients of the unconverted mode are complex. As we might expect from spectra in figure 5.8 the dominant contribution to the total transmitted wavefield observed in the far-

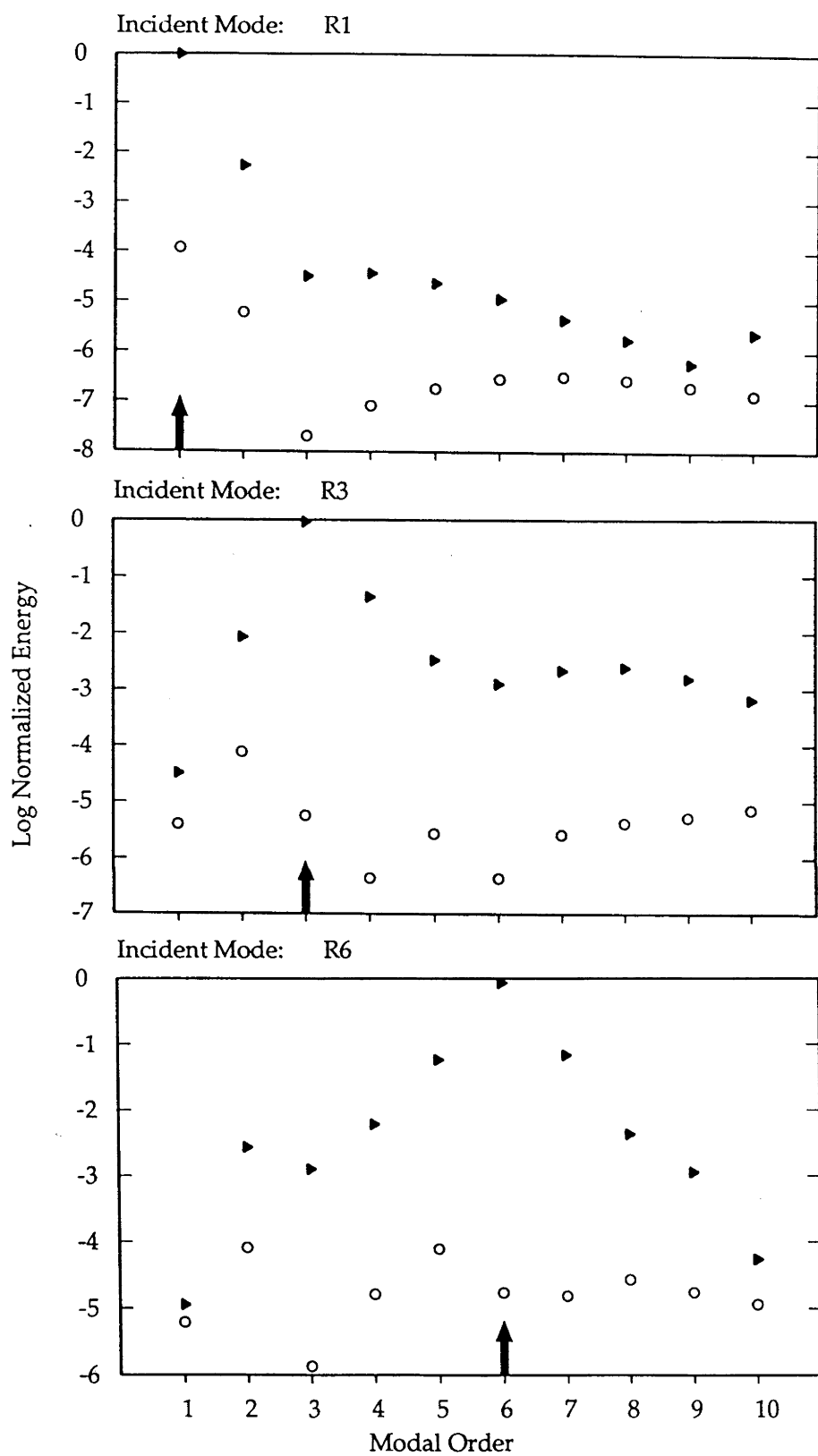


Figure 5.8. Transmitted scattered energy spectra for incident modes $R1$, $R3$ and $R6$ in model 1. Filled triangles and circles denote Rayleigh and Love modes respectively.

for our heterogeneity model comes from the transmitted mode, the sum of the incident and unconverted scattered modes, which is characterized by both a phase shift and change in amplitude relative to the incident mode. As demonstrated by Snieder (1988d) for media smoothly varying with respect to dominant wavelength, the phase is the more sensitive of the two parameters to heterogeneity and, in our model, causes equivalent time delays/advances of up to one half period for wavefronts passing through the regions characterized by negative/positive velocity perturbations. This effect is most pronounced for the higher-order transmitted modes ($R3$ and up) since these contain a higher proportion of their energy over the range of depths where the magnitude of heterogeneity is greatest. The effects on the amplitude of the transmitted mode are a little less dramatic (variations of a few percent) but nonetheless interesting. In all cases, the amplitude of the transmitted mode is greater along the positive x -axis (lowest velocities) than along the negative x -axis (highest velocities). For the low order transmitted modes, $R1, R2$, the maximum amplitude occurs along the positive x -axis and is greater than that of the incident mode (i.e. the amplitude which would be observed in the absence of any heterogeneity). For higher-order transmitted modes however the amplitude is greatest in directions perpendicular to the x axis, and less than that in the incident mode over the entire range of azimuths. This contrast in behaviour between low and high order transmitted modes stems, again, from the fact that higher modes carry a greater proportion of their energy over the interval with the greatest velocity perturbations. As a result the higher modes are more sensitive to the heterogeneity and actually undergo significant reflection from the large radial gradients in heterogeneity along the positive and negative x -axes. Thus energy propagating in this direction is trapped within the internal boundaries of the heterogeneity, which results in a reduction in the amplitude of the transmitted mode observed in the far-field (see for example the radiation pattern for $R6 \rightarrow R6$, figure 5.9). In contrast, that portion of the incident wavefield propagating along the y -axis encounters no large impedance contrast and so remains unaffected. The lower order modes do not experience significant reflection and hence only the effects of

focussing and defocussing, as predicted from geometrical optics, are observed.

5.74 Model 2

The next model exhibits the same azimuthal variation in material properties, that is a $\cos\theta$ dependence but differs in that the heterogeneity shifts to greater depths with radial distance from the source (see figure 5.7). The 'peak' in figure 5.8a is located at 10 km depth (as in model 1) at the inner boundary, $r = 0.5$ km, but falls to 20 km depth at the outer boundary $r = 9.5$ km. Viewed in 3-D this may appear as a somewhat unlikely representation of the real earth but it serves to illustrate, by comparison with model 1, the manner in which the scattered wavefield is affected by heterogeneity which extends systematically through a range of depths.

The scattered energy spectra for modes $R1$, $R3$ and $R6$ are shown in figure 5.10 and do indeed deviate somewhat from the corresponding plots for model 1 where the heterogeneity largely confined to a specific depth level. Energy scattered from incident $R1$ into $R1$ and $R2$ is now much decreased since much of the heterogeneity now occurs at depths below which the fundamental mode carries significant energy. The scattered energy spectra for incident $R3$ differs primarily in the relative distribution of energy in adjacent modes; and it is apparent that this particular choice of heterogeneity model has served to reduce significantly the energy scattered into $R4$ in favour of $R5$. A similar effect is noted for incident $R6$, where the relative proportions of energy in the modes $R4$ and $R8$ is up considerably at the expense of adjacent modes $R5$ and $R7$. Radiation patterns (not shown) are similar in general to those for model 1 and indicate as expected that the major contributing modes to the scattered field contain most of their energy in azimuthal order $m = 1$. The principal differences arise, again in modes where the dominant contributions come from first order scattered converted modes.

5.75 Model 3

Our final model is more complicated in its horizontal description and comprises equal contributions from $\cos\theta$, $\cos2\theta$ and $\cos3\theta$ terms. The net effect of this config-

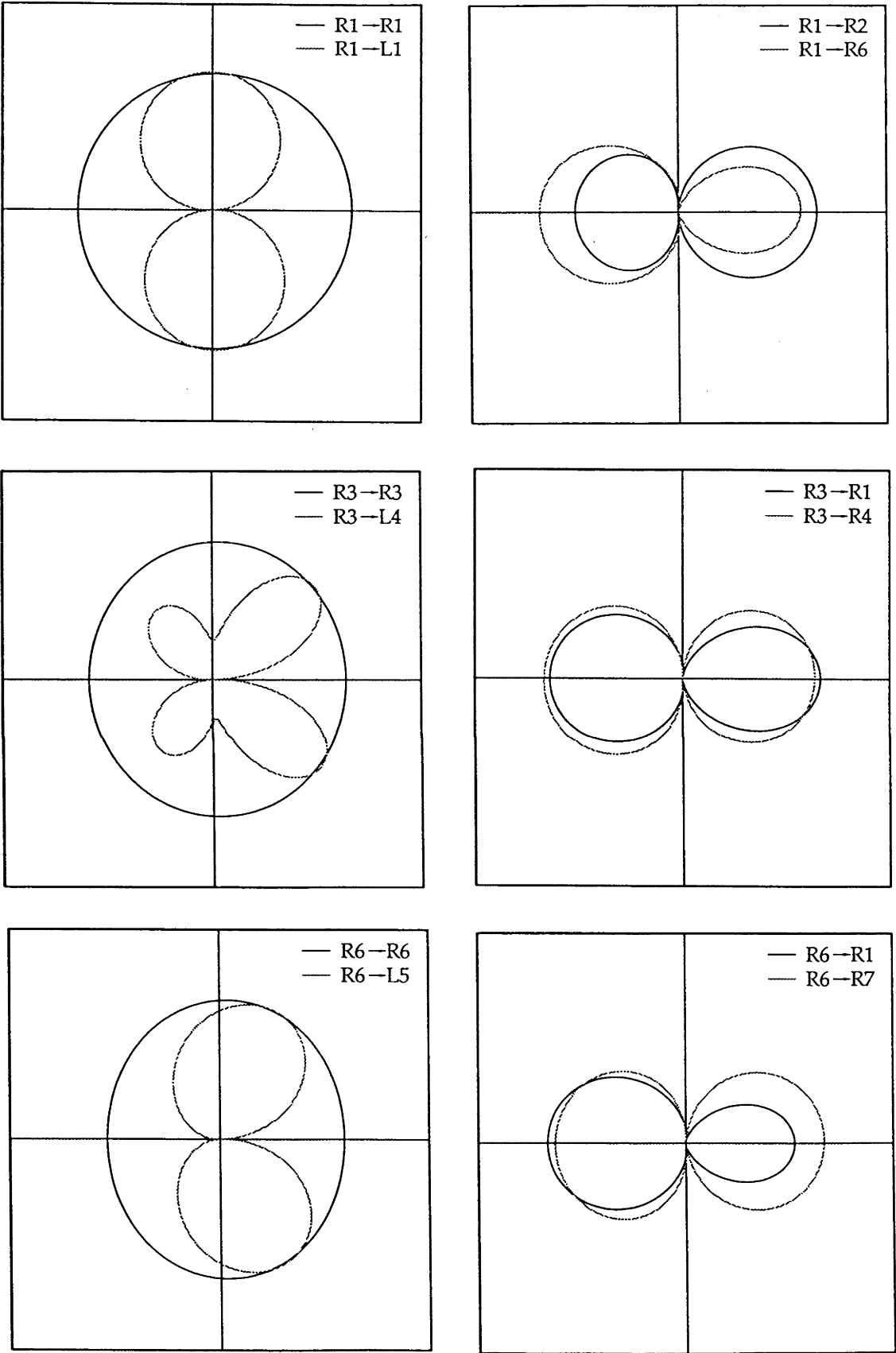


Figure 5.9. Far-field radiation patterns for a selection of scattered modes for incident modes $R1$, $R3$ and $R6$ in model 1.

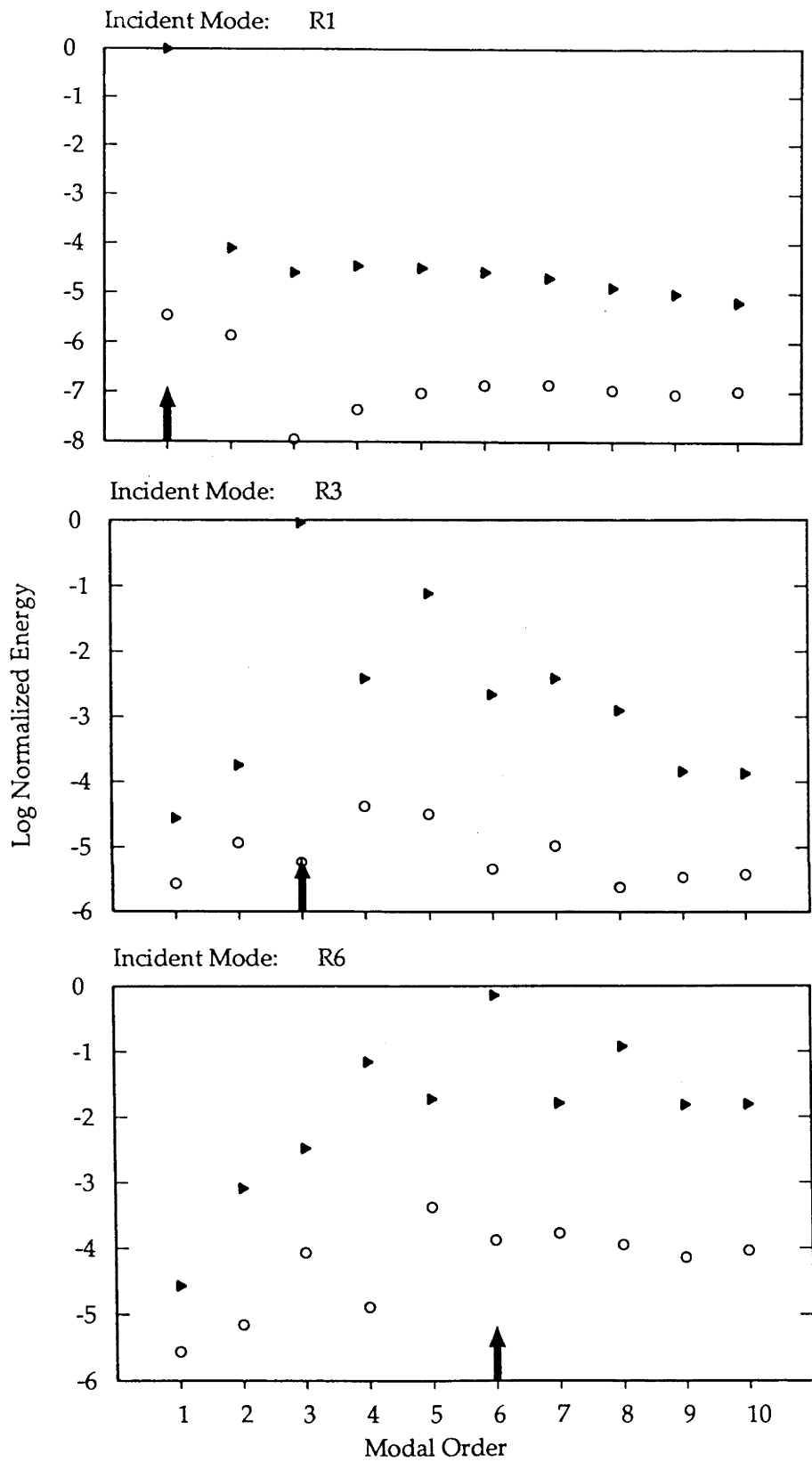


Figure 5.10. Transmitted scattered energy spectra for incident modes $R1$, $R3$ and $R6$ in model 2. Filled triangles and circles denote Rayleigh and Love modes respectively.

uration is the presence of a large negative velocity perturbation from the reference medium straddling the positive x -axis and three less significant, positive perturbations over the remainder of the angular range (see figure 5.7). The depth and radial dependencies are as reported for model 1.

Figure 5.11 shows the scattered energy spectra for the first six scattered Love and Rayleigh modes from an explosive source. In order to accomodate the additional azimuthal orders employed ($m = 12$) to describe scattering from this more complicated model we have been forced to consider, for computational reasons, a more limited group of surface wave modes than for the two previous models. Tests indicate that the effect of this lower level of truncation in modes is not particularly significant, in fact a comparison of figure 5.11 with figure 5.8 indicates that the scattered energy spectra for the first six modes are indeed very similar, the only difference being a slight reduction in energy levels which is attributable to a smaller total volume of significant heterogeneity (note that we have plotted results for $R4$ to illustrate the behaviour of higher order modes to avoid larger truncation levels which might be associated with $R6$). This observation is in accordance with the results of chapter 3 where it was found that change in the horizontal cross section of a body, for example different degrees of ellipticity, had little effect on the relative energy distribution amongst scattered modes, but a profound influence on the directional dependence of the scattered energy. In a similar fashion, the fact that the depth dependence of the perturbations in models 1 and 3 are the same means that the distribution of scattered energy among modes is very similar. Furthermore it is not surprising when we consider radiation patterns of scattered modes (figure 5.12) to find that the directional dependence of the scattered energy in the far-field bears little resemblance to that for model 1 shown in figure 5.9. Several points should be made with regard to figure 5.12. First the radiation patterns indicate that for all combinations of incident and scattered modes the amplitude of the scattered mode is strongest by a factor of at least three in the general direction of the low velocity anomaly, that is along the positive x -direction. This is not unexpected since, as noted, the absolute magnitude of the perturbations is greatest in

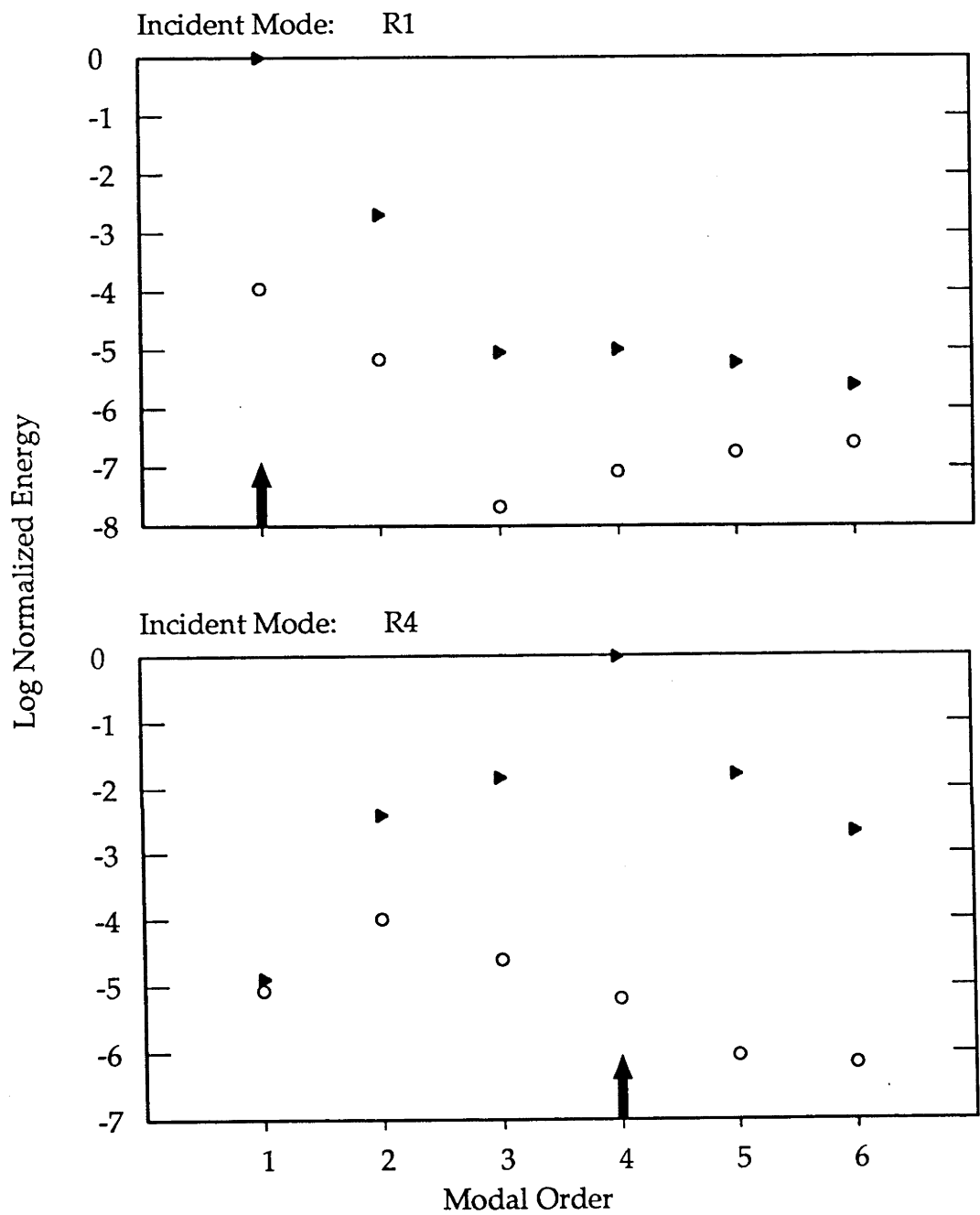


Figure 5.11. Transmitted scattered energy spectra for incident modes *R1* and *R4* in model 3. Filled triangles and circles denote Rayleigh and Love modes respectively.

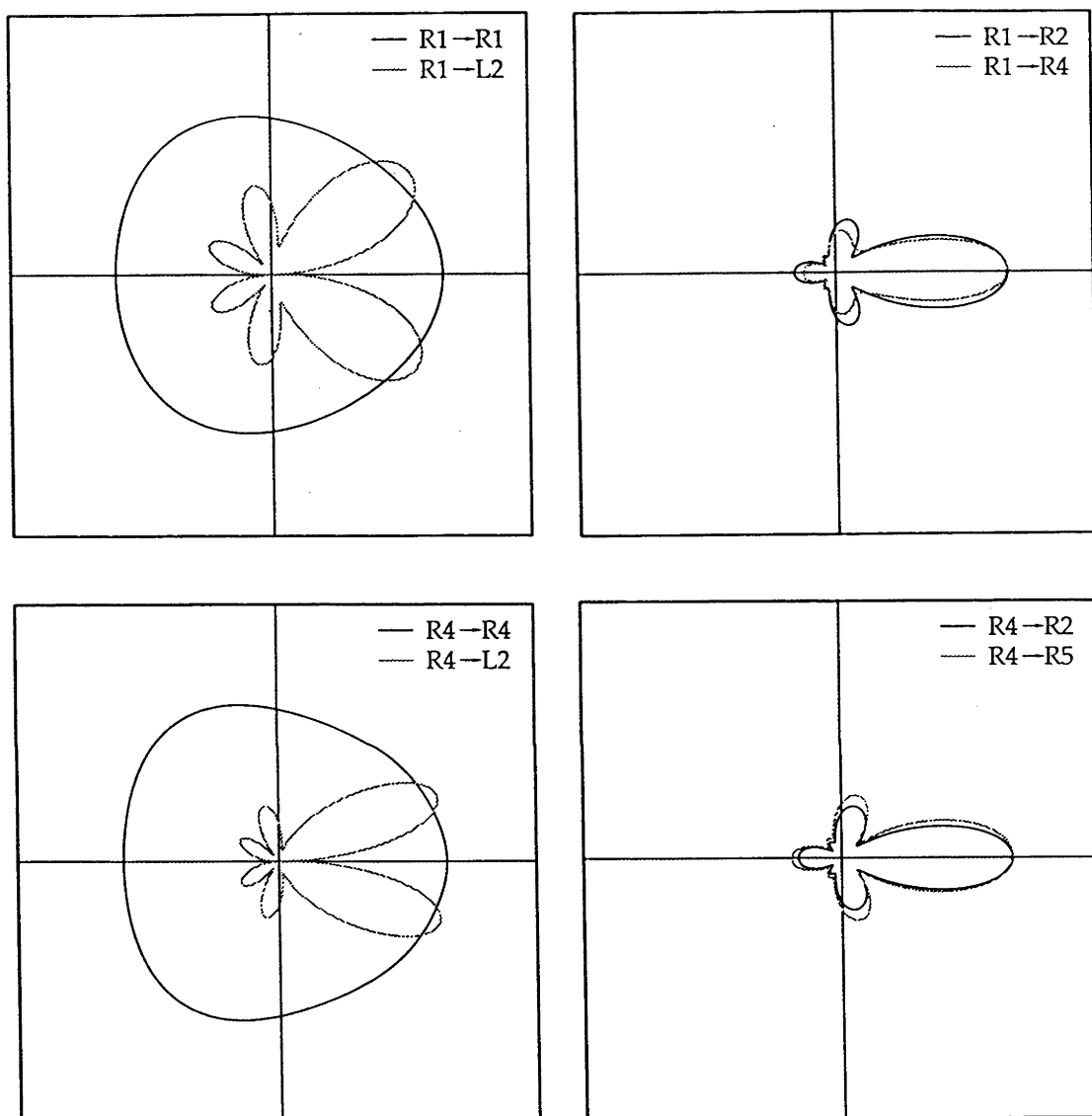


Figure 5.12. Far-field radiation patterns for a selection of scattered modes for incident modes $R1$ and $R4$ in model 3.

this direction and is rather more subdued over the remainder of the heterogeneous band. Second, the amplitude maximum for Rayleigh waves occurs over a broad lobe centered on the x -axis, whereas for Love waves the maximum is distributed symmetrically across two lobes at angles of less than $\pm 30^\circ$ to the x -axis. The angle at which the actual maximum in the transverse gradient of the perturbations occurs is somewhat greater, approximately $\pm 35^\circ$, and this discrepancy is in fact due to the effects of multiple scattering. The characteristics of the transmitted mode (incident plus unconverted scattered modes) over the window are much as expected and include increased amplitudes ($> 10\%$) and increased phase delays over the window encompassed by the low velocity heterogeneity (see figure 5.12, $R1 \rightarrow R1$, $R4 \rightarrow R4$).

5.8 DISCUSSION

In this chapter we have seen that the surface wave T-matrix formulation introduced and developed in chapters 3 and 4 can be viewed in a more general reflection/transmission context akin to that which has been successfully employed in describing plane elastic wave propagation in stratified media. The essential ingredients in our surface wave description are integral equations related to the classical representation theorems of elastodynamics, and a set of orthogonal surface wave basis functions which may be used for the expansion of any field, including the Green's function, in a horizontally stratified medium. The general nature of these requirements implies that the concepts applied here are not restricted to the surface wave problem. This is obvious when we consider that the T-matrix approach to scattering from one or more discrete obstacles was first formulated for acoustic, and later, electromagnetic and elastic cases in both 2-D and 3-D geometries. In a like manner, the present treatment of surface wave propagation through media exhibiting a continuous variation in material properties can be extended to other classes of wave propagation described by some variant of the Helmholtz equation. All that is required is a complete basis function expansion of the Green's function in the form derived in Appendix D. Such expansions are, for example, well known for both

scalar and vector waves in homogeneous media in a spherical coordinate geometry (see Morse & Feshbach, 1953) and were in fact used in the original derivations of the T-matrix (*cf.* Waterman 1969) rather than the somewhat simpler arguments made in chapter 3.

The theory has been applied to investigate the effect of near-source, continuously varying heterogeneity on the transmission of surface waves into the far-field by way of several simple models. By considering an isotropic ($m = 0$) incident wavefield emanating from an explosive source we have been able to examine the scattering of an incident Rayleigh mode into the transmitted mode, Love modes and other Rayleigh modes. For small velocity perturbations $\leq 1\%$ we observe the effects of focussing and defocussing on the incident mode as would be predicted from geometrical optics. That is, portions of the wavefield passing through low velocities regions, for example, exhibit larger amplitudes and positive phase delays relative to the incident field. This behaviour is exemplified by the fundamental Rayleigh mode in our model selection since it interacts only with weaker parts of the heterogeneity. For certain configurations of more strongly defined heterogeneity ($\pm 4\%$) significant reflection of energy may occur resulting in a trapping of energy within the near source zone and a reduction in the amplitude of the transmitted wave over certain azimuths. This effect may be related to variation in signal observed from distant events at nearby stations sited in areas of varying geologic structure (Barker *et al.*, 1981).

Scattering involving wavetype conversion (Rayleigh into Love) is a matter of some interest since regional seismograms from explosive sources frequently reveal large transverse components of displacement. As an explosive source imparts no energy to the Love component of the wavefield, Love waves must be produced through the interaction of the wavefield with heterogeneity somewhere along the propagation path (assuming that we are dealing with isotropic media). A possible candidate for producing transverse energy given an explosive source is the interaction of the wavefield with heterogeneity in the vicinity of the source because Love and Rayleigh waves are, in this regime, no longer separated by coordinate

(e.g. Love waves have both radial and transverse components of displacement) and the outgoing wavefields exhibit a singular rather than sinusoidal behaviour. In addition the interaction with heterogeneity obviously affects a larger portion of the total wavefield in this situation than for heterogeneity at greater distances from the source. The conclusion drawn from our analysis, however, is that reasonably continuous lateral variation in material properties does not contribute significantly to Rayleigh-Love coupling for the magnitude of perturbations involved ($\pm 4\%$ in velocity and density). This does not eliminate the possibility that near-source scattering may play an important role, but rather it is more likely that the form of heterogeneity examined here is less effective than others in this regard. As we have seen in chapter 4, surface wave scattering from discrete obstacles may be an effective way of converting Rayleigh energy to Love energy especially if the scatterers are small $0.01 > ka > 1.0$ and relatively densely distributed such that multiple scattering interactions become significant. The two problems are related in that we can view the first-order multiple scattered field off one obstacle as analogous to the interaction of the source field with local heterogeneity. From this point of view it seems likely that energy transfer between Rayleigh waves and Love waves is most efficient from fairly abrupt changes in material properties, i.e. discontinuous structure, oriented perpendicular to the direction of propagation of the incident wavefield. We could in theory simulate this class of heterogeneity in our perturbation scheme by specifying the heterogeneity in terms of a very large number of azimuthal contributions p , but this would in turn require an even larger number of azimuthal terms m in our truncated basis function expansions, and is impractical from a numerical standpoint. The logical direction in which to proceed would be to employ the T-matrix formulation and specify the incident field to be that of the prescribed, local explosive source. Alternatively we might place the source within a locally, laterally homogeneous cylindrical volume embedded within an otherwise laterally homogeneous half-space and use the formulation of section 5.3.

Finally, we have been restricted to modelling relatively simple forms of lateral heterogeneity in this study owing to limitations in computer hardware. As compu-

tational power increases, the method may be applied to more complicated models which accurately describe situations in the real earth.

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APPENDICES

APPENDIX A: ON THE ORTHOGONALITY OF SURFACE WAVE EIGENFUNCTIONS IN CYLINDRICAL COORDINATES

A.1 INTRODUCTION

In developing an expression for the surface wave Green's function in two dimensions using representation theorems, Herrera (1964) demonstrated that Rayleigh (and Love) wave eigenfunctions for a laterally homogeneous, plane-stratified half-space are orthogonal with respect to a particular depth integral. He noted that the contribution of a given surface wave mode to the Green's function was determined by an appropriate normalization of this integral. McGarr and Alsop (1967) employing a derivation originally used by Takeuchi *et al.* (1962) to compute group velocities, showed that for plane waves the choice of normalization was equivalent to imposing unit energy transport across a plane of unit width and infinite depth perpendicular to the direction of propagation. Herrera's argument was extended to plane waves in three dimensions by Malischewsky (1970) who thereby demonstrated Rayleigh, Love and mutual orthogonality between Love and Rayleigh waves. More recently, Aki and Richards (1980) invoked Herrera's original proof to derive surface wave terms for the two-dimensional Green's function, and recognised that with the appropriate choice of stress/displacement-related quantities the homogenous wave equation in both two and three dimensional problems can be reduced to the same system of first-order differential equations. They suggested that the development of surface wave orthogonality proceeding directly from stress and displacement expressed in a cylindrical coordinate system is not possible. The purpose of this note is to demonstrate that this development, although more complicated algebraically than for the two-dimensional case, does indeed exist and that it gives rise to both a functional orthogonality relation between Love and Rayleigh eigenfunctions and a slightly more specific expression of Rayleigh wave eigenfunction orthogonality. Although not demonstrated here, these relations can be used to derive the surface wave terms to the three-dimensional Green's function in the same fashion as Hererra (1964). In the final section we establish the physical significance of eigenfunction

normalization in a cylindrical coordinate reference frame.

A.2 BETTI'S IDENTITY AND LOVE-RAYLEIGH ORTHOGONALITY

We will begin as Herrera (1964) with Betti's identity and examine first the orthogonality between Love and Rayleigh eigenfunctions and then proceed to orthogonality between Rayleigh modes as the both derivations follow a similar argument but that in the former case is somewhat less involved. Consider any two displacement fields, $\mathbf{u}$ and $\mathbf{v}$ with harmonic time dependence $e^{-i\omega t}$ satisfying the homogeneous elastic wave equation throughout a laterally homogeneous, stratified half-space V . Betti's identity asserts that the two fields and their associated tractions ($\mathbf{t}(\mathbf{u})$, $\mathbf{t}(\mathbf{v})$) over S satisfy

$$\int_S dS [\mathbf{t}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{t}(\mathbf{v}) \cdot \mathbf{u}] = 0, \quad (\text{A.1})$$

where S is a closed surface within V . We note that (A.1) is essentially an expression of energy conservation reflecting the fact that no body forces exist within S . Now let $\mathbf{u}$ and $\mathbf{v}$ represent two individual (standing wave) Fourier-Bessel components of Rayleigh and Love waves, respectively, both of which satisfy the prescribed conditions in a cylindrical coordinate system and which we define as

$$\begin{aligned} u_r &= V_i J'_i e^{im\theta} \\ u_\theta &= V_i \frac{im}{x_i} J_i e^{im\theta} \\ u_z &= U_i J_i e^{im\theta}, \end{aligned} \quad (\text{A.2})$$

and

$$\begin{aligned} v_r &= (-1)^n W_j \frac{-in}{x_j} J'_j e^{-in\theta} \\ v_t &= -(-1)^n W_j J'_j e^{-in\theta} \\ v_z &= 0. \end{aligned} \quad (\text{A.3})$$

Here we have suppressed the harmonic time dependence $e^{-i\omega t}$ common to all physical fields and adopted the notation of Kennett (1983) where V_i , U_i , and W_j are the

eigenfunctions for the i th Rayleigh mode and j th Love mode (in the notation of Aki and Richards [1980] $W = l_1$, $V = r_1$ and $U = -r_2$). The square root of -1 is denoted by a small roman i to distinguish it from the italic index i which represents modal order. We have employed abbreviations of the following form

$$\begin{aligned} x_i &= k_i r \\ J_i &= J_m(k_i r) \\ J_j &= J_n(k_j r) \\ J'_i &= \partial_{x_i} J_i. \end{aligned} \tag{A.4}$$

We will consider these quantities and their respective tractions on a circular cylinder of radius $r = a$ centered at the origin and extending to infinite depth. Since tractions are zero at the free surface and both traction and displacement tend to zero as z approaches infinity we need only consider the components of traction over the vertical surface of the cylinder in evaluating (A.1). These are

$$\begin{aligned} t_r(\mathbf{u}) &= [2\mu k_i V_i J'_i + \lambda J_i (\partial_z U_i - k_i V_i)] e^{im\theta} \\ t_\theta(\mathbf{u}) &= 2\mu \left[k_i V_i \frac{im}{x_i} \left(J'_i - \frac{1}{x_i} J_i \right) \right] e^{im\theta} \\ t_z(\mathbf{u}) &= \mu (\partial_z V_i + k_i U_i) J'_i e^{im\theta}, \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} t_r(\mathbf{v}) &= -(-1)^n 2\mu k_j W_j \frac{in}{x_j} \left(J'_j - \frac{1}{x_j} J_j \right) e^{-in\theta} \\ t_\theta(\mathbf{v}) &= -(-1)^n \mu k_j W_j (2J''_j + J_j) e^{-in\theta} \\ t_z(\mathbf{v}) &= (-1)^n \mu \partial_z W_j \frac{in}{x_j} J_j e^{-in\theta}. \end{aligned} \tag{A.6}$$

We note in applying (A.1) over the specified geometry that the azimuthal dependence contributes a common integral factor $\int_0^{2\pi} d\theta \ e^{i(m-n)\theta}$. For $m \neq n$ this factor vanishes and Betti's identity is satisfied trivially. We therefore consider the case $m = n$ where this integral introduces a factor of 2π . After some algebraic manipu-

lation

$$\begin{aligned}
 & \int_{(r=a)} dS [\mathbf{t}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{t}(\mathbf{v}) \cdot \mathbf{u}] \\
 &= 2\pi a(-1)^m (-im) \int_0^\infty dz \left\{ V_i W_j \left[2\mu \left(\frac{k_i}{x_j} J_i'' J_j - \frac{k_j}{x_i} J_j'' J_i \right) \right. \right. \\
 & \quad \left. \left. + 2\mu \left(\frac{k_i}{x_i} J_i' J_j' - \frac{k_j}{x_j} J_j' J_i' \right) + 2\mu \left(\frac{k_j}{x_j^2} J_j J_i' - \frac{k_i}{x_i^2} J_i J_j' \right) - \mu \frac{k_j}{x_i} J_i J_j - \lambda \frac{k_i}{x_j} J_i J_j \right] \right. \\
 & \quad \left. - \mu U_i \partial_z W_j \frac{1}{x_j} J_i J_j + \lambda W_j \partial_z U_i \frac{1}{x_j} J_i J_j \right\}. \tag{A.7}
 \end{aligned}$$

Note that the terms involving the products $J_i' J_j'$ cancel and by employing Bessel's equation (e.g. $J_i'' + \frac{1}{x_i} J_i' + (1 - \frac{m^2}{x_i^2}) J_i = 0$) for the appropriate values of i and j this may be further reduced to

$$\begin{aligned}
 2\pi a(-1)^m (-im) \int_0^\infty dz \left[-(\lambda + 2\mu) V_i W_j \frac{k_i}{x_j} + \mu V_i W_j \frac{k_j}{x_i} \right. \\
 \left. + \frac{1}{x_j} (\lambda W_j \partial_z U_i - \mu U_i \partial_z W_j) \right] J_i J_j = 0. \tag{A.8}
 \end{aligned}$$

Since this expression must not be dependent on the choice of radius a we have (after multiplying by $-k_i k_j a$)

$$\int_0^\infty dz \left[(\lambda + 2\mu) V_i W_j k_i^2 - \mu V_i W_j k_j^2 + k_i (\mu U_i \partial_z W_j - \lambda W_j \partial_z U_i) \right] = 0. \tag{A.9}$$

which establishes a functional orthogonality relation between Love and Rayleigh wave eigenfunctions.

A.3 RAYLEIGH WAVE ORTHOGONALITY

In deriving the Rayleigh wave orthogonality relation in cylindrical coordinates we apply the same argument used in the previous section and consider $\mathbf{u}$ as before, but in this case choose $\mathbf{v}$ and the corresponding traction over a circular cylindrical surface to be

$$\begin{aligned}
 v_r &= (-1)^m V_j J_j' e^{-im\theta} \\
 v_\theta &= (-1)^m V_j \frac{-im}{x_j} J_j e^{-im\theta} \\
 v_z &= (-1)^m U_j J_j e^{-im\theta},
 \end{aligned} \tag{A.10}$$

and

$$t_r(\mathbf{v}) = (-1)^m 2\mu k_j V_j J_j'' + \lambda J_j (\partial_z U_j - k_j V_j) e^{-im\theta}$$

$$t_\theta(\mathbf{v}) = -(-1)^m 2\mu \left[k_j V_j \frac{im}{x_j} \left(J'_j - \frac{1}{x_j} J_j \right) \right] e^{-im\theta} \quad (A.11)$$

$$t_z(\mathbf{v}) = (-1)^m \mu [\partial_z V_j + k_j U_j] J'_j e^{-im\theta},$$

Note that satisfaction of Betti's identity for two Fourier-Bessel components of different azimuthal order $m \neq n$ will follow trivially as in the previous section. We recognize that the integrals over the horizontal surfaces ($z = 0, z \rightarrow \infty$) in (A.1) vanish so that we may write (A.1) as

$$\begin{aligned} & \int_{(r=a)} dS [\mathbf{t}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{t}(\mathbf{v}) \cdot \mathbf{u}] = 0 \\ & = 2\pi a (-1)^m \int_0^\infty dz \left[2\mu V_i V_j \left(k_i J''_i J'_j - k_j J''_j J'_i \right) - \lambda V_i V_j \left(k_i J_i J'_j - k_j J_j J'_i \right) \right. \\ & \quad + \lambda \left(V_j \partial_z U_i J_i J'_j - V_i \partial_z U_j J_j J'_i \right) + \mu U_i U_j \left(k_i J'_i J_j - k_j J'_j J_i \right) \\ & \quad + \mu \left(U_j \partial_z V_i J'_i J_j - U_i \partial_z V_j J'_j J_i \right) + \frac{m^2}{x_i x_j} 2\mu V_i V_j \left(k_i J'_i J_j - k_j J'_j J_i \right) \\ & \quad \left. - \frac{m^2}{x_i x_j} 2\mu V_i V_j \left(\frac{k_i}{x_i} J_i J_j - \frac{k_j}{x_j} J_i J_j \right) \right], \end{aligned} \quad (A.12)$$

where the abbreviated variables in (A.12) are evaluated at $r = a$ after differentiation. The first term involving second derivatives of Bessel functions can be expressed as two separate terms using Bessel's equation, both containing Bessel functions and their first derivatives and one of which exhibits dependence on azimuthal order. After some algebraic manipulation, those terms in (A.12) depending on azimuthal order can be made to cancel yielding

$$\begin{aligned} & 2\pi a (-1)^m \int_0^\infty dz \left[(\lambda + 2\mu) V_i V_j \left(k_j J_j J'_i - k_i J'_j J_i \right) \right. \\ & \quad + \mu U_i U_j \left(k_i J_j J'_i - k_j J'_j J_i \right) - \lambda \left(V_i \partial_z U_j J_j J'_i - V_j \partial_z U_i J_i J'_j \right) \\ & \quad \left. + \mu \left(U_j \partial_z V_i J_j J'_i - U_i \partial_z V_j J'_j J_i \right) \right]. \end{aligned}$$

All terms exhibit a somewhat similar structure with regard to horizontal dependence and we can rearrange (A.12) such that the depth dependence on either side of the equation is in the form of a single factor

$$\begin{aligned} & \int_0^\infty dz [(\lambda + 2\mu) V_i V_j k_j + \mu U_i U_j k_i - \lambda V_i \partial_z U_j + \mu U_j \partial_z V_i] (J_m(k_j a) J'_m(k_i a)) \\ & = \int_0^\infty dz [(\lambda + 2\mu) V_i V_j k_i + \mu U_i U_j k_j - \lambda V_j \partial_z U_i + \mu U_i \partial_z V_j] \\ & \quad \times (J_m(k_i a) J'_m(k_j a)), \end{aligned} \quad (A.13)$$

where we have shown the horizontal dependence explicitly. Since the horizontal and vertical dependencies have been isolated we may write this as

$$F(k_i, k_j)X_1(a, k_i, k_j) = F(k_j, k_i)X_2(a, k_i, k_j). \quad (A.14)$$

If we fix k_i, k_j with $i \neq j$ this is equivalent to $C_1X_1(a) - C_2X_2(a) = 0$ where C_1 and C_2 are constants. This equation will hold in two distinct situations: i) X_1 and X_2 are linearly dependent functions and hence C_1 and C_2 may be non-zero; or ii) X_1 and X_2 are linearly independent functions so that we require $C_1 = C_2 = 0$. To investigate the relationship between X_1 and X_2 we note that the result in (13) must be independent of the choice of radius a . We may therefore simplify our task by choosing a non-zero such that k_ia is a zero of $J_m(x)$. Equation (A.14) thereby reduces to

$$F(k_i, k_j)X_1(a) = 0. \quad (A.15)$$

In this case $J'_m(k_ia)$ will not be zero and since equation (A.1) must hold for all permissible values of i, j and is independent of the choice of earth model which governs the values of k_i, k_j ; $X_1(a)$ will not, in general, be zero. Therefore we must have

$$\int_0^\infty dz [(\lambda + 2\mu)V_iV_jk_j + \mu U_iU_jk_i - \lambda V_i\partial_z U_j + \mu U_j\partial_z V_i] = 0. \quad (A.16)$$

Also note that the standard expression of Rayleigh wave eigenfunction orthogonality may be retrieved by simply adding (or subtracting) the expression in (A.16) to itself with the indices i, j interchanged.

A.4 NORMALIZATION OF SURFACE WAVE EIGENFUNCTIONS IN CYLINDRICAL COORDINATES

We now wish to investigate the physical significance of the normalization of eigenfunctions in a cylindrical geometry. Again consider a right circular cylinder of finite radius a extending from the surface to infinite depth, and centered about the z -axis of a cylindrical coordinate system in a plane-stratified medium. We will examine the energy flux Σ for a single outgoing Fourier-Bessel component of the

Rayleigh wavefield (modal wavenumber k_i). The energy flux due to the real part of the displacement field through the surface S of the cylinder and averaged over a single period, is given by :

$$\frac{1}{T} \int_0^T dt \int_S dS \Sigma_i n_i = \frac{i\omega}{4} \int_S dS [\mathbf{t}^*(\mathbf{u}) \cdot \mathbf{u} - \mathbf{t}(\mathbf{u}) \cdot \mathbf{u}^*], \quad (\text{A.17})$$

where n_i is the unit normal to the surface S , and $*$ denotes complex conjugation. Since traction vanishes at the free surface and displacement tends to 0 as $z \rightarrow \infty$ we need only consider the flux over the vertical surface ($r = a$). The quantities of interest then are:

$$\begin{aligned} u_r &= V_i H_i^{(1)'} e^{i(m\theta - \omega t)} \\ u_\theta &= V_i \frac{im}{x_i} H_i^{(1)} e^{i(m\theta - \omega t)} \\ u_z &= U_i H_i^{(1)} e^{i(m\theta - \omega t)} \\ t_r &= 2\mu k_i V_i H_i^{(1)''} + \lambda (\partial_z U_i - k_i V_i) H_i^{(1)} e^{i(m\theta - \omega t)} \\ t_\theta &= 2\mu k_i V_i \frac{im}{x_i} \left(H_i^{(1)'} - \frac{H_i^{(1)}}{x_i} \right) e^{i(m\theta - \omega t)} \\ t_z &= \mu (\partial_z V_i + k_i U_i) H_i^{(1)'} e^{i(m\theta - \omega t)}, \end{aligned} \quad (\text{A.18})$$

where we recall that $x = k_i a$, a prime denotes differentiation by x_i , and following our previous convention, $H_i^{(1)} = H_m^{(1)}(k_i a)$. We will consider each of the individual components in (A.18) separately noting that $H_i^{(1)*} = H_i^{(2)}$:

$$\begin{aligned} t_r^* u_r - t_r u_r^* &= 2\mu k_i V_i^2 \left\{ H_i^{(1)'} H_i^{(2)''} - H_i^{(2)'} H_i^{(1)''} \right\} - \lambda V_i \partial_z U_i \left\{ H_i^{(1)} H_i^{(2)'} - H_i^{(2)} H_i^{(1)'} \right\} \\ &\quad + \lambda k_i V_i^2 \left\{ H_i^{(1)} H_i^{(2)'} - H_i^{(2)} H_i^{(1)'} \right\} \\ t_\theta^* u_\theta - t_\theta u_\theta^* &= 2\mu k_i V_i^2 \frac{m^2}{x_i^2} \left\{ H_i^{(1)} H_i^{(2)'} - H_i^{(2)} H_i^{(1)'} \right\} \end{aligned} \quad (\text{A.19})$$

$$t_z^* u_z - t_z u_z^* = \mu \left[U_i \partial_z V_i + k_i U_i^2 \right] \left\{ H_i^{(1)} H_i^{(2)'} - H_i^{(2)} H_i^{(1)'} \right\}.$$

The quantities defining horizontal dependence in the expressions above are Wronskians $\mathcal{W}$ of the first and second Hankel functions and their first derivatives:

$$\begin{aligned} \mathcal{W}(H_i^{(1)}, H_i^{(2)}) &= \left\{ H_i^{(1)} H_i^{(2)'} - H_i^{(2)} H_i^{(1)'} \right\} \\ &= \frac{-4i}{\pi k_i a}, \end{aligned} \quad (\text{A.20})$$

$$\begin{aligned}
\mathcal{W}(H_i^{(1)'}, H_i^{(2)'}) &= \left\{ H_i^{(1)'} H_i^{(2)''} - H_i^{(2)'} H_i^{(1)''} \right\} \\
&= \mathcal{W}(H_i^{(1)}, H_i^{(2)}) \left\{ 1 - \frac{m^2}{x_i^2} \right\} \\
&= \frac{-4i}{\pi k_i a} \left(1 - \frac{m^2}{x_i^2} \right).
\end{aligned} \tag{A.21}$$

The total energy flux from (A.17) then is

$$\begin{aligned}
\frac{i\omega}{4} \int_0^{2\pi} a d\theta \int_0^\infty dz \mathcal{W} \left\{ [(\lambda + 2\mu)k_i V_i^2 + \mu k_i U_i^2 - \lambda V_i \partial_z U_i + \mu U_i \partial_z V_i] \right. \\
\left. + (2\mu - 2\mu)k_i V_i^2 \frac{m^2}{x_i^2} \right\} \\
= \frac{\omega}{k_i} \int_0^\infty dz 2 \left[(\lambda + 2\mu)k_i V_i^2 + \mu k_i U_i^2 - \lambda V_i \partial_z U_i + \mu U_i \partial_z V_i \right].
\end{aligned} \tag{A.22}$$

A similar result holds for the j th Love mode, namely:

$$\frac{i\omega}{4} \int_S dS [\mathbf{t}^*(\mathbf{u}) \cdot \mathbf{u} - \mathbf{t}(\mathbf{u}) \cdot \mathbf{u}^*] = \frac{\omega}{k_j} \int_0^\infty dz 2k_j [\mu W_j^2]. \tag{A.23}$$

In both (A.22) and (A.23) the term on the right hand side is the modal orthogonality relation for waves of similar wavenumber at a given frequency. If we normalize the integrals to the phase slowness (k_i/ω , for Rayleigh, k_j/ω for Love), an outgoing Fourier-Bessel component transports unit energy across the circular cylindrical surface S . Note that this result reflects the fact that we have an effective energy source at the origin arising from the imaginary part of the Hankel function and since we are considering loss-less media, this choice of normalization implies that each outgoing Rayleigh (Love) component carries unit energy out of any body which wholly encompasses the z -axis.

APPENDIX B: NUMERICAL IMPLEMENTATION OF THE TRANSLATION OPERATOR

In our treatment of surface wave scattering from two obstacles we found it convenient to consider three coordinate reference frames, two of these had origins located within the obstacles while a third served as a global reference for the composite T-matrix. It was necessary to employ standing wave translation operators (expressed as A-matrices) to translate the incident wave from the global origin to the origin of either scatterer, and then again to translate the scattered waves associated with each obstacle back to the common global origin. The A-matrix is strictly speaking an infinite quantity (in azimuthal order) and in any numerical implementation must be truncated at some finite maximum dimension M . This can create problems since the elements of the A- and T-matrices interact in such a way that the required M is a function of both the separation distance d and the maximum radius of both obstacles. An alternative approach outlined by Peterson and Ström (1973) is to recognize the A-matrix as simply an expression of a more general translation operator. We can take advantage of this fact when we consider our incident displacement field to be a plane wave. To illustrate the simplification that arises we consider a scalar plane wave which in terms of a set of scalar cylindrical wave functions may be written as

$$\exp(i\mathbf{k} \cdot \mathbf{r}) = \sum_m a^m \hat{Y}^m(kr), \quad (B.1)$$

where

$$a^m = \epsilon_m i^m \begin{pmatrix} \cos m\beta \\ \sin m\beta \end{pmatrix}, \quad (B.2)$$

and β is the angle of the wave vector with respect to the x -axis. Here r is the position vector of a point P relative to the global origin O and we will take r_1 to be the position vector of P relative to an origin at O_1 . As before O and O_1 are separated by the vector d_1 such that $\mathbf{r} = \mathbf{d}_1 + \mathbf{r}_1$. Using the scalar A-matrix of (4.18) we could express (B.1) as

$$\begin{aligned} \sum_m a^m \hat{Y}^m(kr) &= \sum_m a^m \sum_p A^{mp}(d_1) \hat{Y}^p(kr_1) \\ &= \sum_p \left[\sum_m a^m A^{mp}(d_1) \right] \hat{Y}^p(kr_1), \end{aligned} \quad (B.3)$$

Alternatively we may write

$$\begin{aligned}\exp(\mathbf{ik} \cdot \mathbf{r}) &= \exp(\mathbf{ik} \cdot \mathbf{d}_1) \exp(\mathbf{ik} \cdot \mathbf{r}_1) \\ &= \exp(\mathbf{ik} \cdot \mathbf{d}_1) \sum_p a^p \hat{Y}^p(kr_1).\end{aligned}\tag{B.4}$$

Because of the orthogonality of the wave functions $\hat{Y}^p(kr_1)$ we must have

$$\sum_m a^m A^{mp}(\mathbf{d}_1) = \exp(\mathbf{ik} \cdot \mathbf{d}_1) a^p,\tag{B.5}$$

where we have reduced the operation to a single scalar multiplication which is exact while the truncated summation is not.

We can use the result for scalar wave functions and write the translation operator for the incident plane surface wave as a diagonal matrix $\mathbf{D}(\mathbf{d}_1)$ whose non-zero elements are defined as

$$D^{\sigma\sigma}(\mathbf{d}_1) = \exp(\mathbf{ik}_n^l \cdot \mathbf{d}_1),\tag{B.6}$$

where $\mathbf{k}_n^l$ is the wavenumber for the appropriate surface wave partition. This approach is computationally far more efficient than use of the more general implementation of the translation operator through $A^{\sigma\nu}(\mathbf{d}_1)$ (c.f. equation (4.21)). However because the translation must be incorporated within expression in (4.34), the resultant T-matrix is now specific to the incident wave and no longer dependent solely upon the physical characteristics of the scattering obstacle.

The translation of the scattered fields to the global coordinate frame from origins within the respective obstacles can be approached in a similar manner. In this case we exploit the asymptotic behaviour of the scattered wavefield in the far-field regime. Again consider a scalar problem where we can approximate an outgoing wave in the far-field relative to origin O_1 as (c.f. equation (4.4))

$$\sum_m c^m Y^m(kr_1) \approx \sum_m \frac{c^m G^m(\theta)}{\sqrt{r_1}} \exp(\mathbf{ikn}_r \cdot \mathbf{r}_1),\tag{B.7}$$

where $\mathbf{n}_r$ is a unit vector pointing outwards from the origin O_1 and $G^m(\theta)$ describes the azimuthal dependence. Using the same approach as above it is easily shown that in the far-field translation to the global coordinate origin O may be written as

$$\sum_m c^m Y^m(kr_1) \approx \exp(-\mathbf{ikn}_r \cdot \mathbf{d}_1) \sum_m \frac{c^m G^m(\theta)}{\sqrt{r}} \exp(\mathbf{ikn}_r \cdot \mathbf{r}),\tag{B.8}$$

where the action of translation is represented by the factor $\exp(-ik\vec{u}_r \cdot \mathbf{d}_1)$ and the quantity under summation incorporates the asymptotic form of the outgoing wave functions $Y^m(kr)$ in the far-field for an origin at O .

APPENDIX C: BASIS FUNCTION COMPLETENESS AND NUMERICAL CONSIDERATIONS

The T-matrix formulation in principle provides an exact description of the scattered field from one or more obstacles for any incident harmonic wave. The validity of this formulation is however based on the assumption that we have basis sets which are complete in the sense that the physical fields which exist in both the embedding medium and heterogeneity are properly represented by a linear combination of the respective basis functions. Thus, for example, we must exercise caution in applying our surface wave T-matrix formulation to objects exhibiting highly curved or sinuous boundaries since the representation of the scattered field solely in terms of outgoing waves may prove to be inadequate.

In any numerical implementation we are faced with the practical problem of truncation and hence that, in general, we can not obtain ‘complete’ sets of basis functions. Our objective then is to select basis sets which accurately describe the physical fields in as few terms as possible and moreover that the degree of accuracy is comparable for the internal, incident and scattered fields at the particular level of truncation. In the case of surface waves we must assemble basis functions which efficiently represent both the horizontal and vertical dependencies of the wavefield. The vertical dependence is naturally described in terms of the surface wave eigenfunctions and the form which they take depends on how we wish to model the earth (see Malischewsky, 1987). One possibility is to adopt a locked mode approximation (Harvey, 1981) and consider the crust and upper mantle as a strictly conservative waveguide by introducing a perfect reflector at some great depth. In this case we can describe the entire wavefield with an infinite set of surface wave modes. Body wave contributions are then represented through the constructive interference of higher order modes.

In our numerical study we have chosen to model the crust and upper mantle perhaps more realistically with the lowest layer as a homogeneous half-space. For a complete representation of the wavefield we would have to incorporate both the finite set of normal modes represented as poles in the wavenumber domain and the continuous wavenumber

spectrum which accounts for body wave propagation. Although the scattered field from a plane surface wave mode incident upon a single discrete obstacle will undoubtedly contain some body wave component, it is reasonable, for the models adopted here, to assume that the matching of boundary conditions (performed indirectly through the T-matrix formulation) can be accomplished primarily through modal contributions and that errors associated with body waves will be small. Indeed for many purposes we are not particularly concerned with this energy as our interests lie in the far-field. We recall that the continuous spectrum for a given frequency encompasses wavenumbers all of which are smaller in magnitude than the highest order or cut-off mode. Hence we can think of the continuous spectrum as energy radiated into the half-space at subcritical angles or equivalently along steeply inclined rays. Over a given distance each body wave ray will undergo a comparatively large number reflections from the bottom-most interface with each bounce resulting in a considerable loss of energy. Hence at even moderate distances from a scatterer we expect any minor body wave component to have all but disappeared.

One situation however where it is likely that the omission of body wave contributions may be significant arises when considering the scattered field from two point scatterers. Multiple scattering interactions become important at separations equivalent to fractions of a wavelength, for example 0.05 wavelengths for two obstacles of radius $ka = 0.1$ and waves at 1.0Hz. At these distances scattered body waves propagating steeply can be rescattered by the second obstacle without having lost energy through radiation into the half space. One might argue that body wave contributions should still be overshadowed by scattered surface waves, however the numerical problem lies in the proximity of the two multipole expansions representing the scattered fields. We thereby face the prospect of having small errors in the energy distribution across the modal spectrum (which result from the omission of body waves in our truncated T-matrix) amplified through the representation of fields exhibiting near-singular behaviour in terms of the regular basis functions for a different reference frame.

To illustrate in more detail how this can occur we note that each point scatterer lies within the distance range at which the scattered field of the other obstacle, represented

as an expansion in multipoles (i.e. through the Hankel functions in (4.1)), exhibits a singular behaviour. The magnitude of the singularity increases with azimuthal order m . For surface wave displacements this singularity is of the order $1/(kr)$ for $m = 0$, $1/(kr)^2$ for $m = 1$, $1/(kr)^3$ for $m = 2$. Thus the field is changing very rapidly over small distances especially for the $m = 2$ multipole field. As a result the expansion of the primary scattered field (i.e. the field due only to the interaction with the primary incident field) from one obstacle in terms of the regular surface wave basis functions at the origin of the second obstacle involves coefficients which increase very rapidly with azimuthal order p (where p indicates azimuthal orders referred to the expansion about the second origin). In addition a particular coefficient p for a $m = 2$ field, for example, is proportionately greater than that for a $m = 1$ field at very close separations. The number of p terms we need to consider in the expansion depends on the distance from the second obstacle's origin at which the field is to be evaluated and this enters implicitly through the construction of the T-matrix elements for that obstacle. As a result the increase in magnitude of the B-matrix elements (the quantity which describes propagation between obstacles, *cf.* equation (4.22)) for $m, p > 2$ is compensated for by a greater rate of decrease in the the corresponding elements of the T-matrix hence these orders are not required in the wavefield description. However the surface wave T-matrix elements also incorporate the effects of a truncated (and hence incomplete) set of functions describing the depth dependence. This may upset the delicate compensation described above especially for scattered Love waves as they are amplified considerably over Rayleigh waves as a result of the association of a higher order ($m = 2$ versus $m = 0, 1$) dominant multipole representation.

APPENDIX D: A BASIS FUNCTION EXPANSION FOR THE SURFACE WAVE GREEN'S FUNCTION

Our objective in this appendix is to derive an expression for the surface wave Green's function in a laterally homogeneous, stratified medium having as its top boundary the free surface and as its bottom layer a perfect reflector. We find it convenient to consider a fictitious cylindrical volume as shown in figure D.1. This volume is defined by i) a horizontal upper boundary comprising an area over the free surface over which traction must be zero, ii) a horizontal lower boundary coinciding with an area of the perfect reflector at which displacement is zero, and iii) a closed vertical surface S_0 in the form of a right circular cylinder of constant radius R enclosing the z -axis of a cylindrical coordinate reference frame.

The Green's function $G_p(\mathbf{r}|\mathbf{r}')$ for a stratified medium is by definition the response of the medium to a point source in the p direction located at $\mathbf{r}'$, or $\delta_{ip}\delta(\mathbf{r}-\mathbf{r}')$. We will make use of the representation theorems of classical elastodynamics (*cf.* Aki & Richards, 1980; sections 2.5 & 7.4.1) which allow us to represent any function $\mathbf{u}(\mathbf{r})$ which is continuous and contains no sources within a closed surface, for example S_0 , as

$$u_p(r', \theta', z') = \int_{S_0} dS \mathbf{t}(\mathbf{u}(\mathbf{r})) \cdot \mathbf{G}_p(\mathbf{r}|\mathbf{r}') - \mathbf{t}(\mathbf{G}_p(\mathbf{r}|\mathbf{r}')) \cdot \mathbf{u}(\mathbf{r}), \quad (D.1)$$

for any $\mathbf{r}'$ within S_0 . Now consider the task at hand; specifically, we wish to expand the Green's function in terms of our surface wave basis functions. Given the assumptions we have made on the nature of the waveguide these basis functions form a complete orthogonal set and may be used in the expansion of any displacement wavefield. In the first instance we are interested in the form which the Green's function expansion takes when the horizontal distance from the origin to the point of observation $|\mathbf{r}|$ is greater than that from the origin to the source of the Green's function displacement field $|\mathbf{r}'|$. Assume then that $\mathbf{r}$ lies on the surface S_0 which in turn encloses the point source $\delta_{ip}\delta(\mathbf{r}-\mathbf{r}')$. The boundary conditions require that the Green's function exhibit an outgoing character everywhere on S_0 (relative to the designated origin) and in particular at $\mathbf{r}$. Hence, $\mathbf{G}(\mathbf{r}|\mathbf{r}')$ is expanded in terms

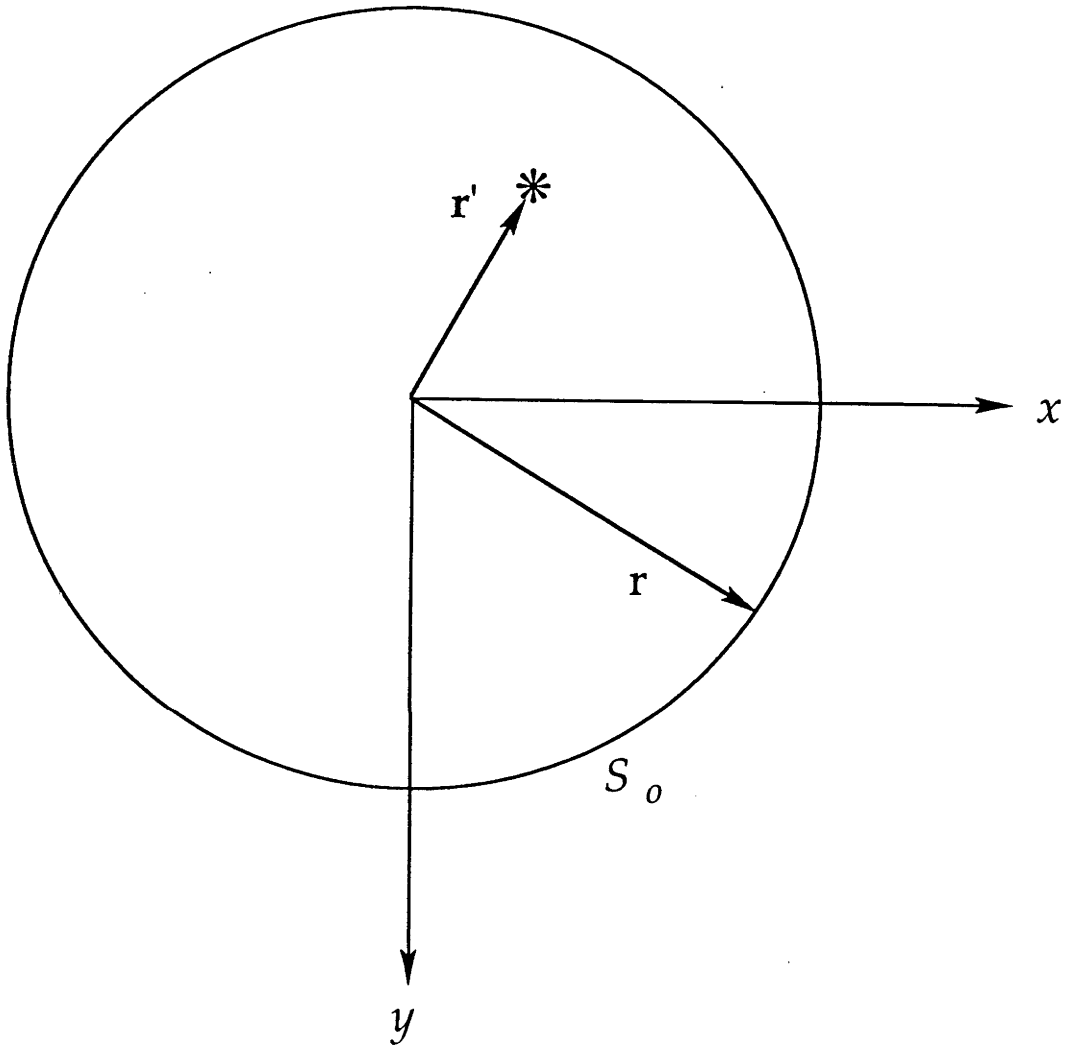


Figure D.1. Geometry used to derive an expression for the surface wave Green's function. In the first case we consider a unit amplitude point source located at $\mathbf{r}'$ within the volume encompassed by the surface S_o and examine the character of the field at a point $\mathbf{r}$ on this surface. We then repeat the procedure for a point source outside S_o to complete the derivation.

of the outgoing basis functions, or

$$\mathbf{G}_p(\mathbf{r}|\mathbf{r}') = \sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \psi^{\sigma}(\mathbf{r}). \quad (D.2)$$

We write the expansion coefficients as $b_p^{\sigma}(\mathbf{r}')$ since these will depend upon the orientation of the point source and its position. Furthermore we will take $\mathbf{u}(\mathbf{r})$ to be one of the regular (standing wave) basis functions, $\hat{\psi}^{\nu}(\mathbf{r})$ which as we note is regular and continuously differentiable within S_0 . This allows us to exploit the orthogonality relations derived in Appendix A and write (D.1) as

$$\begin{aligned} \hat{\psi}_p^{\nu}(\mathbf{r}', \theta', z') &= \int_{S_0} dS \mathbf{t}(\hat{\psi}^{\nu}(\mathbf{r})) \cdot \sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \psi^{\sigma}(\mathbf{r}) - \mathbf{t}(\sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \psi^{\sigma}(\mathbf{r})) \cdot \hat{\psi}^{\nu}(\mathbf{r}) \\ &= \sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \int_{S_0} dS \mathbf{t}(\hat{\psi}^{\nu}(\mathbf{r})) \cdot \psi^{\sigma}(\mathbf{r}) - \mathbf{t}(\psi^{\sigma}(\mathbf{r})) \cdot \hat{\psi}^{\nu}(\mathbf{r}) \\ &= -\frac{2i}{\omega} b_p^{\nu}(\mathbf{r}'). \end{aligned} \quad (D.3)$$

Note that this argument holds for all $\mathbf{r}'$ such that $r' \leq R$ where, recall, R is the radius of S_0 . Hence, in the case that we are evaluating the Green's function at some point $\mathbf{r}$ where $r > r'$ it is appropriate to expand $\mathbf{G}(\mathbf{r}|\mathbf{r}')$ as

$$\mathbf{G}_p(\mathbf{r}|\mathbf{r}') = -\frac{\omega}{2i} \sum_{\sigma} \hat{\psi}_p^{\sigma}(\mathbf{r}') \psi^{\sigma}(\mathbf{r}). \quad (D.4)$$

In the case that the point source lies outside the interior volume enclosed by S_0 we may still employ the representation theorem (D.1) but now we consider the exterior volume which extends from S_0 out towards infinity. The Green's function $\mathbf{G}(\mathbf{r}|\mathbf{r}')$ is required to be continuous throughout the interior volume, in particular along the z -coordinate axis and is therefore expanded in the regular surface wave basis functions

$$\mathbf{G}_p(\mathbf{r}|\mathbf{r}') = \sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \hat{\psi}^{\sigma}(\mathbf{r}). \quad (D.5)$$

We take as our choice for $\mathbf{u}(\mathbf{r})$ a single member of the outgoing surface wave basis function set, $\psi^{\nu}(\mathbf{r})$, which satisfies the prescription that $\mathbf{u}(\mathbf{r})$ have no sources within the exterior volume (i.e. so that we may apply (D.1)). Thus we see

$$\begin{aligned} \psi_p^{\nu}(\mathbf{r}', \theta', z') &= - \int_{S_0} dS \mathbf{t}(\psi^{\nu}(\mathbf{r})) \cdot \sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \hat{\psi}^{\sigma}(\mathbf{r}) - \mathbf{t}(\sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \hat{\psi}^{\sigma}(\mathbf{r})) \cdot \psi^{\nu}(\mathbf{r}) \\ &= - \sum_{\sigma} b_p^{\sigma}(\mathbf{r}') \int_{S_0} dS \mathbf{t}(\psi^{\nu}(\mathbf{r})) \cdot \hat{\psi}^{\sigma}(\mathbf{r}) - \mathbf{t}(\hat{\psi}^{\sigma}(\mathbf{r})) \cdot \psi^{\nu}(\mathbf{r}) \\ &= -\frac{2i}{\omega} b_p^{\nu}(\mathbf{r}'). \end{aligned} \quad (D.6)$$

A negative sign has been applied to the surface integral since the surface normal implicit in the definition of traction now points inward. Combining this result with (D.4) allows us to write

$$\mathcal{G}(\mathbf{r}|\mathbf{r}') = -\frac{\omega}{2i} \sum_{\sigma} \hat{\psi}^{\sigma}(\mathbf{r}_{<}) \psi^{\sigma}(\mathbf{r}_{>}). \quad (D.7)$$

Here we have omitted the index p indicating source direction and written the Green's function as a second rank tensor $\mathcal{G}(\mathbf{r}|\mathbf{r}')$, and we adopt a notation where $r_{>}$, $r_{<}$ are respectively the greater and lesser of r , r' . Note that there is no dot product implied in (D.7). This relation is analogous to expansions of the Green's function for a homogeneous medium in scalar and vector harmonics for the 2-D (cylindrical) and 3-D (spherical) cases (*cf.* Morse & Feshbach, 1953; Boström *et al.*, 1981) and, as in these cases, is uniformly convergent for $r \neq r'$.