Computational Complexity of Electrical Power System Problems

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Except where otherwise indicated, this thesis is my own original work.

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Abstract

The study of the computational complexity of real-world applications, although theoretical, can provide many pragmatic outcomes. For example, demonstrating that some types of algorithms cannot exist to solve the problem; the creation of challenging benchmark examples; and new insights into the underling structure and properties of the problem. In this thesis, we study the computational complexity of several important problems in the application of electrical power systems.

Knowledge of the current state of the power system is important for power network operators. This helps, for example, to predict if the network is trending towards an undesirable state of operation, or if a power line is working at its operational limits. The state of a power system is determined by the demand, the generation and the bus voltage magnitudes and phase angles. The demand of loads can be reliably estimated via forecasts, historic records and/or measurements and the operators of generators report the generation values. Given generation and demand values, the voltage magnitudes and phase angles can be computed. This is what is called the POWER FLOW (PF) problem. Cost for generating power often varies from generator to generator. In the OPTIMAL POWER FLOW (OPF) problem, the aim is to find the cheapest generation dispatch, such that the forecast demand can be satisfied. Disasters, such as storms or floods, and operator errors have to potential to destroy parts of the network. This can make it impossible to satisfy all the demand. In the MAXIMUM POWER FLOW (MPF) problem, the aim is to find a generation dispatch that can satisfy as much demand as possible.

In this thesis, we provide the proofs that the MPF, OPF and the PF problem are NP-hard for: radial networks in the Alternating Current (AC) power flow model and planar networks in the Linear AC Approximation (DC) power flow model with line switching. Furthermore, we show that there does not exist a polynomial approximation algorithm for the OPF problem in any of these settings. We also study the complexity of the Lossless-Sin AC Approximation (SIN) power flow model, showing that the MPF and OPF problem are strongly NP-hard for planar networks.

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Chapter 1

Introduction

An electrical *power system* is a network of power *generators* and *loads*, typically defined on nodes, and transmission *lines* naturally defined on edges. Many interesting questions arise from such a network. For example, how vulnerable is the network to line disconnection, how is the *flow* influenced by the existence of specific edges, and how can we use the knowledge of some part of the network to infer or determine the state of the other parts, specifically those parts that can change, i.e. variables.

Many interesting computational problems in electrical power systems are about determining the values of bus voltages of the system. This is called the POWER FLOW (PF) problem and was introduced by Ward and Hale [1956]. Other computational problems in power systems optimize an objective function, e.g. the optimal generation dispatch problem also called OPTIMAL POWER FLOW (OPF) introduced by Carpentier [1962] and the MAXIMUM POWER FLOW (MPF) as discussed by Adibi [2000]. A variety of optimization applications in power systems also involve adding or removing lines in a power network. These include transmission extension planning Hobbs [1995]; Bent et al. [2010], vulnerability analysis Alsac and Stott [1974]; Bienstock and Verma [2010], and power restoration Yolcu et al. [1983]. The switching of lines may also help to improve "optimal" solutions to the OPF and MPF problems, e.g. Fisher et al. [2008]; Van Hentenryck et al. [2011].

Researchers and engineers around the world are faced with finding algorithms to solve these power system related computational problems. The theory of computational complexity enables these scientist to classify computational problems according to their inherent difficulty. The classes to which a problem belongs to define the types of algorithms that can solve the problem. Conversely, by showing that a problem is not included in a class, we are able to rule out the existence of the corresponding types of algorithms. For example, if a problem is NP-hard, we can rule out the existence of a polynomial time algorithm unless $\mathsf{P} = \mathsf{NP}$. Furthermore, the instances of the problem obtained by the reduction can serve as "hard to solve" test cases.

The flow of power is given by the *Alternating Current* (AC) power flow equations. These describe a non-convex solution set. Furthermore, Klos and Kerner [1975]; Klos and Wojcicka [1991] and Iba et al. [1990] showed that the PF problem can have multiple solutions and Bukhsh et al. [2013] provided examples illustrating that the OPF problem can have locally optimal solutions. This is why computational problems using the AC power flow equations are perceived to be NP-hard[1]. The challenge to solve these problems sparked a lot of research, especially for the OPF problem, see e.g. Alsac et al. [1990]; Huneault et al. [1991]; Momoh et al. [1999a,b]; Baldick [2006]; Pandya and Joshi [2008]; AlRashidi and El-Hawary [2009]; Frank et al. [2012a,b] for overviews.

As solving the AC equations is challenging, current practice in the electricity industry is to use the *Linear* AC *Approximation* (so-called DC model) (O'Neill et al. [2011]). The DC model was presented by Schweppe and Rom [1970] to approximate the PF problem. It exploits the usually tight bounds for *voltage magnitudes* and the small *voltage phase angle* differences in real life network operations. Furthermore, it ignores *reactive power*. This makes the DC model linear and hence easy to solve by design. Armed with an easy-to-solve approximation of AC power flow based problems, researchers have been interested in analyzing the impact of particular complex problems, such as *line switching*. Line switching is the problem of changing the set of lines in the network to achieve a particular goal, i.e. improving on optimal solutions of the OPF problem.

The first proof of NP-hardness for the OPF and the MPF over the AC power flow model was given for a cyclic network structure by Verma [2009]. The proof was done for the *Lossless-Sin* AC *Approximation* (SIN), a variant of DC which uses a sine function around the voltage phase angle difference. From an AC perspective, this means that conductances are 0, voltage magnitudes are all fixed at 1, and reactive power is ignored[2]. The first proof of NP-completeness for the OPF, MPF and the PF problem over the DC model with line switching (called DS model) was given for a series-parallel network structure with an unbounded maximum node degree[3] by Kocuk et al. [2014][4].

In this thesis, we present the first comprehensive study of the computational complexity of the OPF, MPF and the PF problem over the AC, SIN and the DS model. In particular, we improve on the results from Verma [2009] and Kocuk et al. [2014] by presenting reductions with more realistic network structures. We also investigate the complexity of approximating the OPF. We investigate the SIN model separately from the AC model as the SIN model is close to the DC model, property-wise and by appearance, yet it is a special case of the AC model. Hence, any result about the SIN model indicates that any model "in between" the AC and the DC will have similar complexity.

The detailed contributions of the thesis and its organization are as follows. We first

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[1] The non-convexity of the problem does not automatically imply that problems based on the AC power flow are NP-hard. For example, the family of optimization problems $\min y$ such that $0 \leq y \leq \prod_{i=1}^{n} x_i$ where $n \in \mathbb{N}$ has a non-convex constraint and a non-convex solution set but the solution is always $y = 0$. Hence, the problem can be solved in constant time.

[2] "Ignoring" reactive power can be achieved by placing a generator with unbounded reactive power at every bus.

[3] The degree of a node is the number of edges/lines it has.

[4] These results where developed in parallel to the one presented in this thesis.

describe the basic mathematical notations of this thesis in Chapter 2. Then, in Chapter 3, we present a discussion about the properties of our reductions and introduce in an abstract way, the idea on which the majority of reductions in this thesis are based.

In Chapter 4, we present the mathematical definitions and the results regarding the DC model with line switching (DS model). We show that the DS-MPF and DS-OPF for cacti[5] networks with a bounded maximum degree are NP-complete. Furthermore, we show that the DS-OPF for cacti cannot be approximated and that the DS-PF problem is NP-complete for series-parallel networks with a bounded maximum degree. All problems are easy for trees and cacti are a simple extension of trees. Hence, we can derive that any type of network structure will be NP-complete.

In Chapter 5, we present the mathematical definitions and the results regarding the SIN model. We show that the SIN-OPF and SIN-MPF are strongly NP-hard for a planar network structure with a bounded maximum degree. We also show that the SIN-OPF cannot be approximated for a planar network structure with arbitrary maximum degree.

In Chapter 6, we present the mathematical definitions, results and a review of related work regarding the AC model. Here we show that the AC-MPF, AC-OPF and AC-PF problem are NP-hard for a tree network structure. We also show that the AC-OPF cannot be approximated.

In Table 1.1, we present an overview of all major results. The overview contains a selection of properties of our reductions: number of generators (nG), number of loads (nL), maximum bus degree (mD) and network structure (Structure). The table also includes the results from Bienstock and Mattia [2007] (1), Verma [2009] (3) and Kocuk et al. [2014] (2) as well as the results which can be easily derived from these papers. Note that Bienstock and Mattia [2007] did not present a proof. Hence, the properties of the reduction are unknown.

In Chapter 7, we draw conclusions and discuss open problems and questions.

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[5] A graph/network is called a cactus if every edge is part of at most one cycle.

Problem	Result	Structure	mD	nG	nL	Theorem
DS-PF	NP-complete	series-parallel	3	1	1	4.3.4
DS-OPF	not APX	cacti	3	∞	∞	4.4.4
DS-OPF	NP-complete	cacti	3	∞	∞	4.4.5
DS-OPF	NP-complete	series-parallel	3	2	1	4.4.6
DS-MPF	NP-complete	cacti	3	∞	∞	4.5.2
DS-MPF	NP-complete	series-parallel	3	1	1	4.5.3
DS-MPF	not APX	arbitrary	∞	6	6	4.5.5
DS-MPF	strongly NP-complete	planar	3	1	1	4.5.6
DS-PF	NP-complete	?	?	?	?	(1)
DS-PF	NP-complete	series-parallel	∞	1	1	(2)
DS-OPF	NP-complete	series-parallel	∞	1	1	(2)
DS-MPF	NP-complete	series-parallel	∞	1	1	(2)
AC-MPF	NP-hard	tree	∞	∞	1	6.3.3
AC-PF	NP-hard	tree	∞	1	∞	6.4.4
AC-OPF	not APX	tree	∞	2	∞	6.5.1
AC-OPF	NP-hard	tree	∞	2	∞	6.5.2
AC-OPF	not APX	tree	∞	∞	1	6.5.3
AC-OPF	NP-hard	tree	∞	∞	1	6.5.4
VPF	NP-hard	tree	∞	∞	∞	6.6.5
SIN-MPF	strongly NP-hard	planar	6	∞	∞	5.2.5
SIN-OPF	not APX	planar	4	∞	∞	5.3.1
SIN-OPF	strongly NP-hard	planar	6	∞	∞	5.3.2
SIN-OPF	strongly NP-hard	arbitrary	∞	∞	1	(3)
SIN-MPF	strongly NP-hard	arbitrary	∞	∞	1	(3)

Table 1.1: Overview of all results of this thesis and including the results from Bienstock and Mattia [2007] (1), Kocuk et al. [2014] (2), and Verma [2009] (2).

Chapter 2

Background

In the first four sections of this chapter, we present mathematical notations and definitions which are shared among all result chapters (Chapters 4 to 6). These parts are essential to understand the model specific background sections within these chapters. Note that when presenting results regarding computational complexity, we will always have to define specific networks. To that end, the background sections present concepts, such as the extension of a function or the sum of two networks, and notations, for example variants of networks or a function for the phase angle difference. These concepts are not found in academic literature presenting methods to solve (algorithms, heuristics, ...) the problems we study because a solving method has to work for arbitrary networks.

To present our results we use the same identifier for network types, problems and objectives. For example, an OPF *network* is the specific type of network used in the definition of the OPF. The word OPF represents the function that maps every OPF network onto the optimal value of the optimization problem which is finding the generation dispatch that minimizes the generation costs. The OPF *problem* is the decision variant this: decide if the OPF of a given OPF network is less or equal than a given value.

The chapter starts with concepts about functions in Section 2.1 and graphs in Section 2.2. We than present the *general purpose* (GP) network in Section 2.3. In Section 2.4, we introduce the basic networks for our three problem types (MPF, OPF, PF) as special cases of GP networks. Afterwards, in Section 2.5, we introduce the graphical notation used in this thesis. And, finally, in Section 2.7, we provide a short introduction to approximation algorithms.

2.1 Functions

At the heart of all functions in this thesis are the *real numbers* $\mathbb{R}$ and the *rational numbers* $\mathbb{Q}$ as well as functions that map to the real and rational numbers. An *extension* of a function is any function that has the same mapping with a potentially bigger domain. One special extension is the function that extends the domain such that every new element is mapped to the same value.

Definition 2.1.1 (function extension)**.** Let X, Y be sets and $z \in \mathbb{R}$. A function $g : Y \to \mathbb{R}$ *extends* a function $f : X \to \mathbb{R}$ if $X \subseteq Y$ and $\forall x \in X : f(x) = g(x)$. The *$z$ extension of* f on Y is the function $f|_z^Y : Y \to \mathbb{R}$ defined by

$$\forall x \in Y : f|_z^Y(x) := \begin{cases} f(x) & \text{if } x \in X \\ z & \text{if } x \in Y \setminus X. \end{cases}$$

Another concept we use is the *sum* of two functions. The sum of two functions is a function that sums up all values that are shared in the domains of its summons and otherwise keeps the same values.

Definition 2.1.2 (function sum)**.** Let X, Y be sets. The *sum* of the functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ is the function $f + g : X \cup Y \to \mathbb{R}$ with $\forall x \in X \cup Y$:

$$(f+g)(x) := \begin{cases} f(x) + g(x) & \text{if } x \in X \cap Y \\ f(x) & \text{if } x \in X \setminus Y \\ g(x) & \text{if } x \in Y \setminus X. \end{cases}$$

2.2 Graph

The definition of networks is based on *graphs*. Furthermore, in some of our results, we present reductions based on graph problems. To that end, we introduce the definition of a graph as well as some graph concepts and graph structures. For a given set X, let $\mathcal{P}_2(X) := \{Y \subseteq X \mid |Y| = 2\}$ be the set of all two-element sub-sets of X.

Definition 2.2.1 (graph, nodes, degree)**.** A *graph* is a tuple $\mathcal{G} = (N, E)$ where N is the set of *nodes* and $E \subseteq \mathcal{P}_2(N)$ is the set of lines. The *degree* of a node a is the number of edges it belongs to, i.e. $|\{\{\mathsf{a}, \mathsf{d}\} \mid \{\mathsf{a}, \mathsf{d}\} \in E\}|$.

Cycles are a central concept for all problems related to the switching of edges/lines (see Chapter 4). We use the concept of *simple cycles* to define two graph structures important for this thesis.

Definition 2.2.2 (walk, length, path, cycle, simple cycle)**.** A *walk* is a sequence of nodes $\mathsf{a}_1, \mathsf{a}_2 \ldots, \mathsf{a}_{n-1}, \mathsf{a}_n$ such that $\forall 1 \leq i < n : \{\mathsf{a}_i, \mathsf{a}_{i+1}\} \in E$. The *length* of a walk is the number of nodes it passes through, in this case n. A *path* is a walk where $|\{\mathsf{a}_1, \ldots, \mathsf{a}_n\}| = n$. A *cycle* is a walk where $\{\mathsf{a}_1, \mathsf{a}_n\} \in E$. A *simple cycle* is a cycle that is also a path.

The concept of *component* of a graph will be used to present results related to reductions bases on graph problems.

Definition 2.2.3 (connected, sub-graph, component)**.** A graph is called *connected* if for every pair of nodes there exists a path between them. A *sub-graph* of $\mathcal{G}$ is a graph $(\widetilde{N}, E')$ with $\widetilde{N} \subseteq N$ and $E' \subseteq E$. A *component* of $\mathcal{G}$ is a sub-graph $\mathcal{G}'$ which is connected and every other sub-graph of $\mathcal{G}$ who has $\mathcal{G}'$ as sub-graph is not connected.

Some of the reductions presented in this thesis have the structure of *trees*, *cacti* or *series parallel* graphs. The connection between these three is as follows: trees are cacti, cacti are series parallel and series parallel graphs are planar graphs.

Definition 2.2.4 (tree)**.** A connected graph is called a *tree* if it does not have any cycles.

Definition 2.2.5 (cactus)**.** A graph is called a *cactus* if every two distinct simple cycles share at most one node.

Definition 2.2.6 (series-parallel)**.** A graph is called *series parallel* if for every two edges $\{\mathsf{a}_1, \mathsf{d}_1\}$ and $\{\mathsf{a}_2, \mathsf{d}_2\}$ no simple cycles of the form $\mathsf{a}_1, \mathsf{d}_1, \dots, \mathsf{a}_2, \mathsf{d}_2$ and $\mathsf{a}_1, \mathsf{d}_1, \dots, \mathsf{d}_2, \mathsf{a}_2$ exist.

2.3 GP Network

The complexity results presented in this thesis are for three different problems classes: the OPTIMAL POWER FLOW (OPF), MAXIMUM POWER FLOW (MPF) and the POWER FLOW (PF). We investigate these three problem classes on three different power flow models: *Alternating Current* (AC), *Lossless-Sin* AC *Approximation* (SIN) and *Linear* AC *Approximation* (DC). Overall we study eight different problems[1]. An instance of a specific problem is a power network, or *network* for short. In this thesis, we use six different network types for our eight problems. There are only six (and not eight) because the networks/instances of the SIN and the DC model are the same within a fixed problem class.

The network type for the AC based problems can be regarded as an extension of the network for DC and SIN. Hence, in this section, we present the definition of the network for DC/SIN for all three problem classes. We call these networks MPF network, OPF network and PF network. In contrast the networks for the AC power flow model based problems are called AC-MPF network, AC-OPF network and AC-PF network. These are presented in the background section of the chapter about the AC model (Section 6.1).

The three network types MPF network, OPF network and PF network are derived from a generic network type, which is called *general purpose* (GP) network. The three network types for the AC model, AC-MPF network, AC-OPF network and AC-PF network are specializations of the AC network, which itself is a generalization of the GP network.

The definition of GP networks is similar to the definition of graphs. In a GP network, the nodes are called *buses* and the edges are called *lines*.

In the AC power flow equations and for a single line, the current (flow) across the line and the voltage difference of the two ends of this line have to be constant. This constant value is called *admittance*. Since voltage and current are complex numbers, the admittance is complex as well. Its real part is called *conductance* (denoted with g) and its reactive part is called *susceptance* (denoted with b). The susceptance

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[1] In Chapter 5, we explain why the SIN-PF problem is not of interest.

is a negative rational number that is used by all types of flows in this thesis. The conductance, on the other hand, is a parameter that is only used by the AC model. Its value is usually close to 0 for real world transmission lines. Hence, the SIN and DC model approximate the AC model by assuming that the conductance is 0. When defining and graphically presenting lines in any context other than the AC model, we will omit showing the conductance as its value does not matter.

The third parameter of a line is called *line capacity* (denoted with c). Its purpose is to limit the amount of flow along a line. For none of the results about the AC model do we need the "feature" of limiting the flow along a line. Hence, we will ignore this parameter in the context of the AC model. Note that the AC model, however, has a network wide maximum phase angle difference.

Some buses are generators and some buses are loads. These are indicated as subsets of the set of buses. In a network, we also allow to assign values to a subset of these generators and/or loads. These values are later interpreted as given, with fixed generation and/or demand values.

Note that real-world networks also include additional components, such as transformers, bus shunts, line charging or phase shifters (Stott and Alsac [2012]). Our network definition will not include these components because we do not need them for our reductions.

Definition 2.3.1 (GP network)**.** A GP network is a tuple $(N, N_G, N_L, E, G^p, L^p)$ where

- N is the set of *buses*,
- $N_G \subseteq N$ is the set of *generators*,
- $N_L \subseteq N$ is the set of *loads*,
- $E \subset \mathcal{P}_2(N) \times \mathbb{Q}_{\leq 0} \times \mathbb{Q}^2_{\geq 0}$ is the set of lines with

$$\forall(\{\mathsf{a},\mathsf{d}\}, b_1, g_1, c_1), (\{\mathsf{a},\mathsf{d}\}, b_2, g_2, c_2) \in E : b_1 = b_2, g_1 = g_2 \text{ and } c_1 = c_2,$$

- $G^p : N_G^p \to \mathbb{Q}_{\leq 0}$ with $N_G^p \subseteq N_G$ is the (partial) *active power generation*, and
- $L^p : N_L^p \to \mathbb{Q}_{\geq 0}$ with $N_L^p \subseteq N_L$ is the (partial) *active power demand*.

In contrast to the general literature, our generators and loads do not have upper and lower bounds. Having bounds can be regarded as a "feature" which could be used by a reduction. However, only one of our reductions need this feature to work. Therefore, we omit defining it in general.

When defining GP networks (and their derivatives MPF network, OPF network and PF network) we present functions like G^p and L^p in an implicit manner. For example, for the functions $G^p : \{\mathsf{r}\} \to \mathbb{Q}_{\leq 0}$, $G^p(\mathsf{r}) :=$ -1 and $L^p : \{\mathsf{l}\} \to \mathbb{Q}_{\leq 0}$, $L^p(\mathsf{l}) := 12$ we have the two equivalent representations

$$\begin{aligned} \mathcal{N} &:= (\{\mathsf{e},\mathsf{r},\mathsf{l},\mathsf{a},\mathsf{d}\}, \{\mathsf{r},\mathsf{a}\}, \{\mathsf{l},\mathsf{d}\}, E, G^p, L^p) \\ &:= \Big(\{\mathsf{e},\mathsf{r},\mathsf{l},\mathsf{a},\mathsf{d}\}, \{\mathsf{r},\mathsf{a}\}, \{\mathsf{l},\mathsf{d}\}, E, \Big[G^p_{\mathsf{r}}{=}\text{-}1 \Big| L^p_{\mathsf{l}}{=}12\Big]\Big). \end{aligned}$$

In the case where no active power generation and load values are fixed, we write $\emptyset$. Hence, we have

$$\begin{aligned}\mathcal{N} &:= (\{\mathsf{e}, \mathsf{r}, \mathsf{l}, \mathsf{a}, \mathsf{d}\}, \{\mathsf{r}, \mathsf{a}\}, \{\mathsf{l}, \mathsf{d}\}, E, G^p, L^p) \\ &= (\{\mathsf{e}, \mathsf{r}, \mathsf{l}, \mathsf{a}, \mathsf{d}\}, \{\mathsf{r}, \mathsf{a}\}, \{\mathsf{l}, \mathsf{d}\}, E, \emptyset)\end{aligned}$$

where $G^p : \emptyset \to \mathbb{Q}_{\geq 0}$ and $L^p : \emptyset \to \mathbb{Q}_{\leq 0}$.

In some cases, when defining networks, we are given a set of natural numbers, $X := \{x_1, \ldots, x_n\} \subseteq \mathcal{P}(\mathbb{N})$ and the network has one bus per element of X. So $X \subseteq N$ where N is the set of buses. The identifier of these buses are the numbers $x_i \in X$. To aid readability we refer to the value of the number of an $x_i \in X$ with x_i and whenever we refer to the symbol that represents the bus we use the notation x_i.

Given a GP network we sometimes provide proofs of properties for a variant of this network. A variant could be when a bus becomes a generator and load. Another way kind of creating variants is when, for example, we fix the generation of a bus and if that bus was not a generator then we make it one. As these variants are only used temporarily we define a special syntax for them based on the original network.

Definition 2.3.2 (network variant)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network and $\widetilde{G}^p : N_G^p \to \mathbb{Q}_{\leq 0}$ and $\widetilde{L}^p : N_L^p \to \mathbb{Q}_{\geq 0}$ be active power generation and demand functions. We define and denote the GP network variant of $\mathcal{N}$ with respect to $\widetilde{G}^p$ and $\widetilde{L}^p$ via

$$\mathcal{N}[\widetilde{G}^p, \widetilde{L}^p] := (N, N_G \cup N_G^p, N_L \cup N_L^p, E, G^p + \widetilde{G}^p, L^p + \widetilde{L}^p).$$

Let $\mathsf{e} \in N$ be a bus. The GP network variant of $\mathcal{N}$, where e becomes a generator and a load, is defined and denoted as

$$\mathcal{N}[\mathsf{e} \in N_{G/L}] := (N, N_G \cup \{\mathsf{e}\}, N_L \cup \{\mathsf{e}\}, E, G^p, L^p).$$

We use the notations above in the statements of lemmas. To shorten these presentations we never present the functions $\widetilde{G}^p$ and $\widetilde{L}^p$ directly. Instead, we use a similar implicit definition as for the definition of networks. For example, let $\mathcal{N}$ be a GP network, $\widetilde{G}^p$ be a generation function with $\widetilde{G}^p_{\mathsf{r}} :=$ -1 and $\widetilde{L}^p$ be a demand function with $\widetilde{L}^p_{\mathsf{l}} = 1$. Instead of $\mathcal{N}[\widetilde{G}^p, \widetilde{L}^p]$ we write $\mathcal{N}[G^p_{\mathsf{r}}{=}\text{-}1 | L^p_{\mathsf{l}}{=}1]$ omitting the usage and definition of the symbols $\widetilde{G}^p$ and $\widetilde{L}^p$.

In most of our reductions, we define the networks by connecting multiple networks together. The connection of two networks happens along a set of common buses. To ensure that our sum of networks is well defined, we force the condition that both networks do not share any lines.

Definition 2.3.3 (sum of two networks)**.** Let $\mathcal{N} := (N, N_G, N_L, E, G^p, L^p)$ and $\widetilde{\mathcal{N}} := (\widetilde{N}, \widetilde{N}_G, \widetilde{N}_L, \widetilde{E}, \widetilde{G}^p, \widetilde{L}^p)$ be two GP networks with $\{\{\mathsf{a}, \mathsf{d}\} \mid (\{\mathsf{a}, \mathsf{d}\}, b, g, c) \in E\} \cap \{\{\mathsf{a}, \mathsf{d}\} \mid (\{\mathsf{a}, \mathsf{d}\}, b, g, c) \in \widetilde{E}\} = \emptyset$ and we define $N_a := N \cap \widetilde{N}$. The *sum* of $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ with respect to N_a is defined as

$$\mathcal{N} +^{N_a} \widetilde{\mathcal{N}} := (N \cup \widetilde{N}, N_G \cup \widetilde{N}_G, N_L \cup \widetilde{N}_L, E \cup \widetilde{E}, G^p + \widetilde{G}^p, L^p + \widetilde{L}^p).$$

In most cases, the set N_a will consist of only one bus (which we usually denote with e). This bus is called the *connector*. Note that when building the sum of two networks, we always explicitly state at which buses they are connected together. In the case where two networks share buses other than the connector, we assume that these buses get automatically renamed before building the sum network. From the definition above, it is also easy to see that the network sum operator is associative. Hence, we will omit using parentheses when building the sum of multiple networks. If N_a has only one element then we omit the set brackets as well. Let $X = \{x_1, \ldots, x_n\}$ be a set and $\mathcal{N}_{\mathsf{e}}^{x_i}$ be some networks with bus e. We write $\sum_{x \in X}^{\mathsf{e}} \mathcal{N}_{\mathsf{e}}^{x} := \mathcal{N}_{\mathsf{e}}^{x_1} +^{\mathsf{e}} \ldots +^{\mathsf{e}} \mathcal{N}_{\mathsf{e}}^{x_n}$.

In order to later be able to define power flows, we need the set of *directed lines*.

Definition 2.3.4 (directed lines). Let $(N, N_G, N_L, E, G^p, L^p)$ be a GP network. The set of *directed lines* is $E^d := \{(\mathsf{a}, \mathsf{d}, b, g, c) \mid (\{\mathsf{a}, \mathsf{d}\}, b, g, c) \in E\}$

2.3.1 GP Solutions

Before presenting the network types for MPF, OPF and PF, we introduce notations and concepts involving *solutions*. We formulate the problem classes (and their complexity problems) using solutions. A solution is an allocation of values for a given and fixed set of variables: for example, the line flows or the phase angles. Furthermore, a solution has to satisfy given constraints: for example, that all line flows are within their bounds. They also have to match the given values for active power generation and active power demand.

The allocations of variables is always given as a function mapping from the sets of buses, generators, loads or directed lines into the real numbers. The variables important for networks in this thesis are the

- (*voltage*) *phase angles* $\theta : N \to \mathbb{R}$,
- *voltage magnitudes* $v : N \to \mathbb{R}_{>0}$,
- *active power generation* $G^p : N_G \to \mathbb{R}_{\leq 0}$,
- *reactive power generation* $G^q : N_G \to \mathbb{R}$,
- *active power load* $L^p : N_L \to \mathbb{R}_{\geq 0}$,
- *reactive power load* $L^q : N_L \to \mathbb{R}$,
- *active power flow* $p : E^d \to \mathbb{R}$, and
- *reactive power flow* $q : E^d \to \mathbb{R}$.

Note that in contrast to some literature, we define the generation and the load to have opposing sign. The generation is a negative and the load is a positive value. We comment on why we adopted this convention after introducing Kirchhoff's junction law for active power.

In the literature, one can also find active/reactive generation and load being defined for all buses. This is especially common in literature about solving any of our problem classes. The models in the literature, have upper and lower generation and load bounds for every bus. Buses without generator or load are expressed by setting the corresponding upper and lower bounds to 0. Contrarily, our generators and loads do not have bounds. Hence, we have to restrict the generation and load functions to the sets of generators and loads. When presenting Kirchhoff's junction law later we will use the notation of function extension (from Section 2.1) to extend the generation/load function to be 0 for buses which are not generators/loads.

The functions presented above are usually interpreted as vectors in the literature. The assumption is that there exists some fixed order of the set of buses and the set of lines. An element of a vector is usually indicated by an index on the symbol of the vector. This motivates our notation v_{a} (instead of $v(\mathsf{a})$). The same is true for all variables where the domain is a subset of the set of buses: θ, G^p, L^p, L^q, G^q. For the function p, we write p_{ad} instead of $p((\mathsf{a}, \mathsf{d}, b, g, c))$. Similarly for q and any other function with the domain of E^d.

We call the basis of all types of solutions for our problems in this thesis: GP *solution*. These solutions pose the constraint, that at every bus the sum of all flows, the generation and the load balances. Not present in this definition is a power flow constraint. A power flow constraint binds the power flow (active or reactive) to the phase angles (and voltage magnitudes). It is specific for the power flow model we study. Hence, these constraints are introduced in the sections for AC, DC and SIN respectively. These sections also present specific definitions for our problems.

The line parameters susceptance (b) and conductance (g) are only used in the power flow constraint. Hence, they do not appear in the definition of a GP solution. Furthermore, all power flow constraints depend on the phase angle (θ) of the bus. Since all the specific definitions depend on the definition of a GP solution, we include the phase angle in this definition, even though it is not used here.

Definition 2.3.5 (GP solution)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network. A GP *solution* for $\mathcal{N}$ is a tuple $(\theta, \widetilde{G^p}, \widetilde{L^p}, p)$ where $\theta : N \to \mathbb{R}$, $\widetilde{G^p} : N_G \to \mathbb{R}_{\leq 0}$, $\widetilde{L^p} : N_L \to \mathbb{R}_{\geq 0}$ and $p : E^d \to \mathbb{R}$ such that

- $\widetilde{G^p}$ extends G^p,
- $\widetilde{L^p}$ extends L^p,
- and Kirchhoff's junction rule for active power is satisfied,

$$\forall \mathsf{a} \in N : \overline{G^p_{\mathsf{a}}} + \overline{L^p_{\mathsf{a}}} + \sum_{(\mathsf{a},\mathsf{d},b,g,c)\in E^d} p_{\mathsf{ad}} = 0$$

where $\overline{G^p} := \widetilde{G^p}|_0^N$ and $\overline{L^p} := \widetilde{L^p}|_0^N$.

The set of all GP solutions of $\mathcal{N}$ is denoted with $\mathcal{S}^{\mathrm{GP}}(\mathcal{N})$.

In the definition above, we distinguished between the generation function of the network, G^p and the generation function given by the solution $\widetilde{G}^p$. As for solutions, $\widetilde{G}^p$ is always an extension of G^p we will use the same symbol for both from now on. This will happen especially when the domain of G^p is empty. The same applies to L^p. That means for a GP network $(N, N_G, N_L, E, G^p, L^p)$, a solution could be represented by (θ, G^p, L^p, p) where the latter G^p and L^p are extensions of the former.

In the different types of solutions for the models that we have (AC, DC and SIN), the phase angles only occur in a difference of two phase angle values. This motivates us to define a function of phase angle differences.

Definition 2.3.6 (phase angle difference)**.** Let $\mathcal{N}$ be a GP network and (θ, G^p, L^p, p) be a GP solution. We define the function of *phase angle differences* $\Delta : E^d \to \mathbb{R}$ via $\Delta(\mathsf{a}, \mathsf{d}, b, g, c) := \theta_{\mathsf{a}} - \theta_{\mathsf{d}}$.

2.4 Problem Specific Networks

The GP networks will be specialized into three types of networks: MPF, PF and OPF network. These three are underlying networks for all SIN related problems presented in Section 5.1 and DC related problems defined in Section 4.1. They are also the basis for the AC-MPF, AC-PF and AC-OPF networks defined in Section 6.1.

2.4.1 MPF Network

For the MPF, the goal is to find the generation and demand such that we satisfy as much demand as possible. Hence, there must be some loads and generators which do not have fixed demand or generation. There could also be generators or loads which are fixed. For example, there could be the constraint that a hospital must be supplied with power. This demand creates a minimum demand at a bus which could be modeled as having one load with a fixed demand and another load that is free.

In our model, we assume that all generators and loads are not fixed. This is because we do not need fixed values to establish our results. An exception is the result showing that the MPF cannot be approximated in the DC power flow model with line switching. This result needs one load with a fixed demand. The necessary definitions for this case will be presented in the section about the DC model (Section 4.1).

In our definition of MPF networks, generators and loads are disjoint. The MPF could not be well defined otherwise, because having a generator and a load at the same bus would result in a potentially infinite value for the MPF.

Definition 2.4.1 (MPF network)**.** A GP network $(N, N_G, N_L, E, G^p, L^p)$ is called MPF *network* if

- $N_G \cap N_L = \emptyset$,
- $dom(G^p) = \emptyset$, and
- $dom(L^p) = \emptyset$.

2.4.2 OPF Network

The goal of the OPF is to find a generation dispatch such that a given and fixed demand is satisfied and the overall generation cost is minimal. Hence, an OPF network has a *cost function* C which allocates costs for every generator. Also, the demand of all loads is given. In the definition of the OPF problem in the literature, one can usually find a lower and upper bound for the generation. By setting the lower to the upper bounds to the same value one could essentially fix the generation. Although our definition of GP networks does not include lower and upper bounds, it does allow for the fixing of generation values. However, in our results we do not need the feature of fixed generation. Hence, in our definition of OPF network all generators are free. We also add the constraint that generators and loads are disjoint. This is because we do not need the feature of a bus being a generator and a load at the same time.

Definition 2.4.2 (OPF network)**.** An OPF *network* is a tuple $(\mathcal{N}, C)$ where

- $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ is a GP network with
- $N_G \cap N_L = \emptyset$,
- $dom(L^p) = N_L$,
- $dom(G^p) = \emptyset$, and
- $C : N_G \to \mathbb{Q}_{\geq 0}$ is the cost function.

2.4.3 PF Network

In the PF problem we ask the question whether or not a solution exists. It is generally assumed that the demand and the generation of all but one generator is given. This one generator (denoted with s) is called: *slack bus*. The intention of the slack bus is to ensure the existence of a solution. However, as we will show in Section 6.4, this is not guaranteed. We also add the constraint that generators and loads are disjoint. This constraint makes the reductions stronger. We do not need the "feature" of having a generator and a load at the same bus.

Definition 2.4.3 (PF network)**.** A GP network $(N, N_G, N_L, E, G^p, L^p)$ is called PF *network* if

- $N_G \cap N_L = \emptyset$,
- $\exists \mathsf{s} \in N_G : N_G \setminus dom(G^p) = \{\mathsf{s}\}$, and
- $dom(L^p) = N_L$.

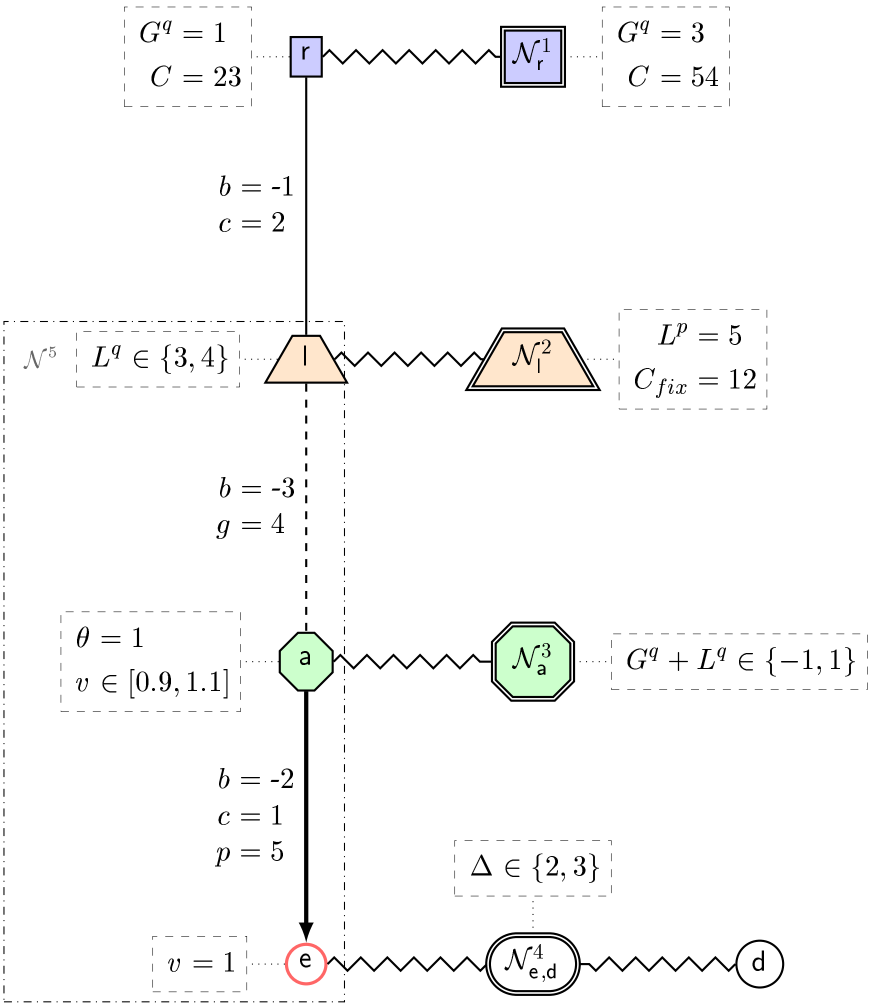

Figure 2.1: The network $\mathcal{N}_{\mathsf{e}}$ showing all graphical features used in this thesis.

2.5 Graphical Representation

Fig. 2.1 presents all graphical features used in this thesis through the network $\mathcal{N}_{\mathsf{e}}$. We deviate from the classical graphical presentation of power system networks. The representation is more similar to the way graphs with their nodes and edges are presented. All graphical features presented are valid for all types of models and networks.

The figures in this thesis are for illustration of networks and their properties. They are not meant as definitions. For example, the figures never show the global maximum phase angle difference and the voltage magnitude bounds of AC networks. Also, the figures contain additional information which represent properties of the network presented and which are proven in some lemmas or theorems. Furthermore, figures do not distinguish between values which are part of the network definition and values which are part of a presented solution[2]. Note that the network $\mathcal{N}_{\mathsf{e}}$ is not of any type of networks we defined earlier. Its purpose is only to present an overview of the graphical features.

Figure 2.1 tells us the following things:

- Bus r is a **generator** (blue rectangle) with a fixed active power generation of $G^p_{\mathsf{r}} = 1$ and (active power) cost of $C_{\mathsf{r}} = 23$. Other variables not specified (e.g. voltage magnitude) are free.
- Bus l is a **load** (orange trapeze) with a reactive power demand of either 3 or 4.
- Bus a is a **generator and a load** (green chamfered) with a fixed phase angle of $\theta_{\mathsf{a}} = 1$ and a voltage magnitude that can range from 0.9 to 1.1.
- Bus e is **neither a generator nor a load** (white ellipse) has a fixed voltage magnitude of 1.
- Bus d is **neither a generator nor a load** and has no fixed values.
- The buses l, a and e together with their lines form the **sub-network** (dashed-dotted box) $\mathcal{N}^5$.
- The line r $\xleftrightarrow[c=2]{b=\text{-}1}$ l has a susceptance of -1 and a capacity of 2. The conductance is not shown, as this line is used in the representation of networks in the chapter about the DC model. The DC model does not use conductance.
- The line l $\xleftrightarrow[g=4]{b=\text{-}3}$ a has a susceptance of -3 and a conductance of 4. The third parameter for this line is not shown as the line is used in the representation of networks in the chapter about the AC model. The AC model does not use any line based capacity or phase angle restriction like the DC or the SIN model.
- The line l$\leftrightarrow$a is **switched off** (dashed).
- The line a $\xleftrightarrow[c=1]{b=\text{-}2}$ e has a susceptance of -2 and a capacity of 1. The conductance is not shown as this line is used in the representation of networks in the chapter about the SIN model. The SIN model does not use conductance.
- The line a$\leftrightarrow$e also has an active **power flow** of $p_{\mathsf{ae}} = 5$ from a to e (indicated by the arrow direction).
- The **networks** $\mathcal{N}^1_{\mathsf{r}}, \mathcal{N}^2_{\mathsf{l}}, \mathcal{N}^3_{\mathsf{a}}$ and $\mathcal{N}^4_{\mathsf{ed}}$ are part of this network but their appearance is hidden (double shaped frame). They have **connector buses** (r, l, a, e, d) which are the same as the ones of network $\mathcal{N}_{\mathsf{e}}$ (indicated by the zigzag edge). Within the symbol of a network the connector bus is presented as under-script.
- The network $\mathcal{N}_{\mathsf{e}}$ pictured in Fig. 2.1 **has the connector** e (red border color).

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[2] The only case where we present solution values in a figure is Fig. 4.1 in Section 4.2.

- The network $\mathcal{N}_{\mathsf{r}}^1$ with connector r **acts as generator** to r (blue rectangle) and generates a fixed amount of 3 reactive power. This implies that there is a total implicit reactive power of 4 at r. Also the network generates active power for cost of 54 per unit.

- The network $\mathcal{N}_{\mathsf{l}}^2$ has the connector l and **acts as load** for l with a fixed active power load of 5. The value for reactive power is a free variable. Furthermore, the network has **fixed costs** of 12. "Fixed" means that this value does not depend on the amount of reactive power demand taken in by the network.

- The network $\mathcal{N}_{\mathsf{a}}^3$ **acts as either generator or load** (green chamfered rectangle) for a where we have the **choice** between either generation of consuming 1 unit of reactive power.

- The network $\mathcal{N}_{\mathsf{e},\mathsf{d}}^4$ has **two connectors** e and d whose phase angle difference can only be either 2 or 3.

2.6 Strongly NP-Completeness

We present a short introduction to *strongly* NP-completeness. More details can be found in Garey and Johnson [1978]. Strongly NP-complete problems are a special case of NP-complete problems. A problem is called *strongly* NP*-hard* if the variant of it where all numerical parameters are bounded by a single polynomial in the size of the input is NP-hard. A problem is called *strongly* NP*-complete* if it is strongly NP-hard and in NP. For example, the problem of finding the longest path in a graph does not have any numerical parameters and hence is strongly NP-complete. The SUBSET SUM PROBLEM, which is widely used in this thesis, is an example of a problem which is NP-complete but not strongly NP-hard.

A way to prove that a problem is strongly NP-hard is to find a polynomial-time reduction of another strongly NP-complete problem such that all numerical parameters of the reduction are bounded by a single polynomial in the size of the input. Such a reduction is called *pseudo-polynomial reduction*.

2.7 Approximation Algorithms

We present a short introduction to approximation algorithms. More details are presented by Vazirani [2013]. An approximation algorithm is an algorithm designed to find a solution of an optimization problem, guaranteeing that the objective value is "not too far" from the optimal value. We are only interested in approximation algorithms which run in polynomial time with respect to the input size. Such algorithms are classified by what guarantees they can provide.

Let W be set of all instances of some minimization problem; and, for an instance $w \in W$, let $OPT(w)$ be the optimal value. Furthermore, let $A : W \to \mathbb{R}$ be an approximation algorithm where $A(w)$ is the objective value of solution found by A for the instance w. The algorithm A is called *ϵ-approximation* algorithm with $\epsilon > 1$ if it runs in polynomial time with respect to the input size; and it can guarantee that the objective value for each instance is not worse than ϵ times the optimal value. That is, if:

$$\forall w \in W : OPT(w) \leq A(w) \leq \epsilon OPT(w).$$

Similarly, for maximization problems we have $0 < \epsilon < 1$ and

$$\forall w \in W : OPT(w) \geq A(w) \geq \epsilon OPT(w).$$

We say that an optimization problem can be approximated by a *constant factor approximation* if there exists at least one ϵ-approximation algorithm for some ϵ. The class of all problems which have a constant factor approximation is called *APX* (an abbreviation for approximable). We say an optimization problem cannot be *approximated within any constant factor* if there does not exist any ϵ-approximation algorithm.

The class APX contains the problems which admit a *Fully Polynomial-Time Approximation Scheme* (FPTAS). These are optimization problems where there exists an ϵ-approximation with a runtime polynomial in the input and $1/\epsilon$ for every ϵ. It is shown in Vazirani [2013] that no strongly NP-hard problem can have an FPTAS. Hence, showing that an optimization problem is strongly NP-hard provides the result that the problem cannot be cheaply arbitrarily approximated.

Chapter 3

Methods

This chapter is about aspects of computational complexity in power systems. We do not present any results here. Also, it is not necessary to read this chapter in order to understand the results in the following chapters. The purpose of this chapter is to provide additional information into the how and why of our reductions.

In Section 3.1 we present an analysis of the "features"[1] of our reductions. A feature could be, for example, that the networks in all reductions of a problem have one generator. We explain which features we focused on and why.

In Section 3.2 we provide an informal introduction into the idea behind most of our reductions. The goal of this section is to aid the understanding of these proofs. Afterwards, in Section 3.3, we give a similar introduction for non-approximability.

3.1 Reduction Features

Overall, there are three models and three classes of problems. This makes for eight different problems[2]. When investigating these problems for the case where there exists an algorithm to solve them, one can wonder: does there exist a fundamentally better method to solve these problems? Studying the computational complexity of these problems can be one way to answer this question. For example, by showing that a problem is NP-hard, we can derive that no polynomial algorithm exists unless $\mathsf{P} = \mathsf{NP}$. This means that the fact that no efficient algorithm was found is simply because none exists and we can stop searching for it.

To prove that a problem is NP-hard we first have to choose an NP-hard problem. Then, we present a *reduction* of this problem into the problem which we study. For example, let the NP-hard problem be the SUBSET SUM PROBLEM (SSP) (defined in the following section) and our problem be the AC-PF problem. A reduction is a mapping assigning each SSP instance a network such that the network has a solution (PF problem) if and only if the SSP instance is solvable. Furthermore, the network must be computed in time polynomial in the size of the SSP instance. This implies that the size of the network must be polynomial in the size of the SSP instance.

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[1] or the absence of them

[2] The SIN-PF problem is trivial to solve, see Chapter 5

Of special interest are the properties that all of these networks share and the properties they do not have. We call such a property a *feature* of the reduction. For example, if in all networks at least one generator has an upper bound for its generation we say that the reduction used the feature of “generation upper bounds”. If on the other hand in all of the networks, there is no load with an upper bound then we say that the reduction does not use the feature of load upper bounds.

If we have a reduction which, for example, uses the feature of generation upper bounds, then it still might be possible that there exists a polynomial algorithm which solves the problem we study for networks without generation upper bounds. Hence, it is desirable to investigate if a reduction exists which does not need the feature of generation upper bounds. Such a reduction is to be considered “stronger”. In a more general sense, we can say that the fewer features are used in a reduction, the “stronger” the reduction is. Note that the word “stronger” here is not a well defined term. The word is an indication that the networks used in the reduction are more likely to be contained in real world cases. Also, some features are not just simply on-off properties but they might have a hierarchy. The structure of the network is such a feature. A reduction which results in only planar networks is considered to be “stronger” then a reduction where the network structures are arbitrary. It is also possible that in order to not use one feature we have to use another one. For example, we might be able to find a reduction where we do not need generation upper bounds, but to make it work we have to use load upper bounds.

We aimed to find reductions which are as strong as possible. To that end, we iteratively improved our reductions . This iterative process also had the side-effect that the reductions themselves became shorter and more elegant over time. Therefore, a major contribution of this thesis is that we present reductions which only use very few features.

In the following, we present an outline of the features we focused on. The choice of features is motivated by the appearance and properties of real world power networks. Note that some features are only relevant for some power models or problem classes. For example, in the PF problem the demand is a given and fixed value. Hence, the feature of load upper bounds is irrelevant as it can be satisfied easily.

Network Structure The feature “network structure” distinguishes between the following from weakest to strongest: arbitrary structure, planar networks, cacti and trees. It is important because, in the real world, power networks cannot have an arbitrary structure. For example, transmission networks are usually (almost) planar networks (Pagani and Aiello [2013]) and distribution networks are trees (Hijazi and Thiebaux [2014]). Hence, presenting a reduction which needs arbitrary structure might not say anything about real world networks.

Bus/Node Degree The degree of a bus is the number of lines connected to it. In real power networks, the degree of a bus is small, usually not more than 10 (Pagani and Aiello [2013]). Hence, we focused on finding reductions where the maximum degree is minimal. At the very least, to be close to realistic networks, it is important to ensure that the maximum degree is bounded by a constant.

Ratio of Generators to Loads Real world power networks usually have few generators and a large number of loads. Hence, in order to be realistic, a reduction where the ratio of generators overs loads is small is desired.

Susceptance to Conductance Ratio (AC model only) Where possible, our reductions do not fix the susceptance and conductance values. Instead, we treat them as external parameters and only present necessary conditions for them to ensure that our proofs work. For example, in some proofs, the only condition we have is that both values cannot be zero at the same time. Having done the proof with abstract parameters has the advantage that essentially we provide reductions for all kinds of ratios (which satisfy the given conditions). Our main focus is to ensure that the range of valid ratios includes the range of realistic ratios. According to Andersson [2004]; Grainger and Stevenson [1994] a ratio should be within the interval $[-30, -1]$.

Realistic Voltage Magnitude Bounds (AC model only) In the AC model, all voltage magnitudes are bounded by one pair of network-wide upper and lower bounds. By making upper and lower bounds equal, we would fix all voltage magnitudes to one value. Some proofs become easier when the voltage magnitudes are fixed. However, doing this is unrealistic. Hence, we made an effort to find reductions where this is not necessary.

Generation Bounds In the literature, authors often assume that generators have lower and upper generation bounds. Therefore, it is not reasonable to use these bounds in a reduction. However, in all of our reductions we do not need this feature. It is an essential part of many of our reductions that generators cannot generate more power than a given amount. This is a result of the line limits of all connected lines. These essentially act as an indirect upper bound. The fact that we do not need this feature implies that the complexity of the problem is already caused by the existence of the line limits.

Load Bounds Similarly to the generation bounds, loads can also have upper and lower bounds in the literature. As the demand is fixed in the OPF and the PF problem, this is only relevant for the MPF. The upper bounds on the loads are essentially given implicitly by the line limits. The feature of lower bounds is not used within this thesis.

3.2 NP-Hardness Reduction

The majority of reductions in this thesis use the SUBSET SUM PROBLEM (SSP). In the SSP, we are given a set of natural numbers and a natural number w. We have to decide if there exists a subset of S which sums up to w. The set of *natural numbers* is denoted with $\mathbb{N}$.

Definition 3.2.1 (SUBSET SUM PROBLEM)**.** A SUBSET SUM PROBLEM (SSP) instance is a tuple (S, w) where $S \subset \mathbb{N}$ is a finite set of natural numbers and $w \in \mathbb{N}_{>0}$ is a number. An instance is called *solvable* if there exists a set $V \subseteq S$ such that $\sum_{x \in V} x = w$. We call the set V a *solution*.

Let (S, w) be an SSP instance, $|S| = n$ and $x \in S$. When trying to solve this instance we are faced with the choice of whether x is in a solution or not. In our reductions this choice is represented by what we call a *choice network*. A reduction of an SSP instance consists of the connection of one choice network per element in S and what we call the *main network*. We have multiple different choice networks depending on the problem class and the power flow model. All types of choice networks, except for one, have something in common that: they have one bus which they share with the main network. We call this bus the *connector* and we typically use the symbol e for it.

In the following, we present an abstract example of what a reduction of a feasibility problem could look like. We assume the existence of choice networks $\mathcal{N}^x_{\mathsf{e}}$ where its superscript indicates a dependency on x. Also, let $\mathcal{N}^{w,n}_{\mathsf{e}}$ be the main network, which depends on w and n. The *reduction network* is defined as $\mathcal{N}_{S,w} := \mathcal{N}^{w,n}_{\mathsf{e}} +^{\mathsf{e}} \sum^{\mathsf{e}}_{x \in S} \mathcal{N}^x_{\mathsf{e}}$. Fig. 3.1 presents network $\mathcal{N}_{S,w}$ for the SSP instance $(\{x_1, x_2, x_3\}, w)$.

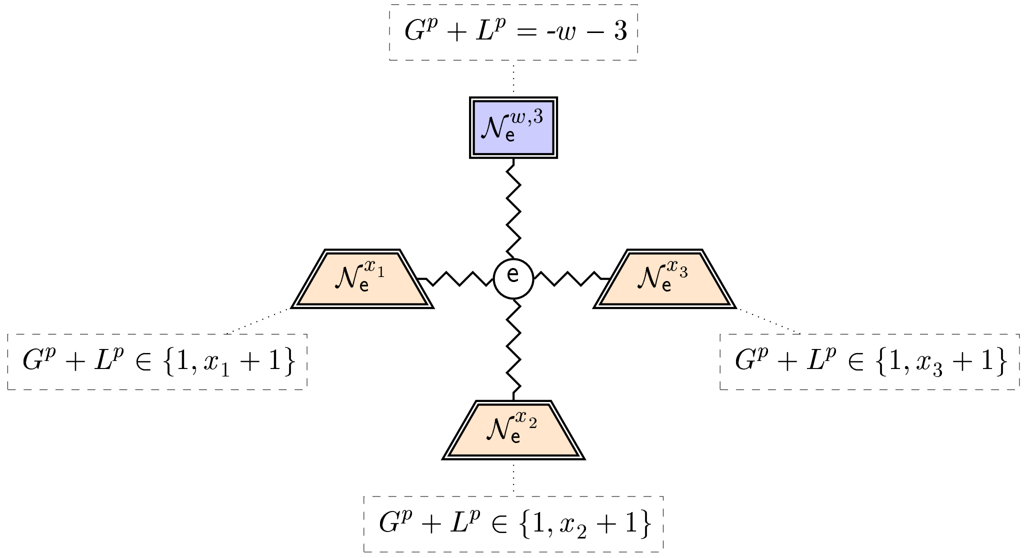


Figure 3.1: The common pattern found in many SSP based reductions.

The index on the choice networks e shows that the connector is called e and the zigzag line between a choice network and the bus e show that the buses e are the same. The superscript x_i indicates that the networks parameters depend on x_i. The shape of the choice networks is an indicator that the bus e acts as virtual load from the outside perspective (and hence as a generator from an inner perspective). The annotated value shows that this virtual load can only have an in/out active power flow of 1 or $x_i + 1$. Since the values are positive, we can derive that the network can only consume power, i.e. it cannot generate power. In other words, it cannot provide any power to the other choice networks and/or the main network. This property will be proven in a lemma by assuming that the bus is generator and load at the same time and showing that in every solution of this network e either generates -1 or -$x_i - 1$ (from an inside point of view).

Also connected to the connector e is the main network $\mathcal{N}_{\mathsf{e}}^{w,n}$ where, in our example, $n = 3$. The superscript indicates that the parameter of this network depend on w and n. The shape of the main network indicates that it acts as generator and the annotated values show that it generates a fixed value of $w + 3$. Note that in some cases the main network simply consists of the bus e only. For example, if the problem class allows for a fixed active power generation we can make e a generator and fix its generation to $w + n$. If the main network is more than just the bus e then the fixed generation of $w + n$ will be proven in a lemma. In some cases, the main network also depends on $m := \sum_{x \in S} x$.

With this pattern, showing NP-hardness works as follows. We call a choice network *active* if it consumes $x_i + 1$. Otherwise, we call it *inactive*. There is a one to one correspondence between the active choice networks of a reduction and the elements of a solution of (S, w). Assume that (S, w) is solvable and V is a solution. We activate all networks $\mathcal{N}_{\mathsf{e}}^{x_i}$ with $x_i \in V$ and keep all other networks inactive. The properties of the choice networks and the property of the main network imply that there are power flow solutions which are consistent within the networks itself. The fact that V is a solution ensures that Kirchhoff's junction law for active power at e is satisfied. Hence, the combination of all these solutions is a solution for $\mathcal{N}_{S,w}$. On the other hand, let $\mathcal{N}_{S,w}$ have a solution. The lemma about the choice network ensures that every choice network is either active or inactive. Furthermore, the main network has to generate $w+n$. Since we have a solution, Kirchhoff's junction law at e must be satisfied. Hence, we have

$$\begin{aligned} w + n &= \sum_{\substack{x \in S \\ \mathcal{N}_{\mathsf{e}}^{x} \text{ active}}} (x+1) + \sum_{\substack{x \in S \\ \mathcal{N}_{\mathsf{e}}^{x} \text{ inactive}}} 1 \\ w &= \sum_{\substack{x \in S \\ \mathcal{N}_{\mathsf{e}}^{x} \text{ active}}} x. \end{aligned}$$

Therefore, $V := \{x \mid \mathcal{N}_{\mathsf{e}}^{x} \text{is active}\}$ is a solution of (S, w).

The pseudo proof above only works if a choice network satisfies two restrictions on the bus e. First, the choice network acting as load does not imply that e is a load (or generator) itself. In fact, the connector of a choice network has to be a bus and neither a load or generator. Otherwise, it might not be possible to connect all choice networks at one bus e and still ensure the desired property. For example, if it were necessary for the connector to be a load with a fixed demand to achieve a certain property, then connecting them all at one point would cause a problem. Furthermore, in case of the AC model, the voltage magnitude of e has to be fixed. It cannot depend on the value of the parameter (x) or the state of the network (active or inactive). Otherwise we might not be able to combine the solutions of the choice networks. In one case we prove the properties of the choice network under the assumption that the voltage magnitude of e is fixed to some value z . Henceforth, in this case the main network has to ensure that the voltage magnitude of e is fixed at z.

3.3 Non-Approximability of OPF

We illustrate how to show that there is no ϵ-approximation algorithm for the OPF problem unless P = NP. The idea presented below works for all flow models. Let (S, w) be an SSP instance. The reduction will be such that the existence of an ϵ-approximation algorithm allows us to decide whether (S, w) is solvable or not. Given that the SSP is NP-complete, the existence of an ϵ-approximation algorithm would hence allow us to decide all problems in NP in polynomial time. However, since we assume P $\neq$ NP, this leads to a contradiction.

In the reduction, we have two types of generators. One which is cheap and one which is expensive. The reduction is such that if the SSP instance is solvable only the cheap generators will be used. Let the cost of this be y. Note that these costs depend on w. Also, the reduction has to ensure that y is the lower bound for all possible solutions.

If the instance is not solvable, then the reduction is such that we have to generate at least one unit of power with the expensive generator. The expensive generator will have cost of $\epsilon y + 1$. Hence, if the instance was not solvable then we have overall cost of at least $\epsilon y + 1$.

Let us assume the existence of an ϵ-approximation algorithm. If (S, w) is solvable then the algorithm returns a solution with cost within $[y, \epsilon y]$. If the instance is not solvable then we have cost of at least $\epsilon y + 1$. Hence, we have that (S, w) is solvable if and only if the cost of the solution returned by the algorithm is less or equal than ϵy.

This shows that the ϵ-approximation algorithm decides the SSP instance. One further restriction on the reduction is that the size of y has to be polynomial in the size the instance (S, w). This is because we need the value of y to decide solvability. Note that the size of ϵ does not matter. It can be considered a constant because there is only one algorithm for all SSP instances.

Chapter 4

The DC Power Flow Model with Line Switching

In this chapter, we present the complexity results regarding the Linear AC Approximation of the Alternating Current (AC) power model. This model is also called DC model due to its visual similarity with the flow equations of the direct current. The DC model approximates the AC model by ignoring the reactive power. The model is build on the assumption that we have unity voltage magnitudes. We also assume that the lines are lossless (conductance is 0) and the sine function within the AC power flow is approximated by a linear function (see Section 4.1).

The DC model is a linear model. Its advantage is that if we have a linear objective, we have a Linear Program. This is true for the OPF and the MPF, in the form we study here.

The fact that we can formulate both optimization problems as Linear Programs makes them polynomial to solve in theory (Karmarkar [1984]; Khachiyan [1980]). Section 4.2 shows that line switching can help improving optimal solutions for the OPF and MPF. Hence, in this chapter we study the complexity of these problems with additional line switching.

We use the acronym DS for the DC model with line switching. An overview of the results of this chapter is presented in Table 4.1. This table also highlights the features: network structure, maximum bus degree (mD), number of generators (nG) and number of loads (nL).

We begin this chapter by introducing formal definitions of our problems in Section 4.1. In Section 4.2 we present an introductory example of the effects of line switching and aim to clarify the reasons which lead to this behavior. We then present results about the PF in Section 4.3, the OPF in Section 4.4 and the MPF in Section 4.5. In Section 4.6 we conclude the chapter with related work.

4.1 Background

In this section, we present the definitions for the DC and DC switching (DS) problems of MPF, OPF and PF. These definitions are based on the definitions presented in Chapter 2, especially GP networks (Definition 2.3.1). We start by introducing DC

Problem	Result	Structure	mD	nG	nL	Theorem
DS-PF	Polynomial	at most k cycles	∞	∞	∞	4.3.1
DS-PF	NP-complete	series-parallel	3	1	1	4.3.4
DS-OPF	Polynomial	at most k cycles	∞	∞	∞	4.4.1
DS-OPF	not APX	cacti	3	∞	∞	4.4.4
DS-OPF	NP-complete	cacti	3	∞	∞	4.4.5
DS-OPF	NP-complete	series-parallel	3	2	1	4.4.6
DS-MPF	Polynomial	at most k cycles	∞	∞	∞	4.5.1
DS-MPF	NP-complete	cacti	3	∞	∞	4.5.2
DS-MPF	NP-complete	series-parallel	3	1	1	4.5.3
DS-MPF$_{L^p}$	not APX	arbitrary	∞	6	6	4.5.5
DS-MPF	strongly NP-complete	planar	3	1	1	4.5.6

Table 4.1: DC Model with Line Switching (DS Model) Result Overview

solutions, which are essentially GP solutions (Definition 2.3.5) where the power flow follows the DC *power flow law*.

Definition 4.1.1 (DC solution, congested)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network. A DC *solution* is a tuple (θ, G^p, L^p) such that (θ, G^p, L^p, p) is a GP solution where $p : E^d \to \mathbb{R}$ and we have to have $\forall(\mathsf{a}, \mathsf{d}, b, g, c) \in E^d$:

$$\begin{aligned} |p_{\mathsf{ad}}| &\leq c \text{ and} \\ p_{\mathsf{ad}} &= b(\theta_{\mathsf{a}} - \theta_{\mathsf{d}}). \end{aligned}$$

A line $(\{\mathsf{a}, \mathsf{d}\}, b, g, c) \in E$ is called *congested* if $|p_{\mathsf{ad}}| = c$. The set of all DC solutions of $\mathcal{N}$ is denoted with $\mathcal{S}^{\mathrm{DC}}(\mathcal{N})$.

We refer to p as the *implied flow* from the DC solution (θ, G^p, L^p). Note that the conductance g is not used in the definition of DC solutions. Hence, when defining lines of GP networks within the context of the DC model, we will omit the conductance. Furthermore, we use the following more compact form when defining lines

$$(\{\mathsf{a}, \mathsf{d}\}, b_1, g_1, c_1) \;\approxeq\; \mathsf{a} \xleftrightarrow[c=c_1]{b=b_1} \mathsf{d}.$$

4.1.1 Line Switching

Switching means disconnecting two previously connected buses. When using the word switching, we always refer to switching lines off, never to switching lines on. This is motivated from the fact that we are always given the network and a solution consists of finding a *sub-network*.

Definition 4.1.2 (sub-network)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network and $E' \subseteq E$. The *sub-network* $\mathcal{N}^{E'}$ is defined as $\mathcal{N}^{E'} := (N, N_G, N_L, E \setminus E', G^p, L^p)$.

We call E' the *set of switched lines*. A *switching solution* (DS solution) is essentially a DC solution for a sub-network of a given network $\mathcal{N}$. The set of switched lines becomes part of the solution.

Definition 4.1.3 (DS solution)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network. A DS *solution* is a tuple (E', θ, G^p, L^p) such that $E' \subseteq E$ and (θ, G^p, L^p) is a DC solution of $\mathcal{N}^{E'}$. The set of all DS solutions for $\mathcal{N}$ is denoted with $\mathcal{S}^{\text{DS}}(\mathcal{N})$.

4.1.2 MAXIMUM POWER FLOW

The DC-MPF returns the maximum demand we can satisfy in a given MPF network (see Definition 2.4.1) whilst satisfying the DC power flow law.

Definition 4.1.4 (DC-MPF, DS-MPF)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a MPF network. The DC-MPF of $\mathcal{N}$ is

$$\text{DC-MPF}(\mathcal{N}) := \max_{(\theta, G^p, L^p) \in \mathcal{S}^{\text{DC}}(\mathcal{N})} \sum_{\mathsf{l} \in N_L} L^p_{\mathsf{l}}.$$

Given an $x \in \mathbb{Q}_{\geq 0}$ the DC-MPF problem is to decide whether $\text{DC-MPF}(\mathcal{N}) \geq x$. The DS-MPF of $\mathcal{N}$ is

$$\text{DS-MPF}(\mathcal{N}) := \max_{E' \subseteq E} \text{DC-MPF}(\mathcal{N}^{E'}).$$

Given an $x \in \mathbb{Q}_{\geq 0}$ the DS-MPF problem is to decide whether $\text{DS-MPF}(\mathcal{N}) \geq x$.

We are not able to show non-approximability of the DS-MPF as defined above. However, we show non-approximability for a variant of the DS-MPF where a single load is fixed. This variant is called DS-MPF_{L^p}.

Definition 4.1.5 (DS-MPF_{L^p})**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network with $N_G \cap N_L = \emptyset$, $|dom(G^p)| = 0$ and $|dom(L^p)| = 1$. The DS-MPF_{L^p} of $\mathcal{N}$ is

$$\text{DS-MPF}_{L^p}(\mathcal{N}) := \max_{(E', \theta, G^p, L^p) \in \mathcal{S}^{\text{DS}}(\mathcal{N})} \sum_{\mathsf{l} \in N_L} L^p_{\mathsf{l}}.$$

4.1.3 OPTIMAL POWER FLOW

The OPF is concerned with assigning values to the generation variables so that the given demand is satisfied and the total generation cost is minimal. Since we have a fixed demand, it is possible that no solution exists. In such a case, the DC-OPF will be infinite.

Definition 4.1.6 (DC-OPF, DS-OPF)**.** Let $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$ be a GP network with $(\mathcal{N}, C)$ be an OPF network. The DC-OPF of $\mathcal{N}$ is

$$\text{DC-OPF}(\mathcal{N}, C) := \min_{(\theta, G^p, L^p) \in \mathcal{S}^{\text{DC}}(\mathcal{N})} \sum_{\mathsf{r} \in N_G} |G^p_{\mathsf{r}}| C_{\mathsf{r}}.$$

Given an $x \in \mathbb{Q}_{\geq 0}$ the DC-OPF problem is to decide whether DC-OPF$(\mathcal{N}, C) \leq x$. The DS-OPF of $(\mathcal{N}, C)$ is

$$\text{DS-OPF}(\mathcal{N}, C) := \min_{E' \subseteq E} \text{DC-OPF}(\mathcal{N}^{E'}, C).$$

Given an $x \in \mathbb{Q}_{\geq 0}$ the DS-OPF problem is to decide whether DS-OPF$(\mathcal{N}, C) \leq x$.

4.1.4 POWER FLOW

Solutions for PF networks only have to find phase angles and a value for the active power generation of the slack bus (see Section 2.4.3). The question whether such values exist defines the DC-PF and DS-PF problem.

Definition 4.1.7 (DC-PF, DS-PF)**.** Let $\mathcal{N}$ be a PF network. The DC-PF problem is to decide whether $\mathcal{S}^{\text{DC}}(\mathcal{N}) \neq \emptyset$. The DS-PF problem is to decide whether $\mathcal{S}^{\text{DS}}(\mathcal{N}) \neq \emptyset$.

We call a DC or DS solution *optimal* if it is a witness for the DC-OPF or DC-MPF or DS-OPF or DS-MPF.

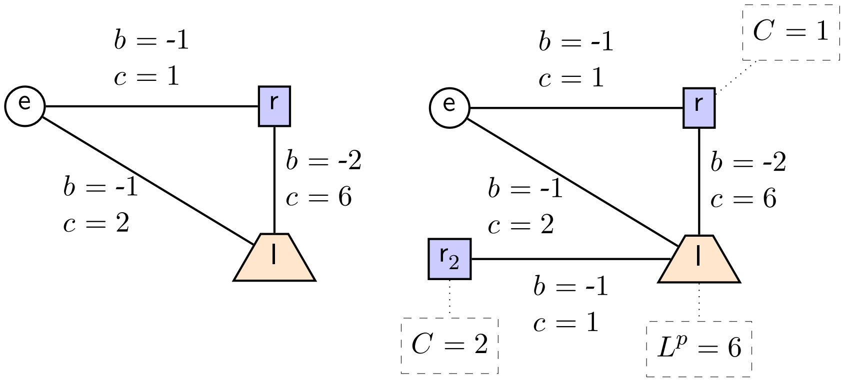
4.2 Example

In the "traditional" maximum flow graph problem, the objective is to find a maximum flow from a source to a sink in a graph. The maximum flow does not benefit from switching off lines. Power flows, on the other hand, do. In this section, we will present this effect via an example and also introduce the reader to the mathematics of the DC model.

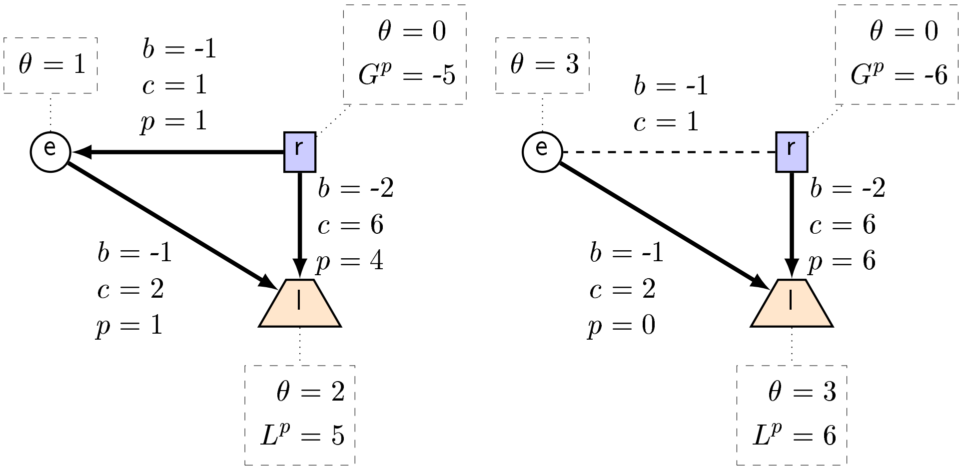
That the switching of lines allows to reduce costs or improve the total power delivered might appear paradoxical at first. Similar effects in traffic networks have been described by Braess [1968], where the addition of a new road to an existing road network leads to an increase in the overall travel time for all participants at the Nash equilibrium. This effect has since been called *Braess Paradox*. We use the example we present in Fig. 4.1 to illustrate the existence and the cause of the "Braess Paradox" in the DC power model.

Figure 4.1(a) presents an MPF network with the structure of a triangle. The "traditional" maximum flow with r being the source and l being the sink of this network is 7. We deliver 6 along the line r$\leftrightarrow$l and 1 along the path via the bus e. The DC-MPF has a value of 5. An optimal solution is presented in Fig. 4.1(c). We can see that the line r$\leftrightarrow$l is not congested. If we want to increase the flow along this line to 6, we would have to increase the phase angle difference between r and l to 3. Kirchhoff's junction law at e implies that this would force the phase angle at e to be 1.5. Hence, we would have a flow of 1.5 along the path r$\rightarrow$e$\rightarrow$l which exceeds the capacity of the line r$\leftrightarrow$e.

In the "traditional" maximum flow the two paths r$\rightarrow$l and r$\rightarrow$e$\rightarrow$l can choose their flow values independently. This is because a flow variable occurs only in the Kirchhoff's junction law of its ends. In contrast, in the DC model, a flow variable also



(a) An MPF network.
(b) An OPF network.



(c) An optimal solution for the DC-MPF.
(d) An optimal solution for the DS-MPF.

Figure 4.1: An example of a network where switching of lines makes a difference for the MPF and the OPF.

depends on the phase angles of the buses. As all flow variables of all lines connected to that bus are depending on the same phase angle, these flow variables are not independent. The capacities of one path can therefore also limit the flow along the other path. By allowing for the switching of lines, we can disable these cyclic dependencies. In our example, switching off the line r↔e allows us to use the line r↔l to its full capacity and hence improve the flow to 6. This is shown in Fig. 4.1(d). The DS solution we present there is also optimal for the DS-MPF. The example also shows that, in general, the DS-MPF value is different from that of the "traditional" maximum flow.

As for the MPF we can observe similar effects for the OPF. Figure 4.1(b) presents an OPF network variant of our example. Here, we have a fixed demand of 6 at l and the generator r has cost of 1. To satisfy this demand in the DC-OPF we have to use the new generator r_2 whose cost is greater than the one of r. Hence, the total cost is 7. In the case of the DS-OPF switching the line r↔e or e↔l (or both) allows us to satisfy the demand with the generator r and therefore decrease our cost to 6.

4.3 POWER FLOW

In this section, we show that the DS-PF problem is NP-complete. An instance of this problem consists of a GP network where all demand and all but one generation variables are fixed (called PF network). The special generator is called: slack bus.

A DC solution for PF network consists of the phase angles and one value for the generation of the slack bus. Since the DC model is lossless, the sum of all generation has to match the sum of load. This implies that the fixed loads and generators determine the generation of the slack bus. For given generation and demand, there exists a unique solution which can be found via solving a linear system. Hence, the DC-PF problem only consists in checking if this solution satisfies the line capacities. The DS-PF problem can also be solved in time polynomial in the input for special cases.

Proposition 4.3.1. *We have:*

1. *the* DC-PF *problem and*
2. *the* DS-PF *problem for* PF *networks with at most* $k \in \mathbb{N}$ *many cycles*

can be solved in time polynomial in the input.

Proof. This result is a consequence of Lemma 4.4.1 and the fact that Fisher et al. [2008] presents a Linear Program where the generators have lower and upper bounds for their generation. Therefore, we can set the upper bound equal to the lower bound in order to fix the generation variables (except for the slack bus). ☐

In the following, we show that the DS-PF problem becomes NP-complete if we allow for arbitrary many cycles in the network. At the core of the proof are the *phase-angle-difference choice networks*, illustrated in Fig. 4.2 and defined below.

Definition 4.3.2 (phase-angle-difference choice network). Let $x \in \mathbb{Q}_{>0}$ be a number. The *phase-angle-difference choice network* with respect to x is defined as

$$\mathcal{A}^{x}_{\mathsf{r},\mathsf{l}} := \left(\{\mathsf{r},\mathsf{l},\mathsf{a}\}, \emptyset, \emptyset, \left\{ \mathsf{r} \xleftrightarrow[c=1]{b=\frac{-1}{x+1}} \mathsf{l} \xleftrightarrow[c=1-\frac{1}{x+1}]{b=2\left(\frac{1}{x+1}-1\right)} \mathsf{a} \xleftrightarrow[c=1-\frac{1}{x+1}]{b=2\left(\frac{1}{x+1}-1\right)} \mathsf{r} \right\}, \emptyset \right).$$

A phase-angle-difference choice network has no generators or loads. However, it has two connectors r and l. Their names indicate that we use the choice network in such a way that power will come into the network via r and leave via l. This network is designed for a power value of 1. The key property of this choice network is that there are exactly two different phase angle differences possible to achieve a flow of 1 from r to l. The first possible phase angle difference is the value 1. This is achieved by switching at least one of the lines $\mathsf{r}\leftrightarrow\mathsf{a}$ or $\mathsf{l}\leftrightarrow\mathsf{a}$. In the case where no line is switched off, we have a phase angle difference of $x + 1$. Two solutions for the first and second case are visualized in Fig. 4.3. Fig. 4.3(a) shows the case where no line is switched and we have a phase angle difference of 1. Fig. 4.3(b) shows the case where the path

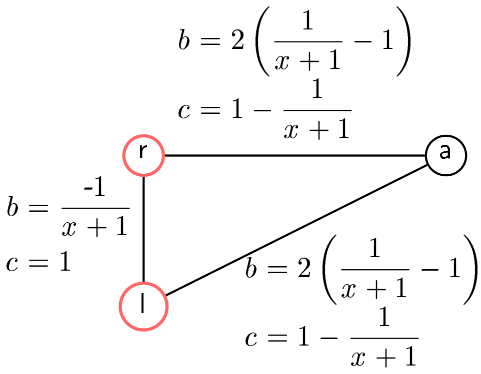


Figure 4.2: The phase-angle-difference choice network $\mathcal{A}^x_{\mathsf{r},\mathsf{l}}$.

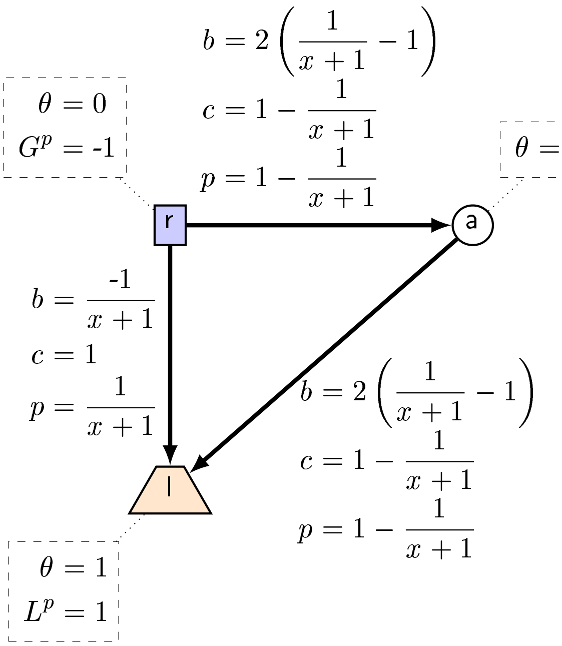
from r to l via a is disconnected. In this solution, we have a phase angle difference of $x+1$. We show this property in Lemma 4.3.3. In the lemma we also show that it is not possible to send more than one unit of power from r to l.

Lemma 4.3.3. *Let $x \in \mathbb{Q}_{>0}$ be a number. We have:*

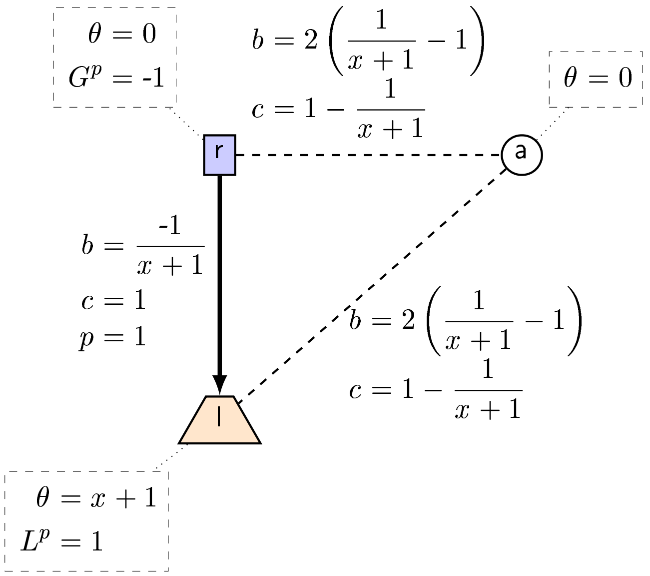
1. $\forall (E, \theta, G^p, L^p) \in \mathcal{S}^{\mathrm{DS}}(\mathcal{A}^x_{\mathsf{r},\mathsf{l}}[G^p_{\mathsf{r}}{=}\text{-}1|L^p_{\mathsf{l}}{=}1]) : \Delta_{\mathsf{lr}} \in \{1, x+1\}$.
2. $\forall (E, \theta, G^p, L^p) \in \mathcal{S}^{\mathrm{DS}}(\mathcal{A}^x_{\mathsf{r},\mathsf{l}}[\mathsf{r}{\in}N_G|\mathsf{l}{\in}N_L]) : L^p_{\mathsf{l}} \leq 1$.
3. *$\mathcal{A}^x_{\mathsf{r},\mathsf{l}}$ has 3 buses and the size of every line parameter is polynomial in the size of x.*

Proof. Let (E, θ, G^p, L^p) be a DS solution of $\mathcal{A}^x_{\mathsf{r},\mathsf{l}}[G^p_{\mathsf{r}}{=}\text{-}1|L^p_{\mathsf{l}}{=}1]$. The generation of 1 at r and the demand of 1 at l imply that there has to be a flow of 1 from r to l. Let us assume that the lines r↔l is switched off. In this case, we have to have a flow of 1 along the line r↔a. Since $x > 0$, the capacity of this edge is $1 - \frac{1}{x+1} < 1$. Hence, a flow of 1 is not possible and r↔l cannot be switched. Therefore, we have only two different cases: either nothing is switched or at least one of the lines r↔a, a↔l is switched off. Let us assume that nothing is switched off. We will now show that no capacities can be violated. Kirchhoff's junction law at a is $0 = p_{\mathsf{ar}} + p_{\mathsf{al}}$ which implies $\Delta_{\mathsf{al}} = \text{-}\Delta_{\mathsf{ar}} = \Delta_{\mathsf{ra}}$ and hence $\Delta_{\mathsf{rl}} = \Delta_{\mathsf{ra}} + \Delta_{\mathsf{al}} = 2\Delta_{\mathsf{ra}}$. Using this and Kirchhoff's junction law at r we can derive

$$\begin{aligned}
0 &= G^g + p_{\mathsf{rl}} + p_{\mathsf{ra}} \\
&= \text{-}1 + \frac{\text{-}1}{x+1}\Delta_{\mathsf{rl}} + 2\left(\frac{1}{x+1} - 1\right)\Delta_{\mathsf{ra}} \\
&= \text{-}1 + \frac{\text{-}1}{x+1}\Delta_{\mathsf{rl}} + \left(\frac{1}{x+1} - 1\right)\Delta_{\mathsf{rl}} \\
&= \text{-}1 - \Delta_{\mathsf{rl}} \\
\Delta_{\mathsf{rl}} &= \text{-}1 \\
\Delta_{\mathsf{lr}} &= 1.
\end{aligned}$$



(a) Solution One.



(b) Solution Two.

Figure 4.3: The two possible solutions of a phase-angle difference choice network $\mathcal{A}^x_{\mathsf{r},\mathsf{l}}[G^p_{\mathsf{r}}{=}\text{-}1|L^p_{\mathsf{l}}{=}1]$.

Since $x > 0$ we have $|p_{\mathsf{lr}}| = |\frac{1}{x+1}| < 1$ and $|p_{\mathsf{ar}}| = 1 - \frac{1}{x+1}$. Hence, no capacity is violated.

Let us assume that at least one of the lines r↔a or a↔l is switched. In this case, we have to have a flow of 1 along the line r↔l which is within its capacity.

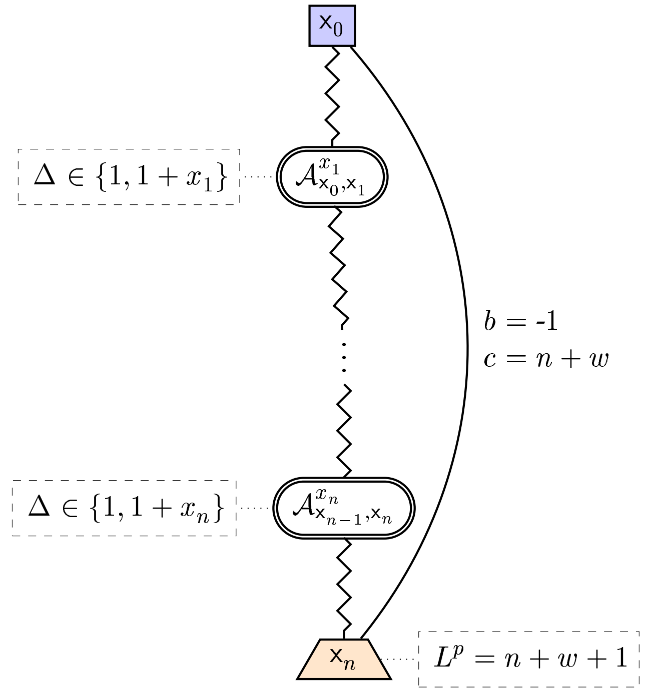


Figure 4.4: The reduction of an SSP instance presented in Theorem 4.3.4.

The fact that the active power flow of a↔r in the first case and r↔l in the second case are at their maximum implies the second part of the lemma.

Part three can be directly derived from Definition 4.3.2. □

Using the phase-angle difference choice network, we can reduce the SSP problem to the DS-PF problem. Let (S, w) be an SSP instance with $S = \{x_1, \dots x_n\}$. Fig. 4.4 presents the reduction into a PF network. Suppose, we have one bus x_i per element of S and there are choice networks $\mathcal{A}^{x_i}_{x_i,x_{i+1}}$ between these buses. Suppose, there is an additional bus x_0 which is a generator. Let the bus x_n be a load with a fixed demand of $n + w + 1$. The generator is connected to the load via a line with susceptance of -1 and capacity of $n + w$. From Lemma 4.3.3, we know that the networks $\mathcal{A}^{x_i}_{x_i,x_{i+1}}$ only allow for a flow of 1 at maximum. Hence, the remaining demand of $n + w$ at x_n can only be satisfied via the line $x_0 \leftrightarrow x_n$. This implies that the phase angle difference between x_0 and x_n is $n+w$. All choice networks have to work together to achieve this phase angle difference where every network has the choice of a difference of either 1 or $1 + x_i$. Hence, we have a solution if and only if the (S, w) is solvable.

Theorem 4.3.4. *Deciding if there exists a* DS-PF *solution for series-parallel* PF *networks with one generator and one load is* NP*-complete.*

Proof. To show the membership in NP we first use the oracle to provide us with the set of switched off lines. The remaining DC-PF problem can be solved in time polynomial in the input according to Proposition 4.3.1. Hence, the problem is in NP.

In the following, we present a reduction of the SSP problem into the DS-PF problem. Let (S, w) be a SSP instance with $S = \{x_1, \ldots, x_n\}$ and x_0 be a symbol not in S. We define the two GP networks

$$\mathcal{N} := \left(\{\mathsf{x}_0, \mathsf{x}_n\}, \{\mathsf{x}_0\}, \{\mathsf{x}_n\}, \left\{\mathsf{x}_0 \xleftrightarrow[c=n+w]{b=\text{-}1} \mathsf{x}_n\right\}, \left[L^p_{\mathsf{x}_n}{=}n + 1 + w\right]\right),$$

$$\mathcal{N}_{S,w} := \mathcal{N} +^{\mathsf{x}_0} \mathcal{A}^{x_1}_{\mathsf{x}_0,\mathsf{x}_1} +^{\mathsf{x}_1} \mathcal{A}^{x_2}_{\mathsf{x}_1,\mathsf{x}_2} +^{\mathsf{x}_2} \ldots +^{\{\mathsf{x}_{n-1},\mathsf{x}_n\}} \mathcal{A}^{x_n}_{\mathsf{x}_{n-1},\mathsf{x}_n}$$

and we are going to show that $\mathcal{N}_{S,w}$ is a PF network with size polynomial in (S, w) and that

$$\mathcal{S}^{\text{DS}}(\mathcal{N}_{S,w}) \neq \emptyset \iff (S, w) \text{ is solvable.}$$

The network $\mathcal{N}_{S,w}$ has exactly one generator which is not fixed and hence is the slack bus. The network also has exactly one load with a fixed demand. This shows that the network is a PF network. Using part three of Lemma 4.3.3 we can derive that $\mathcal{N}_{S,w}$ has $2n + 1$ buses. Furthermore, every line parameter is polynomial in the size of the input. Hence, the size of $\mathcal{N}_{S,w}$ is polynomial in the size of (S, w).

Lemma 4.3.3 shows that we can push a maximum of 1 through $\mathcal{A}^{x_1}_{\mathsf{x}_0,\mathsf{x}_1}$. The line $\mathsf{x}_0{\leftrightarrow}\mathsf{x}_n$ can only handle power of $n + w$. Hence, in order to get the generated power of $n + w + 1$ from the generator x_0 to the load x_n, the line $\mathsf{x}_0{\leftrightarrow}\mathsf{x}_n$ has to be congested and there is a flow of 1 through every network $\mathcal{A}^{x_i}_{\mathsf{x}_{i-1},\mathsf{x}_i}$. The congestion of $\mathsf{x}_0{\leftrightarrow}\mathsf{x}_n$ implies $\Delta_{\mathsf{x}_0\mathsf{x}_n} = n + w$. Hence, the sum of all phase angle differences across the choice networks has to be equal to $\sum_{1 \le i \le n} \Delta_{\mathsf{x}_{i-1}\mathsf{x}_i} = \Delta_{\mathsf{x}_0\mathsf{x}_n} = w + n$. In Lemma 4.3.3 we show that every phase angle difference $\Delta_{\mathsf{x}_{i-1}\mathsf{x}_i}$ is either 1 or $x_i + 1$. We say an i is *active* if $\Delta_{\mathsf{x}_{i-1}\mathsf{x}_i} = x_i + 1$. Our observation above implies

$$n + \sum_{\substack{1 \le i \le n, \\ i \text{ is active}}} x_i = w + n,$$

which shows there exists a DS-PF solution if and only if (S, w) is solvable. □

One can use similar constructions to show that the DS-OPF and the DS-MPF are NP-complete for series parallel networks with one load and two (resp. one) generator. The corresponding results for the DS-OPF can be found in Section 4.4 and for the DS-MPF in Section 4.5.

4.4 OPTIMAL POWER FLOW

In this section, we show that the DS-OPF on cacti networks cannot be arbitrarily approximated with a polynomial algorithm. Recall that a cactus is a network where each line is part of at most one cycle (see Section 2.2). We also show that the DS-OPF problem is NP-complete for cacti networks and series-parallel networks with one load and two generators. The reductions have a bounded maximum bus degree of 3 in both cases.

The DS-OPF problem can be solved in time polynomial in the input if the number of cycles in the network is bounded by some constant. This is a consequence of the fact that the DS-OPF can be solved in polynomial time when the underlying network has a tree structure. In the following, we also show that the problem becomes easy once we have decided which lines to switch.

Lemma 4.4.1. *We have:*

1. *the* DC-OPF *problem and*
2. *the* DS-OPF *problem for* OPF *networks with at most* $k \in \mathbb{N}$ *many cycles*

can be solved in time polynomial in the input.

Proof. Fisher et al. [2008] presents a Mixed-Integer-Linear Program (MILP) to find the DS-OPF. All binary/integer variables in the program are associated with the switching of lines. Hence, by fixing these variables we obtain a Linear Program (LP). This shows that there is an LP which can solve the DC-OPF problem which implies the first point. Khachiyan [1980] and Karmarkar [1984] show that any LP can be solved in time polynomial in the input.

We now investigate the DS-OPF problem. Phase angles only appear in the flow equation as difference. Hence, if we shift the phase angles, we obtain a solution with same the flow, generation, and load. With this in mind, we can show that a line which is not part of a cycle (we call it a *tree line*) does not need to be considered when trying to find the DS-OPF. Assume that we have a tree line $\mathsf{a}\leftrightarrow\mathsf{d}$ and a solution where $\mathsf{a}\leftrightarrow\mathsf{d}$ was switched. Since $\mathsf{a}\leftrightarrow\mathsf{d}$ is a tree line, the network is split in two disjoint parts. The observation above shows that we can create a new solution where the phase angles in the part which contains a are scaled such that θ_{a} becomes equal to θ_{d} and all flows, generation and load are the same. Since both ends of the line have the same phase angle there would be no flow on the line if it where not switched. Hence, there is a solution with the same flows, generation and load where the line is not switched (and has no flow on it). This implies that DC-OPF$(\mathcal{N}, C) \leq$ DC-OPF$(\mathcal{N}^{\{\mathsf{a}\leftrightarrow\mathsf{d}\}}, C)$.

Consider a network with k cycles and n lines. As we do not need to consider tree lines when switching, every line switched destroys one cycle. Hence, we will switch at most k lines. Therefore, we have at most n^k possible networks to which we have to compute the DC-OPF for. Since k is a constant, these are only polynomial many networks with respect to the size of the original network. □

In general, a network can have exponentially many cycles. In between the general and the case of a constant number of cycles lies the case of having a linear number of cycles. As we will show in the following, this case is NP-complete. We reduce the SSP problem to the DS-OPF problem using cacti networks which have only a linear number of cycles with respect to the size of the SSP instance.

We achieve the reduction by using *load-choice networks*, which are presented in Fig. 4.5. Such a network is a triangle built of a generator r, a load l and a connector e.

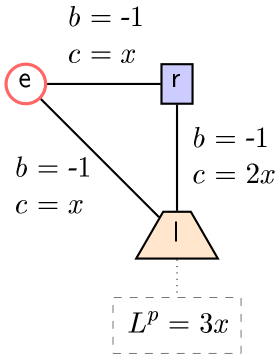
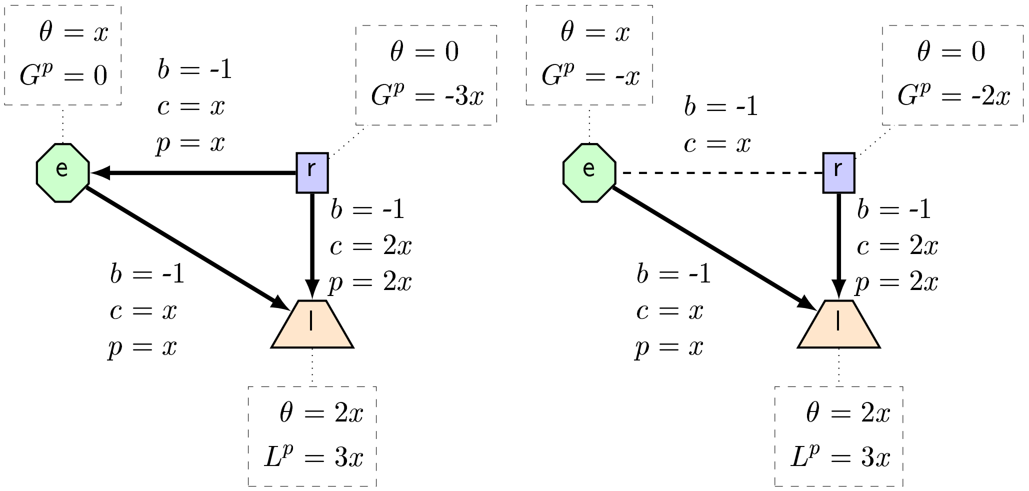


Figure 4.5: The load-choice network $\widetilde{\mathcal{L}}^x_{\mathsf{e}}$.

Definition 4.4.2 (load-choice network)**.** Let $x \in \mathbb{Q}_{>0}$ be a number. We define the *load-choice network* with respect to x by

$$\mathcal{L}^x_{\mathsf{e}} := \left(\{\mathsf{r},\mathsf{l},\mathsf{e}\}, \{\mathsf{r}\}, \{\mathsf{l}\}, \left\{ \mathsf{r} \overset{b=-1}{\underset{c=2x}{\longleftrightarrow}} \mathsf{l} \overset{b=-1}{\underset{c=x}{\longleftrightarrow}} \mathsf{e} \overset{b=-1}{\underset{c=x}{\longleftrightarrow}} \mathsf{r} \right\}, \emptyset \right).$$

Furthermore, let $\widetilde{\mathcal{L}}^x_{\mathsf{e}} := \mathcal{L}^x_{\mathsf{e}}[L^p_{\mathsf{l}}{=}3x]$.



(a) Solution with $G^p_{\mathsf{e}} + L^p_{\mathsf{e}} = 0$. (b) Solution with $G^p_{\mathsf{e}} + L^p_{\mathsf{e}} = x$.

Figure 4.6: Two different solutions of $\widetilde{\mathcal{L}}^x_{\mathsf{e}}$ in the case where the connector e is a generator and a load.

The network $\widetilde{\mathcal{L}}^x_{\mathsf{e}}$ is the variant of $\mathcal{L}^x_{\mathsf{e}}$ with a fixed demand. This is the variant we are going to use in the proof below. Distinguishing between these two cases is necessary because we are going to use load-choice networks for the DS-MPF problem and an MPF network does not have any fixed demand. For the rest of this section the term load-choice network refers to the network $\widetilde{\mathcal{L}}^x_{\mathsf{e}}$. A load-choice network has the property that in the case where e is connected to the outside world it can either consume nothing or x to satisfy its demand. These two different solutions are presented in Fig. 4.6. In Lemma 4.4.3, we will show that these are the only different solutions up to the shifting of phase angles.

Lemma 4.4.3. *Let $x \in \mathbb{Q}_{>0}$ be a number.*

1. *For every* DS *solution of $\widetilde{\mathcal{L}}^{x}_{\mathsf{e}}[\mathsf{e}{\in}N_{G/L}]$ we have $G^{p}_{\mathsf{e}} + L^{p}_{\mathsf{e}} \in \{\text{-}x, 0\}$.*
2. *The networks $\mathcal{L}^{x}_{\mathsf{e}}$ and $\widetilde{\mathcal{L}}^{x}_{\mathsf{e}}$ have 3 buses and the size of every line parameters is polynomial in the size of x.*

Proof. The load at l has a demand of $3x$. The lines connected to l can supply a maximum of $2x$ via the line $\mathsf{l}{\leftrightarrow}\mathsf{r}$ and x via the line $\mathsf{l}{\leftrightarrow}\mathsf{e}$. Hence, both lines have to be congested and cannot be switched in any DS solution. If the line $\mathsf{r}{\leftrightarrow}\mathsf{e}$ is switched then we have to have $G^{p}_{\mathsf{e}} + L^{p}_{\mathsf{e}} = x$ in order to satisfy Kirchhoff's junction law at e. Assume that $\mathsf{r}{\leftrightarrow}\mathsf{e}$ is not switched. The congestion of $\mathsf{l}{\leftrightarrow}\mathsf{r}$ implies that $\Delta_{\mathsf{rl}} = 2$ and the congestion of $\mathsf{l}{\leftrightarrow}\mathsf{e}$ implies $\Delta_{\mathsf{el}} = 1$. Hence, we have $\Delta_{\mathsf{re}} = 1$. This implies a flow of x at $\mathsf{r}{\leftrightarrow}\mathsf{e}$. Hence, Kirchhoff's junction law at e is satisfied if no additional generation or load is present at e.

Part two can be directly derived from Definition 4.4.2. □

Using load-choice networks we can use any ϵ-approximation algorithm to decide the SSP problem. Let (S, w) be an SSP instance. The network based on (S, w) is presented visually in Fig. 4.7. The generator r is cheaper than the generator at x_0. Hence, we prefer to use r to satisfy the demand at l. By doing so we create an implicit generation of w at x_0 (not coming from x_0). This generation has to be absorbed by the choice networks. Lemma 4.4.3 shows that each choice network $\widetilde{\mathcal{L}}^{x_i}_{\mathsf{e}}$ can only consume a value of x_i or nothing. Hence, we have a one-to-one correspondence: the SSP instance is solvable if and only if the choice networks can absorb the power of w. In the case where the instance is not solvable, the triangle has to switch the line $\mathsf{r}{\leftrightarrow}\mathsf{x}_0$. Therefore, there is one unit of power not being satisfied at l which the generator x_0 has to provide. The generator costs are chosen such that generating one unit of power makes this solution more expensive than in the other case, even when we factor in the approximation. Hence, we can decide if the SSP instance is solvable by looking at the value the ϵ-approximation algorithm returns.

Theorem 4.4.4. *There is no ϵ-approximation algorithm for the* DS-OPF *on cacti* OPF *networks with maximum degree of* 3 *unless* $\mathsf{P} = \mathsf{NP}$.

Proof. Assume there exists an ϵ-approximation and (S, w) be an SSP instance where $S = \{x_1, \ldots, x_n\}$ and $m := \sum_{x \in S} x$. We define the set of lines and GP networks:

$$E := \left\{\mathsf{r} \xleftrightarrow[c=2+w]{b=\text{-}1} \mathsf{l} \xleftrightarrow[c=1]{b=\text{-}1} \mathsf{x}_0 \xleftrightarrow[c=1+w]{b=\text{-}1} \mathsf{r}\right\} \cup \left\{\mathsf{x}_{i-1} \xleftrightarrow[c=w]{b=\text{-}1} \mathsf{x}_i \xleftrightarrow[c=x_i]{b=\text{-}1} \mathsf{x}'_i \mid 1 \le i \le n\right\},$$

$$\mathcal{N} := \left(\{\mathsf{r}, \mathsf{l}, \mathsf{x}_0\} \cup \{\mathsf{x}_i, \mathsf{x}'_i \mid 1 \le i \le n\}, \{\mathsf{r}, \mathsf{x}_0\}, \{\mathsf{l}\}, E, \left[L^{p}_{\mathsf{l}}{=}3 + w\right]\right),$$

$$\mathcal{N}_{S,w} := \mathcal{N} +^{\mathsf{x}'_1} \widetilde{\mathcal{L}}^{x_1}_{\mathsf{x}'_1} +^{\mathsf{x}'_2} \ldots +^{\mathsf{x}'_n} \widetilde{\mathcal{L}}^{x_n}_{\mathsf{x}'_n}.$$

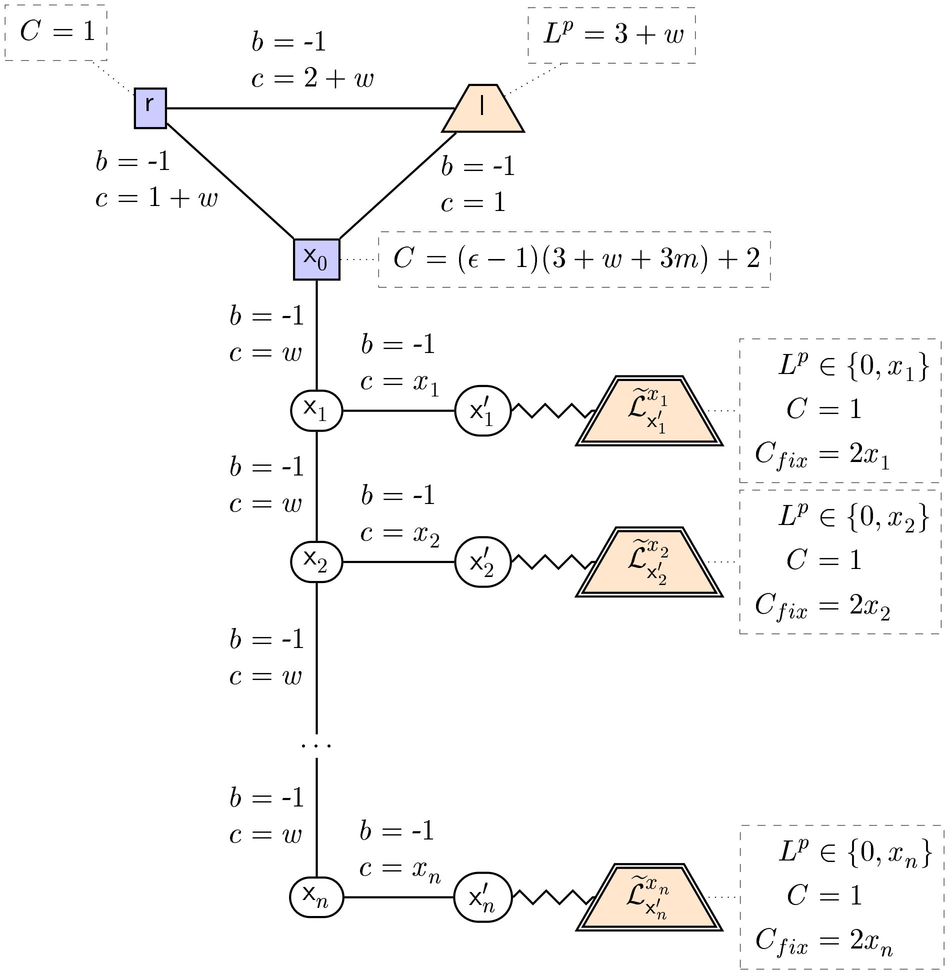


Figure 4.7: The network $\mathcal{N}_{S,w}$ from Theorem 4.4.4 for the SSP instance (S, w) with $S = \{x_1, \ldots, x_n\}$.

For a choice network $\widetilde{\mathcal{L}}^{x_i}_{\mathsf{x}'_i}$, let r_i be the generator within. We define the cost function $C : \{\mathsf{r}, \mathsf{x}_0\} \cup \{\mathsf{r}_i \mid 1 \leq n \leq n\} \rightarrow \mathbb{Q}_{\geq 0}$ with $C_{\mathsf{r}} := C_{\mathsf{r}_1} := \ldots := C_{\mathsf{r}_n} := 1$ and $C_{\mathsf{x}_0} := (\epsilon - 1)(3 + w + 3m) + 2$.

The GP network $\mathcal{N}_{S,w}$ has non-fixed generators only, and all loads are fixed. Using part two of Lemma 4.4.3 we can derive that $\mathcal{N}_{S,w}$ has $4n + 3$ buses. Furthermore, we observe that the size of all line parameters is polynomial in the size of (S, w). The cost is either 1 or they depend on w, m and ϵ (which is a constant). Hence, the tuple $(\mathcal{N}_{S,w}, C)$ is an OPF networks which is polynomial in the size of (S, w). Every bus x_i and x'_i has a degree less than or equal to 3 and every other bus has a degree of 2. Hence, the maximum degree is 3.

To satisfy the demand at l all its lines have to be congested. Lemma 4.4.3 shows that the generators of the networks $\widetilde{\mathcal{L}}^{x_0}_{\mathsf{x}'_0}$ cannot provide any power to l. Since the generator r is cheaper than x_0 the power on the line r↔l comes from r. The power of 1 on the line x_0↔l can either come from x_0 or r. This notion is equivalent to switch the line $\mathsf{r}\leftrightarrow\mathsf{x}_0$. Since $\mathsf{x}_0\leftrightarrow\mathsf{l}$ and $\mathsf{r}\leftrightarrow\mathsf{l}$ are congested we have $\Delta_{\mathsf{r}\mathsf{x}_0} = 1 + w$. Hence, if $\mathsf{r}\leftrightarrow\mathsf{x}_0$ is not switched then r would provide additional power of w to x_0, which can only be absorbed by the networks $\widetilde{\mathcal{L}}^{x_i}_{\mathsf{x}_i'}$. Lemma 4.4.3 shows that these networks can only take in x_i or nothing.

Assume (S, w) is solvable. In this case, we can distribute the power of w. The generator r generates power of $3 + 2w$. A choice network generates $3x_i$ if it consumes nothing, or $2x_i$ if it consumes x_i. Therefore, the generators in the choice networks have a total generation of $2m + m - w$. Hence, we have optimal cost of DS-OPF$(\mathcal{N}_{S,w}, C) = 3 + w + 3m$. That implies that an ϵ-approximation algorithm would return a solution within the interval

$$[3 + w + 3m, \epsilon(3 + w + 3m)].$$

On the other hand, if (S, w) is not solvable then we cannot distribute the power among the choice networks. Hence, we have to switch off the lines $\mathsf{r}\leftrightarrow\mathsf{x}_0$. This leaves the load at l with at least one unit of power short. As the choice networks can only act as loads only the generator x_0 can provide this power. This implies a total cost of $3m$ from the choice networks, $2 + w$ from r and $(\epsilon - 1)(3 + w + 3m) + 2$ from x_0. Hence, an ϵ-approximation algorithm would return a solution within the interval

$$[\epsilon(3 + w + 3m) + 1, \infty).$$

This interval is disjoint from the one above. Hence, (S, w) is solvable if and only if the ϵ-approximation algorithm returns a solution less or equal to $\epsilon(3 + w + 3m)$. We can therefore use the ϵ-approximation algorithm to decide whether (S, w) is solvable by checking its return value against $\epsilon(3 + w + 3m)$. □

We can use the reduction from above to show that the DS-OPF problem is NP-complete. The OPF problem is in NP because we can express it as a Mixed-Integer-Linear Program and hence every solution has to have rational values in size polynomial in the input Karmarkar [1984]; Khachiyan [1980].

Theorem 4.4.5. *The* DS-OPF *problem for* OPF *cacti networks with a maximum degree of* 3 *is* NP*-complete.*

Proof. Let $(\mathcal{N}_{S,w}, C)$ be the network from Theorem 4.4.4 for some $\epsilon > 1$. The theorem is a direct consequence from Theorem 4.4.4 which shows that DS-OPF$(\mathcal{N}_{S,w}, C) = 3 + w + 3m$ if and only if (S, w) is solvable. □

The final result shown in this section is an adaption of the proof from Theorem 4.3.4 in the previous section.

Theorem 4.4.6. *The* DS-OPF *problem for series-parallel networks with two generators and one load is* NP*-complete.*

Proof. Let $\mathcal{N}_{S,w}$ be the network from Theorem 4.3.4 with an additional generator x_{n+1} connected to x_n and the generator x_0 not be fixed. The purpose of this generator is to ensure the existence of a solution. Recall that we defined the DS-OPF problem for networks with at least one solution. We define costs $C : \{\mathsf{x}_0, \mathsf{x}_{n+1}\} \to \mathbb{R}_{\geq 0}$ of $C_{\mathsf{x}_0} := 1$ and $C_{\mathsf{x}_{n+1}} := 2$ and we are going to show that

$$\text{DS-OPF}(\mathcal{N}_{S,w}, C) \leq n + w + 1 \iff (S, w) \text{ is solvable.}$$

The network has one fixed load with a demand of $n + w + 1$. Furthermore, the DC model is lossless. Therefore

$$\text{DS-OPF}(\mathcal{N}_{S,w}, C) \geq n + w + 1. \tag{4.1}$$

If the generator x_{n+1} generates any power, then we have to have $\text{DS-OPF}(\mathcal{N}_{S,w}, C) > n + w + 1$ because its cost is greater than 1. Hence, Theorem 4.3.4 shows that (S, w) is solvable if and only if the generator at x_0 can satisfy the demand. □

4.5 MAXIMUM POWER FLOW

In this section, we present the proof that the DS-MPF problem is NP-complete on cacti networks and strongly NP-complete on planar networks. Additionally, we show that the DS-MPF problem with one fixed load (DS-MPF$_{L^p}$) cannot be approximated within any constant factor. As the DC-OPF and the DC-PF problem, the DC-MPF problem is polynomial. Similarly, switching is easy when the number of cycles is bounded.

Lemma 4.5.1. *We have that*

1. *the* DC-MPF *problem and*
2. *the* DS-MPF *problem for* MPF *networks with at most* $k \in \mathbb{N}$ *many cycles*

can be solved in time polynomial in the input.

Proof. The objective of the MPF can be expressed as a linear function. Hence, the result follows by the same arguments as those from the proof of Lemma 4.4.1. □

Similarly to the DS-OPF, having a linear number of cycles in the network makes the problem NP-complete. We will show this with a reduction which is similar to the one for the DS-OPF (Theorem 4.4.4) using the same choice network: load-choice network (Definition 4.4.2). In Fig. 4.8, we present this reduction. The difference to the encoding for the DS-OPF is that we use the variant of the load-choice network where the loads are not fixed, the demand of l is not fixed, and that there is no generator at x_0.

Theorem 4.5.2. *The* DS-MPF *problem for cacti* MPF *networks with a maximum degree of 3 is* NP-*complete.*

Proof. The DS-MPF problem is in NP because we can formulated it as a Mixed-Integer-Linear Program (see Lemma 4.5.1). Let (S, w) be a SSP instance and we set $m :=$

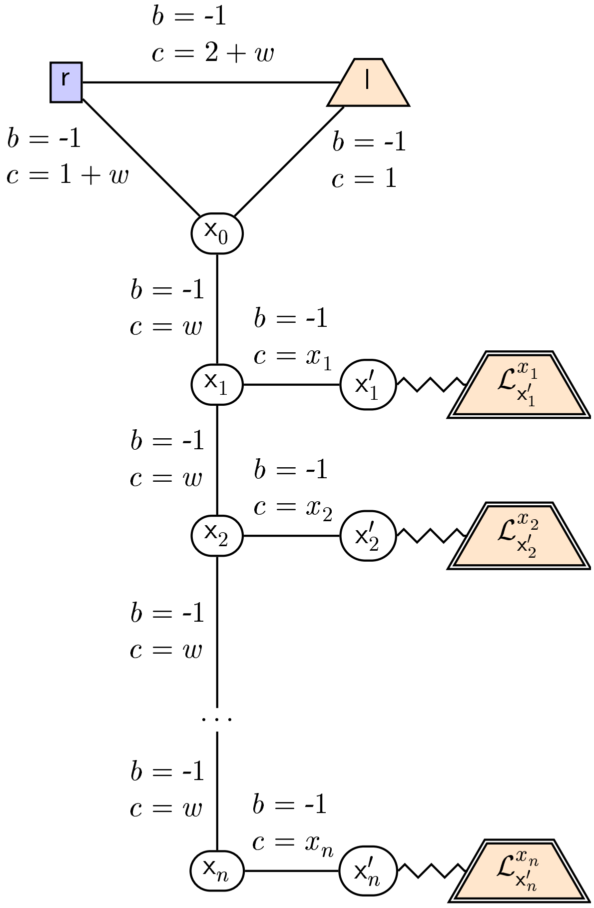


Figure 4.8: The network $\mathcal{N}_{S,w}$ from Theorem 4.5.2 for the SSP instance (S, w) with $S = \{x_1, \ldots, x_n\}$.

$\sum_{x \in S} x$. We define the set of lines and GP networks

$$\begin{aligned}
E &:= \left\{ \mathsf{r} \xleftrightarrow[c=2+w]{b=-1} \mathsf{l} \xleftrightarrow[c=1]{b=-1} \mathsf{x}_0 \xleftrightarrow[c=1+w]{b=-1} \mathsf{r} \right\} \cup \left\{ \mathsf{x}_{i-1} \xleftrightarrow[c=w]{b=-1} \mathsf{x}_i \xleftrightarrow[c=x_i]{b=-1} \mathsf{x}'_i \;\middle|\; 1 \le i \le n \right\}, \\
\mathcal{N} &:= (\{\mathsf{r}, \mathsf{l}, \mathsf{x}_0\} \cup \{\mathsf{x}_i, \mathsf{x}'_i \mid 1 \le i \le n\}, \{\mathsf{r}\}, \{\mathsf{l}\}, E, \emptyset), \\
\mathcal{N}_{S,w} &:= \mathcal{N} +^{\mathsf{x}'_1} \mathcal{L}^{x_1}_{\mathsf{x}'_1} +^{\mathsf{x}'_2} \ldots +^{\mathsf{x}'_n} \mathcal{L}^{x_n}_{\mathsf{x}'_n}.
\end{aligned}$$

The GP network $\mathcal{N}_{S,w}$ has non-fixed generators and non-fixed demand. Using part two of Lemma 4.4.3 we can derive that $\mathcal{N}_{S,w}$ has $4n + 3$ buses. Furthermore, we observe that all line parameters are polynomial depending on values from (S, w). Hence, the network $\mathcal{N}_{S,w}$ is an OPF networks which is polynomial in the size of (S, w). Every bus x_i and x'_i has a degree less or equal to 3 and every other bus has a degree of 2. Hence, the maximum degree is 3. We will now show that

$$\text{DS-MPF}(\mathcal{N}_{S,w}) \ge 3 + w + 3m \iff (S, w) \text{ is solvable.}$$

In the following, we assume that DS-MPF$(\mathcal{N}_{S,w}) \geq 3+w+3m$. This total demand value is equal to the sum of the capacities of all loads of $\mathcal{N}_{S,w}$. Therefore, we can regard the choice networks as having a fixed demand of $3x_i$. This allows us to use Lemma 4.4.3. Which implies that a choice network $\mathcal{L}^{x_i}_{\mathsf{x}'_i}$ can only act as load and has the choice to consume power of x_i. We call a choice network which consumes power *active*.

Since both lines of the load l have to be congested, we have $\Delta_{\mathsf{rl}} = 2 + w$ and $\Delta_{\mathsf{x}_0\mathsf{l}} = 1$. Therefore, $\Delta_{\mathsf{rx}_0} = 1 + w$. Hence, we have an incoming power of $1 + w$ from r at x_0 and an outgoing power of 1 along the line $\mathsf{x}_0 \leftrightarrow \mathsf{l}$. Kirchhoff's junction law at x_0 implies that the rest of the power must be collectively consumed by the load-choice networks $\mathcal{L}^{x_i}_{\mathsf{x}'_i}$. Hence, we have to activate some networks such that

$$w = \sum_{\substack{x \in S \\ \mathcal{L}^{x}_{\mathsf{x}'} \text{ is active}}} x.$$

This shows the one-to-one correspondence between the elements of a solution of (S, w) and the active choice networks. □

Next we show that we can use an idea similar to that used to prove Theorem 4.3.4 to show that, in the case where we have only one generator and one load, series-parallel networks are NP-complete.

Theorem 4.5.3. *The* DS-MPF *problem for series-parallel networks with one generators and one load is* NP*-complete.*

Proof. Let (S, w) be an SSP instance and $\mathcal{N}_{S,w}$ be the network from Theorem 4.3.4 with unfixed generator and load. Theorem 4.3.4 directly implies that DS-MPF$(\mathcal{N}_{S,w}) \geq n + w + 1 \iff (S, w)$ is solvable. □

In the following, we establish that the DS-MPF problem with one fixed-demand load can not be approximated within any constant factor. To that end, we present a reduction of a variant of the LONGEST PATH problem to the DS-MPF$_{L^p}$ problem such that every approximation algorithm for the DS-MPF$_{L^p}$ would imply an approximation result for the LONGEST PATH.

Definition 4.5.4 (Longest Path). Let (N, E) be a graph with $\mathsf{e}, \mathsf{s} \in N$ and $\mathsf{e} \neq \mathsf{s}$. The e-s *Longest Path* problem is to find a path that starts in e, ends in s, visits every bus at most once and is maximal in terms of the number of buses visited.

It is known that for all $\epsilon > 0$ it is not possible to approximate LONGEST PATH to within a factor of $2^{(\log n)^{1-\epsilon}}$ unless NP is contained within quasi-polynomial deterministic time Karger et al. [1997]. This must also be true for our variant with fixed start and end nodes because there are only quadratically many pairs of nodes. If we were to be able to approximate the e-s LONGEST PATH, then we could apply this algorithm to all pairs of nodes and would obtain an approximation algorithm for the LONGEST PATH problem.

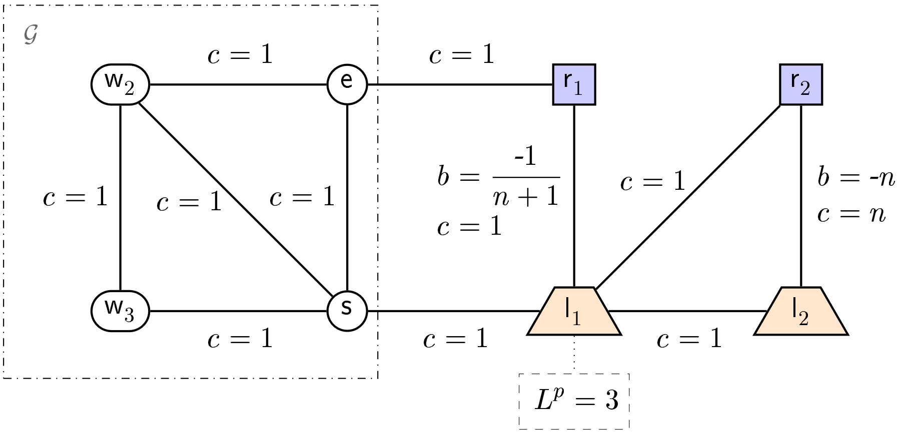


Figure 4.9: Presenting the network $\mathcal{N}_{\mathcal{G}}$ from Theorem 4.5.5 for the graph $\mathcal{G} := (\{\mathsf{e}, \mathsf{w}_2, \mathsf{w}_3, \mathsf{s}\}, \{\mathsf{s}\leftrightarrow\mathsf{e}\leftrightarrow\mathsf{w}_2\leftrightarrow\mathsf{w}_3\leftrightarrow\mathsf{s}\leftrightarrow\mathsf{w}_2\})$.

Let $\mathcal{G}$ be a graph. Fig. 4.9 shows an example of the reduction presented in the theorem below. Before we present the theorem, we outline the idea. The line $\mathsf{r}_2\leftrightarrow\mathsf{l}_2$ is the line that delivers the majority of power in this network. To maximize its flow, we have to maximize the phase angle difference between r_2 and l_2. The phase angle difference is limited by the path $\mathsf{r}_2\leftrightarrow\mathsf{l}_1\leftrightarrow\mathsf{l}_2$.

The load l_1 has fixed demand of 3. Four lines are connected to l_1, all with a capacity of 1. Therefore, we have a total incoming capacity of 4 for l_1. This, together with the minimum demand of 3, implies that at most one of these four lines can be switched off. If we were to switch off the line $\mathsf{s}\leftrightarrow\mathsf{l}_1$ or $\mathsf{r}_1\leftrightarrow\mathsf{l}_1$, the lines $\mathsf{r}_2\leftrightarrow\mathsf{l}_1$ and $\mathsf{l}_1\leftrightarrow\mathsf{l}_2$ would have to have a flow of 1 towards l_1. This is impossible because a congested line $\mathsf{r}_2\leftrightarrow\mathsf{l}_1$ results in a phase angle difference of 1 between r_2 and l_1, and a congested line $\mathsf{l}_1\leftrightarrow\mathsf{l}_2$ results in a phase angle difference of 1 between l_2 and l_1 leaving r_2 and l_2 with the same phase angle. Hence, l_2 could not deliver any power to l_1. It is therefore not possible to break the connection of l_1 and r_1 via the graph $\mathcal{G}$.

As mentioned above, the line $\mathsf{r}_2\leftrightarrow\mathsf{l}_2$ is the line that delivers the majority of power in this network. Its flow depends on the maximum phase angle difference between r_2 and l_2 that we can achieve. This phase angle difference is the bigger, the less power the generator r_2 has to deliver to l_1. Hence, the flow along $\mathsf{r}_2\leftrightarrow\mathsf{l}_2$ is the bigger, the more power we can deliver from r_1 to l_1 to satisfy the fixed demand of 3.

To maximize the flow along the line $\mathsf{r}_1\leftrightarrow\mathsf{l}_1$, we have to maximize the phase angle difference between r_1 and l_1. Switching off lines in $\mathcal{G}$ allows us to increase the phase angle difference. It is maximal when switching off lines in $\mathcal{G}$ results in a sub-graph that is a LONGEST PATH with respect to the component which contains s and e. Any ϵ-approximation for the DS-MPF would produce a switching in $\mathcal{G}$. Using the phase angle difference between r_1 and l_1 we can derive a lower bound for the LONGEST PATH.

Theorem 4.5.5. *It is not possible to approximate the* DS-MPF$_{L^p}$ *problem within a factor of* $2^{(\log n)^{1-\epsilon}}$ *unless* NP *is contained within quasi-polynomial deterministic time.*

Proof. The proof is done by reduction from the $\mathsf{e}-\mathsf{s}$ LONGEST PATH problem. Given a graph $\mathcal{G}=(N^h,E^h)$ with $N^h=\{\mathsf{w}_1,\dots,\mathsf{w}_n\}$, $\mathsf{e}=\mathsf{w}_1$ and $\mathsf{s}=\mathsf{w}_n$. We define the sets of lines and GP network

$$\begin{aligned}
E^1 &:= \left\{\mathsf{s} \xleftrightarrow[c=3]{b=\text{-}1} \mathsf{l}_1 \xleftrightarrow[c=1]{b=\text{-}1} \mathsf{r}_2 \xleftrightarrow[c=n]{b=\text{-}n} \mathsf{l}_2 \xleftrightarrow[c=1]{b=\text{-}1} \mathsf{l}_1 \xleftrightarrow[c=1]{b=\frac{\text{-}1}{n+1}} \mathsf{r}_1 \xleftrightarrow[c=1]{b=\text{-}1} \mathsf{e}\right\},\\
E^2 &:= \left\{\mathsf{a} \xleftrightarrow[c=1]{b=\text{-}1} \mathsf{d} \;\middle|\; \mathsf{a}{\leftrightarrow}\mathsf{d}\in E^h\right\},\\
\mathcal{N}_{\mathcal{G}} &:= \left(N^h\cup\{\mathsf{r}_1,\mathsf{r}_2,\mathsf{l}_1,\mathsf{l}_2\},\{\mathsf{r}_1,\mathsf{r}_2\},\{\mathsf{l}_1,\mathsf{l}_2\},E^1\cup E^2,\left[L^p_{\mathsf{l}_1}{=}3\right]\right).
\end{aligned}$$

The GP network $\mathcal{N}_{\mathcal{G}}$ has $n+4$ buses, all susceptances and capacities are rational numbers and it has no fixed generator and one fixed load.

Let (E',θ,G^p,L^p) be a DS solution and $p':(E^1\cup E^2)\setminus E'\to\mathbb{R}$ be the corresponding flow function. To simplify notations we define the extension p of p' onto all lines, where the value of switched off lines is 0.

The bus l_1 has a fixed demand of 3. This demand, together with Kirchhoff's junction law, implies

$$3+p_{\mathsf{l}_1\mathsf{r}_1}+p_{\mathsf{l}_1\mathsf{s}}+p_{\mathsf{l}_1\mathsf{r}_2}+p_{\mathsf{l}_1\mathsf{l}_2}=0. \tag{4.2}$$

Each line $\mathsf{l}_1{\leftrightarrow}\mathsf{r}_1$, $\mathsf{l}_1{\leftrightarrow}\mathsf{s}$, $\mathsf{l}_1{\leftrightarrow}\mathsf{r}_2$ and $\mathsf{l}_1{\leftrightarrow}\mathsf{l}_2$ has a capacity of 1. Hence, we have $p_{\mathsf{l}_1\mathsf{l}_2}+p_{\mathsf{l}_1\mathsf{r}_2}\geq \text{-}2$. This, together with Eq. (4.2), implies

$$p_{\mathsf{l}_1\mathsf{r}_1}+p_{\mathsf{l}_1\mathsf{s}}\geq \text{-}1.$$

Let us assume that $p_{\mathsf{l}_1\mathsf{r}_1}+p_{\mathsf{l}_1\mathsf{s}}=\text{-}1$. Equation (4.2) implies $p_{\mathsf{l}_1\mathsf{r}_2}+p_{\mathsf{l}_1\mathsf{l}_2}=\text{-}2$. The capacities of $\mathsf{l}_1{\leftrightarrow}\mathsf{r}_2$ and $\mathsf{l}_1{\leftrightarrow}\mathsf{l}_2$ are 1. This implies that the lines $\mathsf{l}_1{\leftrightarrow}\mathsf{r}_2$ and $\mathsf{l}_1{\leftrightarrow}\mathsf{l}_2$ are congested. Therefore, $p_{\mathsf{l}_1\mathsf{r}_2}=p_{\mathsf{l}_1\mathsf{l}_2}$, which is only possible if $\theta_{\mathsf{r}_2}=\theta_{\mathsf{l}_2}$. As the bus l_2 is not a generator and it does not receive power from r_2 it cannot deliver power of 1 to l_1. Therefore, we know that $p_{\mathsf{l}_1\mathsf{r}_1}+p_{\mathsf{l}_1\mathsf{s}}<\text{-}1$. The capacities of $\mathsf{l}_1{\leftrightarrow}\mathsf{r}_1$ and $\mathsf{l}_1{\leftrightarrow}\mathsf{s}$ are 1. This implies that the lines $\mathsf{l}_1{\leftrightarrow}\mathsf{r}_1$ and $\mathsf{l}_1{\leftrightarrow}\mathsf{s}$ cannot be switched off and there is always a flow on both lines towards l_1. The latter implies $\Delta_{\mathsf{l}_1\mathsf{r}_1}\leq 0$.

There are no generators in the buses corresponding to the nodes of the graph $\mathcal{G}$. Hence, there can only be a flow along the line $\mathsf{s}{\leftrightarrow}\mathsf{l}_1$ if the switching leaves a path from s to e within the graph and the line $\mathsf{e}{\leftrightarrow}\mathsf{r}_1$ is not switched off. The switching of lines within $\mathcal{G}$ created a sub-graph $W=(N^h,E^W)$ with $E^W\subseteq E^h$. Our observations above prove that there is at least one path h from e to s in W. Let t be the length of h. In the following, we show upper bounds on the generation of r_1 and r_2 depending on t.

The phase angle difference between r_1 and l_1 is equal to the sum of all phase angle differences along any arbitrary path h' between them. Hence the phase angle difference between r_1 and l_1 can be at most as big as the sum of the maximum phase angle differences of all lines of h'. There are at most t lines between s and e. Henceforth, there are at most $t+2$ lines between r_1 and l_1 within $\mathcal{G}$. Every line in $\mathcal{G}$ has a susceptance of -1. The DC power law implies that every line in $\mathcal{G}$ allows for a maximum phase angle difference of 1. Therefore, the maximum phase angle difference between

r_1 and l_1 with respect to h is bounded by $t+2$, so $0 \leq \Delta_{\mathsf{r}_1\mathsf{l}_1} \leq t+2$. This implies

$$0 \leq p_{\mathsf{r}_1\mathsf{l}_1} = \frac{1}{n+1}\Delta_{\mathsf{r}_1\mathsf{l}_1} \leq \frac{t+2}{n+1}. \tag{4.3}$$

Kirchhoff's junction law at r_1 is $0 = G^p_{\mathsf{r}_1} + p_{\mathsf{r}_1\mathsf{e}} + p_{\mathsf{r}_1\mathsf{l}_1}$. Using the fact that the line $\mathsf{r}_1 \leftrightarrow \mathsf{e}$ has a capacity of 1 and Eq. (4.3), we derive

$$\text{-}G^p_{\mathsf{r}_1} = p_{\mathsf{r}_1\mathsf{e}} + p_{\mathsf{r}_1\mathsf{l}_1} \leq 1 + \frac{t+2}{n+1}. \tag{4.4}$$

We can achieve equality if and only if all lines in the path h are congested. This is true if and only if h is the only path of W from e to s.

The fact that the lines $\mathsf{l}_1 \leftrightarrow \mathsf{s}$ and $\mathsf{l}_1 \leftrightarrow \mathsf{r}_2$ have a capacity of 1 implies $p_{\mathsf{l}_1\mathsf{s}} + p_{\mathsf{l}_1\mathsf{r}_2} \geq \text{-}2$. This, together with Eq. (4.2) and Eq. (4.4), implies

$$\begin{aligned} p_{\mathsf{l}_2\mathsf{l}_1} &= \text{-}p_{\mathsf{l}_1\mathsf{l}_2} \\ &= p_{\mathsf{l}_1\mathsf{r}_1} + p_{\mathsf{l}_1\mathsf{s}} + p_{\mathsf{l}_1\mathsf{r}_2} + 3 \\ &\geq 1 + p_{\mathsf{l}_1\mathsf{r}_1} \\ &\geq 1 - p_{\mathsf{r}_1\mathsf{l}_1} \\ &\geq 1 - \frac{t+2}{n+1}. \end{aligned} \tag{4.5}$$

Kirchhoff's Conservation Law for the triangle $\mathsf{l}_1, \mathsf{r}_2, \mathsf{l}_2$ is $p_{\mathsf{r}_2\mathsf{l}_1} = \frac{p_{\mathsf{r}_2\mathsf{l}_2}}{n} + p_{\mathsf{l}_2\mathsf{l}_1}$. This together with Eq. (4.5), $p_{\mathsf{r}_2\mathsf{l}_1} \leq 1$ and Eq. (4.3) implies

$$\begin{aligned} p_{\mathsf{r}_2\mathsf{l}_2} &= n(p_{\mathsf{r}_2\mathsf{l}_1} - p_{\mathsf{l}_2\mathsf{l}_1}) \\ &\leq n\left(1 - \left(1 - \frac{t+2}{n+1}\right)\right) \\ &\leq n\frac{t+2}{n+1}. \end{aligned} \tag{4.6}$$

Finally, using Eq. (4.6) and $p_{\mathsf{r}_2\mathsf{l}_1} \leq 1$, we have

$$\text{-}G^p_{\mathsf{r}_2} = p_{\mathsf{r}_2\mathsf{l}_1} + p_{\mathsf{r}_2\mathsf{l}_2} \leq 1 + n\frac{t+2}{n+1}, \tag{4.7}$$

where we can achieve equality if and only if $p_{\mathsf{s}\mathsf{l}_1} = 1$.

Combining Eq. (4.4) and Eq. (4.7) shows that

$$\begin{aligned} |G^p_{\mathsf{r}_1}| + |G^p_{\mathsf{r}_2}| &\leq \left(1 + n\frac{t+2}{n+1}\right) + \left(1 + \frac{t+2}{n+1}\right) \\ &\leq 4 + t, \end{aligned} \tag{4.8}$$

and we have equality if and only if h is the only path between s and e in W. Equation (4.8) implies that DS-MPF$(\mathcal{N}_\mathcal{G}) = 4 + t_\mathcal{G}$ where $t_\mathcal{G}$ is the length of a longest path in $\mathcal{G}$.

Let ϵ be such that $1 > \epsilon > 0$ and we assume the existence of an ϵ-approximation algorithm A for DS-MPF($\mathcal{N}_{\mathcal{G}}$). Furthermore, let $A(\mathcal{N}_{\mathcal{G}})$ be the total generation from the solution found by the algorithm. Since A is an ϵ-approximation, we have

$$\epsilon(4 + t_{\mathcal{G}}) = \epsilon\text{DS-MPF}(\mathcal{N}_{\mathcal{G}}) \leq A(\mathcal{N}_{\mathcal{G}}) \leq \text{DS-MPF}(\mathcal{N}_{\mathcal{G}}).$$

Let t be the length of an arbitrary path from e to s through $\mathcal{G}$ in the solution of A that we can find in polynomial time (for example the shortest path). Eq. (4.8) implies that $\epsilon(4 + t_{\mathcal{G}}) \leq A(\mathcal{N}_{\mathcal{G}}) \leq 4 + t$ and hence

$$\epsilon t_{\mathcal{G}} - 4 \leq \epsilon t_{\mathcal{G}} - 4(1 - \epsilon) \leq t.$$

This shows that for every ϵ-approximation algorithm for the DS-MPF problem and every $1 > \theta > 0$ there is an $\epsilon\theta$-approximation algorithm for the longest path problem. □

The final result in this section shows that the DS-MPF problem does not admit a full-polynomial time approximation scheme. This is achieved by showing that the problem is strongly NP-complete which implies the latter as shown in Vazirani [2013]. The proof uses a similar idea as the one above.

Theorem 4.5.6. *The* DS-MPF *problem for planar* MPF *networks is strongly* NP*-complete.*

Proof. The argument given in the proof of Theorem 4.5.2 for the inclusion in NP also works in this case. We now present a reduction of the planar HAMILTONIAN CIRCUIT problem. This problem was shown to be NP-complete by Garey et al. [1976]. Since it is not a number problem it is therefore also strongly NP-complete.

Let (N, E) be a graph. This graph has an HAMILTONIAN CIRCUIT if and only if there exists a pair of adjacent nodes (e, s) where the graph $(N, E \setminus \{\mathsf{e}, \mathsf{s}\})$ has a HAMILTONIAN PATH from e to s. As there are only quadratically many pairs of nodes we only need to present a polynomial reduction of the HAMILTONIAN PATH problem with given end nodes (e and s).

Let $\mathcal{G}$ be a graph with n nodes and e and s be two different nodes from $\mathcal{G}$. Furthermore, let $\widetilde{\mathcal{N}}_{\mathcal{G}}$ be like $\mathcal{N}_{\mathcal{G}}$ from the proof of Theorem 4.5.5 where the buses r_2 and l_2 and their lines being removed and l_1 is a free load (not fixed).

The network $\widetilde{\mathcal{N}}_{\mathcal{G}}$ has the same amount of nodes/buses as $\mathcal{G}$. All lines have a capacity of 1. All lines except for $\mathsf{r}_1 \leftrightarrow \mathsf{l}_1$ have a susceptance of -1, whereas the susceptance of $\mathsf{r}_1 \leftrightarrow \mathsf{l}_1$ is $-{}^{1}/_{n+1}$. Therefore the network has size polynomial in the size of the graph and all line parameters can be bounded by a polynomial based on n.

We will now show that

$$\text{DS-MPF}(\widetilde{\mathcal{N}}_{\mathcal{G}}) \geq 2 \iff \mathcal{G} \text{ has a HAMILTONIAN PATH from } \mathsf{e} \text{ to } \mathsf{s}.$$

The only load in the system is l_1. It has two lines with a capacity of 1 each. Hence, we have DS-MPF($\widetilde{\mathcal{N}}_{\mathcal{G}}$) ≤ 2. Let us assume that equality holds. To achieve equality the line $\mathsf{l}_1 \leftrightarrow \mathsf{r}_1$ cannot be switched and has to be congested. This implies a phase angle difference of $n+1$ between l_1 and r_1. Since the line $\mathsf{l}_1 \leftrightarrow \mathsf{s}$ also has to be congested we have a phase angle difference of $n-1$ between e and s. The phase angle difference between e and s along the graph is bounded from above by the length of the shortest path between them. Every line in the graph has an implicit maximum phase angle difference of 1. Hence, if a DS solution has a phase angle difference of $n-1$ the shortest path in the sub-graph of that solution has a path of length $n-1$. As there are n nodes in the graph, a path of length $n-1$ is a HAMILTONIAN PATH. This shows that the existence of a HAMILTONIAN PATH between s and e allows us to construct a DS solution with a DS-MPF of 2 and if the DS-MPF is 2 then the shortest path from e to s within the sub-graph of the optimal solution is Hamiltonian. Finding the shortest path in a graph is polynomial.

The reduction also works if $\mathcal{G}$ is planar. Since we assumed that their is an edge in $\mathcal{G}$ between e and s, the network $\widetilde{\mathcal{N}}_{\mathcal{G}}$ is also planar. Henceforth, we have shown that planar MPF networks are strongly NP-complete. $\square$

4.6 Related Work

For power network operators it is important to have knowledge about the state of the power system. This helps, for example, to predict if the network is trending towards an undesirable state of operation or if a power line is working at its operational limits. The variables of a system are the demand, the generation and the bus voltage magnitudes and phase angles. The demand of loads can be reliably estimated via forecasts, historic records and/or measurements and the operators of generators report the generation values. Given these values, the voltage magnitudes and phase angles can be computed. This is what we call the POWER FLOW (PF) problem.

The AC-PF provided a big computational challenge in a time when the state-of-the-art algorithms and the available computational capacity where both insufficient. To be able to at least estimate the value of the phase angles, Schweppe and Rom [1970] proposed to use the DC model. This model is still widely used in every day power network operations (O'Neill et al. [2011]). Due to its simplicity, it is also used in academic research to investigate the effects of network topology changes. Fisher et al. [2008] provided the evidence which demonstrated that allowing for line switching can help reduce the generation dispatch cost for the OPF[1], potentially saving millions of dollars. This even holds when N-1 reliability standards are in place (Hedman et al. [2009]). A network is called N-1 reliable if after the failure of any single line there still exists a solution to satisfy all load.

Considering topology changes also helps the unit commitment problem (Hedman et al. [2010]). The unit commitment problem determines an optimal schedule for generators with respect generator start-up and operational cost as well as a load forecast. Changing the topology of the network allows to decrease the amount of generators needed to satisfy the demand. The resulting savings in generator start-up costs trans-

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[1] What we call DS-OPF is called *Optimal Transmission Switching* (OTS) there.

late to overall cost savings. Also, as argued by Hedman et al. [2011], line switching can have benefits aside from economical savings. It also provides greater flexibility in reacting to contingencies and maintaining the N-1 reliability standard. Furthermore, line switching allows us to decrease line losses and provides a mean to maintain operational stability.

The observed decrease of generators needed to satisfy the demand by Hedman et al. [2010] is directly related to the effect that line switching can improve the MPF. This effect can potentially occur on any cyclic network as illustrated in Section 4.2. Using switching to increase the served demand can be used to improve restoration times of a partially destroyed network (Van Hentenryck et al. [2011]). In this problem, called power restoration, we are faced with selecting a sequence of power lines for repair such that the electrical power shortfall due to the blackout is minimized. As power outages are very costly for the economy, faster restoration can translate into a direct economical advantage, potentially saving millions of dollars. By solving a DS-MPF problem every line repair, Van Hentenryck et al. [2011] managed to improve the overall restoration time.

Both the DS-OPF and the DS-MPF problem can formulated as a *Mixed Integer Linear Program* (MILP) using the Big-M method (Nemhauser and Wolsey [1988]). Choosing an approximate Big-M can be crucial to avoid numerical problems. Sometimes these MILP problems are time consuming to solve (Fisher et al. [2008]). Although state-of-the-art solvers have excellent primal heuristics, the search can be sometimes be aided by a custom heuristic based on knowledge of the problem. One techniques is to iterativly identify and switch off the power line which promises the best cost reduction based on a "line profit" heuristic (Ruiz et al. [2011]). A heuristic based on a line-ranking factor computed via the dual problem was presented by Fuller et al. [2012] and shows to further improve over the line profit heuristic. The line outage distribution factor also shows potential to aid the selection of power lines to switch off (Barrows et al. [2013]).

Transmission system expansion planning (also known as power network design problem) also involves a form of line switching and minimizing a cost function. Here, costs are given to every line and the goal is to find a network which is cost-optimal and can satisfy all demand. Various versions of the bounded and unbounded DC transmission system expansion problem are shown to be NP-complete by Lemkens [2015] using reductions from the Steiner Tree, the Spanning Distribution Tree, and the 3-Partition problem.

NP-completeness of the DS-OPF problem via reduction from the SUBSET SUM PROBLEM is shown by Kocuk et al. [2014] and was developed in parallel with our results. The reduction has the feature that the network is series parallel and hence planar, the maximum degree is unbounded, there is one unbounded generator and there is one unbounded load. This result can also easily be adapted to show that the DS-PF and the DS-MPF problem are NP-complete as well. In this thesis we prove the same result with a maximum degree of three as well as some other stronger reductions.

Overall, the results from Lemkens [2015], Kocuk et al. [2014] and ours indicate that using Mixed-Integer-Programming technology and relying on heuristics is the best we can do to solve line switching related problems in the DC model.

Chapter 5

The SIN Power Flow Model

In this chapter, we present complexity results about the MPF and the OPF in the Lossless-Sin AC Approximation (SIN) model. In the SIN model, the active power flow is equal to the product of the sine of the phase angle difference and the susceptance. Also, the reactive power flows are ignored. Hence, the SIN model only differs from the DC model by having the phase angle difference wrapped inside the $\sin$ function. For networks where the conductance of all lines is 0 and every bus has a generator which can supply unlimited amounts of reactive power, the SIN model and the AC model become the same. The non-linearity of the $\sin$ function causes the SIN model to be non-linear. However, it is still lossless[1] as the DC model. In Chapter 4, we demonstrate that the three computational problems, DC-PF, DC-OPF and DC-MPF, can be solved in polynomial time. Allowing for line switching is what makes the problems with DC model hard. In this chapter, we show that by adding the $\sin$ back into the DC model, the OPF and MPF problems are hard without switching.

Here, we do not present results regarding the PF problem with the SIN model. In the PF problem, all demand and generation except for the slack bus are fixed. The SIN model is lossless. Therefore, the generation of the slack bus is implied. Verma [2009] shows that, once all generation and demand are known, there exists unique power flow values. These values form a solution if and only if the power flows are within their bounds. Hence, determining whether there exists a PF solution in the SIN model can be done in polynomial time.

We present the results of this chapter in Table 5.1. We also highlight the features: network structure, maximum bus degree (mD), number of generators (nG) and number of loads (nL). In Theorem 5.2.5 we show that the SIN-MPF problem is strongly NP-hard. Theorem 5.3.1 shows that the SIN-OPF cannot be approximated arbitrarily. A feature of this reductions is that it needs an arbitrary large ratio between the costs of the cheapest and the most expensive generator. This feature makes the reduction less realistic. In Theorem 5.3.2, we show that the SIN-OPF problem is at least strongly NP-hard if we only allow for a bounded generation cost ratio. This reduction is based on the same idea as the reduction in Theorem 5.2.5.

For both problems, showing membership in NP is non-trivial. The standard way

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[1] A model is called lossless if for every line $\mathsf{a}\leftrightarrow\mathsf{d}$ and every solution the line flows have the property $p_{\mathsf{ad}} = -p_{\mathsf{da}}$.

Problem	Result	Structure	mD	nG	nL	Theorem
SIN-MPF	strongly NP-hard	planar	6	∞	∞	Theorem 5.2.5
SIN-OPF	not APX	planar	4	∞	∞	Theorem 5.3.1
SIN-OPF	strongly NP-hard	planar	6	∞	∞	Theorem 5.3.2

Table 5.1: SIN Model Result Overview

to prove membership in NP is to assume that an oracle provides a solution tuple and then check/show that it is feasible (it is a solution) and that its total satisfied demand is above the value of the SIN-OPF/SIN-MPF problem[2] in time polynomial in the input size. This approach would fail here because the values of a solution could potentially be irrational. That means that they cannot be represented in finite space. Hence, we cannot check whether they are valid in time polynomial in the input. This does not imply that either of the problems is not in NP, as there could be a way to utilize the oracle to check whether the SIN-OPF/SIN-MPF is higher or equal than the given value without computing an optimal solution.

Verma [2009] was the first to show that the SIN-MPF problem is strongly NP-hard on arbitrary networks. The proof presented there essentially also works for the SIN-OPF problem. In this chapter, we present results for SIN-OPF and SIN-MPF which use a different set of features. The results in Verma [2009] only work for arbitrary network structures and the maximum degree is unbounded. In contrast we only need a planar network structure and our maximum degree is bounded. On the plus side, the reduction from Verma [2009] only uses one load whereas we need an arbitrary amount of loads. Hence, neither result subsumes the other.

The chapter has four sections. In Section 5.1, we present the SIN model. Afterwards, in Section 5.2 we present the results for the SIN-MPF and in Section 5.3 the results regarding the SIN-OPF. Note that the latter depends on the $\sin$ choice network which is presented in the former. In Section 5.4, we present related work and discuss implications of our results.

5.1 Background

In this section, we present the definitions of SIN-MPF and SIN-OPF. Both are based on the definition of SIN *solutions*, which is based on the definition of GP solutions (Definition 2.3.5). A SIN solution is essentially a GP solution where the power flow p follows the SIN power flow law. The SIN power flow law defines the flow as the product of the susceptance (b) and the sine of the phase angle difference. It therefore differs from the DC power flow law by having the phase angle difference wrapped in a $\sin$. In the SIN model, we also use line capacities and a SIN solutions also limits the phase angle difference to $\pi/2$. Restricting the phase angle differences by $\pi/2$ is commonly to ensure operational stability[3].

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[2] The problem version of an optimization problem, for example the SIN-MPF, is defined as the decision: given $x \in \mathbb{Q}_{\geq 0}$ is SIN-MPF$(\mathcal{N}) \geq x$. We call x the value of the problem.

[3] The real world restrictions are even stricter.

Definition 5.1.1 (SIN solution)**.** Let $(N, N_G, N_L, E, G^p, L^p)$ be a GP network. A SIN *solution* is a tuple (θ, G^p, L^p) such that (θ, G^p, L^p, p) is a GP solution and we have that $\forall(\mathsf{a}, \mathsf{d}, b, g, c) \in E^d$:

$$\begin{aligned} |p_{\mathsf{ad}}| &\leq c, \\ |\theta_{\mathsf{a}} - \theta_{\mathsf{d}}| &\leq \pi/2, \text{ and} \\ p_{\mathsf{ad}} &= b\sin(\theta_{\mathsf{a}} - \theta_{\mathsf{d}}). \end{aligned}$$

A line , $(\{\mathsf{a}, \mathsf{d}\}, b, g, c) \in E$, is called *congested* if $|p_{\mathsf{ad}}| = c$. The set of all SIN solutions of $\mathcal{N}$ is denoted with $\mathcal{S}^{\text{SIN}}(\mathcal{N})$.

We refer to p as the *implied flow* from the SIN solution (θ, G^p, L^p). Note that the conductance g is not used in the definition of SIN solutions. Hence, when we define lines of GP networks within the context of the SIN model we will omit the conductance. Furthermore, we use the following more compact and readable form when defining lines

$$(\{\mathsf{a}, \mathsf{d}\}, b_1, g_1, c_1) \;\approxeq\; \mathsf{a} \xleftrightarrow[c=c_1]{b=b_1} \mathsf{d}.$$

The SIN-MPF is the maximum demand we can satisfy in an MPF network (Definition 2.4.1) using only SIN solutions. The SIN-MPF problem is: given a positive rational number x, decide whether the SIN-MPF is bigger than x.

Definition 5.1.2 (SIN-MPF, SIN-MPF problem)**.** Let $\mathcal{N}$ be an MPF network. The SIN-MPF of $\mathcal{N}$ is defined as

$$\text{SIN-MPF}(\mathcal{N}) := \max_{(\theta, G^p, L^p) \in \mathcal{S}^{\text{SIN}}(\mathcal{N})} \sum_{\mathsf{l} \in N_L} L^p_{\mathsf{l}}.$$

Given an $x \in \mathbb{Q}_{\geq 0}$ the SIN-MPF *problem* is to decide whether SIN-MPF$(\mathcal{N}) \geq x$.

The OPF is concerned with assigning values to the generation variables so that the given demand is satisfied and the total generation cost is minimal. Note that we have to take the absolute value of the generation variables because generation is a negative value in our definition. Since we have a fixed demand, it is possible that no solution exists[4]. In such a case, the SIN-OPF will be infinite.

Definition 5.1.3 (SIN-OPF, SIN-OPF problem)**.** Let $(\mathcal{N}, C)$ be an OPF network with $\mathcal{N} = (N, N_G, N_L, E, G^p, L^p)$. The SIN-OPF of $(\mathcal{N}, C)$ is defined as

$$\text{SIN-OPF}(\mathcal{N}, C) := \min_{(\theta, \widetilde{G}^p, L^p) \in \mathcal{S}^{\text{SIN}}(\mathcal{N})} \sum_{\mathsf{r} \in N_G} |\widetilde{G}^p_{\mathsf{r}}| C_{\mathsf{r}}.$$

Given an $x \in \mathbb{Q}_{\geq 0}$ the SIN-OPF *problem* of $\mathcal{N}$ is to decide whether SIN-OPF$(\mathcal{N}, C) \leq x$.

We call a SIN solution *optimal* if it is a witness for the SIN-OPF or SIN-MPF.

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[4] The simplest example is an OPF network which consist of one bus with a load.

5.2 MAXIMUM POWER FLOW

In this section, we provide the proof that the SIN-MPF problem for planar MPF networks with a maximum degree of 6 is strongly NP-hard. The reduction is using the sin choice network presented below and pictured in Fig. 5.1.

Definition 5.2.1 (sin choice network)**.** Let $x \in \mathbb{Q}_{>0}$ be a number. The sin *choice network* is defined as

$$\mathcal{D}^x_{\mathsf{e}} := (\{\mathsf{r},\mathsf{a},\mathsf{d},\mathsf{l},\mathsf{s},\mathsf{e}\},\{\mathsf{r}\},\{\mathsf{l}\},E,\emptyset) \text{ where}$$

$$E := \left\{ \mathsf{r} \xleftrightarrow[c=x]{b=\text{-}x} \mathsf{a} \xleftrightarrow[c=x]{b=\text{-}\frac{5}{6}x} \mathsf{s} \xleftrightarrow[c=x]{b=\text{-}\frac{5}{6}x} \mathsf{e} \xleftrightarrow[c=x]{b=\text{-}\frac{25}{48}x} \mathsf{d} \xleftrightarrow[c=x]{b=\text{-}x} \mathsf{r}, \mathsf{d} \xleftrightarrow[c=\frac{x}{4}]{b=\text{-}\frac{x}{4}} \mathsf{l} \xleftrightarrow[c=\frac{x}{4}]{b=\text{-}\frac{x}{4}} \mathsf{a} \right\}.$$

The variant of $\mathcal{D}^x_{\mathsf{e}}$ with fixed demand is

$$\widetilde{\mathcal{D}}^x_{\mathsf{e}} := \mathcal{D}^x_{\mathsf{e}}[L^p_{\mathsf{l}}={}^x\!/_2].$$

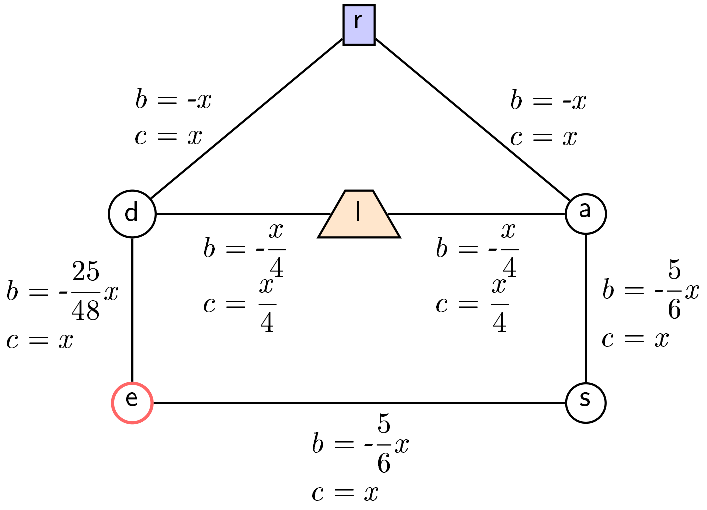

Figure 5.1: The sin choice network $\mathcal{D}^x_{\mathsf{e}}$.

The sin choice network is inspired by the network *System B* presented by Verma [2009]. Both networks have a similar appearance. The key difference is that System B does not have the bus l, and the buses a and d are connected directly. System B has the property that its connector, e, acts as generator (from an outside perspective) whose generation has to be within two disjoint intervals[5].

Whenever the demand at l is ${}^x\!/_2$, the sin choice network allows for generation at e which is either 0 or x where x is the parameter of the network. The motivation to build our choice network comes from the fact that having a choice between two discrete values, rather than two intervals, makes our reductions easier.

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[5] An in-depth proof is presented by Bienstock and Verma [2015].

Before proving the property of $\mathcal{D}^x_{\mathsf{e}}$ formally, we outline the idea. Assume that the load at l has a fixed demand of $^x/_2$. Furthermore, for simplicity, assume that all susceptance are -1 and $x = 1$. Let y be the phase angle difference between e and s. The flow along the line e↔s is $\sin(y)$.

The demand at l can only be satisfied if the lines of l are congested. This congestion implies that the phase angles of a and d have to be the same. The phase angle difference between e and a is $2y$. Hence, the flow along the line e↔d is $\sin(2y)$.

Both buses a and d are connected to r via the same lines and have the same implicit demand from l. Therefore, Kirchhoff's junction law implies that the flow among the path a→s→e has to be equal to the flow of the line d↔e. Hence, we have to have $\sin(2y) = \sin(y)$. This equation has exactly three solutions for the value of $\sin(y)$ as shown in Fig. 5.2. One solution is $\sin(y) = 0$. This corresponds to the case where e does not generate any power. Another solution corresponds to the case where e acts as generator.

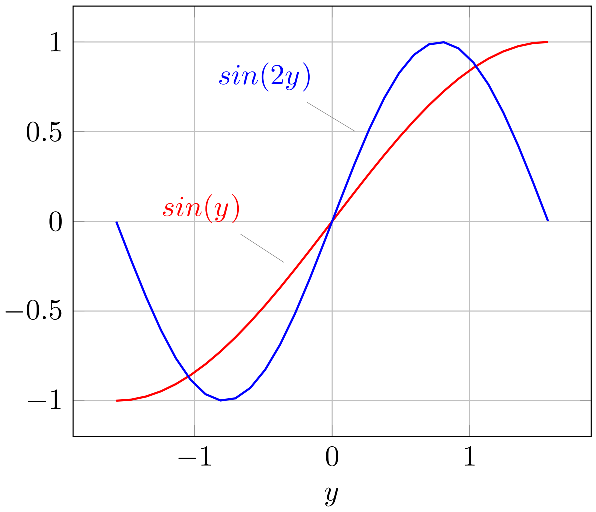


Figure 5.2: The flows along the path e↔d ($\sin(2y)$) and the path e↔s↔a ($\sin(y)$) dependent on y.

In the third solution, e would act as load and hence provide power to a and d. This power is $\sin(^\pi/_2) = \sqrt{3}/2 \approx 0.866$. The implicit demand at a and d created by l is $^1/_4$. . Hence, power of $\sqrt{3}/2 - 0.25 \approx 0.616$ must continue to flow to r. The bus r is not a load and therefore cannot consume power . Therefore, this solution is impossible.

Lemma 5.2.2. *Let* $x \in \mathbb{Q}_{>0}$.

1. *For every* SIN *solution* (θ, G^p, L^p) *of* $\widetilde{\mathcal{D}}^x_{\mathsf{e}}[\mathsf{e}{\in}N_{G/L}]$ *we have*

$$(G^p_{\mathsf{r}}, G^p_{\mathsf{e}} + L^p_{\mathsf{e}}) \in \left\{\left(\text{-}\frac{x}{2}, 0\right), \left(\text{-}\frac{3x}{2}, x\right)\right\}.$$

2. *The networks* $\mathcal{D}^x_{\mathsf{e}}$ *and* $\widetilde{\mathcal{D}}^x_{\mathsf{e}}$ *are planar, have 6 buses and all line parameters are the result of the multiplication of a rational numerical constant with* x.

Proof. Let (θ, G^p, L^p) be a SIN solution of $\widetilde{\mathcal{D}}^x_{\mathsf{e}}[\mathsf{e}{\in}N_{G/L}]$ and p its implied power flow function. First, we observe that the lines of l have to be congested in order to satisfy its demand. Both lines have the same capacity and susceptance. This and the fact that the SIN model has phase angle difference bound of $\nicefrac{\pi}{2}$ implies that $\Delta_{\mathsf{al}} = \Delta_{\mathsf{dl}}$. Hence, we have $\theta_{\mathsf{d}} = \theta_{\mathsf{a}}$. This implies $p_{\mathsf{al}} = p_{\mathsf{dl}}$ and $p_{\mathsf{ar}} = p_{\mathsf{dr}}$. Kirchhoff's junction laws for d and a are $0 = p_{\mathsf{dl}} + p_{\mathsf{dr}} + p_{\mathsf{de}}$ and $0 = p_{\mathsf{al}} + p_{\mathsf{ar}} + p_{\mathsf{as}}$. Both, together with our observation above, imply $p_{\mathsf{de}} = p_{\mathsf{as}}$. Kirchhoff's junction law for s shows $p_{\mathsf{as}} = p_{\mathsf{se}}$. This allows us to derive $p_{\mathsf{de}} = p_{\mathsf{se}}$.

For the bus e we have $0 = G^p_{\mathsf{e}} + L^p_{\mathsf{e}} + p_{\mathsf{ed}} + p_{\mathsf{es}}$ and hence $G^p_{\mathsf{e}} + L^p_{\mathsf{e}} = p_{\mathsf{de}} + p_{\mathsf{se}} = 2p_{\mathsf{se}}$. W.l.o.g we assume that $\theta_{\mathsf{e}} = 0$ and we set $y := \theta_{\mathsf{s}}$. Hence, we have $p_{\mathsf{se}} = \text{-}\nicefrac{5}{6x}\sin(y)$ and $G^p_{\mathsf{e}} + L^p_{\mathsf{e}} = \text{-}\nicefrac{5}{3x}\sin(y)$. In the following, we are going to show that $\sin(y) \in \{0, \text{-}\nicefrac{3}{5}\}$. This will imply the result.

We have: $p_{\mathsf{as}} = p_{\mathsf{se}}$ implies $\sin(\theta_{\mathsf{a}} - y) = \sin(y)$ which implies $\theta_{\mathsf{a}} = 2y$ which shows $\theta_{\mathsf{d}} = 2y$. This implies $p_{\mathsf{de}} = \nicefrac{25}{48x}\sin(2y)$. Our observation $p_{\mathsf{se}} = p_{\mathsf{de}}$ allows us to derive that $\nicefrac{5}{6x}\sin(y) = \nicefrac{25}{48x}\sin(2y)$ or, simplified, $\sin(y) = \nicefrac{5}{8}\sin(2y)$. Applying the half-angle formula $\sin(2y) = 2\sin(y)\cos(y)$ leads us to $\sin(y) = \nicefrac{5}{4}\sin(y)\cos(y)$. One solution of this equation is $\sin(y) = 0$. In this case the generator r only has to satisfy the demand of $\nicefrac{x}{2}$ at l.

Assume that $\sin(y) \neq 0$. By eliminating the term $\sin(y)$ we obtain $\cos(y) = \nicefrac{4}{5}$. Solving the trigonometric identity $\sin(y)^2 + \cos(y)^2 = 1$ towards $\sin(y)$ shows that $\sin(y) = \pm\sqrt{1 - \cos(y)^2} = \pm\sqrt{1 - \nicefrac{16}{25}} = \pm\nicefrac{3}{5}$. If $\sin(y) = \nicefrac{3}{5}$ then $p_{\mathsf{se}} < 0$. This implies that e would deliver power of $\nicefrac{x}{2}$ to the bus a and the same amount to the bus d. The implicit demand from l for each bus is $\nicefrac{x}{4}$. Therefore, each bus a and d, would have to deliver power of $\nicefrac{x}{4}$ to r. This is impossible because r cannot consume this power. Hence, we have $\sin(y) = \text{-}\nicefrac{3}{5}$. In this case the generator r has to provide the power for the demand at l and an additional x units of power to e. Overall this is $G^p_{\mathsf{r}} = \frac{3x}{2}$.

Part two can be directly derived from Definition 5.2.1. □

In the following, we reduce the *3-planar exact cover by 3-set* problem to the SIN-MPF problem using $\sin$ choice networks. The 3-planar exact cover by 3-set problem was shown to be strongly NP complete by Dyer and Frieze [1986].

Definition 5.2.3 (3-planar exact cover by 3-set)**.** Given a set X and a set of subsets S of X with $\forall s \in S : |s| = 3$, the *exact cover by 3-set* (X3C) problem is to decide whether or not there exists a set $V \subseteq S$ such that $\bigcup_{s \in V} s = X$ and $\forall s_1, s_2 \in V : s_1 \neq s_2 \implies s_1 \cap s_2 = \emptyset$. We call (X, S) an *instance* and V a *solution* of (X, S). The *graph* of (X, S) is defined as $(X \cup S, E)$ where $E := \{\{\mathsf{x}, \mathsf{s}\} \mid \mathsf{s} \in S, x \in s\}$.

An instance (X, S) is called *3-planar* if its graph is planar and has a maximum degree of 3. This is equivalent to every element of X is in at most 3 elements of S.

Let (X, S) be an X3C instance. Fig. 5.3 shows an example of our reduction and Fig. 5.4 shows an optimal solution. The reduction will resemble the graph of the given X3C instance by interpreting the sets X and S as buses. Furthermore, we add $\sin$ choice networks $\mathcal{D}^3_{\mathsf{s}}$ to the nodes s from S. A network $\mathcal{D}^3_{\mathsf{s}}$ is called *active* if it generates

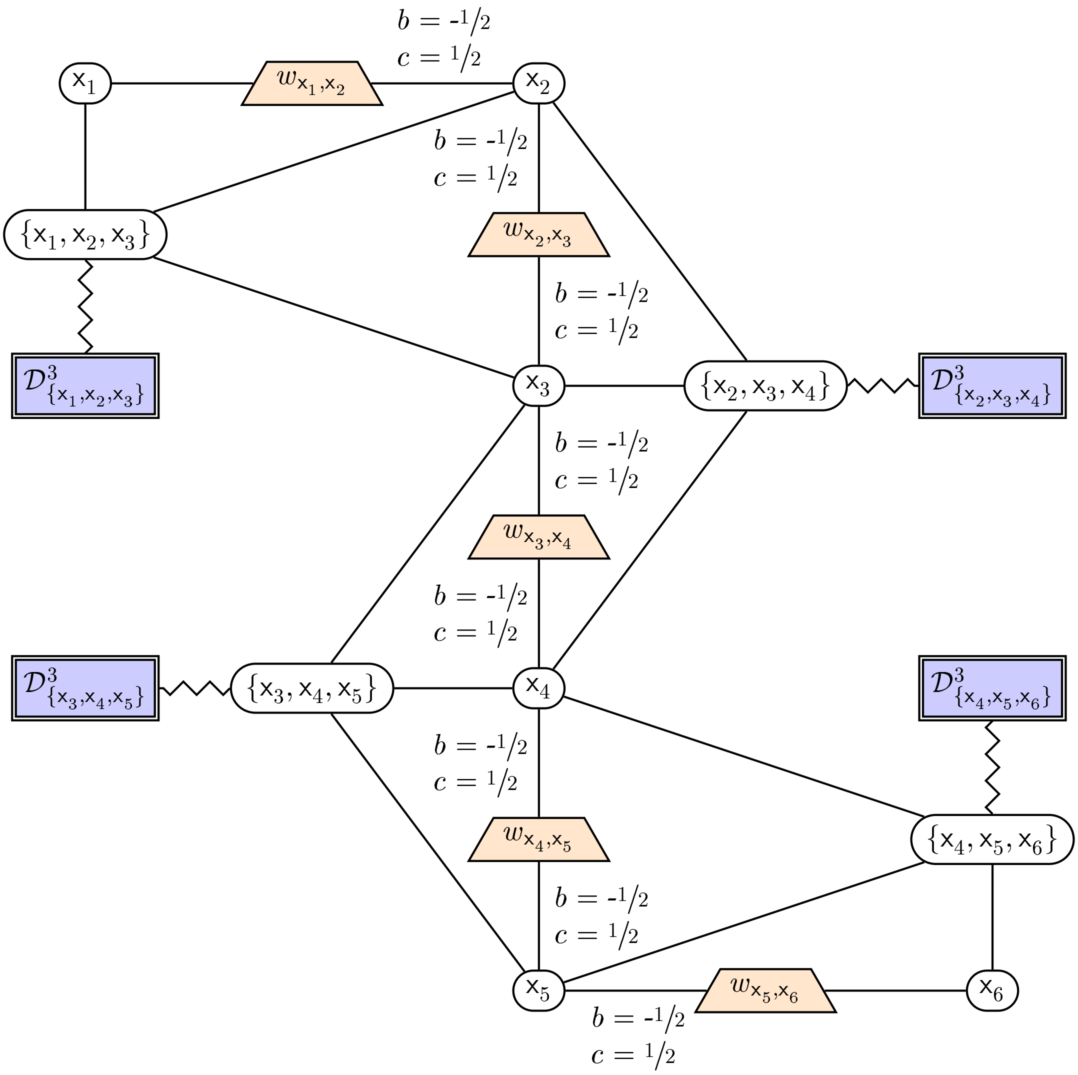


Figure 5.3: An example for the X3C instance (X, S) where $X := \{\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4, \mathsf{x}_5, \mathsf{x}_6\}$ and $S := \{\{\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3\}, \{\mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4\}, \{\mathsf{x}_3, \mathsf{x}_4, \mathsf{x}_5\}, \{\mathsf{x}_4, \mathsf{x}_5, \mathsf{x}_6\}\}$ used in Theorem 5.2.5.

power of 3. The idea is that all buses x will have an (implicit) demand of 1 whereas the choice networks have an implicit generation of either 0 or 3 at s. Therefore, if a choice network $\mathcal{D}^3_{\mathsf{s}}$ is active it will satisfy the implicit demand of all $\mathsf{x} \in \mathsf{s}$.

To have a one-to-one correspondence between active choice networks and the element of a solution of (X, S), it is crucial to ensure that a bus x is connected to exactly one active network. If we were to turn the buses, x, into loads with a demand of 1 then it would be possible to transfer power from one bus x_1 to another bus x_2 via the buses s. Hence, a bus x_1 could be connected to two active choice networks where the power of one of these networks is delivered to another bus x_2.

One way to prevent the sending of power from one bus x_1 to another x_2 is by forcing them all to have the same phase angle. This is achieved by adding additional

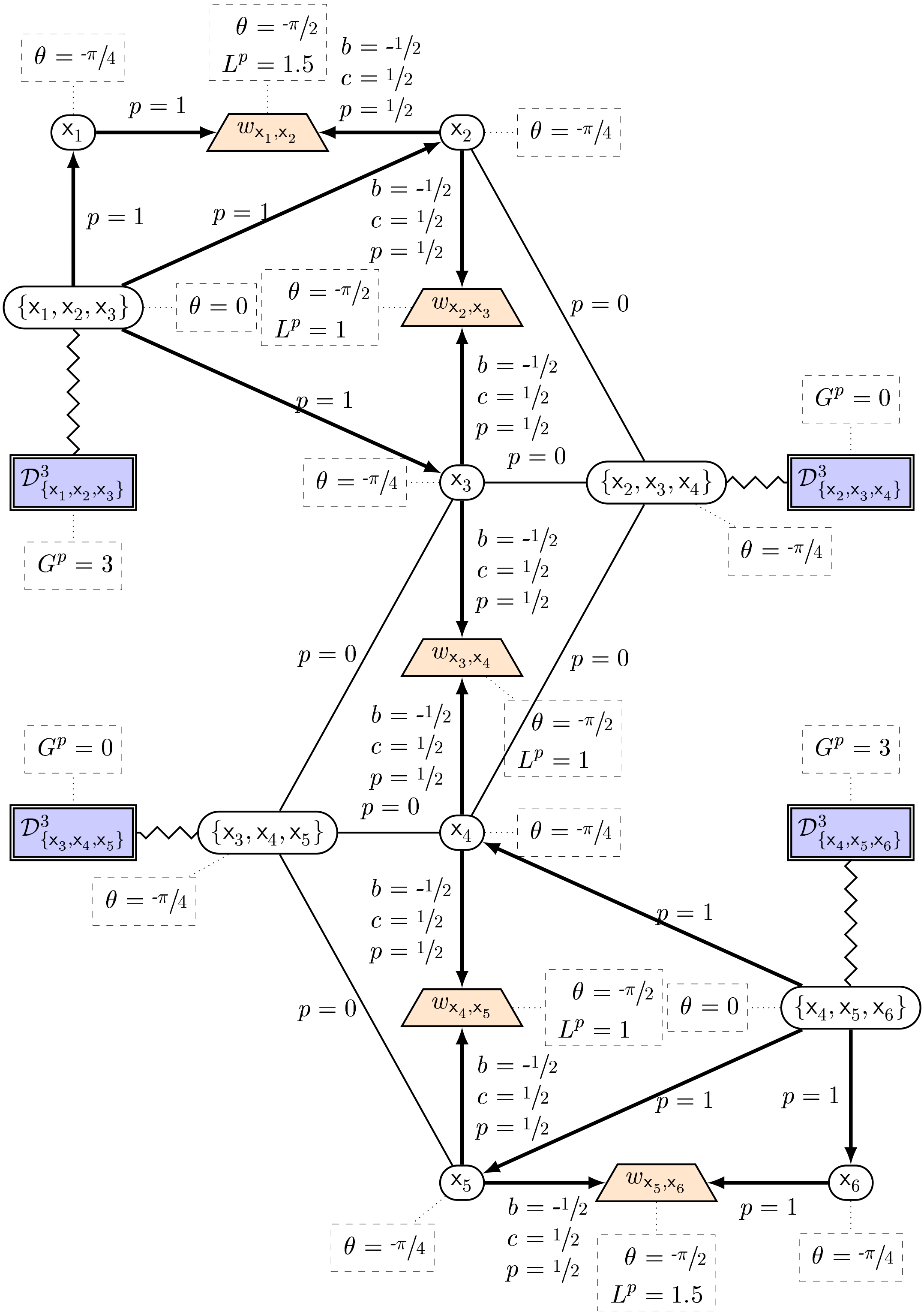
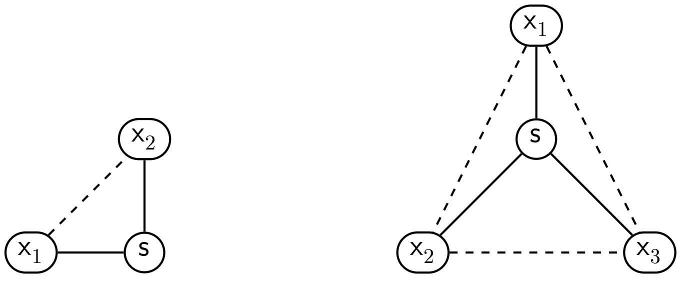


Figure 5.4: An optimal solution for the example from Fig. 5.3.

buses which "connect" all buses X together. These new buses are loads. The lines have parameters such that: to satisfy the demand of the loads, all of their lines have to be congested. This ensures that all buses, x, connected to a load have the same phase angle. Since the new buses connect all buses x, a recursive argument shows that they all have to have the same phase angle. The following lemma shows that it is always possible to find lines/edges which will connect all buses x and still ensure that the result is planar. Our new load will then simply be placed in the middle of each of such line/edge.



(a) Degree two. (b) Degree three.

Figure 5.5: From the proof of Lemma 5.2.4: a node $\mathsf{s} \in S$ of degree two or three; the neighbors of s in some planar layout; and the added edges which bypass s.

Lemma 5.2.4. *Let* $\mathcal{G} := (X \cup S, E)$ *be a connected, bipartite and planar graph with a maximum degree of* 3. *There exists a set of edges* $\widetilde{E} \subseteq \mathcal{P}_2(X)$ *such that* $(X, \widetilde{E})$ *is a connected tree and* $(X \cup S, E \cup \widetilde{E})$ *is a planar graph with a maximum degree of* 6.

Proof. For a node $\mathsf{s} \in S$, let $I(\mathsf{s})$ be the set of its neighbors. Since the graph $\mathcal{G}$ is bipartite all nodes in $I(\mathsf{s})$ belong to X. We define

$$Y := \bigcup_{\mathsf{s} \in S} \mathcal{P}_2(I(\mathsf{s}))$$

as the set of edges built by connecting all neighbors of all nodes, s, with degree two or three (maximum degree of $\mathcal{G}$ is three). This is illustrated in Fig. 5.5.

Let a and d be two different nodes from X. Since $\mathcal{G}$ is connected there must be a path $\mathsf{x}_0\mathsf{s}_1\mathsf{x}_1 \dots \mathsf{x}_{n-1}\mathsf{s}_n\mathsf{x}_n$ in $\mathcal{G}$ where $\mathsf{a} = \mathsf{x}_0$, $\mathsf{d} = \mathsf{x}_n$, $\mathsf{s}_i \in S$, $\mathsf{x}_i \in X$ and $n \geq 1$. As the nodes s_i have a degree of at least two the edges $\{\mathsf{x}_i, \mathsf{x}_{i+1}\}$ must be in Y. Hence, the path $\mathsf{x}_0\mathsf{x}_1 \dots \mathsf{x}_{n-1}\mathsf{x}_n$ must be in (X, Y) which therefore is connected. Consider the graph $\mathcal{G}$ with added edges Y. For every node, s, with degree two, we add an edge bypassing this node. This added edge preserves planarity. For every node, s, with degree three, we add three edges which from a triangle around the node s. This also preserves the planarity. Hence, the new graph is planar.

The set $\widetilde{E}$ can be chosen as the set of edges representing any spanning tree of (X, Y). Every node x originally had a degree of at least 3. Hence, it has at most three neighbors from s and therefore has at most a degree of 3 in (X, Y). Therefore, a node x has a degree of at most 6 in $(X \cup S, E \cup \widetilde{E})$. ☐

Theorem 5.2.5. *The* SIN-MPF *problem for planar networks with a maximum degree of* 6 *is strongly* NP*-hard.*

Proof. Let (X, S) be an instance of 3-planar X3C. W.l.o.g. we assume that its graph is connected. Otherwise we do one reduction per component. Let $\widetilde{E}$ be the set of lines from Lemma 5.2.4 for the graph of (X, S) and let $d(\mathsf{x})$ be the degree of a node x in the graph $(X, \widetilde{E})$. We have one node per edge in $\widetilde{E}$. The set of all of these nodes is $W := \{\mathsf{w}_y \mid y \in \widetilde{E}\}$. And we connect these nodes to the ends of their corresponding edge. This gives us the set of lines

$$E := \left\{ \mathsf{w}_{\{\mathsf{x}_1,\mathsf{x}_2\}} \xleftrightarrow[c=\frac{1}{d(\mathsf{x}_1)}]{b=\frac{-1}{d(\mathsf{x}_1)}} \mathsf{x}_1 \;\middle|\; \{\mathsf{x}_1, \mathsf{x}_2\} \in \widetilde{E} \right\}.$$

We also have one line per edge in the graph of (X, S),

$$E^{'} := \left\{ \mathsf{x} \xleftrightarrow[c=1]{b=-1} \mathsf{s} \;\middle|\; \mathsf{s} \in S, \mathsf{x} \in \mathsf{s} \right\}.$$

Using these sets we define the GP network:

$$\mathcal{N} := (X \cup S \cup W, \emptyset, W, E \cup E^{'}, \emptyset).$$

W.l.o.g we assume that S is of the form $\{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$ and we define $\mathcal{N}_{X,S} := \mathcal{N} +^{\mathsf{s}_1} \mathcal{D}^3_{\mathsf{s}_1} +^{\mathsf{s}_2} \ldots +^{\mathsf{s}_n} \mathcal{D}^3_{\mathsf{s}_n}$.

Part two from Lemma 5.2.2 states that all network $\mathcal{D}^3_{\mathsf{s}_i}$ have 6 buses, and all line parameters are rational constants. Henceforth, the network $\mathcal{N}_{X,S}$ has at most $|X| + |S| + |X|^2 + 6|S|$ buses. All line parameters with respect to $E^{'}$ and the networks $\mathcal{D}^3_{\mathsf{s}_i}$ are rational numerical constants. The capacity and susceptance of a line in E depends on the degree of the corresponding node in the graph $(X, \widetilde{E})$. This degree is upper bounded by $|X|$. Therefore the network has size polynomial in the input and all line parameters can by bounded by a polynomial in the size of the input.

We will now show that

$$\text{SIN-MPF}(\mathcal{N}_{X,S}) \geq |X| + \frac{3}{2}|S| \iff (X, S) \text{ is solvable.}$$

A network $\mathcal{D}^x_{\mathsf{s}}$ has 6 buses. Hence, the network $\mathcal{N}_{X,S}$ has $6|S| + 2|X| - 1$ buses. All susceptances and capacities are rational numbers. Overall, this shows that the network $\mathcal{N}_{X,S}$ can be represented in space polynomial in (X, S). The planarity of (X, S) and its maximum degree of 6 are implied by Lemma 5.2.4.

Assume that we have SIN-MPF$(\mathcal{N}_{X,S}) \geq |X| + \frac{3}{2}|S|$ and (θ, G^p, L^p) is an optimal solution. Every choice network $\mathcal{D}^3_{\mathsf{s}}$ has one load which is denoted with l in Definition 5.2.1. The bus l has two lines with an implicit capacity of $\frac{3}{2}$. Since there are $|S|$ many such networks we have a total maximum demand of $\frac{3}{2}|S|$ from all choice networks.

The only other loads in $\mathcal{N}_{X,S}$ are the buses W. Their total maximum demand is equal to the sum of all the capacities of all of their lines. Each has two lines which are connected to the buses from X. Hence, the line capacities of the buses W are easiest summed up by looking at the buses X. A bus $\mathsf{x} \in \mathcal{N}_{X,S}$ is connected to $d(\mathsf{x})$ many buses from W and each capacity is $1/d(\mathsf{x})$. The bus x can provide power of 1 to the loads W. Hence, the loads, W, have a total maximum demand of $|X|$. Since $\text{SIN-MPF}(\mathcal{N}_{X,S}) \geq |X| + \frac{3}{2}|S|$ we can derive that all lines from all loads in $\mathcal{N}_{X,S}$ must be congested.

Let w be a load from W. The fact that all its lines are congested implies that the buses connected to w have an implicit demand of 1. It also implies that they have the same phase angle. The buses W and their lines are essentially representing the edge $\widetilde{E}$. Since $\widetilde{E}$ is a spanning tree among the buses X, we can infer that in our optimal solution all buses X must have the same phase angle. This implies that no power can flow from one bus x_1 to another bus x_2. Hence, the implicit demand of 1 at every bus x can only be satisfied by the choice networks $\mathcal{D}^3_{\mathsf{s}}$ with $\mathsf{x} \in \mathsf{s}$.

The fact that all lines of all loads are congested implies that the loads of the choice networks $\mathcal{D}^3_{\mathsf{s}}$ must have a demand of $\frac{3}{2}$ each. Hence, we can use Lemma 5.2.2. It shows that in every $\mathcal{D}^3_{\mathsf{s}}$ the bus s acts as implicit generator with a generation of either 0 or 3. We call a network $\mathcal{D}^3_{\mathsf{s}}$ *active* if it generates power of 3.

Since we have an X3C instance every s in S has exactly three elements. This implies that the bus s is connected to exactly three buses from X. If a choice network $\mathcal{D}^3_{\mathsf{s}}$ is active it therefore satisfies the demand of all of its elements $\mathsf{x} \in \mathsf{s}$. This implies that no other $\mathcal{D}^3_{\mathsf{s}_2}$ connected to $\mathsf{x} \in \mathsf{s}$ ($\mathsf{x} \in \mathsf{s}_2$) can be active. Hence, we have a one-to-one correspondence between active choice networks and solutions of X3C. Given a solutions of X3C we activate all choice networks $\mathcal{D}^3_{\mathsf{s}}$ corresponding to that solution. Our observations above show that this allows us to build a solution with a total demand of $|X| + \frac{3}{2}|S|$. If, on the other hand, we have an optimal solution of $\mathcal{N}_{X,S}$ where the SIN-MPF is greater or equal to $|X| + \frac{3}{2}|S|$ then the set $V := \{\mathsf{s} \mid \mathcal{D}^3_{\mathsf{s}} \text{ is active}\}$ is a solution for (X, S). □

5.3 OPTIMAL POWER FLOW

In this section, we show that the SIN-OPF for planar OPF networks with a bounded maximum degree cannot be approximated and is strongly NP-hard. The prove follows the general method described in Section 3.2. We will now show that the existence of an ϵ-approximation allows us to decide the SSP. As choice network, we will use the variant of the sin choice network $\widetilde{\mathcal{D}}^x_{\mathsf{e}}$ from Section 5.2 where the demand of the inner load is fixed. This is necessary because OPF networks have fixed loads.

Fig. 5.6 shows the main idea of the reduction. Let (S, w) be an SSP instance. For every element x in the set S we have a bus x and a network $\widetilde{\mathcal{D}}^x_{\mathsf{x}}$. All these networks are connected to the load l. This load has a demand of w. Additionally, a generator r

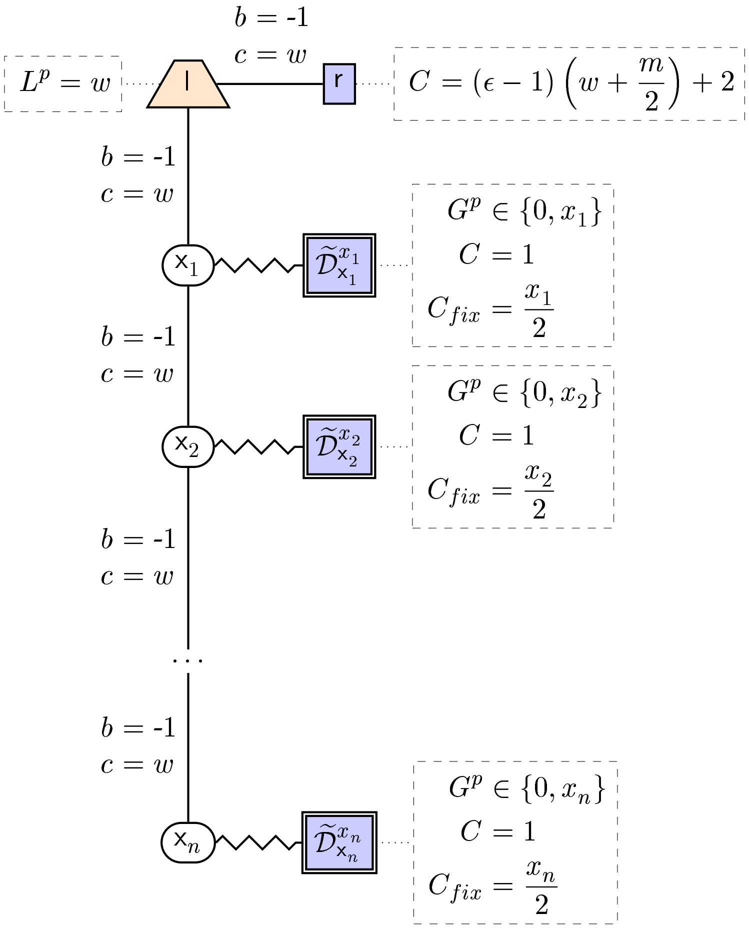


Figure 5.6: The network $\mathcal{N}_{S,w}$ from Theorem 5.3.1 for the SSP instance $(\{x_1, \dots, x_n\}, w)$.

is connected to l with cost of

$$(\epsilon - 1)\left(w + \frac{m}{2}\right) + 2$$

where $m := \sum_{x \in S} x$. The generator inside the network $\widetilde{\mathcal{D}}^{x}_{\mathsf{x}}$ has both the cost of 1. Lemma 5.2.2 implies that the generator of every network $\widetilde{\mathcal{D}}^{x}_{\mathsf{x}}$ has to generate at least $\frac{x}{2}$. Hence, the cost of every solution consists of one part of fix cost $\frac{m}{2}$ and the cost for the generation which satisfies the demand at l. The generator r is more expensive than the generators of the networks $\widetilde{\mathcal{D}}^{x}_{\mathsf{x}}$. Hence, any optimal SIN solution will utilize the generators of $\widetilde{\mathcal{D}}^{x}_{\mathsf{x}}$ first to satisfy the demand of l. Lemma 5.2.2 showed that a network $\widetilde{\mathcal{D}}^{x}_{\mathsf{x}}$ can only generate either nothing or x. If the SSP instance is solvable, then the network $\widetilde{\mathcal{D}}^{x}_{\mathsf{x}}$ can satisfy all the demand. Hence, we have cost of w and any

ϵ-approximation algorithm would deliver a solution whose cost is within the interval

$$\left[w + \frac{m}{2}, \epsilon\left(w + \frac{m}{2}\right)\right].$$

If, on the other hand, the SSP instance is not solvable, then the generator at r has to generate at least one unit of power. Hence, we have cost of at least

$$w + \frac{m}{2} - 1 + (\epsilon - 1)\left(w + \frac{m}{2}\right) + 2 = \epsilon\left(w + \frac{m}{2}\right) + 1$$

and any ϵ-approximation would deliver a solution whose cost is within the interval

$$\left[\epsilon\left(w + \frac{m}{2}\right) + 1, \infty\right).$$

Hence, we can use any ϵ-approximation to decide the SSP simply by checking which interval we are in.

Theorem 5.3.1. *There is no ϵ-approximation algorithm for the* SIN-OPF *on planar* OPF *networks with maximum degree of* 4 *unless* $\mathsf{P} = \mathsf{NP}$.

Proof. Let (S, w) be an SSP instance with $S = \{x_1, \dots, x_n\}$ and $m := \sum_{x \in S} x$. Furthermore, let us assume we have an ϵ-approximation for the SIN-OPF problem. We will now show that we can use this algorithm to decide whether or not (S, w) is solvable. We define

$$\begin{aligned} E &:= \left\{ \mathsf{r} \xleftrightarrow{b=\text{-}1} \mathsf{l} \xleftrightarrow{b=\text{-}1} \mathsf{x}_1 \xleftrightarrow{b=\text{-}1} \mathsf{x}_2 \dots \mathsf{x}_{n-1} \xleftrightarrow{b=\text{-}1} \mathsf{x}_n \right\}, \\ \mathcal{N} &:= \left(\{\mathsf{r}, \mathsf{l}\} \cup S, \{\mathsf{r}\}, \{\mathsf{l}\}, E, \left[L^p_{\mathsf{l}} = w \right] \right) \text{ and} \\ \mathcal{N}_{S,w} &:= \mathcal{N} +^{\mathsf{x}_1} \widetilde{\mathcal{D}}^{x_1}_{\mathsf{x}_1} +^{\mathsf{x}_2} \widetilde{\mathcal{D}}^{x_2}_{\mathsf{x}_2} +^{\mathsf{x}_3} \dots +^{\mathsf{x}_n} \widetilde{\mathcal{D}}^{x^n}_{\mathsf{x}_n}. \end{aligned}$$

For a network $\widetilde{\mathcal{D}}^{x_i}_{\mathsf{x}_i}$, let r_i be the generator from Definition 5.2.1. Hence, the set of generators of $\mathcal{N}_{S,w}$ is $N_G = \{\mathsf{r}\} \cup \{\mathsf{r}_i \mid 1 \leq i \leq n\}$. We define the cost function $C : N_G \to \mathbb{Q}_{>0}$ via $\forall \mathsf{x}_i \in S : C_{\mathsf{r}_i} := 1$, and $C_{\mathsf{r}} := (\epsilon - 1)\left(w + \frac{m}{2}\right) + 2$.

All loads in the network $\mathcal{N}_{S,w}$ have a fixed demand whereas the generators are not fixed. Hence, the tuple $(\mathcal{N}_{S,w}, C)$ is an OPF network. Part two from Lemma 5.2.2 states that all networks $\widetilde{\mathcal{D}}^{x_i}_{\mathsf{x}_i}$ have 6 buses and the size of all line parameters is polynomial in the size of (S, w). The other line parameters are either 1 or -1 and hence constant. Finally, the generation r is formed by sums and products of given numbers ϵ, m and w. Therefore, all numbers of the reduction are polynomial sized with respect to the input size. The network has $2 + 6n$ buses. Hence, the size of the network is polynomial in the input. This shows that the OPF network $(\mathcal{N}_{S,w}, C)$ is a polynomial reduction. Also, the network is planar and the maximum degree is 4.

Lemma 5.2.2 shows that the generators r_i of every network $\widetilde{\mathcal{D}}^{x_i}_{\mathsf{x}_i}$ has to generate power of $\frac{x_i}{2}$ no matter the choice this network makes. Therefore, we have a total fix cost of $\frac{m}{2}$.

We call a choice network $\widetilde{\mathcal{D}}_{\mathsf{x}_i}^{x_i}$ where its connector implicitly generates x_i *active*. Lemma 5.2.2 shows that an active choice network $\widetilde{\mathcal{D}}_{\mathsf{x}_i}^{x_i}$ generates cost of x_i and zero otherwise (ignoring the fixed cost). Since $\epsilon > 1$, we have $C_{\mathsf{r}} > 1 = C_{\mathsf{r}_i}$. Hence, any optimal solution prefers to activate a choice network instead of using the generator r to satisfy the demand of l.

Let us assume that (S, w) is solvable. In this case, all the demand of l can be satisfied by activating the choice networks corresponding to the solution of (S, w). We therefore have cost of $w + \frac{m}{2}$. Our ϵ-approximation algorithm would return a solution with cost within the interval

$$\left[w + \frac{m}{2}, \epsilon\left(w + \frac{m}{2}\right)\right]. \tag{5.1}$$

On the other hand let (S, w) be unsolvable. In this case, the demand at l cannot be satisfied by the choice networks alone. At least one unit of power must be provided by the generator r. Given its cost, this implies that we have cost of at least

$$w + \frac{m}{2} - 1 + (\epsilon - 1)\left(w + \frac{m}{2}\right) + 2 = \epsilon\left(w + \frac{m}{2}\right) + 1.$$

Hence, our ϵ-approximation algorithm must return a solution with cost within the interval

$$\left[\epsilon\left(w + \frac{m}{2}\right) + 1, \infty\right). \tag{5.2}$$

The intervals (5.1) and (5.2) are disjoint. Let A be the approximation algorithm and $A(\mathcal{N}_{S,w})$ be the cost of the solution it returns. The values ϵ and w are given and the value $\frac{m}{2}$ can be computed in polynomial time. Hence, the number $\epsilon\left(w + \frac{m}{2}\right)$ is polynomial in size and can be used in a comparison. All in all, we have

$$A(\mathcal{N}_{S,w}) \leq \epsilon\left(w + \frac{m}{2}\right) \iff (S, w) \text{ is solvable.}$$

□

The cost of the generator r depend on the values of ϵ, w and m. All these values can be arbitrary large. Hence, the ratio between the cost of r and the cost of the other generators can be arbitrarily large. Being able to have an arbitrarily large ratio between the cost of at least two generators is a crucial "feature" necessary to make the proof work. In the following, we show that in case of a limited ratio, the SIN-OPF problem is strongly NP-hard for a bounded maximum degree. The proof is similar to, and is based on, the result of Theorem 5.2.5 about the SIN-MPF.

Theorem 5.3.2. *The* SIN-OPF *problem for planar* OPF *networks with a maximum degree of* 6 *is strongly* NP*-hard.*

Proof. Let $\mathcal{N}_{X,S}$ be as in Theorem 5.2.5 with some modifications. First, we replace the choice networks $\mathcal{D}_{\mathsf{s}}^{x}$ with $\widetilde{\mathcal{D}}_{\mathsf{s}}^{x}$. Second, we fix the demand of every load such that all of its lines (ignoring the new lines we add below) have to be congested to satisfy the demand. Finally, we connect a generator r_{w} to every load $\mathsf{w} \in W$ with a line whose capacity allows to satisfy all demand of w. We define a cost function C as follows. The cost of these new generators is 2. The cost of the generator in the network $\widetilde{\mathcal{D}}_{\mathsf{s}}^{x}$ is 1.

The cost function in this reduction only uses fixed rational numerical constants. The network size is similar to the network $\mathcal{N}_{X,S}$ used in Theorem 5.2.5. Hence, by using similar arguments as in the proof of Theorem 5.2.5, we can derive that this network is polynomial in the size of the input and all line parameters are polynomial bounded as well. We will now show that

$$\text{SIN-OPF}(\mathcal{N}_{X,S}, C) \leq |X| + \frac{3}{2}|S| \iff (X, S) \text{ is solvable.}$$

Theorem 5.2.5 shows that the sum of the total demand is $|X| + \frac{3}{2}|S|$. As the cost of the cheapest generators is 1 we have SIN-OPF$(\mathcal{N}_{X,S}, C) \geq |X| + \frac{3}{2}|S|$. Theorem 5.2.5 shows that the demand at the loads can be satisfied without the new generators, r_{w}, if and only if (X, S) is solvable. As the new generators have cost strictly greater than 1, this implies that using them would result in SIN-OPF$(\mathcal{N}_{X,S}, C) > |X| + \frac{3}{2}|S|$. This implies the result. □

5.4 Related Work

Besides Verma [2009], other works using the SIN model are Donde et al. [2005] and Pinar et al. [2010]. They study the vulnerability problem: how to identify sets of power lines whose failure would lead to major black outs. A motivation for why these models are used was not given.

As illustrated in the introduction, our interest in the SIN model is purely theoretical. The model is “close” to the DC model yet also special case of the AC model. Hence, the results for the SIN model are also valid for the AC model. Our results show that by reintroducing the sine into the DC model, the OPF and the MPF problem become hard to solve. Furthermore, this is true for the realistic class of planar networks with a bounded maximum degree.

The DC model is easy to solve. The price for this is that it is a very broad approximation and can lead to unrealistic results as shown by Stott et al. [2009]; Coffrin et al. [2014b]. A reason for this is that it does not contain any of the non-linear components of the AC model. One way to overcome the inaccuracy of the DC model is to reintroduce some of the non-linearity to the DC. Our results show that there is little hope to find a model that has at least some of the non-linearity of the AC and is as easy to solve as the DC model.

Chapter 6

The AC Power Flow Model

Electrical power obeys the rules of the AC equations. These equations are summarized in the AC model. In contrast, the DC (Chapter 4) and the SIN model (Chapter 5) are approximations of the AC model, and hence approximations of the underlying physics. We present results of this chapter in Table 6.1. Note that the SIN model can be interpreted as a special case of the AC model. Hence, to obtain a full overview of the computational complexity of the AC model, the results from the SIN model have to be taken into account as well.

Problem	Result	Structure	mD	nG	nL	Theorem
AC-MPF	NP-hard	tree	∞	∞	1	6.3.3
AC-PF	NP-hard	tree	∞	1	∞	6.4.4
AC-OPF	not APX	tree	∞	2	∞	6.5.1
AC-OPF	NP-hard	tree	∞	2	∞	6.5.2
AC-OPF	not APX	tree	∞	∞	1	6.5.3
AC-OPF	NP-hard	tree	∞	∞	1	6.5.4
VPF	NP-hard	tree	∞	∞	∞	6.6.5

Table 6.1: AC Model Result Overview

The results presented in Table 6.1 are based on three different types of choice networks[1]. In the reductions for the AC-PF and the first two results for the AC-OPF problem, we use a choice network which is based on the quadratic nature of the voltage magnitudes in the AC power flow together with the coupling of active and reactive power flow. The reduction for the AC-PF with fixed voltages (VPF) uses a choice network which relies on the fact that the sine and cosine functions occurring in the AC power flow are essentially also of quadratic nature. Finally, in the reduction for the AC-MPF, we have a choice network which is based on the uniqueness of the ratio between the active and the reactive power flow of a line with non-zero flow.

The reduction for the AC-MPF is also used in Theorem 6.5.3 and Theorem 6.5.4 for the AC-OPF. This reduction works for almost arbitrary line parameter and maximum phase angle difference values as opposed to the reductions from Theorems 6.5.1 and 6.5.2 which have a limited range of possible parameters. The disadvantage of the

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[1] See Section 3.2 for an introduction into the concept of choice networks.

reduction from Theorem 6.5.3 and Theorem 6.5.4 is that it needs an unlimited number of generators and we have to assume that all voltage magnitudes are fixed at the same point by setting the lower voltage magnitude bound equal to the upper bound which is not the case in Theorems 6.5.1 and 6.5.2.

All proofs contain a reduction of the SSP problem in a way outlined in Section 3.2. The network structure is like a star. That is, we have one center bus connected to multiple leaves. For all these reductions, the degree of the center bus depends on the size of the SSP instance. Hence, the bus degree of the set of all reduction instances is unbounded. At first glance, one might try to change the reduction such that all leaves are connected to a path leading to the center bus to overcome this problem. However, the problem are the line losses, as explained in the corresponding sections. We hypothesize that it is possible to use this trick to bound the maximum degree. However, achieving this could become potentially complex mathematics.

A star is a very simple network structure which every real world power network contains. Hence, if we want to identify a class of power networks which are easy to solve, studying different network structures is a fruitless attempt. Instead, we have to focus on line and voltage parameters which are not covered by any of our proofs.

We begin the chapter with definitions of the relevant problems in Section 6.1. We then present a class of networks called Magical-Tree networks (Section 6.2). It is used to deal with the slack bus in AC-PF networks and is crucial for the reduction regarding the AC-OPF and VPF. We then present the four sections containing the main results of this chapter (see above). Finally, in Section 6.7 we present related work and place our results in context to this work.

6.1 Background

In the following, we define AC *networks* as well as the AC variants of the MPF (Definition 2.4.1), OPF (Definition 2.4.2) and the PF network (Definition 2.4.3). We also define the VPF network, a variant of the AC-PF network where the voltage magnitudes are fixed. In the definition of AC *solutions*, we will introduce the AC power flow equations. Furthermore, we present formal definitions of the computational problems related the AC model.

We start with AC networks. An AC network is a GP network (Definition 2.3.1) which additionally allows the fixing of voltage magnitudes as well as reactive power generation and demand. The AC power flow depends on the phase angle difference and the voltage magnitude. Real world operational bounds for the voltage translate to upper and lower bounds for the voltage magnitude and a maximum phase angle difference bound in the AC network (Glover et al. [1987]). In the AC power flow model, the demand can be regarded as a complex number where the real part is called active power and the imaginary part is called reactive power. For loads, the active power in our model is a positive value. A negative value indicates a generator. The reactive power, on the other hand, can be positive or negative.

The numerical parameters from AC networks are expressions build from rational numbers $\mathbb{Q}$; the binary operators $+\,-\,*/$; the unary operators $\sin(\cdot), \cos(\cdot), \operatorname{atan}(\cdot)$ and $\sqrt{\cdot}$; and the numerical constant π. We denote the set of these numbers with $\widetilde{\mathbb{Q}}$. Although the numbers in $\widetilde{\mathbb{Q}}$ can be irrational, the all can be represented as finite strings over the alphabet $01234567890 + - * \,/\, \sin \cos \operatorname{atan} \pi$ and $\sqrt{\cdot}$. When referencing to GP networks, we use them as if they where defined over $\widetilde{\mathbb{Q}}$ (instead of $\mathbb{Q}$).

The fact that the numbers from $\widetilde{\mathbb{Q}}$ can be represented as finite strings allows the networks of our encodings to have irrational numbers as numerical parameters and still be represented as finite strings. This is important because the size of the networks in our reduction have to be polynomial and hence finite. Each reduction will use a finite amount of irrational numbers. For example, to show that the AC-MPF is NP-hard, we need three irrational numbers. However, we will not specify the exact numbers. We keep the proofs as general as possible.

Definition 6.1.1 (AC network)**.** An AC *network* is a tuple

$$(N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^p, L^q)$$

where

- $(N, N_G, N_L, E, G^p, L^p)$ is a GP network over $\widetilde{\mathbb{Q}}$,
- $\overline{\Delta} \in \widetilde{\mathbb{Q}}_{\geq 0}$ is the *global maximum phase angle difference*,
- $0 \leq \underline{v} \leq \overline{v} \leq \infty$ are the *voltage magnitude bounds*,
- $v : N^v \to \widetilde{\mathbb{Q}}_{>0}$ with $N^v \subseteq N$ are the *voltage magnitudes* ,
- $G^q : N_G^q \to \widetilde{\mathbb{Q}}$ with $N_G^q \subseteq N_G$, is the *reactive power generation*, and
- $L^q : N_L^q \to \widetilde{\mathbb{Q}}$, with $N_L^q \subseteq N_L$ is the *reactive power demand*.

When defining AC networks and their derivatives AC-MPF, AC-OPF, AC-PF and VPF network, we present functions like G^p, L^p, G^q, L^q and v in an implicit manner. For example, the network

$$\mathcal{N} := (\{\mathsf{e}, \mathsf{r}, \mathsf{l}, \mathsf{a}, \mathsf{d}\}, \{\mathsf{r}, \mathsf{a}\}, \{\mathsf{l}, \mathsf{d}\}, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^q, L^q)$$

where E is some set of lines and we have $G^p : \{\mathsf{r}\} \to \widetilde{\mathbb{Q}}_{\leq 0}$ with $G^p_{\mathsf{r}} :=$ -1, $L^p : \{\mathsf{l}\} \to \widetilde{\mathbb{Q}}_{\leq 0}$ with $L^p_{\mathsf{l}} := 12$; $v : \{\mathsf{e}\} \to \widetilde{\mathbb{Q}}_{\geq 0}$ with $v_{\mathsf{e}} := 1.1$; $G^q : \emptyset \to \widetilde{\mathbb{Q}}$ and $L^q : \emptyset \to \widetilde{\mathbb{Q}}$ is presented in a compressed way via

$$\mathcal{N} := \Big(\{\mathsf{e}, \mathsf{r}, \mathsf{l}, \mathsf{a}, \mathsf{d}\}, \{\mathsf{r}, \mathsf{a}\}, \{\mathsf{l}, \mathsf{d}\}, E, \overline{\Delta}, \underline{v}, \overline{v}, \Big[G^p_{\mathsf{r}}{=}\text{-}1 \Big| L^p_{\mathsf{l}}{=}12 \Big| v_{\mathsf{e}}{=}1.1\Big]\Big).$$

Also, similar to GP networks, we define variants of AC networks.

Definition 6.1.2 (network variant)**.** Let $\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^q, L^q)$ be an AC network and $\widetilde{v} : N^v \to \widetilde{\mathbb{Q}}_{\geq 0}$ a voltage magnitude function, and $\widetilde{G}^p : N_G^p \to \widetilde{\mathbb{Q}}_{\leq 0}$, $\widetilde{G}^q : N_G^q \to \widetilde{\mathbb{Q}}$, $\widetilde{L}^p : N_L^p \to \widetilde{\mathbb{Q}}_{\geq 0}$, $\widetilde{L}^q : N_L^q \to \widetilde{\mathbb{Q}}$ be active and reactive generation and demand functions. We define and denote the AC network variant of $\mathcal{N}$ with respect to $\widetilde{G}^p$,$\widetilde{L}^p$,$\widetilde{v}$,$\widetilde{G}^q$ and $\widetilde{L}^q$ via

$$\begin{aligned}\mathcal{N}[\widetilde{G}^p,\widetilde{L}^p,\widetilde{v},\widetilde{G}^q,\widetilde{L}^q] :=&(N, N_G \cup N^p_G \cup N^q_G, N_L \cup N^p_L \cup N^q_L, E,\\ &\overline{\Delta}, \underline{v}, \overline{v}, v+\widetilde{v}, G^p+\widetilde{G}^p, L^p+\widetilde{L}^p, G^q+\widetilde{G}^q, L^q+\widetilde{L}^q).\end{aligned}$$

Let $\mathsf{e} \in N$ be a bus. We define the GP network variant of $\mathcal{N}$ where e becomes a generator and a load as

$$\mathcal{N}[\mathsf{e}{\in}N_{G/L}] := (N, N_G \cup \{\mathsf{e}\}, N_L \cup \{\mathsf{e}\}, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^q, L^q).$$

We use a compressed syntax to express $\mathcal{N}[\widetilde{G}^p,\widetilde{L}^p,\widetilde{v},\widetilde{G}^q,\widetilde{L}^q]$ without the need to define any of the functions $\widetilde{G}^p$, $\widetilde{L}^p$, $\widetilde{v}$, $\widetilde{G}^q$ or $\widetilde{L}^q$ explicitly.

Whenever the voltage magnitude bounds are not of interest, we use the values of $\underline{v} = 0$ and $\overline{v} = \infty$.

As for GP networks, we define a sum operator for AC networks.

Definition 6.1.3 (sum)**.** Let

$$\begin{aligned}\mathcal{N} &= (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^q, L^q) \text{ and}\\ \widetilde{\mathcal{N}} &= (\widetilde{N}, \widetilde{N}_G, \widetilde{N}_L, \widetilde{E}, \overline{\Delta}', \underline{v}', \overline{v}', \widetilde{v}, \widetilde{G}^p, \widetilde{L}^p, \widetilde{G}^q, \widetilde{L}^q)\end{aligned}$$

be AC networks with $N \cap \widetilde{N} = \{\mathsf{e}\}$. The *sum* of $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ is defined as

$$\begin{aligned}\mathcal{N} +^{\mathsf{e}} \widetilde{\mathcal{N}} :=&(N \cup \widetilde{N}, N_G \cup \widetilde{N}_G, N_L \cup \widetilde{N}_L, E \cup \widetilde{E}, \min\{\overline{\Delta}, \overline{\Delta}'\}, \max\{\underline{v}, \underline{v}'\},\\ &\min\{\overline{v}, \overline{v}'\}, v+\widetilde{v}, G^p+\widetilde{G}^p, L^p+\widetilde{L}^p, G^q+\widetilde{G}^q, L^q+\widetilde{L}^q).\end{aligned}$$

6.1.1 AC Solution

An AC solution is an extension of a GP solution (Definition 2.3.5). That is, on top of specifying values for the variables of phase angles and active power generation and demand, we have to find voltage magnitudes and values for reactive power generation and demand. On the constraint side we now have to satisfy Kirchhoff's junction law for active and reactive power. And, additionally to the maximum phase angle constraint, we also have a bound constraint on the voltage magnitudes. Finally, the definition of AC solution defines active and reactive power flows via the AC power flow equations. These equations are the polar form of the complex equation for the power flow.

Definition 6.1.4 (AC solution)**.** Let $\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^p, L^q)$ be an AC network. An AC *solution* is a tuple $(\theta, \widetilde{G}^p, \widetilde{L}^p, \widetilde{v}, \widetilde{G}^q, \widetilde{L}^q)$ where $\theta : N \to \mathbb{R}$, $\widetilde{G}^p : N_G \to \mathbb{R}_{\leq 0}$, $\widetilde{L}^p : N_L \to \mathbb{R}_{\geq 0}$, $\widetilde{G}^q : N_G \to \mathbb{R}$, $\widetilde{L}^q : N_L \to \mathbb{R}$, the active power flow $p : E^d \to \mathbb{R}$ is defined by

$$\forall(\mathsf{a}, \mathsf{d}, b, g, c) \in E^d : p_{\mathsf{ad}} := v_{\mathsf{a}}(g(v_{\mathsf{a}} - v_{\mathsf{d}} \cos(\theta_{\mathsf{a}} - \theta_{\mathsf{d}})) - v_{\mathsf{d}} b \sin(\theta_{\mathsf{a}} - \theta_{\mathsf{d}})),$$

and the reactive power flow $q : E^d \to \mathbb{R}$ is defined by

$$\forall(\mathsf{a},\mathsf{d},b,g,c) \in E^d : q_{\mathsf{ad}} := v_{\mathsf{a}}(\text{-}b(v_{\mathsf{a}} - v_{\mathsf{d}}\cos(\theta_{\mathsf{a}} - \theta_{\mathsf{d}})) - v_{\mathsf{d}}g\sin(\theta_{\mathsf{a}} - \theta_{\mathsf{d}}))$$

and we have

- $(\theta, \widetilde{G}^p, \widetilde{L}^p, p)$ is a GP solution of $\mathcal{N}$,
- the voltage magnitudes are within the given bounds,

  $$\forall \mathsf{a} \in N : \underline{v} \le v_{\mathsf{a}} \le \overline{v},$$

- the phase angle differences are within the given bounds,

  $$\forall(\{\mathsf{a},\mathsf{d}\},b,g,c) \in E : |\theta_{\mathsf{a}} - \theta_{\mathsf{d}}| \le \overline{\Delta},$$

- Kirchhoff's junction rule for reactive power is satisfied,

  $$\forall \mathsf{a} \in N : \sum_{(\mathsf{a},\mathsf{d},b,g,c)\in E^d} q_{\mathsf{ad}} + \overline{G^q_{\mathsf{a}}} + \overline{L^q_{\mathsf{a}}} = 0$$

  where $\overline{G^q} := \widetilde{G}^q|_0^N$ and $\overline{L^q} := \widetilde{L}^q|_0^N$.

The set of all AC solutions for $\mathcal{N}$ is denoted with $\mathcal{S}^{\mathrm{AC}}(\mathcal{N})$.

Let $(\{\mathsf{a},\mathsf{d}\},b,g,c)$ be a line. Note that the value c is not used in the definition of AC solutions. Hence, when defining lines of AC networks within the context of the AC model, we will omit this value. Furthermore, we use the following more compact form when defining lines

$$(\{\mathsf{a},\mathsf{d}\},b_1,g_1,c_1) \approxeq \mathsf{a} \xleftrightarrow[g=g_1]{b=b_1} \mathsf{d}.$$

6.1.2 MAXIMUM POWER FLOW

AC-MPF networks are a special case of AC networks where the variables of active and reactive power generation and demand as well as voltage magnitudes are not fixed. Additionally, an AC-MPF network specifies a value for the active to reactive power demand ratio. This is because active and reactive demand of loads are coupled values. Consuming less active power is similarly correlated with consuming less absolute reactive power. Also, a load which has negative reactive demand cannot just change to positive reactive demand. This is important for AC-MPF networks, as their demand is not specified. Furthermore, an AC-MPF network specifies an upper bound on the active power demand of loads. This "feature" is necessary for our results about the AC-MPF.

Definition 6.1.5 (AC-MPF network)**.** Let

$$\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, G^q, L^p, L^q)$$

be an AC network with

- $dom(v) = \emptyset$,
- $dom(G^p) = \emptyset$,
- $dom(G^q) = \emptyset$,
- $dom(L^p) = \emptyset$,
- $dom(L^q) = \emptyset$.

An AC-MPF *network* is a tuple $(\mathcal{N}, R, \overline{L}^p)$

- $\overline{L}^p : N_L \to \widetilde{\mathbb{Q}}_{\geq 0}$ is the *active power demand upper bound*, and
- $R : N_L \to \widetilde{\mathbb{Q}}$ is the *active to reactive power demand ratio*.

The existence of the active to reactive demand ratio and the active upper bound make it necessary to define a specific AC-MPF solution.

Definition 6.1.6 (AC-MPF solution)**.** Let $(\mathcal{N}, R, \overline{L}^p)$ be an AC-MPF network where

$$\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, G^q, L^p, L^q).$$

An AC solution $(\theta, G^p, L^p, v, G^q, L^q)$ of $\mathcal{N}$ is called AC-MPF *solution* if $\forall \mathsf{l} \in N_L$:

- the active to reactive power ratio is correct, $L^p_{\mathsf{l}} R_{\mathsf{l}} = L^q_{\mathsf{l}}$, and
- the active demand is below its bounds, $L^p_{\mathsf{l}} \leq \overline{L}^p_{\mathsf{l}}$.

The set of all AC-MPF solutions is denoted with $\mathcal{S}^{\mathrm{AC}}_{\mathrm{MPF}}(\mathcal{N}, R, \overline{L}^p)$.

Using the definition of an AC-MPF network and AC-MPF solution, we can now define the problem of maximizing the demand and its decision version.

Definition 6.1.7 (AC-MPF, AC-MPF problem)**.** Let $(\mathcal{N}, R, \overline{L}^p)$ be an AC-MPF network where

$$\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, G^q, L^p, L^q).$$

The AC-MPF of $(\mathcal{N}, R, \overline{L}^p)$ is

$$\text{AC-MPF}(\mathcal{N}, R, \overline{L}^p) := \max_{(\theta, G^p, L^p, v, G^q, L^q) \in \mathcal{S}^{\mathrm{AC}}_{\mathrm{MPF}}(\mathcal{N}, R, \overline{L}^p)} \sum_{\mathsf{l} \in N_L} L^p_{\mathsf{l}}.$$

Given an $x \in \mathbb{Q}_{\geq 0}$, the AC-MPF problem is to decide whether $\text{AC-MPF}(\mathcal{N}, R, \overline{L}^p) \geq x$.

6.1.3 OPTIMAL POWER FLOW

An AC-OPF network is an AC network where all active and reactive power demands are given. Additionally, like OPF network, we have a cost function for all generators.

Definition 6.1.8 (AC-OPF network)**.** An AC-OPF *network* is a tuple $(\mathcal{N}, C)$ such that

- $\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, G^q, L^p, L^q)$ is an AC network,
- $((N, N_G, N_L, E, G^p, L^p), C)$ is an OPF network,
- $dom(v) = \emptyset$,
- $dom(G^q) = \emptyset$, and
- $dom(L^q) = N_L$.

The AC-OPF is about minimizing the generation cost. In this model, we assume linear cost. Linear costs are a special case of the in the literature mostly used quadratic cost functions. Furthermore, linear costs are sufficient to show that the AC-OPF cannot be approximated. Defining an AC-OPF problem is not necessary because we prove results about the approximation of the AC-OPF. Since we have a fixed demand, it is possible that no solution exists. In such a case, the AC-OPF will be infinite.

Definition 6.1.9 (AC-OPF, AC-OPF problem)**.** Let $(\mathcal{N}, C)$ be an AC-OPF network with

$$\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, G^q, L^p, L^q).$$

The AC-OPF of $(\mathcal{N}, C)$ is

$$\text{AC-OPF}(\mathcal{N}, C) := \min_{(\theta, G^p, G^q, v) \in \mathcal{S}^{\text{AC}}(\mathcal{N})} \sum_{\mathsf{r} \in N_G} |G^p_{\mathsf{r}}| C_{\mathsf{r}}.$$

Given an $x \in \mathbb{Q}_{\geq 0}$, the AC-OPF problem is to decide whether AC-OPF$(\mathcal{N}, C) \leq x$.

6.1.4 POWER FLOW

With respect to the difference of OPF and PF networks, an AC-PF network can be regarded as an AC-OPF network where the active power generation is fixed except for the slack bus. Additionally, we also fix the voltage magnitudes of generators. Hence, a solution has to provide a value for the active power generation of the slack bus, the phase angles for all buses and the voltage magnitudes for the buses which are not generators. The AC-PF problem is to decide if a solution exists.

Definition 6.1.10 (AC-PF network)**.** An AC network

$$\mathcal{N} = (N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, G^q, L^p, L^q)$$

is called AC-PF *network* if

- $(N, N_G, N_L, E, G^p, L^p)$ is an PF network and
- $dom(v) = N_G$.

Definition 6.1.11 (AC-PF problem)**.** Let $\mathcal{N}$ be an AC-PF network. The AC-PF *problem* is to decide whether $\mathcal{S}^{\mathrm{AC}}(\mathcal{N}) \neq \emptyset$.

The FIXED-VOLTAGE POWER FLOW (VPF) is further restricting the degrees of freedom in the AC-PF networks by fixing the voltage magnitudes for all buses (not just the generators).

Definition 6.1.12 (VPF network)**.** An AC-PF network

$$(N, N_G, N_L, E, \overline{\Delta}, \underline{v}, \overline{v}, v, G^p, L^p, G^q, L^q)$$

is called VPF *network* if $dom(v) = N$.

Definition 6.1.13 (VPF problem)**.** Let $\mathcal{N}$ be an VPF network. The VPF *problem* is to decide whether $\mathcal{S}^{\mathrm{AC}}(\mathcal{N}) \neq \emptyset$.

We call an AC-OPF or AC-MPF solution *optimal* if it is a witness for the AC-OPF or AC-MPF.

6.2 Magical-Tree network

In this section, we present the Magical-Tree networks, a network class that we used in the reductions for AC-PF (Section 6.4), VPF (Section 6.6) and AC-OPF (Section 6.5). These reductions will consist of the sum of one Magical-Tree network and multiple choice networks[2]. The type of choice network is specific for the studied problem.

The general approach for the reductions of AC-PF, VPF and AC-OPF only work if all generators involved in the reduction have a fixed active power generation. Contradictory to that, AC-PF and VPF networks have to have a slack bus which is an unfixed generator. Furthermore, in AC-OPF networks all generators are unfixed. The purpose of the Magical-Tree network is to provide an implicit fixed generator. Internally, the Magical-Tree network contains one unfixed generator. The line parameters and the network structure imply that this generator can only generate a fixed amount of active power. This amount is given by a free parameter. Hence, the Magical-Tree network can be regarded as "converting" an unfixed generator into a fixed one.

The core component of the Magical-Tree network consist of a sequence of three buses, $\mathsf{e}\leftrightarrow\mathsf{a}\leftrightarrow\mathsf{r}$. We first prove a crucial property of this setup before presenting the entire Magical-Tree network. To that end, assume that the middle bus has some fixed active and reactive power as presented in Fig. 6.1. Under the assumption that all voltage magnitudes are the same, we will show in Lemma 6.2.1 that there are only two possible solutions for the phase angle differences. Furthermore, both solutions are

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[2] The Magical-Tree network is the "main" network from Section 3.2 where we described the general idea of how our reductions work.

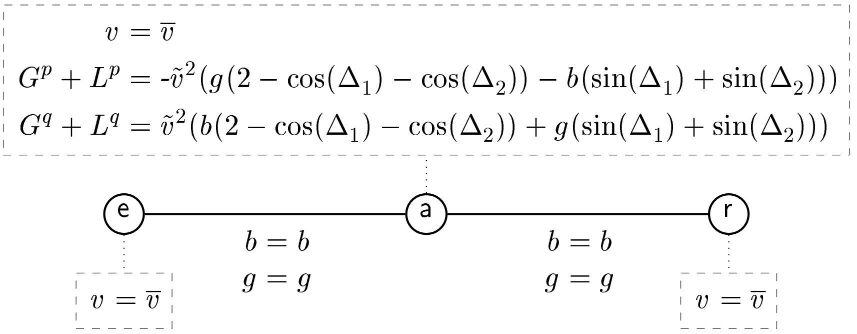

Figure 6.1: The network from Lemma 6.2.1.

essentially the same and only occur because of the symmetry of the problem. These solutions imply two different possible solutions for the line flows. Potentially, one could use this choice between phase angle difference solutions to construct a reduction. We do not use this discreteness in this thesis. This is because we did not find a reduction based on this discreteness which is stronger than the reductions presented in this chapter. For the Magical-Tree network we ensure that only one of these solutions is possible. The generator of the Magical-Tree network will be the bus r. The fact that only one solution exists will force the generator to be fixed.

In Lemma 6.2.1, the bus a has fixed active and reactive power values. In the Magical-Tree network, the active power value will be a demand (positive value). In the lemmas, we present a more general case where the active power could be generation (negative value) as well.

Lemma 6.2.1. *Let* $\tilde{v} \in \widetilde{\mathbb{Q}}_{>0}$*;* $g \in \widetilde{\mathbb{Q}}_{\geq 0}$*;* $b \in \widetilde{\mathbb{Q}}_{\leq 0}$*;* $x_p, y_p, x_q, y_q, \Delta_1, \Delta_2 \in \widetilde{\mathbb{Q}}$ *with* $|\Delta_1|, |\Delta_2| \leq \pi/2$*,* $\{b, g\} \neq \{0\}$*,*

$$\begin{aligned} x_p + y_p = &- \tilde{v}^2(g(1 - \cos(\Delta_1)) - b\sin(\Delta_1)) &(6.1)\\ &- \tilde{v}^2(g(1 - \cos(\Delta_2)) - b\sin(\Delta_2)), \\ x_q + y_q = &- \tilde{v}^2(\text{-}b(1 - \cos(\Delta_1)) - g\sin(\Delta_1)) &(6.2)\\ &- \tilde{v}^2(\text{-}b(1 - \cos(\Delta_2)) - g\sin(\Delta_2)), \\ \mathcal{N} := &\left(\{\mathsf{a},\mathsf{e},\mathsf{r}\}, \{\mathsf{a},\mathsf{e},\mathsf{r}\}, \{\mathsf{a},\mathsf{e},\mathsf{r}\}, \left\{\mathsf{e} \xleftrightarrow[g=g]{b=b} \mathsf{a} \xleftrightarrow[g=g]{b=b} \mathsf{r}\right\}, \frac{\pi}{2}, \tilde{v}, \tilde{v}, \mathcal{T}\right), \\ \mathcal{T} := &\left[G^p_{\mathsf{a}}{=}x_p \middle| G^q_{\mathsf{a}}{=}x_q \middle| L^p_{\mathsf{a}}{=}y_p \middle| L^q_{\mathsf{a}}{=}y_q\right]. \end{aligned}$$

For every AC *solution* $(\theta, G^p, L^p, v, G^q, L^q)$ *of the* AC *network* $\mathcal{N}$ *we have* $\{\Delta_{\mathsf{ae}}, \Delta_{\mathsf{ar}}\} = \{\Delta_1, \Delta_2\}$.

Proof. To simplify notations we define

$$\begin{aligned} c_1 &:= \cos(\Delta_1), & c_2 &:= \cos(\Delta_2), & c_{\mathsf{a}} &:= \cos(\Delta_{\mathsf{ar}}), & c_{\mathsf{e}} &:= \cos(\Delta_{\mathsf{ae}}), \\ s_1 &:= \sin(\Delta_1), & s_2 &:= \sin(\Delta_2), & s_{\mathsf{a}} &:= \sin(\Delta_{\mathsf{ar}}), & s_{\mathsf{e}} &:= \sin(\Delta_{\mathsf{ae}}). \end{aligned}$$

Kirchhoff's junction law for active power at a is

$$\begin{aligned}
0 =& -\tilde{v}^2(g(2-(c_1+c_2)) - b(s_1+s_2)) + p_{\mathsf{ar}} + p_{\mathsf{ae}} \\
=& -\tilde{v}^2(g(2-(c_1+c_2)) - b(s_1+s_2)) \\
& + \tilde{v}^2(g(1-c_{\mathsf{a}}) - bs_{\mathsf{a}}) \\
& + \tilde{v}^2(g(1-c_{\mathsf{e}}) - bs_{\mathsf{e}}) \\
=& -2g + g(c_1+c_2) + b(s_1+s_2) \\
& + g - gc_{\mathsf{a}} - bs_{\mathsf{a}} \\
& + g - gc_{\mathsf{e}} - bs_{\mathsf{e}} \\
0 =& g(c_1+c_2-c_{\mathsf{a}}-c_{\mathsf{e}}) + b(s_1+s_2-s_{\mathsf{a}}-s_{\mathsf{e}}) \\
0 =& bg(c_1+c_2-c_{\mathsf{a}}-c_{\mathsf{e}}) + b^2(s_1+s_2-s_{\mathsf{a}}-s_{\mathsf{e}}).
\end{aligned} \tag{6.3}$$

Kirchhoff's junction law for reactive power at a is

$$\begin{aligned}
0 &= -\tilde{v}^2(-b(2-(c_1+c_2)) - g(s_1+s_2)) + q_{\mathsf{ar}} + q_{\mathsf{ae}} \\
&= -\tilde{v}^2(-b(2-(c_1+c_2)) - g(s_1+s_2)) \\
&\quad + \tilde{v}^2(-b(1-c_{\mathsf{a}}) - gs_{\mathsf{a}}) \\
&\quad + \tilde{v}^2(-b(1-c_{\mathsf{e}}) - gs_{\mathsf{e}}) \\
&= 2b - b(c_1+c_2) + g(s_1+s_2) \\
&\quad - b + bc_{\mathsf{a}} - gs_{\mathsf{a}} \\
&\quad - b + bc_{\mathsf{e}} - gs_{\mathsf{e}} \\
0 &= -b(c_1+c_2-c_{\mathsf{e}}-c_{\mathsf{a}}) + g(s_1+s_2-s_{\mathsf{e}}-s_{\mathsf{a}}) \\
0 &= -bg(c_1+c_2-c_{\mathsf{e}}-c_{\mathsf{a}}) + g^2(s_1+s_2-s_{\mathsf{e}}-s_{\mathsf{a}}).
\end{aligned} \tag{6.4}$$

Combining Eq. (6.3) and Eq. (6.4) and using $\{g, b\} \neq \{0\}$ we have

$$\begin{aligned}
0 &= b^2(s_1+s_2-s_{\mathsf{e}}-s_{\mathsf{a}}) + g^2(s_1+s_2-s_{\mathsf{e}}-s_{\mathsf{a}}) \\
0 &= (b^2+g^2)(s_1+s_2-s_{\mathsf{e}}-s_{\mathsf{a}}) \\
0 &= s_1+s_2-s_{\mathsf{e}}-s_{\mathsf{a}} \\
s_{\mathsf{a}} &= s_1+s_2-s_{\mathsf{e}},
\end{aligned} \tag{6.5}$$

and applying Eq. (6.5) to Eq. (6.3), we get

$$c_{\mathsf{a}} = c_1 + c_2 - c_{\mathsf{e}}. \tag{6.6}$$

Using the trigonometric identity:

$$\sin(\theta)^2 + \cos(\theta)^2 = 1 \tag{6.7}$$

we have

$$1 = s_{\mathsf{a}}^2 + c_{\mathsf{a}}^2 \tag{6.8}$$

$$1 = (s_1 + s_2 - s_{\mathsf{e}})^2 + (c_1 + c_2 - c_{\mathsf{e}})^2. \tag{6.9}$$

Two obvious solutions to Eq. (6.9) are $(s_{\mathsf{e}}, c_{\mathsf{e}}) = (s_1, c_1)$ and $(s_{\mathsf{e}}, c_{\mathsf{e}}) = (s_2, c_2)$. The solutions $(s_{\mathsf{e}}, c_{\mathsf{e}})$ of Eq. (6.9) can be interpreted as all points on a circle with center $(s_1 + s_2, c_1 + c_2)$ and radius one. Equation (6.7) implies that the solutions $(s_{\mathsf{e}}, c_{\mathsf{e}})$ also describe a circle around the center with a radius of one. Therefore, the solutions satisfying both Eq. (6.7) and Eq. (6.9) must be at the intersection of both circles. Since two circles cannot have more than two intersections (unless they are identical), the two previous described solutions are the only solutions possible. Using Eq. (6.5) and Eq. (6.6) we can derive $\{\Delta_{\mathsf{ae}}, \Delta_{\mathsf{ar}}\} = \{\Delta_1, \Delta_2\}$. $\square$

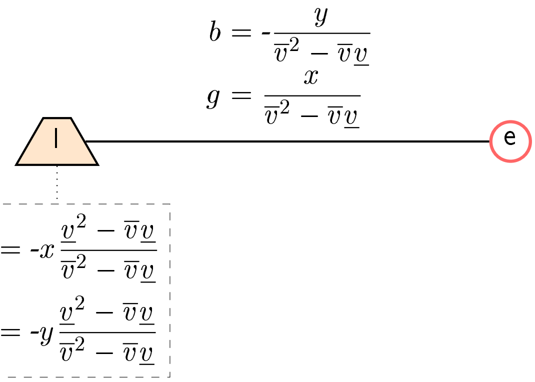

Figure 6.2: The network $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$.

An important condition to make the construction from Lemma 6.2.1 work is the fact that the voltage magnitudes of a, e and r have the same value. In neither AC-PF networks nor in AC-OPF networks can we fix individual voltage magnitudes. The choice network used for the reductions regarding these network classes need free voltage magnitudes. Hence, we cannot fix all voltage magnitudes by setting the lower voltage magnitude bound to the upper bound, i.e. $\underline{v} = \overline{v}$. This motivates the use of the network $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$ presented in Fig. 6.2. This network is used to force the voltage magnitude at its connector e to $\overline{v}$. Each bus of the network from Lemma 6.2.1, Fig. 6.1 (a, e and r) will be connected to one of these networks. The network $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$ further has the feature that the connector will act as a load and that we can choose the values for active and reactive power demand. In the following, we formally define $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$ with respect to some given $\underline{v}$ and $\overline{v}$ and prove its properties.

Definition 6.2.2 ($\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$)**.** Let $x, y, \underline{v}, \overline{v} \in \widetilde{\mathbb{Q}}_{\geq 0}$ be numbers with $\{x, y\} \neq \{0\}$ and

$\underline{v} < \overline{v}$. We define the AC network

$$\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}} := \left(\{\mathsf{e},\mathsf{l}\}, \emptyset, \{\mathsf{l}\}, \left\{ \mathsf{e} \xleftrightarrow[g=\frac{x}{\overline{v}^2-\overline{v}\underline{v}}]{b=-\frac{y}{\overline{v}^2-\overline{v}\underline{v}}} \mathsf{l} \right\}, \frac{\pi}{2}, \underline{v}, \overline{v}, \mathcal{T} \right),$$

$$\mathcal{T} := \left[L^p_{\mathsf{l}}{=}{-x}\frac{\underline{v}^2-\overline{v}\underline{v}}{\overline{v}^2-\overline{v}\underline{v}} \middle| L^q_{\mathsf{l}}{=}{-y}\frac{\underline{v}^2-\overline{v}\underline{v}}{\overline{v}^2-\overline{v}\underline{v}} \right].$$

Lemma 6.2.3. *Let $x, y, \underline{v}, \overline{v} \in \widetilde{\mathbb{Q}}_{\geq 0}$ be numbers with $\{x, y\} \neq \{0\}$, $0 < \underline{v} < \overline{v}$ and $\overline{v} \leq 2\underline{v}$.*

1. *For every AC solution $(\theta, G^p, L^p, v, G^q, L^q)$ of $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}[\mathsf{e}{\in}N_{G/L}]$ we have $v_{\mathsf{e}} = \overline{v}$, $G^p_{\mathsf{e}} + L^p_{\mathsf{e}} = -x$ and $G^q_{\mathsf{e}} + L^q_{\mathsf{e}} = -y$.*

2. *The network $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$ has two buses and the size of every numerical parameter is polynomial in the size of $x, y, \underline{v}$ and $\overline{v}$.*

Proof. Let $(\theta, G^p, L^p, v, G^q, L^q)$ be an AC solution of $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}[\mathsf{e}{\in}N_{G/L}]$. First we observe that

$$L^p_{\mathsf{l}} = -x\frac{\underline{v}^2-\overline{v}\underline{v}}{\overline{v}^2-\overline{v}\underline{v}} = g(\underline{v}^2-\overline{v}\underline{v}) \text{ and}$$

$$L^q_{\mathsf{l}} = -y\frac{\underline{v}^2-\overline{v}\underline{v}}{\overline{v}^2-\overline{v}\underline{v}} = -b(\underline{v}^2-\overline{v}\underline{v})$$

Kirchhoff's junction law at l implies $-L^p_{\mathsf{l}} = p_{\mathsf{le}}$ and $-L^q_{\mathsf{l}} = q_{\mathsf{le}}$. Hence, we have

$$-g(\underline{v}^2-\overline{v}\underline{v}) = gv_{\mathsf{l}}^2 - gv_{\mathsf{l}}v_{\mathsf{e}}\cos(\Delta_{\mathsf{le}}) - bv_{\mathsf{l}}v_{\mathsf{e}}\sin(\Delta_{\mathsf{le}}), \tag{6.10}$$

$$b(\underline{v}^2-\overline{v}\underline{v}) = -bv_{\mathsf{l}}^2 + bv_{\mathsf{l}}v_{\mathsf{e}}\cos(\Delta_{\mathsf{le}}) - gv_{\mathsf{l}}v_{\mathsf{e}}\sin(\Delta_{\mathsf{le}}). \tag{6.11}$$

By multiplying Eq. (6.10) with $-b$ and Eq. (6.11) with g we get

$$gb(\underline{v}^2-\overline{v}\underline{v}) = -gbv_{\mathsf{l}}^2 + gbv_{\mathsf{l}}v_{\mathsf{e}}\cos(\Delta_{\mathsf{le}}) + b^2v_{\mathsf{l}}v_{\mathsf{e}}\sin(\Delta_{\mathsf{le}}), \tag{6.12}$$

$$gb(\underline{v}^2-\overline{v}\underline{v}) = -gbv_{\mathsf{l}}^2 + gbv_{\mathsf{l}}v_{\mathsf{e}}\cos(\Delta_{\mathsf{le}}) - g^2v_{\mathsf{l}}v_{\mathsf{e}}\sin(\Delta_{\mathsf{le}}). \tag{6.13}$$

By subtracting Eq. (6.13) from Eq. (6.12) and using the fact that $\{x, y\} \neq \{0\}$, $0 < \underline{v} \leq v_{\mathsf{l}}, v_{\mathsf{e}}$ and $|\Delta_{\mathsf{le}}| \leq \pi/2$ we derive

$$0 = \left(g^2 + b^2\right) v_{\mathsf{l}}v_{\mathsf{e}}\sin(\Delta_{\mathsf{le}}) \implies 0 = \sin(\Delta_{\mathsf{le}}) \implies 0 = \Delta_{\mathsf{le}}.$$

Hence,

$$L^p_{\mathsf{l}} = g(\underline{v}^2 - \underline{v}\overline{v}) = g(v_{\mathsf{l}}^2 - v_{\mathsf{l}}v_{\mathsf{e}}) \tag{6.14}$$

$$L^q_{\mathsf{l}} = b(\underline{v}^2 - \underline{v}\overline{v}) = b(v_{\mathsf{l}}^2 - v_{\mathsf{l}}v_{\mathsf{e}}). \tag{6.15}$$

The fact $\{x, y\} \neq \{0\}$ implies $\{b, g\} \neq \{0\}$. Therefore, we can use at least one of the equations Eq. (6.14) or Eq. (6.15) to derive $\underline{v}^2 - \underline{v}\overline{v} = v_{\mathsf{l}}^2 - v_{\mathsf{l}}v_{\mathsf{e}}$ and

$$v_{\mathsf{e}} = \frac{v_{\mathsf{l}}^2 + \underline{v}(\overline{v} - \underline{v})}{v_{\mathsf{l}}}.$$

For $v_{\mathsf{l}} = \underline{v}$ we have $v_{\mathsf{e}} = \overline{v}$. To prove that no other solution is possible we show that the function $f(z) = \frac{z^2 + \underline{v}(\overline{v} - \underline{v})}{z}$. is monotonic increasing for $\underline{v} \leq z \leq \overline{v}$. This is done by proving that the derivative $f'(z)$ is positive. We have

$$\begin{aligned}
2\underline{v} &\geq \overline{v} \\
\underline{v} &\geq \overline{v} - \underline{v} \\
z^2 \geq \underline{v}^2 &\geq (\overline{v} - \underline{v})\underline{v} \\
1 &\geq \frac{(\overline{v} - \underline{v})\underline{v}}{z^2} \\
f'(z) = 1 - \frac{(\overline{v} - \underline{v})\underline{v}}{z^2} &\geq 0.
\end{aligned}$$

On the bus e we have $-G^p_{\mathsf{e}} - L^p_{\mathsf{e}} = g\overline{v}(\overline{v} - \underline{v}) = \frac{x}{(\overline{v}^2 - \underline{v}\overline{v})}\overline{v}(\overline{v} - \underline{v}) = x$. Similarly, we derive $G^q_{\mathsf{e}} + L^q_{\mathsf{e}} = -y$.

Part two can be directly derived from Definition 6.2.2. □

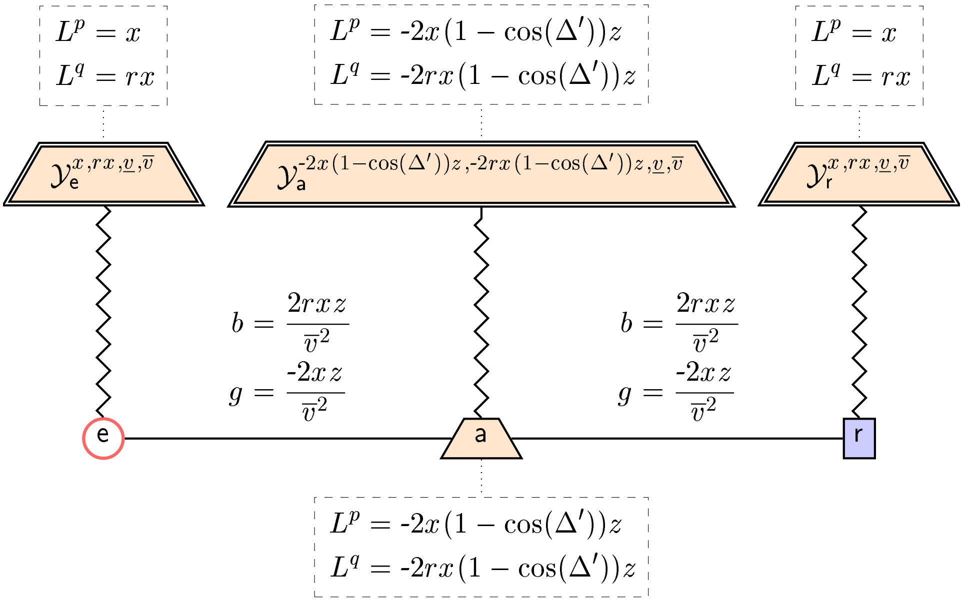

Figure 6.3: The structure of the network $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}$ where $z := \frac{1}{1-\cos(\Delta')-r\sin(\Delta')}$.

We are now in the position to define the class of Magical-Tree networks $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}$ and show that for every $x > 0$ the bus e has an implicit active power generation of -x. We illustrate the Magical-Tree network in Fig. 6.3. It has several parameters besides

x. The parameter Δ' is the phase angle difference of the only possible AC solution of Magical-Tree network. Having this value as a parameter shows that this network will work for any maximum phase angle difference. Another parameter is: r, the ratio between susceptance and conductance. Having this as a free parameter shows that we can adjust the ratio to any value necessary. There is only one restriction on r. The values of r and Δ' have to satisfy the condition that $1 - \cos(\Delta) \leq r\sin(\Delta')$. The last two parameters are the voltage magnitude upper and lower bounds which have to satisfy the conditions in Lemma 6.2.3.

At the core of $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}$ is the network from Fig. 6.1. This network has exactly two solutions. One where power flows along e→a→r and one where the power flows along r→a→e. The latter is the one we desire. Hence, we have to make the former impossible. We achieve this by setting the implicit demand of the network $\mathcal{Y}_{\mathsf{r}}^{x,y,\underline{v},\overline{v}}$ at r such that it is strictly less then the power that would be delivered to r. Hence, there is no way to satisfy Kirchhoff's junction law for active power.

Definition 6.2.4 (Magical-Tree network). Let $x, r, \underline{v}, \overline{v}, \Delta' \in \widetilde{\mathbb{Q}}_{>0}$ be numbers with $0 < \Delta' \leq \pi/2$; $z < 0$; $\underline{v} < \overline{v}$ and $\overline{v} \leq 2\underline{v}$ where $z := \frac{1}{1-\cos(\Delta')-r\sin(\Delta')}$. Furthermore, let

$$\mathcal{N} := \left(\{\mathsf{a},\mathsf{e},\mathsf{r}\}, \{\mathsf{r}\}, \{\mathsf{a}\}, \left\{ \mathsf{e} \xleftrightarrow[g=\frac{-2xz}{\overline{v}^2}]{b=\frac{2rxz}{\overline{v}^2}} \mathsf{a} \xleftrightarrow[g=\frac{-2xz}{\overline{v}^2}]{b=\frac{2rxz}{\overline{v}^2}} \mathsf{r} \right\}, \frac{\pi}{2}, \underline{v}, \overline{v}, \mathcal{T} \right),$$

$$\mathcal{T} := \left[L_{\mathsf{a}}^{p}\text{=-}2x(1-\cos(\Delta'))z \,\middle|\, L_{\mathsf{a}}^{q}\text{=-}2rx(1-\cos(\Delta'))z \right].$$

The *Magical-Tree network* is defined as

$$\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r} := \mathcal{N} +^{\mathsf{e}} \mathcal{Y}_{\mathsf{e}}^{x,rx,\underline{v},\overline{v}} +^{\mathsf{a}} \mathcal{Y}_{\mathsf{a}}^{\text{-}2x(1-\cos(\Delta'))z,\text{-}2rx(1-\cos(\Delta'))z,\underline{v},\overline{v}} +^{\mathsf{r}} \mathcal{Y}_{\mathsf{r}}^{x,rx,\underline{v},\overline{v}}.$$

What follows is the main result of this section.

Lemma 6.2.5. *Let $r, x, \underline{v}, \overline{v}, \Delta' \in \widetilde{\mathbb{Q}}_{>0}$ be numbers satisfying the conditions in Definition 6.2.4. The size of all parameters in $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}$ is polynomial in the size of $r, x, \underline{v}, \overline{v}$ and Δ' and for every* AC *solution $(\theta, G^p, L^p, v, G^q, L^q)$ of the network $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}[\mathsf{e} \in N_{G/L}]$ we have*

1. $v_{\mathsf{e}} = \overline{v}$,
2. $G_{\mathsf{e}}^{p} + L_{\mathsf{e}}^{p} = x$,
3. $G_{\mathsf{e}}^{q} + L_{\mathsf{e}}^{q} = 2xz(r(1-\cos(\Delta')) + \sin(\Delta')) - rx$,
4. $G_{\mathsf{r}}^{p} = \text{-}x + 2xz(1-\cos(\Delta') + r\sin(\Delta'))$, *and*

Proof. Given that every bus of $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}$ is connected to a network of type $\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$, Lemma 6.2.3 implies that $v_{\mathsf{e}} = v_{\mathsf{a}} = v_{\mathsf{r}} = \overline{v}$. Let b be the susceptance and g be the conductance of the lines a↔e and a↔r. For the (implicit) active power demand at a we have:

$$\begin{aligned}
\text{-}2L^p_{\mathsf{a}} &= \text{-}4xz1 - \cos(\Delta') \\
&= 2g\overline{v}^2 1 - \cos(\Delta') \\
&= \overline{v}^2(2g1 - \cos(\Delta') - b\sin(\Delta') + b\sin(\Delta')) \\
&= \overline{v}^2(g1 - \cos(\Delta') - b\sin(\Delta')) + \overline{v}^2(g1 - \cos(\Delta') + b\sin(\Delta')) \\
&= \overline{v}^2(g(1 - \cos(\text{-}\Delta')) + b\sin(\text{-}\Delta')) + \overline{v}^2(g1 - \cos(\Delta') + b\sin(\Delta')).
\end{aligned}$$

Similarly, for the (implicit) reactive power demand at a we have

$$\begin{aligned}
\text{-}2L^q_{\mathsf{a}} &= \text{-}4rxz1 - \cos(\Delta') \\
&= \text{-}2b\overline{v}^2 1 - \cos(\Delta') \\
&= \overline{v}^2(2\text{-}b1 - \cos(\Delta') - g\sin(\Delta') + g\sin(\Delta')) \\
&= \overline{v}^2(\text{-}b1 - \cos(\Delta') - g\sin(\Delta')) + \overline{v}^2(\text{-}b1 - \cos(\Delta') + g\sin(\Delta')) \\
&= \overline{v}^2(\text{-}b(1 - \cos(\text{-}\Delta')) + g\sin(\text{-}\Delta')) + \overline{v}^2(\text{-}b1 - \cos(\Delta') + g\sin(\Delta')).
\end{aligned}$$

This shows that the implicit power at a can be described as in Equations (6.1) and (6.2) from Lemma 6.2.1 where $\Delta_1 = \Delta'$ and $\Delta_2 = -\Delta'$. Let $(\theta, G^p, L^p, v, G^q, L^q)$ be a solution of $\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}[\mathsf{e}{\in}N_{G/L}]$. Lemma 6.2.1 implies that $\{\Delta_{\mathsf{ae}}, \Delta_{\mathsf{ar}}\} = \{\Delta', -\Delta'\}$. Assume that $\Delta_{\mathsf{ar}} = \Delta'$. This implies

$$\begin{aligned}
p_{\mathsf{er}} &= \overline{v}^2(g(1 - \cos(\text{-}\Delta')) - b\sin(\text{-}\Delta')) \\
&= \text{-}2xz1 - \cos(\Delta') - 2xzr\sin(\text{-}\Delta') \\
&= \text{-}2xz(1 - \cos(\Delta') - r\sin(\Delta')) \\
&= \text{-}2x.
\end{aligned}$$

Hence, the line $\mathsf{a}{\leftrightarrow}\mathsf{r}$ generates active power of $2x$ at r. Since the (implicit) active power demand at r is x, Kirchhoff's junction law cannot be satisfied and hence this case is impossible. Therefore, we have to have $\Delta_{\mathsf{ae}} = \Delta'$. Using an equivalent derivation as above, we can derive that this implies $p_{\mathsf{ea}} = \text{-}2x$. Kirchhoff's junction law at e implies

$$\begin{aligned}
G^p + L^p &= \text{-}p_{\mathsf{ea}} - x = 2x - x = x, \\
G^q + L^q &= \text{-}rx - \overline{v}^2(\text{-}b1 - \cos(\Delta') - g\sin(\text{-}\Delta')) \\
&= \text{-}rx + 2rxz1 - \cos(\Delta') + 2xz\sin(\Delta') \\
&= 2xz(r1 - \cos(\Delta') + \sin(\Delta')) - rx, \text{ and} \\
G^p_{\mathsf{r}} &= \text{-}x - p_{\mathsf{ra}} \\
&= \text{-}x - \overline{v}^2(g1 - \cos(\Delta') - b\sin(\text{-}\Delta')) \\
&= \text{-}x + 2xz1 - \cos(\Delta') + 2rxz\sin(\Delta')) \\
&= \text{-}x + 2xz(1 - \cos(\Delta') + r\sin(\Delta')).
\end{aligned}$$

Note that because $x > 0$, $0 < \Delta' \leq {}^{\pi}/_{2}$ and $z < 0$ G^{p}_{r} is negative and hence well defined.

Lemma 6.2.3 and Definition 6.2.4 directly imply that all parameters are polynomial in the size of $x, \underline{v}, \overline{v}, r$ and Δ' with respect to $\widetilde{\mathbb{Q}}$. □

6.3 MAXIMUM POWER FLOW

In this section, we prove that the AC-MPF problem for star networks with one load is NP-hard. Bukhsh et al. [2013] presented a 2-bus example that exhibits disconnected feasibility regions. This example inspired the idea behind the *ratio choice networks* used in this section. A ratio choice network consists of the generator r and the connector e. Its formal definition can be found below, and it is pictured in Fig. 6.4. Both voltage magnitudes are fixed at the same point. As AC-MPF network does not allow the fixing of voltage magnitudes, this is achieved by setting the lower and upper voltage magnitude bounds to the same value[3]. In our reduction the connector will be a load. To ensure that this load is in fact consuming power, it is necessary to enforce that a line flow p_{er} is negative. This introduces a constraint on the values of $\overline{\Delta}$, b and g in the network considered by the proof. Note, however, that this constraint does not remove realistic values for $\overline{\Delta}$, b and g.

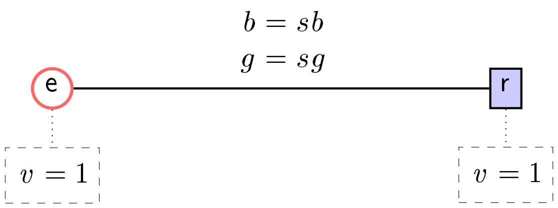

Figure 6.4: The ratio choice network $\mathcal{R}_{\mathsf{e}}^{s,\overline{\Delta},b,g}$ from Definition 6.3.1.

Definition 6.3.1 (ratio-choice network)**.** Let $g \in \widetilde{\mathbb{Q}}_{\geq 0}, b \in \widetilde{\mathbb{Q}}_{\leq 0}, s, \overline{\Delta} \in \widetilde{\mathbb{Q}}_{>0}$ be numbers with $0 < \overline{\Delta} < {}^{\pi}/_{2}$, $\{b, g\} \neq \{0\}$ and $g(1 - \cos(\text{-}\overline{\Delta})) - b\sin(\text{-}\overline{\Delta}) < 0$. A *ratio-choice network* is an AC network

$$\mathcal{R}_{\mathsf{e}}^{s,\overline{\Delta},b,g} = \left(\{\mathsf{e}, \mathsf{r}\}, \{\mathsf{r}\}, \emptyset, \left\{ \mathsf{e} \xleftrightarrow[g=sg]{b=sb} \mathsf{r} \right\}, \overline{\Delta}, 1, 1, \emptyset \right).$$

Ratio choice networks get their name from the fact that the ratio between the active power flow p_{er} and the reactive power flow q_{er} at e is unique. This is pictured in Fig. 6.5 for a line with susceptance of -1 and conductance of 0. Under the assumption that the active power flow is strictly negative at the connector, we can show that the ratio between active and reactive power is strictly monotonically increasing. This is true for any choice of susceptance and conductance, as long as both of them are not 0 at the same time. If there is no flow on the line, then the ratio between active and

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[3] We choose a value of 1 but any other value would work as well.

reactive power can take any value. Hence, if we can enforce a given ratio a ratio choice network has the choice to either generate nothing or exactly the flow corresponding to the given ratio. In our proofs, the given ratio is the ratio obtained when having the maximum phase angle difference. This statement is proven in the following lemma.

Lemma 6.3.2. *Let g, b, s and $\overline{\Delta}$ be like in Definition 6.3.1 and we set*

$$\begin{aligned} p' &:= g(1 - \cos(\text{-}\overline{\Delta})) - b\sin(\text{-}\overline{\Delta}), \\ q' &:= \text{-}b(1 - \cos(\text{-}\overline{\Delta})) - g\sin(\text{-}\overline{\Delta}). \end{aligned}$$

For every AC *solution* $(\theta, G^p, L^p, v, G^q, L^q)$ *of* $\mathcal{R}_{\mathsf{e}}^{s,\overline{\Delta},b,g}[\mathsf{e}{\in}N_L]$ *we have*

$$\Delta_{\mathsf{re}} \geq 0, \tag{6.16}$$

$$q' p_{\mathsf{er}} \leq p' q_{\mathsf{er}}, \tag{6.17}$$

$$q' p_{\mathsf{er}} = p' q_{\mathsf{er}} \iff \Delta_{\mathsf{er}} \in \{0, \overline{\Delta}\}. \tag{6.18}$$

Proof. For $\Delta_{\mathsf{re}} = 0$, we have $g(1 - \cos(\text{-}\Delta_{\mathsf{re}})) - b\sin(\text{-}\Delta_{\mathsf{re}}) = 0$. Assume that $\Delta_{\mathsf{re}} < 0$. We have

$$\begin{aligned} 0 &> g(1 - \cos(\text{-}\overline{\Delta})) - b\sin(\text{-}\overline{\Delta}) \\ 0 &> g(1 - \cos(\overline{\Delta})) + b\sin(\overline{\Delta}) \\ \text{-}b\sin(\overline{\Delta}) &> g(1 - \cos(\overline{\Delta})) \\ \text{-}b &> g\tan(\overline{\Delta}/2) \geq g\tan(\text{-}\Delta_{\mathsf{re}}/2) \\ \text{-}b &> g\tan(\text{-}\Delta_{\mathsf{re}}/2) \\ \text{-}b\sin(\text{-}\Delta_{\mathsf{re}}) &> g(1 - \cos(\text{-}\Delta_{\mathsf{re}})) \\ 0 &> g(1 - \cos(\text{-}\Delta_{\mathsf{re}})) + b\sin(\text{-}\Delta_{\mathsf{re}}) \\ 0 &> g(1 - \cos(\Delta_{\mathsf{re}})) - b\sin(\Delta_{\mathsf{re}}) = p_{\mathsf{re}}. \end{aligned}$$

Since r is a generator, Kirchhoff's junction law at r implies that p_{re} is positive. Hence, we have a contradiction. This implies $\Delta_{\mathsf{re}} > 0$.

To simplify notations, we define $z := \tan(\text{-}\Delta_{\mathsf{re}}/2)$; and $y := \tan(\text{-}\overline{\Delta}/2)$. Assume $\overline{\Delta} > \Delta_{\mathsf{re}} > 0$. Using the fact that the tangent is strongly monotonically increasing within the open interval $(\text{-}\pi/4, 0)$ we have

$$\begin{aligned} y &< z \\ y(b^2 + g^2) &< z(b^2 + g^2) \\ yb^2 - zg^2 &< zb^2 - yg^2 \\ yb^2 - zg^2 + bg(1 - yz) &< zb^2 - yg^2 + bg(1 - yz) \\ (b - zg)(yb + g) &< (b - yg)(zb + g) \\ (zg - b)(-yb - g) &< (yg - b)(-zb - g). \end{aligned}$$

Using the trigonometric identity $\tan(\alpha/2) = \frac{1 - \cos(\alpha)}{\sin(\alpha)}$, and multiplying both sides of the last equation with $s\sin(\text{-}\overline{\Delta})\sin(\text{-}\Delta_{\mathsf{re}})$ (using the fact that $\Delta_{\mathsf{re}} > 0$), we get

$$\begin{aligned}&(sg(1-\cos(\text{-}\Delta_{\mathsf{re}}))-sb\sin(\text{-}\Delta_{\mathsf{re}}))\cdot(\text{-}b(1-\cos(\text{-}\overline{\Delta}))-g\sin(\text{-}\overline{\Delta}))\\ <\ &(sg(1-\cos(\text{-}\overline{\Delta}))-sb\sin(\text{-}\overline{\Delta}))\cdot(\text{-}b(1-\cos(\text{-}\Delta_{\mathsf{re}}))-g\sin(\text{-}\Delta_{\mathsf{re}}))\end{aligned}$$

which is $p_{\mathsf{er}}q' < q_{\mathsf{er}}p'$ for $\overline{\Delta} > \Delta_{\mathsf{re}} > 0$. Eq. (6.17) is true if $\Delta_{\mathsf{re}} = 0$ or $\Delta_{\mathsf{re}} = \overline{\Delta}$. Hence, Eq. (6.17) and Eq. (6.18) are true in general.

$\square$

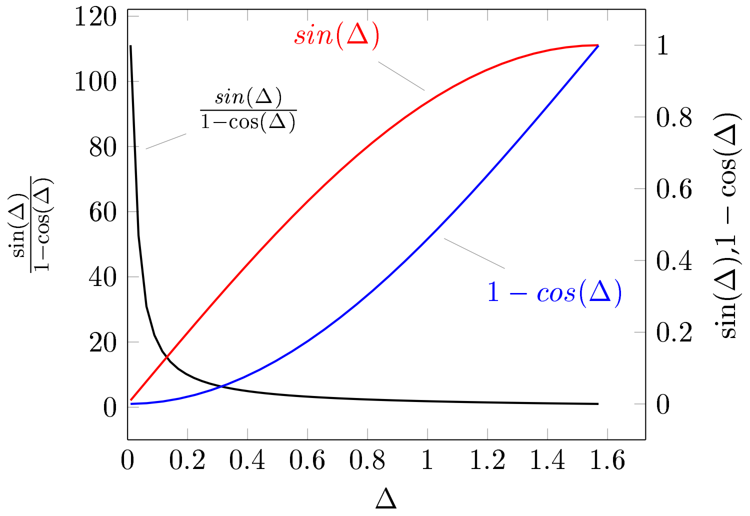

Figure 6.5: The active and reactive power flow and their ratio for a line with susceptance -1 and the conductance 0.

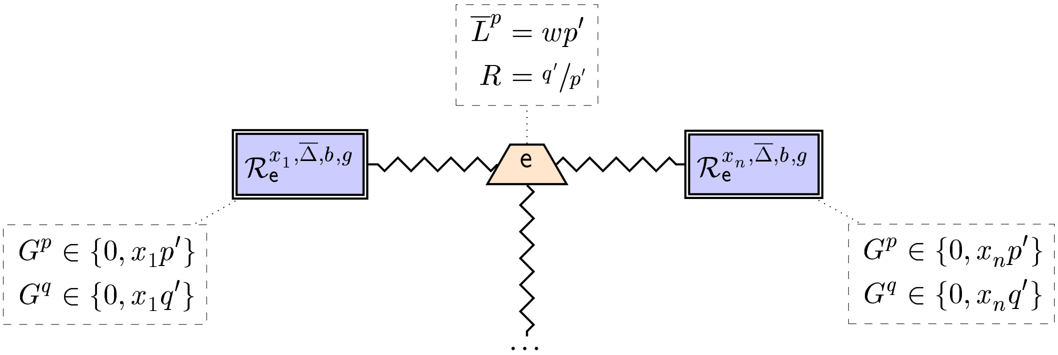

Figure 6.6: The reduction of the SSP instance $(\{x_1,\dots,x_n\}, w)$ used in Theorem 6.3.3.

We are now in the position to prove our main result for this section. Our reduction of the SSP problem is presented in Fig. 6.6.

Theorem 6.3.3. *The* AC-MPF *problem on trees with one load is* NP*-hard.*

Proof. Let g, b, s and $\overline{\Delta}$ be as for Definition 6.3.1; p', q' defined as for Lemma 6.3.2, and, (S, w) be a SSP instance. Both together imply $p' < 0$. We define the AC networks

$$\mathcal{N} := \big(\{\mathsf{e}\}, \emptyset, \{\mathsf{e}\}, \emptyset, \overline{\Delta}, 1, 1, \emptyset\big) \text{ and}$$
$$\mathcal{N}_{S,w} := \mathcal{N} +^{\mathsf{e}} \sum_{x\in S}^{\mathsf{e}} \mathcal{R}_{\mathsf{e}}^{x,\overline{\Delta},b,g}.$$

Furthermore, we define a ratio function $R : \{\mathsf{e}\} \to \mathbb{Q}$ via $R_{\mathsf{e}} := \frac{q'}{p'}$ and a maximum load $\overline{L}^p : \{\mathsf{e}\} \to \mathbb{Q}_{\geq 0}$ via $\overline{L}^p_{\mathsf{e}} := wp'$. The network $\mathcal{N}_{S,w}$ has $|S| + 1$ buses, all line parameters are products and sums of numbers from (S, w) and (possibly irrational) constants g and b; and, the ratio and maximum load depend on the (possible irrational) constants $\overline{\Delta}$ $(\cos(\overline{\Delta}), \sin(\overline{\Delta}))$. Furthermore, there are no fixed variables and we have a fixed active to reactive power ratio for the only load in the network. Hence, $(\mathcal{N}_{S,w}, R, \overline{L}^p)$ is an AC-MPF network with size polynomial in the size of the input. The rest of the proof shows that

$$\text{AC-MPF}(\mathcal{N}_{S,w}, R, \overline{L}^p) \geq wp' \iff (S, w) \text{ is solvable.}$$

First we observe that in all solutions all voltage magnitudes have to be 1. Hence, we ignore them from now on. For the rest of the proof, let x be the generator of the ratio choice network $\mathcal{R}_{\mathsf{e}}^{x,\overline{\Delta},b,g}$.

Case 1: AC-MPF$(\mathcal{N}_{S,w}, R, \overline{L}^p) \geq wp' \impliedby (S, w)$ is solvable.
Let V be a solution for (S, w). We define $\theta_{\mathsf{e}} := 0, \forall \mathsf{x} \in V : \theta_{\mathsf{x}} := \overline{\Delta}, \forall \mathsf{x} \in S \setminus V : \theta_{\mathsf{x}} := 0$. This implies power flows of $\forall \mathsf{x} \in V : p_{\mathsf{ex}} := xp', q_{\mathsf{ex}} := \mathsf{x}q'$. Using the fact that V is a solution for (S, w), the conservation law at l is

$$\sum_{\mathsf{x}\in S} p_{\mathsf{ex}} = \sum_{\mathsf{x}\in V} p_{\mathsf{ex}} = \sum_{\mathsf{x}\in V} xp' = wp' = L^p_{\mathsf{e}} = \text{AC-MPF}(\mathcal{N}_{S,w}, R, \overline{L}^p),$$
$$\sum_{\mathsf{x}\in S} q_{\mathsf{ex}} = \sum_{\mathsf{x}\in V} q_{\mathsf{ex}} = \sum_{\mathsf{x}\in V} xq' = wq' = L^q_{\mathsf{e}}.$$

Moreover, the generation constraints are satisfied because $g(1 - \cos(\overline{\Delta})) - b\sin(\overline{\Delta})$ is always positive for a positive phase angle difference. Hence, we have defined a solution with demand of wp'.

Case 2: AC-MPF$(\mathcal{N}_{S,w}, R, \overline{L}^p) \geq wp' \implies (S, w)$ is solvable.
Let us assume we have the optimal AC-MPF solution $(\theta, G^p, L^p, v, G^q, L^q)$ for $\mathcal{N}_{S,w}$ and p and q are the implied active and reactive power flows. We have

$$\text{AC-MPF}(\mathcal{N}_{S,w}, R, \overline{L}^p) = L^p_{\mathsf{e}} \leq \overline{L}^p_{\mathsf{e}} = wp'.$$

This implies that $L^p_{\mathsf{e}} = wp'$. Eq. (6.16) of Lemma 6.3.2 shows that $\forall \mathsf{x} \in S : \Delta_{\mathsf{xe}} \geq 0$. We define $V := \{\mathsf{x} \in S \mid \Delta_{\mathsf{xe}} > 0\}$. Since we have a solution, Kirchhoff's junction law for active power becomes $\sum_{\mathsf{x}\in S} p_{\mathsf{ex}} = wp'$ and because $R_{\mathsf{e}} = {}^{q'}\!/_{p'}$ reactive power is

$\sum_{\mathsf{x}\in S} q_{\mathsf{ex}} = wq'$. Using $p' < 0$ and $\overline{\Delta} > 0 \implies q' > 0$ we can derive

$$\sum_{\mathsf{x}\in S} \frac{p_{\mathsf{ex}}}{p'} = \sum_{\mathsf{x}\in S} \frac{q_{\mathsf{ex}}}{q'}$$

$$0 = \sum_{\mathsf{x}\in S} \left(\frac{p_{\mathsf{ex}}}{p'} - \frac{q_{\mathsf{ex}}}{q'}\right) = \sum_{\mathsf{x}\in V} \left(\frac{p_{\mathsf{ex}}}{p'} - \frac{q_{\mathsf{ex}}}{q'}\right) = \sum_{\mathsf{x}\in V} (p_{\mathsf{ex}}q' - q_{\mathsf{ex}}p').$$

Equation (6.17) in Lemma 6.3.2 implies that every summand in this equation is non-positive. Hence, all summands must be 0. Given our choice of V and using Eq. (6.18) from Lemma 6.3.2, we have $\forall \mathsf{x} \in V : \Delta_{\mathsf{xe}} = \overline{\Delta}$. This implies $\forall \mathsf{x} \in V : p_{\mathsf{ex}} = xp'$ and hence using Kirchhoff's junction law for active power we have $\sum_{\mathsf{x}\in V} p_{\mathsf{ex}} = \sum_{\mathsf{x}\in V} \mathsf{x}p' = wp'$ which proves $\sum_{\mathsf{x}\in V} \mathsf{x} = w$. ☐

6.4 POWER FLOW

In this section, we present the proof that deciding the AC-PF problem for tree networks with one generator cannot be done in polynomial time unless P = NP. At the core of this proof is the class of *voltage choice networks*. A voltage choice network consists of a line with a load on one end and the connector on the other end. Assume that the voltage magnitude at the connector is fixed. Under some condition for the voltage magnitude at the connector, the line parameters and the active and reactive power demand, we have exactly two different solutions for the voltage magnitude at the load. These two solutions imply two different line flows which then lead to two different implicit active and reactive power demand choices at the connector. As the solutions for the voltage magnitude at the load are rather bulky, we define the voltage choice network with specific line parameters and demand values. And we will use a specific value for the voltage magnitude of the connector. The only remaining free variable in the choice networks will be the scalar s. It allows us to scale the implicit active and reactive power demand at the connector. The following definition presents the voltage choice network. It is also pictured in Fig. 6.7.

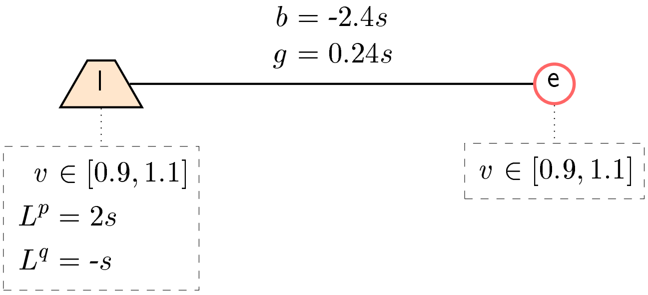

Figure 6.7: The voltage choice network $\mathcal{V}^s_{\mathsf{e}}$ from Definition 6.4.1.

Definition 6.4.1 (voltage choice network)**.** Let $s \in \widetilde{\mathbb{Q}}_{>0}$ be a number. We define the *voltage choice network*

$$\mathcal{V}^s_{\mathsf{e}} := \left(\{\mathsf{e},\mathsf{l}\}, \emptyset, \{\mathsf{l}\}, \left\{\mathsf{a} \xleftrightarrow[g=0.24s]{b=\text{-}2.4s} \mathsf{e}\right\}, \pi/2, 0.9, 1.1, \left[L^p_{\mathsf{l}}=2s \middle| L^q_{\mathsf{l}}=\text{-}s\right]\right).$$

While analyzing the voltage choice network, we can imagine e to be a generator with fixed voltage magnitude of $\overline{v} = 1.1$. This is because e will be connected to the main network of our reduction where it is a generator. The main network will also ensure that the voltage magnitude of e is fixed. Hence, there are two degrees of freedom left: the voltage magnitude at l and the phase angle difference between the two buses. The selection of these two variables defines the active and the reactive power sent and received through the line. Assume $s = 1$. Fig. 6.8 presents the possible values of active (p_{le}, solid blue line) and reactive power flows (q_{le}, dotted green line) at the bus l with respect to the voltage magnitude v_{l} (x-axis) and phase angle difference Δ_{le} (y-axis).

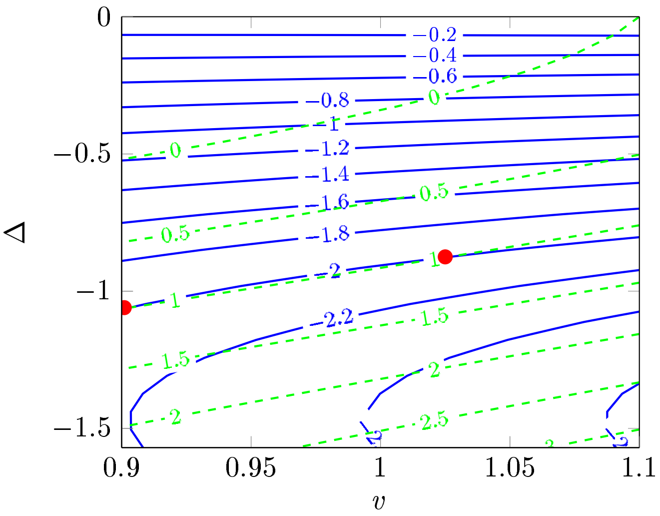

Figure 6.8: The solution space for various active power (blue, dashed) and reactive power (green, solid) values over the phase angle difference (y-axis) and the voltage magnitudes (x-axis). For $(-2, 1)$, both possible solutions are marked.

In order to meet active power demand of 1 at the load l we need an active power flow of -1. Similarly, to meet the reactive power demand of -2 we need a reactive power flow of 2. Fig. 6.8 shows that the curves for active power flow of 1 and reactive power flow of 2 intersect at exactly two points (red squares). This implies that there are exactly two pairs of voltage magnitude v_{l} and phase angle difference Δ_{le} both satisfying Kirchhoff's junction law at l. We can also observe that there are active and reactive power flow values where there is only one solution. Lemma 6.4.2 presents a necessary condition for the existence of two voltage magnitude solutions and characterizes the two solutions in terms of the line parameters and the active and reactive power demand at l.

Lemma 6.4.2. *Let* $b \in \widetilde{\mathbb{Q}}_{\leq 0}; g, p' \in \widetilde{\mathbb{Q}}_{\geq 0}; s, \tilde{v} \in \widetilde{\mathbb{Q}}_{>0}; q' \in \widetilde{\mathbb{Q}}$ *be numbers with* $\{b, g\} \neq \{0\}$*, and*

$$\frac{\tilde{v}^4}{4} + \frac{gp' - bq'}{b^2 + g^2}\tilde{v}^2 - \left(\frac{bp' + gq'}{b^2 + g^2}\right)^2 \geq 0. \tag{6.19}$$

For every AC *solution* $(\theta, G^p, L^p, v, G^q, L^q)$ *of*

$$\mathcal{N} := \left(\{\mathsf{e}, \mathsf{l}\}, \{\mathsf{e}\}, \{\mathsf{e}, \mathsf{l}\}, \left\{\mathsf{e} \xleftrightarrow[g=gs]{b=bs} \mathsf{l}\right\}, \pi/2, 0, \infty, \left[v_\mathsf{e}{=}\tilde{v} \middle| L^p_\mathsf{l}{=}sp' \middle| L^q_\mathsf{l}{=}sq'\right]\right)$$

we have

$$v_\mathsf{l}^2 = \frac{gp' - bq'}{b^2 + g^2} + \frac{v_\mathsf{e}^2}{2} \pm \sqrt{\frac{v_\mathsf{e}^4}{4} + \frac{gp' - bq'}{b^2 + g^2}v_\mathsf{e}^2 - \left(\frac{bp' + gq'}{b^2 + g^2}\right)^2}. \tag{6.20}$$

Proof. The power flow equations together with Kirchhoff's junction law at l imply

$$\text{-}p' = p_{\mathsf{le}} = gv_\mathsf{l}^2 - \tilde{v}v_\mathsf{l}(g\cos(\Delta_{\mathsf{le}}) + b\sin(\Delta_{\mathsf{le}})) \tag{6.21}$$

$$\text{-}q' = q_{\mathsf{le}} = -bv_\mathsf{l}^2 + \tilde{v}v_\mathsf{l}(b\cos(\Delta_{\mathsf{le}}) - g\sin(\Delta_{\mathsf{le}})). \tag{6.22}$$

Since $\{b, g\} \neq \{0\}$ we have $b^2 + g^2 \neq 0$. Let

$$k := \text{-}\frac{p'b + q'g}{b^2 + g^2} \text{ and}$$
$$t := \text{-}\frac{p'g - q'b}{b^2 + g^2}.$$

The sum of Eq. (6.21) multiplied with b and Eq. (6.22) multiplied with g leads to

$$\begin{aligned}
k(b^2 + g^2) &= b(gv_\mathsf{l}^2 - \tilde{v}v_\mathsf{l}(g\cos(\Delta_{\mathsf{le}}) + b\sin(\Delta_{\mathsf{le}}))) \\
&\quad + g(-bv_\mathsf{l}^2 + \tilde{v}v_\mathsf{l}(b\cos(\Delta_{\mathsf{le}}) - g\sin(\Delta_{\mathsf{le}}))) \\
k &= -v_\mathsf{l}\tilde{v}\sin(\Delta_{\mathsf{le}}) \\
\sin(\Delta_{\mathsf{le}}) &= -\frac{k}{v_\mathsf{l}\tilde{v}}.
\end{aligned}$$

Similarly, multiplying Eq. (6.21) with g and Eq. (6.22) with -b shows

$$\begin{aligned}
t(b^2 + g^2) &= g(gv_\mathsf{l}^2 - \tilde{v}v_\mathsf{l}(g\cos(\Delta_{\mathsf{le}}) + b\sin(\Delta_{\mathsf{le}}))) \\
&\quad - b(-bv_\mathsf{l}^2 + \tilde{v}v_\mathsf{l}(b\cos(\Delta_{\mathsf{le}}) - g\sin(\Delta_{\mathsf{le}}))) \\
t &= v_\mathsf{l}^2 - v_\mathsf{l}\tilde{v}\cos(\Delta_{\mathsf{le}}) \\
\cos(\Delta_{\mathsf{le}}) &= -\frac{t - v_\mathsf{l}^2}{v_\mathsf{l}\tilde{v}}.
\end{aligned}$$

Using the Pythagorean identity, $\sin(\Delta_{\mathsf{le}})^2 + \cos(\Delta_{\mathsf{le}})^2 = 1$, we derive

$$\begin{aligned}
1 &= \left(\frac{k}{\tilde{v}v_{\mathsf{l}}}\right)^2 + \left(\frac{t - v_{\mathsf{l}}^2}{v_{\mathsf{l}}\tilde{v}}\right)^2 \\
1 &= \frac{k^2}{\tilde{v}^2 v_{\mathsf{l}}^2} + \frac{t^2 + v_{\mathsf{l}}^4 - 2tv_{\mathsf{l}}^2}{\tilde{v}^2 v_{\mathsf{l}}^2} \\
\tilde{v}^2 v_{\mathsf{l}}^2 &= k^2 + t^2 + v_{\mathsf{l}}^4 - 2tv_{\mathsf{l}}^2 \\
0 &= v_{\mathsf{l}}^4 - v_{\mathsf{l}}^2(2t + \tilde{v}^2) + k^2 + t^2 \\
v_{\mathsf{l}}^2 &= t + \frac{\tilde{v}^2}{2} \pm \sqrt{\frac{\tilde{v}^4}{4} - t\tilde{v}^2 - k^2}.
\end{aligned}$$

Substituting k and t with their definitions gives the desired result. □

The two voltage magnitude solutions presented by Lemma 6.4.2 lead to two different active power flow values p_{el} at the bus e. This choice of active power flow value is what makes this network a choice network. In the following, we use p'_1 and p'_2 to denote these two active power quantities. An important feature of the example of Fig. 6.7 is that it can be scaled. Given a positive real-valued scalar s, the parameters of the network (susceptance and conductance of the line, active and reactive power demand) are multiplied by s again giving us a network with two feasible points sp'_1 and sp'_2. We can choose the parameter s to make the difference $s(p'_2 - p'_1)$ arbitrary. This is summarised in Lemma 6.4.3.

Lemma 6.4.3. *There exists* $p'_1, p'_2 \in \widetilde{\mathbb{Q}}_{\geq 0}$ *with* $p'_1 \neq p'_2$ *and* $q'_1, q'_2 \in \widetilde{\mathbb{Q}}$ *such that for all* AC *solutions of* $\mathcal{V}^s_{\mathsf{e}}[v_{\mathsf{e}}{=}1.1|\mathsf{e}{\in}N_{G/L}]$ *we have* $(G^p_{\mathsf{e}} + L^p_{\mathsf{e}}, G^q_{\mathsf{e}} + L^q_{\mathsf{e}}) \in \{(\text{-}sp'_1, \text{-}sq'_1), (\text{-}sp'_2, \text{-}sq'_2)\}$.

Furthermore the network $\mathcal{V}^s_{\mathsf{e}}$ *has two buses and the size of every numerical parameter is polynomial in* s.

Proof. The chosen values for voltage magnitude, susceptance, conductance, active and reactive power demand satisfy Condition (6.19) of Lemma 6.4.2. The lemma implies that the only two possible positive voltage magnitude values are $v_{\mathsf{l}} \approx 0.9018$ and $v_{\mathsf{l}} \approx 1.028$. These values imply that we have two different possible values for p_{el} and q_{el} which are linked via v_{l}. Furthermore, because the voltage magnitude values are independent from s but the susceptance and conductance of $\mathsf{l}{\leftrightarrow}\mathsf{e}$ are scaled by s, the two active power flow solutions p'_1 and p'_2 as well as the two reactive power solutions q'_1 and q'_2 are scaled by s.

Part two can be directly derived from Definition 6.4.1. □

Using voltage choice networks, it is easy to prove that the AC-PF problem is NP-hard by reducing the SSP. The reduction follows the standard way outlined in Section 3.2 with a Magical-Tree network being the main network. Note that the susceptance to conductance ratio has been chosen here to be -10. Also, we have chosen voltage bounds of 0.9 to 1.1. These values can be changed as long as one can find

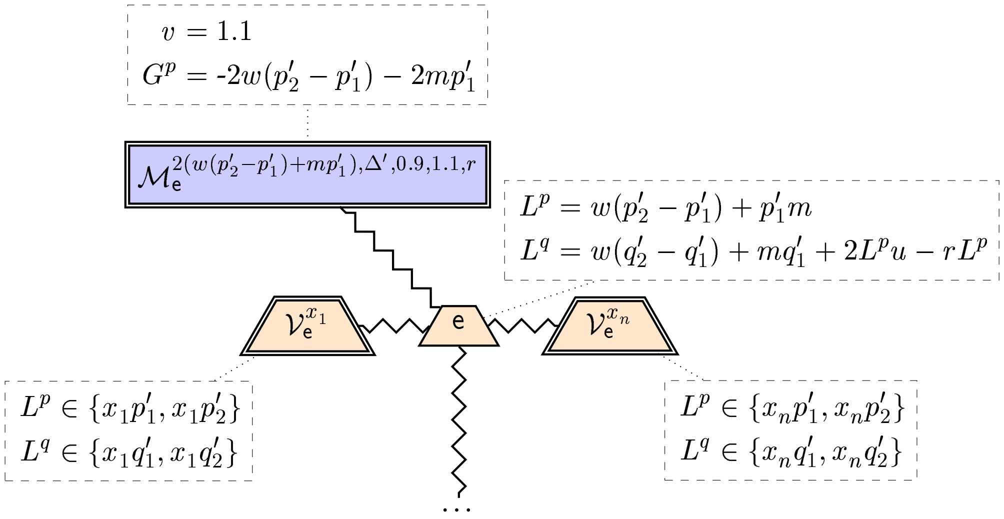

Figure 6.9: The network $\mathcal{N}_{S,w}$ from Theorem 6.4.4 for $(\{x_1, \ldots, x_n\}, w)$ where $m := \sum_{x \in S} x$ and $u := \frac{r(1-\cos(\Delta'))+\sin(\Delta')}{1-\cos(\Delta')-r\sin(\Delta')}$.

active and reactive power demand values for the voltage choice networks such that Condition (6.19) is satisfied.

Our reduction uses a single generator. Also, we need an unbounded maximum degree, but the results also potentially extends to binary trees. An illustration of the reduction is presented in Fig. 6.9. Let (S, w) be an instance of SSP. Furthermore, let p'_1 and p'_2 be the values from Lemma 6.4.3 and we assume that $p'_2 > p'_1$. One Magical-Tree network and one choice network per element of S are all connected at the bus e. The Magical-Tree network forces the voltage magnitude at e to be 1.1 which allows us to use Lemma 6.4.3. The load at e and the Magical-Tree network together imply an implicit active power generation of $w(p'_2 - p'_1) + mp'_1$ at e. A voltage choice network has a demand of either xp'_1 or xp'_2. Hence, the voltage choice networks all have to work together to satisfy Kirchhoff's junction law for active power at e.

The primary function of the load at e is to allow us to satisfy Kirchhoff's junction for reactive power. One could achieve the same implicit active power generation at e by only using the Magical-Tree network. The problem is that the reactive power generated from the Magical-Tree network would, in general, not match the reactive demand from the choice networks. The load at e allows to compensate the implicit reactive power from the Magical-Tree network and having the right amount of reactive power for the choice networks. The latter is based on the values q'_1 and q'_2 from Lemma 6.4.3. An alternative to making e a load would be to make it a generator with a fixed active power generation of half of $w(p'_2 - p'_1) + mp'_1$ and have the Magical-Tree network generate the other half. As the reactive power generation is not fixed, it can compensate for any reactive power coming from the Magical-Tree network.

Theorem 6.4.4. *The* AC-PF *problem is* NP-*hard for tree* AC-PF *networks with a single generator.*

Proof. Let $r, \Delta' \in \widetilde{\mathbb{Q}}_{>0}$, $\Delta' \leq {}^{\pi}\!/_{2}$ and (S, w) be a SSP instance with $m := \sum_{x \in S} x$. Furthermore, let p'_2, p'_1, q'_1 and q'_2 be the values from Lemma 6.4.3 and w.l.o.g. we assume that $p'_2 > p'_1$. We set

$$\begin{aligned} z &:= w(p'_2 - p'_1) + mp'_1, \\ y &:= w(q'_2 - q'_1) + mq'_1 + 2z\frac{r(1 - \cos(\Delta')) + \sin(\Delta')}{1 - \cos(\Delta') - r\sin(\Delta')} - rz \end{aligned}$$

and define the AC networks

$$\begin{aligned} \mathcal{N} &:= \Big(\{\mathsf{e}\}, \emptyset, \{\mathsf{e}\}, \emptyset, \Delta', \underline{v}, \overline{v}, \Big[L^p_{\mathsf{e}}{=}z \Big| L^q_{\mathsf{e}}{=}y\Big]\Big), \text{ and} \\ \mathcal{N}_{S,w} &:= \mathcal{N} +^{\mathsf{e}} \mathcal{M}^{2z,\Delta',0.9,1.1,r}_{\mathsf{e}} +^{\mathsf{e}} \sum_{x \in S}^{\mathsf{e}} \mathcal{V}^x_{\mathsf{e}}. \end{aligned}$$

We will now show that $\mathcal{N}_{S,w}$ is an AC-PF network, and

$$\mathcal{S}^{\mathrm{AC}}(\mathcal{N}_{S,w}) \neq \emptyset \iff (S, w) \text{ is solvable.}$$

The network $\mathcal{N}_{S,w}$ has $|S| + 6$ buses. Part two of Lemma 6.2.5 and Lemma 6.4.3 imply that the networks $\mathcal{M}^{2z,\Delta',0.9,1.1,r}_{\mathsf{e}}$ and $\mathcal{V}^x_{\mathsf{e}}$ are polynomial in the size of (S, w). All numbers involved in the definition of the Magical-Tree network and the network $\mathcal{N}$ are rational numbers except the values p'_1, p'_2, q'_1, q'_2, $\cos(\Delta')$ and $\sin(\Delta')$. The network has one generator which is not fixed and hence can act as a slack bus. Furthermore, the demand of all loads is fixed. Hence, the network $\mathcal{N}_{S,w}$ is an AC-PF network and has size polynomial in the size of (S, w).

Lemma 6.2.5 shows that $v_{\mathsf{e}} = 1.1$. Hence, we can apply Lemma 6.4.3 to the voltage choice networks. The lemma implies that a network $\mathcal{V}^x_{\mathsf{e}}$ has exactly two different active power demands xp'_2 or xp'_1 at e. We call a network $\mathcal{V}^x_{\mathsf{e}}$ *active* if the first solution is true. If no network were active, then we would have an implicit active power demand of mp'_1 at e. For every active network $\mathcal{V}^x_{\mathsf{e}}$, we "gain" the active power demand $x(p'_2 - p'_1)$. Hence, the active power can be characterized as

$$\sum_{\substack{x \in S \\ \mathcal{V}^x_{\mathsf{e}} \text{ is active}}} x(p'_2 - p'_1) + mp'_1.$$

Lemma 6.2.5 shows that the Magical-Tree network has an implicit active power generation of -$2z$ and reactive power generation of -$(2z\frac{r(1-\cos(\Delta'))+\sin(\Delta')}{1-\cos(\Delta')-r\sin(\Delta')} - rz)$. Adding the demand of e to these values leaves us with an implicit active power generation of z at e and a reactive power of $w(q'_2 - q'_1) + mq'_1$. Hence, Kirchhoff's junction law for active power at e is satisfied if and only if (S, w) is solvable. With similar arguments and the us of Lemma 6.4.3, we can show that Kirchhoff's junction law for reactive power is satisfied if and only if (S, w) is solvable. □

6.5 OPTIMAL POWER FLOW

In this section, we show that the AC-OPF for tree AC-OPF networks cannot be approximated arbitrarily close in polynomial time unless $\mathsf{P} = \mathsf{NP}$. As a consequence, the AC-OPF problem for trees is NP-hard. We present two different proofs for these statements. The first one uses an arbitrary number of loads and two generators and works only for limited line parameters. In the second one, there is only one load and an arbitrary number of generators. This proof works for almost arbitrary line parameters and maximum phase angle difference. The downside is that we need to be able to fix the voltage magnitudes by making upper and lower voltage magnitude equal.

The first proof builds on top of the results of Theorem 6.4.4 from Section 6.4. In Fig. 6.10, we show the reduction for an SSP instance (S, w). Let y and z be as in the figure and w.l.o.g. we assume that $\sum_{x\in S} x > w$. The difference to the reduction of Theorem 6.4.4 is that e is a generator with cost of $\epsilon z(1+y)/(p_2' - p_1')$. The generation of the Magical-Tree network is $z(1+y)$ which implies a fixed implicit generation of z. The costs for the generators are chosen such that the generator of the Magical-Tree network is cheap (cost of 1) and e is expensive. Theorem 6.4.4 shows that we can distribute the implicit generation of the Magical-Tree network at e among the demand if and only if the SSP instance is solvable. Hence, if the SSP instance is solvable we do not need the expensive generator and we have an AC-OPF of $z(1+y)$. If, on the other hand, the instance is not solvable, then e has to generate power of at least $p_2' - p_1'$. This means that our cost is at least $z(1+y) + (p_2' - p_1')\epsilon z(1+y)/(p_2' - p_1') = (1+\epsilon)z(1+y)$. If an ϵ-approximation exists and the instance was solvable than we know that the cost of the output of the algorithm is in the interval $[z(1+y), \epsilon z(1+y)]$. Since $\epsilon > 1$, we have $(1+\epsilon)z(1+y) > \epsilon z(1+y)$. Hence, we can derive that the SSP instance is solvable if and only if the ϵ-approximation returns a solution with cost less or equal to $\epsilon z(1+y)$.

Theorem 6.5.1. *There is no ϵ-approximation algorithm for the* AC-OPF *on tree* AC-OPF *networks with two generators unless* $\mathsf{P} = \mathsf{NP}$.

Proof. Assume there exists an ϵ-approximation. We will now show that we can use this algorithm to decide the SSP. Let $r, \Delta' \in \widetilde{\mathbb{Q}}_{>0}$ be numbers with $\Delta' \leq {}^{\pi}\!/_{2}$ and (S, w) be an SSP instance where we define $m := \sum_{x\in S} x$. Furthermore, let p_2', p_1', q_1' and q_2' be the values from Lemma 6.4.3 and w.l.o.g. we assume that $p_2' > p_1'$. We set

$$\begin{aligned} z &:= w(p_2' - p_1') + m p_1', \\ y &:= 2\frac{1 - \cos(\Delta') + r\sin(\Delta')}{1 - \cos(\Delta') - r\sin(\Delta')}, \end{aligned}$$

and define the AC networks

$$\mathcal{N} := \left(\{\mathsf{e}\}, \{\mathsf{e}\}, \emptyset, \emptyset, \Delta', \underline{v}, \overline{v}, \emptyset\right), \text{ and}$$

$$\mathcal{N}_{S,w} := \mathcal{N} +^{\mathsf{e}} \mathcal{M}_{\mathsf{e}}^{z,\Delta',0.9,1.1,r} +^{\mathsf{e}} \sum_{x\in S}^{\mathsf{e}} \mathcal{V}_{\mathsf{e}}^{x}$$

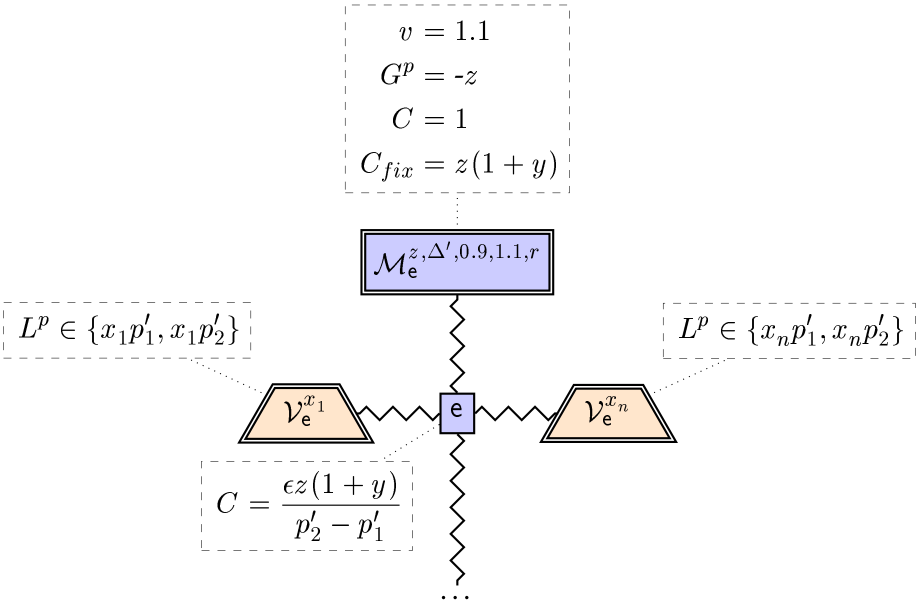

Figure 6.10: The network $\mathcal{N}_{S,w}$ from Theorem 6.5.1 for $(\{x_1, \ldots, x_n\}, w)$, $m := \sum_{x \in S} x$, $z := w(p'_2 - p'_1) + mp'_1$ and $y := 2\frac{1-\cos(\Delta')+r\sin(\Delta')}{1-\cos(\Delta')-r\sin(\Delta')}$.

and a cost function $C : \widetilde{\mathbb{Q}}_{\geq 0} \to \{\mathsf{e}, \mathsf{r}\}$ with $C_{\mathsf{e}} := \frac{\epsilon z(1+y)}{p'_2 - p'_1}$ and $C_{\mathsf{r}} := 1$. As argued in Theorem 6.4.4, the network $\mathcal{N}_{S,w}$ is polynomial in the size of the input. Hence, the tuple $(\mathcal{N}_{S,w}, C)$ is an AC-OPF network polynomial in the size of (S, w).

Lemma 6.2.5 shows that $\mathcal{M}_{\mathsf{e}}^{z,\Delta',\underline{v},\overline{v},r}$ has a fixed and implicit generation of z at e. It also shows that the generator r has to generate a value of $z(1+y)$ to achieve this implicit generation. Hence, we have AC-OPF$(\mathcal{N}_{S,w}, C) \geq z(1+y)$. Theorem 6.4.4 shows that if (S, w) is solvable then we can distribute the generation z among the loads and satisfy all demand. Hence, we have AC-OPF$(\mathcal{N}_{S,w}, C) = z$ if and only if (S, w) is solvable. On the other hand, if (S, w) is not solvable then Theorem 6.4.4 shows there is no solution unless we use the generator at e. Given that all numbers in S and the value w are natural numbers, e has to produce active power of at least $p'_2 - p'_1$. That shows that we have cost of at least $z(1+y) + \epsilon z(1+y)$. Since $\epsilon z(1+y) < z(1+\epsilon)(1+y)$ an SSP instance is solvable if and only if the ϵ-approximation algorithm returns a solution with cost less or equal to $\epsilon z(1+y)$. □

A consequence of the previous result is that deciding whether or not the AC-OPF is less than a given value is NP-hard.

Theorem 6.5.2. *The* AC-OPF *problem for tree networks with two generators is* NP-*hard.*

Proof. Let $(\mathcal{N}_{S,w}, C)$, z and y be like in Theorem 6.5.1 where the cost of e are set to 2. We will now show that

$$\text{AC-OPF}(\mathcal{N}_{S,w}, C) \leq z(1+y) \iff (S, w) \text{ is solvable.}$$

Theorem 6.5.1 shows that if $\mathcal{N}_{S,w}$ is solvable then there exists a solution with cost $z(1+y)$ and hence AC-OPF$(\mathcal{N}_{S,w}, C) \leq z(1+y)$.

On the other hand, let AC-OPF$(\mathcal{N}_{S,w}, C) \geq z(1+y)$. Theorem 6.5.1 shows that AC-OPF$(\mathcal{N}_{S,w}, C) \geq z(1+y)$ which implies that AC-OPF$(\mathcal{N}_{S,w}, C) = z(1+y)$. According to Theorem 6.5.1, this is only possible if the generator at e is not used. The theorem the also shows that in this case $\mathcal{N}_{S,w}$ is solvable. □

The second main reduction of this section is similar to the proof from Theorem 6.3.3 in Section 6.3. In contrast to the first reduction, one it needs an arbitrary amount of generators but only one load and works for almost arbitrary line parameters and maximum phase angle difference.

Theorem 6.5.3. *There is no ϵ-approximation algorithm for the* AC-OPF *on tree* AC-OPF *networks with one load unless* $\mathsf{P} = \mathsf{NP}$.

Proof. Assume there exists and ϵ-approximation algorithm ($\epsilon > 1$). We will now show that we can use this algorithm to decide the SSP. Let $g, b, \overline{\Delta}, p', q', (S, w)$ and $\mathcal{N}_{S,w}$ as in the proof of Theorem 6.3.3. W.l.o.g. we assume that $w \notin S$ Furthermore, we define

$$\begin{aligned} p^* &:= g(1 - \cos(\overline{\Delta})) - b\sin(\overline{\Delta}), \\ \widetilde{\mathcal{N}}_{S,w} &:= \mathcal{N}_{S,w}[L^p_{\mathsf{e}}{=}wp' | L^q_{\mathsf{e}}{=}wq'] +^{\mathsf{e}} \mathcal{R}^{w,\overline{\Delta},b,g}_{\mathsf{e}}. \end{aligned}$$

For a ratio-choice network $\mathcal{R}^{x,\overline{\Delta},b,g}_{\mathsf{e}}$ let r_x be its generator. We define a cost function $C : \{\mathsf{r}_x \mid x \in S \cup \{w\}\} \to \widetilde{\mathbb{Q}}_{\geq 0}$ via $\forall x \in S : C_{\mathsf{r}_x} := 1$ and $C_{\mathsf{r}_w} := 2\epsilon$. Based on the result from Theorem 6.3.3, we can derive that the tuple $(\widetilde{\mathcal{N}}_{S,w}, C)$ is an AC-OPF network and polynomial in the size of the input.

Theorem 6.3.3 shows that the SSP instance (S, w) is solvable if and only if there is a solution that does not use the generator r_w. In this case, we have costs of wp^*. If the instance is not solvable then the ratio-choice network $\mathcal{R}^{w,\overline{\Delta},b,g}_{\mathsf{e}}$ provides all the power to e. Hence, we have costs of $2\epsilon wp^*$. Therefore, we derive that the ϵ-approximation algorithm returns a solution less or equal to ϵwp^* if and only if the SSP instance is solvable. This allows us to decide the SSP in time polynomial in (S, w). □

As with the first proof, the result above also allows us to derive an NP-hardness result.

Theorem 6.5.4. *The* AC-OPF *problem for tree networks with one load is* NP*-hard.*

Proof. This result is a direct consequence of Theorem 6.5.3. □

6.6 FIXED-VOLTAGE POWER FLOW

In this section, we prove that the AC-PF problem for AC-PF networks where all voltage magnitudes are fixed is NP-hard. We call these networks VPF networks (Section 6.1, Definition 6.1.12). In the main result about the AC-PF problem (Section 6.4) we used voltage choice networks. These choice networks cannot be used here as they need a variable voltage magnitude. Hence, in this section we present a new type of choice network called *phase angle choice networks*. They get their name from the fact that we will have the choice between two phase angle differences which ultimately lead to two different active power flows at the connector end. As for voltage choice networks, we analyze one line $\mathsf{l}\leftrightarrow\mathsf{e}$ with susceptance b and conductance g. The maximum phase angle difference is set to $\pi/2$. Furthermore, the voltage magnitudes at l and e are fixed and the active power demand at l is given. This setting is presented in Fig. 6.11 where the active power flow matches the demand of l. The only degree of freedom for the power flow in this system is the phase angle difference, Δ_{le}, between the buses. The selection of a phase angle difference then defines the active and the reactive power sent and received through the line.

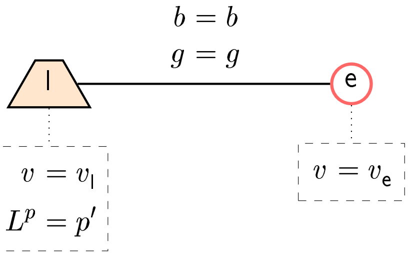

Figure 6.11: The setup from Lemma 6.6.1.

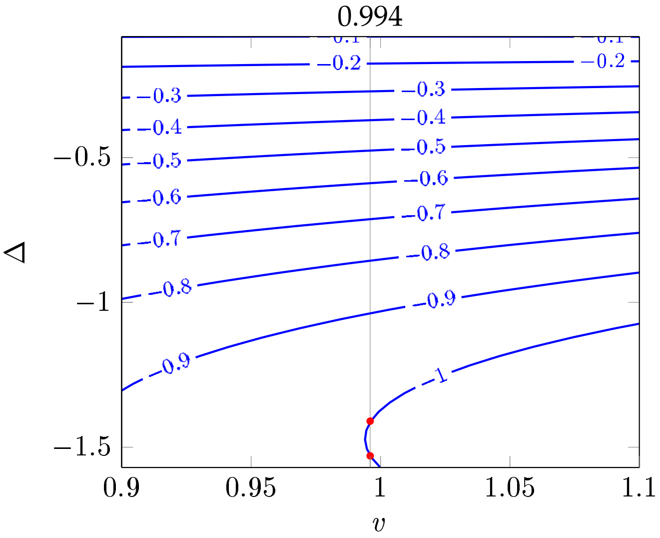

Figure 6.12: The solution space for various active power (dashed) values over the phase angle difference (y-axis) and the voltage magnitudes (x-axis).

Fig. 6.12 presents the solution space of the active power flow p_{le} with respect to the phase angle difference (y-axis) and the (fix) voltage magnitude v_{l} (x-axis) where $v_{\mathsf{e}} = 1.1$. The voltage magnitude bounds are $\underline{v} = 0.9$ and $\overline{v} = 1.1$.

We can observe that for a value of $p_{\mathsf{le}} = \text{-}1$ and a voltage magnitudes of $v_{\mathsf{l}} \approx 1$ there are two possible phase angle differences. This is not for all values the case. For example, for $p_{\mathsf{le}} = \text{-}0.5$, any voltage magnitude corresponds to exactly one phase angle difference within our voltage magnitude bounds. Lemma 6.6.1 presents a sufficient condition on b, g, p', v_{e} and v_{l} under which there are two different phase angle difference solutions possible. The lemma also characterizes these two solutions.

Lemma 6.6.1. *Let* $b \in \widetilde{\mathbb{Q}}_{\leq 0}, g, \tilde{v}, \tilde{v}', p' \in \widetilde{\mathbb{Q}}_{\geq 0}$ *be numbers with*

$$\tilde{v}\tilde{v}'(b^2 + g^2) \geq (p' + g\tilde{v}^2)^2. \tag{6.23}$$

For every AC *solution* $(\theta, G^p, L^p, v, G^q, L^q)$ *of*

$$\mathcal{N} := \left(\{\mathsf{e},\mathsf{l}\}, \{\mathsf{l}\}, \emptyset, \left\{\mathsf{e} \xleftrightarrow[g=g]{b=b} \mathsf{l}\right\}, {}^{\pi}\!/_{2}, 0, \infty, \left[v_{\mathsf{e}}{=}\tilde{v}' \middle| v_{\mathsf{l}}{=}\tilde{v} \middle| L^p_{\mathsf{l}}{=}p'\right]\right)$$

we have

$$\Delta_{\mathsf{le}} = 2\,\mathrm{atan}\left(\frac{\tilde{v}\tilde{v}'b \pm \sqrt{\tilde{v}'^2\tilde{v}^2(b^2+g^2) - (p' - g\tilde{v}^2)^2}}{g(\tilde{v}'\tilde{v} - 1) - p'}\right). \tag{6.24}$$

Proof. Let $x := \text{-}\frac{p' + gv_{\mathsf{l}}^2}{v_{\mathsf{e}}v_{\mathsf{l}}}$; $y := \tan({}^{\Delta_{\mathsf{le}}}\!/_{2})$. Kirchhoff's junction law l implies that $p_{\mathsf{le}} = \text{-}p'$. Using the double angle formulas Weisstein [2000] $\sin(\Delta_{\mathsf{le}}) = \frac{2y}{1+y^2}$ and $\cos(\Delta_{\mathsf{le}}) = \frac{1-y^2}{1+y^2}$ we can derive

$$\text{-}p' = p_{\mathsf{le}} = gv_{\mathsf{l}}^2 - v_{\mathsf{e}}v_{\mathsf{l}}(g\cos(\Delta_{\mathsf{le}}) - b\sin(\Delta_{\mathsf{le}}) \tag{6.25}$$

$$x = g\cos(\Delta_{\mathsf{le}}) - b\sin(\Delta_{\mathsf{le}}) \tag{6.26}$$

$$x = g\frac{1-y^2}{1+y^2} + b\frac{2y}{1+y^2} \tag{6.27}$$

$$x(1+y^2) = g(1-y^2) + 2yb \tag{6.28}$$

$$0 = y^2(x+g) - 2yb + x - g. \tag{6.29}$$

Equation (6.23) implies $b^2 + g^2 \geq x^2$, which allows us to solve this quadratic equation. Its solution is

$$y = \frac{b \pm \sqrt{b^2 - (x+g)(x-g)}}{x+g} \tag{6.30}$$

$$\Delta_{\mathsf{le}} = 2\,\mathrm{atan}\left(\frac{b \pm \sqrt{b^2 + g^2 - x^2}}{x+g}\right) \tag{6.31}$$

$$= 2\,\mathrm{atan}\left(\frac{v_{\mathsf{l}}v_{\mathsf{e}}b \pm \sqrt{v_{\mathsf{e}}^2v_{\mathsf{l}}^2(b^2+g^2) - (p' + gv_{\mathsf{l}}^2)^2}}{g(v_{\mathsf{e}}v_{\mathsf{l}} - 1) - p'}\right). \tag{6.32}$$

□

In the proof above, only the active power demand of l was fixed. If the reactive power demand value would also be fixed, then there would not be a choice for the phase angle difference. This is because the fixed reactive power flow adds another constraint to the system. In VPF, networks the active and reactive demand values of a load have to be fix. Hence, we have to modify the setup from Lemma 6.6.1. To deal with the reactive power problem we add an additional generator r in between l and e. The line parameters of the line l↔r are set such that there is only one possible phase angle difference possible. This difference implies a power flow, p_{lr}, which equals the demand of l. Hence, the active power demand at l becomes an implicit demand at r. The active power generation at r is set to a value smaller than this implicit demand. Therefore, we can regard the bus r as an (implicit) load with fixed active power demand. The reactive power of generators in VPF networks is a free variable. Hence, the reactive power flow value q_{rl} towards r can be compensated by this reactive power generation. Furthermore, as reactive power generation can be positive or negative like reactive demand we can regard r as a load with an implicit and free reactive demand. Overall, we obtain the setup as in Lemma 6.6.1 where the bus l from this lemma is now called r and hence the line l↔e becomes r↔e. An example phase angle choice network is presented in Fig. 6.13.

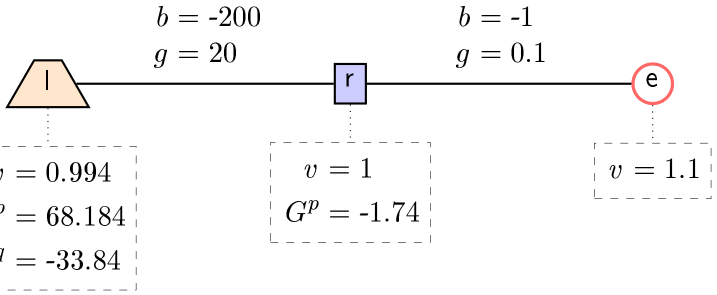

Figure 6.13: The network $\mathcal{T}_{\mathsf{e}}^2$ using the values from Lemma 6.6.3.

In general, a phase angle choice networks has one free parameter, s which acts as a scalar. The function of this parameter is similar to the parameter of the voltage choice network. It will take the value of an element of the SSP instance this phase angle choice network is supposed to represent. On top of that, the phase angle choice network also has several other free parameters. These parameters are b, b', g, g', p, $\underline{v}$, $\tilde{v}$ and $\overline{v}$. Their actual values do not matter for the proof as long as they satisfy the condition presented in Definition 6.6.2. Presenting the proof with these parameters enables us to present the core properties necessary to make the reduction work. Hence, we do not rely on properties of the numbers chosen, other than the Condition 6.33 from Definition 6.6.2.

Definition 6.6.2 (phase-angle choice network)**.** Let $s, p, g, g', \underline{v}, \tilde{v}, \overline{v} \in \widetilde{\mathbb{Q}}_{>0}$ and $b, b', x \in$

$\widetilde{\mathbb{Q}}_{<0}$; be numbers with $\underline{v} < \tilde{v} < \overline{v}$, $x = p - g'(\tilde{v}^2 - \tilde{v}\underline{v})$, $|4\,\mathrm{atan}(b'/g')| > \pi$ and

$$\left| 2\,\mathrm{atan}\left(\frac{b\overline{v}\tilde{v} \pm \sqrt{\overline{v}^2\tilde{v}^2(b^2+g^2) + (p+g\tilde{v}^2)^2}}{g(\overline{v}\tilde{v}-1)-p} \right) \right| < \pi/2. \tag{6.33}$$

We define

$$\begin{aligned} \mathcal{T}^s_{\mathsf{e}} &:= \left(\{\mathsf{e},\mathsf{l},\mathsf{r}\}, \{\mathsf{r}\}, \{\mathsf{l}\}, \left\{ \mathsf{l} \xleftrightarrow[g=g's]{b=b's} \mathsf{r} \xleftrightarrow[g=gs]{b=bs} \mathsf{e} \right\}, {}^{\pi}\!/_{2}, \underline{v}, \overline{v}, \mathcal{T} \right) \\ \mathcal{T} &:= \left[v_{\mathsf{r}}{=}1 \middle| v_{\mathsf{l}}{=}\underline{v} \middle| G^p_{\mathsf{r}}{=}sx \middle| L^p_{\mathsf{l}}{=}sg'(\underline{v}\tilde{v}+\underline{v}^2) \middle| L^q_{\mathsf{l}}{=}sb'(\underline{v}\tilde{v}-\underline{v}^2) \right]. \end{aligned}$$

Note that we only include the parameter s and the connector e into our symbolic representation of phase angle choice networks $\mathcal{T}^s_{\mathsf{e}}$. This is to avoid unnecessary clutter in our syntax. The other parameters are only chosen once and they are the same for all instances of phase angle choice networks within one reduction. For the remainder of this section, we assume that some arbitrary and fixed b, b', g, g', p, $\underline{v}$, $\tilde{v}$ and $\overline{v}$ are given which satisfy Condition 6.33. The next lemma shows that at least one set of values for these variables exists. The phase angle choice network with these values is presented in Fig. 6.13 with $s = 2$.

Lemma 6.6.3. *There exists* $b, b', x \in \widetilde{\mathbb{Q}}_{<0}$; $s, p, g, g', \underline{v}, \tilde{v}, \overline{v} \in \widetilde{\mathbb{Q}}_{>0}$ *such that Condition 6.33 in Definition 6.6.2 is satisfied.*

Proof. We define $\underline{v} := 0.9$; $\tilde{v} := 0.994$; $\overline{v} := 1.1$; $p := 1$; $g' := 20$; $b' := -200$ $b := -1$; and $g := 0.1$. Then we have $|4\,\mathrm{atan}(b'/g')| > 5.88 > \pi$; $p - g'(\tilde{v}^2 - \underline{v}\tilde{v}) < -0.868 < 0$; $1.48 < |\Delta_1| < 1.49 < 1.57 < \pi/2$; and $1.46 < |\Delta_2| < 1.47 < 1.57 < \pi/2$. □

The two phase angle differences implied by Lemma 6.6.1 imply two power flow values p_{er} at e. We denote these values with p'_1 and p'_2. The following lemma shows that in a phase angle choice network the value p_{er} has the choice between p'_1 and p'_2 and that s acts as a scalar for these solutions.

Lemma 6.6.4. *There exists* $p'_1, p'_2 \in \widetilde{\mathbb{Q}}_{>0}$ *such that for all* AC *solutions of* $\mathcal{T}^s_{\mathsf{e}}[v_{\mathsf{e}}{=}\overline{v} | \mathsf{e}{\in}N_{G/L}]$ *we have* $p_{\mathsf{re}} \in \{p'_1 s, p'_2 s\}$. *Furthermore the network* $\mathcal{T}^s_{\mathsf{e}}$ *has three buses and the size of every numerical parameter is polynomial in* s.

Proof. First we look at the line $\mathsf{l}{\leftrightarrow}\mathsf{r}$. Lemma 6.6.1 implies that possible phase angles are $\Delta_1 = 0$ and $\Delta_2 = 2\,\mathrm{atan}({}^{b'}\!/_{g'})$. Since $|4\,\mathrm{atan}({}^{b'}\!/_{g'})| > \pi$, Δ_1 has to be the solution. With a phase angle difference of 0 we have $q_{\mathsf{lr}} + L^q_{\mathsf{l}} = 0$, and hence Kirchhoff's junction law for reactive power at l is satisfied. For the active power at r we get $G^p_{\mathsf{r}} + p_{\mathsf{rl}} = sp$, which implies an implicit demand of sp.

Now we look at the line $\mathsf{r}{\leftrightarrow}\mathsf{e}$. Equation (6.24) from Lemma 6.6.1 together with the Condition (6.33) and the implicit demand of sp shows that there are two different feasible phase angle difference solutions which implies that we have two different possible values for p_{er}. Since both solutions are independent from s but the susceptance and conductance of $\mathsf{r}{\leftrightarrow}\mathsf{e}$ are scaled by s these two active power flow solutions can be scaled arbitrarily. For the reactive power at r we have to satisfy $G^q_{\mathsf{r}} + q_{\mathsf{re}} + q_{\mathsf{rl}} = 0$ which is always possible because G^q_{r} is a free variable.

Part two can be directly derived from Definition 6.6.2. □

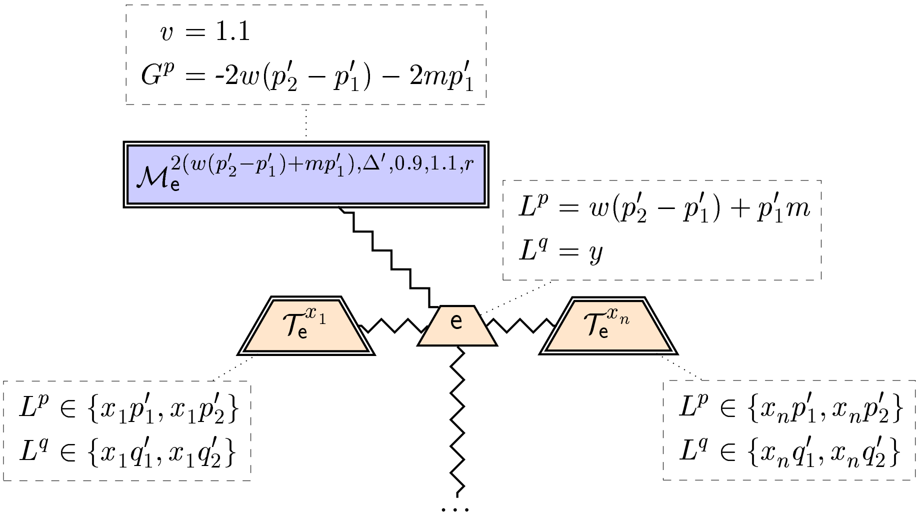

Figure 6.14: The network $\mathcal{N}_{S,w}$ from Theorem 6.6.5 for $(\{x_1, \dots, x_n\}, w)$ where $m := \sum_{x \in S} x$.

With the usage of phase angle choice networks, we show that the VPF problem is NP-hard using a reduction from the SSP problem. Let (S, w) be an SSP instance. The reduction of a given SSP instance is illustrated on Fig. 6.14. The power flow problem instance contains one copy of $\mathcal{T}_{\mathsf{e}}^{x}$ for each element $x \in S$. The bus e has an implicit active power generation of $w(p'_2 - p'_1) + mp'_1$. This generation is build from the active power demand at e and the implicit generation of the Magical-Tree network (see Section 6.2) connected to e. The usage of the Magical-Tree network is necessary to deal with the fact that a VPF network has a slack bus. A Magical-Tree network is not a VPF network by definition. However, the results of Lemma 6.2.3 and Lemma 6.2.5 show that in any solution all voltage magnitudes are fixed values. Hence, we can safely interpret a Magical-Tree network as a VPF network.

It is easy to see that the power flow instance is satisfiable iff the SSP instance is, since each load must choose to draw either xp'_1 or xp'_2 from the generator and the sum of all loads that choose the latter must add up to w. In the following theorem, we present a formal proof of the statements presented above.

Theorem 6.6.5. *The* VPF *problem on tree* VPF *networks is* NP*-hard.*

Proof. Let $r \in \mathbb{R}_{>0}$, $0 < \Delta' \leq \pi/2$ and (S, w) be a SSP instance with $m := \sum_{x \in S} x$. Furthermore, let p'_2, p'_1, q'_1 and q'_2 be the values from Lemma 6.6.4 and w.l.o.g. we assume that $p'_2 > p'_1$. We set

$$\begin{aligned} z &:= w(p'_2 - p'_1) + mp'_1, \\ y &:= w(q'_2 - q'_1) + mq'_1 + 2z\frac{r(1-\cos(\Delta')) + \sin(\Delta')}{1-\cos(\Delta') - r\sin(\Delta')} - rz \end{aligned}$$

and define the AC networks

$$\begin{aligned} \mathcal{N} &:= \Big(\{\mathsf{e}\}, \emptyset, \{\mathsf{e}\}, \emptyset, \Delta', \underline{v}, \overline{v}, \Big[L^p_{\mathsf{e}}{=}z \Big| L^q_{\mathsf{e}}{=}y \Big| v_{\mathsf{e}}{=}\overline{v}\Big]\Big), \text{ and} \\ \mathcal{N}_{S,w} &:= \mathcal{N} +^{\mathsf{e}} \mathcal{M}^{2z,\Delta',0.9,1.1,r}_{\mathsf{e}} +^{\mathsf{e}} \sum_{x\in S}^{\mathsf{e}} \mathcal{T}^x_{\mathsf{e}}. \end{aligned}$$

We will now show that $\mathcal{N}_{S,w}$ is an AC-PF network and

$$\mathcal{S}^{\text{AC}}(\mathcal{N}_{S,w}) \neq \emptyset \iff (S, w) \text{ is solvable.}$$

The network $\mathcal{N}_{S,w}$ has $|S|+6$ buses. Part two of Lemma 6.2.5 and Lemma 6.6.4 imply that the networks $\mathcal{M}^{2z,\Delta',0.9,1.1,r}_{\mathsf{e}}$ and $\mathcal{T}^x_{\mathsf{e}}$ are polynomial in the size of (S, w). All numbers involved in the definition of the Magical-Tree network and the network $\mathcal{N}$ are rational numbers except the values p'_1, p'_2, q'_1, q'_2, $\cos(\Delta')$ and $\sin(\Delta')$. The network has one generator which is not fixed and hence can act as slack bus. Furthermore, all demand of all loads is fixed and all voltage magnitudes are fixed[4]. Hence, the network $\mathcal{N}_{S,w}$ is a VPF network and has size polynomial in the size of (S, w).

Lemma 6.2.5 shows that $v_{\mathsf{e}} = 1.1$. Hence, we can apply Lemma 6.6.4 to the voltage choice networks. It implies that a network $\mathcal{T}^x_{\mathsf{e}}$ has exactly two different active power demands xp'_2 or xp'_1 at e. We call a network $\mathcal{T}^x_{\mathsf{e}}$ *active* if the first solution is true. If no network were active then we would have an implicit active power demand of mp'_1 at e. For every active network $\mathcal{T}^x_{\mathsf{e}}$ we "gain" the active power demand $x(p'_2 - p'_1)$. Hence, the active power can be characterized as

$$\sum_{\substack{x\in S \\ \mathcal{T}^x_{\mathsf{e}} \text{ is active}}} x(p'_2 - p'_1) + mp'_1.$$

Lemma 6.2.5 shows that the Magical-Tree network has an implicit active power generation of -$2z$ and reactive power generation of -$(2z\frac{r(1-\cos(\Delta'))+\sin(\Delta')}{1-\cos(\Delta')-r\sin(\Delta')} - rz)$. Adding the demand of e to these values leaves us with an implicit active power generation of z at e and a reactive power of $w(q'_2 - q'_1) + mq'_1$. Hence, Kirchhoff's junction law for active power at e is satisfied if and only if (S, w) is solvable. With similar arguments and the us of Lemma 6.6.4, we can show that Kirchhoff's junction law for reactive power is satisfied if and only if (S, w) is solvable. □

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[4] Lemma 6.2.5 shows that every bus of $\mathcal{M}^{2z,\Delta',0.9,1.1,r}_{\mathsf{e}}$ has no choice in its voltage magnitude, so we can regard the voltage magnitudes of these buses as fixed without influencing the result.

6.7 Related Work

Solving any AC model based problem has proven itself to be a challenge and numerous different techniques have been applied. The first attempt on the AC-PF[5] was done by Ward and Hale [1956] using the Gauss-Seidel (GS) method. The GS method is easy to implement, has fast iterations and needs only a small amount of memory. However, it suffers from a slow convergence rate and hence a large number of iterations is needed. Furthermore, it tends to diverge and has problems finding solutions for large systems. Tinney and Hart [1967] applied the Newtons-Raphson (NR) method. It has the advantage of a fast (quadratic) convergence when being close to the solution. Convergence for both GS and NR is only guaranteed if the start point is within the convergence region.

For every day operation, NR is still not fast enough. The majority of time per iteration in the NR method is spent on calculating the Jacobian of the system. One common approach to speed up the iterations for the price of accuracy is to use the fast decoupled load flow (FDLF) (Stott and Alsac [1974]). Here, the active and reactive power are decoupled by using an approximation. The approximation is based on the assumptions that: the voltage magnitudes are close to 1; the conductance is much smaller than the absolute value of the susceptance; maximum phase angles are small; and, injected reactive power at a bus is much smaller then the shunt element. These assumptions are assumed to be true for “well behaved” power networks.

The majority of focus in the academic literature is directed towards the AC-OPF. A way to find a solution for the AC-OPF is to model it as a Non-linear Program (NLP). Mathematical optimization techniques to solve the AC-OPF, e.g. spatial branching, are not able to deal with the size of real world power networks and hence fail to converge in a reasonable amount of time. Other techniques based on the Karush-Kuhn-Tucker (KKT) conditions, e.g. interior point methods, are much faster but converge to local optima only (Bertsekas [1999]). Other attempts to solve the AC-OPF have been numerous and various since the introduction of the problem by Carpentier [1962]. Solving methods include sequential linear programming (Kirschen and Van Meeteren [1988]), Newton based methods (Dommel and Tinney [1968]), specialized interior point methods (Jabr et al. [2002]), quadratic programming, evolutionary programming, neural networks, particle swarm optimization, and fuzzy set theory. All of these methods are usually evaluated on IEEE test instances. None of these methods provide a provable guarantee for the quality of the solution. They do, however, present a solution for the AC-OPF problem and hence an upper bound on the optimal costs.

Surveys and overviews are presented by Alsac et al. [1990]; Huneault et al. [1991]; Momoh et al. [1999a,b]; Baldick [2006]; Pandya and Joshi [2008]; AlRashidi and El-Hawary [2009]; Frank et al. [2012a,b].

Algorithms for finding globally optimal solutions are presented by Phan [2012] using branch and bound; and Gopalakrishnan et al. [2012] using spatial branch and

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[5] Also called *load flow* in some literature.

bound. These methods have thus far only proven useful on small instances.

Methods like interior point are able to find a global optimum for convex NLPs (Bertsekas [1999]). Hence, a convex relaxation of the AC power flow equations allows to find a lower bound for the AC-OPF. The difference between this lower bound and the objective value of a solution provides a measure for the quality of the solution. In recent years, multiple convex relaxations have been developed and consecutively improved. These include Second-Order Cone Programming (SOCP) (Jabr [2006, 2008]; Kocuk et al. [2015b]), Semi-Definite Programming (SDP) (Bai et al. [2008]; Low [2014a,b]; Molzahn and Hiskens [2014]; Coffrin et al. [2015c]; Jabr [2013]; Molzahn et al. [2013]; Madani et al. [2015a]; Molzahn and Hiskens [2014], Convex-DistFlow (Farivar et al. [2011]; Coffrin et al. [2015a]), Quadratic Convex relaxation (QC) (Coffrin et al. [2015d,c]), the Network Flow and the Copper Plate relaxation (Coffrin et al. [2015b]), and a linear programming based outer approximation by Bienstock and Munoz [2014]. Furthermore, Madani et al. [2015b] showed that by adding a penalty term based on reactive power to the objective one can obtain high quality solutions using the SDP relaxation.

The QC relaxation with bound tightening was evaluated on 57 test cases (NESTA test case library, Coffrin et al. [2014a]) by Coffrin et al. [2015c]. The results where compared to the objective value of the solution to the AC-OPF problem found by the state of the art interior point NLP solver IPOPT (Wächter and Biegler [2005]). In 34 cases there is no gap between the lower bound found by the relaxation and the upper bound found by IPOPT and in 51 cases the gap is below 1%. This indicates that although, in general, any solution found by IPOPT is only a local solution, most of the solutions found for the test cases are globally optimal.

A convex relaxation for the AC-OPF is called *exact* if its optimal solution is a solution of the AC-OPF. As convex quadratic and conic programs can be solved in time polynomial in the input, any exact relaxation provides an efficient way to solve the AC-OPF. Research shows that exactness of some relaxations can be guaranteed in special cases. Assuming that load over-satisfaction[6] is allowed, Farivar et al. [2011]; Farivar and Low [2013] and Sojoudi and Lavaei [2012] show that the SDP and SOCP relaxation are exact for radial networks. A two bus example is presented by Kocuk et al. [2015a] showing that in the case without load over-satisfaction the SDP relaxation fails to be exact in general. Li et al. [2012] show that the SOCP relaxation is exact for radial networks without voltage magnitude upper bounds and no reference bus. When not considering reactive power, Sojoudi and Lavaei [2013] show that the SDP relaxation is exact if all possible voltage magnitudes have a corresponding solution, and the maximum phase angle difference is bounded by the ratio of susceptance and conductance.

A way to find an approximate-feasible solution which violate the constraints by at most ϵ is presented by Bienstock and Munoz [2015]. The paper shows a way to

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[6] In a model with load over-satisfaction, Kirchhoff's junction law is replaced with similar constraints replacing the equality with and inequality: $\sum_{(\mathsf{a},\mathsf{d},b,g,c)\in E^d} p_{\mathsf{ad}} + \overline{G^p_{\mathsf{a}}} + \overline{L^p_{\mathsf{a}}} \leq 0$ and $\sum_{(\mathsf{a},\mathsf{d},b,g,c)\in E^d} q_{\mathsf{ad}} + \overline{G^q_{\mathsf{a}}} + \overline{L^q_{\mathsf{a}}} \leq 0$.

construct a Linear Program (LP) for a given AC-OPF instance. The optimal solution of the LP allows to derive a solution which violates the constraints by a factor linearly depending on ϵ ($0 < \epsilon < 1/2$). The paper also shows that for every solution for the AC there is a solution of the LP which is potentially underestimating the cost and cannot overestimate the cost more than $\epsilon n/d$. This LP is obtained by first discretizing all variables using a binary representation. The result is then transformed into an equivalent Integer Program (IP). It is then shown that the LP relaxation of this IP is integral[7]. The LP has a size of $\mathcal{O}(2^{\mathcal{O}(\omega)} n d \epsilon^{-1} \log \epsilon^{-1})$ where ω is the tree width of the network, n is the number of nodes, d is the maximum degree and ϵ is the precision of the discretizition. In Section 6.5, we showed that the AC-OPF is NP-hard for tree networks. Tree networks have a tree width (ω) of 1. Therefore, the factor influencing the size of the LP the most is ϵ^{-1}. The smaller ϵ, the higher the precision of the discretization and hence the better the approximation will be, but also the bigger the LP will become.

The only results related to the computational complexity of AC model based problems is from Verma [2009][8]. The paper presents a result about the SIN model which, as we outline in Chapter 5, can be regarded as a special case of the AC model. Therefore, Verma [2009] provides the first proof that the AC-OPF over mesh networks is (strongly) NP-hard. In this chapter, we improved on this result by showing that radial networks are NP-hard. Furthermore, in Chapter 5, we improve on the results from Verma [2009] by showing that the SIN-OPF and SIN-MPF are strongly NP-hard for planar mesh networks with a bounded maximum degree.

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[7] A Linear Program is called integral if its optimal solution is a 0/1 solution.

[8] The paper Lavaei and Low [2012] claims to show NP-hardness for the AC-OPF. To that end, it presents a reduction of a special case of the AC-OPF to quadratic programming in its appendix. This reduction does not imply that the AC-OPF problem is NP-hard. It only shows that the AC-OPF problem is not harder than quadratic programming.

Chapter 7

Conclusion

In this thesis, we study the computational complexity of two optimization problems in electrical power systems: the OPTIMAL POWER FLOW (OPF) and the MAXIMUM POWER FLOW (MPF). Furthermore, we study the complexity of the POWER FLOW (PF) problem: how hard it is to find at least one solution in a given power system. We analyze the complexity for three different types of power flow models: the Alternating Current (AC), the Lossless-Sin AC Approximation (SIN) and the Linear AC Approximation (DC) model (with switching). In the following, we briefly summarize the problems and the models. Afterwards, in Section 7.1, we present an overview of our complexity results and discuss possible extensions. In Section 7.2, we present a final discussion on the area of power systems. For all of these discussions we assume that $\mathsf{P} \neq \mathsf{NP}$.

A power network is described by its set of buses and the lines connecting the buses. The parameter of the power lines are the fixed parameters of the networks. These parameters are the thermal line limits and the line admittances. Power networks also have a global operational limit on the maximum absolute phase angle difference. This limit indirectly implies a flow limit for every power line.

The Models Power network based problems have two types of variables. The first are the voltages. Voltages are complex numbers. Hence, we identify them via their magnitudes and their phase angles. The voltage values of the ends of a power line imply the power flow along the line. In the AC model, the power flow follows the laws of physics, namely the Alternating Current equations.

The DC model approximates the AC model. First, it ignores the imaginary part of the power flows (reactive power). Second, as voltage magnitude bounds in real world networks are tight, the model assumes that all voltage magnitudes are fixed and the same. Third, we also assume that the lines are lossless (conductance is 0). Finally, the sine function within the AC power flow is omitted. Overall, this makes the DC model linear.

To obtain the SIN model, we have to do the first three steps of the DC model. Hence, the only difference to the DC model is the sine function wrapped around the phase angle difference of the lines buses.

The Problems The second set of variables of power networks are the generation of the generators and the demand of the loads. In all three problems (PF, OPF and MPF), the voltages are free variables. The problems differ on whether or not the generation or load is fixed.

In the PF problem, generation and load are given. The problem is to decide if feasible voltages exists. In the OPF, we assume that the load is given and we have a cost value for every generator. The problem is to find a generation dispatch such that the overall generation costs are minimized and there exist feasible voltages. In the MPF problem load and generation are both free variables. Here, the goal is to find generation, load and voltages such that the total load is maximized.

7.1 Results and Possible Extensions

This section presents a brief overview of our results. We start with general observations which apply to all models.

In this thesis, we prove that there does not exist an ϵ-approximation algorithm for the OPF in any of the power flow models. However, all three reductions need the feature that the ratio between the cheapest and the most expensive generator is arbitrary. This raises the question if there exists an approximation algorithm for the OPF with a bounded generation ratio. Such an algorithm would be sufficient to solve real world instances if the run-time dependency on the ratio is not too bad[1].

In all our results it is crucial that some lines operate at their line capacity (thermal line limit or maximum phase angle difference). Without this feature, none of our results would work. However, all results for the DC and SIN model and some of the results for the AC model are flexible such that the actual value of the upper limit is arbitrary. Furthermore, the result for the AC-OPF would work without a line capacity if we could fix the generation and voltage magnitude of at least one generator.

7.1.1 Alternating Current (AC)

Our results show that in the case of the AC model, the OPF, MPF and PF problem are NP-hard. We also show that the PF problem is still hard even when all voltage magnitudes are fixed.

The non-linearity of the AC equations is enough to show that the problems are hard. The dependency of variables introduced by cycles in the network is not necessary. Henceforth, the network structure of all reductions are tree networks. A tree network is as simple as a network structure can be. It therefore will not be possible to find reductions with a simpler network structure.

The reductions work for a wide variety of line parameter settings (global maximum phase angle difference, admittance). However, not all combinations of line parameters are possible. This raises the question if a case which is not covered is "easy"

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[1] A bad dependency for example would be double exponentially with a huge constant.

to solve or if there is another reduction that includes this case. There are other properties which can be used to build choice networks[2]. We therefore believe that stronger results are possible.

In all results we do not need a thermal line limit (line capacity). However, the existence of either voltage magnitude bounds or a maximum phase angle difference is crucial.

In all our reductions the maximum bus degree is unbounded. We believe that it might be possible to do similar reductions with a bounded degree using ideas presented in Chapter 4 about the DC model. However, the mathematics could become very complex.

7.1.2 Lossless-Sin AC Approximation (SIN)

Verma [2009] showed that the OPF and the MPF problem are strongly NP-hard for arbitrary networks and that the PF problem is "easy" to solve. In this thesis, we show that the OPF and MPF problem are strongly NP-hard for planar networks with a bounded maximum degree of 6. The networks in our reductions contain multiple Wheatstone networks. The Wheatstone networks contain a Wheatstone bridge which is crucial for the underlying idea of the reduction. A network without any (implicit) Wheatstone bridge is a series-parallel network. This raises the question if it is possible to show that the OPF and the MPF problem are "easy" for series-parallel networks.

7.1.3 Linear AC Approximation with line switching (DC)

The DC model was originally designed for "easy" solvability. We therefore study the complexity of our problems when reconfiguration via line switching is allowed. We show that, contrary to the general opinion, switching problems are hard to solve even on the easy to solve DC model. Our findings are that the OPF and MPF problems are NP-hard on cacti networks. Cacti networks are a natural extension of tree networks as they allow every edge/line to be in at most one cycle. We demonstrate that tree networks are "easy" to solve. We therefore believe that no stronger reduction exists.

We also show that the PF problem is hard for series-parallel networks. If a further simplification with for example cacti networks exists remains unclear.

The thesis also shows that there does not exist an ϵ-approximation algorithm for the MPF with one fixed load. This raises the question of whether it is possible to approximate the MPF when all loads are free or if a stronger reduction can be found.

7.2 Open Power Network Problems

With this thesis, we have closed several gaps in the literature on the topic of computational complexity in the area of electrical power systems. What remains is to find

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[2] One such property is proved in Section 6.2 but not used within this thesis.

algorithms and/or heuristics to solve either of the problems studied in this thesis or closely related ones. To that end, our results can be used as aids for researchers working in those areas.

One of our aims was to use networks in our reductions which are close to real-world networks. However, in the literature there is only little research in properties of real-world networks. Hence, we focused on the rather obvious properties such as network structure. It might be possible that real-world networks have a property which is yet unknown. Furthermore, there might be an algorithm which can solve any of our problems efficiently for networks with that property. We therefore believe that identifying the properties of real-world networks is a goal worth investigating.

Knowledge of the properties of real-world networks can also aid the design of heuristics. The performance of heuristics is typically evaluated on test cases. The larger and more realistic the test cases, the more we can be confident that the heuristic will perform well in real-world applications. The amount of publicly available power networks is well below 100 (Coffrin et al. [2014a]). Also, many networks are either unrealistically small or variations of each other. Hence, we believe that the amount of test data is insufficient to evaluate heuristic approaches. A good way to obtain new network test cases is to build random networks. However, these random networks are only useful in practice if they resemble real-world networks. To that end, it is imperative to characterize real-world networks with respect to their properties.

All of our work applies to the steady state analysis of power networks. For a current state of the network and an optimal solution of a problem, it is unclear if there is a transition from one state to the other where all transient states are feasible. A transient state could, for example, violate the thermal limits. In real world network analysis, this is tested via simulations. This raises the question if it is possible to incorporate the ideas from the simulation into the solving process such that we can guarantee that all transient states are feasible. Alternatively, another approach at the transient problem could be to find criteria which guarantee that all transient states are feasible. These criteria could potentially interfere with the ideas of our reductions. Hence, the new problems might be easier to solve.

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Glossary

AC	Alternating Current
MPF	Maximum Power Flow
OPF	Optimal Power Flow
PF	Power Flow
DC	Linear AC Approximation
DS	Linear AC Approximation with line switching
SIN	Lossless-Sin AC Approximation
SSP	Subset Sum Problem
VPF	Fixed-Voltage Power Flow
X3C	Exact-Cover by 3-set
$\mathcal{S}^{\text{AC}}_{\text{MPF}}(\mathcal{N})$	set of AC-MPF solutions
$\mathcal{S}^{\text{AC}}(\mathcal{N})$	set of AC solutions
a	bus
d	bus
e	bus, typically the connector
r	bus, typically a generator
l	bus, typically a load
s	bus, typically the slack
N	set of buses
x	bus
c	line capacity
$\mathcal{N}[L^p_{\mathsf{e}}{=}12]$	The variant of $\mathcal{N}$ where e is a load with a fixed active power demand of 12
g	line conductance
C	generation costs
$\mathcal{S}^{\text{DC}}(\mathcal{N})$	set of DC solutions
E^d	set of directed lines
$dom(f)$	domain of function f
DS-MPF$_{L^p}$	DS with a single fixed load
$\mathcal{S}^{\text{DS}}(\mathcal{N})$	set of DS solutions
$\mathsf{a}\leftrightarrow\mathsf{d}$	simplified notation of a line between a and d, line parameters are omitted

$\mathsf{a} \xleftrightarrow[g=3]{b=\text{-}12} \mathsf{d}$	line between a and d with susceptance -12 and conductance 3, used in the AC model
$\mathsf{a} \xleftrightarrow[c=3]{b=\text{-}12} \mathsf{d}$	line between a and d with susceptance -12 and capacity 3, used in the DC model
$\mathsf{a} \xleftrightarrow[c=0.2]{b=\text{-}12} \mathsf{d}$	line between a and d with susceptance -12 and capacity 0.2, used in the SIN model
C_{fix}	the fix costs of a choice network, only used for illustration in the graphical representation
$f\vert_c^X$	constant Extension of function f into X with values c
$f\vert_X$	restriction of Function $f : Y \to Z$ onto domain X
N_G	set of generators
$\mathcal{G}$	graph
E	set of lines
$\mathcal{L}_{\mathsf{e}}^x$	load choice network
$\widetilde{\mathcal{L}}_{\mathsf{e}}^x$	load choice network with fixed demand
N_L	set of loads
$\mathcal{M}_{\mathsf{e}}^{x,\Delta',\underline{v},\overline{v},r}$	Magical-Tree network
$\mathbb{N}$	natural numbers
$\mathcal{N}$	network
$\mathbb{Q}_{>0}$	rational numbers, non-negative
$\mathbb{R}_{>0}$	real numbers, non-negative
$\widetilde{\mathbb{Q}}_{>0}$	rational numbers with $\pi, \sin, \cos, \operatorname{atan}$ and $\sqrt{\cdot}$, non-negative
$\mathbb{R}_{<0}$	real numbers, non-positive
$\widetilde{\mathbb{Q}}_{<0}$	rational numbers with $\pi, \sin, \cos, \operatorname{atan}$ and $\sqrt{\cdot}$, non-positive
$\mathbb{Q}_{\leq 0}$	rational numbers, negative
$\mathbb{R}_{\leq 0}$	real numbers, negative
$\widetilde{\mathbb{Q}}_{\leq 0}$	rational numbers with $\pi, \sin, \cos, \operatorname{atan}$ and $\sqrt{\cdot}$, negative
θ	voltage phase angle
$\mathcal{T}_{\mathsf{e}}^s$	phase angle choice network
$\mathcal{A}_{\mathsf{r},\mathsf{l}}^x$	phase angle difference choice network
$\overline{\Delta}$	maximum phase angle difference
Δ	phase angel difference
p	flow, active power
G^p	generation, active power
L^p	demand, active power
$\overline{L}^p$	demand, upper bound
$\mathcal{P}_2(X)$	set of all 2-element subsets of X
$\mathbb{Q}_{\geq 0}$	rational numbers, positive
$\mathbb{R}_{\geq 0}$	real numbers, positive
$\widetilde{\mathbb{Q}}_{\geq 0}$	rational numbers with $\pi, \sin, \cos, \operatorname{atan}$ and $\sqrt{\cdot}$, positive

q	flow, reactive power
G^q	generation, reactive power
L^q	demand, reactive power
R	active to reactive power ratio
$\mathcal{R}_{\mathsf{e}}^{s,\overline{\Delta},b,g}$	ratio choice newtork
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\widetilde{\mathbb{Q}}$	rational numbers with $\pi, \sin, \cos, \mathrm{atan}$ and $\sqrt{\cdot}$
$\mathcal{S}^{\mathrm{SIN}}(\mathcal{N})$	set of SIN solutions
(S, w)	SSP instance
$\mathcal{N}_{S,w}$	network based on an SSP instance
S	SSP set
V	SSP solution set
m	SSP sum of all elements
w	SSP value
b	line susceptance
E'^d	set of directed switched lines
E'	set of switched lines
$\mathcal{N}^{E'}$	network where the lines E' are switched
$\mathcal{Y}_{\mathsf{e}}^{x,y,\underline{v},\overline{v}}$	network to fix voltage magnitude
$\mathcal{N}[\mathsf{e} \in N_{G/L}]$	The variant of $\mathcal{N}$ where e is a generator and a load
$\mathcal{D}_{\mathsf{e}}^{x}$	$\sin$ choice network
$\widetilde{\mathcal{D}}_{\mathsf{e}}^{x}$	$\sin$ choice network with fixed demand
$\mathcal{V}_{\mathsf{e}}^{s}$	voltage choice network
v	voltage magnitude
$\overline{v}$	voltage magnitude maximum
$\underline{v}$	voltage magnitude minimum