

Solitons, Breathers and Rogue Waves in Nonlinear Media



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*A thesis submitted for the degree
of
Doctor of Philosophy*

August 2015

To my loving Grandmother...

Disclaimer

Except where otherwise indicated, all the research in this thesis is my own work.

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19th August 2015

Acknowledgments

I want to say ‘thank you’ to my supervisor, Professor Nail Akhmediev, for extending his hand to me at the time when I was almost a fallen man, and giving me the chance to rediscover my life again. From the beginning of my Ph.D, I was in poor health, and this affected my mental state as well. I could not have prepared this thesis if it were not for Professor Akhmediev, who took care of me, as only the closest to someone will do. He is my source of inspiration and will remain so, as long as I live. I am grateful to professor Akhmediev for introducing me into the realm of nonlinear world and its unfathomable beauty.

I would like to express my utmost gratitude to Dr. Adrian Ankiewicz for guiding and inspiring me with great patience in the ways of genuine scientific research. During my research time, when I was very ill and ended up in hospital, Adrian visited me and talked with me while I was lying on the bed. His persistent support and help has allowed me to continue. The result is that finally I am able to prepare this thesis. I really do not have any words by which I can express my gratitude properly to him. He, Dr. Adrian Ankiewicz, is my best teacher, mentor and my guide.

My thanks go to the ‘Australian Government’, as it chose me for an ‘Endeavour Postgraduate Award’ and made it possible for me to pursue a Ph.D. in ‘The Australian National University’. I am in debt to ‘Endeavour Award’ case managers, Katharine Hope and Beth Mutton, and I include other members of the Endeavour award management team. They took care of me throughout the entire program duration. My deep appreciation goes to the contributor of the ‘Ben Williams student support grant’ that provided me with funding for an additional six months. Herewith I thank Karen Nulty and Lucia Lu for generously helping me on various administrative tasks.

Special thanks go to my advisor Natasha Devine; she motivated me during the tough times in the Ph.D. pursuit, keeping me on track all the time and checking me over to see if I were alright. I used to get all the information from her – how to keep in good health and maintain a family – this helped me a lot. I will not forget her help, support and visits in hospital which used to be my sources of strength to fight against pain and illness.

I gratefully acknowledge the support of my fellow colleague David Jacob Kedziora. It was a great pleasure to talk with him and discuss various scientific problems. He shared many of his ideas with me on solving nonlinear equations; this eventually contributed to my thesis preparation.

I greatly appreciate Dr. Wonkeun Chang. He spared his valuable time on many occasions to discuss various scientific problems with me. His insightful demonstrations on specific problems like ‘dissipative systems’ and ‘Supercontinuum Generation’ helped me to understand and develop some ideas connected to my own research theme.

I am grateful to Peter Vouzas, Victor Surkus and all the tea-time members for their enlightened discussions and for making the tea time experience a delightful one. It happened many times that I came to my office only to participate in the tea time discussion, nothing else.

My special thanks goes out to my friend Mohammad Ali Nawaz who enlightened me with many of his interesting thoughts and ideas in politics, sciences, arts, history and religion. During my illness and in all the tough times I had, my friend was always with me, never letting me fall down.

I greatly appreciate the varied helps and supports that I received from my friends Harunur Rashid, Sanjoy Nandy and Fauzia Nasreen Sultana during my PhD period. I am grateful to all my friends and colleagues who made my life in Canberra a happy time of my life, filled with joy and all new experiences.

Finally, I want to say ‘Thank you’ to my wife Aysha Siddiqua, my family members, my mother and my father for keeping me alive and providing me with all the ingredients for a man to live happily.

Md Amdadul Huq Chowdury

19th August 2015

Publications in Refereed Journals

1. A. Chowdury, D.J. Kedziora, A. Ankiewicz and N. Akhmediev. “Breather solutions of the integrable quintic nonlinear Schrödinger equation and their interactions”, *Physical Review E*, **91**, 022919 (2015).
2. A. Chowdury, D.J. Kedziora, A. Ankiewicz and N. Akhmediev. “Breather-to-soliton conversions described by the quintic equation of the nonlinear Schrödinger hierarchy”, *Physical Review E*, **91**, 032928 (2015).
3. A. Chowdury, A. Ankiewicz and N. Akhmediev. “Moving breathers and breather-to-soliton conversions for the Hirota equation”, *Proceedings of the Royal Society A*, **471**, 20150130 (2015).
4. A. Chowdury, D.J. Kedziora, A. Ankiewicz and N. Akhmediev. “Soliton solutions of an integrable nonlinear Schrödinger equation with quintic terms”, *Physical Review E* **90**, 032922 (2014).
5. A. Chowdury, A. Ankiewicz and N. Akhmediev. “Solutions of the higher-order Manakov-type continuous and discrete equations”, *Physical Review E*, **90**, 012902 (2014).
6. A. Ankiewicz, A. Chowdury, N. Devine and N. Akhmediev. “Rogue waves of the nonlinear Schrödinger equation with even symmetric perturbations”, *Journal of Optics*, **15**, 064007 (2013).
7. A. Ankiewicz, N. Devine, M. Ünal, A. Chowdury and N. Akhmediev. “Rogue waves and other solutions of single and coupled Ablowitz–Ladik and nonlinear Schrödinger equations”, *Journal of Optics*, **15**, 064008 (2013).
8. A. Ankiewicz, Jose M. Soto-Crespo, A. Chowdury and N. Akhmediev. “Rogue waves in optical fibers in presence of third-order dispersion, self-steepening, and self-frequency shift”, *Journal of the Optical Society of America B*, **30**, 87–94 (2013).
9. A. Chowdury , A. Ankiewicz and N. Akhmediev. “Breather and rational solutions of modified Korteweg-de Vries equation”, (Submitted)

Abstract

In this thesis, the solutions of the Nonlinear Schrödinger equation (NLSE) and its hierarchy are studied extensively. In nonlinear optics, as the duration of optical pulses get shorter, in highly nonlinear media, their dynamics become more complex, and, as a modelling equation, the basic NLSE fails to explain their behaviour. Using the NLSE and its hierarchy, this thesis explains the ultra-short pulse dynamics in highly nonlinear media. To pursue this purpose, the next higher-order equations beyond the basic NLSE are considered; namely they are the third order Hirota equation and the fifth order quintic NLSE. Solitons, breathers and rogue wave solutions of these two equations have been derived explicitly. It is revealed that higher order terms offer additional features in the solutions, namely, ‘Soliton Superposition’, ‘Breather Superposition’ and ‘Breather-to-Soliton’ conversion.

How robust are the rogue wave solutions against perturbations? To answer this question, two types of perturbative cases have been considered; one is odd-asymmetric and the other type is even-symmetric. For the odd-asymmetric perturbative case, combined Hirota and Sasa-Satsuma equations are considered, and for the latter case, fourth order dispersion and a quintic nonlinear term combined with the NLSE are considered. Indeed, this thesis shows that rogue waves survive these perturbations for specific ranges of parameter values.

The integrable Ablowitz-Ladik (AL) equation is the discrete counterpart of the NLSE. If the lattice spacing parameter goes to zero, the discrete AL becomes the continuous NLSE. Similar rules apply to their solutions. A list of corresponding solutions of the discrete Ablowitz-Ladik and the NLSE has been derived. Using associate Legendre polynomial functions, sets of solutions have been derived for the coupled Manakov equations, for both focusing and defocusing cases. They mainly explain partially coherent soliton (PCS) dynamics in Kerr-like media. Additionally, corresponding approximate solutions for two coupled NLSE and AL equations have been derived. For the shallow water case, closed form breathers, rational and degenerate solutions of the modified Kortweg-de Vries equation are also presented.

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CHAPTER 1

Introduction

1.1. Nonlinear Optics

The birth of systematic nonlinear optics took place in the year 1961 with the pioneering work of Franken *et.al* [1] at the University of Michigan right after the invention of first operable laser in the year 1960. In this work they projected a ruby laser at wavelength 694.3 nm through quartz crystal and produced a second harmonic at 347.2 nm. Theoretical explanation for the second harmonic generation was proposed by Nicolaas Bloembergen and P.S. Pershan in 1962 [2]. They derived solutions of Maxwell's equations taking into account the interface between a linear and a nonlinear medium. Additionally, they established various rules for the interaction of light in a nonlinear medium. Though most nonlinear optical observations requires laser radiation, there are other classes of nonlinear optical phenomena which do not require laser radiation and that were found many years before the invention of the laser. Familiar example includes Pockels electro-optic effects, first studied by German physicist Friedrich Carl Alwin Pockles in 1893, and Kerr electro-optic effects, discovered by Scottish physicist John Kerr in 1875 [3]. Seminal work of Franken *et.al* [1] opened up a new horizon for the science of nonlinear optics with the discovery of a rich diversity of nonlinear optical effects. To name few of the known and principal discoveries, we include 'High Harmonic Generation' (HHG), 'Sum Frequency Generation' (SFG), 'Difference Frequency Generation' (DFG), 'Stimulated Raman

Scattering' (SRS), 'Self Focusing', 'Four Wave Mixing', 'Self phase Modulation', 'Stimulated Brillouin Scattering' (SBS), 'Multi Photon Absorption' and many others. Each of these effects has its own distinctive features and can be characterised in terms of a relation between dielectric polarisation density $\tilde{P}(t)$ and electric field $\tilde{E}(t)$. In that case, the relation between dielectric polarisation $\tilde{P}(t)$ and electric field $\tilde{E}(t)$ can be expressed as

$$\tilde{P}(t) = \epsilon_0 \chi \tilde{E}(t) \quad (1.1)$$

This relation explains that the induced polarisation due to the applied optical field is linear, or, in other words, optical properties such as refractive index and absorption coefficient of a medium are independent of the applied field intensity. Additionally, this relation implies that the fundamental rule of classical optics, the principle of superposition, holds. The frequency of the applied electric field remains the same after propagation through a medium in this regime. These features of linear optics result in light that cannot be controlled. The superscripted tilde ' $\sim$ ' implies rapidly varying fields in time with real quantities. ϵ_0 is the permittivity of free space and χ is known as the linear susceptibility, which dominantly contributes to $\tilde{P}(t)$. In Eq.(1.1), electronic polarisation $\tilde{P}(t)$ is the principal parameter in the theory of linear and nonlinear optics. Nonetheless, this is an approximation in the relation of $\tilde{P}(t)$ and $\tilde{E}(t)$ and considered to be sufficient when the applied electric field is relatively weak, as in linear optics.

The invention of the laser, which is the fundamental source of high intensity monochromatic optical fields, enabled us to break the boundary of linear optics, leading us to observe various new optical phenomena; this was not possible in the linear regime. This part of the optics is 'nonlinear optics', where the principle of superposition does not hold anymore. One can control and change the main features, such as speed and frequency, of light in this regime. Now, if we extend the expression Eq. (1.1) by taking a power series expansion of $\tilde{P}(t)$, in principle we are stepping into the realm of nonlinear optics.

$$\tilde{P}(t) = \epsilon_0 \left[\chi^{(1)} \tilde{E}(t) + \chi^{(2)} \tilde{E}^2(t) + \chi^{(3)} \tilde{E}^3(t) + \dots \right] \quad (1.2)$$

$\chi^{(j)}$, for $j = 1, 2, 3 \dots$ is the j^{th} order susceptibility, and the order of j represents the order of the nonlinearity. In general, $\chi^{(j)}$ is a tensor of rank $(j+1)$. Remarkably, each of the higher order susceptibilities contributes to different class of nonlinear effects. $\chi^{(1)}$ explains the light matter interaction in the linear regime. In the nonlinear regime, for example, second order nonlinearity with $\chi^{(2)}$ is responsible for second harmonic generation, sum frequency generation and optical rectification [4]. However, the contribution of $\chi^{(2)}$ is nonzero only for non-centrosymmetric crystals. For example, SiO_2 is a molecule which shows inversion symmetry, thus $\chi^{(2)}$ vanishes for this molecule.

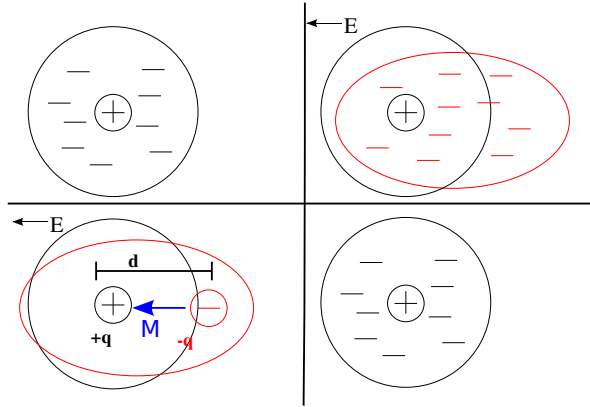


Figure 1.1 Atomic interaction with an applied electric field

The third order nonlinear polarisation is more complex and gives rise to numerous surprising nonlinear effects, for example, four wave mixing, stimulated Raman scattering, stimulated Brillouin scattering, two photon absorption, self focusing and nonlinear refraction [4]. In optical fibres, phenomena of intensity-dependent refractive index arise due to the third order nonlinear susceptibility. The intensity dependent refractive index triggers a large variety of other interesting nonlinear effects, including self phase modulation and cross phase modulation.

The origin of relation Eq. (1.2) lies in the fundamental physics of light-matter interaction. Complex dynamics of atoms and molecules is hidden inside the nonlinear susceptibility $\chi^{(j)}$. Matter usually consists of a collection of positively charged cores surrounded by a cloud of bound negatively charged electrons. At the atomic level, the charge distribution is distorted by an applied electric field. The dynamical picture in this situation can be explained by using a classical model Fig. (1.1) where M

is the atomic dipole moment. In the top-left corner, the atom remains neutral, as there is no external field; in the top-right corner, charge distribution in the atom becomes distorted with a directional electric field. When the field is withdrawn, the atom becomes neutral again – in the bottom-right corner. This is the simplest model for atomic response in an electric field. Behaviour of the dielectric becomes more complex, depending on the particular system. This basic principle is the essence of many physical models described in linear and nonlinear physics.

1.2. Nonlinear Dynamical System

The world around us is seamlessly going through immensely complex dynamical processes at each moment. In most cases, these processes involve entirely non-intuitive mechanisms, often revealing unexpected phenomena governing these complex dynamical systems. Many physical processes around us in our everyday life are nonlinear by nature. Linearity, which is only an approximation of a physical system, allows us to understand or simplify a particular natural phenomenon. The entire framework of linear theory stands on a few mathematical laws generalising the proportionality relation, and that proportionality distorts when the magnitude of the variable becomes large. At this point, linear theory disregards the scale of magnitude, which in turn introduces unrealistic limitations into the system. However, despite the sheer complexity of every natural phenomenon that is hidden inside the system, linear theory, in its essence, is an exploration of ‘order’ in a given system, and hence it remains a central subject in the study of science. This regime is explained by a straight line, of simple proportions in the relations of input and output; this is, at the same time predictable and possesses good behaviour.

To draw a perceptible line between linearity and nonlinearity, we can use some simple example. To begin, we can start with the ‘exponential law’ which plays the lead rule in the linear regime. In its simplest case, the derivation of the exponential function involves linear ‘Ordinary Differential Equations (ODEs)’. For instance, if $x(t)$ is a certain quantity, *e.g.* a number of living organisms, such as bacteria in a observable test sample, the rate of proliferation can be given as

$$\frac{dx}{dt} = c x \quad (1.3)$$

Integrating Eq.(1.3) with an initial condition $x(0) = x_0$, we get $x(t) = x_0 e^{ct}$, which gives the number of bacteria at a certain time following the linear proportionality relation. Clearly, this is an unrealistic situation, with the bacteria population increasing exponentially for all time, which is not possible. In a natural environment, this relation does not hold and we rather see a balance in bacteria population. The exponential increment may continue for a brief period or interval, but cannot continue for all time. The avalanche of bacteria population makes an impact on the surrounding environment, thus limiting the population growth. This point of logical conflict reveals the limitation of linear theory. In the linear system approach, a complex system is viewed as a linear combination of constituent parts. For example, from a linear point of view, the cause of an avalanche or volcanic eruption, an earthquake, tsunami or a tornado, is considered to be the linear sum of the constituent effects.

In reality in each of these cases, series of macroscopic and microscopic systems are involved in a cooperative way, and, at this precise moment, the science of ‘Nonlinearity’ arises to explain the compound effects. Nonlinearity does not allow the decomposition of a complex natural system to express it as the algebraic sum of its constituent parts. In nonlinear science, the system as a whole is quite different from its individual parts. One typical example includes complex living systems. To know how a biological system works, linear theory suggests that we decompose the entire living system into its constituent parts, down to individual cells. By studying individual cells, one should know the compound effects of millions of individual cells, which is how life thrives. But this is not the case. Putting together many living cells does not make a living body and that is precisely the fundamental problem of the linear theory. The expression of life and its nature in a living body is an extremely sophisticated mechanism. Various parts ‘nonlinearly’ contribute to the total system. For example, transmission of information through the nervous system from entire body to the brain is nonlinear propagation of various signals as pulses, by bio-chemical or electrochemical reactions. The functioning of the brain itself is also extremely nonlinear. The theory of ‘nonlinearity’ plays a leading rule from the simple

pendulum to complex lattice systems, fluid dynamics, optics, the chaotic atmosphere, complex living systems, logic of neural networks, gene regulation, planetary motion, population dynamics and many more.

The subject of ‘dynamics’ deals with the time dependent configuration that evolves in time in unpredictable ways. To establish a relation between a natural phenomenon, which is nonlinear by nature, and a physical theory, involves a combination of fundamental physics represented by a nonlinear dynamical system. The term ‘dynamical system’ refers to a multi-component time-dependent configuration represented by a set of numbers termed ‘system variables’. The future state of this configuration is uniquely determined by its present and past configuration through the system variable transformation. In the mid-seventeenth century, Newton formulated the first full dynamical system, introducing calculus and differential equations to describe planetary motion. At first, Newton tried to solve a two-body problem, viz. the dynamics of the earth around the sun using the inverse square law of gravitational attraction between them. Later it was extended to the three-body problem, introducing the motion of moon around the earth into the previous two-body dynamics. Physicists and mathematicians around the globe found this three-body problem impossible to solve. Again, this difficulty arises from the fact that the total gravitational force acting on any planetary object was assumed to be the linear sum of the individual gravitational forces associated with the object under investigation. In reality, there are infinitely many moving celestial bodies floating in space, having associated gravitational forces and they are mutually interacting. This situation naturally poses nonlinear problems. Practically, the three body problem (sun, earth and moon) was the first nonlinear problem to arise in the field of physics and in mathematics.

Toward the end of 19th century, Henry Poincaré, who is considered the father of modern ‘Chaos theory’, made a breakthrough and returned to Newton’s three body problem. Instead of solving the real problem of finding the exact position of a planet in a many-mass planetary system where two point masses never collide, he rather poses the question of the stability of the planetary system having more than two bodies. Poincaré contributed to the problem, initially having serious errors in it. However, he developed various ideas related to many bodies interacting with

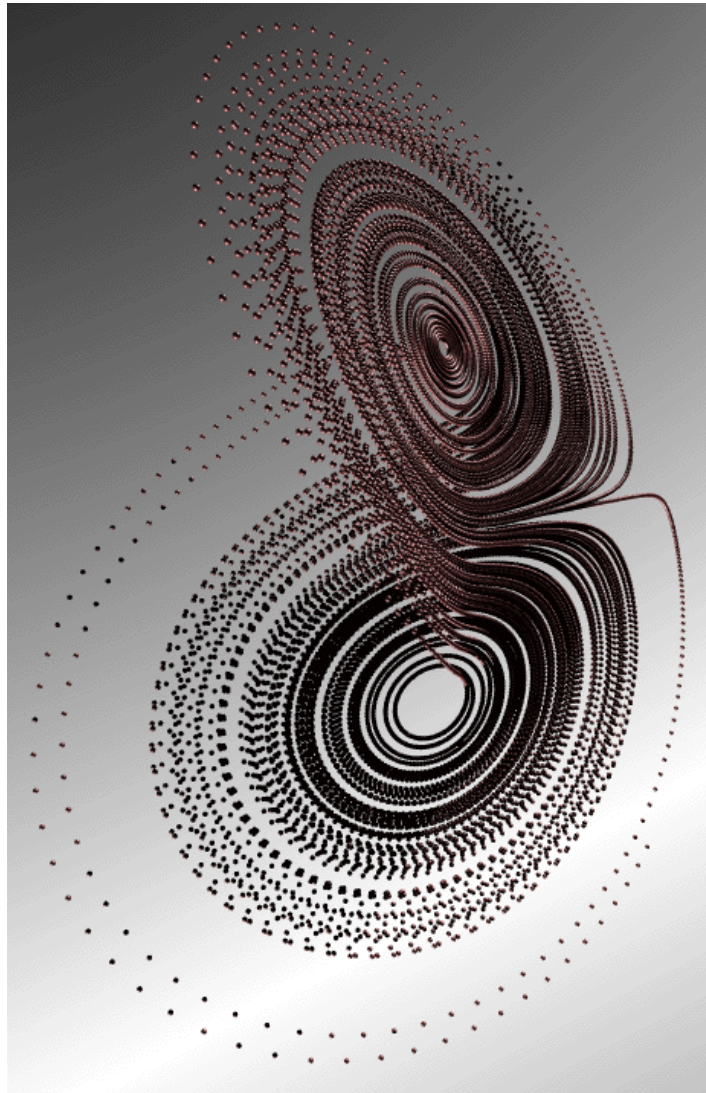


Figure 1.2 A Lorenz-attractor showing the chaotic dynamics

each-other in Newton's gravitational field, which practically leads to the modern chaos theory. His powerful geometric approach predicts the long term behaviour of objects in a deterministic system which exhibits unpredictable behaviour, depending on small changes in the initial conditions, thereby making it impossible to predict longterm dynamics. Later on, the invention of the modern computer in the 1950's opened up a new opportunity to investigate, in more detail, the theory of chaos and nonlinear dynamics.

Edward Lorenz, an American mathematician and meteorologist, pioneered in the chaotic dynamical systems. During the year 1950, he was leading a project in a weather forecasting system based on linear statistical methods, but was skeptical about its validity, as most atmospheric phenomena involved nonlinearity. In the year 1963, he published a well celebrated seminal paper with a system of three ordinary differential equations [5], related with atmospheric behaviour, which is sensitive to the initial condition. This nonlinear dynamical system showed that small changes in the initial conditions can amplify exponentially with time, making the long time behaviour unpredictable. His numerical solutions for two slightly different initial conditions reveal that the trajectory of the solutions look like a two-sheeted surface on a 3-D plot, later named a 'Lorenz attractor', as shown in Fig. (1.2) [6]. To explain the systems sensitivity to initial conditions, Lorenz coined the word 'Butterfly effect' which states that, even a flap of butterfly's wings in Brazil may set off a tornado in Texas. His description of the 'Butterfly effect', developed in the year 1969, followed the work [7, 8]. For atmospheric convection Lorenz uses the following three coupled nonlinear ordinary differential equations:

$$\frac{dX}{dt} = \sigma(Y - X) \quad (1.4)$$

$$\frac{dY}{dt} = X(\rho - Z) - Y \quad (1.5)$$

$$\frac{dZ}{dt} = XY - \beta Z \quad (1.6)$$

where X is proportional to convective intensity, Y is the temperature difference between ascending and descending currents and Z is proportional to the deviation

of the vertical temperature profile from linearity. σ, ρ and β are positive system parameters, representing the physical characteristics of air flow. Lorenz used $\sigma = 10, \rho = 28$ and $\beta = 8/3$ and found chaotic behaviour of the system leading to a Lorenz attractor [5]. Later on, the theory of chaos has evolved in many nonlinear systems. For example, in the 1970s David Ruelle and Floris Takens proposed a theory on the generation of turbulence in dissipative systems [9], Robert M May showed chaotic dynamics in the logistic map [10], in 1975 Mitchell Feigenbaum discovered the famous Feigenbaum constants where he showed that the ratio of the difference between the values at which successive period doubling bifurcations occur is a constant and is 4.6692 [11]. His pioneering discovery enabled mathematicians to understand the intractable random behaviour of chaotic nonlinear dynamical systems. Another influential figure in nonlinear dynamical systems and chaos is Benoît Mandelbrot who studied the ‘Mandelbrot set’ fractal which is named after him. An expanded form of his revolutionary idea on fractals and chaos has been published as a book in 1982 [12].

1.3. Integrable Nonlinear Dynamical Systems

Using a differential equation, we can predict the variation of the dependent variable such as chemical concentration, velocity, diffusion, displacement and population with respect to time or space. A mathematical model should display both regular and systematic, as well as irregular and chaotic, features as two fundamental behaviours of physical phenomena. However, the universal characteristics of nature in its fundamental level are not chaotic. In our everyday life, for nature around us, macroscopic or microscopic, we envisage the harmony and regularity of its ruling ensemble. From intergalactic to the smallest microcosm, when unperturbed by human intervention, we see that a self sustaining and systematic procedure is working underneath. From a scientific point of view, an experimental physicist or chemist has to make sure that their findings are somehow predictable in connection with mathematical laws, and that they can be controlled, so that the same result could be reproduced again. Therefore, it is fundamentally important to study and devise the kind of mathematical model which is not chaotic and allows for solutions

that are independent of initial and boundary conditions and are predictable and regular for all time. This kind of mathematical configuration, in a limited case, is called an integrable system.

In its simplest notion, mathematically, ‘Integrability’ means exact solvability and is a subject of advanced mathematical rigour. Justification of a differential equation – whether it is ‘integrable’ – is a challenging problem. Study reveals that integrability not only means exact solvability, it also implies a connection between geometry, analysis and statistical mechanics [13]. A few nonlinear integrable systems were discovered in the 18th century, without having any explicit understanding of their solution. In the 19th century, mathematicians were trying hard to find integrable systems but were not successful. Works of Henry Poincaré significantly reduced the eagerness and interest on integrable systems, as he mentioned that these systems are exceptions. He pointed that even a small perturbation eliminates its integrability properties. With the discovery of various special functions, the theory for the existence of analytic solution for any differential equation was succeeded by Augustin Cauchy in 1872 and an extension of his result was formulated by Sophie Kowalevski in 1875; it is jointly named as the Cauchy-Kowalevski theorem. This theorem states that a differential equation with an n^{th} order time derivative will have an analytic solution if its time derivatives up to order $n - 1$ in the dependent variable are specified at a single surface [14]. Kowalevski’s work was connected to the singularity analysis of differential equations, where, by definition, the singularities of a complex function are called poles. To be more specific, there are two types of singularities that occur in a differential system; one is ‘fixed’ and the other one is ‘movable’. The ‘fixed’ singularity is determined by the equation itself and the ‘movable’ singularities are determined by the initial conditions. For example, take the ordinary differential equation:

$$(t - c)\frac{du}{dt} = \lambda u \quad (1.7)$$

The solution of Eq. (1.7) reads as $u = K(t - c)^\lambda$ and when λ is a positive integer, the solution is an analytic function, but for λ a negative integer, $u = c$ is a fixed pole of the solution. Moreover, when λ is rational, the solution is an algebraic branch

point, but for other values of λ , the solution is a transcendental branch point [15]. Nonlinear differential equations can have both ‘fixed’ and ‘movable’ singularities. For example, a nonlinear ordinary differential equation is

$$\frac{du}{dt} - \lambda u^{(1-\frac{1}{\lambda})} = 0. \quad (1.8)$$

The solution of Eq. (1.8) is $u = (t - t_0)^\lambda$ where t_0 is arbitrary. Depending on the values of λ , $t = t_0$ is an analytic point ‘movable’ pole or a branch point. Kowalevski observed that all integrable systems, when evolved into the complex time plane, are analytic, except for isolated poles [16]. In 1888 she was awarded the ‘Borden prize’ for her seminal work on the theory of the motion of a rigid body about a fixed point. Her investigation was to determine the choice of parameters for which the equation of motion does not allow movable critical points. After Kowalevski’s work on integrable systems, French mathematician Paul Painlevé investigated more; he enumerated all the first order ordinary differential equations whose movable singularities are poles, which, in other words are singularities that depend on initial values [17, 18]. These kinds of differential equations are called the Painlevé type, having the Painlevé property. It is a generally accepted idea that differential equations that possess the Painlevé property are prime candidates to be integrable equations [19]. Although many integrable systems show the Painlevé property [20, 21], yet, it is unknown how to use singularity analysis to obtain the integrals. However by considering a local singularity analysis in global geometric terms, an intuitive demonstration of the reason why integrability follows from the Painlevé property has been presented in [22]. Painlevé analysis can be applied to both a system of ordinary differential equations (ODEs) and a partial differential equation (PDE) to identify candidature as an integrable system. Algorithmically, Painlevé analysis is one of the most powerful tools to identify an integrable system [23]. One of the big advantages of this method is that a system having the Painlevé properties practically leads to Bäcklund transformation, the Lax Pair and Hirota bilinear method, which are typical mathematical formalisms for integrable systems. Another advantage is that Painlevé properties can be applied equally to exactly solvable models in other branches in physics [23]. The Painlevé algorithm has the ability to reveal some geometric and

algebraic structures that underlie many integrable systems. Additionally, this same Painlevé algorithm applies to extract information about non-integrable systems as well.

The study of the theory of integrability in connection with Painlevé analysis remained dormant until around the mid 1970s and started again with the seminal work of Zabusky and Kruskal in 1965. They observed solitary wave propagation in the numerical solution of the Korteweg-de Vries (KdV) equation [24], and hence coined the term ‘soliton’ for the first time. Here,

$$\psi_t + 6\psi\psi_x + \psi_{xxx} = 0 \quad (1.9)$$

where t is propagation variable and x is the spatial variable, where a subscript shows a partial derivative. The soliton solution of Eq. (1.9) observed is $\psi(x, t) = 2a^2 \text{sech}^2(ax - 4a^3t)$, where a is an arbitrary constant. Their work was inspired by the pioneering work of Enrico Fermi, John Pasta, Stanislaw Ulam and Mary Tsingou where they observed unexpected stability phenomena in an anharmonic lattice system [25]. This is a century old equation used to model propagation of water waves in canals [26]. After this remarkable discovery, in 1967 Miura *et al.* presented an infinite number of conserved quantities for the KdV equation [27]. The discovery of the infinite set of conserved quantities in the KdV aroused the idea that a certain type of nonlinear PDE has special properties. An infinite set of conserved quantities implies that there is an infinite set of symmetries by which the flow is restricted in phase space. Additionally, Gardner *et. al.* shows that there exists a linearizing transformation in the KdV equation and first wrote the solvability condition for the linear system of PDEs [28].

Now,

$$\phi_{xx} + [\lambda + \psi]\phi = 0 \quad (1.10)$$

$$\phi_t + 4\phi_{xxx} + 3\psi_x\phi + 6\psi\phi_x = 0 \quad (1.11)$$

where Eq.(1.10) is the one-dimensional Schrödinger equation with the potential $\psi(x, t)$ and energy λ . Eq. (1.9) was solved using the inverse scattering transform [28].

Adopting this solvability condition of Eq. (1.9), Peter D Lax formulated two linear operators [29]:

$$L = -\frac{\partial^2}{\partial x^2} - \psi \quad (1.12)$$

$$M = -4\frac{\partial^3}{\partial x^3} - 3(2\psi\frac{\partial}{\partial x} + \psi_x) \quad (1.13)$$

He showed that, as a ‘zero curvature condition’, these two operators satisfy the commutation condition $\frac{dL}{dt} = [L, M] = LM - ML$, and proved that the spectrum of L remains invariant in time [29,30]. In 1970 Gardener [31] and in 1971 Zakharov *et.al* [32] separately established the KdV equation as a ‘completely integrable Hamiltonian system’.

Formulation of the Lax Pair for the KdV equation was one big step towards the understanding of integrability properties for nonlinear PDEs, as a Lax Pair allows one to solve a so-called integrable PDE by using the inverse scattering transform (IST). This IST, on the other hand, is considered to be a canonical transformation of action-angle variables [32]. The notion of an ‘angle’ variable is equivalent to the generalised coordinate, where the conjugate momenta are equivalent to ‘actions’, together forming the term ‘action-angle’. Solving the KdV equation by the IST is technically equivalent to an infinite dimensional analogue of Arnol’d-Liouville integrability [33]. Now, the Arnol’d-Liouville theorem states that, a Hamiltonian dynamical system with n degrees of freedom, which also has n known integrals of motion, will have a canonical transformation to ‘action-angle’ coordinates. The transformed Hamiltonian will only depend on ‘action’ coordinates, and angle coordinates will evolve in time linearly. This configuration of a Hamiltonian system will allow solvability in terms of analytic functions when the action-angle map is known explicitly [34]. This formulation of the Arnol’d-Liouville theorem, sometimes referred as Arnol’d-Liouville integrability, is a key concept to justify the ‘complete integrability’ of a nonlinear Hamiltonian PDE systematically. Remarkably, around the same time, while studying a similarity reduction of known integrable PDEs, Ablowitz *et al.* discovered that, after linearising and transforming the PDEs into ODEs,

these ODEs are actually of Painlevé type [35–38]. Following these pioneering works, Ablowitz and Segur, together with Ramani, formulated a conjecture: ‘every ODE that is obtained by reducing a PDE using IST, possesses the Painlevé property [37, 38].

After the discovery of the integrable KdV equation, along with its Lax Pair, many other integrable nonlinear PDEs have been derived. Among them, one of the important and well studied integrable nonlinear PDE is the Nonlinear Schrödinger equation and its hierarchy. Most of this thesis will be dealing with this remarkable integrable system. The realm and universality of this magnificent equation will be clear in the forthcoming chapters.

1.4. Nonlinear Coherent Structures

1.4.1. Solitons

The first observation of a solitary wave on a water surface was reported by the Scottish naval engineer J. Scott Russell back in the nineteenth century [39]. Theoretical analysis of such a wave was presented by British scientist Lord Rayleigh [40] and French scientist Boussinesq [41–44]. In 1895, two Dutch mathematicians, Korteweg and de Vries, formulated a partial differential equation (now known as the KdV equation) whose closed form solution precisely modelled the propagation of shallow water waves [45]. Numerical analysis of this equation by Zabuski and Kruskal [24] showed that a localized initial condition results in the excitation of solitons. Gardner *et al.* [28] were the first to solve an initial value problem involving the KdV equation using the inverse scattering theory with the one-dimensional Schrödinger operator.

During the last few decades, soliton-based research has expanded far beyond optics into a diverse array of fields, including oceanography [46], plasma physics [24], molecular biology [47], meteorology [48], geology [49] and nonlinear field theory [50]. The interdisciplinary spread of this concept is remarkable in its speed, considering that the first theoretical prediction of an optical soliton, subject to several perturbations occurring in a glass fibre, was only given by Hasegawa and Tappert in 1973 [51]. Experimental verification followed several years later [52], and it was shown that

pulse broadening could indeed be counterbalanced by the nonlinearity related to the index of refraction. The major result, from the perspective of theory, was that these ensuing pulse profiles could be well-described by basic sech-type functions. Soliton interactions form a highly nontrivial part of soliton theory [53–56]. The outcome of interactions between solitons depends upon the particular model used in the theory [57–59]. Due to the great variety of possible interactions, their study can be considered as a separate branch of nonlinear physics.

1.4.2. Modulational Instability and Akhmediev Breathers

Modulational instability (MI) is the result of an interaction between a strong carrier harmonic wave with small periodic perturbation, and occurs in various physical systems such as in water waves, plasmas, various biological systems, sand dunes, Supercontinuum Generation (SG) and in fibre optics. The phenomenon of MI is central to the dynamics of nonlinear evolution processes. In the initial stage of the development of MI, a periodic perturbation will amplify exponentially and generate cascades of spectral sidebands [60, 61]. In the developed stage of MI, the dynamics are more complex and involve several stages of energy exchange among the spectral modes. The evolutionary processes of MI can be explained using another similar phenomenon called Fermi-Pasta-Ulam (FPU) recurrence [62].

In nonlinear optics, MI has been the subject of extensive study because of its inherent connection with short pulse dynamics in nonlinear optical media [63], such as generating a train of ultrashort pulses, and also optical parametric amplification [64, 65]. Additionally, it was shown that the underlying mechanism involved in the initial stage of supercontinuum generation development is connected to noise-driven MI, which is usually seeded by continuous wave radiation [66–70]. Recently, it was also shown that the emergence of a rogue wave in the developed stage of SG is also connected with MI [71]. Mostly, the study of MI development in nonlinear optics was conducted using the basic NLSE. Though the analytic solution of the NLSE called the Akhmediev breather (AB) had existed for more than 30 years [72], and it explains MI phenomena in nonlinear media, researchers in similar fields mostly used numerical approaches to investigate MI [73, 74]. Recently, the analytic solution of an

AB [72] was used for quantitative analysis of the evolution of a modulated continuous wave field into a train of ultrashort pulses [69, 70]. In the same work, it was also shown that AB theory also provides insight into the initial phase of supercontinuum generation seeded by noise-driven MI.

The breather solutions of the of the fundamental NLSE as the epitome of MI phenomena have been analysed in detail in [72, 75–82]. The breather solution for the next higher order NLSE, which is well-known as the Hirota equation, has been given in [83]. It describes the complete nonlinear evolution of an initial state that starts with modulation instability [72, 80, 81]. Solutions that have a small non-zero velocity have been dubbed “quasi-Akhmediev breather” [79].

MI and breathers introduce a new dimension into the seemingly well-known world of solutions of integrable equations. They have drawn much attention because of their interactions and energy exchange with a constant background [75, 83, 84]. Such interactions produce unique behaviours that are different from usual soliton dynamics. Breathers are solutions that are periodic in space or in time, and are classified as either Kuznetsov-Ma solitons [85] or Akhmediev breathers [86, 87]. Each of these has been considered as a candidate for describing the notorious rogue waves [88]. The large-period limiting case of each is a Peregrine solution [89–91], which is localised both in space and time, and which serves as a prototype of a rogue wave [92]. Nevertheless MI plays a pivotal role in triggering extreme events and rogue waves, but not every type of MI can excite rogue waves [93].

For the nonlinear Schrödinger equation (NLSE), the interaction of breathers was first considered back in 1985 [80]. N -breather interactions and their possible practical application were studied later in [81]. The relation of breathers to rogue wave formation has recently created renewed interest in this subject [75, 88]. To give an example, the collision of breathers at non-zero angle brings us new unexpected applications [79]. More surprises can be expected when the theory goes beyond the relatively simple NLSE model [94].

1.4.3. Rogue Waves

Rogue waves arise in the zero frequency limit of AB solutions and are rare events, localized in space and time. A rogue wave ‘*appears from nowhere*’ in that it arises from a bulge in a simple ripple; after forming a high-amplitude peak, it decreases and finally ‘*disappears without a trace*’ [75]. In the ocean, it can damage or sink a large ship. An optical rogue wave is a high amplitude bright, narrow pulse of light; applications include fast bio-chemical processes and ‘Supercontinuum generation’ (SG). A rogue wave as a solution of the NLSE has been derived by British applied mathematician Howell Peregrine (1938-2007) more than 30 years ago. The solution, in general, named as a ‘Peregrine soliton’, is localised in both space and in time and can be viewed as a space-time isolated ‘amplitude peak’ [89]; it was discovered in the year 1983. From the point of algebraic structure, the solution is also referred to as a ‘rational’ solution of the NLSE, since the functional form of the solution is nothing but the ratio of two polynomials. Later in the year 1985, the analytic form of this solution as the limiting case of an Akhmediev breather was derived in [80] in the proposal to generate trains of periodic pulses in a optical fiber. In the same work, the second order rogue wave solution, as the limit of a second order breather solution, was also derived for the first time. In these classic works of Akhmediev *et al.*, it was demonstrated how the rise of a rogue wave is connected with the initial periodic perturbation on a constant background and how it emerges in the zero frequency limit of an Akhmediev breather.

In the modelling of wave phenomena on the sea surface, oceanographers used to use linear theory and were unable to explain rogue waves in their linear approach. At the beginning, it was almost impossible to imagine that an ocean liner or a large cargo ship might encounter such a steep wave, having an amplitude three times higher than its background, in the middle of the ocean. There was no direct evidence or witnesses of such extreme occurrences till 1995 when the first real evidence of a rogue ocean wave was recorded [95]. However, the underlying mechanisms for which a wave can have that enormous size is still not clear. Under what circumstances a rogue wave may arise, and a possible explanation of how a rogue wave can be triggered on the ocean surface is presented in the work of Akhmediev *et al.* [96].

Later on, the same group of authors presented an exhaustive theoretical explanation of various aspects of rogue waves in their various works; for example, by modulating a plane wave with weakly random waves, a chaotic wave environment to model the natural oceanic surface is presented in [75]. In the same work, by propagating chaotic initial conditions using the NLSE, rogue waves are produced in the developed stage and explained in terms of analytic solutions. Various higher order rational solutions of the focusing NLSE and their relation with the formation of rogue waves in the ocean and in optics are presented in [97]. By analysing the spectral feature of the Peregrine soliton, a systematic approach of early warning for the appearance of rogue waves is given in [98]. A detailed study of modulational instability, Fermi-Pasta-Ulam recurrence and rogue waves in the discrete Ablowitz-Ladik equation is presented in [99]. Whether rogue waves are robust against perturbation is studied in [100]. Rogue wave solutions for the discrete Hirota equation is given in [101]. Various surprising features of rogue waves as higher order solutions of NLSE were recently discovered by Kedziora *et al.* in the series of works presented in [102–106]. Experimental demonstrations of rogue waves were carried by Amin Chabchoub *et al.* in a series of works in [107–111].

Apart from the existence of rogue waves in the ocean, in optical media also they are equally verified. During the study of supercontinuum generation in optical fibres, Solli *et al.* first observed the existence of optical rogue waves [71]. In the year 2010, Kibler *et al.* observed Peregrine solitons in fibre optics [112]. Since then, research on optical rogue waves has escalated on a large scale and many surprising discoveries have been made.

Research interest in this direction increased, not only because of its multifaceted potential, but also, as a novel phenomenon, it opens the door to innumerable opportunities in the dynamics of stochastic processes. However, at present, research on ‘rogue waves’ is expanding in various directions. Although the NLSE is considered to be the primary equation for deriving rogue wave solutions, recently multicomponent vector rogue wave solutions have attracted considerable attention. They are reported in the work of Baronio *et al.* [113–116]. A complete account of rogue waves and their research potential in various fields is beyond the scope of this thesis. Nonetheless,

the interested reader will have a broad overview about the recent progress in this field from [86, 117, 118].

1.5. Nonlinear Schrödinger Equation and its Hierarchy

The nonlinear Schrödinger equation (NLSE) and a variety of its extensions have been among the most-studied evolution equations. These equations and their solutions have been applied to explain experiments in many branches of physics for more than 50 years. In physics, the applications cover the fields of light-wave technology, plasma physics, water waves, photonics, nonlinear optics, optical wave guide arrays, Bose-Einstein condensation and condensed-matter physics. In mathematics, NLSE research themes include integrability, various aspects of Riemann-Hilbert problems, stability theory of solitary waves, beam dynamics, chaos, and, very recently, extreme events.

The NLSE and its Hierarchy are much-studied equations. They are attractive for modelling various complex physical systems. The NLSE equation is popular among theoretical and experimental scientists in various fields, not only for its physical relevance and rich mathematical structure, but also for its universal nature. After study on the equation for more than 60 years, an astronomical amount of scientific literature has accumulated in relation to nonlinear sciences. However, it is sometimes difficult to get a proper definition of the NLSE in this myriad of papers, as this equation has been derived for numerous physical systems and also in various fields in mathematics. To be specific, from the point of universality, the NLSE arises quite naturally. Within an appropriate limit, any weakly nonlinear, energy preserving and dispersive system will give rise to the NLSE. For example, the NLSE can be derived from the Sine-Gordon, Klein-Gordon and Korteweg-de Vries (KdV) equations, which implies that the NLSE can be derived for any medium which is weakly nonlinear and dispersive. The NLSE provides a systematic description of envelope dynamics of a carrier wave propagating through nonlinear media [119].

The origin of the NLSE can be traced back to the early work of Ginzburg and

Landau in the study of superconductivity [120] , the work of Ginzburg and Pitaevskii in superfluidity [121] and in Landau's work on the theory of the superfluidity of helium-II [122]. The importance of this equation started to become apparent around 1960-1970, when several important physical significances of the NLSE were discovered. For example, in the year 1964, Chiao *et al.* and Talanov discovered that, under certain conditions, an electromagnetic beam can produce its own dielectric waveguide and propagate without spreading [123, 124]. This is the so called self-focusing phenomenon, which plays a central role in beam dynamics in nonlinear optics. Self-focusing phenomena are specifically connected with Kerr-type materials, where the refractive index increases with beam intensity. A circularly polarised optical beam broadens, with an angular divergence (θ), during propagation because of diffraction in a dielectric material. However the refractive index of Kerr-type material depends on optical intensity, following the relation

$$n = n_0 + n_2|E|^2 + \dots . \quad (1.14)$$

Now if the dielectric constant produced by the term $n_2|E|^2$ is so high that the beam's critical angle (θ_c) required for total internal reflection at the beams boundary is greater than (θ), then the propagating beam will not disperse or be broadened. In other words, the beam will be trapped inside the dielectric material.

Apart from the phenomenon of self-focusing, the NLSE is also derived in nonlinear media as it can explain self-compression and self-modulation . For an optical fibre, Hasegawa and Tappert derived the NLSE in their seminal work [51, 125] in the anomalous and normal dispersion regimes. They suggested a new kind of optical communication system where a 'soliton' can be used as a 'bit' for data transmission. In the year 1980 in Bell labs, Mollenauer *et al.* experimentally showed pulse compression and splitting in optical fibres and also showed that at certain critical power levels, soliton characteristics resembled those of a higher-order soliton [52]. However, as in nonlinear optics, for small amplitude water waves with infinite depth, the NLSE was derived by Zakharov [126], and for finite depth, in the next year (1969), the NLSE was derived by Benney and Roskes [127]. Gradually, in subsequent years, scientists in various fields derived the NLSE in various nonlinear physical systems as one of

the main governing equations.

The integrable form of the NLSE can be given as

$$i\frac{\partial\psi}{\partial x} + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi = 0. \quad (1.15)$$

This form is a standard convention used in nonlinear optics, where x is the propagation variable and t is the transverse variable. Throughout the thesis, this particular form will be used. Other forms of expression are also used; in particular, in derivations relating to hydrodynamics, variables x and t are usually swapped. The role of these variables vary, depending on the particular problem involved. For example, to describe a spatial soliton, both of these variables represents spatial dimensions. For an optical soliton, x is the distance along the fibre and t is the coordinate moving with the dispersive pulse. The name of this equation naturally came from the famous one dimensional Schrödinger equation used in quantum mechanics:

$$i\frac{\partial\psi}{\partial x} + \nabla\psi + V\psi = 0. \quad (1.16)$$

Here V is a potential which usually arises in the mean field description of interacting quantum particles and ∇ is a spatial dimension. If one sets $V = |\psi|^2$ in Eq. (1.16), the equation takes the form of the standard NLSE Eq. (1.15).

Nonetheless, the caveat remains that the simplest expressions for solitons, breathers and rogue waves are a consequence of the NLSE, which is still only an approximation to the physics that governs wave evolution in various systems. Thus, to balance the analytic utility of these ‘sech’-type functions with heightened physical realism, better approximations have since been attained via the inclusion of additional phenomena, such as self-steepening, self-frequency shift and third-order dispersion [128–131]. These are the next lowest order terms that have to be taken into account in studies of pulse propagation in optical fibres. These third-order terms which belong to the third member of the NLSE hierarchy are essential in supercontinuum generation [132], Cherenkov radiation by solitons [133] and pulse deforming phenomena [134]. The role of fourth-order terms belonging to the fourth member of NLSE hierarchy in a fibre have also been studied [135–137]. With ever-

increasing intensity of the optical field and further shortening of pulses down to atto-second durations [138], the role of higher-order terms is becoming ever more important. Generally speaking, the role of each of the terms of third, fourth and quintic order should be studied separately. However, there are many more higher-order terms in quintic than cubic and fourth order terms, and these studies cannot be carried out in a single step. Even clarifying the roles of the third-order terms required decades of research effort. The studies of higher order terms, such as quintic terms, which belong to the fifth member of NLSE hierarchy will need more effort, because of the higher number of them and the complexity of their contributions. Thus, it is a better idea first to find integrable models that can be studied analytically, because any other choice would require massive numerical modelling. Importantly, integrable models provide us with general forms of higher order terms, although restricting us in the choice of their relative contributions. Indeed, the new terms cannot be bundled onto the NLSE in any haphazard fashion, but must be incorporated with certain parametric restrictions so as to maintain the integrability. On the other hand, individual terms remain independent of each other. First, the part of the NLSE hierarchy that is most important for applications is considered and is given by:

$$i\psi_x + S[\psi(x, t)] - i\alpha H[\psi(x, t)] + \gamma P[\psi(x, t)] - i\delta Q[\psi(x, t)] = 0 \quad (1.17)$$

where the second-order S is the nonlinear Schrödinger operator (including second-order dispersion),

$$S[\psi(x, t)] = \frac{1}{2}\psi_{tt} + \psi|\psi|^2,$$

the third-order H is the Hirota operator (beginning with third-order dispersion),

$$H[\psi(x, t)] = \psi_{ttt} + 6|\psi|^2\psi_t,$$

the fourth-order P is the Lakshmanan-Porsezian-Daniel (LPD) operator (beginning with fourth-order dispersion),

$$P[\psi(x, t)] = \psi_{tttt} + 8|\psi|^2\psi_{tt} + 6\psi|\psi|^4 + 4\psi|\psi_t|^2 + 6\psi_t^2\psi^* + 2\psi^2\psi_{tt}^*,$$

and the fifth-order Q is the quintic operator (beginning with fifth-order dispersion),

$$\begin{aligned} Q[\psi(x, t)] = & \psi_{ttttt} + 10|\psi|^2\psi_{ttt} + 30|\psi|^4\psi_t + 10\psi\psi_t\psi_{tt}^* + 10\psi\psi_t^*\psi_{tt} + 20\psi^*\psi_t\psi_{tt} \\ & + 10\psi_t^2\psi_t^*. \end{aligned}$$

More succinctly, we can write

$$Q[\psi(x, t)] = \psi_{ttttt} + 10|\psi|^2\psi_{ttt} + 10(\psi|\psi_t|^2)_t + 20\psi^*\psi_t\psi_{tt} + 30|\psi|^4\psi_t. \quad (1.18)$$

Taking only the quintic operator ($Q[\psi(x, t)]$) from Eq. (1.17) with fifth order dispersion makes the quintic NLSE and can be given as

$$i\psi_x + S[\psi(x, t)] - i\delta Q[\psi(x, t)] = 0, \quad (1.19)$$

Here x is the propagation variable and t is the transverse variable ('time' in a moving frame), with the modulus of the complex-valued $\psi(x, t)$ representing the envelope of the waves. Within Eq. (1.17), the integrability of the fundamental NLSE operator is best known. Various solitonic structures and families of other solution types have previously been presented for this system [84, 139]. The simplest exactly-solvable extension then arises when $\gamma = \delta = 0$, known as the Hirota equation [140, 141]. This equation contains third-order terms scaled by an independent real parameter. Accordingly, Painlevé analysis has been done for deformations of both the NLSE and Hirota expressions [142]. Furthermore, when $\alpha = \delta = 0$, the system is also integrable and is known as the Lakshmanan-Porsezian-Daniel equation [143, 144]. In this latter case, the coefficients of the additional terms were found using Painlevé analysis on a model of a Heisenberg spin chain, but were independently derived elsewhere as part of the NLSE hierarchy [145].

A different approach to these operators has since been developed, allowing for the construction of both solitons [146] and rogue waves [137] for arbitrary α and γ values. The next sequential case of $\alpha = \gamma = 0$ where Eq. (1.17) reduces to what will be called the 'quintic NLSE' Eq. (1.19). It has been written out previously by Hoseini and Marchant [147], with emphasis that it is a member of the NLSE integrable hierarchy [145]. They considered soliton solutions and their interaction. An adjusted

version of this equation with real ψ has also already been presented, with Lax pair, conservation laws and N-soliton solutions all generated as well [148]. However, along with its sixth-order relative for which symmetries have been investigated [149], it is regarded in this case as an extended Korteweg-de Vries equation, applicable to shallow-depth fluid studies.

In optics, higher-order terms become progressively more important when pulses of shorter duration propagating along a fibre considered. The general governing equation for pulse propagation in a fibre is given by [150]:

$$i\psi_x = - \sum_{m=1}^{\infty} \frac{i^m \beta_m}{m!} \frac{\partial^m \psi}{\partial t^m} - \gamma \left(1 + is \frac{\partial}{\partial t} \right) \left(\psi \int_0^{\infty} R(t') |\psi(t-t')|^2 dt' \right), \quad (1.20)$$

where the first term on the right-hand-side is the Taylor series expansion for linear dispersion, with corresponding coefficients β_m , while the second term describes the nonlinear terms, taking into account their dispersion. Here, s is the self-steepening coefficient, the nonlinear term γ depends on effective core area and $R(t)$ includes instantaneous (electronic) and delayed nuclear (Raman) contributions of the nonlinear material response. The integral in Eq. (1.20) is often approximated by taking the series to first-order only, *viz.* $|\psi|^2 - \tau_R \frac{\partial}{\partial t} (|\psi|^2)$, for Raman delay τ_R . However, in reality, one need higher-order terms, and these involve higher order time derivatives of intensity, $|\psi|^2$. In particular, $\psi \frac{\partial^3}{\partial t^3} (|\psi|^2)$ produces terms similar to those in Q . The coefficients for each term depend on the function $R(t')$ that, in turn, depends on the material parameters. The quintic operator, Q , contains all essential terms that may appear when one moves beyond the basic NLSE approximation. The first term in Eq. (1.18) describes the linear fifth-order dispersion with $\delta = \beta_5/120$. The other four terms describe the nonlinear dispersion. There is no clear way to associate any individual term in this equation with a single isolated physical phenomenon, as can be done for the third-order equation. Moreover, these terms can be rearranged in various forms. However, Eq. (1.19) contains the complete set of terms describing physical phenomena appropriate to that order. The relevance to a particular physical situation is also restricted by the choice of fixed coefficients in the operator. In reality, the coefficients may differ from those in Eq. (1.19). The coefficient δ in

front of the whole operator ‘ Q ’ can be varied and set close to an experimental value. This flexibility allows us to make reasonable adjustments for the actual physical phenomenon to be approximated in future experiments. Importantly, the whole equation (1.17) is integrable and admits exact solutions. Dealing with several free parameters at the very beginning is a cumbersome exercise. Therefore, one has to take a more cautious approach and first consider solutions with each operator separately before dealing with the whole equation, as it may produce excessively complicated solutions.

Considering the quintic term separately may be a reasonable approach, even from practical point of view. As it is known, modern optical fibres can be designed with various dispersion characteristics, and the lowest-order dispersion terms can be suppressed, making higher-order terms pivotal. Thus, the idea that certain predictions may allow experimental confirmations is not as outrageous as it may initially appear. Talking realistically, fifth-order dispersion does play a significant role in present laser experiments when pulses are close to 20 ps in duration [82].

1.6. Ablowitz-Ladik Equation

In most cases, the NLSE deals with continuous systems, but to handle a physical system that is discrete by nature requires an appropriate discrete model. Ablowitz and Ladik first derived a discrete model that is an integrable discrete counterpart of the NLSE [151–154]. Like the NLSE itself, this discrete analog has a number of applications. A few examples can be mentioned in this regard: modeling a moving quasi-particle inside a periodic crystal and its interaction with lattice vibrations [155], temperature correlations of quantum spins [156], observing the dynamics of Bloch oscillations in an NLSE chain in the presence of static electric field [157] and soliton dynamics in discrete molecular chains [158]. Additional applications of the AL model include arrays of optical waveguides, electronic circuits and the dynamics of nonlinear lattices [159–161]. A more detailed overview, applications and relevant discretizations have been given in the review article in Ref [162].

The NLSE and Ablowitz-Ladik (AL) [152, 153] equations are amongst the most

studied examples in the world of integrable equations. The latter can be considered as a special form of discretisation of the former. This special form allows us to maintain the integrability of the AL equation. Integrability provides an infinite number of exact solutions for each case. As a consequence of the correspondence between the two cases, we can reveal pairs of solutions that can represent a special discretisation of continuous solutions and their discrete analogs. In principle, such a correspondence should allow us to find the solution of one equation if the solution of the other equation is known. However, this linkage is not trivial and requires knowledge of at least several examples before we can find the rules of such correspondence.

CHAPTER 2

Soliton with Quintic Terms

2.1. Introduction

In this chapter, the fifth-order Eq. (1.19) of the nonlinear Schrödinger hierarchy is solved. This integrable partial differential equation contains fifth-order dispersion and nonlinear terms related to it. Using Darboux transformations, expressions for the most representative soliton solutions are derived. This set includes two-soliton collisions and the degenerate case of the two-soliton solution, as well as beating structures composed of two or three solitons. Ultimately, the new quintic operator ‘Q’ and the terms it adds to the standard nonlinear Schrödinger equation (NLSE) are found to primarily affect the velocity of solutions, with complicated flow-on effects. Furthermore, a new structure, composed of coincident equal-amplitude solitons, which cannot exist for the standard NLSE, is also presented.

To derive exact expressions for both single and double soliton solutions in Sec. (2.4), a Lax pair for the full complex-valued quintic NLSE in Sec. (2.2), and the Darboux transformation scheme Sec. (2.3), are used. A brief investigation into triple soliton solutions is also included. In fact, the hierarchy of integrable NLSE equations is actually well-defined [145], where each sequential ‘order’ attaches a collection of fixed-ratio terms to all those of the previous orders. The fifth-order equation of the nonlinear Schrödinger hierarchy is given in Eq. (1.17). Here we take $\gamma = \alpha = 0$,

retaining only the fifth order operator ‘Q’, Eq. (1.19) from the explicit expression Eq. (1.17) with the δ coefficient, giving us the ‘quintic NLSE’.

2.2. Lax Pairs

The inverse scattering technique for the ordinary NLSE was developed in 1972 by Zakharov and Shabat [163]. This technique allows us to write the NLSE in terms of two matrix operators. Later studies have shown that a number of other integrable equations can be represented in similar form. The creation of the world of integrable evolution equations can be considered as one of the main achievements in mathematical physics of the twentieth century [164]. Remarkably, Eq. (1.17) with quintic terms can also be structured in Lax pair format, meaning that it is integrable. Just as in the case of the NLSE, this form can be written as

$$\begin{aligned}\frac{\partial R}{\partial t} &= U \cdot R, \\ \frac{\partial R}{\partial x} &= V \cdot R,\end{aligned}\tag{2.1}$$

such that the ‘zero-curvature’ condition

$$U_x - V_t + [U, V] = 0,\tag{2.2}$$

returns the original system. Naturally, while this transformation has the downside of introducing new variables via the vector R , the upside is that the nonlinear system is now expressed as linear differential equations that are more readily solved by standard methods. Importantly, Lax pairs also serve as the basis for many solution-generating procedures, ranging from the inverse scattering method to the Darboux scheme. In this work, we employ the latter. Given this, we can consider two equations to be in the same Lax pair hierarchy if they share the same U matrix. For instance, the standard NLSE is linked to the 2×2 Ablowitz-Kaup-Newell-Segur (AKNS)

matrix [165], given by

$$U = i \begin{bmatrix} \lambda & \psi(x, t)^* \\ \psi(x, t) & -\lambda \end{bmatrix}. \quad (2.3)$$

The general form of our quintic NLSE, which is thus named because the V matrix is a fifth-order polynomial in eigenvalue λ , also uses Eq. (2.3) as its U matrix. Specifically, the appropriate corresponding V matrix is written as

$$V = \sum_{j=0}^5 \lambda^j V_j, \quad V_j = i \begin{bmatrix} A_j & B_j^* \\ B_j & -A_j \end{bmatrix}, \quad (2.4)$$

$$\begin{aligned} A_5 &= 16\delta, B_5 = 0, & A_4 &= -8\gamma, B_4 = 16\delta\psi, \\ A_3 &= -4\alpha - 8\delta|\psi|^2, & B_3 &= -8\gamma\psi + 8i\delta\psi_t, \\ A_2 &= 1 + 4\gamma|\psi|^2 + 4i\delta(\psi_t^*\psi - \psi_t\psi^*), & B_2 &= -4\alpha\psi - 8\delta|\psi|^2\psi - 4i\gamma\psi_t - 4\delta\psi_{tt}, \\ A_1 &= f_1 + 2\alpha|\psi|^2 + 6\delta|\psi|^4 - 2i\gamma(\psi_t^*\psi - \psi_t\psi^*) + 2\delta(\psi_{tt}^*\psi - |\psi_t|^2 + \psi_{tt}\psi^*), \\ B_1 &= \psi + 4\gamma|\psi|^2\psi - 2i\alpha\psi_t - 12i\delta|\psi|^2\psi_t + 2\gamma\psi_{tt} - 2i\delta\psi_{ttt}, \\ A_0 &= \frac{1}{2}f_0 - \frac{1}{2}|\psi|^2 - 3\gamma|\psi|^4 - i\alpha(\psi_t^*\psi - \psi_t\psi^*) - \gamma(\psi_{tt}^*\psi - |\psi_t|^2 + \psi_{tt}\psi^*) \\ &\quad - i\delta(\psi_{ttt}^*\psi - \psi_{tt}^*\psi_t + \psi_{tt}\psi_t^* - \psi_{ttt}\psi^*) - 6i\delta(\psi_t^*\psi - \psi_t\psi^*)|\psi|^2, \\ B_0 &= f_1\psi + 2\alpha|\psi|^2\psi + 6\delta|\psi|^4\psi + i\frac{1}{2}\psi_t + 6i\gamma|\psi|^2\psi_t + \alpha\psi_{tt} + 2\delta\psi_{tt}^*\psi^2 + 4\delta|\psi_t|^2\psi \\ &\quad + 6\delta(\psi_t)^2\psi^* + 8\delta\psi_{tt}|\psi|^2 + i\gamma\psi_{ttt} + \delta\psi_{tttt}, \end{aligned}$$

where coefficients f_0, f_1 are arbitrary real constants, while α, γ and δ are coefficients of the higher-order terms in the Eq. (1.17). Thus, the V -matrix contains all necessary evolution data for the solutions of this equation. Naturally, when $\alpha = 0, \gamma = 0$ and $\delta = 0$, the V -matrix describes the evolution data for the NLSE. With these U and V matrices, the zero-curvature equation reproduces

$$\begin{aligned} &i\psi_x - f_0\psi - if_1\psi_t + \frac{1}{2}(\psi_{tt} + 2|\psi|^2\psi) - i\alpha(\psi_{ttt} + 6|\psi|^2\psi_t) \\ &+ \gamma(\psi_{tttt} + 6|\psi|^4\psi + 2\psi^2\psi_{tt}^* + 4\psi|\psi_t|^2 + 6\psi^*(\psi_t)^2 + 8|\psi|^2\psi_{tt}) \\ &- i\delta(\psi_{ttttt} + 30|\psi|^4\psi_t + 10\psi\psi_t\psi_{tt}^* + 10\psi_t|\psi_t|^2 + 10\psi\psi_t^*\psi_{tt} + 20\psi^*\psi_t\psi_{tt} \\ &+ 10|\psi|^2\psi_{ttt}) = 0. \end{aligned} \quad (2.5)$$

The first two coefficients, f_0 and f_1 , are generally set to zero, since the corresponding terms simply relate to phase and velocity transformations applied to solutions. Then, Eq. (2.5) reproduces Eq. (1.17). So as to eliminate any possible confusion, we also emphasize here that all the subscripted A_j and B_j expressions (for $0 \leq j \leq 5$) denote nothing more than the coefficients of the eigenvalue polynomial represented by the V matrix. They have no direct relation to the ‘hierarchical order’ of operators that was mentioned in Sec. (2.1). For instance, the hierarchically fourth-order LPD coefficient γ is not limited to A_4 and B_4 , arising instead in all V_j (for $j \leq 4$) so that the zero-curvature equation can adequately reproduce all the LPD terms. This clarification aside, it is evident that setting $\alpha = \gamma = 0$ within the Lax pair will reduce generalised Eq. (2.5) into the specific form of integrable ‘quintic’ Eq. (1.19) that we study here, complete with the fundamental NLSE operator.

2.3. The Darboux Scheme

Referring back to the Lax pair formalism in Eq. (2.1), the elementary version of the Darboux method [166] attempts to find an operator T so that $T \cdot R$ is compatible with a new $\tilde{U}$ and $\tilde{V}$ Lax pair of interest. In general, these matrices must be of the same structure as the old U and V Lax pair, except with wavefunction ψ replaced by $\tilde{\psi}$. The idea is that if one possesses a solution ψ to the nonlinear system, along with compatible R , then one should also be able to construct a new wavefunction $\tilde{\psi}$ using the new Lax pair solution $T \cdot R$. This procedure is already well established in the case of the standard NLSE, providing utility for multi-soliton [167] and multi-breather [81] construction, and nothing appears to change for higher-order equations. Indeed, the simplest ‘seeding’ solution of the quintic NLSE is still $\psi = \psi_0 = 0$, for which vector R can easily be determined. This is particularly true as all β_j in Eq. (2.4) become zero, meaning that both U and V are purely diagonal in form. Consequently, Eq. (2.1) produces what we call ‘first-order component functions’,

$$R = \begin{bmatrix} r_{1,j} \\ s_{1,j} \end{bmatrix}. \quad (2.6)$$

Here we note that j is an indexation variable, to be used shortly in constructing high-order solutions, and each set of component functions $(r_{1,j}, s_{1,j})$ is linked with an individual set of parameters also indexed by j , namely eigenvalue λ_j and axial shifts (x_j, t_j) .

As the transformation matrix T can be applied multiple times, so too can solutions of ever higher order be generated. For the quintic equation, a solution of order n is recursively constructed via

$$\psi_n = \psi_{n-1} + \frac{2(\lambda_n^* - \lambda_n)s_{n,1}r_{n,1}^*}{|r_{n,1}|^2 + |s_{n,1}|^2}. \quad (2.7)$$

For $(n, j) = (1, 1)$, the substitution of $r_{1,1}$ and $s_{1,1}$ from Eq. (2.6) into Eq. (2.7) reveals that $\psi_0 = 0$ is a base on which to construct a soliton solution. For $n > 1$, higher-order component functions must first be produced by

$$\begin{aligned} r_{n,j} &= J \left[(\lambda_{n-1}^* - \lambda_{n-1}) s_{n-1,1}^* r_{n-1,j+1} \right. \\ &\quad + (\lambda_{j+n-1} - \lambda_{n-1}) |r_{n-1,1}|^2 r_{n-1,j+1} \\ &\quad \left. + (\lambda_{j+n-1} - \lambda_{n-1}^*) |s_{n-1,1}|^2 r_{n-1,j+1} \right], \\ s_{n,j} &= J \left[(\lambda_{n-1}^* - \lambda_{n-1}) s_{n-1,1} r_{n-1,1}^* r_{n-1,j+1} \right. \\ &\quad + (\lambda_{j+n-1} - \lambda_{n-1}) |s_{n-1,1}|^2 s_{n-1,j+1} \\ &\quad \left. + (\lambda_{j+n-1} - \lambda_{n-1}^*) |r_{n-1,1}|^2 s_{n-1,j+1} \right], \end{aligned} \quad (2.8)$$

where $J = (|r_{n-1,1}|^2 + |s_{n-1,1}|^2)^{-1}$. In this way, it is possible to nonlinearly superimpose n solitons with n sets of parameters to produce one compound wavefunction.

2.4. Exact Quintic NLSE Solutions

2.4.1. Single Soliton Solution

As previously stated, all solitonic structures for a governing equation within the NLSE hierarchy can be constructed via the Darboux method upon a zero background, as given by $\psi = \psi_0 = 0$. Substituting this into the matrices within Eqs. (2.3) and (2.5), remembering that $f_0 = f_1 = \alpha = \gamma = 0$, the first-order component functions

in Eq. (2.6) are readily derived from the linear Lax pair system in Eq. (2.1). These functions of x and t are specifically given as

$$\begin{aligned} r_{1,1} &= \exp \left[ix\lambda_1^2 (16\delta\lambda_1^3 + 1) + i\lambda_1 t \right], \\ s_{1,1} &= \exp \left[-ix\lambda_1^2 (16\delta\lambda_1^3 + 1) - i\lambda_1 t \right], \end{aligned} \quad (2.9)$$

where it is interesting to note the reciprocity with $s_{1,1} = 1/r_{1,1}$. Additionally, an arbitrary phase term can be inserted into the r and s functions, arising as a constant of integration, but it is ignored here for convenience of expression. As mentioned in Sec. (2.3), the integration constant can also take the form of axial shifts, via the replacement of x and t with $x - x_1$ and $t - t_1$, respectively. In the related case of breathers, the effect of these axial shifts can be unintuitive and dramatic [103, 139], but they are henceforth ignored in this work, as they do not significantly affect the solitons. We also note that eigenvalue λ_1 is allowed to be complex, unless a restriction is explicitly stated.

It is now possible to use the seed $\psi_0 = 0$ and the component functions in Eq. (2.9) to generate the first non-trivial solution ψ_1 via the transformation in Eq. (2.7). The resulting structure happens to be a first-order soliton of Eq. (1.19), recalling that only δ is non-zero for our quintic equation. Explicitly, with $\lambda_1 = a + ib$, the solution is

$$\psi_1(x, t) = 2ib \exp[-2i\Omega(x, t)] \operatorname{sech}[2b(t + x\vartheta)], \quad (2.10)$$

with the coefficient

$$\vartheta = 2 \left[8\delta (5a^4 - 10a^2b^2 + b^4) + a \right] \quad (2.11)$$

and the phase factor

$$\Omega(x, t) = x \left[a^2 - b^2 + 16a\delta (a^4 - 10a^2b^2 + 5b^4) \right] + at. \quad (2.12)$$

The soliton amplitude is simply $2|b|$, i.e. the same as for the NLSE soliton. Its width along the t -axis also does not change. However, now the soliton velocity is

$$v = -\vartheta = -2 \left[8\delta (5a^4 - 10a^2b^2 + b^4) + a \right],$$

which differs from the velocity of the NLSE soliton. When $\delta = 0$, Eq. (2.10) is a soliton of the fundamental NLSE. In that scenario, its velocity, $-2a$, is completely determined by the real part, a , of eigenvalue λ_1 . Thus, a purely imaginary eigenvalue corresponds to straight propagation of the soliton along the x -axis. For the quintic equation, the soliton has non-zero velocity even for $a = 0$, *viz.* $v = -16\delta b^4$. Then the phase term $\exp(2ib^2x)$ evolves along the x -axis and does not depend on t . Thus, the directions of phase evolution and soliton propagation generally differ. This creates interference patterns for colliding solitons that would be unusual in the NLSE case (e.g., see Fig. (2.1b)).

For purely imaginary eigenvalues, terms arising from hierarchical orders that are even (e.g. the NLSE and LPD operators) solely affect the phase factor, while terms related to odd hierarchical orders (e.g. the Hirota and quintic operators) affect the group velocity alone. An example of a first-order soliton of the quintic equation is shown in Fig. (2.1a), with $\delta = \frac{1}{64}$, $a = 1$ and $b = 1$. The imaginary part of eigenvalue λ_1 determines the height of the structure, given as $|\psi_1(0, 0)| = 2|b|$, so the displayed wavefunction has an amplitude of 2. The soliton velocity in this case is -1 , which corresponds to the structure in Fig. (2.1a) propagating left along the t axis (with $|v| = 1$) for increasing values of evolution variable x . As can be seen, even a very small δ significantly influences the soliton velocity.

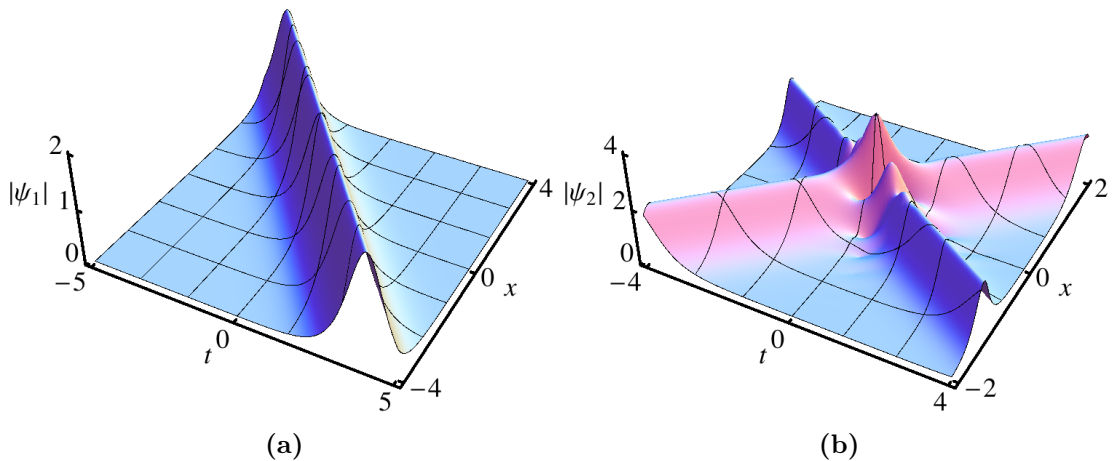


Figure 2.1 (a) A soliton solution of the quintic equation as given by Eq. (2.10). Parameters are $\delta = 2^{-6}$ and $\lambda = 1 + i$ ($a = 1$ and $b = 1$). Propagation direction is along the x -axis. (b) A two-soliton solution of the quintic equation as given by Eq. (2.14). Parameters are $\delta = 0.001$, $\lambda_1 = -3.5 + i$ and $\lambda_2 = -1 + i$, so as to depict a collision of two solitons. Propagation direction is along the x -axis.

2.4.2. General Second-order Soliton

As detailed in Sec. (2.3), it is possible to combine two first-order solitons in nonlinear superposition to create a second-order soliton. Because this involves two ‘components’, the first-order component functions from Eq. (2.9) must be generalised as

$$\begin{aligned} r_{1,j} &= \exp \left[ix\lambda_j^2 (16\delta\lambda_j^3 + 1) + i\lambda_j t \right], \\ s_{1,j} &= \frac{1}{r_{1,j}}, \end{aligned} \quad (2.13)$$

where $j = 1$ or 2 , depending on the first-order soliton that is being referenced. Mathematically, the superposition procedure involves using Eq. (2.8) to generate second-order component functions $r_{2,1}$ and $s_{2,1}$, before employing Eq. (2.7) with ψ_1 from Eq. (2.10) to generate the new solution ψ_2 .

The result is

$$\psi_2(x, t) = 2e^{-2i(\Omega_1 + \Omega_2)} \frac{N_2}{D_2} \quad (2.14)$$

where

$$\begin{aligned} N_2 &= ib_2 [(a_1 - a_2)^2 - b_1^2 + b_2^2] e^{2i\Omega_1} \cosh(2\Theta_1) \\ &\quad + ib_1 [(a_1 - a_2)^2 + b_1^2 - b_2^2] e^{2i\Omega_2} \cosh(2\Theta_2) - 2(a_1 - a_2)b_1b_2S_1, \\ D_2 &= [(a_1 - a_2)^2 + b_1^2 + b_2^2] S_2 - 2b_1b_2S_3, \\ S_1 &= e^{2i\Omega_1} \sinh(2\Theta_1) - e^{2i\Omega_2} \sinh(2\Theta_2), \\ S_2 &= \cosh(2\Theta_1) \cosh(2\Theta_2), S_3 = \cos[2(\Omega_1 - \Omega_2)] + \sinh(2\Theta_1) \sinh(2\Theta_2), \end{aligned}$$

with

$$\begin{aligned} \lambda_j &= a_j + ib_j, \\ \Omega_j &= a_j t + x [a_j^2 - b_j^2 + 16a_j\delta(a_j^4 - 10a_j^2b_j^2 + 5b_j^4)], \\ \Theta_j &= b_j [t + 2x[a_j + 8\delta(5a_j^4 - 10a_j^2b_j^2 + b_j^4)]] . \end{aligned} \quad (2.15)$$

This solution, $\psi_2(x, t)$ depends on the fifth-order equation coefficient δ and thus differs from the known NLSE result. While complicated in appearance, the form of

this expression shares similarities with that of the first-order soliton. For instance, the phase factor of each component, as given by Eq. (2.12), is easily discernible as Ω_j . Each factor Θ_j is also essentially the ‘sech’ argument in Eq. (2.10), complete with indexed versions of the coefficient ϑ shown in Eq. (2.11). Thus Eq. (2.14) describes two first-order solitons passing through the origin, generally colliding at an angle for arbitrary λ_j . The height of this intersection is the amplitude sum or difference of both individual solitons, depending on constructive or destructive interference, and is written as $|\psi_2(0,0)| = 2|b_1 + b_2|$. In the case of positive equal unitary values, $b_1 = b_2 = 1$, the maximum amplitude is 4. This result is indeed confirmed by Fig. (2.1b), which depicts two solitons with arbitrary velocities colliding. The interference pattern here is more complicated than the case involving the collision of NLSE solitons. This happens because the direction of phase evolution and pulse propagation for each soliton may differ significantly.

2.4.3. Beating Second-order Soliton

It is worth examining the second-order quintic soliton in Eq. (2.14) with greater care. For instance, when $\delta = 0$, the expression reduces to a solution of the standard NLSE. For distinct purely imaginary eigenvalues $\lambda_1 = ib_1$ and $\lambda_2 = ib_2$, this ‘beating’ structure consists of two solitons in perfect alignment along evolution axis x . This is the well-known ‘higher-order soliton’ of Satsuma and Yajima [168]. As discussed previously, this overlap arises because each soliton velocity becomes solely dependent on the real part, a_j , of the eigenvalue, as is made explicit in Eq. (2.11). Thus the two components are only coincident when $a_1 = a_2$.

Soliton alignment is no longer simple when the quintic operator from Eq. (1.19) comes into play, considering that the axial alignment angle of each individual component is now dependent on its amplitude. While general soliton collisions of finite but significant duration still impart degrees of modulation [146], we restrict the ‘beating soliton’ label to components that are perfectly aligned for all x . This is why the inclusion of a_1 and a_2 is important, so that velocity adjustments can be

made to realize coincident first-order solitons. This alignment is only possible if

$$5a_1^4 - 10a_1^2b_1^2 + b_1^4 + \frac{a_1}{8\delta} = 5a_2^4 - 10a_2^2b_2^2 + b_2^4 + \frac{a_2}{8\delta}, \quad (2.16)$$

which happens to confirm the aforementioned standard NLSE condition of $a_1 = a_2$ when $\delta = 0$. But even when $\delta \neq 0$, it is possible to first select the amplitudes of the two component solitons by fixing b_1 and b_2 , then choose the velocity of one structure via the selection of a_2 , and finally solve Eq. (2.16), so as to determine the appropriate a_1 value that achieves solitonic alignment.

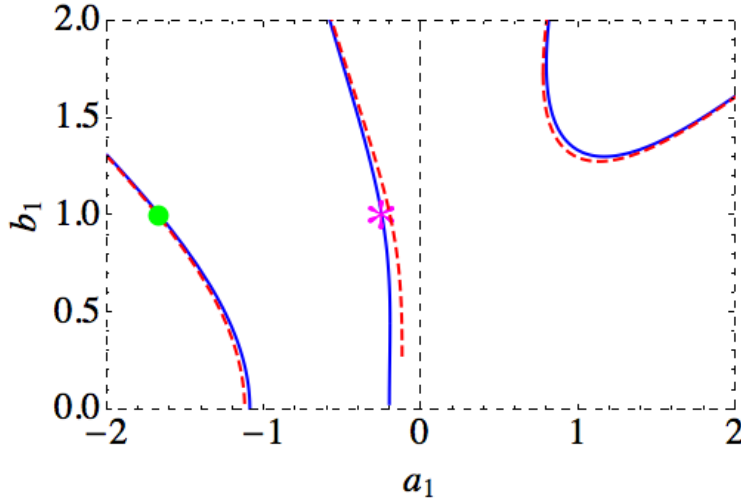


Figure 2.2 Locus of points on the complex plane for eigenvalues λ_1 and λ_2 . Second eigenvalue $\lambda_2 = -1/5 + i/2$ (solid blue line) and $\lambda_2 = -1/5 + i$ (red dashed line) are fixed while $\lambda_1 = a_1 + ib_1$ is arbitrary. For both λ_2 , $\delta = 1/64$. The magenta star on blue solid line is for $a_1 = -0.251892$ which is Fig. (2.3) and solid green dot on red dashed line $a_1 = -1.67702$ presented in Fig. (2.4)

Because of the quartic polynomials involved, the function $a_1(b_1, a_2, b_2)$ is omitted here, due to its complexity, and we take the opportunity to solve Eq. (2.16) numerically. Remarkably, the numerical solution reveals that there is a certain locus of parameters on the complex plane of eigenvalues λ_1 and λ_2 , where two solitons superpose, as shown in Fig. (2.2); it not always arbitrary. To get the locus, we can solve Eq. (2.16) in two ways; thus, we can solve for b_1 with specific values of λ_2 in terms of a_1 , or we can solve for a_1 with specific values of λ_2 in terms of b_1 . We choose the former way to solve this equation. Only these values of corresponding a_1 and b_1 satisfy the equal velocity condition ‘ $a_1 = a_2$ ’ on the complex plane. We do not get any aligned two soliton solutions outside of these loci for specific values of λ_2 . To be more specific,

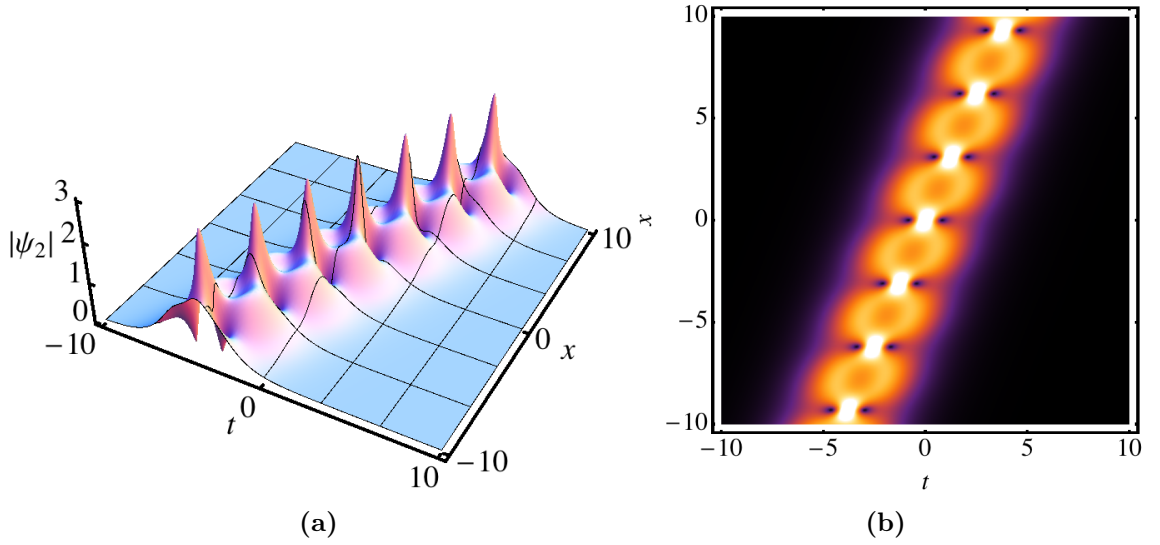


Figure 2.3 A second-order periodic soliton of the quintic equation as given by Eq. (2.14). Parameters are $\delta = 2^{-6}$, $\lambda_1 \approx -0.251892 + i$ and $\lambda_2 = -1/5 + i/2$, so as to depict beating between two solitons. Propagation direction is along the x -axis. (a) Surface diagram. (b) Density diagram.

here, in Fig. (2.2), the solid blue line is for $\lambda_2 = -1/5 + i/2$ and the red dashed line is for $\lambda_2 = -1/5 + i$ with arbitrary a_1 and b_1 and $\delta = 1/64$. At this point, if we solve Eq. (2.16) for $b_1 = 0.5$, we get four possible solutions for each set of parameters, two of which are generally real. As a_1 cannot have an imaginary part, by definition, only those two values are acceptable for attaining soliton alignment. The two real solutions are $a_1 = -1.66474, -0.251892$. A two-aligned-soliton solution is presented in Fig. (2.3) for $a_1 = -0.251892$. This same point is presented on the complex plane of the eigenvalue as magenta star. From the aligned soliton structure, Fig. (2.3), we can see that the second-order soliton in Eq. (2.14) becomes an angled beating structure, with the choice of a_1 typically determining whether the beating frequency is high or low. An example of the latter case is depicted in Fig. (2.3), comprised of a $b_1 = 1$ soliton coincident with a $b_2 = 1/2$ soliton. Correspondingly, the peaks that denote constructive interference have an amplitude of 3.

An interesting case now arises when considering the nonlinear superposition of two aligned solitons with equal amplitudes (i.e. $b_1 = b_2$). In the standard NLSE case, the alignment condition and this restriction require the eigenvalues to be exactly identical (i.e. $\lambda_1 = \lambda_2$). The Darboux process does not allow this and a ‘degenerate’ limit needs to be taken, the result of which will be detailed shortly. In contrast, the additional

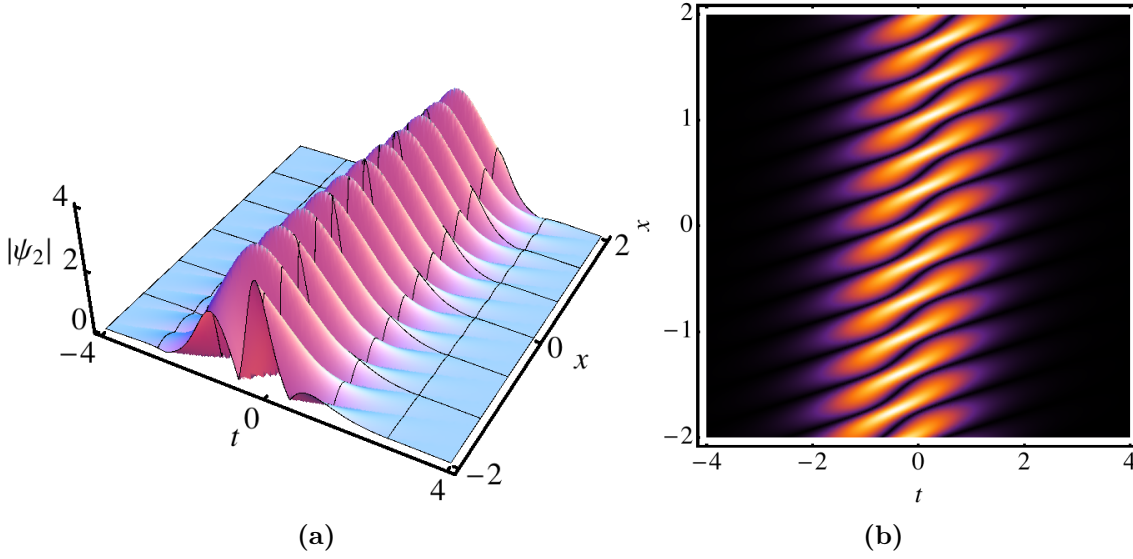


Figure 2.4 A second-order periodic soliton of the quintic equation as given by Eq. (2.14). Parameters are $\delta = 2^{-6}$, $\lambda_1 \approx -1.67702 + i$ and $\lambda_2 = -1/5 + i$, so as to depict beating between two equal-amplitude solitons. Propagation direction is along the x -axis. (a) Surface diagram. (b) Density diagram.

quintic operator allows for two ‘aligned’ scenarios on account of the two real solutions to Eq. (2.16). One is the upcoming degenerate case with $a_1 = a_2$, but the alternative involves $a_1 \neq a_2$. Indeed, for an arbitrary δ value, this is generally nothing more than a high beating-frequency second-order soliton, as shown in Fig. (2.4). However, we will soon discuss what happens as δ approaches zero.

2.4.4. Degenerate Two-Soliton Solution

Degenerate solutions are those that occur when two eigenvalues coincide [84]. However, the Darboux scheme does not give an explicit answer when eigenvalues are strictly equal, a result related to the requirement for asymmetry [106]. To surmount this, a $\lambda_2 \rightarrow \lambda_1$ limit must be applied to the general second-order soliton solution in Eq. (2.14). The result is

$$\psi_2(x, t) = -8ibe^{-2i\Omega} \frac{N_2}{D_2}, \quad (2.17)$$

$$\begin{pmatrix} N_2 &= & P_1 \sinh(2\Theta) - (1 + iP_2) \cosh(2\Theta) \\ D_2 &= & 1 + 2(P_1^2 + P_2^2) + \cosh(4\Theta) \\ P_1 &= & 2b \{t + 2x [a + 40\delta (a^4 - 6a^2b^2 + b^4)]\}, \\ P_2 &= & 4b^2x [1 + 160a\delta (a^2 - b^2)] \end{pmatrix}$$

with

$$\begin{aligned}\lambda &= a + ib, \\ \Omega &= at + x [a^2 - b^2 + 16a\delta(a^4 - 10a^2b^2 + 5b^4)], \\ \Theta &= b \{t + 2x [a + 8\delta(5a^4 - 10a^2b^2 + b^4)]\},\end{aligned}\tag{2.18}$$

where the subscripted index has been dropped from all parameters, as the soliton components are identical in eigenvalue. Given this expression, an example of a degenerate soliton is shown in Fig. (2.5). Both components in this image possess the same eigenvalue as the single first-order structure in Fig. (2.1a), which means that the two solitons also share the same velocity. It is thus unusual that the solitons seem spatio-temporally parallel instead of coincident, diverging from this behaviour only once to intersect at the origin. However, this degenerate structure is actually nothing more than an extreme case of a beating soliton. In fact, closer inspection of the wavefunction in Fig. (2.3) reveals ‘craters’ between every pair of peaks. It follows then that, as λ_2 approaches λ_1 and the beating frequency decreases, the points of constructive interference spread apart and the craters themselves stretch out. In the $\lambda_2 \rightarrow \lambda_1$ limit, only one intersection peak remains in the finite domain, leaving behind crater structures of infinite length that appear as parallel solitons. As a side note, it is also worth mentioning that a deep relationship exists between the peak of a second-order degenerate soliton and a first-order ‘rogue wave’ [169], but we leave that discussion limited here.

This degenerate case can be considered one way that the NLSE hierarchy ‘cheats’ the restriction on intersecting equal-amplitude solitons having the same velocity. As mentioned in Sec. (2.4.3), the other way is to have the two first-order solitons beat, generally with extremely high frequency. Indeed, as δ becomes substantially smaller, the periodicity of the resulting interference pattern seems to decrease. It follows that, in a numerically-applied $\delta \rightarrow 0$ limit, the envelope of the beating structure appears to approximate a first-order soliton with double amplitude, in a way that is similar to that of a $|\cos(\omega x)|$ envelope approaching the constant 1 for large ω values. In effect, the two solitons are attempting to linearly superimpose within the restrictions of nonlinear superposition, and this situation is shown in Fig. (2.6). The

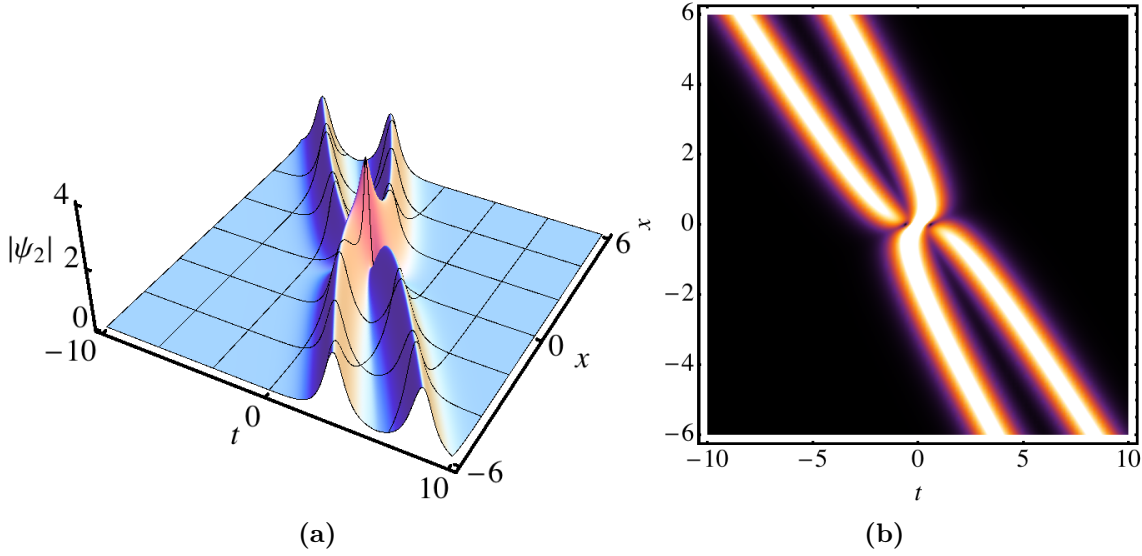


Figure 2.5 A two-soliton degenerate solution of the quintic equation as given by Eq. (2.17). Parameters are $\delta = 2^{-6}$ and $\lambda = 1 + i$ ($a = 1$ and $b = 1$). Propagation direction is along the x -axis. (a) Surface diagram. (b) Density diagram.

depicted oscillations in Fig. (2.6a) and (2.6b) are cross-sectional profiles along x and t , respectively, of such a wavefunction, the surface of which would be a higher-frequency version of Fig. (2.4). This structure is significant as it appears to have no analogue in the standard NLSE case. Presumably, when $\delta = 0$, the beating frequency becomes infinite and the solution can no longer exist.

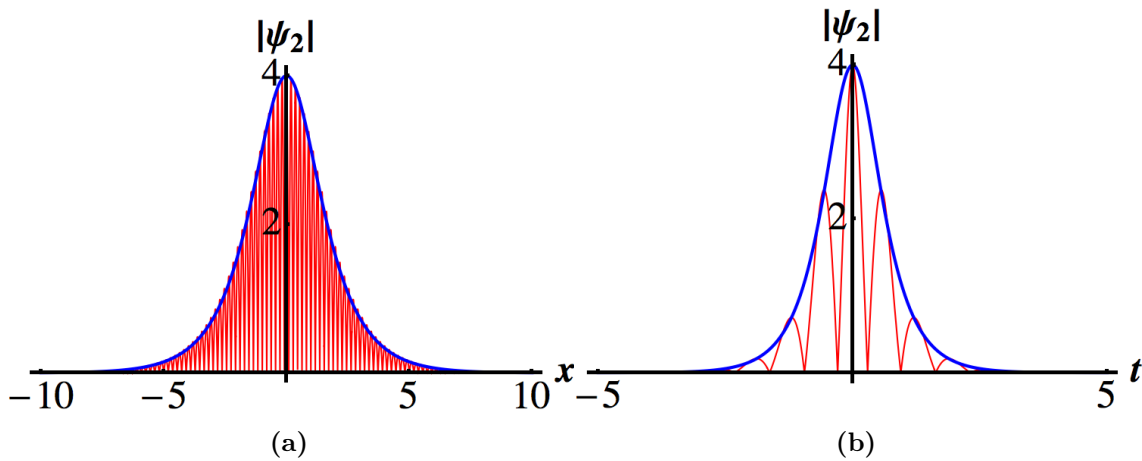


Figure 2.6 Cross-sectional cuts of both a double-amplitude first-order soliton (in blue) and a second-order equal-amplitude beating soliton (in red). Quintic parameter is $\delta = 2^{-12}$. Eigenvalue of double-amplitude soliton is $\lambda = -1/5 + i$. Eigenvalues of beating structure are $\lambda_1 \approx -4.75617 + i$ and $\lambda_2 = -1/5 + i$. (a) Cross-sections along x . (b) Cross-sections along t .

However, we do note here that the addition of the quintic operator alone may not be necessary for the existence of this new solution. Any terms that modify velocity in such a manner that Eq. (2.16) is left non-trivial should allow for the construction of this special case. This means that the combination of the standard NLSE and the Hirota operator should also allow for a beating structure composed of coincident equal-amplitude solitons.

2.4.5. Third-order Solitons

We do not present the explicit expression for a third-order soliton here due to its written length. Nonetheless, it is standard fare to generate these solutions via the recursive formulae of the Darboux scheme, as detailed in Eqs. (2.7) and (2.8). Consequently, a typical three-soliton collision is displayed in Fig. (2.7). As shifts are still ignored in this work, all components constructively interfere at the origin and attain a maximum amplitude of $\psi_3(0, 0) = 2(b_1 + b_2 + b_3)$, provided that $b_j \geq 0$ for all j . Given the component eigenvalues of the depicted wavefunction, the intersection peak of the structure reaches a height of 4.5.

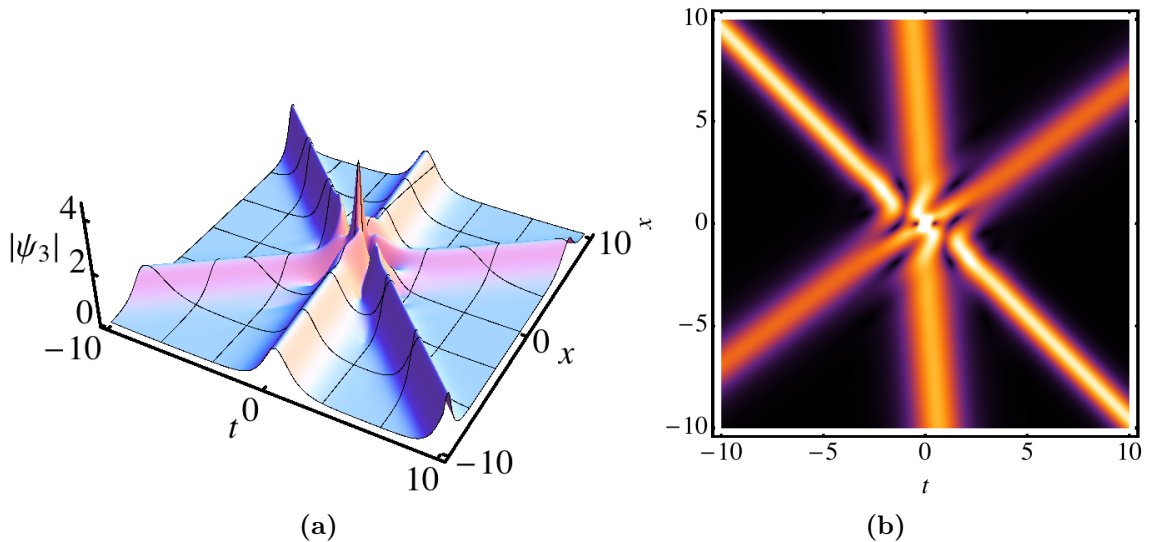


Figure 2.7 Three-soliton solution, with parameters $\delta = 2^{-6}$, $\lambda_1 = -1 + i/2$, $\lambda_2 = 3i/4$ and $\lambda_3 = 1 + i$. Propagation direction is along the x -axis. (a) Surface diagram. (b) Density diagram.

It is of course possible to align three solitons by use of the same strategy in Sec. (2.4.3). An example of a third-order beating structure is shown in Fig. (2.8),

where a_1 and a_3 were determined following the fixing of a_2 and b_2 . The wavefunction thus has the same velocity as the second-order solution in Fig. (2.3), but now features an additional component. With a value of $b_3 = 3/2$, this third soliton adds to the central peak via constructive interference, so as to reach a maximum amplitude of 6.

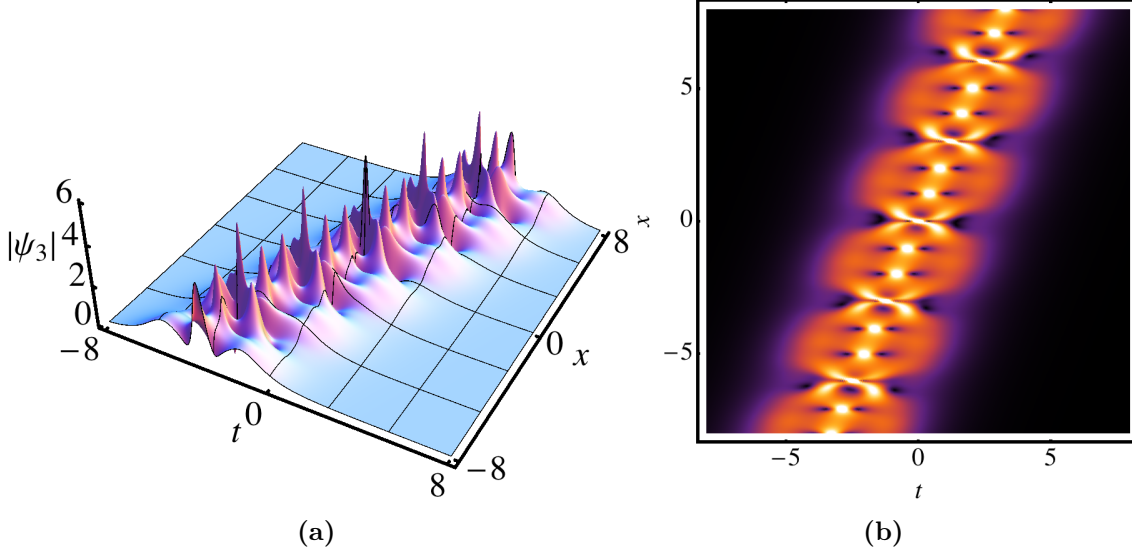


Figure 2.8 Third-order beating soliton, with parameters $\delta = 2^{-6}$, $\lambda_1 \approx -0.251892 + i$, $\lambda_2 = -1/5 + i/2$ and $\lambda_3 \approx -0.400809 + 3i/2$. Propagation direction is along the x -axis. (a) Surface diagram. (b) Density diagram.

As previously hinted, the points of intersection for an order n soliton are actually order $n - 1$ structures from a so-called ‘breather’ family of solutions [169]. Indeed, a chain of first- and second-order rogue waves are clearly visible in Fig. (2.8b). It should also be noted that, while the appearance of a first-order breather in the intersection of two beating solitons is periodic by definition, as shown in Fig. (2.3), the regularity of this second-order breather chain in Fig. (2.8) appears to be gradually breaking down away from the origin. Indeed, a more arbitrary selection of b_j values makes this explicit in Fig. (2.9). Points of intersection adjacent to and near the central fused peak take the form of non-uniform second-order rogue wave triplets [102], as best visualised in Fig. (2.9a). The implication here is that there is a low likelihood of any new intersection down the line ever fusing perfectly again.

It follows that structural periodicity is difficult to attain for a third-order soliton. In the case of second-order breathers, modulation frequencies need to be in integer ratio for this to occur [106]. Similarly, to achieve this type of regularity with three

coincident solitons, it is likely that the beating frequency between components 1 and 2 needs to be in integer ratio with the beating frequency between components 2 and 3. To complicate matters, the quintic operator generally twists solitonic interference patterns with respect to the path along which they occur, as is most apparent with the angled striations in Fig. (2.4b). This means that the spatio-temporal direction of solitonic phase oscillation in Eq. (2.12) needs to be projected onto the spatio-temporal direction in which the soliton travels, with its velocity given by Eq. (2.11). Simply speaking, the phase exponent and the group velocity of the solitons for higher-order equations are not the same.

As this is somewhat involved, we do not pursue the issue further here. It is worth noting though that the components of the structure in Fig. (2.9) possess similar eigenvalues. Accordingly, the beating frequencies are small and the solitons appear to intersect less often than in the case of Fig. (2.8). Moreover, from a localized point of view, the shape of a degenerate third-order soliton is evident. Three components with parallel appearance are expected, colliding only once, so as to give a ‘pitchfork’ arrangement on either side of the origin.

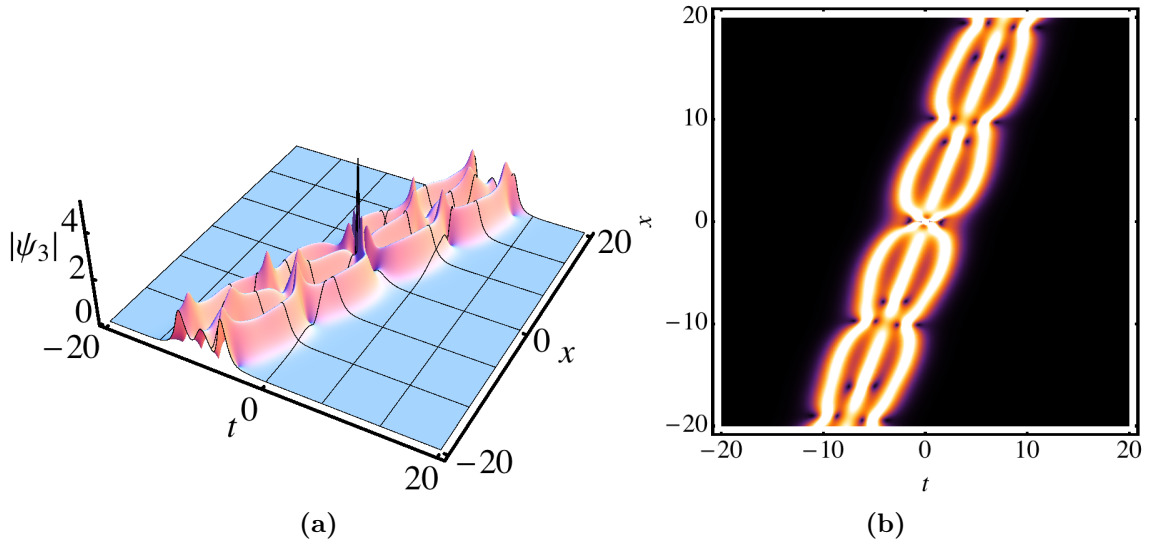


Figure 2.9 Third-order beating soliton, with parameters $\delta = 2^{-6}$, $\lambda_1 \approx -0.244981 + i$, $\lambda_2 = -1/5 + i/\sqrt{2}$ and $\lambda_3 \approx -0.219388 + i\sqrt{3}/2$. Propagation direction is along the x -axis. (a) Surface diagram. (b) Density diagram.

2.5. Conclusion

We have studied the fifth-order nonlinear equation in the NLSE hierarchy. Its integrability is a corollary of its ability to be expressed in Lax pair form, while the system itself arises naturally when the zero-curvature equation is applied to both a 2-by-2 AKNS matrix and a corresponding matrix that is quintic in eigenvalue. Accordingly, we have provided explicit expressions for the Lax pair of this equation.

By using the Darboux scheme, we have then derived the basic one-soliton solution and have nonlinearly superimposed two such components into a general eigenvalue-dependent structure. Appropriate choice of parameters allows for the alignment of individual solitons, despite amplitude-dependence arising with velocities due to the additional quintic NLSE operator. Consequently, we have examined so-called second-order beating structures and their extreme cases. In the limit of zero beating frequency, the second-order degenerate soliton forms with only one intersection at the origin, despite both components sharing the same velocity. Notably, the quintic operator also allows for a high beating-frequency wavefunction composed of equal-amplitude solitons, which does not appear to have an analogue in the standard NLSE. Furthermore, this investigation has involved higher-order soliton structures that are conveniently constructed with the Darboux procedure. It is evident that the peaks formed from solitonic intersection in these cases resemble a related breather family of solutions. Subsequently, it is indicated that an integer ratio between beating frequencies is key to ensuring periodic regularity of the resulting structures.

CHAPTER 3

Conversion of an Akhmediev Breather into a Soliton

3.1. Introduction

We analyse the quintic integrable equation of the nonlinear Schrödinger hierarchy. It includes fifth-order dispersion with matching higher-order nonlinear terms Eq. (1.17). Preliminary studies and a Darboux transformation technique for this equation have been given in the work [170]. The same Lax pair and systematic Darboux transformation for Eq. (1.17) are given in the previous chapter (2) in Sec. (2.2) and in Sec. (2.3). We show that a breather solution of this equation can be converted into a non-pulsating soliton solution on a background. We calculate the locus of the eigenvalues on the complex plane which convert breathers into solitons. This transformation does not have an analogue in the standard nonlinear Schrödinger equation. We also study the interaction between the new type of solitons, as well as between breathers and these solitons.

Preliminary analysis of this equation and its soliton solutions has been presented in [170]. Here, we present first, second and third-order breather solutions of this equation, and the possibility of transforming them into continuous (non-periodic) solitons on a background. The existence of such solutions is an unexpected feature

of breathers. Various examples of collisions of these solitons and their interactions with ordinary breathers are given.

3.2. Breather Solution with Complex Eigenvalue

The periodicity plays an important role in the dynamics of the breather solution. The addition of higher-order odd terms to the NLSE results in oblique propagation of waves. Effectively, these terms change the velocity of the waves. The velocity change is proportional to the coefficient δ in the equation. Thus, a breather is subjected to re-orientation in the (x, t) -plane. Below, we will explicitly demonstrate how higher-order terms, specifically, the ‘quintic’ operator, change the overall profile of breather solutions. The details of the mathematical calculations are given in Appendix (3.6). To begin with, we present the first-order breather solution with complex eigenvalue $\lambda = a + ib$:

$$\psi_1 = \left[1 + 2b \frac{G_1 + iH_1}{D} \right] e^{ix}. \quad (3.1)$$

Here the functions G_1 , H_1 and D are combinations of trigonometric and hyperbolic functions:

$$\begin{aligned} G_1 &= \cos(xV_T + t\kappa_r) \cosh(2\chi_i) - \cosh(xV_H + t\kappa_i) \sin(2\chi_r), \\ H_1 &= \cos(2\chi_r) \sinh(xV_H + t\kappa_i) + \sin(xV_T + t\kappa_r) \sinh(2\chi_i), \\ D &= \cosh(xV_H + t\kappa_i) \cosh(2\chi_i) - \cos(xV_T + t\kappa_r) \sin(2\chi_r), \end{aligned}$$

with

$$\begin{aligned} \kappa &= 2\sqrt{1 + \lambda^2} = \kappa_r + i\kappa_i, \\ \chi &= \frac{1}{2} \cos^{-1} \left(\frac{\kappa}{2} \right) = \chi_r + i\chi_i, \end{aligned}$$

and

$$\begin{aligned} \Omega &= 16ab(1 - 4a^2 + 4b^2) \\ \mathcal{U} &= 2[3 + 8a^4 + 4b^2 + 8b^4 - 4a^2(1 + 12b^2)] \\ V_T &= -b\kappa_i + a\kappa_r + \delta(\Omega\kappa_i + \mathcal{U}\kappa_r) \\ V_H &= a\kappa_i + b\kappa_r + \delta(\mathcal{U}\kappa_i - \Omega\kappa_r) \end{aligned}$$

This solution, for a fixed set of parameters, $\lambda = 0.08 + 0.9i$ and $\delta = \frac{1}{64}$, is illustrated

in Fig. (3.1a). As we will see in the next section, the actual value of the parameter δ does not change the effects that we observe qualitatively. It has to be sufficiently small in order to reflect the smallness of the higher-order terms in the quintic NLSE. Just as in the NLSE case, the imaginary part of the eigenvalue λ is responsible for the period of the breather along the t -axis. This real part a of the eigenvalue is equivalent to the velocity in the case of a soliton and thus induces a rotation of the total pulse train in the (x, t) -plane. The quintic terms controlled by the parameter δ introduce skewing of each individual peak profile in the breather. In the limiting case $\delta = 0$, Eq. (1.19) reduces to the NLSE and the solution Eq. (3.1) reduces to the breather solution of the NLSE with a complex eigenvalue. As the equation we are dealing with is of fifth order with higher-order derivative terms in it, the solutions are extremely sensitive to the coefficient δ . Therefore, we choose a fairly small δ , e.g. $\frac{1}{16}$ or $\frac{1}{64}$, to generate visible effects in the solution.

3.3. Breather-to-Soliton Conversion of the Quintic NLSE Solutions

In the NLSE case, Kuznetsov-Ma solitons ($b > 1$) and Akhmediev breathers ($0 < b < 1$) can be transformed into each other by a change of the eigenvalue λ . Solutions stay periodic unless the period itself is taken to infinity; this occurs for $\lambda = i$, i.e. $a = 0$ and $b = 1$. In this case, each of these solutions is transformed into a Peregrine rational solution [171]. There are no solutions that describe continuous non-periodic soliton propagation similar to standard solitons on a zero background. The situation changes when we add the quintic terms into equation. Then, specific values of the eigenvalue may transform a breather into a solution which is similar to an ordinary soliton solution. An example of such solution is shown in Fig. (3.1b). Below, we refer to all solitary waves which continuously propagate in t as “solitons”. In terms of collisions, they have the same properties as ordinary solitons of the NLSE. As will be shown in the following sections, they maintain the same velocity after a collision as they had before the collision. However, we should keep in mind that the solutions are on a constant background. The plane wave background is modulationally unstable

and its distortion may distort the solutions formed on top of it. A complete study of the stability of such composite solutions is an issue that requires special investigation.

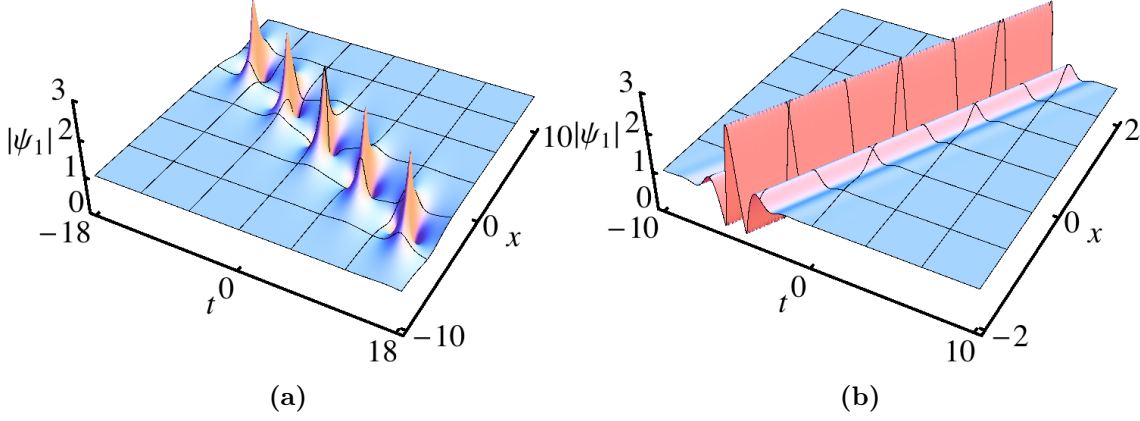


Figure 3.1 (a) First order quintic breather solution given by Eq. (3.1) with complex eigenvalue $\lambda = 0.08 + 0.9i$ and $\delta = \frac{1}{64}$. (b) A breather transformed into a soliton. This solution is given by Eq. (3.1), with the eigenvalue $\lambda = -1.34329 + 0.9i$. The propagation direction is x .

A transformation of a quintic NLSE breather, Eq. (3.1), into a continuous soliton occurs when we take

$$\frac{V_H}{\kappa_i} = \frac{V_T}{\kappa_r}. \quad (3.2)$$

In this case, the extrema of ‘cosine’ and ‘cosh’ functions in Eq. (3.1) are located along the same straight lines in the (x, t) -plane causing transformation of the breather into a continuous soliton. Solutions of Eq. (3.2) can be represented on the complex plane (a, b) for any particular δ . After simple calculation, we find that the condition of Eq. (3.2) can be reduced to a polynomial equation in a and b :

$$16a\delta(1 - 4a^2 + 4b^2) = 1, \quad (3.3)$$

i.e. $\Omega\delta = b$. For an arbitrary real parameter δ , Eq. (3.3) defines curves on the complex plane of λ , so that any point on these curves corresponds to a breather-to-soliton conversion. The easiest way to plot these curves is to take a as the independent variable and find the solution for b :

$$b = \pm \sqrt{a^2 - \frac{1}{4} + \frac{1}{64a\delta}}. \quad (3.4)$$

Curves calculated for positive b for three different values of δ are presented in Fig. 3.2.

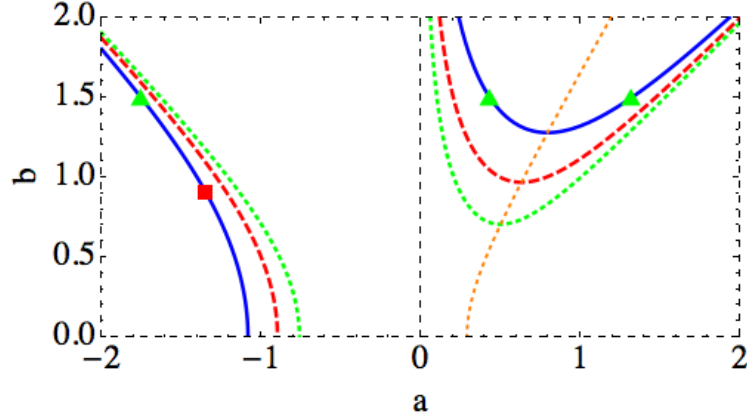


Figure 3.2 Solutions of Eq. (3.3) on a complex plane of λ for (blue solid curves) $\delta = 1/64$, (red long-dashed curves) $\delta = 1/32$ and (green dotted curves) $\delta = 1/16$. When $a > 0$, part of a branch moves into the region of breathers ($b < 1$) when $\delta > \frac{3}{80}\sqrt{\frac{3}{5}} \approx 0.029\dots$. The large (red rectangle and green triangle) points on solution curves are the specific points where breathers become solitons. The (orange dotted) curve is the locus of the bifurcation points when δ changes (see Eq. (3.5)).

Alternatively, we can take b as the independent variable and find the possible values of a by solving a cubic algebraic equation Eq. (3.4). There are three branches of solution if b is above the bifurcation point:

$$a(1) = \frac{-8 \times 3^{1/3} (4b^2 + 1) \delta^2 - 2^{1/3} m^{2/3}}{4 \times 6^{2/3} \delta m^{1/3}},$$

$$a(2) = \frac{8 (\sqrt{3} + 3i) (4b^2 + 1) \delta^2 + 2^{1/3} 3^{1/6} (1 - i\sqrt{3}) m^{2/3}}{8 \times 2^{2/3} 3^{5/6} \delta m^{1/3}},$$

and

$$a(3) = \frac{8 (\sqrt{3} - 3i) (4b^2 + 1) \delta^2 + 2^{1/3} 3^{1/6} (1 + i\sqrt{3}) m^{2/3}}{8 \times 2^{2/3} 3^{5/6} \delta m^{1/3}},$$

where $m = \delta^2 \left(\sqrt{81 - 768 (4b^2 + 1)^3 \delta^2} + 9 \right)$. Only one of these solutions, namely $a(1)$, is real if b is below the bifurcation point. At the bifurcation point, the value of b is:

$$b_0 = \frac{1}{2} \sqrt{\frac{3}{4(2\delta)^{2/3}} - 1}.$$

This is the point where $db/da = 0$ from Eq. (3.4), and also $a(2) = a(3)$. The value of a at this point is:

$$a_0 = \frac{1}{4(2\delta)^{1/3}}.$$

The curve giving the locus of bifurcation points, as δ changes, is

$$3a_0^2 - b_0^2 = \frac{1}{4}. \quad (3.5)$$

For the particular value of $\delta = 1/64$, the bifurcation point is given by $a_0 = 0.793701$ and $b_0 = 1.28058$. This is the lowest point on the (blue) right-hand-side solid curve in Fig. (3.2). To give an example, using the above solutions, we find that, for the same imaginary part as in Fig. (3.1a), i.e. $b = 0.9$, and for the same $\delta = 1/64$, the real part of the eigenvalue is $a \approx -1.34329....$ This point is indicated by the lower solid red rectangle on the left-most (blue) branch of the curves shown in Fig. (3.2). For this point (λ) on the complex plane, a breather becomes a continuous soliton on a background. This soliton is shown in Fig. (3.1b). For a better understanding of this phenomenon, let us start with a simple case. When $a = 0$, the velocity is zero for the entire breather. The breather is parallel to the t axis, while individual peak profiles can be skewed when δ is nonzero. An example is shown in Fig. (3.3). Here, in order to see more clearly the skewed position of each component, we have chosen a higher value, *viz.* $\delta = 1/16$.

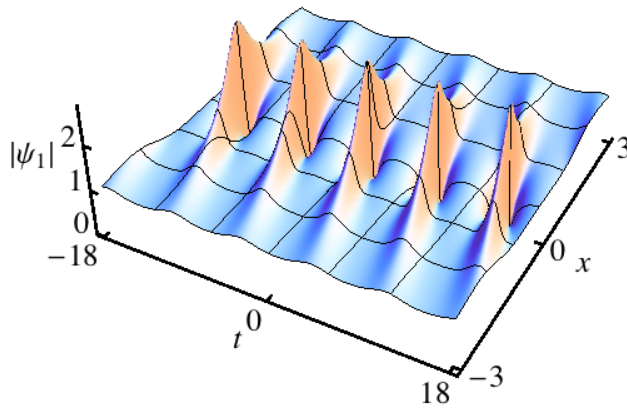


Figure 3.3 A breather given by Eq. (3.1) with the eigenvalue $\lambda = 0 + 0.9i$ and $\delta = 1/16$. The propagation direction is x .

Deviations of ‘ a ’ from zero change the angle at which the breather as a whole is positioned relative to the t -axis. These deviations can be positive or negative. When this angle is equal to the angle of skewness of each peak, the breather becomes a soliton. For the parameters $b = 0.9$ and $\delta = 1/64$ this happens when $a = -1.34329....$

This is exactly the case shown in Fig. (3.1b). Here, only one central peak of the periodic sequence remains. It becomes infinitely stretched along the direction of propagation. This stretched pulse appears as a soliton, as can be seen from Fig. (3.1b). In this example, the soliton profile is symmetric. However, introducing translations which change the phase of the trigonometric functions may change the shape into an asymmetric or double-peak profile. The cosine function provides oscillations of the envelope across the soliton, thus resulting in the oscillating tails in the orthogonal direction. Consequently, the soliton has tail oscillations in the form of a modulation of the background, as is clearly visible in Fig. (3.1b). These oscillations form a distinctive feature of the solution, being different from the features of ordinary solitons of the NLSE on a zero background. When $\delta \rightarrow 0$, Eqs. (3.2) and (3.3) do not have any solutions, showing that the NLSE does not allow conversion of breathers into new types of solitons. Only higher-order equations of the NLSE hierarchy admit such solutions. All of our examples are presented for the value of $\delta = 1/64$ (except for the one in Fig. (3.3)). This is a representative case that allows us to illustrate new solutions in the best way. Higher values of δ do not bring any new features in comparison to those shown in the plots given in this work.

3.3.1. Second-order Breather-to-Soliton Conversions

If a first-order solution admits a breather to soliton transformation, we can expect similar transformations occurring in second-order solutions. Indeed, all the higher-order solutions are nonlinear superpositions of first-order solutions. Thus, if one or two components are transformed into solitons, we should obtain patterns similar to soliton interactions. For the second-order solution, we need two complex eigenvalues, $\lambda_j = a_j + ib_j$, where $j = 1, 2$. Details of the calculation are given in Appendix (3.6). We use the same representation of the Lax-pair and the Darboux scheme for the quintic NLSE, Eq. (1.19), as in our previous chapter (2). Fig. (3.4a) shows an example, with $\delta = 1/64$, of a two-breather collision. In the particular case $\delta = 0$, the solution reduces to the two-breather solution of the NLSE given previously in Ref. [75] by equation (22) of Appendix B. For comparison, the two-breather collision described by the NLSE can be found in Fig.10 of [75].

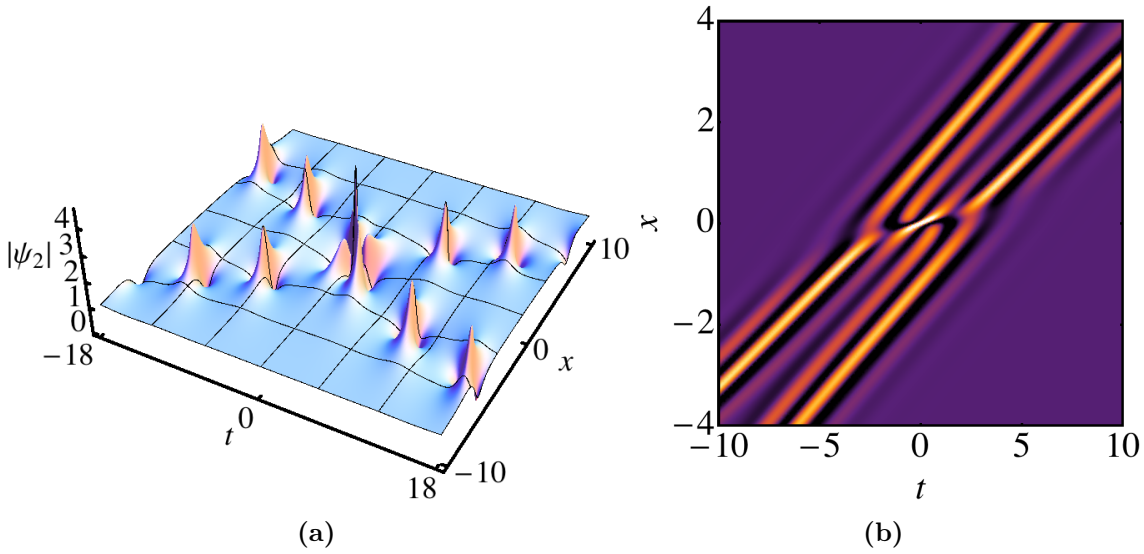


Figure 3.4 (a) Two-breather solution with complex eigenvalues $\lambda_1 = 0.05 + 0.9i$ and $\lambda_2 = -0.05 + 0.9i$, with $\delta = \frac{1}{64}$. Here x is the propagation direction. (b) Contour plot of a second order breather-to-soliton transformation. Here, the eigenvalues are $\lambda_1 = -1.34329 + 0.9i$ and $\lambda_2 = -1.2903 + 0.8i$, while $\delta = \frac{1}{64}$.

In order to convert a two-breather solution into a two-soliton solution, we choose two arbitrary points $\lambda_1 = (a_1, b_1)$ and $\lambda_2 = (a_2, b_2)$ on the curves in Fig. (3.2). In the following numerical example, we take, as the first point $\lambda_1 = -1.34329 + 0.9i$, as already used in the previous section. Then, we take as the second point λ_2 with $b_2 = 0.8$ and $a_2 = -1.2903$. We have chosen λ_2 different from λ_1 . Equality would give a degenerate case which introduces additional complications which we do not want at the moment. An example of a breather collision converted to a soliton collision with these eigenvalues is shown in Fig. (3.4b). This plot is indeed a two-soliton collision with the collision point located at the origin. The two solitons propagate in slightly different directions, due to the differences in λ . Each soliton has an oscillating tail, thus creating a complicated collision pattern.

We have also the choice of converting only one of the components into a soliton, while allowing the other component to remain as a breather. Then we choose only one λ located on a curve in Fig. (3.2), while the second λ can be chosen arbitrarily on the complex plane. An example of a mixed collision of a breather and a soliton is shown in Fig. (3.5). The relative positions of the soliton and the breather can be controlled, not only by the eigenvalues, but also by translations of each component and changes in the phase of the breather. We omitted these from our calculations

for simplicity. The highest amplitude, $|\psi_2|$, reached in the collision is $|\psi_2(0, 0)| = 4.4$. It is the same in both Fig. (3.4b) and Fig. (3.5). In our examples, the maximum is located at the origin.

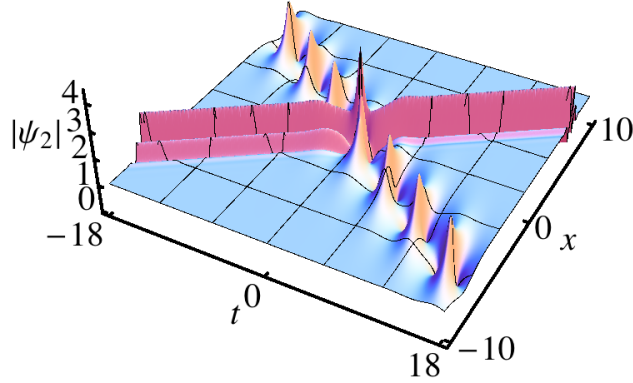


Figure 3.5 Collision between a breather with $\lambda_2 = 0.2 + 0.8i$ and a soliton with $\lambda_1 = -1.34329 + 0.9i$, when $\delta = \frac{1}{64}$.

3.3.2. Higher-order Solutions

Continuing the process of constructing higher-order breather-to-soliton converted solutions requires additional eigenvalues, λ_j , located on any of the curves in Fig. (3.2). For example, in order to build a three-breather solution converted to a three-soliton solution, we have to choose three points on the curves in Fig. (3.2) or, equivalently, solve algebraic Eq. (3.3) for three different eigenvalues. To be specific, if we have a_1 with b_1 , and a_2 with a different b_2 , we should generate one more a_3 with a different b_3 . The choice of all three of these specific eigenvalues converts the three-breather solution into a three-soliton solution. Again, none of the eigenvalues can be identical if we want to avoid degenerate cases. The latter requires special consideration [84,106] that is not presented here.

An example of a three-soliton collision built this way is shown in Fig. (3.6a). This plot is constructed for three complex eigenvalues, $\lambda_1 = -1.3432 + 0.9i$, $\lambda_2 = -1.2903 + 0.8i$ and $\lambda_3 = -1.31616 + 0.85i$. The highest amplitude at the point of collision here is $\psi_3(0, 0) = 6.1$. For a fourth-order soliton solution, we choose one more eigenvalue, λ_4 with $b_4 = 0.82$ and $a_4 = -1.30051$, located on a curve in

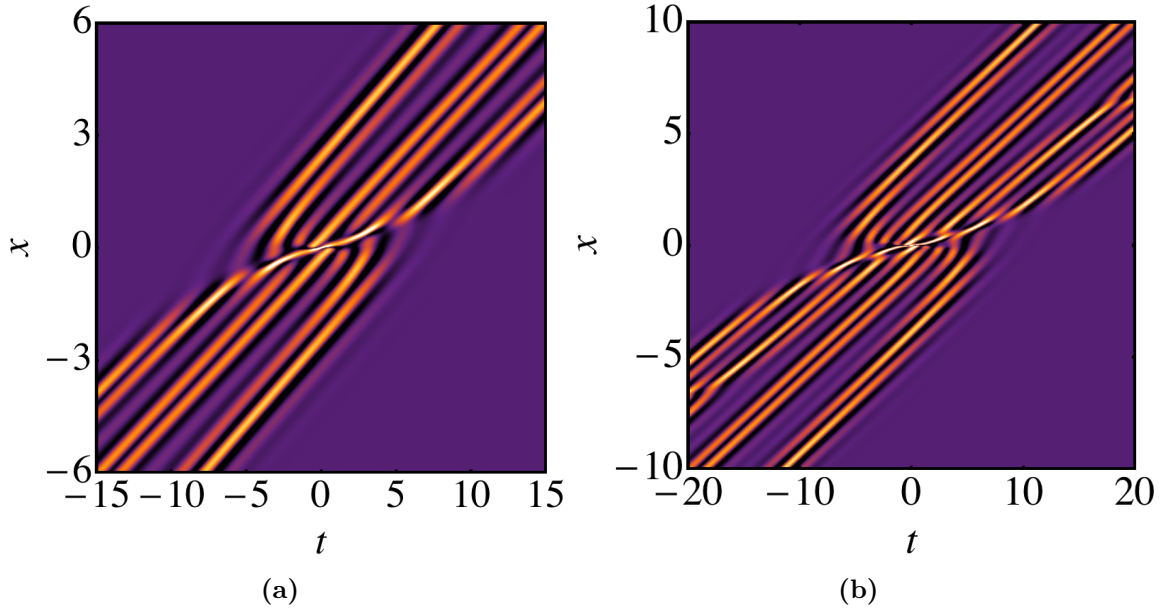


Figure 3.6 (a) Contour plot of a third-order breather-to-soliton converted solution with three eigenvalues $\lambda_1 = -1.3432 + 0.9i$, $\lambda_2 = -1.2903 + 0.8i$ and $\lambda_3 = -1.31616 + 0.85i$, with $\delta = \frac{1}{64}$. (b) Contour plot of a fourth-order breather converted to four-soliton collision for $\lambda_1 = -1.3432 + 0.9i$, $\lambda_2 = -1.2903 + 0.8i$, $\lambda_3 = -1.3161 + 0.85i$ and $\lambda_4 = -1.30051 + 0.82i$ with $\delta = \frac{1}{64}$

Fig. (3.2). Using the previous three eigenvalues, and this fourth eigenvalue, we can produce a fourth-order soliton solution from the four-order breather solution. It is presented in Fig. (3.6b).

3.4. Kuznetsov-Ma Breather Conversion to Soliton

In the previous sections, we mainly presented the conversion of Akhmediev breathers to soliton solutions for the parameter range $0 < b < 1$. Apart from Akhmediev breathers, the conversion scenario also holds for Kuznetsov-Ma solitons when $b > 1$. In this case, solving Eq.(3.3) leads to three real solutions for a . For given $b_{1,2,3} = 1.5$, the solutions are $a_1 = -1.75233$, $a_2 = 0.43232$, and $a_3 = 1.32001$. These eigenvalues are shown in Fig. (3.2) by the large green triangle located at $b = 1.5$ one on left branch (solid blue line) other two on right branch (solid blue line). Each eigenvalue produces a continuous soliton. For example, the choice $a = a_2 = 0.43232$ produces the soliton which is shown in Fig. (3.7). This soliton has only one, infinitely stretched, peak. Two of these solitons propagate to the right while the one with $a = a_2 = 0.43232$

propagates to the left. Tail oscillations are damped for the latter case, and only one zero remains on each side of the soliton. This is clearly seen in Fig. (3.7). For lower values of δ , the right-hand-side branch of the solutions in Fig. (3.2) moves down and three eigenvalues can exist, even for the case $b < 1$.

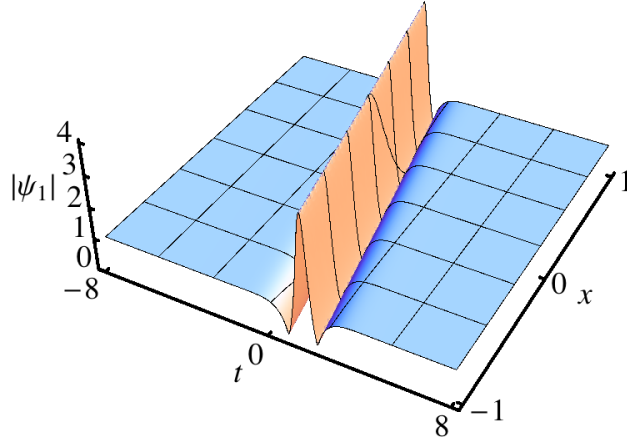


Figure 3.7 A Kuznetsov-Ma breather to soliton conversion for $\lambda = 0.43232 + 1.5i$ with $\delta = \frac{1}{64}$.

Solitons of the right-hand-side branch can collide with each other, or with solitons of the left-hand-side branch. An example of a collision of two solitons of the right-hand-side branch is shown in Fig. (3.8a). Comparing this case with the plot in Fig. (3.6a), we can notice that, firstly, the solitons propagate in a direction different from the one in Fig. (3.6a). Secondly, the oscillatory tails are less visible. These features are in accordance with the properties of single solitons discussed above. The highest amplitude of the collision here is 7.2 at $(x, t) \rightarrow (0, 0)$. An example of a three-soliton collision is shown in Fig. (3.8b). All eigenvalues here also belong to the right-hand-side branch of eigenvalues in Fig. (3.2).

3.5. Conclusion

We have derived first-order breather solutions of the quintic NLSE, and have shown that they can be converted into solitons. The reasons and the conditions for conversion have been given explicitly. We calculated the locus of eigenvalues on the complex plane which convert the breathers into solitons. The solitons with these eigenvalues

can collide as normal solitons on a zero background. We have provided several illustrations of such special solitons and patterns of their collisions.

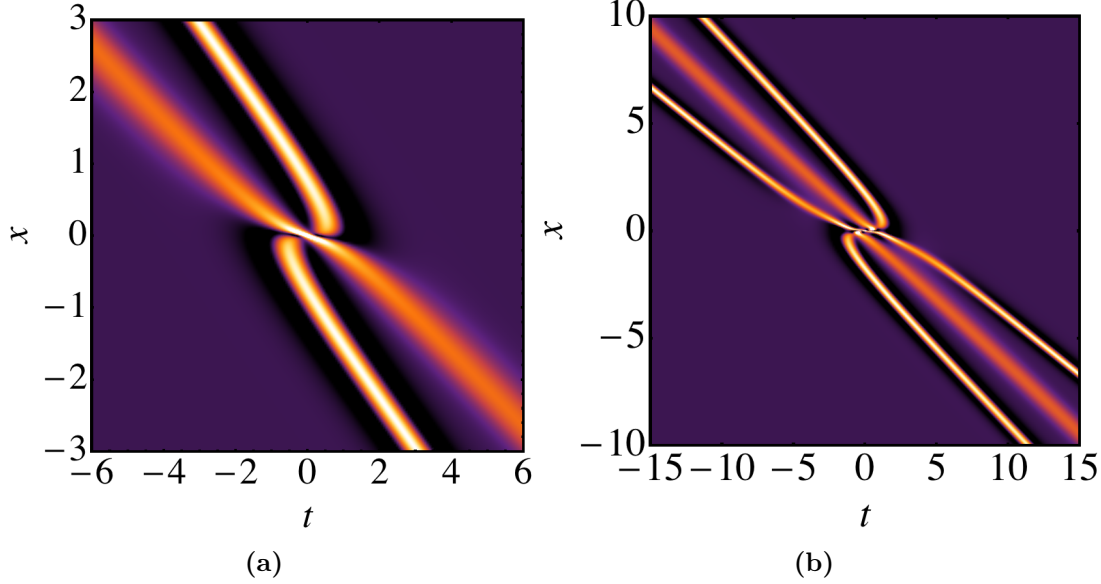


Figure 3.8 (a) Collision of two solitons with eigenvalues $\lambda_1 = 0.43232 + 1.5i$ and $\lambda_2 = 0.37457 + 1.6i$, when $\delta = \frac{1}{64}$ (b) Three-soliton collision, where the eigenvalues are on a right-hand-side branch in Fig. (3.2), $\lambda_1 = 0.43232 + 1.5i$ and $\lambda_2 = 0.37457 + 1.6i$, $\lambda_3 = 0.413193 + 1.53i$. Here, $\delta = \frac{1}{64}$.

3.6. Appendix: Second-order Quintic Breather Solutions with Complex Eigenvalues

Here, we follow our previous calculations, given explicitly in Appendices A and B of [75], except that the notation is adjusted to the present work. We separate the real and imaginary parts of κ and χ , so that $\kappa = \kappa_{jr} + i\kappa_{ji}$ and $\chi = \chi_{jr} + i\chi_{ji}$. We can express the linear r and s functions of [75] either in terms of exponential functions or trigonometric functions. To be specific, we will follow the latter case. Then $\lambda_j = a_j + ib_j$, $\kappa_j = 2\sqrt{1 + \lambda_j^2}$, and $\chi_j = \cos^{-1}(\frac{\kappa_j}{2})$. We define $\kappa_{jr} = \Re(\kappa_j)$, $\kappa_{ji} = \Im(\kappa_j)$, $\chi_{jr} = \Re(\chi_j)$, $\chi_{ji} = \Im(\chi_j)$. Then linear functions r and s are:

$$r_{1,1} = 2ie^{-ix/2} \sin(G), s_{1,1} = 2e^{ix/2} \cos(H).$$

We can split the argument of the trigonometric functions as $G = A_r + iA_i$ and

$H = B_r + iB_i$, where the real and imaginary parts are:

$$\begin{aligned} A_r &= \chi_{1r} + \frac{1}{2}(\kappa_{1r}t + d_{1r}x) - \frac{\pi}{4} \\ A_i &= \chi_{1i} + \frac{1}{2}(\kappa_{1i}t + d_{1i}x) \\ B_r &= -\chi_{1r} + \frac{1}{2}(\kappa_{1r}t + d_{1r}x) - \frac{\pi}{4} \\ B_i &= -\chi_{1i} + \frac{1}{2}(\kappa_{1i}t + d_{1i}x) \end{aligned}$$

Here $d_1 = d_{1r} + id_{1i}$ is the coefficient of x in the argument. Unlike the case of the NLSE, for the quintic equation d_{1r} and d_{1i} can be written in the form:

$$\begin{aligned} \Omega_1 &= 16a_1b_1(1 - 4a_1^2 + 4b_1^2) \\ \mathcal{U}_1 &= 2[3 + 8a_1^4 + 4b_1^2 + 8b_1^4 - 4a_1^2(1 + 12b_1^2)] \\ d_{1r} &= -b_1\kappa_{1i} + a_1\kappa_{1r} + \delta(\Omega_1\kappa_{1i} + \mathcal{U}_1\kappa_{1r}) \\ d_{1i} &= a_1\kappa_{1i} + b_1\kappa_{1r} + \delta(\mathcal{U}_1\kappa_{1i} - \Omega_1\kappa_{1r}) \end{aligned}$$

Thus we obtain the solution:

$$\psi = e^{ix} \left[1 + \frac{8}{D_1} ib_1 \cosh(B_i - iB_r) \sinh(A_i + iA_r) \right] \quad (3.6)$$

where $D_1 = \cos(2B_r) + \cosh(2A_i) + \cosh(2B_i) - \cos(2A_r)$. The form of this solution looks similar to Eq.(18) of Appendix A in [75]. However, the arguments of the functions are quite different. Similarly, for the second order solution, the linear functions $r_{1,2}$ and $s_{1,2}$ are

$$r_{1,2} = 2ie^{-ix/2} \sin(C), \quad s_{1,2} = 2e^{ix/2} \cos(D),$$

where

$$\begin{aligned} d_2 &= d_{2r} + id_{2i} \\ C_r &= \chi_{2r} + \frac{1}{2}(\kappa_{2r}t + d_{2r}x) - \frac{\pi}{4} \\ C_i &= \chi_{2i} + \frac{1}{2}(\kappa_{2i}t + d_{2i}x) \\ D_r &= -\chi_{2r} + \frac{1}{2}(\kappa_{2r}t + d_{2r}x) - \frac{\pi}{4} \\ D_i &= -\chi_{2i} + \frac{1}{2}(\kappa_{2i}t + d_{2i}x). \end{aligned}$$

Here d_2 is the coefficient of x in the argument of the cos and sin functions in $r_{1,2}$ and $s_{1,2}$, while the arguments are separated into real and imaginary parts: $C = C_r + iC_i$

and $D = D_r + iD_i$. Now, in order to derive the second order quintic breather solution, we finally need to derive $r_{2,1}$ and $s_{2,1}$ in terms of $r_{1,1}$, $s_{1,1}$, $r_{1,2}$ and $s_{1,2}$. The coefficient of x , which is d_2 , will have the same set of variables Ω_2 and $\mathcal{U}_2$ in its real and imaginary parts (d_{2i}, d_{2r}) , as in the first-order solution:

$$\begin{aligned}\Omega_2 &= 16a_2b_2(1 - 4a_2^2 + 4b_2^2) \\ \mathcal{U}_2 &= 2[3 + 8a_2^4 + 4b_2^2 + 8b_2^4 - 4a_2^2(1 + 12b_2^2)] \\ d_{2r} &= -b_2\kappa_{2i} + a_2\kappa_{2r} + \delta(\Omega_2\kappa_{2i} + \mathcal{U}_2\kappa_{2r}) \\ d_{2i} &= a_2\kappa_{2i} + b_2\kappa_{2r} + \delta(\mathcal{U}_2\kappa_{2i} - \Omega_2\kappa_{2r}).\end{aligned}$$

On taking these variables into account, the new linear functions $r_{2,1}^*$ and $s_{2,1}$ are:

$$\begin{aligned}r_{2,1}^* &= -\frac{2}{D_1}e^{ix/2}(4ib_1 \cosh(B_i - iB_r) \cosh(D_i + iD_r) \sinh(A_i + iA_r) + \sinh(C_i + iC_r) \\ &\times \{\cosh(2A_i)[-a_1 + a_2 + i(b_1 - b_2)] + \cos(2A_r)[a_1 - a_2 - i(b_1 - b_2)] \\ &+ [\cos(2B_r) + \cosh(2B_i)][-a_1 + a_2 - i(b_1 + b_2)]\}) \\ s_{2,1} &= -\frac{2}{D_1}e^{ix/2}(4ib_1 \cosh(B_i - iB_r) \sinh(A_i + iA_r) \sinh(C_i - iC_r) + \cosh(D_i - iD_r) \\ &\times \{\cosh(2A_i)[a_1 - a_2 - i(b_1 + b_2)] + \cos(2A_r)[-a_1 + a_2 + i(b_1 + b_2)] \\ &+ [\cos(2B_r) + \cosh(2B_i)][a_1 - a_2 + i(b_1 - b_2)]\})\end{aligned}$$

Using a recurrence relation for higher-order solutions in the Darboux scheme, we can derive the second-order quintic breather solution, including the real part of the eigenvalue. With $\delta = 0$, this solution is a direct analogue of the solution given in the Appendix B, Eq.(22) of [75].

CHAPTER 4

Quintic Breather Dynamics

4.1. Introduction

We present breather solutions of the quintic integrable equation of the Schrödinger hierarchy Eq. (1.19). This equation has terms describing fifth-order dispersion and matching nonlinear terms. Using a Darboux transformation, we derive first-order and second-order breather solutions. In the various limits, we derive first and second order quintic rogue wave solutions, the quintic triplet and, in the equal eigenvalue limit, we also derive the degenerate two breather solution. To some extent, these solutions are analogous with the corresponding NLSE solutions. The Lax pair and a systematic Darboux transformation for the integrable quintic equation Eq. (1.17) were given in chapter (2) in Sec. (2.2) and in Sec. (2.3). To eliminate any confusion, we again mention that our ‘quintic NLSE’ is Eq. (1.19) taking coefficients $\alpha = \gamma = 0$ in Eq. (1.17) and we derive solutions of Eq. (1.19) as the quintic NLSE, not Eq. (1.17), where α and γ can be non-zero. However, the presence of a free parameter in the equation results in specific solutions that have no analogues in the NLSE case. We analyze new features of these solutions. We provide general solutions for the first- and second-order breather cases and consider several particular cases of interest. For deductive purposes, we separate the cases of purely imaginary and complex eigenvalues. For the complex eigenvalue case, an interesting new feature, namely, ‘breather-superposition’, is observed.

In the case of the NLSE, these solutions have a complex eigenvalue λ of the inverse scattering transform theory, where the real part of the eigenvalue is responsible for the velocity, while the imaginary part characterizes the frequency of the modulation. Simple solutions like single-soliton or single-breather solutions have one eigenvalue, while higher-order solutions may involve several eigenvalues. When using Darboux transformation schemes, each step adds one additional eigenvalue to the complex solution.

In this chapter, we extend breather theory to the quintic equation of the nonlinear Schrödinger hierarchy. The sequence of equations in this hierarchy can be found in the classical work of Kano [145]. In reality, it is difficult to study the infinite set of these equations. We can only move step-by-step in attempting to appreciate the significance of the whole hierarchy. Breathers comprise a different class of solutions of nonlinear evolution equations. Thus, the study of interactions between them can potentially open a new area of research [79, 172].

4.2. Breather Solutions with Imaginary Eigenvalues

The axially aligned Akhmediev breather solutions of the fundamental NLSE have been given explicitly in [75–79], while the breather solution for the Hirota equation was firstly presented in [173]. The basic property of this special solution is that it is localized in ‘ x ’ and periodic in ‘ t ’, making it fundamentally different from soliton solutions. Breathers are closely related to the process of modulation instability of plane waves with periodic perturbations. Similar breather solutions can be derived for the quintic Eq. (1.19). In terms of inverse scattering theory, they correspond to having purely imaginary eigenvalues. They can be obtained in various ways. Our derivation is based upon a Darboux transformation technique given in the Appendix (4.6). Following this technique, for the first order, we can write a one-parameter family of breather solutions in the form:

$$\psi_1 = \left[\frac{\kappa^2 \cosh(dx) + 2id \sinh(dx)}{2 \{ \cosh(dx) - b \cos[\kappa(t + V_b x)] \}} - 1 \right] e^{ix}, \quad (4.1)$$

where $\kappa = 2\sqrt{1 + \lambda^2}$ is the frequency of modulation, $d = \kappa\sqrt{1 - \frac{\kappa^2}{4}}$ is the growth rate of periodic modulation at the tails of the breather, and $V_b = \delta(\kappa^4 - 10\kappa^2 + 30)$ defines the skewing angle. Here the eigenvalue is purely imaginary $\lambda = ib$ and b plays the role of the parameter of the family. Thus the frequency of the breather can be written as $\kappa = 2\sqrt{1 - b^2}$, and consequently the period along the t -axis is $T = \pi/\sqrt{1 - b^2}$. This breather solution is presented in Fig. (4.1a) for the case $b = 0.9$. It is identical to the Akhmediev breather (AB) of the NLSE [88, 172, 174–176], except for the skewed profiles of individual peaks determined by V_b .

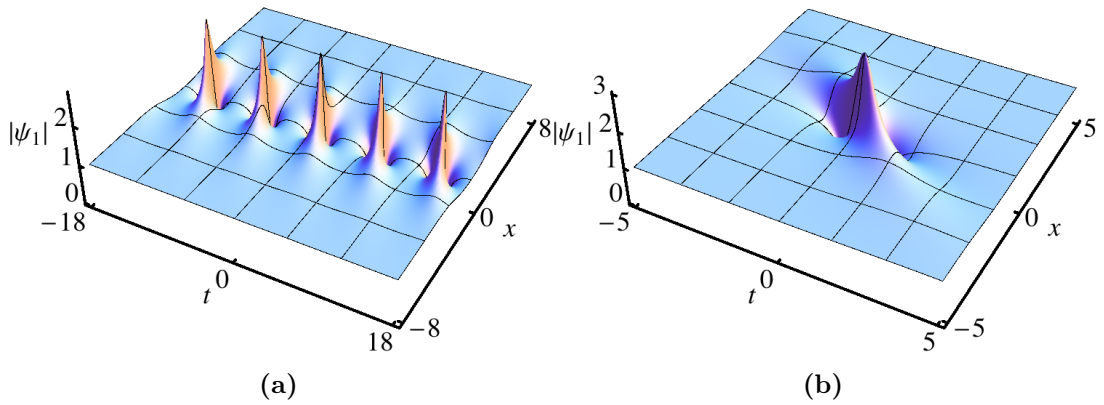


Figure 4.1 (a) A first-order Akhmediev breather of the quintic equation with eigenvalue $\lambda = 0.9i$. It is given by Eq. (4.1) with $\delta = \frac{1}{32}$. Here x is the evolution variable and t is the transverse one. (b) A first-order rogue-wave solution of the quintic equation for $\delta = \frac{1}{32}$

Breathers exist over the range of eigenvalues $0 < b < 1$. The frequency remains within the interval $0 < \kappa < 2$. The growth rate, d , in Eq. (4.1) is same as in the case of the NLSE [72]. The skewing angle is clearly visible, even for the small value of $\delta = \frac{1}{32}$ that we use here. The parameter δ is arbitrary and can take large values. The smallness of δ only signifies the sensitivity of the higher order terms in the operator $Q[\psi(x, t)]$. Otherwise, variations of δ do not make qualitative changes in the solution profile, apart from skewing and stretching of the localization. The parameter V_b is around 1 when $b \approx 1$ and $\kappa \approx 0$. The effect is relatively large due to the higher-order terms in the quintic operator. A special case occurs when the eigenvalue $\lambda = i$, i.e. $b \rightarrow 1$. The frequency κ becomes zero and the period T goes to infinity. The solution (4.1) becomes undefined as the numerator and denominator are zero. However, taking the limit $b \rightarrow 1$ in the breather solution (4.1), we obtain the rogue-wave solution of

the quintic equation:

$$\psi_1 = \left[\frac{4(1 + 2ix)}{4(t + 30\delta x)^2 + 4x^2 + 1} - 1 \right] e^{ix}. \quad (4.2)$$

Only one localized peak remains from the periodic train. This rational solution is presented in Fig. (4.1b). It is an equivalent of the Peregrine solution of the NLSE [89, 92, 112]. Indeed, in the limit of $\delta \rightarrow 0$, it is transformed into the actual Peregrine solution.

When the imaginary eigenvalue is above i (i.e. $b > 1$), in the case of the NLSE, an Akhmediev breather [88, 92, 175, 177, 178] is transformed into a Kuznetsov-Ma (KM) soliton [85, 179]. A similar transformation occurs in the case of the quintic equation. Indeed, converting the imaginary expressions into real ones in Eq. (4.1), we obtain:

$$\psi_m = \left(\frac{-f^2 \cos(gx) - 2ig \sin(gx)}{2 \{ \cos(gx) - b \cosh[f(t + V_m x)] \}} - 1 \right) e^{ix}, \quad (4.3)$$

where $V_m = \delta(30 + 10f^2 + f^4)$, while $g = fb$ and $f = 2\sqrt{b^2 - 1}$. This solution is shown in Fig. (4.2). It is periodic along the oblique line in the (x, t) -plane.

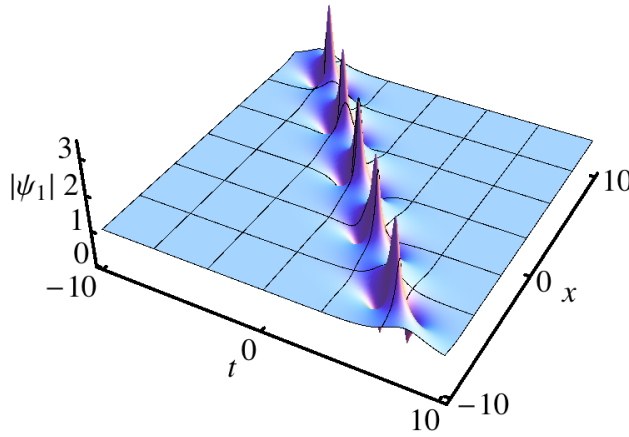


Figure 4.2 A first-order Kuznetsov-Ma breather of the quintic equation with eigenvalue $\lambda = 1.2i$. It is given by Eq. (4.3) with $\delta = \frac{1}{64}$. Here x is the evolution variable and t is the transverse one.

In contrast to the NLSE case, the Kuznetsov-Ma solitons have an oblique angle of propagation due to the δ -dependent velocity component V_m . This can be clearly seen in Fig. (4.2). The soliton propagates along the x -axis when $\delta = 0$. Each of the three types of breather solutions admits trivial translations along the x and t axes;

these are not explicitly shown in the above equations.

4.3. Two-Breather Solutions with Imaginary Eigenvalues

Two-breather solutions depend on two imaginary eigenvalues $\lambda_1 = ib_1$ and $\lambda_2 = ib_2$. They can be obtained in the second step of the Darboux transformation given in Appendix (4.6). The two frequencies κ_j and eigenvalues are related by $\kappa_j = 2\sqrt{1 - b_j^2}$. An example of a solution consisting of two breathers translated in x by $\Delta x = \pm 5$ is shown in Fig. (4.3a). The two breathers are well-separated and almost independent. They have different periods and skewing angles.

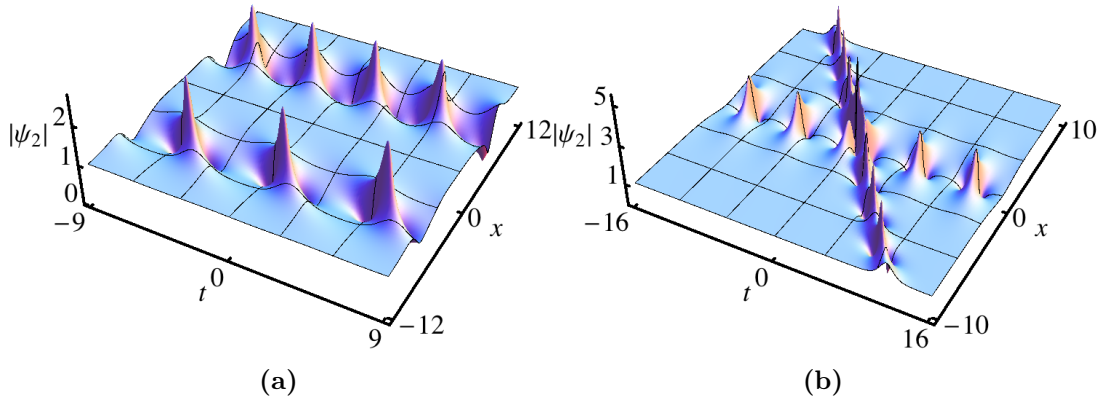


Figure 4.3 (a) A two-breather solution with eigenvalues $\lambda_1 = 0.65i$ and $\lambda_2 = 0.85i$, and $\delta = \frac{1}{32}$. The two breathers are shifted in x due to translations $\Delta x_1 = 5$ and $\Delta x_2 = -5$. (b) Two-breather solution with one being a Kuznetsov-Ma soliton with eigenvalue $\lambda_2 = 1.35i$ and the other one being an AB with $\lambda_1 = 0.85i$. The parameter $\delta = \frac{1}{64}$

Another example of a two-breather solution is presented in Fig. (4.3b). One imaginary part of the eigenvalues here (b_1) is smaller than 1 while the other one (b_2) is larger than 1. Thus the solution gives a crossing of an AB with a KM soliton. Their periods are defined by the frequencies $\kappa_1 = 2\sqrt{1 - b_1^2}$ and $f_2 = 2\sqrt{b_2^2 - 1}$. Various combinations are possible. Similar examples of the NLSE breather collisions are given in [106].

For two ABs located at the same position in x , the solution becomes more involved.

Patterns depend on the relative frequencies. For example, if the ratio $\kappa_1 : \kappa_2$ is $2 : 1$, then the superposition pattern of two ABs is a periodic sequence of merged triplets of Peregrine solutions [102]. If the ratio $\kappa_1 : \kappa_2$ is $3 : 2$, the periodic pattern consists of Peregrine pairs alternating with second-order merged triplets [106]. A few examples of superimposed ABs are shown in Fig. (4.4). The best way to observe these patterns would be in optical experiments with higher-order modulation instability [180–182].

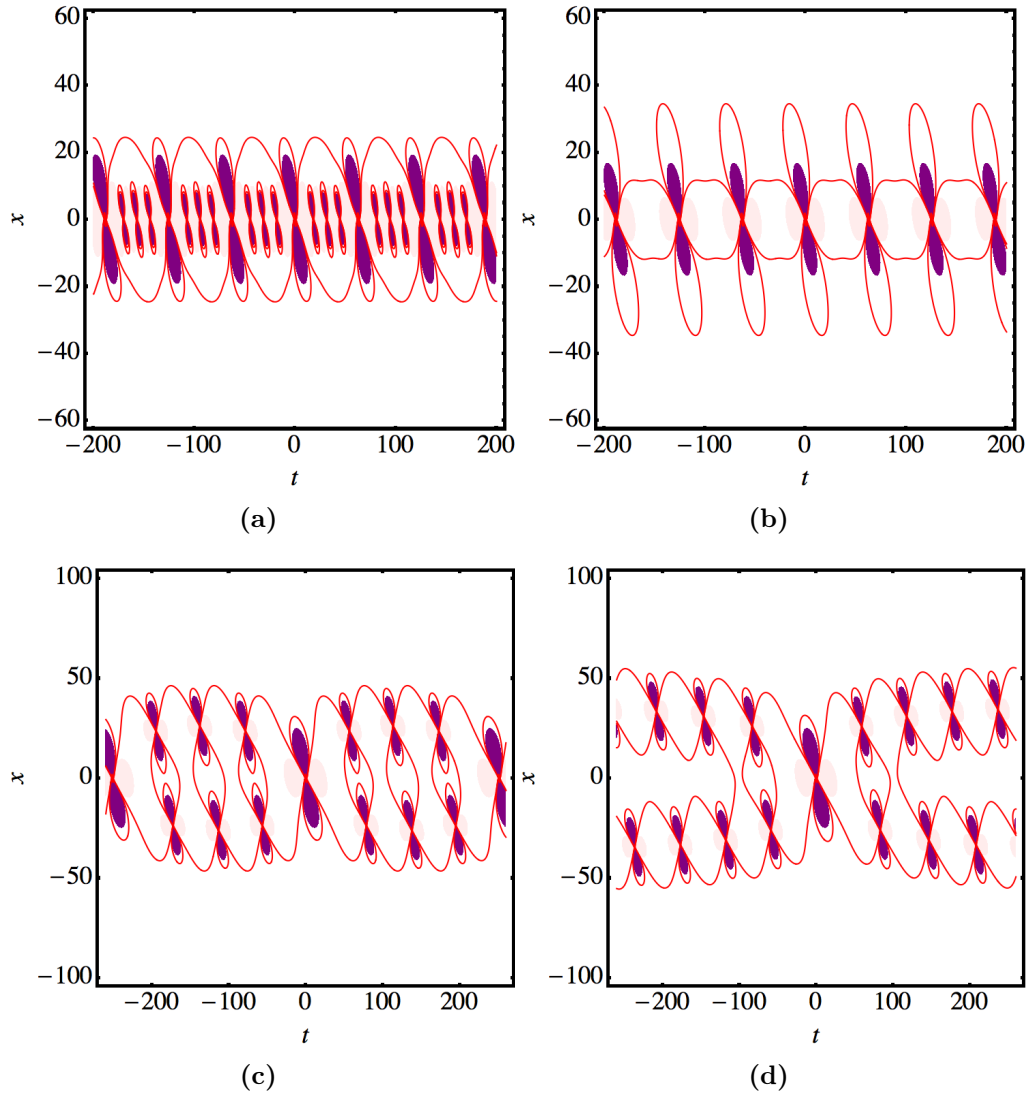


Figure 4.4 Contour plots of two superimposed ABs with $\delta = \frac{1}{64}$ and various frequency ratios: (a) $\kappa_1 = \frac{1}{2}$ and $\kappa_2 = \frac{1}{10}$; (b) $\kappa_1 = \frac{1}{5}$ and $\kappa_2 = \frac{1}{10}$; (c) $\kappa_1 = \frac{1}{8}$ and $\kappa_2 = \frac{1}{10}$; (d) $\kappa_1 = \frac{1}{9}$ and $\kappa_2 = \frac{1}{10}$.

In Fig. (4.4a), the frequency ratio $\kappa_1 : \kappa_2$ is $1 : 5$. There are six individual pulse components in each period. Three of them are merged and the other three remain separated. In Fig. (4.4b), the ratio of $\kappa_1 : \kappa_2$ is $1 : 2$. The pattern consists of a periodic

sequence of second-order peaks. When the frequency ratio is close to 1, the beating period of the pattern increases. This can be clearly seen from Fig. (4.4c), where $\kappa_1:\kappa_2$ is 4 : 5. In the degenerate case, $\kappa_1:\kappa_2 = 1$, the beating period is infinite and only one intersecting point between the two breathers remains. A near-degenerate case is shown in Fig. (4.4d). Here $\kappa_1:\kappa_2$ is 9 : 10. Similar patterns appear for second-order Kuznetsov-Ma solitons. The frequencies in this case are defined by id_1 and id_2 . From an experimental point of view, the solutions in Fig. (4.4) can be excited in the form of higher-order modulation instability, i.e. the same way as in [181]. Higher frequencies of the instability can be achieved with higher intensities of initial excitation. The specific implementation of terms in the quintic equation can be done by the special design of the optical fibre. The zero limit of κ_2 transforms one of the ABs into a rogue wave. It can be positioned at any point on the (x, t) -plane. An example of a superposition of an AB with a rogue wave, shifted relative to each other, is shown in Fig. (4.5a). The closed-form solution for this case is given by Eq. (4.13) of Appendix (4.6). This diagram is similar to Fig.3 of [106] for the NLSE case. For zero shifts, the superposition is shown in Fig. (4.5b). This is an example of a rogue wave on top of an AB.

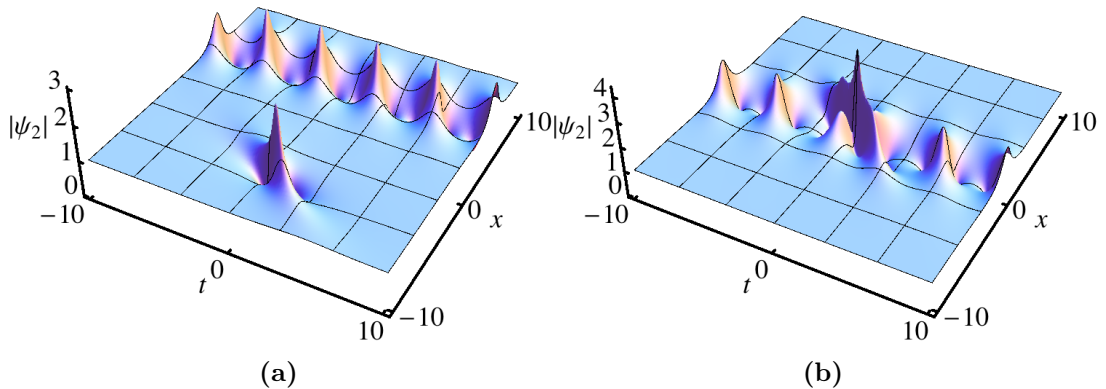


Figure 4.5 (a) A separated superposition of a rogue wave with an AB with eigenvalue $\lambda_1 = 0.65i$ and $\delta = \frac{1}{32}$. (b) rogue wave on top of AB with eigenvalue $\lambda_1 = 0.65i$ and $\delta = \frac{1}{32}$

Taking, additionally, the limit, $\kappa_1 \rightarrow 0$, we obtain the second order rogue wave shown in Fig. (4.6). It has a maximum amplitude of 5, located at the origin. The solution

can be written in the form

$$\psi = \left[1 + \frac{P + iQ}{D} \right] e^{ix} \quad (4.4)$$

where

$$\begin{aligned} P &= -12(-3 + 72x^2 + 80x^4 + 8\{2t^4 + 3t^2(1 + 4x^2\Gamma) + 240t^3x\delta \\ &\quad + 900x^2\delta^2[19 + 12x^2(1 + 150\delta^2)] + 60tx\delta[11 + 12x^2(1 + 300\delta^2)]\}), \\ Q &= -24x\{-15 + 8x^2 + 8[-3t^2 + 2t^4 + 4t^2x^2 + 2x^4\Gamma^2 \\ &\quad + 60tx(5 + 4t^2 + 4x^2\Gamma)\delta + 900(13 + 12t^2)x^2\delta^2]\}, \\ D &= 9 + 64t^6 + 64x^6\Gamma^3 + 11520t^5x\delta - 432x^4(-1 + 300\delta^2)(1 + 2900\delta^2) \\ &\quad + 36x^2(11 + 37900\delta^2) + 720tx\delta[25 + 8x^2(5 + 2x^2\Gamma^2 - 6300\delta^2)] \\ &\quad + 1920t^3x\delta[-5 + 12x^2(1 + 1500\delta^2)] + 48t^4[1 + 4x^2(1 + 4500\delta^2)] \\ &\quad + 12t^2\{9 + 8x^2[-3 + 2x^2\Gamma + 900(-13 + 10x^2\Gamma)\delta^2]\}, \end{aligned}$$

and where $\Gamma = 1 + 900\delta^2$. The solution has a variable equation parameter δ that controls the skew angle of the pattern. If $\delta = 0$, this solution coincides with the second-order Akhmediev-Peregrine solution of the NLSE, presented earlier in [80,183]. It has been experimentally observed in hydrodynamics [108].

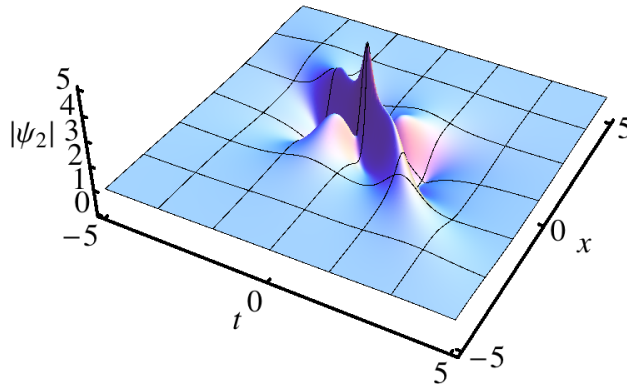


Figure 4.6 A second-order quintic rogue wave with $\delta = \frac{1}{64}$ given by Eq. (4.4).

4.3.1. Degenerate two-Breather Solution

The common inverse scattering technique [163] does not allow two eigenvalues to be equal. When this happens, the solution is undefined. It needs additional effort to unveil such solutions in explicit form [84]. In order to do that, we rewrite the solution, making the change $\kappa_2 \rightarrow \kappa_1 + h$. Then, we take a Taylor series expansion of this expression in terms of h and retain only the lowest-order terms. The resulting solution becomes well-defined. Solutions obtained in this way are known as degenerate solutions [84].

In our case, the resulting solution contains only one frequency κ_1 , which we denote as κ . This solution is given by Eq. (4.14) in the Appendix. It is a combination of rational, hyperbolic and trigonometric functions of x and t . The free parameter of the solution is κ . Two illustrations are given in Fig. (4.7). One case is for real $\kappa = 0.8$, while the other one is for imaginary $\kappa = 0.8i$. The former case is a superposition of two ABs while the latter case is a superposition of two KM solitons with equal eigenvalues. In each case, there is only one point of crossing between the two breathers. For the NLSE case, when $\delta = 0$, the solutions have been given earlier in [84, 106, 184]. In the NLSE case, the solutions have mirror symmetry relative to the axes x and t . For the quintic equation, this symmetry is lost. Instead, there is an inversion symmetry relative to the origin.

4.3.2. Rogue Wave Triplet

A second-order rogue-wave solution can have additional free parameters which split the superposition into its fundamental components. For the NLSE case, such a solution has been found in [185] and studied in detail in [102]. Surprisingly, the second-order rogue-wave solution is split into three fundamental rogue waves. Each of them is a Peregrine solution. This splitting feature is different from higher-order soliton solutions. The second-order soliton solution consists of 2 solitons.

A similar splitting can be observed in the case of the second-order rogue-wave solution of the quintic equation. In order to derive this solution, we have to take into account translations that may depend on κ . Namely, we represent the translations

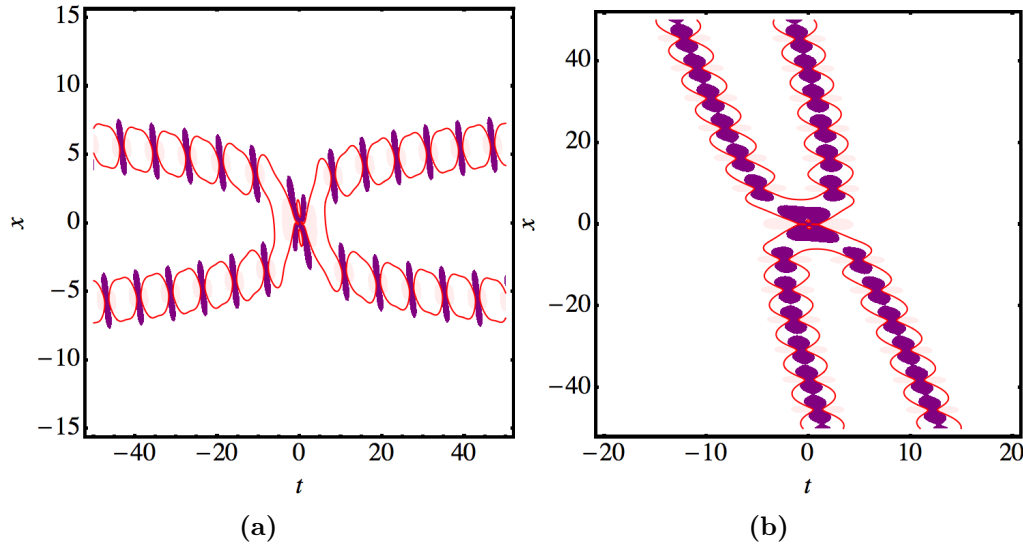


Figure 4.7 Degenerate two-breather solutions. (a) A two-AB solution with $\kappa = 0.8$ and $\delta = \frac{1}{64}$. (b) A two-KM soliton solution with $\kappa = 0.8i$ and $\delta = \frac{1}{256}$.

in the form of a series in κ :

$$x_j = \sum_{m=1}^N \kappa^{2(m-1)} X_{mj}, t_j = \sum_{m=1}^N \kappa^{2(m-1)} T_{mj},$$

with $1 \leq j \leq m$. At the very point $\kappa = 0$, the translations are given by X_{11} , T_{11} , X_{12} and T_{12} . If these are zero, the solution is centred at the origin. However, the coefficients of higher-order terms in these series become the free parameters of the solution when we take the exact limit of $\kappa \rightarrow 0$. The number, N , of terms needed in these series, depends upon the order of the solution. For the second-order solution, two lowest-order terms are sufficient. For the solution centred at the origin, this leaves us with four parameters: X_{21} , T_{21} , X_{22} and T_{22} . These coefficients enter the solution as the differences $x_d = X_{22} - X_{21}$ and $t_d = T_{22} - T_{21}$. Thus, only these two free independent parameters are involved in the solution:

$$\psi = \left[1 + \frac{G_{tr} + iH_{tr}}{D_{tr}} \right] e^{ix}, \quad (4.5)$$

where

$$\begin{aligned} G_{tr} = & -12\{-3 + 16t^4 + 24t^2(1 + 4x^2\Gamma) + 1920t^3x\delta + 8x^2[9 + 10x^2\Gamma \\ & + 900(19 + 2x^2\Gamma)\delta^2] + 480tx\delta[11 + 12x^2(1 + 300\delta^2)] - 128(t + 30x\delta)t_d \\ & - 128(x + 30t\delta + 900x\delta^2)x_d\}, \end{aligned}$$

$$\begin{aligned}
H_{tr} &= -24\{x[-15 + 16t^4 + 1920t^3x\delta + 480tx(5 + 4x^2\Gamma)\delta + 8t^2[-3 \\
&\quad + 4x^2(1 + 2700\delta^2)] + 8x^2(1 + 11700\delta^2 + 2x^2\Gamma^2)] - 128x(t + 30x\delta)t_d \\
&\quad + 16(1 + 4t^2 - 4x^2\Gamma)x_d\}, \\
D_{tr} &= 9 + 64t^6 + 64x^6\Gamma^3 + 11520t^5x\delta - 432x^4(-1 + 300\delta^2)(1 + 2900\delta^2) \\
&\quad + 36x^2(11 + 37900\delta^2) + 1920t^3x\delta[-5 + 12x^2(1 + 1500\delta^2)] \\
&\quad + 48t^4[1 + 4x^2(1 + 4500\delta^2)] + 12t^2\{9 + 8x^2[-3 + 2x^2\Gamma \\
&\quad + 900(-13 + 10x^2\Gamma)\delta^2]\} + 720tx\delta[25 + 8x^2(5 - 6300\delta^2 + 2x^2\Gamma^2)] \\
&\quad + 128\{8t_d^2 + x_d[x(-9 + 12t^2 - 4x^2) + 30t(-3 + 4t^2 + 12x^2\Gamma)\delta \\
&\quad + 900(-19 + 12t^2)x\delta^2 + 3240000x^3\delta^4 + 8\Gamma x_d] + t_d[4t^3 + 360t^2x\delta \\
&\quad + 30x\delta[-19 + 12x^2(-1 + 300\delta^2)] + 3t(-1 + 4x^2[-1 + 900\delta^2]) + 480\delta x_d]\},
\end{aligned}$$

and $\Gamma = 1 + 900\delta^2$. An example of this solution is shown in Fig. (4.8). It is a triplet of first-order rogue waves similar to the rogue wave triplet of the NLSE [102]. For zero values of x_d and t_d , the three components merge and the solution reduces to Eq. (4.4). In the NLSE case, the individual components of the triplet are located at the vertices of an equilateral triangle [102]. The quintic terms shift these relative positions. The triangle becomes skewed in the (x, t) -plane, in addition to the skewing of each component of the solution.

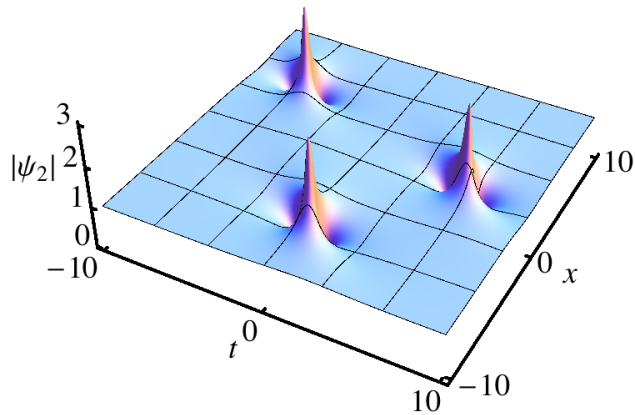


Figure 4.8 Quintic rogue wave triplet given by Eq.(4.5) with $x_d = -25$, $t_d = -50$ and $\delta = \frac{1}{64}$.

4.4. Two-Breather Solutions with Complex Eigenvalues

The one-soliton solution of the NLSE with complex eigenvalue $\lambda = a + ib$ has an amplitude b and velocity a . Thus, the inclusion of a real part of λ results in the soliton propagating at an angle to the x -axis. A one-breather solution with a complex eigenvalue has the same property. It has a propagation direction that is oblique to the x -axis [75–79]. Velocity appears in the one-breather solutions of the quintic equation, as well. However, there is an additional tilt to this direction caused by the quintic terms. This can be seen from Fig. (4.2). Even the Kuznetsov-Ma soliton with $a = 0$, which would be propagating along the x -axis in the case of the NLSE, has a non-zero tilt for the quintic equation. The explicit form of the one-breather solution is:

$$\psi_1 = \left[1 + 2b \frac{G_1 + iH_1}{D_1} \right] e^{ix}, \quad (4.6)$$

where the functions G_1 , H_1 and D_1 are combinations of trigonometric and hyperbolic functions:

$$\begin{aligned} G_1 &= \cos(xV_T + t\kappa_r) \cosh(2\chi_i) - \cosh(xV_H + t\kappa_i) \sin(2\chi_r), \\ H_1 &= \cos(2\chi_r) \sinh(xV_H + t\kappa_i) + \sin(xV_T + t\kappa_r) \sinh(2\chi_i), \\ D_1 &= \cosh(xV_H + t\kappa_i) \cosh(2\chi_i) - \cos(xV_T + t\kappa_r) \sin(2\chi_r), \end{aligned}$$

with

$$\kappa = 2\sqrt{1 + \lambda^2} = \kappa_r + i\kappa_i, \chi = \frac{1}{2} \cos^{-1} \left(\frac{\kappa}{2} \right) = \chi_r + i\chi_i,$$

and

$$\begin{aligned} V_T &= -b\kappa_i + a\kappa_r + \delta(\Omega\kappa_i + \mathfrak{U}\kappa_r) \\ V_H &= a\kappa_i + b\kappa_r + \delta(\mathfrak{U}\kappa_i - \Omega\kappa_r) \\ \Omega &= 16ab(1 - 4a^2 + 4b^2) \\ \mathfrak{U} &= 2[3 + 8a^4 + 4b^2 + 8b^4 - 4a^2(1 + 12b^2)]. \end{aligned}$$

Again, if we take $\delta = 0$, Eq. (4.6) reduces to the breather solution of the NLSE with complex eigenvalue, presented in [75]. The one-breather solution is similar to the breather solution given in Fig. (4.2). However, the direction of propagation of the

whole breather and the skewing of each peak are now defined by the expressions V_H/κ_i and V_T/κ_r respectively.

Having the one-breather solution, and using Darboux transformations, we can construct two-breather solutions, as described in the Appendix (3.6). One example is shown in Fig. (4.9). In analogy with multi-soliton solutions, this pattern can be called a “collision of two breather”. The choice of parameters is such that the maximum of collision is located at the origin. However, translation of the components may convert the maximum into zero. Except for the area of the collision, the two breathers are roughly the same as given by Eq. (4.6). In addition, there are small shifts imposed at the sides of each breather by the collision.

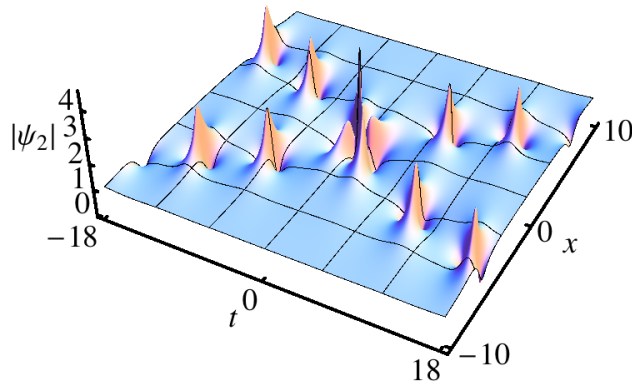


Figure 4.9 Two-breather collision. The two complex eigenvalues are $\lambda_1 = 0.05 + 0.9i$ and $\lambda_2 = -0.05 + 0.9i$. The parameter $\delta = \frac{1}{64}$.

A collision of general type is shown in Fig. (4.9). An interesting possibility is the alignment of the two breathers along the same direction in the (x, t) -plane. Having an additional parameter, δ , in the quintic equation, such an alignment can be arranged without equalizing the real parts of the two eigenvalues. Below, we consider various special cases of this solution that are possible only for the quintic equation. These cases do not have analogues among the solutions of the NLSE equation.

Indeed, we now consider the case when the directions of propagation of the two colliding breathers coincide. This happens if

$$\frac{V_{H1}}{\kappa_{1i}} = \frac{V_{H2}}{\kappa_{2i}} . \quad (4.7)$$

The notation here is the same as in Eq. (4.6), with the addition of a subscript that corresponds to a particular breather, one or two. For example, $\lambda_j = a_j + ib_j$, where $j = 1, 2$. Similarly:

$$\begin{aligned} V_{Tj} &= -b_j \kappa_{ji} + a_j \kappa_{jr} + \delta(\Omega_j \kappa_{ji} + \mathcal{U}_j \kappa_{jr}) \\ V_{Hj} &= a_j \kappa_{ji} + b_j \kappa_{jr} + \delta(\mathcal{U}_j \kappa_{ji} - \Omega_j \kappa_{jr}) \\ \Omega_j &= 16a_j b_j (1 - 4a_j^2 + 4b_j^2), \\ \mathcal{U}_j &= 2[3 + 8a_j^4 + 4b_j^2 + 8b_j^4 - 4a_j^2 (1 + 12b_j^2)]. \end{aligned}$$

With this notation, the condition (4.7) can be written as:

$$\begin{aligned} &[a_1 \kappa_{1i} + b_1 \kappa_{1r} + \delta(\mathcal{U}_1 \kappa_{1i} - \Omega_1 \kappa_{1r})] / \kappa_{1i} \\ &= [a_2 \kappa_{2i} + b_2 \kappa_{2r} + \delta(\mathcal{U}_2 \kappa_{2i} - \Omega_2 \kappa_{2r})] / \kappa_{2i}, \end{aligned} \tag{4.8}$$

This algebraic equation cannot be solved analytically. However, we can solve it numerically. Fig. (4.10) shows the locus of points on the complex plane of λ_1 for fixed λ_2 and δ , found by solving Eq. (4.8). The plot demonstrates a rich variety of possibilities that lead to the parallel superimposed breathers. Indeed, the locus of points is a set of continuous curves of solutions rather than isolated points. In addition, there are three such curves. Consequently, we can have a rich variety of specific two-breather solutions. By choosing different points on the curves, we can have various patterns.

To begin, we solve the two-breather alignment condition for the specific values of eigenvalues $\lambda_1 = a_1 + 0.8i$ with a_1 to be identified and $\lambda_2 = 0.08 + 0.9i$. The plot in Fig. (4.10) provides three real solutions for a_1 shown by the solid red dots. They are: $a_1 = -1.70693$, $a_1 = 0.141238$ and $a_1 = 1.1181$. These three values of a_1 correspond to three qualitatively different cases. They are shown in three panels of Fig. (4.11). Fig. (4.11a) shows the complex pattern of two superimposed breathers with different periods. In this case, we observe the beating on top of the original periods of the two breathers. The transverse widths of the two breathers are comparable. This results in a “single mode” pattern along the combined breather. The peaks of the breather remain single peaks. They are higher at the points of

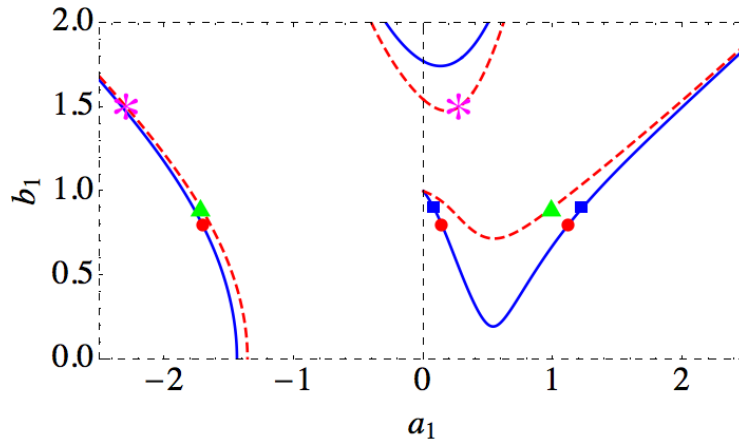


Figure 4.10 Locus of points on the plane of complex eigenvalues λ_1 that lead to parallel breathers. The second eigenvalue is fixed and equal to $\lambda_2 = 0.08 + 0.9i$ for solid (blue) lines and equal to $\lambda_2 = 0.08 + 1.5i$ for the dashed (red) lines. Parameter $\delta = \frac{1}{64}$. Solid red dots, blue rectangles, green triangles and magenta stars on the curves are chosen to signify the examples in the figures below.

constructive interference of the two components with “ radiation” around them. Fig. (4.11b) shows the beating pattern of two breathers for $\lambda_1 = 0.141238 + 0.8i$. The periods of two breathers are incommensurate. Consequently, the beating period is not well pronounced. Significant differences in the widths of the two first-order breathers set up a complex “multimode” transverse pattern of the superposition. The skew angles V_T/κ_r of the two breathers are also different, and this is an additional reason for the complex pattern.

In Fig. (4.11c), the beating period is comparable with the periods of each breather. The widths of the two components are also slightly different. Thus, we have another complex pattern different from the two previous cases. The highest peak in each of the three superpositions is always at the point (0,0). This is due to the fact that individual breathers are centred at the origin. If we add translations, the two breathers can be separated in space and time.

We can use another value of λ_2 when solving Eq. (4.8). For example, the dashed curves in Fig. (4.10) are calculated for $\lambda_2 = 0.08 + 1.5i$. If we choose $b_1 = 0.9$, we again obtain three solutions for a_1 . These are: $a_1 = -1.71834$, $a_1 = 0.200181$ and $a_1 = 0.990488$. Two of these points are shown by the green triangles on the dashed red curves.

Fig. (4.12) shows the patterns of breather superposition for two of the above solutions,

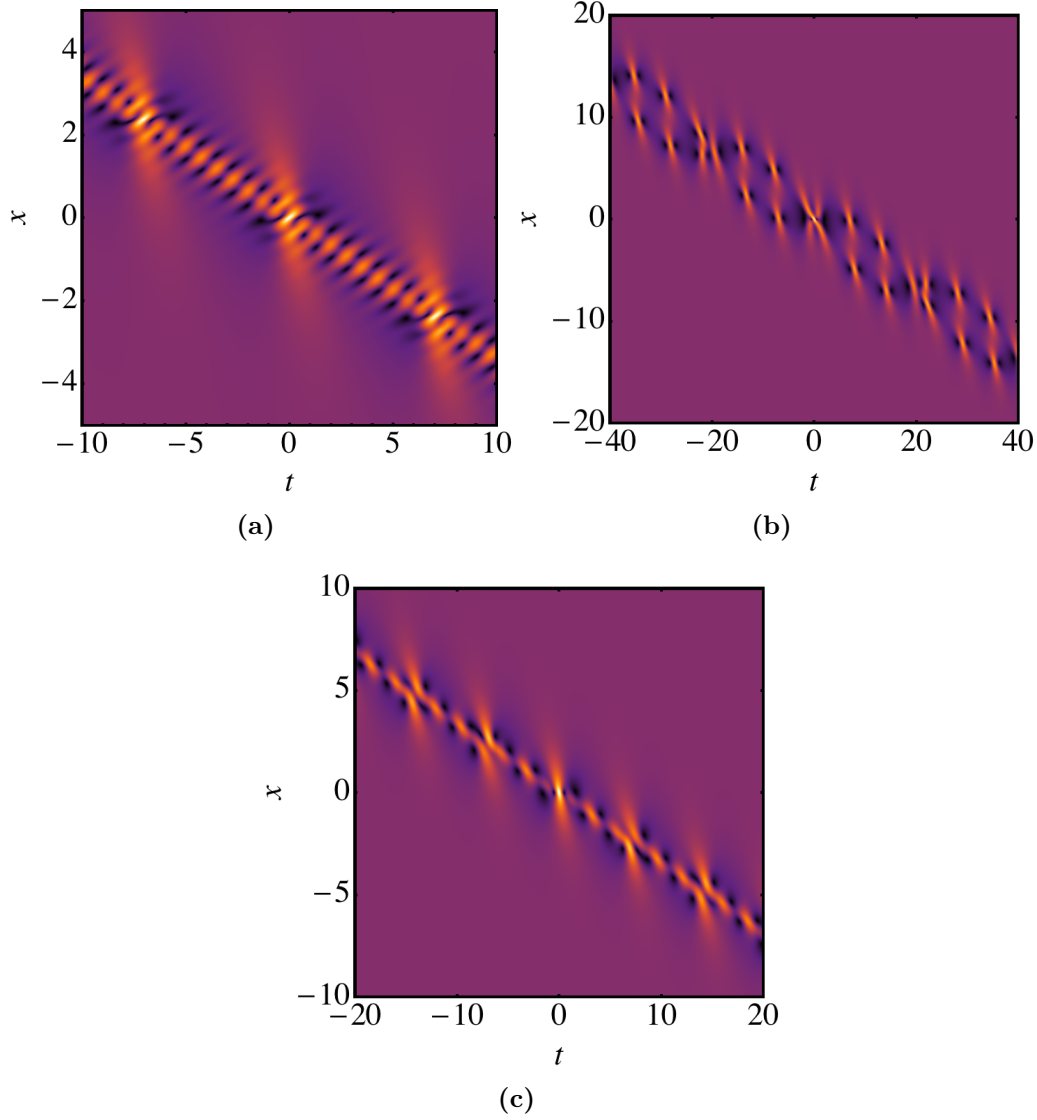


Figure 4.11 Two-breather superposition with the condition Eq. (4.8), marked by solid red dots in Fig. (4.10). Here $\lambda_2 = 0.08 + 0.9i$ and $\delta = \frac{1}{64}$. (a) $\lambda_1 = -1.70693 + 0.8i$; (b) $\lambda_1 = 0.141238 + 0.8i$; (c) $\lambda_1 = 1.1181 + 0.8i$.

$a_1 = -1.71834$ and $a_1 = 0.990488$. There are no fundamental differences between these composite breathers and those shown in Figs. (4.11). The patterns remain quasi-periodic along the breather. The angles of propagation in the (x, t) -plane vary, of course. Clearly, we can observe a wide variety of patterns, taking into account that the condition of Eq. (4.8) admits an infinite number of possibilities. Among all of them, we can choose some special cases. Namely, solving Eq. (4.8), we have the chance to select equal values b_1 and b_2 . For example, if the eigenvalue $\lambda_2 = 0.08 + 0.9i$ is chosen, three eigenvalues λ_1 satisfy Eq. (4.8). They are: $\lambda_1 = -1.77613 + 0.9i$,

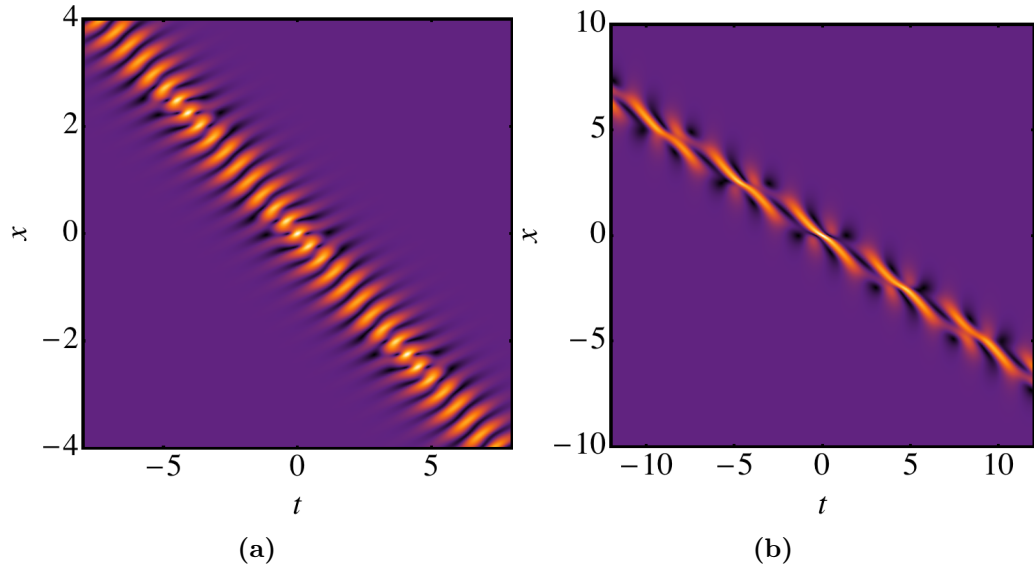


Figure 4.12 Two-breather superposition with the condition of Eq. (4.8), marked by green triangles in Fig. (4.10). Here, $\lambda_2 = 0.08 + 1.5i$ and $\delta = \frac{1}{64}$. (a) $\lambda_1 = -1.71834 + 0.9i$; (b) $\lambda_1 = 0.990488 + 0.9i$.

$\lambda_1 = 0.08 + 0.9i$ and $\lambda_1 = 1.22549 + 0.9i$. Two of them are shown by blue rectangles in Fig. (4.10). Clearly, the second of these eigenvalues coincides with λ_2 and the two-breather solution is then degenerate. Two other choices show similar patterns, as before, except for the completely periodic pattern along the breather. This happens due to the equal imaginary parts of the two eigenvalues $b_1 = b_2$. One example is shown in Fig. (4.13a). The other choice, $\lambda_1 = -1.77613 + 0.9i$, provides a similar periodic beating pattern. However, the eigenvalue $\lambda_1 = 0.08 + 0.9i$ results in the degenerate solution shown in Fig. (4.13b). The two breathers split, apart from the origin, which is the point of their “collision”. This pattern is similar to degenerate solutions considered in the previous sections.

Special cases can be found for other values of λ_2 . For example, if we choose $\lambda_2 = 0.08 + 1.5i$, we obtain four values for the first eigenvalue: $\lambda_1 = -2.29479 + 1.5i$, $\lambda_1 = 0.08 + 1.5i$, $\lambda_1 = 0.273178 + 1.5i$ and $\lambda_1 = 1.93937 + 1.5i$ that satisfy Eq. (4.8). The second of these solutions, $\lambda_1 = 0.08 + 1.5i$, corresponds to the degenerate case. The pattern of this solution (not shown here) is similar to the case shown in Fig. (4.7b). However, two other patterns for $\lambda_1 = -2.29479 + 1.5i$ and $\lambda_1 = 0.273178 + 1.5i$ are shown in Figs. (4.14a) and (4.14b), respectively. They correspond to the magenta stars in Fig. (4.10). These figures demonstrate once again the huge variety of

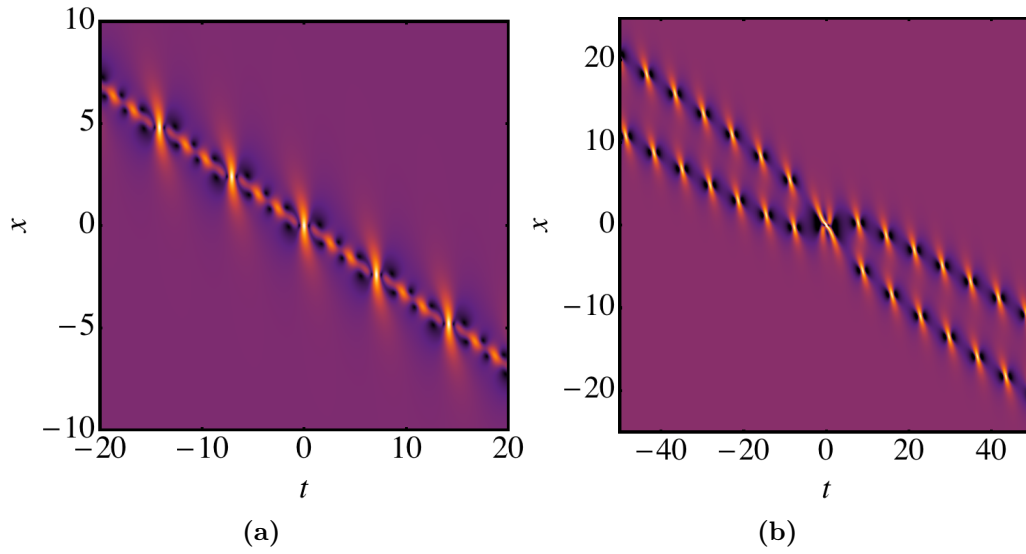


Figure 4.13 Two-breather superposition with the condition of Eq. (4.8), marked by blue rectangles in Fig. 4.10. Here (a) $\lambda_1 = 1.22549 + 0.9i$, $\lambda_2 = 0.08 + 0.9i$; (b) $\lambda_1 = 0.08 + 0.9i$, $\lambda_2 = 0.08 + 0.9i$ (degenerate case). Parameter $\delta = \frac{1}{64}$.

possibilities in designs of two-breather solutions. Each of these patterns is strictly periodic along the breather, because of the equal imaginary parts of the eigenvalues.

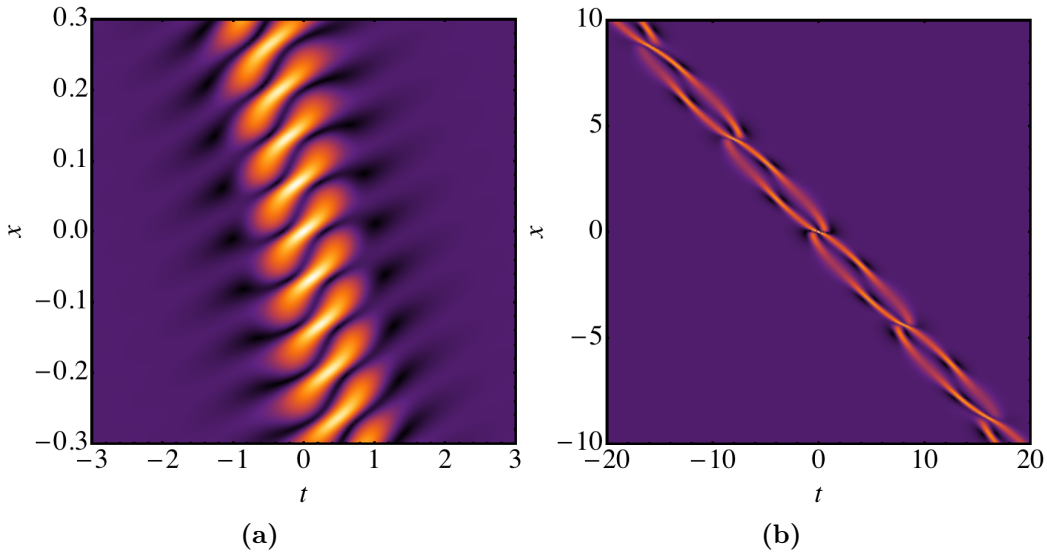


Figure 4.14 Two-breather superposition with the condition Eq.(4.8), marked by magenta stars in Fig. 4.10. Here, $\lambda_2 = 0.08 + 1.5i$ and $\delta = \frac{1}{64}$. (a) $\lambda_1 = -2.29479 + 1.5i$; (b) $\lambda_1 = 0.273178 + 1.5i$.

4.5. Conclusion

Here, we have presented breather solutions of the quintic integrable equation of the Schrödinger hierarchy. This equation has terms describing fifth-order dispersion and additional matching nonlinear terms with a free real parameter δ . These terms transform the breathers of the NLSE, adding a skewing angle into the breather patterns. Significant differences from the NLSE case can be observed for second-order breathers. In particular, two breathers can be aligned in parallel without matching their eigenvalues. These beating superpositions generate an infinite variety of patterns. Some of them have been illustrated graphically, and conditions for their appearance have been discussed in detail.

Practical realization of new solutions in optical fibres will require special arrangements. Firstly, we need to operate with pulses with durations below 20 ps in order to activate quintic terms in the fibre [82]. This, in turn, will require higher intensities of the optical radiation. The use of specially designed fibres is another requirement that may help to obtain a special relation between the higher-order terms. On the bright side, we have the exact solutions that can be used for preparing initial conditions in the experiments. The remaining question to answer is the robustness of the new solutions. Several papers [79, 100, 186] already posed and partially answered this question for the breather solutions of the NLSE. The answer for the solutions of higher order equations is still needs to be given. We leave this for future work.

4.6. Appendix: Various one-and two-Breather Solutions

The Darboux transformation technique has been developed in general form by Matveev and Salle [166], and applied for breather solutions of the NLSE in [81]. The quintic integrable equation requires a few modifications in comparison to the NLSE case. These are related to the inclusion of quintic terms into the 2×2 matrices of the Lax-pair formalism. Multi-breather solutions require the use of a plane wave as the seeding solution of the technique [81]. This is different from the multisoliton

solutions, which only require the zero solution in order to start the process [166]. Substituting the seeding plane wave $\psi_0 = e^{ix}$ and the eigenvalue $\lambda_{1,2}$ into the set of linear differential equations relative to r and s (compatibility condition) and solving them, at the first step, we obtain the following functions

$$r_{1,j} = -2ie^{-ix/2} \cos(A_j + iB_j + \chi_j) \quad (4.9)$$

$$s_{1,j} = 2e^{ix/2} \sin(A_j + iB_j - \chi_j), \quad (4.10)$$

where, depending on the eigenvalue that we use, $j = 1$ or 2 . The variables in this solution are:

$$A_j = d_j G_j x_{sj} + \kappa_j t_{sj}/2 + \frac{\pi}{4}, B_j = d_j x_{sj}/2, \chi_j = \frac{1}{2} \cos^{-1} \left(\frac{\kappa_j}{2} \right), G_j = V_j/v_j, \\ V_j = \delta(\kappa_j^4 - 10\kappa_j^2 + 30), v_j = \sqrt{4 - \kappa_j^2}, d_j = \frac{1}{2} \kappa_j v_j, \kappa_j = 2\sqrt{1 + \lambda_j^2}.$$

When we admit translations along x and t , we use the shifted spatial $x_{sj} = x - x_j$ and temporal $t_{sj} = t - t_j$ variables. In this section, we use the purely imaginary eigenvalue, $\lambda_j = ib_j$. Solutions of the quintic equation, at the first and second steps of calculation, can be found from the same equation

$$\psi_n = \psi_{n-1} + \frac{2(\lambda_n^* - \lambda_n)s_{n,1}r_{n,1}^*}{|r_{n,1}|^2 + |s_{n,1}|^2}, \quad (4.11)$$

where n is the order of the solution. In particular, it provides the one- and the two-breather solutions of the quintic equation. The main advantage of the Darboux transformation technique is that the second-order linear functions $r_{2,1}$ and $s_{2,1}$ do not require solution of differential equations. Instead, we use recursive algebraic relations derived in [166]. A simple diagram for such calculations is given in Fig.2.2 of [84] (see also [81]). The calculations require a knowledge of $r_{1,1}$, $s_{1,1}$, $r_{1,2}$ and $s_{1,2}$ from Eqs. (4.9) and (4.10). Thus, we obtain

$$r_{2,1} = \frac{1}{D_2} \left\{ -2 \cos(A_2 + iB_2 + \chi_2) [b_1 \cos(2A_1) \cos(2\chi_1) - b_2 \cosh(2B_1) \right. \\ + b_2 \sin(2A_1) \sin(2\chi_1)] - 4b_1 \cos(A_1 + iB_1 + \chi_1) \sin(A_1 - iB_1 - \chi_1) \\ \times \left. \sin(A_2 + iB_2 - \chi_2) \right\}$$

$$\begin{aligned}
s_{2,1} = & \frac{1}{D_2} \left\{ -2ib_1 \cos(A_2 + iB_2 + \chi_2) \sin(2A_1) + 2i[b_1 \cos(2A_1) \cos(2\chi_1) \right. \\
& + b_2 \cosh(2B_1) - b_2 \sin(2A_1) \sin(2\chi_1)] \sin(A_2 + iB_2 - \chi_2) \\
& \left. + 2b_1 \cos(A_2 + iB_2 + \chi_2) \sinh(2(B_1 + i\chi_1)) \right\}, \tag{4.12}
\end{aligned}$$

where $D_2 = \cosh(2B_1) - \sin(2A_1) \sin(2\chi_1)$. Substituting these $r_{2,1}$ and $s_{2,1}$ into Eq.(4.11), we obtain the two-breather solution of quintic equation. The calculations are trivial and not presented here.

The case when $\lambda_2 \rightarrow i$ is special. In this case $\kappa_2 \rightarrow 0$, the period of breather becomes infinite and the breather is transformed into a rogue wave. After applying L'Hôpital's rule to the ratio, the second-order solution takes the form:

$$\psi = \left[1 + \frac{N_k}{D_k} \right] e^{ix}, \tag{4.13}$$

where

$$\begin{aligned}
N_k = & \kappa_1^2 \{ -8t_{s2}(t_{s2} + 60\delta x_{s2}) [\cosh(2B_1) - \sin(2A_1) \sin(\chi_1)] \\
& - v_1 [\sin(2A_1) + i \sinh(2B_1 + i\chi_1)] (1 + 4y^2 + 4x_{s2}^2) \\
& - 2[\cosh(2B_1) - \sin(2A_1) \sin(\chi_1)] [-1 + 4x_{s2} (-i + x_{s2} + 900\delta^2 x_{s2})] \}, \\
D_k = & 4v_1 \{ 4 \cos(\chi_1) [y \cos(2A_1) - x_{s2} \sinh(2B_1)] + [\cosh(2B_1) \sin(\chi_1) \\
& - \sin(2A_1)] (1 - 4y^2 - 4x_{s2}^2) \} - [\cosh(2B_1) - \sin(2A_1) \sin(\chi_1)] \\
& \times (1 + 4y^2 + 4x_{s2}^2) (-8 + \kappa_1^2).
\end{aligned}$$

As expected, it is a combination of rational, trigonometric and hyperbolic functions.

Degenerate two-breather solution: $\kappa_2 = \kappa_1 = \kappa$.

When the two eigenvalues λ_1 and λ_2 coincide, the solution again becomes undefined. It has to be recovered by using L'Hôpital's rule on the numerator and denominator of the resulting expression. In this way, we obtain:

$$\psi = \left[1 + \frac{N_d}{D_d} \right] e^{ix}, \tag{4.14}$$

where N_d and D_d are:

$$\begin{aligned} N_d &= -4\kappa c_2 \{ -2[\kappa(-(-3 \cosh(2B) + \cosh(2(B + i\chi))) \sin(2A) + (-2 + \cos(4A)) \\ &\quad \times \sin(\chi) + i \sinh(4B + i\chi)) + 2 \cos(2A) \cos(\chi) \cosh(2B + i\chi) y_1 (t + 5x\delta y_2)] \\ &\quad + 2c_2 [\cos(4A) \cos(\chi) + \cosh(4B + i\chi) - 2i \sin(2A) \sinh(2B) + 2x \cos(\chi) \\ &\quad \times (-i + \sin(2A) \sinh(2B + i\chi)) y_3] \}, \end{aligned}$$

$$\begin{aligned} D_d &= 2\{ (8 - 3\kappa^2) \cos(4A) + 8 \cosh(4B) + \kappa^2 [3 - 2 \cos(2\chi) \sin^2(2A)] + 8 \cosh(2B) \\ &\quad \times [c_1 t \cos(2A) \cos(\chi) - \kappa^2 \sin(2A) \sin(\chi)] - 8x (8 - 6\kappa^2 + \kappa^4) \cos(\chi) \\ &\quad \times \sin(2A) \sinh(2B) + 4 \cos(\chi) [10c_1 x \delta \cos(2A) \cosh(2B) y_2 \\ &\quad + \cos(\chi) y_1 (y_1 (t + 5x\delta y_2)^2 - x^2 y_3^2)] \}. \end{aligned}$$

Here $c_1 = v^3$, $v = c_2 = \sqrt{4 - \kappa^2}$, $d = \frac{1}{2}\kappa v$, and

$$\begin{aligned} A &= \frac{1}{4} [\pi + 2(t + Vx)\kappa], B = \frac{dx}{2}, V = \delta (30 - 10\kappa^2 + \kappa^4), \\ \chi &= \cos^{-1} \left(\frac{\kappa}{2} \right), y_1 = \kappa^2 - 4, y_2 = 6 - 6\kappa^2 + \kappa^4, y_3 = \kappa^2 - 2. \end{aligned}$$

CHAPTER 5

Breather-to-Soliton Conversion for Hirota Equation

5.1. Introduction

Recent publications have revealed the intricate relation between the soliton and breather solutions of a certain class of evolution equations [174, 187, 188]. Breathers can be converted into solitons with special perturbations [174]. Solitons themselves can evolve into the form of breathers [188]. Even the Peregrine solution can be converted into an infinitely-elongated structure under certain conditions [187]. In this chapter, we show that there are special cases where breather oscillations can be suppressed, and in that case, breathers may be transformed into solitons. Mutual conversions between the two become possible when the evolution equation contains a sufficient number of free parameters which can be used to control their solutions. One of the simplest equations of this type is the so-called Hirota equation (HE) [83, 140]. It can be written in the following operator form with two free parameters:

$$i\psi_x + \eta S[\psi(x, t)] - i\alpha H[\psi(x, t)] = 0, \quad (5.1)$$

where S is the nonlinear Schrödinger operator $S[\psi(x, t)] = \psi_{tt} + 2\psi|\psi|^2$, while H is the Hirota operator, featuring third-order linear dispersion (ψ_{ttt}) and nonlinear

dispersion ($6|\psi|^2\psi_t$) terms. Thus: $H[\psi(x, t)] = \psi_{ttt} + 6|\psi|^2\psi_t$.

In Eq. (5.1), x is the propagation variable and t is the transverse variable (time in a moving frame), with the complex function $\psi(x, t)$ describing the envelope of the waves. Coefficients α and η are arbitrary real numbers. When both coefficients are non-zero, we have the 'plain' integrable Hirota equation. If $\psi(x, t)$ is a real function and $\eta = 0$, Eq. (5.1) is a modified Korteweg-de Vries equation. Below, we assume that the function $\psi(x, t)$ is always complex. The limit of $\alpha = 0$ leaves us with the classic nonlinear Schrödinger equation (NLSE). Conveniently, we can take $\eta = 1/2$ [84], but this is not essential. The choice of $\eta = 1/2$ only becomes important when we want to keep circular symmetry of rogue wave clusters in the (x, t) -plane [102, 103]. Remarkably, Eq. (5.1) remains integrable for any values of the two parameters. Thus, every solution of Eq. (5.1) will contain free parameters that can be used to control certain features of the solutions. One of these features involves pulsations of the breather solutions. If such pulsations of a localised solution disappear, we can talk about conversion of a breather into a soliton. As we usually fix one of the parameters, namely η , we can study such transformations in the simplest situation by changing just one parameter, α . However, keeping both coefficients variable allows us to consider more general cases.

5.2. Lax Pairs

The inverse scattering technique for the NLSE was developed in 1972 by Zakharov and Shabat [163]. This technique is based on the fact that the NLSE can be written in terms of two matrix operators which are known as Lax operators [29]. The latter can be found, not only for the NLSE, but for many other integrable equations [30]. The Lax pair for the Hirota equation (5.1) has been given in [137, 146, 170] with the $\eta = 1/2$ normalization. Below, we provide a more general formulation for the Lax pair with arbitrary η and α . Following [170], we can write the Lax equations in the form:

$$\frac{\partial R}{\partial t} = U \cdot R, \quad \frac{\partial R}{\partial x} = V \cdot R$$

such that the ‘zero-curvature’ condition $U_x - V_t + [U, V] = 0$ will reproduce Eq. (5.1). Here U and V are 2×2 matrices with U given by

$$U = i \begin{bmatrix} \lambda & \psi(x, t)^* \\ \psi(x, t) & -\lambda \end{bmatrix} \quad (5.2)$$

while V is a matrix polynomial in eigenvalue λ . For the NLSE, it is a simple quadratic polynomial. The order of the polynomial increases when there is an additional higher-order operator. With the Hirota operator, it becomes a cubic polynomial which can be written in general form $V = \sum_{j=0}^3 \lambda^j V_j$, where submatrices V_j are

$$V_j = i \begin{bmatrix} A_j & B_j^* \\ B_j & -A_j \end{bmatrix}, \quad (5.3)$$

with

$$\begin{aligned} A_0 &= -\eta|\psi|^2 - i\alpha(\psi_t^* \psi - \psi_t \psi^*), \quad B_0 = 2\alpha|\psi|^2 \psi + i\eta\psi_t + \alpha\psi_{tt}, \\ A_1 &= 2\alpha|\psi|^2, \quad B_1 = 2\eta\psi - 2i\alpha\psi_t, \quad A_2 = 2\eta, \quad B_2 = -4\alpha\psi, \\ A_3 &= -4\alpha, \quad B_3 = 0. \end{aligned}$$

It is easy to check that substitution of the matrices U and V into the ‘zero-curvature’ condition leads directly to Eq. (5.1).

5.3. Breather-to-Soliton Conversions in the Hirota Equation

Taking purely imaginary eigenvalues allows us to derive the first-order Akhmediev breather (AB) solution of the Hirota equation. It was first presented in [173]. In [141], more general solutions of the Hirota equation were presented. In the present work, we generalize the procedure for arbitrary coefficients α and η and we use complex eigenvalues, $\lambda = a + ib$. The real part of λ leads to oblique propagation of the breather.

The first-order breather solution of the Hirota equation with non-zero η and α and with complex eigenvalue is given by

$$\psi_1 = \left[1 + 2b \frac{G_1 + iH_1}{D_1} \right] e^{i2\eta x}. \quad (5.4)$$

Here the functions G_1 , H_1 and D_1 are combinations of trigonometric and hyperbolic functions:

$$\begin{aligned} G_1 &= \cos(xV_T + t\kappa_r) \cosh(2\chi_i) - \cosh(xV_H + t\kappa_i) \sin(2\chi_r), \\ H_1 &= \cos(2\chi_r) \sinh(xV_H + t\kappa_i) + \sin(xV_T + t\kappa_r) \sinh(2\chi_i), \\ D_1 &= \cosh(xV_H + t\kappa_i) \cosh(2\chi_i) - \cos(xV_T + t\kappa_r) \sin(2\chi_r), \end{aligned}$$

with $\kappa = 2\sqrt{1 + \lambda^2}$, $\chi = \frac{1}{2} \arccos\left(\frac{\kappa}{2}\right)$ while

$$V_T = 2\eta(-b\kappa_i + a\kappa_r) + \alpha(\Omega\kappa_i + \mathcal{U}\kappa_r), \quad V_H = 2\eta(a\kappa_i + b\kappa_r) + \alpha(\mathcal{U}\kappa_i - \Omega\kappa_r).$$

where $\Omega = 8ab$ and $\mathcal{U} = 2(1 - 2a^2 + 2b^2)$, with a and b being arbitrary real numbers. An example of this solution is presented in Fig. (5.1a). We can clearly see the non-zero angle of propagation of the breather on the (x, t) -plane, its periodicity along the propagation direction and also the oblique orientation of each peak on the plane.

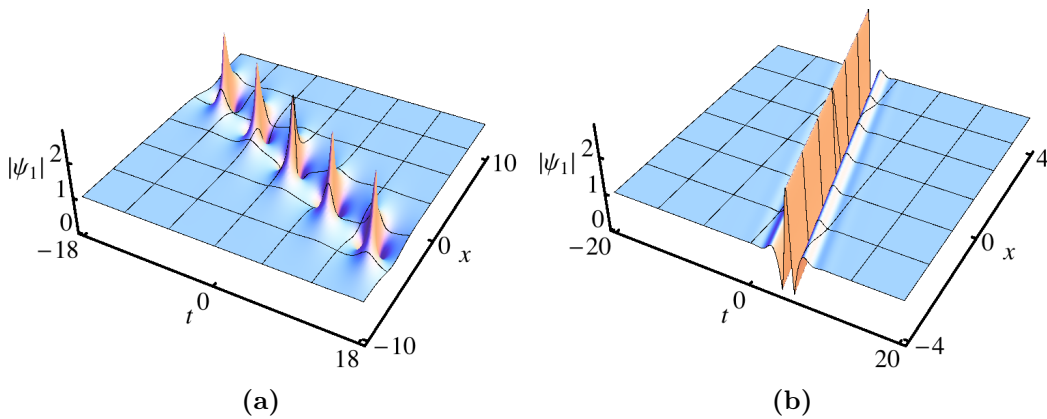


Figure 5.1 (a) Single breather solution Eq. (5.4) of Hirota equation with $\alpha = \frac{1}{12}$, $\eta = \frac{1}{2}$ with eigenvalue $\lambda = 0.08 + 0.9i$ (b) A breather solution Eq. (5.4) of the Hirota equation converted to a soliton when the eigenvalue is $\lambda = 0.75 + 0.9i$. Equation parameters are $\eta = \frac{1}{2}$ and $\alpha = \frac{1}{6}$.

One of the new features of this breather is that it can be converted into a non-periodic solution. This occurs when the lines of extrema of the hyperbolic and the

trigonometric functions in Eq. (5.4) on the (x, t) -plane coincide. For this to happen, the following condition:

$$\frac{V_H}{\kappa_i} = \frac{V_T}{\kappa_r}, \quad (5.5)$$

must be satisfied. Solving Eq. (5.5) leads to a simple expression for the real part of the eigenvalue

$$a = \frac{\eta}{4\alpha}. \quad (5.6)$$

This condition, in effect, enables conversion of the breather into a soliton. The slope, i.e. tangent of the propagation angle (relative to the x-axis) of the soliton on the (x, t) -plane is given by

$$\frac{V_T}{\kappa_r} = \frac{V_H}{\kappa_i} = \frac{\eta^2}{4\alpha} + 2\alpha(1 + 2b^2).$$

The imaginary part of the eigenvalue, b , is arbitrary while the coefficient α cannot be zero. This means that the conversion can happen only for the Hirota equation. NLSE breathers cannot be converted into solitons. An example of a converted solution is shown in Fig. (5.1b). The solution is indeed non-periodic, like an ordinary soliton. However, it is located on a non-zero background and has oscillating tails. The latter appear due to the trigonometric functions involved in the solution Eq. (5.4). Oscillations are minimal when $a \rightarrow 0$, while for large a they increase in amplitude. The directions of the oscillations and their decay are orthogonal to the line of the soliton maximum. The expression for the second-order breather solution is given in the Appendix 5.8. One general example calculated using Eq. (5.11) is shown in Fig. (5.2). The solution is a nonlinear superposition of two first-order breathers. Their parameters are given by the two eigenvalues $\lambda_1 = a_1 + ib_1$ and $\lambda_2 = a_2 + ib_2$. Each of the breathers involved in this solution can individually be converted into a soliton. The condition of conversion is the same as before, *i.e.*

$$\frac{V_{Hj}}{\kappa_{ji}} - \frac{V_{Tj}}{\kappa_{jr}} = 0, \quad (5.7)$$

where $j = 1, 2$ is an integer labelling the breathers. From Eq. (5.7), we have the

same expression as before for each real part:

$$a_{1,2} = \frac{\eta}{4\alpha}.$$

This expression is valid for every breather in a multi-component solution for an arbitrary number of breathers. The angular orientations of individual breathers in the superposition may differ, depending on the real parts of the eigenvalues, a_j .

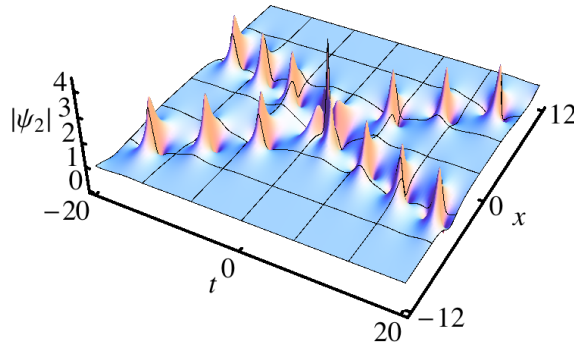


Figure 5.2 Collision of two breathers of Hirota equation with $\eta = 1/2$, $\alpha = \frac{1}{16}$ and eigenvalues $\lambda_1 = -0.08 + 0.9i$, $\lambda_2 = 0.08 + 0.8i$. This is an in-phase superposition with maximum amplitude at the origin, $|\psi_2(0, 0)| = 4.4$.

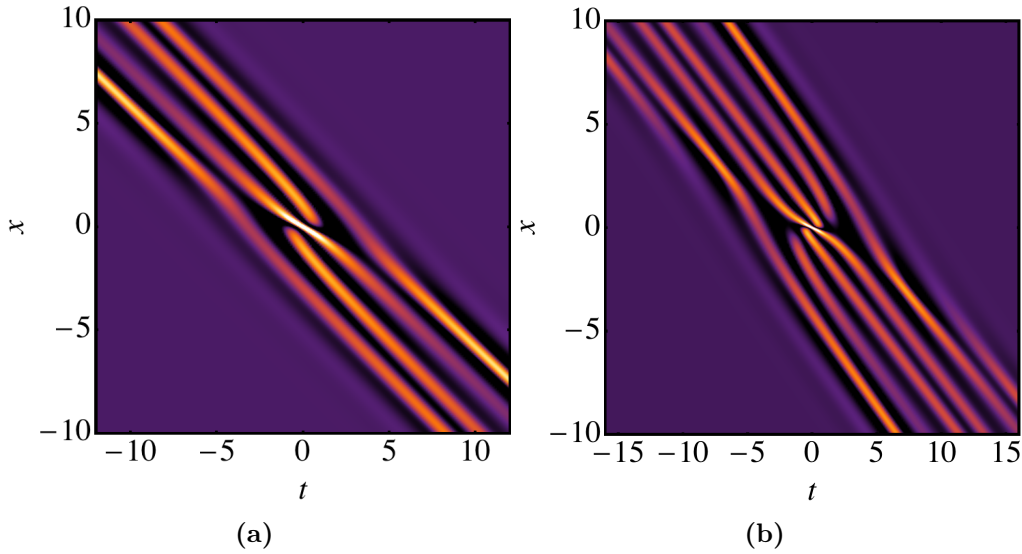


Figure 5.3 (a) A second-order breather converted to a two-soliton solution with $\lambda_1 = 0.75 + 0.9i$ and $\lambda_2 = 0.75 + 0.85i$. (b) A third-order breather converted to a three-soliton solution with the eigenvalues $\lambda_1 = 0.75 + 0.9i$, $\lambda_2 = 0.75 + 0.95i$ and $\lambda_3 = 0.75 + 0.8i$. Equation parameters are $\eta = \frac{1}{2}$ and $\alpha = \frac{1}{6}$.

Fig. (5.3a) and (5.3b) show two- and three-component breather solutions with each converted to a soliton. For the eigenvalues, we use the same value for the real parts, $a = a_1 = a_2 = a_3 = \frac{1}{8\alpha}$, but have different b_j . In these examples, we have chosen the relatively high value $\alpha = 1/6$. This reduces the values of a_j , resulting in fewer tail oscillations. The converted solitons have maximum amplitudes in the middle and quickly decaying oscillations on each tail.

5.4. Higher-order Breather Superpositions

Another interesting possibility when dealing with multi-component solutions is aligning the directions of propagation for different breathers. The condition of alignment is

$$\frac{V_{H1}}{\kappa_{i1}} = \frac{V_{H2}}{\kappa_{i2}} \quad (5.8)$$

Then the lines of maxima for the two breathers on the (x, t) -plane become parallel. In this case, each breather remains periodic. We do not introduce spatial shifts for the breathers, and let them overlap. The solution of Eq. (5.8) is complicated and has to be found numerically. We can fix one of the eigenvalues, say λ_2 and find the other one, λ_1 , using Eq. (5.8). For each λ_2 , we obtain a curve on the complex plane for λ_1 . The results are presented in Fig. (5.4).

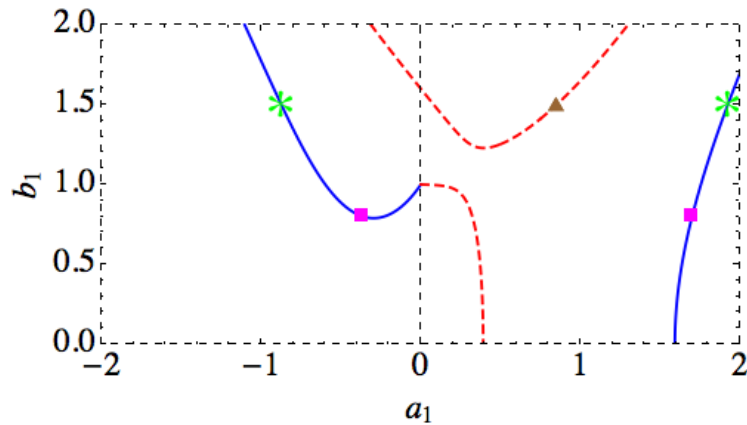


Figure 5.4 Locus of the eigenvalues $\lambda_1 = a_1 + ib_1$ found as solutions of Eq. (5.8) for three fixed eigenvalues, λ_2 . Solid blue curves correspond to $\lambda_2 = -0.08 + 0.9i$ while the dashed red curves correspond to $\lambda_2 = 0.08 + 1.5i$. Equation parameters are $\alpha = \frac{1}{6}$ and $\eta = \frac{1}{2}$. Green star, brown triangle and magenta squares on the curves are chosen for the illustrations below.

Here, the solid blue curve is the locus of points λ_1 found for fixed $\lambda_2 = -0.08 + 0.9i$. Moving along the curve allows us to change the common angle of propagation of the two superposed breathers. Another similar curve is calculated for $\lambda_2 = 0.08 + 1.5i$. It is shown by the red coloured dashed curve. Having the freedom of moving along the curves offers infinite possibilities for generating various breather structures. Here, we give only a few examples indicated by the green stars, brown triangle and magenta squares in Fig. (5.4). Solving Eq. (5.8) for fixed $\lambda_2 = -0.08 + 0.9i$ and fixed imaginary part of $\lambda_1 = a_1 + 0.8i$ provides us with three solutions for the real part of λ_1 , i.e. $a_1 = 1.69952$, -0.36428 and -0.228223 . Two of them are shown in Fig. (5.4) by the thick magenta squares. These are $a_1 = 1.69952$ and $a_1 = -0.36428$. We omit the illustration for the case $a_1 = -0.228223$, as it is similar to the other cases. The first two breather profiles are presented in Figs. (5.5a) and (5.5b). There is a beating pattern on top of regular periodicity. This is related to the presence of two frequencies of modulation of the two breathers. The beating periods are generally incommensurate. This is to be expected, as we did not match the basic periods. As the phases of the two breathers have not been altered, the two breathers have constructive interference at the origin. Thus, the absolute maximum of the whole pattern in each case is reached at the centre, *i.e.* $|\psi(0,0)|$ is the maximum amplitude. This can be seen by comparing its brightness with one of other peaks.

Two other examples correspond to the two green stars in Fig. (5.4) with $b_1 = 1.5$. The real parts of the first eigenvalue in this case are $a_1 = -0.874443$ and 1.92315 , while $\lambda_2 = -0.08 + 0.9i$. These green stars are located in the upper parts of the blue curves in Fig. (5.4). The corresponding wave profiles are shown in Figs. (5.5c) and (5.5d). These are similar to the previous two cases, apart from the longer beating periods in comparison to the basic periods of individual breathers. One more example corresponds to the eigenvalues $\lambda_1 = a_1 + 1.5i$ and $\lambda_2 = 0.08 + 1.5i$. Of the two solutions for a_1 ($a_1 = 0.08$ and $a_1 = 0.850848$), we choose only one, namely $a_1 = 0.850848$. This choice corresponds to the brown triangle in Fig. (5.4). The breather profile for this case is shown in Fig. (5.6). The breather has a slope that is opposite to the previous cases. In addition, it has a ‘chain-shaped’ structure. The choice $a_1 = 0.08$ represents a degenerate case, as $\lambda_1 = \lambda_2$. It needs a special consideration [106] which will be handled elsewhere.

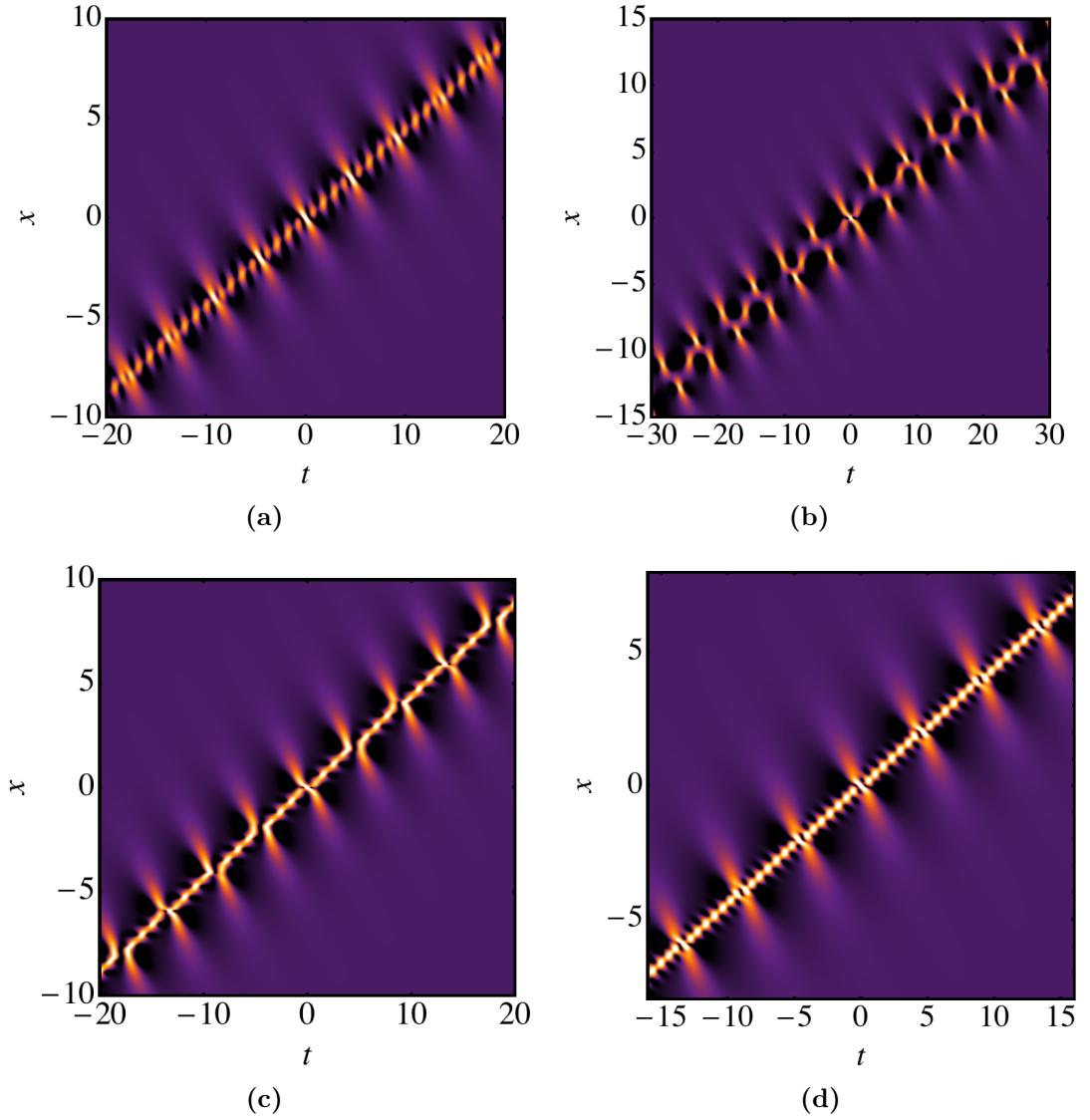


Figure 5.5 Breather superpositions for (a) $\lambda_1 = 1.69952 + 0.8i$ and $\lambda_2 = -0.08 + 0.9i$ and (b) $\lambda_1 = -0.36428 + 0.8i$ and $\lambda_2 = -0.08 + 0.9i$. The points corresponding to these cases are indicated by the two magenta squares on the blue curves in Fig. (5.4). (c) Breather superpositions for (a) $\lambda_1 = -0.874443 + 1.5i$ and $\lambda_2 = -0.08 + 0.9i$. (d) $\lambda_1 = 1.92315 + 1.5i$ and $\lambda_2 = -0.08 + 0.9i$. These choices correspond to the green stars in Fig. (5.4) located on the blue curves.

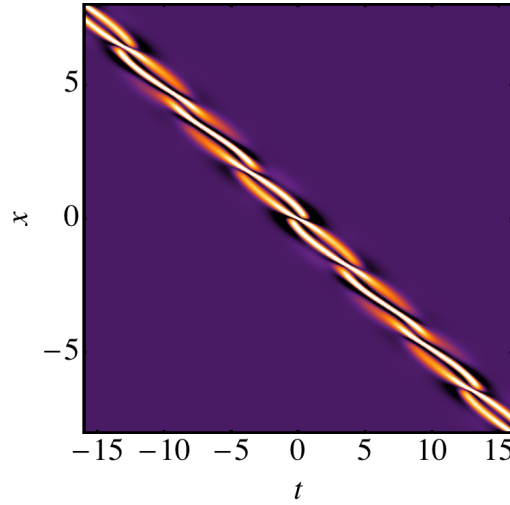


Figure 5.6 Breather superposition for $\lambda_1 = 0.850848 + 1.5i$ and $\lambda_2 = 0.08 + 1.5i$. This choice corresponds to the solid brown triangle in Fig. (5.4) located on the upper red dashed curve.

5.5. Moving Breathers of the NLSE

The real part of the eigenvalue that follows from the conditions Eq. (5.5) and Eq. (5.6) does not exist when $\alpha = 0$. This shows that the type of the breather-to-soliton conversion discussed above cannot happen for solutions of the NLSE. However, there is a transition from breather-to-soliton when the imaginary part of the eigenvalue changes from values below 1 to values above 1 when the real part of the eigenvalue is zero [87]. The non-zero real part of the eigenvalue, a_j , introduces velocity both to the breather and to the soliton. The mathematical analysis of moving breathers has been presented earlier in [75]. Here, we provide a few more clarifications.

A standard AB of the NLSE with $\lambda = 0.6i$ is shown in Fig. (5.7a). It starts with the plane wave solution at $x = -\infty$. The solution reaches its highest modulation depth at $x = 0$. This stage of evolution is shown in Fig. (5.7a). This periodic structure exists over an infinite range, $-\infty < t < \infty$. Adding a small real part to the eigenvalue, $a = 0.08$, transforms the infinite periodic structure into a localized one which is still located on a constant background. It is shown in Fig. (5.7b). This solution was called a “quasi-Akhmediev breather” in [79]. The presence of the real part of the eigenvalue also leads to a movement of the modulated part of the solution with constant velocity. A further increase of the real part of the eigenvalue increases the velocity of this motion, leading to stronger localization and a smaller oscillation

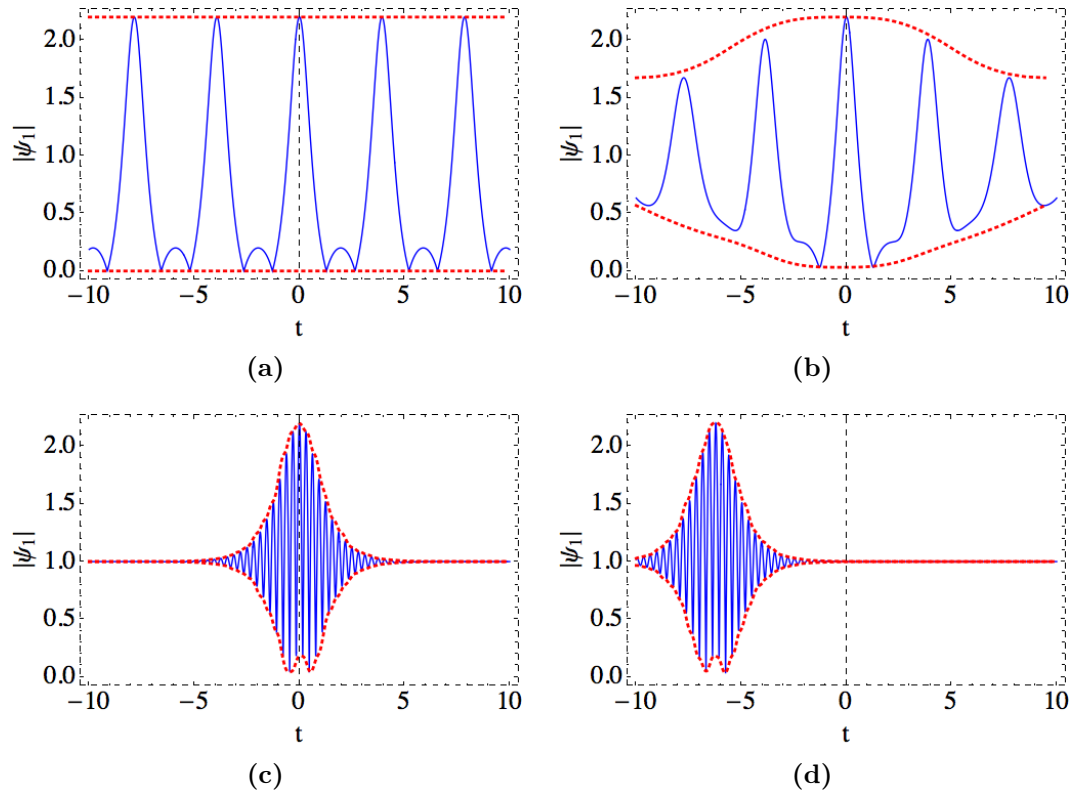


Figure 5.7 (a) Akhmediev breather of the NLSE at maximum amplitude stage $x = 0$ (solid-blue curve). Equation parameters are: $\alpha = 0$, $\eta = 1/2$. The eigenvalue $\lambda = 0.6i$. The envelope (red-dashed line) extends to infinity. (b) When a small real part is added to the eigenvalue, giving $\lambda = 0.08 + 0.6i$, the breather becomes localized. (c) Increasing the real part of the eigenvalue, giving $\lambda = 10 + 0.6i$, leads to a higher degree of localization. (d) Same as in (c) but $x = 0.31$. The breather acquires velocity and moves to the left.

period. This can be seen in Figs. (5.7c) and (5.7d). A two-dimensional plot of the breather evolution is shown in Fig. (5.8a). This plot shows that the periodic structure within the breather moves with a velocity that is different from the velocity of the breather itself. A collision of two such breathers is shown in Fig. (5.8b). As was shown in [79], this solution may be highly interesting from a practical point of view. In particular, it describes modulation instability on a variable background.

5.6. Superposition of moving Breathers of the NLSE

In this section, we consider an interaction of two moving breathers that is more specific than the straight collision shown in Fig. (5.8b). Namely, we study the superposition of two breathers propagating at the same angle on the (x, t) -plane.

The condition for such a superposition is the following:

$$\frac{V_{H1}}{\kappa_{i1}} = \frac{V_{H2}}{\kappa_{i2}}. \quad (5.9)$$

Here, we assume that $\eta = \frac{1}{2}$ and $\alpha = 0$. Acting the same way as before, we fix the eigenvalue λ_2 and find the eigenvalues λ_1 that satisfy the condition of Eq. (5.9). Now, Eq. (5.9) can be solved analytically for b_1 . We find that $b_1 = \pm \sqrt{\frac{p_1}{p_2}}$ where

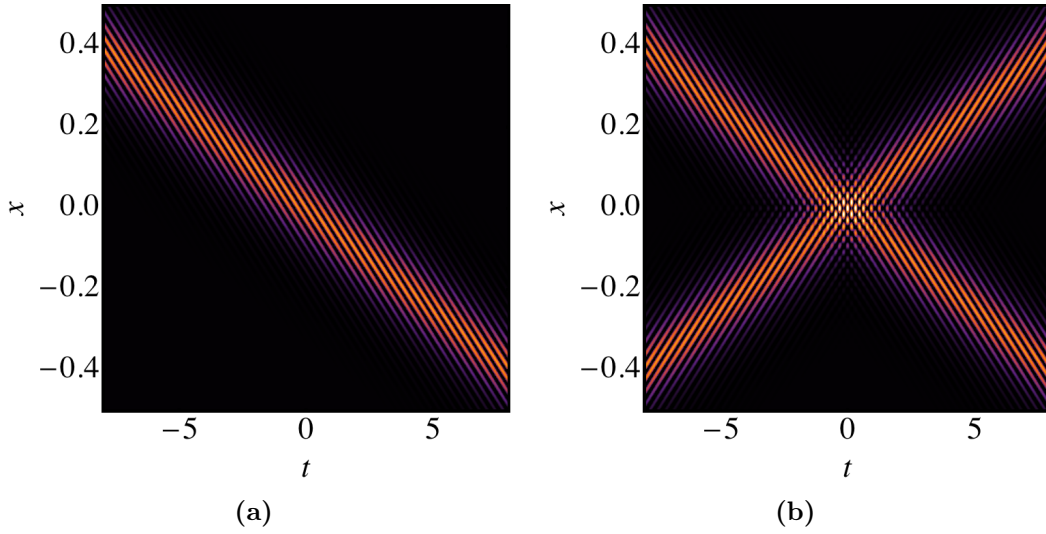


Figure 5.8 (a) Moving breather of NLSE described by Eq. (5.4) with $\lambda = 10 + 0.6i$. (b) Collision of two moving breathers of the NLSE with eigenvalues $\lambda_1 = -10 + 0.6i$ and $\lambda_2 = 10 + 0.6i$. Equation parameters $\eta = 1/2$ and $\alpha = 0$. Black areas correspond to the amplitude level $|\psi| = 1$.

$$\begin{aligned} p_1 &= 4a_1^4 a_2^2 + a_2^2 (1 + 2a_2^2 - 2b_2^2) - 2a_1^3 a_2 (2 + 6a_2^2 - 2b_2^2 + m_1 m_2) \\ &\quad - a_1 a_2 [3a_2^4 + (-1 + b_2^2) (-2 + b_2^2 - m_1 m_2) + a_2^2 (7 - 4b_2^2 + m_1 m_2)] \\ &\quad + a_1^2 [11a_2^4 (-1 + b_2^2) (-1 + b_2^2 - m_1 m_2) + a_2^2 (9 - 8b_2^2 + 3m_1 m_2)] \end{aligned}$$

and

$$p_2 = a_2 [a_2 + 4a_1^2 a_2 + 2a_2^3 - 2a_2 b_2^2 + 2a_1 (-1 - 3a_2^2 + b_2^2)],$$

with $m_1^2 = a_2^2 + (1 - b_2)^2$ and $m_2^2 = a_2^2 + (1 + b_2)^2$. The solutions for three fixed values of λ_2 are shown on the complex plane of λ_1 presented in Fig. (5.9a). These solutions appear as complex conjugate pairs. The sign reversal also produces a solution. In order to obtain co-propagating superposition, we must choose the solution in such

a way that the signs of real parts, a_1 and a_2 , of the two eigenvalues coincide. For opposite signs of a_1 and a_2 , the moving breathers will cross each other and form a breather collision rather than breather superposition.

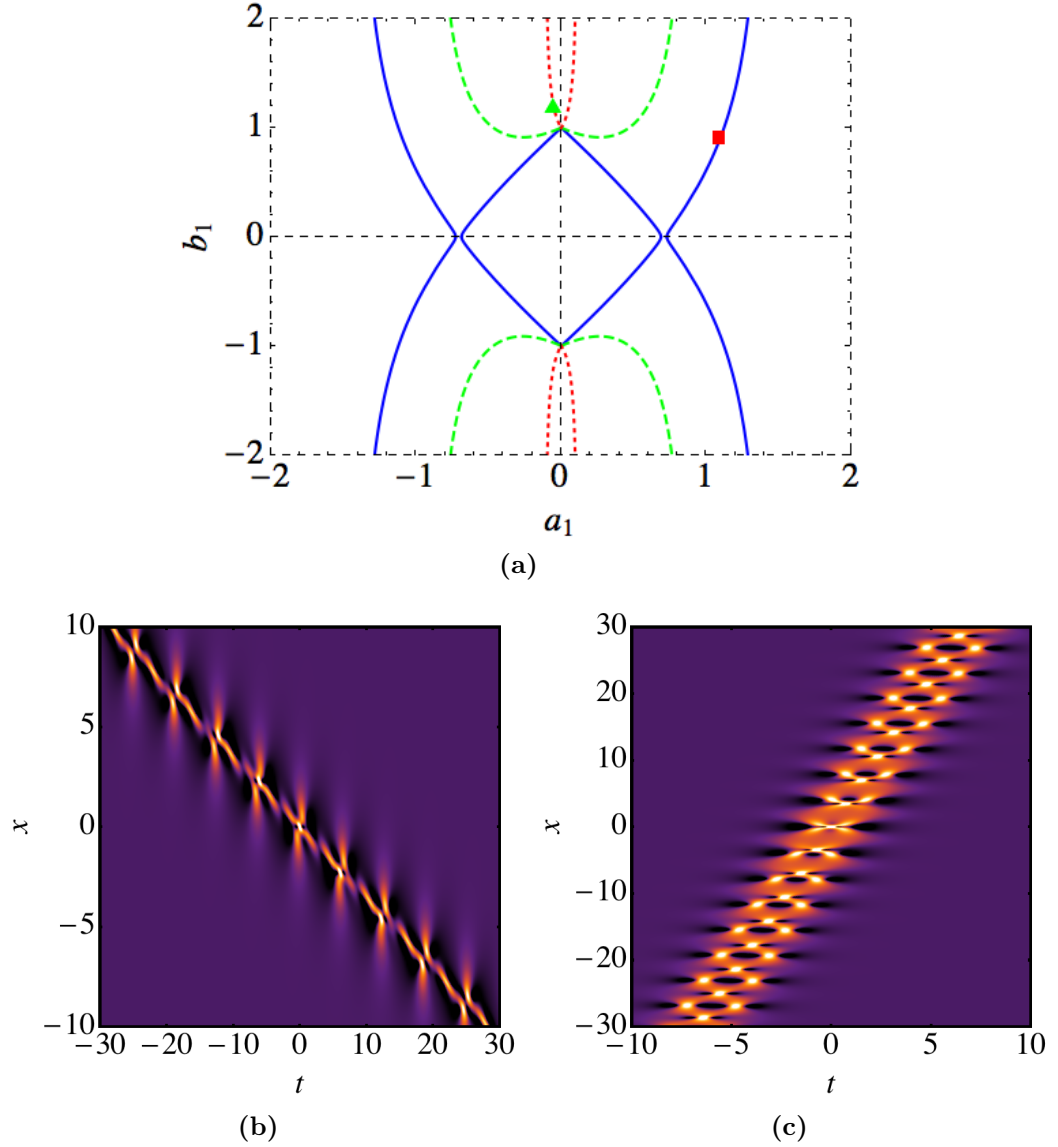


Figure 5.9 (a) Locus of points on the complex plane, $\lambda_1 = a_1 + ib_1$, that lead to co-propagating moving breathers for fixed $\lambda_2 = 0.08 + 0.9i$ (solid blue curves), $\lambda_2 = -0.08 + 1.5i$ (red dotted curves) and $\lambda_2 = -0.1 + 0.95i$ (green dashed curves). The thick green triangle and red square are chosen to illustrate the moving breather superpositions. (b) Two examples of moving breather superpositions of the NLSE. The eigenvalues are $\lambda_1 = 1.09458 + 0.9i$ and $\lambda_2 = 0.08 + 0.9i$ (red square in Fig. (5.9a)) and (c) $\lambda_1 = -0.0528537 + 1.2i$ and $\lambda_2 = -0.08 + 1.5i$ (green triangle in Fig. (5.9a)).

There is an unlimited number of combinations that allow us to construct moving breather superpositions. Here, we present only two examples that correspond to the thick coloured green triangle and red square in Fig. (5.9a). They are given in Fig. (5.9b-5.9c). The left panel in this figure corresponds to $\lambda_1 = 1.09458 + 0.9i$

and $\lambda_2 = 0.08 + 0.9i$ (red thick square). The right panel corresponds to $\lambda_1 = -0.0528537 + 1.2i$ and $\lambda_2 = -0.08 + 1.5i$ (green thick triangle).

A common feature of these two examples is the beating pattern of the superposition. As the periods of the two breathers are generally incommensurate, none of the patterns ever exactly repeats itself. One of the differences between the two cases is that the two superpositions are propagating in opposite directions, as they should because of the opposite signs of the real parts of the eigenvalues in these two examples. Another difference is the width of the superpositions. This is also caused by the differences in eigenvalues.

5.7. Conclusion

We have considered various cases of transformations of breather solutions into moving solitons and breathers. This relates to the Hirota equation with two arbitrary real parameters. The presence of free parameters allows us to treat the general case, as well as particular cases. The latter includes the NLSE. We have shown that there is a specific relationship between the coefficients of the equation and the real part of the eigenvalue that causes transformations of breather solutions to solitons. We have also found the equation which connects the eigenvalues of separate breathers that allows us to transform them into moving breather superpositions.

5.8. Appendix: Second-order Breather Solution with Complex Eigenvalues

The Darboux transformation technique was first applied to breather solutions in [81]. Here, we will follow our previous calculations, given explicitly in Appendices A and B of [75], except that we keep arbitrary coefficients of the Hirota equation in the results. Also, some of the notations will be changed in the present calculation. We separate the real and imaginary parts of variables and the function arguments where necessary. For example, $\kappa_j = 2\sqrt{1 + \lambda_j^2} = \kappa_{jr} + i\kappa_{ji}$ with $\lambda_j = a_j + ib_j$

and $\chi_j = \arccos\left(\frac{\kappa_j}{2}\right) = \chi_{jr} + i\chi_{ji}$. We can write the r and s functions in terms of trigonometric functions of $G = A_r + iA_i$ and $H = B_r + iB_i$:

$$r_{1,1} = 2ie^{-i\eta x} \sin(G), \quad s_{1,1} = 2e^{i\eta x} \cos(H).$$

where:

$$\begin{aligned} A_r &= \chi_{1r} + \frac{1}{2}(\kappa_{1r}t + d_{1r}x) - \frac{\pi}{4}, \quad A_i = \chi_{1i} + \frac{1}{2}(\kappa_{1i}t + d_{1i}x) \\ B_r &= -\chi_{1r} + \frac{1}{2}(\kappa_{1r}t + d_{1r}x) - \frac{\pi}{4}, \quad B_i = -\chi_{1i} + \frac{1}{2}(\kappa_{1i}t + d_{1i}x). \end{aligned}$$

while $d_1 = d_{1r} + id_{1i}$ is the complex coefficient in front of the argument x with

$$\begin{aligned} d_{1r} &= 2\eta(-b_1\kappa_{1i} + a_1\kappa_{1r}) + \alpha(\Omega_1\kappa_{1i} + \mathcal{U}_1\kappa_{1r}) \\ d_{1i} &= 2\eta(a_1\kappa_{1i} + b_1\kappa_{1r}) + \alpha(\mathcal{U}_1\kappa_{1i} - \Omega_1\kappa_{1r}). \end{aligned}$$

where $\Omega_1 = 8a_1b_1$ and $\mathcal{U}_1 = 2 - 4a_1^2 + 4b_1^2$. Then, the first order result is

$$\psi_1 = e^{2i\eta x} \left[1 + \frac{8}{D_1} ib_1 \cosh(B_i - iB_r) \sinh(A_i + iA_r) \right],$$

where $D_1 = \cos(2B_r) + \cosh(2A_i) + \cosh(2B_i) - \cos(2A_r)$. Two other linear functions $r_{1,2}$ and $s_{1,2}$ can be written similarly, *viz.* $C = C_r + iC_i$ and $D = D_r + iD_i$:

$$r_{1,2} = 2ie^{-i\eta x} \sin(C), \quad s_{1,2} = 2e^{i\eta x} \cos(D),$$

where

$$\begin{aligned} C_r &= \chi_{2r} + \frac{1}{2}(\kappa_{2r}t + d_{2r}x) - \frac{\pi}{4}, \quad C_i = \chi_{2i} + \frac{1}{2}(\kappa_{2i}t + d_{2i}x) \\ D_r &= -\chi_{2r} + \frac{1}{2}(\kappa_{2r}t + d_{2r}x) - \frac{\pi}{4}, \quad D_i = -\chi_{2i} + \frac{1}{2}(\kappa_{2i}t + d_{2i}x) \end{aligned}$$

while $d_2 = d_{2r} + id_{2i}$ with

$$\begin{aligned} d_{2r} &= 2\eta(-b_2\kappa_{2i} + a_2\kappa_{2r}) + \alpha(\Omega_2\kappa_{2i} + \mathcal{U}_2\kappa_{2r}) \\ d_{2i} &= 2\eta(a_2\kappa_{2i} + b_2\kappa_{2r}) + \alpha(\mathcal{U}_2\kappa_{2i} - \Omega_2\kappa_{2r}) \end{aligned}$$

where $\Omega_2 = 8a_2b_2$ and $\mathcal{U}_2 = 2 - 4a_2^2 + 4b_2^2$. In these notations, the second order linear functions $r_{2,1}$ and $s_{2,1}$ are (see Eqs.(7) of [81]):

$$\begin{aligned} r_{2,1}^* &= -\frac{2}{D_1} e^{i\eta x} \left(4ib_1 \cosh(B_i - iB_r) \cosh(D_i + iD_r) \times \sinh(A_i + iA_r) + \sinh(C_i + iC_r) \right. \\ &\quad \times \left\{ \cosh(2A_i) [-a_1 + a_2 + i(b_1 - b_2)] + \cos(2A_r) [a_1 - a_2 - i(b_1 - b_2)] \right. \\ &\quad \left. \left. + [\cos(2B_r) + \cosh(2B_i)] [-a_1 + a_2 - i(b_1 + b_2)] \right\} \right) \end{aligned} \quad (5.10)$$

$$\begin{aligned} s_{2,1} &= -\frac{2}{D_1} e^{i\eta x} \left(4ib_1 \cosh(B_i - iB_r) \sinh(A_i + iA_r) \times \sinh(C_i - iC_r) + \cosh(D_i - iD_r) \right. \\ &\quad \times \left\{ \cosh(2A_i) [a_1 - a_2 - i(b_1 + b_2)] + \cos(2A_r) [-a_1 + a_2 + i(b_1 + b_2)] \right. \\ &\quad \left. \left. + [\cos(2B_r) + \cosh(2B_i)] [a_1 - a_2 + i(b_1 - b_2)] \right\} \right) \end{aligned}$$

The second order solution of the Hirota equation follows from here directly:

$$\psi_2 = \psi_1 + \frac{2(\lambda_2^* - \lambda_2)r_{2,1}^*s_{2,1}}{|r_{2,1}|^2 + |s_{2,1}|^2}. \quad (5.11)$$

CHAPTER 6

Breather and Rational Solutions of the mKdV Equation

6.1. Introduction

The modified Korteweg-de Vries (mKdV) equation can be obtained from the Korteweg-de Vries (KdV) equation by using the Miura transformation [189]. Consequently, the solutions of the KdV equation can be transformed into the mKdV solutions using the same transformation. Both equations are integrable and have infinitely many conserved quantities [190]. The mKdV equation and its further modifications are found to describe pulses consisting of a few optical cycles [191–193] and in modelling supercontinuum generation in optical fibres [194]. The mKdV equation has many other applications in various fields such as soliton propagation in lattices [195], nonlinear Alfvén waves propagating in plasma [196] and meandering ocean currents [197]. It can also be applied to the dynamics of traffic flow [198–200]. Furthermore, the mKdV equation is related to Schottky barrier transmission lines [201], ion acoustic soliton experiments in plasmas [202] and fluid mechanics [203].

We can also mention the Gardner equation that has an intimate connection with the KdV and mKdV equations. The one-dimensional form of this equation can be

written as:

$$\frac{\partial \psi}{\partial x} + c_1 \psi \frac{\partial \psi}{\partial t} + \frac{\partial^3 \psi}{\partial t^3} = c_2 \psi^2 \frac{\partial \psi}{\partial t} \quad (6.1)$$

When $c_2 = 0$, Eq. (6.1) reduces to the KdV equation while for $c_1 = 0$ it gives the mKdV equation [46, 204]. Because of this close connection, sometimes the extended Korteweg-de Vries equation is considered as a Gardner equation, and with appropriate scaling and variable transformations, all the solutions of the Gardner equation become solutions of the KdV or mKdV equation [205, 206]. For convenience, we write the mKdV equation in the following way:

$$\frac{\partial \psi}{\partial x} - \alpha \left(\frac{\partial^3 \psi}{\partial t^3} + 6\psi^2 \frac{\partial \psi}{\partial t} \right) = 0, \quad (6.2)$$

where $\psi = \psi(x, t)$ is a real function with evolution variable x and transverse variable t . We explicitly include an arbitrary real parameter, α , in the equation in order to have consistent notation with our previous results [170, 207]. It can be removed by rescaling the x -variable: $\alpha x \rightarrow x'$. The equation containing the Hirota operator alone [137, 146] differs from Eq. (6.2) in that ψ^2 in the third term is replaced by $|\psi|^2$. (This Hirota operator equation is sometimes called the ‘complex modified Korteweg-de Vries equation’ [208, 209], but this label is misleading, since it is in the NLSE family, not the KdV family). A real solution of the Hirota equation will also solve the mKdV. The mKdV admits a scaling transformation. Namely, if $\psi = \psi(x, t)$ is a solution of Eq. (6.2), then

$$\psi' = q\psi(q^3x, qt) \quad (6.3)$$

is also a solution of Eq. (6.2) for arbitrary real q . For example, for the basic soliton solution $\psi = \text{sech}(t + \alpha x)$, the extended one-parameter family of solutions can be written as $\psi = q \text{sech}(qt + \alpha q^3 x)$, where q is an arbitrary real number. This scaling applies to all mKdV solutions including those given in this work. We will provide solutions with $q = 1$ as unit-background solutions. The mKdV can also be written in the form of the continuity equation [210]:

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial t} [\alpha (\psi_{tt} + 2\psi^3)].$$

In contrast to KdV, the mKdV equation is not invariant under Galilean transformation. Explicit solutions of the mKdV have been derived using various methods. For instance, Hirota derived the exact envelope soliton for the mKdV equation in [140] and for multiple collision of solitons with different amplitudes in [211]. The inverse scattering technique (IST) has been developed to solve a range of nonlinear evolution differential equations, including the mKdV equation in [212]. The IST for the mKdV equation was given by Tanaka [213]. Wadati used the IST to obtain the exact N -soliton solution for the mKdV [214]. Direct methods can also be used to derive certain solutions. Some classes of periodic solutions of mKdV have been derived in [215]. In this work, using a Darboux transformation with seeding solution $\psi = 1$, we derive exact breather and rational solutions. These correspond to imaginary eigenvalues of the IST.

The Lax-pair for the mKdV equation was given by Wadati [214]. We note that Wadati used the opposite sign for α . Our notation below is adapted to Eq. (6.2). Now, two linear equations

$$\begin{aligned}\frac{\partial R}{\partial t} &= U \cdot R, \\ \frac{\partial R}{\partial x} &= V \cdot R,\end{aligned}\tag{6.4}$$

are such that the ‘zero-curvature’ condition:

$$U_x - V_t + [U, V] = 0,\tag{6.5}$$

will reproduce Eq. (6.2). Here U and V are 2×2 matrices with U given by

$$U = i \begin{bmatrix} \lambda & \psi(x, t)^* \\ \psi(x, t) & -\lambda \end{bmatrix},\tag{6.6}$$

while V is a matrix polynomial in eigenvalue λ . For the mKdV, it is a simple cubic polynomial that can be written in general form $V = \sum_{j=0}^3 \lambda^j V_j$, where submatrices

V_j are

$$V_j = i \begin{bmatrix} A_j & B_j^* \\ B_j & -A_j \end{bmatrix}, \quad (6.7)$$

with

$$\begin{aligned} A_0 &= -i\alpha(\psi_t^*\psi - \psi_t\psi^*), \quad B_0 = \alpha(2\psi^2\psi + \psi_{tt}), \\ A_1 &= 2\alpha\psi^2, \quad B_1 = -2i\alpha\psi_t, \quad A_2 = 0, \quad B_2 = -4\alpha\psi, \\ A_3 &= -4\alpha, \quad B_3 = 0. \end{aligned}$$

It is easy to check that substitution of the matrices U and V into Eq. (6.5) leads directly to Eq. (6.2)

6.2. Periodic Breather and First-order Rational Solutions

Using the the seeding solution $\psi = 1$, the imaginary eigenvalue $\lambda = ib$ and the same basic steps as in [81], we obtain the periodic breather of the mKdV:

$$\begin{aligned} \psi_1 &= -1 + \frac{\kappa^3}{2\kappa - 2\delta \cos[\kappa(t + vx)]} \\ &= -1 + \frac{\kappa^2}{2 - \sqrt{4 - \kappa^2} \cos[\kappa(t + vx)]}, \end{aligned} \quad (6.8)$$

where $\kappa = 2\sqrt{1 + \lambda^2}$, $\delta = \kappa\sqrt{1 - \frac{\kappa^2}{4}}$ and $v = \alpha(6 - \kappa^2)$. Thus we need $\kappa < 2$. This solution is shown in Fig. (6.1a). In contrast to the Akhmediev breather of the nonlinear Schrödinger equation (NLSE) [72, 80], which is localized in x and periodic in t , the mKdV breather is periodic in t but maintains constant amplitude of oscillations *viz.* $\sqrt{4 - \kappa^2}$ everywhere. The oscillations move with velocity v in the (x, t) -plane. The solution of Eq. (6.8) is real and contains a trigonometric function only. Here, κ is the frequency of the periodic function while δ defines the amplitude of oscillations, although it has the same form as the growth rate of modulation instability in the

NLSE case [72]. In our case, the modulation instability is absent. The solutions of the KdV and mKdV equations are modulationally stable [216–219], so there is no amplitude increase on propagation direction, x . This is clearly seen from Fig. (6.1a), where amplitude remains constant. Indeed, Fig. (6.1a), shows a periodic pattern with the frequency $\kappa = 2\sqrt{1-b^2} < 2$. The eigenvalue is $\lambda = ib$ where b is real. Then $\delta = b\kappa$ and the breather period is $T = \pi/\sqrt{1-b^2}$ along the transverse direction t . Breathers exist for $0 < b < 1$ and consequently their frequencies remain within $0 < \kappa < 2$. The longest period of breather oscillations occur in the limit $\kappa \rightarrow 0$.

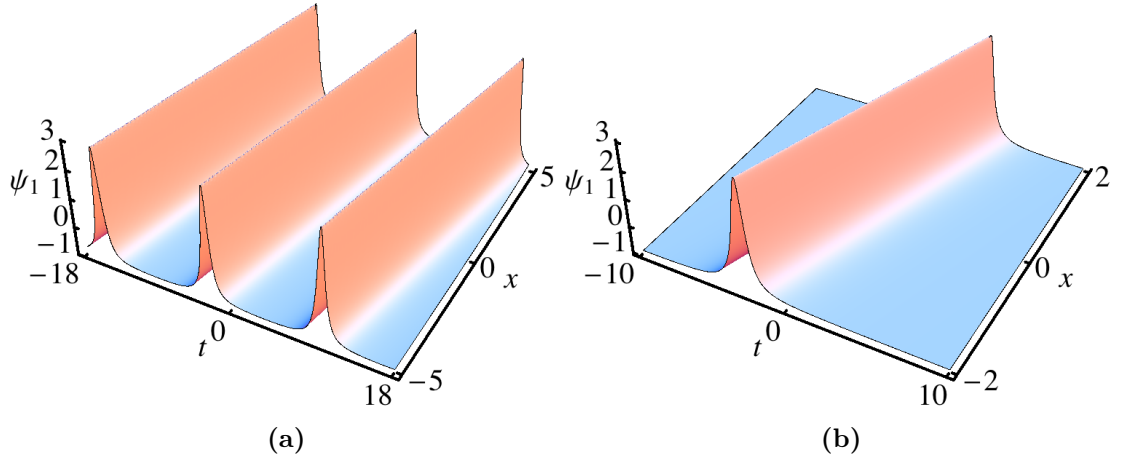


Figure 6.1 (a) A first order mKdV breather solution Eq. (6.8) with $\lambda = 0 + 0.97i$ (or $\kappa = 0.48621$) and $\alpha = -\frac{1}{10}$. (b) A first order mKdV rational solution Eq. (6.9) with $\alpha = -\frac{1}{4}$

Similar to the case of an Akhmediev breather that becomes a rogue wave [97, 171] in the limit $\kappa \rightarrow 0$, the mKdV breather of Eq. (6.8) in the same limit, $\kappa \rightarrow 0$, becomes a rational soliton:

$$\psi_1 = \left[-1 + \frac{4}{1 + 4(t + 6x\alpha)^2} \right], \quad (6.9)$$

This corresponds to the Peregrine solution of the NLSE [112, 220]. Adding to the similarity, the highest amplitude of the solution is 3, just as occurs with the Peregrine solution. However, instead of being doubly localized, the solution Eq. (6.9) is localized only in the t -direction and looks more like a soliton on a constant background, $\psi = -1$, moving with velocity v in the (x, t) -plane. Both solutions, Eq. (6.8) and Eq. (6.9) lack phase evolution which is an essential part of any NLSE solution. This is in accordance with the fact that the mKdV directly describes the wave profile rather than a wave envelope function.

6.2.1. Second order Breather Solution

The second step in the Darboux transformation scheme [81] provides us with the second-order breather of the mKdV equation:

$$\psi_2 = 1 + \frac{N_2}{D_2} \quad (6.10)$$

where

$$\begin{aligned} N_2 &= v_2(v_1^2 - v_2^2) \left[\frac{v_1}{2} \sin C_1 - 1 \right] \left[\sin C_2 - \frac{v_2}{2} \right] + v_1 \left[\sin C_1 - \frac{v_1}{2} \right] \left[\frac{v_2}{2} \sin C_2 - 1 \right] \\ &\quad \times (2v_2^2 + y_1) \\ D_2 &= -2v_1v_2 \left\{ \frac{\kappa_1\kappa_2}{4} \cos C_1 \cos C_2 + \left[\sin C_1 - \frac{v_1}{2} \right] \left[\sin C_2 - \frac{v_2}{2} \right] \right\} + y_1 \left[1 - \frac{v_1}{2} \sin C_1 \right] \\ &\quad \times \left[\frac{v_2}{2} \sin C_2 - 1 \right] \end{aligned}$$

and

$$\begin{aligned} C_1 &= \frac{1}{2} \left[\pi + 2x\alpha\kappa_1(6 - \kappa_1^2) + 2t\kappa_1 \right], \quad C_2 = \frac{1}{2} \left[\pi + 2x\alpha\kappa_2(6 - \kappa_2^2) + 2t\kappa_2 \right], \\ v_1 &= \sqrt{4 - \kappa_1^2}, \quad v_2 = \sqrt{4 - \kappa_2^2}, \quad y_1 = -8 + \kappa_1^2 + \kappa_2^2. \end{aligned}$$

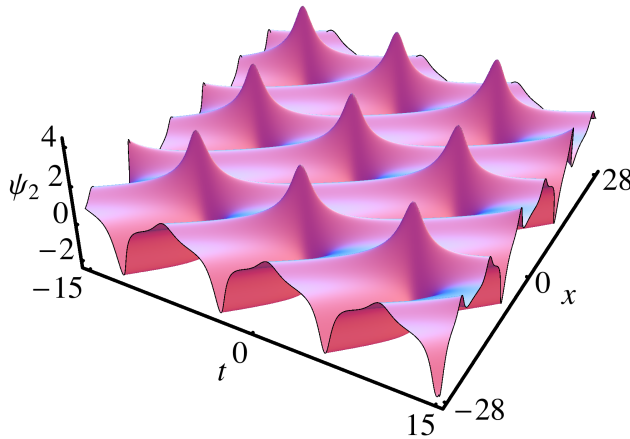


Figure 6.2 A second order mKdV breather solution Eq. (6.10) with $\kappa_1 = 0.7$, $\kappa_2 = 1.4$ and $\alpha = -\frac{1}{6}$.

Here, the maximum value is $1 + v_1 + v_2$ and the minimum value is $1 - v_1 - v_2$. So, the sum of the maximum and minimum values is always 2, independent of α ,

and the total structure always has height of $2(v_1 + v_2)$. The solution is shown in Fig. (6.2). This is the nonlinear superposition of the set of two periodic breathers obtained in the previous step crossing each other. Thus, the solution creates the two-dimensional ‘lattice’. Each individual wave train has specific values of frequencies, κ_1 and κ_2 , respectively. The maximum amplitude of the superposition is 4.30, while the minimum one is -2.30 . The maxima of the lattice appear at the positions of intersection of troughs of one periodic structure with the maxima of the other one. This happens with all second-order solutions below. Fig. (6.3) shows another example of a second order doubly-periodic breather with different values of κ_1 and κ_2 . The velocities of propagation of the two individual breathers, $\alpha(6 - \kappa_1^2)$ and $\alpha(6 - \kappa_2^2)$, are now closer to each other. Thus, the two periodic structures comprising the lattice are located at smaller angles to each other. Their periods differ significantly. In Fig. (6.3), the maximum amplitude is 4.57 and the minimum is -2.57 . The frequencies of the individual breathers, κ_1 and κ_2 , in the above solutions must remain within the range $0 < \kappa_1, \kappa_2 \leq 2$. When we have $\kappa_i \rightarrow 2$ for either one, the corresponding breather reduces to a constant amplitude background wave.

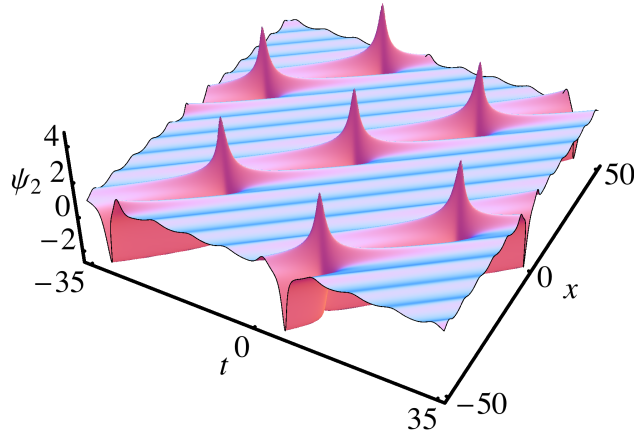


Figure 6.3 A second order mKdV breather solution Eq. (6.10) with $\kappa_1 = 0.98$, $\kappa_2 = 0.8$ and $\alpha = -\frac{1}{6}$.

Either of the frequencies, κ_1 or κ_2 , in the breather can be zero. When $\kappa_2 \rightarrow 0$, the solution is:

$$\psi_2 = 1 - \frac{\kappa_1^2 (-8 + 4 \cos(R_1) v_1 + (1 + 4p_1^2) \kappa_1^2)}{D_3} \quad (6.11)$$

with

$$\begin{aligned} D_3 = & -16p_1^2(-4 + v_1^2) - 16\sin(R_1)p_1v_1\kappa_1 + 4[4 - 4\cos(R_1)v_1 + v_1^2] \\ & + (1 + 4p_1^2)[-2 + \cos(R_1)v_1]\kappa_1^2 \end{aligned}$$

where

$$R_1 = (t - V_1x)\kappa_1, \quad p_1 = t + 6x\alpha, \quad V_1 = \alpha(-6 + \kappa_1^2).$$

The solution is significantly simplified and contains a rational component and only one wave frequency. This solution is shown in Fig. (6.4). The background here has only one periodic component. The presence of the rational component is revealed in a single ridge of higher amplitudes crossing the origin. The solution is also periodic along this line, as the ridge crosses the periodic background at a finite angle. The periodicity of the maxima along the ridge is defined by this angle. The maximal amplitude of periodic field variation along this line is 4.2.

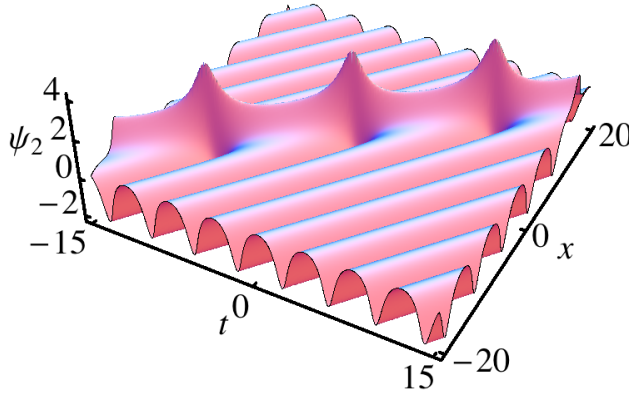


Figure 6.4 A second order mKdV breather solution Eq. (6.11) with $\kappa_1 = 1.6$ while $\kappa_2 \rightarrow 0$ and $\alpha = -\frac{1}{6}$.

6.2.2. Degenerate Breather Solution

Generally, the inverse scattering technique does not allow the presence of two equal eigenvalues in the solution. The solution becomes undefined when $\lambda_1 \rightarrow \lambda_2$, or,

equivalently $\kappa_1 \rightarrow \kappa_2$. Despite having this restriction, it is still possible to derive the solution in this limiting case. To proceed with the derivation, we set $\kappa_2 \rightarrow \kappa_1 + \epsilon$. Then, we take a Taylor series expansion of this expression in terms of ϵ and retain only the lowest order terms. This leaves us with a real second order degenerate mKdV solution:

$$\psi_2 = 1 + \frac{N_d}{D_d} \quad (6.12)$$

where

$$N_d = 4\kappa^2 v_a^2 \times \{ \kappa v_a^2 [t - 3\alpha(\kappa^2 - 2)x] \sin R_b - (\kappa^2 - 8) \cos R_b - 4v_a \}$$

and

$$\begin{aligned} D_d = & -8\kappa v_a^4 [t - 3\alpha(\kappa^2 - 2)x] \sin R_b - v_a^5 \cos(2R_b) - 8\kappa^2 v_a^2 \cos R_b \\ & - v_a \{ -2\kappa^6 (t^2 + 60\alpha t x + 468\alpha^2 x^2) + 12\alpha\kappa^8 x(t + 18\alpha x) \\ & + \kappa^4 [16(t + 6\alpha x)(t + 18\alpha x) + 1] - 8\kappa^2 [4(t + 6\alpha x)^2 + 1] \\ & - 18\alpha^2 \kappa^{10} x^2 - 16 \}, \end{aligned}$$

with

$$v_a = \sqrt{4 - \kappa^2}, V_b = \alpha(\kappa^2 - 6), R_b = (t - V_b x)\kappa$$

Here, $\kappa = \kappa_1 = \kappa_2$. An example of this degenerate solution of the mKdV is presented in Fig. (6.5a). The wave pattern of this solution consists of a single periodic ridge of high amplitude peaks on a periodic wave background. The background modulation is much weaker than the modulation along the periodic ridge. The maximum amplitude of the peaks along the ridge is $(\psi_2)_{max} = 5$. The line of peaks crosses the origin. The positions of the peaks are synchronised with the periodic wave structure of the background.

We can further simplify the solution of Eq. (6.11) by taking the other frequency $\kappa_1 \rightarrow 0$. Trigonometric functions then disappear and we obtain the second-order

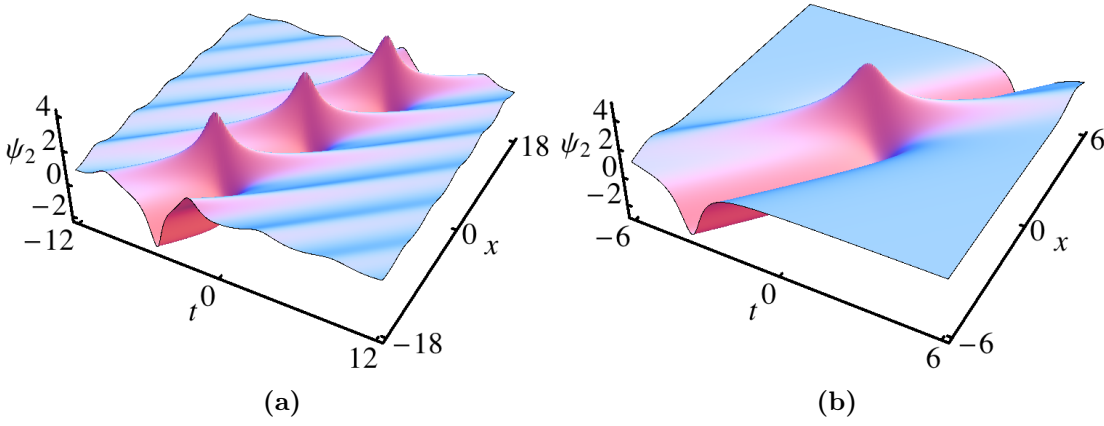


Figure 6.5 (a) A second order degenerate mKdV breather solution, Eq. (6.12), with $\kappa = 1.2$ and $\alpha = -\frac{1}{6}$. (b) A second order mKdV rational solution Eq. (6.13) with $\alpha = -\frac{1}{6}$.

purely rational solution of mKdV equation:

$$\psi_2 = 1 + 12 \frac{G_2}{D_2}, \quad (6.13)$$

where

$$G_2 = 3 - 8(6\alpha x + t) [2(6\alpha x + t)^3 + 3(22\alpha x + t)]$$

and

$$\begin{aligned} D_2 = & 48\alpha x \left[62208\alpha^4 x^4 (\alpha x + t) + 432\alpha^3 (60t^2 - 13) x^3 + 288\alpha^2 t (20t^2 - 9) x^2 \right. \\ & + 3\alpha (240t^4 - 120t^2 + 139) x + t (48t^4 - 8t^2 + 51) \left. \right] \\ & + 64t^6 + 48t^4 + 108t^2 + 9. \end{aligned}$$

This solution is shown in Fig. (6.5b). The solution is real without any periodicity and is located on a flat background, $\psi = 1$. The solution is seemingly a result of a collision of a dark and a bright solitons with slightly different velocities. The collision point is at $(0, 0)$ and has a single peak with the highest amplitude 5 at the origin. Remarkably, this value is the same as for the second-order rogue wave of the NLSE [75, 80, 183]. This solution is a proof that the mKdV also has a high-amplitude rational solution localized in two dimensions. The dark soliton part of this solution has a minimum around -2.75 , thus making the central part of the solution significantly higher than

the bottom of the trough.

Returning to the simple scaling of Eq. (6.3), we note that the rational solution can be transformed to arbitrary background a and arbitrary maximal amplitude $5a$. Then it is given by

$$\psi_2 = a \left(1 + 12 \frac{G'_2}{D'_2} \right) \quad (6.14)$$

where each x is replaced by $a^3 x$ and each t by $a t$ in the above expressions for G_2 and D_2 . Thus,

$$G'_2 = 3 - 8a (6a^2 \alpha x + t) \left[2a^3 (6a^2 \alpha x + t)^3 + 3a (22a^2 \alpha x + t) \right]$$

and similar modification applies for D'_2 .

6.2.3. Second order Rational Solution with Differential Shift

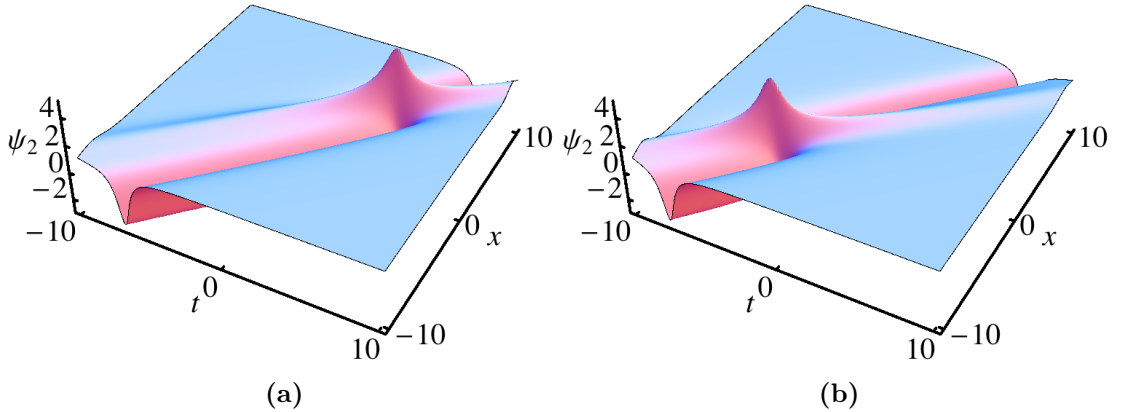


Figure 6.6 (a) A second order mKdV rational solution, Eq. (6.15), with differential shifts $x_d = -1$ and $t_d = 1$, when $\alpha = -\frac{1}{6}$. (b) A second order mKdV rational solution, Eq. (6.15), with differential shifts $x_d = 1$, $t_d = -1$, where $\alpha = -\frac{1}{6}$

Continuing the comparison with the rogue wave solutions of the NLSE, we recall that, for the NLSE equation, a form of the second-order rogue wave solution appears as a triplet of well-separated first-order rogue waves [102, 185]. This form requires two additional free parameters that move the rogue wave components out from the centre of the structure. We call these parameters “differential shifts” [103, 106]. They should not be confused with trivial translations along the x and t co-ordinates. In

contrast to the NLSE equation, the mKdV rational solution cannot be split into three components, as there are no first-order rogue waves here. Applying the “differential shifts”, x_d and t_d , to the solution of Eq. (6.13) in the same way as in [106, 207]. We set $c_a = x_d + 6\alpha t_d$ as the combined constant.

We find:

$$\psi_2 = 1 + \frac{N_t}{D_t} \quad (6.15)$$

where

$$\begin{aligned} N_t = & -12\{8(t + 6\alpha x)[-16c_a + 2t^3 + 6\alpha(6t^2 + 11)x \\ & + 216\alpha^2 tx^2 + 3t + 432\alpha^3 x^3] - 3\}, \end{aligned}$$

and

$$\begin{aligned} D_t = & 4\{256c_a^2 + 3456\alpha^3 x^3(8c_a + 20t^3 - 9t) \\ & + 36\alpha^2 x^2[24t(16c_a + 10t^3 - 5t) + 139] \\ & + 128c_a t^3 + 12\alpha x(192c_a t^2 - 176c_a + 48t^5 - 8t^3 + 51t) \\ & - 96c_a t + 16t^6 + 12t^4 + 5184\alpha^4(60t^2 - 13)x^4 \\ & + 27t^2 + 3(12\alpha x)^5(t + \alpha x)\} + 9. \end{aligned}$$

Now, the new solution, Eq. (6.15), has two more free real parameters, x_d and t_d . For zero values of x_d and t_d , the solution Eq.(6.15) coincides with the second-order rational solution of Eq. (6.13). Two examples of this solution with nonzero values of x_d and t_d are shown in Figs. (6.6a) and (6.6b). As we can see, the action of these parameters on the rational solution results in a shift of the peak along the “dark soliton” trough. This is different from elementary translations of the solution along the x and t axes. In the present case, the translation is eigenvalue-dependent.

6.3. Conclusion

In our work, we have found new breather solutions of the mKdV equation. A second order nonlinear superposition of them, constructed with the use of the Darboux

transformation, leads to doubly-periodic “lattice”-type structures. Limiting cases of these structures include a high-amplitude oscillating ridge on a single periodic background, soliton-like structures and, most remarkably, a rational solution on a constant background with an infinitely extended “dark soliton” trough. Breathers and rogue waves solutions of the NLSE are used mainly to explain wave dynamics in deep water and optical pulse dynamics in nonlinear optics. The corresponding classes of solutions of the mKdV equation derived in this work are fundamentally different from those of the NLSE. The solutions of the mKdV are related to the wave profile directly. These solutions reveal that the phenomena of breathers and rogue waves not only occur in the deep ocean. Being the rational solutions of the mKdV equation, they may appear in electromagnetic waves in quantized films, internal waves for some density stratifications, elastic media [221], etc., as well. Remarkably, in the deep ocean, modulation instability contributes to the formation of rogue waves through NLSE breather dynamics, while in the shallow water case we obtain rational solutions and solitons in the zero frequency limit ($\kappa \rightarrow 0$) of mKdV breathers. Hence, the deep and shallow water cases provide two different descriptions in hydrodynamics. In addition to shallow water wave dynamics, these new solutions will undoubtedly be useful and will give insight in modelling nonlinear wave processes in various other physical systems where mKdV is relevant.

CHAPTER 7

Rogue Wave Perturbations (Odd-Asymmetric)

7.1. Introduction.

Recent advances in supercontinuum generation in optical fibres [67, 132, 222–228] have led to the discovery of “optical rogue waves” [71]. A strict definition of the latter is still a subject of intense discussion [229]. Various mechanisms of optical rogue wave formation have been considered [69, 174, 230–232]. The mathematical description is mainly based on the nonlinear Schrödinger equation and its extensions [100, 233, 234]. Significant progress has been made on the experimental side with the recent observation of the Peregrine soliton in an optical fibre [112]. The latter is an idealized mathematical representation of a rogue wave based on the NLSE [89]. The question arises as to how this idealization changes if we add higher order terms, representing effects that are always present, into the equations describing nonlinear wave propagation in an optical fibre [129, 235].

The standard, dimensionless focusing NLSE can be written in the form [84]:

$$i\frac{\partial\psi}{\partial x} + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi = 0, \quad (7.1)$$

where x is the normalized propagation distance along the fibre, t is the retarded time

in a reference frame moving with the group velocity [51,131] and the function $|\psi(x, t)|$ is the field envelope. Essential modifications that have to be taken into account in realistic cases include a self-steepening term, $s_a \frac{\partial}{\partial t}(|\psi|^2\psi)$, a term related to the self-frequency shift, $a_f \psi \frac{\partial}{\partial t}(|\psi|^2)$, and a 3-rd order dispersion term, $\gamma_3 \psi_{ttt}$ [100,131]. When these terms are added, the NLSE takes the form

$$i \frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{\partial^2 \psi}{\partial t^2} + |\psi|^2 \psi + i s \left[a_f \psi \frac{\partial}{\partial t}(|\psi|^2) + s_a \frac{\partial}{\partial t}(|\psi|^2 \psi) - \gamma_3 \psi_{ttt} \right] = 0. \quad (7.2)$$

The above-mentioned terms include the main general effects in the primary order corrections to the NLSE, with the coefficients s_a , a_f and γ_3 being three independent parameters (of order 1) controlling the relative contribution of each effect. An overall multiplicative factor, s , plainly represents the smallness of the three terms. Thus, effectively, only three parameters are independent. When $s = 0$, the equation reduces to the NLSE, Eq. (7.1). Each additional term introduces asymmetry along the t -axis and represents a higher-order correction to the group velocity dispersion term. These three terms have been considered by various authors, e.g. [134,140,236–238]. If a rogue wave solution, even an approximate one, could be found for arbitrary values of these three parameters, the problem could be considered solved for many cases of interest. However, this is not the case.

It turns out that Eq. (7.2), is integrable for a few combinations of the 3 parameters [140,236,239]. In each case, the rogue wave solution can be found analytically [94,141]. However, these are exceptional cases. Extending the solutions beyond the integrable limits should significantly widen the applicability of the solutions. The fundamental rational solution [89,183] of the NLSE (7.1)

$$\psi = \left[4 \frac{1 + 2ix}{1 + 4x^2 + 4t^2} - 1 \right] e^{ix}, \quad (7.3)$$

is commonly known as the Peregrine soliton [112]. This solution focuses the energy of the background carrier wave into a spatio-temporally localized formation due to the nonlinear properties of the fibre. In this work, we provide a few approximations relevant to the extended Eq. (7.2) and, using numerical simulations, we show that these approximations do describe the propagation. Of course, the accuracy depends

on the values of s and the three parameters chosen in the r.h.s. of Eq. (7.2). Approximate results around the NLSE, to first order in s , have been presented earlier in [100]. Here, we consider solutions close to other integrable cases, namely the Hirota equation and the Sasa-Satsuma equation. Obtaining approximate rational solutions close to these two cases still requires certain relations between the coefficients which we illustrate graphically. Our numerical simulations confirm that the new approximations are indeed valid.

7.1.1. Rogue Waves in the Integrable Sasa-Satsuma case

We start with the integrable Sasa-Satsuma case [236] which arises from Eq. (7.2) when $\gamma_3 = 1$, $a_f = 3$ and $s_a = -6$. An exact expression for its rogue wave solution has been presented recently in [94]. The solution is quite involved and we will not rewrite it here. The exact solution for two sets of parameters ($c = 0.18$, $\epsilon = s = 0.09$) and ($c = 0.24$, $\epsilon = s = 0.12$) is illustrated in Fig. (7.1). It has a distinct double peak structure with the contrast between the maxima and the central amplitude depending on ϵ and s . Interestingly, the possibility of such structures had been also predicted using approximate solutions [100].

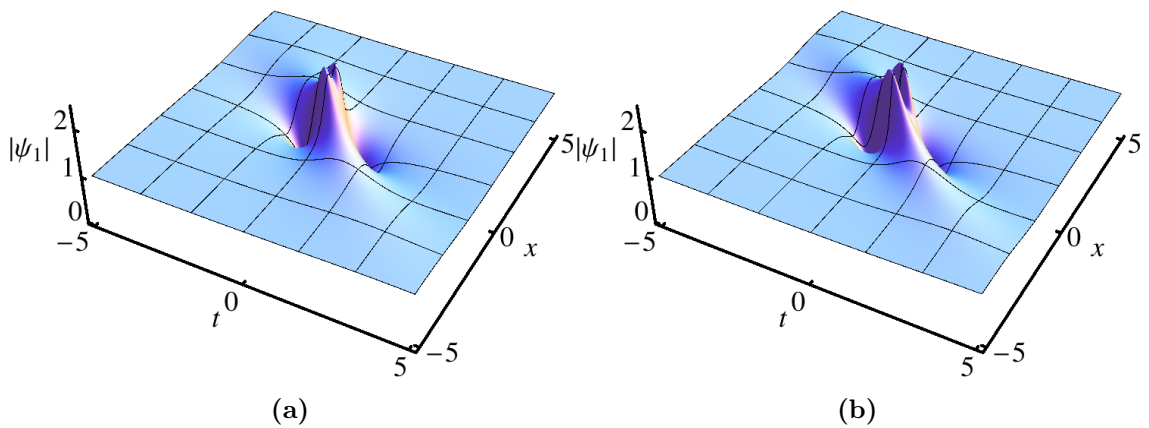


Figure 7.1 Two-dimensional profile of the exact rogue wave solution of the Sasa-Satsuma equation based on the formulae (10) and (11) of [94]). Here $k = 0$ and (a) $c = 0.18$, $\epsilon = s = 0.09$, (b) $c = 0.24$, $\epsilon = s = 0.12$.

In order to show that this rogue wave solution may appear in a real wave field, we have numerically solved Eq. (7.2), using a split step Fourier method. Fig. (7.2) shows

the evolution of the maximum field amplitude $|\psi(t)|_{\max}$ vs. x , for the special case of the Sasa-Satsuma equation, taking as initial conditions a small bump on a constant background. The initial shape of the bump is not crucial. To be specific, in this example we used the Peregrine breather, Eq. (7.3), as the initial condition at $x = -6$. The values of the equation parameters used in this simulation are written inside Fig. (7.2). The figure clearly shows that the initial evolution up to $x = 9$ describes the exact SSE rogue wave solution with its distinctive double peak structure at the top of the solution. The maximal amplitude is also very close to the exact value of 2.5. This first stage of evolution is shown in Fig. (7.3a). This figure confirms that the wave profile is indeed very similar to the shape of the exact solution presented in Fig. (7.1).

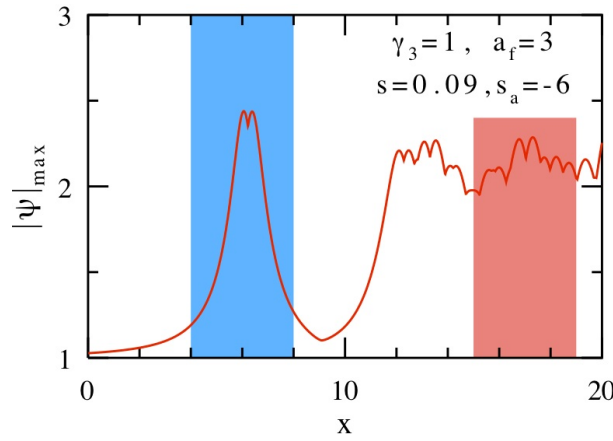


Figure 7.2 Excitation of the rogue wave of the Sasa-Satsuma equation (within the blue region) in propagation starting from the initial condition given by Eq. (7.3) at $x = -6$. Further evolution is chaotic with repeated double peaks, characteristic of the wave fields governed by the SSE.

After $x = 10$ the field evolves chaotically with several double peak structures developed in propagation. As expected, each nearest pair of maxima are centered at different values of t . The amplitude profile of one of them is shown in Fig. (7.3b). As the rest of the field is chaotic, the rogue wave is immersed in a sea of smaller waves.

Various initial conditions with a small perturbation of the constant background led to a similar chaotic evolution. Fig. (7.4) shows a small part of the contour plot of the numerical solution of the SSE (7.2) with the same parameters, but with initial conditions in the form of a small random function $R(t)$ added to a constant background:

$$\psi(t, 0) = 1 + \mu R(t) \quad (7.4)$$

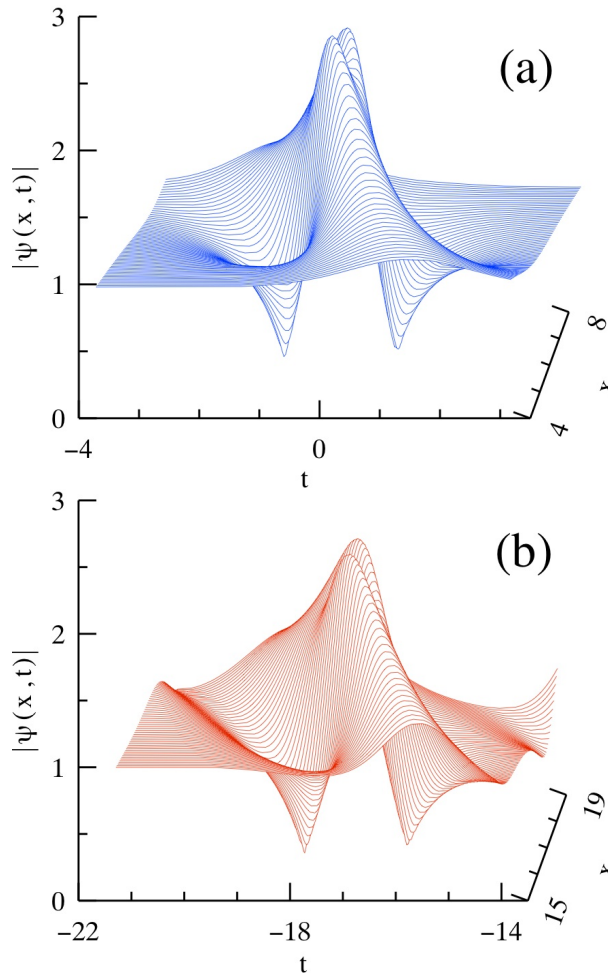


Figure 7.3 Wave profiles of the rogue waves of the Sasa-Satsuma equation within (a) the blue and (b) red regions of Fig.(7.2). Each one has a shape very close to that given by the exact solution and shown in Fig. (7.3)(a).

where μ is a small parameter. After the initial exponential growth of perturbations, the field evolves straight into the chaotic regime without a rogue wave being excited first. However, rogue waves appear in the random field generated this way. Fig.(7.4a) contains two of them that are marked by the black circles. Magnified contour plots of these two parts of the solution are shown in Figs.(7.4b) and (7.4c). Their profiles are similar to the exact rogue wave shape but distorted by the surrounding parts of the chaotic field.

7.1.2. Approximations around the Sasa-Satsuma Equation

The SSE is a very particular way to extend the NLSE for a more realistic description of waves in an optical fibre. However, the integrability of this equation imposes

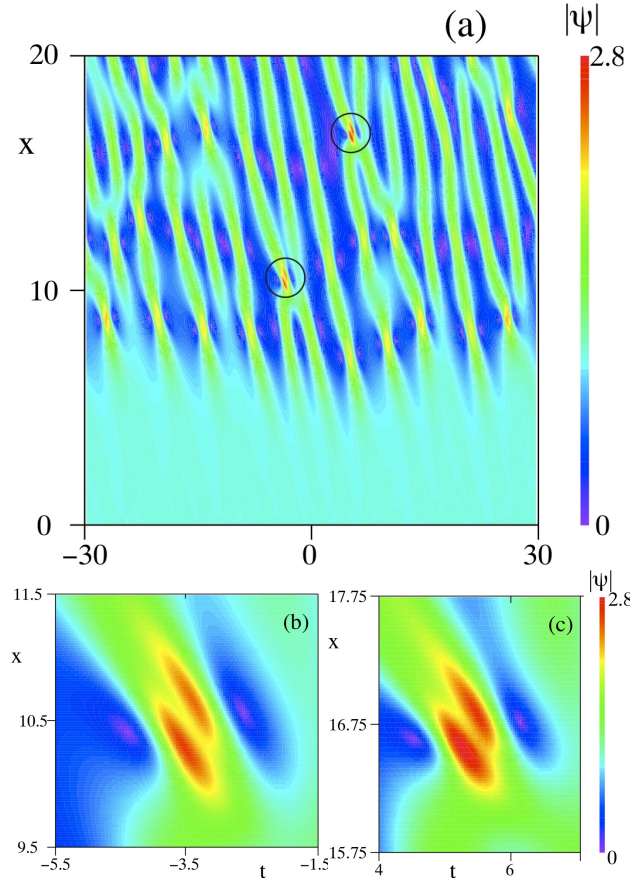


Figure 7.4 (a) Small part of the chaotic wave field generated by the SSE for an initial condition consisting of a plane wave randomly perturbed. Two rogue waves are marked by black circles. Panels (b) and (c) represent the magnification of the encircled parts of the plot in (a). The value of the parameters is the same as in the preceding figure, namely: $\gamma_3 = 1$, $a_f = 3$, $s = 0.09$ and $s_a = -6$.

strict values for the equation parameters. In order to partially lift this restriction, we will find approximate solutions of Eq. (7.2) in terms of polynomials, when the parameters are on a line that includes the SSE. However, this cannot be done easily for an arbitrary relation between the parameters of the extended Eq. (7.2) but only on certain lines in the space of parameters. For convenience, we define a constant k_n by:

$$k_n = 6\gamma_3 + 3a_f + 5s_a. \quad (7.5)$$

Without loss of generality, we will take $\gamma_3 = 1$ and consider the plane of the two other parameters (s_a, a_f) . This plane is shown in Fig. (7.5). The integrable case of the SSE is shown on this plane by the blue diamond. The dashed line with the negative slope which admits approximate solutions, including the SSE, is shown in blue color.

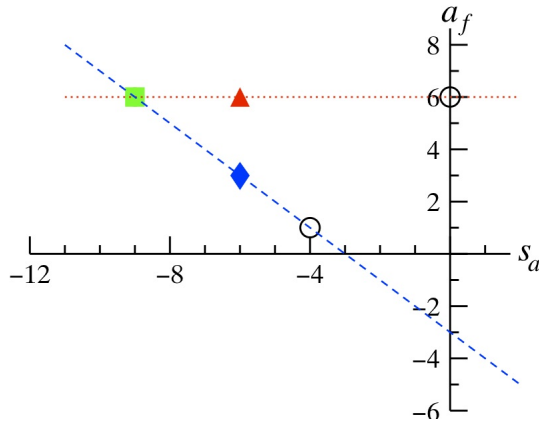


Figure 7.5 (Color online) The plane of the parameters (s_a, a_f) considered in this chapter. With no loss of generality, we have set $\gamma_3 = 1$. The blue diamond on the sloping dashed line corresponds to the parameters of the Sasa-Satsuma equation while the triangle directly above it indicates the parameters of the Hirota equation. The dashed blue line with negative slope gives the extension of the Sasa-Satsuma equation while the horizontal dotted red line gives the extension of the Hirota equation. The intersection point of the lines is marked by the green square.

It happens that we can approximate the solution around the integrable SSE case by taking s_a to be arbitrary and considering the line $a_f = -3 - s_a$. Using the parameters satisfying this relation allows us to find approximate solutions. These cases are not integrable unless $s_a = -6$. We can write the approximate solution to second order in s in the form:

$$\begin{aligned} \frac{\psi(x, t)}{e^{ix}} &= -1 + \frac{4}{D}(1 + 2ix) + \frac{8ts}{D^2}[4k_n x + i f_a(x, t)] \\ &+ \frac{4s^2}{D^3}[k_b(x, t) + 2ix f_b(x, t)], \end{aligned} \quad (7.6)$$

where $D = 1 + 4x^2 + 4t^2$. We now find that $k_n = 2s_a - 3$ and the polynomials $f_a(x, t)$, $k_b(x, t)$ and $f_b(x, t)$ are:

$$\left(\begin{array}{lcl} f_a(x, t) & = & 3 - s_a + 4t^2(s_a + 3) + 12x^2(s_a - 1) \\ k_b(x, t) & = & 9[-1 + 64t^4 + 44x^2 + 192x^4 + 4t^2(3 + 80x^2)] \\ & & + 3s_a[1 + 48t^4 + 24x^2 + 80x^4 - 16t^2(3 + 8x^2)] \\ & & + 4s_a^2[4t^4 + t^2(28x^2 - 3) - 6x^2(1 + 4x^2)] \\ f_b(x, t) & = & 9[-9 + 32t^4 - 20x^2 + 64x^4 + 4t^2(40x^2 - 7)] \\ & & - 6s_a[1 - 16x^4 + 4t^2(5 + 12x^2)] \\ & & + s_a^2[3 - 48x^4 + 4t^2(4x^2 - 9)] \end{array} \right)$$

The shape of this approximate rogue wave solution is shown in Fig. (7.6). Clearly, the shape is similar to the exact solution shown in Fig. (7.1). However, it changes noticeably as we vary s_a . The solution has a single peak when $s_a > 0$, two peaks when $s_a < 0$ (including the Sasa-Satsuma case) and up to 4 peaks when $s_a \approx -9$. The latter does not happen in the SSE case.

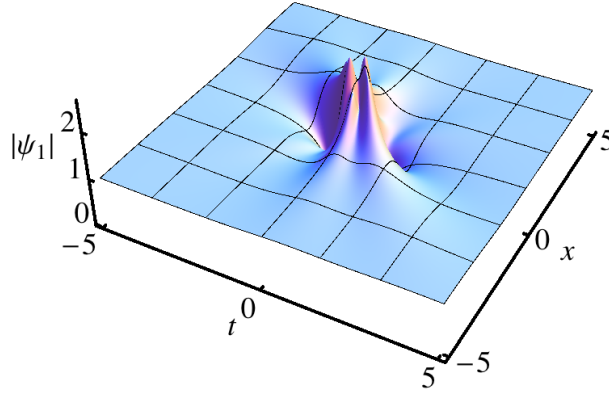


Figure 7.6 Approximation of the rogue wave near the Sasa-Satsuma equation given by Eq. (7.6). Here $s = 0.09$.

For the Sasa-Satsuma equation, where $s_a = -6$, we can add a third order term to the r.h.s. of Eq. (7.6), viz:

$$\frac{216}{D^4} s^3 t [8x k_d(x, t) - i f_d(x, t)], \quad (7.7)$$

with

$$\begin{pmatrix} k_d(x, t) &= & (4x^2 + 1)(344x^2 + 55) - 32t^4 - 4t^2(312x^2 + 35) \\ f_d(x, t) &= & 16t^4(16x^2 + 23) + 8t^2(736x^4 - 212x^2 - 13) \\ && -8192x^6 - 1872x^4 + 104x^2 + 15. \end{pmatrix}$$

Generally, this rogue wave makes an angle θ with the x -axis, which means that we have an effective ‘velocity’, $v = \tan(\theta)$. For small values of s and s_a , an approximation for the velocity is

$$v \approx \left(\frac{4}{3} s_a - 2 \right) s. \quad (7.8)$$

For the Sasa-Satsuma equation $s_a = -6$ and we thus have $v \approx -10s$. This is supported by the wave directions in Figs. (7.1a) and (7.6), where we can estimate that $v \approx -0.9$.

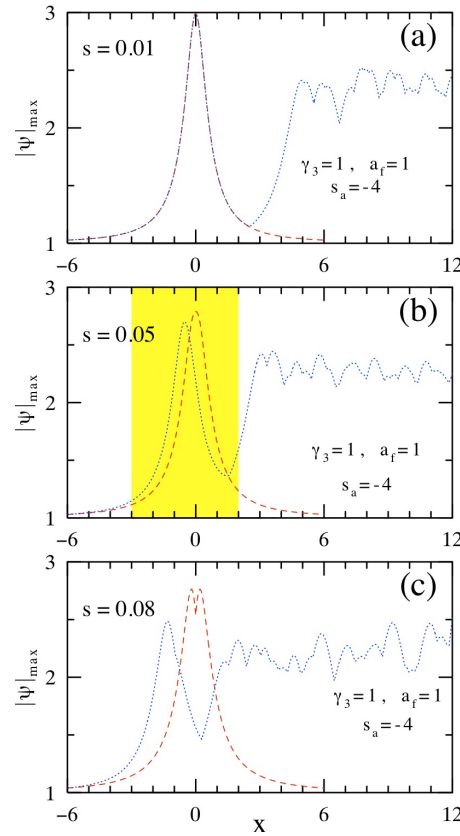


Figure 7.7 Wave propagation starting with the approximate rogue wave solution given by Eq. (7.6) at $x = -6$ for the parameters: $s_a = -4$ and $a_f = 1$. The three plots are made for (a) $s = 0.01$, (b) $s = 0.05$, (c) $s = 0.08$. The blue dotted line corresponds to the maximum amplitude of the field obtained in the numerical simulation while the red dashed line corresponds to the approximate solution Eq. (7.6).

In order to check the accuracy of our approximation, we numerically solve Eq. (7.2) using the approximate solution, namely Eq. (7.6), as the initial condition at $x = -6$. The results are shown in Fig. (7.7). The blue dotted line in each panel represents the maximum of the field amplitude $|\psi(t)|_{max}$ at each x , while the red dotted line follows the maximum of the amplitude of the approximate solution Eq. (7.6). For small values of s , simulations almost coincide with the approximate solution, as we can see from Fig. (7.7b). Small deviations start at $s = 0.05$ as Fig. (7.7b) shows. Comparison of the two rogue wave profiles for the case (b) of the x -interval located in the yellow panel is shown in Fig. (7.8). The two contour plots show very similar shapes and identical ‘velocities’. The correspondence becomes poor above the value $s = 0.08$, as can be seen from Fig. (7.7c). We can conclude, from these results, that approximations on the extended line which includes the SSE are well justified for values of s below 0.05.

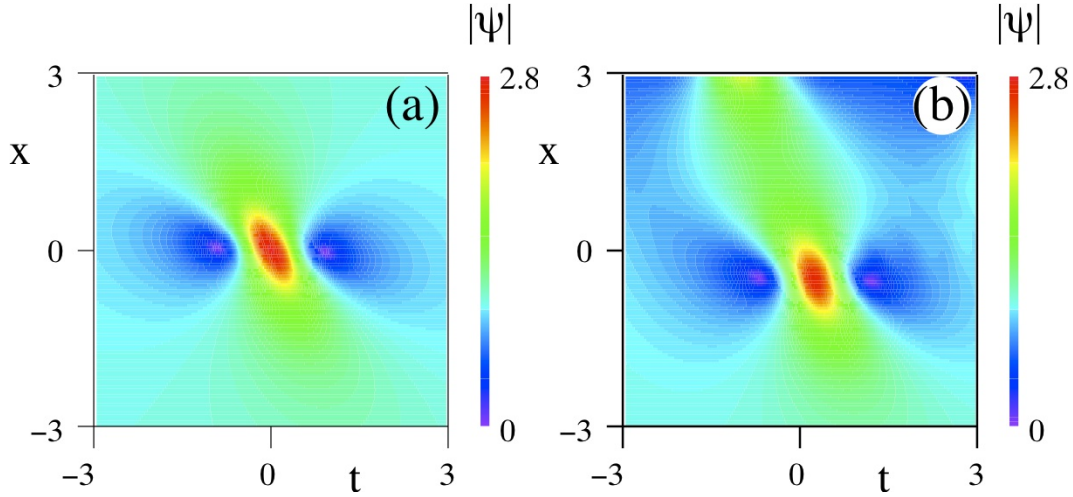


Figure 7.8 Contour plots of (a) approximate and (b) numerical profiles located within the yellow panel of Fig. (7.7b).

7.2. Approximations around the Integrable Hirota case.

The Hirota equation [140] is another integrable case when the rogue wave solution can be written in exact form [141, 173]:

$$\frac{\psi_{ex}(x, t)}{e^{ix}} = -1 + 4 \frac{(1 + 2ix)}{1 + 4(t + 6sx)^2 + 4x^2} \quad (7.9)$$

This is a special case corresponding to the point $s_a = -6$. We plot Eq. (7.9) in Fig. (7.9a) for $s = 0.12$. In order to consider cases beyond the integrable one, we take arbitrary s_a and consider the line $a_f = 6$ (with $\gamma_3 = 1$). This case is depicted in Fig. (7.5) by the horizontal dotted red line. Then we can write the approximate solution in the form of polynomials:

$$\begin{aligned} \frac{\psi(x, t)}{e^{ix}} = & -1 + \frac{4}{D}(1 + 2ix) + \frac{32ts}{D^2}[k_n x + i f_e(x, t)] \\ & + \frac{16s^2}{D^3}[k_c(x, t) + 2ix f_c(x, t)] + \dots, \end{aligned} \quad (7.10)$$

where the constant k_n in Eq. (7.10) is $k_n = 24 + 5s_a$ and the polynomials $f_e(x, t)$, $k_c(x, t)$ and $f_c(x, t)$ are:

$$\left(\begin{array}{lcl} f_e(x, t) & = & (s_a + 6)(2t^2 - 1) + 6(s_a + 4)x^2 \\ k_c(x, t) & = & s_a^2(32t^4 + 12t^2(17x^2 - 1) - 21x^2(4x^2 + 1)) \\ & & + 3s_a(144t^4 + 8t^2(80x^2 - 9) - 208x^4 - 48x^2 + 1) \\ & & + 18[1 + 80t^4 + 16t^2(16x^2 - 3) - 8(6x^4 + x^2)] \\ f_c(x, t) & = & s_a^2(8t^4 + 18t^2(2x^2 - 3) - 36x^4 + 3x^2 + 3) \\ & & + 3s_a(32t^4 + 4t^2(20x^2 - 53) - 80x^4 + 16x^2 + 9) \\ & & + 18(16t^4 + 8t^2(4x^2 - 13) - 16x^4 + 8x^2 + 3) \end{array} \right)$$

For this extended solution of the “near-Hirota equation” we find that at small s , say $|s| < 0.03$:

$$v \approx \left(\frac{8}{3}s_a + 10 \right) s. \quad (7.11)$$

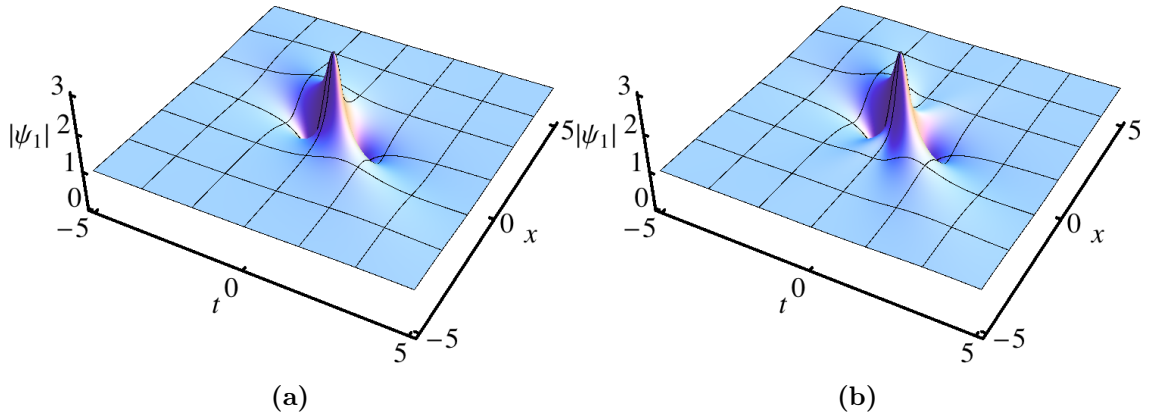


Figure 7.9 Profiles of (a) the exact rogue wave solution of the Hirota equation (7.9). (b) the approximation of the rogue wave of “near-Hirota” equation defined by Eq. (7.12). In each case $s=0.12$.

The special case of the Hirota equation is still included in this approximation. In this special case $s_a = -6$, the “velocity” $v \approx -6s$, and we can easily obtain more terms in the approximation. Then Eq. (7.10) can be extended to

$$\begin{aligned} \frac{\psi(x, t)}{e^{ix}} &= -1 + \frac{4}{D}(1 + 2ix) \left[1 - \frac{48xts}{D} - \frac{144x^2s^2}{D^2}(1 - 12t^2 + 4x^2) \right. \\ &\quad \left. + \frac{13824tx^3s^3}{D^3}(1 - 4t^2 + 4x^2) + \dots \right]. \end{aligned} \quad (7.12)$$

We plot the profile defined by Eq. (7.12) in Fig. (7.9b) for $s = 0.12$. To check the

validity of the expansion in Eq. (7.12), we can take the exact result Eq. (7.9) and also expand it:

$$\begin{aligned} \frac{\psi_{ex}(x,t)}{e^{ix}} = & -1 + \frac{4}{D}(1 + 2ix) \left[1 - \frac{48xts}{D} - \frac{144x^2s^2}{D^2}(1 - 12t^2 + 4x^2) \right. \\ & + \frac{13824tx^3s^3}{D^3}(1 - 4t^2 + 4x^2) + \frac{20736x^4s^4}{D^4}(1 - 40t^2 + 80t^4 \\ & \left. + 8x^2 - 160t^2x^2 + 16x^4) + \dots \right]. \end{aligned} \quad (7.13)$$

Comparison with Eq. (7.12) shows that the approach is indeed valid. It is not surprising that Figs. (7.9a) and (7.9b) are very similar. Here the velocity is exactly $v = -6s$, in agreement with the estimate given above. Consistency is seen from the fact that, at the point of intersection of the two lines shown in Fig. (7.5), i.e. the point $(a_f = 6, s_a = -9)$, the velocity approximations, Eqs. (7.8) and (7.11) both give the same result, *viz.* $v = -14s$.

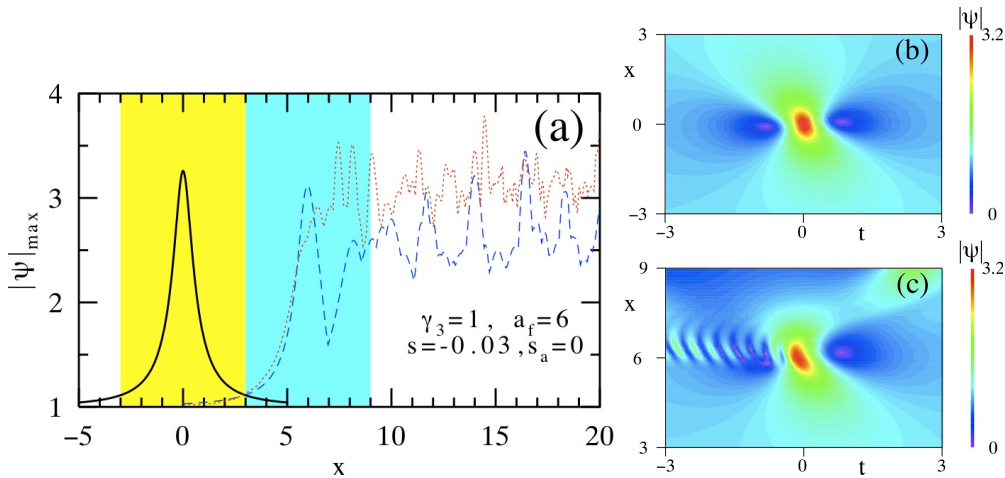


Figure 7.10 (a) Evolution of the maximum field along x governed by the "extended-Hirota" equation. The solid black curve shows the maximum amplitude of the approximate solution centered at $x = 0$. The dashed blue curve shows the evolution of the maximum value of the field with the initial condition Eq. (7.10) at $x = -6$ while the dotted red curve is for the initial condition (7.4). (b) Contour plot of the approximate solution in the (x,t) -plane (yellow part in (a)). (c) Contour plot of the first peak of the blue dashed curve (light blue colored part in (a)).

In Fig. (7.10), we present the results of numerical simulations that start with a perturbed constant background. The perturbation is either the approximate rogue wave solution (7.10) at $x = -6$ (dashed blue curve) or a chaotically perturbed unit background (7.4) (red dotted curve). The approximate rogue wave solution is shown by the solid black curve for comparison. We can see that the approximate rogue wave

solution is excited at the first stage of evolution in case of the blue curve. The contour plot of this part is presented in Fig. (7.10b). Further evolution becomes chaotic, as we would expect. When the initial condition is weakly chaotic, the evolution is also chaotic after the exponential growth of the chaotic perturbation. This case is shown by the red dotted curve. These simulations confirm again that the approximate polynomial solutions describe the rogue waves reasonably well at values of $|s| < 0.05$.

7.2.1. The case with no Third order Dispersion

One more case when the solution can be approximated by polynomials is when $\gamma_3 = 0$. In this case we take s_a arbitrary and consider the line $a_f = -s_a$. In Fig. (7.5), this would correspond to a straight line passing through the origin. However, we should remember that here $\gamma_3 = 0$ and the above line is at a different level in the three-dimensional parameter space. The Eq. (7.2) has now effectively only one physical effect added to the NLSE:

$$i \frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{\partial^2 \psi}{\partial t^2} + |\psi|^2 \psi + i s s_a \psi_t |\psi|^2 = 0. \quad (7.14)$$

The approximate polynomial solution becomes then a function of ss_a only:

$$\begin{aligned} \frac{\psi(x, t)}{e^{ix}} = & -1 + \frac{4}{D}(1 + 2ix) + \frac{8 t s s_a}{D^2}[8x + i f_r(x, t)] \\ & + \frac{8(ss_a)^2}{D^3}[2k_g(x, t) + ix f_g(x, t)] \\ & + \frac{32 t (ss_a)^3}{D^4}[2x k_h(x, t) + i f_h(x, t)] \\ & + \frac{16(ss_a)^4}{D^5}[k_j(x, t) + 2x i f_j(x, t)] + \dots, \end{aligned} \quad (7.15)$$

where the polynomials $f_r(x, t)$, $k_g(x, t)$, $f_g(x, t)$, $k_h(x, t)$, $f_h(x, t)$, $k_j(x, t)$ and $f_j(x, t)$ are:

$$\left(\begin{array}{rcl}
 f_r(x, t) & = & 4t^2 + 12x^2 - 1 \\
 k_g(x, t) & = & 4t^4 + t^2(28x^2 - 3) - 6x^2(4x^2 + 1) \\
 f_g(x, t) & = & 4t^2(4x^2 - 9) - 48x^4 + 3, \\
 k_h(x, t) & = & -16t^4 - 8(4t^2 + 3)x^2 - 24t^2 - 144x^4 + 3, \\
 f_h(x, t) & = & -16t^6 - 8t^4(8x^2 + 1) + t^2(-80x^4 - 56x^2 + 7) \\
 & & + 2x^2(1 + 4x^2)(19 - 20x^2), \\
 k_j(x, t) & = & -192t^8 - 32t^6(38x^2 + 1) - 4t^4(544x^4 + 84x^2 - 17) \\
 & & - 12t^2x^2(400x^4 - 88x^2 - 47) + (92x^2 - 13)x^2(4x^2 + 1)^2 \\
 f_j(x, t) & = & 304t^6 + 8t^4(8x^4 + 110x^2 + 29) \\
 & & + t^2(-640x^6 + 2480x^4 + 496x^2 - 41) \\
 & & + (20x^2 - 31)x^2(4x^2 + 1)^2.
 \end{array} \right)$$

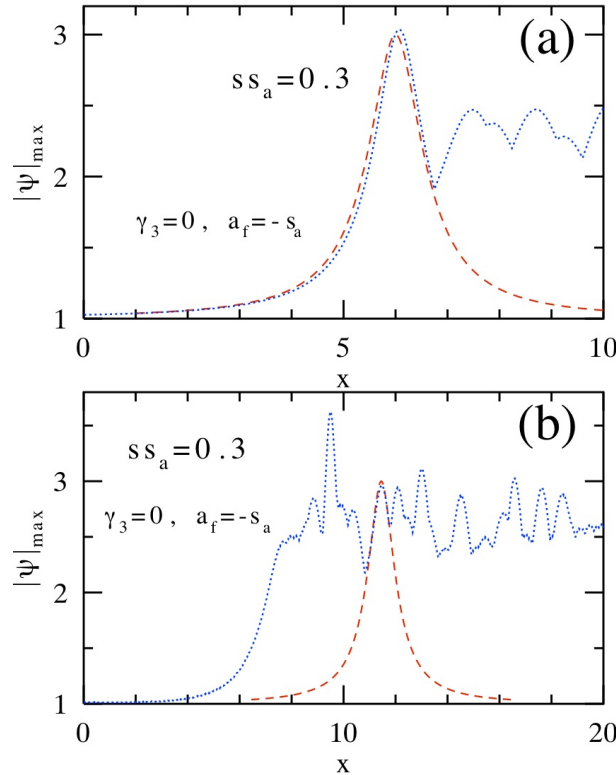


Figure 7.11 (a) Maximum of the field governed by Eq. (7.14), with the initial condition Eq. (7.15) at $x = -6$. (b) Maximum of the field governed by Eq. (7.14), with the initial condition as small amplitude chaotic perturbation of a constant background. Here $a_f = -s_a$, $\gamma_3 = 0$ and $s s_a = 0.3$.

We present the results of numerical simulations of Eq. (7.14) in Fig. (7.11) for the case $s s_a = 0.3$. Despite the relatively large deviation from the NLSE, the expression (7.15)

provides a good approximation for the solution. Fig. (7.11a) shows the evolution of the field maximum when the initial condition is Eq. (7.15) at the point $x = -6$. The red dashed line in this figure corresponds to the approximate solution while the blue dotted line shows the results of simulation. As we can see, the initial stages of evolution up until $x \approx 6.7$ are very close to the approximate solution. Beyond the line $x \approx 6.7$, new waves emerge from the surrounding area and the total field becomes chaotic. Still the first rogue wave observed in the simulations is very close to the approximate one well above $x = 6.7$. The dotted blue curve deviates from the dashed red one because we are representing the maximum of the total field amplitude and above $x \approx 6.7$, new waves emerging somewhere else from the background exceed the size of the tail of the rogue wave.

Fig. (7.11b) presents the results of numerical simulations with the initial condition in the form of a constant background perturbed slightly with the chaotic field (7.4). The evolution of the maximum in this case is chaotic with exponential growth of perturbation at the beginning. Just as in the two previous cases, some of the peaks can be identified as rogue waves very accurately described by the approximate solution of Eq. (7.15). To give an example, the approximate solution shown by the red dashed line in Fig. (7.11b) is adjusted close to one of these peaks.

7.3. Conclusions

In conclusion, we have presented approximate polynomial rogue wave solutions for the nonlinear Schrödinger equation, taking into account third order dispersion, self-steepening and self-frequency shift. These solutions can be found for special relations between the coefficients of the additional terms which include the cases when the equation is integrable (*viz.* the Sasa-Satsuma and Hirota equations). Our numerical simulations confirm the validity of the approximations when these terms are relatively small. Approximate solutions can potentially be helpful for experimentalists working on rogue waves in optical fibres [69] when short pulses are involved. Then, the additional terms play an important role and must be taken into account.

CHAPTER 8

Rogue Wave Perturbations (Even-Symmetric)

8.1. Introduction

Rogue wave research has progressed well for ocean waves [46, 91, 107, 240, 241], and has now moved to the physics of optical fibres [71, 112, 177]. Gradually, this concept finds applications in other branches of optics, and, more generally, in other branches of physics [229, 242–244]. The most common mathematical description of rogue waves is based on rational solutions of the nonlinear Schrödinger equation (NLSE) [92, 185, 233, 245–247]. However, the NLSE is only a very basic model that just takes into account group velocity dispersion and nonlinearity to the lowest-order approximations [84, 126, 131]. Other physical effects are important if we want to describe wave propagation phenomena more realistically. More accurate models [84, 248–250] include higher-order effects such as third and fourth order dispersion, self-steepening and other effects that should be taken into account in real applications.

First, we consider the focusing NLSE [84]

$$i\frac{\partial\psi}{\partial x} + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi = 0. \quad (8.1)$$

It is well-known [89,92] that the NLSE has a rational solution that is located on a non-zero background and is localized both in the x and t directions:

$$\psi(x, t) = \psi_0 e^{ix} = - \left[1 - 4 \frac{1 + 2ix}{D(x, t)} \right] e^{ix} \quad (8.2)$$

where we define a polynomial in x and t , namely $D(x, t) = 1 + 4x^2 + 4t^2$, and use this notation throughout this chapter. The maximum of the envelope $|\psi(x, t)|$ occurs at the origin, where $|\psi(0, 0)| = 3$. There are also higher order rogue waves – see [80,183] and [251] for an interactive demonstration.

In chapter 7 we extended the NLSE and included a self-steepening term, $\sim s_a \frac{\partial}{\partial t}(|\psi|^2\psi)$, a term related to the self-frequency shift, $\sim a_f \psi \frac{\partial}{\partial t}(|\psi|^2)$, and 3^{rd} order dispersion, $\sim \gamma_3 \psi_{ttt}$, as higher-order effects. The analysis showed that rogue waves can 'survive' these particular perturbations. A specific feature of these terms is that they are of odd symmetry, i.e. changing the sign of t changes the sign of the perturbation. Hence the pulses propagate at an angle, i.e. they have some velocity [252]. Clearly, this is a limitation which allows us to consider rogue waves only in a certain class of solutions. As we consider solutions to the lowest order of the perturbation, the corresponding equations are linear. This means that perturbations are additive and corresponding extensions of solutions are also additive. Thus, the solutions of [100] could be included here, but we omit them for clarity of presentation. Adding these terms is just a matter of simple algebra.

In the present chapter, we consider even terms, namely 4^{th} order dispersion and the quintic nonlinearity. These perturbations are of even symmetry, which means that changing the sign of t does not change the sign of the perturbation. Thus, dealing with these perturbations, we consider a specific class of rogue wave solutions. Taking these terms into account, we expect the pulses to propagate at zero angle to the x -axis, i.e. to have zero velocity.

Specifically, we are dealing with the equation:

$$i \frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{\partial^2 \psi}{\partial t^2} + |\psi|^2 \psi + \nu \left(|\psi|^4 \psi - h \frac{\partial^4 \psi}{\partial t^4} \right) = 0, \quad (8.3)$$

Several higher order NLSE equations has been developed, namely the Dysthe model

and the ‘even broadband Dysthe’ (EBD) model. Schober [253,254] has studied the dynamics of rogue waves in the NLSE, Dysthe and EBD equations. These have various degrees of robustness against perturbations. The EBD equation has a fourth order term but no quintic term, and it still supports a rogue wave. The Dysthe model is an extension of the NLSE, where other higher order terms have been included to make the fundamental NLSE into a more realistic model. The above-mentioned papers propose details that make the equations model more realistic rogue waves. Then, to answer the question of rogue evolution, various numerical approaches have been made, especially where the initial sea state is taken to be white Gaussian noise. On the other hand, the Dysthe and EBD models use JONSWAP (Joint North Sea Wave Project) spectra.

The chapter is organized as follows. In Sec. (8.2), we introduce an analytical procedure for finding the solutions of Eq. (8.3). Section 3 details the numerical procedure to solve Eq. (8.3) by using a random initial condition and approximate analytic solution. We then note that rogue waves exist up to certain values of ν .

8.2. Perturbation Analysis

In order to find the rogue waves of Eq.(8.3), we apply a perturbation analysis and seek a solution of the form

$$\psi(x, t) = [\psi_0(x, t) + \nu \psi_p(x, t)] e^{ix} \quad (8.4)$$

The zeroth order solution here, $\psi_0(x, t)$, is given in Eq. (8.2). Substituting this form into Eq. (8.3), taking into account the smallness of the parameter ν , and separating the higher order terms, we obtain 2 coupled partial differential equations with variable coefficients for the real and imaginary parts of the perturbation, $\psi_p(x, t) = R(x, t) + i J(x, t)$.

$$\begin{cases} A(R_{tt} - 2J_x) + BR + GJ + C - Y = 0, \\ A(J_{tt} + 2R_x) + GR + HJ + Q - 2xY = 0, \end{cases} \quad (8.5)$$

where the coefficients A, B, C, G, H and Q are polynomials of higher order than those in Eq.(8.2). They are given by:

$$\left(\begin{array}{lcl} A & = & D^5(x, t) \\ B & = & 4[16t^4 + 8t^2(F - 6) + 16x^4 - 8x^2 + 13] D^3(x, t) \\ C & = & -2[1024t^{10} + 1280t^8(F - 4) + 128t^6(80x^4 - 56x^2 + 45) \\ & & + F^4(F + 12) + 32t^4(F + 4)(101 - 88x^2 + 80x^4) \\ & & + 4t^2F(320x^6 + 496x^4 - 324x^2 - 2155)] \\ G & = & -32x(D - 4)D^3(x, t) \\ H & = & -16(D - 6F + 4)D^3(x, t) \\ Q & = & +2[8x[256t^8 + 256t^6(F - 4) + 32t^4(F - 8)(3F + 16) \\ & & + 16t^2F(16x^4 + 24x^2 + 133) + F^3(F + 16)]] \\ Y & = & 512(6h - 1)(80t^4 - 40t^2 F + F^2) \end{array} \right)$$

where $F = 4x^2 + 1$.

Now, h only appears in the coefficient Y . We choose $h = 1/6$ to set $Y = 0$, as it provides tractable equations and physically allows for the existence of a new type of rogue wave. The set of Eqs. (8.5) is a linear coupled set of diffusion equations with variable coefficients. The difficulty of solving them is that each coefficient depends on two variables, viz. x and t . Thus, known techniques cannot be applied here. However, we can find at least some solutions by using polynomial representations, and adjusting the coefficients in these polynomials. Here, we find :

$$\psi_p(x, t) = \frac{1}{2} - \frac{2}{D}(17 + 14ix) + \frac{8}{D^2}f(x, t) + \frac{256}{D^3}(1 + 2ix)t^2,$$

where $f(x, t) = 3 + 10t^2 + 2ix(3 + 2t^2)$. Here, we consider the quintic nonlinearity, ν , to be small. The ratio coefficient h (which we assume to be of order of 1) shows the amount of 4th order dispersion relative to the quintic nonlinearity. When $\nu = 0$, the equation reduces to the standard NLSE, Eq. (8.1).

In the Fig. (8.1) for positive value of $\nu = 0.05$, the maximum amplitude is 2.52. It is lower than for the Peregrine breather which is the solution of the NLSE. This rogue wave has been reproduced in our numerical simulations as well. In fact, rogue

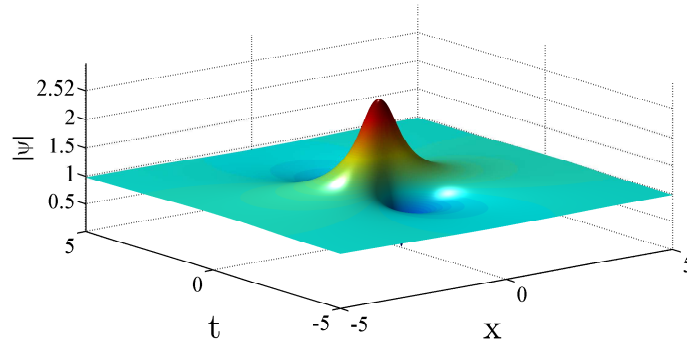


Figure 8.1 Plot of perturbed rogue wave found by solving Eqs. (8.4) and (8.6). Here parameter values are $h = 1/6$ and $\nu = 0.05$.

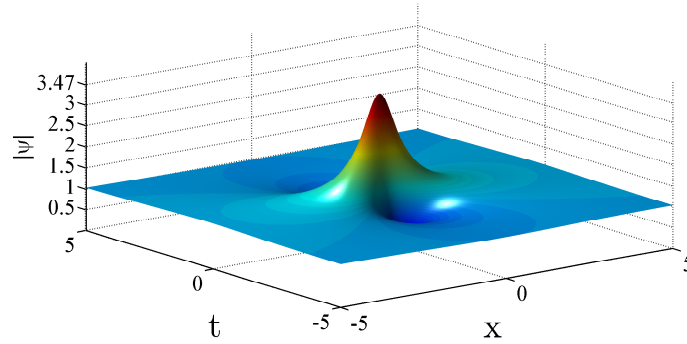


Figure 8.2 Plot of perturbed rogue wave found by solving Eqs. (8.4) and (8.6). The parameter values are $h = 1/6$ and $\nu = -0.05$.

waves can still exist when h is not equal to $1/6$. For the solution, given by Eqs. (8.4), and (8.6), we have

$$\psi(0,0) = 3 - \frac{19}{2}\nu \quad (8.6)$$

The amplitude at the centre is lower than 3, so positive ν reduces the amplitude at the centre of the rogue wave. On the other hand, a negative ν makes the centre of the rogue wave even higher than for the Peregrine breather. This increase is clearly seen in Fig. (8.2). This rogue wave solution of perturbed NLSE, given by Eqs. (8.4) and (8.6), is still a particular case of a more general solution. Indeed, we can extend it to a 1-parameter family of solutions:

$$\begin{aligned} \psi(x,t) = & \left[-1 + \frac{4(1+2ix)}{D} + (1-r)\frac{\nu}{2} - \frac{4\nu}{D}(5+r) \right. \\ & \left. + \frac{4\nu(1+2ix)}{D}k(x,t) + \frac{64\nu}{D^2}t^2 \right] e^{ix(1+r\nu)}, \end{aligned} \quad (8.7)$$

where r is a free real parameter and the function $k(x, t)$ is:

$$k(x, t) = \frac{2(1 + 2t^2)(1 + r) + 4}{D} + \frac{64}{D^2} t^2 - \frac{(7 + r)}{2}.$$

The previous expressions, given by Eqs. (8.4) and (8.6), represent a particular case of (8.7) when $r = 0$. It can be shown that the parameter r controls the asymmetry of the solution on the (x, t) -plane. With the new parameter, the amplitude of the solution at the centre of rogue wave is given by

$$\psi(0, 0) = 3 + \frac{\nu}{2}(3r - 19).$$

It can be either slightly higher or lower than the Peregrine value of 3. The 'background' level also deviates from the normalised Peregrine case:

$$|\psi(|x|, |t| \rightarrow \infty)| = \left| 1 - \frac{\nu}{2}(1 - r) \right|.$$

It can also be either slightly higher or lower than 1.

8.2.1. Generalized solution

The number of parameters in the solution can also be increased using a scaling transformation. So far, we have assumed a normalized background level of 1 for the Peregrine solution. Now we can allow for an arbitrary background of amplitude q by multiplying Eq. (8.7) by q and using a scaling which is similar to the one used for the NLSE. Namely, we scale the parameter $\nu \rightarrow \nu q^2$, and the variable $t \rightarrow q t$, as well as $x \rightarrow q^2 x$. With this transformation, $D(x, t) = 1 + 4 q^4 x^2 + 4 q^2 t^2$ and we obtain the solution with scaled amplitude:

$$\begin{aligned} \psi(x, t) &= q e^{iq^2 x(1 + r\nu q^2)} \\ &\times \left[-1 + \frac{4(1 + 2i q^2 x)}{D(x, t)}(1 + \nu q^2 j(x, t)) + \nu q^2 p(x, t) \right], \end{aligned} \quad (8.8)$$

where r is the same free parameter as before and the polynomials $j(x, t)$ and $p(x, t)$ are:

$$j(x, t) = \frac{2}{D(x, t)} [(1 + 2q^2 t^2)(1 + r) + 2] + \frac{64}{D^2} q^2 t^2 - \frac{(7 + r)}{2},$$

$$p(x, t) = \frac{1 - r}{2} + \frac{64}{D^2} q^2 t^2 - 4 \frac{5 + r}{D}.$$

Thus, Eq. (8.8) represents the most general solution of our perturbed NLSE. Clearly, it is valid within the approximation of small perturbation ($\nu \ll 1$). Moreover, the two perturbation terms in the equation must be related through $h = 1/6$ for solutions to be presented in this explicit form. The situation here is similar to the case of the Hirota equation where rogue wave (and other) solutions can be found only for a specific relation between the two terms of odd symmetry [141]. Nevertheless, our analysis gives further evidence that localized rogue events can be observed for models of ocean waves and optical fibre pulses that are more general than the one provided by the basic NLSE.

8.3. Numerical Analysis

In order to confirm that the modified solutions are real we reproduced them in numerical simulations. We use a symmetrised split step method in the simulations. We also used an initial condition, Eqs. (8.4) and (8.6), at $x = -x_0$ where x_0 varied in different runs. Numerical simulations allowed us to determine the maximum value of ν for solution validity. The rogue wave solution for Eq. (8.3), is symmetric and our propagation of distance stepping is in the forward direction. The blue solid lines in Figs. (8.3) - (8.5) are the approximate analytic solutions where the maximum values are set at $x = 5$ for comparison purposes, and the red dotted lines are the solutions obtained from numerical simulations. We took the initial condition and start the propagation at $x = -5$ and propagate up to $x = 10$.

Although we take our initial condition at $x = -5$ of the analytic expression, the propagation is shifted in x by five units so that simulations start at $x = 0$. Therefore in the numerical runs, the maximum amplitude occurs at $x = 5$ instead of $x = 0$. We have made a translation along the axis to ensure that the maximum of $|\psi|$ occurs

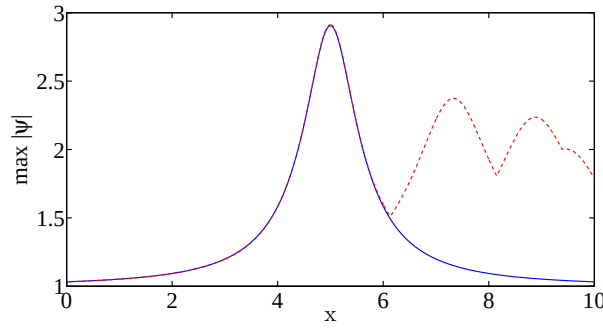


Figure 8.3 Numerical results for the evolution of the center of the rogue wave for $h = 1/6$ and $\nu = 0.01$

at $x = 5$ in each case. At the beginning of the propagation, the rogue wave starts to build up from the background to its maximum value and then returns to the background. Due to modulation instability, the background becomes unstable near $x = 6$, thus disturbing the background.

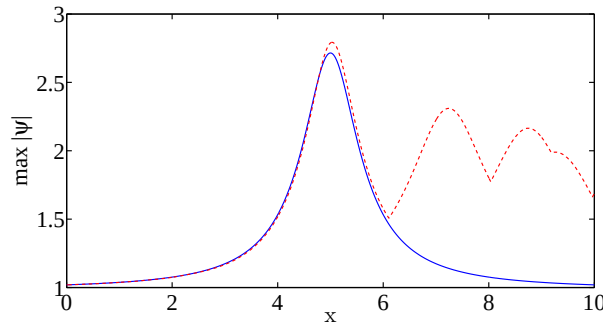


Figure 8.4 Numerical results for the evolution of the center of the rogue wave for $h = 1/6$ and $\nu = 0.03$

From Figs. (8.3) -(8.5), it is clear that the solution is valid up to approximately $\nu = 0.05$. For higher values of ν , the numerics shows that the amplitude at the centre of the rogue wave diverges from its analytical prediction given by Eq. (8.6). This can be seen from Fig. (8.6) which shows in red dots the maximum amplitude of the rogue wave obtained in numerical simulations and maximum amplitude predicted by the analytical expression (blue line). Significant deviations start at $\nu = 0.04$. Otherwise, our approach gives reasonable approximation for the maximum amplitude.

Another observation from this figure is that the amplitude in numerical simulations is higher than predicted by the analytic expressions. Thus, in reality, the additional terms in the equation lead to a stronger rogue wave than that obtained in approximations. In particular, Fig. (8.5) for $\nu = 0.05$ shows that the analytic

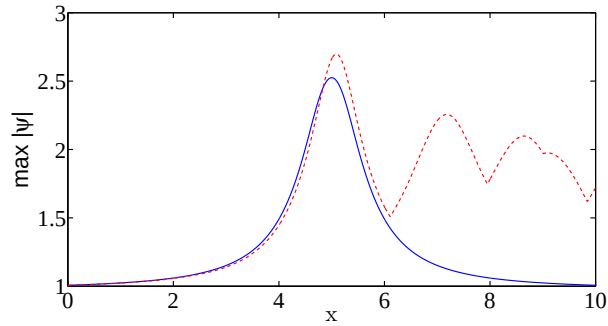


Figure 8.5 Numerical results for the evolution of the center of the rogue wave for $h = 1/6$ and $\nu = 0.05$.

solution predicts the maximum amplitude to be 2.52. But in the case of numerics, if we look at Fig. (8.5), the red curve for $\nu = 0.05$ gives a higher maximum amplitude.

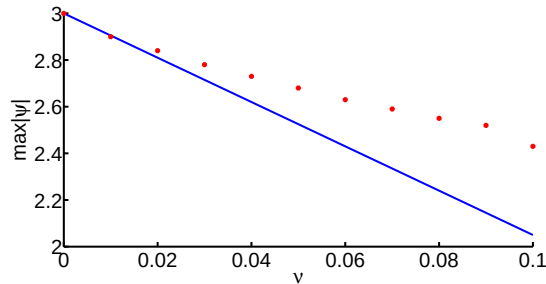


Figure 8.6 Maximum amplitude reached by the rogue wave in numerical simulations as a function of ν . Solid blue line is according to the analytical solution given by Eq. (8.6). Red dots are obtained in numerical simulations started with initial condition Eq. (8.4). Accuracy is lost when ν is larger than about 0.04. Remarkably, the amplitude in simulations is higher than predicted analytically.

The value $h = 1/6$ is exceptional as it allows us to write an approximate solution analytically. However, our numerical simulations show that rogue waves also exist for $0 < h < 1/6$. The h -dependence of the full solution is complicated and could be presented elsewhere.

8.4. Conclusion

We have shown that localized rogue waves described by the rational solutions of the NLSE continue to exist when the equation is perturbed by a combination of higher-order dispersive and nonlinear terms. We have presented an approximate solution which is accurate to first-order of the perturbing function and confirmed, numerically, that the solution can be excited from small amplitude initial conditions.

CHAPTER 9

Corresponding Solutions of Ablowitz-Ladik and NLSE

9.1. Introduction

In regard to the correspondence between discrete and continuous equations, we should mention a few previous works that relate to various examples [255–262]. Hirota [255] constructed a partial difference analogue of the Korteweg - de Vries equation and found its N-soliton solutions. Date *et al.* [256] provided a general method for generating discrete soliton equations and applied it to several physical models, including the Heisenberg ferromagnet equation [256], a single component Kadomtsev-Petviashvili (KP) equation [257], Boussinesq and sine-Gordon equations [258], as well as the KP family [259]. Hasakado [261] considered discretization of the Davey-Stewartson system, double Kadomtsev-Petviashvili system and Ablowitz-Ladik hierarchy in general form. Sadakane [262] showed, in terms of free fermion operators, that the AL hierarchy and the Ruijsenaars - Toda hierarchy arise from a reduction of the two-component Toda lattice. Most of these papers deal with generating the discrete analogue of a continuous integrable equation and do not consider any direct correspondence between the solutions. The correspondence between the solutions of discrete and continuous equations can also be quite complicated. This can be even more confusing when we deal with families of solutions that involve free parameters.

Formalising this correspondence and illustrating it with relatively simple examples is the aim of this chapter.

Thus, this chapter provides simple rules and some sets of corresponding solutions. We have restricted ourselves to just the simplest examples of the periodic and localized solutions that are most frequently used in the literature. Following these rules, one can extend the results to more elaborate cases. As an example of new solutions, we provide a few corresponding pairs for coupled Manakov-type sets of NLSE and AL equations.

In order to make our message clear, we start with the rules of transformation, which at first glance appear to be trivial. Nevertheless, these rules allow us to operate with highly complicated solutions. In fact, we start with simple solutions of the two equations, and turn to more complex solutions later in the work. Such a gradual increase in complexity of the solutions allows us to see the similarity of the solutions at all levels of the construction.

We start the analysis with a single Ablowitz-Ladik equation that has only one component ψ_n [263]:

$$i \frac{d\psi_n^{(1)}}{dt} + \frac{1}{2h^2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} - 2\psi_n^{(1)} \right) + \frac{1}{2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} \right) |\psi_n^{(1)}|^2 = 0 \quad (9.1)$$

In contrast to the common form of the AL equation, a real coefficient, $1/(2h^2)$, is included here. This leaves the integrability of the equation intact. One way to convert the equation to a more common form is to use the transformations

$$t \rightarrow T h^2 \quad \text{and} \quad \psi_n \rightarrow \Psi_n/h. \quad (9.2)$$

Then, we have

$$i \frac{d\Psi_n^{(1)}}{dT} + \frac{1}{2} \left(\Psi_{n-1}^{(1)} + \Psi_{n+1}^{(1)} \right) (1 + |\Psi_n^{(1)}|^2) - \Psi_n^{(1)} = 0 \quad (9.3)$$

On the other hand, we obtain the same form of Eq. (9.3) if we take the arbitrary coefficient h to be $h = 1$ in Eq. (9.1). Note that the normalization on T differs by a factor of 2 from that in the forms given in [99, 152, 153, 264]. This allows the solutions

of the AL equation to line up directly with those of the traditional form of the NLSE (9.4).

We can use inverse relations (9.2) to transform a general solution of AL equation (9.3) with $h = 1$ to a solution of (9.1) with arbitrary h . This transformation allows us to link the discrete derivative of the AL equation with the continuous derivative of the NLSE. The transformation seems to be simple and natural. However, it does not work without additional assumptions in cases when the solutions are families with one or several parameters. An important point is that, when doing this, we also need to transform the solution parameters. Normally, we achieve this by multiplying each independent parameter of a solution by h . This procedure turns out to be the key for linking the discrete and continuous solutions. To give an example, in each solution, we commonly have a phase advance term $\exp[i q^2 t]$. Setting the parameter $q \rightarrow qh$ and having $t \rightarrow t/h^2$ clearly leaves this exponential term invariant. Similar conversions can be made with all other free parameters of a solution. Eq. (9.1) can be considered as a special discretisation of the NLSE equation (Eq.(2.1) of [84]):

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi = 0 \quad (9.4)$$

Clearly, transforming the discrete variable n into continuous one through $n = x/h$ and taking the limit $h \rightarrow 0$ reduces Eq. (9.1) to Eq. (9.4) [265]. Using this fact we can obtain NLSE solutions once we have the appropriate solution of the AL equation (9.3). The reverse operation cannot be applied directly. Having an exact solution of the NLSE does not allow us to obtain the solution of the AL equation in a straightforward way. Nevertheless, if we have a certain solution of NLSE, then we can often guess the form of the AL solution with relative ease and try to transform it back to the NLSE solution.

The problem is that the NLSE can be discretised in many ways [266]. The AL equation (9.3) is only one of them. However, it is important to note that the AL equation, like the NLSE, is integrable. This means that, as with the NLSE, it has an infinite number of exact solutions. For example, the discrete NLSE

$$i \frac{d\psi_n^{(1)}}{dt} + \frac{1}{2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} - 2\psi_n^{(1)} \right) + |\psi_n^{(1)}|^2 \psi_n^{(1)} = 0 \quad (9.5)$$

(labelled DNLS) is also a discretisation of the NLSE but its solutions cannot be obtained on a regular basis. Thus, we leave the DNLS and other forms of discretisation alone and consider only the AL of Eq. (9.3), comparing its solutions with those of NLSE.

The above considerations show that there should be a correspondence between the solutions of the two equations, NLSE and AL. Some part of this correspondence has been found in previous works [99, 264, 267]. The present chapter gives a direct link between the two sets of solutions and provides more examples of such correspondence. Having pairs of corresponding solutions is important from both a fundamental point of view as well as for finding solutions which are known for one case and are not known for the other. At the same time, we show that the equivalence between the solutions of the two equations cannot be complete. The AL equation has periodic solutions where the period is commensurate with the discrete lattice. When this period is small, the solution of the AL equation cannot be directly transformed into a solution of the NLSE. However, we can find the solution of the NLSE that may describe the corresponding solution of the AL. These special cases will also be considered in this chapter. The technique can be applied, not only to a single equation, but to sets of coupled AL and NLSE equations.

9.2. Simple Example: Travelling Soliton Solution

In finding pairs of solutions, we start with a simple example. The well-known travelling soliton solution of a single AL (9.1) is:

$$\psi_n^{(1)} = \frac{\sinh(hp)}{h} \operatorname{sech}(nhp - vt) \exp[i(kt + bnh)], \quad (9.6)$$

where the velocity v and the frequency k is given by

$$\begin{aligned} v &= \frac{1}{h^2} \sin(hb) \sinh(hp) \\ k &= \frac{1}{h^2} [\cos(hb) \cosh(hp) - 1]. \end{aligned}$$

Here, independent real parameters b and p are arbitrary. Taking $x = nh$ and $h \rightarrow 0$ in Eq. (9.6), we get $v = bp$ and $k = \frac{1}{2}(p^2 - b^2)$. Thus, we obtain

$$\psi^{(1)} = p \operatorname{sech}[p(x - bt)] \times \exp \left[i \left(bx + \frac{1}{2} (p^2 - b^2) t \right) \right], \quad (9.7)$$

which is the basic result for an NLSE soliton of amplitude p moving with velocity b . When the velocity is zero ($b = 0$), the solution (9.6) reduces to

$$\psi_n^{(1)} = \frac{\sinh(hp)}{h} \operatorname{sech}(hnp) \exp \left[\frac{2it}{h^2} \sinh^2 \left(\frac{hp}{2} \right) \right]. \quad (9.8)$$

Taking $x = nh$ and then the limit as $h \rightarrow 0$ reduces Eq. (9.8) to

$$\psi = p \operatorname{sech}(xp) \exp \left(ip^2 \frac{t}{2} \right) \quad (9.9)$$

which is also a solution of the NLSE, Eq. (9.4). This simplest example demonstrates the technique of finding the pairs of solutions and serves as proof of the principle.

9.3. Energy Considerations

For localized solutions with zero background when $n \rightarrow \pm\infty$, there is an “energy invariant” that is conserved. If the background is not zero but has a finite value q , we can re-define energy, since, with the usual definition, it would become infinite. It is also conserved. Namely, for a solution ψ_n , located on a zero background field, $q = 0$, we define [101]:

$$Q = \sum_{n=-\infty}^{\infty} Q_n, \quad (9.10)$$

where

$$Q_n = \frac{h}{2} (\psi_n \psi_{n+1}^* + \psi_n^* \psi_{n+1}). \quad (9.11)$$

Here, $q = 0$ and

$$Q_n = \frac{1}{h} \sinh^2(hp) \operatorname{sech}(hnp) \operatorname{sech}[(n+1)hp] \quad (9.12)$$

and

$$Q = \frac{2}{h} \sinh(hp). \quad (9.13)$$

For the NLSE soliton, the energy, Q , is the $h \rightarrow 0$ limit of this, viz. $Q = 2p$. It agrees with the value found by integrating $|\psi|^2$ from Eq. (9.9) over all x .

9.3.1. Special Cases: Strictly Periodic Solutions in terms of Sine or Cosine Functions

The AL equation has “periodic” solutions with periods that can be incommensurate with the lattice period. Generally, the period in these solutions is a continuous variable that may coincide with the period of the lattice. In this case, the solution becomes strictly periodic. In this section, we present three special results for strictly periodic solutions of the AL equation which can be expressed in terms of sine or cosine functions.

Case (a):

$$\psi_n^{(1)} = c \sin\left(\frac{n\pi}{2}\right) \exp(-it/h^2), \quad (9.14)$$

which can be presented as the pattern

$$= [0, 1, 0, -1, 0, 1, 0, -1, \dots] \times c \exp(-it/h^2)$$

with the lattice period 4.

Case (b). Another strictly periodic solution

$$\psi_n^{(1)} = c \sin\left(\frac{n\pi}{3}\right) \exp\left[i\left(\frac{3c^2}{8} - \frac{1}{2h^2}\right)t\right],$$

has the pattern

$$[0, 1, 1, 0, -1, -1, 0, 1, 1, \dots] \times \frac{\sqrt{3}}{2} c \exp\left[i\left(\frac{3c^2}{8} - \frac{1}{2h^2}\right)t\right]$$

with the lattice having period 6.

Case (c). Finally, the third strictly periodic solution

$$\psi_n^{(1)} = c \sin\left(\frac{2n\pi}{3}\right) \exp\left[-\frac{3i}{8}\left(c^2 + \frac{4}{h^2}\right)t\right], \quad (9.15)$$

has the pattern

$$[1, -1, 0, 1, -1, 0, 1, -1, \dots] \times \frac{\sqrt{3}}{2} c \exp\left[-\frac{3i}{8}\left(c^2 + \frac{4}{h^2}\right)t\right],$$

with the lattice period being 3. An arbitrary (positive real) parameter c simultaneously defines the amplitude and the propagation constant of the solutions. The strictly periodic solutions presented in this section do not have limits at $h \rightarrow 0$ because the phase variation in t would be infinitely rapid. In turn, the NLSE does not have simple sinusoidal solutions with finite amplitude. The above expressions exist only in the discrete limit. When talking about similar solutions of the two equations, we can only consider those AL solutions with variations along n much bigger than the mesh-size of the grid, h .

9.3.2. Periodic Solutions in terms of Elliptic Jacobi Functions

More general periodic stationary solutions can be written in terms of Jacobi elliptic functions. The variable period in these solutions may or may not coincide with the lattice period. We give three example solutions in this category [see [266] for example] with Jacobi elliptic functions with modulus m . The first one is in terms of the dn -function:

$$\psi_n^{(1)} = \sqrt{a} \operatorname{dn}(bnh, m) \exp(ik t) \quad (9.16)$$

where m and b are arbitrary and

$$a = \frac{1 - 2 \operatorname{dn}^2(bh, m) + \operatorname{dn}(2bh, m)}{h^2 \operatorname{dn}^2(bh, m) [1 - \operatorname{dn}(2bh, m)]} \quad (9.17)$$

while

$$k = h^{-2}[(1 + ah^2) \operatorname{dn}(bh, m) - 1]. \quad (9.18)$$

We stress that, although the elliptic Jacobi functions are periodic, the solution can be considered as periodic only when the period of the Jacobi function is commensurate with the period of the lattice. Otherwise, the periodicity is lost. When we take the limit $m = 1$, we obtain the soliton solution (9.8) above. To obtain the NLSE solution from (9.16), we take the limits of a and k from Eqs. (9.17) and (9.18), as $h \rightarrow 0$. This transformation gives $a = b^2$ and $k = (1 - \frac{m}{2}) b^2$. This provides the correct NLSE solution, viz.

$$\psi^{(1)}(x, t) = b \operatorname{dn}(bx, m) \exp \left[i \left(1 - \frac{m}{2} \right) b^2 t \right], \quad (9.19)$$

where m and b are arbitrary. A second type of periodic solution can be expressed in terms of the sn-function:

$$\psi_n^{(1)} = \sqrt{a} \operatorname{sn}(bnh, m) \exp(ik t) \quad (9.20)$$

where m and b are arbitrary and

$$a = \frac{1}{h^2 \operatorname{sn}(b, m) \operatorname{sn}(3b, m)} - \frac{1 + \frac{\operatorname{sn}(bh, m)}{\operatorname{sn}(3bh, m)}}{h^2 \operatorname{sn}^2(bh, m)} \quad (9.21)$$

while

$$k = \frac{a}{2} \operatorname{sn}(bh, m) \operatorname{sn}(2bh, m) - \frac{1}{h^2} + \frac{\operatorname{sn}(2bh, m)}{2 h^2 \operatorname{sn}(bh, m)}.$$

Here we need $a > 0$ but

$$\lim_{m \rightarrow 1} a < 0,$$

so this limit does not give any valid solution. The limit $h \rightarrow 0$ just gives the trivial zero solution of the NLSE. Finally, what can be called “cnoidal wave” is the following:

$$\psi_n^{(1)} = \sqrt{a} \operatorname{cn}(bnh, m) \exp(ik t) \quad (9.22)$$

where m, b are arbitrary and

$$a = \frac{1 - 2 \operatorname{cn}^2(bh, m) + \operatorname{cn}(2bh, m)}{h^2 \operatorname{cn}^2(bh, m) [1 - \operatorname{cn}(2bh, m)]} \quad (9.23)$$

while

$$k = h^{-2} [(1 + a h^2) \operatorname{cn}(bh, m) - 1]. \quad (9.24)$$

When we take the limit $m = 1$, we again get, as the limiting case, Eq. (9.8) above. To get the NLSE solution from this, we take the limits of a and k from Eqs. (9.23,9.24), as $h \rightarrow 0$. This gives $a = m b^2$ and $k = (m - \frac{1}{2}) b^2$ and provides the correct NLSE solution, viz.

$$\psi^{(1)}(x, t) = b\sqrt{m} \operatorname{cn}(bx, m) \exp \left[i \left(m - \frac{1}{2} \right) b^2 t \right], \quad (9.25)$$

where (m, b) are arbitrary.

In general, the period of elliptic Jacobi functions is not commensurate with the period of the lattice. However, it can be chosen to be commensurate. In this case we can obtain the strictly periodic solutions presented in the Sec. (9.3.1). For example, let us consider the ‘cn’ example given by Eqs. (9.23) and (9.24). The period of the solution is $T = \frac{4}{bh} K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. Now $K(m) > \frac{\pi}{2}$ so $T > \frac{2\pi}{bh}$. If we choose $bh = K(m)$, then the period is $T = 4$. Then the parameters are $k = -1/h^2$ and $a = \frac{m}{h^2(1-m)}$. The amplitude at $t = 0$ in this case can be presented as the following sequence:

$$\psi_n(0) = \frac{1}{h} \sqrt{\frac{m}{1-m}} \times [0, -1, 0, 1, 0, -1, 0, 1, \dots].$$

This solution can also be written in the form

$$\psi_n(t) = \frac{1}{h} \sqrt{\frac{m}{1-m}} \cos \left(\frac{n\pi}{2} \right) e^{-it/h^2}.$$

which is just the solution (9.14) given in Sec. (9.3.1) case (a) above, but shifted along n . Here the amplitude changes with h and m and can be arbitrary, just as in (9.14). The sequence $\psi_n(0)$, calculated for $h = 1$ and $b = 3$, is shown in Fig. (9.1). It is labelled as f_4 . If we take $h = 1$ and $b = 4$, and choose m to get $T = 3$, we obtain

$$\psi_n = f_3 = 0.72 \times [-1, -1, 4.255, -1, -1, 4.255, \dots].$$

We label this expression as f_3 . Finally, $b = 12/7$ gives $T = 7$ and a sequence labelled f_7 . All solutions are plotted in Fig. (9.1). If we take bh slightly above $2\pi/3$, and set

$T = 3$, we get $m \approx 0$ so that

$$\text{cn}(bn, m) \rightarrow \cos(2\pi n/3).$$

This way, we obtain the solution in the form presented in Sec. (9.3.1) case (c) which is shifted in n .

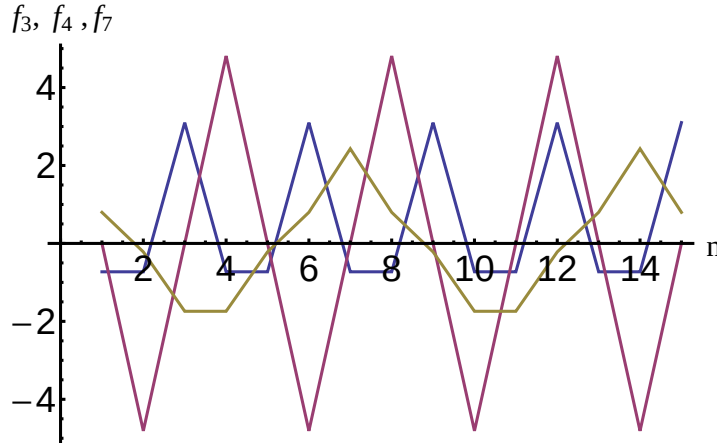


Figure 9.1 Plot of the strictly periodic sequences f_3 (blue), f_4 (mauve), and f_7 (olive green) with periods 3, 4 and 7, respectively.

9.3.3. Modulation Instability and Heteroclinic Breather Solutions

It is well-known that the NLSE has a plane-wave solution that is modulationally unstable. The family of exact solutions of the NLSE that includes both the growth and decay cycles of modulation instability is known as the ‘Akhmediev breather’ family [88, 92, 268]. Equivalent heteroclinic solutions of the AL equation (9.3) have been presented and analyzed in detail in [99]. The whole family of solutions has a free parameter that corresponds to the arbitrary frequency of modulation. A particular solution of this family is related to the case when the growth rate of instability takes its maximum value. First, let us consider the latter case. To find the heteroclinic solution of Eq. (9.1) for the complete growth-return cycle of modulation instability at the maximum growth rate, we modify Eq.(15) of [264] to include the parameter h .

Then, we will have:

$$\psi_n(t) = q \frac{\mp \sqrt{1+h^2q^2} \cos(vnh) + i\sqrt{2+h^2q^2} \sinh(q^2t)}{\sqrt{2+h^2q^2} \cosh(q^2t) \pm \sqrt{1+h^2q^2} \cos(vnh)} e^{iq^2t} \quad (9.26)$$

where

$$v = \frac{1}{h} \arccos \left[\frac{1}{1+h^2q^2} \right] = \frac{1}{h} \operatorname{arcsec} [1+h^2q^2].$$

This solution describes the growth of a periodic perturbation, due to the modulation instability of the constant background, and its return to the background with a shifted phase. A qualitatively similar solution of the NLSE has been derived in [72,80]. To obtain it from the AL solution (9.26), we set $nh \rightarrow x$ and then take $h \rightarrow 0$. We have

$$\lim_{h \rightarrow 0} v = q\sqrt{2}$$

so that:

$$\psi(x, t) = q \frac{\mp \cos(\sqrt{2}qx) + i\sqrt{2} \sinh(q^2t)}{\sqrt{2} \cosh(q^2t) \pm \cos(\sqrt{2}qx)} e^{iq^2t} \quad (9.27)$$

The above solution is a particular case of a heteroclinic orbit with an arbitrary modulation frequency, κ . The latter parameter is assumed to be within the modulation instability band. This generalization provides us with:

$$\psi_n(t) = \frac{1}{h} \exp(iq^2t) \times \left[\frac{2(1+h^2q^2)w \cosh\left(\frac{\delta t}{2h^2}\right) + i\frac{\delta}{2} \sinh\left(\frac{\delta t}{2h^2}\right)}{qh \cosh\left(\frac{\delta t}{2h^2}\right) \pm c_1 \cos(\kappa qnh)} - qh \right] \quad (9.28)$$

where

$$w = \sin^2 \left(\frac{1}{2} hq \kappa \right),$$

while

$$c_1 = \sqrt{\frac{q^2 h^2 (1-w) - w}{1-w}}$$

and the growth rate of instability, δ , is

$$\delta = 4\sqrt{(1+h^2q^2)w[(h^2q^2(1-w)-w)]}.$$

Clearly, Eq. (9.28) generalizes the solution with $h = 1$ found earlier in [99]. To derive the NLSE solution from (9.28), we take $nh = x$ and note that the limits of

the coefficients are:

$$\begin{pmatrix} \lim_{h \rightarrow 0} \left(\frac{w}{h^2} \right) & = & \left(\frac{q\kappa}{2} \right)^2 \\ \lim_{h \rightarrow 0} \left(\frac{c_1}{h} \right) & = & \frac{q}{2} \sqrt{4 - \kappa^2} \\ \lim_{h \rightarrow 0} \left(\frac{\delta}{h^2} \right) & = & \kappa q^2 \sqrt{4 - \kappa^2} \end{pmatrix}$$

These limits allow us to write the generalization of Eq. (9.27) for an arbitrary modulation frequency, κ , in the range $0 < \kappa < 2$, as the $h \rightarrow 0$ limit of Eq. (9.28):

$$\psi(x, t) = q \left[\kappa \frac{\kappa^2 \cosh(\delta_1 q^2 t) + 2i\delta_1 \sinh(\delta_1 q^2 t)}{2\kappa \cosh(\delta_1 q^2 t) \pm 2\delta_1 \cos(\kappa q x)} - 1 \right] e^{iq^2 t}. \quad (9.29)$$

where the instability growth rate δ_1 is:

$$\delta_1 = \frac{\kappa}{2} \sqrt{4 - \kappa^2} \quad (9.30)$$

in standard form. This solution satisfies the NLSE for any frequency κ in the instability range. It was first derived in [72, 80]. In the particular case $\kappa = \sqrt{2}$, where the growth rate (9.30) is maximal, the expression (9.29) coincides with Eq. (9.27) above.

9.3.4. Soliton on Background

Another important solution of the NLSE is a soliton on a background. It has been found by Kuznetsov [269] and Ma [179]. There is a similar solution of the AL equation [264]. In order to include the parameter h in the solution, we allow k_2 to have an arbitrary value and define

$$u = \sinh(hk_2/2)$$

for convenience, so that $k_2 = \frac{2}{h} \operatorname{arcsinh}(u)$. Generalizing the solution given in [264], we find:

$$\psi_n(t) = \frac{1}{h} \exp(iq^2 t) \times \left[\frac{\cos\left(\frac{w_1 t}{2h^2}\right) + is_6 \sin\left(\frac{w_1 t}{2h^2}\right)}{s_3 \left[-\cos\left(\frac{w_1 t}{2h^2}\right) \pm s_5 \cosh(k_2 n h) \right]} - q h \right]. \quad (9.31)$$

where

$$\begin{pmatrix} s_3 = & \frac{qh}{2(q^2 h^2 + 1)u^2} \\ s_5 = & \sqrt{\frac{q^2 h^2 + (1+q^2 h^2)u^2}{q^2 h^2 (1+u^2)}} \\ s_6 = & \sqrt{1 + \frac{q^2 h^2}{(q^2 h^2 + 1)u^2}} \\ w_1 = & 4(q^2 h^2 + 1) u \sqrt{u^2 + \frac{q^2 h^2}{q^2 h^2 + 1}} \end{pmatrix}$$

Thus, in this formulation, q , h and k_2 are the arbitrary real parameters. Once they are given, we can then find s_3, s_5, s_6 and w_1 . In order to transform it to the equivalent NLSE solution, we first give the limiting values of the coefficients:

$$\begin{pmatrix} \lim_{h \rightarrow 0} \left(\frac{u}{h} \right) = & \frac{k_2}{2} \\ \lim_{h \rightarrow 0} (s_3 h) = & \frac{2q}{k_2^2} \\ \lim_{h \rightarrow 0} (s_5) = & \frac{1}{2} \sqrt{4 + \frac{k_2^2}{q^2}} \\ \lim_{h \rightarrow 0} (s_6) = & \sqrt{1 + \frac{4q^2}{k_2^2}} \\ \lim_{h \rightarrow 0} \left(\frac{w_1}{h^2} \right) = & k_2 \sqrt{k_2^2 + 4q^2} \end{pmatrix}$$

Substituting these values into (9.31), we obtain the solution of the continuous equation which is also a soliton on a background q :

$$\psi(x, t) = e^{iq^2 t} \left[k_2 \frac{k_2 \cos\left(\frac{k_2}{2} dt\right) + id \sin\left(\frac{k_2}{2} dt\right)}{\pm d \cosh(k_2 x) - 2q \cos\left(\frac{k_2}{2} dt\right)} - q \right], \quad (9.32)$$

where $d = \sqrt{k_2^2 + 4q^2}$.

9.4. Rogue Waves

9.4.1. First order Rogue Wave

Rogue waves are doubly localized, rational solutions of the AL and NLSE equations. There is an infinite hierarchy of these solutions with increasing order. For the NLSE, the first-order solution has been presented by Peregrine [89] while the second order solution has been given in [80] and [183]. Rogue wave solutions of the AL equation have been presented in [99] and [264].

We consider the rogue wave solution of the AL, Eq. (9.1), modified from the form presented in [99]. We define $N(n) = 2qhn$ and $T(t) = 2q^2 \sqrt{1 + q^2 h^2} t$. As explained in the Introduction, the two h^2 terms in $T(t)$ cancel each other. The j -th order rogue wave solution in general form can be written as:

$$\psi_n^{(j)}(t) = q \left[\frac{T^2}{q^4 t^2} \frac{G_n^{(j)}(t) + i 2q^2 t H_n^{(j)}(t)}{D_n^{(j)}(t)} + (-1)^j \right] e^{iq^2 t} \quad (9.33)$$

where $G_n^{(j)}(t)$ etc., are polynomials. For $j = 1$, we have $G_n^{(1)}(t) = 1$, $H_n^{(1)}(t) = 1$ and $D_n^{(1)}(t) = 1 + N^2 + T^2$. In other words:

$$\psi_n^{(1)}(t) = q \left(\frac{4(1 + h^2 q^2)(1 + 2iq^2 t)}{1 + 4h^2 q^2(n^2 + q^4 t^2) + 4q^4 t^2} - 1 \right) e^{iq^2 t} \quad (9.34)$$

We can use this expression to obtain the NLSE rogue wave solution by taking $x = nh$, i.e. $N(n) = 2qhn = 2qx = X$, and then take h to be small:

$$\psi(x, t) = q \left(\frac{4(1 + 2iq^2 t)}{1 + 4q^2 x^2 + 4q^4 t^2} - 1 \right) e^{iq^2 t}. \quad (9.35)$$

The parameter q can be any real number in Eq. (9.35). This first order rogue wave is usually called the ‘‘Peregrine soliton’’ [92, 97, 112], as it was first obtained by Peregrine [89] as a limiting case of the soliton on a background (9.32).

9.4.2. Second-order Rogue Wave

Modifying the second-order rogue wave solution presented in [101] for the case of Eq. (9.1) that involves arbitrary h , we find a one-parameter family with arbitrary background level, q , which has a form similar to Eq. (9.33):

$$\psi_n^{(2)}(t) = q \left[\frac{T^2}{q^4 t^2} \frac{G_n^{(2)}(t) + i 2q^2 t H_n^{(2)}(t)}{D_n^{(2)}(t)} + 1 \right] e^{iq^2 t} \quad (9.36)$$

where

$$\begin{aligned} G_n^{(2)}(t) &= 3[3 - 6(N^2 - 2q^2 h^2) + N^2(4q^2 h^2 - N^2) \\ &\quad - 6(3 + 2q^2 h^2 + N^2)T^2 - 5T^4], \end{aligned}$$

$$\begin{aligned}
H_n^{(2)}(t) &= 3[15 + 6(N^2 + 4q^2 h^2) + N^2(16q^2 h^2 - N^2) \\
&\quad - 2(1 + N^2)T^2 - T^4],
\end{aligned}$$

while

$$\begin{aligned}
D_n^{(2)}(t) &= 9 + (27 + 24q^2 h^2 + 16q^4 h^4)N^2 + N^6 + 3(33 + 72q^2 h^2 - 6N^2 \\
&\quad + 48q^4 h^4 - 16q^2 h^2 N^2 + N^4)T^2 + 3(9 + 8q^2 h^2 + N^2)T^4 \\
&\quad + T^6 + N^4(3 - 8q^2 h^2).
\end{aligned}$$

In order to link this solution to the second-order rogue wave of the NLSE, in accordance with our technique, we set $nh = x$ and take h to be small. Thus, as above, we denote $X = 2qx$ and $T(t) = 2q^2t$. We have

$$\psi^{(2)}(x, t) = q \left[4 \frac{G^{(2)}(x, t) + iT H^{(2)}(x, t)}{D^{(2)}(x, t)} + 1 \right] e^{iT/2} \quad (9.37)$$

where

$$\left(\begin{array}{lcl}
G^{(2)}(x, t) & = & 3[3 - 6X^2 - X^4 - 6(3 + X^2)T^2 - 5T^4] \\
H^{(2)}(x, t) & = & 3[15 + 6X^2 - X^4 - 2(1 + X^2)T^2 - T^4] \\
D^{(2)}(x, t) & = & 9 + 27X^2 + 3X^4 + X^6 + T^6 \\
& + & 3(33 - 6X^2 + X^4)T^2 + 3(9 + X^2)T^4
\end{array} \right)$$

This satisfies the NLSE and agrees with the solution in [97] and [80].

9.5. Coupled AL and NLSE (Manakov) Equations

The technique of finding equivalent solutions that we suggest in this work can be readily extended to the case of coupled AL and NLSE equations. In order to demonstrate this, we write the set of two AL equations in the form:

$$\begin{aligned} i \frac{d\psi_n^{(1)}}{dt} + \frac{1}{2h^2}(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} - 2\psi_n^{(1)}) + \frac{1}{2}(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)}) (|\psi_n^{(1)}|^2 + |\psi_n^{(2)}|^2) &= 0, \\ i \frac{d\psi_n^{(2)}}{dt} + \frac{1}{2h^2}(\psi_{n-1}^{(2)} + \psi_{n+1}^{(2)} - 2\psi_n^{(2)}) + \frac{1}{2}(\psi_{n-1}^{(2)} + \psi_{n+1}^{(2)}) (|\psi_n^{(1)}|^2 + |\psi_n^{(2)}|^2) &= 0, \end{aligned} \quad (9.38)$$

where, just as before, t is the continuous evolution variable (time or longitudinal spatial variable) and $n = 0, \pm 1, \pm 2, \dots$ are integers labelling each cite. This set is an analog of the Manakov equations in the continuous NLSE limit:

$$\begin{aligned} i \frac{\partial \psi^{(1)}}{\partial t} + \frac{1}{2} \psi_{xx}^{(1)} + (|\psi^{(1)}|^2 + |\psi^{(2)}|^2) \psi^{(1)} &= 0 \\ i \frac{\partial \psi^{(2)}}{\partial t} + \frac{1}{2} \psi_{xx}^{(2)} + (|\psi^{(1)}|^2 + |\psi^{(2)}|^2) \psi^{(2)} &= 0 \end{aligned} \quad (9.39)$$

9.6. Strictly Periodic Solutions

For the set of the two AL equations, we can also write down a number of strictly periodic solutions. We can classify three of them. Namely,

Case (a): The solution with period 6. It is given by:

$$\begin{aligned} \psi_n^{(1)} &= r \exp \left[i \left(\frac{3}{h^2} + 4r^2 \right) t \right] \left(3 - 2 \cos \left(\frac{2n\pi}{3} \right) \right), \\ \psi_n^{(2)} &= \frac{2}{\sqrt{5}} \sqrt{\frac{9}{h^2} - 16r^2} \exp \left[\frac{it}{5h^2} (11 + 16h^2 r^2) \right] \sin \left(\frac{n\pi}{3} \right), \end{aligned} \quad (9.40)$$

for arbitrary r in the range $0 < r < \frac{3}{4h}$. Thus, we have

$$\psi_n^{(1)} = r \exp \left[i \left(\frac{3}{h^2} + 4r^2 \right) t \right] f_1(n),$$

(9.41)

$$\psi_n^{(2)} = \sqrt{\frac{3}{5}} \sqrt{\frac{9}{h^2} - 16r^2} \exp \left[\frac{it}{5h^2} (11 + 16h^2 r^2) \right] f_2(n),$$

where the sequences $f_1(n)$ and $f_2(n)$ are

$$\begin{aligned} f_1(n) &= [4, 4, 1, 4, 4, 1, \dots] \\ f_2(n) &= [1, 1, 0, -1, -1, 0, 1, 1, \dots]. \end{aligned}$$

These sequences are strictly periodic, but the period of the second sequence is twice the period of the first one. The overall period is then 6. They are shown in Fig. (9.2). Again, there is no physical limit for $h \rightarrow 0$, and so no corresponding NLSE solution pair.

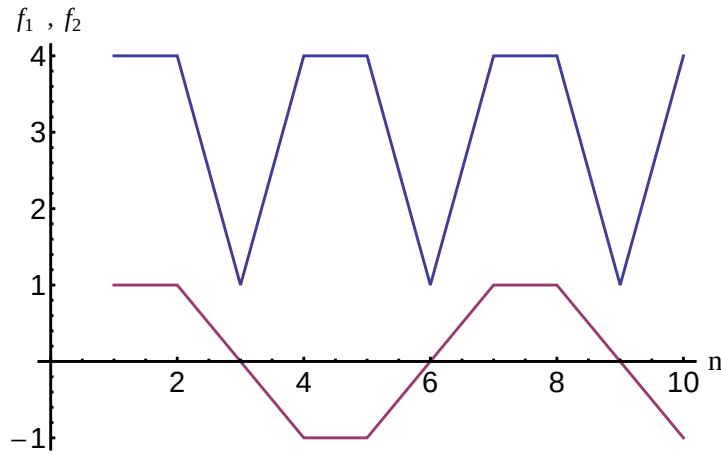


Figure 9.2 Plot of f_1 (upper curve) and f_2 (lower curve) for the pair of solutions in case (a). Period is 6.

Case (b): The solution with the period 4. The second family is given by:

$$\begin{aligned} \psi_n^{(1)} &= d \exp(ibt) (3 - 2g(-1)^n), \\ \psi_n^{(2)} &= \frac{2\sqrt{6g}}{h(3-2g)} \exp\left(\frac{-it}{h^2}\right) \sin\left(\frac{n\pi}{2}\right), \end{aligned} \tag{9.42}$$

where $b = \frac{4g}{(3-2g)h^2} + d^2(9-4g^2)$, for arbitrary parameters d and g . The second component (which has period 4) does not contain d because $\psi_{n-1}^{(2)} + \psi_{n+1}^{(2)}$ is always

zero. So the pattern is:

$$\begin{aligned}\psi_n^{(1)} &= d e^{i b t} \times f_1(n), \\ \psi_n^{(2)} &= \frac{2}{h(3-2g)} \sqrt{6g} e^{-i t/h^2} \times f_2(n)\end{aligned}$$

Here, we have two sequences

$$\begin{aligned}f_1(n) &= [3+2g, 3-2g, 3+2g, 3-2g, 3+2g, \dots], \\ f_2(n) &= [1, 0, -1, 0, 1, 0, -1, \dots],\end{aligned}\tag{9.43}$$

with the period of the second one being twice the period of the first one. The two sequences $f_1(n)$ and $f_2(n)$ are shown, for a particular $g = 5/4$ in Fig. (9.3).

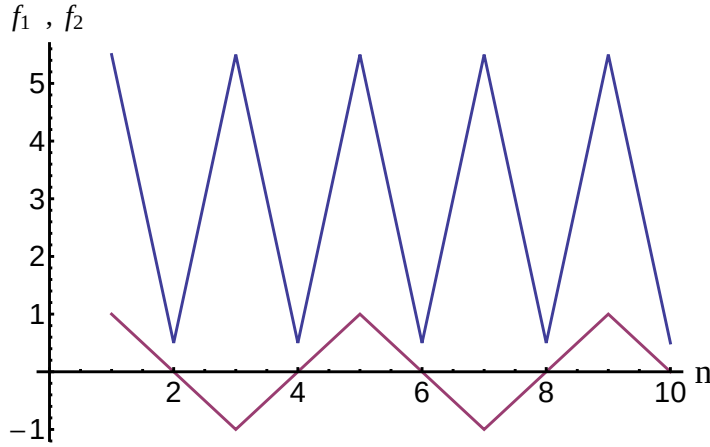


Figure 9.3 Plot of $f_1(n)$ (upper curve) and $f_2(n)$ (lower curve) for the pair (b) of Eq. (9.43). The cumulative period is 4.

Case (c): The solution with the period 3. The 3^{rd} family of periodic solutions has period 3 and is given by:

$$\psi_n^{(1)} = r \exp \left[i \left(\frac{3}{h^2} + 4r^2 \right) t \right] \left(3 - 2 \cos \left(\frac{2n\pi}{3} \right) \right), \tag{9.44}$$

$$\psi_n^{(2)} = \frac{2}{h} \sqrt{\frac{9 - 16h^2r^2}{5}} \exp \left[-it \frac{(21 + 16h^2r^2)}{5h^2} \right] \sin \left(\frac{2n\pi}{3} \right), \tag{9.45}$$

for arbitrary r in the range $0 < r < \frac{3}{4h}$. These expressions can be written in an

explicit form:

$$\begin{aligned}\psi_n^{(1)} &= r \exp \left[i \left(\frac{3}{h^2} + 4r^2 \right) t \right] f_1(n), \\ \psi_n^{(2)} &= \sqrt{\frac{3}{5}} \sqrt{9 - 16h^2r^2} \exp \left[-it \frac{(21 + 16h^2r^2)}{5h^2} \right] \frac{f_2(n)}{h},\end{aligned}\tag{9.46}$$

where the sequences are

$$f_1(n) = [4, 4, 1, 4, 4, 1, \dots]$$

which is same as in the case (a), while

$$f_2(n) = [1, -1, 0, 1, -1, 0, 1, -1, 0, \dots]$$

which is different from the one in (a). The two latter sequences are shown in Fig. (9.4).

Symmetry relative to the exchange of variables $\psi_n^{(1)}(t) \leftrightarrow \psi_n^{(2)}(t)$ allows us to write

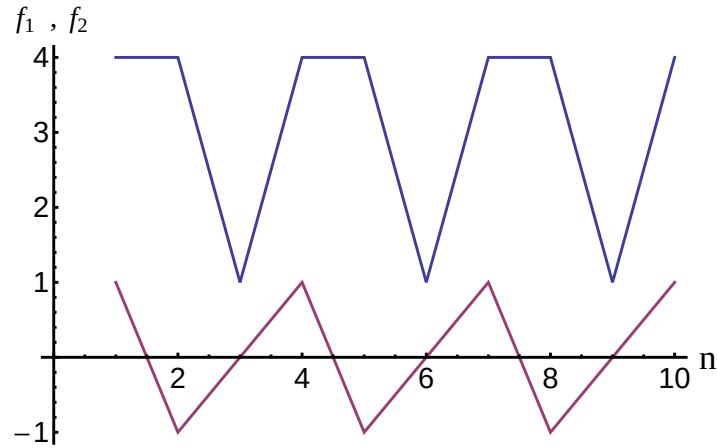


Figure 9.4 Plot of the discrete functions f_1 (upper) and f_2 (lower) for the solution pair (c) of Eq. (9.46). The period here is 3.

down three more similar solutions. Clearly, in all these cases, there is no analogous coupled NLSE solution, because the phase goes to ∞ as $h \rightarrow 0$.

9.7. Localized solution

The localized solution of the set of coupled AL equations, Eq. (9.38), can be written in the form:

$$\begin{aligned}\psi_n^{(1)} &= \frac{\sqrt{r}}{\cosh(hnp)} \exp\left(it \frac{[(1 + h^2 r) \operatorname{sech}(hp) - 1]}{h^2}\right), \\ \psi_n^{(2)} &= \sqrt{k_2} \tanh(hnp) \exp(ik_2 t).\end{aligned}\quad (9.47)$$

where $k_2 = h^{-2} [(1 + h^2 r) \operatorname{sech}^2(hp) - 1]$, for arbitrary parameters r and p . The above solution has its analog in the form of the solution of $N = 2$ Manakov (coupled NLSE) set of equations. In order to see the correspondence, we take $nh = x$ and then $h \rightarrow 0$ and obtain:

$$\begin{aligned}\psi^{(1)}(x, t) &= \sqrt{r} \operatorname{sech}(xp) \exp\left[it \left(r - \frac{p^2}{2}\right)\right], \\ \psi^{(2)}(x, t) &= \sqrt{r - p^2} \tanh(xp) \exp\left[it \left(r - p^2\right)\right].\end{aligned}\quad (9.48)$$

This effectively reproduces the solution found in [270] (see Eq.(6)).

For travelling waves, we allow non-zero velocity (i.e. $b \neq 0$). Then the solution pair is:

$$\begin{aligned}\psi_n^{(1)} &= \sqrt{r} \operatorname{sech}(hnp - vt) \exp[i(k_1 t + bnh)], \\ \psi_n^{(2)} &= \sqrt{d} \tanh(hnp - vt) \exp[i(k_2 t + b_2 nh)].\end{aligned}\quad (9.49)$$

where

$$\begin{aligned}k_1 &= \frac{1}{h^2} [-1 + (1 + h^2 r) \cos(bh) \operatorname{sech}(ph)], \\ k_2 &= \frac{1}{h^2} [-1 + (1 + h^2 r) \operatorname{sech}^2(ph) \sqrt{1 - \operatorname{sech}^2(ph) \sin^2(bh)}], \\ v &= \frac{1}{h^2} (1 + h^2 r) \operatorname{sech}(ph) \sin(bh) \tanh(ph), \\ b_2 &= \frac{1}{h} \arcsin[\operatorname{sech}(ph) \sin(bh)], \\ d &= \frac{1}{h^2} [-1 + (1 + h^2 r) \operatorname{sech}^2(ph)],\end{aligned}$$

for arbitrary parameters b , p and r . We take $nh = x$ and then $h \rightarrow 0$ and find that $b_2 = b$ and $v = bp$, so the $N = 2$ solution pair for the Manakov set becomes:

$$\begin{aligned}\psi^{(1)}(x, t) &= \sqrt{r} \operatorname{sech}[p(x - bt)] \exp[i(k_1 t + bx)], \\ \psi^{(2)}(x, t) &= \sqrt{r - p^2} \tanh[p(x - bt)] \exp[i(k_2 t + bx)].\end{aligned}\quad (9.50)$$

where

$$\begin{aligned}k_1 &= -\frac{1}{2} (b^2 + p^2 - 2r), \\ k_2 &= +r - \frac{b^2}{2} - p^2,\end{aligned}\quad (9.51)$$

for arbitrary b , p and r .

9.8. Conclusion

To conclude, in this chapter, we have provided a direct way to establish the correspondence between the solutions of the Ablowitz-Ladik and nonlinear Schrödinger equations. This technique is illustrated with a few examples and exceptions. A few pairs of important solutions are presented in Table I. In addition to the solutions considered in the text, we have also included in the table doubly-periodic solutions of the AL and NLSE equations in terms of elliptic Jacobi functions. These are periodic generalizations of the heteroclinic orbits given by solutions describing modulation instability. We have also shown that the technique can be easily extended to the cases of coupled AL and NLSE equations. We provide three examples of corresponding solutions of the coupled sets in Table II.

9.9. Appendix: Corresponding Solution of NLSE and AL Equation

Table 1: Comparison of selected exact solutions of the NLSE and the AL equations. Below, each group represents corresponding sets of exact solutions of a single equation of NLSE and AL equation

- Nonlinear Schrödinger equation (NLSE):

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi = 0$$

Ablowitz-Ladik equation(AL) with a free parameter h :

$$i \frac{\partial \psi_n^{(1)}}{\partial t} + \frac{1}{2h^2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} - 2\psi_n^{(1)} \right) + \frac{1}{2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} \right) |\psi_n^{(1)}|^2 = 0$$

- One soliton solution of NLSE:

$$\psi(x, t) = p \operatorname{sech}(xp) \exp \left(ip^2 \frac{t}{2} \right)$$

One soliton solution of AL:

$$\psi_n(t) = \frac{\sinh(hp)}{h} \operatorname{sech}(hnp) \exp \left[\frac{2it}{h^2} \sinh^2 \left(\frac{hp}{2} \right) \right]$$

- Growth-return cycle of modulation instability for NLSE (Akhmediev breather):

$$\psi(x, t) = q \left[\kappa \frac{\kappa^2 \cosh(\delta_1 q^2 t) + 2i\delta_1 \sinh(\delta_1 q^2 t)}{2\kappa \cosh(\delta_1 q^2 t) \pm 2\delta_1 \cos(\kappa q x)} - 1 \right] e^{iq^2 t}$$

Growth-return cycle of modulation instability for AL (Akhmediev breather):

$$\psi_n(t) = \frac{1}{h} \left[\frac{2(1+h^2 q^2)w \cosh\left(\frac{\delta t}{2h^2}\right) + i\frac{\delta}{2} \sinh\left(\frac{\delta t}{2h^2}\right)}{qh \cosh\left(\frac{\delta t}{2h^2}\right) \pm c_1 \cos(\kappa q n h)} - qh \right] e^{iq^2 t}$$

- Growth-return cycle of modulation instability with maximum growth rate [80] for NLSE:

$$\psi(x, t) = q \frac{\cos(\sqrt{2} qx) + i\sqrt{2} \sinh(q^2 t)}{\sqrt{2} \cosh(q^2 t) - \cos(\sqrt{2} qx)} e^{iq^2 t}$$

Growth-return cycle of modulation instability with maximum growth rate [264]; Eq. (9.26) for AL:

$$\psi_n(t) = q \frac{\sqrt{1+h^2 q^2} \cos(v n h) + i\sqrt{2+h^2 q^2} \sinh(q^2 t)}{\sqrt{2+h^2 q^2} \cosh(q^2 t) - \sqrt{1+h^2 q^2} \cos(v n h)} e^{iq^2 t}$$

- First order rational solution [89] for NLSE:

$$\psi(x, t) = q \left(\frac{4(1+2iq^2 t)}{1+4q^2 x^2 + 4q^4 t^2} - 1 \right) e^{iq^2 t}$$

First order rational solution for AL equation [99], [264] :

$$\psi_n^{(1)}(t) = q \left(\frac{4(1+h^2 q^2)(1+2iq^2 t)}{1+4h^2 q^2(n^2+q^4 t^2)+4q^4 t^2} - 1 \right) e^{iq^2 t}$$

- Soliton on a background for NLSE [179]:

$$\psi(x, t) = e^{iq^2 t} \left[k_2 \frac{k_2 \cos\left(\frac{k_2}{2} dt\right) + id \sin\left(\frac{k_2}{2} dt\right)}{\pm d \cosh(k_2 x) - 2q \cos\left(\frac{k_2}{2} dt\right)} - q \right]$$

Soliton on a background [264]; Eq. (9.31) for AL:

$$\psi_n(t) = \frac{1}{h} e^{iq^2 t} \left[\frac{\cos\left(\frac{w_1 t}{2h^2}\right) + i s_6 \sin\left(\frac{w_1 t}{2h^2}\right)}{s_3 \left[-\cos\left(\frac{w_1 t}{2h^2}\right) \pm s_5 \cosh(k_2 n h) \right]} - q h \right]$$

- Doubly-periodic breather of NLSE equation [72, 171]:

$$\psi(x, t) = \frac{r}{\sqrt{2}} \exp\left(ik_1 \frac{r^2}{2} t\right) \times \frac{\left(\frac{1-k_1}{2}\right)^{1/4} \operatorname{dn}\left(\frac{r^2}{2} t, m_1\right) \operatorname{cn}\left(r x, \frac{1-k_1}{2}\right) + i\sqrt{k_1} k_t \operatorname{sn}\left(\frac{r^2}{2} t, m_1\right)}{k_t - \left(\frac{1-k_1}{2}\right)^{1/4} \sqrt{k_1} \operatorname{cn}\left(\frac{r^2}{2} t, m_1\right) \operatorname{cn}\left(r x, \frac{1-k_1}{2}\right)}$$

where r and k_1 are independent parameters; $m_1 = k_1^2$.

Doubly-periodic breather of AL equation [264]:

$$\psi_n(t) = q \exp(i k_1 q^2 t) \times \frac{\sqrt{k} \operatorname{dn}(q^2 t, m_1) \operatorname{cn}(r n h, m) + i\sqrt{k_1} k_t \operatorname{sn}(q^2 t, m_1)}{k_t - \sqrt{k k_1} \operatorname{cn}(q^2 t, m_1) \operatorname{cn}(r n h, m)},$$

- Periodic stationary solutions of NLSE equation :

$$\psi(x, t) = \sqrt{a} \operatorname{dn}(bx, m) \exp(ik t), \text{ Eq. (9.19)}$$

$$\psi(x, t) = 0$$

$$\psi(x, t) = \sqrt{a} \operatorname{cn}(bx, m) \exp(ik t), \text{ Eq. (9.25)}$$

Periodic stationary solutions of AL equation :

$$\psi_n^{(t)} = \sqrt{a} \operatorname{dn}(bnh, m) \exp(ik t), \text{ Eqs. (9.16) – (9.18)}$$

$$\psi_n^{(t)} = \sqrt{a} \operatorname{sn}(bnh, m) \exp(ik t), \text{ Eq. (9.20)}$$

$$\psi_n^{(t)} = \sqrt{a} \operatorname{cn}(bnh, m) \exp(ik t), \text{ Eqs. (9.22) – (9.24)}$$

- Strictly periodic solutions of NLSE equation:

There is no continuous analog

There is no continuous analog

There is no continuous analog

Strictly periodic solutions of AL equation:

$$\begin{aligned} \psi_n^{(1)} &= c \sin\left(\frac{n\pi}{2}\right) \exp(-it/h^2), \\ \psi_n^{(1)} &= c \sin\left(\frac{n\pi}{3}\right) \exp\left[i\left(\frac{3c^2}{8} - \frac{1}{2h^2}\right)t\right], \\ \psi_n^{(1)} &= c \sin\left(\frac{2n\pi}{3}\right) \exp\left[-\frac{3i}{8}\left(c^2 + \frac{4}{h^2}\right)t\right], \end{aligned}$$

Table 2: *Comparison of exact solutions of the coupled set of NLSE and the coupled set of A-L equations. Corresponding sets of exact solutions of a set of coupled NLSE and AL equations*

- Coupled set of nonlinear Schrödinger equations (NLSE) (see [171], [270], [271])($j = 1, 2$):

$$i \frac{\partial \psi_j}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi_j}{\partial x^2} + \psi_j \left(\sum_{m=1}^N |\psi_m|^2 \right) = 0,$$

Coupled Ablowitz-Ladik set of equations (present work) Eqs. (9.38), ($j = 1, 2$)

$$i \frac{\partial \psi_n^{(j)}}{\partial t} + \frac{1}{2h^2} \left(\psi_{n-1}^{(j)} + \psi_{n+1}^{(j)} - 2\psi_n^{(j)} \right) + \frac{1}{2} \left(\psi_{n-1}^{(j)} + \psi_{n+1}^{(j)} \right) \left(\sum_{m=1}^N |\psi_n^{(m)}|^2 \right) = 0$$

- Manakov pair solution [270]; Eqs. (9.48) :

$$\begin{aligned} \psi^{(1)}(x, t) &= \sqrt{r} \operatorname{sech}(x p) \exp \left[i t \left(r - \frac{p^2}{2} \right) \right], \\ \psi^{(2)}(x, t) &= \sqrt{r - p^2} \tanh(x p) \exp [i t (r - p^2)]. \end{aligned}$$

AL Discrete pair solution - Eqs. (9.47) of this work:

$$\begin{aligned} \psi_n^{(1)} &= \sqrt{r} \operatorname{sech}(h n p) \exp \left(\frac{it}{h^2} [(1 + h^2 r) \operatorname{sech}(h p) - 1] \right), \\ \psi_n^{(2)} &= \sqrt{k_2} \tanh(h n p) \exp(it k_2). \end{aligned}$$

CHAPTER 10

Solution of Manakov and Coupled Ablowitz-Ladik Equation

10.1. Introduction

The Ablowitz- Ladik (AL) equation and the NLSE are both integrable systems. Each can be solved using the well-known inverse scattering technique [272]. As the AL equation is an integrable discretization of the NLSE, their solutions are also linked to each other. The most-studied common solutions are soliton solutions on a zero background [273] or dark solitons [274], depending on the sign of the coefficients in the equation. Numerous techniques have been used to derive exact solutions of the AL equation with reference to its related continuous (NLSE) partner solutions [275–277]. The explicit connection between the exact solutions of the AL equation and NLSE has been given in the previous chapter (9) . The latest new solutions of the AL equation are rogue waves. Their nature, mathematical interpretation and occurrences have been discussed in Refs. [99,264]. The phenomena of modulation instability, Fermi-Pasta-Ulam recurrence, rogue waves, nonlinear phase shift, and the corresponding exact solutions of the Ablowitz-Ladik equation have been discussed in [278]. A convenient approach to first-order exact solutions of the AL equation has been presented in [267]. The 2^{nd} order discrete rogue wave solution of the Ablowitz-Ladik equation has been given in [101].

Our aim here is to find solutions of coupled AL equations by extending a Manakov model to a discrete case. A basic 2-component ‘*tanh*’ and ‘*sech*’ solution for discrete coupled AL equations was derived in Ref. [278]. The coupled set of AL equations for the defocusing case has been solved in terms of Lamé and Chebyshev polynomials in [279, 280]. Our present endeavour is to find different solutions of the AL equations using ideas from Ref. [270], where an N -coupled set of focusing NLSEs has been solved in terms of Legendre polynomials. In the continuous limit, these solutions converge to the solutions of the set of Manakov equations in both the focusing and defocusing forms.

10.2. Discrete and Continuous Coupled Equations

The standard normalized integrable AL equation can be written as

$$i \frac{d\psi_n^{(1)}}{dt} + \frac{1}{2h^2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} - 2\psi_n^{(1)} \right) + \frac{\alpha}{2} \left(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} \right) |\psi_n^{(1)}|^2 = 0 \quad (10.1)$$

where $n = \dots - 2, -1, 0, 1, 2, \dots$ is the discrete variable and h is the discretization parameter. The equation in this form includes defocusing ($\alpha = -1$, where the nonlinear response decreases with increasing intensity) and focusing ($\alpha = +1$, where the nonlinear response increases with increasing intensity) cases. Equation (10.1) is a second-order finite-difference approximation of the standard NLSE [Eq.(2.1) of [84]]:

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \psi_{xx} + \alpha |\psi|^2 \psi = 0. \quad (10.2)$$

This particular integrable discretization has been discussed in [263, 267, 278, 281]. Complex functions $\psi_n(t)$ and $\psi(t, x)$ in Eqns. (10.1) and (10.2) are the wave functions for the discrete and continuous cases, respectively. If we set $n = x/h$, the limit of $h \rightarrow 0$ reduces the discrete AL equation to the continuous NLSE. For both AL equation and NLSE, $\alpha = 1$ means focusing and $\alpha = -1$ means a defocusing system. A detailed and systematic procedure for lining up the AL equation and NLSE and their corresponding solutions by using standard transformations and scaling of the t and h variables was given in [278].

Equations (10.1) and (10.2), when applied to light-wave technology, usually deal with unidirectional polarized light consisting of a single component. More complicated systems can be described by the multicomponent NLSEs and AL equations. Partially coherent solitons (PCSs) form one example where multicomponent coupled NLSEs are needed to provide a detailed understanding of the formation of PCSs, their shape, and collision properties [270, 271, 282, 283]. The standard generalized coupled AL equation system, which is basically a discretized set of NLSEs, can be written as:

$$\begin{aligned}
 i \frac{d\psi_n^{(1)}}{dt} + \frac{1}{2h^2}(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)} - 2\psi_n^{(1)}) + \frac{\alpha}{2}(\psi_{n-1}^{(1)} + \psi_{n+1}^{(1)}) (|\psi_n^{(1)}|^2 + |\psi_n^{(2)}|^2) &= 0. \\
 i \frac{d\psi_n^{(2)}}{dt} + \frac{1}{2h^2}(\psi_{n-1}^{(2)} + \psi_{n+1}^{(2)} - 2\psi_n^{(2)}) + \frac{\alpha}{2}(\psi_{n-1}^{(2)} + \psi_{n+1}^{(2)}) (|\psi_n^{(1)}|^2 + |\psi_n^{(2)}|^2) &= 0.
 \end{aligned}
 \tag{10.3}$$

In the case of just two sets, we can call them “discretized Manakov equations”. Indeed, in the limit $nh = x$ and $h \rightarrow 0$, the above system reduces to the continuous coupled Manakov equations with two components [278], which, for the focusing case ($\alpha = 1$), can be written as

$$\begin{aligned}
 i \frac{\partial \psi^{(1)}}{\partial t} + \frac{1}{2} \psi_{xx}^{(1)} + (|\psi^{(1)}|^2 + |\psi^{(2)}|^2) \psi^{(1)} &= 0 \\
 i \frac{\partial \psi^{(2)}}{\partial t} + \frac{1}{2} \psi_{xx}^{(2)} + (|\psi^{(1)}|^2 + |\psi^{(2)}|^2) \psi^{(2)} &= 0.
 \end{aligned}
 \tag{10.4}$$

Higher-order sets of coupled equations describe so-called incoherent solitons. The concept of incoherent solitons or random phase nonlinear waves was introduced by Hasegawa [284–286] more than 35 years ago. The experiment carried out by Mitchell *et al.* [287] extended this concept to the case of spatial solitons. This experiment confirmed that incoherent solitons do exist and are observable. Photorefractive materials turned out to be ideal for experimental observations of self-trapping of partially coherent light, due to their slow response time [288] and low optical power requirement for the development of strong nonlinear effects [289–293]. Christodoulides *et al.* [294, 295] have shown that incoherent self-trapped light inside a photorefractive material can be represented by an infinite set of coupled nonlinear Schrödinger equations (NLSEs) [294]. The theory of incoherent solitons within nonlinear Kerr-like

media has been developed in [283]. This theory allowed researchers to study the shapes of incoherent solitons and their changes after collisions. They are described by multi-parameter families of solutions, leading to arbitrary shapes of solitons in the limit of complete incoherence [283]. We will provide approximate solutions to Eq. (10.3) in Sec. (10.6).

The stationary solutions for incoherent solitons, their dynamics, and collisions, have also been analyzed in [296–300]. A comprehensive description of partially coherent solitons has been given in [271]. This work, which follows [301], elucidates soliton shapes and their collisions in terms of exact analytic solutions. It was found that for equally-spaced eigenvalues, a symmetric solution of M coupled NLSE equations can be derived in terms of associated Legendre functions, representing symmetric partially coherent solitons on zero background. They are based on the modes of the ‘sech-squared’ waveguide [282]. Here, the symmetry and ‘sech-squared’ guide reduce the solution families from multi-parameter to single parameter solutions [299, 300]. As the amplitude of partially coherent solitons provides another parameter to the solution, finally a two-parameter symmetric solution has been derived in [270] for arbitrary M coupled NLSEs in terms of associated Legendre functions; it is on a finite background. Here, we generalise [270] presenting solutions for both defocusing ($\alpha = -1$) and focusing ($\alpha = 1$) cases.

10.3. Self-Trapped Incoherent Solitons: Defocusing ($\alpha = -1$) and Focusing ($\alpha = 1$) cases

Propagation of an M -component self-trapped mutually incoherent soliton set in photorefractive media can be expressed in terms of the solution of a set of focusing NLSEs given by

$$i\frac{\partial\psi_j(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2\psi_j(x,t)}{\partial x^2} + \alpha\delta n\psi_j(x,t) = 0, \quad (10.5)$$

where $\psi_j(x,t)$ is the j^{th} component of the beam ($j = 1, \dots, M$), x is the spatial dimension, t is time variable or propagation direction and α is a coefficient denoting

the strength of the nonlinearity. When $\alpha = 1$, Eq. (10.5) is the focusing NLSE and, when $\alpha = -1$, it becomes the defocusing NLSE. The total refractive index change is

$$\delta n = \sum_{j=1}^M |\psi_j|^2. \quad (10.6)$$

We are looking for solutions of Eq. (10.5) with $\alpha = \pm 1$ in the form

$$\psi_j(x, t) = u_j(x) e^{ik_j t} \quad (10.7)$$

where $j = 1, \dots, M$. Propagation constants k_j are different for each j ; this makes the components mutually incoherent. This ansatz reduces Eqs. (10.5) to the set of ordinary differential equations (ODEs):

$$u_j''(x) + 2\alpha u_j(x) \sum_{i=1}^M u_i^2(x) = 2k_j u_j(x) \quad (10.8)$$

for $j = 1, \dots, M$. The two- and three-component cases of Eqs. (10.5) have been considered in [302]. In particular, Hioe [302] derived a dark-bright soliton solution of Eqs. (10.5). The higher-order solution sets for arbitrary M have been found in [270]. However, only the focusing case, $\alpha = 1$ has been considered so far. The solutions comprise families with the eigenvalues arranged in decreasing order. We now generalize these solutions, allowing both positive and negative coefficients, i.e. $\alpha = \pm 1$. Even for positive $\alpha = +1$, we present these solutions in a simpler form which involves two arbitrary parameters. Namely, the solution of Eqs. (10.8) can be presented in terms of associated Legendre polynomials with the total intensity in Eqs. (10.8) set to

$$\delta n = \sum_{i=1}^M u_i^2(x) = b_1(M) a_1 - \alpha (M-1) p^2 \left[M-1 - \frac{M}{2} \text{sech}^2(px) \right]$$

where $b_1(M) = \frac{1}{30} (4 + 15M - M^2)$ and a_1 and p are two arbitrary parameters. Clearly, the first term in (10.9) has to be positive, i. e. we need $b_1(M) a_1 > 0$. If we assume that M is an arbitrary integer, change variable to $y = px$, and rearrange

Eqs. (10.8) to the form:

$$p^2 \frac{d^2 u_j}{dy^2} + 2\alpha \delta n u_j = 2k_j u_j, \quad (10.9)$$

then the solution of (10.9) can be expressed in terms of associated Legendre polynomials:

$$u_j(y) = \pm \sqrt{c_{M,j}} P_{M-1}^{M-j} \tanh(y) \quad (10.10)$$

for $j = 1, \dots, M$. The coefficients of the polynomial $c_{M,j}$ must be chosen in such a way that (10.10) is a solution of (10.9). This happens when

$$c_{M,j} = \frac{2(j-1)!}{(2M-j-2)!(\delta_{j,M}+1)} \times \left[\frac{a_1 b_1(M)}{2M-j-1} - \alpha(j-1)p^2 \right],$$

where the Kronecker delta function is $\delta_{j,M} = 0$ for $M \neq j$ and 1 for $M = j$. The propagation constants in (10.9) that correspond to the above solutions are:

$$k_j = \alpha a_1 b_1(M) - p^2 \left[(M-1)^2 - \frac{1}{2}(M-j)^2 \right]. \quad (10.11)$$

The expressions under the square root in (10.10) must not be negative, starting with the higher-order one $j = M$, i.e. $c_{M,M} \geq 0$. This means that

$$a_1(4 + 15M - M^2) \geq 30\alpha(M-1)^2 p^2. \quad (10.12)$$

This ensures that all lower-order coefficients with $j = 1, 2, \dots, M-1$ are also non-negative $c_{M,j} \geq 0$. For example, if $\alpha = +1$ and $M \geq 16$, then $b_1(M) < 0$ and we need to choose $a_1 < 0$. The number of components, M , controls the overall profile of the solution. For the lowest order nontrivial case $M = 2$, the general solution is

$$\begin{aligned} u_1(x) &= -\sqrt{a_1} \operatorname{sech}(px), \\ u_2(x) &= \sqrt{a_1 - p^2 \alpha} \tanh(px) \end{aligned} \quad (10.13)$$

while the total intensity δn is given by

$$\sum_{i=1}^2 u_i^2(x) = a_1 - \alpha p^2 \tanh^2(px) \quad (10.14)$$

For $M = 3$, general solution can be written as

$$\begin{aligned} u_1(x) &= \sqrt{a_1} \operatorname{sech}^2(px) \\ u_2(x) &= -\sqrt{4a_1 - 9p^2\alpha} \tanh(px) \operatorname{sech}(px) \\ u_3(x) &= \frac{1}{\sqrt{3}} \sqrt{a_1 - 3p^2\alpha} \tanh(px) [3 \tanh^2(px) - 1] \end{aligned} \quad (10.15)$$

while the intensity for this case is

$$\sum_{i=1}^3 u_i^2(x) = \frac{4a_1}{3} - \alpha p^2 [4 - 3 \operatorname{sech}^2(px)]. \quad (10.16)$$

The solutions for higher values of M can be written in a similar way. They all contain two arbitrary real parameters a_1 and p . A particular soliton with $M = 5$ is illustrated in Fig. (10.1a,10.1b) for both focusing and defocusing cases.

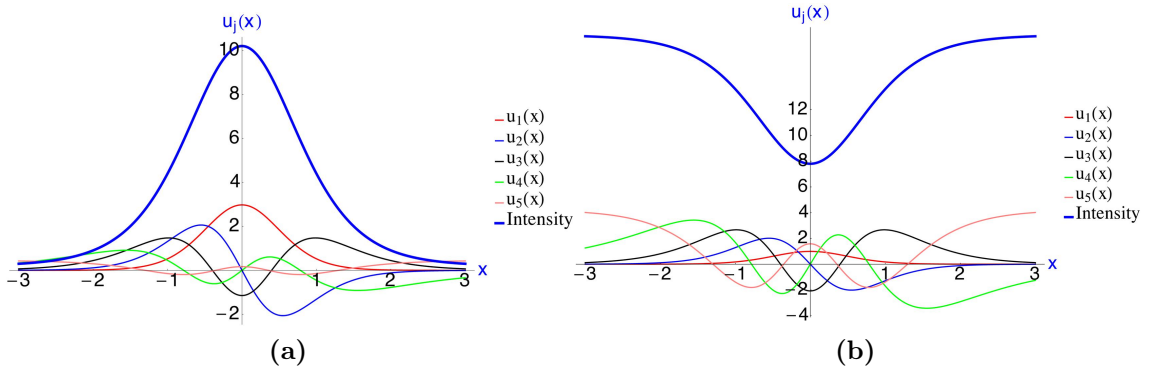


Figure 10.1 An example of an incoherent soliton for $M = 5$ in the (a) focusing ($\alpha = +1$) and (b) defocusing ($\alpha = -1$) cases. Parameters are chosen to be: for the focusing case $a_1 = 9, p = 1$ while for the defocusing case $a_1 = 1, p = 1$. The overall intensity profiles δn are shown by the solid (blue) curves. The five modes of the solution are shown by the thin coloured curves with the legends presented on the right hand side of each figure.

10.4. Reduction to the zero-Background set of order $M - 1$

In the general case, the solution has a nonzero background. The background level of the whole intensity is defined by the last component of the solution in Eq. (10.11). The background intensity can be set to zero if we set the last component coefficient,

$c_{M,M}$, to be zero, i.e.

$$a_1 b_1(M) - \alpha p^2 (M-1)^2 = 0 \quad (10.17)$$

This can only be done for the focusing case ($\alpha = 1$), as $\sqrt{c_{M,j}}$ remains real for all M whether $b_1(M)$ is positive or negative. By solving (10.17) for focusing case we get

$$a_1 = \frac{30(M-1)^2 p^2}{4 + 15M - M^2}. \quad (10.18)$$

Inserting this value of a_1 into Eqs. (10.10) and (10.11), we obtain each real-valued component $u_{M,j}$ forming the solution set. Then, the last component is zero and so the overall background is also zero. For example, in order to obtain the solution with zero background for $M = 3$, we find $a_1 = 3p^2$ from Eq. (10.18) and substitute it into Eq. (10.11). The result is:

$$\begin{aligned} u_1(x) &= \sqrt{3} \, p \operatorname{sech}^2(px), \\ u_2(x) &= \sqrt{3} \, p \tanh(px) \operatorname{sech}(px). \end{aligned} \quad (10.19)$$

Clearly, this is the solution set from [271] given by the equations (29) and (30). The coefficient of the j^{th} component of the zero background solution for any M is then given by $\sqrt{c_{M,j}}$ from Eq. (10.11). It now takes this simpler form:

$$c_{M,j} = 2 \frac{(j-1)!(M-j)^2}{(2M-j-1)!} p^2, \quad (10.20)$$

where $j = 1, 2, 3, \dots, M$. This expression is always ≥ 0 , as required. The propagation constant (10.11), of the j^{th} component of the solution set is now given by

$$k_j = \frac{1}{2} (M-j)^2 p^2. \quad (10.21)$$

For any M , the total intensity is given by

$$\sum_{j=1}^M u_j^2 = \frac{M}{2} (M-1) p^2 \operatorname{sech}^2(px).$$

For example, the zero-background solution set for $M = 6$ and $\alpha = 1$ is

$$\begin{aligned}
 u_1(x) &= -\frac{15}{4}\sqrt{7} p \operatorname{sech}^5(px) \\
 u_2(x) &= \frac{3}{2} p \sqrt{35} \operatorname{sech}^4(px) \tanh(px) \\
 u_3(x) &= -\frac{3}{8}\sqrt{\frac{35}{2}} p \operatorname{sech}(px) \operatorname{sech}^4(px) [4 \cosh(2px) - 5] \\
 u_4(x) &= \frac{1}{4}\sqrt{105} p \operatorname{sech}^5(px) [\sinh(3px) - 5 \sinh(px)] \\
 u_5(x) &= -\frac{1}{8}\sqrt{15} p \operatorname{sech}(px) [21 \tanh^4(px) \\
 &\quad - 14 \tanh^2(px) + 1] \\
 u_6(x) &= 0.
 \end{aligned}$$

This solution is presented in Fig. (10.2).

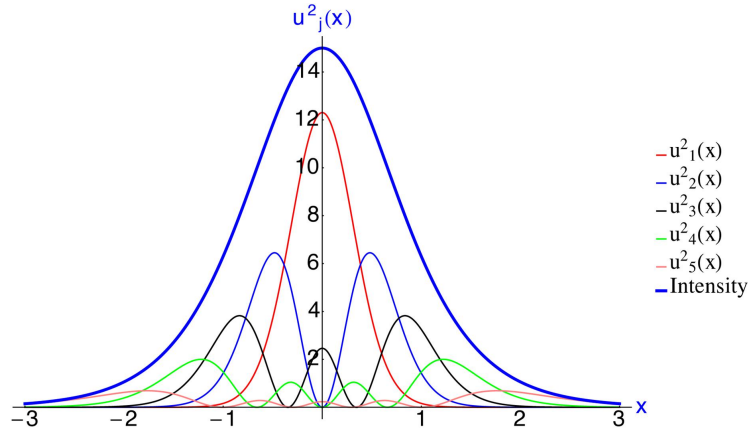


Figure 10.2 An example of an incoherent soliton for $M = 6$ in the focusing ($\alpha = +1$) case. The parameter $p = 1$. The overall intensity profile δn is shown by the solid (blue) curve. The individual intensities of the components of the solution are shown by the thin coloured curves with the legends presented on the right hand side of the figure.

The total intensity background (i.e. δn at $x = \pm \infty$) equals the background intensity of the last component, since all other components approach zero as $x \rightarrow \pm \infty$. So, setting the background part of the total intensity to zero by using the condition (10.18) also makes the last component of the solution set zero and provides the full set with $M - 1$ zero-background components.

10.5. Higher-order Solutions of the Ablowitz-Ladik Focusing Equations

The Ablowitz-Ladik model ($h = 1$) presented in [279, 280] is slightly different from the set of equations (10.3) considered here. A solution of this coupled defocusing AL set has been expressed in [280] using Chebyshev polynomials of the first and second kinds and making a connection with Lamé polynomials. Generally, the solution of equations (10.3) for arbitrary h, p can be written as a pair:

$$\begin{aligned}\psi_n^{(1)} &= \frac{1}{h} F(hnp) e^{it/h^2} \\ \psi_n^{(2)} &= \frac{1}{h} \sqrt{1 - F^2(hnp)} e^{it/h^2}\end{aligned}\tag{10.22}$$

for an arbitrary real function $F(hnp)$. The Chebyshev polynomials fit this form. However, any other function can also be used. The solution of the order $M = 3$ admits two arbitrary functions:

$$\begin{aligned}\psi_n^{(1)} &= \frac{1}{h} F(hnp) e^{it/h^2} \\ \psi_n^{(2)} &= \frac{1}{h} G(hnp) e^{it/h^2} \\ \psi_n^{(3)} &= \frac{1}{h} \sqrt{1 - F^2(hnp) - G^2(hnp)} e^{it/h^2}\end{aligned}\tag{10.23}$$

These arbitrary functions are $F(hnp), G(hnp)$. The common property of these solutions is that the total intensity is always 1. Clearly we can have a solution with an arbitrary number of components of the general form

$$\psi_n^{(j)} = \frac{1}{h} F_j(hnp) e^{it/h^2}$$

where $j = 1, 2, \dots, k-1$, and with the last component

$$\psi_n^{(k)} = \frac{1}{h} \sqrt{1 - \sum_{j=1}^{k-1} F_j^2(hnp)} e^{it/h^2}.$$

The phase factor e^{it/h^2} shows that we cannot have a continuous limit $h \rightarrow 0$ for this defocusing case.

10.6. Expansion

In the continuous example (10.5), solutions of the focusing case, when $\alpha = 1$, differ significantly from the defocusing one, $\alpha = -1$. In the work [271], exact solutions have been given in terms of Legendre polynomials for N coupled Manakov equations. This solution describes partially coherent solitons (PCSs) with zero background in a Kerr medium. Now, if we turn to the discretized version of Eq. (10.5) from [271] for $N=2$, we obtain a coupled AL system, Eq. (10.3). Chapter (9) makes it clear that there is a correspondence between the solutions of the integrable AL equation system and the NLSE. Based on this newly-established principle, we assume that, for the focusing case, there may be some solution set for the AL equation system which has its $h \rightarrow 0$ limit in equations (29,30) of [271], given as Eqs. (10.19) in this present paper :

$$\begin{aligned}\psi^{(1)} &= \sqrt{3} \operatorname{sech}^2(x) \exp(2it) \\ \psi^{(2)} &= \sqrt{3} \operatorname{sech}(x) \tanh(x) \exp(it/2),\end{aligned}\tag{10.24}$$

where x is the transverse spatial variable, t is the propagation variable, and we have taken $k_1 = 2$ and $k_2 = 1/2$.

In the introduction, we mentioned general optical and electrical applications of the AL equation. In Section (10.2), we explained the application of this Manakov-type formalism to incoherent light propagation in photorefractive media. Another physical application can be a set of two arrays of optical waveguides located one above the other. Then the overall solution has 2 component sets, one for the lower array and one for the upper. For sufficiently close arrays, there will be coupling between them. As another example, this particular coupled AL equation set can be used to model sophisticated soliton dynamics and energy exchange between two ladder-lattices [303–305] in the presence of inter-site impurities. This application may require investigation of energy transfer in the ‘exciton-vibron’ system [306]. Now

we represent the corresponding coupled AL equation solution of Eq. (10.3) with an associated multiplying function as:

$$\begin{aligned}\Psi_n^{(1)} &= \sqrt{3} u_1(n, h) \operatorname{sech}^2(nh) \exp(ik_1 t) \\ \Psi_n^{(2)} &= \sqrt{3} u_2(n, h) \operatorname{sech}(nh) \tanh(nh) \exp(ik_2 t).\end{aligned}\quad (10.25)$$

Here the unknown functions $u_1(n, h)$, $u_2(n, h)$ and the constants in the exponent, k_1 and k_2 , have to be determined so that the set will solve Eq. (10.3). So we require $u_1(n, 0)=1, u_2(n, 0) = 1$. When $h = 0$, we need $k_1 = 2$ and $k_2 = 1/2$. We expand the functions as

$$\begin{aligned}u_1(n, h) &= 1 + h^2 y_1(n) + h^4 y_2(n) + h^6 y_3(n) + \cdots \\ u_2(n, h) &= 1 + h^2 z_1(n) + h^4 z_2(n) + h^6 z_3(n) + \cdots,\end{aligned}\quad (10.26)$$

and the constants as

$$\begin{aligned}k_1 &= 2 + h^2 a_1 + h^4 a_2 + h^6 a_3 + h^8 a_4 + \cdots \\ k_2 &= 1/2 + h^2 b_1 + h^4 b_2 + h^6 b_3 + h^8 b_4 + \cdots.\end{aligned}\quad (10.27)$$

Now, substituting equations (10.26) and (10.27) into Eq. (10.3), taking an expansion in terms of h up to high order, and solving for the undetermined coefficients, we find that k_1 and k_2 turn out to be

$$\begin{aligned}k_1 &= \frac{2 \sinh^2(h)}{h^2} \\ k_2 &= \frac{2 \sinh^2\left(\frac{h}{2}\right)}{h^2}.\end{aligned}\quad (10.28)$$

10.7. Approximate Solution

Generally, the functions $u_1(n, h)$ and $u_2(n, h)$ are infinite even-order polynomials in h and $\tanh(nh)$. The contribution terms become smaller with increasing order of h , so we can ignore terms that are of very high order in h . These infinite polynomials can be approximated in a simpler form in terms of a finite specific function, which

we call $D(n)$. We find that a convenient, yet simple, approximation to the solution of the pair of AL equations can be written in terms of this function, $D(n) = H/(1 - H)$, where $H = h^2 \tanh^2(hn)$:

$$\begin{aligned}\frac{u_1(n, h)}{1 + D(n)} &\approx \frac{3}{3 + 2h^2 D(n)} + f(h) + D(n)u(h) \\ \frac{u_2(n, h)}{1 + D(n)} &\approx \frac{3}{3 + 2h^2 D(n)} + g(h) + D(n)y(h)\end{aligned}\quad (10.29)$$

The cumbersome functions $f(h)$, $u(h)$, $g(h)$, $y(h)$, which are polynomials in h , are given in the Appendix as Sec. (10.10). Thus, we have the complete form of the approximate solution:

$$\begin{aligned}\Psi_n^{(1)} &\approx \sqrt{3} u_1(n, h) \operatorname{sech}^2(hn) \exp\left(\frac{2it \sinh^2(h)}{h^2}\right) \\ \Psi_n^{(2)} &\approx \sqrt{3} u_2(n, h) \tanh(hn) \operatorname{sech}(hn) \exp\left(\frac{2it \sinh^2\left(\frac{h}{2}\right)}{h^2}\right)\end{aligned}\quad (10.30)$$

The factors $u_1(n, h)$ and $u_2(n, h)$ can be approximated with simple functions

$$\begin{aligned}u_1(n, h) &\approx \cosh^{2/3}(h) \cosh^{(h^2)}(nh) \\ u_2(n, h) &\approx \cosh^{(h^2/2)}(nh).\end{aligned}\quad (10.31)$$

This means that

$$\begin{aligned}\Psi_n^{(1)} &\approx \sqrt{3} \cosh^{2/3}(h) \operatorname{sech}^{(2-h^2)}(nh) \exp(ik_1 t), \\ \Psi_n^{(2)} &\approx \sqrt{3} \tanh(nh) \operatorname{sech}^{(1-h^2/2)}(nh) \exp(ik_2 t).\end{aligned}\quad (10.32)$$

with k_1, k_2 given by Eqs. (10.28) above. Of course, Eqs. (10.31) and (10.32) provide the shapes but are not as accurate as Eqs. (10.29) and (10.30). The functions (10.32) are approximations of Eq. (10.30).

The solutions are accurate within the limits $0 < h < \frac{3}{4}$. The expansion is correct up to the order h^{13} . In Fig. (10.3), the perturbation curves have their minima slightly above and below 1, respectively. In effect, these alter the amplitude of the total solution to some extent. For example, $\Psi_n^{(1)}$ increases from an unperturbed value 1.73

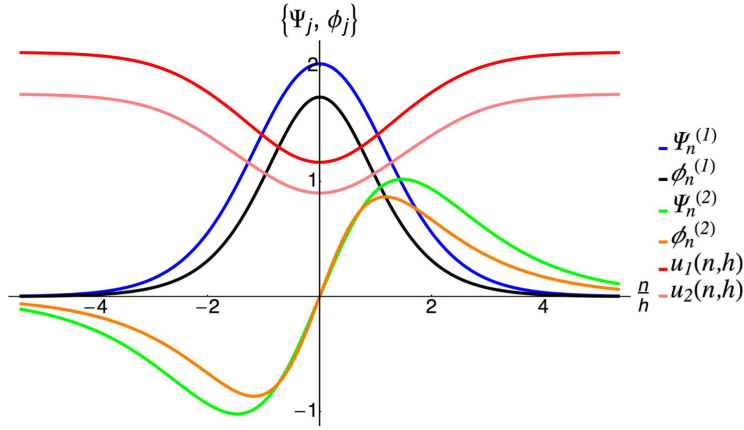


Figure 10.3 Profiles of the solution for coupled AL with $h = 0.75$. Perturbed and unperturbed ($h \rightarrow 0$) forms of the solution are presented separately with different color profiles. Here, real functions $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ (with $t = 0$), with blue and green colours, represent the total solution. From the figure, the scaling of the transverse variable n with inter-lattice spacing h is also clear. For clarity, we have taken the total range between $n/h = -4$ and 4 . The full solution profile deviates from that of the unperturbed ($h \rightarrow 0$) solution profiles $\phi_n^{(1)}$ and $\phi_n^{(2)}$ (with $t = 0$), which are shown in black and orange color, respectively. Red ($u_1(n, h)$) and pink ($u_2(n, h)$) colour curves show the perturbation factors (10.31) applied to the total solution.

to 2.12, while the other component $\Psi_n^{(2)}$ maximum increases from an unperturbed value 0.87 to 1. Another important observation is that if we want to increase the accuracy of the solution, we have to take into account the higher-order terms in h . However, we find that there is very little contribution to the solution profile from high-order terms in h . As an example, the contribution is of the order of 10^{-6} for $h = 0.75$ for terms of the order of h^{13} .

Now, if we put $n = x/h$ and take the limit $h \rightarrow 0$, the perturbed functions $u_1(n, h)$ and $u_2(n, h)$ both reduce to 1, and for the phase factors in the solution, we have $k_1 = \frac{2 \sinh^2(h)}{h^2} \rightarrow 2$ and $k_2 = \frac{2 \sinh^2(\frac{h}{2})}{h^2} \rightarrow \frac{1}{2}$. Hence, the approximate solution of Eq. (10.30) will approach the solution set for the continuous coupled NLSEs, which is the coupled Manakov set. In the limit $h \rightarrow 0$, it reduces to the solution given in Eq. (10.24). Our plots of the solutions give an idea of how much the solution of the coupled AL equation set deviates from that of the continuous Manakov set for small values of h .

10.8. Numerics and Approximations

In order to justify the analytical theory of our solution, we used numerical calculations. In the numerics, the solution represented by equation (10.30) has been used as the initial condition in the beam propagation scheme. The propagation was coupled and consistent, and the results have been verified from different perspectives. Values for the variable n , which is the discrete lattice point, are chosen between -12 and 12 . The distance between two consecutive lattice points has been scaled with the inter-lattice distance variable, h , so that we can see how each component of the solution actually scales with h . The pattern evolves with t which is the propagation variable, ranging from 0 to 100 . We have checked that the propagation profile is intact for even longer propagation distances. If we look at the propagation profile of the solution in Figs. (10.4a) and (10.4b) with $h = 0.75$, then for both components of the solution, it becomes obvious that the soliton profiles are intact, as predicted in the theoretical explanation above. The lower profiles in Figs. (10.4c) and (10.4d) show the same solution profile with $h = 0.85$. In that case, we observe that solitons actually are not in their steady state condition, and they start to scatter with propagation. This verifies that our approximate solutions are valid up to $h = 0.75$.

To verify the analytic approximate solution of the coupled AL set, we have extended our numerical investigation further. The phase parameters of the solution (see equation (10.28)), *viz.* k_1 and k_2 , for components $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$, respectively, are functions of h only. We have compared the numerically determined variations of k_1, k_2 with the analytical expressions $k_1 = \frac{2\sinh^2(h)}{h^2}$ and $k_2 = \frac{2\sinh^2(\frac{h}{2})}{h^2}$. In principle, k_1 and k_2 should be independent of n and t . Now, if we look at Fig. (10.5a), the red line represents the analytic expression for the phase of the first component, k_1 , as a function of h and the blue dots are the numeric values. It can be seen that the blue dots start to deviate from the analytic line around $h=0.75$, which is our estimated maximum value for which our coupled approximate solution is valid. In the same way, Fig. (10.5b), shows the phase profile for the second component, k_2 , where the blue dots start to deviate from the analytic line at $h=0.75$.

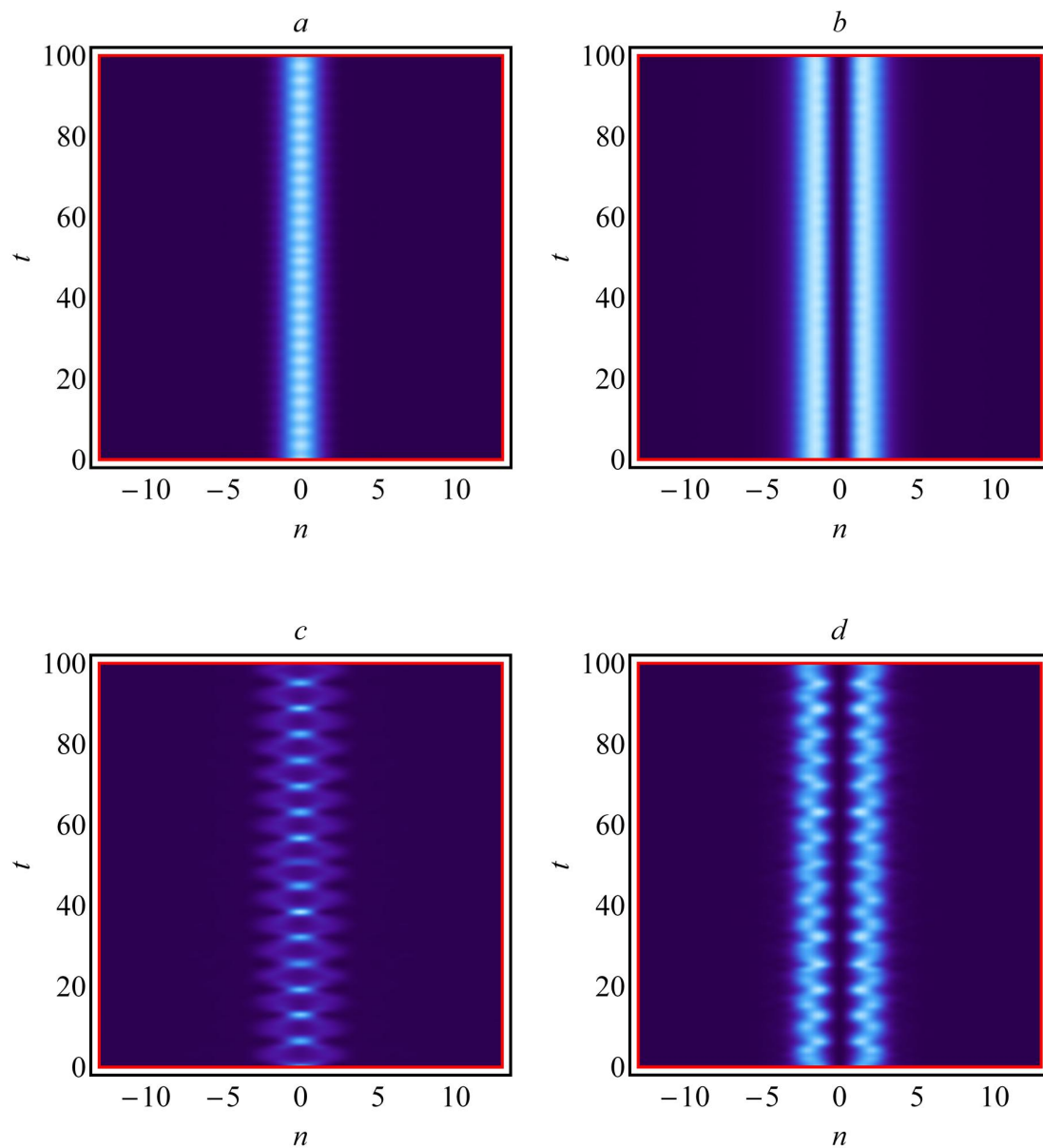


Figure 10.4 Soliton propagation for $|\Psi_1(n)|$ and $|\Psi_2(n)|$. Here at the top, figures (a) and (b) show first and second components of the solution for $h = 0.75$, while (c) and (d) show the propagation for $h = 0.85$

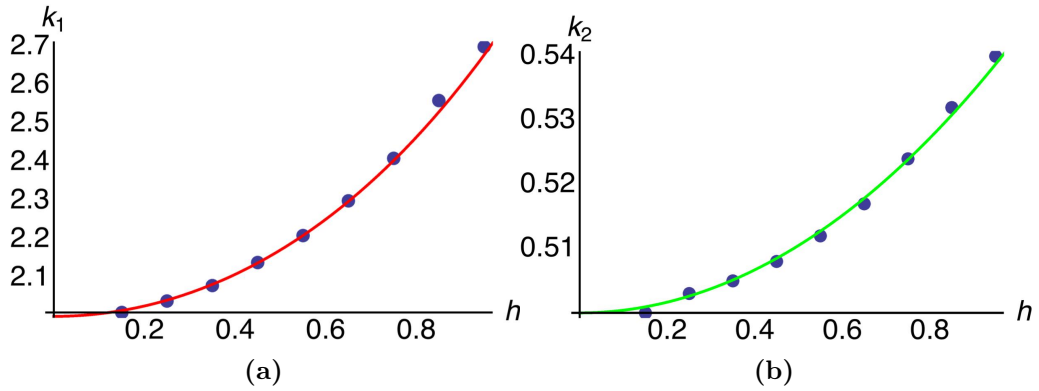


Figure 10.5 Comparison of phase parameters (a) k_1 and (b) k_2 , found numerically (dots) with their analytic forms (curves).

10.9. Conclusion

In this chapter, we have shown that the solution of the coupled Manakov system, with any number of modes, can be expressed in terms of Legendre polynomials. This can be done for both focusing and defocusing systems. Remarkably, the rational coefficients of the polynomials can be expressed using just a single function in each case. We have also derived an approximate solution for the two-component AL equation. We have shown that, in the limit $h \rightarrow 0$, the solution of the two-component AL equation becomes the solution of the two-component Manakov system. By using numerical simulations, we have verified that the approximate solution is valid for a wide range of the parameter values, *viz.* $0 < h < 0.75$.

The main idea behind this work is that, since the AL equation reduces to the NLSE in the limit $h \rightarrow 0$, the solutions should also show the same correspondence. For various cases, we have seen that this particular procedure is useful, following on from our previous work [278]. Using this analogy here, we have worked to find a related solution of the Manakov system of the coupled AL equations, and have provided an approximate solution. It remains a computational challenge to find a systematic way to derive exact solutions for infinite-component coupled AL equations, and we look forward to resolution of this challenge in the future.

10.10. Appendix

The functions used in Eqs. (10.29) are:

$$\begin{aligned}
 f(h) &= \frac{2h^2}{3!} - 2\frac{(2h)^4}{3 \times 5!} + \frac{61(2h)^6}{3 \times 4 \times 7!} - \frac{59(2h)^8}{9!} + 3 \times 19 \times 431 \frac{(2h)^{10}}{4 \times 11!} \\
 &+ 3^5 \frac{119923h^{12}}{112 \times 13!} + 2^8 \frac{32552473157h^{14}}{21 \times 15!} + \dots \\
 u(h) &= 8h^4 \left(\frac{7}{3 \times 5!} - 2\frac{11h^2}{3 \times 7!} - 7 \times 2^4 \frac{8 \times 11 \times 19h^4}{5 \times 9!} - 3\frac{99509(2h)^6}{4 \times 7 \times 11!} \right. \\
 &\left. - 2^5 \frac{26030019779h^8}{3 \times 5 \times 7^2 \times 13!} \right) + \dots \\
 g(h) &= -\frac{h^2}{3!} - \frac{17h^4}{3 \times 5!} + \frac{1417h^6}{3 \times 7!} - \frac{31319h^8}{9!} - 3\frac{3843187h^{10}}{11!} + 3\frac{1957250863h^{12}}{13!} \\
 &- \frac{3995485914943h^{14}}{3 \times 15!} + \dots \\
 y(h) &= 4h^4 \left(4\frac{11}{3 \times 5!} - \frac{863h^2}{3 \times 7!} - 2^4 \frac{67 \times 89h^4}{5 \times 9!} - \frac{39503543h^6}{7 \times 11!} \right. \\
 &\left. + 2^2 \times 11 \frac{15939718901h^8}{3 \times 5 \times 7^2 \times 13!} \right) + \dots
 \end{aligned}$$

CHAPTER 11

Summary

Solitons, breathers and rogue waves are well-known as nonlinear coherent structures. Over the last few decades, their unequivocal presence in nature as a consequence of nonlinearity has been confirmed by scientists throughout the world from various fields. As nonlinear media, water and optical fibres are at the frontiers among those that have been studied rigorously. Generally, there are two paradigms of this research field. One is ‘Discrete Nonlinear Dynamical Systems’ and the other one is ‘Continuous Nonlinear Dynamical systems’. In this thesis, both continuous and discrete nonlinear systems have been studied. To model a nonlinear discrete system, an integrable discrete nonlinear ‘Differential-difference Equation (DDE)’, viz. the Ablowitz-Ladik (AL) equation, has been considered. To model continuous systems, the integrable ‘Nonlinear Schrödinger equation’ (NLSE) (and its non-integrable variants) and its hierarchy have been considered. A brief summary of the main results achieved in this thesis are listed in the following few points.

1. The fifth member of the NLSE hierarchy, which is the ‘quintic NLSE’ has been derived. An explicit expression for the Lax pair and Darboux transformation was derived for this equation. Following the Darboux transformation technique and considering the complex eigenvalue case, one, two and three soliton solutions have been presented. For the first time, it was noted that the real part of the eigenvalue not only introduces velocity, but also can play a central role

- in soliton superposition. Loci of parameters on the complex plane, indicating where soliton superposition can be achieved, were presented and various beating structures of soliton superposition observed. In the equal eigenvalue limit, the degenerate two soliton solution was also achieved.
2. Considering the complex eigenvalue case and using a Darboux transformation, one and two breather solutions of the ‘quintic NLSE’ have been derived explicitly. The importance of higher order terms present in the ‘quintic NLSE’ was realised. Beyond specific optical pulse duration, the basic NLSE fails to model ultra-short pulse dynamics. One has to consider higher-order terms, including higher order dispersion, to model optical short pulse dynamics in a highly nonlinear medium. From this point of view, the connection between the ‘quintic NLSE’ and the ‘Generalised nonlinear Schrödinger Equation (GNLSE)’ has been established. Again in the derivation of breather solutions using complex eigenvalues, a new phenomenon in breather dynamics was revealed; this is called ‘breather-to-soliton conversion’. New insight into the breather-soliton interaction has been achieved. With a specific setting of the parameters, an AB becomes a non-pulsating soliton. Beginning from the first order breather solution, this transformation applies to all orders of breather solutions. Thus, in other words, all orders of breather solutions can be converted into higher-order soliton solutions. The loci of parameters on the complex plane where a breather becomes a soliton was presented explicitly. It was found that each point on these loci indicates a normal soliton; it is even possible to involve multiple points to obtain a many-soliton collision. The procedure of ‘breather-to-soliton conversion’ has been considered for both an AB and Ma breather. It was also revealed that similar features do not exist for the breather solutions of the basic NLSE.
 3. Breather dynamics of the ‘quintic NLSE’ was studied by using a Darboux transformation and considering both complex and imaginary eigenvalues. Various closed form solutions of rogue waves and breathers have been presented explicitly. With imaginary eigenvalues, various limiting cases of second-order breather solutions of the ‘quintic NLSE’ were studied. For the two-breather

solution, by varying the frequency components with respect to each other, the breather interaction mechanism was revealed. In the equal eigenvalue condition, the degenerate two-breather solution was derived. In the infinite frequency limit, the second order quintic rogue wave solution with eigenvalue-dependent differential shift and the quintic rogue wave triplet solution have been derived. In the complex eigenvalue case, the quintic two-breather solution allows breather superposition, as was the case for the soliton in the ‘quintic NLSE’. An explicit expression for the loci of parameters on the complex eigenvalue field has been identified. Each point on the locus represents two superposed breathers and none of these are similar to each other, thus offering an infinite variety of breather structures.

4. Breather dynamics has been studied for the third member of the NLSE hierarchy, which is the Hirota equation. The explicit expression for the Lax pair for this equation has been presented. Using a Darboux transformation and taking complex eigenvalues, a closed-form two-breather solution of the Hirota equation was derived. A simple parametric relation for which a Hirota breather becomes a soliton was recognized. Using this parametric relation, various higher-order breather solutions have been converted into non-pulsating solitons. Additionally, the loci of parameters on the complex eigenvalue field, for which breather superposition occurs, was revealed; various different structured superposed breathers have been presented. With a similar procedure, the loci of parameters for superposed breathers of the NLSE were presented. It was revealed that the parametric relation, for which a breather becomes a soliton, does not hold for the NLSE; instead, in the same circumstances, a moving breather is produced and it propagates with a certain velocity.
5. Rogue waves and breather dynamics in the shallow water case have been studied. Closed-form solutions of the modified Korteweg-de Vries (mKdV) equation have been presented. Remarkably, it was revealed that the mKdV breather is localized in t and maintains a constant amplitude in x , indicating a soliton. In the infinite frequency limit, it becomes a rational soliton. The limiting case of the two-breather solution revealed various interesting structures that were

quite unexpected for the simple mKdV equation. In the second order rational solution, it was found that the centre rogue profile is nothing but the collision profile between a bright and a dark soliton. Introducing a differential shift does not produce NLSE-type rogue-triplets; instead, it translates the rogue profile along the trough of the dark soliton. The rogue wave and the breathers for the shallow water case have been recognized for the first time.

6. Rogue waves in various perturbative cases have been considered in nonlinear fibre optics. In the presence of third-order dispersion, self-steepening and self-frequency shift, various rogue wave approximate solutions have been derived. For these cases, a combined Sasa-Satsuma with Hirota equation has been considered. These third-order terms apply an odd-symmetry perturbation to the rogue profile. It was found that the rogue profile indeed survives these perturbation when the associated parameter value is small. Derived approximate solutions have been verified with numerical simulations.
7. To investigate the robustness of rogue waves, an even-symmetric perturbation was applied. As perturbative terms, fourth-order dispersion and quintic nonlinearity were combined with the NLSE with an arbitrary coefficient. An approximate solution, accurate up to first order, has been derived. Numerical simulations have been used to check the validity of the solution and to identify the range of parameter values for which the solution is accurate.
8. A list of corresponding solutions of the discrete AL and continuous NLSE has been drawn up. The solutions include breathers, solitons, rogue waves, periodic ones, doubly-periodic ones, Manakov pairs and travelling Manakov pair solutions.
9. The discrete Ablowitz-Ladik (AL) equation and the NLSE are both integrable and can be solved by using an inverse scattering technique. The AL is the integrable discretization of the NLSE. Similar integrable properties are shared by each of these equations. In most cases, families of solutions which exist for the NLSE also exist for the AL equation, and, depending on a parameter relation, all these solutions correspond to one another. On the other hand, a

Manakov system is formed from a number of coupled NLSEs. Solutions have been derived for the coupled Manakov system using an associated Legendre polynomial formalism, for both focusing and de-focusing cases. Similarly, corresponding approximate solutions have been derived for two coupled AL equations. These approximate solutions have been verified using numerical simulations.

CHAPTER 12

Future Challenges

As we are immersed in a world of complicated microscopic and macroscopic physical activities, both linear and nonlinear equations are possible approaches for studying and understanding all the varieties of complex natural systems. However, it is at the same time not trivial to engineer an equation and solve it for a specific physical system. As the natural process itself and the relevant system have their inherent complexity, very often it is challenging to solve them by using traditional methods and mathematical tools. New ways of solving and interpreting techniques are always in high demand. *The aim of this thesis was not only to solve equations and illustrate their physical relevances, but also to show a path into the future, and describe how the developed tools and techniques could be used to resolve future challenges in the relevant fields.* Of the myriad of potential applications and future challenges, the following examples illustrate a few of them. Here they are:

1. The NLSE is a nonlinear variation of the linear Schrödinger equation, derived in many areas of physics. Because of its universal character, it attracts attention from scientific communities. The importance of the NLSE is not only for its physical relevance, but also for its mathematically rich structure. It possesses a true nonlinearity and can be connected to various other nonlinear systems and relevant equations. For example, by applying certain rules, it is possible to derive the whole hierarchy of Korteweg-de Vries (KdV) and modified Korteweg-

- de Vries (mKdV) equations. Each of these equations is important to model various wave phenomena in nonlinear media. Even though the NLSE has been studied for more than 50 years, innumerable opportunities still lie ahead. Using the NLSE as a key nonlinear equation, it is possible to study other complicated nonlinear equations and, subsequently, to reveal their mysterious properties.
2. Additionally, understanding the dynamics of the light-matter interaction covers a vast range of research fields from inter-galactic to inter-atomic scales. For example, various chemical processes occur so fast that it is impossible for the human eye to observe their progress. However, controlling a particular complex chemical reaction, such as splitting molecules into atoms, requires use of extremely short pulses in a controlled manner. Moreover, a precise control of extremely short pulses is important for understanding and interpreting various experimental data. This enables one to unveil the true nature of particular processes at atomic to electronic scales. To achieve these goals, theoretical and computational knowledge are required. For example, 'Supercontinuum Generation (SG)' and 'High Harmonic Generation (HHG)' are two nonlinear phenomena helping us to understand short pulse dynamics; they are connected with the integrable NLSE and its hierarchy. The better we understand the integrable NLSE and its hierarchy, the better insight we will have into SG, HHG and ultra-short pulse dynamics.
 3. The NLSE and its family can be studied in combined operator form, gradually adding one member at a time, in ascending order. Each operator represents an individual family member of the NLSE hierarchy. The NLSE itself is the first and leading member of the entire family. Remarkably, individual members possess different characteristics. It is worthwhile to know what will be the solution structure if one takes different combinations among the members. For example, in the SSE, only the first member of the NLSE is needed, and this provides the basic structure of this equation. However, if we remove the NLSE operator and plug in another member of the NLSE hierarchy, this will certainly lead us to a new equation and it will have different nonlinear characteristics. There is an excellent future opportunity to study these particular combinations

of equations and derive their solutions. This same procedure also applies to the Complex Ginzburg-Landau equation (CGLE), which is a representative equation for explaining dissipative systems.

4. After the theoretical prediction of a rogue wave as a rational solution of the NLSE more than 30 years ago, experimentally it was first observed in an optical system. As an extreme event, this solution may explain financial disasters, oceanic rogue waves, stampedes, unpredictable climate catastrophes and various other extreme occurrences in nature. Research in this direction is still in its infancy, and a complete characterization of rogue wave dynamics has not yet been achieved. Experimental and theoretical studies, based on mathematical results, are fundamental requirements for getting insight into the nature of rogue waves. A complete characterization of rogue waves will allow control of their dynamics, so that they can be prevented or created for use. For future research on rogue waves, analytical and numerical tools developed in this thesis will be useful. In this thesis, three new nonlinear phenomena have been discovered, namely, ‘Breather-to-Soliton’ conversion, ‘Breather Superposition’ and ‘Soliton-Superposition’. These dynamical interactions do not occur arbitrarily; they occur for a specific ‘locus’ of parameters on the complex plane. Can there be other undiscovered nonlinear mechanisms hidden on this complex plane? At present these ‘loci’ represent dynamics up to the second-order solutions of the ‘quintic NLSE’ and the Hirota equation. How do the higher-order solutions and their dynamics behave on the complex plane? The physical significance of the higher-order terms in the ‘quintic NLSE’ is not known. What role do they play in a physical system? Furthermore, all the terms in the integrable ‘quintic NLSE’ have specific integer coefficients for which the equation is solvable; these allow coherent structures such as optical rogue waves, ABs and solitons as solutions. A small deviation from these integer values may make the system chaotic and unpredictable. Why is the system integrable only for these specific integer coefficients? Is it a mathematical quirk? Or has it a natural cause, in that it supports harmony and consistency, rather than chaotic and unpredictable behaviour? Answering these questions is challenging.

A step-by-step and systematic study and research will reveal the true nature of this mathematical model for rogue wave and short pulse dynamics in nonlinear media. Existence and dynamics of solitons, ABs and rogue waves have been verified as unequivocal mechanisms in various nonlinear media such as in water and in optical fibres. But these newly discovered mechanisms of ‘Breather-to-soliton’ conversion, ‘Breather Superposition’ and ‘soliton-superposition’, which I discovered during my PhD research, have not yet been verified experimentally. Without doubt, there are many excellent future opportunities, theoretically and experimentally, in this direction.

5. All the solutions and newly-discovered nonlinear dynamics of ‘Breather-to-Soliton’ conversion, ‘Breather Superposition’ and ‘Soliton-Superposition’ of the integrable ‘quintic NLSE’ are directly connected to an exotic nonlinear phenomenon, viz. SG. Applications of SG include achieving single or sub-cycle pulses by compression, wideband tunable wavelength conversion, optical coherence tomography and optical frequency metrology. Among the myriad of nonlinear optical phenomena, SG is a stunning effect, visually, as well as from an application perspective. How does an initial narrow-band laser pulse gradually transform into a dazzling rainbow of colours spanning the entire visible and near infrared, simply by passing through a fibre? A supercontinuum itself is a result of collective and complex effects of a number of nonlinear mechanisms within an applied field that is propagating through a highly nonlinear medium. The detailed processes and dynamics behind the SG phenomenon are complicated, as a cascade of various nonlinear effects is involved.

After propagation, at the output, the applied field transforms into a broad spectral continuum, typically ranging over 3 to 4 octaves. Although this unique phenomenon has been studied for a decade now, the reasons for spectral broadening are still not fully understood. Scientific research reveals that the primary reason for SG is the mutual interaction of several intrinsic nonlinear mechanisms, namely, self-phase modulation, modulation instability, soliton fission, Raman-induced frequency shift and self-steepening. Recent scientific study reveals that the developed stage of SG is connected to ABs,

which are analytic solutions of the NLSE hierarchy. At this moment, there is no existing analytic theory to explain and characterize SG. Numerical solution of the GNLSE is the only means to explain SG approximately.

In this thesis it was found that there is a significant similarity between the integrable ‘quintic NLSE’ and the modeling equation of SG, the ‘Generalized Nonlinear Schrödinger Equation’ (GNLSE), which is non-integrable. As there exist families of analytic solutions for the integrable ‘quintic NLSE’, there is an excellent opportunity to characterize SG from a more deterministic point of view. Combining the analytic solution with numerical simulation to study the systematic progress of SG will help us to understand the process of wide spectrum development.

This will thus allow controlling the relative importance of individual non-linear phenomena, giving researchers control over the coherence and bandwidth properties of SG, and effectively allowing this novel nonlinear phenomenon to be used in ever more sophisticated ways.

Another major issue in SG is that, in a highly nonlinear medium, e.g. a photonic crystal fiber, the envelope equation (GNLSE) fails to model pulse propagation and one has to use a field equation instead. Why does the GNLSE fail? The question has to be answered. There is an excellent experimental and theoretical opportunity to predict the underlying physics to find the regime where the GNLSE fails. Undoubtedly, this will present many opportunities for new and exciting discoveries.

6. ‘Diametric drive’ is a theoretical concept to make a space propulsion system without any propellant. The underlying physics of this hypothetical system came from exploiting the idea of ‘negative mass’. A classical system with a block of ‘negative mass’ attached to a positive mass forms an unintuitive dynamical picture, defying Newton’s third law of motion. Following Newton’s third law of motion, instead of exerting equal and opposite forces on each other, a negative and a positive mass system will constantly accelerate in the same direction – the phenomenon called ‘diametric drive’, proposed by NASA scientist Marc G. Millis. In principle, in a classical system, mass entities are

always positive and ‘negative mass’ is forbidden. However, with the existence of ‘negative mass’ and its interaction with a positive mass, there is a breakdown of the action-reaction symmetry. However, a classical realistic ‘diametric drive’ has never been reported. A ‘diametric drive’ is not possible in a classical system, but, by adopting the same idea, experimentally it has already been shown that a self-accelerating ‘Optical Diametric Drive’ is indeed possible.

In optics, the classical ‘negative mass’ has been replaced by the ‘negative effective mass’ of a short optical pulse that is free from all the limitations of a classical mass system. A localized optical pulse in an optical medium plays the role of a classical object in free space. Depending on the sign of the group velocity dispersion, photon effective mass can be positive or negative, thus providing the ultimate opportunity to engineer a self-accelerating pulse system in nonlinear fiber optics. To achieve the ‘negative mass’, one of the pulses was considered to be in the normal dispersion regime. At present, only soliton-type pulses have been considered. There is an excellent opportunity for future research to relate the ideas of ‘negative mass’ to ABs and rogue waves, and investigate their dynamical features as a ‘self-accelerating’ system. Research in this context may contribute to particle dynamics in crystal lattices, faster communication systems, frequency conversion and pulse steering processes.

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