

ON THE COHOMOLOGY THEORY
OF KNOT GROUPS.

by

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SUMMARY

This thesis deals with the cohomology theory of groups, attempting, in particular to find a proof of the following conjecture.

CONJECTURE : All knot groups have cohomological dimension 2.

An introduction is given to the theory of knots and knot groups, together with the cohomology theory of a group.

A proof of the conjecture for all knot groups has not been found. However, using known results from the theory of groups in general and knot groups in particular, the following classes of knot groups are shown to satisfy the conjecture.

- a) The class of knot groups which have a presentation involving only one relator.
- b) The class of groups of alternating knots.
- c) The class of groups of non-alternating knots with less than ten (10) crossings.
- d) The class of knot groups whose commutator subgroup is free.

A method developed by myself, under the instigation of my supervisor Dr. I.M.S. Dey, shows a necessary and sufficient condition for a knot group to satisfy the conjecture. Using the condition, the following knots are shown to have groups with cohomological

dimension 2, provided the assumption made in section 4.6 is correct.

- e) The group of the Kinoshita Terasaka knot.
- f) The groups of knots as shown in Fig.1 below :

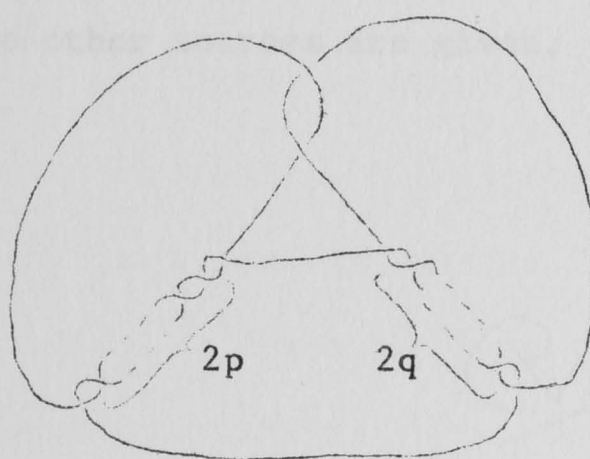


Fig. 1

- g) The groups of knots as shown in Fig. 2 below.

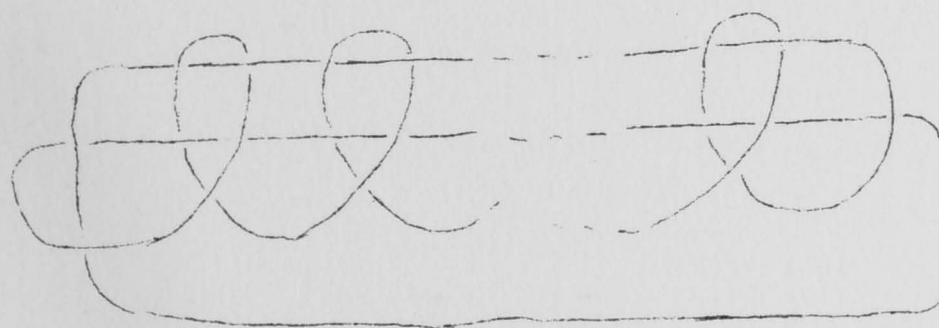


Fig.2

A proof for the assumption referred to above has not yet been found. The assumption seems plausible, and can probably be proved using advanced techniques of algebraic topology, with which the author is not familiar at this time.

STATEMENT.

The work reported in this thesis is due to the author, except when references to other sources are given.

R. V. Lagarias

INTRODUCTION.

0.1 The aim of this thesis is to prove the conjecture :

CONJECTURE : All knot groups have cohomological dimension two.

We introduce the background material in chapters 1 and 2. These chapters cover the theory of knot groups and the subject of the cohomology theory of a group. Most proofs are omitted, being readily available in the references given. The notation used in the thesis is introduced in these chapters.

The important definitions in these chapters are those of :

- a) a knot
- b) the knot group
- c) the cohomology theory of a group
- and d) the cohomological dimension of a group.

These concepts explain the conjecture made above.

Other topics developed in chapters 1 and 2, for later use in chapter 4 are :

- a) the method of representing a knot on a plane.
- b) the determination of a presentation for the knot group from the plane representation of a knot.
- c) the Gruenberg Resolution, from which the cohomology theory of a group may be found.

A knowledge of the basic theory of groups, free groups and presentations of groups is taken as understood by readers of this thesis.

0.2 By using some results from the literature of group theory and knot theory, we are able to find a set of sufficient conditions for a knot group to have cohomological dimension 2.

To give an adequate background for all the results would consume far too much space. Therefore, again, references to the appropriate literature is provided.

The knot groups covered by these methods are those denoted a) - d) in the summary on p.ii.

Also in this chapter a proof due to G.Baumslag of the conjecture in section 0.1 is shown to be incorrect.

0.3 The work done in the final chapter is on a method of showing the cohomological dimension of a knot group to be 2, indicated to me by my supervisor I.M.S. Dey. The method involves the use of the Gruenberg Resolution and the Wirtinger presentation of a knot group to find a necessary and sufficient condition on a knot group for it to have cohomological dimension 2.

In attempting to show that all knot groups satisfy this condition I have been unsuccessful. It appears that rather difficult topological methods would be required to obtain a positive

result. However by using the assumption given in section 4.6, and simple topological methods, I show that some further classes of knot groups (e) - g) in the summary) satisfy the conjecture in section 0.1. If the assumption is incorrect then more work needs to be done for these classes of knot groups.

Although the method seems in general to be unsuccessful, I have been unable to find any counterexamples to the conjecture, in as much as I have found no knot groups which do not satisfy the necessary and sufficient condition arrived at in chapter 4.

The thesis therefore ends rather unsatisfactorily with no positive result on the conjecture in general. I believe, however, that more research, involving some advanced topological methods, would bring the investigations in this thesis to a satisfactory conclusion.

CHAPTER 1

KNOT GROUPS

1.0 INTRODUCTION.

In this section we shall make a mathematical analogy or interpretation of our physical notion of a knot. Using this analogy we shall associate with each knot a group, unique up to isomorphism, by considering the placement of the knot in 3-dimensional Euclidean space.

A detailed exposition of the procedure for finding such groups is given in the book by Crowell & Fox [2], chapters 1 - 6. All proofs omitted in the brief survey given here may be found in the book.

1.1 DEFINITIONS.

1.1.1 KNOT. We consider a knot tied in a single piece of string whose ends are joined together. The interpretation of this concept is :

DEFINITION: A figure K in 3-dimensional space R^3 is a knot if there exists a homeomorphism of the unit circle C into the space R^3 whose image is K .

(The unit circle C is the set of points (x,y) in R^2 satisfying $x^2 + y^2 = 1$).

1.1.2 EQUIVALENCE OF KNOTS. We interpret the notion of deforming a knot in a piece of string into another knot, without having to cut and rejoin the string at any point.

DEFINITION: Two knots K_1 and K_2 are equivalent if there exists a homeomorphism of R^3 onto R^3 which maps K_1 onto K_2 .

If we had made our definition of a knot to interpret the situation where the piece of string in which the knot were tied to have two ends, then the above definition of equivalence would be trivial, since all knots would be equivalent.

1.1.3 POLYGONAL KNOTS.

DEFINITION: A figure in R^3 is a closed line segment if it is homeomorphic to the closed interval $[0,1]$ in R^1 .

DEFINITION: If a knot is the union of a finite number of straight closed line segments, called edges, then it is called a polygonal knot.

1.1.4 TAME KNOT.

DEFINITION: A knot is tame if it is equivalent to a polygonal knot.

We shall deal exclusively with tame knots, for the group associated with non-tame knots (usually called wild knots) is too difficult to consider in this thesis.

1.2 THE REPRESENTATION OF A KNOT IN A PLANE.

1.2.1 In order to easily draw a knot on paper we need a standard method of representing a knot in a plane. We do this in an obvious way, so that certain properties of the knot are reflected in the plane representation.

Briefly, we project the knot in 3-dimensional space, R^3 , into 2-dimensional space, R^2 . We require the following properties for this projection :

- a) No more than two points of the knot have the same image under projection into the plane.
- b) No two open segments of the knot have the same image under projection into the plane.

[An open segment of the knot is a subset of points of the knot which is homeomorphic to the open interval $(0,1)$.]

- c) The configuration shown in Fig. 1 does not occur in the projection.

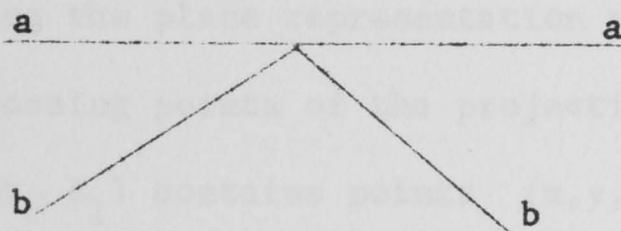


Fig. 1.

(Where aa and bb are distinct segments of the projection)

1.2.2 Consider the projection $P : R^3 \rightarrow R^2$ where :

$$P(x,y,z) = (x,y,0).$$

For a knot K , the image of K under P may have the properties 1.2.1 a), b), c). If it does not then we can make sure all the properties are satisfied by obtaining another projection in the following way :

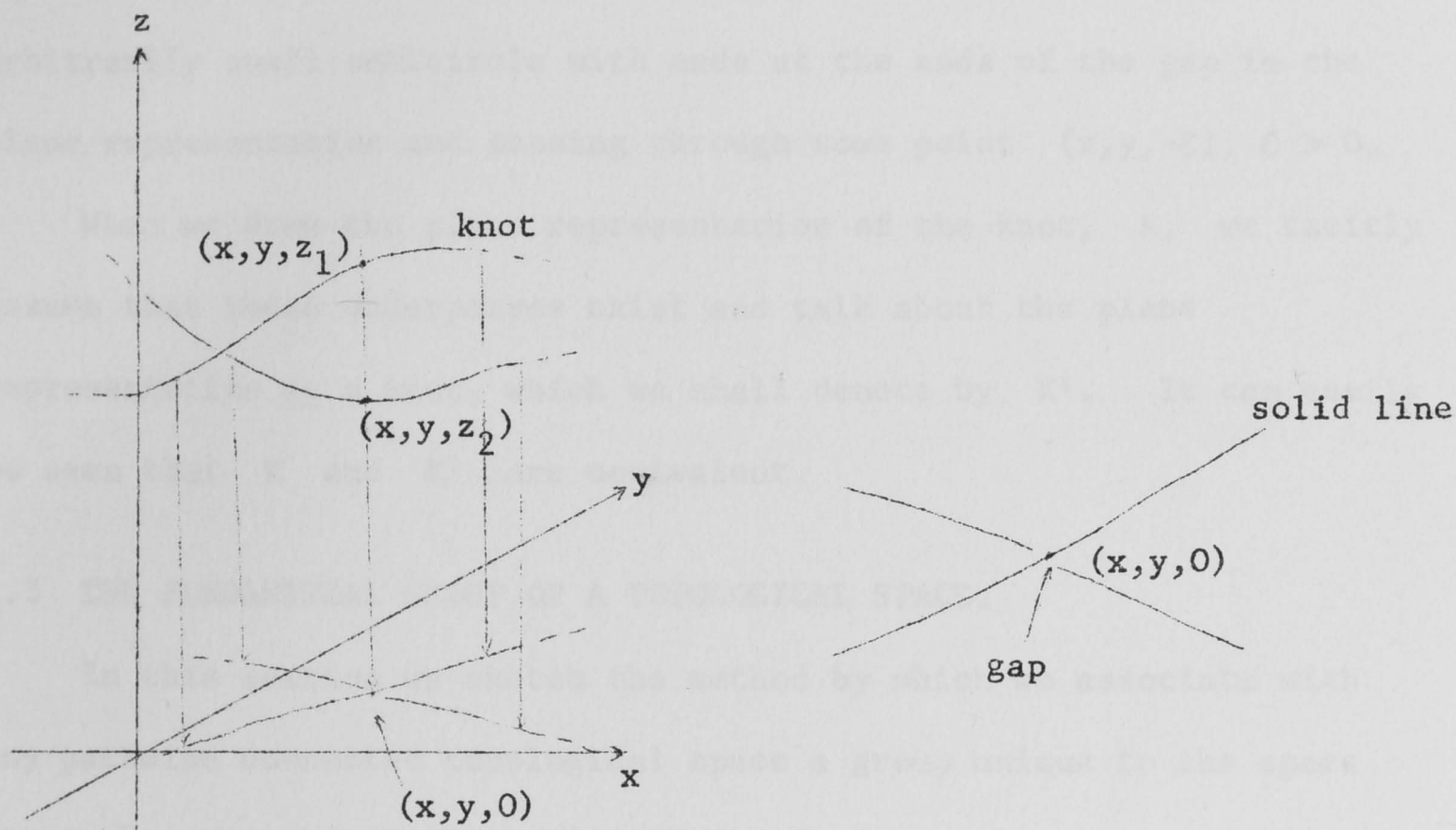
By finding some homeomorphism $h : R^3 \rightarrow R^3$ which will take the knot K onto an equivalent knot $K_1 = h(K)$, so that the projection $P(h(K)) = P(K_1)$ satisfies the conditions.

Such a homeomorphism need be quite trivial and can easily be determined by looking at the projection of the knot K to find how this needs to be altered to satisfy a), b), c). Crowell & Fox [23] p.7 give a precise proof of this.

We therefore define a plane representation of a knot K to be a projection of the knot, K , or an equivalent knot, which satisfies the conditions 1.2.1 a), b), c).

1.2.3 In drawing the plane representation we adopt the following convention for the crossing points of the projection. If the knot K (or an equivalent knot K_1) contains points (x, y, z_1) and (x, y, z_2) , where $z_1 > z_2$, then we draw the segment of the projection which is the image of the segment of the knot containing (x, y, z_1) as a solid line, and leave a small gap in the projection of the segment containing (x, y, z_2) , as shown in the diagram.

The crossing appears as shown.



When we have such a plane representation for a knot, we can consider this as a knot if we add small underpasses where the gaps mentioned previously exist. The diagram illustrates this :



plane representation of a crossing.

addition of underpass to make the lines continuous.

The underpass is not in the plane of the representation but is an arbitrarily small semicircle with ends at the ends of the gap in the plane representation and passing through some point $(x, y, -\varepsilon)$, $\varepsilon > 0$.

When we draw the plane representation of the knot, K , we tacitly assume that these underpasses exist and talk about the plane representation as a knot, which we shall denote by K' . It can easily be seen that K and K' are equivalent.

1.3 THE FUNDAMENTAL GROUP OF A TOPOLOGICAL SPACE.

In this section we sketch the method by which we associate with any pathwise connected topological space a group unique to the space (up to homeomorphism) which we call its fundamental group.

1.3.1 DEFINITION. Path.

A Path in a topological space X is a continuous map of some closed interval into the space X .

[For any two real numbers m, n where $m < n$, the closed interval $[m, n]$ is the set $\{x \in \mathbb{R} : m \leq x \leq n\}$.]

We call the images under the continuous map of the end points m, n of the interval, the end points of the path. We say the path joins these end points.

Further we call the image of m under the map the initial point and the image of n the terminal point. We give the path a direction in the sense from the initial to the terminal point along the path.

1.3.2 DEFINITION. Loop.

In a topological space X containing a point p , we define a p -based loop as a path with initial and terminal point the same at p .

Thus a p -based loop, a_p , is a continuous map

$$a : [m,n] \rightarrow X, \text{ where } a(m) = a(n) = p.$$

We call two loops a_p, b_q equivalent, where $p, q \in X$ if there exists a continuous map

$$f : [m,n] \rightarrow [0,1] \rightarrow X,$$

such that :

$$\begin{aligned} f[x,0] &= a(x) \\ f[x,1] &= b(x) \end{aligned} \quad x \in [m,n]$$

and $f[m,y] = p = q = f[n,y]$ for $y \in [0,1]$.

We denote the equivalence by $a_p \cong b_q$ (noting $p = q$).

It can be shown that this is in fact an equivalence relation on the loops in the top space X .

Intuitively we say that a_p can be continuously deformed into b_q in the space X .

1.3.3 DEFINITION. Product of Loops.

Let a_p, b_q be two loops in X .

Assume a_p is $a_p : [m,n] \rightarrow X$

and b_q is $b_q : [n,t] \rightarrow X$.

Suppose $p = q$.

Then we define the product $a_p \cdot b_q$ of the loops by

$$a \cdot b_p(x) = \begin{cases} a_p(x) & : x \in [m, n] \\ b_p(x) & : x \in [n, t]. \end{cases}$$

This is a map $a \cdot b_p : [m, t] \rightarrow X$.

Oviously this is again a p -based loop in the space X .

Further this multiplication can be shown to be associative.

DEFINITION. Identity loop.

The identity loop e_p is the constant map :

$$e_p : [m, n] \rightarrow X \text{ where } e_p(x) = p \quad \forall x \in [m, n].$$

This is both a right and left multiplicative identity for any p -based loop.

1.3.4 THE GROUP OF EQUIVALENCE CLASSES OF LOOPS.

From section 1.3.2 and 1.3.3 we have that the set of p -based loops in a top. space X has a multiplication operation which is closed and associative, and also contains a mult. identity.

Thus the set is a semigroup with identity.

Now consider the set of equivalence classes of the p -based loops of the topological space X .

Denote the equivalence class of the loop q_p by $[a_p]$.

Define multiplication in this set by :

$$[a_p] \cdot [b_p] = [a_p \cdot b_p] = [a \cdot b_p].$$

This can be shown to be well defined.

Again the multiplication is closed and associative.

The class $[e_p]$ is an identity for the multiplication.

Consider the path $a_p : [m,n] \rightarrow X$.

The path $a_p^* : [m,n] \rightarrow X$, defined by

$$a_p^*(m) = a_p(n)$$

$$a_p^*(n) = a_p(m)$$

$$a_p^*(x) = a_p(n-x) \quad \text{for } x \in [m,n],$$

is again a p -based loop and can be shown to have the property that :

$$[a_p^*] \cdot [a_p] = [e_p] = [a_p] \cdot [a_p^*].$$

Thus $[a_p^*] = [a_p]^{-1}$, the mult. inverse class to that of the loop a_p .

For any equivalence class we can construct such a multiplicative inverse.

Thus the set of equivalence classes of p -based loops, together with the multiplication defined is a group.

We call this group the fundamental group of the space X at the point p and denote it by $\pi_1(X,p)$.

1.3.5 SOME PROPERTIES OF THE FUNDAMENTAL GROUP.

DEFINITION. Pathwise connected topological space.

A top. space X is pathwise connected if any two points of the space are joined by a path.

THEOREM 1 : Let X be a pathwise connected topological space.

Then $\pi_1(X, p) \cong \pi_1(X, p')$, $p, p' \in X$.

A proof may be found in Massey [11] p.62. This is quite easy.

From this theorem we see that the fundamental group of such a space is independent, up to isomorphism, of the base point. We therefore denote the fundamental group by $\pi_1(X)$.

THEOREM 2 : Let X, Y be homeomorphic pathwise connected top. spaces. Then $\pi_1(X) \cong \pi_1(Y)$.

The proof is a corollary of Th.8.3 p.82 Massey [11].

We shall use this result in the next section to show that the groups we define as "knot groups" are isomorphic for equivalent knots.

1.4 KNOT GROUPS.

1.4.1 DEFINITION. KNOT GROUP.

The knot group G of a knot K in 3-dimensional space R^3 is the fundamental group of the topological space $R^3 - K$.

$$\text{i.e. } G = \pi_1(R^3 - K).$$

This is well defined since $R^3 - K$ is pathwise connected. This can be shown using topological methods developed by Crowell & Fox [2] Ch.5. Then by TH. 1, 1.3.5 the fundamental group is independent of base point.

From the definition 1.1.2 of equivalent knots we have that if knots K_1 and K_2 are equivalent, then K_2 is the image of K_1

under a homeomorphism $h : R^3 - R^3$. The map h will also map the complementary space $R^3 - K_1$ homeomorphically onto $R^3 - K_2$.

i.e. $R^3 - K_1$ is homeomorphic to $R^3 - K_2$.

Thus by TH.2,1.3.5 $\pi_1(R^3 - K_1) \cong \pi_1(R^3 - K_2)$.

Hence the knot groups of equivalent knots are unique up to isomorphism.

1.4.2 A PRESENTATION FOR THE KNOT GROUP.

A presentation of a group G is the expression of G as being isomorphic to a quotient group Y/R of a free group Y by a normal subgroup R of Y .

Consider the map $p : Y \rightarrow G$, where $p = \rho i$, the composition of the natural map $\rho : Y \rightarrow Y/R$ and the isomorphism $i : Y/R \rightarrow G$.

Then briefly, the images under p of the set $\underline{y}$ of generators of the free group Y will generate the group G . Further, since R is the kernel of p , the elements of R being products of elements of $\underline{y}$ give relations between the elements of G .

Usually we consider R as the normal closure of a set $\underline{r}$ of elements of Y , and denote this presentation of G as $G = \langle \underline{y}; \underline{r} \rangle$.

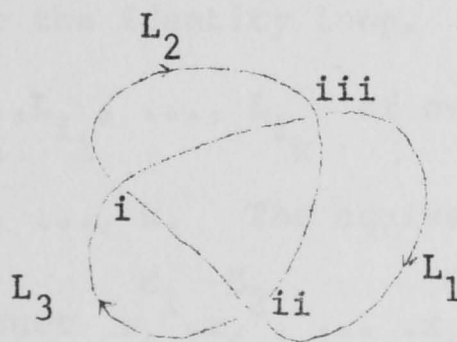
1.4.3 THE GENERATORS OF THE KNOT GROUP.

Consider the representation of a knot, K , in the plane. The representation has a finite set of crossing points at which two segments of the plane projection intersect. That there are only a

finite number is obvious from the properties of the projection and the fact that we are dealing with tame knots. We have established a criterion (1.2.3) for this crossing configuration wherein one segment is an overcrossing, and the other (represented by a gap) is an undercrossing.

There will be a set of disjoint continuous lines called overpasses which make up the plane representation. The number of these will be the same as the number of crossings. This can easily be seen if we assign a direction along the knot and transfer this in a natural way to the plane representation, noticing then that each overpass ends at one crossing, and each crossing corresponds to the end of one overpass.

e.g. The "trefoil knot" has the following representation with 3 crossings and 3 overpasses.



Now take any point p in R^3 "above" the plane of the representation of the knot. Consider the p -based loops each of which travels under just one of the overpasses in the direction indicated in Fig.2

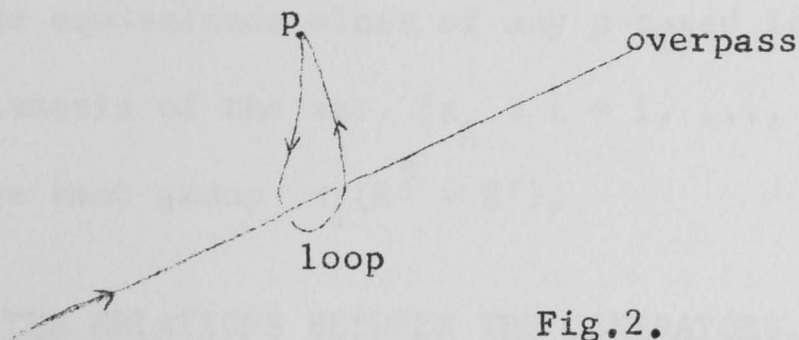


Fig.2.

Number the overpasses consecutively along the direction of the knot to obtain the set $\{L_1, \dots, L_n\}$ of overpasses. We can begin this enumeration at any overpass of the projection. Denote by x_i the equivalence class in $R^3 - K'$ of a p -based loop around the overpass L_i . Then we have :

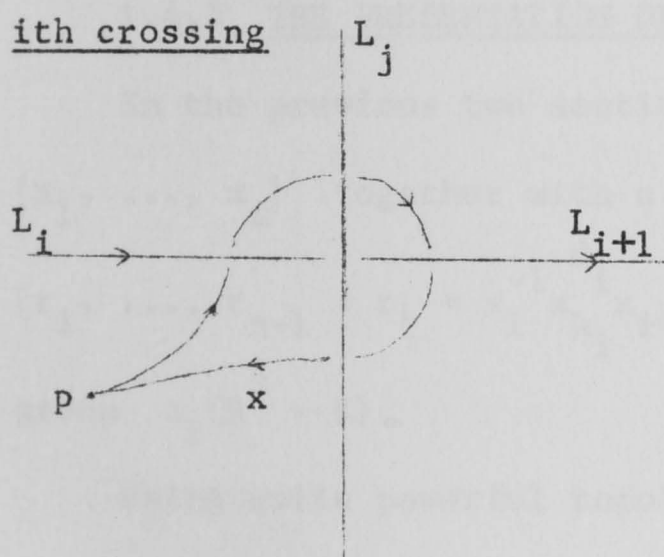
LEMMA: The set $\{x_i; i = 1, \dots, n\}$ generates the knot group $\pi_1\{R^3 - K'\}$.

This can be seen by considering any p -based loop in $R^3 - K'$. If the loop does not pass under any of the overpasses of the knot then it is equivalent to the identity loop. If it does then it does so in some sequence $L_{i_1}, L_{i_2}, \dots, L_{i_k}$ of overpasses, where $i_j \in \{1, \dots, n\}; j = 1, \dots, k$. The equivalence class of the loop can be seen to be a product $x_{i_1}^{\epsilon_1} \cdot x_{i_2}^{\epsilon_2} \cdot \dots \cdot x_{i_k}^{\epsilon_k}$ of elements of the set $\{x_i : i = 1, \dots, n\}$, where $\epsilon_j = \pm 1$ depending upon whether the direction of the loop under the overpass L_{i_j} is the same or opposite to the direction of a loop in the equivalence class x_{i_j} .

Thus the equivalence class of any p -based loop is a product of powers of elements of the set $\{x_i : i = 1, \dots, n\}$. i.e. the set generates the knot group $\pi_1(R^3 - K')$.

1.4.4 THE RELATIONS BETWEEN THE GENERATORS.

We can find n relations between the n generators of the group $\pi_1(R^3 - K)$ by considering the n loops as shown each corresponding to one crossing of the knot, representation.



It can be seen that the loop x is in the equivalence class of the trivial loop.

Also it can be seen that

$$[x] = x_i^{-1} \cdot x_j^{\varepsilon} \cdot x_{i+1} x_j^{-\varepsilon}, \text{ where}$$

$\varepsilon = \pm 1$ depending on the direction of L_j .

Thus $x_i^{-1} \cdot x_j^{\varepsilon} \cdot x_{i+1} x_j^{-\varepsilon} = 1$ in the group $\pi_1(R^3 - K')$.

The overpass L_j will be determined by the overpass L_i , since each overpass ends at one and only one other. Thus the subscript j is dependent on the subscript i . We shall put $j = k_i$. Hence k is in effect a function from the set $\{1, \dots, n\}$ into itself.

Since ε depends on the direction of L_j , which in turn depends on the overpass L_i , we put $\varepsilon = \varepsilon_i$ for the i th crossing. In this way we obtain n relations one from each crossing of the form :

$$x_i^{-1} x_{k_i}^{\varepsilon_i} x_{i+1} x_{k_i}^{-\varepsilon_i} = 1; \quad i = 1, \dots, n, \quad \text{for the group } \pi_1(R^3 - K).$$

It can be shown, [2], that all other relations of the group are the products of a finite number of these relations or their conjugates.

Further it may be shown (Crowell & Fox [2] p.84) that any one relation is a consequence of the others.

1.4.5 THE PRESENTATION OF THE GROUP.

In the previous two sections we have given a set of generators $\{x_1, \dots, x_n\}$ together with a set of relations

$$\{r_1, \dots, r_{n-1} : r_i = x_i^{-1} x_{k_i}^{\varepsilon_i} x_{i+1} x_{k_i}^{-\varepsilon_i}, \quad i = 1, \dots, n-1\} \quad \text{for the group } \pi_1(R^3 - K).$$

Using quite powerful topological methods Crowell & Fox [2] Ch. VI prove that a presentation for the knot group $\pi_1(R^3 - K)$ is $\langle x_1, \dots, x_n; r_1, \dots, r_{n-1} \rangle$.

This presentation is called the Wirtinger presentation of the knot group. We shall make use of it in ch.4 of this thesis.

CHAPTER 2

THE COHOMOLOGY THEORY OF GROUPS

2.0 We present here a brief survey of the cohomology of groups, first defining most of the concepts used in the theory. An introduction to the Gruenberg Resolution concludes the chapter. We shall use this resolution in Ch.4 in the investigation of the cohomology of knot groups.

2.1 DEFINITIONS.

2.1.1 Category: A Category C is a collection of objects which we denote $A, B, C, \dots$, and mappings between them, denoted $\alpha, \beta, \gamma, \dots$ which we call morphisms, and written as right hand operations, such that the following conditions hold :

- i) If $\alpha, \beta \in C$ then $\alpha\beta$ is defined if domain $\beta = \text{range } \alpha$.
- ii) If $(\alpha\beta)\gamma$ is defined then so is $\alpha(\beta\gamma)$ and the two are equal.
- iii) Define an identity morphism i as one for which $i\alpha = \alpha$ and $\beta i = \beta$ whenever these products are defined.

Then there exists an identity morphism i_A corresponding to each object A such that if $\alpha : A \rightarrow B$ is a morphism, then

$$i_A \alpha = \alpha = \alpha i_B.$$

2.1.2 Functor: A Functor F is a mapping from a category $\mathcal{C}$ to a category $\mathcal{D}$, which maps objects into objects and morphisms into morphisms, such that :

- i) If $\gamma : C_1 \rightarrow C_2$ in $\mathcal{C}$ then $F(\gamma) : F(C_1) \rightarrow F(C_2)$ in $\mathcal{D}$.
- ii) $F(i)$ is an identity in $\mathcal{D}$ whenever i is an identity in $\mathcal{C}$.

We call a functor F covariant if $F(\gamma_1)\gamma_2$, and contravariant if $F(\gamma_1\gamma_2) = F(\gamma_2) \cdot F(\gamma_1)$.

2.1.3 We define a relationship between functors as follows :

Let F, H be two covariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$.

Then a natural transformation $\tau : F \rightarrow H$ is a collection of morphisms of $\mathcal{D}$, $\tau(C) : F(C) \rightarrow H(C)$, one for each object $C \in \mathcal{C}$, such that if $\gamma : C_1 \rightarrow C_2$ in $\mathcal{C}$ then :

$$\begin{array}{ccc} F(C_1) & \xrightarrow{F(\gamma)} & F(C_2) \\ \downarrow \tau(C_1) & & \downarrow \tau(C_2) \\ H(C_1) & \xrightarrow{H(\gamma)} & H(C_2) \end{array} \quad \text{is a commutative diagram in } \mathcal{D}.$$

If all the $\tau(C)$ are invertible in $\mathcal{D}$, then τ is called a natural equivalence, and the functors F and H are called naturally equivalent.

2.1.4 Modules: Given a group G , we define a left- [or right] G -module as an abelian group $(A,+)$ together with a mapping $G \times A \rightarrow A$, [resp. $A \times G \rightarrow A$] given by $(g,a) \rightarrow ga$, $[(a,g) \rightarrow ag]$ satisfying :

- i) $g(a_1 + a_2) = ga_1 + ga_2$ $[(a_1 + a_2)g = a_1g + a_2g]$
- ii) $(g_1g_2)a = g_1(g_2a)$ $[a(g_1g_2) = (ag_1)g_2]$
- iii) $1a = a$, where 1 is the $[a1 = a]$
identity in G .

We notice that i) shows the map $a \mapsto ga$ to be a homomorphism; and that ii) and iii) show that each map has an inverse. Hence the action of g on A is automorphic.

Often we deal with double G -modules where G acts on both sides of the abelian group A , and :

$$(ga)g' = g(ag') \quad \forall a \in A, g, g' \in G.$$

The following work can all be done with slight modifications for the respective cases of left, right and double G -modules. We shall develop the cohomology theory using left modules.

We define a G -homomorphism to be an abelian group homomorphism $\alpha : A \rightarrow B$ (A, B abelian groups) such that if A, B are both left G -modules then :

$$(ga)\alpha = g(a\alpha) \quad \forall a \in A, g \in G.$$

2.1.5 TWO EXAMPLES OF CATEGORIES.

- i) The set of all abelian groups, together with the set of all abelian group homomorphisms is a category. We denote it by $\mathcal{A}$ and call it the category of abelian groups.
- ii) The set of all left G -modules together with the set of all G -homomorphisms between them is a category. We denote it by $\mathcal{A}(G)$ and call it the category of left G -modules. Each group G will give us a different category of this type.

2.1.6 Projective & Injective Modules.

In a category we define a null sequence (at the object B) to be a sequence of morphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \quad \text{such that } \alpha\beta = 0.$$

We say the sequence is exact at B if $\text{image } \alpha = \text{kernel } \beta$.

We may then define, for any group G , the following :

- a) a G -module Q is G -injective if every diagram in $\mathcal{A}(G)$ of the form :

$$\begin{array}{ccccc} & & Q & & \\ & & \uparrow & \swarrow & \\ 0 & \longrightarrow & A & \longrightarrow & B, \end{array}$$

with exact row, can be

completed along the dotted line so as to be commutative.

- b) a G -module P is G -projective if every diagram in $A(G)$ of the form :

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ A & \longrightarrow & B \longrightarrow 0 \end{array}, \quad \text{with exact row, can be}$$

completed along the dotted line so as to be commutative.

- c) A G -module is free if it possesses a set $\underline{x}$ of elements, such that each element of the module may be expressed in the form $\sum_i (\sum_j m_{ij} g_{ij}) x_j$ where $x_j \in \underline{x}$, $g_{ij} \in G$, $m_{ij} \in \mathbb{Z}$ (the ring of integers), and further that if the x_j are distinct the sum is zero if and only if each coefficient is zero.

We call the set $\underline{x}$ a free generating set for the module.

2.1.7 For the modules defined above the following may easily be verified :

LEMMA: i) Every free G -module is G -projective.

ii) Every G -module is the image in $A(G)$ of a free G -module.

2.2 THE COHOMOLOGY THEORY OF A GROUP.

2.2.1 DEFINITION. The cohomology theory of a group G is a sequence of covariant functors,

$$H^0(G, \cdot), H^1(G, \cdot), \dots, H^n(G, \cdot), \dots,$$

from the category $\mathcal{A}(G)$ into the category $\mathcal{A}$ whose values for a given G -module A satisfy :

$$i) \quad H^0(G, A) = A^G. \quad (A^G \text{ is the subgroup of } A \text{ invariant under the action of } G).$$

ii) To every short exact sequence in $\mathcal{A}(G)$

$$\underline{SE} : 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

there exists a connecting homomorphism in $\mathcal{A}$:

$$H^n(G, A'') \longrightarrow H^{n+1}(G, A') ; \quad n \in \mathbb{Z}, \quad n \geq 0.$$

such that the sequence

$$\begin{aligned} \underline{LE} : 0 \longrightarrow H^0(G, A') \longrightarrow H^0(G, A) \longrightarrow H^0(G, A'') \longrightarrow H^1(G, A') \longrightarrow \dots \\ \dots \longrightarrow H^n(G, A'') \longrightarrow H^{n+1}(G, A') \longrightarrow H^{n+1}(G, A) \longrightarrow \dots \end{aligned}$$

is exact everywhere, and the correspondence $\underline{SE} \rightarrow \underline{LE}$ is functorial. (see note b) below).

iii) $H^n(G, Q) = 0 \quad \forall n \in \mathbb{Z}, n > 0$ whenever Q is G -injective.

Note : a) a short exact sequence is one of the form

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \text{ which is exact at } A, B \text{ and } C.$$

b) The correspondence $\underline{SE} \rightarrow \underline{LE}$ is functorial means that if, for two short exact sequences $\underline{SE}_1$ and $\underline{SE}_2$ in $\mathcal{A}(G)$, the diagram :

$$\begin{array}{ccccccc} \underline{SE}_1 & , & \text{that is :} & \underline{SE}_1 : & 0 & \rightarrow & A' & \rightarrow & A & \rightarrow & A'' & \rightarrow & 0 \\ \downarrow & & & & & & \downarrow & & \downarrow & & \downarrow & & \\ \underline{SE}_2 & & & \underline{SE}_2 : & 0 & \rightarrow & B' & \rightarrow & B & \rightarrow & B'' & \rightarrow & 0 \end{array}$$

commutes, then the corresponding diagram :

$$\begin{array}{c} \underline{LE}_1 \\ \downarrow \\ \underline{LE}_2 \end{array}$$

involving the long exact sequences also commutes everywhere.

2.2.2 UNIQUENESS OF THE COHOMOLOGY THEORY.

If the cohomology theory of a group, G , exists, then it is unique in the following sense :

THEOREM : Let $H^0(G,.)$, $H^1(G,.)$, ..., $H^n(G,.)$, and
and $K^0(G,.)$, $K^1(G,.)$, $K^2(G,.)$, ..., $K^n(G,.)$,

be cohomology theories for the group G .

Then the functors $H^i(G,.)$ and $K^i(G,.)$ are naturally equivalent for each $i \in \mathbb{Z}$, $i \geq 0$.

We say the cohomology theory is unique to within natural equivalence.

For proof see Northcott [14] ch.10.

2.2.3 THE EXISTENCE OF A COHOMOLOGY THEORY.

We shall briefly sketch the construction of a cohomology theory for a group G . We begin by defining :

- a) A projective resolution of a G -module C is an exact sequence

$$\underline{R} : \dots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow C \rightarrow 0,$$

where the X_i are projective G -modules in $A(G)$.

If the X_i are free G -modules we call the sequence a free resolution of C . (By 2.1.7, i), it is then also projective).

- b) Let A, B be G -modules. Define $\text{hom}_G(A, B)$ to be the set of all G -homomorphisms from A to B . It can be shown quite easily that $\text{hom}_G(A, B)$ is an abelian group.
- c) Define the functor $\text{Hom}_G(., A)$ for fixed A in $\mathcal{A}(G)$, to associate with each G -module, X , the group $\text{hom}_G(X, A)$. This is a functor from $\mathcal{A}(G)$ into $\mathcal{A}$, and can be shown to be contravariant.

The mapping of morphisms under the functor is defined as follows :
for a given G -homomorphism $\alpha : B \rightarrow C$, define

$$\text{Hom } \alpha : \text{Hom}_G(C, A) \rightarrow \text{Hom}_G(B, A)$$

by $\text{Hom } \alpha (\eta) = \eta \alpha$ for each $\eta \in \text{Hom}_G(C, A)$.

We construct the cohomology theory for G as follows :

Take any G -module C . We may construct of free resolution for C by using 2.1.7 ii). We know that C is the image of a free G -module, X_0 , say. Hence $C \cong X_0/Y_0$, where Y_0 is a sub G -module of X_0 . We obtain the exact sequence :

$$0 \rightarrow Y_0 \rightarrow X_0 \rightarrow C \rightarrow 0 ,$$

with the obvious morphisms.

Similarly $Y_0 \cong X_1/Y_1$, where X_1 is a free G -module, and Y_1 a sub- G -module, and we get the sequence :

$$0 \rightarrow Y_1 \rightarrow X_1 \rightarrow X_0 \rightarrow C \rightarrow 0, \text{ which is again exact.}$$

We may continue this process indefinitely to obtain a free resolution :

$$\underline{R} : \dots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow C \rightarrow 0.$$

If we now apply the functor $\text{Hom}_G(., A)$, for some G -module A , to the terms of $\underline{R}$, we get the sequence :

$$\begin{aligned} 0 \rightarrow \text{Hom}_G(C, A) \xrightarrow{\varepsilon} \text{Hom}_G(X_0, A) \xrightarrow{\delta_0} \text{Hom}_G(X_1, A) \xrightarrow{\delta_1} \dots \\ \dots \xrightarrow{\delta_{n-1}} \text{Hom}_G(X_n, A) \xrightarrow{\delta_n} \dots \end{aligned}$$

with the corresp. morphisms labelled as shown.

This can be shown to be a null sequence at each point.

Now put $\text{Ext}_G^n(C, A) = \ker \delta_n / \text{im } \delta_{n-1}$, for $n > 0$. (These quotient groups are independent of the resolution $\underline{R}$.)

Further replace C by Z , where Z ^{is} ~~are~~ the additive group of the integers considered as a trivial G -module, i.e. $gn = n$, $\forall n \in Z, g \in G$.

Then we have :

$$\underline{\text{THEOREM}} : \text{Put } H^0(G, A) = \text{Ext}_G^0(Z, A) = \text{Hom}_G(Z, A)$$

$$\text{and } H^n(G, A) = \text{Ext}_G^n(Z, A), \quad n > 0.$$

Then the sequence $H^i(G, \cdot)$ is the cohomology theory for G .
The proof is given in Northcott [14] ch.10.

2.3 COHOMOLOGICAL DIMENSION OF A GROUP.

DEFINITION. A group G , has cohomological dimension n if $H^m(G, A) = 0 \quad \forall A \in \mathcal{A}(G), \quad m \geq n+1$; and if there exists a G -module B such that $H^n(G, B) \neq 0$.

We recall that the aim of this thesis is to find which knot groups have cohomological dimension 2, in an attempt to prove that all knot groups have this property.

2.4 THE GRUENBERG RESOLUTION.

This free resolution for a group was developed by Gruenberg [5]. We shall use it to attempt an algebraic method for determining the cohomological dimension of a knot group as outlined in Ch.4. Here we state some lemmas necessary to the development of the resolution and conclude with the resolution itself.

In this section we deal mainly with right modules.

2.4.1 Let a group G , have presentation $\langle \underline{y}, r \rangle$.

Then $G \cong Y/R$, where $Y = F[\underline{y}]$ (the free group on the set $\underline{y}$), and $R = \{\underline{r}\}^Y$, (the normal closure of the set $\underline{r}$ in Y).

DEFINITION: Let Y be a group and let $\alpha : ZY \rightarrow Z$ be the mapping which takes $\sum_i m_i y_i$ into $\sum_i m_i$. We define the

augmentation ideal, $\mathcal{V}$, of the group ring ZY to be the kernel of the map α .

We have :

LEMMA: Let Y be a free group freely generated by the set $\underline{y}$, then the augmentation ideal $\mathcal{V}$ is freely generated as right [or left] Y -module by the set $\{Y-1 : Y \in \underline{y}\}$.

A proof of this lemma may be found in Cartan-Eilenberg [22] ch. 10. §5.

2.4.2 NOTATION. Let Y be a free group, and R a normal subgroup of Y . Write $R = \ker \sigma$, where $\rho : ZY \rightarrow Z[Y/R]$ is the mapping

$$\sum m_i Y_i \mapsto \sum m_i (Y_i R) .$$

In this thesis, whenever $G \cong Y/R$ we shall identify ZG and $Z[Y/R]$ and use the same map $\rho : ZY \rightarrow ZG$ to mean the composition of the map $\rho : ZY \rightarrow Z[Y/R]$ and the isomorphism $Z[Y/R] \rightarrow ZG$.

Then :

LEMMA : If R is freely generated by the set $\underline{k}$, then R is freely generated as right [or left] Y -module by the set $\{r-1 : r \in \underline{r}\}$.

Gruenberg [5] proves this as Lemma 2 of his paper.

2.4.3 LEMMA : Let S and T be right [or left] Y -modules generated as such by the sets $\underline{s}$ and $\underline{t}$ respectively. Then if T [or S] is a 2-sided Y -module, the product module ST is generated as a right [or left] Y -module by the set $\underline{st} = \{st : s \in \underline{s}, t \in \underline{t}\}$.

This is Lemma 3 of Gruenberg's paper [5].

2.4.4 LEMMA : Let $G \cong Y/R$, where Y is a free group. Let S be a sub Y -module of ZY freely generated as such by the set $\underline{s}$. Then the quotient module S/SR is in a natural way a right G -module freely generated as such by the set of cosets $\{s + SR : s \in \underline{s}\}$.

Proof : Since this lemma is used again in 4.3.3 we shall furnish a proof here.

Let $\underline{x}$ be a set in 1:1 correspondence with $\underline{s}$. Let X be the right ZG -module freely generated by $\underline{x}$. Define $\sigma : S \rightarrow X$ by $\sigma : \sum_i s_i u_i \mapsto \sum_i x_i (u_i \rho)$, $s_i \in \underline{s}$, $x_i \in \underline{x}$, $u_i \in ZG$. This is an epimorphism of abelian groups. Its kernel consists of all $\sum_i s_i u_i$ with $\sum_i x_i (u_i \rho) = 0$, i.e. with $u_i \rho = 0$ for all i . This means $u_i \in R$ for all i . Hence $\ker \sigma = SR$. Thus σ induces an isomorphism $S/SR \rightarrow X$ of abelian groups. Since X is by definition a free ZG module, i.e. a free G -module, then S/SR becomes one by transferring the action of G in the following way :

$$(s + SR)g = sz + SR, \text{ where } z \in Y \text{ such that } z\rho = g.$$

2.4.5 GRUENBERG'S RESOLUTION.

We may now prove Gruenberg's result :

THEOREM Let G be a group, with $G \cong Y/R$, where Y is free.

Then the following is a free resolution of the trivial G -module Z in $A(G)$:

$$\dots \rightarrow R^{n-1}/R^n \rightarrow YR^{n-2}/YR^{n-1} \rightarrow \dots \rightarrow R/R^2 \rightarrow Y/YR \rightarrow ZG \rightarrow Z \rightarrow 0$$

where the morphism $Y/YR \rightarrow ZG$ is that induced by the map

$\rho : ZY \rightarrow Z[Y/R]$, the morphism $R^{n-1}/R^n \rightarrow YR^{n-2}/YR^{n-1}$ is induced

by the natural injection $R^{n-1} \hookrightarrow YR^{n-2}$, and the morphism

$YR^{n-2}/YR^{n-1} \rightarrow R^{n-1}/R^n$ is induced by the injection $YR^{n-2} \hookrightarrow R^{n-2}$.

Proof : By 2.4.1 and 2.4.2 Y and R are free right

Y -modules and R is a two sided Y -module. Using 2.4.3 repeatedly

we may show that YR^{n-1} and R^n are free right ZY -modules. Hence

by 2.4.4 the quotients R^{n-1}/R^n and YR^{n-1}/YR^n are free right

G -modules.

Further, since $R \subseteq Y$ as Y -modules, then $R^{n-1} \subseteq YR^{n-2}$
and $YR^{n-2} \subseteq R^{n-2}$.

We show the sequence is exact by noticing that the kernel of $YR^{n-2}/YR^{n-1} \rightarrow R^{n-2}/R^{n-1}$ is R^{n-1}/YR^{n-1} , and that this is the image of R^{n-1}/R^n under the morphism $R^{n-1}/R^n \rightarrow YR^{n-2}/YR^{n-1}$. We can easily see that this is true at each point.

The resolution

$$\dots \rightarrow R^{n-1}/R^n \rightarrow YR^{n-2}/YR^{n-1} \rightarrow \dots \rightarrow R/R^2 \rightarrow Y/YR \rightarrow ZG \rightarrow Z \rightarrow 0$$

is called Gruenberg's Resolution.

CHAPTER 3

ON METHODS OF DETERMINING THE COHOMOLOGICAL DIMENSION OF SOME CLASSES OF KNOT GROUPS.

3.0 This chapter is mainly a survey of the literature which is applicable to our topic. We state without proof the relevant theorems from the literature, showing how these may be combined to show that certain classes of knot groups satisfy our conjecture. In Ch.4 I develop a method which holds promise of giving the result for more classes of knot groups.

At the end of this chapter is a comment on the proof of our conjecture given by Baumslag [1].

3.1 THE SEMIGROUP OF KNOTS.

3.1.1 We first make a remark about knot groups and the knots which we deal with in the following sections.

LEMMA : The only knot group which is free is that of the trivial knot.

Proof : It is well known (see for example Hall [6] p.175) that for a free group F_n of rank n , the factor group F_n/F'_n , where F'_n is the commutator subgroup, is free abelian of rank n .

Now suppose G is a knot group. We know see 3.2.3 that G/G' is an infinite cyclic group, i.e. is free abelian of rank 1.

Therefore if G is free it must be a free group of rank 1, i.e. free cyclic.

Now Papakyriakopolous [15] shows that G is free cyclic if and only if the knot K , where $G = \pi_1(R^3 - K)$, is the trivial knot.

Also Stallings [20] has shown that if a finitely generated group has cohomological dimension 1 then it is a free group.

Hence only the trivial knot has knot group with cohomological dimension 1, since all groups of tame knots are finitely generated.

In the following work we shall consider only non-trivial knots, for which the cohomological dimension of the knot group must be at least 2.

3.1.2 THE SEMIGROUP OF KNOTS.

We define the product of two knots to be the knot formed by tying one knot after the other on the same piece of string. This multiplication may be shown to be associative, and commutative. Hence the set of knots together with this multiplication form a commutative semigroup (Schubert [17]).

Concerning the knot group of the product of knots we have that :

LEMMA : Let K_1, K_2 be knots, with knot groups G_1, G_2 respectively. Then the knot group G of the product $K_1 K_2$ is given by $G = G_1 *_C G_2$, the free product of G_1 and G_2 with the amalgamated subgroup being infinite cyclic C .

A proof may be found in Neuwirth [12] p.76.

3.1.3 Lyndon [8] in the paper on the cohomology of one relator groups uses group presentations and techniques of free differential calculus (Fox [4]) to introduce a method of proving the following lemma.

LEMMA : Let a group G be a free product $G_1 \ast_A G_2$ of groups G_1 and G_2 with amalgamated subgroup A . Let the cohomological dimension of the group A be p . Then for all $n > p$ we have

$$H^n(G, B) = H^n(G_1, B) \oplus H^n(G_2, B) \quad \text{for all modules } B.$$

It is well known that all free groups have cohomological dimension 1, this being a trivial consequence of Gruenberg's Resolution. Thus we have :

COROLLARY: Let K_1, K_2 be knots whose groups have cohomological dimension 2. Then so does the knot group of the product knot $K_1 K_2$.

We see that the set of all knots whose groups satisfy our conjecture form a commutative semigroup under the multiplication defined.

3.2 ON SOME CLASSES OF KNOTS.

3.2.1 KNOT GROUPS WITH ONE RELATOR.

The main theorem of the paper by Lyndon [8] is :

THEOREM : All groups which have a presentation with only one relator, the relator not being a power in the group, have cohomological dimension 2.

Crowell & Fox [2] give many examples of knots whose groups are one relator groups of this type. For example the torus knots have knot groups with presentations of the type $\langle a, b; a^p = b^q \rangle$, where $(p, q) = 1$. These knots are curves on the surface of the torus $(r-2)^2 + z^2 = 1$, which cut a meridian in p points and a longitude in q points, with no intersection of the curve by itself. Hence these knots have groups with cohomological dimension 2.

LEMMA : Let G be a torsion free group. Let $\langle \underline{y}, r^n \rangle$, where $n \in \mathbb{Z}$, be a presentation of G . Then $n = \pm 1$ or $n = 0$.

Proof : Suppose $|n| > 1$.

Let $\rho : F[\underline{y}] \rightarrow G$ be the mapping corresponding to the presentation $\langle \underline{y}, r^n \rangle$.

Then $(r^n)\rho = (r\rho)^n = 1$.

Now Magnus [10] §4.4 shows that $r\rho$ is not a consequence of r^n in G . Hence $r\rho$ is a non-trivial element in G with finite order, which contradicts our assumption.

Now all knot groups are torsion free. This is intuitively clear and can be demonstrated using topological methods as in Massey [11].

Thus a one-relator presentation of a knot group cannot involve a power of an element of the group. Hence any knot group which can be presented as a one-relator group has cohomological dimension 2.

3.2.2 ALTERNATING KNOTS AND KNOTS WITH LESS THAN TEN CROSSINGS.

Neuwirth [13] makes the following conjecture :

CONJECTURE: The group of any knot can be represented as the free product $F_n *_{F_{2n-1}} F_n$ of two free groups of rank n with amalgamated subgroup free of rank $2n-1$.

In his paper [13] he proves the conjecture for all alternating knots, i.e. those whose projections have crossings which are alternately under and over around the knot. Also the non-alternating knots with less than ten crossings, as given by Reidemeester [16]. Zieschang [21] proves the lemma for the knot denoted 9_{46} by Reidemeester, which was omitted by Neuwirth.

The method used is to consider the knots as curves on a solid pretzel and to find a representation of the knot group from this configuration.

The lemma 3.1.3 ensures that all such knot groups have cohomological dimension 2. If the conjecture is true for all knot groups then our main conjecture is an obvious corollary.

We note in passing that the torus knots are of this type having a knot group which can be represented as the free product of free cyclic groups with free amalgamation.

3.2.3 KNOT GROUPS WHOSE COMMUTATOR SUBGROUP IS FREE.

Using methods of algebraic topology, Neuwirth [12] shows that :

THEOREM: Let G be a knot group. Then its commutator subgroup G' is either a) free

or b) of the form, $\dots * A \underset{F}{*} A \underset{F}{*} A \underset{F}{*} A * \dots$,

i.e. the direct limit of free products with free amalgamations.

Now it is well known that the abelianized knot group G/G' is free. See for example Crowell & Fox [2] p.111.

The theory of spectral sequences as developed in MacLane [9] ch.11 gives us the following result as a corollary of theorem 10.1 p.351 in his book.

LEMMA: Let H be a group. Let K be a subgroup of H with cohomological dimension n . Suppose the factor group H/K has cohomological dimension m . Then H has cohomological dimension at most $m+n$.

We see that in the case of knot groups with free commutator subgroup we have :

COROLLARY: Let G be a knot group with free commutator subgroup G' . Then G has cohomological dimension 2.

I note here that it is usually quite difficult to determine whether a given knot has knot group with free commutator subgroup.

3.3 ON A PROOF OF OUR CONJECTURE BY BAUMSLAG.

In a paper presented to the Internat. Conf. Theory of Groups, A.N.U., by G. Baumslag he sketches a proof of our conjecture using Dehn's lemma.

DEHN'S LEMMA: Let K be a knot in R^3 . Then the space $R^3 - K$ is aspherical.

(that a space is aspherical means that all its homotopy groups except the first are zero)

This lemma was stated by Dehn [3] in 1910 and finally proved by Papakyriakopolous [15] in 1957.

Baumslag interprets MacLane [9] p.137 as saying that the fundamental groups of all aspherical spaces have cohomological dimension 2.

However this is not true, for consider the following result derived from Spanier [19] pp.424-6.

LEMMA: Let G be any group. Then we may construct an aspherical space X , such that $\pi_1(X) = G$.

If the interpretation by Baumslag were correct then every group would have cohomological dimension 2, which is not so. Thus the proof by Baumslag is incorrect.

CHAPTER 4

A NECESSARY & SUFFICIENT CONDITION FOR KNOT GROUPS TO HAVE COHOMOLOGICAL DIMENSION 2, AND ITS APPLICATION.

4.0 In this chapter we shall use Gruenberg's Resolution to find a condition for the cohomological dimension of a group to be 2. We shall then attempt to show that certain knot groups satisfy this condition.

4.1 THE APPLICATION OF GRUENBERG'S RESOLUTION.

4.1.1 From theorem 2.4.6 we have the Gruenberg Resolution :

$$\dots \rightarrow R^{n-1}/R^n \rightarrow YR^{n-2}/YR^{n-1} \rightarrow \dots \rightarrow R/R^2 \rightarrow Y/YR \rightarrow ZG \rightarrow Z \rightarrow 0.$$

Consider the last four terms of the resolution, i.e.

$$\dots \rightarrow R/R^2 \rightarrow Y/YR \rightarrow ZG \rightarrow Z \rightarrow 0.$$

The kernel of the mapping $Y/YR \rightarrow ZG$ is R/YR , since the mapping is induced by $\rho : ZY \rightarrow ZG$, and $\ker \rho = R$.

Now Y/YR is a free G -module, and R/YR can be considered as a sub G -module by restriction the action of G to it as follows :

$$[(r-1) + YR]g = (r-1)z + YR \quad \text{where } r \in R \text{ and } z \in Y \text{ s.t. } z\rho = g.$$

Since R is a right Y module in ZY we have $(r-1)z \in R$. Thus restriction of the action of G to R/YR makes R/YR a sub G -module.

4.1.2 If we are able to show that R/YR is a free G -module then the sequence :

$$0 \rightarrow R/VR \rightarrow V/VR \rightarrow ZG \rightarrow Z \rightarrow 0 ,$$

with mappings induced by those of the Gruenberg Resolution, is a free resolution of Z .

Further, applying the functor $\text{Hom}_G(.,A)$ for $A \in A(G)$, as in 2.2.3 we obtain :

$$0 \rightarrow \text{Hom}_G(Z,A) \xrightarrow{\varepsilon} \text{Hom}_G(ZG,A) \xrightarrow{\delta_0} \text{Hom}_G(V/VR,A) \xrightarrow{\delta_1} \text{Hom}_G(R/VR,A) \xrightarrow{\delta_2} 0.$$

Now $H^3(G,A) = \ker \delta_3 / \text{im } \delta_2 = 0$, since δ_3 is in effect $\delta_3 : 0 \rightarrow 0$.

Thus by the definition 2.3 the cohomological dimension of the group G is at most 2.

4.2 In order to be able to show the G -module R/VR to be free we need to find a set of free generators for it. We use the following lemma.

LEMMA : Let $R = \{\underline{r}\}^Y$ where Y is a free group.

Then R/VR is generated as G -module by the set $\{(r-1) + VR : r \in \underline{r}\}$.

Proof Certainly R/VR is generated as G -module by the set $\{(r'-1) + VR : r' \in R\}$.

Since $R = \{\underline{r}\}^Y$ any $r' \in R$ must be of the form :

$$r' = \prod_{i=1}^s h_i r_i h_i^{-1}, \text{ where } h_i \in Y \text{ and } r_i \in \underline{r}.$$

$$\begin{aligned}
\text{Thus } (r'-1) + VR &= \left(\prod_{i=1}^s h_i r_i h_i^{-1} - 1 \right) + VR \\
&= \sum_{j=1}^s \left[(h_j r_j h_j^{-1} - 1) \prod_{i=j+1}^s h_i r_i h_i^{-1} \right] + VR \\
&= \sum_{j=1}^s [h_j (r_j - 1) h_j^{-1} + VR] \left(\prod_{i=j+1}^s h_i r_i h_i^{-1} \right) \rho .
\end{aligned}$$

[recall ρ is the mapping $ZY \rightarrow Z[Y/R]$ and $G \cong Y/R$].

Now putting $g_j = \left(\prod_{i=j+1}^s h_i r_i h_i^{-1} \right) \rho$, and noting that $g_s = 1$, we have

$$(r'-1) + VR = \sum_{j=1}^s [h_j (r_j - 1) h_j^{-1} + VR] g_j . \quad (1)$$

Consider the term $h_j (r_j - 1) h_j^{-1} + VR$.

We see that :

$$h_j (r_j - 1) h_j^{-1} + VR = (h_j - 1)(r_j - 1) h_j^{-1} + (r_j - 1) h_j^{-1} + VR.$$

Certainly $(h_j - 1)(r_j - 1) h_j^{-1} \in VR$, so that

$$\begin{aligned}
h_j (r_j - 1) h_j^{-1} + VR &= (r_j - 1) h_j^{-1} + VR . \\
&= [(r_j - 1) + VR] g_j' , \text{ where } g_j' = (h_j^{-1}) \rho .
\end{aligned} \quad (2)$$

Combining (1) and (2) we get :

$$(r'-1) + VR = \sum_{j=1}^s [(r_j - 1) + VR] g_j' g_j .$$

Thus R/VR is generated as G -module by $\{(r-1) + VR : r \in \underline{r}\}$

4.3 THE APPLICATION TO KNOT GROUPS.

In 1.4.5 we gave a presentation for a knot group. We shall now use this to find a generating set for the module R/VR derived from the knot group, and attempt to show that the generating set is free.

4.3.1 We have the Wirtinger presentation for a knot group G .

$$G = \langle x_1, \dots, x_n; r_1, \dots, r_{n-1} \rangle \text{ with } r_i = x_i^{-1} x_{k_i}^{\varepsilon_i} x_{i+1} x_{k_i}^{-\varepsilon_i}, i=1, \dots, n-1,$$

$$\varepsilon_i = \pm 1.$$

We put $Y = F[x_1, \dots, x_n]$ the free group on $\{x_1, \dots, x_n\}$ and

$$R = \{r_1, \dots, r_{n-1}\}^Y.$$

Then Y, R are the modules as defined in the Gruenberg Resolution.

Hence the G -module R/VR is generated as such by the set

$$\{(r_i - 1) + VR : i = 1, \dots, n-1\}.$$

For this to be a free generating set we need to show that

$$\sum_{i=1}^{n-1} [(r_i - 1) + VR] z_i = 0 \quad (\text{where } z_i \in ZG)$$

implies $z_i = 0, i = 1, \dots, n-1$.

4.3.2 We alter the form of the problem by noticing that

$$\begin{aligned} (r_i - 1) &= (x_i^{-1} x_{k_i}^{\varepsilon_i} x_{i+1} x_{k_i}^{-\varepsilon_i} - 1) \\ &= -(x_i - 1)r_i + (x_{k_i}^{\varepsilon_i} - 1)x_{i+1} x_{k_i}^{-\varepsilon_i} + (x_{i+1} - 1)x_{k_i}^{-\varepsilon_i} - (x_{k_i}^{\varepsilon_i} - 1)x_{k_i}^{-\varepsilon_i}. \end{aligned}$$

Then identities $(xx'-1) = (x-1)x' + (x-1)$

and $(x^{-1}-1) = -(x-1)x'$, for $x, x' \in Y$, have

been used above.

Collecting terms we have

$$\begin{aligned} (r_i-1) &= -(x_i-1)r_i + (x_{k_i}^{\varepsilon_i}-1)(x_{i+1}-1)x_{k_i}^{-\varepsilon_i} + (x_{i+1}-1)x_{k_i}^{-\varepsilon_i} \\ &= -(x_i-1)(r_i-1) - (x_i-1) + (x_{k_i}^{\varepsilon_i}-1)(x_{i+1}-1)x_{k_i}^{-\varepsilon_i} + (x_{i+1}-1)x_{k_i}^{-\varepsilon_i} \end{aligned}$$

$$\text{Thus } (r_i-1) + yR = -(x_i-1) + (x_{k_i}^{\varepsilon_i}-1)(x_{i+1}-1)x_{k_i}^{-\varepsilon_i} + (x_{i+1}-1)x_{k_i}^{-\varepsilon_i} + yR.$$

Further

$$\sum_{i=1}^{n-1} [(r_i-1) + yR] z_i =$$

$$\sum_{i=1}^{n-1} [-(y_i-1)y_i + (x_{k_i}^{\varepsilon_i}-1)(x_{i+1}-1)x_{k_i}^{-\varepsilon_i}y_i + (x_{i+1}-1)x_{k_i}^{-\varepsilon_i}y_i] + yR$$

[Here $y_i \in ZY$, s.t. $y_i \rho = z_i$].

We distinguish between the cases $\varepsilon_i = \pm 1$.

$$\text{If } \varepsilon_i = +1, (x_{k_i}^{\varepsilon_i}-1)(x_{i+1}-1)x_{k_i}^{-\varepsilon_i} = (x_{k_i}-1)(x_{i+1}-1)x_{k_i}^{-1}$$

$$\text{and } \varepsilon_i = -1, (x_{k_i}^{\varepsilon_i}-1)(x_{i+1}-1)x_{k_i}^{-\varepsilon_i} = -(x_{k_i}-1)(x_{k_i}^{-1}x_{i+1}x_{k_i}-1).$$

We shall write

$$m_i' = \begin{cases} (x_{i+1}-1)x_{k_i}^{-1} & \text{whenever } \varepsilon_i = +1 \\ -(x_{k_i}^{-1}x_{i+1}x_{k_i}-1) & \text{whenever } \varepsilon_i = -1. \end{cases}$$

Substituting in (A) we obtain

$$(B) \quad \sum_{i=1}^{n-1} [(r_i - 1) + \gamma R] z_i = \sum_{i=1}^{n-1} [-(x_i - 1)y_i + (x_{k_i} - 1)m'_i y_i + (x_{i+1} - 1)x_{k_i}^{\varepsilon_i} y_i] + \gamma R.$$

We notice that for each $j \in \{1, \dots, n\}$ we have terms of the following types :

- a) $-(x_j - 1)y_j \quad j \in \{1, \dots, n-1\}$
- b) $(x_j - 1)m'_i y_i$ for the j for which there exists some $i \in \{1, \dots, n-1\}$ such that $k_i = j$.
- c) $(x_j - 1)x_{k_{j-1}}^{-\varepsilon_{j-1}} y_{j-1} \quad j \in \{2, \dots, n\}$

Notice that for given j there may be none, one, or more terms of type b). This will be seen more clearly in Ch.5.

Rearranging the terms of the equation (B) we have

$$(C) \quad \sum_{i=1}^{n-1} [(r_i - 1) + \gamma R] z_i = \sum_{j=1}^n (x_j - 1) [-y_j + \sum_{k_i=j} m'_i y_i + x_{k_{j-1}}^{-\varepsilon_{j-1}} y_{j-1}] + \gamma R, \text{ where } y_0 = y_n = 0.$$

We know that $R/\gamma R$ is a sub G -module of $V/\gamma R$.

Therefore writing the R.H.S. of equation (C) as a G -action on the generators of $V/\gamma R$ we find :

$$(D) \quad \sum_{i=1}^{n-1} [(r_i - 1) + \gamma R] z_i = \sum_{j=1}^n [(x_j - 1) + \gamma R] (-y_j + \sum_{k_i=j} m'_i y_i + x_{k_{j-1}}^{-\varepsilon_{j-1}} y_{j-1}) \rho \quad \text{with } y_n = y_0 = 0.$$

4.3.3 Now $\sum_{i=1}^{n-1} [(r_i - 1) + \gamma R] z_i = 0$ implies

$$(E) \quad \sum_{j=1}^n [(x_j - 1) + \gamma R] (-y_j + \sum_{k_i=j} m_i! y_i + x_{k_{j-1}}^{-\varepsilon_{j-1}} y_{j-1}) \rho = 0$$

From lemma 2.3.5 we know that $\gamma/\gamma R$ is freely generated as right G -module by the set $\{(x_i - 1) + \gamma R : i = 1, \dots, n\}$.

Hence the equation (E) shows

$$(F) \quad (-y_j + \sum_{k_i=j} m_i! y_i + x_{k_{j-1}}^{-\varepsilon_{j-1}} y_{j-1}) \rho = 0, \quad j = 1, \dots, n. \quad (y_0 = y_n = 0).$$

We have $y_j \rho = z_j$, $j = 0, 1, \dots, n$.

Now write $x_j \rho = g_j$, $j = 1, \dots, n$

and notice that for $\varepsilon_j = -1$, $m_j! = -(x_j r_j - 1)$.

Since

$$r_j \rho = 1, \quad j = 1, \dots, n$$

we have

$$m_j! \rho = \begin{cases} (g_{j+1} - 1) g_{k_j}^{-1} & \text{when } \varepsilon_j = +1 \\ -(g_j - 1) & \varepsilon_j = -1. \end{cases}$$

Write $m_j = m_j! \rho$; $j = 1, \dots, n-1$.

The condition (F) is therefore

$$(G) \quad -z_j + \sum_{k_i=j} m_i z_i + g_{k_{j-1}}^{-\varepsilon_{j-1}} z_{j-1} = 0, \quad j = 1, \dots, n; \quad z_0 = z_n = 0.$$

We see from the above that the condition for the G-module R/VR to be free on the generators $\{(r_i - 1) + VR : r_i \in \underline{r}\}$, is that the only solution to the set of equations (G) is the trivial solution, i.e. $z_i = 0$, $i = 1, \dots, n-1$.

Thus a necessary and sufficient condition for a knot group to have cohomological dimension 2 is that the only solution to the set of equations (G), derived from the presentation of the group, is the trivial solution.

4.4' A NECESSARY CONDITION FOR THE SOLUTIONS OF (G).

A necessary condition on the solutions of the set of equations (G) may be found by simply adding all the equations together.

We recall the equation (B) from which we derived the set (G), and notice from the form of (B) that each z_i will occur in (G) in exactly three places, with the following coefficients :

- a) -1
- b) $-e_i$
- c) g_{k_i}
- d) m_i

Hence if we sum all the equations of (G) we obtain the condition :

$$(H) \quad \sum_{i=1}^{n-1} [-1 + g_{k_i}^{-e_i} + m_i] z_i = 0.$$

Now we have,

$$\begin{aligned} \text{if } \varepsilon_i = +1, \quad -1 + g_{k_i}^{-\varepsilon_i} + m_i &= -1 + g_{k_i}^{-1} + (g_{i+1}^{-1})g_{k_i}^{-1} \\ &= (g_{i+1}g_{k_i}^{-1} - 1) \end{aligned}$$

$$\begin{aligned} \text{and if } \varepsilon_i = -1, \quad -1 + g_{k_i}^{-\varepsilon_i} + m_i &= -1 + g_{k_i} - (g_i - 1) \\ &= g_{k_i} - g_i. \end{aligned}$$

$$\text{Write } a_i = \begin{cases} g_{i+1}g_{k_i}^{-1} - 1 & \text{when } \varepsilon_i = +1 \\ g_{k_i} - g_i & \text{when } \varepsilon_i = -1. \end{cases}$$

Then the condition (H) is :

$$(K) \quad \sum_{i=1}^{n-1} a_i z_i = 0.$$

Certainly any solution of (G) is a solution of (K). If we can find all solutions of (K) then by substitution in (G) we can determine all solutions of (G). We shall attempt this in the following section.

4.4 ON THE EQUATIONS IN (G) & (K).

I have been unable to solve the system of equations (G) by using purely algebraic methods. The method holding the most promise appears to be that of finding the physical interpretation of the equations using the plane representation of the knot and proceeding by

considering the topological properties of the knot and the complementary space from which the knot group is derived.

4.4.1 INTERPRETATION OF THE EQUATIONS IN (G).

The set of equations (G) is :

$$-z_i + k_{j_\rho} \sum_{\rho=1}^{m_j} z_{j_\rho} + g_{k_{i-1}}^{-\epsilon_{i-1}} z_{i-1} = 0, \quad i = 1, \dots, n.$$

where $z_0 = z_n = 0$, $\rho = 1, 2, 3, \dots$, $j_\rho \in \{1, \dots, n-1\}$.

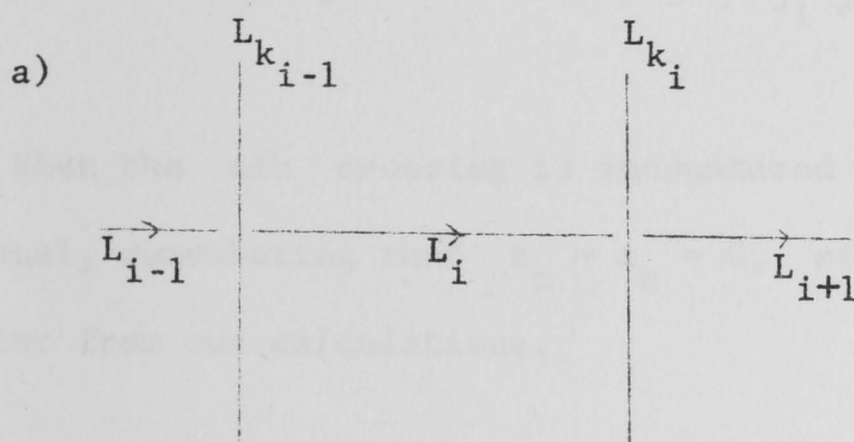
We shall show how each equation is related to an overpass of the plane representation of the knot. Recall that the i th equation in the set (G) is found by considering the coefficient of $(x_i - 1) + VR$ in the sum on the LHS of the equation (E) in section 4.3.3. From the expansion of each term $(r_j - 1) + VR$ in 4.3.2 it is easily seen that we obtain $(x_i - 1) + VR$ on the left of a term in the sum $\sum_{k=1}^{n-1} [(r_k - 1) + VR] z_k$ only if x_i occurs in one of the relators r_j . This occurs when the j th crossing of the representation, from which we find the j th relator of the presentation, is incident with the i th overpass of the representation.

The three types of terms on the i th equation arise as follows :

- i) $-z_i$ comes from the relator $r_i = x_i^{-1} x_{k_i}^{\epsilon_i} x_{i+1} x_{k_i}^{-\epsilon_i}$
- ii) $g_{k_{i-1}}^{-\epsilon_{i-1}} z_{i-1} \dots \dots \dots r_{i-1} = x_{i-1}^{-1} x_{k_{i-1}}^{\epsilon_{i-1}} x_i x_{k_{i-1}}^{-\epsilon_{i-1}}$
- iii) $m_{j_\rho} z_{j_\rho} \dots \dots \dots r_{j_\rho} = x_{j_\rho}^{-1} x_i^{\epsilon_{j_\rho}} x_{j_\rho+1} x_i^{-\epsilon_{j_\rho}}$

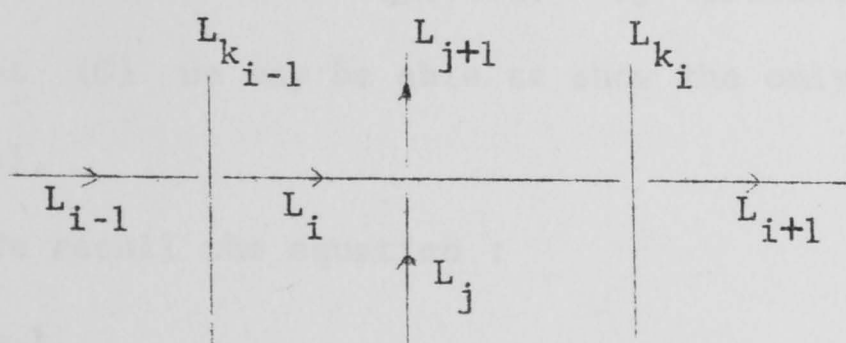
In any equation there may be none, one or more terms of type iii) depending on whether the overpass has none, one or more undercrossings. We use the subscript ρ to distinguish between these terms.

We give some examples of the types of overpass and the equations corresponding to them :

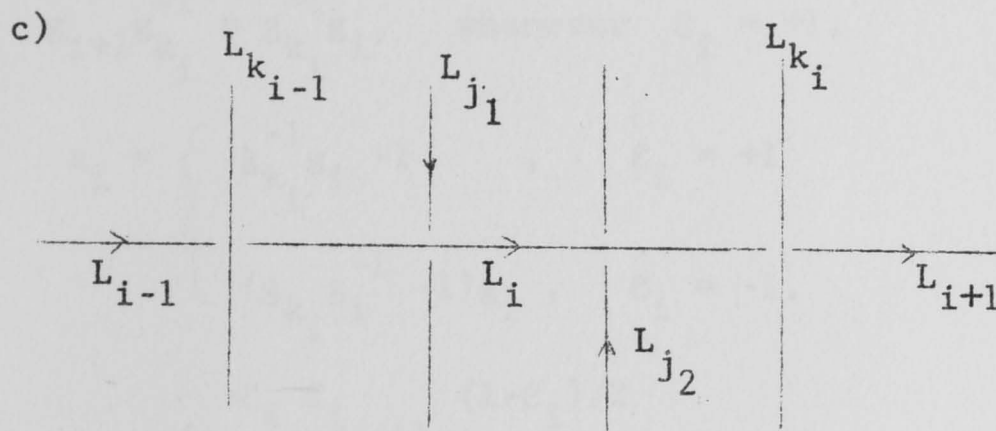


The equation corresponding to L_i is : $-z_i + g_{k_{i-1}}^{i-1} z_{i-1} = 0$.

b)



Corresponding to L_i we have the equation : $-z_i + m_j z_j + g_{k_{i-1}}^{i-1} z_{i-1} = 0$.



Corresponding to L_i we have : $-z_i + m_{j_1} z_{j_1} + m_{j_2} z_{j_2} + g_{k_{i-1}}^{-\varepsilon_{i-1}} z_{i-1} = 0$.
etc.

When the n th crossing is encountered we write the equation as usual, remembering that $z_0 = z_n = 0$, since we omit the n th relator from our calculations.

4.4.2 PHYSICAL INTERPRETATION OF THE TERMS IN (K).

By finding a physical interpretation of the terms in the equation (K), we can find by inspection of the knot rep. a set of solutions for this equation. By substituting these solutions in the set (G) we may be able to show the only solution of (G) to be trivial.

We recall the equation :

$$(K) \quad \sum_{i=1}^{n-1} a_i z_i = 0, \quad \text{where } z_i \in ZG \quad \text{and} \quad a_i = \begin{cases} g_{i+1} g_{k_i}^{-1}, & \varepsilon_i = +1 \\ g_{k_i}^{-1} g_i, & \varepsilon_i = -1. \end{cases}$$

We know that in G , $g_i^{-1} g_{k_i} g_{i+1} g_{k_i}^{-1} = 1$, when $\varepsilon_i = +1$.

Hence $g_{i+1} g_{k_i}^{-1} = g_{k_i}^{-1} g_i$, whenever $\varepsilon_i = +1$.

$$\text{Thus } a_i = \begin{cases} g_{k_i}^{-1} g_i^{-1}, & \varepsilon_i = +1 \\ (g_{k_i} g_i^{-1} - 1) g_i, & \varepsilon_i = -1. \end{cases}$$

$$\text{i.e. } a_i = (g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i} - 1)^{(1-\varepsilon_i)/2} g_i$$

$$\text{Write } z'_i = g^{(1-\varepsilon_i)/2} z_i, \quad i = 1, \dots, n-1.$$

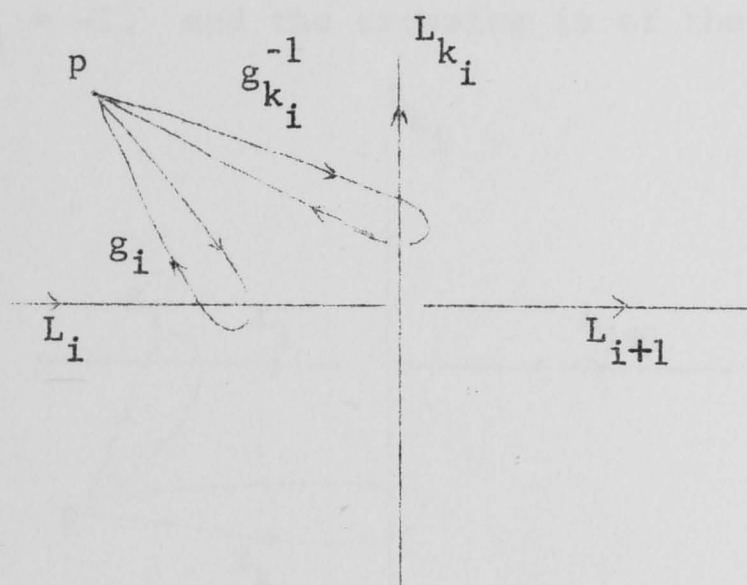
Then (K) becomes :

$$(L) \quad \sum_{i=1}^{n-1} (g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i} - 1) z'_i = 0.$$

Now the set $\{g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i} : i = 1, \dots, n-1\}$ represents a set of equivalence classes of paths in the space $R^3 - K$. We will give an interpretation of these group elements by one path from each equivalence class.

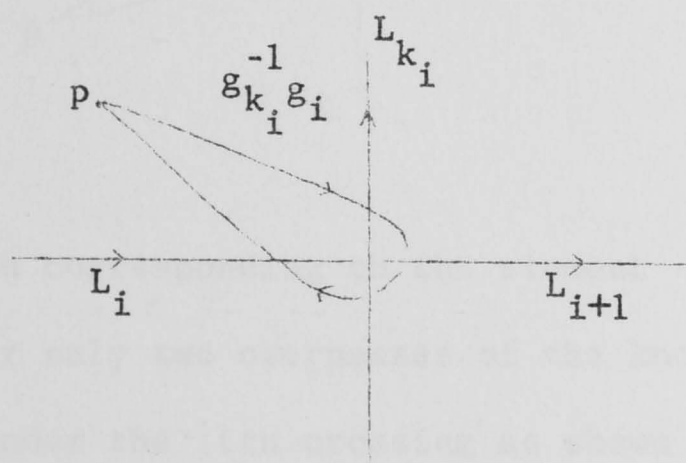
Consider the group element $g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i}$. At the i th crossing of the knot we have the following situation :

a) either a) $\varepsilon_i = +1$ and the crossing is represented :

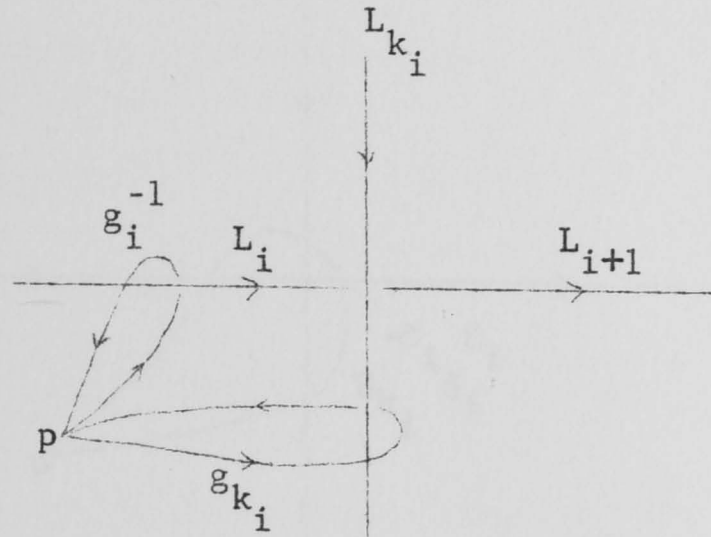


Take a point p "above" the plane of the representation.
 The p -based loops corresponding to $g_{k_i}^{-1}$ and g_i are
 as shown.

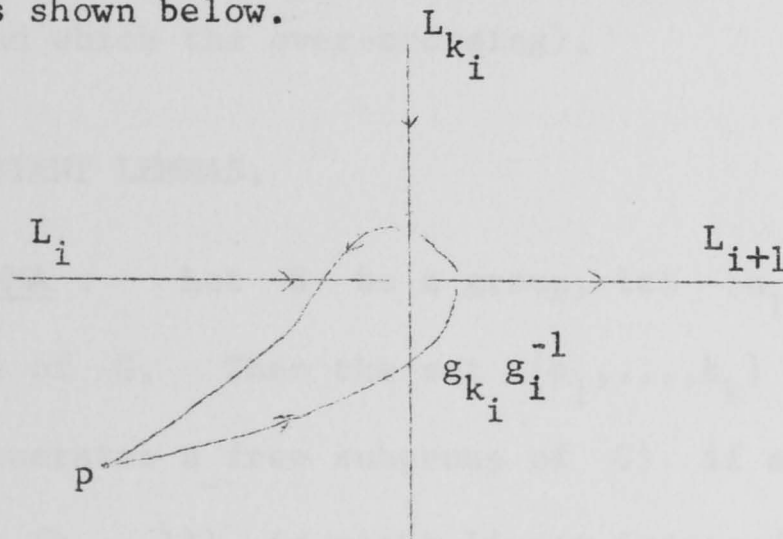
Their product is therefore the path as shown below
 corresponding to $g_{k_i}^{-1} g_i$.



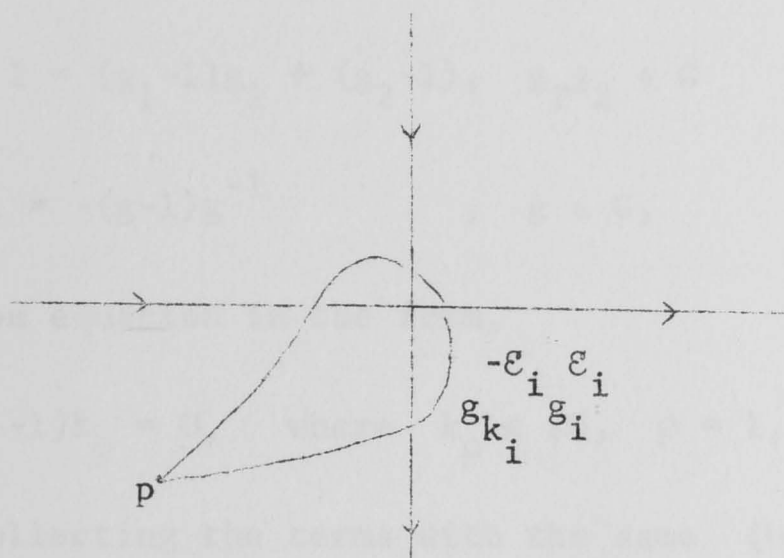
or b) $\varepsilon_i = -1$. and the crossing is of the form :



The corresponding to g_{k_i} and g_i^{-1} are as shown. Their product is shown below.



Thus the path corresponding to the element $g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i}$ is a loop passing under only two overpasses of the knot in a direction diagonally under the i th crossing as shown below.



ith crossing.

(Only the direction of the overpasses is important - not which is the under-and which the over-crossing).

4.5 SOME IMPORTANT LEMMAS.

4.5.1 LEMMA : Let G be a group, let $\{h_1, \dots, h_k\}$ be a set of elements of G . Then the set $\{h_1, \dots, h_k\}$ is free in G (i.e. freely generates a free subgroup of G) if and only if the set $\{(h_1 - 1), \dots, (h_k - 1)\}$ is right linear independent in ZG .

Proof: a) Suppose the set $\{h_1, \dots, h_k\}$ is not free in G .

Then $\prod_{\rho=1}^m h_{i_\rho}^{\varepsilon_\rho} = 1$, where $i_\rho \in \{1, \dots, k\}$,

$\rho = 1, \dots, m$, $\varepsilon_\rho = \pm 1$ for some product of the elements of $\{h_1, \dots, h_k\}$.

Then in ZG , $\prod_{\rho=1}^m h_{i_\rho}^{\varepsilon_\rho} - 1 = 0$.

We expand this, using the identities ,

$$g_1 g_2 - 1 = (g_1 - 1)g_2 + (g_2 - 1), \quad g_1, g_2 \in G$$

$$g^{-1} - 1 = -(g - 1)g^{-1}, \quad g \in G,$$

and obtain an equation in the form,

$$\sum_{\rho=1}^m (h_{i_\rho} - 1)k_\rho = 0, \quad \text{where } k_\rho \in ZG, \quad \rho = 1, \dots, m.$$

Then collecting the terms with the same $(h_j - 1)$ factor for each $j \in \{1, \dots, k\}$ we have ,

$$\sum_{j=1}^k (h_j - 1)m_j = 0, \quad \text{where } m_j \in ZG, \quad j = 1, \dots, k.$$

If all the m_j are zero, then it is easily seen that the product

$$\prod_{\rho=1}^m h_{i_\rho}^{\varepsilon_\rho} \text{ contains at least one subproduct } \prod_{\rho=1}^{m'} h_{i_\rho}^{\varepsilon_\rho}, \text{ such that}$$

$$\prod_{\rho=1}^{m'} h_{i_\rho}^{\varepsilon_\rho} = 1, \text{ and for which, by applying the same procedure as before}$$

to the supproduct, we may find a sum

$$\sum_{j=1}^k (h_j - 1)m'_j = 0, \quad \text{where the } m'_j \in ZG \text{ are not all zero.}$$

Thus the set $\{(h_1 - 1), \dots, (h_k - 1)\}$ is not right linear independent in ZG .

The contrapositive inference therefore gives us that right linear independence in ZG implies freeness in G .

- b) The converse is a corollary of the lemma 2.3.1, concerning the generation of the augmentation ideal of the free group.

4.5.2 LEMMA : Let G be a knot group, and ZG its group ring. Then a) if $g \in G$, $z \in ZG$ and $gz = 0$, then $z = 0$

- b) if $g \in G$, $g \neq 1$, $z \in ZG$ and $(g-1)z = 0$, then $z = 0$.

Proof : a) $g \in G \Rightarrow g^{-1} \in G$

$$\text{thus } gz = 0 \Rightarrow g^{-1}(gz) = (g^{-1}g)z = z = 0.$$

- b) We recall that all knot groups are torsion free 3.2.2, and that in a group ring ZH ,

$$\sum_{i=1}^k m_i h_i = 0 \Leftrightarrow m_i = 0, \quad i = 1, \dots, k, \quad \text{whenever} \\ \text{the } h_i \text{ are distinct in the group } H.$$

Suppose $z \neq 0$.

Then we may write $z = \sum_{i=0}^k m_i g_i$, where k is finite, $m_i \neq 0 \quad \forall i$, and the g_i are $k+1$ distinct elements of G . In dealing with the equation $(g-1)z = 0$ we may assume that $g_0 = 1$; for if it is not then

$$(g-1)z = 0 \Rightarrow (g-1)zg_0^{-1} = 0,$$

and writing $z' = zg_0^{-1}$, and $z' = \sum m_i g_i'$ we have $g_0' = 1$.

On the L.H.S. of $(g-1)z = 0$ we therefore have a term

$$(g-1)m_0 = m_0(g-1).$$

Thus there must be another term of the sum $\sum_{i=1}^k m_i(g-1)g_i$ which contains a term in g .

i.e. $\exists g_\ell, \ell \neq 0$, s.t. $m_\ell(g-1)g_\ell = m_\ell g g_\ell - m_\ell g_\ell$ contains a term in g . Now $g_\ell \neq g_0 \Rightarrow g g_\ell \neq g$. Thus $g_\ell = g$, and the summand $m_\ell(g-1)g_\ell$ is the only one other than $m_0(g-1)$ containing a term in g (otherwise $g_k = g$, for some $k \neq \ell$, and g_k and g_ℓ are not distinct). For the term in g to vanish we must therefore have $m_0 = m_\ell$.

Now rearrange the sum $\sum_{i=0}^k m_i(g_i-1)g_i$ so that $\ell = 1$, i.e. $g_\ell = g_1 = g$.

Then $(g-1)m_1g_1 = m_1(g-1)g$, gives us a term in g^2 .

Thus there must be another term in g^2

We continue sorting thru the g_i 's to find the summand with the required term in each case, rearranging the sum at each stage, until we use up all $k+1$ of the g_i 's in the sum z .

We find that :

$$g_i = g^i$$

$$\text{and } m_i = m_0, \quad i = 0, 1, \dots, k.$$

The equation $(g-1)z = 0$ becomes $(g-1)\sum_{i=0}^k m_i g^i = 0$

i.e.

$$m_k g^{k+1} + \sum_{i=0}^k (m_i - m_{i+1}) g^{i+1} - m_0 = 0.$$

Thus we have $m_0 (g^{k+1} - 1) = 0$ in ZG .

Now this can only occur if $m_0 = 0$ or if $g^{k+1} - 1 = 0$, i.e. $g^{k+1} = 1$.

But by our assumption that $m_0 \neq 0$, and since G is torsion free we have a contradiction.

Hence $z = 0$.

We shall not use part b) of this lemma in the remaining parts of this thesis. However it can be used as is part a), in section 4.7. We notice that for products of the type

$$m_j z_j \quad (\text{here } m_j \text{ is as defined in section 4.3.3}).$$

we have
$$m_j = \begin{cases} (g_{j+1} - 1)g_{k_j}^{-1} & \text{when } \varepsilon_j = +1 \\ -(g_j - 1) & \dots \varepsilon_j = -1, \end{cases}$$

so that $m_j z_j$ is of the form $(g-1)z$ where $g \in G$, $z \in ZG$.

Thus $m_j z_j = 0 \Rightarrow z_j = 0$ for any $j \in \{1, \dots, n-1\}$.

4.6 ON THE SOLUTIONS OF (L).

By lemma 4.4.1 it is obvious that we may find solutions of the equation (L) by considering the freeness of the set

$$\{g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i}, i = 1, \dots, n-1\} \text{ in } G. \quad \text{We shall write } p_i = g_{k_i}^{-\varepsilon_i} g_i^{\varepsilon_i},$$

$i = 1, \dots, n-1$.

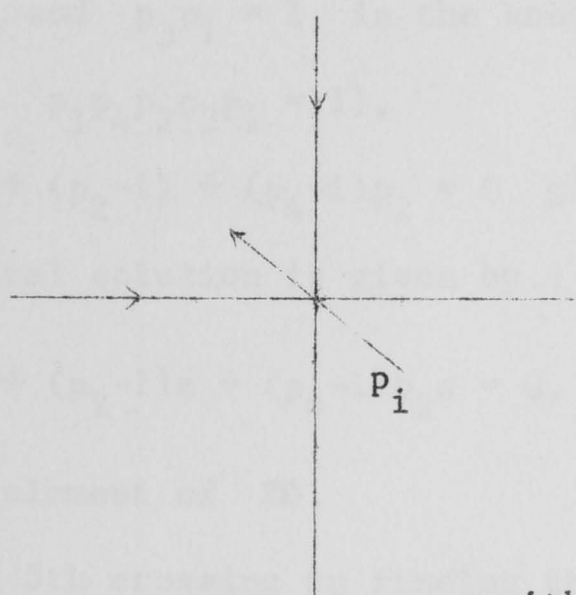
Inspection of the plane representation of the knot will show quite readily whether any subset of $\{p_1, \dots, p_{n-1}\}$ is not free in the knot group. For whenever we have

$$\prod_{\rho=1}^s p_{i_\rho} = 1, \quad p_{i_\rho} \in \{p_1, \dots, p_{n-1}\}, \quad \text{in the knot group we have}$$

a solution of the equation (L).

As we shall see the difficulty lies in finding all the solutions of (L). I have been unable to resolve this difficulty.

In the plane representation of the knot we shall draw the paths p_i as small arrows diagonally across each crossing in the direction determined in 4.4.2 i.e.

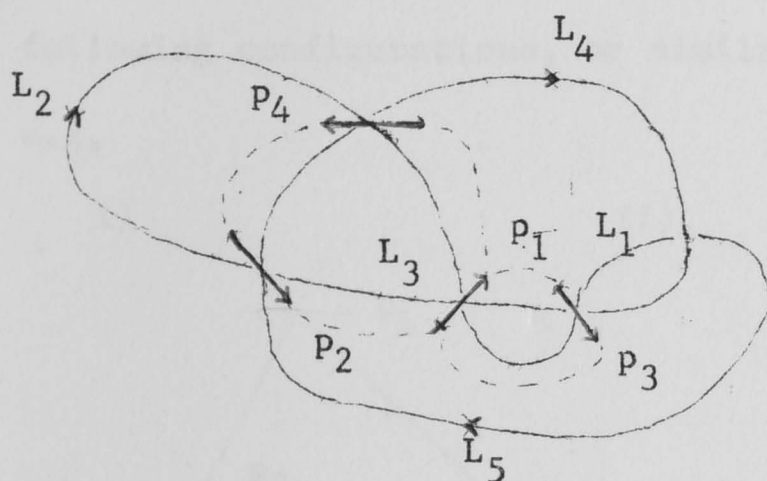


ith crossing.

We give two general types of solution to the problem of the non-freeness of the p_i 's in the knot group.

- a) When we can draw a closed line through a subset of the p_i 's such that the path does not cross any line of the plane representation except along the p_i 's.

e.g.



we may draw a path through p_1, p_4 and p_2 , and also p_5 and p_3 .
From the interpretation of the paths in 4.4.2 we have that :

$$p_1 p_4 p_2 = 1, \text{ and } p_3 p_1 = 1 \text{ in the knot group.}$$

(Notice also that $p_1 p_4 p_2 p_3 p_1 = 1$).

Thus $(p_1 - 1)p_4 p_2 + (p_2 - 1) + (p_4 - 1)p_2 = 0$ gives a solution of (L).

Hence a more general solution is given by :

$$(p_1 - 1)p_4 p_2 z + (p_2 - 1)z + (p_4 - 1)p_2 z = 0,$$

where z is any element of ZG .

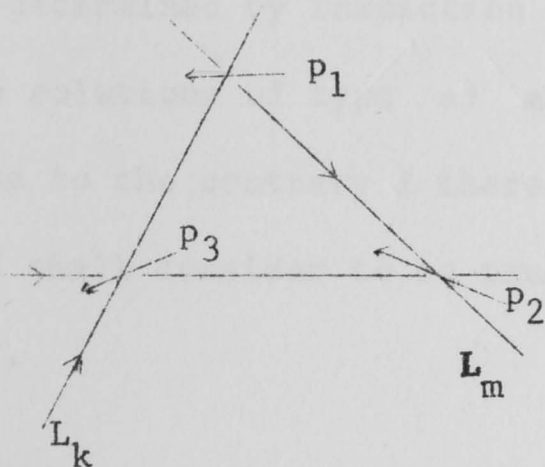
Since we omit the 5th crossing in finding the Wirtinger presentation for this knot group, it appears that the general solution of the equation (L) is :

$$[z_1', \dots, z_4'] = [p_4 p_2 z + z', z, p_1 z', p_2 z], \quad z, z' \in ZG.$$

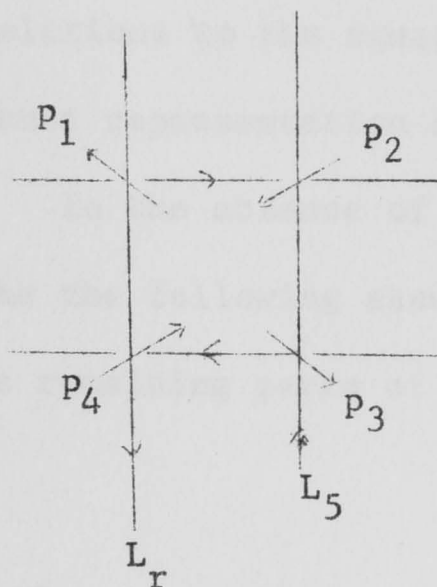
b) Other solutions to the equation (L) may arise if the following configurations, or similar ones, occur :

e.g.

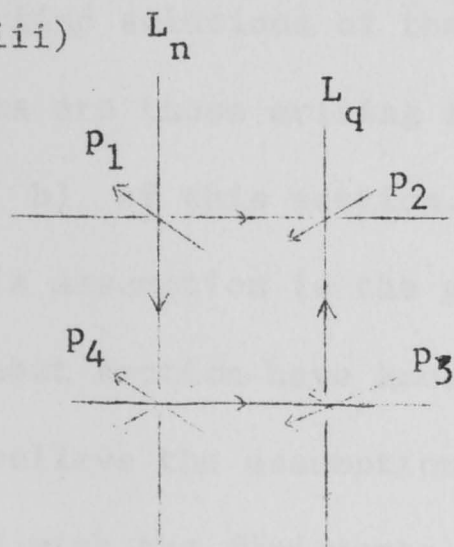
i)



ii)



iii)



In i) we have $p_1 = p_2 p_3 (= x_m x_k^{-1})$. The product represents the loop under the segments L_m and L_k respectively.

In ii) we have $p_2 p_1 = (p_3 p_4)^{-1} (= x_s^{-1} x_r)$.

In iii) we have $p_2 p_1 = p_3 p_4 (= x_q^{-1} x_n)$.

There are further variations of this type of solution, all arising when two or more consecutive over-or under-crossings occur along a segment of the knot.

I am fairly certain that all the solutions to the equation (L) may be determined by inspection of the knot representation and finding all the solutions of type a) and b). In the absence of any evidence to the contrary I therefore make the following assumption, which I shall consider to be true in the remaining parts of the chapter.

ASSUMPTION : That in inspecting the plane representation of a knot to find solutions of the equation (L) for the knot, the only solutions are those arising from configurations given in the parts a) and b) of this section.

This assumption is the gap in the proofs that the knots given in the next section have knot groups with cohomological dimension 2.

I believe the assumption may be proved using topological methods involved with the fundamental groups of spaces or covering spaces of knots, however I have been unable to show it true myself. On the other hand I have also been unable to find any solutions not of type a) or b) for any knot.

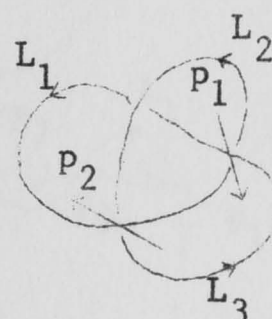
Even if the assumption is not true, the work in the following section may still be valid if the extra solutions to the equation (L) are dealt with in similar fashion when they occur.

Notice that for alternating knots only solutions of type a) can occur, under the assumption. These solutions bear a close relationship to the Seifert circles [18] of the representation.

4.7 SOME EXAMPLES OF SOLUTIONS OF (A).

4.7.1 The trefoil knot.

This knot is represented :



We omit the relator gained from the 3rd crossing.

Then there are no solutions to the equation (L) for this knot of type a) or b). Hence the solution of (L) and therefore (K) is trivial.

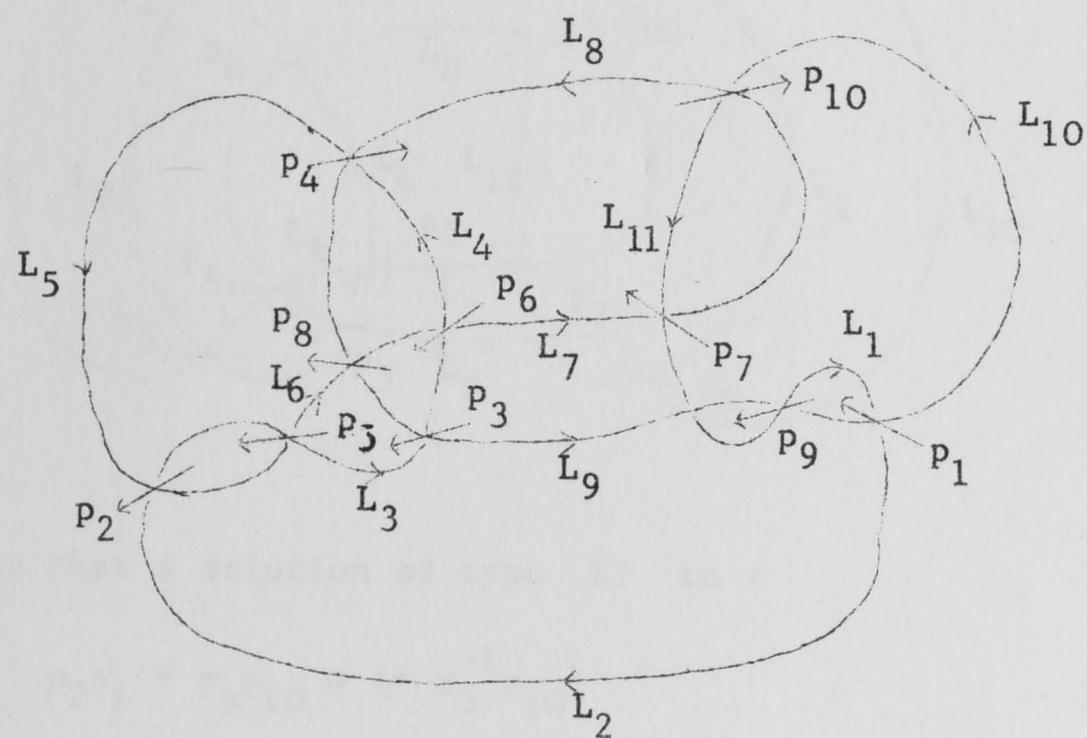
Thus the knot group has cohomological dimension 2.

(That this is so is of course well known, the knot being alternating, and with knot group easily presented as a one-relator group.)

4.7.2 The Kinoshita-Terasaka knot.

This knot is referred to in the paper by Kinoshita & Terasaka [7].

The Kinoshita-Terasaka knot.

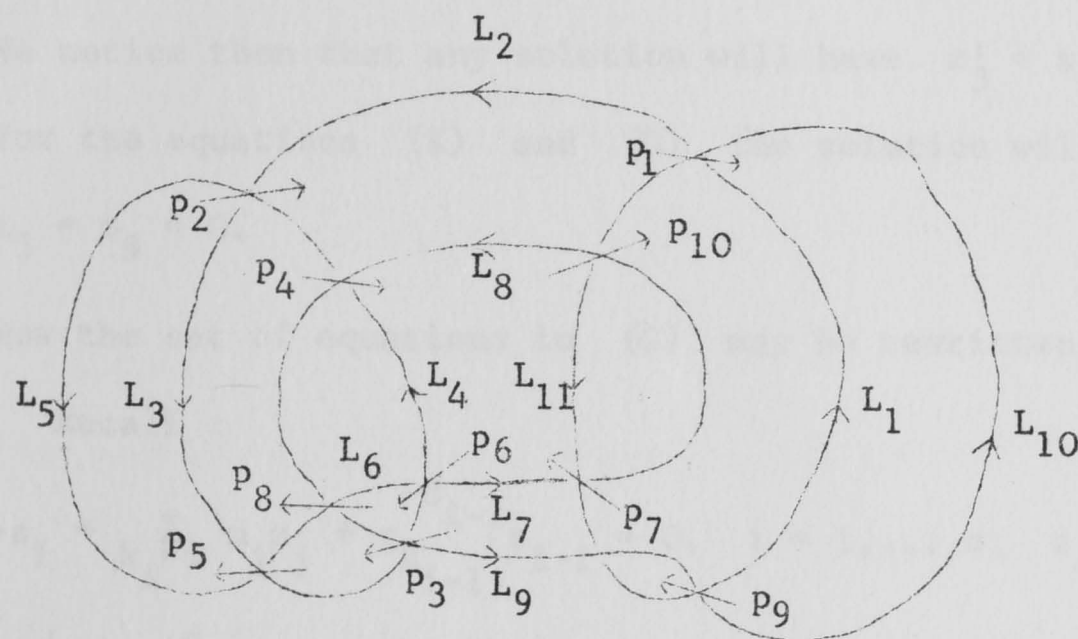


We can see that solutions of type a) are $p_6 p_8 p_4 = 1$,

and $p_{10} p_7 = 1$.

Since we omit the 11th relator, these are all the solutions of type a).

Now redraw the representation as :



We notice that a solution of type b) is :

$$p_2 p_1 = p_4 p_{10} \neq (= x_5^{-1} x_{10})$$

Looking at the products $p_7 p_6$ and $p_5 p_2 p_1 p_9 (= (p_{11} p_3)^{-1})$, we see that these will not be involved in other solutions of type b).

Therefore solutions of the equation (L) for the K-T knot are

- i) $(p_6 - 1)p_8 p_4 z + (p_8 - 1)p_4 z + (p_4 - 1)z = 0$
- ii) $(p_{10} - 1)p_7 z' + (p_7 - 1)z' = 0$
- iii) $(p_2 - 1)p_4 p_{10}^{-1} p_4^{-1} z'' + (p_1 - 1)p_{10}^{-1} p_4^{-1} z'' - (p_{10} - 1)p_{10}^{-1} p_4^{-1} z''$
 $- (p_4 - 1)p_4^{-1} z'' = 0, \quad z, z', z'' \in ZG,$

and any sum of these solutions.

Under our assumption these are all the types of solutions of (L).

We notice then that any solution will have $z'_3 = z'_5 = z'_9 = 0$.
 Thus for the equations (K) and (G) the solution will have
 $z_3 = z_5 = z_9 = 0$.

Now the set of equations in (G) may be rewritten in matrix form. Recall :

$$(G) \quad -z_i + \sum_{k_i=i} m_j z_j + g_{k_{i-1}}^{-\epsilon_{i-1}} z_{i-1} = 0, \quad i = 1, \dots, n; \quad z_0 = z_n = 0.$$

$$\text{Write } g'_i = g_{k_{i-1}}^{-\epsilon_{i-1}}.$$

Then in matrix form we have :

$$\begin{bmatrix} -1 & & & & & \\ g'_1 & -1 & & & & \\ & g'_2 & -1 & & & \\ & & g'_3 & & & \\ & m_1 & & & & \\ & & m_3 & & & \\ & & & m_2 & & \\ & & & & -1 & \\ & & & & & g'_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Call the matrix $A = (a_{rs})$. Then A is an $n \times (n-1)$ matrix

and $a_{rr} = -1$, $r = 1, \dots, n-1$

$$a_{r,r-1} = g_i' , r = 2, \dots, n$$

$$a_{r,s} = m_s , \text{ whenever } k_s = r$$

and $a_{r,s} = 0$ elsewhere.

The i th row of this matrix corresponds to the i th equation of the set (G) and therefore the i th overpass of the knot representation, for $i = 1, \dots, n$.

The j th column corresponds to the j th crossing of the representation, in as much as it includes the coefficients in the equations (G) which arise from the j th relator of the presentation. For the K.T. knot it may be shown that the matrix A is :

$$\begin{bmatrix} -1 & & & & & & & & m_9 & 0 \\ g_1' & -1 & & & & & & & & \\ & g_2' & -1 & 0 & m_5 & & & & & \\ & 0 & g_3' & -1 & 0 & m_6 & & & & \\ & m_2 & 0 & g_4' & -1 & 0 & & & & \\ & & 0 & g_5' & -1 & 0 & m_8 & & & \\ & & 0 & & g_6' & -1 & 0 & & & \\ & & & m_4 & & g_7' & -1 & 0 & m_{10} & \\ & & & & & 0 & g_8' & -1 & 0 & \\ & & & & & & 0 & g_9' & -1 & \\ m_1 & 0 & 0 & & & & & & & \\ 0 & & 0 & & & m_7 & & & g_{10}' & \end{bmatrix}$$

Now in finding the solution of the set (G) we have shown that we must have $z_3 = z_5 = z_9 = 0$. Thus the matrix may be reduced to appear as in the following equations:

$$\begin{bmatrix}
 -1 & & & & & & & & & \\
 g'_1 & -1 & & & & & & & & \\
 & g'_2 & -1 & m_6 & & & & & & \\
 & m_2 & g'_4 & 0 & & & & & & \\
 & & & & -1 & 0 & m_8 & & & \\
 & & & & g'_6 & -1 & 0 & & & \\
 & & m_4 & & & g'_7 & -1 & m_{10} & & \\
 & & 0 & & & & g'_8 & 0 & & \\
 m_1 & 0 & 0 & & & 0 & & -1 & & \\
 0 & 0 & 0 & & & 0 & & g'_{10} & &
 \end{bmatrix}
 \cdot
 \begin{bmatrix}
 z_1 \\
 z_2 \\
 z_4 \\
 z_6 \\
 z_7 \\
 z_8 \\
 z_{10}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot
 \end{bmatrix}$$

We notice immediately that $-z_1 = 0$, (row 1)

$$g'_2 z_2 = 0 \quad (\text{row 3})$$

$$g'_8 z_8 = 0 \quad (\text{row 9}).$$

Thus we have $z_1 = z_2 = z_8 = 0$ (from Lemma 4.5.2, a)).

Then from row 5 we have $g'_4 z_4 = 0, \Rightarrow z_4 = 0$

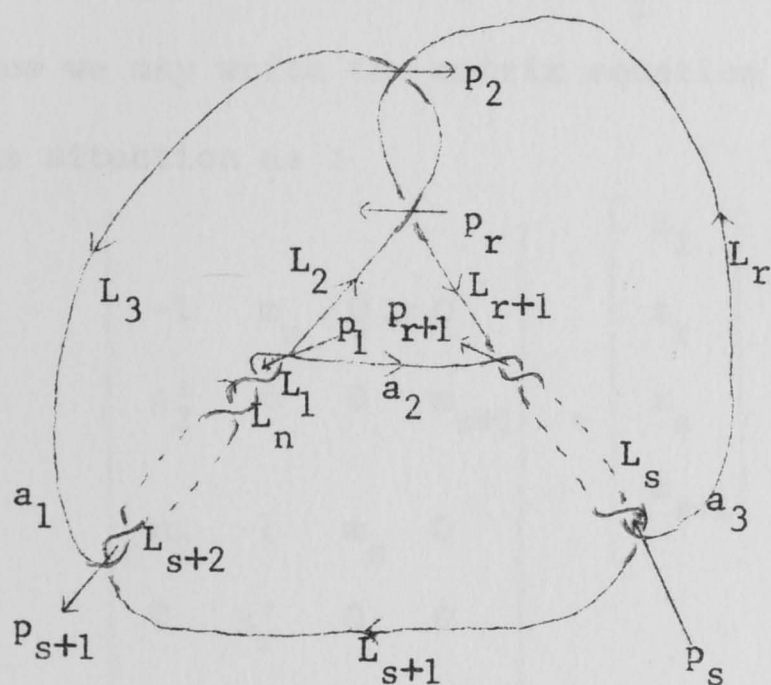
$$\dots \text{row 6} \dots \dots -z_6 = 0, \Rightarrow z_6 = 0$$

$$\dots \text{row 7} \dots \dots -z_7 = 0, \Rightarrow z_7 = 0$$

$$\dots \text{row 10} \dots \dots -z_{10} = 0, \Rightarrow z_{10} = 0.$$

Hence the only solution of the equation (G) is the trivial solution. i.e. The knot group of the Kinoshita Terasaka knot has cohomological dimension 2.

4.7.3 Consider the knots as represented in the diagram :



The letters a_1, a_2, a_3 denote that these segments are "consecutive" over the two series of "twists".

Write the overpasses as shown, and assume $L_{s+2} \neq L_1$.

Since we omit the n th relator, derived from the n th crossing, we see that the only solution for (L) of type a) is $p_r p_2 = 1$.

A solution of type b) is $p_s^{-1} p_{s+1}^{-1} = p_r$.

We see that the product $p_{r+1} p_1$ does not give rise to another solution of this type, thus exhausting the configurations which give rise to solutions of type b).

Thus the only solutions of (L) involve the paths p_2, p_r, p_s and p_{s+1} .

Hence the only solution of (K) will have z_2, z_r, z_s and z_{s+1} unequal zero, with all the other z_i vanishing.

Now we may write the matrix equation for (G) corresponding to this situation as :

$$\begin{array}{c}
 2 \\
 3 \\
 \cdot \\
 \cdot \\
 r \\
 r+1 \\
 \cdot \\
 \cdot \\
 s \\
 s+1 \\
 s+2
 \end{array}
 \begin{bmatrix}
 -1 & m_r & 0 & 0 \\
 g'_2 & 0 & 0 & m_{s+1} \\
 m_2 & -1 & m_s & 0 \\
 0 & g'_r & 0 & 0 \\
 0 & 0 & -1 & 0 \\
 0 & 0 & g'_s & -1 \\
 0 & 0 & 0 & g'_{s+1}
 \end{bmatrix}
 \begin{bmatrix}
 z_2 \\
 z_r \\
 z_s \\
 z_{s+1}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 0
 \end{bmatrix}$$

(certainly $r > 3, s > r+1$).

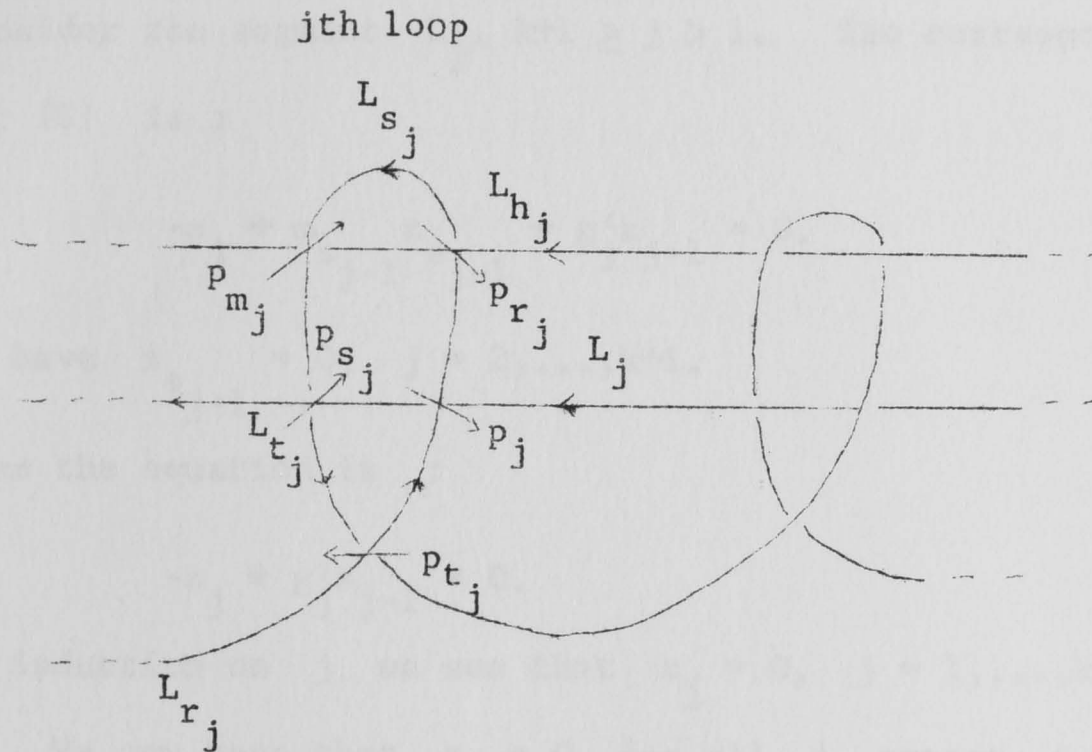
(This is so since we can easily see $k_r = 2, k_2 = r, k_s = r, k_{s+1} = 3$).

Then from row s , $-z_s = 0 \Rightarrow z_s = 0$

from row $r+1$, $g'_r z_r = 0 \Rightarrow z_r = 0$

and from row 2 , $-z_2 = 0 \Rightarrow z_2 = 0$, row $s+1 \Rightarrow z_{s+1} = 0$.

Hence the only solution of (G) is trivial. Thus given our assumption, we have that the knot groups for this class of knots have cohomological dimension 2.



Consider the equation in (G) corresponding to the overpass L_{s_j} .

$$\text{i.e. } -z_{s_j} + m_{h_j} z_{h_j} + g'_{s_j} z_{r_j} = 0.$$

$$z_{h_j} = z_{r_j} = 0 \Rightarrow z_{s_j} = 0, \quad j = 1, \dots, k.$$

Further the equation in (G) corresponding to the overpass L_{t_j} is :

$$-z_{t_j} + g'_{t_j} z_{s_j} = 0.$$

$$\Rightarrow z_{t_j} = 0.$$

Now for the overpass L_1 we have the equation :

$$-z_1 + g'_1 z_0 = 0.$$

Now $z_0 = 0$. Thus $z_1 = 0$.

Consider the segment L_j , $k+1 \geq j > 1$. The corresponding equation in (G) is :

$$-z_j + m_{s_{j-1}} z_{s_{j-1}} + g'_j z_{j-1} = 0.$$

We have $z_{s_{j-1}} = 0$, $j = 2, \dots, k+1$.

Thus the equation is :

$$-z_j + g'_j z_{j-1} = 0.$$

By induction on j we see that $z_j = 0$, $j = 1, \dots, k$.

We now have that $z_i = 0$ for all i except $i = n-1$.

The equation in (G) corresponding to the overpass L_n is :

$$-z_n + g'_n z_{n-1} = 0.$$

But $z_n = 0 \Rightarrow z_{n-1} = 0$.

Thus the z_i in the solution of (G) are all zero.

Hence the knot group has cohomological dimension 2.

4.7.5 CONCLUSION.

We have seen that under the assumption in 4.6, the knots in the previous section all have knot groups with cohomological dimension 2. The techniques used in 4.7 may of course be extended to many other classes of knots, and be used for particular knots.

However in the absence of a satisfactory proof of the assumption I shall conclude the thesis at this point.

Using the methods developed in this chapter, I have been unable to find a knot which does not satisfy the necessary and sufficient condition. On the contrary I believe that the method will lead to a proof of our conjecture if more research is done in the subject.

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