(Finite) presentations of Bi-Zassenhaus loop algebras

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We prove that Bi-Zassenhaus loop algebras are finitely presented up to central and second central elements by showing an explicit finite presentation for a Lie algebra whose factor over its second centre is isomorphic to a Bi-Zassenhaus loop algebra.

Key Words: Graded Lie algebras of maximal class, finite presentation.

1. INTRODUCTION

The family of simple modular Lie algebras nowadays known as Hamiltonian algebras was discovered in the fifties by A.A. Albert and M.S. Frank (see [1]) and is named after them. By their very definition, they are graded over a non-cyclic elementary abelian p-group. In his paper [13], A. Shalev noticed that they possess a cyclic grading as well and they admit a non-singular derivation that cycles over this grading. This property allowed him to build an infinite-dimensional loop algebra starting from the original simple one: the new structure is of maximal class, *i.e.* it is $\mathbf{N}$-graded with all homogeneous components of dimension one except the first one which has dimension two and generates the entire algebra. These new algebras are known as Albert-Frank-Shalev (AFS, for short) algebras.

In the following years there has been a growing interest in these algebras: A. Caranti, S. Mattarei and M.F. Newman developed new techniques to construct more algebras (see [2]) and later Caranti and Newman achieved a classification theorem for odd characteristic fields of definition in their paper [3]. The remarkable result they reached shows that every infinite-dimensional $\mathbf{N}$-graded Lie algebras of

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maximal class generated by its first homogeneous component can be built starting from an AFS-algebra.

To get the claim they need a theorem proved by C. Carrara in [4], [5], which states that every AFS algebra is uniquely determined by a suitable finite dimensional quotient; she obtained this result as a corollary of a more general property she discovered: AFS-algebras are finitely presented up to central and second central elements.

A similar classification for characteristic two is still valid (see [9]), but in this case another family of algebras is involved: they are called Bi-Zassenhaus loop algebras (B_l for short). The loop construction process for them is exactly the same as for Albert-Frank-Shalev algebras, but the simple finite-dimensional algebra at the beginning is different: their definition is in [8].

Here we want to show that the property described above for AFS-algebras is satisfied by B_l-algebras, too; the fact that B_l-algebras are uniquely determined by a suitable finite-dimensional quotient again follows as a corollary, although in [9] we prove it directly.

In particular, the result we prove is the following

THEOREM 1.1. *For every Bi-Zassenhaus loop algebra* $\mathrm{B}_l(g,h)$*, there exists a finitely presented graded Lie algebra* $M(g,h)$ *such that*

$$M(g,h)/Z_2(M(g,h)) \cong \mathrm{B}_l(g,h)\ .$$

The construction of the central extension M by means of cohomological arguments is shown in [8]. Note that, since the centre of M is infinite-dimensional, a group-theoretical result of B.H. Neumann [12, pp. 52-53] transferred to Lie algebra shows that this implies that B_l itself is *not* finitely presented.

Since the relations in the presentation of the algebra M are retrieved by expanding several suitable generalized Jacobi identities, machine computations took a main rôle throughout the work. These computations were performed by the software p-Quotient Program [6] developed at the Australian National University in Canberra. Nevertheless, all the proofs are independent from such calculations.

2. PRELIMINARIES AND NOTATION

We refer to the above cited papers for background on graded Lie algebras of maximal class and the construction of B_l algebras. This section is meant to give a brief survey of the main facts concerning the topics involved in the present discussion.

A graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i\ ,$$

defined over a field of positive characteristic p and generated by L_1 is said to be of *maximal class* if $\dim(L_1) = 2$ and $\dim(L_i) \leq 1$ for $i > 1$.

Its elements are written left-normed and in exponential form:

$$[uv^n] = [[[u\underbrace{v]v]\cdots v}_{n}] \,.$$

The *two-step centralizers* C_i of L are defined as the one-dimensional subspaces $C_{L_1}(L_i)$ of L_1 centralizing the homogeneous components L_i when $i > 1$, while $C_1 = C_2$ is formally assumed. Following the classical notation, we define the element y by means of $C_2 = \mathbf{F}y$ and we choose another element $x \in L_1 \setminus \mathbf{F}y$ such that the pair $\{x, y\}$ is a set of generators for L_1. When the class of L is large enough, all the two-step centralizers coincide with C_2 apart from isolated occurrences of different subspaces, so it is possible to define a *constituent* of L as a subsequence $(C_i, \ldots, C_j)$ where all centralizers coincide with C_2 but the last one, and either $i = 1$ or $C_{i-1} \neq C_2$. Homogeneous elements lying in class $i - 1$ and j are respectively said to be *at the beginning* and *at the end* of the constituent and they are not centralized by C_2.

A graded Lie algebras of maximal class is determined by its sequence of two-step centralizers, and, when only two of them are distinct, by its sequence of constituent lengths. The first constituent of a large enough graded Lie algebra of maximal class is always of length $2q$ for some power $q = p^h$ called its *parameter*, and all other constituents can only be *short* (*i.e.* of length q) or *long* ($2q$) or *intermediate* ($2q - p^s - 1$, where $0 \leq s \leq h - 1$).

From now on, unless explicitly stated, we will assume $p = 2$. This allows us to ignore signs safely so that the generalized Jacobi identity reads as follows in the two equivalent forms:

$$\begin{aligned}
[v[wu^\lambda]] &= \sum_{i=0}^{\lambda} \binom{\lambda}{i} [vu^i wu^{\lambda-i}] \\
&= \sum_{i=0}^{\lambda} \binom{\lambda}{i} [vu^{\lambda-i} wu^i] \,;
\end{aligned}$$

We will introduce the element $z = x + y$ to apply the above formula more effectively: if $0 \neq v \in L_i$, for some $i > 1$, and if

$$C_i, C_{i+1}, \ldots, C_{i+n-1} \in \{\mathbf{F}x, \mathbf{F}y\},$$

then for every non-zero commutator $[vx_1x_2 \ldots x_n]$ with $x_i \in \{x, y\}$ we have that

$$[vx_1x_2 \ldots x_n] = [vz^n] \,.$$

Moreover, Lucas' Theorem (see [11], [10]) will be used several times: if $a = \sum_{i=0}^{n} a_i \cdot 2^i$ and $b = \sum_{i=0}^{n} b_i \cdot 2^i$ are the 2-adic expansions of two integers, then

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod 2 .$$

Finally, the following identity will be useful

$$2^a - 2^b = \sum_{i=b}^{a-1} 2^i \quad \forall\, 0 \leq b < a ;$$

note that, as a consequence of the two formulas above, we have that

$$\binom{2^w - 1}{i} \equiv 1 \pmod 2 \quad \forall\, 0 \leq i \leq 2^w - 1 . \tag{1}$$

The algebras $\mathrm{B}_l(g, h)$, defined when the characteristic of the underlying field is two and $(h, g-1) \in \mathbf{N} \times \mathbf{N}$, form a family of elementary objects among infinite-dimensional graded Lie algebras of maximal class. If exponential notation is employed in order to indicate consecutive occurrences of patterns, the sequence of constituent lengths of the algebra $\mathrm{B}_l(g, h)$ reads as

$$2q, 2q-1, \left((2q)^{\eta-1}, (2q-1)^2\right)^{\infty} , \tag{2}$$

where $q = 2^h$ and $\eta = 2^g - 1$. Thus in a B_l-algebra the only intermediate constituents are those of maximal length $2q-1$ and there are no short constituents.

Suppose $M = \bigoplus_{i=1}^{\infty} M_i$ is a graded Lie algebra such that $M/Z_2(M)$ is a graded Lie algebra of maximal class. We define a *constituent* of M as a constituent of $M/Z_2(M)$, ignoring the central or second central elements of M.

In order to lighten notations, we introduce here some shorthands that will be used in what follows. First of all, we define the elements

$$v_n = [yx^{2q-1}(yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2})^n] ,$$

which lie in the homogeneous components of indices $2q+dn$ where $d = 2^{g+h+1} - 2$ is just the dimension of the simple algebra $\mathrm{B}(g, h)$ and n ranges over the non-negative integers. When the two parameters a, b lie in the intervals $2 \leq a \leq h+1$ and $h+2 \leq b \leq g+h$, we can define the following elements (the notation is consistent with the one in [8])

$$\begin{aligned}
\theta_n^1 &= [v_n x] && (2q+1+dn)\\
\theta_n^a &= [v_n y x^{2q-2^{h+2-a}-1} y] && (4q-2^{h+2-a}+1+dn)\\
\theta_n^b &= [v_n y x^{2q-2} (y x^{2q-1})^{\eta-2^{g+h+1-b}} y x^{2q-2} y] && \\
& && (2q(\eta+3-2^{g+h+1-b})-1+dn)\\
\theta_n^\omega &= [\theta_{2n+1}^1 y] && (2q+2+d(2n+1))\ .
\end{aligned}$$

The number in parentheses is the *weight* of the element, *i.e.* the index i of the homogeneous component M_i in which it lies. Finally, we define the elements

$$\chi_t = [y x^{2q-1} y x^{2q-2} (y x^{2q-1})^t y x^{2q-2} y]\ .$$

Note that $\chi_{\eta-2^{g+h+1+b}} = \theta_0^b$ when $h+2 \le b \le g+h$.

3. THE FINITE PRESENTATION

Fix a particular B_l algebra L and then consider a finite set R' of homogeneous relations such that $M' = \langle x, y : R' \rangle$ is a presentation of a suitable m-dimensional Lie algebra such that $M'/Z_2(M')$ is isomorphic to a graded quotient of L and remove the relations of degree $m+1$.

We will show that what we obtain is a finitely presented, infinite-dimensional graded Lie algebra M such that $M/Z_2(M) \cong L$. Adding to R' the set of relations used to annihilate the central and second central elements, we will obtain a graded Lie algebra of maximal class M isomorphic to L and uniquely determined by a suitable finite-dimensional quotient M'.

When all the homogeneous relations that define the algebra L up to central and second central elements up to class $m = 2q(\eta+2)$ are added to R, then the resulting algebra M starts as a graded Lie algebra of maximal class with initial segment of constituent lengths

$$2q, 2q-1, 2q^{\eta-1}, 2q-1$$

and $C_i \in \{\mathbf{F}x, \mathbf{F}y\}$ for every $1 \le i \le m$.

We will show that most of the $m-2$ relations that in every class $2 \le i \le m$ set the two-step centralizer C_i are redundant: a suitable set of $q+h+\eta$ of them is actually enough, and we now explain which are the chosen relations and what they mean in terms of the classification of graded Lie algebra of maximal class in characteristic two.

The first set of q relations defines the parameter $q = 2^h$:

$$[y x^{2j+1} y] \quad \text{for } 0 \le j \le q-2\ , \qquad [y x^{2q+1}]\ .$$

This makes all homogeneous components have the same two-step centralizer up to weight $2q$. Moreover, they force all further constituents to be short, long or intermediate. The next relation

$$[yx^{2q-1}yx^{q-1}yx]$$

states that the second constituent is not short: in particular, this implies that no more short constituents or two-step centralizers other than the first two will be involved. A standard argument shows that the second constituent cannot be long, so it can be only intermediate. The following $h-1$ relations

$$[yx^{2q-1}yx^{2q-2^s-1}yx] \quad \text{for } 1 \leq s \leq h-1$$

establish its length as the maximal possible $2q-1$. So the algebra is not inflated and starts moving on the branch of the limit algebra $\mathrm{AFS}(h, h+1, \infty, 2)$: furthermore, we know that from now on its sequence of constituent lengths will contain only long and maximal intermediate constituents. The remaining η relations determine the algebra: again by the classification we know that this may happen only when the number of long constituents before two intermediate ones is a power of two minus one or minus two. These relations enable recognition of the latter case (the B_l pattern) rather than the former one (the AFS one):

$$\begin{array}{ll} [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^t yx^{2q-2}y(x)] & \text{for } 0 \leq t \leq \eta-2\ , \\ [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-1}yx]\ , & \end{array}$$

where the (x) appears only when $\eta - t$ is a two-power. Rewriting the entire set R of relations in terms of the above defined shorthands, it reads

$$\begin{array}{ll} [yx^{2j+1}y], & 0 \leq j \leq q-2\ , \\ [\theta_0^t x], & 1 \leq t \leq g+h,\ \omega\ , \\ \chi_t, & 0 \leq t \leq \eta-2 \text{ and } \eta - t \neq 2^\alpha, \text{ for } 1 \leq \alpha \leq g-1\ . \end{array}$$

We will prove that the sequence of constituent lengths of (M, R) and L are the same and that the two algebras differ only in central or second central elements; in particular, we claim that the elements θ_n^t are the only central and second central elements that may occur: the construction shown in [8] completes the picture.

4. THE PROOF: THE STRUCTURE

For clearness' sake, the expansion of the following Jacobi identities which are not immediate is carried out in the next section.

4.1. Using the first group of relations

The first homogeneous component is obviously two-dimensional:

$$M_1 = \langle x, y \rangle\ .$$

The first group of relations written above shows the structure of M up to class $2q+1$:

$$M_i = \langle [yx^{i-1}] \rangle \quad \text{for } 2 \le i \le 2q \; , \qquad M_{2q+1} = \langle [yx^{2q-1}y], [yx^{2q}] = \theta_0^1 \rangle \; .$$

In fact, when j ranges between 0 and $2q-2$, the element

$$[yx^j y]$$

vanishes, by the relations above in the case j odd and by the following inductive argument when j is even:

$$0 = [[yx^{\frac{j}{2}}][yx^{\frac{j}{2}}]] = \binom{\frac{j}{2}}{\frac{j}{2}} [yx^j y] \; .$$

Finally, the relation in class $2q+2$

$$[yx^{2q+1}] = [\theta_0^1 x] = 0$$

and the expansions

$$\begin{aligned} 0 &= [yx^{2q-2}[yxy]] = [yx^{2q-1}yy] \; , \\ 0 &= [[yx^q][yx^q]] = \binom{q}{q-1}[yx^{2q-1}yx] + \binom{q}{q}[yx^{2q}y] = [\theta_0^1 y] \end{aligned}$$

show that the first constituent of M has length $2q$, that $C_{2q} = \mathbf{F}x$ and that θ_0^1 is central:

$$M_{2q+2} = \langle [yx^{2q-1}yx] \rangle \; .$$

4.2. The constituent lengths

By the relations just obtained, the standard result [2, Proposition 5.6] can be proved: for a (large enough) algebra of parameter q,

the only possible constituent lengths are $2q$ *and* $2q - 2^s$, $\quad 0 \le s \le h$. $(\mathcal{CL})$

When central and second central elements occur, the proof of $(\mathcal{CL})$ needs a small refinement, already pointed out in [5]; following the convention on two-step centralizers, when $C_i = \mathbf{F}y$ in M and $w \in M_i$ we can only say that $[wyzz]$ vanishes, which is sufficient to prove that

$$0 = [w[yxy]] = [wxyy] + [wyyx] = [wxyy] \; ,$$

this lets the proposition work in this new context, too.

The above property implies that, if v is not centralized by y, then

$$[v_n y x^k y] = 0 \ ,$$

when $k \neq 2q - 2^s - 1$ for $0 \leq s \leq h$. By using this property, it is not difficult to show that the second constituent of M is of maximal intermediate length $2q-1$, that its last element is centralized by x and that the elements θ_0^a are central, for $2 \leq a \leq h+1$:

$$M_{2q+i} = \begin{cases} \langle [yx^{2q-1}yx^{i-1}] \ , & \text{for } 3 \leq i \leq 2q-1 \ , \\ \ [yx^{2q-1}yx^{i-2}y] = \theta_0^a \rangle & i = 2q - 2^{h+2-a} + 1 \ , \\ & 2 \leq a \leq h+1 \ , \\ & \\ \langle [yx^{2q-1}yx^{i-1}] \rangle & \text{otherwise} \ , \end{cases}$$

$$M_{4q} = \langle [yx^{2q-1}yx^{2q-2}y] \rangle \ .$$

Since two consecutive values of i cannot be both of the type $2q - 2^{h+2-a} - 1$, the claim is given by the fact that every element

$$[yx^{2q-1}yx^jy]$$

vanishes by the property $(\mathcal{CL})$ when $0 \leq j \leq 2q-3$ and $j \neq 2q - 2^s - 1$ for $1 \leq s \leq h$ and otherwise, by the relation

$$[yx^{2q-1}yx^jyx] = [\theta_0^{h+2-s}x]$$

and the expansions

$$0 = [yx^{2q-1}yx^{j-1}[yxy]] = [yx^{2q-1}yx^jyy] = [\theta_0^{h+2-s}y] \ ,$$

$$0 = [[yx^{2q-1}][yx^{2q-1}]] = \binom{2q-1}{0}[yx^{2q-1}yx^{2q-1}] \ . \tag{3}$$

4.3. The number of long constituents

Now we deal with the remaining structure of M up to class $m = 2q(\eta+2)$, by showing that when i runs between zero and $\eta - 1$ and k between zero and $2q-1$, with $k \neq 2q-1$ for $i = \eta - 1$, we have that

$$M_{4q+2qi+k} = \begin{cases} \langle [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^i yx^{2q-1}] \ , & \text{for } k = 2q-1 \ , \\ \ [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^i yx^{2q-2}y] & i = \eta - 2^{g+h+1-b} \ , \\ \ = \theta_0^b \rangle & h+2 \leq b \leq g+h \ , \\ & \\ \langle [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^i yx^k] \rangle & \text{otherwise} \ . \end{cases}$$

To prove the above result, we consider separately some cases:

- when $k = 0$,
  - and $i = 0$, then the claim is just equation (3);
  - and $i > 0$, first of all we have the relation $[\theta_0^1 x]$:

$$
\begin{aligned}
0 &= [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-1}yx^{2q-2}[\theta_0^1 x]] \\
&= [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-1}yx^{2q-2}[yx^{2q+1}]] \\
&= \binom{2q+1}{1}[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i}yx^{2q-1}x] \; ;
\end{aligned}
\tag{4}
$$

– in particular, if $i = \eta - 2^{g+h+1-b}$, for $h+2 \leq b \leq g+h$, then the previous homogeneous component has θ_0^b as a generator, too; to show that this element is central, we can use the relation

$$[\theta_0^b x] = 0$$

and the standard expansion

$$0 = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-1}yx^{2q-3}[yxy]] = [\theta_0^b y] \; ;$$

- when $k = 2q - 2^s$, for $1 \leq s \leq h$,
  - and $i = 0$, we can use the following Jacobi expansion:

$$
\begin{aligned}
0 &= [[yx^{2q-1}yx^{q-2^{s-1}-1}][yx^{2q-1}yx^{q-2^{s-1}-1}]] \\
&= [yx^{2q-1}yx^{2q-2}yx^{k-1}y] \; ;
\end{aligned}
\tag{5}
$$

– and $i > 0$, by using the relation $[\theta_0^{s+1}x]$ in the following identity, which is the case $n = 0$ of the most general one (16):

$$
\begin{aligned}
0 &= [\mu_0[\theta_0^{s+1}x]] \\
&= [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i}yx^{k-1}y] \; ,
\end{aligned}
\tag{6}
$$

where we have defined

$$\mu_0 = \begin{cases} [yx^{2q-1}yx^{2q-3}] & \text{for } i = 1 \\ [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{i-2}yx^{2q-2}] & \text{otherwise} \; ; \end{cases}$$

- when $k = 2q - 1$ and $i \leq \eta - 2$, the claim is explicitly stated by the relation χ_i, when $i \neq \eta - 2^{g+h+1-b}$, while otherwise there is nothing to prove since the homogeneous component is two-dimensional.
- by proposition ($\mathcal{CL}$) in the remaining cases.

So we have proved that M has the same structure of L up to class m, apart from some central elements.

4.4. From the factor algebra to the whole algebra

Now we have to prove that the finite-dimensional factor we built determines M as a B_l-algebra, apart from the central and second central elements θ_n^t, for any integer $n \geq 1$.

The last equation in the previous section shows that

$$M_{2q+d} = \langle v_1 \rangle \; ;$$

by induction the homogeneous component in class $2q + dn$ is one-dimensional too, generated by the element

$$M_{2q+dn} = \langle v_n \rangle \; .$$

In what follows we describe the structure of the next d homogeneous components of M, *i.e.* of an entire period of the algebra. We will use the notation w^{-c} for an element such that

$$[w^{-c} x^c] = w \; .$$

The homogeneous component in class $2q + 1 + dn$ is two-dimensional:

$$M_{2q+1+dn} = \langle [v_n x] = \theta_n^1, [v_n y] \rangle \; ,$$

where θ_n^1 can be central or second central, since in the next class

$$M_{2q+2+dn} = \begin{cases} \langle [v_n y x] \rangle & \text{when } n \text{ is even} \; , \\ \langle [v_n y x], [v_n x y] = \theta^{\omega}_{\frac{n-1}{2}} \rangle & \text{when } n \text{ is odd} \; . \end{cases}$$

In fact, in addition to the standard identities

$$0 = [v_n^{-1}[yxy]] = [v_n yy]$$

and

$$\begin{aligned} 0 &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [\theta_0^1 x]] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [y x^{2q+1}]] \\ &= \binom{2q+1}{1} [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1} y x^{2q}] \\ &= [\theta_n^1 x] \\ &= [v_n x x] \; , \end{aligned} \tag{7}$$

when $n = 2s$ is even we have the equation

$$[v_n xy] = [\theta_n^1 y] = 0 \ ,$$

given by the expansion

$$[[v_s^{-(q-1)}][v_s^{-(q-1)}]] = 0 \ , \tag{8}$$

that together with the (7) shows θ_1^1 to be central in those cases.

Now we consider class $2q + 3 + dn$: if $n \geq 1$ is odd, first we have the equation

$$0 = [v_n[yxy]] = [\theta^{\omega}_{\frac{n-1}{2}} y] = [v_n xyy] \ ;$$

then , if $n = 1$ we have the relation

$$0 = [\theta_0^{\omega} x] = [v_n xyx] \ ,$$

otherwise we can use the expansion

$$0 = [v_{n-2} yx^{2q-2}(yx^{2q-1})^{\eta-2} yx^{2q-2}[\theta_0^{\omega} x]] = [\theta^{\omega}_{\frac{n-1}{2}} x] = [v_n xyx] \ , \tag{9}$$

to show that the element $\theta^{\omega}_{\frac{n-1}{2}}$ is central. Finally, if $q = 2$ the element $[v_n yxy]$ is just θ_n^2, while when $q > 2$ the relation $[yxxxy] = 0$ holds, so we can expand

$$0 = [v_n^{-2}[yxxxy]] = \binom{3}{2}[v_n yxy] = [v_n yxy] \ .$$

Summarizing, we have

$$M_{2q+3+dn} = \begin{cases} \langle [v_n yxx], [v_n yxy] = \theta_n^2 \rangle & \text{when } q = 2 \ , \\ \\ \langle [v_n yxx] \rangle & \text{otherwise .} \end{cases}$$

Now we focus the attention on the homogeneous components up to class $4q + dn$.

If $q > 2$, then $2q + 3 + dn < 4q - 1 + dn$ so we can study what happens in classes $2q + 1 + k + dn$, where $3 \leq k \leq 2q - 3$. We claim that

$$M_{2q+1+k+dn} = \begin{cases} \langle [v_n yx^k], [v_n yx^{k-1} y] = \theta_n^a \rangle & \text{when } k = 2q - 2^{h+2-a} \ , \\ & \text{for some } 2 \leq a \leq h+1 \ , \\ \\ \langle [v_n yx^k] \rangle & \text{otherwise .} \end{cases}$$

If $k = 2q - 2^{h+2-a}$ for some $2 \leq a \leq h + 1$, by hypothesis $M_{2q+1+k+dn}$ is generated by both $[v_n yx^k]$ and $[v_n yx^{k-1} y]$: since $h + 2 - a > 0$ the number $k + 1$

cannot be of the form $2q - 2^{h+2-a'}$ for any $2 \leq a' \leq h+1$, so $[v_n y x^k y] = 0$ by $(\mathcal{CL})$; moreover, the following equation (10)

$$0 = [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [\theta_0^a x]] = [\theta_n^a x] \ , \tag{10}$$

and the immediate one (11)

$$0 = [v_n y x^{2q-2^{h+2-a}-2} [yxy]] = [\theta_n^a y] \tag{11}$$

show that the element $[v_n y x^{k-1} y] = \theta_n^a$ is central.

The remaining case is covered by the proposition $(\mathcal{CL})$.

Now look at the class $4q-1+dn$, which in the case $q=2$ (*i.e.* $h=1$) coincide with the class $2q+3+dn$:

$$M_{4q-1+dn} = \langle [v_n y x^{2q-2}], [v_n y x^{2q-3} y] = \theta_n^{h+1} \rangle \ .$$

The two equations above (10) and (11) show the centrality of the element θ_n^{h+1}; moreover, if we define the element

$$\Xi = \begin{cases} v_s & \text{for } n = 2s \ , \\ & \\ [v_s y x^{2q-2} (y x^{2q-1})^{\frac{\eta-1}{2}}] & \text{for } n = 2s+1 \ , \end{cases}$$

then the expansion

$$0 = [\Xi\Xi] = [v_n y x^{2q-1}] \tag{12}$$

shows that

$$M_{4q+dn} = \langle [v_n y x^{2q-2} y] \rangle = \langle [v_n y x^{2q-2} y] \rangle \ . \tag{13}$$

Finally, the last homogeneous components are generated as follows, where i runs between zero and $\eta-1$ and k between zero and $2q-1$, with $k \neq 2q-2$ for $i = \eta - 1$:

$$M_{4q+2qi+k+dn} = \begin{cases} \langle [v_n y x^{2q-2} (y x^{2q-1})^i y x^{2q-1}] \ , & \text{for } k = 2q-1 \ , \\ \ [v_n y x^{2q-2} (y x^{2q-1})^i y x^{2q-2} y] & i = \eta - 2^{g+h+1-b} \ , \\ \ = \theta_n^b \rangle & h+2 \leq b \leq g+h \ , \\ & \\ \langle [v_n y x^{2q-2} (y x^{2q-1})^i y x^k] \rangle & \text{otherwise} \ . \end{cases}$$

To prove this result, we distinguish some cases.

- when $k = 0$,

– and $i=0$, then the claim is just (13);

– and $1 \leq i \leq \eta-1$, then first we have the generalization of (4):

$$\begin{aligned} 0 &= [v_n yx^{2q-2}(yx^{2q-1})^i \theta_0^1] \\ &= [v_n yx^{2q-2}(yx^{2q-1})^i [yx^{2q}]] \\ &= \binom{2q}{0}[v_n yx^{2q-2}(yx^{2q-1})^i yx^{2q-1}x] \ ; \end{aligned}$$

in particular, in the case $i=\eta-2^{g+h+1-b}$ for some integer $h+2 \leq b \leq g+h$, two more identities are required, since $M_{4q+2qi+dn-1}$ is two-dimensional: the former reads as

$$\begin{aligned} 0 &= [v_n yx^{2q-2}(yx^{2q-1})^i yx^{2q-3}[yxy]] \\ &= [v_n yx^{2q-2}(yx^{2q-1})^i yx^{2q-2}yy] \\ &= [\theta_n^b y] \ , \end{aligned}$$

while the latter uses the relation $[\theta_0^b x]$:

$$\begin{aligned} 0 &= [v_{n-1} yx^{2q-2}(yx^{2q-1})^{\eta-2} yx^{2q-2}[\theta_0^b x]] \\ &= [v_n yx^{2q-2}(yx^{2q-1})^i yx^{2q-2}yx] \\ &= [\theta_n^b x] \ ; \end{aligned} \tag{14}$$

- when $k=2q-2^s$ for $1 \leq s \leq h$,

– and $i=0$, then the relation χ_0 can be used:

$$\begin{aligned} 0 &= [v_{n-1} yx^{2q-2}(yx^{2q-1})^{\eta-2} yx^{2q-2^s-1}\chi_0] \\ &= [v_n yx^{2q-2}yx^{k-1}y] \ ; \end{aligned} \tag{15}$$

– and $i \geq 1$, we can use the relation $[\theta_0^{s+1}x]$:

$$\begin{aligned} 0 &= [\mu_n[\theta_0^{s+1}x]] \\ &= [v_n yx^{2q-2}(yx^{2q-1})^i yx^{k-1}y] \ , \end{aligned} \tag{16}$$

where

$$\mu_n = \begin{cases} [v_n yx^{2q-3}] & \text{for } i=1 \\ [v_n yx^{2q-2}(yx^{2q-1})^{i-2} yx^{2q-2}] & \text{otherwise} \ ; \end{cases}$$

- when $k=2q-1$,

– and $i=\eta-2^{g+h+1-b}$ for some $h+2 \leq b \leq g+h$, then $M_{4q+2qi+2q-1+dn}$ is two dimensional;

– and i is not one of the above values, then define λ as the exponent of the higher power of two dividing $i+1$ and use the relation $\chi_{i+2^\lambda-1}$ as follows:

$$\begin{aligned} 0 &= [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-1-2^\lambda}yx^{2q-2}\chi_{i+2^\lambda-1}] \\ &= [v_n yx^{2q-2}(yx^{2q-1})^i yx^{2q-2}y]\ ; \end{aligned} \tag{17}$$

- when k is not one of the above values, then proposition $(\mathcal{CL})$ proves the claim.

Now to conclude look at the homogeneous component occurring for $k=2q-2$ and $i=\eta-1$:

$$\begin{aligned} M_{4q+2q(\eta-1)+2q-2+dn} = M_{2q+d(n+1)} &= \langle [v_n yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}]\rangle \\ &= \langle [v_{n+1}]\rangle\ . \end{aligned}$$

5. THE PROOF: THE EXPANSIONS

The first equation we deal with is (5), whose expansion gives

$$\begin{aligned} 0 &= [[yx^{2q-1}yx^{q-2^{s-1}-1}][yx^{2q-1}yx^{q-2^{s-1}-1}]] \\ &= [[yx^{2q-1}yx^{q-2^{s-1}-1}][yz^{3q-2^{s-1}-1}]] \\ &= [yx^{2q-1}yx^{q-2^{s-1}-1}z^{3q-2^{s-1}-1}y] \\ &\quad +[yx^{2q-1}yx^{q-2^{s-1}-1}z^{3q-2^{s-1}}]\cdot\binom{3q-2^{s-1}-1}{q+2^{s-1}-1} \\ &= [yx^{2q-1}yx^{q-2^{s-1}-1}z^{3q-2^{s-1}-1}y] \\ &= [yx^{2q-1}yx^{2q-2}yx^k y]\ , \end{aligned}$$

since the binomial coefficient is equivalent to

$$\binom{3q-2^{s-1}-1}{q+2^{s-1}-1} \equiv \binom{2}{1}\binom{q-2^{s-1}-1}{2^{s-1}-1} \equiv 0$$

modulo two.

The next expansion we show is related to both equation (6) and (16), respectively in the case $n=0$ and $n>0$, where $1\le s\le h$:

$$\begin{aligned} 0 &= [\mu_n[\theta_0^{s+1}x]]] \\ &= [\mu_n[yx^{2q-1}yx^{2q-2^{h+2-(s+1)}}x]] \\ &= [\mu_n[yz^{4q-2^{h+1-s}}x]] \end{aligned}$$

$$
\begin{aligned}
&= [\mu_n[yz^{4q-2^{h+1-s}}]x] + \\
&\quad +[\mu_n x[yz^{4q-2^{h+1-s}}]] \\
&= [\mu_n z^{4q-2^{h+1-s}+1}x] \\
&\qquad \cdot \left(\binom{4q-2^{h+1-s}}{1} + \binom{4q-2^{h+1-s}}{2q+1} \right) \\
&\quad +[\mu_n x z^{4q-2^{h+1-s}} y] \\
&\quad +[\mu_n x z^{4q-2^{h+1-s}+1}] \\
&\qquad \cdot \left(\binom{4q-2^{h+1-s}}{0} + \binom{4q-2^{h+1-s}}{2q} \right) \ ;
\end{aligned}
$$

via Lucas' Theorem the involved binomial coefficients can be evaluated as follows, where $\alpha, \beta \in \{0, 1\}$

$$
\binom{4q-2^{h+1-s}}{2q\alpha+\beta} \equiv \binom{1}{\alpha}\binom{2q-2^{h+1-s}}{\beta} \equiv (1-\beta) \pmod 2
$$

and thus the above expansion reduces to

$$
0 = [\mu_n[\theta_0^{s+1}x]] = [\mu_n x z^{4q-2^{h+1-s}} y] = [v_n y x^{2q-2}(yx^{2q-1})^i y x^{k-1} y] \ .
$$

Then we have the equation (9), needed to prove the centrality of $\theta^{\omega}_{\frac{n-1}{2}}$: note that, in this case, the relation has to be written pointing out the two last occurrences of the generators,

$$
[\theta_0^{\omega} x] = [yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-1}yx] = [yz^{2q(\eta+1)+2q-2}yx] \ ,
$$

since the element $[yz^{2q(\eta+1)+2q-3}yxx]$ is non zero. The expansion begins as:

$$
\begin{aligned}
0 &= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[\theta_0^{\omega}x]]] \\
&= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yx^{2q-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-1}yx]] \\
&= [[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yz^{2q(\eta+1)+2q-2}yx]] \\
&= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yz^{2q(\eta+1)+2q-2}y]x] \\
&\quad +[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}x[yz^{2q(\eta+1)+2q-2}y]] \\
&= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yz^{2q(\eta+1)+2q-2}]yx] \\
&\quad +[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}x[yz^{2q(\eta+1)+2q-2}]y] \\
&\quad +[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xy[yz^{2q(\eta+1)+2q-2}]] \ ;
\end{aligned}
$$

the resulting right-hand side reads as follows:

$$
[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}z^{2q(\eta+1)+2q-1}yx]
$$

$$
\begin{aligned}
&\left(\binom{2q(\eta+1)+2q-2}{1}+\binom{2q(\eta+1)+2q-2}{2q}\right.\\
&\qquad\left.+\binom{2q(\eta+1)+2q-2}{2q+2q-1}+\sum_{i=1}^{\eta-1}\binom{2q(\eta+1)+2q-2}{2q(i+1)+2q-1}\right)\\
&+[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xz^{2q(\eta+1)+2q-1}y]\\
&\left(\binom{2q(\eta+1)+2q-2}{0}+\binom{2q(\eta+1)+2q-2}{2q-1}\right.\\
&\qquad\left.+\binom{2q(\eta+1)+2q-2}{2q+2q-2}+\sum_{i=1}^{\eta-1}\binom{2q(\eta+1)+2q-2}{2q(i+1)+2q-2}\right)\\
&+[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xyz^{2q(\eta+1)+2q-2}y]\\
&+[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xyz^{2q(\eta+1)+2q-4}yxz]\\
&\left(\binom{2q(\eta+1)+2q-2}{2q-2}+\binom{2q(\eta+1)+2q-2}{2q+2q-3}\right.\\
&\qquad\left.+\sum_{i=1}^{\eta-1}\binom{2q(\eta+1)+2q-2}{2q(i+1)+2q-3}+\binom{2q(\eta+1)+2q-2}{2q(\eta+1)+2q-4}\right)\\
&+[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xyz^{2q(\eta+1)+2q-4}xyz]\\
&\left(\binom{2q(\eta+1)+2q-2}{2q-2}+\binom{2q(\eta+1)+2q-2}{2q+2q-3}\right.\\
&\qquad\left.+\sum_{i=1}^{\eta-1}\binom{2q(\eta+1)+2q-2}{2q(i+1)+2q-3}+\binom{2q(\eta+1)+2q-2}{2q(\eta+1)+2q-3}\right) \;;
\end{aligned}
$$

by applying Lucas' Theorem to the above binomial coefficients we get, for $0 \le a \le \eta+1$ and $1 \le b \le 2q$:

$$
\begin{aligned}
\binom{2q(\eta+1)+2q-2}{2qa+2q-b} &\equiv \binom{\eta+1}{a}\binom{2q-2}{2q-b} \pmod 2\\
&\equiv \binom{2^g}{a}\binom{2q-2}{2q-b}\\
&\equiv 1 \pmod 2 \iff a=0,\eta+1 \text{ and } b=2,4,2q \;,
\end{aligned}
$$

so that the above equation turns out to be

$$
\begin{aligned}
0 &= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[\theta_0^{\omega}x]]\\
&= [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xz^{2q(\eta+1)+2q-1}y]
\end{aligned}
$$

$$
\begin{aligned}
&+[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xyz^{2q(\eta+1)+2q-2}y] \\
&+[v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xyz^{2q(\eta+1)+2q-4}xyz] \\
=\;& [v_{n-2}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}xyz^{2q(\eta+1)+2q-4}xyx] \\
=\;& [\theta^{\omega}_{\frac{n-1}{2}}x] \, .
\end{aligned}
$$

To prove equation (10), let $a = h + 2 - s$, for $1 \leq s \leq h$ and expand as follows:

$$
\begin{aligned}
0 \;=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[\theta_0^{h+2-s}x] \\
=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yx^{2q-1}yx^{2q-2^s-1}yx]] \\
=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yz^{4q-2^s}x]] \\
=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[yz^{4q-2^s}]x] \\
&+[v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}x[yz^{4q-2^s}]] \\
=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}z^{4q-2^s}yx] \\
&\quad \cdot \left(\binom{4q-2^s}{1} + \binom{4q-2^s}{2q} + \binom{4q-2^s}{2q+2q-2^s} \right) \\
&+[v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}x[yz^{4q-2^s}]] \\
=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}z^{4q-2^s}xx] \\
&\quad \cdot \left(\binom{4q-2^s}{1} + \binom{4q-2^s}{2q} \right) \\
&+[v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}yz^{4q-2^s-1}yx] \\
&\quad \cdot \left(\binom{4q-2^s}{0} + \binom{4q-2^s}{2q-1} + \binom{4q-2^s}{2q+2q-2^s-1} \right) \\
&+[v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}yz^{4q-2^s-1}xx] \\
&\quad \cdot \left(\binom{4q-2^s}{0} + \binom{4q-2^s}{2q-1} \right) \; ;
\end{aligned}
$$

the non-trivial binomial coefficients above can be evaluated as elements of the underlying field by means of Lucas' Theorem as follows:

$$
\begin{aligned}
\binom{4q-2^s}{2q} &\equiv \binom{4q-2^s}{2q+2q-2^s} \equiv 1 \pmod 2 \, , \\
\binom{4q-2^s}{2q-1} &\equiv \binom{4q-2^s}{2q+2q-2^s-1} \equiv 0 \pmod 2 \, ,
\end{aligned}
$$

and then the equation reads as

$$
\begin{aligned}
0 \;=\;& [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}[\theta_0^{h+2-s}x]] \\
=\;& [v_nyx^{2q-2^s-1}xx]
\end{aligned}
$$

$$\begin{aligned}
&+[v_n y x^{2q-2^s-1} yx] \\
&+[v_n y x^{2q-2^s-1} xx] \\
= \ &[v_n y x^{2q-2^s-1} yx] \\
= \ &[\theta_1^{h+2-s} x] \ .
\end{aligned}$$

Then we proceed with (14), where $i = \eta - 2^{g+h+1-b}$:

$$\begin{aligned}
0 \ = \ &[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [\theta_0^b x]] \\
= \ &[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [y x^{2q-1} y x^{2q-2} (y x^{2q-1})^{i} y x^{2q-2} yx]] \\
= \ &[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [y z^{2q-2+2q(i+2)} x]] \\
= \ &[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [y z^{2q-2+2q(i+2)}] x] \\
&+[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} x [y z^{2q-2+2q(i+2)}]] \\
= \ &[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} z^{2q-3+2q(i+2)} yx] \\
&\quad \cdot \left(\binom{2q(i+2)+2q-2}{1} + \binom{2q(i+2)+2q-2}{2q} \right. \\
&\quad + \binom{2q(i+2)+2q-2}{2q+2q-1} + \sum_{j=1}^{i} \binom{2q(i+2)+2q-2}{2q(j+1)+2q-1} \\
&\quad \left. + \binom{2q(i+2)+2q-2}{2q(i+2)+2q-2} \right) \\
&+[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1} z^{2q-2+2q(i+2)} y] \\
&+[v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1} z^{2q-1+2q(i+2)}] \\
&\quad \cdot \left(\binom{2q(i+2)+2q-2}{0} + \binom{2q(i+2)+2q-2}{2q-1} \right. \\
&\quad + \binom{2q(i+2)+2q-2}{2q+2q-2} + \sum_{j=1}^{i} \binom{2q(i+2)+2q-2}{2q(j+1)+2q-2} \\
&\quad \left. + \binom{2q(i+1)+2q-2}{2q(i+2)+2q-3} \right) \ .
\end{aligned}$$

Since the binomial coefficients whose denominator is odd vanish, the non-zero and non-trivial binomial coefficients above result

$$\begin{aligned}
\binom{2q(i+2)+2q-2}{2q} \ &\equiv \ \binom{i+2}{1} = \eta - 2^{g+h+1-b} + 2 \equiv 1 \pmod 2 \ , \\
\binom{2q(i+2)+2q-2}{2q(j+1)+2q-2} \ &\equiv \ \binom{i+2}{j+1} \pmod 2 \ ,
\end{aligned}$$

remembering the property

$$\sum_{j=0}^{\alpha} \binom{\alpha}{j} = (1+1)^{\alpha} \equiv 0 \pmod 2 ,$$

the above identity reads

$$\begin{aligned} 0 &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-2} [\theta_0^b x]] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1} z^{2q-2+2q(i+2)} y] \\ &\quad + [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1} z^{2q-1+2q(i+2)}] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1} z^{2q-3+2q(i+2)} y x] \\ &= [\theta_n^b x] . \end{aligned}$$

The next two expansions involve the relations χ_t. The former is related to equation (15):

$$\begin{aligned} 0 &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} \chi_0] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} [y x^{2q-1} y x^{2q-2} y x^{2q-2} y]] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} [y z^{6q-3} y]] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} [y z^{6q-3}] y] \\ &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} z^{6q-2} y] \\ &\qquad \left(\binom{6q-3}{2^s} + \binom{6q-3}{2q+2^s-1} + \binom{6q-3}{4q+2^s-2} \right) ; \end{aligned}$$

the evaluation of the binomial coefficients can be done as usual by Lucas' Theorem first

$$\begin{aligned} \binom{6q-3}{2^s} &\equiv \binom{4q}{0}\binom{2q-3}{2^s} \equiv \binom{2q-3}{2^s} \pmod 2 , \\ \binom{6q-3}{2q+2^s-1} &\equiv \binom{4q}{0}\binom{0}{2q}\binom{2q-3}{2^s-1} \equiv 1 \cdot 0 \cdot \binom{2q-3}{2^s-1} \equiv 0 \pmod 2 , \\ \binom{6q-3}{4q+2^s-2} &\equiv \binom{4q}{4q}\binom{2q-3}{2^s-2} \equiv \binom{2q-3}{2^s-2} \pmod 2 , \end{aligned}$$

and then using their elementary properties, the characteristic of the field and the observation (1):

$$\begin{aligned} \binom{2q-3}{2^s} + \binom{2q-3}{2^s-2} &\equiv \binom{2q-3}{2^s} + 2\binom{2q-3}{2^s-1} + \binom{2q-3}{2^s-2} \pmod 2 \\ &= \left(\binom{2q-3}{2^s} + \binom{2q-3}{2^s-1} \right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\binom{2q-3}{2^s-1} + \binom{2q-3}{2^s-2} \right) \\
&= \binom{2q-2}{2^s} + \binom{2q-2}{2^s-1} \\
&= \binom{2q-1}{2^s} \\
&\equiv 1 \pmod 2 ,
\end{aligned}
$$

thus the expansion itself reads as

$$
\begin{aligned}
0 &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} \chi_0] \\
&= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1-2^s} [y x^{2q-1} y x^{2q-2} y x^{2q-2} y]] \\
&= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-2} y x^{2q-1} y x^{2q-2} y x^{2q-2} y x^{2q-2^s-1} y] \\
&= [v_n y x^{2q-2} y x^{k-1} y] .
\end{aligned}
$$

The latter equation (17) deserves more attention: consider the index $0 \leq i \leq \eta - 3$ and let λ be the exponent of the higher power of two dividing $i+1$, hence $i + 1 = 2^\lambda \cdot r$ where $\lambda \geq 0$ (with equality when i is even) and r is positive *odd* integer. Due to the bounds on i, we have

$$
r \leq \left[\frac{2^g - 2}{2^\lambda} \right] \leq 2^{g-\lambda} - 1,
$$

where the first becomes an equality if and only if $i = \eta - 2^{g+h+1-b}$ for some $h + 2 \leq b \leq g + h$, or, equivalently, $i = 2^g - 2^\gamma - 1$ for some $1 \leq \gamma \leq g - 1$ so that $\lambda = \gamma$. When we are not in the above case, then

$$
i + 2^\lambda - 1 = 2^\lambda \cdot (r+1) - 2 < 2^\lambda \cdot (2^{g-\lambda} - 1 + 1) - 2 \leq 2^g - 2^\lambda - 2 \leq \eta - 2 .
$$

Moreover, if i is even, then $\lambda = 0$ and then $i + 2^\lambda - 1$ is even, too; otherwise, if i is odd, then $\lambda \geq 1$, and then $i + 2^\lambda - 1$ is still even. But $\eta - 2^\alpha$ is always even, for $1 \leq \alpha \leq g - 1$, hence $\eta - (i + 2^\lambda - 1)$ is never 2^α, for $1 \leq \alpha \leq g - 1$. Then $\chi_{i+2^\lambda-1} = 0$ and we can expand the Jacobi identity

$$
\begin{aligned}
0 &= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1-2^\lambda} y x^{2q-2} \chi_{i+2^\lambda-1}] \\
&= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1-2^\lambda} y x^{2q-2} [y z^{2q(i+1+2^\lambda)+2q-3} y]] \\
&= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1-2^\lambda} y x^{2q-2} [y z^{2q(i+1+2^\lambda)+2q-3}] y] \\
&= [v_{n-1} y x^{2q-2} (y x^{2q-1})^{\eta-1-2^\lambda} y x^{2q-2} z^{2q(i+1+2^\lambda)+2q-3} y] \\
&\qquad \cdot \left(\sum_{j=0}^{2^\lambda - 1} \binom{2q(i+1+2^\lambda) + 2q - 3}{2qj + 1} + \binom{2q(i+1+2^\lambda) + 2q - 3}{2q \cdot 2^\lambda} \right.
\end{aligned}
$$

$$+\sum_{j=0}^{i}\binom{2q(i+1+2^\lambda)+2q-3}{2q(2^\lambda+j)+2q-1}\Bigg)\ ;$$

As usual, the terms in the last sum vanish since $2q-1>2q-3$, while the remaining terms can be rewritten together as follows:

$$\begin{aligned}\sum_{j=0}^{2^\lambda}\binom{2q(i+1+2^\lambda)}{2qj} &\equiv \sum_{j=0}^{2^\lambda}\binom{2^\lambda\cdot r+2^\lambda}{j} \pmod 2\\ &\equiv 1+\sum_{j=1}^{2^\lambda-1}\binom{2^\lambda\cdot(r+1)}{j}+\binom{r+1}{1} \pmod 2\\ &\equiv 1+0+r+1 \pmod 2\\ &\equiv 1 \pmod 2\ ,\end{aligned}$$

since r is odd. Then the above expansion reads as

$$\begin{aligned}0 &= [v_{n-1}yx^{2q-2}(yx^{2q-1})^{\eta-1-2^\lambda}yx^{2q-2}[yz^{2q-3+2q(i+1+2^\lambda)}y]]\\ &= [v_nyx^{2q-2}(yx^{2q-1})^iyx^{2q-2}y]\ ,\end{aligned}$$

so that $M_{4q+2qi+2q-2+dn}$ is one-dimensional. Nothing can be said when i is one of the excluded values, and in those cases that homogeneous component has two generators.

We are left to prove the Jacobi identities (8) and (12): they need a different approach, already employed in [5]. First of all, for any index $i\geq 1$ define the element

$$\gamma_i=[v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}]=[yz^{d-1}]\ .$$

Its weight is an integer multiple di of the period of the algebra M and then it satisfies the identity

$$[v_nz^\lambda\gamma_i]=[v_{n+i}z^\lambda]\ .$$

Furthermore, if the relation $[v_nx]=0$ holds for the element v, then the relation $[v\gamma_ix]=0$ holds, too. As a consequence, if $[wxy]=[wyx]=0$ for some element w, then

$$[w\gamma_ixy]=[wx\gamma_iy]=[wy\gamma_ix]=[wyx\gamma_i]=0\ ,$$

and thus $[w[\gamma_ixy]]=0$. Now we apply the above property to the expansion of the three identities we have to prove.

The identity (8) is the easiest to expand in terms of the elements γ_i:

$$0 = [[v_s^{-(q-1)}][v_s^{-(q-1)}]]$$

$$
\begin{aligned}
&= [[v_s^{-(q-1)}][\gamma_s xyx^{q-1}]] \\
&= \sum_{i=0}^{q-1} \binom{q-1}{i} [v_s^{-(q-1-i)}[\gamma_s xy]x^{q-1-i}] \qquad (18) \\
&= \binom{q-1}{q-1}[v_s[\gamma_s xy]] + \binom{q-1}{q-2}[(v_s)^{-1}[\gamma_s xy]x] \\
&= [v_s[\gamma_s x]y] + [v_s y[\gamma_s x]] + [v_s^{-1}[\gamma_s x]yx] \ .
\end{aligned}
$$

The expansion (12) for n even is straightforward, too:

$$
\begin{aligned}
0 &= [\Xi\Xi] \\
&= [v_s v_s] \\
&= [v_s[\gamma_s xyx^{2q-2}]] \qquad (19) \\
&= \sum_{i=0}^{2q-2} \binom{2q-2}{i} [v_s x^i[\gamma_s xy]x^{2q-2-i}] \\
&= [v_s[\gamma_s xy]x^{2q-2}] \\
&= [v_s[\gamma_s x]yx^{2q-2}] + [v_s y[\gamma_s x]x^{2q-2}] \ .
\end{aligned}
$$

Finally, the equation (12) for n odd requires a slightly more careful treatment; let $\alpha = 2q\left(\dfrac{\eta+1}{2}\right) + 2q - 3$ and expand:

$$
\begin{aligned}
0 &= [\Xi\Xi] \\
&= [[v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}][v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}]] \\
&= [[v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}][\gamma_s xyx^{2q-2}yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}]] \qquad (20) \\
&= [[v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}][\gamma_s xyz^{\alpha}]] \\
&= \sum_{i=0}^{\alpha} \binom{\alpha}{i} [v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} z^i[\gamma_s xy]z^{\alpha-i}] \ .
\end{aligned}
$$

The only non-vanishing elements in the above sum are, together with their coefficients:

$$
\begin{aligned}
&\binom{\alpha}{0} && [v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_s xy]z^{\alpha}] \\
\sum_{j=0}^{\frac{\eta-3}{2}} &\binom{\alpha}{2qj+2q-1} && [v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}+j}yx^{2q-2}[\gamma_s xy]z^{\alpha-(2qj+2q-1)}] \\
\sum_{j=0}^{\frac{\eta-3}{2}} &\binom{\alpha}{2q(j+1)} && [v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}+(j+1)}[\gamma_s xy]z^{\alpha-2q(j+1)}]
\end{aligned}
$$

$$\begin{array}{rl}
\binom{\alpha}{2q\frac{\eta-1}{2}+2q-2} & [v_s yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-3}[\gamma_s xy]z^{\alpha-(2q\frac{\eta-1}{2}+2q-2)}] \\
\binom{\alpha}{2q\frac{\eta-1}{2}+2q-1} & [v_s yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}[\gamma_s xy]z^{\alpha-(2q\frac{\eta-1}{2}+2q-1)}] \\
\binom{\alpha}{\alpha} & [v_s yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}yx^{2q-3}[\gamma_s xy]] \,,
\end{array}$$

but all the involved binomial coefficients other the first and the last ones vanish, given the bounds $1 \leq j+1 \leq \dfrac{\eta-1}{2} = 2^{g-1}-1$ and the obvious inequalities $2q-1, 2q-2 > 2q-3$: in fact,

$$\begin{array}{rcl}
\binom{\alpha}{2qj+2q-1} & \equiv & \binom{\frac{\eta+1}{2}}{j}\binom{2q-3}{2q-1} \equiv 0 \pmod 2 \,, \\
\binom{\alpha}{2q(j+1)} & \equiv & \binom{\frac{\eta+1}{2}}{j+1}\binom{2q-3}{0} \equiv \binom{2^{g-1}}{j+1} \equiv 0 \pmod 2 \,, \\
\binom{\alpha}{2q\frac{\eta-1}{2}+2q-2} & \equiv & \binom{\frac{\eta+1}{2}}{\frac{\eta-1}{2}}\binom{2q-3}{2q-2} \equiv 0 \pmod 2 \,, \\
\binom{\alpha}{2q\frac{\eta-1}{2}+2q-1} & \equiv & \binom{\frac{\eta+1}{2}}{\frac{\eta-1}{2}}\binom{2q-3}{2q-1} \equiv 0 \pmod 2 \,,
\end{array}$$

and thus the expansion (20) eventually reduces to the terms

$$\begin{aligned}
0 &= [\Xi\Xi] \\
&= [v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_s xy]z^{\alpha}] \\
&\quad +[v_s yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}yx^{2q-3}[\gamma_s xy]] \\
&= [v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_s x]yz^{\alpha}] \\
&\quad +[v_s yx^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}y[\gamma_s x]z^{\alpha}] \\
&\quad +[v_s yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}yx^{2q-3}[\gamma_s x]y] \,.
\end{aligned} \tag{21}$$

Now, to complete the three expansions (18), (19) and (21) we use the following six equations (a)-(f) which show in a more broad sense the behaviour of the elements involved in the last step of the previous expansions. All the equations we state will be proven by induction and by using the following identity, which in some sense can be viewed as the key one:

$$\begin{aligned}
[\gamma_i x\gamma_1] &= [v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}x[yz^{d-1}]] \\
&= [v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}z^{d-1}y]
\end{aligned}$$

$$\begin{aligned}
&+[v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}z^d]\cdot\left(\binom{d-1}{0}+\binom{d-1}{2q-1}\right.\\
&\qquad\left.+\binom{d-1}{4q-2}+\sum_{j=0}^{\eta-2}\binom{d-1}{2q(j+2)-2}\right)\\
=\ &[v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}z^{d-1}y]\\
&+[v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}z^d] \qquad (22)\\
=\ &[v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}z^{d-1}x]\\
=\ &[v_{i-1}yx^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}yx^{2q-2}(yx^{2q-1})^{\eta-1}]\\
=\ &[v_iyx^{2q-2}(yx^{2q-1})^{\eta-2}yx^{2q-2}x]\\
=\ &[\gamma_{i+1}x]\ ,
\end{aligned}$$

since $d-1=2q-3+2q\eta$ and $2q-1, 2q-2>2q-3$.

The first identity is the following:

$$[v_i^{-1}[\gamma_jx]]=[v_{i+j}]\ . \qquad \text{(a)}$$

First of all we must compute two preliminary expansions. The former one reads as:

$$\begin{aligned}
[v_i^{-1}\gamma_1]\ =\ &[v_i^{-1}[yz^{d-1}]]\\
=\ &[v_i^{-1}z^{d-1}y]\\
&+[v_i^{-1}z^d]\cdot\left(\binom{d-1}{1}+\binom{d-1}{2q}+\sum_{j=1}^{\eta-1}\binom{d-1}{2q(j+1)}\right)\ .
\end{aligned}$$

The evaluation of the binomial coefficients as

$$\begin{aligned}
\binom{d-1}{1}&\equiv\binom{2q-3}{1}\binom{\eta}{0}\pmod 2\ ,\\
\binom{d-1}{2q}&\equiv\binom{2q-3}{0}\binom{\eta}{1}\pmod 2\ ,\\
\binom{d-1}{2q(j+1)}&\equiv\binom{2q-3}{0}\binom{\eta}{j+1}\pmod 2
\end{aligned}$$

shows that all the terms can be evaluated together as

$$\sum_{j=0}^{\eta}\binom{\eta}{j}=(1+1)^\eta\equiv 0\pmod 2\ ,$$

and so the entire equation reduces to

$$\begin{aligned}[v_i^{-1}\gamma_1] &= [v_i^{-1}z^{d-1}y] \\ &= [v_i y x^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-4}y] \\ &= 0\ .\end{aligned}$$

The latter expansion is similar to the previous one

$$\begin{aligned}[v_i\gamma_1] &= [v_i[yz^{d-1}]] \\ &= [v_i z^{d-1}y] \\ &\quad +[v_i z^d]\cdot\left(\binom{d-1}{0}+\binom{d-1}{2q-1}+\sum_{j=1}^{\eta-1}\binom{d-1}{2qj+2q-1}\right)\ ,\end{aligned}$$

but now, since $2q-1>2q-3$, only the first binomial coefficient is different from zero, so that we get

$$\begin{aligned}[v_i\gamma_1] &= [v_i z^{d-1}y] \\ &\quad +[v_i z^d] \\ &= [v_i z^{d-1}x] \\ &= [v_i y x^{2q-2}(yx^{2q-1})^{\eta-1}yx^{2q-2}] \\ &= [v_{i+1}]\ .\end{aligned}$$

These two equations give (a), for $j=1$:

$$\begin{aligned}[v_i^{-1}[\gamma_1 x]] &= [(v_i)^{-1}\gamma_1 x]+[v_i\gamma_1] \\ &= 0+[v_{i+1}]\ .\end{aligned}$$

To complete the claim, we have to prove the inductive step: so suppose (a) holds for j and use again the two previous expansions, together with (22):

$$\begin{aligned}[v_i^{-1}[\gamma_{j+1}x]] &= [v_i^{-1}[\gamma_j x\gamma_1]] \\ &= [v_i^{-1}[\gamma_j x]\gamma_1]+[v_i^{-1}\gamma_1[\gamma_j x]] \\ &= [v_{i+j}\gamma_1]+0 \\ &= [v_{i+j+1}]\ .\end{aligned}$$

The second identity we need to prove is the following:

$$[v_i[\gamma_j x]]=\theta^1_{i+j}\ . \tag{b}$$

To get this one, we use the relation $[\gamma_j xx]=0$

$$\begin{aligned}0 &= [v_i^{-1}[\gamma_j xx]] \\ &= [v_i^{-1}[\gamma_j x]x]+[v_i[\gamma_j x]]\ ,\end{aligned}$$

and the equation (a)

$$\begin{aligned}
[v_i[\gamma_j x]] &= [v_i^{-1}[\gamma_j x]x] \\
&= [v_{i+j}x] \\
&= \theta^1_{i+j}\ .
\end{aligned}$$

For the third identity (and for the following three ones, too)

$$[v_i y[\gamma_j x]] = [v_{i+j}yx] \tag{c}$$

the proof proceeds like it did in the first case, so first we have to expand two intermediate identities. The former is

$$\begin{aligned}
[v_i y\gamma_1] &= [v_i y[yz^{d-1}]] \\
&= [v_i yz^{d-1}y] \\
&\quad +[v_i yz^d]\cdot\left(\sum_{j=0}^{\eta-1}\binom{d-1}{2qj+2q-2}\right) \\
&= [v_i yz^{d-1}y] \\
&= [v_{i+1}y]\ ,
\end{aligned}$$

since $2q-2 > 2q-1$ and then the involved binomial coefficients vanish. The latter is

$$\begin{aligned}
[v_i yx\gamma_1] &= [v_i yx[yz^{d-1}]] \\
&= [v_i yxz^{d-1}y] \\
&\quad +[v_i yxz^d]\cdot\left(\sum_{j=0}^{\eta-1}\binom{d-1}{2qj+2q-3}+\binom{d-1}{2q\eta+2q-4}\right)\ ;
\end{aligned}$$

Lucas' Theorem evaluates the coefficients:

$$\begin{aligned}
\sum_{j=0}^{\eta-1}\binom{d-1}{2qj+2q-3} &\equiv \sum_{j=0}^{\eta-1}\binom{\eta}{j}\binom{2q-3}{2q-3} \pmod 2 \\
&\equiv \sum_{j=0}^{\eta-1}\binom{\eta}{j} \pmod 2 \\
&\equiv (1+1)^\eta-\binom{\eta}{\eta} \pmod 2 \\
&\equiv 1 \pmod 2\ ,
\end{aligned}$$

and

$$\begin{aligned}\binom{d-1}{2q\eta+2q-4} &= \binom{d-1}{d-1-(2q-4+2q\eta)} \\ &= \binom{d-1}{1} \\ &= d-1 \\ &\equiv 1 \pmod 2,\end{aligned}$$

and thus we have the identity

$$\begin{aligned}[v_i y x \gamma_1] &= [v_i y x z^{d-1} y] \\ &= [v_{i+1} y y] \\ &= 0\,.\end{aligned}$$

Again, the two equations above give the basis for the induction process

$$\begin{aligned}[v_i y[\gamma_1 x]] &= [v_i y \gamma_1 x] + [v_i y x \gamma_1] \\ &= [v_{i+1} y x] + 0\,,\end{aligned}$$

and the identities needed to prove the inductive step, where we suppose the formula holds for j:

$$\begin{aligned}[v_i y[\gamma_{j+1} x]] &= [v_i y[\gamma_j x \gamma_1]] \\ &= [v_i y[\gamma_j x]\gamma_1] + [v_i y \gamma_1[\gamma_j x]] \\ &= [v_{i+j} y x \gamma_1] + [v_{i+1} y[\gamma_j x]] \\ &= 0 + [v_{i+1+j} y x]\,.\end{aligned}$$

The next three identities deal with the elements involved in the expansion (21). The fourth equation is

$$[v_i y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}}[\gamma_j x]] = 0\,. \tag{d}$$

The only auxiliary identity required to prove the claim is the following:

$$\begin{aligned}[v_i y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}}\gamma_1] &= [v_i y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}}[y z^{d-1}]] \\ &= [v_i y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}} z^{d-1} y] \\ &\quad + [v_i y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}} z^d] \\ &\qquad \cdot \left(\sum_{j=0}^{\frac{\eta-1}{2}} \binom{d-1}{2qj} + \binom{d-1}{2q\left(\frac{\eta-1}{2}\right)+2q-1}\right)\end{aligned}$$

$$+\sum_{j=0}^{\frac{\eta-3}{2}}\binom{d-1}{2q\left(\frac{\eta+1}{2}+j\right)+2q-2}\Bigg)\ ;$$

since $2q-2>2q-3$ only the binomial coefficients in the very first sum are not zero: anyway, the sum itself yields

$$\begin{aligned}\sum_{j=0}^{\frac{\eta-1}{2}}\binom{d-1}{2qj} &\equiv \sum_{j=0}^{\frac{\eta-1}{2}}\binom{\eta}{j}\binom{2q-3}{0} \pmod 2\\ &\equiv \sum_{j=0}^{2^{g-1}-1}\binom{2^g-1}{j} \pmod 2\\ &\equiv \sum_{j=0}^{2^{g-1}-1} 1 \pmod 2\\ &= 2^{g-1}\\ &\equiv 0 \pmod 2\ ,\end{aligned}$$

so the identity reduces to

$$\begin{aligned}[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}\gamma_1] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} z^{d-1} y]\\ &= [v_{i+1} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-3}{2}} y x^{2q-2} y]\\ &= 0\ .\end{aligned}$$

The above relation together with the fact

$$[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} x] = 0$$

give the case $j=1$ of the claim

$$\begin{aligned}[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_1 x]] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}\gamma_1 x]\\ &= 0\ ,\end{aligned}$$

and the inductive step follows immediately:

$$\begin{aligned}[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_{j+1} x]] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_j x \gamma_1]]\\ &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}[\gamma_j x]\gamma_1]\\ &\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}\gamma_1[\gamma_j x]]\\ &= 0\ .\end{aligned}$$

The fifth identity deals with elements one class deeper than the previous ones:

$$[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y[\gamma_j x]] = [v_{i+j} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx]\,. \tag{e}$$

This time the auxiliary expansions needed are two. The former is

$$\begin{aligned}
[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y\gamma_1] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y[yz^{d-1}]] \\
&= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yz^{d-1}y] \\
&\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yz^{d}] \\
&\qquad \cdot\left(\sum_{j=0}^{\frac{\eta-3}{2}} \binom{d-1}{2qj+2q-1}\right. \\
&\qquad\quad + \binom{d-1}{2q\left(\frac{\eta-1}{2}\right)+2q-2} \\
&\qquad\quad \left. + \sum_{j=0}^{\frac{\eta-3}{2}} \binom{d-1}{2q\left(\frac{\eta+1}{2}+j\right)+2q-3}\right)\ ;
\end{aligned}$$

now the only non vanishing binomial coefficients are those in the last sum, that becomes

$$\begin{aligned}
\sum_{j=0}^{\frac{\eta-3}{2}} \binom{d-1}{2q\left(\frac{\eta+1}{2}+j\right)+2q-3} &\equiv \sum_{j=0}^{\frac{\eta-3}{2}} \binom{\eta}{\frac{\eta+1}{2}+j}\binom{2q-3}{2q-3} \pmod 2 \\
&\equiv \sum_{j=0}^{2^{g-1}-2} \binom{2^g-1}{2^{g-1}+j} \pmod 2 \\
&\equiv \sum_{j=0}^{2^{g-1}-2} 1 \pmod 2 \\
&= 2^{g-1}-1 \\
&\equiv 1 \pmod 2,
\end{aligned}$$

and then we have

$$\begin{aligned}
[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y\gamma_1] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yz^{d-1}y] \\
&\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yz^{d}] \\
&= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yz^{d-1}x] \\
&= [v_{i+1} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} x] \\
&= 0\,.
\end{aligned}$$

The latter identity is similar to the previous one:

$$\begin{aligned}
[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx\gamma_1] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx[yz^{d-1}]] \\
&= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yxz^{d-1}y] \\
&\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yxz^{d}] \\
&\qquad \cdot \left(\sum_{j=0}^{\frac{\eta-3}{2}} \binom{d-1}{2qj+2q-2}\right. \\
&\qquad\quad + \binom{d-1}{2q\left(\frac{\eta-1}{2}\right)+2q-3} \\
&\qquad\quad \left.+ \sum_{j=0}^{\frac{\eta-1}{2}} \binom{d-1}{2q\left(\frac{\eta+1}{2}+j\right)+2q-4}\right) .
\end{aligned}$$

The addenda of the first sum are zero since $2q-2 > 2q-3$; the other binomial coefficients involved can be evaluated as

$$\begin{aligned}
\binom{d-1}{2q\left(\frac{\eta-1}{2}\right)+2q-3} &\equiv \binom{\eta}{\frac{\eta-1}{2}}\binom{2q-3}{2q-3} \pmod 2 \\
&\equiv 1 \pmod 2, \\
\sum_{j=0}^{\frac{\eta-1}{2}} \binom{d-1}{2q\left(\frac{\eta+1}{2}+j\right)+2q-4} &\equiv \sum_{j=0}^{\frac{\eta-1}{2}} \binom{\eta}{\frac{\eta+1}{2}+j}\binom{2q-3}{2q-4} \pmod 2 \\
&\equiv \sum_{j=0}^{2^{g-1}-1} \binom{2^g-1}{2^{g-1}+j} \pmod 2 \\
&\equiv \sum_{j=0}^{2^{g-1}-1} 1 \pmod 2 \\
&= 2^{g-1} \\
&\equiv 0 \pmod 2,
\end{aligned}$$

and the whole expression turns out to be

$$\begin{aligned}
[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx\gamma_1] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yxz^{d-1}y] \\
&\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yxz^{d}] \\
&= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yxz^{d-1}x] \\
&= [v_{i+1} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx] .
\end{aligned}$$

Now we can deduce the basis for the induction

$$\begin{aligned}[][v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y[\gamma_1 x]] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y\gamma_1 x] \\ &\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx\gamma_1] \\ &= [v_{i+1} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx] \, ,\end{aligned}$$

and the inductive step follows:

$$\begin{aligned}[][v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y[\gamma_{j+1} x]] &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y[\gamma_j x\gamma_1]] \\ &= [v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y[\gamma_j x]\gamma_1] \\ &\quad +[v_i y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} y\gamma_1[\gamma_j x]] \\ &= [v_{i+j} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx\gamma_1] \\ &= [v_{i+j+1} y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} yx] \, .\end{aligned}$$

The sixth identity is the last one needed to complete expansion (21):

$$[v_i y x^{2q-3}[\gamma_j x]] = [v_{i+j} y x^{2q-2}] \, . \tag{f}$$

Again we first deal with two preliminary equations. The former is

$$\begin{aligned}[][v_i y x^{2q-3}\gamma_1] &= [v_i y x^{2q-3}[yz^{d-1}]] \\ &= [v_i y x^{2q-3} z^{d-1} y] \\ &\quad +[v_i y x^{2q-3} z^{d}] \\ &\qquad \cdot \left(\binom{d-1}{1} + \sum_{j=0}^{\eta-1} \binom{d-1}{2q(j+1)} \right) \ ;\end{aligned}$$

coefficient evaluations give

$$\begin{aligned}\binom{d-1}{1} &= d-1 \\ &\equiv 1 \pmod 2 \, , \\ \sum_{j=0}^{\eta-1} \binom{d-1}{2q(j+1)} &\equiv \sum_{j=1}^{\eta} \binom{\eta}{j} \pmod 2 \\ &\equiv (1+1)^\eta - \binom{\eta}{0} \pmod 2 \\ &\equiv 1 \pmod 2 \, ,\end{aligned}$$

and thus

$$[v_i y x^{2q-3}\gamma_1] = [v_i y x^{2q-3}[yz^{d-1}]]$$

$$\begin{aligned} &= [v_i y x^{2q-3} z^{d-1} y] \\ &= [v_{i+1} y x^{2q-4} y] \\ &= 0\,. \end{aligned}$$

When investigating what happens just one class deeper, we obtain

$$\begin{aligned} [v_i y x^{2q-2}\gamma_1] &= [v_i y x^{2q-2}[y z^{d-1}]] \\ &= [v_i y x^{2q-2} z^{d-1} y] \\ &\quad +[v_i y x^{2q-2} z^d] \\ &\qquad \cdot \left(\binom{d-1}{0} + \sum_{j=0}^{\eta-1} \binom{d-1}{2qj+2q-1} \right) \\ &= [v_i y x^{2q-2} z^{d-1} y] \\ &\quad +[v_i y x^{2q-2} z^d] \\ &= [v_i y x^{2q-2} z^{d-1} x] \\ &= [v_{i+1} y x^{2q-2}]\,, \end{aligned}$$

since only the first binomial coefficient gives a non-zero contribution, while the remaining ones vanish since $2q-1 > 2q-3$.

Now the claim can be proved; first of all, the case $j=1$:

$$\begin{aligned} [v_i y x^{2q-3}[\gamma_1 x]] &= [v_i y x^{2q-3}\gamma_1 x] + [v_i y x^{2q-2}\gamma_1] \\ &= 0 + [v_{i+1} y x^{2q-2}]\,; \end{aligned}$$

then, the induction step:

$$\begin{aligned} [v_i y x^{2q-3}[\gamma_{j+1} x]] &= [v_i y x^{2q-3}[\gamma_j x \gamma_1]] \\ &= [v_i y x^{2q-3}[\gamma_j x]\gamma_1] + [v_i y x^{2q-3}\gamma_1[\gamma_j x]] \\ &= [v_{i+j} y x^{2q-2}\gamma_1] + 0 \\ &= [v_{i+j+1} y x^{2q-2}]\,. \end{aligned}$$

At this point, by using equations (a)-(f), we can complete the expansions (18), (19) and (21).

The first one becomes

$$\begin{aligned} 0 &= [[v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1}][v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1}]] \\ &= [v_s^{-(q-1)} v_s^{-(q-1)}] \\ &= [v_s[\gamma_s x]y] + [v_s y[\gamma_s x]] + [v_s^{-1}[\gamma_s x] y x] \\ &= [\theta^1_{2s} y] + [v_{2s} y x] + [v_{2s} y x] \\ &= [\theta^1_n y]\,; \end{aligned}$$

the expansion (19) reduces to

$$\begin{aligned} 0 &= [\Xi\Xi] \\ &= [v_s v_s] \\ &= [v_s[\gamma_s x] y x^{2q-2}] + [v_s y[\gamma_s x] x^{2q-2}] \\ &= 0 + [v_{2s} y x x^{2q-2}] \\ &= [v_n y x^{2q-1}]\ ; \end{aligned}$$

and, finally, identity (21) reads as follows:

$$\begin{aligned} 0 &= [\Xi\Xi] \\ &= [[v_s y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}}][v_s y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}}]] \\ &= [v_s y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}}[\gamma_s x] y z^{2q-3+2q\left(\frac{\eta+1}{2}\right)}] \\ &\quad +[v_s y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}} y[\gamma_s x] z^{2q-3+2q\left(\frac{\eta+1}{2}\right)}] \\ &\quad +[v_s y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{2q-2} y x^{2q-3}[\gamma_s x] y] \\ &= 0 + [v_{2s} y x^{2q-2}(y x^{2q-1})^{\frac{\eta-1}{2}} y x z^{2q-3+2q\left(\frac{\eta+1}{2}\right)}] + [v_{2s+1} y x^{2q-2} y] \\ &= [v_n y x^{2q-1}]\ . \end{aligned}$$

The expansions above prove the relations (8) and (12) and complete the proof of the theorem.

6. SOME ODD EXPANSIONS

If we consider the three equations (18), (19) and (21) and we compare the result obtained expanding them as we did before rather than applying the generalized Jacobi identity straightforwardly, we can derive three identities involving binomial coefficients over $\mathbf{F}_2$ otherwise hard to be proven directly.

First we deal with the identity (8), where we start expanding from the back:

$$\begin{aligned} 0 &= [[v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1}][v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1}]] \\ &= [[v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1}][y z^{ds+q}]] \\ &= [v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1} z^{ds+q} y] \\ &\quad +[v_{s-1} y x^{2q-2}(y x^{2q-1})^{\eta-1} y x^{q-1} z^{ds+q+1}] \cdot \left(\sum_{i=1}^{\eta} \binom{ds+q}{2qi} \right. \\ &\qquad + \binom{ds+q}{2q(\eta+1)-1} + \sum_{j=0}^{s-2} \left(\binom{ds+q}{dj-2+2q(\eta+2)} \right. \\ &\qquad \left.\left. + \sum_{i=1}^{\eta-1} \binom{(2^{g+h+1}-2)s+2^h}{dj-2+2q(\eta+2+i)} + \binom{(2^{g+h+1}-2)s+2^h}{dj-3+2q(2\eta+2)} \right) \right)\ . \end{aligned}$$

The second and the last term in the above coefficient are equivalent to zero modulo two in view of Lucas' Theorem, since their denominator is odd while the numerator is even. The remaining terms can be rewritten in terms of g and h as follows:

$$\begin{aligned}
\binom{ds+q}{2qi} &= \binom{(2^{g+h+1}-2)s+2^h}{2^{h+1}i} \\
\binom{ds+q}{dj-2+2q(\eta+2)} &= \binom{ds+q}{(2^{g+h+1}-2)(j+1)+2^{h+1}} \\
\binom{ds+q}{dj-2+2q(\eta+2+i)} &= \binom{(2^{g+h+1}-2)s+2^h}{(2^{g+h+1}-2)(j+1)+2^{h+1}(i+1)} ,
\end{aligned}$$

so they all can be arranged in the unique sum

$$\sum_{j=0}^{s-1}\sum_{i=1}^{2^g-1}\binom{(2^{g+h+1}-2)s+2^h}{(2^{g+h+1}-2)j+2^{h+1}i} ,$$

equivalent by Lucas' Theorem to the following:

$$\sum_{j=0}^{s-1}\sum_{i=1}^{2^g-1}\binom{(2^{g+h}-1)s+2^{h-1}}{(2^{g+h}-1)j+2^h i} \pmod 2 .$$

Since the result must be $[\theta_n^1 y]$ by the expansion in the previous section, we get the first binomial identity:

$$\sum_{j=0}^{s-1}\sum_{i=1}^{2^g-1}\binom{(2^{g+h}-1)s+2^{h-1}}{(2^{g+h}-1)j+2^h i} \equiv 0 \pmod 2 .$$

Analogously we can expand completely the two equations (12),

$$0 = [\Xi\Xi] = [v_n y x^{2q-1}] .$$

First we expand the n even one:

$$\begin{aligned}
0 &= [v_s v_s] \\
&= [v_s[yz^{2q-1+ds}]] \\
&= [v_s z^{2q-1+ds} y] \\
&\quad +[v_s z^{2q+ds}]\cdot\left(\sum_{l=0}^{s-1}\left(\binom{2q-1+ds}{dl}+\right.\right. \\
&\qquad \left.\left. +\sum_{j=0}^{\eta-1}\binom{2q-1+ds}{2q-1+2qj+dl}\right)+\binom{2q-1+ds}{ds}\right) .
\end{aligned}$$

The involved binomial coefficients can be rewritten first as

$$\binom{2q-1+ds}{0}+\sum_{l=1}^{s}\left(\binom{2q-1+ds}{dl}+\sum_{j=0}^{\eta-1}\binom{2q-1+ds}{2q-1+2qj+d(l-1)}\right)\ ,$$

and then, by using binomial properties, as

$$1+\sum_{l=1}^{s}\left(\binom{2q-1+ds}{dl}+\binom{2q-1+ds}{d(s-l+1)}+\sum_{j=1}^{\eta-1}\binom{2q-1+ds}{d(s-l+1)-2qj}\right)\ ;$$

but now,

$$\sum_{l=1}^{s}\binom{2q-1+ds}{dl}=\sum_{l=1}^{s}\binom{2q-1+ds}{d(s-l+1)}\ ,$$

and so the remaining coefficient is just

$$1+\sum_{l=1}^{s}\sum_{j=1}^{\eta-1}\binom{2q-1+ds}{d(s-l+1)-2qj}=1+\sum_{l=1}^{s}\sum_{j=1}^{\eta-1}\binom{2q-1+ds}{dl-2qj}\ .$$

Since the expansion (19) of the same formula eventually gives $[v_n y x^{2q-1}]$, this means that the coefficient above must be odd, *i.e.*

$$\sum_{l=1}^{s}\sum_{j=1}^{\eta-1}\binom{2q-1+ds}{dl-2qj}=0\ ,$$

which yields

$$\sum_{l=1}^{s}\sum_{j=1}^{2^g-2}\binom{(2^{g+h+1}-2)s+2^{h+1}-1}{(2^{g+h+1}-2)l-2^{h+1}j}\equiv 0 \pmod 2\ .$$

Finally, we completely expand equation (21):

$$\begin{aligned}
0 &= [[v_s y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}][v_s y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}]]\\
&= [[v_s y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}}][y z^{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}]]\\
&= [v_s y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} z^{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds} y]\\
&\quad +[v_s y x^{2q-2}(yx^{2q-1})^{\frac{\eta-1}{2}} z^{2q\left(\frac{\eta+1}{2}\right)+2q-1+ds}]\ \cdot\\
&\qquad \cdot\left(\sum_{j=0}^{\frac{\eta-1}{2}}\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{2qj}+\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{2q\left(\frac{\eta-1}{2}\right)+2q-1}\right)
\end{aligned}$$

$$+\sum_{l=0}^{s-1}\left(\sum_{j=0}^{\eta-1}\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{2q\left(\frac{\eta+1}{2}+j\right)+2q-2+dl}\right.$$
$$\left.\left.+\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{2q\left(\frac{\eta+1}{2}\right)+d(l+1)-1}\right)\right).$$

The second and the last binomial coefficients vanish since they have an even numerator and an odd denominator, while the third ones can be rewritten as

$$\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{2q\left(\frac{\eta+1}{2}+j\right)+2q-2+dl}=\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{d(s-l)-2qj}.$$

Hence, since the alternative expansion in the previous section forces the total coefficient of the terms ending in z to be one, we get the equivalence modulo two of the two sums:

$$\sum_{j=1}^{\frac{\eta-1}{2}}\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{2qj}\ \text{and}\ \sum_{l=1}^{s}\sum_{j=0}^{\eta-1}\binom{2q\left(\frac{\eta+1}{2}\right)+2q-2+ds}{dl-2qj},$$

and then the identity

$$\sum_{l=1}^{s}\sum_{j=0}^{2^g-2}\binom{(2^{g+h+1}-2)s+2^{g+h}+2^{h+1}-2}{(2^{g+h+1}-2)l+2^{h+1}((-1)^{1-\mathrm{sgn}(l)}j-1+\delta_{l,0})}\equiv 0 \pmod 2.$$

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