

# Morse homology

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# Declaration

The work in this thesis is my own except where otherwise stated.

乔竹格



# Abstract

Morse homology were developed during the first half of the twentieth century. The underlying idea and various infinite dimensional versions, such as Floer homology, continue to be of interest to researchers in mathematics and theoretical physics today.

In the first chapter of this thesis, we briefly describe the finite dimensional Morse theory and its cellular singular homology . Our main focuses are stable/unstable manifolds, the associated CW-complex. Suppose  $M$  is a smooth compact finite dimensional Riemannian manifold and  $f$  is a real valued function defined on  $M$  with all critical points nondegenerate. We consider the gradient flow line, that is, a smooth curve  $\gamma : \mathbb{R} \rightarrow M$  satisfying the following differential equation:

$$\frac{d\gamma(t)}{dt} + \nabla f(\gamma(t)) = 0$$

Using these gradient flow lines, we can decompose  $M$  into cells and construct a CW-complex associated to  $f$ .

In chapter two, we introduce Morse homology, whose chain complex is generated by critical points of  $f$  and the boundary operators is defined by counting certain gradient flow lines connecting two critical points of relative index one. In order to check that this boundary is well-defined, We need to study the analysis of moduli spaces of gradient flow lines. These moduli spaces can be identified as zero sets of some Fredholm operators between infinite dimensional Banach manifolds. Since the moduli spaces may fail to be compact, we will discuss about the compactification of the moduli space. Then we obtain the fact that there are finitely many flow lines between critical points with relative index 1. After defining an coherent orientation on moduli spaces, we can construct boundary operator of Morse complex by counting flow lines with sign. This approach is very useful in studying a generalization of Morse theory on certain infinite-dimensional manifolds.

In chapter 3, we show that the two kinds of homology we construct in the

previous two chapters are isomorphic. In fact, we show that two chain complexes are identical.

In the final chapter, we give an overview of Floer theory and very briefly introduce Lagrangian-Floer theory and instanton Floer theory.

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# Notation

$H^{1,2}(\mathbb{R}, \mathbb{R}^n)$	Sobolev space of curves square-integrable together with the first derivative
$C_{x,y}^\infty$	smooth curves with ends $x, y$
$L(X, Y)$	space of continuous linear Banach space operators
$K(X, Y)$	space of compact operators
$F(X, Y)$	space of Fredholm operators between Banach spaces
$C_{loc}^\infty$	space of smooth function with $C^\infty$ -compact-open topology



# Chapter 1

## Cellular Homology

### 1.1 Brief review of Morse theory

The main goal of Morse theory is to show how to construct a CW-complex that is homotopy equivalent to a smooth manifold  $M$  using “Morse function”. More precisely, we can uncover the information about the topology of  $M$  by discussing the critical point of Morse function. Morse theory plays an important role in classifying smooth manifolds. Floer in 1980s developed an infinite dimensional version of Morse theory and our understanding of low dimensional manifolds and symplectic topology. This will be introduced briefly in chapter 4.

We first introduce some definitions.

**Definition 1.1.** Let  $M$  be a smooth compact finite dimensional manifold and

$$f : M \rightarrow \mathbb{R}$$

is a smooth map. Then a point  $p$  is called a *critical point* if the following differential map

$$df_p : T_p M \rightarrow T_{f(p)} \mathbb{R} \cong \mathbb{R}$$

is zero.

If  $p$  is a critical point of  $f$ , we can define a symmetric bilinear form  $Hess_p f$  on  $T_p M$ , which is called the Hessian of  $f$  at  $p$ .

$$Hess_p f : T_p M \times T_p M \rightarrow \mathbb{R}$$

In terms of local coordinate chart  $\{x_1, x_2, \dots, x_n\}$  near  $p$ ,  $Hess_p f$  is the familiar matrix of second partial derivatives

$$Hess_p f = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} (p) \right)$$

A critical point  $p$  is called non-degenerate if  $Hess_p f$  is nonsingular. It is easy to check that the non-degeneracy does not depend on the coordinate system. A function with all its critical point nondegenerate is called Morse function. We can now talk about the index of  $f$  at  $p$ .

**Definition 1.2.** The *index* of a critical point  $p$ , denoted as  $\lambda(p)$ , is the maximal dimension of a subspace of  $T_p M$  on which  $Hess_p f$  is negative definite.

From linear algebra, it is obvious that the index of a critical point  $p$  is exactly the number of the negative eigenvalue of the Hessian matrix at  $p$ . Now we introduce the Morse lemma, which shows that the behaviour of  $f$  near  $p$  can completely described by the index.

**Lemma 1.3. (Morse lemma)** *Suppose  $p$  is a non-degenerate critical point of  $f$ , then  $\exists$  local coordinate chart  $(x_1, x_2, \dots, x_n)$  near  $p$  s.t.  $x_i(p) = 0$  for all  $i$  and*

$$f = f(p) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2$$

where  $\lambda = \lambda(p)$ ,  $n = \dim M$ .

Then we have the following theorem describing the CW-complex associated to a Morse function.

**Theorem 1.4.** *If  $f \in C^\infty(M, \mathbb{R})$  is a Morse function and  $M$  is compact smooth and finite dimensional. Then  $M$  has the homotopy type of a CW-complex, with one cell of dimension  $\lambda$  for each critical point of index  $\lambda$ , i.e.,  $M \simeq \bigcup_{x \in \text{crit}_k f} e_x^{\lambda(x)}$ .*

We denote the set of all points whose function value is less or equal to  $a$  by  $M^a$ , i.e.,  $M^a = \{x : f(x) \leq a\}$ . The key observation which finally proved this theorem is the change of the homotopy type of  $M^a$  when  $a$  crosses a critical value. The proof of this theorem and Morse lemma is given in Milnor's "Morse theory". As a consequence, if we denote the Betti number and numbers of critical points of index  $k$  by:

$$b_k = \dim H_k(M, \mathbb{R}) \quad k = 0, 1, \dots, n = \dim M$$

$$c_k = \# \text{crit}_k f \quad \text{crit}_k f = \{x \in \text{crit} f : \lambda(x) = k\}$$

Then the following "Morse inequalities" hold:

$$c_k - c_{k-1} + \dots \pm c_0 \geq b_k - b_{k-1} + \dots \pm b_0 \quad k = 0, \dots, n-1$$

$$\chi(M) = c_n - c_{n-1} + \dots \pm c_0$$

## 1.2 The CW complex associated to a Morse-Smale function

Morse theory has described how a compact finite dimensional manifold  $M$  is homotopy equivalent to a CW complex. It would be nice if we could describe the manifold as a CW complex directly, instead of homotopy equivalence. We will see that only Morse function is not enough to accomplish this, and we also need a transversality condition which is called Morse-Smale condition. To do this, we first show how the cells of the CW complex come from.

### 1.2.1 The unstable manifold theorem

Suppose  $M$  is a finite dimensional compact Riemannian manifold and  $f$  is a Morse function defined on  $M$  and  $x \in M$ . Let  $\gamma_x(t)$  be a flow line through  $x$ , i.e., it satisfies the following differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}(\gamma) &= -\nabla f(\gamma) \\ \gamma_x(0) &= x\end{aligned}$$

Then for any critical point  $p$ , we can have the stable and unstable manifolds defined by:

$$\begin{aligned}W^s(p) &= \{x \in M : \lim_{t \rightarrow +\infty} \gamma_x(t) = p\} \\ W^u(p) &= \{x \in M : \lim_{t \rightarrow -\infty} \gamma_x(t) = p\}\end{aligned}$$

In other words, the stable manifold is the set of points in  $M$  that flow down to  $p$ , and the unstable manifold is the set of points that flow up to  $p$ . We now state a theorem that explains the term “manifold” in these two definitions.

**Theorem 1.5.** *Let  $f$  be a Morse function and  $(M, g)$  be a compact smooth and  $n$ -dimensional Riemannian manifold. Then for any critical point  $p \in M$  of  $f$  with  $\lambda(p) = \lambda$ , we have*

$$T_p M = T_p^s M \oplus T_p^u M$$

where the Hessian is positive definite on  $T_p^s M$  and negative definite on  $T_p^u M$ . Moreover,  $W^s(p)$  and  $W^u(p)$  are smoothly embedded open disks of dimension equal to  $n - \lambda$  and  $\lambda$  respectively.

The proof of theorem 1.5 is given in chapter 4 of [1].

We now introduce the following proposition of unstable manifold.

**Proposition 1.6.** *If  $(M, g)$  is a compact Riemannian manifold, and  $f$  is a Morse function on  $M$ , then*

$$M = \bigsqcup_{p \in \text{crit}(f)} W^u(p)$$

*Proof.* For any  $x \in M$ , consider the flow line  $\gamma_x$  (The existence and uniqueness of  $\gamma_x$  come from the result of Lemma 2.4 in Milnor's "Morse theory"). Then  $f \circ \gamma_x$  is a function from  $\mathbb{R}$  to  $\mathbb{R}$ .

By the fundamental theorem of calculus,

$$(f \circ \gamma_x)(b) - (f \circ \gamma_x)(a) = \int_a^b \frac{d}{dt}(f \circ \gamma_x)(t) dt$$

$f \circ \gamma_x$  is bounded since  $M$  is compact, so the left-hand side of the formula is bounded. For right-hand side, we have

$$\begin{aligned} \frac{d}{dt}(f \circ \gamma_x) &= \langle \nabla(f \circ \gamma_x), \gamma'_x \rangle \\ &= \langle \nabla(f \circ \gamma_x), -\nabla(f \circ \gamma_x) \rangle \\ &= -|\nabla(f \circ \gamma_x)|^2 \\ &\leq 0 \end{aligned} \tag{1.1}$$

Therefore,

$$\lim_{t \rightarrow \pm\infty} \frac{d}{dt}(f \circ \gamma_x)(t) = 0$$

So,

$$\lim_{t \rightarrow \pm\infty} -|\nabla f(\gamma_x(t))|^2 = 0$$

Since  $M$  is compact,  $M - \bigcup_{p \in \text{crit}(f)} U_p$  is compact, where  $U_p$  is a small neighbourhood of  $p$ . So  $|\nabla f|^2$  has its minimum on  $M - \bigcup_{p \in \text{crit}(f)} U_p$ , which is positive since there is no critical point on  $M - \bigcup_{p \in \text{crit}(f)} U_p$ . We have proved that the limit of  $|\nabla f(\gamma_x(t))|^2$  is zero, so when the absolute value of  $t$  is sufficiently large,  $\gamma_x(t) \in \bigcup_{p \in \text{crit}(f)} U_p$ .

The limits of  $\gamma_x(t)$  are critical points since we can make  $U_p$  arbitrary small. As a consequence,

$$M = \bigcup_{p \in \text{crit}(f)} W^u(p)$$

$W^u(p) \cap W^u(q) = \emptyset$  when  $p \neq q$  comes from the uniqueness of  $\gamma_x$ .  $\square$

The structures of the unstable manifold have shown how the cells of CW-complex come from. Unfortunately, the immediate cell decomposition is not enough to represent a CW-complex. We need a transversality condition, which defined by Smale in the 50's.

### 1.2.2 Morse-Smale condition

**Definition 1.7.** Suppose  $(M, g)$  is a finite dimensional smooth Riemannian manifold, and  $f : M \rightarrow \mathbb{R}$  is a Morse function.  $f$  is called a Morse-Smale function if for all  $p, q \in \text{crit}(f)$

$$W^s(p) \pitchfork W^u(q).$$

Given Morse-Smale condition, we can see how unstable and stable manifolds of different critical points intersect. For any two critical points  $p, q$ , we will denote

$$W^u(p) \cap W^s(q) = W(p, q)$$

$W(p, q)$  is the space of points that flow from  $p$  to  $q$ . Then we have the following proposition.

**Proposition 1.8.** *Suppose  $(M, g)$  is an  $n$ -dimensional compact smooth manifold, and  $f$  is a Morse-Smale function defined on  $M$ . Then  $\forall p, q \in \text{crit}(f)$ , if  $W(p, q) \neq \emptyset$ ,  $W(p, q)$  is a submanifold of  $M$  of dimension  $\lambda(p) - \lambda(q)$ .*

**Lemma 1.9.** *The intersection of two transversal submanifolds of  $Y$ , say  $X$  and  $Z$ , is again a submanifold. Moreover,*

$$\text{codim}(X \cap Z) = \text{codim}(X) + \text{codim}(Z)$$

The proof of lemma 1.9 can be found in section 1.5 of [3].

*Proof.* (of proposition 1.8) From theorem 1.5 we know  $W^u(p)$  and  $W^s(q)$  are smooth embedded submanifold of dimension  $\lambda(p)$  and  $n - \lambda(q)$ . Since  $W^s(p) \pitchfork W^u(q)$ , by lemma 1.9, we know  $W(p, q)$  is a submanifold and

$$\begin{aligned} \text{codim}(W(p, q)) &= \text{codim}(W^u(p)) + \text{codim}(W^s(q)) \\ \Rightarrow \text{dim}(W(p, q)) &= \lambda(p) - \lambda(q) \end{aligned}$$

□

If  $W(p, q) \neq \emptyset$ , then it contains at least one flowline from  $p$  to  $q$ , which means that  $W(p, q)$  can not be zero-dimensional. Thus, we can immediately have the following corollary.

**Corollary 1.10.** *Suppose  $M, f, p, q$  are defined the same as above, then  $W(p, q) \neq \emptyset$  implies that  $\lambda(p) > \lambda(q)$ .*

We refer to the number  $\lambda(p) - \lambda(q)$  as the relative index of  $p$  and  $q$ .

In 1963, Kupka and Smale proved that Morse-Smale gradient vector fields are generic. We now state the theorem but the proof of this theorem will be shown later in chapter 2 after we construct the flow line space by Fredholm system. Once we have this property of Morse-Smale condition, we can change a Morse function to a Morse-Smale function by perturb the metric a little bit.

**Theorem 1.11.** *Suppose  $M$  is a smooth compact finite-dimensional manifold and  $f$  is a Morse function defined on  $M$ , then for a generic  $C^k$ -smooth metric  $g$  on  $M$ , the submanifold  $W^u(p)$  and  $W^s(q)$  intersect transversally, i.e.,*

$$W^u(p) \pitchfork W^s(q) \quad \forall p, q \in \text{crit}(f)$$

### 1.2.3 The attaching map

We have seen that  $W^u(a)$  is  $\lambda(a)$ -cell of  $M$  for each critical point  $a$ , while to define the attaching map, it is convenient to treat a little neighbourhood of  $a$  in  $W^u(a)$  as the  $\lambda(a)$ -cell rather than the whole unstable submanifold. To see this, we derive the following theorem.

**Theorem 1.12.** *Suppose  $(M, g)$  is a Riemannian manifold and  $f : M \rightarrow \mathbb{R}$  is a Morse function. Let  $a$  be a critical point of  $f$  with index  $\lambda$  and  $\tilde{f} : W^u(a) \rightarrow \mathbb{R}$  be the restriction of  $f$  on  $W^u(a)$ . Then*

- i)  $a$  is the unique critical point of  $\tilde{f}$  and  $\tilde{f}(a)$  is the absolute maximum.*
- ii)  $\tilde{f}^{-1}((\tilde{f}(a) - \epsilon, \tilde{f}(a)))$  is diffeomorphic to a  $\lambda$ -disk for some small  $\epsilon$ .*
- iii)  $\partial \tilde{f}^{-1}((\tilde{f}(a) - \epsilon, \tilde{f}(a))) = \tilde{f}^{-1}(\tilde{f}(a) - \epsilon) \cong S^{\lambda-1}$*

*Proof.*  $\forall x \in W^u(a) \setminus \{a\}$ , we have

$$\lim_{t \rightarrow -\infty} \gamma_x(t) = a$$

Then from (1.1) we know that

$$\frac{d}{dt}(f \circ \gamma_x) = -|\nabla(f \circ \gamma_x)|^2 < 0$$

Since  $x \neq a$ , the image of  $\gamma_x$  does not contain critical points of  $f$ , so the derivative can not be zero. Thus  $f(\gamma_x(t))$  is strictly decreasing. By the continuity of  $f$ , we have

$$\lim_{t \rightarrow -\infty} f(\gamma_x(t)) = f(a)$$

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Therefore,  $f(a) > f(x)$ , i.e.,  $a$  is the absolute maximum of  $\tilde{f}$ .

For all  $t \in \mathbb{R}$ ,  $\gamma_x(t) \in W^u(a)$ , so  $\gamma'_x(0) \in T_x W^u(a)$ . Since  $f(\gamma_x(t))$  is strictly decreasing,  $\gamma'_x(0) \neq 0$ , so  $\nabla_x f \neq 0$ . Therefore  $x \notin \text{crit}(\tilde{f})$ , i.e.,  $a$  is the only critical point of  $\tilde{f}$ .

Now, we want to show  $\tilde{f}$  is a Morse function. Choose a coordinate chart of  $M$  near  $a$  such that  $x_{\lambda+1} = \dots = x_n = 0$  on  $W^u(a)$ . Since  $Hess(f)$  is negative definite on  $T_x W^u(a)$ , the matrix

$$\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{ij}$$

is negative definite on  $T_x W^u(a)$ . Since  $x_{\lambda+1} = \dots = x_n = 0$ , then for  $i, j \leq \lambda$ , the Hessian matrix of  $f$  is the same as

$$\left( \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j} \right)_{ij}$$

So  $Hess(\tilde{f})$  is negative definite. We can now conclude that  $\tilde{f}$  is a Morse function.

Apply Morse lemma on  $\tilde{f}$ , there exists a neighbourhood  $U \subseteq W^u(a)$  near  $a$  and a coordinate chart  $(x_1, \dots, x_\lambda)$  such that

$$\tilde{f}(b) = \tilde{f}(a) - x_1^2(b) - \dots - x_\lambda^2(b)$$

$$x_1(a) = \dots = x_\lambda(a) = 0$$

Let  $\epsilon > 0$  be small enough so that  $D^\lambda(\epsilon) = \{(x_1, \dots, x_\lambda) \mid x_1^2 + \dots + x_\lambda^2 < \epsilon\} \subseteq U$ . It is clear that  $D^\lambda(\epsilon) \subseteq \tilde{f}^{-1}((\tilde{f}(a) - \epsilon, \tilde{f}(a)))$ . Now we want to show  $\tilde{f}^{-1}((\tilde{f}(a) - \epsilon, \tilde{f}(a))) \subseteq D^\lambda(\epsilon)$ .

If  $x \in W^u(a)$ , and  $x \notin D^\lambda(\epsilon)$ , since  $a \in D^\lambda(\epsilon)$  and  $\lim_{t \rightarrow -\infty} \gamma_x(t) = a$ ,  $\gamma_x(t) \in D^\lambda(\epsilon)$  for some  $t < 0$ . Then  $\gamma_x(T) \in \partial D^\lambda(\epsilon)$  for some  $T < 0$  (By the Jordan-Brouwer separation theorem). Since

$$f(x) = f(\gamma_x(0)) < f(\gamma_x(T)) = f(a) - \epsilon$$

then

$$x \notin \tilde{f}^{-1}((\tilde{f}(a) - \epsilon, \tilde{f}(a)))$$

So  $\tilde{f}^{-1}((\tilde{f}(a) - \epsilon, \tilde{f}(a))) \subseteq D^\lambda(\epsilon)$ . This shows ii) and iii).  $\square$

For each critical point, the corresponding stable manifold has similar properties. Then we denote  $W^s(a) \cap f^{-1}((f(a), f(a) + \epsilon))$  as  $d_a^{n-\lambda(a)}$  and  $W^u(a) \cap f^{-1}((f(a) - \epsilon, f(a)))$  as  $e_a^{\lambda(a)}$ , which is the  $\lambda(a)$ -cell of the CW-complex we are constructing. Now we introduce a theorem which is crucial for defining the attaching map.

**Theorem 1.13.** *Let  $(M, g)$  be a smooth compact  $n$ -dimensional Riemannian manifold and  $f : M \rightarrow \mathbb{R}$  be a Morse-Smale function.  $p, q \in \text{crit}(f)$  and  $\lambda(p) = k + 1$ ,  $\lambda(q) = k$ ,  $W(p, q) \neq \emptyset$ , i.e.,  $\exists \gamma : \mathbb{R} \rightarrow M$  such that*

$$\lim_{t \rightarrow -\infty} \gamma(t) = p \quad \lim_{t \rightarrow +\infty} \gamma(t) = q$$

*Denote  $\gamma(\mathbb{R}) \cap \partial e_p^{k+1}$  as  $\alpha$ . Then there exists a neighbourhood  $O_p^\gamma$  of  $\alpha$  in  $\partial e_p^{k+1}$ , i.e.,  $\alpha \in O_p^\gamma \subseteq \partial e_p^{k+1}$  and a homeomorphism*

$$\varphi_{p,q}^\gamma : O_p^\gamma \rightarrow e_q^k$$

To prove this theorem, we first introduce a lemma.

**Lemma 1.14.** *Let  $(M, g), p, q$  be defined the same as those in theorem 1.13, then there exists a neighbourhood  $U$  of  $q$  and a coordinate chart  $x = (x_1, \dots, x_n)$  such that  $x(U) = D^k(\epsilon) \times D^{n-k}(\epsilon)$  and*

$$\begin{aligned} x|_{e_q^k} & : e_q^k \rightarrow D^k \times \{0\} \\ x|_{d_q^{n-k}} & : d_q^{n-k} \rightarrow \{0\} \times D^{n-k} \end{aligned}$$

*are diffeomorphisms.*

*Proof.* By Morse lemma, we know that there exists a neighbourhood  $V$  of  $q$  and a coordinate chart  $(x_1, \dots, x_n)$  on  $V$  such that

$$f(p) = f(q) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

$$x_1(q) = \dots = x_n(q) = 0$$

Let  $\epsilon > 0$  be small enough such that

$$D^k(\epsilon) \times D^{n-k}(\epsilon) \subseteq V$$

Let  $U = x^{-1}(D^k(\epsilon) \times D^{n-k}(\epsilon))$ , then

$$\begin{aligned} x^{-1}(D^k(\epsilon) \times \{0\}) & = \{p \in U \mid x(p) = (x_1, \dots, x_k, 0, \dots, 0) \text{ } x_1^2 + \dots + x_k^2 < \epsilon\} \\ & = W^u(q) \cap f^{-1}((f(q) - \epsilon, f(q))) \\ & = e_q^k \end{aligned}$$

Similarly,

$$x^{-1}(\{0\} \times D^{n-k}(\epsilon)) = d_q^{n-k}$$

□

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*Proof.* (of theorem 1.13) We construct  $\varphi_{p,q}^\gamma$  by two steps: first, we define a map from the plaque  $O_\alpha^\gamma \ni \alpha$  diffeomorphically to a plaque  $O_\beta^\gamma \ni \beta$  (readers can see what  $\beta$  is intuitively in the figure 1.1, we will define  $\beta$  rigorously in the following proof); Second, we define a map from  $O_\beta^\gamma$  to  $e_q^k$ .

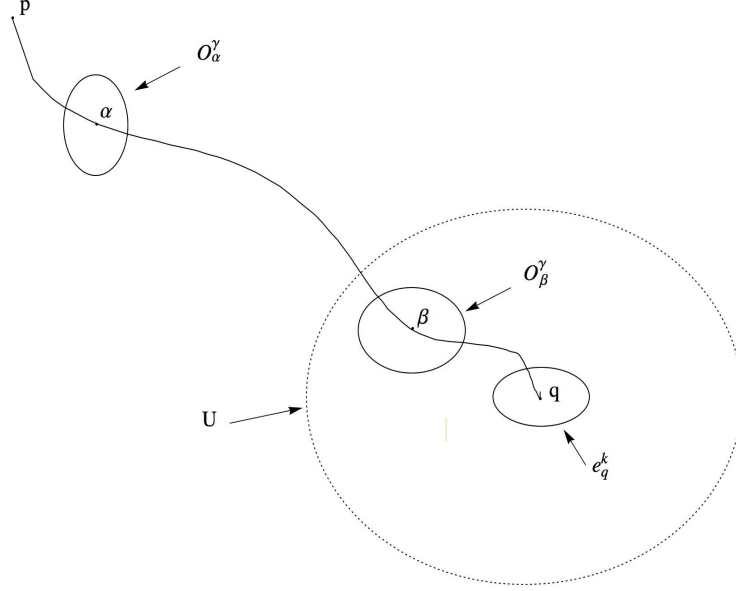


Figure 1.1

Since  $\gamma$  is a flow line from  $p$  to  $q$ , then  $\gamma(\mathbb{R}) \cap d_q^{n-k} \neq \emptyset$ . Let  $\beta \in \gamma(\mathbb{R}) \cap d_q^{n-k}$ .

Step 1: Choose a little plaque  $O_\alpha^\gamma$  of  $\alpha$  in  $\partial e_p^{k+1}$ . Since  $\partial e_p^{k+1} \cong S^k$ , then  $O_p^\gamma \cong D^k$ . Since  $\alpha, \beta \in \gamma(\mathbb{R})$ ,  $\beta = \gamma_\alpha(T)$  for some  $T > 0$ . Then we define  $\varphi_1$  on  $O_\alpha^\gamma$  by

$$\varphi_1(r) = \gamma_r(T)$$

Then  $\varphi_1$  is a diffeomorphism (Since  $\gamma_x(t)$  is generated by a smooth vector field  $\nabla f$ ). Thus  $O_\beta^\gamma = \varphi_1(O_\alpha^\gamma)$  is diffeomorphic to  $D^k$ .

Step 2: Choose a neighbourhood  $\mathcal{T}_{\gamma([T, \infty])}$  of  $\gamma([T, \infty])$  such that

- i)  $\mathcal{T}_{\gamma([T, \infty])} \subseteq U$ , where  $U$  is defined in lemma 1.14.
- ii) There exists an diffeomorphism  $\psi : \mathcal{T}_{\gamma([T, \infty])} \rightarrow [0, 1] \times D^{n-k-1}(\epsilon) \times D^k(\epsilon)$  such that for all  $t \geq T$

$$\begin{aligned} \psi(\gamma(t)) &= \left( \frac{t-T}{t}, 0, 0 \right) \in [0, 1] \times D^{n-k-1}(\epsilon) \times D^k(\epsilon) \\ \psi(O_\beta^\gamma) &= \{0\} \times \{0\} \times D^k \\ \psi(e_q^k) &= \{1\} \times \{0\} \times D^k \end{aligned}$$

Then we define  $\varphi_2 : O_\beta^\gamma \rightarrow e_q^k$  by the following diagram

$$\begin{array}{ccc}
 O_\beta^\gamma & \xrightarrow{\psi} & \{0\} \times \{0\} \times D^k \\
 \downarrow \varphi_2 & & \downarrow (0,0,*) \\
 e_q^k & \xrightarrow{\psi} & \{1\} \times \{0\} \times D^k \\
 & & \downarrow (1,0,*)
 \end{array}$$

This diagram commutes, and it is obvious that  $\varphi_2$  is a diffeomorphism.  $\square$

Suppose  $\text{crit}_k(f) = \{q_1, \dots, q_s\}$ , then we define the attaching map

$$\varphi : \partial e_p^{k+1} \rightarrow \bigcup_{i=1}^s e_{q_i}^k$$

by

$$\varphi(r) = \begin{cases} \varphi_{p,q_i}^\gamma(r) & r \in O_\alpha^\gamma \text{ for some } \gamma \text{ that flows from } p \text{ to } q_i \\ * & \text{otherwise} \end{cases}$$

### 1.3 Cellular homology of M

Denote the  $k$ -skeleton by  $X^k$ , then

$$X^k = \bigsqcup_{p \in \text{crit}_i(f), i \leq k} e_p^i / \sim$$

where the equivalence relationship comes from the attaching map. Then we define the boundary map by

- i) When  $k = 1$ , the boundary map  $\partial_1^C : H_1(X^1, X^0) \rightarrow H_0(X^0)$  is the same as the simplicial boundary map.
- ii) When  $k > 1$ ,

$$\partial_k^C(e_p^k) = \sum_{q \in \text{crit}_{k-1}(f)} \text{deg}(\alpha_{pq}) e_q^{k-1}$$

where  $\alpha_{pq}$  is the map

$$\partial e_p^k \rightarrow X^{k-1} \rightarrow S_q^{k-1}$$

that is, the composition of the attaching map of  $e_p^k$  with the quotient map collapsing  $X^{k-1} \setminus e_q^{k-1}$  to a point.

Now we have the cellular chain complex:

$$(C_k = \bigoplus_{p \in \text{crit}_k(f)} \mathbb{Z} \langle e_p^k \rangle, \partial^C)$$



# Chapter 2

## Morse Homology

### 2.1 The space of gradient flow line $\mu_{x,y}$

In the previous chapter, we have introduced the dynamical system given by the negative gradient flow of a Morse-Smale function. Now we continue to study more about the gradient flow line space, which is needed for the definition of Morse homology.

**Definition 2.1.** Let  $M$  be an  $n$ -dimensional compact manifold,  $f : M \rightarrow \mathbb{R}$  a Morse-Smale function. Then we define the gradient flow line space as follow

$$\mu_{x,y} = \{\gamma \in C^\infty(\mathbb{R}, M) \mid \lim_{t \rightarrow -\infty} \gamma(t) = x, \lim_{t \rightarrow +\infty} \gamma(t) = y, \gamma'(t) = -\nabla f(\gamma)\}$$

$\mu_{x,y}$  can be seen as the space of zeros of a section of a fredholm system. To see this, we first define the Banach space and Banach bundle.

**Definition 2.2.** Let  $E$  be a smooth, finite-dimensional vector bundle over  $\bar{\mathbb{R}}$  (the compactification of  $\mathbb{R}$ ) and let  $\phi : E \rightarrow \bar{\mathbb{R}} \times \mathbb{R}^n$  be a smooth trivialization. Then  $\phi_*$  is the induced one-to-one map between the associated vector space, i.e.

$$\phi_* : \Gamma(\bar{\mathbb{R}}, E) \rightarrow C^0(\bar{\mathbb{R}}, \mathbb{R}^n)$$

Then we define

$$H^{1,2}(E) = \phi_*^{-1}(H^{1,2}(\bar{\mathbb{R}}, \mathbb{R}^n))$$

Since the Sobolev space  $H^{1,2}(\bar{\mathbb{R}}, \mathbb{R}^n)$  is a Banach space,  $\phi_*$  induces a Banach space topology on the vector space  $H^{1,2}(E)$ . The following proposition shows that this topology is independent of the choice of the trivialization  $\phi$ , for the change of trivialization from  $\phi$  to  $\tilde{\phi}$  can be represented by some  $A = \tilde{\phi} \circ \phi \in C^1(\bar{\mathbb{R}}, GL(n, \mathbb{R}))$ .

**Proposition 2.3.**  $\forall A \in C^1(\bar{\mathbb{R}}, GL(n, \mathbb{R})), \exists c > 0$  s.t.

$$\|As\|_{1,2} \leq c \|s\|_{1,2} \quad \forall s \in H^{1,2}(\mathbb{R}, \mathbb{R}^n)$$

where  $(As)(t) = A(t) \cdot s(t), t \in \mathbb{R}$ .

*Proof.* It is sufficient to show  $A_{ij}$  and  $A'_{ij}$  are bounded.

$\forall f \in C^1(\bar{\mathbb{R}}, \mathbb{R})$ ,  $f$  is obviously bounded. Define  $h : \bar{\mathbb{R}} \rightarrow [-1, 1]$  by  $h(t) = \frac{t}{\sqrt{1+t^2}}$  then  $f \circ h^{-1} \in C^1([-1, 1], \mathbb{R})$ . Then

$$c(f) = \sup_{[-1,1]} |(f \circ h^{-1})'| < \infty$$

Since

$$((h^{-1})' \circ h)(t) = (1+t^2)^{\frac{3}{2}}$$

$$(f \circ h^{-1})' = f' \circ h^{-1} \cdot (h^{-1})'$$

then  $c(f) \geq |f'(t)| (1+t^2)^{\frac{3}{2}}$ , i.e.  $|f'(t)| \leq \frac{c(f)}{(1+t^2)^{\frac{3}{2}}}$ , which implies  $f'$  is bounded.  $\square$

We now give a brief definition of the Banach manifold in the Fredholm system we have previously mentioned. We start with the exponential map on the complete Riemannian manifold  $M$ .

Let  $D$  be an open convex neighbourhood in the zero section of  $TM$ . Then  $\forall h \in C^\infty(\bar{\mathbb{R}}, M)$ ,  $h^*D$  stands for the induce open convex neighbourhood of the zero section in the pull-back bundle  $h^*TM$ . By the Sobolev embedding, we define

$$P_{x,y}^{1,2} = \{exp \circ s \in C^0(\bar{\mathbb{R}}, M) \mid s \in H^{1,2}(h^*D), h \in C_{x,y}^\infty(\bar{\mathbb{R}}, M)\}$$

where  $exp \circ s(t) = exp_{h(t)}s(t)$ . Then  $P_{x,y}^{1,2}$  is equipped with a Banach manifold structure via the atlas  $\{H^{1,2}(h^*D), exp_h\}_{h \in C_{x,y}^\infty(\bar{\mathbb{R}}, M)}$ . For more details of the fact, see Appendix A of [7].

Similarly, we can define

$$L^2(E) = \phi_*^{-1}(L^2(\mathbb{R}, \mathbb{R}^n))$$

which is also a Banach space since  $L^2(\mathbb{R}, \mathbb{R}^n)$  is a Banach space. This leads to the Banach bundle over  $P_{x,y}^{1,2}$

$$L^2(P_{x,y}^{1,2*}TM) = \bigcup_{s \in P_{x,y}^{1,2}} L^2(s^*TM)$$

Then the following theorem shows that the flow line space we are studying can be seen as the zeros of a section in the Banach bundle that we defined above.

**Theorem 2.4.** *Given a section*

$$\begin{aligned} F : P_{x,y}^{1,2} &\rightarrow L^2(P_{x,y}^{1,2*}TM) \\ s &\mapsto \dot{s} + \nabla f \circ s \end{aligned}$$

Then  $F^{-1}(0) = \mu_{x,y}$

*Proof.* We prove  $\mu_{x,y} \subseteq F^{-1}(0)$  first. By assumption, elements of  $F^{-1}(0)$  are continuous curves with  $x, y$  as its end points whose weak derivative can be expressed as

$$\dot{s} = -\nabla f \circ s$$

It follows that  $\mu_{x,y} \subseteq F^{-1}(0)$ .

To see  $F^{-1}(0) \subseteq \mu_{x,y}$ , we introduce the following lemma:

**Lemma 2.5.** *(Exponential decay to critical points) There exists an  $\epsilon > 0$  such that  $\forall s \in F^{-1}(0), \exists c > 0$  and  $t_0 > 0$  satisfying*

$$d(s(t), y) \leq ce^{-\epsilon t} \quad \forall t \geq t_0 \quad d(s(t), x) \leq ce^{-\epsilon t} \quad \forall t \leq -t_0$$

*Proof.* (of lemma) By Morse lemma, we can find a neighbourhood  $U$  of  $y$  and equip a Morse chart on it. Without loss of generality, assume  $f(y) = 0$ , then

$$f(z) = \sum_{i=1}^k z_i^2 - \sum_{i=k+1}^n z_i^2 \quad \forall z \in U$$

and accordingly:

$$\nabla f(z) = (2z_1, \dots, 2z_k, -2z_{k+1}, \dots, -2z_n)$$

$$\|\nabla f(z)\|^2 = 4 \sum_{i=1}^n z_i^2 \geq 4 |f(z) - f(y)|$$

Since  $\lim_{t \rightarrow +\infty} s(t) = y$ ,  $z_i(t) = 0$  for  $i = k+1, \dots, n$ , it is clear that for such an  $s$ , it must exponentially decay to 0 as  $t \rightarrow +\infty$  since  $\frac{\partial s(t)}{\partial t} = -\nabla f(s(t))$ , i.e. for  $t > 0$

$$\begin{cases} \frac{\partial z_i(t)}{\partial t} = -2z_i(t) & i = 1, \dots, k \\ \frac{\partial z_i(t)}{\partial t} = 2z_i(t) & i = k+1, \dots, n \end{cases}$$

The proof of the result of  $x$  is similar. □

From lemma 2.5, we conclude that each derivative of  $s$  exponentially decrease to zero and  $s$  is infinitely differentiable (i.e. smooth). □

Now, we have managed to represent the flow line space as a zero set of a smooth section of a Banach bundle over a Banach manifold. To proceed to the manifold property of  $\mu(x, y)$ , we need to develop the Fredholm property of a certain linear first order differential operator  $F_\gamma$ , which is defined in the following section.

## 2.2 Fredholm property

Suppose  $s$  is a vector field along  $\gamma \in C_{x,y}^\infty(\bar{\mathbb{R}}, M)$ , and  $s(t) \in T_{\gamma(t)}M$ , we define

$$\begin{aligned} F_\gamma : H^{1,2}(\mathbb{R}, \gamma^*TM) &\rightarrow L^2(\mathbb{R}, \gamma^*TM) \\ s &\mapsto \nabla_t s + \nabla_s \nabla f(\gamma) \end{aligned}$$

where  $\nabla$  is the covariant derivative.

**Theorem 2.6.**  *$F_\gamma$  is a Fredholm operator and*

$$\text{ind}(F_\gamma) = \text{ind}(x) - \text{ind}(y)$$

The proof of this theorem requires the following observation:

**Lemma 2.7.** *Let  $X, Y, Z$  be Banach spaces, suppose  $F \in L(X, Y)$  and  $K \in L(Y, Z)$ . If  $\exists c > 0, s.t. \forall x \in X$*

$$\|x\|_X \leq c(\|Fx\|_Y + \|Kz\|_Z)$$

*then  $\dim \ker F < \infty$  and the image of  $F$  is closed.*

*Proof.* The proof consists of two parts:

i) We first show that  $\dim \ker F < \infty$ .

By Riesz's lemma, it suffices to show that the unit ball in the kernel of  $F$  is compact.

Let  $\{x_n\}_{n=1}^\infty$  be a sequence in the unit ball of  $\ker F$ , i.e.

$$\|x\|_X \leq 1 \quad F(x_n) = 0 \quad \forall n \in \mathbb{N}$$

then the hypothesis becomes:

$$\|x\|_X \leq c \|Kz\|_Z$$

By the compactness of  $K$ , the image of the unit ball under  $K$  is precompact. Then we may assume with out loss of generality that  $\{Kx_n\}$  converges in  $Z$  (since  $Z$  is

complete). Then  $\{x_n\}$  is Cauchy and hence converges in  $X$  and therefore in the unit ball. This proves the compactness of the unit ball in  $\ker F$ .

ii) We now show that  $\text{Im}(F)$  is closed in  $Y$ .

Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$  s.t.  $\lim_{n \rightarrow \infty} F(x_n) = y \in Y$ , we want to show that  $y \in \text{Im}(F)$ .

If  $\{x_n\}_{n=1}^{\infty}$  is bounded, then  $\{Kx_n\}_{n=1}^{\infty}$  remains in a compact subset of  $Y$ . Along with  $L(x_n) \rightarrow y$  and the inequality

$$\|x\|_X \leq c(\|Fx\|_Y + \|Kz\|_Z)$$

it follows that  $\{x_n\}_{n=1}^{\infty}$  has a Cauchy subsequence, hence converges and the limit  $x$  satisfies  $y = F(x)$ . So  $y \in \text{Im}(F)$ .

If  $\{x_n\}_{n=1}^{\infty}$  is unbounded. Since  $\ker F$  is finite-dimensional, we can find a closed subspace  $E \subseteq X$  s.t.  $\ker F \oplus E = X$ . Consider the sequence  $u_n = \frac{x_n}{\|x_n\|}$ , without loss of generality, we can assume

$$\{u_n\} \subseteq E \quad \|Fu_n\|_Y \rightarrow 0 \text{ as } n \rightarrow \infty$$

Since  $\{u_n\}$  is bounded, there exists  $u$  such that

$$u_n \rightarrow u \quad \|u\| = 1 \quad Fu = 0$$

So  $u \in \ker F \cap E = \{0\}$ , which contradicts  $\|u\| = 1$ .  $\square$

Equipped with the lemma, we can now prove theorem 2.6.

*Proof.* i) We first prove that  $\dim \ker F_\gamma < \infty$ .

Let  $X = H^{1,2}(\mathbb{R}, \gamma^*TM)$ ,  $Y = Z = L^2(\mathbb{R}, \gamma^*TM)$ ,  $F = F_\gamma$ ,  $K : s \mapsto s \cdot \delta_T$ , where

$$\delta_T = \begin{cases} 1 & t \in [-T, T] \\ 0 & t \notin [-T, T] \end{cases}$$

Then  $X, Y, Z, F, K$  satisfy lemma 2.7. So it is sufficient to show that  $\exists c, T > 0$  s.t.

$$\int_{\mathbb{R}} (|s|^2 + |\nabla s|^2) dt \leq c \left( \int_{\mathbb{R}} |F_\gamma s|^2 dt + \int_{-T}^T |s|^2 dt \right) \quad (2.1)$$

We first trivialize the bundles  $X, Y, Z$ . Let  $e_1(t), e_2(t), \dots, e_n(t)$  be an orthonormal basis of  $T_{\gamma(t)}M$  for all  $t \in \mathbb{R}$  which satisfies that  $\nabla e_i(t) \equiv 0$  for  $i = 1, \dots, n$ . Let  $(s_1, \dots, s_n) \in \mathbb{R}^n$  be the coordinates of  $s$  with respect to this basis. Then

$$\begin{aligned} F_\gamma(s) &= F_\gamma \sum_i s_i e_i \\ &= \sum_i \left( \frac{ds_i}{dt} + \sum_j \langle e_i, \nabla_{e_j} \nabla f(\gamma) \rangle s_j \right) e_i \end{aligned}$$

Let  $a_{ij}(t) = \langle e_i, \nabla_{e_j} \nabla f(\gamma) \rangle$  and  $A = (a_{ij})_{n \times n}$ , then  $A$  is a symmetric matrix.

We denote  $F_\gamma$  by

$$\begin{aligned} F_\gamma^{triv} : H^{1,2}(\mathbb{R}, \mathbb{R}^n) &\rightarrow L^2(\mathbb{R}, \mathbb{R}^n) \\ s = (s_1, \dots, s_n) &\mapsto \frac{ds}{dt} + A \cdot s \triangleq \varphi \end{aligned}$$

Note that  $A_\pm = \lim_{t \rightarrow \pm\infty} A(t)$  represents the Hessian of  $f$  at  $y$  and  $x$  respectively. Then  $A_+$  and  $A_-$  are nonsingular for  $f$  is a Morse function. Consider the maps

$$F_{\gamma, \pm}^{triv} : s \mapsto \frac{ds}{dt} + A_\pm s \triangleq \varphi_\pm$$

By Fourier transformation,

$$\hat{s}(i\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega t} s(t) dt$$

then

$$\begin{aligned} \hat{\varphi}_\pm(i\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega t} s(t) dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega t} \left( \frac{ds}{dt} + A_\pm s \right) dt \\ &= (i\omega I + A_\pm) \hat{s}(i\omega) \end{aligned}$$

Since  $A_\pm$  are nonsingular,  $\exists c \in \mathbb{R}$  s.t.

$$(1 + |\omega|^2) |\hat{s}(i\omega)|^2 \leq c |\hat{\varphi}_\pm(i\omega)|^2$$

So  $\|s\|_{H^{1,2}}^2 \leq c \|\varphi_\pm\|_{L^2}^2$ . This property holds for sufficiently small perturbation of  $F_{\gamma, \pm}^{triv}$ .

For sufficiently large  $T$ , if  $s$  satisfies that  $s(t) = 0$  for all  $t \in [-T, T]$ , we then have the following estimate

$$\|s\|_{H^{1,2}}^2 \leq c \|\varphi\|_{L^2}^2$$

On  $[-T-1, T+1]$ ,

$$\begin{aligned} \int_{-T-1}^{T+1} |\varphi|^2 &= \int_{-T-1}^{T+1} \left( \left| \frac{ds}{dt} \right|^2 + 2 \left\langle \frac{ds}{dt}, As \right\rangle + |As|^2 \right) dt \\ &\geq \int_{-T-1}^{T+1} \left( \frac{1}{2} \left| \frac{ds}{dt} \right|^2 - |As|^2 \right) dt \end{aligned}$$

Since  $\det A(t)$  is bounded on  $[-T-1, T+1]$ , we obtain the following estimate for large enough  $c$  that:

$$\int_{-T-1}^{T+1} \left( |s|^2 + \left| \frac{ds}{dt} \right|^2 \right) dt \leq c \int_{-T-1}^{T+1} \left( |s|^2 + |\varphi|^2 \right) dt$$

Now, let  $\beta$  be a cut-off function:

$$\beta(t) = \begin{cases} 1 & |t| \leq T \\ 0 & |t| \geq T+1 \end{cases}$$

and choose  $c$  large enough so that  $|\beta'| < c$ , then

$$\begin{aligned} \|s\|_{H^{1,2}} &\leq \|\beta s\|_{H^{1,2}} + \|(1-\beta)s\|_{H^{1,2}} \\ &\leq c(\|\beta s\|_{L^2} + \|F_\gamma^{triv}(\beta s)\|_{L^2} + \|F_\gamma^{triv}((1-\beta)s)\|_{L^2}) \end{aligned}$$

Since

$$\begin{aligned} F_\gamma^{triv}(\beta s) &= \frac{d(\beta s)}{dt} + A \cdot \beta s \\ &= \frac{d\beta}{dt}s + \beta F_\gamma^{triv}(s) \end{aligned}$$

we have:

$$\begin{aligned} \|s\|_{H^{1,2}} &\leq c(\|\beta s\|_{L^2} + 2\|\frac{d\beta}{dt}s\|_{L^2} + \|\beta F_\gamma^{triv}(s)\|_{L^2} + \|(1-\beta)F_\gamma^{triv}(s)\|_{L^2}) \\ &\leq 3c^2(\|\beta s\|_{L^2} + \|F_\gamma^{triv}(s)\|_{L^2}) \end{aligned}$$

This proves (2.1).

ii) It remains to show that  $F_\gamma$  has finite codimension and  $ind F_\gamma = ind(x) - ind(y)$ .

To see this, we compute the dimension of  $ker F$  directly and  $dim coker F_\gamma$  via the adjoint operator of  $F_\gamma$ , relying on the result from part one that  $F_\gamma$  has closed range.

Consider

$$F_\gamma^{triv} : s \mapsto \nabla s + As$$

$$(F_\gamma^{triv})^* : s \mapsto \nabla s - As$$

Since  $F(X, Y)$  is an open subset of  $L(X, Y)$  and the index is a locally constant function on  $F(X, Y)$ , we can perturb  $F_\gamma^{triv}$  a little without changing its index.

Define

$$\begin{aligned} P : L^2(\mathbb{R}, \mathbb{R}^n) &\rightarrow L^2(\mathbb{R}, \mathbb{R}^n) \\ s &\mapsto P \cdot s \end{aligned}$$

where

$$P = \begin{cases} A(t) - A_+ & t \in [T, +\infty) \\ (t - T)(A(t) - A_+) & t \in [T - 1, T] \\ 0 & t \in [-T + 1, T - 1] \\ (-T - t)(A(t) - A_-) & t \in [-T, -T + 1] \\ A(t) - A_- & t \in (-\infty, -T] \end{cases}$$

The norm of  $P$  can be arbitrarily small by making  $T$  large enough. Thus, we can find a  $T > 0$  s.t.

$$\text{ind}F_\gamma^{\text{triv}} = \text{ind}(F_\gamma^{\text{triv}} + P)$$

Since  $A_-, A_+$  are the Hessian of  $f$  at  $x, y$  respectively, then there exists  $C_+, C_- \in GL(n, \mathbb{R})$  and a curve  $C \in C^\infty(\mathbb{R}, GL(n, \mathbb{R}))$  s.t.

$$\begin{cases} \det(C_\pm) > 0 \\ C_\pm A_\pm C_\pm^{-1} = \text{diag}(\lambda_1^\pm, \dots, \lambda_n^\pm) \text{ where } \text{sgn}\lambda_i^\pm \geq \lambda_{i+1}^\pm \\ \text{for some } T > 0, C(t) = C_+ \text{ when } t \geq T \text{ and } C_- \text{ when } t \leq -T \end{cases}$$

Define  $C : s \mapsto C \cdot s$ , then

$$\begin{aligned} C(F_\gamma^{\text{triv}} + P)C^{-1} &= C(\nabla_t + A + P)C^{-1} \\ &= \nabla_t + C(\nabla_t C^{-1}) + C(A + P)C^{-1} \end{aligned}$$

Since  $C, C^{-1}$  are invertible, the maps  $C, C^{-1}$  are isomorphisms and we have

$$\text{ind}C(F_\gamma^{\text{triv}} + P)C^{-1} = \text{ind}F_\gamma$$

For any initial value  $s_0$ , the ODE

$$\begin{cases} (C(F_\gamma^{\text{triv}} + P)C^{-1})s = 0 \\ s(0) = s_0 \end{cases}$$

has unique solution on  $[-T, T]$ .

On  $(-\infty, -T)$  and  $(T, +\infty)$ , the equation becomes

$$\nabla_t s = -\text{diag}(\lambda_1^-, \dots, \lambda_n^-)s \quad \nabla_t s = -\text{diag}(\lambda_1^+, \dots, \lambda_n^+)$$

respectively. Thus,  $s$  satisfies

$$s_i(t) = \begin{cases} c_i e^{-\lambda_i^- t} & t < -T \\ d_i e^{-\lambda_i^+ t} & t > T \end{cases}$$

for  $i = 1, \dots, n$ . So  $s \in \ker C(F_\gamma^{triv} + P)C^{-1}$  if and only if  $\lambda_i^- < 0$  and  $\lambda_i^+ > 0$ . Since we have ordered the sign of eigenvalues and the number of negative eigenvalue of  $A^\pm$  are the index of  $y, x$  respectively, we can compute

$$\begin{aligned} \dim \ker F_\gamma &= \#\{i \in \{1, \dots, n\} \mid \lambda_i^- < 0, \lambda_i^+ > 0\} \\ &= \max\{\text{ind}(x) - \text{ind}(y), 0\} \end{aligned}$$

Similarly,

$$\begin{aligned} \dim \text{coker } F_\gamma &= \#\{i \in \{1, \dots, n\} \mid \lambda_i^- > 0, \lambda_i^+ < 0\} \\ &= \max\{\text{ind}(y) - \text{ind}(x), 0\} \end{aligned}$$

Therefore,  $\text{ind} F_\gamma = \text{ind}(x) - \text{ind}(y)$ . □

## 2.3 Transversality theorem

In this section we show that  $\mu_{x,y}$  is a finite-dimensional manifold using the Fredholm property we discussed in last section. However this property is true only for a generic set of metrics. Also, Morse function turns out to be Morse-Smale function for a generic set of metric.

**Theorem 2.8.** *Suppose  $G$  is the metric set on  $M$ , then there exists a generic subset  $\Sigma \subseteq G$  s.t. for all  $g \in \Sigma$ , the section*

$$\begin{aligned} F_g : P_{x,y}^{1,2} &\rightarrow L^2(P_{x,y}^{1,2*} TM) \\ s &\mapsto \dot{s} + \nabla f \circ s \end{aligned}$$

*is transversal to its zero section and  $F_g^{-1}(0)$  is a closed submanifold of dimension  $\text{ind}(x) - \text{ind}(y)$ .*

Theorem 1.11 follows from theorem 2.8.

*Proof.* We extend  $F_g$  to a section

$$\begin{aligned} \Phi : G \times P_{x,y}^{1,2} &\rightarrow L^2(P_{x,y}^{1,2*} TM) \\ (g, s) &\mapsto (g, s, F_g(s)) \end{aligned}$$

Then from the previous discussion,  $F_g = \Phi(g, \cdot)$  is a Fredholm map with index  $\text{ind}(x) - \text{ind}(y)$  for all  $g \in G$ .

i) We first prove the manifold property of  $F_g^{-1}(0)$  (i.e.  $\mu_{x,y}$ ).

To do this, we need the following two lemmas.

**Lemma 2.9.**  $0$  is a regular value of  $\Phi$ .

*Proof.* We want that  $\forall (g, s) \in \Phi^{-1}(0)$ , the vertical differential

$$\begin{aligned} D\Phi(g, s) : T_{(g,s)}(G \times B) &\rightarrow E_s \\ (\eta, \xi) &\mapsto D_1\Phi(g, s) \cdot \eta + D_2\Phi(g, s) \cdot \xi \end{aligned}$$

is surjective, where  $B = P_{x,y}^{1,2}$ ,  $E_s = L^2(s^*TM)$ .

Since  $D_2\Phi(g, s) = DF_g(s)$  is a Fredholm operator, the image of  $D\Phi(g, s)$  is closed in  $L^2$  with finite codimension.

Suppose  $h \in L^2(s^*TM)$  is orthogonal to the image of  $D\Phi(g, s)$ , i.e.

$$\int_{\mathbb{R}} \langle D\Phi(g, s)(\eta, \xi), h \rangle dt = 0 \quad \forall (\eta, \xi) \in T_{(g,s)}(G \times B)$$

Then it is sufficient to show that  $h = 0$ .

Let  $\eta = 0$ , we get

$$\int_{\mathbb{R}} \langle DF_g \cdot \xi, h \rangle dt = 0 \quad \forall \xi \in T_s B$$

Then  $h$  can be expressed as the solution of some linear differential equation

$$\dot{h}(t) = X(t)h(t)$$

If  $h \neq 0$ , due to the uniqueness of solution of the ODE,  $h(t) \neq 0$  for all  $t$ . So, letting  $\xi = 0$ , we have

$$\int_{\mathbb{R}} \langle D_1\Phi(g, s) \cdot \eta, h \rangle dt = 0 \quad h \in C^\infty, \quad \forall \eta \in T_g G \quad (2.2)$$

Fix  $t_0 \in \mathbb{R}$ , choose  $\xi$  s.t.  $D_1\Phi(g, s) \cdot \eta = h(t_0)$ , then  $\langle D_1\Phi(g, s) \cdot \eta, h \rangle > 0$  at  $t_0$ . By the smoothness of  $h$ , (2.2) can not be zero, which leads to a contradiction.  $\square$

**Lemma 2.10.** Let  $\pi : \Phi^{-1}(0) \rightarrow G$  be the projection map, then  $\pi$  is a Fredholm map of index  $ind(x) - ind(y)$ .

*Proof.* All we need to show is that  $\forall (g, s) \in \Phi^{-1}(0)$ , the following map

$$\begin{aligned} D\pi(g, s) : T_{(g,s)}\Phi^{-1}(0) &\rightarrow T_g G \\ (\eta, \xi) &\mapsto \eta \end{aligned}$$

is a Fredholm operator with index  $ind(x) - ind(y)$ . We get this result by showing that the kernel and cokernel of  $D\pi(g, s)$  are isomorphic to the kernel and cokernel of  $D_2\Phi(g, s)$  respectively.

The isomorphism of kernels is shown by

$$\begin{aligned} \ker(D\pi(g, s)) &= \{(0, \xi) \in T_g G \times T_s B \mid (0, \xi) \in T_{(g,s)}\Phi^{-1}(0), \text{ i.e. } D_2\Phi(g, s) \cdot \xi = 0\} \\ &\cong \{\xi \in T_s B \mid D_2\Phi(g, s) \cdot \xi = 0\} \\ &= \ker D_2\Phi(g, s) \end{aligned}$$

For the cokernel part, consider the homomorphism induced by  $D_1\Phi(g, s)$

$$\begin{aligned} \widetilde{D}_1\Phi : \text{coker}(D\pi(g, s)) = T_g G / \text{Im}(D\pi(g, s)) &\rightarrow \text{coker}(D_2\Phi(g, s)) = E_s / \text{Im}(D_2\Phi(g, s)) \\ [\eta] &\mapsto [D_1\Phi(g, s) \cdot \eta] \end{aligned}$$

Claim 1:  $\widetilde{D}_1\Phi$  is a surjective.

For all  $v \in E_s$ , by the surjectivity of  $D\Phi(g, s)$  from lemma 2.9, there is  $(\eta, \xi) \in T_{(g,s)}G \times B$ , such that  $D_1\Phi(g, s) \cdot \eta + D_2\Phi(g, s) \cdot \xi = v$ . Then

$$\widetilde{D}_1\Phi([\eta]) = [D_1\Phi(g, s) \cdot \eta] = [D_1\Phi(g, s) \cdot \eta + D_2\Phi(g, s) \cdot \xi] = [v]$$

Claim 2:  $\widetilde{D}_1\Phi$  is an injective.

Suppose  $v \in \text{Im}(D_2\Phi(g, s))$ , then  $v = D_2\Phi(g, s) \cdot \xi$  for some  $\xi \in T_s B$ . If  $\eta \in T_g G$  satisfies  $D_1\Phi(g, s) \cdot \eta = v$ , we have  $D_1\Phi(g, s) \cdot \eta = D_2\Phi(g, s) \cdot \xi$ , i.e.  $(\eta, -\xi) \in T_{(g,s)}\Phi^{-1}(0)$ .

So  $\eta = D\pi(g, s)(\eta, -\xi) \in \text{Im}(D\pi(g, s))$ .

Therefore  $\widetilde{D}_1\Phi$  is an isomorphism.  $\square$

Now we continue the proof of theorem 2.8.

Finally, by the Sard-Smale theorem, we can conclude that there is a generic set  $\Sigma \in G$  being regular values of  $\pi$ .

ii) We now show the transversal property.

Let  $g \in \Sigma$ , assume there exists  $s \in B$  such that  $\Phi(g, s) = 0$  (for otherwise, 0 is trivially a regular value of  $\Phi_g$ ). We want to show that  $D_2\Phi(g, s)$  is onto.

For all  $v \in E_s$ , since 0 is a regular value of  $\Phi$ ,  $\exists(\eta, -\xi) \in T_{(g,s)}(G \times B)$  s.t.

$$v = D_1\Phi(g, s) \cdot \eta + D_2\Phi(g, s) \cdot \xi \tag{2.3}$$

Since  $g$  is regular value of  $\pi$ , there exists  $(\eta', \xi') \in T_{(g,s)}\Phi^{-1}(0)$  such that  $\eta = D\pi(g, s)(\eta', \xi')$ . Thus  $\eta = \eta'$ , therefore  $(\eta, \xi') \in \ker D\Phi(g, s)$ , i.e.

$$D_1\Phi(g, s) \cdot \eta + D_2\Phi(g, s) \cdot \xi' = 0 \tag{2.4}$$

(2.3) and (2.4) together imply that  $v = D_2\Phi(g, s)(\xi - \xi')$ .

$\square$

Now, we have shown that  $W(x, y)$  and  $\mu_{x,y}$  are both manifolds with the same dimension. We can now show that these two space are equivalent.

**Proposition 2.11.**  $W(x, y) \cong \mu_{x,y}$ .

*Proof.* Consider the following map

$$\begin{aligned} \Phi : P_{x,y}^{1,2} &\rightarrow M \\ \gamma &\mapsto \gamma(0) \end{aligned}$$

This map is smooth because  $D\Phi : \xi \mapsto \xi(0)$  is continuously linear. We claim  $\Phi|_{\mu_{x,y}} : \mu_{x,y} \hookrightarrow M$  is an embedding. Because  $\Phi$  is an injective and so is the differential:

$$\begin{aligned} D\Phi|_{\gamma} : T_{\gamma}\mu_{x,y} &\rightarrow T_{\gamma(0)}M \\ \xi &\mapsto \xi(0) \end{aligned}$$

To see this, we can consider  $\xi$  as the solutions of ODE with initial values  $\xi(0)$ . If  $\xi \in \ker D\Phi|_{\gamma}$  i.e.  $\xi(0) = 0$ . Then the uniqueness of the solution implies  $\xi \equiv 0$ . Therefore the differential is an injective. As a consequence,  $\Phi$  is an diffeomorphism between  $\mu_{x,y}$  and  $\Phi(\mu_{x,y}) = W(x, y)$ .  $\square$

## 2.4 Moduli space of $\mu_{x,y}$

Suppose  $\gamma_1, \gamma_2 \in \mu_{x,y}$ , and there is a  $t \in \mathbb{R}$  such that  $\gamma_1(t) = \gamma_2(0)$ , then for all  $s \in \mathbb{R}$ ,  $\gamma_1(t+s) = \gamma_2(s)$ . That is to say,  $\gamma_1(\mathbb{R}) = \gamma_2(\mathbb{R})$ . Then from geometrical viewpoint,  $\gamma_1$  and  $\gamma_2$  are the same. In fact, the following proposition properly addresses this observation.

**Proposition 2.12.** *For different critical points  $x, y$ ,  $(\mathbb{R}, +)$  acts smoothly, freely and properly on  $\mu_{x,y}$  by*

$$\begin{aligned} \Psi : \mathbb{R} \times \mu_{x,y} &\rightarrow \mu_{x,y} \\ (s, \gamma) &\mapsto \Psi_s(\gamma) = \gamma(s + \cdot) \end{aligned}$$

*Proof.* Define

$$\begin{aligned} \Theta : \mathbb{R} \times W(x, y) &\rightarrow W(x, y) \\ (s, p) &\mapsto \Theta_s(p) = \gamma_p(s) \end{aligned}$$

where  $\gamma_p \in \mu_{x,y}$  and  $\gamma_p(0) = p$ . Then  $\Psi_s = \Phi^{-1} \circ \Theta_s \circ \Phi$ , where  $\Phi$  is defined in proposition 2.11. So it is sufficient to show that  $\mathbb{R}$  acts smoothly, freely and properly on  $W(x,y)$  by  $\Theta$ .

$\Theta$  is obviously smooth and since there is no critical point in  $W(x,y)$ , the action is free. It remains to show that the action is proper.

Let  $\{p_n\} \subseteq W(x,y)$ ,  $\{s_n\} \subseteq \mathbb{R}$  such that  $p_n \rightarrow p$  and  $\gamma_{p_n}(s_n) \rightarrow q$ , then it is sufficient to show that  $\{s_n\}$  has a convergent subsequence.

Suppose not, then  $\{s_n\}$  is unbounded. Without loss of generality, assume  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then  $\lim_{n \rightarrow \infty} \gamma_{p_n}(s_n) = y \neq q$ . It is a contradiction.  $\square$

Now we define the moduli space  $\hat{\mu}_{x,y} = \mu_{x,y}/\mathbb{R}$ . Choose  $a \in (f(y), f(x))$  and  $a$  is not a critical point, let  $\mu_{x,y}^a = \{\gamma \in \mu_{x,y} \mid \gamma(0) = a\}$ . Then  $\mu_{x,y}^a$  is a  $ind(x) - ind(y) - 1$  dimensional submanifold of  $\mu_{x,y}$ . In order to describe the differential structure of  $\hat{\mu}_{x,y}$ , we have the following proposition.

**Proposition 2.13.** *Restricting  $\Psi$  (defined in proposition 2.12) to  $\mathbb{R} \times \mu_{x,y}^a$ , we have*

$$\begin{aligned} \Psi^a : \mathbb{R} \times \mu_{x,y}^a &\rightarrow \mu_{x,y} \\ (s, \gamma) &\mapsto \Psi_s(\gamma) = \gamma(s + \cdot) \end{aligned}$$

*This is an  $\mathbb{R}$ -equivariant diffeomorphism.*

*Proof.* It is obvious that  $\Psi^a(s_1 + s_2, \gamma) = \gamma(s_1 + s_2 + \cdot) = \Psi_{s_1} \circ \Psi_{s_2}(\gamma)$ . So it remains to show  $\Psi^a$  is a diffeomorphism.

The bijectivity is trivial. Thus it is sufficient to show

$$D\Psi^a(s, \gamma) : \mathbb{R} \times T_\gamma \mu_{x,y}^a \rightarrow T_{\Psi_s(\gamma)} \mu_{x,y}$$

is an isomorphism.

For all  $(v, \xi) \in \ker D\Psi^a(0, \gamma)$ ,  $D\Psi^a(0, \gamma)(v, \xi) = \gamma' \cdot v + \xi = 0$ . We get  $\xi(0) = 0$  since the initial value of flow lines in  $\mu_{x,y}^a$  is constantly  $a$ . Then  $v \cdot \gamma'(0) = 0$ , which implies  $v = 0$ . So  $D\Psi^a(0, \gamma)$  is injective.

The surjectivity follows from the equality of the dimension of the domain and range. Finally, the fact that  $\Psi_s$  is a diffeomorphism and  $\Psi^a(s, \gamma) = \Psi_s \circ \Psi(0, \gamma)$  finishes the proof.  $\square$

From this observation we have

$$\hat{\mu}_{x,y} = \mu_{x,y}/\mathbb{R} \cong \mu_{x,y}^a$$

So  $\hat{\mu}_{x,y}$  is a manifold with dimension equals to  $ind(x) - ind(y) - 1$ .

## 2.5 Compatification of $\hat{\mu}_{x,y}$

Given the definition and differential structure of  $\hat{\mu}_{x,y}$ , we now study the compactness of the moduli space. Unfortunately,  $\hat{\mu}_{x,y}$  is not always compact. As a counterexample, suppose there is a sequence  $\{\hat{u}_n\} \subseteq \hat{\mu}_{x,y}$  sketched in Figure 2.1.

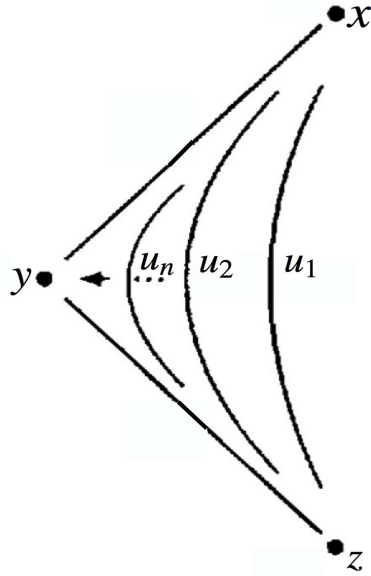


Figure 2.1

Then  $\{\hat{u}_n\}$  does not have a subsequence that is convergent in  $\hat{\mu}_{x,y}$  (Consider the example of  $M$  being a tilted torus and  $f$  being the height function). To solve this problem, we add some elements called broken flow line to  $\hat{\mu}_{x,y}$  to compatify the moduli space.

**Definition 2.14.** A sequence  $\{\hat{u}_n\} \subseteq \hat{\mu}_{x,y}$  converges to  $(\hat{v}_1, \dots, \hat{v}_i) \in \hat{\mu}_{x,y_1} \times \hat{\mu}_{y_1,y_2} \times \dots \times \hat{\mu}_{y_{i-1},y}$  if for any lift  $u_n$  of  $\hat{u}_n$  and  $v_j$  of  $\hat{v}_j$ , there are shifts  $\tau_{n,j}$  such that

$$u_n \cdot \tau_{n,j} \xrightarrow{C_{loc}^\infty} v_j \quad \forall j \in \{1, \dots, i-1\}$$

where  $y_1, \dots, y_{i-1}$  are critical points of  $f$ .

Let

$$\tilde{\mu}_{x,y} = \hat{\mu}_{x,y} \bigsqcup \left( \bigcup_{y_1, \dots, y_{i-1}} \hat{\mu}_{x,y_1} \times \hat{\mu}_{y_1,y_2} \times \dots \times \hat{\mu}_{y_{i-1},y} \right)$$

then we have the following proposition.

**Proposition 2.15.**  $\tilde{\mu}_{x,y}$  is compact with the definition 2.14.

**Lemma 2.16.** *Suppose  $\{u_n\} \subseteq \mu_{x,y}$ , then there exists a subsequence  $\{u_{n_j}\}$  s.t.*

$$u_{n_j} \xrightarrow{C_{loc}^\infty} v$$

where  $v$  is also a gradient flow line.

*Proof.* We first show  $\{u_n\}$  is equicontinuous in  $C^0(\mathbb{R}, M)$ . For all  $t_1, t_2 \in \mathbb{R}$  with  $t_1 \leq t_2$

$$\int_{t_1}^{t_2} |\dot{u}_n(t)|^2 dt = \int_{t_1}^{t_2} \langle \dot{u}_n, -\nabla f \circ u_n \rangle dt \leq f(x) - f(y)$$

then

$$\begin{aligned} d(u_n(t_1), u_n(t_2)) &\leq \int_{t_1}^{t_2} |\dot{u}_n(t)| dt \\ &\leq \sqrt{(t_2 - t_1) \int_{t_1}^{t_2} |\dot{u}_n(t)|^2 dt} \quad (\text{Holder inequality}) \\ &\leq \sqrt{(t_2 - t_1)(f(x) - f(y))} \end{aligned}$$

which implies the equicontinuity.

The compactness of  $M$  implies that  $u_n$  are uniformly bounded. By Arzela-Ascoli theorem, there is a subsequence  $\{u_{n_j}\}$  and  $v \in C^0(\mathbb{R}, M)$  such that

$$v(t) = \lim_{j \rightarrow \infty} u_{n_j}(t) \quad (\forall t \in \mathbb{R}) \quad u_{n_j} \Big|_{[-R,R]} \xrightarrow{C^0([-R,R])} v \Big|_{[-R,R]} \quad (\forall R > 0)$$

Since  $\dot{u}_n = -\nabla f \circ u_n$  and the gradient vector field of  $f$  is smooth, we have the  $C^k$ -convergence

$$u_{n_j} \Big|_{[-R,R]} \xrightarrow{C^k([-R,R])} v \Big|_{[-R,R]} \quad (\forall R > 0) \quad \text{with} \quad \dot{v} = -\nabla f \circ v$$

iteratively for all  $k \in \mathbb{N}$ . □

**Lemma 2.17.** *Suppose  $\{u_n\} \subseteq \mu_{x,y}$ ,  $v \in \mu_{x,y}$  and  $u_n \xrightarrow{C_{loc}^\infty} v$ , then*

$$u_n \xrightarrow{P_{x,y}^{1,2}} v$$

*Proof.* First we prove  $u_n$  converge uniformly to  $x$  and  $y$ , i.e. for any neighbourhood  $U_x, U_y$  of  $x$  and  $y$ ,  $\exists T > 0$  s.t.  $\forall n \in \mathbb{N}$

$$u_n(t) \in U_x \quad (\forall t < -T) \quad u_n(t) \in U_y \quad (\forall t > T)$$

We only show the latter fomula and the proof of the former is similar.

Suppose not, i.e. there exists  $U_y \ni y$ ,  $t_n \rightarrow +\infty$  such that  $u_n(t_n) \notin U_y$ . Consider the Morse coordinate near  $y$ . Without loss of generality, assume  $U_y$  is a ball centered at  $y$  with radius  $\epsilon$ . Then for all  $p \in W(x, y)$ , if  $p$  is in the local chart of  $y$ ,  $x(p) = (0, \dots, 0, x_{\lambda+1}, \dots, x_n)$ , then  $f(p) - f(y) = x_{\lambda+1}^2 + \dots + x_n^2 < \epsilon \Leftrightarrow p \in U_y$ . If  $p$  is out of the local chart of  $y$ , then  $f(p) - f(y) \geq \epsilon$ . Therefore

$$u_n(t_n) \notin U_y \Rightarrow f(u_n(t_n)) - f(y) \geq \epsilon$$

Suppose  $f(v(t_0)) = \frac{\epsilon}{2} + f(y)$ , then by making  $n$  large enough  $t_n > t_0$ , we have

$$f(u_n(t_0)) - f(y) \geq f(u_n(t_n)) - f(y) \geq \epsilon$$

which contradicts  $u_n(t_0) \rightarrow v(t_0)$ .

Having the uniform convergence of  $u_n$ , along with the exponential decay we have shown in lemma 2.5, we can say that there are  $\epsilon, C, T > 0$  such that for all  $n \in \mathbb{N}$

$$\begin{aligned} d(u_n(t), x) &\leq Ce^{\epsilon t} \quad (\forall t < -T) \\ d(u_n(t), y) &\leq Ce^{-\epsilon t} \quad (\forall t > T) \end{aligned}$$

On  $(-\infty, -T]$  and  $[T, +\infty)$ , the exponential convergence implies  $H^{1,2}$ -convergence. On the compact set  $[-T, T]$ , convergence in  $C_{loc}^\infty$  implies  $H^{1,2}$ -convergence.  $\square$

*Proof.* (of proposition 2.15)

For  $\{u_n\} \subseteq \mu_{x,y}$ , lemma 2.16 tells us there is a subsequence  $u_{n_j} \xrightarrow{C_{loc}^\infty} v$ , where  $v$  is a gradient flow line. However the problem is  $v$  may not lie in  $\mu_{x,y}$ . Suppose  $v \in \mu_{x',y'}$ , then  $f(v(\mathbb{R})) \subseteq [f(y), f(x)]$  implies that  $f(y) \leq f(y') \leq f(x') \leq f(x)$ .

Now we discuss different cases of  $x'$  and  $y'$ .

Case 1:  $x = x'$ ,  $y = y'$ , then lemma 2.17 gives us  $H^{1,2}$ -convergent subsequence directly.

Case 2:  $x = x'$ ,  $y \neq y'$ . Choose  $a \in (f(y), f(y'))$  and  $a$  is a regular value of  $f$  and a sequence of shifting  $\{\tau_j\}$  such that

$$f(u_{n_j}(\tau_j)) = a$$

Denote  $\tilde{u}_j = u_{n_j} \cdot \tau_j$ . Apply lemma 2.16 to  $\{\tilde{u}_j\}$ , we have a subsequence such that  $\tilde{u}_{j_k} \xrightarrow{C_{loc}^\infty} \tilde{v}$  and  $f(y) \leq f(\tilde{v}(+\infty)) \leq f(\tilde{v}(-\infty)) \leq f(y')$ .

Iterating these procedures finitely many times leads to the limit.  $\square$

We are immediately led to the conclusion:

**Corollary 2.18.** *If  $ind(x) - ind(y) = 1$ ,  $\hat{\mu}_{x,y}$  is a finite set.*

## 2.6 Gluing

We have shown that the space of flow lines  $\tilde{\mu}_{x,y}$  is compact and  $\hat{\mu}_{x,y}$  is a manifold of dimension  $\text{ind}(x) - \text{ind}(y) - 1$ . Now we study what happened in the neighbourhood of the broken flow lines. This is described in the following so-called gluing theorem.

**Theorem 2.19.** *Let  $x, y, z$  be critical points with  $\text{ind}(x) > \text{ind}(y) > \text{ind}(z)$ , given a compact set  $K \subseteq \mu_{x,y} \times \mu_{y,z}$ , there is a lower bound  $\rho_0 > 0$  and a smooth map*

$$\sharp : K \times [\rho_0, +\infty) \rightarrow \mu_{x,z}$$

such that

1.  $\sharp$  induces a smooth embedding

$$\hat{\sharp} : \hat{K} \times [\rho_0, +\infty) \hookrightarrow \hat{\mu}_{x,z}$$

2. As  $\rho \rightarrow +\infty$ ,  $\hat{\sharp}(\hat{u}, \hat{v}, \rho) \xrightarrow{C_{loc}^\infty} (\hat{u}, \hat{v})$
3. If  $\{\hat{l}_n\} \subseteq \hat{\mu}_{x,z}$  converges to  $(\hat{u}, \hat{v})$ , then  $\hat{l}_n$  are in the range of  $\hat{\sharp}$  for sufficiently large  $n$ .

The entire proof is rather complicated so we first give a brief outline of the proof. The construction of the gluing map is carried out in three steps:

- (1) Establishing a “pre-gluing map”  $\sharp_\rho^0$ , which glue broken flow lines  $(u, v)$  and end up with a smooth curve in  $P_{x,z}^{1,2}$ . This curve is an approximate solution of the negative gradient flow from  $x$  to  $z$  in the following sense:  $(u \sharp_\rho^0 v)(\mathbb{R})$  differs from  $u(\mathbb{R}) \cup v(\mathbb{R})$  by a sufficiently small amount and the convergence

$$(u \sharp_\rho^0 v) \xrightarrow{C_{loc}^\infty} (\hat{u}, \hat{v})$$

is provided as  $\rho \rightarrow +\infty$ .

- (2) Forming a procedure of associating nearby flow lines from  $\mu_{x,z}$  to those pre-glued curves in a unique way for large  $\rho$ . We will construct a map, which is denoted by:

$$\text{exp}_{\omega_\rho}((\gamma(u, v, \rho)))$$

for some  $\gamma \in H^{1,2}(\omega_\rho^* TM) = T_{\omega_\rho} P_{x,z}^{1,2}$  where  $\omega_\rho = u \sharp_\rho^0 v$ . We will show it is a real gradient flow line.

- (3) Verifying the embedding property of  $\hat{\sharp}$ .

### 2.6.1 Pre-gluing

Throughout the whole section we choose two cut-off functions  $\beta^\pm \in C^\infty(\mathbb{R}, [0, 1])$  such that

$$\beta^-(t) = \begin{cases} 1 & t \leq -1 \\ 0 & t \geq 0 \end{cases} \quad \text{and} \quad \beta^+(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t \geq 1 \end{cases}$$

which is shown schematically in the figure 2.2.

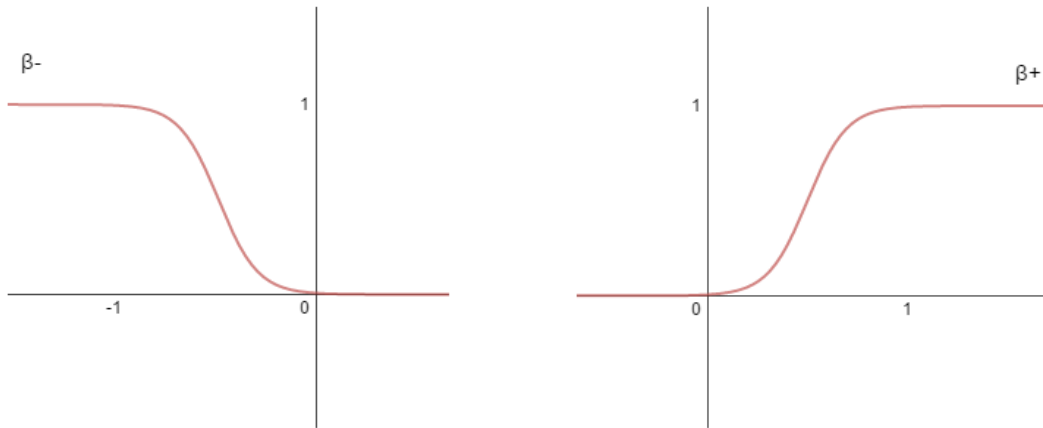


Figure 2.2

We define the pre-gluing map by

$$(u \#_\rho^0 v)(t) = \begin{cases} u_\rho(t) & t \leq -1 \\ \exp_y(\beta^- \exp_y^{-1}(u_\rho) + \beta^+ \exp_y^{-1}(v_{-\rho})) & |t| \leq 1 \\ v_{-\rho}(t) & t \geq 1 \end{cases}$$

where  $u_\rho = u(\cdot + \rho)$ ,  $v_{-\rho} = v(\cdot - \rho)$ . Note that this formula makes sense only for  $|t| \leq 1$ ,  $u_\rho$  and  $v_{-\rho}$  are in the image of exponential. Fortunately, these conditions are true for  $\rho$  large enough since  $\lim_{t \rightarrow +\infty} u(t) = y = \lim_{t \rightarrow -\infty} v(t)$ .

$\omega_\rho (= u \#_\rho^0 v)$  has the following properties:

- (1)  $\omega_\rho \in C_{x,z}^\infty$ .
- (2)  $\omega_\rho(0) = y$ .
- (3) For  $t \leq \rho - 1$ ,  $\omega_\rho(t - \rho) = u(t)$ . And likewise, if  $t \geq 1 - \rho$ ,  $\omega_\rho(t + \rho) = v(t)$ .

(4)  $\sharp^0$  is smooth.

(5)  $\lim_{\rho \rightarrow +\infty} \omega_\rho(t) = y$ .

Figure 2.3 sketches what happens:

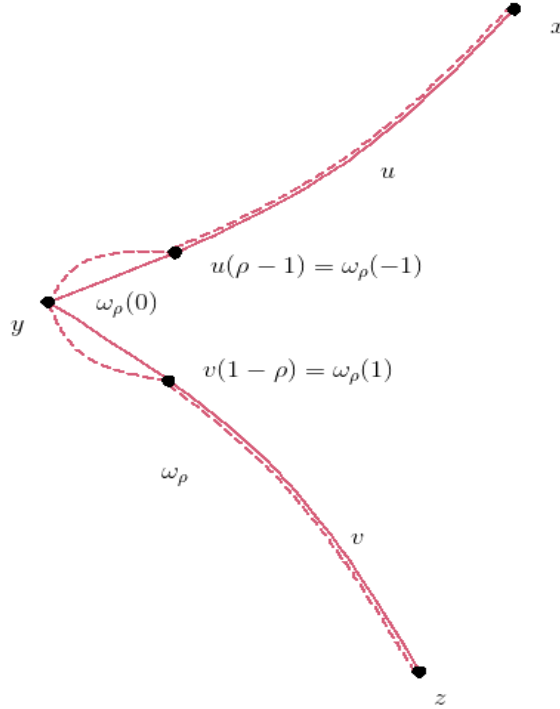


Figure 2.3

We will also need the linearized version. First, we collect some background material. Let  $\tau : TM \rightarrow M$  denote the projection map,  $K : TTM \rightarrow TM$  be the Levi-Civita connection map defined by the Riemannian metric  $g$  and  $exp : \mathcal{O} \subseteq TM \rightarrow M$  be the corresponding exponential map. As a consequence,  $K$  and  $D\tau : TTM \rightarrow TM$  yield a decomposition of  $TTM$  into a horizontal bundle and a vertical bundle:  $T_{\xi,v}(TM) = \ker(D\tau(\xi))$ ,  $T_{\xi,h}(TM) = \ker(K)$ ,  $\xi \in TM$ . We then obtain two isomorphisms:

$$\nabla_1 exp(\xi) = Dexp(\xi) \circ (D\tau |_{T_{\xi,h}(TM)})^{-1} : T_{\tau(\xi)}M \xrightarrow{\cong} T_{exp(\xi)}M$$

$$\nabla_2 exp(\xi) = Dexp(\xi) \circ (K |_{T_{\xi,v}(TM)})^{-1} : T_{\tau(\xi)}M \xrightarrow{\cong} T_{exp(\xi)}M$$

We are able to define the linear version of the pre-gluing operation: Let  $\xi \in T_u P_{x,y}^{1,2}$  and  $\eta \in T_v P_{y,z}^{1,2}$

$$(\xi \#_{\rho}^l \eta)(t) = \begin{cases} \xi_{\rho}(t) & t \leq -1 \\ \nabla_2 \exp(\beta^- \nabla_2 \exp^{-1}(\xi_{\rho}) + \beta^+ \nabla_2 \exp^{-1}(\eta_{-\rho}))(t) & |t| \leq 1 \\ \eta_{-\rho}(t) & t \geq 1 \end{cases}$$

In the process of construction of the gluing map, we deduce the linearisation of the gluing map from the linear version of pre-gluing. By the linearisation of the actual gluing map we mean a projection onto the tangent bundle of  $\mu_{x,z}$  as a subbundle of  $H^{1,2}$ .

**Definition 2.20.** Since  $D_u = DF_u$  and  $D_v = DF_v$  are surjective Fredholm operators,

$$(ker)^2 = \bigcup_{(u,v,\rho)} ker D_u \times ker D_v \subseteq (H^{1,2})^2$$

is a subbundle with constant and finite dimension. Then we can define

$$L_{(u,v,\rho)}^{\perp} = \{v_{\rho} \in H^{1,2}(\omega_{\rho}^* TM) \mid \langle v_{\rho}, \xi \#_{\rho}^l \eta \rangle_{0,2} = 0 \text{ for all } (\xi, \eta) \in ker D_u \times ker D_v\}$$

The importance of  $L_{(u,v,\rho)}^{\perp}$  lies in the following sense: one would like to glue the flow lines  $u$  and  $v$  in such a way to get a flow line from  $x$  to  $z$ . Now the result  $\omega_{\rho}$  of the pre-gluing operation lies in  $P_{x,z}^{1,2}$  but fail to be a flow line. However, by perturbing  $\omega_{\rho}$  with a small correction term  $\gamma \in L_{(u,v,\rho)}^{\perp}$  one does obtain a flow line.

Since  $\omega_{\rho} \in P_{x,z}^{1,2}$ , we can consider the local representation:

$$F_{\omega_{\rho}} = \nabla_2 \exp_{\omega_{\rho}}^{-1} \circ F \circ \exp_{\omega_{\rho}} : \mathcal{O}_{\omega_{\rho}} \subseteq H^{1,2}(\omega_{\rho}^* TM) \rightarrow L^2(\omega_{\rho}^* TM)$$

We also have the identity:  $F_{\omega_{\rho}}(0) = F(\omega_{\rho})$  (One can see [7,P74] for details of this result).

The following results holds:

**Proposition 2.21.** ([7, Proposition 2.50]) *If  $\rho_0$  is large enough, for any  $\rho > \rho_0$ ,*

$$DF_{\omega_{\rho}}(0) : H^{1,2}(\omega_{\rho}^* TM) \rightarrow L^2(\omega_{\rho}^* TM)$$

*is surjective and the map*

$$Proj_{\rho} \circ \#_{\rho}^l : ker DF_u \times ker DF_v \xrightarrow{\cong} ker DF_{\omega_{\rho}}(0)$$

*is bijective, where  $Proj_{\rho}$  is the  $L^2$ -orthogonal projection onto  $ker DF_{\omega_{\rho}}(0)$ .*

Then from proposition 2.21,

$$\ker D = \bigcup_{(u,v) \in K, \rho \geq \rho_0} \ker DF_{\omega_\rho}(0)$$

is a vector bundle on  $K \times [\rho_0, \infty)$  with dimension  $\text{ind}(x) - \text{ind}(z)$  and

$$L^\perp = \bigcup_{(u,v) \in K, \rho \geq \rho_0} L^\perp_{(u,v,\rho)}$$

is finite-codimensional bundle such that

$$\ker DF_{\omega_\rho}(0) \oplus L^\perp_{(u,v,\rho)} = H^{1,2}(\omega_\rho^* TM)$$

Hence

$$DF_{\omega_\rho}(0) |_{L^\perp_{(u,v,\rho)}} : L^\perp_{(u,v,\rho)} \rightarrow L^2(\omega_\rho^* TM)$$

is bijective and we denote the inverse map by

$$G_{\omega_\rho} = (DF_{\omega_\rho}(0) |_{L^\perp_{(u,v,\rho)}})^{-1} : L^2(\omega_\rho^* TM) \xrightarrow{\cong} L^\perp_{(u,v,\rho)}$$

### 2.6.2 The construction of gluing map $\sharp$

Given the pre-gluing map  $\sharp^0$ , which is an approximate version of the gluing operation, we now try to find a correction term for  $\sharp^0$  using the normal bundle  $L^\perp$  constructed above. In other words, we want to find a section  $\gamma$  of  $H^{1,2}(P_{x,z}^{1,2*} TM)$  over  $K \times [\rho_0, \infty)$  so that

$$\psi(\rho) = \exp_{\omega_\rho}((\gamma(u, v, \rho)))$$

is a real gradient flow line. Therefore we need the section

$$\gamma : K \times [\rho_0, \infty) \rightarrow H^{1,2}(P_{x,z}^{1,2*} TM)$$

satisfies  $F \circ \gamma = 0$ . Besides, we want  $\exp_{\omega_\rho}((\gamma(u, v, \rho)))$  converge to the broken flow lines  $(u, v)$  as  $\rho$  tends to infinity. From the previous section we can see that the convergence property holds for pre-gluing map, so the correction term should converges to zero, i.e.  $\gamma(u, v, \rho) \rightarrow 0$  as  $\rho \rightarrow \infty$ .

Now we introduce a basic analytic machinery of this section.

**Lemma 2.22.** *Let  $X$  and  $Y$  be two Banach spaces and  $f : X \rightarrow Y$  be a smooth map of the form*

$$f(x) = f(0) + Df(0) \cdot x + N(x)$$

*with  $\dim \ker Df(0) < \infty$ . And we assume  $G : Y \rightarrow X$  satisfies*

$$(1) Df(0) \circ G = Id.$$

(2) There exists a constant  $C > 0$  such that the contraction property

$$\|GN(x) - GN(y)\| \leq C(\|x\| + \|y\|) \|x - y\|$$

holds for all  $x, y$  in  $B_\epsilon(0)$ , where  $\epsilon = \frac{1}{5C}$ .

$$(3) \|Gf(0)\| \leq \frac{\epsilon}{2}.$$

Then there exists a unique  $\alpha \in \text{Im}(G) \cap B_\epsilon(0)$  such that  $f(\alpha) = 0$ . Moreover,  $\|\alpha\| \leq 2\|Gf(0)\|$ .

*Proof.* We prove this lemma using the contraction mapping principle. Define  $\phi : X \rightarrow X$  by setting

$$\phi(x) = G(Df(0)x - f(x)) = -G(f(0) + N(x))$$

We first show that  $x$  is a zero point of  $f$  if and only if  $\phi(x) = x$ . Note that

$$Df(0)\phi(x) = Df(0)G(Df(0)x - f(x)) = Df(0)x - f(x)$$

Hence  $\phi(x) = x$  implies  $f(x) = 0$ . Conversely, if  $f(x) = 0$  and  $x = G(y)$ , then

$$\phi(x) = GDf(0)(x) = GDf(0)G(y) = G(y) = x$$

We will therefore show that  $\phi$  has a unique fixed point in  $B_\epsilon(0)$ .

First,  $\phi$  sends  $B_\epsilon(0)$  to itself: If  $\|x\| \leq \epsilon$ , then

$$\begin{aligned} \|\phi(x)\| &= \|G(f(0) + N(x))\| \leq \|Gf(0)\| + \|GN(x)\| \\ &\leq \frac{\epsilon}{2} + C\|x\|^2 \quad (\text{since } N(0) = 0) \\ &\leq \frac{\epsilon}{2} + C\epsilon^2 = \frac{\epsilon}{2} + \frac{\epsilon}{5} < \epsilon \end{aligned}$$

Next, if  $x$  and  $y$  are two points of  $B_\epsilon(0)$ , then

$$\begin{aligned} \|\phi(x) - \phi(y)\| &= \|GN(x) - GN(y)\| \leq C(\|x\| + \|y\|) \|x - y\| \\ &\leq 2C\epsilon \|x - y\| = \frac{2}{5} \|x - y\| \end{aligned}$$

By the contraction mapping theorem, there is a unique  $\alpha \in B_\epsilon(0)$  satisfying  $\phi(\alpha) = \alpha$ .

It remains to show the inequality stated in the lemma.

We have

$$\| \alpha \| - \| \phi(0) \| \leq \| \phi(\alpha) - \phi(0) \| \leq \frac{2}{5} \| \alpha \| \leq \frac{1}{2} \| \alpha \|$$

It follows that

$$\| \alpha \| \leq 2 \| \phi(0) \| = 2 \| Gf(0) \| .$$

□

We are going to apply lemma 2.22 on

$$F_{\omega_\rho} = \nabla_2 \exp_{\omega_\rho}^{-1} \circ F \circ \exp_{\omega_\rho} : H^{1,2}(\omega_\rho^* TM) \rightarrow L^2(\omega_\rho^* TM)$$

which can be express explicitly as

$$F_{\omega_\rho}(x) = \nabla_t x + \Theta(x) \cdot \dot{\omega}_\rho + \nabla_2 \exp_{\omega_\rho}^{-1} \circ \nabla f \circ \exp_{\omega_\rho}$$

where  $\Theta(x) = (\nabla_2 \exp(x))^{-1} \circ \nabla_1 \exp(x) : T_{\tau(\xi)} M \xrightarrow{\cong} T_{\tau(\xi)} M$ . Consider the expansion of  $F_{\omega_\rho}$  as the form of the formula of lemma 2.22:

$$F_{\omega_\rho}(x) = F_{\omega_\rho}(0) + DF_{\omega_\rho}(0) \cdot x + N_{\omega_\rho}(0, x)$$

Compute the difference of the nonlinear term at two different points:

$$\begin{aligned} N_{\omega_\rho}(x) - N_{\omega_\rho}(y) &= F_{\omega_\rho}(x) - F_{\omega_\rho}(y) - DF_{\omega_\rho}(0) \cdot (x - y) \\ &= DF_{\omega_\rho}(y)(x - y) + N_{\omega_\rho}(y, x - y) - DF_{\omega_\rho}(0) \cdot (x - y) \\ &= (DF_{\omega_\rho}(y) - DF_{\omega_\rho}(0)) \cdot (x - y) + N_{\omega_\rho}(y, x - y) \end{aligned}$$

Since  $K$  is compact, we have the following estimate for some constant  $C$ :

$$\| N_{\omega_\rho}(x) - N_{\omega_\rho}(y) \|_{0,2} \leq C(\| x \|_{1,2} + \| y \|_{1,2}) \| x - y \|_{1,2} \quad (2.5)$$

To get the required estimate in lemma 2.22, we need the following lemma.

**Lemma 2.23.** (*[7, lemma 2.51]*) *If  $\rho_0$  is large enough, there exists a constant  $C > 0$  such that the isomorphism  $G_{\omega_\rho}$  satisfies*

$$\| G_{\omega_\rho} \xi \|_{1,2} \leq C \| \xi \|_{0,2} \quad \forall \rho \in [\rho_0, \infty), \quad \xi \in L^2(\omega_\rho^* TM)$$

Then (2.5) together with lemma 2.23 implies that there exists a constant  $C > 0$  such that

$$\begin{aligned} \| G_{\omega_\rho} N_{\omega_\rho}(x) - G_{\omega_\rho} N_{\omega_\rho}(y) \|_{1,2} &\leq C(\| x \|_{1,2} + \| y \|_{1,2}) \| x - y \|_{1,2} \\ \| G_{\omega_\rho} F_{\omega_\rho}(0) \|_{1,2} &= \| G_{\omega_\rho} F(\omega_\rho) \|_{1,2} \leq C \| F(\omega_\rho) \|_{0,2} \end{aligned}$$

for all  $(u, v, \rho) \in K \times [\rho_0, \infty)$ .

Now, in order to apply lemma 2.22, it remains to show that  $\|F(\omega_\rho)\|_{0,2}$  tends to zero as  $\rho$  goes to infinity. This result follows from the next lemma.

**Lemma 2.24.** *There are constants  $c, \delta > 0$  such that*

$$\|F(\omega_\rho)\|_{0,2} \leq ce^{-\delta\rho}$$

for all  $(u, v, \rho) \in K \times [\rho_0, \infty)$ .

*Proof.* By the definition of  $\sharp^0$ , we have the identity:

$$F(\omega_\rho)(t) = 0 \quad \text{for } |t| > 1$$

So, without loss of generality,

$$\begin{aligned} u(t) &= \exp_y \xi(t) & t \leq -\rho - 1 \\ v(t) &= \exp_y \eta(t) & t \geq \rho + 1 \end{aligned}$$

Consider the local coordinates near  $y$ ,  $F = \frac{\partial}{\partial t} + X : H^{1,2}(\mathbb{R}, \mathbb{R}^n) \rightarrow L^2(\mathbb{R}, \mathbb{R}^n)$ , where  $X : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $DX(0)$  is the Hessian of  $f$  at  $y$ . Then we have

$$\omega_\rho = \beta^- \cdot \xi_\rho + \beta^+ \eta_{-\rho}$$

Hence

$$\begin{aligned} \|F(\omega_\rho)\|_{L^2} &= \|F(\omega_\rho)\|_{L^2([-1,1])} \leq \text{const} \|\omega_\rho\|_{H^{1,2}([-1,1])} \\ &\leq \text{const} (\|\xi_\rho\|_{H^{1,2}([-1,0])} + \|\eta_{-\rho}\|_{H^{1,2}([0,1])}) \end{aligned}$$

Since  $\xi$  and  $\eta$  satisfy exponential decay at the critical point 0:

$$|\xi(t)| \leq \text{const} e^{-\delta t} \quad \text{and} \quad |\eta(-t)| \leq \text{const} e^{-\delta t} \quad \text{for } t \geq 0$$

The constant  $\delta$  depends only on the Hessian of  $f$  at  $y$ , and we can make  $c$  independent of  $\xi$  and  $\eta$  by compactness of  $K$ . Since

$$\|\xi_\rho\|_{H^{1,2}([-1,0])} \leq \sup_{t \in [-1,0]} \{|\xi(t+\rho)|\} \leq \text{const} e^{-\delta\rho}$$

Similarly,  $\|\eta_{-\rho}\|_{H^{1,2}([0,1])} \leq \text{const} e^{-\delta\rho}$ . These complete the proof.  $\square$

Up to now, we have  $F_{\omega_\rho} : H^{1,2}(\omega_\rho^* TM) \rightarrow L^2(\omega_\rho^* TM)$  and  $G_{\omega_\rho} : L^2(\omega_\rho^* TM) \xrightarrow{\cong} L^{\perp}_{(u,v,\rho)} \subseteq H^{1,2}(\omega_\rho^* TM)$  satisfy the assumptions of lemma 2.22, therefore we can

conclude that there exists a unique correction term  $\gamma(u, v, \rho) \in B_\epsilon(0) \cap L_{(u,v,\rho)}^\perp$  such that

$$F_{\omega_\rho}(\gamma(u, v, \rho)) = 0$$

and

$$\begin{aligned} \|\gamma(u, v, \rho)\|_{1,2} &\leq 2 \|G_{\omega_\rho} F_{\omega_\rho}(0)\|_{1,2} \\ &\leq 2C \|F_{\omega_\rho}(0)\|_{0,2} = 2C \|F(\omega_\rho)\|_{0,2} \end{aligned}$$

Hence

$$\|\gamma(u, v, \rho)\|_{1,2} < 2cCe^{-\delta\rho}$$

which verifies the assertion we made at the beginning of the section ( $\gamma(u, v, \rho) \rightarrow 0$  as  $\rho \rightarrow \infty$ ). And the smoothness of  $\gamma$  follows from the implicit function together with the smoothness of  $F$ .

Now we have finished the definition of the gluing map

$$\begin{aligned} \sharp : K \times [\rho_0, \infty) &\rightarrow \mu_{x,z} \\ (u, v, \rho) &\mapsto \exp_{\omega_\rho} \gamma(u, v, \rho) \end{aligned}$$

The following two propositions that states the embedding property and the last two assertions in theorem 2.19 help us finish the proof of theorem 2.19.

**Proposition 2.25.** *If  $\rho_0$  is large enough, the unparametrized gluing map*

$$\begin{aligned} \hat{\sharp} : \hat{K} \times (\rho_0, \infty) &\rightarrow \hat{\mu}_{x,z} \\ (\hat{u}, \hat{v}, \rho) &\mapsto \hat{u} \hat{\sharp}_\rho \hat{v} \end{aligned}$$

*is a smooth embedding.*

**Proposition 2.26.** *Suppose  $(\hat{u}, \hat{v}) \in \hat{\mu}_{x,y} \times \hat{\mu}_{y,z}$ , and  $\{\rho_n\}$  is an increasing sequence tends to infinity. then*

$$\hat{u} \hat{\sharp}_{\rho_n} \hat{v} \xrightarrow{C_{loc}^\infty} (\hat{u}, \hat{v})$$

*And the converse is also true, that is, any sequence of unparametrized flow line converging to  $(\hat{u}, \hat{v})$  finally lies in the range of  $\hat{\sharp}$ .*

One can see the proof of these two propositions in [7, proposition 2.56] and [7, proposition 2.57].

### 2.6.3 Conclusion

We have stated and proved the gluing theorem under general condition, in fact, we only need to consider broken flow lines with one break and the relative Morse index  $ind(x) - ind(z) = 2$ , which is sufficient to construct the Morse complex. Since the space of broken flow lines appears as a compactification of the space of flow lines, the space of flow lines  $\tilde{\mu}_{x,z}$  is compact and  $\hat{\mu}_{x,z}$  is a manifold of dimension 1. After study about what happens in the neighbourhood of broken flow lines, we have the following theorem.

**Theorem 2.27.** *Let  $x$  and  $z$  be two critical points with relative index equals to 2, then  $\tilde{\mu}_{x,z}$  is a compact manifold of dimension 1 with boundary and*

$$\partial\tilde{\mu}_{x,z} = \bigcup_{ind(x)=ind(y)+1} \hat{\mu}_{x,y} \times \hat{\mu}_{y,z}$$

Hence we are able to classify the component of  $\hat{\mu}_{x,z}$  by two types: one is diffeomorphic to  $S^1$  and the other one is diffeomorphic to an open interval. For the latter type, there are exactly two different broken flow lines as its boundary.

## 2.7 Orientation

In section 2.5 we have shown that  $\hat{\mu}_{x,y}$  is consist of finitely many unparametrised flow lines when the relative index of  $x$  and  $y$  is one. The definition of the boundary operator  $\partial$  in Morse-Smale complex requires us to count the elements in  $\hat{\mu}_{x,y}$  with sign. These signs are determined by the orientation of the  $\hat{\mu}_{x,y}$  and  $\hat{u} \in \hat{\mu}_{x,y}$ . The orientation of  $\hat{u}$  is the canonical orientation of the flow line. So we need to define a coherent orientation for the moduli space.

### 2.7.1 The determinant line bundle

Consider two finite dimensional real vector spaces  $E, F$ . We define a one-dimensional vector space by

$$Det(E, F) = (\wedge^{max} E) \otimes (\wedge^{max} F)^*$$

where  $\wedge^{max} E = \wedge^{dim E} E$  and  $\wedge^0 E = \mathbb{R}$  for any  $E$ .

**Lemma 2.28.** *Given an exact sequence of finite-dimensional real vector spaces*

$$0 \rightarrow E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} \dots \xrightarrow{d_{k-1}} E_k \rightarrow 0$$

There exists a canonical isomorphism

$$\phi : \bigotimes_{i \text{ even}} (\wedge^{\max} E_i) \xrightarrow{\cong} \bigotimes_{i \text{ odd}} (\wedge^{\max} E_i)$$

*Proof.* Suppose  $e_{11}, \dots, e_{1n_1}$  is a basis of  $E_1$ . By the exactness of the sequence,  $d_1$  is injective, so we can extend the linear independent vectors  $d_1(e_{11}), \dots, d_1(e_{1n_1})$  to a basis of  $E_2$ , suppose is  $d_1(e_{11}), \dots, d_1(e_{1n_1}), e_{21}, \dots, e_{2n_2}$ .

Claim:  $d_2(e_{21}), \dots, d_2(e_{2n_2})$  are linearly independent in  $E_3$ .

Suppose not, then  $x_1 d_2(e_{21}) + \dots + x_{n_2} d_2(e_{2n_2}) = 0$  for some  $x_1, \dots, x_{n_2} \in \mathbb{R}$  with  $x_1 \cdot \dots \cdot x_{n_2} \neq 0$ . Hence

$$\begin{aligned} x_1 e_{21} + \dots + x_{n_2} e_{2n_2} &\in \ker(d_2) = \text{Im}(d_1) \\ \Rightarrow x_1 e_{21} + \dots + x_{n_2} e_{2n_2} &= y_1 d_1(e_{11}) + \dots + y_{n_1} d_1(e_{1n_1}) \\ \Rightarrow x_1 = \dots = x_{n_2} = y_1 = \dots = y_{n_1} &= 0 \end{aligned}$$

It is a contradiction.

Then we can extend  $d_2(e_{21}), \dots, d_2(e_{2n_2})$  to a basis of  $E_3$ , say,  $d_2(e_{21}), \dots, d_2(e_{2n_2}), e_{31}, \dots, e_{3n_3}$ . Similarly, for each  $1 < i \leq k-1$ , we can find a basis of  $E_{i+1}$  of the form

$$d_i(e_{i1}), \dots, d_i(e_{in_i}), e_{i+1,1}, \dots, e_{i+1, n_{i+1}}$$

where  $n_k = 0$  since  $d_k$  is surjective. Then we define  $\phi$  by

$$\begin{aligned} &(d_1(e_{11}) \wedge \dots \wedge d_1(e_{1n_1}) \wedge e_{21} \dots \wedge e_{2n_2} \otimes \dots) \\ \mapsto &(e_{11} \wedge \dots \wedge e_{1n_1} \otimes d_2(e_{21}) \wedge \dots \wedge d_2(e_{2n_2}) \wedge \dots) \end{aligned}$$

This map is independent of the choice of the bases of the vector spaces, since any change of bases operates by multiplication with same determinant of the transformation on both sides of  $\phi$ .  $\square$

Suppose now  $X$  is a topological space and

$$f : X \rightarrow F(H_0, H_1)$$

be a continuous map, where  $H_0, H_1$  are Banach spaces. We define the determinant of a Fredholm operator  $F$  as

$$\text{Det}(F) := \text{Det}(\ker F, \text{coker} F)$$

Besides, we define

$$\text{Det}(f) := \bigcup_{x \in X} \{x\} \times \text{Det}(f(x))$$

If  $\dim(\ker(f(x)))$  is locally constant, since Fredholm index is locally constant, so is  $\dim(\operatorname{coker}(f(x)))$ . Then  $\ker(f) := \bigcup_{x \in X} \{x\} \times \ker f(x)$  and  $\operatorname{coker}(f) := \bigcup_{x \in X} \{x\} \times \operatorname{coker} f(x)$  can be seen as sub-bundles of  $X \times H_0$  and  $X \times H_1$  respectively over  $X$ . Therefore we have  $\operatorname{Det}(f) = \wedge^{\max} \ker(f) \otimes (\wedge^{\max} \operatorname{coker}(f))^*$ , which implies that  $\operatorname{Det}(f)$  is a real line bundle. However, without the assumption, we still have the property.

**Proposition 2.29.**  $\operatorname{Det}(f) = \bigcup_{x \in X} \{x\} \times \operatorname{Det}(f(x)) \rightarrow X$  is a real line bundle.

*Proof.* The proof follows from [4, Appendix].

Let  $\psi : \mathbb{R}^n \rightarrow H_1$  be a linear map such that  $\hat{f}_\psi(x)$  is a surjective Fredholm operator for some  $x \in X$ , where  $\hat{f}_\psi$  is defined as

$$\begin{aligned} \hat{f}_\psi : X &\rightarrow L(\mathbb{R}^n \times H_0, H_1) \\ x &\mapsto ((h, k) \mapsto \psi(h) + f(x)k) \end{aligned}$$

Then we can find an open neighbourhood  $U(x) \ni x$  such that  $\hat{f}_\psi(y)$  is surjective for all  $y \in U(x)$  and the Fredholm index of  $\hat{f}_\psi(y)$  constantly equals to  $\operatorname{ind}(\hat{f}_\psi(x))$  on  $U(x)$ . Hence we define  $f_\psi(y)$  for  $y \in U(x)$  as follows:

$$\begin{aligned} f_\psi(y) : \mathbb{R}^n \times H_0 &\rightarrow \mathbb{R}^n \times H_1 \\ (h, k) &\mapsto (0, \hat{f}_\psi(y)(h, k)) \end{aligned}$$

Therefore  $\dim(\ker f_\psi(y))$  and  $\dim(\operatorname{coker} f_\psi(y))$  are constant on  $U(x)$ . Thus, we have a determinant bundle

$$\operatorname{Det}(f_\psi) \xrightarrow{\pi_\psi} U(x)$$

as long as  $(x, \psi, U(x))$  satisfies conditions described above.

Consider the exact sequence for  $y \in U(x)$ :

$$0 \rightarrow \ker f(y) \xrightarrow{d_1} \ker f_\psi(y) \xrightarrow{d_2} \mathbb{R}^n \xrightarrow{d_3} \operatorname{coker} f(y) \rightarrow 0$$

defined by

$$\begin{aligned} d_1(k) &= (0, k) \\ d_2(h, k) &= h \\ d_3(h) &= [\psi(h)]_{R(f(y))} \end{aligned}$$

According to lemma 2.28, we have a canonical isomorphism

$$\phi_{y,\psi} : \wedge^{\max} \ker(f(y)) \otimes \wedge^{\max} \mathbb{R}^n \xrightarrow{\cong} \wedge^{\max} \ker(f_\psi(y)) \otimes \wedge^{\max} (\operatorname{coker} f(y))$$

Tensor both side of  $\phi_{y,\psi}$  by  $(\wedge^{\max} \mathbb{R}^n)^* \otimes (\wedge^{\max} (\operatorname{coker} f(y)))^*$ . Since for all finite-dimensional real vector space  $E$ ,  $(\wedge^{\max} E) \otimes (\wedge^{\max} E)^* \cong \mathbb{R}$  via the map  $\xi \otimes \eta^* \mapsto \eta^*(\xi)$ . We obtain an isomorphism:

$$\operatorname{Det}(f(y)) \xrightarrow{\cong} \wedge^{\max} \ker(f_\psi(y)) \otimes (\wedge^{\max} \mathbb{R}^n)^* = \operatorname{Det}(f_\psi(y))$$

The equality comes from the identity  $\operatorname{coker} f_\psi(y) = (H_1 \times \mathbb{R}^n)/H_1 \times \{0\} = \mathbb{R}^n$ . Then we have isomorphisms on fibres of  $\operatorname{Det}(f_\psi)$  and  $\operatorname{Det}(f)|_{U(x)}$ :

$$\begin{array}{ccc} \operatorname{Det}(f_\psi) & \xrightarrow{\cong} & \operatorname{Det}(f)|_{U(x)} \\ \pi_\psi \downarrow & \swarrow \pi & \\ U(x) & & \end{array}$$

It remains to show that for two admissible triples  $(x, \psi, U(x))$  and  $(x', \psi', U(x'))$ , the transition map

$$\operatorname{Det}(f_\psi)|_V \xrightarrow{\text{transition}} \operatorname{Det}(f_{\psi'})|_V$$

is a vector bundle isomorphism, where  $V = U(x) \cap U(x')$ .

Assume  $\psi' : \mathbb{R}^{n'} \rightarrow H_1$ , we define

$$\begin{aligned} \psi \oplus \psi' : \mathbb{R}^n \times \mathbb{R}^{n'} &\rightarrow H_1 \\ (h, k) &\mapsto \psi(h) + \psi'(k) \end{aligned}$$

Then for  $y \in V$ , we have an exact sequence

$$0 \rightarrow \ker f_\psi(y) \rightarrow \ker f_{\psi \oplus \psi'}(y) \rightarrow \mathbb{R}^{n'} \rightarrow \operatorname{coker} f_\psi(y) \rightarrow 0$$

By similar calculation as before, we obtain an isomorphism

$$\operatorname{Det}(f_\psi(y)) \xrightarrow{\cong} \operatorname{Det}(f_{\psi \oplus \psi'}(y))$$

Then it is obvious that  $\operatorname{Det}(f_\psi) \xrightarrow{\cong} \operatorname{Det}(f_{\psi \oplus \psi'})$  is a vector bundle isomorphism over  $V$ . We have the same result for  $\operatorname{Det}(f_{\psi' \oplus \psi})$ . And we can easily induce a vector bundle isomorphism  $\operatorname{Det}(f_{\psi \oplus \psi'} \rightarrow \operatorname{Det}(f_{\psi' \oplus \psi})$  from the vector bundle isomorphisms between kernel and cokernel bundles of the type  $(h_1, h_2, k) \mapsto (h_2, h_1, k)$ . Then the results follows from the commutative diagram:

$$\begin{array}{ccc} \operatorname{Det}(f_\psi) & \xrightarrow{\text{transition}} & \operatorname{Det}(f_{\psi'}) \\ \cong \downarrow & & \downarrow \cong \\ \operatorname{Det}(f_{\psi \oplus \psi'}) & \xrightarrow{\cong} & \operatorname{Det}(f_{\psi' \oplus \psi}) \end{array}$$

□

**Definition 2.30.** An orientation of a family of Fredholm operators  $f : X \rightarrow F(H_0, H_1)$  is given by a non-vanishing section of the determinant bundle  $\operatorname{Det}(f)$ .

### 2.7.2 Orientation on glued Fredholm operators

Let  $A$  be an element in  $C(\bar{\mathbb{R}}, \text{End}(\mathbb{R}^n))$  such that  $A^\pm = A(\pm\infty) \in GL(n, \mathbb{R})$  are nondegenerate self-adjoint operators on  $\mathbb{R}^n$ . Then we define a set of special class of Fredholm operators:

$$\Theta(A^-, A^+) = \left\{ \frac{d}{dt} + A \mid A \in C(\bar{\mathbb{R}}, \text{End}(\mathbb{R}^n)), A^\pm = A(\pm\infty) \right\}$$

Elements in  $\Theta(A^-, A^+)$  are Fredholm operators defined from  $H^{1,2}(\mathbb{R}, \mathbb{R}^n)$  to  $L^2(\mathbb{R}, \mathbb{R}^n)$ . And we also have the following index property:

$$\text{ind}(F) = \#\{\text{negative eigenvalue of } A^-\} - \#\{\text{negative eigenvalue of } A^+\} \quad (2.6)$$

We also have the following property for  $\Theta(A^-, A^+)$ :

**Proposition 2.31.** ([7, lemma 2,15])

$\Theta(A^-, A^+)$  is contractible in  $\Sigma = \bigcup \Theta(M_1, M_2)$ , where  $M_1, M_2$  are nondegenerate self-adjoint operators on  $\mathbb{R}^n$ .

Suppose  $K = \frac{d}{dt} + A \in \Theta(A^-, A^+)$  and  $L = \frac{d}{dt} + B \in \Theta(B^-, B^+)$  are asymptotically constant operators, i.e.

$$A(t) = \begin{cases} A^- & t \leq -T \\ A^+ & t \geq T \end{cases} \quad \text{and} \quad B(t) = \begin{cases} B^- & t \leq -T \\ B^+ & t \geq T \end{cases}$$

**Definition 2.32.** The gluing operation for asymptotically constant operators  $K, L$  with  $A^+ = B^-$  is defined by

$$K \#_\rho L = \frac{d}{dt} + C \in \Theta(A^-, B^+)$$

where

$$C(t) = \begin{cases} A(t + \rho) & t \leq 0 \\ B(t - \rho) & t \geq 0 \end{cases}$$

for all  $\rho \geq T$ .

From (2.6), we have the following formula

$$\text{ind}(K \#_\rho L) = \text{ind}(K) + \text{ind}(L)$$

Now, the crucial question is whether there is a concept of orientation of  $K \#_\rho L$  associate to orientations of  $K$  and  $L$  in a canonical way.

**Proposition 2.33.** *Let  $K, L$  be Fredholm operators defined as above, then there exists an isomorphism*

$$\text{Det}K \otimes \text{Det}L \xrightarrow{\cong} \text{Det}(K \#_{\rho} L)$$

for  $\rho$  large enough.

*Proof.* In the following proof we denote  $H^{1,2}(\mathbb{R}, \mathbb{R}^n)$  and  $L^2(\mathbb{R}, \mathbb{R}^n)$  as  $H^{1,2}$  and  $L^2$  respectively.

Choose a linear map  $\psi : \mathbb{R}^k \rightarrow L^2$  such that the maps

$$\begin{aligned} \mathbb{R}^k \times H^{1,2} &\rightarrow L^2 \\ \hat{K}_{\psi} : (a, u) &\mapsto \psi(a) + K(u) \\ \hat{L}_{\psi} : (a, u) &\mapsto \psi(a) + L(u) \end{aligned}$$

are surjective. Then we define

$$\begin{aligned} \mathbb{R}^k \times H^{1,2} &\rightarrow \mathbb{R}^k \times L^2 \\ K_{\psi} : (a, u) &\mapsto (0, \hat{K}_{\psi}(a, u)) \\ L_{\psi} : (a, u) &\mapsto (0, \hat{L}_{\psi}(a, u)) \end{aligned}$$

Then from the proof of 2.29, we have

$$\text{Det}K_{\psi} \xrightarrow{\cong} \text{Det}K \quad \text{and} \quad \text{Det}L_{\psi} \xrightarrow{\cong} \text{Det}L$$

With out loss of generality, assume for all  $a \in \mathbb{R}^k$ ,  $\psi(a)$  is compact supported, i.e., there exists  $R > 0$ ,  $\forall a$ ,  $\text{supp}(\psi(a)) \subseteq [-R, R]$ , because

$$\hat{L}_{\beta_R \cdot \psi} \rightarrow \hat{L}_{\psi} \text{ in } L(H^{1,2}, L^2) \text{ as } R \rightarrow \infty$$

( $\beta_R$  is a cut-off function with support in  $[-R, R]$ ) and the set of surjective Fredholm operators is open in  $L(H^{1,2}, L^2)$ .

We define the glued operator

$$\begin{aligned} \hat{K}_{\psi} \#_{\rho} \hat{L}_{\psi} : \mathbb{R}^k \times \mathbb{R}^k \times H^{1,2} &\rightarrow L^2 \\ (a, b, u) &\mapsto (K \#_{\rho} L)u + \psi(a)(\cdot + \rho) + \psi(b)(\cdot - \rho) \end{aligned}$$

for  $\rho \geq R + 1 + T$ . Then we introduce the following lemma.

**Lemma 2.34.** *There exists  $\rho_0 \geq R + 1 + T$  such that*

$$\begin{aligned} \phi_{\rho} : \ker \hat{K}_{\psi} \times \ker \hat{L}_{\psi} &\xrightarrow{\cong} \ker(\hat{K}_{\psi} \#_{\rho} \hat{L}_{\psi}) \\ ((a, u), (b, v)) &\mapsto P_{\rho}(a, b, u_{\rho}, -v_{\rho}) \end{aligned}$$

is an isomorphism, where  $P_\rho$  is the orthogonal projection in  $\mathbb{R}^k \times \mathbb{R}^k \times L^2$  onto  $\ker(\hat{K}_\psi \#_\rho \hat{L}_\psi)$ .

The proof of this lemma can be found in [4], Proposition 9.

Define

$$\begin{aligned} K_\psi \#_\rho L_\psi : \mathbb{R}^k \times \mathbb{R}^k \times H^{1,2} &\rightarrow \mathbb{R}^k \times \mathbb{R}^k \times L^2 \\ (a, b, u) &\mapsto (0, 0, \hat{K}_\psi \#_\rho \hat{L}_\psi(a, b, u)) \end{aligned}$$

Then  $K_\psi \#_\rho L_\psi = (K \#_\rho L)_{\psi_\rho \oplus \psi_{-\rho}}$ , from the proof of proposition 2.29 we have a natural isomorphism:

$$\text{Det}(K_\psi \#_\rho L_\psi) \xrightarrow{\cong} \text{Det}(K \#_\rho L)$$

By lemma 2.34, we have an isomorphism

$$(\wedge^{\max} \ker \hat{K}_\psi) \otimes (\wedge^{\max} \ker \hat{L}_\psi) \xrightarrow{\cong} \wedge^{\max} \ker \hat{K}_\psi \#_\rho \hat{L}_\psi$$

Therefore, if  $\rho$  is large enough,

$$\text{Det}(K_\psi) \otimes \text{Det}(L_\psi) \xrightarrow{\cong} \text{Det}(K_\psi \#_\rho L_\psi)$$

Finally, we have

$$\text{Det}K \otimes \text{Det}L \cong \text{Det}K_\psi \otimes \text{Det}L_\psi \cong \text{Det}(K_\psi \#_\rho L_\psi) \cong \text{Det}(K \#_\rho L)$$

□

It follows from proposition 2.31 that  $\text{Det}(\Theta(A^-, A^+))$  is trivial, so  $\Theta(A^-, A^+)$  is orientable. A choice of orientation for  $\Theta(A^-, A^+)$  will be denoted by  $o(A^-, A^+)$ . At the end of this section we introduce the following theorem.

**Theorem 2.35.** *Suppose  $A^+ = B^-$ , then a choice of orientations of  $\Theta(A^-, A^+)$  and  $\Theta(B^-, B^+)$ , say  $o(A^-, A^+)$ ,  $o(B^-, B^+)$ , determines an orientation  $o(A^-, A^+) \#_\rho o(B^-, B^+)$  on  $\Theta(A^-, B^+)$ . Moreover, if  $B^+ = C^-$ , the associativity rule is fulfilled:*

$$(o(A^-, A^+) \#_\rho o(B^-, B^+)) \#_\rho o(C^-, C^+) = o(A^-, A^+) \#_\rho (o(B^-, B^+) \#_\rho o(C^-, C^+))$$

This theorem follows immediately from proposition 2.33. More precisely,  $o(A^-, A^+)$  and  $o(B^-, B^+)$  induce orientations  $o(K)$  and  $o(L)$  respectively for  $K \in \Theta(A^-, A^+)$ ,  $L \in \Theta(B^-, B^+)$ . Then by proposition 2.33, we have an orientation  $o(K \#_\rho L)$  which determines the desired glued orientation  $o(A^-, A^+) \#_\rho o(B^-, B^+)$  on  $\Theta(A^-, B^+)$ .

### 2.7.3 Coherent orientation on $\mu_{x,y}$

Recall that for each pair of critical points  $x, y$ ,  $\mu_{x,y} = F^{-1}(0)$  (by theorem 2.4). For  $u \in \mu_{x,y}$  we have

$$T_u\mu_{x,y} = \ker(DF(u))$$

Since  $DF(u)$  is surjective for generic metric  $g$ , we also have

$$\wedge^{\max} \ker(DF(u)) \cong \text{Det}(DF(u))$$

Given  $u \in \mu_{x,y}$  we have a trivialization  $\phi_u : u^*TM \rightarrow \mathbb{R} \times \mathbb{R}^n$  such that under this trivialization  $DF(u) = \frac{d}{dt} + A$ , where  $A^-, A^+$  are Hessian of  $f$  at  $x, y$  respectively. So we can consider  $DF(u)$  as an element of  $\Theta(A^-, A^+)$ . Then an orientation  $o(A^-, A^+)$  together with the trivialization  $\phi_u$  determines an orientation for  $T_u\mu_{x,y}$ . From this, we can induce an orientation on  $u$ . Next, we want to orient  $\mu_{x,y}$ .

Since  $\phi_u : u^*TM \rightarrow \mathbb{R} \times \mathbb{R}^n$  is an isomorphism, we can deduce from  $\phi_u$  the following two isomorphisms between Hilbert spaces

$$H^{1,2}(\mathbb{R}, u^*TM) \rightarrow H^{1,2}(\mathbb{R}, \mathbb{R}^n) \quad \text{and} \quad L^2(\mathbb{R}, u^*TM) \rightarrow L^2(\mathbb{R}, \mathbb{R}^n)$$

which are again denoted by  $\phi_u$ . Therefore,  $\phi_u \circ DF(u) \circ \phi_u^{-1} = \frac{d}{dt} + A$ .

For all  $v \in \mu_{x,y}$ , there exists a trivialization  $\phi_v : v^*TM \rightarrow \mathbb{R} \times \mathbb{R}^n$  such that  $\phi_v \circ DF(u) \circ \phi_v^{-1} = \frac{d}{dt} + B$ , where  $B(\pm\infty) = A(\pm\infty)$  and  $\phi_u \circ DF(u) \circ \phi_u^{-1}$  and  $\phi_v \circ DF(u) \circ \phi_v^{-1}$  are homotopic in  $\Theta(A^-, A^+)$ , which is denoted by  $\phi_u \circ DF(u) \circ \phi_u^{-1} \sim \phi_v \circ DF(u) \circ \phi_v^{-1}$ . Then  $\phi_v$  determine orientation for the tangent space  $T_v\mu_{x,y}$ .

Now we introduce the following proposition:

**Proposition 2.36.** (*[4, lemma 15]*) *For all  $u, v \in \mu_{x,y}$ , let  $\text{Det}(DF(u))$  and  $\text{Det}(DF(v))$  be oriented by  $o_1, o_2$ . Then  $\phi_u, \phi_v$  induce orientations  $\phi_u(o_1), \phi_v(o_2)$  on  $\text{Det}(\phi_u \circ DF(u) \circ \phi_u^{-1})$  and  $\text{Det}(\phi_v \circ DF(v) \circ \phi_v^{-1})$  respectively. Suppose  $\phi_v, \phi'_v$  are two trivializations such that*

$$\phi_u \circ DF(u) \circ \phi_u^{-1} \sim \phi_v \circ DF(u) \circ \phi_v^{-1} \quad \text{and} \quad \phi_u \circ DF(u) \circ \phi_u^{-1} \sim \phi'_v \circ DF(u) \circ (\phi'_v)^{-1}$$

*then  $\phi_u(o_1)$  and  $\phi_v(o_2)$  are related by continuation in  $\Theta(A^-, A^+)$  if and only if  $\phi_u(o_1)$  and  $\phi'_v(o_2)$  are related by continuation in  $\Theta(A^-, A^+)$ .*

Proposition 2.36 gives a tool to orient  $\mu_{x,y}$ . If we fix a trivialization  $\phi_u$  for some  $u \in \mu_{x,y}$ , then an orientation  $o(A^-, A^+)$  of  $\Theta(A^-, A^+)$  yields an orientation for

each  $T_v\mu_{x,y}$ . For all  $v \in \mu_{x,y}$ , we induce an orientation  $o(v)$  from the orientation of  $T_v\mu_{x,y}$ .

For the collection of flow line spaces

$$\bigcup_{x,y \in \text{Crit}(f)} \mu_{x,y}$$

We have a corresponding set

$$\bigcup_{x,y \in \mu_{x,y}} \Theta(A_x, A_y)$$

where  $A_x$  is the Hessian of  $f$  at  $x$ .

**Definition 2.37.** A collection of orientations  $o(A_x, A_y)$  on  $\Theta(A_x, A_y)$  is called coherent if it is compatible with the gluing operation, i.e.

$$o(A_x, A_y) \# o(A_y, A_z) = o(A_x, A_z)$$

**Theorem 2.38.** *Coherent orientations exist.*

*Proof.* To show the existence, we construct one.

First, fix a critical point  $x_0$  and choose an orientation  $o(A_{x_0}, A_{x_0})$  on  $\Theta(A_{x_0}, A_{x_0})$ .

Second, for each critical point  $x$ , choose orientations  $o(A_{x_0}, A_x)$  on  $\Theta(A_{x_0}, A_x)$ .

The coherent condition determines the orientations of  $\Theta(A_x, A_{x_0})$  by

$$o(A_{x_0}, A_x) \# o(A_x, A_{x_0}) = o(A_{x_0}, A_{x_0})$$

Third, the orientation of  $\Theta(A_x, A_y)$  is determined by

$$o(A_{x_0}, A_x) \# o(A_x, A_y) \# o(A_y, A_{x_0}) = o(A_{x_0}, A_{x_0})$$

which is well-defined due to associativity. Moreover, associativity guarantees that this construction provides a coherent orientation.  $\square$

## 2.7.4 Orientation on $W(x, y)$

We have discussed how to orient the flow line space  $\mu_{x,y}$ . Since  $\mu_{x,y} \cong W(x, y)$ , there is another way to orient  $\mu_{x,y}$ , that is, define an orientation on  $W(x, y)$ .

Theorem 1.5 tells us that  $W^u(x) \cong \mathbb{R}^{\text{ind}(x)}$  for all  $x \in \text{Crit}(f)$ . Hence the unstable manifolds are orientable. We choose orientations  $o_x$  for these unstable

manifolds. Given that  $M$  is orientable and oriented, these orientations impose positive orientations on  $T_x W^s(x)$  for each  $x$  through the identity

$$T_x W^u(x) \oplus T_x W^s(x) = T_x M.$$

Thus, all the stable and unstable manifolds are oriented.

Next we want to define orientations on  $W(x, y) = W^u(x) \cap W^s(y)$ . We will make use of the following fact: Given a short exact sequence of vector bundles

$$0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

the orientation of  $E_x$  (a fibre of  $E$ ) followed by the orientations of  $F_x$  and  $G_x$ . This fact follows trivially from lemma 2.28.

Suppose  $p \in W(x, y)$  is a non-critical point, there is a short exact sequence

$$\begin{aligned} 0 \rightarrow T_p W(x, y) \xrightarrow{i} T_p W^u(x) \xrightarrow{j} T_p W^u(y) \rightarrow 0 \\ (\dot{u}) \quad \mapsto (\dot{u}, e_1, \dots, e_{k-1}) \mapsto (e_1, \dots, e_{k-1}) \end{aligned}$$

where  $\text{ind}(x) = k$ ,  $\text{ind}(y) = k - 1$  and  $u$  is a flow line from  $x$  to  $y$ , thus  $(\dot{u})$  is a basis of  $T_p W(x, y)$  (since  $W(x, y)$  is a 1-dimensional manifold), and  $(\dot{u}, e_1, \dots, e_{k-1})$  is a complete positive basis for  $T_p W^u(x)$ . The exactness of the sequence follows from the injectivity of  $i$  and surjectivity of  $j$ .

Since

$$T_u \mu_{x,y} = \ker(DF(u)) \quad \text{and} \quad DF(u)(\dot{u}) = 0$$

$\dot{u}$  spans  $T_u \mu_{x,y}$ . Now, we introduce the following mechanism for orienting  $T_u \mu_{x,y}$ :  $(\dot{u})$  is a positive basis if  $(e_1, \dots, e_{k-1})$  is a positive basis of  $T_p W^u(y)$ ,  $(\dot{u})$  is a negative basis if  $(e_1, \dots, e_{k-1})$  is a negative basis of  $T_p W^u(y)$ .

## 2.8 Morse complex

We define the Morse complex  $(C_*^M(f), \partial^M)$  as follows. Let  $\text{Crit}_k(f)$  denote the set of index  $k$  critical points of  $f$ . The chain element  $C_k$  is the free  $\mathbb{Z}$ -module generated by  $\text{Crit}_k(f)$ :

$$C_k^M = \bigoplus_{x \in \text{Crit}_k(f)} \mathbb{Z} \langle x \rangle$$

The boundary map  $\partial^M : C_k \rightarrow C_{k-1}$  counts flow lines, that is for all  $x \in \text{Crit}_k(f)$

$$\partial^M(x) = \sum_{y \in \text{Crit}_{k-1}(f)} \# \hat{\mu}_{x,y} \cdot y$$

Note that  $\hat{\mu}_{x,y}$  is a 0-dimensional manifold consists with finite point, here  $\#\hat{\mu}_{x,y}$  denotes the number of points in  $\hat{\mu}_{x,y}$ , counted with signs given by the orientation on  $\hat{\mu}_{x,y}$ .

We now show that  $(C_*^M(f), \partial^M)$  is a chain complex.

**Theorem 2.39.**  $(\partial^M)^2 = 0$ .

*Proof.* From theorem 2.27, we know that if  $\text{ind}(x) = \text{ind}(z) + 2$ ,  $\bar{\mu}_{x,z}$  is a compact 1-dimensional manifold and

$$\partial\bar{\mu}_{x,z} = \bigcup_{\text{ind}(x)=\text{ind}(y)+1} \hat{\mu}_{x,y} \times \hat{\mu}_{y,z}$$

Now we compute  $(\partial^M)^2$  explicitly,

$$\begin{aligned} (\partial^M)^2(x) &= \partial^M\left(\sum_{y \in \text{Crit}_{k-1}(f)} \#\hat{\mu}_{x,y} \cdot y\right) \\ &= \sum_{y \in \text{Crit}_{k-1}(f)} \left(\#\hat{\mu}_{x,y} \sum_{z \in \text{Crit}_{k-2}(f)} \#\hat{\mu}_{y,z} \cdot z\right) \\ &= \sum_{z \in \text{Crit}_{k-2}(f)} \sum_{y \in \text{Crit}_{k-1}(f)} \#\hat{\mu}_{x,y} \#\hat{\mu}_{y,z} \cdot z \end{aligned}$$

So for every critical point  $z$  with index equals to  $\text{ind}(x) - 2$ , the coefficient of  $z$  in  $(\partial^M)^2(x)$  is

$$\#\sum_{y \in \text{Crit}_{k-1}(f)} \hat{\mu}_{x,y} \times \hat{\mu}_{y,z} = \#\partial\bar{\mu}_{x,z}$$

which is zero since the boundary of a compact oriented 1-dimensional manifold has zero points counted with sign(1-manifold has two kinds of forms, one is  $S^1$  and the other is interval.  $S^1$  has no boundary point and the two boundary points of an interval has opposite sign).  $\square$

## Chapter 3

# Morse complex agrees with cellular complex

We have introduced two approaches to construct chain complex of  $(M, f)$  and thus obtain two kinds of homology. In this chapter, we aim to show that Morse complex agrees with cellular complex.

**Theorem 3.1.** *Suppose*

$$(C_k(f) = \bigoplus_{x \in \text{crit}_k(f)} \mathbb{Z} \langle e_x^k \rangle, \partial^C) \quad \text{and} \quad (C_k^M(f) = \bigoplus_{x \in \text{Crit}_k(f)} \mathbb{Z} \langle x \rangle, \partial^M)$$

are the cellular chain complex and Morse complex defined at the end of chapter 1 and chapter 2 respectively. Then the following diagram commutes.

$$\begin{array}{ccc} C_k(f) & \xrightarrow{\partial^C} & C_{k-1}(f) \\ \cong \downarrow & & \downarrow \cong \\ C_k^M(f) & \xrightarrow{\partial^M} & C_{k-1}^M(f) \end{array}$$

Here, the left vertical map maps  $e_x^k$  to  $x$  and the right one is similar.

*Proof.* Since

$$\partial_k^C(e_x^k) = \sum_{y \in \text{crit}_{k-1}(f)} \text{deg}(\alpha_{xy}) e_y^{k-1}$$

and

$$\partial^M(x) = \sum_{y \in \text{Crit}_{k-1}(f)} \# \hat{\mu}_{x,y} \cdot y$$

The commutativity of the diagram follows immediately from the identity

$$\text{deg}(\alpha_{xy}) = \# \hat{\mu}_{x,y}.$$

Now we begin to prove this identity.

First, we study the map defined in section 1.3:

$$\alpha_{xy} : \partial e_x^k \xrightarrow{\varphi} X^{k-1} \xrightarrow{q} S_y^{k-1}$$

where  $\varphi$  is the attaching map and  $q$  is the quotient map collapsing  $X^{k-1} \setminus e_x^{k-1}$  to a point. Since there is a finite number of flow lines from  $x$  to  $y$ ,  $\partial e_x^k$  contains finite points in  $W(x, y)$ . According to the definition of  $\varphi$ ,  $\varphi^{-1}(y)$  is a finite set and  $\varphi$  is a local diffeomorphism at each  $p \in \varphi^{-1}(y)$ .

Since

$$\deg(\alpha_{xy}) = \sum_{p \in \varphi^{-1}(y)} \deg(\alpha_{xy}, p)$$

where  $\deg(\alpha_{xy}, p)$  is the local degree of  $\alpha_{xy}$  at  $p$ .  $\deg(\alpha_{xy}, p)$  is either 1 or -1 because  $\varphi$  is a local diffeomorphism at  $p$ . Moreover, we have

$$\deg(\alpha_{xy}, p) = \begin{cases} 1 & \text{if } d\varphi(p) \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

From the discussion of orientation on  $W(x, y)$  in section 2.7.4, we can give an equivalent definition of the sign of flow lines from  $x$  to  $y$ . Consider the short exact sequence

$$\begin{aligned} 0 \rightarrow T_p W(x, y) \xrightarrow{i} T_p W^u(x) \xrightarrow{j} T_p W^u(y) \rightarrow 0 \\ (\dot{u}) \quad \mapsto (\dot{u}, e_1, \dots, e_{k-1}) \mapsto (e_1, \dots, e_{k-1}) \end{aligned}$$

the sign of  $u$  is positive if  $(e_1, \dots, e_{k-1})$  is a positive basis of  $T_p W^u(y)$  and negative otherwise.

$T_p W^u(x)$  can be written as  $T_p W(x, y) \oplus T_p \partial e_x^k$ , also,  $T_{\alpha(p)} S_y^{k-1}$  can be taken as  $T_p W^u(y)$ . Now,  $T_p \partial e_x^k$  is attached to  $T_{\alpha(p)} S_y^{k-1}$  via  $\alpha$ . Choose  $(\dot{u}, e_1, \dots, e_{k-1})$  as a positive basis of  $T_p W(x, y) \oplus T_p \partial e_x^k$ , then  $(d\varphi(p)(e_1, \dots, e_{k-1}))$  is a basis of  $T_{\alpha(p)} S_y^{k-1}$ . Then it is obvious that  $(d\varphi(p)(e_1, \dots, e_{k-1}))$  is a positive basis if  $d\varphi(p)$  preserves orientation, and negative basis otherwise.

So the sign of  $u$  is positive if  $d\varphi(p)$  preserves orientation, and negative otherwise.

Figure 3.1 is a visualisation of the reasoning above.

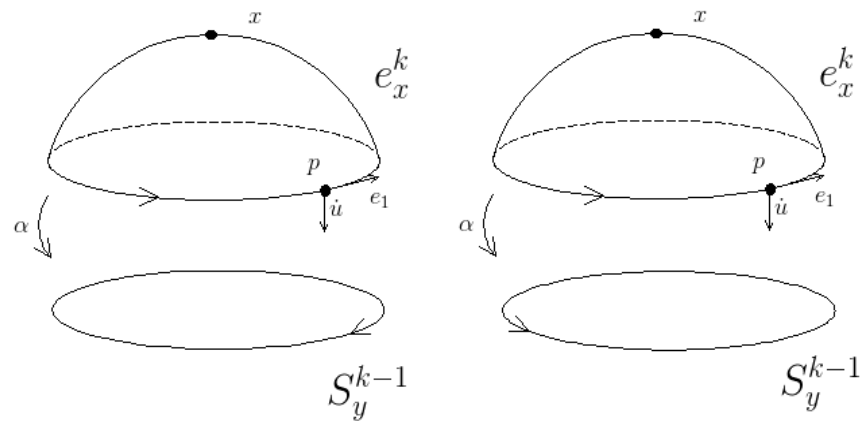


Figure 3.1

The left one is the case when  $d\varphi(p)$  does not preserve orientation, and the right one is the orientation preserving case.

Therefore,  $\deg(\alpha_{xy}) = \#\hat{\mu}_{x,y}$  □

From theorem 3.1, we can conclude:

**Corollary 3.2.**  $H_*^M(f) \cong H_*^C(X)$ .



# Chapter 4

## An overview of Floer theory

Floer theory attempts to build the equivalent of Morse complex in infinite dimensions. The finite dimensional manifold  $M$  is replaced by an infinite dimensional manifold and the Morse-Smale function  $f$  is replaced by some functional. Floer homology is a useful tool to study many problems in various areas, such as symplectic geometry, gauge theory and the low dimensional topology. There are several Floer homology theories and we discuss two of them in this chapter.

### 4.1 Lagrangian-Floer theory

Andreas Floer introduced Floer homology in symplectic geometry in order to solve the Arnold's conjecture concerning the fixed points of a Hamiltonian diffeomorphism  $\phi : M \rightarrow M$  of a compact symplectic manifold  $(M, \omega)$ .

The conjecture states that if all of the fixed points are nondegenerate, then

$$\#fix(\phi) \geq \sum_k rank H_k(M).$$

A Lagrangian submanifold  $L$  of a symplectic manifold  $(M, \omega)$  is a submanifold whose dimension equals to half of the dimension of  $M$  and satisfies that  $i^*\omega = 0$ , where  $i$  is the inclusion map. Suppose  $L_0, L_1 \subseteq (M, \omega)$  are two compact Lagrangian submanifolds and  $L_0$  is transversal to  $L_1$ . Floer homology, denoted by  $HF(L_0, L_1)$ , is a homology group of the chain complex that is given by

$$CF_*(L_0, L_1) = \sum_{p \in L_0 \cap L_1} \mathbb{Z}[p].$$

Floer defined  $HF(L_0, L_1)$  as the homology of the chain complex  $(CF_*, \partial)$  where the boundary operator  $\partial$  is defined by counting the number of pseudo-holomorphic

discs with half of its boundary lying on  $L_0$  and the other half lying on  $L_1$  and with its asymptotic limits given by  $p, q \in L_0 \cap L_1$ . The main task is to prove  $\partial^2 = 0$ .

Consider the infinite dimensional manifold of smooth paths

$$\mathcal{M}(L_0, L_1) = \{l : [0, 1] \rightarrow M \mid l(0) \in L_0, l(1) \in L_1\}.$$

On this space we have the action one-form  $\alpha$

$$\alpha(l)(\xi) = \int_0^1 \omega(\xi(t), \dot{l}) dt \quad \forall \xi \in T_l \mathcal{M}$$

Since  $L_0, L_1$  are Lagrangian submanifolds, this form is closed. Note that  $\mathcal{M}(L_0, L_1)$  is not necessarily connected but has countably many connected components. We denote the component containing  $l_0$  by  $\mathcal{M}(L_0, L_1, l_0)$  and its Novikov covering space by  $\pi : \widetilde{\mathcal{M}}(L_0, L_1, l_0) \rightarrow \mathcal{M}(L_0, L_1, l_0)$ .

Given  $l \in \mathcal{M}(L_0, L_1)$ , there exists a smooth map  $\Gamma : [0, 1] \times [0, 1] \rightarrow M$  such that

$$\begin{cases} u(s, t) = l_s(t) \\ l_s \in \mathcal{M}(L_0, L_1) \\ u(1, t) = l(t) \end{cases}$$

We define the action functional

$$\begin{aligned} \mathcal{A} : \widetilde{\mathcal{M}}(L_0, L_1, l_0) &\rightarrow \mathbb{R} \\ (l, u) &\mapsto \int_{[0,1] \times [0,1]} u^* \omega \end{aligned}$$

,the differential of which is

$$(d\mathcal{A}(l, u))(\xi) = \int_0^1 \omega(\xi(t), \dot{l}(t)) dt \quad \forall \xi \in T_l \mathcal{M}(L_0, L_1)$$

The critical points of  $\mathcal{A}$  are constant maps. If  $l(0) = l(1) = p \in L_0 \cap L_1$ , then  $l$  is a constant map on intersection points of  $L_0$  and  $L_1$ . Moreover,  $l$  is non-degenerate if and only if  $L_0$  and  $L_1$  intersect transversally at  $p$ .

Suppose  $J_0$  and  $J_1$  is a pair of compatible almost complex structures on  $(M, \omega)$  and  $\{J_t\}_{t \in [0,1]}$  is a path of compatible almost complex structures joining them. Then we can define a metric on  $\mathcal{M}(L_0, L_1)$  by

$$\langle \xi_1, \xi_2 \rangle_{\{J_t\}} = \int_0^1 \omega(\xi_1(t), J_t \xi_2(t)) dt$$

The gradient equation of  $\mathcal{A}$  becomes

$$\begin{cases} \frac{du}{ds} + J_t \frac{du}{dt} = 0 \\ u(s, \cdot) = l_s \in \mathcal{M}(L_0, L_1) \end{cases} \quad (4.1)$$

where  $u : \mathbb{R} \times [0, 1] \rightarrow M$  is a path  $s \mapsto l_s$  in  $\mathcal{M}(L_0, L_1)$ .

Now we define the space of bounded solution of (4.1) by

$$\mu(L_0, L_1) = \left\{ u \mid u \text{ satisfies (4.1) and } \int_{\mathbb{R} \times [0, 1]} u^* \omega < \infty \right\}$$

Then we have the following proposition.

**Proposition 4.1.** *For all  $u \in \mu(L_0, L_1)$ , we have the following limit*

$$\lim_{s \rightarrow -\infty} u(s, \cdot) = l_p, \quad \lim_{s \rightarrow \infty} u(s, \cdot) = l_q$$

where  $p, q \in L_0 \cap L_1$  and  $l_p, l_q$  are constant paths at  $p, q$  respectively.

Given this proposition, we can define

$$\mu(p, q) = \left\{ u \in \mu(L_0, L_1) \mid \lim_{s \rightarrow -\infty} u(s, \cdot) = l_p, \lim_{s \rightarrow \infty} u(s, \cdot) = l_q \right\}$$

for  $p, q \in L_0 \cap L_1$ .  $\mu(p, q)$  is the space of connecting orbits. It plays the role of gradient flow lines in the finite dimensional Morse homology.

Floer homology is defined by

$$\begin{cases} HF(L_0, L_1) = \frac{\ker \partial}{\text{Im} \partial} \\ \partial : CF(L_0, L_1) \rightarrow CF(L_0, L_1) \\ CF(L_0, L_1) = \bigoplus \mathbb{Z}[l_p] \text{ where } p \in L_0 \cap L_1 \\ \partial l_p = \sum_q \langle \partial l_p, l_q \rangle l_q \\ \langle \partial l_p, l_q \rangle = \# \text{gradient lines of } \mathcal{A} \text{ joining } l_p \text{ and } l_q \end{cases}$$

**Theorem 4.2.** (*Floer*)

Suppose  $L_0 = L$  and  $L_1 = \phi(L)$  for  $L$  satisfying  $\pi_2(M, L) = 0$  and  $\phi : M \rightarrow M$  is a Hamiltonian diffeomorphism. Then  $\partial$  is well-defined and  $\partial^2 = 0$  and Floer homology is invariant under the Hamiltonian diffeomorphisms, that is to say

$$HF_*(L_0, L_1) \cong HF_*(\varphi_0 L_0, \varphi_1 L_1)$$

where  $\varphi_0, \varphi_1$  are Hamiltonian diffeomorphisms.

## 4.2 Instanton Floer theory

Another area where Floer homology has been very useful is low dimensional topology. In instanton Floer theory, we study the space of connections on a principal  $SU(2)$  bundle over a 3-manifold  $Y$  (modulo the gauge group). The role of the Morse function in classical Morse theory is played by the Chern-Simons function.

Let  $\pi : P \rightarrow Y$  be a principal  $G$ -bundle where  $G = SU(2)$  and  $Y$  is a closed 3-manifold. Then there is a Hermitian  $\mathbb{C}^2$  bundle  $E = P \times_{SU(2)} \mathbb{C}^2$ . Suppose  $\mathfrak{g}$  is the Lie algebra of  $G$ , then we denote the bundle  $P \times_G \mathfrak{g}$  by  $adP$ .

A connection  $A$  on  $P$  is a covariant derivative

$$\nabla_A : C^\infty(Y; E) \rightarrow C^\infty(Y; T^*Y \otimes E)$$

satisfying  $\nabla_A(f\sigma) = f\nabla_A(\sigma) + df \otimes \sigma$ . For the associated bundle  $adP$ , we write  $d_A : C^\infty(Y; adP) \rightarrow C^\infty(Y; T^*Y \otimes adP)$ , and we have the higher forms

$$d_A : \Omega^k(Y; adP) \rightarrow \Omega^{k+1}(Y; adP).$$

Although vector bundles are locally trivializable, not every connection is locally trivial. We define the curvature 2-form on  $P$ , denoted by  $F_A$ .

$$F_A = dA + \frac{1}{2}[A \wedge A]$$

Note that  $F_A \in C^\infty(Y; \Lambda^2 \otimes adP)$  is 0 when  $A$  is trivial connection. The set of connections on  $P$ , denoted by  $\mathcal{A}$ , forms an affine space over  $\Omega^1(Y; adP)$ . A connection  $A$  satisfying  $F_A = 0$  is called a flat connection and we denote the set of flat connections by  $flat(\mathcal{A})$ .

A gauge transformation is a  $G$ -equivariant automorphism of  $P$ . The set of gauge transformations is called the gauge group and denoted by  $\mathcal{G}$ . We define

$$\mathcal{M} = \mathcal{A}/\mathcal{G}$$

Now we discuss about the Chern-Simons function. Suppose  $X$  is an oriented 4-manifold with boundary  $Y$ , and extend the bundle  $P$  and connection  $A$  on  $X$ , denoted as  $\mathbf{P}$  and  $\mathbf{A}$  respectively. Then the quantity

$$\frac{1}{8\pi^2} \int_X tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \pmod{\mathbb{Z}}$$

depends only on the gauge equivalence class of  $A$ .

For a connection  $A$  on  $P \rightarrow Y$ , we define the Chern-Simons invariant  $CS(A)$  to be the residue of  $\frac{1}{8\pi^2} \int_X tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \pmod{\mathbb{Z}}$ . The freedom of choosing the extension leads to an alternative construction. Let  $X = Y \times [0, 1]$  and  $\mathbf{A}$  is a connection on  $\mathfrak{p}$  over  $X$ , which is trivial product connection over  $Y \times \{0\}$  and agrees with  $A$  on  $Y \times \{1\}$ . Then we have

$$CS(A) = \frac{1}{8\pi^2} \int_{Y \times [0,1]} tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}})$$

On the other hand, in a trivialization of our bundle over  $X$ , we have

$$d(tr(\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A})) = tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}})$$

By Stokes' theorem,

$$\begin{aligned} CS(A) &= \frac{1}{8\pi^2} \int_{Y \times [0,1]} tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \\ &= \frac{1}{8\pi^2} \int_Y tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \end{aligned}$$

The function  $CS : \mathcal{M} \rightarrow S^1$  is called the Chern-Simons functional.

The critical points of  $CS$  turns out to be flat connections. Let  $\mathcal{R} \subseteq \mathcal{M}$  denote the space of gauge equivalence classes of the flat connections. The gradient flow lines of the Chern-Simons functional are smooth maps  $\mathbb{R} \rightarrow \mathcal{A} : t \mapsto A(t)$  satisfying

$$\frac{dA(t)}{dt} + *F_A = 0$$

where  $*$  is the Hodge star operator.

Now we consider the Hessian operator

$$\begin{aligned} Q_A : \ker d_A^* &\rightarrow \ker d_A^* \\ a &\mapsto *d_A(a) \end{aligned}$$

To understand more about the Hessian, we introduce another operator which is equivalent to it. For any connection  $A$  on  $P$ , let

$$L_A : \Omega^0(Y, adP) \oplus \Omega^1(Y, adP) \rightarrow \Omega^0(Y, adP) \oplus \Omega^1(Y, adP)$$

be defined by the matrix

$$\begin{bmatrix} 0 & -d_A^* \\ -d_A & *d_A \end{bmatrix}$$

The operator  $L_A$  is self-adjoint, elliptic. To relate  $L_A$  with  $Q_A$ , we decompose the 1-form as

$$\Omega^1(Y, adP) = \ker d_A^* \oplus \text{Im} d_A$$

Then if  $A$  is flat,  $L_A = Q_A \oplus S_A$ , where  $S_A$  is

$$\begin{bmatrix} 0 & -d_A \\ -d_A^* & 0 \end{bmatrix}$$

Since the spectrum of  $L_A$  is real and discrete, we obtain the spectrum of  $Q_A$  by discarding the spectrum of  $S_A$ . For a pair of flat connections  $(A_0, A_1)$ , we define the relative index  $\text{ind}(A_0, A_1)$  by counting the net number of eigenvalues crossing 0.

Assume that  $CS$  is a Morse function, we define the Floer chains to be

$$C_* = \bigoplus_{A \in \text{flat}(A)} \mathbb{Z} \langle [A] \rangle$$

The definition of the boundary operator is similar to that in the Morse homology, we want to count gradient flow lines between gauge equivalence classes of flat connections that have relative index one. To do this, we introduce Yang-Mills equation.

Consider the 4-dimensional manifold  $X = Y \times \mathbb{R}$  and let the bundle  $P \times \mathbb{R} \rightarrow X$  be a trivial extension of  $P \rightarrow Y$ . A connection  $A$  on  $P \rightarrow Y$  extends to a connection  $\mathbf{A}$  on  $P \times \mathbb{R}$ , where the  $\mathbb{R}$  direction in  $\mathbb{R} \times P$  is horizontal. Then the gradient flow equation takes the form

$$F_{\mathbf{A}} = - * F_{\mathbf{A}}$$

This is the anti-self-dual Yang-Mills equation.

Let  $\alpha$  and  $\beta$  be flat connections, define  $\mathcal{M}(\alpha, \beta)$  to be the set of connections  $\mathbf{A}$  quotient by  $\mathcal{G}_X$  satisfying

$$\begin{cases} F_{\mathbf{A}} = - * F_{\mathbf{A}} \\ \lim_{t \rightarrow -\infty} \mathbf{A}|_{Y \times t} = \alpha \\ \lim_{t \rightarrow +\infty} \mathbf{A}|_{Y \times t} = \beta \\ \int_X \text{tr}(F_{\mathbf{A}} \wedge * F_{\mathbf{A}}) < \infty \text{ (}\mathbf{A} \text{ has finite energy)} \end{cases}$$

Note that shifting the  $t$  variable defines an  $\mathbb{R}$  action on  $\mathcal{M}(\alpha, \beta)$ . We define the moduli space  $\hat{\mathcal{M}}(\alpha, \beta) = \mathcal{M}(\alpha, \beta)/\mathbb{R}$ .

The boundary operator of the Floer chain complex is given by

$$\partial\alpha = \sum_{\beta \in \text{flat}(\mathcal{A}), \text{ind}(\alpha, \beta) = 1} \#\hat{\mathcal{M}}(\alpha, \beta) \cdot \beta$$

The choice of orientation is another subtle point. Besides, we also need the condition that the moduli space consists of finite points when  $\text{ind}(\alpha, \beta) = 1$ . The proof that the chain complex is well-defined follows the same basic scheme as in Morse homology but is a lot more strenuous.



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