

PROLONGATION ON CONTACT MANIFOLDS

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ABSTRACT. On contact manifolds we describe a notion of (contact) finite-type for linear partial differential operators satisfying a natural condition on their leading terms. A large class of linear differential operators are of finite-type in this sense but are not well understood by currently available techniques. We resolve this in the following sense. For any such D we construct a partial connection ∇_H on a (finite rank) vector bundle with the property that sections in the null space of D correspond bijectively, and via an explicit map, with sections parallel for the partial connection. It follows that the solution space of D is finite dimensional and bounded by the corank of the holonomy algebra of ∇_H . The treatment is via a uniform procedure, even though in most cases no normal Cartan connection is available.

1. INTRODUCTION

The prolongations of a k^{th} order linear differential operator between vector bundles arise by differentiating the given operator $D : E \rightarrow F$, and forming a new system comprising D along with auxiliary operators that capture some of this derived data. To exploit this effectively it is crucial to determine what part of this information should be retained, and then how best to manage it. With this understood, for many classes of operators the resulting prolonged operator can expose key properties of the original differential operator and its equation. Motivated by questions related to integrability and deformations of structure, a theory of overdetermined equations and prolonged systems was developed during the 1950s and 1960s by Goldschmidt, Spencer, and others [2, 17]. Generally, results in these works are derived abstractly using jet bundle theory, and are severely restricted in the sense that they apply most readily to differential operators satisfying involutivity conditions. These features mean the theory can be difficult to apply.

In the case that the given partial differential operator $D : E \rightarrow F$, has surjective symbol there is an effective algorithmic approach to this problem. The prolongations are constructed from the leading symbol $\sigma(D) : \odot^k \Lambda^1 \otimes E \rightarrow F$, where $\odot^k \Lambda^1$ is the bundle of symmetric covariant tensors on M of rank k . At a point of M , denoting by K the kernel of $\sigma(D)$, the spaces $K^\ell = (\odot^\ell \Lambda^1 \otimes K) \cap (\odot^{k+\ell} \Lambda^1 \otimes E)$, $\ell \geq 0$, capture spaces of new variables to be introduced, and the system closes up if $K^\ell = 0$ for

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sufficiently large ℓ . In this case the operator D is said to be of *finite-type* (following [17]). The equation is *regular* if the spaces K^ℓ have constant rank over the manifold. The leading symbol determines whether or not an equation is of finite-type and/or regular. If it is both, then the final prolonged system is a linear connection on $\bigoplus_{\ell=0}^{k-1} (\bigodot^k \Lambda^1 \otimes E) \oplus \bigoplus_{\ell=0}^{\infty} K^\ell$ with the property that its covariant constant sections are in 1–1 correspondence with solutions of $D\sigma = 0$. In general, prolonged systems are complicated. In [1] Kostant’s algebraic Hodge theory [11] led to an explicit and uniform treatment of prolongations for a large class of overdetermined partial differential equations (in fact, semilinear equations are also treated in [1]).

On a connected manifold, a solution of a finite-type differential operator is evidently determined by its finite jet at any point, that is by a finite part of its Taylor series data. However on contact manifolds a large class of differential operators that have the latter property nevertheless fail to be of finite-type, in the sense above. For example even the operation of taking the differential of a function in contact directions is not of finite-type. This signals that the general prolongation theory is not adequate. If the underlying manifold has a structure from the class of parabolic geometries [4, §4.2] (e.g. hypersurface type CR geometry) then, for a special class of natural operators, the methods of the Bernstein-Gelfand-Gelfand machinery [3, 5] may be applied. However, these methods are not applicable in general.

Drawing on Tanaka’s notion of a filtered manifold, Morimoto initiated a programme for studying differential equations on contact manifolds and their generalisations [12] via a notion of weighted jet bundles that are adapted to the structure. This provides a formal framework for treating these structures and, in particular, leads to a notion of weighted finite-type. For example, using this notion of weighted jets, Neusser [13] has recently and usefully adapted to the filtered manifold setting, some tools of Goldschmidt [7] sufficient to show quite easily that the solution space of a weighted finite-type system is finite-dimensional.

Despite this progress a significant gap remains. Ideally we would have a uniform approach that, when applied to any specific equation from the class, yields an explicit prolonged system from which obstructions to solution can be calculated directly. In this article we provide a solution to this problem. In particular we develop a new prolongation theory for contact structures which, on the one hand, maintains a transparent and useable link with the weighted jet picture of [12, 13], and which on the other hand is effective and practically applicable. The main result is as follows. Corresponding to weighted jets, on a contact manifold there is a notion of contact symbol. For (suitably regular) partial differential operators $D : E \rightarrow F$ with surjective contact symbol we describe an explicit iterative scheme for treating the contact prolongation problem. The operator is said to be of (*contact*) *finite-type* if the prolongations stabilise after a finite number of steps, and in this case we obtain a partial connection on the prolonged system with the property that its parallel sections correspond 1–1 and explicitly to solutions of D . This partial connection canonically promotes to a connection on the same bundle. It follows that the dimension of the solution space for D is bounded by the rank of the bundle supporting this partial connection and the existence of solutions is equivalent to a rank reducing holonomy

reduction of the connection in the obvious way. Since the connection is constructed concretely it is possible directly to use this to construct explicit curvature obstructions to solutions of the D equation.

For first order operators, our main result may be stated as follows. Let H denote the contact distribution and Λ_H^1 its dual. There is a canonical surjection $\Lambda^1 \rightarrow \Lambda_H^1$. A first order differential operator $E \rightarrow F$ is said to be *compatible* with the contact structure if and only if its symbol $\Lambda^1 \otimes E \rightarrow F$ factors through $\Lambda^1 \otimes E \rightarrow \Lambda_H^1 \otimes E$. It means that the operator D only differentiates in the contact directions. In this case the resulting homomorphism $\Lambda_H^1 \otimes E \rightarrow F$ is called the *partial symbol* of D . We shall suppose that it is surjective and write $K_H \subseteq \Lambda_H^1 \otimes E$ for its kernel. There are canonical subbundles of $S_\perp^\ell \subseteq \bigotimes^\ell \Lambda_H^1$ defined via the *Levi form*, as follows. In terms of a locally chosen contact form ϕ , the Levi form may be regarded as $d\phi|_H$ and, from this point of view, is well-defined up to scale. Adopting Penrose's *abstract index notation* [15] for sections of H and its associated bundles, let us write L_{ab} for the Levi form. Then, it is clear that $S_\perp^\ell \subseteq \bigotimes^\ell \Lambda_H^1$ defined as

$$(1.1) \quad S_\perp^\ell \equiv \left\{ \underbrace{X_{abcde\dots f}}_{\ell \text{ indices}} \text{ s.t. } \begin{array}{l} X_{[ab]cde\dots f} = X_{c[ab]de\dots f} = X_{cd[ab]e\dots f} \\ = \dots = L_{ab}Y_{cde\dots f} \end{array} \text{ for some } Y_{cde\dots f} \right\}$$

does not see the scale of L_{ab} (enclosing a pair of indices in square brackets means to take the skew part in those indices). Certainly, $S_\perp^\ell \supseteq \bigodot^\ell \Lambda_H^1$ but, in fact, is strictly bigger (3.13) for $\ell \geq 2$. Now we define

$$(1.2) \quad K_H^\ell \equiv (S_\perp^\ell \otimes K_H) \cap (S_\perp^{\ell+1} \otimes E), \quad \text{for } \ell \geq 0.$$

Theorem 1.1. *Suppose that K_H^ℓ are vector bundles for all ℓ and that $K_H^\ell = 0$ for ℓ sufficiently large. Then there is a connection on the bundle $\mathbb{T} \equiv E \oplus \bigoplus_{\ell \geq 0} K_H^\ell$ so that the projection $\mathbb{T} \rightarrow E$ induces an isomorphism between the covariant constant sections of $\mathbb{T}$ and the solution space $\{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}$.*

Following a simplified treatment of the general prolongation theory for first order operators in Section 2, Theorem 1.1 is proved in Section 3 (cf. Theorem 3.17). Then, following a simplified treatment of the general prolongation theory for higher order operators in Section 4, Theorem 1.1 is generalised to higher order operators on contact manifolds in Section 5. The construction is reasonably straightforward in dimensions $2n+1$ for $n \geq 2$. Theorem 5.1 is used to replace the given operator with an equivalent contact compatible first order prolonged system. It is used to construct a first order contact compatible differential operator with surjective contact symbol, at which point we are able to apply an iterative procedure developed for first order operators in proving Theorem 3.17. For 3-dimensional contact structures, however, one expects rather different phenomena to occur [14, 16], and this is indeed the case. Nevertheless, Proposition 5.3 provides a more general iterative scheme, and finally the main result takes the same form in all dimensions. This is Theorem 5.4. For these theorems to be useful, of course, one needs to compute spaces of the form (1.2) (and more generally (5.3)). Although this is, in principle, a simple matter of multilinear algebra, in practise these spaces are difficult to identify. In particular, it would be useful to know some *a priori* bounds on their dimension so that the dimension of the

solution space $\{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}$ can thereby be bounded. For a large class of geometrically arising linear differential operators on contact manifolds, all this is possible and Section 6 is devoted to the computation of the spaces (1.2) and (5.3) for these operators. It reduces to the computation of certain Lie algebra cohomologies for the Heisenberg algebra. This cohomology is, in turn, already known as a special case of Kostant's algebraic Bott-Borel-Weil Theorem [11] and the resulting bounds on the dimension of the solution space are sharp.

2. GENERAL PROLONGATION FOR FIRST ORDER OPERATORS

Suppose $D : E \rightarrow F$ is a first order linear differential operator and suppose that its symbol $\Lambda^1 \otimes E \rightarrow F$ is surjective. Write π for this symbol and K for its kernel. Define the vector bundle E' as the kernel of $D : J^1 E \rightarrow F$. We obtain a commutative diagram

$$(2.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & K & \rightarrow & E' & \rightarrow & E \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \Lambda^1 \otimes E & \rightarrow & J^1 E & \rightarrow & E \rightarrow 0 \\ & & \pi \downarrow & & D \downarrow & & \\ & & F & = & F & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns.

Lemma 2.1. *We can find a connection ∇ on E so that D is the composition*

$$(2.2) \quad E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\pi} F.$$

Proof. From diagram (2.1), a splitting of

$$(2.3) \quad 0 \rightarrow K \rightarrow E' \rightarrow E \rightarrow 0$$

gives rise to a splitting of

$$0 \rightarrow \Lambda^1 \otimes E \rightarrow J^1 E \rightarrow E \rightarrow 0.$$

Interpreted as a connection on E , it has the required property. In fact, the connections with this property correspond precisely to splittings of (2.3). $\square$

Let us fix a splitting of (2.3) and therefore a connection on E in accordance with Lemma 2.1. Having done this, the following theorem and its proof describe the crucial step in classical prolongation.

Theorem 2.2. *There is a first order differential operator*

$$\tilde{D} : E' = \begin{array}{c} E \\ \oplus \\ K \end{array} \longrightarrow \begin{array}{c} \Lambda^1 \otimes E \\ \oplus \\ \Lambda^2 \otimes E \end{array}$$

so that the canonical projection $E' \rightarrow E$ induces an isomorphism

$$(2.4) \quad \{\Sigma \in \Gamma(E') \text{ s.t. } \tilde{D}\Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

Proof. Define $\tilde{D}$ by

$$(2.5) \quad \begin{bmatrix} \sigma \\ \mu \end{bmatrix} \xrightarrow{\tilde{D}} \begin{bmatrix} \nabla\sigma - \mu \\ \nabla\mu - \kappa\sigma \end{bmatrix},$$

where ∇ acting on μ denotes the differential operator $\Lambda^1 \otimes E \rightarrow \Lambda^2 \otimes E$ induced by the connection $\nabla : E \rightarrow \Lambda^1 \otimes E$ and $\kappa : E \rightarrow \Lambda^2 \otimes E$ denotes the curvature of ∇ . From (2.2) it is clear that $D\sigma = 0$ if and only if

$$\nabla\sigma = \mu \quad \text{for some } \mu \in \Gamma(K).$$

Having thus rewritten $D\sigma = 0$, applying the differential operator $\nabla : \Lambda^1 \otimes E \rightarrow \Lambda^2 \otimes E$ to both sides of this equation implies that $\nabla\mu = \kappa\sigma$. In other words, this component of $\tilde{D}\Sigma$ is an optional extra arising as an obvious compatibility requirement. $\square$

Remark. Actually, there is no need to choose a connection in order to define $\tilde{D}$. Following Goldschmidt [7, Proposition 3], the target bundle can be invariantly defined as $J^1 J^1 E / J^2 E$ and $\tilde{D}$ may then be obtained by restricting the tautological first order differential operator $J^1 E \rightarrow J^1 J^1 E / J^2 E$ to $E' \subseteq J^1 E$. The main reason for choosing ∇ is that it makes prolongation into an effective and computable procedure.

Maintaining our chosen splitting of (2.3) and induced connection, it is evident that the symbol of $\tilde{D}$ is

$$(2.6) \quad \begin{array}{ccc} \Lambda^1 \otimes E & & \Lambda^1 \otimes E \\ \oplus & \begin{bmatrix} \text{Id} & 0 \\ 0 & \partial \end{bmatrix} & \oplus \\ \Lambda^1 \otimes K & \xrightarrow{\quad} & \Lambda^2 \otimes E \end{array},$$

where ∂ is the composition

$$\Lambda^1 \otimes K \hookrightarrow \Lambda^1 \otimes \Lambda^1 \otimes E \xrightarrow{\wedge \otimes \text{Id}} \Lambda^2 \otimes E.$$

Let us suppose that ∂ has constant rank, write F' for the subbundle

$$\begin{array}{ccc} \Lambda^1 \otimes E & & \Lambda^1 \otimes E \\ \oplus & \subseteq & \oplus \\ \partial(\Lambda^1 \otimes K) & & \Lambda^2 \otimes E \end{array},$$

and define $D' : E' \rightarrow F'$ by

$$\begin{bmatrix} \sigma \\ \mu \end{bmatrix} \xrightarrow{D'} \begin{bmatrix} \nabla\sigma - \mu \\ \delta(\nabla\mu - \kappa\sigma) \end{bmatrix},$$

where δ is an arbitrary splitting of $\partial(\Lambda^1 \otimes K) \hookrightarrow \Lambda^2 \otimes E$, equivalently an arbitrary choice of complementary bundle.

Theorem 2.3. *The canonical projection $E' \rightarrow E$ induces an isomorphism*

$$\{\Sigma \in \Gamma(E') \text{ s.t. } D'\Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

Proof. We follow exactly the same reasoning as for Theorem 2.2. The only difference is that the $\delta(\nabla\mu - \kappa\sigma)$ records only some part of the optional first order differential consequences of the equation $\nabla\sigma = \mu$. $\square$

Remark. Although the bundle F' is canonically defined just from $\pi : \Lambda^1 \otimes E \rightarrow F$, the construction of D' does involve a choice of splitting δ . In practise, there is often a natural choice for δ but, from the point of view adopted in this article, the main reason for introducing D' is that its symbol is surjective by construction.

From (2.6), the kernel of the symbol of D' is precisely $\ker \partial \subseteq \Lambda^1 \otimes K$. Equivalently, it is the intersection

$$K' \equiv (\Lambda^1 \otimes K) \cap (\odot^2 \Lambda^1 \otimes E)$$

inside $\Lambda^1 \otimes \Lambda^1 \otimes E$. If K' is trivial, then D' is a connection. If not, we can iterate this procedure, at the next stage identifying

$$(2.7) \quad K'' \equiv (\odot^2 \Lambda^1 \otimes K) \cap (\odot^3 \Lambda^1 \otimes E)$$

as the kernel of the symbol of D'' . The details are left to the reader. Eventually, if

$$(\odot^\ell \Lambda^1 \otimes K) \cap (\odot^{\ell+1} \Lambda^1 \otimes E) = 0 \quad \text{for } \ell \text{ sufficiently large,}$$

then D is said to be of *finite-type* in the sense of Spencer [17] and we have constructed a vector bundle with connection whose covariant constant sections are in one-to-one correspondence with the solutions of $D\sigma = 0$.

3. CONTACT PROLONGATION FOR FIRST ORDER OPERATORS

Let us firstly establish some notation. We shall denote the contact distribution by H and its annihilator line-bundle $H^\perp \hookrightarrow \Lambda^1$ by L . We have a short exact sequence

$$0 \rightarrow L \rightarrow \Lambda^1 \rightarrow \Lambda_H^1 \rightarrow 0,$$

which determines the contact structure. The de Rham sequence begins

$$(3.1) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & L & & \Lambda_H^1 \otimes L & \\ & & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \dots \\ & & & & \downarrow & & \downarrow & & \\ & & & & \Lambda_H^1 & & \Lambda_H^2 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where Λ_H^2 denotes $\Lambda^2(\Lambda_H^1)$ and the columns are exact. Let us denote by $\mathcal{L}$, the composition

$$L \rightarrow \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \Lambda_H^2.$$

It is a homomorphism of vector bundles. We shall refer to it as the *Levi form* and the contact condition implies that $\mathcal{L}$ is injective. For an arbitrary vector bundle E with connection ∇ it is easily verified that the composition

$$(3.2) \quad L \otimes E \rightarrow \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E \rightarrow \Lambda_H^2 \otimes E$$

is simply $\mathcal{L} \otimes \text{Id}$. Let us denote by d_H the composition $\Lambda^0 \xrightarrow{d} \Lambda^1 \rightarrow \Lambda_H^1$ and, following Pansu [14], say that a differential operator $\nabla_H : E \rightarrow \Lambda_H^1 \otimes E$ is a *partial connection* if and only if

$$(3.3) \quad \nabla_H(f\sigma) = f\nabla_H\sigma + d_H f \otimes \sigma \quad \text{for any smooth function } f \text{ and } \sigma \in \Gamma(E).$$

If ∇ is a connection on E , then the composition $E \xrightarrow{\nabla} \Lambda^1 \otimes E \rightarrow \Lambda_H^1 \otimes E$ is a partial connection.

The operator $d_H : \Lambda^0 \rightarrow \Lambda_H^1$ on a contact manifold is the natural replacement for the exterior derivative $d : \Lambda^0 \rightarrow \Lambda^1$, the point being that, although d_H sees only the contact directions, these operators have the same kernel. With reference to the diagram (3.1), if $d_H f = 0$ then df is actually a section of L . But then $d^2 = 0$ implies that $\mathcal{L}df = 0$ and then $df = 0$ because $\mathcal{L}$ is supposed to be injective. There is also a replacement for $d : \Lambda^1 \rightarrow \Lambda^2$, defined as follows. Again with reference to diagram (3.1), for $\omega \in \Gamma(\Lambda_H^1)$, lift to $\tilde{\omega} \in \Gamma(\Lambda^1)$ and project $d\tilde{\omega} \in \Gamma(\Lambda^2)$ into $\Gamma(\Lambda_H^2)$. Of course, this is ill-defined owing to the choice of lift but the freedom so entailed is precisely in the image of $\mathcal{L}$ in Λ_H^2 . Thus, we obtain a well-defined first order differential operator

$$d_H : \Lambda_H^1 \rightarrow \Lambda_{H\perp}^2 \equiv \Lambda_H^2 / \mathcal{L}(L).$$

Furthermore, in dimension 5 or more a little diagram chasing in (3.1) and injectivity of $\text{Id} \wedge \mathcal{L} : \Lambda_H^1 \otimes L \rightarrow \Lambda_H^3$ shows that

$$(3.4) \quad 0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \xrightarrow{d_H} \Lambda_H^1 \xrightarrow{d_H} \Lambda_{H\perp}^2$$

is locally exact just as the de Rham sequence is. It is the first part of the Rumin complex [16]. In dimension 3, however, the Levi form $\mathcal{L} : L \rightarrow \Lambda_H^2$ is an isomorphism so (3.4) breaks down. For the remainder of this section we shall suppose that our contact manifold has dimension at least 5, postponing the 3-dimensional case until §3.1.

The arguments in dimension 5 or more closely follow the general procedure outlined in §2. Suppose $D : E \rightarrow F$ is a first order linear differential operator and that D only differentiates in the contact directions. Precisely, we shall suppose that D is *compatible* with the contact structure meaning that its symbol factors as

$$(3.5) \quad \Lambda^1 \otimes E \rightarrow \Lambda_H^1 \otimes E \xrightarrow{\pi_H} F$$

and, in this case, refer to π_H as the *partial* symbol of D . As in the general case, we shall suppose that π_H is surjective and write K_H for its kernel. Factoring (2.1) by

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & L \otimes E & \rightarrow & L \otimes E & \rightarrow & 0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & L \otimes E & \rightarrow & L \otimes E & \rightarrow & 0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & = & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & K_H & \rightarrow & E'_H & \rightarrow & E \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 (3.6) & 0 \rightarrow & \Lambda_H^1 \otimes E & \rightarrow & J_H^1 E & \rightarrow & E \rightarrow 0 \\
 & & \pi_H \downarrow & & D \downarrow & & \\
 & & F & = & F & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

with exact rows and columns and, in particular, hereby define E'_H . A splitting of

$$0 \rightarrow K_H \rightarrow E'_H \rightarrow E \rightarrow 0$$

gives rise to a partial connection ∇_H such that $D = \pi_H \circ \nabla_H$. Any partial connection $\nabla_H : E \rightarrow \Lambda_H^1 \otimes E$ gives rise to an operator

$\nabla_H : \Lambda_H^1 \otimes E \rightarrow \Lambda_{H\perp}^2 \otimes E$ characterised by $\nabla_H(\omega \otimes \sigma) = d_H \omega \otimes \sigma - \omega \wedge \nabla_H \sigma \bmod \mathcal{L}(L)$, mimicking the case of ordinary connections. From the partial Leibniz rule (3.3), it follows that the composition

$$E \xrightarrow{\nabla_H} \Lambda_H^1 \otimes E \xrightarrow{\nabla_H} \Lambda_{H\perp}^2 \otimes E$$

is a homomorphism of vector bundles, which we shall denote by κ_H (being the natural *curvature* of a partial connection [14]). Parallel to Theorem 2.2 we have:–

Theorem 3.1. *There is a first order differential operator*

$$\tilde{D}_H : E'_H = \begin{array}{c} E \\ \oplus \\ K_H \end{array} \longrightarrow \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_{H\perp}^2 \otimes E \end{array}$$

so that the canonical projection $E'_H \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(E'_H) \text{ s.t. } \tilde{D}_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

Proof. Define $\tilde{D}_H$ by

$$(3.7) \quad \begin{bmatrix} \sigma \\ \mu \end{bmatrix} \xrightarrow{\tilde{D}_H} \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H \mu - \kappa_H \sigma \end{bmatrix}$$

and argue as before. $\square$

Notice that $\tilde{D}_H$ is again compatible with the contact structure. Indeed, the symbol of $\tilde{D}_H$ factors through

$$(3.8) \quad \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_H^1 \otimes K_H \end{array} \xrightarrow{\begin{bmatrix} \text{Id} & 0 \\ 0 & \partial_H \end{bmatrix}} \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_{H^\perp}^2 \otimes E \end{array},$$

where ∂_H is the composition

$$\Lambda_H^1 \otimes K_H \hookrightarrow \Lambda_H^1 \otimes \Lambda_H^1 \otimes E \xrightarrow{\wedge \otimes \text{Id}} \Lambda_H^2 \otimes E \rightarrow \Lambda_{H^\perp}^2 \otimes E,$$

which we shall suppose to be of constant rank. Again shadowing the general case, let us write F'_H for the subbundle

$$\begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \partial_H(\Lambda_H^1 \otimes K_H) \end{array} \subseteq \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_{H^\perp}^2 \otimes E \end{array},$$

choose a splitting δ_H of $\partial_H(\Lambda_H^1 \otimes K_H) \hookrightarrow \Lambda_{H^\perp}^2 \otimes E$, and define $D'_H : E'_H \rightarrow F'_H$ by

$$\begin{bmatrix} \sigma \\ \mu \end{bmatrix} \xrightarrow{D'_H} \begin{bmatrix} \nabla_H \sigma - \mu \\ \delta_H(\nabla_H \mu - \kappa_H \sigma) \end{bmatrix}.$$

The counterpart to Theorem 2.3 follows immediately:–

Theorem 3.2. *The canonical projection $E'_H \rightarrow E$ induces an isomorphism*

$$\{\Sigma \in \Gamma(E'_H) \text{ s.t. } D'_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

The operator D'_H is compatible with the contact structure and, by design, has surjective symbol. Thus, we are in a position to iterate this construction. We begin by observing from (3.8) that the kernel of the partial symbol of D'_H is

$$\ker \partial_H : \Lambda_H^1 \otimes K_H \rightarrow \Lambda_{H^\perp}^2 \otimes E,$$

which may be viewed as the intersection

$$K'_H \equiv (\Lambda_H^1 \otimes K_H) \cap (S_\perp^2 \otimes E) \quad \text{inside } \Lambda_H^1 \otimes \Lambda_H^1 \otimes E$$

where

$$(3.9) \quad S_\perp^2 = \odot^2 \Lambda_H^1 \oplus \mathcal{L}(L) = \left\{ \phi_{ab} \in \otimes^2 \Lambda_H^1 \text{ s.t. } \begin{array}{l} \phi_{ab} = P_{ab} + QL_{ab} \\ \text{where } P_{ab} = P_{(ab)} \end{array} \right\},$$

where L_{ab} is (a representative of) the Levi form. That we are confined to 5 or more dimensions also shows up algebraically as follows. Let us write $2n+1$ for the dimension of our contact manifold.

Lemma 3.3. *If $n \geq 2$, then*

$$(\Lambda_H^1 \otimes S_\perp^2) \cap (S_\perp^2 \otimes \Lambda_H^1) = \left\{ \phi_{abc} \in \bigotimes^3 \Lambda_H^1 \text{ s.t. } \begin{array}{l} \phi_{abc} = P_{abc} + Q_a L_{bc} + Q_b L_{ac} + Q_c L_{ab} \\ \text{where } P_{abc} = P_{(abc)} \end{array} \right\}.$$

Proof. According to elementary representation theory for $\mathrm{Sp}(2n, \mathbb{R})$, we can uniquely decompose $\phi_{abc} \in \Lambda_H^1 \otimes S_\perp^2$ as

$$(3.10) \quad \phi_{abc} = P_{abc} + R_{abc} + T_b L_{ac} + T_c L_{ab} + Q_a L_{bc},$$

where

$$P_{abc} = P_{(abc)} \quad R_{abc} = R_{a(bc)} \quad R_{(abc)} = 0 \quad L^{ab} R_{abc} = 0,$$

and $L^{ab} L_{ac} = \delta^b_c$, the Kronecker delta. The image

$$\phi_{abc} - \phi_{bac} - \frac{1}{n} L^{de} \phi_{dec} L_{ab}$$

of this element in $\Lambda_{H\perp}^2 \otimes \Lambda_H^1$ is

$$(3.11) \quad R_{abc} - R_{bac} + (Q_a - T_a) L_{bc} - (Q_b - T_b) L_{ac} + \frac{1}{n} (Q_c - T_c) L_{ab}$$

and transvecting with L^{bc} gives

$$\frac{(2n+1)(n-1)}{n} (Q_a - T_a).$$

Therefore, if ϕ_{abc} is also in $S_\perp^2 \otimes \Lambda_H^1$, then it follows that $T_a = Q_a$, from (3.11) that $R_{abc} = 0$, and from (3.10) the stated result. $\square$

Similarly, if we inductively define

$$(3.12) \quad S_\perp^\ell = (\Lambda_H^1 \otimes S_\perp^{\ell-1}) \cap (S_\perp^{\ell-1} \otimes \Lambda_H^1), \quad \forall \ell \geq 3,$$

or intrinsically as in (1.1), then as $\mathrm{Sp}(2n, \mathbb{R})$ -bundles

$$(3.13) \quad S_\perp^\ell \cong \odot^\ell \Lambda_H^1 \oplus \odot^{\ell-2} \Lambda_H^1 \oplus \odot^{\ell-4} \Lambda_H^1 \oplus \dots$$

with explicit decompositions such as

$$(3.14) \quad S_\perp^4 = \left\{ \begin{array}{l} P_{abcd} + Q_{ab} L_{cd} + Q_{ac} L_{bd} + Q_{ad} L_{bc} + Q_{bc} L_{ad} + Q_{bd} L_{ac} + Q_{cd} L_{ab} \\ + R L_{ab} L_{cd} + R L_{ac} L_{bd} + R L_{ad} L_{bc} \\ \text{where } P_{abcd} = P_{(abcd)} \text{ and } Q_{ab} = Q_{(ab)} \end{array} \right\}.$$

The counterpart to (2.7) is

$$(3.15) \quad K_H'' \equiv (S_\perp^2 \otimes K_H) \cap (S_\perp^3 \otimes E)$$

as the symbol of D_H'' and, more generally, if

$$(S_\perp^\ell \otimes K_H) \cap (S_\perp^{\ell+1} \otimes E) = 0 \quad \text{for } \ell \text{ sufficiently large,}$$

then, by iteration of the construction leading to Theorem 3.2, we may construct a vector bundle $\mathbb{T}$ with partial connection ∇_H such that

$$\{\Sigma \in \Gamma(\mathbb{T}) \text{ s.t. } \nabla_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

It particular, in this case it is clear that the solution space of D is finite-dimensional with dimension bounded by the rank of $\mathbb{T}$, namely

$$\dim E + \dim K_H + \sum_{\ell \geq 1} \dim ((S_\perp^\ell \otimes K_H) \cap (S_\perp^{\ell+1} \otimes E)).$$

The details are left to the reader.

3.1. The 3-dimensional case. On 3-dimensional contact manifolds the Levi form $\mathcal{L} : L \rightarrow \Lambda_H^2$ is an isomorphism and so (3.4) breaks down. The Rumin complex [16] provides a perfectly satisfactory replacement as follows.

Lemma 3.4. *On a 3-dimensional contact manifold, there is a canonically defined second order differential operator $d_H^{(2)} : \Lambda_H^1 \rightarrow \Lambda_H^1 \otimes L$ so that*

$$(3.16) \quad 0 \rightarrow \mathbb{R} \rightarrow \Lambda^0 \xrightarrow{d_H} \Lambda_H^1 \xrightarrow{d_H^{(2)}} \Lambda_H^1 \otimes L$$

is locally exact just as the de Rham sequence is.

Proof. With reference to diagram (3.1), if ω is a local section of Λ_H^1 , choose an arbitrary lift $\tilde{\omega}$ to Λ^1 and consider $\tilde{\omega} - \mathcal{L}^{-1} q d\tilde{\omega}$, where q is the natural projection $\Lambda^2 \rightarrow \Lambda_H^2$. By diagram chasing, this is independent of choice of $\tilde{\omega}$ and canonically defines a differential operator $\Lambda_H^1 \rightarrow \Lambda^1$ splitting the natural projection $\Lambda^1 \rightarrow \Lambda_H^1$. By design, it also has the property that the composition $d(\tilde{\omega} - \mathcal{L}^{-1} q d\tilde{\omega})$ actually takes values in $\Lambda_H^1 \otimes L$. This defines $d_H^{(2)}$ and further diagram chasing ensures that (3.16) is locally exact. $\square$

Just as the de Rham sequence couples with any connection on a vector bundle, so (3.16) couples with any partial connection. To see this we can proceed as follows. Firstly, some linear algebra. Not only is the Levi form $\mathcal{L} : L \rightarrow \Lambda_H^2$ injective, but also its range consists of non-degenerate forms. If, as in (3.9), we choose L_{ab} in the range of $\mathcal{L}$ and, as in the proof of Lemma 3.3, write L^{ab} for its inverse, then we obtain a complement

$$\{\omega_{ab} \in \Lambda_H^2 \text{ s.t. } L^{ab}\omega_{ab} = 0\}$$

to the range of $\mathcal{L}$, independent of the choice of L_{ab} . We may identify this complement with $\Lambda_{H\perp}^2$. Let us write $s : \Lambda_H^2 \rightarrow L$ for the canonical splitting of $\mathcal{L}$ so obtained.

Proposition 3.5. *Suppose that $\nabla_H : E \rightarrow \Lambda_H^1 \otimes E$ is a partial connection on a contact manifold of arbitrary dimension. Then ∇_H extends to a connection ∇ , uniquely characterised by the vanishing of the composition*

$$(3.17) \quad E \xrightarrow{\kappa} \Lambda^2 \otimes E \xrightarrow{q \otimes \text{Id}} \Lambda_H^2 \otimes E \xrightarrow{s \otimes \text{Id}} L \otimes E,$$

where κ is the curvature of ∇ . Moreover, for this connection $\nabla_H \sigma = 0 \iff \nabla \sigma = 0$.

Proof. Pick an arbitrary extension ∇ of ∇_H . Any homomorphism $\Phi : E \rightarrow \Lambda^1 \otimes E$ gives rise another connection $\hat{\nabla} = \nabla + \Phi$ with curvature $\hat{\kappa} = \kappa + \nabla \Phi - \Phi \wedge \Phi$, where $\nabla : \Lambda^1 \otimes \text{End}(E) \rightarrow \Lambda^2 \otimes \text{End}(E)$ is the natural differential operator derived from the induced connection $\nabla : \text{End}(E) \rightarrow \Lambda^1 \otimes \text{End}(E)$ and $\Phi \wedge \Phi$ is the composition

$$E \xrightarrow{\Phi} \Lambda^1 \otimes E \xrightarrow{\text{Id} \otimes \Phi} \Lambda^1 \otimes \Lambda^1 \otimes E \xrightarrow{\wedge \otimes \text{Id}} \Lambda^2 \otimes E.$$

If $\hat{\nabla}$ is to extend ∇_H , however, then Φ must have range in $L \otimes E \subset \Lambda^1 \otimes E$. In this case, the term $\Phi \wedge \Phi$ does not arise in the formula for $\hat{\kappa}$. Also recall (3.2) that the composition

$$L \otimes \text{End}(E) \hookrightarrow \Lambda^1 \otimes \text{End}(E) \xrightarrow{\nabla} \Lambda^2 \otimes \text{End}(E) \xrightarrow{q \otimes \text{Id}} \Lambda_H^2 \otimes \text{End}(E)$$

is always $\mathcal{L} \otimes \text{Id}$. Hence, the curvature in the contact directions

$$E \xrightarrow{\kappa} \Lambda^2 \otimes E \xrightarrow{q \otimes \text{Id}} \Lambda_H^2 \otimes E$$

of a connection ∇ extending a given partial connection ∇_H is determined up to

$$(\mathcal{L} \otimes \text{Id})\Phi, \text{ for } \Phi : E \rightarrow L \otimes E \text{ an arbitrary homomorphism.}$$

Thus, its further composition with $\Lambda_H^2 \otimes E \xrightarrow{s \otimes \text{Id}} L \otimes E$, as in (3.17), is determined up to Φ and may be precisely eliminated. For the last statement, it is clear that $\nabla\sigma = 0 \implies \nabla_H\sigma = 0$. Conversely, if $\nabla_H\sigma = 0$ then $\nabla\sigma$ is a section of $L \otimes E$ whence $(\mathcal{L} \otimes \text{Id})\nabla\sigma = (q \otimes \text{Id})\kappa\sigma$. The vanishing of (3.17) now implies that

$$\nabla\sigma = (s \otimes \text{Id})(\mathcal{L} \otimes \text{Id})\nabla\sigma = (s \otimes \text{Id})(q \otimes \text{Id})\kappa\sigma = 0,$$

as required. $\square$

Corollary 3.6. *Suppose $\nabla_H : E \rightarrow \Lambda_H^1 \otimes E$ is a partial connection on a 3-dimensional contact manifold. Then ∇_H extends to a unique connection ∇ characterised by being flat in the contact directions, i.e.*

$$(3.18) \quad \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma = \nabla_{[X,Y]} \sigma$$

for all $X, Y \in \Gamma(H)$ and $\sigma \in \Gamma(E)$.

Proof. In 3-dimensions $\mathcal{L}$ is an isomorphism and $s = \mathcal{L}^{-1}$. Equation (3.18) is an explicit rendering of the vanishing curvature (3.17). $\square$

Now, to couple (3.16) with ∇_H we simply extend to a full connection ∇ on E in accordance with Corollary 3.6. Then, bearing in mind that the composition (3.2) is simply $\mathcal{L} \otimes \text{Id}$, the construction just given in the proof of Lemma 3.4 goes through almost unchanged. This can be seen by chasing the following diagram

$$(3.19) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & L \otimes E & & \Lambda_H^1 \otimes L \otimes E & \\ & & & \downarrow & & \downarrow & \\ 0 & \rightarrow & E & \xrightarrow{\nabla} & \Lambda^1 \otimes E & \xrightarrow{\nabla} & \Lambda^2 \otimes E \xrightarrow{\nabla} \dots \\ & & \searrow \nabla_H & & \downarrow & & \downarrow \\ & & & & \Lambda_H^1 \otimes E & & \Lambda_H^2 \otimes E \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

obtained by coupling (3.1) with the connection ∇ provided by Corollary 3.6. Let us write $\nabla_H^{(2)}$ for the resulting operator. Of course, it is no longer the case that the composition

$$(3.20) \quad E \xrightarrow{\nabla_H} \Lambda_H^1 \otimes E \xrightarrow{\nabla_H^{(2)}} \Lambda_H^1 \otimes L \otimes E$$

vanishes. Instead, since the connection ∇ is characterised by having its curvature compose with $\Lambda^2 \otimes E \rightarrow \Lambda_H^2 \otimes E$ to give zero, it follows immediately from (3.19) that the composition (3.20) is precisely this curvature, which we shall now write as κ_H .

(The contrast between 3-dimensions and higher regarding the notion of curvature of a partial connection is also noted and explored in [14].)

We may now establish a counterpart to Theorem 3.1 in the 3-dimensional setting.

Theorem 3.7. *There is a differential operator*

$$\tilde{D}_H : E'_H = \begin{matrix} E \\ \oplus \\ K_H \end{matrix} \longrightarrow \begin{matrix} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_H^1 \otimes L \otimes E \end{matrix}$$

so that the canonical projection $E'_H \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(E'_H) \text{ s.t. } \tilde{D}_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

Proof. Define $\tilde{D}_H$ by

$$(3.21) \quad \begin{bmatrix} \sigma \\ \mu \end{bmatrix} \xrightarrow{\tilde{D}_H} \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H^{(2)} \mu - \kappa_H \sigma \end{bmatrix},$$

noting that $\nabla_H \sigma = \mu \implies \nabla_H^{(2)} \mu = \kappa_H \sigma$ by applying $\nabla_H^{(2)}$. Apart from this natural adjustment, the remainder of the proof is as for Theorem 2.2. $\square$

There is, of course, a significant difference between Theorems 3.1 and 3.7 stemming from the significantly different behaviour of the Rumin complex. The operator $\tilde{D}_H$ in dimension 5 and higher is again first order. But $\tilde{D}_H$ in Theorem 3.7 is second order. Instead, we would like a first order prolonged operator and an analogue of Theorem 3.2.

To remedy this we may proceed as follows. Firstly, we shall present an argument involving special local coördinates and then we shall indicate how to remove this choice to obtain a global result. For any contact distribution in 3 dimensions there are well-known local coördinates (x, y, z) due to Darboux such that the contact distribution is spanned by $X \equiv \partial/\partial x$ and $Y \equiv \partial/\partial y + x\partial/\partial z$. Notice that

$$(3.22) \quad [X, Y] = Z \quad [X, Z] = 0 \quad [Y, Z] = 0$$

where $Z \equiv \partial/\partial z$. The vector fields X, Y, Z span the tangent vectors near the origin. Dually, the cotangent vectors are spanned by $dx, dy, dz - x dy$ and we may split the projection $\Lambda^1 \rightarrow \Lambda_H^1$ by decreeing that dx, dy span the lift of Λ_H^1 . It is then a simple exercise to write out $d_H^{(2)}$ of Lemma 3.4 using these local coördinates. Firstly, we compute the Levi form:–

$$L \ni dz - x dy \mapsto -dx \wedge dy \in \Lambda_H^2.$$

Following the recipe in the proof of Lemma 3.4, by writing $\omega = g dx + h dy$ we have already lifted $\omega \in \Lambda_H^1$ to a 1-form $\tilde{\omega}$. Therefore,

$$\mathcal{L}^{-1} q d\tilde{\omega} = \mathcal{L}^{-1}(Xh - Yg) dx \wedge dy = (Xh - Yg)(x dy - dz)$$

so

$$\tilde{\omega} - \mathcal{L}^{-1} q d\tilde{\omega} = g dx + (h - xXh + xYg) dy + (Xh - Yg) dz.$$

Computing $d(\tilde{\omega} - \mathcal{L}^{-1} q d\tilde{\omega})$ now yields

$$((X^2 h - XYg - Zg) dx + (YXh - Y^2 g - Zh) dy) \wedge (dz - x dy).$$

Regarding this a section of $\Lambda_H^1 \otimes L$ allows us to write out (3.16) explicitly:–

$$f \xrightarrow{d_H} \begin{bmatrix} Xf \\ Yf \end{bmatrix} \quad \begin{bmatrix} g \\ h \end{bmatrix} \xrightarrow{d_H^{(2)}} \begin{bmatrix} X^2h - XYg - Zg \\ YXh - Y^2g - Zh \end{bmatrix},$$

where Λ_H^1 is trivialised using dx, dy and L is trivialised using $dz - xdy$. As a check, notice that the composition is easily seen to be zero by dint of (3.22). The coupled operators are given by essentially the same formulæ. Specifically, a partial connection on a vector bundle E is determined by ∇_X and ∇_Y . Corollary 3.6 promotes this to a full connection by $\nabla_Z \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X$ and (3.20) becomes

$$(3.23) \quad \sigma \xrightarrow{\nabla_H} \begin{bmatrix} \nabla_X \sigma \\ \nabla_Y \sigma \end{bmatrix} \quad \begin{bmatrix} \tau \\ v \end{bmatrix} \xrightarrow{\nabla_H^{(2)}} \begin{bmatrix} \nabla_X \nabla_X v - \nabla_X \nabla_Y \tau - \nabla_Z \tau \\ \nabla_Y \nabla_X v - \nabla_Y \nabla_Y \tau - \nabla_Z v \end{bmatrix}.$$

A second order linear differential operator $V \rightarrow W$ on a contact manifold is said to be *compatible* with the contact structure if and only if its symbol $\odot^2 \Lambda^1 \otimes V \rightarrow W$ factors through the canonical projection $\odot^2 \Lambda^1 \otimes V \rightarrow \odot^2 \Lambda_H^1 \otimes V$. From (3.23), the operator $\nabla_H^{(2)}$ of (3.20) evidently has this property and hence so does its restriction to K_H . This is the key observation needed to re-express (3.21) as a first order system. We proceed as follows. Pick any partial connection on K_H and extend to a full connection according to Corollary 3.6. Pick local Darboux coördinates (x, y, z) as above and write

$$(3.24) \quad \nabla_1 \equiv \nabla_X \quad \nabla_2 \equiv \nabla_Y \quad \nabla_0 \equiv \nabla_Z = \nabla_1 \nabla_2 - \nabla_2 \nabla_1.$$

To say that the second order operator $\nabla_H^{(2)}$ is compatible with the contact structure means that we can write it uniquely as

$$\mu \xrightarrow{\nabla_H^{(2)}} S^{ab} \nabla_a \nabla_b \mu + \Gamma^0 \nabla_0 \mu + \Gamma^a \nabla_a \mu + \Theta \mu,$$

where S^{ab} , Γ^0 , Γ^a , Θ all take values in $\text{Hom}(K_H, \Lambda_H^1 \otimes L \otimes E)$ and S^{ab} is symmetric. Therefore, we can write the equation $\nabla_H^{(2)} \mu = \kappa_H \sigma$ as

$$(3.25) \quad \mathcal{P} \rho + \Theta \mu = \kappa_H \sigma,$$

where

$$\mathcal{P} \rho \equiv S^{ab} \nabla_a \rho_b + \Gamma^0 (\nabla_1 \rho_2 - \nabla_2 \rho_1) + \Gamma^a \rho_a \quad \text{and} \quad \rho_a = \nabla_a \mu.$$

Overall, if we define a first order operator

$$(3.26) \quad E_H'' \equiv \begin{array}{ccc} E & & \Lambda_H^1 \otimes E \\ \oplus & & \oplus \\ K_H & \xrightarrow{\widehat{D}_H} & \Lambda_H^1 \otimes K_H \\ \oplus & & \oplus \\ \Lambda_H^1 \otimes K_H & & \Lambda_H^1 \otimes L \otimes E \end{array} \quad \text{by} \quad \begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H \mu - \rho \\ \mathcal{P} \rho + \Theta \mu - \kappa_H \sigma \end{bmatrix},$$

then we have proved

Theorem 3.8. *The projection $E_H'' \rightarrow E$ induces an isomorphism*

$$\{\Sigma \in \Gamma(E_H'') \text{ s.t. } \widehat{D}_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

This is the claimed remedy for Theorem 3.7. Certainly, the new operator $\hat{D}_H$ is first order. It is compatible with the contact structure because the same is true of $\mathcal{P}$. Indeed, from the formula for $\hat{D}_H$, it is clear that on the first two components of E_H'' the symbol is induced by the canonical projection $\Lambda^1 \rightarrow \Lambda_H^1$. Therefore, the symbol of $\hat{D}_H$ is carried by the symbol of $\mathcal{P}$

$$\Lambda^1 \otimes \Lambda_H^1 \otimes K_H \rightarrow \Lambda_H^1 \otimes L \otimes E,$$

which is, in turn, carried by the tensors S^{ab} and Γ^0 , which we now compute.

Lemma 3.9. *For any second order operator $V \rightarrow W$ compatible with a three-dimensional contact structure and written in Darboux local coördinates as*

$$\mu \longmapsto S^{ab} \nabla_a \nabla_b \mu + \Gamma^0 \nabla_0 \mu + \Gamma^a \nabla_a \mu + \Theta \mu \quad \text{with } S^{ab} = S^{ba}$$

for some partial connection ∇_a on V , the vector bundle homomorphisms

$$S^{ab} \in \text{Hom}(V, W) \quad \text{for } a, b = 1, 2 \quad \text{and} \quad \Gamma^0 \in \text{Hom}(V, W)$$

are independent of choice of partial connection.

Proof. Any other partial connection has the form

$$(3.27) \quad \hat{\nabla}_a \mu = \nabla_a \mu - \Xi_a \mu \quad \text{for } \Xi_1, \Xi_2 \in \text{End}(V)$$

and the required conclusion follows by substitution. $\square$

Lemma 3.10. *Given the hypotheses of Lemma 3.9 and a subbundle $U \subseteq V$, the corresponding homomorphisms for the differential operator restricted to U are simply $S^{ab}|_U$ and $\Gamma^0|_U$.*

Proof. Now that we know by Lemma 3.9 that these homomorphisms are well-defined, we can start with a partial connection on U and extend it to V . $\square$

Remark. As far as the homomorphisms S^{ab} are concerned, Lemmata 3.9 and 3.10 are merely saying that the symbol $S : \odot^2 \Lambda_H^1 \otimes V \rightarrow W$ is invariantly defined and behaves well when restricted to a subbundle. This is completely standard. The new aspect is that, on a 3-dimensional contact manifold, the particular lower order coefficient Γ^0 behaves just as well. This is a familiar feature of contact geometry whereby derivatives transverse to the contact distribution should “count double”. Later, in Proposition 3.12, we shall see this feature more precisely and find that there is an enhanced symbol in all dimensions, best regarded as a homomorphism $S_\perp^2 \otimes V \rightarrow W$.

Recall that we wanted to compute the symbol of $\mathcal{P}$. From Lemmata 3.9 and 3.10 it follows that we may do this by performing the analogous computation for the operator $\nabla_H^{(2)} : \Lambda_H^1 \otimes E \rightarrow \Lambda_H^1 \otimes L \otimes E$, for which we have a formula (3.23), and then restrict the result to $K_H \subseteq \Lambda_H^1 \otimes E$. Using (3.24) we may re-write (3.23):–

$$\nabla_H^{(2)} \begin{bmatrix} \tau \\ v \end{bmatrix} = \begin{bmatrix} \nabla_1 \nabla_1 v - \frac{1}{2}(\nabla_1 \nabla_2 + \nabla_2 \nabla_1) \tau - \frac{3}{2} \nabla_0 \tau \\ \frac{1}{2}(\nabla_2 \nabla_1 + \nabla_1 \nabla_2) v - \nabla_2 \nabla_2 \tau - \frac{3}{2} \nabla_0 v \end{bmatrix}.$$

This is of the form required in Lemma 3.9 with E as a passenger and, otherwise,

$$\begin{aligned} S^{11} dx &= 0 & S^{12} dx &= -\frac{1}{2} dx \wedge (dz - x dy) \\ S^{11} dy &= dx \wedge (dz - x dy) & S^{12} dy &= \frac{1}{2} dy \wedge (dz - x dy) \\ S^{22} dx &= -dy \wedge (dz - x dy) & \Gamma^0 dx &= -\frac{3}{2} dx \wedge (dz - x dy) \\ S^{22} dy &= 0 & \Gamma^0 dy &= -\frac{3}{2} dy \wedge (dz - x dy). \end{aligned}$$

Therefore, the symbol of the corresponding first order operator

$$\Lambda_H^1 \otimes \Lambda_H^1 \otimes E \longrightarrow \Lambda_H^1 \otimes L \otimes E$$

has E as a passenger and otherwise factors through the homomorphism

$$\Lambda_H^1 \otimes \Lambda_H^1 \otimes \Lambda_H^1 \longrightarrow \Lambda_H^1 \otimes L$$

given by

$$\begin{aligned} dx \otimes dx \otimes dx &\mapsto 0 \\ dx \otimes dx \otimes dy &\mapsto dx \wedge (dz - x dy) \\ dx \otimes dy \otimes dx &\mapsto -2 dx \wedge (dz - x dy) \\ dx \otimes dy \otimes dy &\mapsto -dy \wedge (dz - x dy) \\ dy \otimes dx \otimes dx &\mapsto dx \wedge (dz - x dy) \\ dy \otimes dx \otimes dy &\mapsto 2 dy \wedge (dz - x dy) \\ dy \otimes dy \otimes dx &\mapsto -dy \wedge (dz - x dy) \\ dy \otimes dy \otimes dy &\mapsto 0. \end{aligned} \tag{3.28}$$

The most important attribute of this homomorphism is its kernel:–

Proposition 3.11. *The kernel of the homomorphism (3.28) is*

$$\odot^3 \Lambda_H^1 \oplus \text{span}\{dx \otimes dx \otimes dy - dy \otimes dx \otimes dx, dx \otimes dy \otimes dy - dy \otimes dy \otimes dx\}.$$

Proof. Clearly (3.28) is surjective and it is easy to check that the given elements are sent to zero. $\square$

Evidently, there is another way of writing this kernel:–

$$(3.29) \quad \{P_{abc} + Q_a L_{bc} + Q_b L_{ac} + Q_c L_{ab} \in \bigotimes^3 \Lambda_H^1, \text{ such that } P_{abc} = P_{(abc)}\},$$

where L_{ab} is the Levi form. In all dimensions, we shall write this space as $S_\perp^3$. Lemma 3.3 shows that it extends our previous definition (3.12) and that it coincides with the definition (1.1) given in the introduction. The main import of Theorem 3.8 stems from the kernel of the symbol of $\widehat{D}_H$, which we have now identified as

$$(\bigotimes^2 \Lambda_H^1 \otimes K_H) \cap (S_\perp^3 \otimes E) = (S_\perp^2 \otimes K_H) \cap (S_\perp^3 \otimes E)$$

just as we found for the kernel of the second prolongation in higher dimensions (3.15).

Before constructing yet higher prolongations, we pause to eliminate the use of Darboux coördinates. In Lemma 3.9, we can view $\nabla_a \mu$ as employing abstract indices in the sense of Penrose [15]. Thus, a section of Λ_H^1 is written as ω_a with no implied choice of frame. Darboux coördinates were used, however, to define $\nabla_a \nabla_b \mu$. The natural remedy is to choose a partial connection on Λ_H^1 and view $\nabla_a \nabla_b \mu$ as applying

the coupled connection on $\Lambda_H^1 \otimes V$ to $\nabla_b \mu$. There are now two checks that must be performed in order to see that

$$S^{ab} : \odot^2 \Lambda_H^1 \otimes V \rightarrow W \quad \text{and} \quad \Gamma^0 : L \otimes V \rightarrow W$$

are well-defined. Firstly, if we change the partial connection on V by means of (3.27) for $\Xi_a \in \Lambda_H^1 \otimes \text{End}(V)$, then the second order terms in a compatible second order operator change according to

$$S^{ab} \nabla_a \nabla_b \mu = S^{ab} (\hat{\nabla}_a + \Xi_a) (\hat{\nabla}_b + \Xi_b) \mu = S^{ab} \hat{\nabla}_a \hat{\nabla}_b \mu + 2(S^{ab} \Xi_a) \hat{\nabla}_b \mu + (S^{ab} \hat{\nabla}_a \Xi_b) \mu.$$

In particular, the induced change in the first order terms is to replace

$$\Gamma^a \nabla_a \mu \quad \text{by} \quad \hat{\Gamma}^a \hat{\nabla}_a \mu, \quad \text{where} \quad \hat{\Gamma}^a = \Gamma^a + 2S^{ab} \Xi_b$$

and, hence, if we interpret the first order coefficients as specifying a homomorphism $\Lambda^1 \otimes V \rightarrow W$ then the change in such a homomorphism is only in $\Gamma^a : \Lambda_H^1 \otimes V \rightarrow W$. In summary, the composition

$$L \otimes V \rightarrow \Lambda^1 \otimes V \rightarrow W$$

does not depend on the choice of partial connection on V and we denote it by Γ^0 . Secondly, we must check that the same is true if we change the partial connection on Λ_H^1 . The general such change is

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - \Omega_{ab}^c \omega_c \quad \text{for} \quad \Omega_{ab}^c \in \Lambda_H^1 \otimes \text{End}(\Lambda_H^1)$$

and $\hat{\Gamma}^a = \Gamma^a + S^{bc} \Omega_{bc}^a$ is the only change in first order coefficients. Again Γ^0 is unaffected. Also note that, with this interpretation, it is unnecessary that the contact manifold be 3-dimensional. Thus, we have proved the following.

Proposition 3.12. *On a contact manifold of arbitrary dimension, a second order linear differential operator $V \rightarrow W$ compatible with the contact structure gives rise to invariantly defined homomorphisms*

$$S : \odot^2 \Lambda_H^1 \otimes V \rightarrow W \quad \text{and} \quad \Gamma^0 : L \otimes V \rightarrow W,$$

in other words, an enhanced symbol $S_\perp^2 \otimes V \rightarrow W$.

For use in §5, it is worthwhile recording the reasoning employed in deriving (3.25) and Proposition 3.12 as the following.

Lemma 3.13. *Suppose $D : V \rightarrow W$ is a second order differential operator compatible with the contact structure on a 3-dimensional contact manifold. Suppose ∇_H is a partial connection on V . Then we can find a first order operator $\mathcal{P} : \Lambda_H^1 \otimes V \rightarrow W$ compatible with the contact structure and a homomorphism $\Theta : V \rightarrow W$ such that*

$$D = \mathcal{P} \circ \nabla_H + \Theta.$$

Moreover, the restricted symbol of $\mathcal{P}$

$$\Lambda_H^1 \otimes (\Lambda_H^1 \otimes V) \rightarrow W$$

coincides with the enhanced symbol of D

$$\otimes^2 \Lambda_H^1 \otimes V = S_\perp^2 \otimes V \rightarrow W.$$

Now that we know by Proposition 3.12 that the homomorphisms S and Γ^0 are invariantly defined, we may compute them for the operator

$$(3.30) \quad \nabla_H^{(2)} : \Lambda_H^1 \otimes E \rightarrow \Lambda_H^1 \otimes L \otimes E$$

on a 3-dimensional contact manifold by using Darboux coördinates. We already did this in deriving (3.28) and the following proposition simply writes the result in a globally well-defined manner.

Proposition 3.14. *Let $\Sigma : \bigotimes^3 \Lambda_H^1 \rightarrow \Lambda_H^1 \otimes L$ denote the operator*

$$(3.31) \quad \bigotimes^3 \Lambda_H^1 \ni \phi_{abc} \longmapsto L^{ab}(\phi_{abc} - \phi_{cab}) \in \Lambda_H^1 \otimes L.$$

Then the enhanced symbol of the operator (3.30) is

$$S_\perp^2 \otimes \Lambda_H^1 \otimes E = \bigotimes^3 \Lambda_H^1 \otimes E \xrightarrow{\Sigma \otimes \text{Id}} \Lambda_H^1 \otimes L \otimes E.$$

Remark. One can readily verify that the kernel of (3.31) is $S_\perp^3$, as expected. Indeed, if we regard the homomorphism Σ as

$$\Sigma^{ab} : \bigotimes^2 \Lambda_H^1 \rightarrow \text{Hom}(\Lambda_H^1, \Lambda_H^1 \otimes L),$$

then Proposition 3.14 implies that we may write the operator $\mathcal{P}$ as

$$(3.32) \quad \mathcal{P}\rho = \Sigma^{ab} \nabla_a \rho_b + \Gamma^a \rho_a$$

and so the partial symbol of $\mathcal{P}$ is $(\Sigma \otimes \text{Id})|_{\Lambda_H^1 \otimes \Lambda_H^1 \otimes K_H}$ with kernel

$$(\Lambda_H^1 \otimes \Lambda_H^1 \otimes K_H) \cap \ker(\Sigma \otimes \text{Id}) = (\Lambda_H^1 \otimes \Lambda_H^1 \otimes K_H) \cap (S_\perp^3 \otimes E).$$

Remark. Contact geometry is often developed by supposing that the bundle L is trivial. A trivialising section α is then referred to as a *contact form*. Such a contact form gives rise to a preferred vector field Z transverse to the contact distribution and characterised by $Z \lrcorner \alpha = 1$ and $Z \lrcorner d\alpha = 0$. It is called the *Reeb* vector field. In Darboux coördinates on a 3-dimensional contact manifold $Z = \partial/\partial z$. We obtain an alternative global point of view in which $\nabla_0 = Z \lrcorner \nabla$.

Remark. Although an unnecessary restriction in choosing a partial connection on Λ_H^1 above, it is interesting to note that there are preferred connections having a convenient relationship with the Levi form as follows (cf. [4, Proposition 4.2.1]). We work in arbitrary dimension. Let us say a linear connection ∇ on the tangent bundle TM to our contact manifold M is *adapted* if it preserves the distribution H . Adapted connections can be constructed by splitting the sequence

$$(3.33) \quad 0 \rightarrow H \rightarrow TM \rightarrow L^* \rightarrow 0$$

and choosing separate connections on H and L . An adapted connection ∇ on TM restricts to a partial connection ∇_H . We shall use the same notation for the induced connections and partial connections on H and L .

Let T^∇ be the torsion of an adapted connection ∇ on TM . From the formula

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \text{ for } X, Y \in \Gamma(TM)$$

it follows that if $X, Y \in \Gamma(H)$, then $T(X, Y) \bmod H$ is precisely $-\mathcal{L}(X, Y)$, where $\mathcal{L}$ is the Levi form. In particular, adapted connections cannot be torsion free. On the

other hand, there is a related and well-suited condition available in the presence of a splitting of the sequence (3.33), equivalently a splitting of the first column of the diagram (3.1). Then we may take the Levi-form $\mathcal{L}(\cdot, \cdot)$ to be TM -valued and define

$$T_H^\nabla \equiv T^\nabla + \mathcal{L},$$

as a section of $\Lambda^2 \otimes TM$. For this *adapted torsion*, if $X, Y \in \Gamma(H)$, then we have $T_H^\nabla(X, Y) \in \Gamma(H)$. In fact we shall be mainly interested in this part of T_H so let us write τ^∇ for the restriction of T_H^∇ to $H \wedge H$ and call it *partial torsion*.

Note that if we modify ∇ to ∇' , so that the difference $\nabla - \nabla'$ is $\frac{1}{2}T_H^\nabla$, then ∇' is again adapted but is also adapted torsion free, i.e. $T_H^{\nabla'} = 0$. The full torsion of ∇' is then $-\mathcal{L}$. In particular, ∇' is partially torsion-free, i.e. $\tau^{\nabla'} = 0$.

Let ∇' be any adapted connection such that $\tau^{\nabla'} = 0$ and let us write $R^{\nabla'}$ for its curvature, and ∇'_H for the associated partial connection on TM . By Proposition 3.5 (with TM as E there) there is modification ∇'' of ∇ , so that ∇'_H remains fixed, and so that $(q \otimes Id) \circ R^{\nabla''}$ is a section of $\Lambda_{H\perp}^2 \otimes \text{End}(TM)$. From the proof of that proposition (in particular that the Φ involved takes values in $L \otimes TM$) it follows at once that $\tau^{\nabla''}$ is a property of ∇'_H , whence $\tau^{\nabla''} = \tau^{\nabla'} = 0$. Using also how the uniqueness statement is established in the proposition, it follows easily that ∇'' preserves H . Using this and the uniqueness statement itself we have, in summary, the following.

Proposition 3.15. Given a contact distribution H and a splitting of the short exact sequence (3.33), there is a partial connection ∇_H on TM with partial torsion zero, i.e. $\tau^{\nabla_H} = 0$. The connection ∇_H admits a unique extension to a connection ∇ on TM characterised by the property that

$$(q \otimes Id) \circ R^\nabla$$

is a section of $\Lambda_{H\perp}^2 \otimes \text{End}(TM)$. The connection ∇ preserves H and, viewed as a connection on H , is the (unique) extension of the partial connection ∇_H on H , as given by Proposition 3.5.

Finally concerning adapted connections, let us note that if L is trivial (in which case a splitting of (3.33) can be obtained from the Reeb field associated to any trivialising section) then one can also arrange that the induced connection on L is flat. If we work locally, then the best we can do for an adapted connection is to construct a flat connection ∂ with torsion from Darboux local coördinates. Using abstract indices $a, b, \dots$ to adorn sections of Λ_H^1 and the index 0 to indicate a section of the line bundle L (now trivialised), we have a flat connection with

$$(3.34) \quad \nabla_{[a} \nabla_{b]} f = -L_{ab} \nabla_0 f, \quad \text{for all smooth functions } f$$

and we shall refer to it as a *Darboux connection*.

The main remaining task in this subsection is to construct higher prolongations of a first order operator $D : E \rightarrow F$ compatible with a 3-dimensional contact structure. Unfortunately, for this task it is not sufficient to modify and iterate Theorem 3.8 as one might expect. The problem is that

$$(\otimes^2 \Lambda_H^1 \otimes S_\perp^3) \cap (S_\perp^3 \otimes \otimes^2 \Lambda_H^1) \not\subseteq S_\perp^5$$

whereas we shall soon see that $S_\perp^5$ is attained with a more efficient prolongation. But firstly, we shall encounter $S_\perp^4$ as follows. Write $K'_H \equiv \Lambda_H^1 \otimes K_H$ and ∂_H for the composition

$$\Lambda_H^1 \otimes K'_H = \bigotimes^2 \Lambda_H^1 \otimes K_H \hookrightarrow \bigotimes^3 \Lambda_H^1 \otimes E \xrightarrow{\Sigma \otimes \text{Id}} \Lambda_H^1 \otimes L \otimes E.$$

It is the partial symbol of $\mathcal{P} : K'_H \rightarrow \Lambda_H^1 \otimes L \otimes E$ with kernel

$$K''_H \equiv (\Lambda_H^1 \otimes \Lambda_H^1 \otimes K_H) \cap (S_\perp^3 \otimes E).$$

Write F''_H for the subbundle

$$\begin{array}{ccc} \Lambda_H^1 \otimes E & & \Lambda_H^1 \otimes E \\ \oplus & & \oplus \\ \Lambda_H^1 \otimes K_H & \subseteq & \Lambda_H^1 \otimes K_H \\ \oplus & & \oplus \\ \partial_H(\Lambda_H^1 \otimes K'_H) & & \Lambda_H^1 \otimes L \otimes E \end{array},$$

recall the definition of E''_H in (3.26), choose a splitting δ_H of

$$\partial_H(\Lambda_H^1 \otimes K'_H) \hookrightarrow \Lambda_H^1 \otimes L \otimes E,$$

and consider the operator obtained from $\widehat{D}_H$ by using this splitting to delete some of the consistency equations, namely

$$D''_H : E''_H \rightarrow F''_H \quad \text{defined by} \quad \begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H \mu - \rho \\ \delta_H(\mathcal{P}\rho + \Theta\mu - \kappa_H \sigma) \end{bmatrix}.$$

The following theorem is a true analogue of Theorems 2.3 and 3.2 (note that D''_H is evidently compatible with the contact structure and has surjective symbol).

Theorem 3.16. *The projection $E''_H \rightarrow E$ induces an isomorphism*

$$\{\Sigma \in \Gamma(E''_H) \text{ s.t. } D''_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

Proof. Immediate from Theorem 3.8. □

Let us now lift the splitting δ_H so that it maps to $\Lambda_H^1 \otimes K'_H$. In other words, let us consider the exact sequence

$$0 \rightarrow K''_H \rightarrow \Lambda_H^1 \otimes K'_H \xrightarrow{\partial_H} \Lambda_H^1 \otimes L \otimes E$$

and choose $\delta_H : \Lambda_H^1 \otimes K'_H \leftarrow \Lambda_H^1 \otimes L \otimes E$ such that $\partial_H \delta_H \partial_H = \partial_H$ and $\delta_H \partial_H \delta_H = \delta_H$. Then we can rewrite the kernel of D''_H as the system of equations

$$(3.35) \quad \begin{aligned} \nabla_H \sigma &= \mu \\ \nabla_H \mu &= \rho \\ \delta_H \mathcal{P}\rho &= \delta_H \kappa_H \sigma - \delta_H \Theta \mu \quad \text{mod } K''_H. \end{aligned}$$

Consider the operator

$$K'_H \ni \rho \mapsto \delta_H \mathcal{P}\rho \quad \text{mod } K''_H \in \frac{\Lambda_H^1 \otimes K'_H}{K''_H}.$$

Its symbol is surjective by design. Therefore, there is a partial connection ∇_H on K'_H such that

$$(3.36) \quad \delta_H \mathcal{P}\rho = \nabla_H \rho \mod K''_H$$

and we may rewrite the last equation of (3.35) as

$$(3.37) \quad \nabla_H \rho = \delta_H \kappa_H \sigma - \delta_H \Theta \mu + \tau \quad \text{for some } \tau \in \Gamma(K''_H).$$

Note that previously, in order to define $\mathcal{P}$, we already chose a partial connection on Λ_H^1 and hence on $K'_H = \Lambda_H^1 \otimes K_H$ but henceforth we shall always prefer to use our new partial connection chosen so that (3.36) holds. More concretely, we are choosing the partial connection on K'_H so as to eliminate the effect of the Γ^a terms in (3.32).

Now consider the equation $\nabla_H \mu = \rho$ from (3.35). As usual, Corollary 3.6 extends the partial connection ∇_H to a full connection on K_H whose curvature κ'_H appears as the composition

$$(3.38) \quad K_H \xrightarrow{\nabla_H} \Lambda_H^1 \otimes K_H \xrightarrow{\nabla_H^{(2)}} \Lambda_H^1 \otimes L \otimes K_H.$$

Therefore, we may add another equation

$$\nabla_H^{(2)} \rho = \kappa'_H \mu$$

to the system (3.35) without disturbing its solutions. As we did with the equation $\nabla_H^{(2)} \mu = \kappa_H \sigma$ in (3.25), we may write this second order equation as a first order system

$$(3.39) \quad \begin{aligned} \nabla_H \rho &= \nu \\ \mathcal{Q}\nu + \Omega\rho &= \kappa'_H \mu \end{aligned}$$

where

- ∇_H is our preferred partial connection on K'_H ;
- $\mathcal{Q} : \Lambda_H^1 \otimes K'_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$ is a first order differential operator compatible with the contact structure and whose partial symbol is

$$\Lambda_H^1 \otimes \Lambda_H^1 \otimes K'_H = \bigotimes^3 \Lambda^1 \otimes K_H \xrightarrow{\Sigma \otimes \text{Id}} \Lambda_H^1 \otimes L \otimes K_H$$

with kernel $S_\perp^3 \otimes K_H$;

- $\Omega : K'_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$ is some homomorphism.

We may write $\mathcal{Q}$ explicitly as

$$(3.40) \quad \mathcal{Q}\nu = (\Sigma \otimes \text{Id})\nabla_H \nu + \Gamma\nu$$

where the partial connection ∇_H on $\Lambda_H^1 \otimes K'_H$ is induced by choosing any partial connection on Λ_H^1 and $\Gamma : \Lambda_H^1 \otimes K'_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$ is some homomorphism. But instead of the system (3.39) we may substitute from (3.37) to eliminate ν and obtain

$$(3.41) \quad \begin{aligned} \nabla_H \rho &= \delta_H \kappa_H \sigma - \delta_H \Theta \mu + \tau \\ \mathcal{Q}(\tau + \delta_H \kappa_H \sigma - \delta_H \Theta \mu) &= \kappa'_H \mu - \Omega\rho. \end{aligned}$$

If we regard $\delta_H \kappa_H \sigma$ as $(\delta_H \kappa_H) \lrcorner \sigma$ obtained by pairing

$$\delta_H \kappa_H \in \text{Hom}(E, \Lambda_H^1 \otimes K'_H) = \Gamma(\Lambda_H^1 \otimes K'_H \otimes E^*) \quad \text{with } \sigma \in \Gamma(E),$$

then we may use the Leibniz rule to write

$$\nabla_H(\delta_H \kappa_H \sigma) = (\nabla_H(\delta_H \kappa_H))\sigma + \nabla_H \sigma \lrcorner (\delta_H \kappa_H)$$

and, furthermore, substitute from (3.35) to obtain

$$\nabla_H(\delta_H \kappa_H \sigma) = (\nabla_H(\delta_H \kappa_H))\sigma + \mu \lrcorner (\delta_H \kappa_H).$$

Similarly,

$$\nabla_H(\delta_H \Theta \mu) = (\nabla_H(\delta_H \Theta))\mu + \rho \lrcorner (\delta_H \Theta).$$

In other words, these are known linear expressions in σ, μ, ρ . Bearing in mind the formula (3.40) for $\mathcal{Q}$, the same conclusion applies to $\mathcal{Q}(\delta_H \kappa_H \sigma)$ and $\mathcal{Q}(\delta_H \Theta \mu)$. Hence, we may combine (3.35) with (3.41) to conclude that our original equation $D\sigma = 0$ is equivalent to the prolonged system

$$(3.42) \quad \begin{aligned} \nabla_H \sigma &= \mu \\ \nabla_H \mu &= \rho \\ \nabla_H \rho &= \delta_H \kappa_H \sigma - \delta_H \Theta \mu + \tau \\ \mathcal{Q}\tau &= L(\sigma, \mu, \rho) \end{aligned}$$

for some explicit linear function $L(\sigma, \mu, \rho)$ defined in terms of the chosen connections on the three vector bundles E, K_H, K'_H . Recall that σ, μ, ρ, τ are sections of the bundles E, K_H, K'_H, K''_H , respectively. The operator $\mathcal{Q}$ is initially defined on $\Lambda^1_H \otimes K'_H$ but in (3.42) we see that its action is confined to $K''_H \subset \Lambda^1_H \otimes K'_H$. As such, its partial symbol therefore has as its kernel

$$K'''_H \equiv (S^3_\perp \otimes K_H) \cap (\Lambda^1_H \otimes K''_H) = (S^3_\perp \otimes K_H) \cap (S^4_\perp \otimes E),$$

where

$$S^4_\perp = (\Lambda^1_H \otimes S^3_\perp) \cap (S^3_\perp \otimes \Lambda^1_H)$$

as is the case in dimension ≥ 5 . Furthermore, is straightforward to verify that (3.14) also holds in dimension 3. Hence, although the method of building prolongations on a contact manifold is quite different in dimension 3, the criteria for being of finite-type are identical so far. This phenomenon continues for higher prolongations. Although rather complicated in practise, it is clear enough how to continue with higher order prolongations in principle. With reference to the prolonged system (3.42), at the next stage one rewrites $\mathcal{Q}\tau$ at the expense of introducing a partial connection on K''_H , a suitable splitting δ_H , and a new variable taking values in K'''_H . There is a second order constraint

$$\nabla_H^{(2)}(\delta_H \kappa_H \sigma - \delta_H \Theta \mu + \tau) = \kappa''_H \rho$$

obtained from the third equation in (3.42) where κ''_H is the curvature arising from our chosen partial connection on K'_H . As usual, one rewrites this as a first order system using the various partial connections and the Leibniz rule, organising the result in terms of a first order operator $\mathcal{R}$ on K'''_H . The details are left to the reader. In fact, this scheme is obtained by taking the deceptively simple iterative scheme from §2, writing it out in detail, and then making adjustments to account for the relevant integrability conditions in the Rumin complex being of second order in dimension 3. An iterative scheme in dimension 3 is presented in §5. For convenience we record the final conclusion in all dimensions as follows.

Theorem 3.17. *Suppose that $D : E \rightarrow F$ is a first order linear differential operator between smooth vector bundles on a contact manifold. Suppose its symbol $\Lambda^1 \otimes E \rightarrow F$ is surjective and descends to a homomorphism $\Lambda_H^1 \otimes E \rightarrow F$ whose kernel we shall denote by K_H . Let*

$$S_\perp^\ell \cong \odot^\ell \Lambda_H^1 \oplus \odot^{\ell-2} \Lambda_H^1 \oplus \odot^{\ell-4} \Lambda_H^1 \oplus \cdots \subset \bigotimes^\ell \Lambda_H^1$$

be defined in terms of the Levi form L_{ab} by (3.9), (3.29), (3.14), and generally by (1.1). Suppose

$$K_H^\ell \equiv (S_\perp^\ell \otimes K_H) \cap (S_\perp^{\ell+1} \otimes E)$$

are vector bundles for all ℓ (we say that D is ‘regular’) and that $K_H^\ell = 0$ for ℓ sufficiently large (we say that D is ‘finite-type’). Then there is a partial connection ∇_H on the bundle

$$\mathbb{T} \equiv E \oplus K_H \oplus \bigoplus_{\ell \geq 1} K_H^\ell$$

such that taking the first component $\mathbb{T} \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(\mathbb{T}) \text{ s.t. } \nabla_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

In particular, the solution space of D is finite-dimensional with dimension bounded by the rank of $\mathbb{T}$.

Remark. A uniform approach to contact prolongation in all dimensions is provided by the theory of *weighted jets* developed by Morimoto [12] in the much more general context of *filtered manifolds*. As far as contact manifolds are concerned, the bundle $J_H^1 E$ appearing in (3.6) is the first *weighted jet bundle* and, more generally and in all dimensions, there are higher weighted jet bundles and weighted jet exact sequences

$$(3.43) \quad 0 \rightarrow S_\perp^\ell \otimes E \rightarrow J_H^\ell E \rightarrow J_H^{\ell-1} \rightarrow 0.$$

We shall return to these sequences in §5 but here we just remark that one can modify, without too much trouble, the usual theory of prolongation and finite-type linear differential operators due to Goldschmidt [7], Spencer [17], et alia, and usually expressed in terms of ordinary jet bundles, so as to apply to filtered manifolds simply by systematically replacing ordinary jets by weighted jets. This is the spirit of [12]. Although the partial connection in Theorem 3.17 seems to be out of reach from this point of view, Neusser [13] has used weighted jets to obtain the same final bound on the dimension of the solution space of D .

Example. As a simple example of Theorem 3.17 in action, let us consider the system of partial differential equations on $\mathbb{R}^3$ given by

$$(3.44) \quad Xf = 0, \quad Xg + Yf = 0, \quad Yg = 0$$

in Darboux coördinates (with $X = \partial/\partial x$ and $Y = \partial/\partial y + x\partial/\partial z$, as before). Recall that we may take $\Lambda_H^1 = \text{span}\{dx, dy\}$. As a differential operator

$$E = \mathbb{R}^2 \ni \begin{bmatrix} f \\ g \end{bmatrix} \mapsto \begin{bmatrix} Xf \\ Xg + Yf \\ Yg \end{bmatrix} \in \mathbb{R}^3 = F$$

with partial symbol given by

$$dx \otimes \begin{bmatrix} f \\ g \end{bmatrix} \mapsto \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} \quad dy \otimes \begin{bmatrix} f \\ g \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ f \\ g \end{bmatrix}.$$

We see that

$$K_H = \text{span} \left\{ dx \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - dy \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

has rank 1 and so $K'_H = \Lambda^1_H \otimes K_H$ (because we are in 3 dimensions) has rank 2. However, it is easy to use the description of S^3_1 in Darboux coördinates given as the kernel of (3.28) to check that $K''_H = 0$. Theorem 3.17 implies that the dimension of the solution space is bounded by

$$\text{rank } E + \text{rank } K_H + \text{rank } K'_H = 2 + 1 + 2 = 5.$$

In fact, taking

$$(3.45) \quad f = 2pz + qy^2 + ry + s \quad g = 2q(z - xy) - px^2 - rx + t$$

for arbitrary constants p, q, r, s, t solves (3.44) and so this bound is sharp with (3.45) the general solution. We shall return to this example in §6. For the system

$$Xf = 0, \quad Xg + Yf = 0, \quad Xh + Yg = 0, \quad Yh = 0,$$

we find that $K_H^\ell = 0$ for $\ell \geq 4$ and that

$$\text{rank } E + \text{rank } K_H + \text{rank } K'_H + \text{rank } K''_H + \text{rank } K'''_H = 3 + 2 + 4 + 2 + 3 = 14.$$

In fact, with the machinery of §6 we shall be able to see that the pattern of bounds for systems of this type in dimension 3 continues as

$$5, 14, 30, 55, 91, 140, \dots, \frac{(k+1)(k+2)(2k+3)}{6}, \dots$$

4. GENERAL PROLONGATION FOR HIGHER ORDER OPERATORS

The initial steps in prolonging a higher order operator closely follow the first order case detailed in §2. Suppose $D : E \rightarrow F$ is a k^{th} order linear differential operator and suppose that its symbol $\odot^k \Lambda^1 \otimes E \rightarrow F$ is surjective. Write π for this symbol and K for its kernel. Define the vector bundle E' as the kernel of $D : J^k E \rightarrow F$. We obtain, generalising (2.1), a commutative diagram

$$(4.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & K & \rightarrow & E' & \rightarrow & J^{k-1}E \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \odot^k \Lambda^1 \otimes E & \rightarrow & J^k E & \rightarrow & J^{k-1}E \rightarrow 0 \\ & & \pi \downarrow & & D \downarrow & & \\ & & F & = & F & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns. To replace Lemma 2.1, we need the following notion.

Definition. A k^{th} order connection on a smooth vector bundle E is a linear differential operator $\nabla^k : E \rightarrow \odot^k \Lambda^1 \otimes E$ whose symbol $\odot^k \Lambda^1 \otimes E \rightarrow \odot^k \Lambda^1 \otimes E$ is the identity. Equivalently, such a higher order connection is a splitting of the jet exact sequence

$$(4.2) \quad 0 \rightarrow \odot^k \Lambda^1 \otimes E \rightarrow J^k E \xrightarrow{p} J^{k-1} E \rightarrow 0.$$

Lemma 4.1. *There is a k^{th} order connection ∇^k on E so that D is the composition*

$$E \xrightarrow{\nabla^k} \odot^k \Lambda^1 \otimes E \xrightarrow{\pi} F.$$

Proof. Choose a splitting of the short exact sequence

$$(4.3) \quad 0 \rightarrow K \rightarrow E' \rightarrow J^{k-1} E \rightarrow 0$$

and then mimic the proof of Lemma 2.1. $\square$

A connection $\nabla : E \rightarrow \Lambda^1 \otimes E$ induces an operator $\nabla : \Lambda^1 \otimes E \rightarrow \Lambda^2 \otimes E$ and allows us to define the its curvature. To extend this to higher order connections, let us denote by $\odot^{k+1} \Lambda^1$ the bundle of covariant tensors $\phi_{abcd\dots e}$ with $k+1$ indices satisfying

$$\phi_{abcd\dots e} = \phi_{[ab](cd\dots e)} \quad \text{and} \quad \phi_{[abc]d\dots e} = 0.$$

Notice that there is a canonical projection

$$\Lambda^1 \otimes \odot^k \Lambda^1 \ni \phi_{abcd\dots e} \xrightarrow{Y} \phi_{[ab]cd\dots e} \ni \odot^{k+1} \Lambda^1$$

corresponding to the decomposition of irreducible tensor bundles

$$(4.4) \quad \Lambda^1 \otimes \odot^k \Lambda^1 = \square \otimes \square \square \square \square = \square \square \square \square \square \oplus \square \square \square \square \square \parallel \odot^{k+1} \Lambda^1 \oplus \odot^{k+1} \Lambda^1.$$

Proposition 4.2. *A k^{th} order connection $\nabla^k : E \rightarrow \odot^k \Lambda^1 \otimes E$ canonically induces a first order operator $\nabla : \odot^k \Lambda^1 \otimes E \rightarrow \odot^{k+1} \Lambda^1 \otimes E$ such that*

- its symbol $\Lambda^1 \otimes \odot^k \Lambda^1 \otimes E \rightarrow \odot^{k+1} \Lambda^1 \otimes E$ is $Y \otimes \text{Id}$
- the composition $E \rightarrow \odot^k \Lambda^1 \otimes E \rightarrow \odot^{k+1} \Lambda^1 \otimes E$ is a differential operator of order $k-1$, which we shall denote by κ .

Proof. Choose an arbitrary connection on E and an arbitrary torsion-free connection on the tangent bundle and hence on all tensor bundles coupled with E . Denoting all resulting connections by ∂_a , the operator ∇^k has the form

$$\sigma \xrightarrow{\nabla^k} \overbrace{\partial_{(b} \partial_c \partial_d \dots \partial_e)}^k \sigma + \Gamma_{bcd\dots e}^{fg\dots h} \overbrace{\partial_f \partial_g \dots \partial_h}^{k-1} \sigma + \text{lower order terms}$$

for a uniquely defined tensor $\Gamma_{bcd\dots e}^{fg\dots h}$ symmetric in both its lower and upper indices and having values in $\text{End}(E)$. But then

$$\sigma_{bcd\dots e} \xrightarrow{\nabla} \partial_{[a} \sigma_{b]cd\dots e} + \Gamma_{cd\dots e[a}^{fg\dots h} \sigma_{b]fg\dots h}$$

is forced by the two characterising properties of ∇ (and, in particular, does not depend on choice of ∂_a). $\square$

Remark. The operator ∇ can also be constructed in a rather tautological but less explicit fashion as follows. As a special case of [7, Proposition 3], there is a canonically defined first order differential operator

$$J^k E \xrightarrow{\mathcal{G}} \frac{\Lambda^1 \otimes J^k E}{\odot^{k+1} \Lambda^1 \otimes E},$$

where $\odot^{k+1} \Lambda^1 \otimes E$ is regarded as a subbundle of $\Lambda^1 \otimes J^k E$ by means of the inclusion $\odot^{k+1} \Lambda^1 \otimes E \hookrightarrow \Lambda^1 \otimes \odot^k \Lambda^1 \otimes E$ and the jet exact sequence (4.2). But a k^{th} order connection splits (4.2) whence there is a homomorphism of vector bundles

$$\frac{\Lambda^1 \otimes J^k E}{\odot^{k+1} \Lambda^1 \otimes E} \longrightarrow \frac{\Lambda^1 \otimes \odot^k \Lambda^1 \otimes E}{\odot^{k+1} \Lambda^1 \otimes E} = \oslash^{k+1} \Lambda^1 \otimes E,$$

where the last identification comes from (4.4). The operator ∇ is the composition

$$\odot^k \Lambda^1 \otimes E \rightarrow J^k E \xrightarrow{\mathcal{G}} \frac{\Lambda^1 \otimes J^k E}{\odot^{k+1} \Lambda^1 \otimes E} \rightarrow \oslash^{k+1} \Lambda^1 \otimes E.$$

The *Spencer operator* [17] is a canonically defined first order linear differential operator $\mathcal{S} : J^k E \rightarrow \Lambda^1 \otimes J^{k-1} E$ uniquely characterised by

- its symbol $\Lambda^1 \otimes J^k E \rightarrow \Lambda^1 \otimes J^{k-1} E$ is induced by the projection $J^k E \xrightarrow{p} J^{k-1} E$
- the sequence $E \xrightarrow{j^k} J^k E \xrightarrow{\mathcal{S}} \Lambda^1 \otimes J^{k-1} E$ is exact,

where j^k is the universal k^{th} order differential operator. As done in the proof of Proposition 4.2, it is straightforward to write down a formula for $\mathcal{S}$ in terms of arbitrarily chosen connections ∂_a . If $k = 2$, for example, then

$$(4.5) \quad \sigma \xrightarrow{j^2} \begin{bmatrix} \sigma \\ \partial_b \sigma \\ \partial_{(b} \partial_{c)} \sigma \end{bmatrix} \text{ forces } \begin{bmatrix} \sigma \\ \sigma_b \\ \sigma_{bc} \end{bmatrix} \xrightarrow{\mathcal{S}} \begin{bmatrix} \partial_a \sigma - \sigma_a \\ \partial_a \sigma_b - K_{ab} \sigma - \sigma_{ab} \end{bmatrix}$$

where K_{ab} is the curvature of ∂_a . Formulæ for higher k have more complicated terms involving higher covariant derivatives of curvature but, clearly, the result is forced and when the connections are flat, as can always be supposed locally, the general component of $\mathcal{S}$ is simply $\partial_a \sigma_{bc \dots d} - \sigma_{abc \dots d}$.

The Spencer operator can be combined with a k^{th} order connection ∇^k on E to yield an ordinary connection on $J^{k-1} E$. Specifically, we view ∇^k as a splitting of (4.2) and compose

$$J^{k-1} E \hookrightarrow J^k E \xrightarrow{\mathcal{S}} \Lambda^1 \otimes J^{k-1} E,$$

noting that the result is a connection because its symbol is the identity. Also denoting this connection by ∇ , it is clear that the composition

$$J^{k-1} E \xrightarrow{\nabla} \Lambda^1 \otimes J^{k-1} E \xrightarrow{\text{Id} \otimes p} \Lambda^1 \otimes J^{k-2} E$$

is simply the Spencer operator $J^{k-1} E \xrightarrow{\mathcal{S}} \Lambda^1 \otimes J^{k-2} E$ one degree lower down. A detailed investigation into the relationship between k^{th} order connections on E and ordinary connections on $J^{k-1} E$ is undertaken in [6].

Proposition 4.3. *Let $\mu \in \Gamma(\odot^k \Lambda^1 \otimes E)$ and also regard μ as a section of $\Lambda^1 \otimes J^{k-1}E$ by means of the inclusions*

$$\odot^k \Lambda^1 \otimes E \hookrightarrow \Lambda^1 \otimes \odot^{k-1} \Lambda^1 \otimes E \hookrightarrow \Lambda^1 \otimes J^{k-1}E.$$

Then, the canonical projection $J^{k-1}E \rightarrow E$ induces an isomorphism

$$\{\tilde{\sigma} \in \Gamma(J^{k-1}E) \text{ s.t. } \nabla \tilde{\sigma} = \mu\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } \nabla^k \sigma = \mu\}.$$

Proof. The crucial observation is that

$$\nabla \tilde{\sigma} = \mu \implies \mathcal{S}\tilde{\sigma} = 0 \iff \tilde{\sigma} = j^{k-1}\sigma, \quad \text{for some } \sigma \in \Gamma(E).$$

The remainder of the proof is just a matter of untangling a couple of definitions. $\square$

Remark. It is illuminating to view Proposition 4.3 in terms of arbitrarily chosen connections ∂_a as above. Suppose, for example, that $k = 2$. Then we can write

$$\sigma \xrightarrow{\nabla^2} \partial_{(a}\partial_{b)}\sigma + \Gamma_{ab}{}^c \partial_c \sigma + \Theta_{ab}\sigma$$

for certain uniquely determined tensors $\Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c$ and $\Theta_{ab} = \Theta_{(ab)}$ having values in $\text{End}(E)$, in which case

$$\nabla_a \begin{bmatrix} \sigma \\ \sigma_b \end{bmatrix} = \mathcal{S} \begin{bmatrix} \sigma \\ \sigma_b \\ -\Gamma_{bc}{}^d \sigma_d - \Theta_{bc}\sigma \end{bmatrix} = \begin{bmatrix} \partial_a \sigma - \sigma_a \\ \partial_a \sigma_b - K_{ab}\sigma + \Gamma_{bc}{}^d \sigma_d + \Theta_{bc}\sigma \end{bmatrix}$$

in accordance with (4.5). Therefore,

$$\nabla \tilde{\sigma} = \mu \iff \begin{cases} \partial_a \sigma - \sigma_a &= 0 \\ \partial_a \sigma_b - K_{ab}\sigma + \Gamma_{bc}{}^d \sigma_d + \Theta_{bc}\sigma &= \mu_{ab}. \end{cases}$$

But the first of these equations implies that

$$\begin{aligned} \partial_a \sigma_b - K_{ab}\sigma + \Gamma_{bc}{}^d \sigma_d + \Theta_{bc}\sigma &= \partial_a \partial_b \sigma - K_{ab}\sigma + \Gamma_{bc}{}^d \partial_d \sigma + \Theta_{bc}\sigma \\ &= \partial_{(a}\partial_{b)}\sigma + \Gamma_{bc}{}^d \partial_d \sigma + \Theta_{bc}\sigma = \nabla_{ab}^2 \sigma \end{aligned}$$

and so the second equation maybe rewritten as $\nabla^2 \sigma = \mu$.

Remark. The abstract approach and results expressed in terms of jets are due to Goldschmidt and Spencer, e.g. [7, 17]. It is often the case, however, that the operator $D : E \rightarrow F$ in question has a geometric origin, in which case there are associated connections that one is almost obliged to use in writing down an effective prolongation scheme. This is the approach adopted, for example, in [1].

The following result generalises Theorem 2.2.

Theorem 4.4. *There is a first order differential operator*

$$\tilde{D} : E' \equiv \begin{array}{c} J^{k-1}E \\ \oplus \\ K \end{array} \longrightarrow \begin{array}{c} \Lambda^1 \otimes J^{k-1}E \\ \oplus \\ \odot^{k+1} \Lambda^1 \otimes E \end{array}$$

so that the canonical projection $E' \rightarrow J^{k-1}E \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(E') \text{ s.t. } \tilde{D}\Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

Proof. Choose a k^{th} order connection ∇^k in accordance with Lemma 4.1 so that $D\sigma = 0$ if and only if $\nabla^k\sigma = \mu$ for some $\mu \in \Gamma(K)$. Define $\tilde{D}$ by

$$\begin{bmatrix} \tilde{\sigma} \\ \mu \end{bmatrix} \xrightarrow{\tilde{D}} \begin{bmatrix} \nabla\tilde{\sigma} - \mu \\ \nabla\mu - \kappa\tilde{\sigma} \end{bmatrix}.$$

It is the same formula as used in the proof of Theorem 2.2 but the meaning of the terms have been generalised:–

- $\tilde{\sigma} \mapsto \nabla\tilde{\sigma}$ is the connection on $J^{k-1}E$ associated to ∇^k
- $\mu \mapsto \nabla\mu$ is the restriction to K of the operator provided by Proposition 4.2
- $\tilde{\sigma} \mapsto \kappa\tilde{\sigma}$ is the homomorphism of vector bundles

$$J^{k-1}E \longrightarrow \bigcirc^{k+1}\Lambda^1 \otimes E$$

induced by the $(k-1)^{\text{st}}$ order $\kappa : E \rightarrow \bigcirc^{k+1}\Lambda^1 \otimes E$ in Proposition 4.2.

We have already seen in Proposition 4.3 that

$$\nabla^k\sigma = \mu \iff \nabla\tilde{\sigma} = \mu \quad \text{for some uniquely determined } \tilde{\sigma},$$

namely $\tilde{\sigma} = j^{k-1}\sigma$. The equation $\nabla\mu = \kappa\tilde{\sigma}$ is an optional differential consequence obtained by applying $\nabla : \bigcirc^k\Lambda^1 \otimes E \rightarrow \bigcirc^{k+1}\Lambda^1 \otimes E$ to the equation $\nabla^k\sigma = \mu$. $\square$

To obtain a suitable generalisation of Theorem 2.3 we consider the homomorphism ∂ obtained as the composition

$$(4.6) \quad \Lambda^1 \otimes K \hookrightarrow \Lambda^1 \otimes \bigcirc^k\Lambda^1 \otimes E \xrightarrow{Y \otimes \text{Id}} \bigcirc^{k+1}\Lambda^1 \otimes E$$

and choose a splitting δ of $\partial(\Lambda^1 \otimes E) \hookrightarrow \bigcirc^{k+1}\Lambda^1 \otimes E$. Then, if we define

$$D' : E' \equiv \begin{matrix} J^{k-1}E \\ \oplus \\ K \end{matrix} \longrightarrow \begin{matrix} \Lambda^1 \otimes J^{k-1}E \\ \oplus \\ \partial(\Lambda^1 \otimes K) \end{matrix} \equiv F' \quad \text{by} \quad \begin{bmatrix} \tilde{\sigma} \\ \mu \end{bmatrix} \xrightarrow{D'} \begin{bmatrix} \nabla\tilde{\sigma} - \mu \\ \delta(\nabla\mu - \kappa\tilde{\sigma}) \end{bmatrix},$$

then, from Theorem 4.4, we evidently obtain

Theorem 4.5. *The canonical projection $E' \rightarrow E$ induces an isomorphism*

$$\{\Sigma \in \Gamma(E') \text{ s.t. } D'\Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

The operator D' is manifestly first order with symbol

$$\Lambda^1 \otimes E' = \begin{matrix} \Lambda^1 \otimes J^{k-1}E \\ \oplus \\ \Lambda^1 \otimes K \end{matrix} \xrightarrow{\begin{bmatrix} \text{Id} & 0 \\ 0 & \partial \end{bmatrix}} \begin{matrix} \Lambda^1 \otimes J^{k-1}E \\ \oplus \\ \partial(\Lambda^1 \otimes K) \end{matrix}.$$

In particular, the symbol is surjective and its kernel is carried by the kernel of ∂ , which we shall write as K' . In fact, from (4.6) and (4.4) we see that

$$K' = (\Lambda^1 \otimes K) \cap (\bigcirc^{k+1}\Lambda^1 \otimes E).$$

We conclude that if $K' = 0$ then D' is a connection. Otherwise we are now in the realm of first order operators and may construct higher prolongations as §2. We have proved the following prolongation theorem.

Theorem 4.6. *Suppose that $D : E \rightarrow F$ is a k^{th} order linear differential operator between smooth vector bundles. Suppose its symbol $\odot^k \Lambda^1 \otimes E \rightarrow F$ is surjective and write K for its kernel. Suppose that*

$$K^\ell \equiv (\odot^\ell \Lambda^1 \otimes K) \cap (\odot^{k+\ell} \Lambda^1 \otimes E)$$

are vector bundles for all ℓ (we say that D is ‘regular’) and that $K^\ell = 0$ for ℓ sufficiently large (we say that D is ‘finite-type’). Then there is a connection ∇ on the bundle

$$\mathbb{T} \equiv J^{k-1}E \oplus K \oplus \bigoplus_{\ell \geq 1} K^\ell$$

such that taking the first component $\mathbb{T} \rightarrow J^{k-1}E \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(\mathbb{T}) \text{ s.t. } \nabla \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

In particular, the solution space of D is finite-dimensional with dimension bounded by the rank of $\mathbb{T}$.

It is shown in [1] that there is an extensive class of geometrically defined symbols both on manifolds with no further structure and on Riemannian manifolds, which belong to operators necessarily of finite-type and for which the bundles K^ℓ can be computed and their dimensions determined. In §6 we shall derive corresponding results for our modified prolongation procedure on contact manifolds.

Although the classical approach by means of jets [17] does not reach Theorem 4.6, it is useful to see how far it goes. Firstly, there is a canonical inclusion $J^{k+1}E \hookrightarrow J^1 J^k E$ for any smooth vector bundle E corresponding to the composition of differential operators

$$E \xrightarrow{j^k} J^k E \xrightarrow{j^1} J^1 J^k E.$$

Secondly, as we already observed following Proposition 4.2, the jet exact sequence (4.2) induces a canonical inclusion

$$\odot^{k+1} \Lambda^1 \otimes E \hookrightarrow \Lambda^1 \otimes \odot^k \Lambda^1 \otimes E \hookrightarrow \Lambda^1 \otimes J^k E.$$

Goldschmidt [7, Proposition 3] shows that there is a canonical isomorphism

$$(4.7) \quad \frac{J^1 J^k E}{J^{k+1} E} \cong \frac{\Lambda^1 \otimes J^k E}{\odot^{k+1} \Lambda^1 \otimes E}.$$

Let us write $W^k E$ for the vector bundle defined by either side of this isomorphism. The operator $\mathcal{G} : J^k E \rightarrow W^k E$ is then induced by the universal differential operator $j^1 : J^k E \rightarrow J^1 J^k E$ and the differential operator $\tilde{D}$ in Theorem 4.4 invariantly defined as the restriction of $\mathcal{G}$ to E' where $E' \subset J^k E$ in accordance with (4.1). To proceed further, the classical approach is either to assume that the range of $\tilde{D}$ is the same as the range of its symbol (this is the first criterion for the system D to be *compatible* or *formally integrable* in the sense of Goldschmidt [7]) in which case there is no need to choose a splitting δ in order to obtain Theorem 4.5 or, instead, to prolong the original operator $D : E \rightarrow F$ of order k to an operator $D^\ell : E \rightarrow J^\ell F$ of order $k + \ell$ and then use (4.7) to construct a first order operator with injective symbol in the finite-type case for ℓ sufficiently large. This latter approach is carried out by Neusser [13] on general filtered manifolds, including contact manifolds as a special case.

It is illuminating to write out the Goldschmidt operator $\mathcal{G}$ using a connection on E coupled with a flat torsion-free connection on Λ^1 as can be arranged locally (whilst maintaining a preferred connection on E). Writing ∇_a for all these connections and using them to trivialise the jet bundles $J^k E$, the second Spencer operator (4.5) yields

$$(4.8) \quad J^1 E \ni \begin{bmatrix} \sigma \\ \mu_b \end{bmatrix} \xrightarrow{\mathcal{G}} \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_{[a} \mu_{b]} - \kappa_{ab} \sigma \end{bmatrix} \in W^1 E,$$

as is familiar (2.5), whilst the third Spencer operator is straightforwardly computed to be

$$J^3 E \ni \begin{bmatrix} \sigma \\ \mu_b \\ \rho_{bc} \\ \tau_{bcd} \end{bmatrix} \xrightarrow{\mathcal{S}} \begin{bmatrix} \nabla_a - \mu_a \\ \nabla_a \mu_b - \kappa_{ab} \sigma - \rho_{ab} \\ \nabla_a \rho_{bc} - \kappa_{ab} \mu_c - \kappa_{ac} \mu_b - \frac{1}{3}(\nabla_c \kappa_{ab})\sigma - \frac{1}{3}(\nabla_b \kappa_{ac})\sigma - \tau_{abc} \end{bmatrix}$$

and yields

$$(4.9) \quad J^2 E \ni \begin{bmatrix} \sigma \\ \mu_b \\ \rho_{bc} \end{bmatrix} \xrightarrow{\mathcal{G}} \begin{bmatrix} \nabla_a - \mu_a \\ \nabla_a \mu_b - \kappa_{ab} \sigma - \rho_{ab} \\ \nabla_{[a} \rho_{b]c} - \kappa_{ab} \mu_c + \kappa_{a[b} \mu_{c]} - \frac{1}{3}(\nabla_c \kappa_{ab})\sigma + \frac{1}{3}(\nabla_{[b} \kappa_{c]a})\sigma \end{bmatrix}.$$

5. CONTACT PROLONGATION FOR HIGHER ORDER OPERATORS

Our first task is to explain what it means for a higher order differential operator to be *compatible* with a contact structure. For 1st or 2nd order operators, compatibility was defined in §3 as a restriction on the symbol, namely that it factor through

$$\Lambda^1 \otimes E \rightarrow \Lambda_H^1 \otimes E \quad \text{or} \quad \odot^2 \Lambda^1 \otimes E \rightarrow \odot^2 \Lambda_H^1 \otimes E,$$

respectively. For a k^{th} order operator, having the symbol factor through

$$\odot^k \Lambda^1 \otimes E \rightarrow \odot^k \Lambda_H^1 \otimes E,$$

is a necessary but not sufficient condition for compatibility. To proceed, let us recall [17] the definition of the fibre of the k^{th} order jet bundle $J^k E$ at a point x as the space of germs of smooth sections of E at x modulo those that vanish to order $k+1$. Also recall that the notion of vanishing to a certain order is defined componentwise with respect to any local trivialisation of E and that a function f vanishes to order $k+1$ at x if and only if $X_1 X_2 \cdots X_\ell f|_x = 0$ for any vector fields $X_1, \dots, X_\ell$ defined near x and for any $\ell \leq k+1$. Following Morimoto [12], we define the weighted jet bundles $J_H^k E$ in exactly the same manner except that we require the vector fields $X_1, \dots, X_\ell$ to lie in the contact distribution. As a less stringent requirement, this defines a larger subspace of the germs and so there is a natural surjection of bundles $J^k E \rightarrow J_H^k E$. We now define compatibility of a k^{th} order linear differential operator $D : E \rightarrow F$ with the contact structure to mean that the corresponding homomorphism of vector bundles $D : J^k E \rightarrow F$ factor through $J^k E \rightarrow J_H^k E$. The usual jet exact sequence

$$0 \rightarrow \odot^k \Lambda^1 \otimes E \rightarrow J^k E \rightarrow J^{k-1} E \rightarrow 0$$

is derived from the canonical isomorphisms

$$(\odot^k \Lambda^1)_x \cong \frac{\{f \text{ s.t. } X_1 \cdots X_\ell f|_x = 0, \forall \text{ vector fields and } \forall \ell \leq k\}}{\{f \text{ s.t. } X_1 \cdots X_\ell f|_x = 0, \forall \text{ vector fields and } \forall \ell \leq k+1\}}$$

induced by $\phi_{ab\dots d} \mapsto \phi_{ab\dots d} x^a x^b \cdots x^d$ for any local coördinates $(x_1, x_2, \dots, x_m)$ centred on x . We have already mentioned (3.43) that there is a corresponding exact sequence

$$0 \rightarrow S_\perp^k \otimes E \rightarrow J_H^k E \rightarrow J_H^{k-1} \rightarrow 0.$$

for weighted jets, where $S_\perp^k$ is defined by (1.1) and has the form given by (3.9), (3.29), (3.14), and so on. It is derived from the canonical isomorphisms

$$(S_\perp^k)_x \cong \frac{\{f \text{ s.t. } X_1 \cdots X_\ell f|_x = 0, \forall \text{ contact fields and } \forall \ell \leq k\}}{\{f \text{ s.t. } X_1 \cdots X_\ell f|_x = 0, \forall \text{ contact fields and } \forall \ell \leq k+1\}}$$

induced by using Darboux local coördinates $(x^1, x^2, \dots, x^{2n}, z)$ instead (for example, $(P_{abcd}, Q_{ab}, R) \mapsto P_{abcd} x^a x^b x^c x^d + Q_{ab} x^a x^b z + R z^2$ for $S_\perp^4$ as in (3.14)).

The commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \Lambda^1 \otimes E & \rightarrow & J^1 E & \rightarrow & E & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \rightarrow & \Lambda_H^1 \otimes E & \rightarrow & J_H^1 E & \rightarrow & E & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with exact rows and columns shows that a first order differential operator $D : E \rightarrow F$ is compatible with the contact structure as defined above if and only if its symbol factors through $\Lambda_H^1 \otimes E$ as defined in §3. Similarly, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & \odot^2 \Lambda^1 \otimes E & \rightarrow & J^2 E & \rightarrow & J^1 E & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \rightarrow & \odot^2 \Lambda_H^1 \otimes E & \rightarrow & J_H^2 E & \rightarrow & J^1 E & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with exact rows and columns shows that a second order $D : E \rightarrow F$ is compatible with the contact structure as defined above if and only if its symbol factors through $\odot^2 \Lambda_H^1 \otimes E$ as defined in §3. For higher order operators there is no such equivalence because $J^k E \rightarrow J^{k-1} E$ does not factor through $J_H^k E$ for $k \geq 3$.

For a k^{th} order operator $D : E \rightarrow F$ compatible with the contact structure, the *enhanced symbol* of D is defined to be the composition

$$S_\perp^k \otimes E \rightarrow J_H^k E \xrightarrow{D} F.$$

Its invariance extends Proposition 3.12 for second order operators. In line with (3.5), we shall write π_H for the enhanced symbol of D and suppose that it is surjective.

Our next task is to generalise Theorems 3.1 and 3.7 for contact compatible higher order operators in the same way that Theorem 4.4 generalises Theorem 2.2 for higher order operators in the absence of extra structure. The following commutative diagram

with exact rows and columns extends (3.6) and defines the bundle E'_H parallel to the definition of E' via (4.1) in the general case.

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_H & \rightarrow & E'_H & \rightarrow & J_H^{k-1}E \rightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \rightarrow & S_\perp^k \otimes E & \rightarrow & J_H^k E & \rightarrow & J_H^{k-1}E \rightarrow 0 \\
& & \pi_H \downarrow & & D \downarrow & & \\
& & F & = & F & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Let us first approach the contact version of Theorem 4.4 via weighted jet constructions and then make these constructions more explicit by means of partial connections.

Proposition 1 of [13] may be interpreted as the existence of a canonical isomorphism

$$\frac{J_H^1 J_H^k E}{J_H^{k+1} E} \cong \frac{\Lambda_H^1 \otimes J_H^k E}{S_\perp^{k+1} \otimes E}$$

parallel to (4.7) in the general case and we shall denote by $W_H^k E$ the vector bundle defined by either side of this isomorphism. It follows that there is a canonically defined compatible first order linear differential operator $\mathcal{G}_H : J_H^k E \rightarrow W_H^k E$ induced by the universal first order contact compatible operator $j_H^1 : J_H^k E \rightarrow J_H^1 J_H^k E$. If we define an operator $\tilde{D}_H : E'_H \rightarrow W_H^k E$ as the restriction of $\mathcal{G}_H$, then we might expect the following result analogous to Theorem 4.4.

Theorem 5.1. *There is a contact compatible first order linear differential operator*

$$\tilde{D}_H : E'_H \longrightarrow W_H^k E$$

so that the canonical projection $E'_H \rightarrow J_H^{k-1} E \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(E'_H) \text{ s.t. } \tilde{D}_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

This theorem is essentially proved in [13] by reasoning with jets. In order to make the definition of $\tilde{D}_H$ and this reasoning more explicit, we might expect to use higher partial connections by analogy with the constructions of §4. However, it turns out that this is not quite sufficient and we shall also need the adapted connections of Proposition 3.15 or, locally, the Darboux connections (3.34).

Before carrying this out, let us observe that Theorem 5.1 is sufficient to start an inductive contact prolongation in dimension ≥ 5 . The restricted symbol of $\mathcal{G}_H$

$$\Lambda_H^1 \otimes J_H^k E \longrightarrow W_H^k E = \frac{\Lambda_H^1 \otimes J_H^k E}{S_\perp^{k+1} \otimes E}$$

is the canonical projection with $S_\perp^{k+1} \otimes E$ as kernel. Therefore, the restricted symbol of $\tilde{D}_H$ is the composition

$$(5.1) \quad \Lambda_H^1 \otimes E'_H \hookrightarrow \Lambda_H^1 \otimes J_H^k E \longrightarrow \frac{\Lambda_H^1 \otimes J_H^k E}{S_\perp^{k+1} \otimes E} = W_H^k E$$

with $(\Lambda_H^1 \otimes K_H) \cap (S_{\perp}^{k+1} \otimes E)$ as kernel. But this is what we should define to be K'_H , suppose that it has constant rank, and, following the method of §3,

- write F'_H for range of the composition (5.1);
- choose a complementary subbundle to $F'_H \subseteq W_H^k E$;
- define $D'_H : E'_H \rightarrow F'_H$ as the resulting projection of $\tilde{D}_H$.

Then D'_H is a first order operator compatible with the contact structure, having

$$(5.2) \quad K'_H \equiv (\Lambda_H^1 \otimes K_H) \cap (S_{\perp}^{k+1} \otimes E)$$

as the kernel of its restricted symbol. As in §3, further prolongation of this first order operator gives first order operators $D_H^\ell : E_H^\ell \rightarrow F_H^\ell$, $\forall \ell \geq 1$ such that

$$(5.3) \quad K_H^\ell \equiv (S_{\perp}^\ell \otimes K_H) \cap (S_{\perp}^{k+\ell} \otimes E)$$

is realised as the kernel of the restricted symbol of D_H^ℓ .

Clearly, we are heading for contact version of Theorem 4.6 where the vanishing of (5.3) for ℓ sufficiently large is the contact version of finite-type. The final statement is Theorem 5.4. It remains to sort out the 3-dimensional case but, before we do so, let us make more explicit the constructions given above with weighted jets.

We shall content ourselves with a formula for $\mathcal{G}_H : J_H^2 E \rightarrow W_H^2 E$ written in terms of a general partial connection on E and a local Darboux connection (3.34) on Λ^1 . The partial connection on E canonically lifts to a full connection by Proposition 3.5 and the remaining commutation relation for the tensor connection on $\Lambda^1 \otimes E$ is that

$$(5.4) \quad \nabla_{[a} \nabla_{b]} \sigma = \kappa_{ab} \sigma - L_{ab} \nabla_0 \sigma \quad \text{for all } \sigma \in \Gamma(E).$$

The first Goldschmidt operator $\mathcal{G}_H : J_H^1 E \rightarrow W_H^1 E$ is then given by

$$\begin{bmatrix} \sigma \\ \mu_b \end{bmatrix} \mapsto \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_{[a} \mu_{b]} - \kappa_{ab} \sigma \end{bmatrix}$$

and its restriction to E'_H coincides with (3.7) from the proof of Theorem 3.1. The only difference from (4.8) is that the second entry on the right hand side has values in $\Lambda_{H\perp}^2$ and therefore does not see L_{ab} from (5.4). Similarly, the second Goldschmidt operator $\mathcal{G}_H : J_H^2 E \rightarrow W_H^2 E$ is a simple variation on (4.9) given by

$$\begin{bmatrix} \sigma \\ \mu_b \\ \rho_{bc} + L_{bc} \nu \end{bmatrix} \mapsto \begin{bmatrix} \nabla_a - \mu_a \\ \nabla_a \mu_b - \kappa_{ab} \sigma + L_{ab} \nu - \rho_{ab} \\ \nabla_{[a} \rho_{b]c} - \kappa_{ab} \mu_c + \kappa_{a[b} \mu_{c]} - \frac{1}{3} (\nabla_c \kappa_{ab}) \sigma + \frac{1}{3} (\nabla_{[b} \kappa_{c]a}) \sigma \\ + L_{c[a} \nabla_{b]} \nu - L_{[ab} \nabla_{c]} \nu \end{bmatrix}.$$

In the three-dimensional case, we have already seen in §3.1 how to prolong a first order operator compatible with the contact structure. If $D : E \rightarrow F$ is a compatible operator of order $k \geq 2$, let us still consider the first order compatible operator $D'_H : E'_H \rightarrow F'_H$ constructed above. For this construction, there is nothing special about the three-dimensional case and the kernel of its restricted symbol is K'_H as

in (5.2). Now if we choose partial connections on E and Λ_H^1 , then we have

$$(5.5) \quad D'_H : \begin{array}{c} J_H^{k-2}E \\ \oplus \\ S_\perp^{k-1} \otimes E \\ \oplus \\ K_H \end{array} \longrightarrow \begin{array}{c} \Lambda_H^1 \otimes J_H^{k-2}E \\ \oplus \\ \Lambda_H^1 \otimes S_\perp^{k-1} \otimes E, \\ \oplus \\ (\Lambda_H^1 \otimes K_H)/K'_H \end{array}$$

a first order differential operator of the form

$$\begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \xrightarrow{D'_H} \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H \mu + L(\sigma) - \rho \\ \nabla_H \rho + M(\sigma, \mu) \pmod{K''_H} \end{bmatrix}$$

for some smooth homomorphisms

$$L : J_H^{k-2}E \rightarrow \Lambda_H^1 \otimes S_\perp^{k-1} \otimes E \quad \text{and} \quad M : J_H^{k-2}E \oplus (S_\perp^{k-1} \otimes E) \rightarrow \Lambda_H^1 \otimes K_H.$$

But, since $K_H \subseteq S_\perp^k \otimes E$ and

$$(\Lambda_H^1 \otimes S_\perp^k) \cap (S_\perp^3 \otimes S_\perp^{k-2}) = S_\perp^{k+1}, \quad \text{for } k \geq 2$$

we see that we can rewrite K'_H as

$$(\Lambda_H^1 \otimes K_H) \cap (S_\perp^3 \otimes J_H^{k-2}E).$$

Therefore, viewed as in (5.5), the operator D'_H is precisely of the type to which Proposition 5.3 below may be applied. Thus starts an iterative process for building contact prolongations of D , crucially employing that D is of order at least 2.

The following lemma is convenient for the proof of Proposition 5.3.

Lemma 5.2. *Suppose $\mathcal{P} : V \rightarrow W$ is a first order operator on a contact manifold compatible with the contact structure. Suppose its partial symbol $\pi : \Lambda_H^1 \otimes V \rightarrow W$ has constant rank. Then we can find a partial connection ∇_H on V and a homomorphism $\theta : V \rightarrow W$ such that*

$$\mathcal{P} = \pi \circ \nabla_H + \theta.$$

Proof. Choose a complement C to the range R of π in W and take $\theta : V \rightarrow W$ to be the composition of $\mathcal{P}$ with projection to C . It is a homomorphism of vector bundles and $\mathcal{P} - \theta$ is a compatible first order operator $V \rightarrow R$, which therefore may be written as $\pi \circ \nabla_H$ for an appropriate choice of partial connection ∇_H . $\square$

As presaged in §3.1, we now formulate and prove a result that can be iterated in 3-dimensions to give the prolongations we require. The method of proof is a variation on the discussion in §3.1 but there are some extra points to be borne in mind, namely

- that equality $K'_H = \Lambda_H^1 \otimes K_H$ is weakened to inclusion $K'_H \subseteq \Lambda_H^1 \otimes K_H$;
- that an extra linear term $L(\sigma)$ is allowed in the definition of D_H .

Proposition 5.3. *Suppose E is a smooth vector bundle on a 3-dimensional contact manifold. Suppose we are given subbundles $K_H \subseteq \Lambda_H^1 \otimes E$ and $K'_H \subseteq \Lambda_H^1 \otimes K_H$ such*

that $K_H'' \equiv (\Lambda_H^1 \otimes K_H') \cap (S_\perp^3 \otimes E) \subset \bigotimes^3 \Lambda_H^1 \otimes E$ is also a subbundle. Suppose

$$D_H : \begin{array}{c} E \\ \oplus \\ K_H \\ \oplus \\ K_H' \end{array} \longrightarrow \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_H^1 \otimes K_H \\ \oplus \\ (\Lambda_H^1 \otimes K_H')/K_H'' \end{array}$$

is a first order differential operator of the form

$$\begin{bmatrix} \sigma \\ \mu \\ \rho \end{bmatrix} \xrightarrow{D_H} \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H \mu + L(\sigma) - \rho \\ \nabla_H \rho + M(\sigma, \mu) \pmod{K_H''} \end{bmatrix}$$

for some partial connections on E , K_H , and K_H' and smooth homomorphisms

$$L : E \rightarrow \Lambda_H^1 \otimes K_H \quad \text{and} \quad M : E \oplus K_H \rightarrow \Lambda_H^1 \otimes K_H'.$$

Finally suppose that $K_H''' \equiv (\Lambda_H^1 \otimes K_H'') \cap (S_\perp^4 \otimes E) \subset \bigotimes^4 \Lambda_H^1 \otimes E$ is a subbundle. Then we can find a differential operator

$$D_H' : \begin{array}{c} E \\ \oplus \\ K_H \\ \oplus \\ K_H' \\ \oplus \\ K_H'' \end{array} \longrightarrow \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_H^1 \otimes K_H \\ \oplus \\ \Lambda_H^1 \otimes K_H' \\ \oplus \\ (\Lambda_H^1 \otimes K_H'')/K_H''' \end{array}$$

of the form

$$\begin{bmatrix} \sigma \\ \mu \\ \rho \\ \tau \end{bmatrix} \xrightarrow{D_H'} \begin{bmatrix} \nabla_H \sigma - \mu \\ \nabla_H \mu + L(\sigma) - \rho \\ \nabla_H \rho + M(\sigma, \mu) - \tau \\ \nabla_H \tau + N(\sigma, \mu, \rho) \pmod{K_H'''} \end{bmatrix}$$

with the same kernel as D_H .

Proof. Certainly, we may rewrite the last line of $D_H \Phi = 0$ as

$$(5.6) \quad \nabla_H \rho + M(\sigma, \mu) - \tau = 0, \quad \text{for some } \tau \in \Gamma(K_H'')$$

and it suffices to derive a differential equation on τ of the given form. A suitable equation arises from the second line of $D_H \Phi = 0$, namely

$$\nabla_H \mu = \rho - L(\sigma),$$

by an application of the coupled Rumin operator $\nabla_H^{(2)} : \Lambda_H^1 \otimes K_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$ discussed in §3.1. We conclude that

$$(5.7) \quad \nabla_H^{(2)} \rho = \kappa_H' \mu + \nabla_H^{(2)} (L(\sigma)).$$

We would like to rewrite both $\nabla_H^{(2)} \rho$ and $\nabla_H^{(2)} (L(\sigma))$, noting that $\rho \in \Gamma(K_H')$ whereas $L(\sigma) \in \Gamma(\Lambda_H^1 \otimes K_H)$. Dealing with ρ first, we may use Lemma 3.13 to write

$$\nabla_H^{(2)} \rho = \mathcal{P} \nabla_H \rho + \Theta \rho$$

where ∇_H is the given partial connection on K'_H . From (5.6) we may substitute for $\nabla_H \rho$ and conclude that

$$\nabla_H^{(2)} \rho = \mathcal{P}(\tau - M(\sigma, \mu)) + \Theta \rho.$$

If we now use Lemma 5.2 to rewrite $\mathcal{P} : \Lambda_H^1 \otimes K'_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$, then

$$(5.8) \quad \nabla_H^{(2)} \rho = \pi \nabla_H \tau - \pi \nabla_H (M(\sigma, \mu)) + \theta(\tau - M(\sigma, \mu)) + \Theta \rho$$

where ∇_H is a partial connection on $\Lambda_H^1 \otimes K'_H$ and π is the composition

$$(5.9) \quad \Lambda_H^1 \otimes \Lambda_H^1 \otimes K'_H \hookrightarrow \Lambda_H^1 \otimes \Lambda_H^1 \otimes \Lambda_H^1 \otimes K_H \xrightarrow{\Sigma \otimes \text{Id}} \Lambda_H^1 \otimes L \otimes K_H$$

because $\Sigma \otimes \text{Id}$ is the enhanced symbol of $\nabla_H^{(2)}$, as noted in Proposition 3.14. Recall that $M : E \oplus K_H \rightarrow \Lambda_H^1 \otimes K'_H$ is a smooth homomorphism between bundles on which we already have defined various partial connections. We may therefore use the partial Leibniz rule to expand

$$\nabla_H(M(\sigma, \mu)) = (\nabla_H M)(\sigma, \mu) + M(\nabla_H \sigma, \nabla_H \mu)$$

and substitute from the first and second lines of $D_H \Phi = 0$ for $\nabla_H \sigma$ and $\nabla_H \mu$ to rewrite (5.8) as

$$(5.10) \quad \nabla_H^{(2)} \rho = \pi \nabla_H \tau + \theta \tau + S(\sigma, \mu, \rho),$$

where $\theta : K''_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$ and $S : E \oplus K_H \oplus K'_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$ are smooth homomorphisms. Unravelling $\nabla_H^{(2)}(L(\sigma))$ is a similar exercise except that in using Lemma 3.13 we are obliged to choose a partial connection on $\Lambda_H^1 \otimes K_H$. Schematically, the calculation reads

$$\begin{aligned} \nabla_H^{(2)}(L(\sigma)) &= \mathcal{P}(\nabla_H(L(\sigma))) + \Theta(L(\sigma)) \\ &= \mathcal{P}((\nabla_H L)(\sigma) + L(\mu)) + \Theta(L(\sigma)) \\ &= \pi \nabla_H((\nabla_H L)(\sigma) + L(\mu)) + \theta((\nabla_H L)(\sigma) + L(\mu)) + \Theta(L(\sigma)) \\ &= \pi(\nabla_H(\nabla_H L))(\sigma) + 2\pi(\nabla_H L)(\mu) + \pi L(\rho - L(\sigma)) \\ &\quad + \theta((\nabla_H L)(\sigma) + L(\mu)) + \Theta(L(\sigma)) \end{aligned}$$

and the result is that

$$(5.11) \quad \nabla_H^{(2)}(L(\sigma)) = T(\sigma, \mu, \rho)$$

for some smooth homomorphism $T : E \oplus K_H \oplus K'_H \rightarrow \Lambda_H^1 \otimes L \otimes K_H$. Substituting (5.10) and (5.11) into (5.7), we find that

$$(5.12) \quad \pi \nabla_H \tau + \theta \tau + (S - T)(\sigma, \mu, \rho) - \kappa'_H \mu = 0.$$

The left hand side is a section of $\Lambda_H^1 \otimes L \otimes K_H$. Let us consider the differential operator

$$K''_H \ni \tau \longmapsto \pi \nabla_H \tau + \theta \tau \in \Lambda_H^1 \otimes L \otimes K_H.$$

According to (5.9), the partial symbol of this operator is $(\Sigma \otimes \text{Id})|_{\Lambda_H^1 \otimes K''_H}$, where $K''_H \subseteq \Lambda_H^1 \otimes K'_H \subseteq \Lambda_H^1 \otimes \Lambda_H^1 \otimes K_H$ and, according to (3.31), its kernel is

$$(\Lambda_H^1 \otimes K''_H) \cap (S_\perp^3 \otimes K_H).$$

However, recalling that

- $K''_H = (\Lambda_H^1 \otimes K'_H) \cap (S_\perp^3 \otimes E);$

- $K_H \subseteq \Lambda_H^1 \otimes E$;
- $(\Lambda_H^1 \otimes S_\perp^3) \cap (S_\perp^3 \otimes \Lambda_H^1) = S_\perp^4$,

we see that

$$(\Lambda_H^1 \otimes K_H'') \cap (S_\perp^3 \otimes K_H) = (\Lambda_H^1 \otimes K_H'') \cap (S_\perp^4 \otimes E),$$

which is precisely how K_H''' is defined in the statement of the proposition. In summary, the kernel of the symbol of $\tau \mapsto \pi \nabla_H \tau + \theta \tau$ is K_H''' . We shall use $\Sigma \otimes \text{Id}$ to identify

$$\Lambda_H^1 \otimes L \otimes K_H \supseteq \text{range}((\Sigma \otimes \text{Id})|_{\Lambda_H^1 \otimes K_H''}) = (\Lambda_H^1 \otimes K_H'')/K_H''',$$

choose a smooth splitting

$$\delta : \Lambda_H^1 \otimes L \otimes K_H \rightarrow (\Lambda_H^1 \otimes K_H'')/K_H''',$$

and consider

$$(5.13) \quad \delta(\pi \nabla_H \tau + \theta \tau) + \delta((S - T)(\sigma, \mu, \rho) - \kappa'_H \mu)$$

whose vanishing is a consequence of (5.12). We have just arranged that the symbol of the operator $\tau \mapsto \delta(\pi \nabla_H \tau + \theta \tau)$ is the canonical projection

$$\Lambda_H^1 \otimes K_H'' \rightarrow (\Lambda_H^1 \otimes K_H'')/K_H'''$$

and the remaining part of (5.13) is some homomorphism $K_H'' \rightarrow (\Lambda_H^1 \otimes K_H'')/K_H'''$. It remains to choose a partial connection on K_H'' and a lift of this homomorphism to $\Lambda_H^1 \otimes K_H''$ to write (5.13) in the form in the form

$$\nabla_H \tau + N(\sigma, \mu, \rho) \mod K_H''',$$

as required. $\square$

It remains to see why Proposition 5.3 can be iterated. The point is that we may regroup the output from this Proposition as follows.

$$\begin{aligned} \tilde{E} = \left\{ \begin{array}{c} E \\ \oplus \\ K_H \\ \oplus \\ \tilde{K}_H = K'_H \\ \oplus \\ \tilde{K}'_H = K''_H \end{array} \right\} \ni \left[\begin{array}{c} \sigma \\ \mu \\ \rho \\ \tau \end{array} \right] \xrightarrow{D'_H} \left[\begin{array}{c} \nabla_H \sigma - \mu \\ \nabla_H \mu + L(\sigma) - \rho \\ \vdots \end{array} \right] \in \left\{ \begin{array}{c} \Lambda_H^1 \otimes E \\ \oplus \\ \Lambda_H^1 \otimes K_H \\ \oplus \\ \Lambda_H^1 \otimes K'_H \\ \oplus \\ (\Lambda_H^1 \otimes K''_H)/K_H''' \end{array} \right\} = \Lambda_H^1 \otimes \tilde{E} \end{aligned}$$

In order to substitute back into Proposition 5.3, the crux is to note that

- $\left[\begin{array}{c} \sigma \\ \mu \end{array} \right] \mapsto \left[\begin{array}{c} \nabla_H \sigma - \mu \\ \nabla_H \mu + L(\sigma) \end{array} \right]$ is a partial connection on $\tilde{E}$;
- $\tilde{K}_H'' \equiv (\Lambda_H^1 \otimes \tilde{K}'_H) \cap (S_\perp^3 \otimes \tilde{E}) = (\Lambda_H^1 \otimes K''_H) \cap (S_\perp^4 \otimes E) \equiv K_H'''$.

In order to interpret the new output, the crux is to note that

$$\tilde{K}_H''' \equiv (\Lambda_H^1 \otimes \tilde{K}_H'') \cap (S_\perp^4 \otimes \tilde{E}) = (\Lambda_H^1 \otimes K_H''') \cap (S_\perp^5 \otimes E) \equiv K_H'''.$$

For completeness, here is the final conclusion in all dimensions and for operators of arbitrary order.

Theorem 5.4. *Suppose $D : E \rightarrow F$ is a k^{th} order linear differential operator between smooth vector bundles on a contact manifold. Suppose that it is compatible with the contact structure and that its enhanced symbol $S_{\perp}^k \otimes E \rightarrow F$ is surjective with kernel K_H . Suppose that*

$$K_H^{\ell} \equiv (S_{\perp}^{\ell} \otimes K_H) \cap (S_{\perp}^{k+\ell} \otimes E)$$

are vector bundles for all ℓ (we say that D is ‘regular’) and that $K_H^{\ell} = 0$ for ℓ sufficiently large (we say that D is ‘finite-type’). Then there is a computable partial connection ∇_H on the bundle

$$\mathbb{T} \equiv J_H^{k-1}E \oplus K_H \oplus \bigoplus_{\ell \geq 1} K_H^{\ell}$$

such that taking the first component $\mathbb{T} \rightarrow J_H^{k-1}E \rightarrow E$ induces an isomorphism

$$\{\Sigma \in \Gamma(\mathbb{T}) \text{ s.t. } \nabla_H \Sigma = 0\} \cong \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\}.$$

In particular, the solution space of D is finite-dimensional with dimension bounded by the rank of $\mathbb{T}$.

6. GEOMETRIC OPERATORS

Although their definition is simple enough and determined purely in terms of the given subbundle $K_H \subset S_{\perp}^k \otimes E$, the spaces

$$(6.1) \quad K_H^{\ell} \equiv (S_{\perp}^{\ell} \otimes K_H) \cap (S_{\perp}^{k+\ell} \otimes E)$$

can be hard to understand. For a wide class of geometrically natural examples, however, these bundles can be sensibly computed. The corresponding operators are seen to be finitely determined and we obtain sharp bounds on the dimension of their solution spaces. The key ingredient is Kostant’s computation of certain Lie algebra cohomologies [11] and our approach follows [1] where similar reasoning was used in the case of classical prolongation.

To proceed, let us recall from §3 that we are writing the dimension of our contact manifold as $2n + 1$ and let us realise the Lie algebras $\mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{sp}(2(n + 1), \mathbb{R})$ as matrices of the form

$$\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & -q^t & p^t & \alpha \\ r & A & B & p \\ s & C & -A^t & q \\ \beta & s^t & -r^t & -\lambda \end{bmatrix}$$

respectively, where B and C are symmetric $n \times n$ real matrices, A is an arbitrary $n \times n$ real matrix, λ, α, β are real numbers, and p, q, r, s are real n -vectors. As the notation suggests, we have $\mathfrak{sp}(2n, \mathbb{R}) \hookrightarrow \mathfrak{sp}(2(n + 1), \mathbb{R})$ an embedding of Lie algebras. The adjoint action of the ‘grading element’

$$(6.2) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

splits $\mathfrak{g} \equiv \mathfrak{sp}(2(n+2), \mathbb{R})$ into eigenspaces

$$(6.3) \quad \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

containing elements with the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & * & * & 0 \end{bmatrix} \quad \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \quad \begin{bmatrix} 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively. Let us denote by $\mathfrak{g}_-$ the Lie subalgebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$. As a Lie algebra is its own right, it is usually referred to as the *Heisenberg algebra*. Suppose $\mathbb{V}$ is a finite-dimensional representation of $\mathfrak{g}_-$. Then we may define linear transformations

$$(6.4) \quad 0 \rightarrow \mathbb{V} \xrightarrow{\partial} \text{Hom}(\mathfrak{g}_-, \mathbb{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{g}_-, \mathbb{V})$$

by $(\partial v)(X) \equiv Xv$ and $(\partial \phi)(X \wedge Y) \equiv \phi([X, Y]) - X\phi(Y) + Y\phi(X)$, respectively. It is easily verified that $\partial^2 = 0$ and we define the *Lie algebra cohomologies*

$$H^0(\mathfrak{g}_-, \mathbb{V}) \equiv \ker \partial : \mathbb{V} \rightarrow \text{Hom}(\mathfrak{g}_-, \mathbb{V})$$

and

$$H^1(\mathfrak{g}_-, \mathbb{V}) \equiv \frac{\ker \partial : \text{Hom}(\mathfrak{g}_-, \mathbb{V}) \rightarrow \text{Hom}(\Lambda^2 \mathfrak{g}_-, \mathbb{V})}{\text{im } \partial : \mathbb{V} \rightarrow \text{Hom}(\mathfrak{g}_-, \mathbb{V})}.$$

Lemma 6.1. *Suppose $n = 1$. Then $H^0(\mathfrak{g}_-, \mathbb{V})$ and $H^1(\mathfrak{g}_-, \mathbb{V})$ may be computed using the complex*

$$(6.5) \quad 0 \rightarrow \mathbb{V} \xrightarrow{\partial_H} \text{Hom}(\mathfrak{g}_{-1}, \mathbb{V}) \xrightarrow{\partial_H} \text{Hom}(\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}, \mathbb{V})$$

instead of (6.4), where $(\partial_H v)(X) \equiv Xv$ and

$$(\partial_H \phi)(X \otimes [Y, Z]) \equiv [Y, Z]\phi(X) - X(Y\phi(Z) - Z\phi(Y))$$

for $X, Y, Z \in \mathfrak{g}_{-1}$.

Proof. The composition

$$(6.6) \quad \text{Hom}(\mathfrak{g}_{-2}, \mathbb{V}) \rightarrow \text{Hom}(\mathfrak{g}_-, \mathbb{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{g}_-, \mathbb{V}) \rightarrow \text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V})$$

is given by $\phi \mapsto (X \wedge Y \mapsto \phi([X, Y]))$. In other words, we may use the isomorphism

$$(6.7) \quad \Lambda^2 \mathfrak{g}_{-1} \ni X \wedge Y \mapsto [X, Y] \in \mathfrak{g}_{-2}$$

to eliminate the composition (6.6) from

$$\begin{array}{ccc} \text{Hom}(\mathfrak{g}_{-2}, \mathbb{V}) & & \text{Hom}(\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}, \mathbb{V}) \\ \oplus & = \text{Hom}(\mathfrak{g}_-, \mathbb{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{g}_-, \mathbb{V}) = & \oplus \\ \text{Hom}(\mathfrak{g}_{-1}, \mathbb{V}) & & \text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V}) \end{array}$$

to leave the complex (6.5), as required. $\square$

Lemma 6.2. *Suppose $n \geq 2$. Then $H^0(\mathfrak{g}_-, \mathbb{V})$ and $H^1(\mathfrak{g}_-, \mathbb{V})$ may be computed using the complex*

$$(6.8) \quad 0 \rightarrow \mathbb{V} \xrightarrow{\partial_H} \text{Hom}(\mathfrak{g}_{-1}, \mathbb{V}) \xrightarrow{\partial_H} \text{Hom}(\Lambda^2_{\perp} \mathfrak{g}_{-1}, \mathbb{V})$$

induced from (6.4), where $\Lambda^2_{\perp} \mathfrak{g}_{-1}$ is the kernel of the Lie bracket homomorphism (6.7).

Proof. A crucial difference for $n \geq 2$ is that the complex (6.4) may be replaced by

$$(6.9) \quad 0 \rightarrow \mathbb{V} \xrightarrow{\partial} \text{Hom}(\mathfrak{g}_-, \mathbb{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V})$$

without changing the cohomology. In other words, the kernel of the homomorphism $\partial : \text{Hom}(\mathfrak{g}_-, \mathbb{V}) \rightarrow \text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V})$ is the same as the kernel of the composition

$$\text{Hom}(\mathfrak{g}_-, \mathbb{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{g}_-, \mathbb{V}) \rightarrow \text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V}).$$

To see this, suppose that $\phi \in \text{Hom}(\mathfrak{g}_-, \mathbb{V})$ and $(\partial\phi)(\omega) = 0$ for all $\omega \in \Lambda^2 \mathfrak{g}_{-1}$. Then, according to the decomposition

$$\Lambda^2 \mathfrak{g}_- = (\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-1}) \oplus \Lambda^2 \mathfrak{g}_{-1},$$

we must show that $(\partial\phi)(\omega) = 0$ for all $\omega \in \mathfrak{g}_{-2} \wedge \mathfrak{g}_{-1}$. As $n \geq 2$, the homomorphism

$$\Lambda^3 \mathfrak{g}_{-1} \ni X \wedge Y \wedge Z \mapsto [X, Y] \otimes Z + [Y, Z] \otimes X + [Z, X] \otimes Y \in \mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$$

is surjective and we find

$$\begin{aligned} & (\partial\phi)([X, Y] \wedge Z + [Y, Z] \wedge X + [Z, X] \wedge Y) = \\ & Z\phi([X, Y]) - [X, Y]\phi(Z) + X\phi([Y, Z]) - [Y, Z]\phi(X) + Y\phi([Z, X]) - [Z, X]\phi(Y) = \\ & X(\partial\phi)(Y \wedge Z) + Y(\partial\phi)(Z \wedge X) + Z(\partial\phi)(X \wedge Y) = 0, \end{aligned}$$

as required. Having replaced (6.4) by (6.9), we may now argue as in the proof of Lemma 6.1. Specifically, we may cancel $\text{Hom}(\mathfrak{g}_{-2}, \mathbb{V})$ inside $\text{Hom}(\mathfrak{g}_-, \mathbb{V})$ with its image in $\text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V})$, leaving the complex (6.8). This completes our proof. $\square$

Remark. The Killing form on $\mathfrak{g} = \mathfrak{sp}(2(n+1), \mathbb{R})$ canonically identifies the duals of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ with $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. It is sometimes convenient to rewrite the complexes (6.5) and (6.8) as

$$(6.10) \quad 0 \rightarrow \mathbb{V} \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathbb{V}$$

and

$$(6.11) \quad 0 \rightarrow \mathbb{V} \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial_H} \Lambda_{\perp}^2 \mathfrak{g}_1 \otimes \mathbb{V}$$

respectively, where $\Lambda_{\perp}^2 \mathfrak{g}_1$ is the kernel of the Lie bracket $\Lambda^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$.

Remark. The reader will have noticed that the distinction between the cases $n = 1$ and $n \geq 2$ resembles the distinction found earlier in contact geometry, namely the algebraic complex (6.11) closely resembles the complex (3.4) of differential operators whilst (6.10) follows (3.16). This observation continues into higher cohomology: the Lie algebra cohomology $H^q(\mathfrak{g}_-, \mathbb{V})$ for any representation $\mathbb{V}$ of the Heisenberg Lie algebra $\mathfrak{g}_-$ is defined by an algebraic complex resembling the de Rham complex but may be computed by an alternative algebraic complex following the Rumin complex [16] in contact geometry. A precise explanation for this observation may be obtained from the usual interpretation [10] of $H^q(\mathfrak{g}_-, \mathbb{V})$ as equivariant de Rham cohomology on the Heisenberg group.

When the representation $\mathbb{V}$, of $\mathfrak{g}_-$, is obtained from an irreducible finite-dimensional representation of $\mathfrak{g}$ by restriction to $\mathfrak{g}_-$, the Lie algebra cohomology $H^q(\mathfrak{g}_-, \mathbb{V})$ is computed by a theorem of Kostant [11]. If we characterise such representations of $\mathfrak{g}$ by means of their highest weight written as an integral combination of the fundamental weights in the usual way, then for $k \geq 1$ and non-negative integers $a, b, c, \dots, d, e$,

$$(6.12) \quad H^0(\mathfrak{g}_-, \overset{k-1}{\bullet} \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet}) = \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet}$$

and

$$(6.13) \quad H^1(\mathfrak{g}_-, \overset{k-1}{\bullet} \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet}) = \overset{a+k}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet},$$

where these equalities are interpreted as isomorphisms of $\mathfrak{sp}(2n, \mathbb{R})$ -modules. More precisely, it is easy to check that the various complexes used to define and compute the Lie algebra cohomology are complexes of $\mathfrak{g}_0$ -modules. Hence, the cohomologies are $\mathfrak{g}_0$ -modules and, in particular, $\mathfrak{sp}(2n, \mathbb{R})$ -modules under restriction $\mathfrak{sp}(2n, \mathbb{R}) \subset \mathfrak{g}_0$.

Remark. In fact, with more care, the complexes used in defining and computing the Lie algebra cohomology of a representation of $\mathfrak{g}$ restricted to $\mathfrak{g}_-$ are all complexes of $\mathfrak{p}$ -modules, where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

The point of these considerations is that (6.12) and (6.13) allow us to compute the spaces K_H^ℓ for a large class of geometrically natural operators on contact manifolds. Recall that K_H^ℓ are defined as intersections (6.1) but we shall see that they also occur in the Lie algebra cohomology that we have been discussing. To proceed, let us consider the action of the grading element (6.2) from $\mathfrak{g}_0$ on $\mathbb{V}$. The representation $\mathbb{V}$ thereby splits as a direct sum of eigenspaces, each of which is a $\mathfrak{g}_0$ -module and, following [1], it is convenient for our purposes to write this decomposition as

$$(6.14) \quad \mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots \oplus \mathbb{V}_N, \quad \text{in which } \mathfrak{g}_i \mathbb{V}_j \subseteq \mathbb{V}_{i+j}.$$

Standard representation theory [9] allows us to conclude that

$$\mathbb{V} = \overset{k-1}{\bullet} \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet} \Rightarrow \mathbb{V}_0 = \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet} \text{ and } N = 2(k-1+a+b+c+\cdots+d+e).$$

Now let us suppose that $n \geq 2$ and consider the complex (6.11) with a view to computing the Lie algebra cohomologies $H^0(\mathfrak{g}_{-1}, \mathbb{V})$ and $H^1(\mathfrak{g}_{-1}, \mathbb{V})$. Because the differentials ∂_H are compatible with the action of the grading element, it follows that ∂_H must be compatible with the grading (6.14), whence (6.11) splits into a series of complexes

$$(6.15) \quad 0 \rightarrow \mathbb{V}_j \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathbb{V}_{j-1} \xrightarrow{\partial_H} \Lambda_\perp^2 \mathfrak{g}_1 \otimes \mathbb{V}_{j-2}.$$

Let

$$\mathbb{E} \equiv \mathbb{V}_0 = \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet} \quad \text{and} \quad \mathbb{F} \equiv \overset{a+k}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet}.$$

Kostant's Theorem [11] implies that $H^0(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{E}$ and $H^1(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{F}$. The action of the grading element forces the first cohomology $\mathbb{F}$ to arise from the complex (6.15) when $j = k$. The identification $\mathbb{V}_0 = \mathbb{E}$ is also built into (6.15) as the trivial case $j = 0$. Otherwise, we conclude that for $j = 1, 2, \dots, \hat{k}, \dots, N$ the complexes (6.15)

are exact. To state and interpret the algebraic consequences of these statements we make some preliminary observations and introduce some suggestive notation. Notice that the action of $\mathfrak{sp}(2n, \mathbb{R}) \subset \mathfrak{g}_0$ on $\mathfrak{g}_2$ is trivial. Therefore, we may identify $\mathfrak{g}_2 = \mathbb{R}$, view the Lie bracket $\Lambda^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ as a non-degenerate skew form, and thereby $\mathfrak{g}_1$ as the defining representation of $\mathfrak{sp}(2n, \mathbb{R})$, in which case we shall denote it by $\mathbb{S}_\perp$. As representations of $\mathfrak{sp}(2n, \mathbb{R})$, we have the irreducible decomposition

$$(6.16) \quad \bigotimes^2 \mathbb{S}_\perp = \underbrace{\bigodot^2 \mathbb{S}_\perp \oplus \mathbb{R}}_{\equiv \mathbb{S}_\perp^2} \oplus \Lambda_\perp^2 \mathbb{S}_\perp$$

defining $\mathbb{S}_\perp^2$, where $\Lambda_\perp^2 \mathbb{S}_\perp \subset \Lambda^2 \mathbb{S}_\perp$ is the kernel of the symplectic form $\Lambda^2 \mathbb{S}_\perp \rightarrow \mathbb{R}$. Let

$$(6.17) \quad \mathbb{S}_\perp^\ell \equiv (\mathbb{S}_\perp \otimes \mathbb{S}_\perp^{\ell-1}) \cap (\mathbb{S}_\perp^{\ell-1} \otimes \mathbb{S}_\perp) \quad \forall \ell \geq 3.$$

Proposition 6.3. *For $n \geq 2$, the algebraic consequences alluded to above are*

$$\begin{aligned} \mathbb{V}_j &= \mathbb{S}_\perp^j \otimes \mathbb{E}, & \forall j < k \\ \mathbb{V}_k &= \mathbb{K}_H \\ \mathbb{V}_j &= (\mathbb{S}_\perp^{j-k} \otimes \mathbb{K}_H) \cap (\mathbb{S}_\perp^j \otimes \mathbb{E}), & \forall j > k, \end{aligned}$$

where $\mathbb{K}_H$ is the kernel of the natural projection $\mathbb{S}_\perp^k \otimes \mathbb{E} \rightarrow \mathbb{F}$.

Proof. If $j = 1 < k$, then the exact sequence (6.15) reduces to

$$0 \rightarrow \mathbb{V}_1 \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathbb{V}_{j-1} \rightarrow 0, \quad \text{equivalently } \mathbb{V}_1 = \mathbb{S}_\perp \otimes \mathbb{E}.$$

For $j = 2, \dots, k-1$, suppose by induction that $\mathbb{V}_{j-1} = \mathbb{S}_\perp^{j-1} \otimes \mathbb{E}$. Then (6.15) reads

$$(6.18) \quad 0 \rightarrow \mathbb{V}_j \rightarrow \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-1} \otimes \mathbb{E} \xrightarrow{\partial_H} \Lambda_\perp^2 \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-2} \otimes \mathbb{E}.$$

Tracing back through the definitions, it may be verified that the homomorphism ∂_H in (6.18) does not see $\mathbb{E}$, i.e. there is a homomorphism

$$\tilde{\partial} : \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-1} \rightarrow \Lambda_\perp^2 \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-2}$$

so that $\partial_H = \tilde{\partial} \otimes \text{Id}$. Explicitly, $\tilde{\partial}$ is the composition

$$\mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-1} \hookrightarrow \mathbb{S}_\perp \otimes \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-2} \xrightarrow{\wedge_\perp \otimes \text{Id}} \Lambda_\perp^2 \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-2}$$

where $\wedge_\perp$ is the projection onto $\Lambda_\perp^2 \mathbb{S}_\perp$ visible in the symplectic decomposition (6.16). More explicitly, these statements easily follow from the observation that the operator $\partial_H : \mathfrak{g}_1 \otimes \mathbb{V} \rightarrow \Lambda_\perp^2 \mathfrak{g}_1 \otimes \mathbb{V}$ in (6.11) may itself be written as the composition

$$(6.19) \quad \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\text{Id} \otimes \partial_H} \mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\wedge_\perp^2 \otimes \text{Id}} \Lambda_\perp^2 \mathfrak{g}_1 \otimes \mathbb{V}.$$

When $j = 2$ it follows immediately from (6.16) that the kernel of $\tilde{\partial}$ is $\mathbb{S}_\perp^2$. More generally, it follows from the definition (6.17) that

$$0 \rightarrow \mathbb{S}_\perp^j \rightarrow \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-1} \xrightarrow{\tilde{\partial}} \Lambda_\perp^2 \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-2}$$

is exact. Therefore $\mathbb{V}_j = \mathbb{S}_\perp^j \otimes \mathbb{E}$, for $j < k$. Now we encounter the complex

$$0 \rightarrow \mathbb{V}_k \rightarrow \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{k-1} \otimes \mathbb{E} \xrightarrow{\partial_H = \tilde{\partial} \otimes \text{Id}} \Lambda_\perp^2 \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{k-2} \otimes \mathbb{E}$$

but it is no longer exact. Instead, the kernel of ∂_H is $\mathbb{S}_\perp^k \otimes \mathbb{E}$ and we have a short exact sequence

$$0 \rightarrow \mathbb{V}_k \rightarrow \mathbb{S}_\perp^k \otimes \mathbb{E} \rightarrow \mathbb{F} \rightarrow 0$$

in accordance with (6.13.) It follows that $\mathbb{V}_k = \mathbb{K}_H$ and the remaining statements follow by induction from the exactness of (6.15) for $j > k$. $\square$

Although the proofs are slightly different, with suitable caveats the results for $n = 1$ are essentially the same. The main difference is that $\mathbb{S}_\perp^3$ must be defined separately. Since $\Lambda_\perp^2 \mathbb{S}_\perp = 0$,

$$\mathbb{S}_\perp^2 = \bigotimes^2 \mathbb{S}_\perp = \bigodot^2 \mathbb{S}_\perp \oplus \mathbb{R}$$

in accordance with (6.16), but instead of (6.17), we define

$$\mathbb{S}_\perp^3 \equiv \bigodot^3 \mathbb{S}_\perp \oplus \{Q_a L_{bc} + Q_b L_{ac} + Q_c L_{ab} \text{ s.t. } Q_a \in \mathbb{S}_\perp\},$$

where $L_{ab} \in \Lambda^2 \mathbb{S}_\perp$ is the symplectic form inverse to the Lie bracket

$$\Lambda^2 \mathbb{S}_\perp = \Lambda^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 = \mathbb{R}.$$

With this definition in place and

$$\mathbb{S}_\perp^\ell \equiv (\mathbb{S}_\perp \otimes \mathbb{S}_\perp^{\ell-1}) \cap (\mathbb{S}_\perp^{\ell-1} \otimes \mathbb{S}_\perp) \quad \forall \ell \geq 4,$$

the conclusions are unchanged:–

Proposition 6.4. *For $n = 1$,*

$$\begin{aligned} \mathbb{V}_j &= \mathbb{S}_\perp^j \otimes \mathbb{E}, & \forall j < k \\ \mathbb{V}_k &= \mathbb{K}_H \\ \mathbb{V}_j &= (\mathbb{S}_\perp^{j-k} \otimes \mathbb{K}_H) \cap (\mathbb{S}_\perp^j \otimes \mathbb{E}), & \forall j > k, \end{aligned}$$

where $\mathbb{K}_H$ is the kernel of the natural projection $\mathbb{S}_\perp^k \otimes \mathbb{E} \rightarrow \mathbb{F}$.

Proof. The proof of Proposition 6.3 need only be modified as follows. We are obliged to use (6.10) rather than (6.11). The homomorphisms ∂_H must again respect gradings but $\mathfrak{g}_2$ has weight 2 with respect to the action of the grading element so we obtain complexes

$$(6.20) \quad 0 \rightarrow \mathbb{V}_j \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathbb{V}_{j-1} \xrightarrow{\partial_H} \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathbb{V}_{j-3}$$

replacing (6.15). These complexes makes themselves felt through a new version of the homomorphism $\tilde{\partial}$. Specifically, the short exact sequence

$$0 \rightarrow \mathbb{S}_\perp^3 \rightarrow \bigotimes^3 \mathbb{S}_\perp \xrightarrow{\Sigma} \mathbb{S}_\perp \rightarrow 0,$$

where Σ is the homomorphism

$$\bigotimes^3 \mathbb{S}_\perp = \bigotimes^3 \mathfrak{g}_1 \ni \phi_{abc} \mapsto L^{ab}(\phi_{abc} - \phi_{cab}) \in \mathfrak{g}_1 \otimes \mathfrak{g}_2 = \mathbb{S}_\perp \otimes \mathbb{R} = \mathbb{S}_\perp$$

and $L : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is Lie bracket, replaces

$$0 \rightarrow \mathbb{S}_\perp^2 \rightarrow \bigotimes^2 \mathbb{S}_\perp \xrightarrow{\pi_\perp} \Lambda_\perp^2 \mathbb{S}_\perp \rightarrow 0$$

and $\tilde{\partial}$ is defined as the composition

$$\mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-1} \hookrightarrow \bigotimes^3 \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-3} \xrightarrow{\Sigma \otimes \text{Id}} \mathbb{S}_\perp \otimes \mathbb{S}_\perp^{j-3}.$$

Parallel to (6.19) is composition

$$\mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\text{Id} \otimes \partial_H} \mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\text{Id} \otimes \text{Id} \otimes \partial_H} \mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\Sigma \otimes \text{Id}} \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathbb{V}$$

as a way of writing $\partial_H : \mathfrak{g}_1 \otimes \mathbb{V} \rightarrow \mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathbb{V}$ in (6.10). Apart from this key alteration, the proof follows exactly the same lines and details are left to the reader. $\square$

Propositions 6.3 and 6.4 have immediate geometric consequences as follows. Fix a contact manifold M of dimension $2n + 1$ with contact distribution H as usual. Any finite-dimensional representation $\mathbb{E}$ of $\text{Sp}(2n, \mathbb{R})$, gives rise to a vector bundle E on M by induction from the bundle of symplectic co-frames for H . We shall refer to such E as *symplectic* vector bundles. In particular, the defining representation, which we have been writing above as $\mathbb{S}_\perp$, gives rise to the bundle Λ_H^1 dual to H . More generally, the representations $\mathbb{S}_\perp^\ell$ give rise to bundles that we have already been writing as $S_\perp^\ell$ in §3 and §5. We shall refer to the bundle E as *irreducible* if the representation $\mathbb{E}$ is irreducible. Thus, the irreducible symplectic bundles on M can be parameterised exactly as for irreducible representations of $\mathfrak{sp}(2n, \mathbb{R})$, namely as

$$\overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \cdots \cdots \overset{d}{\bullet} \overset{e}{\bullet} \quad \text{for } a, b, c, \dots, d, e \in \mathbb{Z}_{\geq 0}.$$

In particular,

$$\mathbb{S}_\perp = \overset{1}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \cdots \cdots \cdots \overset{0}{\bullet} \overset{0}{\bullet} \quad \odot^k \mathbb{S}_\perp = \overset{k}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \cdots \cdots \cdots \overset{0}{\bullet} \overset{0}{\bullet}$$

and

$$\mathbb{E} = \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \cdots \cdots \overset{d}{\bullet} \overset{e}{\bullet} \implies \overset{a+k}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \cdots \cdots \overset{d}{\bullet} \overset{e}{\bullet} = \odot^k \mathbb{S}_\perp \odot \mathbb{E},$$

where $\odot$ denotes the *Cartan product*, namely the unique irreducible summand of the tensor product whose highest weight is the sum of the highest weights of the two factors. We shall use the same notation for the corresponding Cartan product of irreducible symplectic bundles. Finally let us notice that

$$(6.21) \quad \mathbb{S}_\perp^k \cong \odot^k \mathbb{S}_\perp \oplus \odot^{k-2} \mathbb{S}_\perp \oplus \odot^{k-4} \mathbb{S}_\perp \oplus \cdots$$

as the algebraic counterpart of (3.13). A key geometric consequence alluded to above is as follows.

Theorem 6.5. *Suppose E is an irreducible symplectic bundle on a contact manifold of dimension $2n + 1$ corresponding to the irreducible representation*

$$\mathbb{E} = \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \cdots \cdots \overset{d}{\bullet} \overset{e}{\bullet} \quad \text{of } \text{Sp}(2n, \mathbb{R}).$$

Let $F = \odot^k \Lambda_H^1 \odot E$. Suppose $D : E \rightarrow F$ is a k^{th} order linear differential operator compatible with the contact structure whose enhanced symbol is the composition

$$S_\perp^k \otimes E \rightarrow \odot^k \Lambda_H^1 \otimes E \xrightarrow{\odot} \odot^k \Lambda_H^1 \odot E = F.$$

Then D is of finite-type and the dimension of its solution space is bounded by the dimension of the representation

$$\overset{k-1}{\bullet} \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \cdots \cdots \overset{d}{\bullet} \overset{e}{\bullet}$$

of $\mathfrak{sp}(2(n+1), \mathbb{R})$.

Remark. On a symplectic manifold of dimension $2n$ the structure group of the full co-frame bundle is reduced to $\mathrm{Sp}(2n, \mathbb{R})$. Hence, one can similarly define symplectic vector bundles on symplectic manifolds as those induced by the finite-dimensional representations of $\mathrm{Sp}(2n, \mathbb{R})$. Suppose E is such a bundle, induced by the irreducible representation

$$\mathbb{E} = \overset{a}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet} \quad \text{of } \mathrm{Sp}(2n, \mathbb{R}).$$

Let $F = \odot^k \Lambda^1 \odot E$, in other words F is induced by the representation $\overset{a+k}{\bullet} \overset{b}{\bullet} \overset{c}{\bullet} \cdots \overset{d}{\bullet} \overset{e}{\bullet}$, which we shall denote by $\mathbb{F}$. Suppose $D : E \rightarrow F$ is a k^{th} order linear differential operator whose symbol is the composition

$$\odot^k \Lambda^1 \otimes E \xrightarrow{\odot} \odot^k \Lambda^1 \odot E = F.$$

Then the hypotheses of Theorem 4.6 apply owing to the following observations.

- Λ^1 is induced by $\mathbb{S}_\perp$, the defining representation of $\mathrm{Sp}(2n, \mathbb{R})$
- K is induced by $\mathbb{K} \equiv \ker \odot : \odot^k \mathbb{S}_\perp \otimes \mathbb{E} \rightarrow \mathbb{F}$
- K^ℓ is induced by $\mathbb{K}^\ell \equiv (\odot^\ell \mathbb{S}_\perp \otimes \mathbb{K}) \cap (\odot^{k+\ell} \mathbb{S}_\perp \otimes \mathbb{E})$
- $\mathbb{K}^\ell \subseteq (\mathbb{S}_\perp^\ell \otimes \mathbb{K}_H) \cap (\mathbb{S}_\perp^{k+\ell} \otimes \mathbb{E}) \forall \ell \geq 0$, where $\mathbb{K}_H \equiv \ker \odot : \mathbb{S}_\perp^k \otimes \mathbb{E} \rightarrow \mathbb{F}$.

In other words, the consequences of Kostant's Theorem detailed in Propositions 6.3 and 6.4 are clearly stronger than needed (for D to be finite-type in the sense of Theorem 4.6) by dint of (6.21). In particular, we obtain a bound on the dimension of the solution space for D . Presumably, this bound is not at all sharp.

It is also possible to adapt the theory to deal with contact manifolds endowed with certain additional structures following a similar set of variations concerning general prolongations. In the general case, the main examples are affine manifolds and Riemannian manifolds. However, as detailed in [1], the theory also applies to geometries derived from any $|1|$ -graded simple Lie algebra. For contact manifolds, the corresponding results are as follows.

Recall the decomposition (6.3) of $\mathfrak{sp}(2(n+1))$:–

$$\mathfrak{sp}(2(n+1)) = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension } 2n+1} \oplus \underbrace{\mathbb{R} \oplus \mathfrak{sp}(2n)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

These salient features pertain for any simple Lie algebra other than $\mathfrak{sl}(2)$. For the remaining classical Lie algebras

$$\mathfrak{sl}(n+2) = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension } 2n+1} \oplus \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{sl}(n)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

$$\mathfrak{so}(n+4) = \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension } 2n+1} \oplus \underbrace{\mathbb{R} \oplus \mathfrak{sl}(2) \oplus \mathfrak{so}(n)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

and for the exceptional Lie algebras:–

$$\begin{aligned}
E_6 &= \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension 21}} \oplus \underbrace{\mathbb{R} \oplus \mathfrak{sl}(6)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\
E_7 &= \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension 33}} \oplus \underbrace{\mathbb{R} \oplus \mathfrak{so}(12)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\
E_8 &= \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension 57}} \oplus \underbrace{\mathbb{R} \oplus E_7}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\
F_4 &= \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension 15}} \oplus \underbrace{\mathbb{R} \oplus \mathfrak{sp}(6)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\
G_2 &= \underbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\text{Heisenberg algebra of dimension 5}} \oplus \underbrace{\mathbb{R} \oplus \mathfrak{sl}(2)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.
\end{aligned}$$

In each case, the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is compatible with the Lie bracket having values in the 1-dimensional $\mathfrak{g}_{-2}$. We obtain a subalgebra of the symplectic Lie algebra and corresponding subgroups by exponentiation:–

$$\begin{aligned}
\mathrm{Sp}(2n, \mathbb{R}) &\supset \mathrm{Sp}(2n, \mathbb{R}) \quad \mathrm{SL}(n, \mathbb{R}) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R}) \quad \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(n) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R}) \\
\mathrm{SL}(6, \mathbb{R}) &\hookrightarrow \mathrm{Sp}(20, \mathbb{R}) \quad \mathrm{Spin}(12) \hookrightarrow \mathrm{Sp}(32, \mathbb{R}) \quad E_7 \hookrightarrow \mathrm{Sp}(56, \mathbb{R}) \\
\mathrm{Sp}(6, \mathbb{R}) &\hookrightarrow \mathrm{Sp}(14, \mathbb{R}) \quad \mathrm{SL}(2, \mathbb{R}) \hookrightarrow \mathrm{Sp}(4, \mathbb{R}).
\end{aligned}$$

In most cases, it is straightforward to describe these embeddings by explicit formulæ. For these purposes, let us realise

$$\mathrm{Sp}(2n, \mathbb{R}) = \{S \in \mathrm{SL}(2n, \mathbb{R}) \text{ s.t. } SJS^t = J\}, \quad \text{where } J \equiv \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}.$$

Then, for example,

$$\begin{aligned}
\mathrm{SL}(n, \mathbb{R}) \ni M &\mapsto \begin{bmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) \quad (6.22) \\
\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(n) \ni \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, M \right) &\mapsto \begin{bmatrix} aM & bM \\ cM & dM \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) \\
\mathrm{SL}(2, \mathbb{R}) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\mapsto \begin{bmatrix} a^3 & \sqrt{3}a^2b & -b^3 & \sqrt{3}ab^2 \\ \sqrt{3}a^2c & 2abc + a^2d & -\sqrt{3}b^2d & 2abd + b^2c \\ -c^3 & -\sqrt{3}c^2d & d^3 & -\sqrt{3}cd^2 \\ \sqrt{3}ac^2 & 2acd + bc^2 & -\sqrt{3}bd^2 & 2bcd + ad^2 \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R}).
\end{aligned}$$

There are alternative real forms of some of these embeddings. For example, instead of $\mathrm{SL}(n, \mathbb{R}) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ we may consider the standard embedding $\mathrm{SU}(n) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ obtained by regarding J as a complex structure.

For each of these subgroups $G \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ we may define an additional structure on a contact manifold of dimension $2n + 1$ by reducing the structure group on the contact distribution to G . Often there is a simple geometric interpretation of such an additional reduction. For example, it is clear from the embedding (6.22) that such a reduction corresponds to choosing a pair of transverse Lagrangian subdistributions of

the contact distribution. A contact manifold equipped with a reduction of structure group to $SU(n) \subset Sp(2n, \mathbb{R})$ is the same as an almost CR-structure together with a choice of contact form, i.e. an almost pseudo-Hermitian structure. There are many natural differential operators compatible with these various structures. Here are some simple examples.

Example. For any partial connection ∇_H on Λ_H^1 on a contact manifold, consider the differential operator D defined as the composition

$$\odot^k \Lambda_H^1 \xrightarrow{\nabla_H} \Lambda_H^1 \otimes \odot^k \Lambda_H^1 \xrightarrow{\odot} \odot^{k+1} \Lambda_H^1.$$

Using abstract indices as in §3.1,

$$\sigma_{bc\dots d} \xrightarrow{D} \nabla_{(a} \sigma_{bc\dots d)}.$$

This is an operator between symplectic bundles whose symbol in dimension $2n + 1$ is induced by the homomorphism of $Sp(2n, \mathbb{R})$ -modules

$$\begin{array}{c} 1 \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} \otimes \begin{array}{c} k \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} \longrightarrow \begin{array}{c} k+1 \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array}.$$

Theorem 6.5 applies and we conclude that

$$\dim\{D\sigma = 0\} \leq \dim \begin{array}{c} 0 \ k \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} = \frac{(2n+k)!(2n+k-1)!(2n+2k+1)}{(2n-1)!(2n+1)!k!(k+1)!}.$$

The case $n = 1$ gives the series of bounds presented at the end of §3.

Example. Suppose that ∇_H is a partial connection on Λ_H^1 and consider the differential operator

$$\sigma_{cde} \xrightarrow{D} \nabla_{(a} \nabla_b \sigma_{cde)}$$

for σ_{cde} symmetric. It is a differential operator

$$D : \begin{array}{c} 3 \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} \longrightarrow \begin{array}{c} 5 \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array}$$

to which Theorem 6.5 applies, whence

$$\dim\{D\sigma = 0\} \leq \dim : \begin{array}{c} 1 \ 3 \ 0 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} = \frac{4n(n+1)^2(n+2)(n+4)(2n+1)(2n+3)}{45}.$$

Example. Suppose that ∇_H is a partial connection on Λ_H^1 and consider the differential operator

$$\sigma_{bc} \xrightarrow{D} \nabla_{(a} \sigma_{b)c} + \frac{1}{2n+1} (L^{de} \nabla_d \sigma_{e(a)} L_{b)c} + \frac{1}{2n+1} L_{c(a} L^{de} \nabla_b) \sigma_{de}$$

for σ_{bc} skew and trace-free with respect to the Levi form, i.e. a section of $\Lambda_{H^\perp}^2$ (we must suppose that $n \geq 2$). It is designed as an operator

$$D : \begin{array}{c} 0 \ 1 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} \longrightarrow \begin{array}{c} 1 \ 1 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array}$$

to which Theorem 6.5 applies, whence

$$\dim\{D\sigma = 0\} \leq \dim : \begin{array}{c} 0 \ 0 \ 1 \ 0 \ \dots \ 0 \ 0 \\ \bullet \bullet \bullet \ \dots \ \bullet \bullet \end{array} = \frac{2(n-1)(n+1)(2n+3)}{3}.$$

Example. In [8], the authors consider a second order differential operator compatible with a contact Lagrangian structure [4, §4.2.3] in 3-dimensions whose leading terms in Darboux local coördinates are

$$f \mapsto (X^2 f, Y^2 f).$$

By taking $\mathbb{E} = \overset{1}{\bullet} \overset{1}{\bullet}$ in the contact Lagrangian counterpart of Theorem 6.5, it follows that the solutions space has dimension bounded by 8. This dimension bound agrees with [8, Theorem 3.1], which the authors establish by an ad hoc prolongation.

Remark. There is a close parallel between the methods used in this article and the methods of *parabolic geometry* as described in [4]. These methods are informally and collectively known as the *Bernstein-Gelfand-Gelfand* machinery and Kostant's computation [11] of Lie algebra cohomologies is a key ingredient in this machinery. The *homogeneous models* and their first *Bernstein-Gelfand-Gelfand operators* show that the dimension bounds in Theorem 6.5 and its parabolic variants are sharp. The prolongations of [1] compare to $|1|$ -graded parabolic geometries [4, §4.1] in an entirely analogous fashion.

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