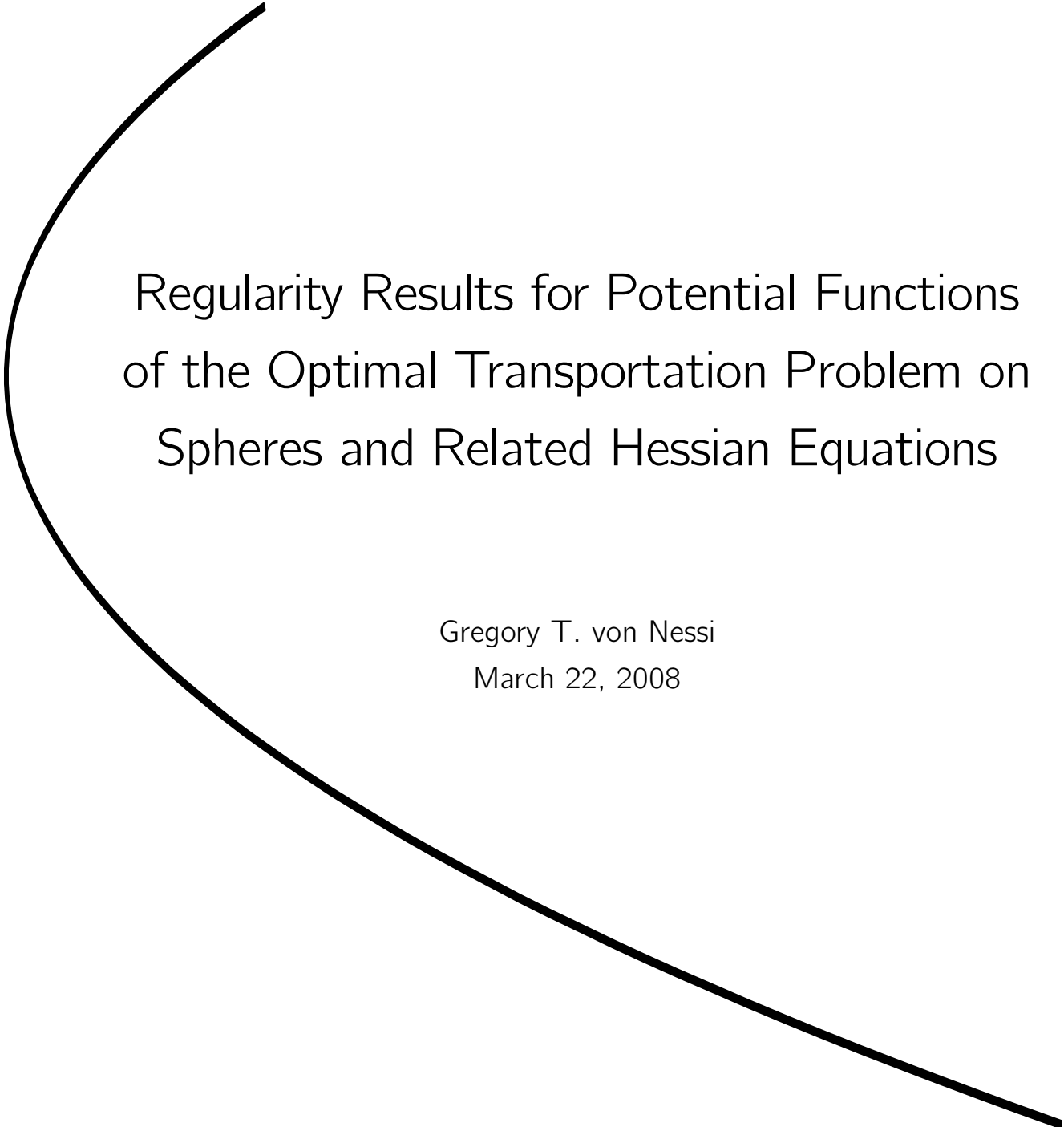




Regularity Results for Potential Functions of the Optimal Transportation Problem on Spheres and Related Hessian Equations

Gregory T. von Nessi

March 22, 2008

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University



The work in this thesis is my own except where otherwise stated.



Gregory T. von Nessi Jr.

This thesis is dedicated to my sister Virginia L. von Nessi and my wife
Kassetra M. von Nessi; without their love and support this thesis would not have been
written.



Abstract

In this thesis, results will be presented that pertain to the global regularity of solutions to boundary value problems having the general form

$$\begin{aligned} F [D^2 u - A(\cdot, u, Du)] &= B(\cdot, u, Du), \quad \text{in } \Omega^-, \\ T_u(\Omega^-) &= \Omega^+, \end{aligned} \tag{1}$$

where A , B , T_u are all prescribed; and Ω^- along with Ω^+ are bounded in $\mathbb{R}^n$, smooth and satisfying notions of c -convexity and c^* -convexity relative to one another (see [MTW05] for definitions). In particular, the case where F is a quotient of symmetric functions of the eigenvalues of its argument matrix will be investigated. Ultimately, analogies to the global regularity result presented in [TW06] for the Optimal Transportation Problem to this new fully-nonlinear elliptic boundary value problem will be presented and proven. It will also be shown that the (A3w) condition (first presented in [MTW05]) is also necessary for global regularity in the case of (1). The core part of this research lies in proving various *a priori* estimates so that a method of continuity argument can be applied to get the existence of globally smooth solutions. The *a priori* estimates vary from those presented in [TW06], due to the structure of F , introducing some complications that are not present in the Optimal Transportation case.

In the final chapter of this thesis, the (A3) condition will be reformulated and analysed on round spheres. The example cost-functions subsequently analysed have already been studied in the Euclidean case within [MTW05] and [TW06]. In this research, a stereographic projection is utilised to reformulate the (A3) condition on round spheres for a general class

of cost-functions, which are general functions of the geodesic distance as defined relative to the underlying round sphere. With this general expression, the (A3) condition can be readily verified for a large class of cost-functions that depend on the metrics of round spheres, which is tantamount (combined with some geometric assumptions on the source and target domains) to the classical regularity for solutions of the Optimal Transportation Problem on round spheres.



Contents

Abstract	iv
Contents	vi
Preface	x
I Background Material	1
Chapter 1: The Optimal Transportation Problem	2
1.1 Introduction	2
1.2 Monge's Original Problem	3
1.3 Obstacles to Monge's Problem	5
Chapter 2: Potential Functions	10
2.1 Introduction	10
2.2 Lagrange Multipliers	10
2.3 Kantorovich's Formulations	13
2.3.1 Relaxed Formulation	13
2.3.2 Dual Formulation	14
2.4 Key Conditions on Cost-functions	15
2.4.1 Some Properties of the (A3) Condition	16
2.4.2 The Degenerate (A3) Condition	18
2.5 c -convexity	19
2.6 Existence of Potential Functions	20
2.6.1 Construction of Convex Sequences	21

2.6.2	Existence of an Optimal Mass Transfer Plan	24
2.7	The Optimal Transportation Equation	26
2.8	Regularity Results	29
2.8.1	Global Regularity	29
2.8.2	Interior Regularity	29
II	New Results	31
Chapter 3:	The Second B.V.P. for modified-Hessian Equations	32
3.1	Introduction	32
3.2	A Class of Modified-Hessian Equations	33
3.2.1	The Structure of F	33
3.2.2	Conditions from Optimal Transportation	36
3.2.3	Conditions on Inhomogeneity	37
3.2.4	A Barrier Condition	39
3.2.5	The Quotient Transportation Equation	39
3.3	Main Results	39
3.4	Preliminary Lemmas	41
3.5	Solution Estimates	45
3.5.1	Obliqueness Estimate	45
3.5.2	C^0 Estimate	53
3.5.2.1	Supremum Solution Bound	54
3.5.2.2	Infimum Solution Bound	54
3.5.3	Global Second Derivative Estimates	56
3.5.4	Boundary estimates for second derivatives	62
3.6	Method of Continuity	66
3.6.1	Application of the L-S Theorem	68
3.7	Conclusions	69
Chapter 4:	The (A3) Condition on $\mathbb{S}^n$	72
4.1	Introduction	72
4.2	Main Results	73
4.3	Analysis of the (A3) Condition on $\mathbb{S}^n$	74
4.3.1	Stereographic Projection	75
4.3.1.1	The Half-Sphere Stereographic Projection	75
4.3.1.2	Stereographic Reformulation of the (A3) Condition	77
4.4	Examples	81
4.4.1	$c(x, y) = \frac{1}{2}d^2(x, y)$	81
4.4.2	$c(x, y) = 2R^2 \sin^2\left(\frac{d(x, y)}{2R}\right)$	84
4.4.3	$c(x, y) = \sqrt{1 - d^2(x, y)}$	85
4.4.4	$c(x, y) = -\sqrt{1 - d^2(x, y)}$	86

4.4.5	$c(x, y) = \sqrt{1 + d^2(x, y)}$	86
4.4.6	$c(x, y) = -\sqrt{1 + d^2(x, y)}$	86
4.4.7	$c(x, y) = \pm \frac{1}{m} d^m(x, y)$	87
4.4.7.1	(+), $m < 0$:	87
4.4.7.2	(-), $m < 0$:	88
4.4.7.3	(+), $m = 0$:	90
4.4.7.4	(-), $m = 0$:	90
4.4.7.5	(+), $0 < m < 1$	91
4.4.7.6	(+), $1 < m < 2$	92
4.4.7.7	(+), $2 < m < \infty$	92
4.4.7.8	(-), $0 < m < 1$	92
4.4.7.9	(-), $1 < m < 2$	93
4.4.7.10	(-), $2 \leq m < \infty$	93
4.4.7.11	Remarks on the $\mathbf{c}(\mathbf{x}, \mathbf{y}) = \pm \frac{1}{m} \mathbf{d}^m(\mathbf{x}, \mathbf{y})$ Example	93
4.4.8	Final Remarks on Examples	93
4.5	Conclusions	94
4.5.1	Geometric Implications	94
4.5.2	Further Studies	95

Bibliography	98
---------------------	-----------

Index	102
--------------	------------



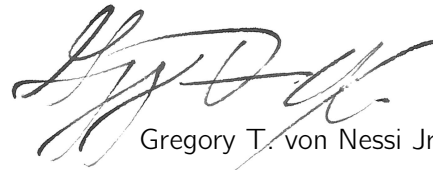
Preface

The research presented in this thesis correlates to two different projects which both have their roots in the Optimal Transportation Problem. The first project, presented in Chapter 6, deals with the regularity of boundary value problems that are related to the Monge-Ampère equations emerging from the theory of Optimal Transportation. For these boundary value problems, regularity results analogous to the ones presented in [TW06] for the Optimal Transportation Equation will be proven. These proofs have resemblance to those presented in [TW06]; but there are significant differences in the obliqueness and C^2 solution estimates, as the newly defined class of boundary value problems do not have some of the structural benefits present with the analogous Optimal Transportation Equation. In particular, tools will be used from both [Ger96] and [Urb01] to prove the necessary *a priori* bounds needed for the final regularity results.

In the Chapter 7, the research focus changes to the regularity study of Optimal Transportation potential functions on round spheres. These results extend the research presented in [McC01], [DL06] and [Loe05], in a sense that regularity results for more general costs than the ones presented in those papers are proven. The method thus employed centres around the novel use of stereographic projections to reformulate the expression for the (A3) condition presented in [TW97, MTW05]: a condition which is tantamount to potential function regularity in the Euclidean case. The caveat to this approach is that it is geometrically non-intrinsic (unlike the methods presented in [McC01], [DL06] and [Loe05]) and thus, incurs some extraneous geometric conditions for the method to be valid. At the end of this chapter, possibilities for further studies are discussed.

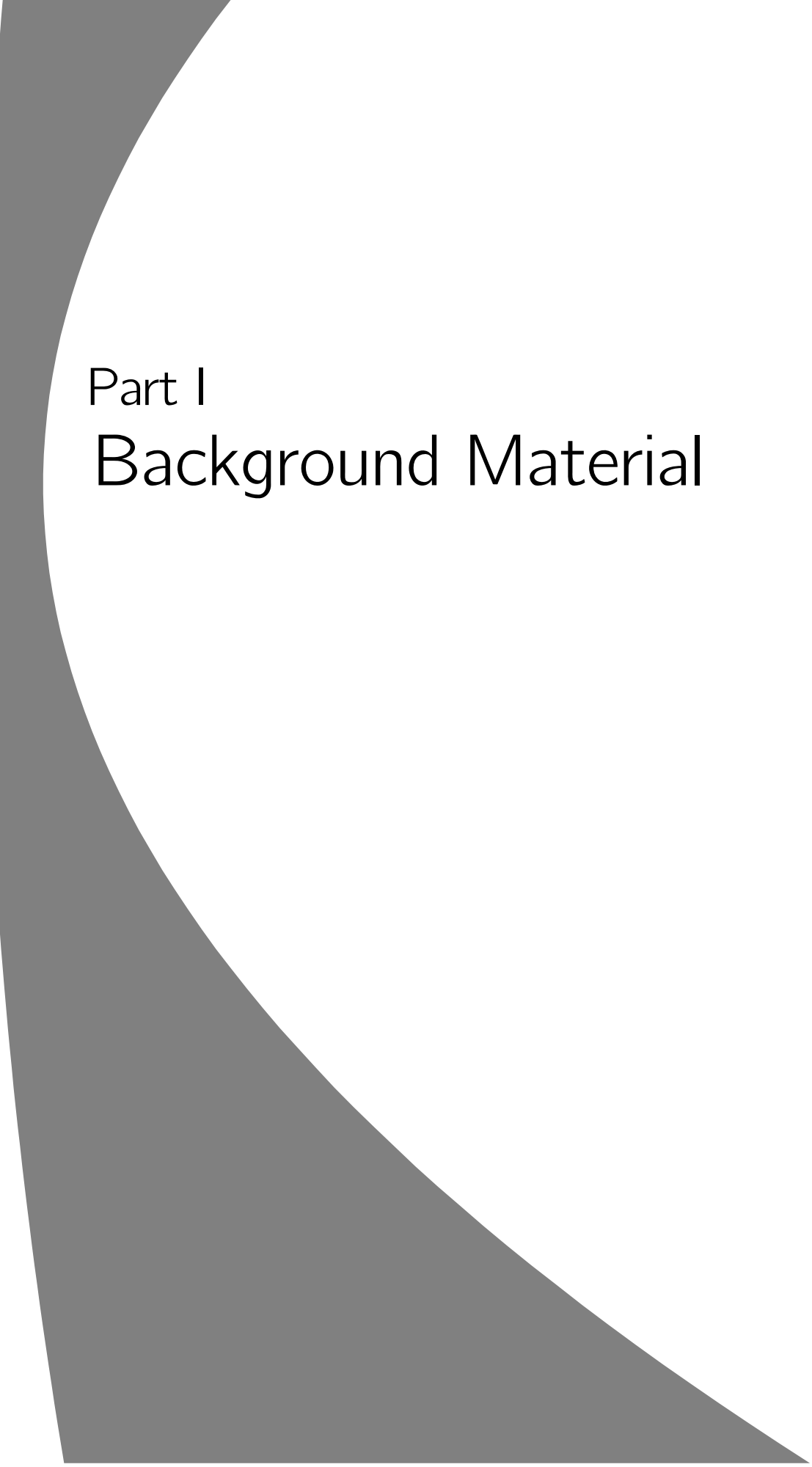
Given that Optimal Transportation is of central importance to the results presented in this thesis, Part I focuses on presenting the Optimal Transportation Problem up to the derivation of the Optimal Transportation Equation, whose solutions correspond to potentials that ultimately yield mappings that solve the original Optimal Transportation Problem as stated by Monge in the late 1700's [Mon81]. Through that exposition, the necessary definitions and ideas will be recalled so that the regularity results presented in [TW06], [MTW05], [TWar], and [Loe05] can be stated at the conclusion of Part I. These are the results that motivated the new research subsequently presented in Part II of this thesis.

I would like to thank Professors Neil Trudinger, Xu-Jia Wang, and Ben Andrews for supervising my Ph.D. studies. Their guidance, suggestions and discussions proved invaluable to the development of the research contained in this thesis. I would also like to thank Professor Phillipe Delanoë and Dr. Huy Nguyen for their insightful discussions on Optimal Transportation and Riemannian Geometry.



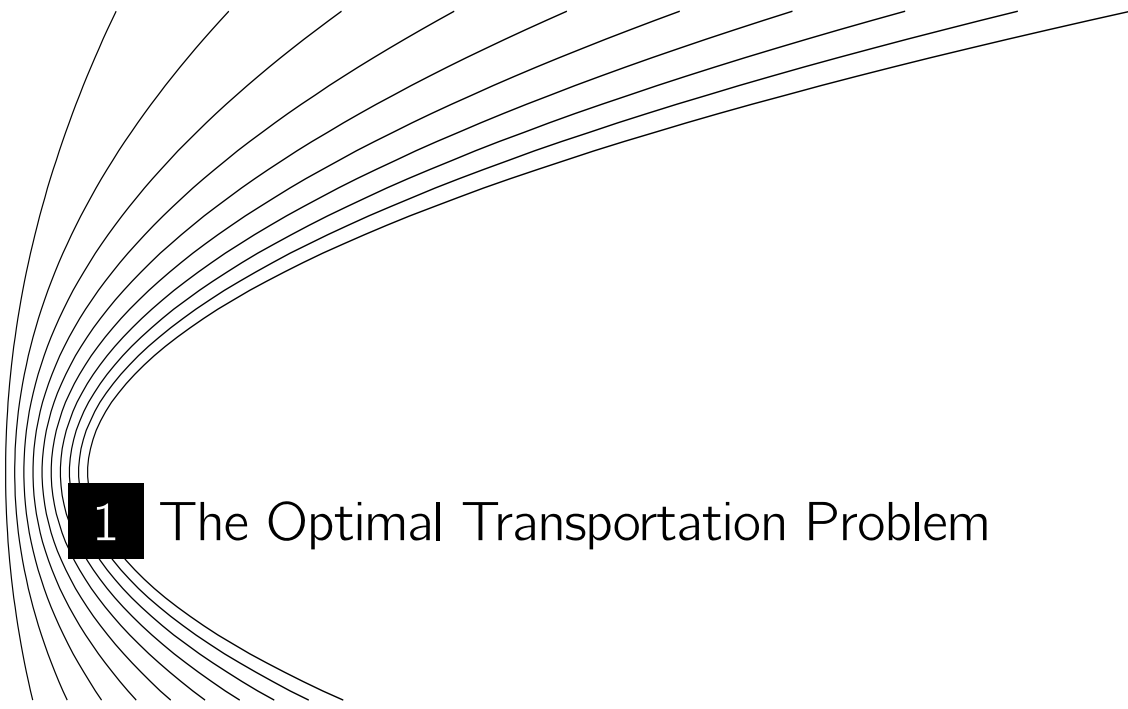
Gregory T. von Nessi Jr.

The Australian National University



Part I

Background Material



1 The Optimal Transportation Problem

Contents

1.1	Introduction	2
1.2	Monge's Original Problem	3
1.3	Obstacles to Monge's Problem	5

1.1 Introduction

The subject of Optimal Transportation has recently been an area of intense study, which has subsequently yielded many important results that have been applied to a broad spectrum of subjects including Ricci flow [Stu06, LV04], fluid dynamics, plasma physics, cosmology [SF04] and public transportation, just to name a few. Indeed, the Optimal Transportation Problem is a generalised variational problem whose parameters can be set to replicate a great number of physical phenomena.

In this chapter, the Optimal Transportation Problem will be introduced as it originally was by Monge in the late 1700's [Mon81]. Specifically, some of the mathematical difficulties will be presented that make analysing the Optimal Transportation Problem still interesting, even after being first stated over 200 years ago. There are many sources that introduce the theory of Optimal Transportation; and here, many of the ideas and observations presented in [Urb98a] and [Eva01] are taken to formulate the following exposition, which will introduce subsequent chapters that relay the new results of this thesis.

1.2 Monge's Original Problem

In 1781 Monge [Mon81] put forth a problem, loosely stated as follows (see Figure 1.1 below for a profile visualisation): *Given a pile of soil and an excavation or pit, how does one move the soil from the pile to the pit so as to expend the least amount of energy?*

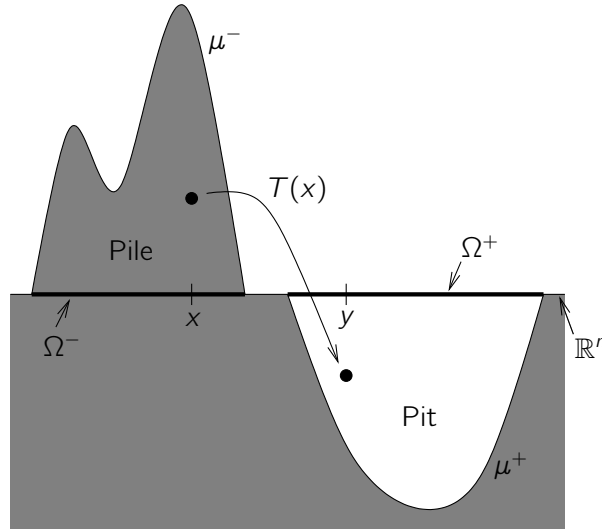


Figure 1.1: A profile view of Monge's transportation problem

Although simply stated, there are a few parameters one has to consider before setting out to solving the problem. First, the notion of “energy” (in the context of Monge's question) needs to be somehow quantified. Next, the physical configuration of both the pile and pit will affect a transportation solution; and so, these too must be quantified. To mathematically formalise and quantify this problem, the following definition is needed:

Definition 1.2.1: Given two radon measures μ^- , μ^+ on $\mathbb{R}^n$ with $\Omega^- := \text{supp}(\mu^-)$ and $\Omega^+ := \text{supp}(\mu^+)$, a Borel measurable map $T : \Omega^- \rightarrow \Omega^+$ is called a measure-preserving map if

$$\mu^-(T^{-1}(E)) = \mu^+(E), \quad (1.1)$$

for any Borel set $E \subset \Omega^+$. In other words, the change of variables formula:

$$\int_{T^{-1}(\Omega^+)} (h \circ T) d\mu^- = \int_{\Omega^+} h d\mu^+, \quad (1.2)$$

is valid for all $h \in C^0(\mathbb{R}^n)$.

The space of all measure-preserving mappings, relative to μ^- and μ^+ , will be denoted by

$$\mathcal{T} = \mathcal{T}(\mu^-, \mu^+).$$

Remark: The criterion stated in (1.1) is often written as

$$T_{\#}\mu^{-} = \mu^{+};$$

that is, μ^{+} is said to be the push-forward measure of μ^{-} by the map T . While this notation is useful in intrinsic calculations, this thesis relies more on the intuition behind the validity of (1.2) in its exposition and calculations.

Corresponding to the pit and pile having equal volume, it is required that $\mu^{\pm}$ satisfy a mass-balance condition:

$$\mu^{-}(\mathbb{R}^n) = \mu^{+}(\mathbb{R}^n) < \infty.$$

To analogue the notion of “energy” or “work” associated with moving a particular grain of soil from the pit to the pile, the *cost-function* is defined as a continuous mapping, specifically denoted as

$$c : \Omega^{-} \times \Omega^{+} \rightarrow \mathbb{R}.$$

Essentially, the cost-function returns the energy required to move a piece of soil between two points whose positions correspond to the arguments of the cost-function itself.

Upon defining the *Monge Cost Functional*:

$$\mathcal{C}[T] := \int_{\Omega^{-}} c(x, T(x)) d\mu^{-}, \quad (1.3)$$

the solution of Monge’s original problem can be denoted as T^{*} , which is defined as solving

$$\mathcal{C}[T^{*}] = \min_{T \in \mathcal{T}} \mathcal{C}[T]. \quad (1.4)$$

The original cost-function considered by Monge was

$$c(x, y) = |x - y|. \quad (1.5)$$

That is, the work to move a grain of soil from pile to pit was simply proportional to the distance the grain of soil was moved. It turns out that the mathematical formulation embodied in (1.4) of Monge’s problem is problematic to work with in a mathematically rigorous sense. Monge’s original cost-function (1.5) is especially difficult to handle when compared with cost-functions having some notion of strict convexity/concavity. Before moving on to discussing potential functions, the mathematical difficulties involved with solving (1.4) will be elaborated upon.

Remark: It is not necessary to consider $\Omega^\pm$ to be subsets of $\mathbb{R}^n$. Indeed, $\Omega^\pm$ can simply be taken to be arbitrary open sets in a Hausdorff topological space for the above formulation to make sense. As this thesis focuses on using methods from partial differential equations to understand the Optimal Transportation Problem, this level of generality will not be pursued in the forthcoming exposition. Instead, it is assumed that the underlying topological vector space (on which the Optimal Transportation Problem will be analysed) is $\mathbb{R}^n$.

1.3 Mathematical Obstacles to Monge's Problem

As formulated in the previous section, existence and uniqueness of Monge's problem are by no means mathematically obvious. Indeed, Optimal Transportation is currently the focus of intense research in the area of partial differential equations, despite being over 200 years old. In this section various heuristics will be presented that give indications as to why Monge's formulation is mathematically difficult to handle and why a new formulation is needed to make mathematical headway in solving this problem.

First, a simple discrete analogue of Monge's problem will be considered. Taking $\Omega^- = \{x_1, \dots, x_k\}$ and $\Omega^+ = \{y_1, \dots, y_k\}$ with $\mathcal{T}$ denoting all one-to-one maps from Ω^- to Ω^+ , $T^* \in \mathcal{T}$ is map which minimises

$$\mathcal{C}[T] := \sum_{i=1}^k c(x_i, T(x_i))$$

among all $T \in \mathcal{T}$. The following example shows that minimisers of this problem are not unique in general.

Example 1: Let $\Omega^- = \{(0, 0), (1, 1)\}$ and $\Omega^+ = \{(0, 1), (1, 0)\}$ with the original Monge cost of $c(x, y) = |x - y|$. In this case, there are only two competing maps

$$T_1 : (0, 0) \mapsto (0, 1), \quad T_1 : (1, 1) \mapsto (1, 0)$$

$$T_2 : (0, 0) \mapsto (1, 0), \quad T_2 : (1, 1) \mapsto (0, 1).$$

It is clear that $\mathcal{C}[T_1] = \mathcal{C}[T_2]$; and thus, the minimiser is not unique.

From this example in the discrete case, it is demonstrated that the question of uniqueness of T in Monge's continuous problem is not obvious even in a heuristic sense.

Going back to the continuous version of Monge's problem, $\mathcal{T}$ will now be further analysed. First, for $\Omega^\pm \subset \mathbb{R}^n$ bounded it is an elementary observation that shows $\mathcal{T}$ is non-empty. Indeed, decomposing $\Omega^\pm$ into cubes and constructing measure-preserving (generally discontinuous) maps using this decomposition, ensures that $\mathcal{T} \neq \emptyset$. As T is assumed to be Borel measurable, analogising from classical real analysis, one can see that maps built off of cu-

bic decompositions of $\Omega^\pm$ can approximate any element of $\mathcal{T}$. An elementary estimation indicates that

$$\int_{\Omega^-} |T|^p d\mu^- \leq \mu^-(\Omega^-) \cdot \sup_{x \in \Omega^+} |x|^p < \infty;$$

thus (by Lebesgue-dominated convergence applied to the cubic decomposition approximations of T) it is deduced that $\mathcal{T}$ is bounded in $L^p(\Omega^+, \Omega^-)$ for $1 \leq p \leq \infty$.

Even though $\mathcal{T}$ is bounded in L^p , it is neither convex nor weakly-compact. This is seen by considering two more simple examples (this time in a continuous formulation).

Example 2: Let $\Omega^\pm = [0, 1]$ with $\mu^\pm$ standard Lebesgue measures on $[0, 1]$. Both

$$\begin{aligned} T_1(x) &:= x \\ T_2(x) &:= \min\{2x, 2 - 2x\} \end{aligned}$$

are both measure-preserving, but $\frac{1}{3}(2T_1 + T_2) = \frac{1}{3} \min\{4x, 2\}$ is not (see Figure 1.2). This shows that $\mathcal{T}$ is not convex in this case.

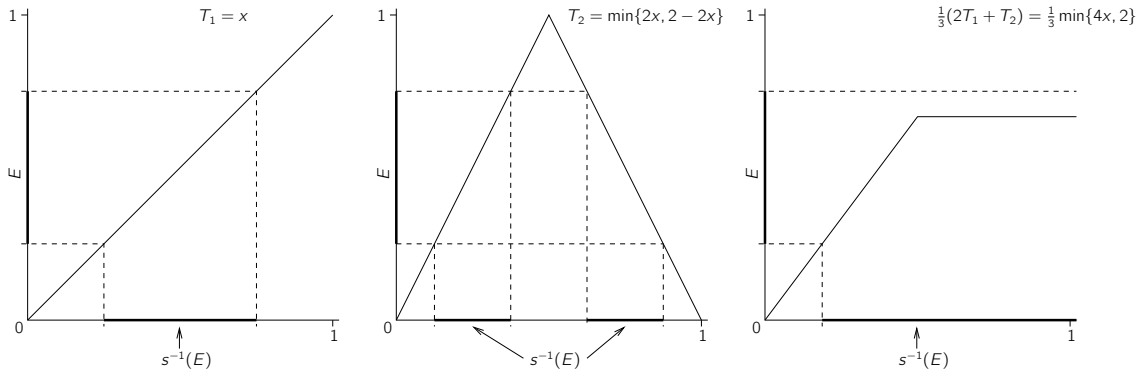


Figure 1.2: Plot of transport maps for Example 1

Example 3: Taking $\Omega^\pm$ and $\mu^\pm$ as in Example 2, the following defines a sequence of maps T_k :

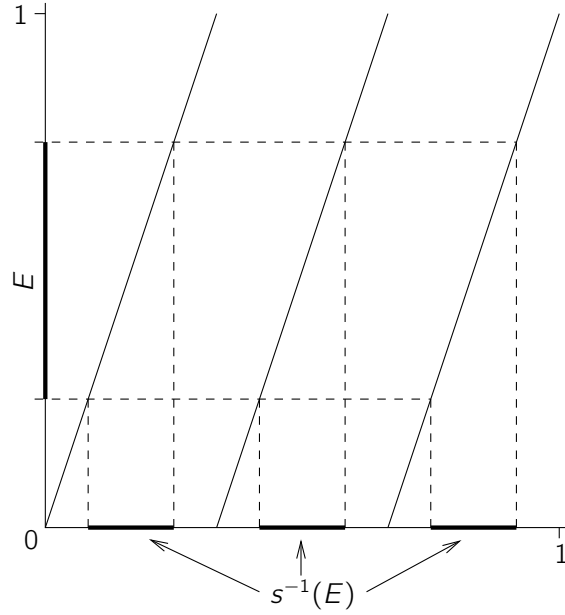
$$T_k(x) := \sum_{i=0}^{k-1} (kx - i) \chi_{I_i}, \quad \text{where } I_i := \left[\frac{i}{k}, \frac{i+1}{k} \right].$$

Figure 1.3 below shows a plot of $T_3(x)$.

It is calculated that

$$\frac{1}{2} \sum_{i=0}^{k-1} \frac{1}{k} \min_{x \in \left[\frac{i}{k}, \frac{i+1}{k} \right]} \{g(x)\} \leq \int_0^1 g \cdot T_k dx \leq \frac{1}{2} \sum_{i=0}^{k-1} \frac{1}{k} \max_{x \in \left[\frac{i}{k}, \frac{i+1}{k} \right]} \{g(x)\}. \quad (1.6)$$

Thus, it is seen that the central term is bounded by two Riemann sums converging to the same Riemann (that is, Lebesgue) integral-value as $k \rightarrow \infty$. Namely, (1.6) demonstrates

Figure 1.3: Plot of $T_3(x)$ for Example 3

that

$$\int_0^1 g \cdot T_k dx \rightarrow \frac{1}{2} \int_0^1 g dx, \quad \text{as } k \rightarrow \infty,$$

for any $g \in L^1([0, 1])$ (readily deduced by correction of a Lebesgue integrable function on set of measure zero to be Riemann integrable). So, each T_k is measure preserving, while $T_k \rightharpoonup \frac{1}{2}$ is not.

From this example, one sees that $\mathcal{T}$ is not closed even under weak limits. These counter examples are readily analogised to more general $\Omega^\pm$. Thus, it is clear that $\mathcal{T}$ can not be expected to be convex nor even weakly-compact in the general case.

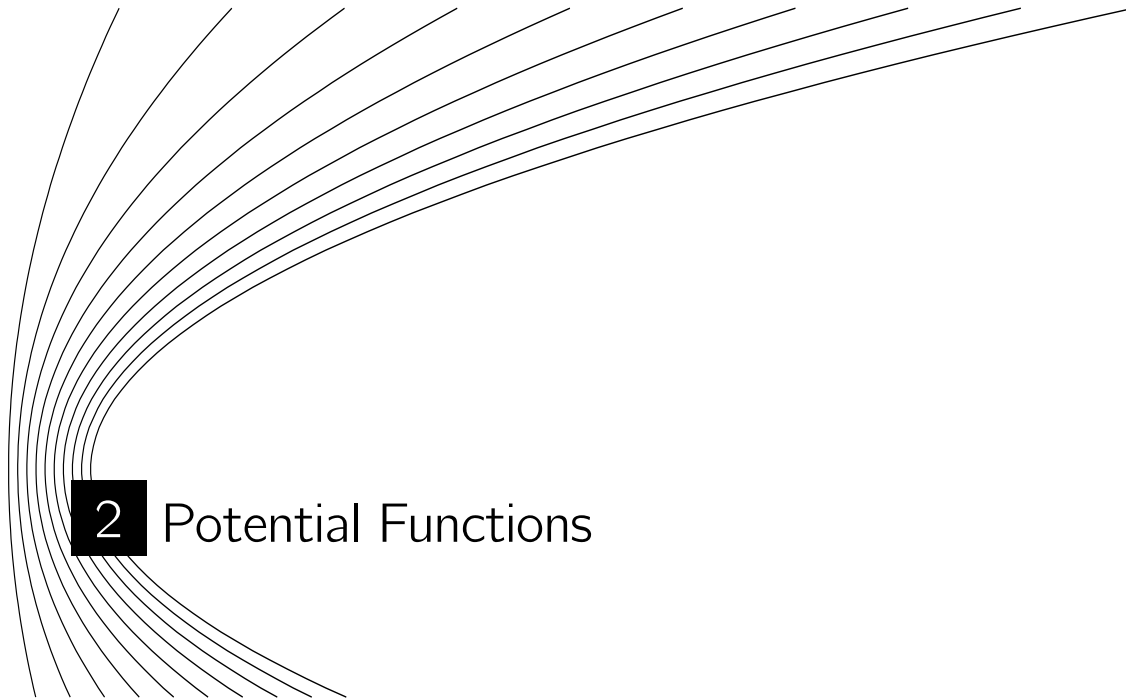
In addition to the difficulties presented by the space $\mathcal{T}$, the structure of the Monge cost-function (1.3) itself also presents difficulties to analysing Monge's problem. First, there are no gradient terms in (1.3), which means the Monge cost functional is not coercive in any Sobolev space. Thus, we are unable to apply the Lax-Milgram theorem to get existence of a Sobolev solution. Indeed, there are no terms present in (1.3) that give any compactness to any potential solution subspace of $\mathcal{T}$. This combined with the weak non-compactness of $\mathcal{T}$ prevents using direct methods in the calculus of variations to solve the Monge Minimisation Problem. Lastly, the measure-preserving criterion for the Monge variational problem is highly non-linear. Indeed, when taking $d\mu^- = f dx$ and $d\mu^+ = g dy$, the measure-preserving criterion corresponds to the pointwise constraint

$$(g \circ T)|\text{Det}[DT]| = f, \quad \text{in } \Omega^-, \quad (1.7)$$

where $[DT]_{ij}(x)$ is a matrix whose elements are given by $D_j T^i(x)$.

Given the above issue of $\mathcal{T}$ being non-compact, there is no way of ascertaining information regarding limits of minimising sequences of the Monge Cost Functional. In addition, the lack of convexity of $\mathcal{T}$ and the highly non-linear nature of (1.7) make construction of such minimising sequences another obstacle to using standard variational methods.

Given these mathematical obstacles to rigorously analysing the original Monge Optimal Transportation Problem, a new formulation of Monge's original question is needed. Kantorovich, in the 1942, came up with such a formulation [Kan42]; although, the link to Monge's problem was not formally noted until later in 1948 [Kan48]. It is this reformulation of Monge's original transportation problem that enables methods from elliptic partial differential equation theory to be used to deduce existence and regularity of minimisers of Monge's original cost functional.



2 Potential Functions

Contents

2.1	Introduction	10
2.2	Lagrange Multipliers	10
2.3	Kantorovich's Formulations	13
2.4	Key Conditions on Cost-functions	15
2.5	c -convexity	19
2.6	Existence of Potential Functions	20
2.7	The Optimal Transportation Equation	26
2.8	Regularity Results	29

2.1 Introduction

Heuristically, the Kantorovich formulation of the Monge problem correlates to finding a maximiser to a special functional over a class of coupled, potential functions. As these potential functions play a central role in both Optimal Transportation and the research presented here, it is helpful to gain some intuition about their geometric structure before relating the full Kantorovich Dual Formulation.

2.2 Lagrange Multipliers

Remark: From this point forward, it will be assumed that $\Omega^\pm$ are open and bounded in $\mathbb{R}^n$ with $d\mu^- = f \, dx$ and $d\mu^+ = g \, dy$.

The goal of this section is to give an intuitive idea of the structure of potential functions coming from the Optimal Transportation Problem and how that structure can be heuristically derived from ideas based in variational calculus with linear constraints. The exposition here follows the one presented in [Tru07], which subsequently is a generalisation of the heuristics written in [Eva01].

To proceed, a restatement of Monge's original problem into the language of variational calculus is needed. Indeed, Monge's original problem is equivalent to seeking a minimiser of the functional

$$\mathcal{C}[T] := \int_{\Omega^-} f \cdot c(x, T(x)) \, dx \quad (2.1)$$

over $\mathcal{T}(f, g)$, whose constituents are elements of $L^p(\Omega^-, \Omega^+)$, satisfying the pointwise constraint

$$(g \circ T)|\text{Det}[DT]| = f, \quad \text{in } \Omega^-. \quad (2.2)$$

As non-linear constraints pose significant obstacles to direct variational methods, it is desirable to somehow incorporate the constraint in (2.2) into the functional itself. Thus, (2.1) is modified into the *Augmented Monge Cost Functional*:

$$\mathcal{C}_\lambda[T] := \int_{\Omega^-} f \cdot c(x, T(x)) - \lambda(x) \{f - (g \circ T)|\text{Det}[DT]|\} \, dx. \quad (2.3)$$

Remarks:

- (1) (2.3) is not motivated by heuristics alone. Indeed, the idea of augmenting functionals in the manner depicted in (2.3) comes from linear variational theory. In the case where the pointwise constraint is linear, there exists a $\lambda(x)$ such that a minimiser of the original functional (with constraint) is also a minimiser of the linear analogy of the augmented functional presented in (2.3) (see [GF63, Section 2.12]). Conventionally, $\lambda(x)$ is called a Lagrange multiplier.
- (2) It is important to note that (2.2) correlates to a pointwise constraint and not an integral constraint. Indeed, (1.2) needs to be valid for any $h \in C^0(\Omega^-)$. This distinction manifests itself in (2.3) as λ being a function of x and not just a constant, as in the case with an integral constraint (see [GF63, Section 2.12], [Eva98, Section 8.4]). Again, this is following ideas taken from linear variational theory.

Calculating the first variation of (2.3) yields

$$\begin{aligned} \left. \frac{d}{d\epsilon} \mathcal{C}_\lambda[T + \epsilon S] \right|_{\epsilon=0} &= \int_{\Omega^-} \{f \cdot c_{y_k}(x, T(x)) + \lambda(x) \cdot (g_k \circ T) \cdot |\text{Det}[DT]|\} S_k \, dx \\ &\quad - \int_{\Omega^-} \lambda_i(x) \cdot (g \circ T) \cdot \text{Cof}[DT]_{ij} \cdot S_j \, dx \\ &\quad - \int_{\Omega^-} \lambda(x) \{ (g_k \circ T)[DT]_{ik} \cdot \text{Cof}[DT]_{ij} \\ &\quad \quad + (g \circ T) \cdot D_i \text{Cof}[DT]_{ij} \} S_j \, dx, \end{aligned} \quad (2.4)$$

where integration by parts along with the assumption that $S(x) = 0$ for $x \in \partial\Omega$ have both been used. In (2.4), $\text{Cof}[DT]_{ij}$ represents the cofactor matrix of $[DT]_{ij}$. From elementary linear algebra it is recalled that $[DT]_{ik} \cdot \text{Cof}[DT]_{ij} = \delta_{jk} \cdot \text{Det}[DT]$ and $D_i \text{Cof}[DT]_{ij} = 0$. Using these relations, (2.4) may be reduced to

$$\begin{aligned} \left. \frac{d}{d\epsilon} \mathcal{C}_\lambda[T + \epsilon S] \right|_{\epsilon=0} &= \int_{\Omega^-} \{ f \cdot c_{y_k}(x, T(x)) + \lambda(x) \cdot (g_k \circ T) \cdot |\text{Det}[DT]| \} S_k dx \\ &\quad - \int_{\Omega^-} \lambda_i(x) \cdot (g \circ T) \cdot \text{Cof}[DT]_{ij} \cdot S_j dx \\ &\quad - \int_{\Omega^-} \lambda(x) \{ (g_k \circ T) \cdot \delta_{jk} \cdot \text{Det}[DT] \} S_j dx \\ &= \int_{\Omega^-} \{ f \cdot c_{y_k}(x, T(x)) - \lambda_i(x) \cdot (g \circ T) \cdot \text{Cof}[DT]_{ik} \} S_k dx. \end{aligned}$$

Thus, T is a maximum if

$$f \cdot c_{y_k}(x, T(x)) = \lambda_i(x) \cdot (g \circ T) \cdot \text{Cof}[DT]_{ik}.$$

Multiplying this by $[DT]_{jk}$, then summing on k and using the measure-preserving criterion (2.2) yields

$$c_{y_k}(x, T(x)) [DT]_{jk} = \lambda_j(x),$$

which can be written as

$$c_y(x, T(x)) = \frac{\partial}{\partial T} \lambda(x), \quad (2.5)$$

via elementary implicit differentiation. Defining $v := \lambda \circ T^{-1}$, (2.5) becomes

$$c_y(x, T(x)) = D_T v(T). \quad (2.6)$$

One may interchange Ω^- and Ω^+ in the above variational calculations to derive (in analogy to (2.6)) that

$$c_x(x, T(x)) = D_x u(x). \quad (2.7)$$

Integrating both (2.6) and (2.7) yields

$$u(x) + v(T(x)) = c(x, T(x)) + C_0, \quad (2.8)$$

for arbitrary constant C_0 . It will be assumed from now on that both $c_x(x, \cdot)$ and $c_y(\cdot, y)$ are both invertable for all $x \in \Omega^-$, $y \in \Omega^+$ respectively. This assumption ensures the mapping T is uniquely determined from either potential u or v . This condition will be formalised later in Section 2.4.

Remarks:

- (1) In the case of the Monge cost, these derived potentials do not uniquely determine

the mapping T . Indeed, in this case, (2.6) and (2.7) both become degenerate in the sense that only the direction of the transport mapping T can be determined and not its magnitude.

- (2) Another way to show the Optimal Transport map is a gradient of some potential is by using cyclic monotonicity and a theorem of Rockafeller [Roc66] to show that T lies in the subdifferential of a convex potential. This method is largely due to Brenier in [Bre91]. Although not reviewed here, the method is summarised nicely in [Eva01].

Even though the above variational calculations give an intuition that a solution of an Optimal Transportation problem is the gradient of some potential, the existence of that potential is by no means obvious, as the currently considered variational problem has a *non-linear* pointwise constraint. Unfortunately, generalising the analogous existence proof for $\lambda(x)$ in the case of linear constraints to the current situation is not readily possible. It is because of this situation that the Kantorovich Dual Formulation is thus needed to prove the existence of these potentials in the case of strictly convex/concave costs.

2.3 Kantorovich's Formulations

As mentioned in Section 1.3, Kantorovich introduced a relaxed variant of and a dual variational principle corresponding to Monge's original transportation problem in 1942 [Kan42]. The actual correlation of these constructions to Monge's original question was not noted until 1948 in [Kan48]. These two constructions alleviate the problems stated in Section 1.3 corresponding to Monge's original formulation; that is, these Kantorovich formulations may be readily analysed by classical direct methods in variational calculus.

2.3.1 Relaxed Formulation

Kantorovich's "Relaxed" Formulation transforms the problem of minimising (1.3) into a linear variational problem. The term "relaxed" in name of this construction will be justified later in this subsection.

Considering two arbitrary Radon probability measures μ^- and μ^+ on $\mathbb{R}^n$, Kantorovich introduced the class $\Gamma(\mu^-, \mu^+)$ of Radon probability measures γ on $\mathbb{R}^n \times \mathbb{R}^n$, which have the following properties:

$$\mu^-(E) = \gamma(E \times \mathbb{R}^n) \quad \text{and} \quad \mu^+(E) = \gamma(\mathbb{R}^n \times E),$$

for any Borel set $E \subset \mathbb{R}^n$. That is, $\mu^\pm$ are the *projections* of γ . It is clear that $\Gamma(\mu^-, \mu^+)$ is a convex subset of $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$: the class of Radon probability measures on $\mathbb{R}^n \times \mathbb{R}^n$. Given

this, the *Kantorovich Relaxed Functional* is defined as

$$\mathcal{C}_R[\gamma] := \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) d\gamma(x, y). \quad (2.9)$$

One may see how this relates to the original Monge problem by considering $T \in \mathcal{T}(\mu^-, \mu^+)$, $E \subset \mathbb{R}^n \times \mathbb{R}^n$ Borel and defining a measure γ_T as follows:

$$\gamma_T(E) := \mu^-(\{x : (x, T(x)) \in E\}).$$

Taking an arbitrary Borel set $F \subset \mathbb{R}^n$, it is observed that

$$\gamma_T(F \times \mathbb{R}^n) = \mu^-(\{x : (x, T(x)) \in F \times \mathbb{R}^n\}) = \mu^-(F)$$

and

$$\gamma(\mathbb{R}^n \times F) := \mu^-(\{x : (x, T(x)) \in \mathbb{R}^n \times F\}) = \mu^-(T^{-1}(F)) = \mu^+(F),$$

since $T \in \mathcal{T}(\mu^-, \mu^+)$. Thus, it is clear that

$$\mathcal{C}_R[\gamma_T] = \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) d\gamma_T = \int_{\mathbb{R}^n} c(x, T(x)) d\mu^- = \mathcal{C}[T],$$

which subsequently indicates the relation

$$\inf_{\gamma \in \Gamma(\mu^-, \mu^+)} \mathcal{C}_R[\gamma] \leq \inf_{T \in \mathcal{T}(\mu^-, \mu^+)} \mathcal{C}[T].$$

Thus, the problem of minimising (2.9) over $\Gamma(\mu^-, \mu^+)$ is justified as being a “relaxed” version of Monge’s Original Problem.

This relaxed formulation has the benefit of making the existence of a minimiser of (2.9) a straight-forward application of classical compactness arguments (see [Urb98a]). Indeed, minimisers for the relaxed functional can be shown to exist for general measures $\mu^\pm$ on topologies as general as Hausdorff spaces. The trade-off for this direct existence theory is that little can be directly gained in the way of higher regularity of transport maps using this formulation. Thus, while this is a very crucial development in Optimal Transportation, it will not be elaborated upon further in this thesis.

2.3.2 Dual Formulation

The second formulation Kantorovich presented is his dual formulation of the Optimal Transportation problem. In this construction, one seeks to maximise the *Kantorovich Dual Functional*:

$$J[\mu^-, \mu^+] := \int_{\mathbb{R}^n} u d\mu^- + \int_{\mathbb{R}^n} v d\mu^+, \quad (2.10)$$

over the set

$$K := \{(u, v) : u, v \in C^0(\mathbb{R}^n), u(x) + v(y) \leq c(x, y), x, y \in \mathbb{R}^n\}. \quad (2.11)$$

This is related to the original Monge Transportation Problem by the fact that

$$\sup_{(u,v) \in K} J[u, v] = \inf_{T \in \mathcal{T}(\mu^-, \mu^+)} \mathcal{C}[T]. \quad (2.12)$$

The first thing to note is that the space K does not have a direct dependence on $\mu^\pm$, which is clearly not the case for $\mathcal{T}(\mu^-, \mu^+)$. Indeed, $\mu^\pm$ only comes into the formulation via the definition of dual functional, J . Further insight into why (2.12) holds will be presented in Subsection 2.6.2 via the proof of existence of potential functions corresponding to maximisers of the Kantorovich Dual Functional. In this proof, the linear structure of Kantorovich Dual Formulation plays the central role; and it is there that this construction's importance to Optimal Transportation will become evident.

2.4 Key Conditions on Cost-functions

Before proceeding with the more formal analysis of potential functions, some key conditions are needed on the cost-function, which are ultimately tantamount to existence and higher regularity of associated potential functions. The following conditions are presented as they appeared in [MTW05] and [TWar], as the presentation and notation therein have become somewhat standard in current research pertaining to Optimal Transportation.

Unless otherwise noted, for some open set U in $\mathbb{R}^n \times \mathbb{R}^n$ containing $\Omega^- \times \Omega^+$, the following conditions on the cost-function c will be assumed for the rest of this thesis.

(A1) For any $x, y \in U$ and $(p, q) \in D_x c(U) \times D_y c(U)$, there exists a unique $Y = Y(x, p)$, $X = X(q, y)$, such that $c_x(x, Y) = p$, $c_y(X, y) = q$.

(A2) For any $(x, y) \in U$,

$$\text{Det}[D_{xy}^2 c] \neq 0, \quad (2.13)$$

where $D_{xy}^2 c$ is the matrix whose elements at the i^{th} row and j^{th} column is $\frac{\partial^2 c}{\partial x_i \partial y_j}$.

(A3) There exists a constant $C_0 > 0$ such that for any $(x, y) \in U$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$ such that

$$(c^{q,r} c_{ij,q} c_{r,st} - c_{ij,st}) c^{s,k} c^{t,l} \xi_i \xi_j \eta_k \eta_l \geq C_0 |\xi|^2 |\eta|^2, \quad (2.14)$$

where $c_{i,j}(x, y) = \frac{\partial^2 c(x, y)}{\partial x_i \partial y_j}$, and $[c^{i,j}]$ is the inverse matrix of $[c_{i,j}]$.

It will be assumed that the convex hulls of the sets $c_x(x, \Omega^+)$ and $c_y(\Omega^-, y)$ lie in $D_x c(U)$ and $D_y c(U)$ respectively. This condition is automatic when Ω^- and Ω^+ are both c -convex relative to each other (this notion of set convexity will be introduced in Section 2.5).

Conditions (A1) and (A2) are required in the existence proof of potential functions which maximise the Kantorovich Dual Functional. Condition (A3) was first used in [MTW05] as a sufficient condition for classical regularity of the potential function u . In the next subsection, it will be demonstrated that (2.14) is symmetric, which subsequently dictates that the analogous condition for v is (again) just (2.14).

2.4.1 Some Properties of the (A3) Condition

Before moving on, several properties of the (A3) condition can be derived from the expression in (2.14). These properties will be used in calculations presented later in this thesis.

To start of, it will be shown that the (A3) condition is symmetric in x and y . To demonstrate this, $\tilde{\xi}$ is defined as

$$\tilde{\xi}_k := c_{q,k} \xi_q.$$

Using this definition to rewriting (2.14) with ξ replaced by $\tilde{\xi}$, one sees that

$$(c^{p,q} c_{ij,p} c_{q,rs} - c_{ij,rs}) c^{i,t} c^{j,h} c^{r,k} c^{s,l} \tilde{\xi}_t \tilde{\xi}_h \eta_k \eta_l \geq C_0 |\xi|^2 |\eta|^2, \quad (2.15)$$

with a modified orthogonality criterion of

$$\eta_q c^{q,r} \tilde{\xi}_r = 0. \quad (2.16)$$

The symmetry of x and y in the (A3) criterion is now evident in the rewritten form embodied in (2.15) and (2.16).

Next, the expression depicted in (2.14) will be reduced to a less-cumbersome form, that also gives a better intuition as to what the (A3) condition itself actually means. Calculating, one see that

$$\begin{aligned} D_{p_k} c_{ij}(x, Y(x, p)) &= c_{ij,q} \cdot D_{p_k} Y^q \\ &= c_{ij,q} c^{q,k}, \end{aligned} \quad (2.17)$$

where the definition of Y (as stated in (A1)) has been utilised to gain the second equality. Differentiation of (2.17) subsequently yields

$$D_{p_i p_k}^2 c_{ij}(x, Y(x, p)) = (c_{ij,qr} c^{q,k} + c_{ij,q} c_{,r}^{q,k}) c^{r,l}. \quad (2.18)$$

To proceed, a reduction of the term $c_{,r}^{q,k}$ is required. From previous notational definitions, it is understood that

$$c_{i,q} c^{q,k} = \delta_i^k.$$

Differentiating this relation immediately produces the following relations:

$$\begin{aligned}
 c_k^{i,j}(x, y) &= D_{x_k} c^{i,j}(x, y) \\
 &= -c^{i,q} c^{r,j} c_{kq,r}(x, y), \\
 c_{,k}^{i,j}(x, y) &= D_{y_k} c^{i,j}(x, y) \\
 &= -c^{i,q} c^{r,j} c_{q,kr}(x, y).
 \end{aligned} \tag{2.19}$$

Combining (2.18) with (2.19), it is now observed that

$$D_{p_l p_k} c_{ij}(x, Y(x, p)) = -(c^{q,r} c_{ij,q} c_{r,st} - c_{ij,st}) c^{s,k} c^{t,l}.$$

Thus, the (A3) condition is equivalent to

$$D_{p_l p_k} c_{ij}(x, y) \xi_i \xi_j \eta_k \eta_l \leq -C_0 |\xi|^2 |\eta|^2, \tag{2.20}$$

for a positive constant C_0 .

With (2.20), it is now a straight-forward calculation to verify that the (A3) condition is also invariant under coordinate transformations. Fixing y , consider an arbitrary change of coordinates in x given by

$$g(x) = x'.$$

An elementary calculation shows that

$$\begin{aligned}
 c_i(x, y) &= [D_i g^q] c_q(x', y') \\
 c_{ij}(x, y) &= ([D_i g^q][D_j g^r] c_{qr}(x', y') + [D_{ij} g^s] c_s(x', y'))
 \end{aligned} \tag{2.21}$$

From (A1) and (2.21), it is observed that

$$\begin{aligned}
 p'_k &= c_k(x', y') \\
 &= [D_k g^q]^{-1} c_q(x, y).
 \end{aligned} \tag{2.22}$$

Using this in (2.21) yields

$$c_{ij}(x', y') = ([D_i g^q][D_j g^r] c_{qr}(x', y') + [D_{ij} g^s] p'_s). \tag{2.23}$$

From here, one can use the chain-rule along with the relation in (2.22) to deduce

$$D_{p_k} = [D_k g^q]^{-1} D_{p'_q}. \tag{2.24}$$

Thus, from (2.23) and (2.24), it has been shown that the left-hand side of (2.20) is transformed to

$$D_{p_l p_k} c_{ij}(x, y) [D_i g^q][D_j g^r][D_k g^s]^{-1} [D_l g^t]^{-1} \xi_q \xi_r \eta_s \eta_t.$$

Redefining ξ_i as $[D_i g^q] \xi_q$ and η_i as $[D_i g^q]^{-1} \eta_q$, finally shows that the (A3) condition is invariant under any change of coordinates, as the orthogonality criterion $\xi \perp \eta$ is preserved.

The (A3) criterion being invariant under change of coordinates will be of central importance in Chapter 7, when a stereographic projection is used to change coordinates to analyse the (A3) condition on round spheres.

2.4.2 The Degenerate (A3) Condition

While the (A3) condition is a requirement for the local regularity of potential functions, it was shown in [TW06] that only a degenerate form of the (A3) condition is needed to prove the global regularity of potential functions in Optimal Transportation problems. In the literature, the degenerate form of the (A3) condition is typically labelled (A3w) and is stated as follows:

(A3w) For any $(x, y) \in U$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$, the following inequality holds:

$$(c^{q,r} c_{ij,q} c_{r,st} - c_{ij,st}) c^{s,k} c^{t,l} \xi_i \xi_j \eta_k \eta_l \geq 0. \quad (2.25)$$

(2.25) is associated with the potential function u . It is clear from the calculations in the previous subsection that the (A3w) condition is symmetric; and hence, (2.25) also represents the (A3w) criterion for v . One also has from the calculations in Subsection 2.4.1 that the (A3w) condition is invariant under coordinate transformations and can be represented as

$$D_{p_l p_k} c_{ij}(x, y) \xi_i \xi_j \eta_k \eta_l \leq 0. \quad (2.26)$$

Recently, Loeper showed in [Loe05] that the (A3w) condition was also *necessary* for classical regularity of potential functions. To do this, Loeper derived a geometric reformulation of the (A3w) condition, and subsequently used this to construct potentials that were not C^1 around a point where the (A3w) condition was violated by the corresponding cost-function. Thus, Loeper completed the demonstration of the (A3w) condition being not only sufficient but indeed *necessary* for potential function regularity. These statements will be elaborated upon in Chapter 7.

At the end of this chapter, the current theorems pertaining to the regularity of maximisers of the Kantorovich Dual Formulation will be restated from works outside of this thesis. It is these theorems that have motivated the new research presented in Part II. In addition to this, in Chapter 6 the (A3w) condition will be analogised to the Hessian equations considered there.

2.5 c -convexity

By using cost-functions, one can naturally generalise the classical notions of convexity. This generalisation is a powerful tool whose related notions and definitions will be used throughout the rest of this thesis.

The following definitions have become standard in the literature surrounding Optimal Transportation (see [Caf96, GM95, MTW05]).

Definition 2.5.1: A c -support function of ϕ at x_0 is a function of the form $c(x, y_0) + a$, where $y_0 \in \Omega^+$ and $a = a(x_0, y_0)$ is a constant, such that

$$\begin{aligned}\phi(x_0) &= c(x_0, y_0) + a, \\ \phi(x) &\leq c(x, y_0) + a, \quad \forall x \in \Omega^-. \end{aligned} \tag{2.27}$$

Using the above definition, one may also define the notion of c^* -support functions in a similar manner by switching x and y , Ω^- and Ω^+ . Now, the notion of support functions can be used to generalise the notion of convexity in the context of cost-functions.

Definition 2.5.2: An upper semi-continuous function ϕ defined on Ω^- is c -concave if for any point $x_0 \in \Omega^-$, there exists a c -support function at x_0 . Similarly, an upper semi-continuous function ψ defined on Ω^+ is c^* -concave if for any point $y_0 \in \Omega^+$, there exists a c^* -support function at y_0 .

From these definitions, one can also easily ascertain the notion of a c -convex function by simply switching the direction of the inequality shown in (2.27). The definition for a c^* -convex function is also thus obtained by replacing c -support functions with c^* -support functions in Definition 2.5.2 and reversing the inequality in (2.27).

Before proceeding onto other definitions, some properties of c -concave functions can be readily observed through facts in classical analysis. As it is assumed that the cost-function c is smooth, any c -concave function ϕ is *semi-concave*; that is, there exists a constant C such that $\phi(x) - C|x|^2$ is concave. It is a classical result that now shows ϕ is twice differentiable almost everywhere. In addition to this, it is readily demonstrated (via a Perron process) that if (ϕ_k) is a sequence of c -concave functions and $\phi_k \rightarrow \phi$, then ϕ is c -concave. Clearly, there are analogous results for c -convex and c^* -convex/concave functions.

Remark: It is important to note that the above definitions correspond with the classical notions of concavity in the special case where $c(x, y) = x \cdot y$. In this situation, a c -support function is simply a support hyperplane at a particular point. This special case provides a good basis of intuition for more general cost-functions.

In addition to using generalised definitions for concavity of functions, one also needs to consider analogous generalisations of concavity with respect to sets. In particular, such notions will be of key importance in the construction of barriers for boundary gradient estimates, in addition to being necessary for making the statements of current regularity theorems in Optimal Transportation. With that, the notion of c -segments (which generalise the notion of line-segments in classical convex analysis) is next defined:

Definition 2.5.3: A c -segment in Ω^- with respect to a point y is a solution set $\{x\}$ to $D_y c(x, y) = z$ for z on a line segment in $\mathbb{R}^n$. A c^* -segment in Ω^+ with respect to a point x is a solution set $\{y\}$ to $D_x c(x, y) = z$ for z on a line segment in $\mathbb{R}^n$.

By conditions (A1) and (A2), it is clear that a c -segment is a smooth curve; and for any two points $x_0, x_1 \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$, there exists a unique c -segment connecting x_0 and x_1 relative to y .

Definition 2.5.4: A set E^- is c -convex relative to a set E^+ if for any two points $x_0, x_1 \in E^-$ and any $y \in E^+$, the c -segment relative to y connecting x_0 and x_1 lie in E^+ . Analogously, it is said that E^+ is c^* -convex relative to E^- if for any two points $y_0, y_1 \in E^+$ and any $x \in E^-$, the c^* -segment relative to x connecting y_0 and y_1 lies in E^- .

In general the notion of c -convexity, with respect to sets, is stronger than the classical notion of convexity. This is most readily observed when taking a ball for the set in question. Indeed, a ball may not be c -convex (relative to another given domain) at an arbitrary location. However, a sufficiently small ball will be c -convex if c is C^3 smooth. In light of this example, it is insightful to consider an alternate statement of Definition 2.5.4:

E^- is c -convex with respect to E^+ if for each $y \in E^+$, the image $D_y c(E^-, y)$ is convex in $\mathbb{R}^n$.

An analogous statement may be made for a set that is c^* -convex relative to another set.

With these definitions, it is now possible to move on to the existence proof of Optimal Transportation potentials and the statements of higher regularity pertaining to such potentials.

2.6 Existence of Potential Functions

In the Section 2.2, a notion of potential function was deduced from heuristics associated with Lagrange multipliers in classical linear variational theory. In Section 2.3, the dual formulation due to Kantorovich and associated potential functions were introduced without derivation. At this stage, it is evident from (2.8) and (2.11) that the potential functions presented in both these formulations are one-in-the-same.

While the Lagrange heuristics provide an intuition for what these potential functions are, Kantorovich's Dual Formulation is needed to formally prove their existence. With the definitions from Section 2.5 and conditions from Section 2.4, it is now possible to proceed with proving the existence of these potential functions. The method of proof employed here uses the ideas presented in [Gan94, Caf96] and is a slight generalisation of the expositions in [Eva01, Urb98a].

2.6.1 Construction of Convex Sequences

Kantorovich's Dual Formulation allows one to use standard techniques from direct methods in the calculus of variations to prove the existence of a pair of functions which maximise (2.10). Heuristically, it has been indicated in Section 2.2 that taking a gradient of either of these potentials should produce a solution to the original Optimal Transportation Problem. That is, it is expected that $T = Du$ a.e., where T is an Optimal Transportation map and u is one of the aforementioned potential functions. This will now be shown rigorously using the Kantorovich Dual Formulation introduced in Section 2.3.

From this point onward, it will be assumed that

$$d\mu^- = f \, dx, \quad \text{and} \quad d\mu^+ = g \, dy,$$

where f, g are bounded, non-negative, have compact support in Ω^-, Ω^+ (which are assumed both to be bounded and open) respectively and satisfy the mass-balance condition:

$$\int_{\Omega^-} f \, dx = \int_{\Omega^+} g \, dy. \quad (2.28)$$

Moreover, it will also be assumed that $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is uniformly convex in both arguments.

Remark: The requirement of $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ being uniformly convex in both arguments is not the weakest condition one can have on the cost-function in what is to follow. Indeed, having a cost-function satisfying (A1) on $\overline{\Omega^-} \times \overline{\Omega^+}$ is enough to prove the existence of optimal Kantorovich potentials. [Lev99] and [Lev04] both discuss the existence of optimal Kantorovich potentials under weaker assumptions than the ones assumed in this thesis. In the following exposition the stronger convexity condition is placed on the cost-function, as it allows for a simpler (and more intuitive) existence proof than the ones having weaker assumptions placed on the cost-function.

With the above assumptions, one has to prove the existence of maximisers to the Kantorovich Dual Functional:

$$J[u, v] := \int_{\Omega^-} u(x)f(x) \, dx + \int_{\Omega^+} v(y)g(y) \, dy, \quad (2.29)$$

among all pairs (u, v) in the set

$$K := \{(u, v) : u, v \in C^0(\mathbb{R}^n), u(x) + v(y) \leq c(x, y), x, y \in \mathbb{R}^n\}.$$

To do this, one first needs to show that subsequences of function pairs belonging to K , which are bounded, Lipschitz and satisfying some notion of convexity (described below) can be extracted from any sequence of function pairs converging to a maximiser of (2.29). This idea is embodied in the following lemma.

Lemma 2.6.1: *J has a maximising pair $(\hat{u}, \hat{v}) \in K$. $\hat{u}$ and $\hat{v}$ are both bounded and are uniformly Lipschitz in Ω^-, Ω^+ respectively. Moreover, this pair of functions satisfy the dual relation*

$$\begin{aligned} \hat{u}(x) &= \inf_{y \in \Omega^+} [c(x, y) - \hat{v}(y)], \quad x \in \Omega^-, \\ \hat{v}(y) &= \inf_{x \in \Omega^-} [c(x, y) - \hat{u}(x)], \quad y \in \Omega^+. \end{aligned} \quad (2.30)$$

That is, $\hat{u}$ and $\hat{v}$ are said to be dual-convex functions of each other relative to the cost-function c .

Proof. Taking $(u, v) \in K$, $\hat{u}$ is defined by

$$\hat{u}(x) := \inf_{y \in \Omega^+} [c(x, y) - v(y)], \quad x \in \Omega^-. \quad (2.31)$$

Clearly, $\hat{u} \leq c(x, y) - v(y)$ for all $(x, y) \in \Omega^- \times \Omega^+$ and thus, $(\hat{u}, v) \in K$. Moreover, given that c and v are both assumed to be continuous, it is known that for every $x \in \Omega^-$, there exists some $y \in \overline{\Omega^+}$ such that

$$\hat{u}(x) = c(x, y) - v(y) \geq u(x),$$

where the fact that $(u, v) \in K$ has been used to ascertain the right inequality. Now, since $f \geq 0$, it is readily observed that $J(\hat{u}, v) \geq J(u, v)$. Analogously for v , one can make the following definition:

$$\hat{v}(y) := \inf_{x \in \Omega^-} [c(x, y) - \hat{u}(x)], \quad y \in \Omega^+. \quad (2.32)$$

Following the same reasoning in the analysis of $\hat{u}$ above, it is concluded that $\hat{v} \geq v$ and $(\hat{u}, \hat{v}) \in K$ with

$$J(\hat{u}, \hat{v}) \geq J(\hat{u}, v) \geq J(u, v).$$

Since $\hat{v} \geq v$, (2.31) indicates that

$$\hat{u}(x) \geq \inf_{y \in \Omega^+} [c(x, y) - \hat{v}(y)];$$

but since $(\hat{u}, \hat{v}) \in K$, it is observed that indeed

$$\hat{u}(x) = \inf_{y \in \Omega^+} [c(x, y) - \hat{v}(y)], \quad x \in \Omega^-.$$

Thus, it has been demonstrated that $J[u, v]$ does not decrease when replacing (u, v) by $(\hat{u}, \hat{v})$: a dual-convex pair of functions relative to the cost c .

As c is uniformly convex on $\mathbb{R}^n \times \mathbb{R}^n$, there exists a constant C such that

$$\sup_{y \in \Omega^+} \left[\sup_{x \in \Omega^-} \frac{|c(x_0, y) - c(x, y)|}{|x_0 - x|} \right] \leq C < \infty, \quad \forall x_0 \in \Omega^-;$$

that is, given $x \in \Omega^-$, the Lipschitz constant for c at x for a fixed y is uniformly bounded for all $y \in \Omega^+$. Thus, from (2.31), it is clear that $\hat{u}$ is Lipschitz with a Lipschitz constant bounded by C .

Finally, it is also observed that $(u + \theta, v - \theta) \in K$ for an arbitrary constant θ . Moreover, the mass-balance condition (2.28) shows that $J[u, v] = J[u + \theta, v - \theta]$. Thus, one may transparently consider such renormalisation that add any constant to u provided that constant is subsequently subtracted from v in the function pair. With that, a θ can now be chosen such that for some $x_0 \in \Omega^-$, $\hat{u}(x_0) = 0$. Relabelling our function pair with this renormalisation and then using the fact that $(\hat{u}, \hat{v}) \in K$, it is seen that

$$\hat{v}(y) \leq c(x_0, y), \quad \forall y \in \Omega^+;$$

that is, $\hat{v}$ is bounded from above on Ω^+ . Furthermore, since $\hat{u}$ is locally Lipschitz on $\mathbb{R}^n$ (again due to the assumed uniform convexity of c on $\mathbb{R}^n \times \mathbb{R}^n$), it is ascertained that

$$|\hat{u}(x)| \leq R \cdot \sup_{y \in \Omega^+} \left[\sup_{x \in B_R(x_0)} \frac{|c(x_0, y) - c(x, y)|}{|x_0 - x|} \right] \leq C < \infty, \quad \forall x \in \Omega^-,$$

where $R = \text{diam}(\Omega^-)$. From this, it is deduced that $|\hat{u}|$ is bounded on Ω^- , which subsequently implies that $|\hat{v}|$ is also bounded in Ω^+ , via (2.32).

With this, it has been demonstrated that any pair $(u, v) \in K$ may be replaced by a bounded, Lipschitz, dual-convex pair $(\hat{u}, \hat{v})$ without decreasing the functional J . Thus, maximising sequences may be restricted to those that only contain bounded, uniformly Lipschitz pairs of functions as elements. As the bounds on values and Lipschitz constants only depend on c , the convergence of a maximising sequence goes uniformly to a bounded, Lipschitz, dual-convex, maximising pair. $\square$

2.6.2 Existence of an Optimal Mass Transfer Plan

(2.32) is now used to define $\hat{u}(x)$ and $\hat{v}(y)$ for all $x, y \in \mathbb{R}^n$. From the previous subsection, it is known that both $\hat{u}$ and $\hat{v}$ are c -convex and c^* -convex respectively and, thus, are both semi-convex, as noted in Section 2.5. Subsequently, both these functions are known to be differentiable a.e. It will now be shown that $T_{\hat{u}}(x)$, defined by the relation:

$$D\hat{u}(x) = c_x(x, T_{\hat{u}}(x)), \quad \text{a.e. } x \in \Omega^-, \quad (2.33)$$

minimises the Monge Cost Functional, (1.3).

Theorem 2.6.2: *If one defines $T_{\hat{u}}(x)$ as the solution of (2.33), then the following are true:*

(i) $T_{\hat{u}}(x) : \Omega^- \rightarrow \Omega^+$ is essentially one-to-one and onto.

(ii) $T_{\hat{u}}(x)$ is such that

$$u(x) + \hat{v}(T(x)) = c(x, T(x)). \quad (2.34)$$

(iii) $T_{\hat{u}}(x)$ satisfies the measure-preserving criterion; that is

$$\int_{\Omega^-} h(T_{\hat{u}}(x)) d\mu^- = \int_{\Omega^+} h(y) d\mu^+, \quad \forall h \in C(\Omega^+). \quad (2.35)$$

(iv) $T_{\hat{u}}(x)$ minimises the Monge Cost Functional; that is

$$\sup_{(u,v) \in K} J[u, v] = \inf_{S \in \mathcal{T}} \mathcal{C}[S], \quad (2.36)$$

for all $S : \Omega^- \rightarrow \Omega^+$, such that $S_{\#}(\mu^-) = \mu^+$.

Proof.

(i) From the proof of Lemma 2.6.1, it is already known that $\hat{u}(x)$ is Lipschitz for all $x \in \Omega^-$. By Rademacher's Theorem (see [Fed69, Section 3.1.6]) $\hat{u}(x)$ is thus differentiable a.e. in Ω^- . This fact combined with (A1) ensure the definition of $T_{\hat{u}}(x)$ in (2.33) corresponds to a mapping that is one-to-one and onto a.e.

(ii) As $(\hat{u}, \hat{v}) \in K$ represent a maximising pair, one has that

$$\hat{u}(x) + \hat{v}(y) = c(x, y).$$

Differentiating with respect to x , it is calculated that

$$Du(x) = c_x(x, y)$$

a.e.; such a differentiation is validated by the proof of item one. The result follows immediately from the the definition of $T_{\hat{u}}(x)$ in (2.33).

(iii) Fixing $\epsilon > 0$, one defines the variations

$$\begin{cases} v_\epsilon(y) := \hat{v}(y) + \epsilon h(y), & y \in \Omega^+, \\ u_\epsilon(x) := \inf_{y \in \Omega^+} (c(x, y) - \hat{v}(y)), & x \in \Omega^-. \end{cases} \quad (2.37)$$

From this definition, it is clear that

$$u_\epsilon(x) + v_\epsilon(y) \leq c(x, y), \quad x \in \Omega^-, y \in \Omega^+;$$

and so,

$$I[\epsilon] := J[u_\epsilon, v_\epsilon] \leq J[\hat{u}, \hat{v}].$$

Thus, the mapping $\epsilon \mapsto I[\epsilon]$ has a maximum at $\epsilon = 0$, which subsequently indicates that

$$\begin{aligned} 0 &\leq \frac{J[\hat{u}, \hat{v}] - J[u_\epsilon, v_\epsilon]}{\epsilon} \\ &= \int_{\Omega^-} \left[\frac{\hat{u}(x) - u_\epsilon(x)}{\epsilon} \right] f \, dx - \int_{\Omega^+} h(y) g \, dy. \end{aligned} \quad (2.38)$$

Next, the claim that $\left| \frac{\hat{u} - u_\epsilon}{\epsilon} \right| \leq \|h\|_{L^\infty}$ needs to be proven. To show this, $y_\epsilon \in \Omega^+$ is taken such that

$$u_\epsilon(x) = c(x, y_\epsilon) - v_\epsilon(y_\epsilon).$$

Then it is calculated that

$$\hat{u}(x) - u_\epsilon(x) = \hat{u}(x) - c(x, y_\epsilon) + \hat{v}(y_\epsilon) + \epsilon h(y_\epsilon) \leq \epsilon h(y_\epsilon).$$

On the other hand, if one selects $y \in \Omega^+$ such that

$$\hat{u}(x) = c(x, y) - \hat{v}(y), \quad (2.39)$$

then

$$\hat{u}(x) - u_\epsilon(x) \geq \hat{u}(x) - c(x, y) + \hat{v}(y) + \epsilon h(y) = \epsilon h(y),$$

via the definition of u_ϵ in (2.37). Thus, it is ascertained that

$$h(y) \leq \frac{\hat{u}(x) - u_\epsilon(x)}{\epsilon} \leq h(y_\epsilon),$$

which proves the claim of

$$\left| \frac{\hat{u} - u_\epsilon}{\epsilon} \right| \leq \|h\|_{L^\infty}. \quad (2.40)$$

Now, if one takes a point $x \in \Omega^-$ where $\hat{u}(x)$ is differentiable, then (2.39) implies $y = T_{\hat{u}}(x)$. Moreover, it is clear that as $\epsilon \rightarrow 0$, $y_\epsilon \rightarrow T_{\hat{u}}(x)$. Thus, from (2.38),

(2.40) and the Lebesgue dominated convergence theorem, it is ascertained that

$$\int_{\Omega^-} h(T_{\hat{u}}(x))f(x) dx \leq \int_{\Omega^+} h(y)g(y) dy.$$

Finally, by replacing h by $-h$, one can conclude that the equality in (2.36) holds.

(iv) If S is taken to be any admissible mapping, then

$$\int_{\Omega^-} \hat{v}(S(x))f(x) dx = \int_{\Omega^+} \hat{v}(y)g(y) dy.$$

Since $\hat{u}(x) + \hat{v}(S(x)) \leq c(x, S(x))$, it is calculated that

$$\begin{aligned} J[\hat{u}, \hat{v}] &= \int_{\Omega^-} \hat{u}(x)f(x) dx + \int_{\Omega^+} \hat{v}(y)g(y) dy \\ &= \int_{\Omega^-} \hat{u}(x)f(x) dx + \int_{\Omega^-} \hat{v}(S(x))f(x) dx \\ &\leq \int_{\Omega^-} c(x, S(x))f(x) dx; \end{aligned}$$

that is,

$$J[\hat{u}, \hat{v}] \leq \mathcal{C}[S]. \quad (2.41)$$

By the proofs for the second and third item, one ascertains that equality holds in (2.41) when $S(x) = T_{\hat{u}}(x)$. Thus, T is indeed optimal. $\square$

With this proof, the link between Kantorovich's Dual Formulation and Monge's original minimisation problem has now been rigorously demonstrated.

2.7 The Optimal Transportation Equation

At this stage, the relationship between potential functions of the Optimal Transportation problem and elliptic partial differential equations is ready to be demonstrated. From this point forward, the hat notation on potentials will be dropped, and it will be assumed that any potentials represent a maximising pair of the Kantorovich Dual Functional.

Denoting $(u, v) \in K$ as a maximising pair of the Kantorovich Dual Functional, it is recalled from the last section that the transportation map, T_u , is defined as the mapping that solves

$$Du(x) = c_x(x, T_u(x)), \quad (2.42)$$

for almost every $x \in \Omega^-$. Similarly, there is a mapping T_v such that for almost all $y \in \Omega^+$, one has that

$$Dv(y) = c_y(T_v(y), y).$$

It is clear that T_v has analogous results to the ones conveyed in Theorem 2.6.2. Given (2.34), it is readily deduced that

$$T_u(x) = y \iff T_v(y) = x;$$

thus, T_u and T_v are inverse mappings of one another.

Given that

$$\begin{aligned} u(x) &= c(x, T_u(x)) - v(T_u(x)), \\ u(x) &\leq c(x, y) - v(y), \quad y \neq T_u(x), \quad \forall x \in \Omega^-, \end{aligned}$$

one observes that if $u \in C^2(\Omega^-)$, then

$$D^2u(x) \leq c_{xx}(x, T_u(x)), \quad \forall x \in \Omega^-, \quad (2.43)$$

where $[c_{xx}]$ is the Hessian matrix with respect to the x -variable. Similarly, if $v \in C^2(\Omega^+)$, then

$$D^2v(y) \leq c_{yy}(T_v(y), y), \quad \forall y \in \Omega^+.$$

Next, differentiating (2.42) yields

$$D^2u(x) = c_{xx}(x, T_u(x)) + D_{xy}^2c \cdot DT_u,$$

which subsequently leads to

$$\begin{aligned} \text{Det} [c_{xx}(x, T_u(x)) - D^2u(x)] &= \text{Det} [-c_{xy}(x, T_u(x))] \text{Det} [DT_u(x)] \\ &= |\text{Det} [c_{xy}(x, T_u(x))]| \frac{f(x)}{g(T_u(x))}, \quad x \in \Omega^-. \end{aligned} \quad (2.44a)$$

Given (2.43), one has that the above equation is *degenerate elliptic* for c -concave functions. Moreover, for the Optimal Transportation Problem, one has a *natural boundary condition*:

$$T_u(\Omega^-) = \Omega^+. \quad (2.44b)$$

Remark: The boundary condition (2.44b) is also sometimes referred to as a boundary condition of second type for the associated equation (2.44a). This term was originally coined by Pogorelov in [Pog64].

Analogously, for v it is ascertained that

$$D^2v(y) = c_{yy}(T_v(y), y) + D_{xy}^2c \cdot DT_v,$$

which subsequently yields

$$\text{Det} [c_{yy}(T_v(y), y) - D^2v] = |\text{Det} [-c_{xy}(T_v(y), y)]| \frac{g(y)}{f(T_v(y))}, \quad y \in \Omega^+, \quad (2.45a)$$

with the corresponding boundary condition

$$T_v(\Omega^+) = \Omega^-. \quad (2.45b)$$

Equations (2.44a) and (2.45a) are commonly referred to as the *Optimal Transportation Equations*. They are Monge-Ampère-type equations that are degenerate elliptic for u, v being c -concave, c^* -concave respectively. It is a trivial observation that both the Optimal Transportation Equations are of the exact same structure and type. Thus, the Optimal Transportation Equations will often be referred to in the singular without loss of clarity.

The Optimal Transportation Equation and its associated natural boundary condition are the basis for the studies under-taken in this thesis. In Part II, global regularity results will be presented for a class of Hessian equations closely related to the Optimal Transportation Equation. This is then followed by a study on the regularity of solutions to the Optimal Transportation Equation for various cost-functions that are dependent on the Riemannian metrics corresponding to round spheres.

Remark: In Chapter 6, equations related to

$$\text{Det} [D^2u - D_x^2c(\cdot, T_u)] = |\text{Det} [D_{xy}^2c(\cdot, T_u)]| \frac{f}{g(T_u)}, \quad \text{in } \Omega^-, \quad (2.46a)$$

$$T_u(\Omega^-) = \Omega^+, \quad (2.46b)$$

will actually be the structures used to analogue the theory of the Optimal Transportation Equation to a new class of modified-Hessian equations. In the case of (2.46a)–(2.46b), u being c -convex will ensure that (2.46a) is degenerate elliptic. In reality, such a u corresponds to a minimiser of the Kantorovich Dual Functional over the set

$$K^* := \{(u, v) : u, v \in C^0(\mathbb{R}^n), u(x) + v(y) \geq c(x, y), x, y \in \mathbb{R}^n\}.$$

The theory of such potentials is exactly analogous to what has already been presented. While considering maximiser to the Kantorovich Dual Functional is more insightful with regard to solving Monge's original problem, it is becoming standardised in the literature to consider the convention embodied in (2.46a), as it is easier to associate with the classical notions of elliptic equations, with $[D^2u] \geq 0$ being a common criterion for ellipticity. As this essentially amounts to nothing more than a sign change in the analysis surrounding this subject, this convention of analysing c -convex functions will also be adopted in this thesis.

In Chapter 7, the notational convention will switch back to the one depicted in (2.44a)–(2.44b); as most of the literature pertaining to explicit verification of the (A3) condition already use this convention. This is done with the intention of making comparative studies between this thesis and other works a less cumbersome endeavour.

2.8 Regularity Results

In this final section of the chapter, theorems that represent the current state-of-the-art in regularity theory for solutions of the Optimal Transportation Equation will be stated. As already indicated, these results are tantamount to regularity theories for potentials that minimise the Kantorovich Dual Functional.

2.8.1 Global Regularity

The first theorem is the global regularity result of Trudinger and Wang published in [TW06]:

Theorem 2.8.1 (Global Regularity [TW06]): *Let c be a cost-function satisfying (A1), (A2) and (A3w) and let Ω^- and Ω^+ be bounded C^∞ domains which are respectively uniformly c , c^* -convex with respect to one another. If these two assumptions hold and the densities $f, g \in C^\infty(\overline{\Omega^-}), C^\infty(\overline{\Omega^+})$ respectively are both positive, then there exists an a.e. unique optimal diffeomorphism $T_u \in [C^\infty(\overline{\Omega^-})]^n$, where $u \in C^\infty(\overline{\Omega^-})$ is an elliptic solution of the Optimal Transportation Equation (2.46a) satisfying the secondary boundary condition (2.46b).*

This is the theorem that will be analogised in Chapter 6 to a new class of modified-Hessian equations.

Remark: *In regards to Theorem 2.8.1, one only needs $c \in C^{3,1}$, $f, g \in C^{1,1}(\overline{\Omega^-}), C^{1,1}(\overline{\Omega^+})$ respectively and $\Omega^-, \Omega^+ \in C^{3,1}$ to ensure that $T_u \in [C^{2,\alpha}(\overline{\Omega^-})]^n$, $\forall \alpha < 1$.*

2.8.2 Interior Regularity

Finally, the current interior classical and partial regularity results for the Optimal Transportation Equation will be reviewed. The theorem of classical interior regularity was the main result in [MTW05], but the interior estimate used in the proof contained a gap in the comparison argument used therein. This situation was remedied with the preliminary analysis recently presented in [TWar]. Thus, as a result of both of these papers, one has the following theorem:

Theorem 2.8.2 (Classical Interior Regularity [MTW05, TWar]): Let $c \in C^\infty$ be a cost-function satisfying (A1), (A2) and (A3). If Ω^+ is c^* -convex with respect to Ω^- with $f, g \in C^\infty(\overline{\Omega^-}), C^\infty(\overline{\Omega^+})$ respectively both positive, then there exists an a.e. unique optimal mapping $T_u \in [C^\infty(\Omega^-)]^n$, where $u \in C^\infty(\overline{\Omega^-})$ is an elliptic solution of the Optimal Transportation Equation (2.46a).

Remark: Again, the assumptions in Theorem 2.8.1 may be relaxed to only needing $c \in C^{3,1}$, $f, g \in C^{1,1}(\overline{\Omega^-}), C^{1,1}(\overline{\Omega^+})$ respectively to ensure that $T_u \in [C^{2,\alpha}(\Omega^-)]^n$, $\forall \alpha < 1$.

In addition to classical interior regularity, an interesting partial interior regularity result was recently proven by Loeper in [Loe05]:

Theorem 2.8.3 (Partial Interior Regularity [Loe05]): Taking the cost-function c and the domains Ω^- and Ω^+ under the assumptions of Theorem 2.8.2, if one has that $f \in L^p(\Omega^-)$ for $p > n$ and $\inf g > 0$, then there exists an a.e. unique optimal mapping $T_u \in [C^{0,\alpha}(\Omega^-)]^n$ for some $\alpha > 0$, where $u \in C^{1,\alpha}(\overline{\Omega^-})$ is an elliptic solution of the Optimal Transportation Equation (2.46a).

The latest in partial regularity is another result of Trudinger and Wang, which was first presented in [TWar]:

Theorem 2.8.4 (Interior C^1 Regularity [TWar]): If Ω^+ is c -convex with respect to Ω^- and the cost-function c satisfies (A1)–(A3), then the potentials u and v are fully c -concave. Moreover, if the densities f, g , together with their reciprocals are bounded, then g is strictly c -concave and u is C^1 smooth, where u is an elliptic solution of the Optimal Transportation Equation (2.46a).

Remark: By approximation and the uniqueness of potential functions (when $f, g > 0$), the boundedness conditions in Theorem 2.8.4 can be weakened to $0 \leq \frac{f}{g \circ T} < C$. From the C^1 smoothness of u , it follows that the optimal mapping T is continuous and is a homeomorphism if both Ω^- and Ω^+ are c -convex relative to each other. In addition, if f and g are $C^{1,1}$ then T is a diffeomorphism by the analysis presented in [MTW05].

These last three results show the important role played by the (A3) condition in determining the regularity of Optimal Transportation Maps, and it will be the focus of Chapter 7 to verify this condition for general costs that depend on Riemannian metrics corresponding to round spheres.

Part II
New Results

3

The Second Boundary Value Problem for modified-Hessian Equations

Contents

3.1	Introduction	32
3.2	A Class of Modified-Hessian Equations	33
3.3	Main Results	39
3.4	Preliminary Lemmas	41
3.5	Solution Estimates	45
3.6	Method of Continuity	66
3.7	Conclusions	69

3.1 Introduction

In this chapter, an analogy of the result stated in Theorem 2.8.1 will be proven for a class of modified-Hessian equations. These modified-Hessian equations are similar to the Optimal Transportation Equation presented in (2.46a) and satisfy the same kind of boundary condition as the one stated in (2.46b). Specifically, it will be shown that solutions to this class of equations are globally smooth, using the method of continuity. This, in turn, requires that various *a priori* estimates be made on admissible solutions of these modified-Hessian equations. These *a priori* estimates represent the majority of the new research presented in this chapter, as the actual application of the method of continuity will follow the procedure already presented in [TW06].

While the general methods used to make the following *a priori* estimates follow the classical ideas of Pogorelov (see [Pog64, GT01]), the forth-coming calculations are a result of adapting the ideas presented in [TW06] to the aforementioned modified-Hessian equations. This adaptation requires new ideas and estimates influenced by those put forth in both [Urb01] and [SUW04], with a few key modifications that accommodate the particular nature of the presently-considered set of non-linear partial differential operators. This will be explained in more detail within the following sections.

3.2 A Class of Modified-Hessian Equations

To begin, a new class of modified-Hessian equations that are closely related to the Optimal Transportation Equation will be presented; it is this class of equations that will be the focus of the research contained in this chapter. The equations thus considered, have the general form

$$F [D^2u - A(\cdot, u, Du)] = B(\cdot, u), \quad \text{in } \Omega^-, \quad (3.1a)$$

where A is a given $n \times n$ matrix-valued function and B is a given scalar-valued function, defined on $\Omega^- \times \mathbb{R} \times \mathbb{R}^n$ and $\Omega^- \times \mathbb{R}$ respectively. Associated with (3.1a) is the boundary condition

$$T_u(\Omega^-) = \Omega^+, \quad (3.1b)$$

which (as one may recall from Section 2.7) is often called a *natural boundary condition*. The exact conditions on A , B , T_u , Ω^- and Ω^+ will be stated in the following sections, after some notational conventions and definitions are presented to convey the specific form of F .

3.2.1 The Structure of F

First, the left-hand side of (3.1a) can be represented as

$$f(\lambda_1, \dots, \lambda_n) := F [D^2u - A(\cdot, u, Du)], \quad \text{in } \Omega^-, \quad (3.2)$$

where f is a suitably-defined (detailed below), symmetric function of λ_i , which are the eigenvalues of the *modified-Hessian matrix*: $[D^2u - A(\cdot, u, Du)]$.

In order to start formulating a regularity theory for solutions solving (3.1a), some conditions need to be placed on f (in addition to being a symmetric function), as it is denoted in (3.2). To do this, the following set definitions are required:

$$\Gamma = \Gamma(f) := \{\lambda : 0 < f(\lambda)\}, \quad (3.3a)$$

$$\Gamma^* = \Gamma^*(f) := \{\lambda \in \Gamma(f) : f(\lambda) \text{ is concave}\}, \quad (3.3b)$$

$$\Gamma_{\mu_1, \mu_2} = \Gamma_{\mu_1, \mu_2}(f) := \{\lambda \in \Gamma^*(f) : \mu_1 \leq f(\lambda) \leq \mu_2\}, \quad (3.3c)$$

for any given μ_1 and μ_2 such that $0 \leq \mu_1 \leq \mu_2$. The relevance of these sets will now be made clear.

Given definition (3.3b), it is assumed that $f \in C^2(\Gamma^*) \cap C^0(\overline{\Gamma^*})$ is a symmetric function such that $\Gamma^* \subset \mathbb{R}^n$ is an open, convex, symmetric domain, with $0 \in \partial\Gamma^*$ and having the property that $\Gamma^* + \Gamma_+ \subset \Gamma$, where Γ_+ is the positive cone in $\mathbb{R}^n$. It will be assumed that f satisfies the following hypotheses:

$$f > 0 \text{ in } \Gamma^*, \quad f = 0 \text{ on } \partial\Gamma^*, \quad (3.4a)$$

$$f \text{ is concave in } \Gamma^*, \quad (3.4b)$$

$$\sum_i f_i \geq \sigma_0 \text{ on } \Gamma_{\mu_1, \mu_2}(f), \quad (3.4c)$$

and

$$\sum_i f_i \lambda_i \geq \sigma_1 \text{ on } \Gamma_{\mu_1, \mu_2}(f), \quad (3.4d)$$

where σ_0, σ_1 are positive constants depending on μ_1 and μ_2 , where $0 < \mu_1 \leq \mu_2$.

Before moving on, some relevant remarks based on the ones made in [SUW04] are now conveyed.

Remarks 3.2.1:

- (1) It is clear that (3.4a) and (3.4b) are a trivial consequence from definition of $\Gamma^*(f)$. These two conditions together imply the degenerate ellipticity condition:

$$f_i = \frac{\partial f}{\partial \lambda_i} \geq 0, \quad \text{in } \Gamma^* \text{ for } i = 1, \dots, n.$$

This combined with the concavity assumption on f subsequently implies that $F[D^2u - A(\cdot, u, Du)]$ is a concave function of $D^2u - A(\cdot, u, Du)$, which is required to apply the $C^{2,\alpha}$ estimates of Lieberman and Trudinger presented in [LT86].

- (2) $\Gamma^*(f)$ enables the definition of an admissible solution to be made corresponding to (3.1a). That is, a solution $u \in C^2(\Omega^-)$ is admissible if

$$D^2u - A(\cdot, u, Du) \in \Gamma^*(f), \quad \text{in } \Omega^-. \quad (3.5)$$

It is clear that (3.5) combined with an assumption that $B(x, u) > 0$, ensures that (3.1a) is elliptic with respect to a solution $u \in C^2(\Omega^-)$. In this context, ellipticity and admissibility of functions are indeed equivalent and will be used interchangeably for the remainder of this thesis. Moreover, a solution to (3.1a) is admissible if and only if it is a viscosity solution (see [Tru90]). Indeed, one may substitute the term viscosity solution for admissible solution anywhere in this thesis and vice-versa. It is through this equivalence that the notion of viscosity solutions will be understood for the rest of the forth-coming exposition.

- (3) Assuming (3.4a) and (3.4b) to be true, conditions (3.4c) and (3.4d) can be shown to be equivalent to other criterion. For instance, if

$$f(t, \dots, t) \rightarrow \infty, \quad \text{as } t \rightarrow \infty, \quad (3.6)$$

then from the concavity of f and the fact that $0 \leq \sum_i f_i \lambda_i$, it is readily calculated that

$$f(t, \dots, t) \leq f(\lambda) + \sum_i f_i \cdot (t - \lambda_i) \leq f(\lambda) + \sum_i f_i. \quad (3.7)$$

From this, (3.4c) follows on any $\Gamma_{0, \mu_2}(f)$ taking t large enough. Indeed, (3.4c) and (3.6) are equivalent. Moreover, recalling (3.4a) and $0 \in \partial \Gamma^*$, (3.7) indicates that

$$\sum_i f_i \lambda_i \leq f(\lambda) \leq \mu_2, \quad (3.8)$$

on $\Gamma_{0, \mu_2}(f)$ when taking $t = 0$. Similarly, if for any μ_1 and μ_2 with $0 < \mu_1 \leq \mu_2$, there is a constant $\theta = \theta(\mu_1, \mu_2)$ such that

$$\theta + f(\lambda) \leq f(2\lambda), \quad \forall \lambda \in \Gamma_{\mu_1, \mu_2}, \quad (3.9)$$

then the concavity of f can once again be used to directly show

$$f(2\lambda) \leq f(\lambda) + \sum_i f_i \lambda_i.$$

This subsequently implies (3.4c) with $\sigma_0 = \theta$. It is clear that (3.9) is satisfied with $\theta = (2^\alpha - 1)\mu_1$ if f is homogeneous of degree $\alpha \in (0, 1]$.

- (4) The main examples of functions f satisfying (3.4a)–(3.4d) are those corresponding to

$$f(\lambda) = S_k^{1/k}(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{m=i_1}^{i_k} \lambda_m \right) \quad (3.10)$$

and the quotients $\sigma_{k,l}$, $1 \leq \dots \leq l < k \leq \dots \leq n$, for which it is denoted that

$$f(\lambda) = \sigma_{k,l} := \left(\frac{S_k(\lambda)}{S_l(\lambda)} \right)^{1/(k-l)}. \quad (3.11)$$

With the specific form of both (3.10) and (3.11), it is readily observed that $\Gamma^*(f) = \Gamma(f)$ in both these cases. As $\Gamma(S_k) \subseteq \Gamma(S_l)$ for $l \leq k$, $\Gamma^*(f) = \Gamma(f) = \Gamma(S_k)$ holds when f is either $S_k^{1/k}$ or $\sigma_{k,l}$. For these examples, the concavity condition (3.4c) is verified in [CNS88, Tru90, Urb01]; and (3.4d) is clear from the argument presented in Remark 3.2.1(2), as both (3.10) and (3.11) depict functions which are both homogeneous of degree 1. Lastly, the degenerate ellipticity condition in (3.6) is strengthened to a strict ellipticity condition of

$$f_i > 0, \quad \text{in } \Gamma, \quad \text{for } i = 1, \dots, n, \quad (3.12)$$

in the case where f is defined by either (3.10) or (3.11).

- (5) Conditions (3.4a)–(3.4d) combined with (3.12) are essentially the ones used in [Urb01] to prove the existence of smooth solutions to a class of Hessian equations closely related to (3.1a), satisfying a simpler version of the natural boundary condition depicted in (3.1b).

At this stage, $S_k^{1/k}$ and $\sigma_{k,l}$ represent the two prime candidates for f , for which the existence of globally-smooth solutions may be able to be proven. Next, the structure from the Optimal Transportation Problem will be used to ascertain criterion on both $A(\cdot, u, Du)$ and T_u that will subsequently further reduce the possible forms of F .

3.2.2 Conditions from Optimal Transportation

From here, reasonable criterion need to be placed on $A(\cdot, u, Du)$, $B(\cdot, u)$ in (3.1a) and T_u in (3.1b), if any headway is to be made in proving a global regularity result. The regularity theory presented regarding solution to the Optimal Transportation Problem in the last chapters gives strong indication that $A(\cdot, u, Du)$ and T_u need to satisfy certain relations to guarantee that globally smooth solutions to (3.1a) exist, given the forms of f depicted in (3.10) and (3.11).

The calculations in [TW06] that will be analogised here, essentially require that both $A(\cdot, u, Du)$ and T_u satisfy conditions laid down by Optimal Transportation theory in order to make the necessary obliqueness estimate, which is subsequently required to apply the theory presented in [LT86] to make a $C^{2,\alpha}$ *a priori* estimate. The research in this chapter also requires this structure for the same reasons, in addition to being necessary to make a C^0 estimate in Subsection 3.5.2.

As with the Optimal Transportation Problem, associated with the class of modified-Hessian equations depicted in (3.1a) will be a cost-function:

$$c : \Omega^- \times \Omega^+ \rightarrow \mathbb{R}^n,$$

assumed to satisfy conditions (A1), (A2) and (A3w) as defined in Section 2.4. With this, T_u is defined as solving

$$Du = D_x c(\cdot, T_u), \quad \text{in } \Omega^-, \quad (3.13)$$

with $A(\cdot, u, Du)$ defined as

$$A(\cdot, u, Du) := D_x^2 c(\cdot, T_u), \quad \text{in } \Omega^-. \quad (3.14)$$

Thus, (2.26) indicates that the (A3w) condition in the current situation can be represented as

$$D_{p_l p_k}^2 A(x, u, Du) = D_{p_l p_k} c_{ij}(x, y) \xi_i \xi_j \eta_k \eta_l \geq 0. \quad (3.15)$$

It is important to recall that the notational convention stated in the remark at the end of Section 2.7 dictates that the inequality in (3.15) be reversed from the one in (2.26). The structure of T_u and $A(\cdot, u, Du)$ embodied in (3.13) and (3.14) respectively, will be assumed along with c satisfying (A1), (A2) and (A3w) for the rest of this chapter, unless otherwise indicated.

Remark: Since admissible solutions of (3.22a)–(3.22b) must be c -convex, it is readily verified that Loeper's counter-example in [Loe05] is also valid in the case of the Quotient Transportation Equation. This (combined with the role played by the (A3w) condition in the C^2 estimate reduction to the boundary in Subsection 3.5.3) indicates that the (A3w) condition is both necessary and sufficient for the existence of globally smooth solutions solving the Quotient Transportation Equation.

In the forthcoming obliqueness estimate, the local invertability of the matrix $[D^2u - D_x^2c(\cdot, T_u)]$ will be of central importance to the calculations presented therein. As this invertability is tantamount to

$$\text{Det} [D^2u - D_x^2c(\cdot, T_u)] > 0, \quad \text{in } \Omega^-, \quad (3.16)$$

it is required that $\Gamma^*(f)$ be contained in the cone defined by (3.16). Subsequently, the only functions in (3.10) or (3.11) that have this property are $\sigma_{n,l}$ for $l < n$. In fact, by Remark 3.2.1(3) in Subsection 3.2.1, $\Gamma^*(\sigma_{n,l})$ is equivalent to the cone defined by (3.16), when considering (3.13) and (3.14) applied to (3.1a).

Along with specifying the forms of $A(\cdot, u, Du)$ and T_u , conditions need to be placed on Ω^- and Ω^+ in order to make the forthcoming obliqueness and boundary C^2 estimates. Again, the necessity of these conditions mirrors that of the Optimal Transportation Equation. Specifically, it is required that Ω^- and Ω^+ both be bounded and C^4 ; with Ω^- , Ω^+ being uniformly c -convex, c^* -convex (respectively) relative to each other (recalling the definitions in Section 2.5). This notion of Ω^- being uniformly c -convex relative to Ω^+ is explicitly stated as $\partial\Omega^- \in C^2$ with a positive constant δ_0^- such that

$$[D_i\gamma_j(x) - c^{l,k}c_{ij,l}(x, y)\gamma_k(x)]\tau_i\tau_j \geq \delta_0^-, \quad \forall x \in \partial\Omega^-, y \in \Omega^+, \quad (3.17)$$

where τ is a unit tangent vector of $\partial\Omega^-$ at x with the outer unit normal γ . An analogous representation holds for Ω^+ being c^* -convex relative to Ω^- . This notion of Ω^- , Ω^+ being c -convex, c^* -convex (respectively) relative to one another plays a key role in the forthcoming obliqueness estimate in Subsection 3.5.1.

3.2.3 Conditions on Inhomogeneity

Next, conditions on the inhomogeneity of (3.1a) will be presented. These conditions mirror those stated in [Urb01] for classes of Hessian equations satisfying a natural boundary

condition. In order to ensure uniform ellipticity of (3.1a), it is required that

$$B(x, z) > 0, \quad x \in \Omega^-, z \in \mathbb{R}. \quad (3.18)$$

In addition, it will also be assumed for all $x \in \Omega^-$ that

$$\begin{aligned} B(x, z) &\rightarrow \infty, \quad \text{as } z \rightarrow \infty, \\ B(x, z) &\rightarrow 0, \quad \text{as } z \rightarrow -\infty, \end{aligned} \quad (3.19)$$

which will be necessary for making the C^0 solution estimate in Subsection 3.5.2. Lastly, in order to apply the method of continuity, unique solvability of the linearised problem is required, which requires that

$$B_z(x, z) > 0, \quad x \in \Omega^-, z \in \mathbb{R}. \quad (3.20)$$

This condition may be relaxed to the following:

$$B_z(x, z) \geq 0, \quad x \in \Omega^-, z \in \mathbb{R}, \quad (3.20w)$$

via the application of the Leray-Schauder theorem [GT01, Theorem 11.6] (as done in [Urb95]), at the expense of uniqueness of an admissible solution. This procedure will be reviewed in Subsection 3.6.1.

Remarks:

- (1) If (3.18) is relaxed to merely requiring that $B(\cdot, u)$ be non-negative rather than positive, the eigenvalues of the modified-Hessian matrix may not lie in a compact subset of $\Gamma^*(f)$, even if D^2u is bounded. Thus, it is not possible to deduce the uniform ellipticity of (3.1a) in this scenario.
- (2) (3.20w) is also required in the boundary C^2 estimate.
- (3) In general, $B(x, z)$ can not be allowed to have a dependence on Du for the case where $f = \sigma_{n,l}$. In [Urb95, Section 6], Urbas constructs an example where a solution corresponding to $\sigma_{n,l}$ with B having a Du dependence has its second derivatives blowing up at the boundary in the two dimensional case; this example can be readily adapted to the current class of modified-Hessian equations with $f = \sigma_{n,l}$. Theoretically, in the general case, such a dependence prohibits the existence of barriers which are required for boundary gradients estimates subsequently used in the forth-coming obliqueness and boundary C^2 estimates. This restriction is again reflected in [Urb01] for a class of Hessian equations for the same reason. However, it is possible to have B dependent on Du if the minimum eigenvalue of the second fundamental form of $\partial\Omega^-$ is large enough for all $x \in \partial\Omega^-$. In this scenario, it is possible to construct barriers for the subsequent boundary gradient estimates; this is remarked upon in Subsection 3.5.1. In addition to this, B also needs to be convex in its gradient argument in order for the C^2

estimate in Subsection 3.5.3 to be applied; this will also be remarked upon further in that subsection. As B having any dependence on Du introduces a dependence between B and Ω^- , it is not a valid criterion for a general result.

3.2.4 A Barrier Condition

If A is only assumed to satisfy the weak (A3w) condition, then a technical barrier condition is also needed to make the global C^2 *a priori* estimates in Subsection 3.5.3. Indeed, if A is such that the strong (A3) criterion holds, then no such extraneous condition will be needed; this is discussed in Remark 3.5.5 at the end of Subsection 3.5.3.

Taking A as in (3.1a), it will be assumed that there exists a function $\tilde{\phi} \in C^2(\overline{\Omega^-})$ satisfying

$$[D_{ij}\tilde{\phi}(x) - D_{p_k}A_{ij}(x, z, p) \cdot D_k\tilde{\phi}(x)]\xi_i\xi_j \geq \tilde{\delta}|\xi|^2 \quad (3.21)$$

for some positive $\tilde{\delta} > 0$ and for all $\xi \in \mathbb{R}^n$, $(x, z, p) \in U \subset \Omega^- \times \mathbb{R} \times \mathbb{R}^n$, with $\text{proj}_{\Omega^-}(U) = \Omega^-$.

This condition places a relatively minor restriction on Ω^- when A is assumed to satisfy the (A3w) condition. In [MTW05], this condition was needed in order to make the global C^2 estimates for the Optimal Transportation Equation; but it was stated in that paper that this condition was removable via a duality argument. For the currently considered class of modified-Hessian equations, no such duality exists; thus, (3.21) must be kept as a separate condition in order to make the following calculations go through.

3.2.5 The Quotient Transportation Equation

With the preceding justification regarding the form of F , $A(\cdot, u, Du)$ and T_u , along with the hypotheses placed on $B(\cdot, u)$, attention will now be focused on the following set of boundary value problems:

$$\left(\frac{S_n}{S_l}\right)^{\frac{1}{n-l}} [D^2u - D_x^2c(\cdot, T_u)] = B(\cdot, u), \quad \text{in } \Omega, \quad (3.22a)$$

$$T_u(\Omega^-) = \Omega^+, \quad (3.22b)$$

where T_u is defined by (3.13); c satisfies conditions (A1), (A2) and (A3w); $B(\cdot, u)$ satisfies (3.18)–(3.20); and both Ω^- and Ω^+ are bounded and C^4 with Ω^- , Ω^+ c -convex, c^* -convex (respectively) relative to each other. (3.22a)–(3.22b) will be referred to as the *Quotient Transportation Equation* for the rest of this thesis.

3.3 Main Results

We now are able to state the main results of this thesis.

Theorem 3.3.1: *Let c be a cost-function satisfying hypotheses (A1) and (A2) with two bounded C^4 domains $\Omega^-, \Omega^+ \subset \mathbb{R}^n$ both uniformly c -convex, c^* -convex (respectively) with respect to each other, in addition to either*

- *the barrier condition stated in Subsection 3.2.4 holding for c and Ω^- , with c satisfying the weak (A3w) condition or*
- *c satisfying the strong (A3) condition.*

If B is a strictly positive function in $C^2(\overline{\Omega^-} \times \mathbb{R})$ satisfying (3.19) and (3.20w) with T_u defined by (3.13), then any elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (3.22a)–(3.22b) satisfies the a priori estimate

$$|D^2 u| \leq C, \quad (3.23)$$

where C depends on c, B, Ω^-, Ω^+ and $\sup_{\Omega^-} |u|$.

From the theory of linear elliptic equations, higher regularity automatically follows from better regularity on c, Ω^-, Ω^+ and B . For example, if c, Ω^-, Ω^+ and B are all C^∞ , then one has that $u \in C^\infty(\overline{\Omega^-})$.

Remark: *The dependence of the estimate (3.23) on $\sup_{\Omega^-} |u|$ may be removed if B is independent of u .*

As a consequence of Theorem 3.3.1, the method of continuity of continuity will be applied in Section 3.6 to prove the existence of classical solutions of (3.22a)–(3.22b).

Theorem 3.3.2: *If the hypotheses in Theorem 3.3.1 hold, then there exists an elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (3.22a), (3.22b). If, in addition, (3.20) is satisfied, then the elliptic solution is unique.*

The plan for the proof of Theorem 3.3.2 is as follows. First, some technical results and inequalities from other works will be reviewed in Section 3.4. From there, various solution estimates will be proven in Section 3.5. The first such estimate will be the obliqueness estimate proven in Subsection 3.5.1. This calculation will show that the boundary condition (3.22b) with T_u defined by (3.13), where the cost-function is assumed to satisfy (A1) and (A2), is strictly oblique for functions u where $[DT_u]$ is non-singular. In this obliqueness estimate, the convexity assumptions on both Ω^- and Ω^+ , play a critical role. This particular estimates differs from the one in [TW06], as calculations are made without the use of a dual formulation of (3.22a)–(3.22b). Instead, the property of T_u being a local diffeomorphism is exploited to make the argument go through.

Following the obliqueness estimate, C^0 bounds for solutions to (3.22a) are then derived in Subsection 3.5.2. This calculation mimics the corresponding estimate in [Urb01] but with a new analogy of a parabolic subsolution discovered in [TW06]. Without this newly-discovered function, it would not be possible to prove the C^0 estimate using the current methods.

In Subsection 3.5.3, it is proven that second derivatives of solutions of (3.1a) can be estimated in terms of their boundary values, if $A(\cdot, u, Du)$ obeys the (A3w) condition (3.15). Again, this estimation deviates from the corresponding calculation in [TW06], as the case when $f = \sigma_{n,l}$ requires the use of specific technical inequalities in order to bound the second derivatives in terms of boundary values. This particular argument is carried out for general, symmetric $A(\cdot, u, Du)$ satisfying the (A3w) condition and not necessarily having the specific form dictated by (3.14). Finally, in Subsection 3.5.4, the boundary estimate for second derivatives is proven in a similar manner to [TW06]; but a key lemma from [Urb01] is needed to make the estimation go through. This directly leads to the global second derivative bounds stated in Theorem 3.3.1.

Following these *a priori* estimates, the method of continuity is applied in Section 3.6 to ascertain the first part of the result in Theorem 3.3.1, regarding the existence of globally smooth solutions uniquely solving (3.22a)–(3.22b). The application of the method of continuity follows the procedure in [TW06]; but instead of simply integrating the equation to get solution bounds, the C^0 estimate from Subsection 3.5.2 is used instead. From there, the Leray-Schauder fixed point theorem is then applied to prove existence of globally smooth (albeit not necessarily unique) solutions when one only has $B_z(\cdot, u) \geq 0$ in Ω^- .

The final section of the chapter discusses possible directions in research under which to proceed from the current set of results.

3.4 Preliminary Lemmas

In this section, some technical relations and inequalities will be recalled from other works that will be subsequently used in the *a priori* estimates presented in the next section.

To gain the scope of generality of these lemmas, the following notation will be assumed. Given an arbitrary $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ will be defined by

$$f(\lambda_1, \dots, \lambda_n) := F[M],$$

where M is an arbitrary $n \times n$ matrix having $\lambda_1, \dots, \lambda_n$ as its eigenvalues. In addition to this, derivatives on F will be denoted by

$$F^{ij} := \frac{\partial F}{\partial M_{ij}} \quad \text{and} \quad F^{ij,kl} := \frac{\partial^2 F}{\partial M_{ij} \partial M_{kl}},$$

with derivatives on f correspondingly written as

$$f_i := f_{\lambda_i} = \frac{\partial f}{\partial \lambda_i} \quad \text{and} \quad f_{i,j} := f_{\lambda_i, \lambda_j} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}.$$

Lastly, in the literature, the *trace* of the operator F is frequently denoted by

$$\mathcal{T} = \mathcal{T}(F) := F^{ii} = \sum_i f_i.$$

With this notation and that from Section 3.2, it is now possible to state the supporting lemmas that will be used in the forthcoming *a priori* estimates.

This first lemma states a technical relation that will be used in the calculations of the C^2 estimate reduction to the boundary. This lemma is proven in [Ger96]; but this proof will not be recalled here as it does not aid in the understanding of the *a priori* estimates in Section 3.5.

Lemma 3.4.1: *For any $n \times n$ symmetric matrix $\Xi = [\Xi_{ij}]$, one has that that*

$$F^{ij,kl} \Xi_{ij} \Xi_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \Xi_{ii} \Xi_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \Xi_{ij}^2. \quad (3.24)$$

The second term on the right-hand side is non-positive if f is concave, and is interpreted as a limit if $\lambda_i = \lambda_j$.

Remark: Lemma 3.4.1 was also used in [SUW04] to make a C^2 *a priori* estimate similar to the one presented in Subsection 3.5.3.

The following lemma is due to Urbas (see [Urb01]); and while the statement of this lemma will be used in the boundary C^2 estimate of Subsection 3.5.4, the proof itself contains some valuable technical relations that will subsequently be used in various locations within the forthcoming *a priori* estimates.

Lemma 3.4.2 [Urb01]: *If $f = \sigma_{n,l}$ with $l \in \{1, \dots, n-1\}$, then there exists a positive constant $C(\epsilon)$ such that*

$$\sum_i f_i \lambda_i^2 \leq (C(\epsilon) + \epsilon |\lambda|) \sum_i f_i, \quad \text{on } \Gamma_{\mu_1, \mu_2}(f), \quad (3.25)$$

for any $\epsilon > 0$ and $0 < \mu_1 \leq \mu_2$, with $C(\epsilon)$ depending only on μ_1, μ_2 and ϵ .

Proof. Denoting

$$S_{k-1;i}(\lambda) := S_{k-1}(\lambda) \Big|_{\lambda_i=0} = \frac{\partial S_k(\lambda)}{\partial \lambda_i},$$

it is calculated that

$$f_i = \frac{1}{n-l} \left(\frac{S_n}{S_l} \right)^{\frac{1}{n-l}-1} \left(\frac{S_{n-1;i}}{S_l} - \frac{S_n S_{l-1;i}}{S_l^2} \right). \quad (3.26)$$

Next, it is readily observed that

$$\sum_i S_{k;i}(\lambda) = (n-k)S_k(\lambda), \quad (3.27)$$

for $k = 0, \dots, n$. Summing (3.26) across i and applying (3.27) yields

$$\sum_i f_i = \frac{1}{n-l} \left(\frac{S_n}{S_l} \right)^{\frac{1}{n-l}-1} \left(\frac{S_{n-1}S_l - (n-l+1)S_n S_{l-1}}{S_l^2} \right). \quad (3.28)$$

Next, using the fact that

$$S_k(\lambda) = S_{k-1;i}(\lambda)\lambda_i + S_{k;i}(\lambda) \text{ for each } i = 1, \dots, n, \quad (3.29)$$

along with (3.27), it is deduced that

$$\begin{aligned} \sum_i S_{k-1;i}(\lambda)\lambda_i &= n \cdot S_k(\lambda) - \sum_i S_{k;i}(\lambda) \\ &= k \cdot S_k(\lambda). \end{aligned} \quad (3.30)$$

On the other hand, (3.29) can also be used to show that

$$\begin{aligned} \sum_i S_{k-1;i}(\lambda)\lambda_i^2 &= \sum_i S_k(\lambda)\lambda_i - \sum_i S_{k;i}(\lambda)\lambda_i \\ &= S_1(\lambda)S_k(\lambda) - (k+1)S_{k+1}(\lambda), \end{aligned}$$

where $S_{k+1}(\lambda)$ is defined to be zero if $k = n$ and (3.30) has subsequently been used to produce the second equality. Using this with (3.27) and summing, it is next calculated that

$$\sum_i f_i \lambda_i^2 = \frac{l+1}{n-l} \left(\frac{S_n(\lambda)}{S_l(\lambda)} \right)^{\frac{1}{n-l}} \frac{S_{l+1}(\lambda)}{S_l(\lambda)}. \quad (3.31)$$

In the special case where $l = n-1$, (3.31) reduces to

$$\sum_i f_i \lambda_i^2 = n \left(\frac{S_n(\lambda)}{S_{n-1}(\lambda)} \right)^2 = nB^2,$$

which completes the proof for the case when $l = n-1$.

To proceed further, the *Newton inequality* is next recalled:

$$\frac{S_k(\lambda)}{\binom{n}{k}} \frac{S_{l-1}(\lambda)}{\binom{n}{l-1}} \leq \frac{S_{k-1}(\lambda)}{\binom{n}{k-1}} \frac{S_l(\lambda)}{\binom{n}{l}},$$

which is valid for any $1 \leq l \leq k \leq n$ and any $\lambda \in \Gamma_+$ (see [Mit70, Section 2.15]). Taking $k = n$, Newton's inequality yields

$$(n - l + 1)S_n(\lambda)S_{l-1}(\lambda) \leq \frac{l}{n}S_{n-1}(\lambda)S_l(\lambda);$$

and therefore, from (3.28) it is ascertained that

$$\frac{1}{n} \left(\frac{S_n(\lambda)}{S_l(\lambda)} \right)^{\frac{1}{n-l}} \frac{S_{n-1}(\lambda)}{S_n(\lambda)} \leq \sum_i f_i \leq \frac{1}{n-l} \left(\frac{S_n(\lambda)}{S_l(\lambda)} \right)^{\frac{1}{n-l}} \frac{S_{n-1}(\lambda)}{S_n(\lambda)}. \quad (3.32)$$

Given (3.31), (3.32) and the fact that $S_n(\lambda)/S_l(\lambda)$ is bounded between two positive constants, the following inequality now follows:

$$C_0 \frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)} \leq \frac{\sum_i f_i \lambda_i^2}{\sum_i f_i} \leq C_1 \frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)}, \quad (3.33)$$

for some positive constants C_1 and C_2 . The ratio $S_{l+1}(\lambda)/S_{n-1}(\lambda)$ is trivially bounded if $l = n - 2$, so the lemma holds in this case as well.

Now, the remaining cases are considered, which (in light of (3.33)) entail bounding the quantity $S_{l+1}(\lambda)/S_{n-1}(\lambda)$ from above. Without loss of generality, it is assumed that $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n$. Since all λ_i are positive, it is seen that

$$\prod_{i=1}^k \lambda_i \leq S_k(\lambda) \leq C_k \prod_{i=1}^k \lambda_i,$$

for $j = 1, \dots, n$ and some positive constants C_j . Therefore, one now has the following:

$$\frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)} \leq C_0 \frac{\prod_{i=1}^{l+1} \lambda_i}{\prod_{i=1}^{n-1} \lambda_i} \leq C_1 \frac{S_l(\lambda)}{S_n(\lambda)} \lambda_{l+1} \lambda_n \leq C_2 \lambda_{l+1} \lambda_n. \quad (3.34)$$

Next, it is calculated that

$$C_0 \lambda_n^{n+1} \leq C_1 \prod_{i=l+1}^n \lambda_i \leq \frac{S_n(\lambda)}{S_l(\lambda)} \leq C_2;$$

and hence,

$$\lambda_n \leq C. \quad (3.35)$$

Now, an arbitrary $\epsilon > 0$ is chosen. If $\lambda_{l+1} \leq \epsilon \lambda_1$, then by (3.34) and (3.35)

$$\frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)} \leq C \lambda_{l+1} \leq C \epsilon \lambda_1. \quad (3.36)$$

If $\lambda_{l+1} > \epsilon \lambda_1$, two cases require consideration. First, if $\lambda_n \leq \epsilon$, then by (3.34), it is ascertained that

$$\frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)} \leq C\epsilon\lambda_{l+1} \leq C\epsilon\lambda_1, \quad (3.37)$$

while if $\lambda_n > \epsilon$, then

$$C_0\epsilon^{n-l-1}\lambda_1 \leq C_1 \prod_{i=l+1}^n \lambda_i \leq \frac{S_n}{S_l}(\lambda) \leq C_2.$$

Thus, one has the following relation:

$$\lambda_1 \leq \frac{C}{\epsilon^{n-l-1}}.$$

Since $S_n(\lambda)/S_l(\lambda)$ is assumed to be bounded between two positive constants, and $S_n/S_l = 0$ on $\partial\Gamma_+$, a positive lower bound is now implied:

$$C(\epsilon) \leq \lambda_n.$$

It then follows that

$$\frac{S_{l+1}(\lambda)}{S_{n-1}(\lambda)} \leq C(\epsilon). \quad (3.38)$$

By applying (3.36), (3.37) and (3.38) to (3.33) and replacing ϵ by ϵ/C for a suitably large constant C , it is finally derived that

$$\sum_i f_i \lambda_i^2 \leq (C(\epsilon) + \epsilon|\lambda|) \sum_i f_i$$

for any $\epsilon > 0$ as required. □

Remark: The proof of Lemma 3.4.2 comes straight from [Urb01]. As mentioned before, this particular exposition of proof has been recalled here, as it depicts key calculations and technical relations that will subsequently be of use in the forthcoming *a priori* estimates.

3.5 Solution Estimates

In this section, various *a priori* estimates to elliptic solutions of (3.22a), (3.22b) will be presented. These estimates will subsequently be used to prove (via the method of continuity) the existence of globally smooth solutions to the Quotient Transportation Equation.

3.5.1 Obliqueness Estimate

In this section, it will be proven that the boundary condition (3.22b) implies a strict oblique boundary condition. This estimate will subsequently be used in the continuity estimate in Subsection 3.5.2, the boundary C^2 estimate in Subsection 3.5.4, in addition to justifying the

use of the results from [LT86] that yield $C^{2,\alpha}$ estimates from the forthcoming C^2 *a priori* bound.

To begin, a boundary condition of the form

$$G(\cdot, u, Du) = 0, \quad \text{on } \partial\Omega^-, \quad (3.39)$$

for a second order partial differential equation in a domain Ω^- is called *oblique* if

$$G_p \cdot \gamma > 0$$

for all $(x, z, p) \in \partial\Omega^- \times \mathbb{R} \times \mathbb{R}^n$, where γ denotes the unit outer normal to $\partial\Omega^-$.

Next, it is assumed that ϕ^- and ϕ^+ are C^2 defining functions for Ω^- and Ω^+ respectively; with $\phi^-, \phi^+ < 0$ near $\partial\Omega^-, \partial\Omega^+$ respectively; $\phi^- = 0$ on $\partial\Omega^-$, $\phi^+ = 0$ on $\partial\Omega^+$; and $\nabla\phi^-, \nabla\phi^+ \neq 0$ near $\partial\Omega^-, \partial\Omega^+$ respectively. A possible case for these assumptions is depicted in Figure 3.1 below. If $u \in C^2(\overline{\Omega^-})$ is an elliptic solution of the second boundary value problem (3.22a)–(3.22b), then the following relations hold:

$$\phi^+ \circ T_u = 0 \text{ on } \Omega^-, \quad \phi^+ \circ T_u < 0 \text{ near } \partial\Omega^-.$$

By tangential differentiation, it is ascertained that

$$\phi_i^+(D_j T^i) \tau_j = 0, \quad (3.40)$$

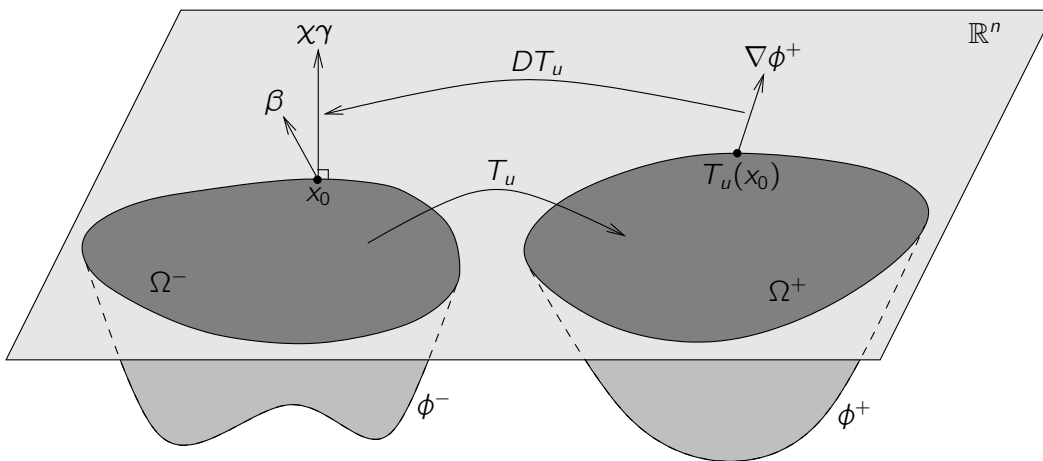


Figure 3.1: Visualisation of obliqueness estimate setup

for all unit tangent vectors τ . Note that the subscript on T_u has been dropped without loss of clarity. From (3.40), it follows that

$$\phi_k^+(D_i T^k) = x \gamma_i,$$

for some $\chi \geq 0$ and γ is again an outer, unit normal of $\partial\Omega^-$. Consequently, one has that

$$\phi_k^+ c^{k,l} w_{il} = \chi \gamma_i, \quad (3.41)$$

where

$$w_{ij} = u_{ij} - c_{ij}.$$

The relation in (3.41) is geometrically depicted in Figure 3.1.

At this point, it is observed that $\chi > 0$ on $\partial\Omega^-$ since $|\nabla\phi^+| \neq 0$ on $\partial\Omega^-$ and $\det DT \neq 0$, which is subsequently implied by the ellipticity of u and the (A2) condition. Moreover, since u is assumed to be an elliptic solution to (3.22a), it is observed that

$$\phi_k^+ c^{k,i} = \chi w^{ki} \gamma_k, \quad (3.42)$$

where $[w^{ij}]$ denotes the inverse matrix of $[w_{ij}]$. Upon writing

$$G(x, p) := \phi^+ \circ Y(x, p), \quad (3.43)$$

one has from the (A1) condition that

$$\begin{aligned} \beta_k &:= G_{p_k}(\cdot, Du) = \phi_i^+ D_{p_k} Y^i \\ &= \phi_k^+ c^{i,k} \\ &= \chi w^{ki} \gamma_k, \end{aligned}$$

which subsequently indicates that

$$\begin{aligned} \beta \cdot \gamma &= \chi w^{ij} \gamma_i \gamma_j \\ &> 0 \end{aligned} \quad (3.44)$$

on $\partial\Omega^-$. On the other hand, from (3.42), it is calculated that

$$\begin{aligned} \phi_i^+ c^{i,k} w_{jk} \phi_l^+ c^{l,j} &= \chi \phi_i^+ c^{i,j} \gamma_j \\ &= \chi (\beta \cdot \gamma). \end{aligned} \quad (3.45)$$

Eliminating χ from (3.44) and (3.45), yields

$$(\beta \cdot \gamma)^2 = (w^{ij} \gamma_i \gamma_j) (w_{kl} c^{q,k} c^{r,l} \phi_q^+ \phi_r^+). \quad (3.46)$$

(3.46) is referred to as a formula of *Urbas type*, as it was proven in [Urb97] for the Monge-Ampère equation with a natural boundary condition.

Remark: In the above calculations, the fact that w_{ij} is invertable has been used; and this is a consequence of u being an elliptic solution of (3.22a) and c satisfying the (A2) condition.

It is this invertability that requires the equations under consideration to have an Optimal Transportation structure combined with a structure that automatically implies that elliptic solutions are c -convex.

Now, $\beta \cdot \gamma$ needs to be estimated from below. This calculation mimics the one in [TW06] and [Urb97] for the Monge-Ampère equation, but with a modification to avoid the use of a dual formulation to (3.22a). These calculations start with estimating the double normal derivatives of a solution to a Dirichlet problem related to (3.22a); this follows the key idea from [Tru95]. Specifically, a point $x_0 \in \partial\Omega^-$ is fixed, where $\beta \cdot \gamma$ is minimised for an elliptic solution $u \in C^3(\overline{\Omega^-})$. From there a comparison argument is used to estimate $\gamma \cdot D(\beta \cdot \gamma)$ from above. Given that $\beta \cdot \gamma$ does not have any assumed concavity criterion in the gradient argument, the quantity itself needs to be modified so that a workable differential inequality can be derived. Thus, the following auxiliary function is defined:

$$v := \beta \cdot \gamma - \kappa(\phi^+ \circ T),$$

with a point x_0 on $\partial\Omega^-$ fixed, where $\beta \cdot \gamma$ is minimised for an elliptic solution $u \in C^3(\overline{\Omega^-})$ for sufficiently large κ , with the function ϕ^+ now chosen so that

$$[D_{ij}(\phi^+ \circ T) - c^{k,l} c_{l,ij}(\cdot, T) \cdot D_k(\phi^+ \circ T)] \xi_i \xi_j \geq \delta_0^+ |\xi|^2 \quad (3.47)$$

near $\partial\Omega^-$, $\forall \xi \in \mathbb{R}^n$ and some positive constant δ_0^+ . Inequality (3.47) is possible via the uniform c^* -convexity of Ω^+ with respect to Ω^- and taking ϕ^+ to be of the form

$$\phi^+ = a(d^+)^2 - bd^+, \quad (3.48)$$

where

$$d^+(y_0) = \inf_{y \in \partial\Omega^-} |y - y_0| \quad (3.49)$$

with a and b taken to be positive constants, [GT01, TW06].

Remark: (3.48) represents only one particular example of ϕ^+ that may be chosen that satisfy (3.49). Indeed, there are several alternative barrier functions listed for convex domains in [GT01, Section 14.2]. (3.48) is a generalised construction of a barrier, in that only derivatives up to second order of a barrier work into the subsequent boundary gradient estimates. As a bound on the gradient is only needed at it's maximum point on the boundary, only a neighbourhood of such a point is considered. In this neighbourhood, a diffeomorphic mapping can be applied to straighten the boundary, to see that (3.48) indeed represents a second-order Taylor expansion about the extremal point x_0 , along the inner-normal of the boundary (see [Eva98] for examples of such calculations).

For clarity in the forthcoming calculations, the following definition is made:

$$H(x, p) := G_p(x, p) \cdot \gamma(x) - \kappa G(x, p), \quad (3.50)$$

where $G(x, p)$ is defined by (3.43); that is,

$$v(x) = H(x, Du(x)).$$

Calculating, it is seen that

$$\begin{aligned} D_i v &= D_i H + (D_{p_k} H) u_{ki} \\ D_{ij} v &= D_{ij} H + (D_{i,p_k} H) u_{kj} + (D_{j,p_k} H) u_{ki} + (D_{p_k} H) D_k u_{ij} \\ &\quad + (D_{p_k p_l} H) u_{ki} u_{lj}. \end{aligned} \quad (3.51)$$

Next, taking $F := \sigma_{n,l}$, equation (3.22a) in the general form (3.1a) is differentiated to ascertain that

$$F^{ij} [D_k u_{ij} - D_k A_{ij} - (D_{p_l} A_{ij}) u_{lk}] = B_k + B_z D_k u.$$

Introducing the linearised operator $\mathcal{L}$:

$$\mathcal{L}v = F^{ij} [D_{ij} v - (D_{p_k} A_{ij}) D_k v]; \quad (3.52)$$

and using (3.51) and (3.52) with some simple estimation, it is calculated that

$$\begin{aligned} \mathcal{L}v &= F^{ij} [D_{ij} H + 2(D_{i,p_k} H) u_{kj} + (D_{p_k p_l} H) u_{ki} u_{lj} - (D_{p_k} A_{ij}) D_k H] \\ &\quad + (D_{p_k} H) (B_{x_k} + B_z D_k u + F^{ij} D_k A_{ij}) \\ &\leq F^{ij} u_{ik} u_{jl} [D_{p_k p_l} H + \delta_{kl}] + F^{ij} [(D_{i,p_k} H) (D_{j,p_l} H) \delta_{kl} - (D_{p_k} A_{ij}) D_k H] \\ &\quad + (D_{p_k} H) (B_{x_k} + B_z D_k u + F^{ij} D_k A_{ij}) \\ &\leq F^{ij} u_{ik} u_{jl} [D_{p_k p_l} H + \delta_{kl}] + C(F^{ii} + 1). \end{aligned} \quad (3.53)$$

In the above estimation, the gradient bound on the solution u (which is implied by (3.13); the boundedness of Ω^+ , Ω^- ; and the continuity of c) has been used. Now, using the formulae in (2.19), along with condition (A1) applied to (3.43), one now has the following:

$$\begin{aligned} D_{p_i p_j} G &= D_{p_j} (\phi_k^+ c^{k,i}) \\ &= \phi_{kl}^+ c^{k,i} c^{lj} - \phi_k^+ c^{s,j} c_{s,l}^{k,i} \\ &= c^{k,i} c^{lj} [\phi_{kl}^+ - \phi_r^+ c^{r,s} c_{s,kl}] \end{aligned}$$

Utilising the criterion for ϕ^+ in (3.47), it is subsequently calculated that

$$\begin{aligned} [D_{p_i p_j} G(x, Du)] \xi_i \xi_j &\geq \delta_0^+ \sum_i |c^{ij} \xi_j|^2 \\ &\geq \delta_1^+ |\xi|^2 \end{aligned} \quad (3.54)$$

for a further positive constant δ_1^+ . Thus, by choosing κ sufficiently large, one has

$$[D_{p_i p_j} H(x, Du)] \xi_i \xi_j \leq -\frac{1}{2} \kappa |\xi|^2$$

holding true near $\partial\Omega^-$. Substituting this into (3.53) yields

$$\mathcal{L}v \leq -\frac{1}{4}\kappa\delta_{kl}F^{ij}u_{ik}u_{jl} + C(F^{ii} + 1) \quad (3.55)$$

where C is a constant depending on c , B , Ω^- , Ω^+ and κ .

Before proceeding, a technicality in the above argument must be addressed. The calculations that lead to the inequality in (3.54) depend on the explicit structure of ϕ^+ . Indeed, unless ϕ^+ extends to all of Ω^+ such that (3.47) holds for all $T \in \Omega^+$, there is no control near $\partial\Omega^-$ to validate (3.54) and thus (3.55). To get around this, one can simply modify the definition of G in (3.50) by a function satisfying (3.54) in all of Ω^- and agreeing with (3.43) near $\partial\Omega^-$. One example of such a G (suggested in [TW06]) is given by:

$$G(x, p) := \rho_\epsilon (\max\{\phi^+ \circ Y(x, p), C_0(|p|^2 - C_1^2)\}) ,$$

where C_0 and C_1 are positive constants with C_0 sufficiently small and $C_1 > \max|Du|$ and with ϵ being sufficiently small with ρ_ϵ being a standard mollification.

A suitable barrier is now provided by the uniform c -convexity of Ω^- which implies analogously to the case of Ω^+ above, that there exists a defining function ϕ^- for Ω^- satisfying

$$[D_{ij}\phi^- - c^{l,k}c_{ij,l}(\cdot, T)D_k\phi^-]\xi_i\xi_j \geq \delta_0^-|\xi|^2, \quad (3.56)$$

near $\partial\Omega^-$. Specifically, ϕ^- is defined in a similar manner to ϕ^+ in (3.48):

$$\phi^- := a(d^-)^2 - bd^-, \quad (3.57)$$

for some positive constants a and b . Given the elementary fact that $D(d^-) = -\gamma$, the definition above can be combined with (3.17) and the definition of $\mathcal{L}$ in (3.52) to yield

$$\mathcal{L}\phi^- \geq F^{ii}\delta_0(b - 2ad^-) + F^{ij}\frac{2a}{(b - 2ad^-)^2}D_i\phi^-D_j\phi^-, \quad (3.58)$$

in a sufficiently small neighbourhood of $\partial\Omega^-$; that is,

$$\mathcal{L}\phi^-(x_0) \geq F^{ii}\delta_0b + F^{ij}2a\gamma_i\gamma_j. \quad (3.59)$$

Now the fact that F^{ii} is bounded from below on Γ_{0,μ_2} is recalled from (3.4c) and Remark 3.2.1(3). With this, (3.55) and (3.59) indicate that one can pick a and b in (3.57) such that

$$\mathcal{L}\phi^- \geq \mathcal{L}v,$$

in a small enough, fixed neighbourhood of $\partial\Omega^-$. From this, it is inferred by the standard

barrier argument (see [GT01, Chapter 14]) that

$$\gamma \cdot Dv(x_0) \leq C,$$

where again C is a constant depending on c, Ω^-, Ω^+ and B . From (3.56) and since x_0 is a minimum point of v on $\partial\Omega^-$, it can be written that

$$Dv(x_0) = \tau\gamma(x_0) \quad (3.60)$$

where $\tau \leq C$. Next, it is calculated that

$$\begin{aligned} D_i(\beta \cdot \gamma) &= D_i[\phi_k^+ c^{k,j} \gamma_j] \\ &= \phi_{kl}^+ D_i(T^l) c^{k,j} \gamma_j + \phi_k^+ c^{k,j} D_i \gamma_j + \phi_k^+ \gamma_j (c_i^{k,j} + c_{,l}^{k,j} D_i T^l) \\ &= \phi_k^+ c^{k,j} (D_i \gamma_j - c^{s,r} c_{ij,s} \gamma_r) + (\phi_{kl}^+ - \phi_r^+ c^{r,s} c_{s,kl}) c^{k,j} \gamma_j D_i T^l \end{aligned}$$

Multiplying by $\phi_t^+ c^{t,i}$ and summing over i , one subsequently has the inequality

$$\begin{aligned} \phi_i^+ c^{i,j} D_j(\beta \cdot \gamma) &= \phi_k^+ \phi_t^+ c^{k,j} c^{t,i} (D_i \gamma_j - c^{s,r} c_{ij,s} \gamma_r) \\ &\quad + \phi_t^+ c^{t,i} c^{k,j} \gamma_j c^{l,q} w_{iq} (\phi_{kl}^+ - \phi_r^+ c^{r,s} c_{s,kl}) \\ &\geq \delta_0^- \sum_j |\phi_i^+ c^{i,j}|^2, \end{aligned} \quad (3.61)$$

by virtue of the uniform c -convexity of Ω^- , the c^* -convexity of Ω^+ and (3.42). From (3.50) and (3.60), it is derived that at x_0 that the following relation holds:

$$-\kappa w_{kl} c^{i,k} c^{j,l} \phi_i^+ \phi_j^+ \leq C(\beta \cdot \gamma) - \tau_0,$$

for positive constants C and τ_0 . Hence if $\beta \cdot \gamma \leq \tau_0/2C$, one has the lower bound

$$w_{kl} c^{i,k} c^{j,l} \phi_i^+ \phi_j^+ \geq \frac{\tau_0}{2\kappa}. \quad (3.62)$$

At this point in the argument, a Legendre transform (or in the case of Optimal Transportation a c -transform) would usually be invoked to derive a dual equation to (3.22a), with the intention to bound $w^{ij} \gamma_i \gamma_j$ at x_0 from below. In the current situation, such a dual formalism can not be explicitly constructed; indeed, the transformed Quotient Transportation Equation would not have the same form as (3.22a)–(3.22b). Instead of doing this, (3.60) is manipulated another way via the fact that T_u is a local diffeomorphism (In fact, it's a global diffeomorphism.).

At x_0 , (3.60) indicates that

$$\begin{aligned} w^{ij} \gamma_i D_j v &= \tau w^{ij} \gamma_i \gamma_j \\ &\leq C w^{ij} \gamma_i \gamma_j. \end{aligned} \quad (3.63)$$

Using (3.41) and following the same vein as the calculation done in (3.61), one calculates that

$$\begin{aligned}
w^{ij}\gamma_i D_j(\beta \cdot \gamma) &= \chi^{-1} \phi_k^+ c^{k,i} D_i(\phi_l^+ c^{l,j} \gamma_j) \\
&= \chi^{-1} \phi_k^+ \phi_l^+ c^{k,i} c^{l,j} (D_i \gamma_j - c^{s,t} c_{ij,s} \gamma_t) \\
&\quad + \chi^{-1} \phi_k^+ c^{k,i} w_{iq} c^{l,q} \gamma_j c^{k,j} (\phi_{kl}^+ - \phi_r^+ c^{r,s} c_{s,kl}) \\
&\geq \chi^{-1} \delta_0^- \sum_j |\phi_i^+ c^{i,j}|^2 + \gamma_j \gamma_i c^{i,l} c^{j,k} (\phi_{kl}^+ - \phi_r^+ c^{r,s} c_{s,kl}) \\
&\geq \chi^{-1} \delta_0^- \sum_j |\phi_i^+ c^{i,j}|^2 + \delta_0^+ \sum_l |\gamma_k c^{l,k}|^2 \\
&\geq \delta_0^+ \sum_l |\gamma_k c^{l,k}|^2,
\end{aligned} \tag{3.64}$$

where δ_0^- and δ_0^+ are the constants associated with the uniform c -convexity of Ω^- and the c^* -convexity of Ω^+ respectively.

Combining (3.63) with (3.64) and using the definition for v along with (3.45), it is ascertained that

$$\begin{aligned}
w^{ij}\gamma_i D_j v &\geq \delta_0^+ \sum_l |\gamma_k c^{k,l}|^2 - \chi^{-1} \kappa \phi_i^+ \phi_j^+ c^{i,k} c^{j,l} w_{kl} \\
&= \delta_0^+ \sum_l |\gamma_k c^{k,l}|^2 - \kappa(\beta \cdot \gamma),
\end{aligned}$$

at x_0 . Setting $\tau_0^+ := \delta_0^+ \sum_{k,l} |\gamma_k c^{k,l}|^2$ at x_0 , the above reduces to

$$w^{ij}\gamma_i \gamma_j \geq \frac{\tau_0^+}{C} - \frac{\kappa}{C}(\beta \cdot \gamma).$$

Thus, if $(\beta \cdot \gamma) \leq \frac{\tau_0^+}{2\kappa}$, then it follows that

$$w^{ij}\gamma_i \gamma_j \geq \frac{\tau_0^+}{2C}. \tag{3.65}$$

Remark: In order to estimate $\phi_i^+ \phi_j^+ c^{i,k} c^{j,l} w_{kl}$, only the positivity of the second term of the right-hand side of (3.61) (due to c^* -convexity) was used. That is, the estimate for $\phi_i^+ \phi_j^+ c^{i,k} c^{j,l} w_{kl}$ only depends on the value of δ_0^- and not-necessarily uniform c^* -convexity of Ω^+ . On the other hand, the estimate for $w^{ij}\gamma_i \gamma_j$ depends on δ_0^+ and the not-necessarily uniform c -convexity of Ω^- .

Combining (3.46), (3.62) and (3.65), the uniform obliqueness estimate is thus derived:

$$G_p \cdot \gamma > \delta, \tag{3.66}$$

for some positive constant δ which depends on Ω^- , Ω^+ , c and B . This estimate is now stated formally in the following result.

Theorem 3.5.1: *Let $c \in C^3(\mathbb{R} \times \mathbb{R}^n)$ be a cost-function satisfying conditions (A1) and (A2) with respect to bounded C^3 domains Ω^- , $\Omega^+ \in \mathbb{R}^n$, which are respectively, uniformly c -convex, c^* -convex with respect to each other. In this case, if $B(x, z)$ is a positive, bounded function in $C^1(\overline{\Omega^-} \times \mathbb{R}^n)$ and T_u a mapping satisfying (3.13), then any elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (3.22a)–(3.22b) satisfies the obliqueness estimate (3.66).*

Remarks 3.5.2:

- (1) *The key difference in this obliqueness proof is that a dual formulation to (3.22a) is not used anywhere in the estimate. Instead, the fact that T_u is a local diffeomorphism is exploited to make the estimate (3.62) and (3.65) go through.*
- (2) *As mentioned earlier, the restriction that B be independent of the gradient in the general case is unattractive. However, a similar criterion was required by Urbas in [Urb01] in the case where equations involved the operator $\sigma_{n,l}$ applied to Hessian matrices. The reason for this restriction is due to (3.58) holding for general barriers and the fact that F^{ii} can remain bounded even as $|D^2u| \rightarrow \infty$. This observation is clear, recalling the relation (3.28) derived in Section 3.4 and applying MacLauren's Inequality.*
- (3) *This estimate can be derived in special cases where B has a Du dependence provided that the second fundamental form of $\partial\Omega^-$ has a large enough minimum eigenvalue. Specifically, it is required that*

$$2F^{ir} [G_{i,p_k p_s} \gamma_k - \kappa G_{i,p_s}] + (G_{p_k p_r} \gamma_k - \kappa G_{p_r}) B_{p_s} \leq -2F^{ir} G_{p_k p_s} D_i \gamma_k. \quad (3.67)$$

This can be realized by examining (3.53) and noting the appearance of a $H_{p_r} B_{p_s} u_{rs}$ term when B has a Du dependence. If (3.67) holds, this new term can be removed immediately from the calculation; and thus, it would not affect the subsequent barrier construction. As (3.67) implies a dependence between B and Ω^- , this situation can not be assumed for the general case.

3.5.2 C^0 Estimate

In [MTW05], C^0 solution estimates are naturally handled as the Optimal Transportation problem corresponds to a Monge-Ampère equation, which subsequently correlates to the determinant of a Jacobian corresponding to a measure-preserving coordinate transform. Thus, simply integrating the equation will lead directly to uniform integral bounds that, in turn, directly lead to C^0 estimates of the solution. In the case of the Quotient Transportation Equation, there are no such simplification given that $\sigma_{n,l}$ has no direct correspondence to a

Jacobian of a non-singular coordinate transform. Instead, the estimate set forth by Urbas in [Urb01] will be adapted to the current situation, with the use of a special auxiliary function first introduced in [TW06].

3.5.2.1 Supremum Solution Bound

First, an upper-bound for an elliptic solution u of (3.22a)–(3.22b) will be proven. The argument for this part of the estimate is the same as the one [Urb01] and needs a trivial modification for the current scenario. Recalling (3.7), the concavity of f implies

$$f(\lambda) \leq f(1, \dots, 1) + \sum_i f_i(1, \dots, 1)(\lambda_i - 1);$$

and thus,

$$F[u] \leq C_1 + \sigma \Delta_x(u - c), \quad (3.68)$$

where $\sigma = \sum_i f_i(1, \dots, 1)$. Integrating (3.68) and applying the divergence theorem, it is readily calculated that

$$\begin{aligned} \int_{\Omega^-} B(x, u) dx &\leq C_1 \text{Vol}(\Omega^-) + \sigma \int_{\Omega^-} \Delta_x(u - c) dx \\ &\leq C_2 \text{Vol}(\Omega^-) + \sigma \int_{\partial\Omega^-} (\gamma \cdot Du) dx \\ &< C_3, \end{aligned} \quad (3.69)$$

where γ is the unit outer-normal vectorfield relative to $\partial\Omega^-$. The last inequality comes from the gradient solution bound implied by (3.13); the continuity of c ; and the boundedness of both Ω^- and Ω^+ . From (3.69) and (3.19), it is clear that u is bounded from above somewhere in Ω^- ; thus, by the aforementioned bound on Du , it is ascertained that

$$\sup_{\Omega^-} u < C,$$

where C depends on B , c , Ω^- , and Ω^+ .

3.5.2.2 Infimum Solution Bound

Before proceeding to the infimum estimate, the following definition is needed.

Definition 3.5.3: A c -convex function ϕ on an arbitrary domain U is said to be uniformly c -convex on U , if ϕ satisfies the following inequality

$$[\phi_{ij} - c_{ij}(\cdot, T_u)] > 0, \quad \text{in } U, \quad (3.70)$$

where T_u is defined by (3.13).

Remark: It is clear that in the case where $f := \sigma_{n,l}$, (3.70) corresponds to the ellipticity criterion (3.5).

To get the lower bound for u , a specific type of auxiliary function $u_0(x)$ is considered, which is uniformly c -convex. This auxiliary function was first used as the basis for an alternate proof of Lemma 5.1 in [TW06]. Let y_0 be a point in Ω^+ and u_0 be the c^* -transform of the function

$$\psi(y) = -\sqrt{r^2 - |y - y_0|^2},$$

given by

$$u_0(x) = \sup_{y \in B_r(y_0)} [c(x, y) - \psi(y)], \quad (3.71)$$

for sufficiently small $r > 0$. u_0 is a locally uniformly c -convex function defined in some ball $B_R(0)$, with $R \rightarrow \infty$ as $r \rightarrow 0$; and the image of its c -normal mapping satisfies

$$T_{u_0}(\Omega^-) \subset B_r(y_0),$$

where T_{u_0} is a diffeomorphism between Ω^- and $T_{u_0}(\Omega^-)$. The reader is referred to [TW06, TWar] for the derivation and discussion of these properties of u_0 .

With the above construction, it is clear that for given $x_0 \in \Omega^-$ and $y_0 \in \Omega^+$, $r > 0$ can be fixed small enough so that one has u_0 is uniformly c -convex on $\Omega^- \Subset B_R(0)$ and

$$T_{u_0}(\Omega^-) \subset B_r(y_0) \subset \Omega^+. \quad (3.72)$$

Since, $B_R(0) \Subset \Omega^-$ (by the smoothness of both u_0 and c combined with (3.70)), it is clear that

$$\left(\frac{S_n}{S_l}\right)^{\frac{1}{n-l}} [D^2 u_0 - D_x^2 c(\cdot, T_{u_0})] \geq C(r), \quad \text{in } \Omega^-,$$

for some positive constant $C(r)$. Next, it is supposed that

$$B(\cdot, u) < C(r), \quad \text{in } \Omega^-. \quad (3.73)$$

From this, one has that

$$\left(\frac{S_n}{S_l}\right)^{\frac{1}{n-l}} [D^2 u_0 - D_x^2 c(\cdot, T_{u_0})] > B(\cdot, u) = \left(\frac{S_n}{S_l}\right)^{\frac{1}{n-l}} [D^2 u - D_x^2 c(\cdot, T_u)],$$

so $u_0 - u$ obtains its maximum on $\partial\Omega^-$; this maximal point is denoted x_0 . With that, it is calculated that

$$D_\gamma(u_0 - u)(x_0) \geq 0 \quad \text{and} \quad \delta(u_0 - u)(x_0) = 0,$$

where δ denotes the tangential part of the gradient; that is, $\delta := D - \gamma(\gamma \cdot D)$. By a suitable rotation of coordinate axes, it is assumed that $\gamma = e_1$ at x_0 ; so one now has the following:

$$\begin{aligned} D_1 u_0(x_0) &\geq D_1 u(x_0), \\ D_\alpha u_0(x_0) &= D_\alpha u(x_0), \quad \text{for } \alpha = 2, \dots, n. \end{aligned} \quad (3.74)$$

From here, the obliqueness estimate from Subsection 3.5.1 is now used. Specifically, it is observed that since u is an elliptic solution to (3.22a)–(3.22b), with c being convex in both arguments and satisfying condition (A2), one has that $[DT_u] > 0$. Using this fact and the obliqueness estimate (3.66) to analyse the result when (3.41) is multiplied by ϕ^+ yields

$$\gamma(x) \cdot \gamma^+(T_u(x)) > 0, \quad x \in \partial\Omega^-,$$

where γ and γ^+ are the outward pointing unit normal vectorfields to $\partial\Omega^-$ and $\partial\Omega^+$ respectively. From this, it is observed that $p_0 = T_u(x_0)$ must lie in the set $\partial\Omega^+ := \{p \in \partial\Omega^+ : \gamma_1^+(p) \geq 0\}$. Since Ω^+ is c -convex (hence convex), it follows from (3.74) that $T_{u_0}(x_0)$ lies outside Ω^+ , which contradicts (3.72). Therefore, (3.73) is false, and so it must be that $B(x, u) > \sigma(r)$ somewhere in Ω^- . Thus, the bound on the gradient of u implies a lower bound on the solution:

$$C < \inf_{\Omega^-} u.$$

With this, the following result is thus proven:

Theorem 3.5.4: *Let $c \in C^3(\mathbb{R} \times \mathbb{R}^n)$ be a cost-function satisfying conditions (A1) and (A2) with respect to bounded C^3 domains $\Omega^-, \Omega^+ \in \mathbb{R}^n$, which are respectively, uniformly c -convex, c^* -convex with respect to each other. In this case, if $B(x, z)$ is a positive function in $C^1(\overline{\Omega^-} \times \mathbb{R}^n)$ satisfying (3.19) and T_u a mapping satisfying (3.13), then any elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (3.22a)–(3.22b) satisfies the solution bound:*

$$\sup_{\Omega^-} |u| < C, \tag{3.75}$$

where C depends on c , B , Ω^- and Ω^+ .

Remarks:

- (1) *In the infimum estimate the obliqueness estimate was used, which subsequently utilised the fact that $\sum_i f_i$ is bounded away from zero on Γ_{0, μ_2} . It is important to note that the obliqueness estimate does not depend on μ_1 as defined in (3.3c). Obviously, if this were the case then the infimum estimate above would fall victim to circular logical argument.*
- (2) *Applying (3.75) to the condition (3.19), indicates that B is bounded from above and away from zero. Thus, the forthcoming calculations pertaining to (3.22a)–(3.22b) will be carried out on $\Gamma_{\mu_1, \mu_2}(\sigma_{n,l})$, where $0 < \mu_1 \leq \mu_2$ without loss of generality.*

3.5.3 Global Second Derivative Estimates

In this section, global bounds for second derivatives of elliptic solutions of equation (3.22a) will be shown to be estimated in terms of their boundary values. The following arguments

mimic those presented in [TW06] and [MTW05], which are subsequent modification of the arguments presented in [GT01, Section 17.6].

For this estimate, it suffices to consider the more general form of the Quotient Transportation Equation depicted in (3.1a), under the assumption that the matrix valued function $A \in C^4(\Omega^- \times \mathbb{R} \times \mathbb{R}^n)$ satisfies condition (A3w); that is

$$D_{p_k p_l}^2 A_{ij}(x, z, p) \xi_i \xi_l \eta_k \eta_j \geq 0,$$

for all $(x, z, p) \in \Omega^- \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$ and $\xi \perp \eta$. It is also assumed that A is symmetric, hence diagonalisable. Lastly, the barrier condition from Subsection 3.2.4 is also assumed to hold for A . This condition is recalled here for convenience: it is assumed that there exists a function $\tilde{\phi} \in C^2(\overline{\Omega^-})$ satisfying

$$[D_{ij}\tilde{\phi}(x) - D_{p_k} A_{ij}(x, z, p) \cdot D_k \tilde{\phi}(x)] \xi_i \xi_j \geq \tilde{\delta} |\xi|^2, \quad (3.76)$$

for some positive $\tilde{\delta} > 0$ and for all $\xi \in \mathbb{R}^n$, $(x, z, p) \in U \subset \Omega^- \times \mathbb{R} \times \mathbb{R}^n$, with $\text{proj}_{\Omega^-}(U) = \Omega^-$. Remark 3.5.5, at the end of this section, will discuss how this barrier condition can be removed if A satisfies the *strong* (A3) criterion.

To begin, $u \in C^4(\Omega^-)$ is assumed to be an elliptic solution of (3.1a) with $(x, u(x), Du(x)) \in U$ for $x \in \Omega^-$ and $\xi \in \mathbb{S}^n$. The auxiliary function v is now defined as

$$v := \log(w_{ij} \xi_i \xi_j) + \kappa \tilde{\phi},$$

where $w_{ij} := u_{ij} - A_{ij}$. Differentiation (3.1a) yields

$$F^{ij} [D_\xi u_{ij} - D_\xi A_{ij} - (D_z A_{ij}) D_\xi u - (D_{p_l} A_{ij}) u_{\xi l}] = B_\xi + B_z D_\xi u.$$

Another differentiation subsequently produces

$$\begin{aligned} F^{ij,kl} D_\xi w_{ij} D_\xi w_{kl} + F^{ij} [D_{\xi\xi} u_{ij} - D_{\xi\xi} A_{ij} - (D_{zz} A_{ij})(D_\xi u)^2 - (D_{p_k p_l} A_{ij}) u_{\xi k} u_{\xi l} \\ - (D_z A_{ij}) u_{\xi\xi} - (D_{p_l} A_{ij}) D_\xi u_{\xi l} - 2(D_{\xi z} A_{ij}) D_\xi u - 2(D_{\xi p_k} A_{ij}) u_{\xi k} \\ - 2(D_{z p_k} A_{ij}) u_{\xi k} D_\xi u] = D_{\xi\xi} B + (D_{zz} B)(D_\xi u)^2 + 2(D_{\xi z} B) D_\xi u + (D_z B) u_{\xi\xi}. \end{aligned}$$

Recalling that a gradient bound for an elliptic solution u exists (implied by the boundary condition (3.22b) and (3.13)), one can write the above as

$$\begin{aligned} F^{ij,kl} D_\xi w_{ij} D_\xi w_{kl} + F^{ij} [D_{\xi\xi} u_{ij} - (D_{p_k p_l} A_{ij}) u_{\xi k} u_{\xi l} - (D_{p_k} A_{ij}) D_k u_{\xi\xi}] \\ \geq -C [1 + (1 + w_{jj}) F^{ii}] \\ \geq -C [(1 + w_{jj}) F^{ii}], \end{aligned} \quad (3.77)$$

where the last inequality comes from the fact that $0 < \sigma \leq F^{ii}$ on Γ_{0,μ_2} . To further reduce

this, the (A3w) condition is now used. Specifically, a point $x_0 \in \Omega^-$ is fixed and coordinates are rotated such that $[w_{ij}]$ (and thus F^{ij}) is diagonal. With such a rotation, it can also be enforced that the eigenvalues of $[w_{ij}]$ are ordered as $0 < \lambda_1 \leq \dots \leq \lambda_n$. Estimating, one sees that

$$\begin{aligned}
 F^{ij} D_{p_k p_l} A_{ij} D_{k\xi} u D_{l\xi} u &= F^{ij} A_{ij,kl} (w_{k\xi} + A_{k\xi}) (w_{l\xi} + A_{l\xi}) \\
 &\geq F^{ij} A_{ij,kl} w_{k\xi} w_{l\xi} - C(1 + w_{jj}) F^{ii} \\
 &\geq \sum_{k \text{ or } l=r} f_r A_{rr,kl} (\lambda_k \xi_k) (\lambda_l \xi_l) - C(1 + w_{jj}) F^{ii} \\
 &\geq -C(F^{ii} + w_{jj} F^{ii} + |w| f_i \lambda_i) \\
 &\geq -C(F^{ii} + |w| + |w| F^{ii}), \tag{3.78}
 \end{aligned}$$

where $|w| = \sum_i w_{ii}$; and the fact that $f_i \lambda_i$ is bounded on Γ_{μ_1, μ_2} (as shown in (3.8)) has been used to ascertain the last inequality. Using this, (3.77) can now be rewritten as

$$F^{ij} [D_{ij} u_{\xi\xi} - (D_{p_k} A_{ij}) D_{k\xi} u_{\xi\xi}] \geq -F^{ij,kl} D_{\xi} w_{ij} D_{\xi} w_{kl} - C(F^{ii} + w_{ii} + w_{jj} F^{ii}). \tag{3.79}$$

(3.79) is used to guide the definition of the linear operator that will be used to analyse v :

$$\mathcal{L} u_{\xi\xi} := F^{ij} [D_{ij} u_{\xi\xi} - (D_{p_k} A_{ij}) D_{k\xi} u_{\xi\xi}].$$

From (3.78), it is calculated that (recalling $w_{ij} = u_{ij} - A_{ij}$)

$$\mathcal{L} w_{\xi\xi} \geq -F^{ij,kl} D_{\xi} w_{ij} D_{\xi} w_{kl} - C(F^{ii} + w_{ii} + w_{jj} F^{ii}),$$

for a further constant C . Next, (3.76) is differentiated to get

$$\begin{aligned}
 D_i v &= \frac{D_i w_{\xi\xi}}{w_{\xi\xi}} + \kappa D_i \tilde{\phi} \\
 D_{ij} v &= \frac{D_{ij} w_{\xi\xi}}{w_{\xi\xi}} - \frac{D_i w_{\xi\xi} D_j w_{\xi\xi}}{w_{\xi\xi}^2} + \kappa D_{ij} \tilde{\phi}. \tag{3.80}
 \end{aligned}$$

Given (3.76), it is clear that $\mathcal{L} \tilde{\phi} \geq F^{ii} \tilde{\delta}$. Using this fact along with (3.80), the following inequality emerges:

$$\begin{aligned}
 \mathcal{L} v &= \frac{\mathcal{L} w_{\xi\xi}}{w_{\xi\xi}} - F^{ij} \frac{D_i w_{\xi\xi} D_j w_{\xi\xi}}{w_{\xi\xi}^2} + \kappa \mathcal{L} \tilde{\phi} \\
 &\geq -\frac{1}{w_{\xi\xi}} F^{ij,kl} D_{\xi} w_{ij} D_{\xi} w_{kl} - F^{ij} \frac{D_i w_{\xi\xi} D_j w_{\xi\xi}}{w_{\xi\xi}^2} - \frac{C}{w_{\xi\xi}} (F^{ii} + w_{ii} + w_{jj} F^{ii}) \\
 &\quad + \kappa \tilde{\delta} F^{ii}. \tag{3.81}
 \end{aligned}$$

Next, it is supposed that v takes its maximum point at $x_0 \in \Omega^-$ in a direction corresponding to the vector ξ . Coordinates are then relabelled so that $\xi = e_1$. To proceed, the first two terms on the final line of (3.81) need to be estimated at this maximum point. Specifically,

it is claimed that

$$-\frac{1}{w_{11}}F^{ij,kl}D_1w_{ij}D_1w_{kl} - F^{ij}\frac{D_iw_{11}D_jw_{11}}{w_{11}^2} \geq 0; \quad (3.82)$$

that is,

$$-w_{11}F^{ij,kl}D_1w_{ij}D_1w_{kl} \geq F^{ij}D_iw_{11}D_jw_{11}. \quad (3.83)$$

To show (3.82), Lemma 3.4.1 and the concavity of f are first used to estimate the left-hand side of (3.83) from below:

$$-w_{11}F^{ij,kl}D_1w_{ij}D_1w_{kl} \geq -\lambda_1 \left[\frac{\partial^2 f}{\partial \lambda_1^2} |D_1w_{11}|^2 + 2 \sum_{r=2}^n \frac{f_1 - f_r}{\lambda_1 - \lambda_r} |D_rw_{11}|^2 \right]. \quad (3.84)$$

Using this, it is observed that (3.83) is true if

$$-\lambda_1 \left[\frac{\partial^2 f}{\partial \lambda_1^2} |D_1w_{11}|^2 + 2 \sum_{r=2}^n \frac{f_1 - f_r}{\lambda_1 - \lambda_r} |D_rw_{11}|^2 \right] \geq \sum_{r=1}^n f_r |D_rw_{11}|^2$$

holds, where coordinates have been rotated so that F^{ij} is diagonal. By breaking this inequality up term-by-term and dividing each resulting inequality by $|D_rw_{11}|^2$, one sees that if

$$-\lambda_1 \frac{\partial^2 f}{\partial \lambda_1^2} \geq f_1, \quad (3.85a)$$

$$-2\lambda_1 \frac{f_1 - f_r}{\lambda_1 - \lambda_r} \geq f_r, \quad \text{for } r \in \{2, \dots, n\} \quad (3.85b)$$

are shown to be true, then (3.84) and hence (3.81) will be proven. For the following calculations, coordinates are relabelled so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, keeping in mind that $\lambda_n \geq \mu_1 > 0$ for $\lambda \in \Gamma_{\mu_1, \mu_2}$. (3.85a) will first be verified. To begin, (3.26) is differentiated to ascertain

$$\begin{aligned} \frac{\partial^2 f}{\partial \lambda_l^2} &= \left(\frac{1}{n-l} \right) \left(\frac{1}{n-l} - 1 \right) \left(\frac{S_n}{S_l} \right)^{\frac{1}{n-l}-2} \left(\frac{S_l S_{n-1;i} - S_{l-1;i} S_n}{S_l^2} \right)^2 \\ &\quad + \left(\frac{2}{n-l} \right) \left(\frac{S_n}{S_l} \right)^{\frac{1}{n-l}-1} \left(\frac{S_{l-1;i}^2 S_{n-1;i} - S_l S_{l-1;i} S_{n-1;i}}{S_l^3} \right). \end{aligned}$$

Using this with (3.26) along with (3.27), it is calculated that

$$\begin{aligned} -\frac{\lambda_1}{f_1} \frac{\partial^2 f}{\partial \lambda_1^2} &= \frac{2\lambda_1 S_{l-1;1}}{S_l} - \left(\frac{1}{n-l} - 1 \right) \left(\frac{S_l S_{n-1;1} - S_n S_{l-1;1}}{S_n S_l} \right) \lambda_1 \\ &= \frac{2\lambda_1 S_{l-1;1}}{S_l} - \left(\frac{1}{n-l} - 1 \right) \left(\frac{S_{l;1}}{S_l} \right) \\ &= \frac{2\lambda_1 S_{l-1;1} + S_{l;1}}{S_l} - \frac{1}{n-l} \frac{S_{l;1}}{S_l} \\ &= 1 + \frac{1}{S_l} \left(\lambda_1 S_{l-1;1} - \frac{S_{l;1}}{n-l} \right). \end{aligned} \quad (3.86)$$

Next, the observation is made that

$$S_{l;1} \leq (n-l)\lambda_2 S_{l-1;1}, \quad l \in \{1, \dots, n-1\},$$

with equality holding when all λ_i equal and $l = n-1$. Thus, it is seen from (3.86) that

$$\begin{aligned} -\frac{\lambda_1}{f_1} \frac{\partial^2 f}{\partial \lambda_1^2} &\geq 1 + \frac{1}{S_l} \left(\lambda_1 S_{l-1;1} - \frac{(n-l)\lambda_2 S_{l-1;1}}{n-l} \right) \\ &= 1 + \frac{S_{l-1;1}}{S_l} (\lambda_1 - \lambda_2) \\ &\geq 1, \end{aligned}$$

which directly implies (3.85a) as $f_i > 0$ on Γ_{μ_1, μ_2} .

Next, (3.85b) is proven for arbitrary $r \in \{2, \dots, n\}$. Again, using (3.26) and (3.27), one finds the following:

$$\begin{aligned} \frac{\lambda_1 f_1}{\lambda_j f_j} &= \frac{\lambda_1 (S_{n-1;1} S_l - S_n S_{l-1;1})}{\lambda_j (S_{n-1;j} S_l - S_n S_{l-1;j})} \\ &= \frac{(S_n - S_{n;1}) S_l - (S_l - S_{l;1}) S_n}{(S_n - S_{n;j}) S_l - (S_l - S_{l;j}) S_n} \\ &= \frac{S_{l;1} S_n - S_{n;1} S_l}{S_{l;j} S_n - S_{n;j} S_l} \\ &= \frac{S_{l;1}}{S_{l;j}} \\ &= \frac{\lambda_j S_{l-1;1;j} + S_{l;1;j}}{\lambda_1 S_{l-1;1;j} + S_{l;1;j}}, \end{aligned} \tag{3.87}$$

where the fact that $S_{n;i} = 0$ for any i has been used to gain the fourth equality. From the current selection of coordinates, it is clear that the following inequalities hold:

$$\begin{aligned} 2\lambda_j^2 &\leq \lambda_1^2 + \lambda_1 \lambda_j, \\ 2\lambda_j &\leq \lambda_1 + \lambda_j. \end{aligned}$$

This combined with (3.87) yields

$$\begin{aligned} \left(\frac{2\lambda_j}{\lambda_1 + \lambda_j} \right) \frac{\lambda_1 f_1}{\lambda_j f_j} &\leq \frac{2\lambda_j^2 S_{l-1;1;j} + 2\lambda_j S_{l;1;j}}{(\lambda_1^2 + \lambda_1 \lambda_j) S_{l-1;1;j} + (\lambda_1 + \lambda_j) S_{l;1;j}} \\ &\leq 1, \end{aligned}$$

which subsequently implies (3.85b) via an elementary algebraic manipulation, using the fact that $f_i > 0$ on Γ_{μ_1, μ_2} .

In light of the above calculation, the claim in (3.82) has been proven. With this, (3.81)

is further reduced to find that

$$\begin{aligned}
 \mathcal{L}v &\geq -\frac{C}{w_{11}}(F^{ii} + w_{ii} + w_{ii}F^{ii}) + \kappa F^{ii} \\
 &\geq -C \left[1 + \frac{F^{ii}}{w_{11}} + F^{ii} \right] + \kappa F^{ii} \\
 &\geq -C \frac{F^{ii}}{w_{11}} + (\kappa - C)F^{ii},
 \end{aligned} \tag{3.88}$$

where the fact from Remark 3.2.1(2) that $F^{ii} > \sigma > 0$ on Γ_{0,μ_2} for any μ_2 has been used to gain the last inequality. On the other hand, since x_0 is a maximum of v , one has that

$$\begin{aligned}
 \mathcal{L}v(x_0) &= F^{ij}[D_{ij}v(x_0) - (D_{p_k}A_{ij})D_k v(x_0)] \\
 &= F^{ij}D_{ij}v(x_0) \\
 &\leq 0.
 \end{aligned}$$

This, combined with (3.88) yields

$$\frac{C}{\kappa - C} \geq w_{11}(x_0),$$

provided $\kappa > C$, where C depends on $\sup |u|$, $\sup |Du|$, A , $\sup |B|$, Ω^- and Ω^+ .

If v does not take a maximum at an interior point, then it must take a maximum on $\partial\Omega^-$. Thus completes the reduction of the C^2 estimate to the boundary.

Remark 3.5.5: In the case where A satisfies the strong (A3) condition, the above estimate is much simpler; and does not require the barrier condition (3.76). Specifically, one only needs to consider the simpler auxiliary function $v := w_{\xi\xi}$ in this case, with the definition of $\mathcal{L}$ remaining unaltered. Upon noting that the (A3) condition directly implies that

$$F^{ij}D_{p_k p_l}A_{ij}D_{k\xi}uD_{l\xi}u \geq CF^{ii}|D^2u|^2,$$

for some positive constant C , it is a straight-forward calculation that shows

$$\mathcal{L}v \geq F^{ii}(C_1|D^2u|^2 - C_2(|D^2u| + 1)).$$

From this, the second derivative estimate immediately falls out as $\mathcal{L}v \leq 0$.

With this remark, the following has now been proven:

Theorem 3.5.6: Let u be an elliptic solution to (3.1a) in Ω^- , with $x, u(x), Du(x) \in U$ for all $x \in \Omega^-$ and U bounded. If (3.76) is true, with $B(x, u)$ a positive, bounded function in $C^2(\overline{\Omega^-} \times \mathbb{R})$, with either

- A satisfying the weak (A3w) condition plus A and Ω^- satisfying the barrier condition stated in Subsection 3.2.4 or

- A satisfying the strong (A3) condition;

then one has the following estimate:

$$\sup_{\Omega^-} |D^2 u| \leq C \left(1 + \sup_{\partial\Omega^-} |D^2 u| \right), \quad (3.89)$$

where C depends on $\sup |u|$, $\sup |Du|$, A , $\sup |B|$, Ω^- and Ω^+ .

From these calculations, it is seen that the (A3w) criterion for A on the set U is indeed sufficient to make the argument go through. Also, it is clear that the calculation holds for the Quotient Transportation Equation (3.22a); thus, the calculations of this subsection provide a crucial step to the proof of Theorem 3.3.1.

Remark: It is possible to generalise Theorem 3.5.6 to the case where $B(x, z, p) > 0$ with

$$B_{p_k p_l}(x, z, p) \xi_k \xi_l > 0, \quad (3.90)$$

for all $(x, z, p) \in U$. To do this generalisation, (3.76) needs to be strengthened. Specifically, it is required that A and B be such that there exists a $\tilde{\phi}$ satisfying

$$[D_{ij}\tilde{\phi}(x) - D_{p_k}(A_{ij}(x, z, p) + B(x, z, p)) \cdot D_k\tilde{\phi}(x)] \xi_i \xi_j \geq \tilde{\delta} |\xi|^2,$$

for all $(x, z, p) \in U$. With that, the calculations proceed precisely as above with the linear operator $\mathcal{L}$ now being defined as

$$\mathcal{L}u := F^{ij} [D_{ij}u - (D_{p_k}A_{ij})D_k u] - (D_{p_k}B)D_k u.$$

This combined with Remark 3.5.2(3), indicates that the only way for B to have a dependence on the gradient of u and still allow for the existence of globally smooth solutions to the Quotient Transportation Equation, is if B satisfies (3.90) with Ω^- of sufficiently high normal curvature so that (3.67) holds.

3.5.4 Boundary estimates for second derivatives

In this subsection, the C^2 *a priori* bound is completed by proving second derivative bounds for solutions of the Quotient Transportation Equation on $\partial\Omega^-$. This treatment is similar to the one presented in [MTW05, LTU86, Urb98b], but requires some modification to accommodate the particular situation where $f = \sigma_{n,l}$. Specifically, Lemma 3.4.2 is required in order to derive differential inequalities from which barriers can be constructed. A key point to the following argument is that for oblique boundary conditions of the form (3.39), where the function G is uniformly convex in the gradient, the twice tangential differentiation of (3.39) yields quadratic terms in second derivatives which compensate for the deviation of $\beta = G_p$ from the geometric normal.

First, non-tangential second derivatives are treated. Taking $\Psi \in C^2(\overline{\Omega^-} \times \mathbb{R})$, the following definition is made:

$$v := \Psi(x, Du).$$

Defining the linear operator $\mathcal{L}$ by (3.52) and calculating as was done for (3.53), one observes that

$$\begin{aligned} |\mathcal{L}v| &\leq C_1 F^{ij} \delta_{kl} u_{ik} u_{jl} + C_2 (F^{ii} + 1), \\ &\leq C \left(1 + \sum_{i=1}^n f_i (\lambda_i^2 + 1) \right). \end{aligned} \quad (3.91)$$

In the last inequality coordinates have been rotated so that F^{ij} is diagonal, without loss of generality. Using Lemma 3.4.2 and defining

$$M := \sup_{\Omega^-} |D_x^2(u - c)|,$$

(3.91) is further reduced to

$$\begin{aligned} |\mathcal{L}v| &\leq (C(\epsilon) + \epsilon M F^{ii}) + C(1 + F^{ii}), \\ &\leq C(C(\epsilon) + \epsilon M F^{ii}) \\ &\leq (C(\epsilon) + \epsilon M) F^{ii}, \end{aligned}$$

where the fact that F^{ii} is bounded away from zero has been used along with a rescaling of ϵ by a factor of C , again without a loss of generality.

Using the construction of ϕ^+ from Subsection 3.5.4, Ψ is now set in the following manner:

$$\Psi(x, Du) = G(x, Du) = \phi^+ \circ T_u(x, Du). \quad (3.92)$$

(3.57) can be used to construct both an upper and lower barriers (using different choices of a and b), as (3.92) indicates $v = 0$ on $\partial\Omega^-$. By using the same barrier argument from the obliqueness estimate, the following boundary estimate is derived:

$$|DG| \leq (C(\epsilon) + \epsilon M), \quad \text{on } \partial\Omega^-,$$

for any $\epsilon > 0$. This, in turn, implies that

$$|D(\beta \cdot Du)| \leq C(\epsilon) + \epsilon M, \quad \text{on } \partial\Omega^-,$$

where it is recalled that $\beta := G_{p_k}$. With this, $u_{\beta\beta}$ is thus estimated on the boundary, which is equivalent to

$$w_{\beta\beta} \leq C(\epsilon) + \epsilon M, \quad \text{on } \partial\Omega^-. \quad (3.93)$$

Remark: By the strict obliqueness estimate (3.66), it is clear that

$$w_{\gamma\gamma} \leq C w_{\beta\beta}, \quad (3.94)$$

where γ is the outer unit normal to $\partial\Omega^-$.

To proceed, an explicit representation of any given vector ξ is written in terms of a tangential component, $\tau(\xi)$, and β :

$$\xi = \tau(\xi) + \frac{\xi \cdot \gamma}{\beta \cdot \gamma} \beta, \quad (3.95)$$

where

$$\tau(\xi) = \xi - (\xi \cdot \gamma) \gamma - \frac{\xi \cdot \gamma}{\beta \cdot \gamma} \beta^T,$$

and

$$\beta^T = \beta - (\beta \cdot \gamma) \gamma.$$

From this, it is calculated that

$$\begin{aligned} |\tau(\xi)|^2 &= 1 - 2 \frac{\beta^T \cdot \xi}{\beta \cdot \gamma} (\xi \cdot \gamma) - \left(1 - \frac{|\beta^T|^2}{(\beta \cdot \gamma)^2} \right) (\xi \cdot \gamma)^2 \\ &\leq 1 - 2 \frac{\beta^T \cdot \xi}{\beta \cdot \gamma} (\xi \cdot \gamma) + C (\xi \cdot \gamma)^2. \end{aligned}$$

It is now assumed that the maximal tangential second derivative of w over $\partial\Omega^-$ occurs at a point at $x_0 \in \partial\Omega^-$ in the direction which is taken to be e_1 . Denoting $\tau = \tau(e_1)$ and utilising (3.95), it is calculated that

$$w_{11} = w_{\tau\tau} + \frac{2\gamma_1}{\beta \cdot \gamma} w_{\tau\beta} + \frac{\gamma_1^2}{(\beta \cdot \gamma)^2} w_{\beta\beta}. \quad (3.96)$$

Next, the boundary condition

$$G(x, Du) = 0, \quad \text{on } \partial\Omega^-,$$

is differentiated in the tangential direction to find that

$$\begin{aligned} 0 &= G_{x_\tau} + \tau_i G_{p_k} u_{ik} \\ &= G_{x_\tau} + u_{\beta\tau}; \end{aligned} \quad (3.97)$$

that is,

$$\begin{aligned} w_{\beta\tau} &\leq -G_{x_\tau} + C \\ &\leq C. \end{aligned}$$

Subsequently, this combined with (3.96) and (3.93) yields

$$\begin{aligned} w_{11} &\leq |\tau|^2 w_{11}(x_0) + C \frac{\xi \cdot \gamma}{\beta \cdot \gamma} + (C(\epsilon) + \epsilon M) \gamma_1^2 \\ &\leq \left(1 - 2 \frac{\beta^T \cdot \xi}{\beta \cdot \gamma} (\xi \cdot \gamma) + C(\xi \cdot \gamma)^2 \right) w_{11}(x_0) + C \frac{\xi \cdot \gamma}{\beta \cdot \gamma} \\ &\quad + (C(\epsilon) + \epsilon M) \gamma_1^2. \end{aligned} \quad (3.98)$$

Now the term $D_\beta w_{11}(x_0)$ will be estimated. To do this, (3.1a) is differentiated twice to find that

$$\begin{aligned} F^{ij} D_{ij} w_{11} &= -F^{ij,kl} D_1 w_{ij} D_1 w_{kl} + B_{11} + 2(B_z u_{11} + B_{1z} D_1 u) + B_{zz} (D_1 u)^2 \\ &\geq C, \end{aligned}$$

where the concavity of F has been used along with the assumed condition of $B_z \geq 0$. Using this and the barrier construction in [GT01, Corollary 14.5], with a linear operator defined by

$$\mathcal{L}u := F^{ij} D_{ij} u,$$

it is surmised that

$$-D_\beta w_{11}(x_0) \leq C. \quad (3.99)$$

Next, (3.97) is differentiated again in the tangential direction to get

$$G_{p_k p_l} u_{k\tau} u_{l\tau} + G_{p_k} u_{k\tau\tau} + G_{x_\tau x_\tau} + 2G_{x_\tau p_k} u_{k\tau} = 0.$$

This combined with (3.99) and (3.54) indicates that at x_0 one has that

$$C_0 u_{11}^2(x_0) - C_1 u_{11}(x_0) \leq C_3,$$

for positive constants C_0 , C_1 and C_2 . That is,

$$w_{11}(x_0) \leq C$$

Using this relation in (3.98) thus yields

$$w_{11} \leq C(\epsilon) + \epsilon M, \quad \text{on } \partial\Omega^-,$$

for new constant $C(\epsilon)$ and a rescaling of ϵ .

Combining this last relation with (3.93) and (3.94) allows one to bound the second derivatives of u on the boundary of Ω^- in the following manner:

$$\sup_{\partial\Omega^-} |D^2 u| \leq C(\epsilon) + \epsilon M.$$

Now utilising (3.89) to eliminate M from the above inequality finally yields

$$\sup_{\partial\Omega^-} |D^2u| \leq \frac{C(\epsilon)}{1 - \epsilon C},$$

for a new C , $C(\epsilon)$ and ϵ scaling. Fixing $0 < \epsilon < \frac{1}{C}$, the following has been proven:

Theorem 3.5.7: *Let c be a cost-function satisfying hypotheses (A1), (A2), (A3w) with respect to bounded C^4 domains $\Omega^-, \Omega^+ \in \mathbb{R}^n$ which are respectively uniformly c -convex, c^* -convex with respect to each other. Let B be a strictly positive function in $C^2(\overline{\Omega^-} \times \mathbb{R} \times \mathbb{R}^n)$ satisfying (3.19) and (3.20w). Then any elliptic solution $u \in C^3(\overline{\Omega^-})$ of the second boundary value problem (3.22a), (3.22b) satisfies the a priori estimate*

$$\sup_{\partial\Omega^-} |D^2u| \leq C, \quad (3.100)$$

where C depends on c, B, Ω^-, Ω^+ and $\sup_{\Omega^-} |u|$.

This combined with Theorem 3.5.6 proves the main result Theorem 3.3.1. Once the second derivatives are bound, (3.22a) is effectively uniformly elliptic. This combined with the obliqueness estimate yields global $C^{2,\alpha}$ estimates for solution of (3.22a)–(3.22b), from the theory of oblique boundary value problems for uniformly elliptic equations presented in [LT86]. Moreover, by the theory of linear elliptic equations with oblique boundary conditions (see [GT01]) and the assumed smoothness of data, one also has $C^{3,\alpha}(\overline{\Omega^-})$ bounds for elliptic solutions, for any $\alpha < 1$.

3.6 Method of Continuity

To complete the proof of Theorem 3.3.2, the standard method of continuity for nonlinear oblique boundary value problems is used, which is presented in [GT01, Section 17.9] and subsequently applied in [Urb97] and [Urb01]. This procedure was modified in [TW06] in order to be applied to the Optimal Transportation Equation; and it is this method that is recalled here with trivial modification. Specifically, two procedures for applying the method of continuity were used in [TW06]. The first method discussed there utilised foliations of both Ω^- and Ω^+ to construct a family of suitable boundary value problems; this procedure will not be used here. The second method proved the existence of a function approximately satisfying the associated boundary condition, which subsequently allowed for the method of continuity to be applied without a domain variation construction. This is the procedure that will be recalled here (for the sake of completeness) with only trivial modification, as the Quotient Transportation Equation is constructed off the archetype of Optimal Transportation (see Subsection 3.2.2).

To start off, a key lemma from [TW06] is recalled without proof:

Lemma 3.6.1 [TW06]: *Let the domains Ω^- and Ω^+ and cost-function c satisfy the hypothesis of Theorem 3.3.1. Then for any $\epsilon > 0$, there exists a uniformly c^* -convex approximating domain, Ω_ϵ^+ , lying within distance ϵ of Ω^+ , and satisfying the corresponding condition (3.17) for fixed δ_0^+ , together with a function $u_0 \in C^4(\overline{\Omega})$ satisfying the ellipticity condition (3.5), for $f = \sigma_{n,l}$ and the boundary condition (3.22b) for Ω_ϵ^+ .*

Remark: *As $\Gamma^*(S_n) = \Gamma^*(\sigma_{n,l})$, the above lemma carries over trivially to the case of the Quotient Transportation Equation.*

With this in mind and making the denotation

$$F[u] := \left(\frac{S_n}{S_l} \right)^{\frac{1}{n-1}} [D^2 u - D_x^2 c(\cdot, T_u)], \quad (3.101)$$

the following family of boundary value problems are now defined:

$$F[u_t] = tB(\cdot, u_t) + (1-t)e^{u_t-u_0}F[u_0], \quad \text{in } \Omega_t^- \quad (3.102a)$$

$$T_{u_t}(\Omega^-) = \Omega_t^+, \quad (3.102b)$$

for $t \in [0, 1]$ with u_0 taken as the function indicated in Lemma 3.6.1. It is clear that u_0 is the unique elliptic solution of (3.102a)–(3.102b) at $t = 0$. In this family of equations, Ω_t^+ is such that $\Omega_0^+ = \Omega_\epsilon^+$ (corresponding to u_0 as depicted in Lemma 3.6.1) and $\Omega_1^+ = \Omega^+$. Given the assumed smoothness of c and the definition of T_u depicted in (3.13), an $\epsilon > 0$ can be chosen in Lemma 3.6.1 small enough to guarantee that Ω_t^- and Ω_t^+ are uniformly c -convex and c^* -convex (respectively) relative to one another with corresponding uniform convexity constants independent of t , as Ω^- and Ω^+ are assumed to be uniformly c -convex and c^* -convex (respectively) relative to one another.

From Subsection 3.5.1, it is understood that the boundary condition (3.102b) is equivalent to the oblique condition

$$G_t(\cdot, Du) := \phi_t^+(T_u(\cdot)) = 0, \quad \text{on } \partial\Omega_t^-,$$

where ϕ_t^+ are defined for Ω_t^+ analogous to the construction of ϕ^+ for Ω^+ in Subsection 3.5.1. From the observations in the previous paragraph, it is clear that the family of boundary value problems correlate to uniformly oblique boundary value problems with a uniform constant of obliqueness independent of t .

To adapt the method of continuity from [GT01, Section 17.9], an $\alpha \in (0, 1)$ is fixed with Σ set to denote the subset of $[0, 1]$ for which $t \in \Sigma$ implies the problem (3.102a)–(3.102b) is solvable for an elliptic solution $u_t \in C^{2,\alpha}(\overline{\Omega_t^-})$, with T_{u_t} invertable. It is clear that the boundary condition (3.102b) implies a uniform bound for Du_t with respect to t . Upon noting that the inhomogeneity of (3.102a) is uniformly bounded in t and satisfies (3.18), (3.19) and

(3.20), uniform estimates in $C^{2,1}(\overline{\Omega^+})$ immediately follow, as all the solution estimates of Section 3.5 are clearly independent of $t \in [0, 1]$. By compactness, it is then inferred that Σ is closed via the Heine-Borel Theorem. To show Σ is open, the implicit function theorem is used along with the linear theory of oblique boundary value problems, as in [GT01, Chapter 17]. As Σ is open, closed and non-trivial, it is known that $\Sigma = [0, 1]$; that is, there exists a unique elliptic solution $u \in C^3(\overline{\Omega^-})$ of the boundary value problem

$$F[u] = B(\cdot, u), \quad \text{in } \Omega^- \quad (3.103a)$$

$$T_u(\Omega^-) = \Omega^+. \quad (3.103b)$$

Thus, the part of Theorem 3.3.2 corresponding to unique solutions is proven.

Remark: (3.20) guarantees uniqueness of the linearised boundary value problem via a straightforward application of the Hopf boundary point lemma (see [GT01, Lemma 3.4]). Without the (3.20) condition, one can not apply the method of continuity directly, as solutions will not be unique in general in this scenario. This will be discussed further in the next subsection.

3.6.1 Application of the Leray-Schauder Fixed Point Theorem

To conclude this section, the Leray-Schauder fixed point theorem (see [GT01, Theorem 11.6]) will be used to relax the monotonicity criterion on the inhomogeneity required by the method of continuity. The Leray-Schauder theorem has been used to similar ends in [LT86], [Urb95] and [Urb01].

Using the notation in (3.101), u_0 is defined to be the unique admissible solution of

$$F[u_0] = e^{u_0}, \quad \text{in } \Omega^-,$$

$$T_{u_0}(\Omega^-) = \Omega^+.$$

Theorem 3.3.2 indicates that such a unique u_0 exists. Moreover, by elliptic regularity theory, $u_0 \in C^\infty(\overline{\Omega^-})$. For $t \in [0, 1]$ and $\psi \in C^3(\overline{\Omega^-})$, the following family of problems are considered:

$$F[u_t] = t(\tilde{B}(\cdot, u_0 + \psi) + e^{u_t - u_0 - \psi} - 1) + (1 - t)e^{u_t}, \quad \text{in } \Omega^- \quad (3.105a)$$

$$T_{u_t}(\Omega^-) = \Omega^+, \quad (3.105b)$$

where $\tilde{B}$ is only assumed to satisfy (3.18), (3.19) and (3.20w). By Theorem 3.3.2 and elliptic regularity theory, for each ψ and t (3.105a)–(3.105b) has a unique admissible solution $u_t \in C^{3,\alpha}(\overline{\Omega^-})$ for any $\alpha < 1$. Consequently, the map $\mathcal{T} : C^3(\overline{\Omega^-}) \times [0, 1] \rightarrow C^3(\overline{\Omega^-})$ defined by $\mathcal{T}(\psi, t) = u_t - u_0$ is continuous and compact; and $\mathcal{T}(\cdot, 0) = 0$ for all $\psi \in C^3(\overline{\Omega^-})$. If it can also be shown for $t \in [0, 1]$, that for all the fixed points of $\mathcal{T}(\cdot, t)$ — that is, for any

admissible solution u_t of

$$F[u_t] = t\tilde{B}(\cdot, u_t) + (1-t)e^{u_t}, \quad \text{in } \Omega^- \quad (3.106a)$$

$$T_{u_t}(\Omega^-) = \Omega^+ \quad (3.106b)$$

— the estimate

$$\|u_t\|_{C^3(\overline{\Omega^-})} \leq C, \quad (3.107)$$

is satisfied with C independent of t , then by the Leray-Schauder fixed point theorem, $\mathcal{T}(\cdot, 1)$ has a fixed point. Subsequently, this is equivalent to

$$F[u] = \tilde{B}(\cdot, u), \quad \text{in } \Omega^-, \quad (3.108a)$$

$$T_u(\Omega^-) = \Omega^+, \quad (3.108b)$$

having an admissible solution u belonging to $C^3(\overline{\Omega^-})$. As the solution estimates from Section 3.5 clearly apply to (3.106a)–(3.106b) independent of t , results from [LT86] and linear theory for oblique elliptic boundary value problems can be applied to deduce (3.107). Subsequently, the monotonicity criterion on B may be relaxed to $B_z \geq 0$ relative to the previous existence criterion, thus finishing the proof of Theorem 3.3.2.

Remarks:

- (1) *As mentioned before, the admissible solution to (3.108a)–(3.108b) may not be unique as $B_z > 0$ is needed for the application of the Hopf Boundary point lemma.*
- (2) *It is recalled that the condition of $B_z \geq 0$ is used in the boundary C^2 estimate in Subsection 3.5.4, and thus, is still a required criterion, as estimates on solution of (3.106a)–(3.106b) need to hold independent of t .*

3.7 Conclusions

While (3.22a)–(3.22b) differ significantly from the Monge-Ampère equations coming from Optimal Transportation, the theory regarding their regularity is very similar. Specifically, the (A3w) represents both a necessary and sufficient condition for higher regularity of (3.22a)–(3.22b). The calculations in Subsection 3.5.3 indicate that the (A3w) condition is sufficient for Theorem 3.3.2. Given that the ellipticity criterion for (3.22a)–(3.22b) corresponds to solutions being c -convex, the counter-example that Loeper presented in [Loe05] still applies in the current situation; and thus, (A3w) is also a necessary criterion for our higher regularity theory.

Unfortunately the Quotient Transportation Equation differs from the Optimal Transportation significantly, in that the restrictions on the inhomogeneity, B , prevent a formulation of

a generalised solution of (3.22a)–(3.22b) from being readily derived. Moreover, the Quotient Transportation Equation is not the direct result of either a variational or change of variable principle; and so, it is difficult to gain even a heuristic idea as to how to formulate a generalised solution for this equation. It may be possible that a generalised solution to (3.22a)–(3.22b) will be realized via T_u satisfying some geometric criterion, beyond that of it being simply a local diffeomorphism; but this is unclear as of the writing of this thesis.

From these observations, it seems that the only three possible courses for further study of (3.22a)–(3.22b) will either come from verifying the (A3w) condition for various cost functions, searching for ways to articulate the notion of a generalised solution, or finding a way to remove the technical barrier condition stated in Subsection 3.2.4.

4

The (A3) condition on $\mathbb{S}^n$ with constant curvature metrics.

Contents

4.1	Introduction	72
4.2	Main Results	73
4.3	Analysis of the (A3) Condition on $\mathbb{S}^n$	74
4.4	Examples	81
4.5	Conclusions	94

4.1 Introduction

As stated earlier in Section 2.8, the (A3) condition introduced in [MTW05] is needed in order to prove the classical interior regularity of the potential functions corresponding to solutions of the Optimal Transportation Problem. Subsequently, in [TW06] the degenerate form of the (A3) condition, (A3w), was proven to be tantamount to the existence of globally smooth potentials solving the Optimal Transportation Equation.

The (A3w) condition was recently realized to be more fundamental to the regularity of potential functions, as Loeper showed in [Loe05] that the (A3w) condition was not only sufficient but also *necessary* for global regularity of Optimal Transportation potentials. In that paper, Loeper showed that if (A3w) is violated, one can build a pair of C^∞ , strictly positive measures, supported on sets with the usual smoothness and convexity assumptions, so that the optimal potential is not even C^1 ; and thus, the corresponding optimal map is discontinuous. Before this work, it was not known whether the (A3w) condition was truly

fundamental for potential function regularity or if it was simply a technical condition to make the *a priori* estimates in [MTW05] and [TW06] work.

In addition to showing the (A3w) condition to be sharp, in [Loe05] Loeper also studied the correlation between curvature and the Optimal Transportation Problem on Riemannian manifolds when the cost-function was of the form

$$c(x, y) = \frac{d^2(x, y)}{2}, \quad (4.1)$$

where $d(x, y)$ represents the geodesic distance between points $x, y \in \mathbb{M}^n$. One of the interesting aspects of this research was that Loeper verified the theory therein (in the case where $\mathbb{M}^n$ was taken to be a round sphere) by explicitly calculating that the cost-function depicted by (4.1) satisfies the *non*-degenerate form of the (A3) condition on round spheres. Specifically, it was shown that

$$D_{p_l p_k}^2 c_{ij}(x, y) \xi_i \xi_j \eta_k \eta_l \rightarrow -\frac{2}{3}, \quad \text{as } x \rightarrow y, \quad (x, y) \in \Omega^- \times \Omega^+ \subset \mathbb{S}^n. \quad (4.2)$$

This is particularly interesting as the cost-function in (4.1) only satisfies the (A3w) condition in the Euclidean case but not the stronger (A3) condition (see [MTW05, TW06]).

The disadvantage of Loeper's calculation in [Loe05] is that it relies heavily upon the specific geometric properties of the specific cost-function $c(x, y) = \frac{1}{2}d^2(x, y)$. In this chapter, this calculation will be generalised on $\mathbb{S}^n$ to include cost-functions which are arbitrary functions of the geodesic distance with respect to the constant curvature metric on $\mathbb{S}^n$.

Remark: As mentioned in Section 2.7, the notational convention embodied in (2.44a)–(2.44b) will be used in this chapter, as it correlates to the current literature that pertain to the verification of the (A3) condition in various scenarios. Hence, the left-hand side of (4.2) being less than zero corresponds to the (A3w) being satisfied in this last part of the thesis.

4.2 Main Results

The main results of this chapter centre around the verification of the (A3) criterion for various cost-functions (on round spheres) having the general form depicted in (4.3) below. The most important example thus considered is when $c(x, y) = \frac{1}{2}d^2(x, y)$. In this particular case, the following theorem (originally proven by Loeper in [Loe05]) is verified:

Theorem 4.2.1 [Loe05]: Given $\mathbb{S}^n$ be an embedded sphere in $\mathbb{R}^{n+1}$ with arbitrary radius, equipped with the round metric with an associated Riemannian geodesic distance d , if $c(x, y) = \frac{1}{2}d^2(x, y)$, then one has that c satisfies the strong (A3) condition on $\mathbb{S}^{n+1} \times \mathbb{S}^{n+1} \setminus \{(x, x) : x \in \mathbb{S}^n\}$.

The results in this chapter extend the calculations in [Loe05]. Indeed, the precise constant for which the (A3) condition is satisfied, for the situation where $c(x, y) = \frac{1}{2}d^2(x, y)$, will be calculated for round spheres of arbitrary radius. Loeper calculated this constant for the case where $R = 1$; and the results in this chapter indeed verify this.

Closely related to $c(x, y) = \frac{1}{2}d^2(x, y)$, is the cost-function of $c(x, y) = 2R^2 \sin^2\left(\frac{d(x, y)}{2R}\right)$. This scenario corresponds to the situation where d is taken to be the chordal distance between two points in $\mathbb{S}^{n+1}$ relative to the ambient Euclidean space, into which the round sphere is embedded. For this case, the following analogous result to Theorem 4.2.1 will be proven:

Theorem 4.2.2: *Given $\mathbb{S}^n$ an embedded sphere in $\mathbb{R}^{n+1}$ with arbitrary radius, equipped with the round metric with an associated Riemannian geodesic distance d , if $c(x, y) = 2R^2 \sin^2\left(\frac{d(x, y)}{2R}\right)$, then one has that c satisfies the strong (A3) condition on $\mathbb{S}^n \times \mathbb{S}^n \setminus \{(x, x) \mid x \in \mathbb{S}^n\}$.*

In addition to the two above situations, a few other examples of cost-functions are studied later in this chapter.

It should be noted at this point that a cost-function satisfying the (A3) condition is not generally enough to guarantee higher regularity for Optimal Transportation potentials. Indeed, the Optimal Transportation Equation becomes singular for $y \in \text{cut}(x)$, for any cost-function depending on the geodesic distance between points x and y (specifically, $|\text{Det}[D_{x,y}^2 c]|$ blows up). Thus, in order to have potential function regularity, bounds on transport vectors, that prevent optimal transport maps from mapping points to their cut-locus, are also needed in addition to the (A3) verification. This geometric criterion will have implications in the forthcoming analysis, which will be discussed in Section 4.3.

4.3 Analysis of the (A3) Condition on $\mathbb{S}^n$

The following formulation relies on several key observations and simplifications that are unique in the specific case of analysing the (A3) condition on round spheres. First, Chapter 4 indicates that the derivation of the Optimal Transportation Problem (and hence the Optimal Transportation Equation) only depends on the measure-space structure associated with Ω^- and Ω^+ . Indeed, the only way geometry can come into the formulation of the Optimal Transportation Problem is if the cost-function is defined as having some explicit geometric dependence. The cost-functions analysed in this chapter all depend on the geodesic distance between points on $\mathbb{S}^n$, equipped with a round Riemannian metric. Thus, the underlying goal of the following calculation is to derive an explicit expression for the geodesic distance between two arbitrary, fixed points x and y lying on a round sphere. As will be discussed later, the round sphere is one of the very few manifolds (to the author's knowledge) where the

geodesic distance between two arbitrary points can be explicitly represented. The rest of the simplifying observations will be stated as needed in the following formulation.

Ultimately, the analysis of the (A3) condition will be carried out in $\mathbb{R}^n$ (that is, in a local chart) with an associated, modified cost-function that yields an equivalent Optimal Transportation Problem, as compared to the one originally defined on $\mathbb{S}^n$. As mentioned earlier, it can not be assumed that Optimal Transportation maps do not move points to their cut-locus. Thus, in order to analyse the Optimal Transportation Problem on a manifold, via a single local chart, requires another geometric criterion or observation. If bounds on transport vectors exists such that optimal maps stay away from the cut-locus of particular points, then no additional criterion need be placed on the problem. As a broad class of cost-functions will be analysed in this chapter, no such bounds have been proven for all the examples thus contained. Therefore, it will be assumed that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$, in addition to the standard Optimal Transportation hypotheses placed on Ω^- and Ω^+ (see Chapter 5), unless stated otherwise.

Since only cost-functions will be considered that have the form

$$c(x, y) = f(d(x, y)), \quad (4.3)$$

where $d(x, y)$ represents the geodesic distance between x and y on the round sphere, it is possible to analyse the Optimal Transportation Problem in a local chart with a modified cost-function whose associated Optimal Transportation Problem in Euclidean space is equivalent to that of (4.3) on $\mathbb{S}^n$. Such a local chart and modified cost-function are formulated via a stereographic projection of $\mathbb{S}^n$ embedded in $\mathbb{R}^{n+1}$ onto the some arbitrary tangent space of $\mathbb{S}^n$. With this projection, an explicit expression for $d(x, y)$ can be derived in terms of $\hat{x}$ and $\hat{y}$: the projected coordinates on an arbitrary tangent space of $\mathbb{S}^n$.

4.3.1 Stereographic Projection

Given that it is assumed that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$, there will be no geometric issues with the forthcoming calculations being carried out in a single chart of $\mathbb{S}^n$. The possibility of relaxing this criterion will be discussed at the end of this chapter, in Section 4.5. As the following stereographic formulation is tantamount to analysing the (A3) condition through a specific coordinate transformation, it is recalled that the (A3) condition is invariant under coordinate transformation according to the calculations in Subsection 2.4.1; and thus, the result of the following calculations will hold in general.

4.3.1.1 The Half-Sphere Stereographic Projection

To begin, the modified form of (4.3) is first derived using the stereographic projection on the half-sphere depicted in Figure 4.1. Following this derivation, it will then be described

how half-sphere formulation is equivalent to the full-sphere projection with the underlying assumption that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$.

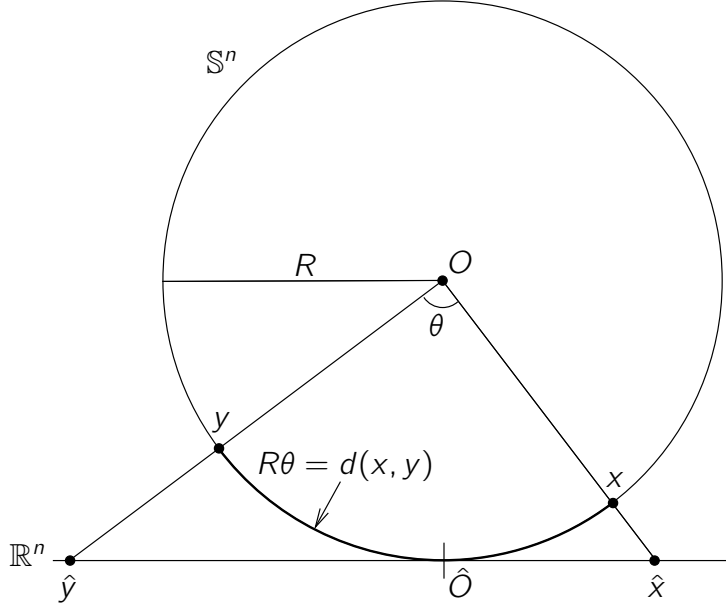


Figure 4.1: Stereographic projection for the half-sphere

Utilising the ambient Euclidean geometry of $\mathbb{R}^{n+1}$, it is an elementary calculation that yields

$$\begin{aligned} d(x, y) &= R \cdot \theta \\ &= R \cdot \arccos \left(\frac{R^2 + \hat{x} \cdot \hat{y}}{\sqrt{R^2 + |\hat{x}|^2} \sqrt{R^2 + |\hat{y}|^2}} \right), \end{aligned} \quad (4.4)$$

where the origin of $\mathbb{R}^{n+1}$ is set to coincide with the centre of the sphere without any loss of generality.

From this calculation, the analysis of (4.3) on $\mathbb{S}^n$ is reduced to studying the Optimal Transportation Problem associated with the cost-function

$$c(\hat{x}, \hat{y}) = f \left(R \cdot \arccos \left(\frac{R^2 + \hat{x} \cdot \hat{y}}{\sqrt{R^2 + |\hat{x}|^2} \sqrt{R^2 + |\hat{y}|^2}} \right) \right) \quad (4.5)$$

between $\widehat{\Omega}^-$ and $\widehat{\Omega}^+$ on our local chart; and thus, (4.5) will now be analysed in the Euclidean setting.

Remark: Even though the ambient Euclidean geometry of $\mathbb{R}^{n+1}$ is used to derive (4.4), having the sphere embedded in $\mathbb{R}^{n+1}$ is not technically required nor is the association of the local chart with a particular tangent plane of $\mathbb{S}^n$. Indeed, these notions are intuitive conveniences, in the sense that the equivalence of the Euclidean formulation of the Optimal Transportation Problem, to the original problem on $\mathbb{S}^n$, is immediate by this explicit stereographic projection.

A key point to the above calculation is that there is a simple relationship between θ and the geodesic distance between two *arbitrary* points on $\mathbb{S}^n$. If one were to project rays from a point on $\mathbb{S}^n$ instead of its centre (as in the case of a full-sphere stereographic projection), there would be no simple relation between the angle of the projected rays and the associated geodesic distance between two arbitrary points. The strength of (4.4) is that it is independent of the particular choice of tangent plane on which the stereographic projection is performed.

To get an explicit representation corresponding to a full-sphere stereographic projection, one would need to specify a particular tangent plane to project onto (specifically, either $T_x(\mathbb{S}^n)$ or $T_y(\mathbb{S}^n)$). However, in this situation the result corresponding to (4.4) can not be differentiated as the relation will vary for perturbations of x or y . This is the central obstruction that prohibits this method from being applied to the full-sphere case; but as $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$ is already assumed, any two points x and y can be contained in a half-sphere. This combined with the fact that the above calculation is independent of which half-sphere is actually projected, shows that the current situation with the half-sphere projection is equivalent to the full-sphere case with $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$.

4.3.1.2 Stereographic Reformulation of the (A3) Condition

With the relation (4.5), it is now possible to formulate a new expression for the (A3) criterion in the current scenario. To do this, the geodesic distance on $\mathbb{S}^n$ is first differentiated with respect to the projected Euclidean coordinate:

$$\begin{aligned} d_{\hat{x}_i}(x, y) &= \frac{d}{d\hat{x}_i} \left[R \cdot \arccos \left(\frac{R^2 + \hat{x} \cdot \hat{y}}{\sqrt{R^2 + |\hat{x}|^2} \sqrt{R^2 + |\hat{y}|^2}} \right) \right] \\ &= \frac{-R}{\sqrt{(R^2 + |\hat{x}|^2)(R^2 + |\hat{y}|^2) - (R^2 + \hat{x} \cdot \hat{y})^2}} \left(\hat{y}_i - \frac{(R^2 + \hat{x} \cdot \hat{y})\hat{x}_i}{R^2 + |\hat{x}|^2} \right). \end{aligned}$$

Without loss of generality, $\hat{x}$ is defined to be the origin of the local chart which reduces the above to

$$d_{\hat{x}_i}(x, y) = \frac{-\hat{y}_i}{|\hat{y}|}. \quad (4.6)$$

Differentiating again with respect to the projected, Euclidean coordinates yields

$$\begin{aligned} d_{\hat{x}_i \hat{x}_j}(x, y) &= \frac{R [(R^2 + |\hat{y}|^2)\hat{x}_j - (R^2 + \hat{x} \cdot \hat{y})\hat{y}_j]}{[(R^2 + |\hat{x}|^2)(R^2 + |\hat{y}|^2) - (R^2 + \hat{x} \cdot \hat{y})^2]^{3/2}} \left(\hat{y}_i - \frac{(R^2 + \hat{x} \cdot \hat{y})\hat{x}_i}{R^2 + |\hat{x}|^2} \right) \\ &\quad + \frac{R}{(R^2 + |\hat{x}|^2) \sqrt{(R^2 + |\hat{x}|^2)(R^2 + |\hat{y}|^2) - (R^2 + \hat{x} \cdot \hat{y})^2}} \\ &\quad \cdot \left(\hat{x}_i \hat{y}_j + (R^2 + \hat{x} \cdot \hat{y}) \delta_{ij} - 2 \frac{(R^2 + \hat{x} \cdot \hat{y})\hat{x}_i \hat{x}_j}{R^2 + |\hat{x}|^2} \right). \end{aligned}$$

Again, choosing $\hat{x}$ to coincide with the origin of the local chart reduces the above to

$$d_{\hat{x}_i \hat{x}_j}(x, y) = -\frac{\hat{y}_i \hat{y}_j}{|\hat{y}|^3} + \frac{\delta_{ij}}{|\hat{y}|}. \quad (4.7)$$

At this point, it has been shown that,

$$c_{\hat{x}_i \hat{x}_j} = \frac{\hat{y}_i \hat{y}_j}{|\hat{y}|^2} \left(f''(d) - \frac{f'(d)}{|\hat{y}|} \right) + \delta_{ij} \frac{f'(d)}{|\hat{y}|}, \quad (4.8)$$

where $\hat{x} = 0$ has been assumed without loss of generality (since the choice was made *after* differentiation).

To proceed, both $\hat{y}$ and d need to be represented in terms of the transportation vector, $\vec{p}$, which is recalled from Chapter 5 to be defined in the formulation of the Optimal Transportation Problem by

$$\vec{p} := \nabla_{\hat{x}} c(\hat{x}, \hat{y}),$$

where $c(\hat{x}, \hat{y})$ is recalled from (4.5). Note that the vector notation on p will be suppressed from this point forward without loss of clarity. From this expression and (4.6), it is ascertained that

$$p_i = -f'(d) \frac{\hat{y}_i}{|\hat{y}|};$$

that is,

$$|p| = |f'(d)|, \quad \text{and} \quad \frac{p_i}{|p|} = -\frac{\hat{y}_i}{|\hat{y}|}. \quad (4.9)$$

On the other hand, it is clear from Figure 4.1 that

$$R \tan \left(\frac{d}{R} \right) = |\hat{y}|. \quad (4.10)$$

Using both (4.9) and (4.10), one may rewrite (4.7) as

$$c_{\hat{x}_i \hat{x}_j}(p, d) = \frac{p_i p_j}{|p|^2} \left(f''(d) - \frac{f'(d)}{R \tan(R^{-1}d)} \right) + \delta_{ij} \frac{f'(d)}{R \tan(R^{-1}d)}. \quad (4.11)$$

Defining the origin of the local chart to coincide with $\hat{x}$ effectively shifts all p dependence onto the $\hat{y}$; and thus, differentiations with respect to p variables may subsequently be applied directly to (4.11) with the understanding $\hat{x} = 0$. From (4.9) it is seen that d does indeed have p dependence; the differentiation of the d variable with respect to p will now be elaborated upon.

To differentiate d , the first expression in (4.9) is rewritten as

$$d = f'^{(-1)}(\text{sgn}(f')|p|).$$

Differentiating implicitly, it is readily calculated that

$$d_{p_i} = \frac{\text{sgn}(f') p_i}{f''|p|}, \quad (4.12)$$

where the argument on f has been suppressed without loss of clarity.

Using (4.12), (4.11) is now differentiated twice with respect to p to yield

$$\begin{aligned}
 D_{p_i p_k}^2 c_{\hat{x}_i \hat{x}_j}(p, d) = & \left(\delta_{ik} \delta_{jl} - 2 \frac{\delta_{ik} p_j p_l}{|p|^2} + \delta_{il} \delta_{jk} - 2 \frac{\delta_{jk} p_i p_l}{|p|^2} - 2 \frac{\delta_{il} p_j p_k}{|p|^2} - 2 \frac{\delta_{jl} p_i p_k}{|p|^2} \right. \\
 & \left. - 2 \frac{\delta_{kl} p_i p_j}{|p|^2} + 8 \frac{p_i p_j p_k p_l}{|p|^4} \right) \left(\frac{f''}{f'^2} - \frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} \right) + \left(\frac{\delta_{ik} p_j p_l}{|p|^2} + \frac{\delta_{jk} p_i p_l}{|p|^2} \right. \\
 & \left. + \frac{\delta_{il} p_j p_k}{|p|^2} + \frac{\delta_{jl} p_i p_k}{|p|^2} + \frac{\delta_{kl}}{|p|^2} - 5 \frac{p_i p_j p_k p_l}{|p|^4} \right) \left(\frac{f'''}{f' f''} - \frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} \right. \\
 & \left. + \frac{1}{R^2 f'' \sin^2(R^{-1}d)} \right) + \frac{p_i p_j p_k p_l}{|p|^4} \left(\frac{f''''}{f'^2} - \frac{f'''^2}{f'^3} - 2 \frac{f' \cos(R^{-1}d)}{R^3 f''^2 \sin^3(R^{-1}d)} \right. \\
 & \left. + 2 \frac{1}{R^2 f'' \sin^2(R^{-1}d)} - \frac{f' f'''}{R^2 f''^3 \sin^2(R^{-1}d)} \right) + \left(\delta_{ij} \delta_{kl} - \frac{\delta_{ij} p_k p_l}{|p|^2} \right) \\
 & \cdot \left(\frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} - \frac{1}{R^2 f'' \sin^2(R^{-1}d)} \right) + \frac{\delta_{ij} p_k p_l}{|p|^2} \left(2 \frac{f' \cos(R^{-1}d)}{R^3 f''^2 \sin^3(R^{-1}d)} \right. \\
 & \left. - 2 \frac{1}{R^2 f'' \sin^2(R^{-1}d)} + \frac{f' f'''}{R^2 f''^3 \sin^2(R^{-1}d)} \right). \tag{4.13}
 \end{aligned}$$

Remarks:

- (1) It is readily observed that all terms involving $\text{sgn}(f')$ have been cancelled out in (4.13). Thus, the singularities in (4.13) corresponding to (4.12) being undefined where $f'(d) = 0$ are removable as it is an underlying assumption of this thesis that our cost-function is C^4 . Nonetheless, the situation where $f'(d) = 0$ can not happen, as the (A1) condition is assumed on c .
- (2) (4.13) gives the explicit, radial-scale dependency of the (A3) term in the current scenario. This has the interesting implication that it is possible to design cost-functions of the geodesic distance corresponding to the round sphere, so that for certain radii, the cost is strictly (A3) and for radii, it is not (A3) at all. This radial scale dependence is demonstrated explicitly in some of the forthcoming examples in Section 4.4.
- (3) The implicit differentiation in (4.12) is the key to gaining an explicit representation of the (A3) term for the general cost-function $f(d(x, y))$. Avoiding the use of (4.12) in favour of an expression that is well defined everywhere forces the (A3) term being expressed in terms of f'^{-1} , which clearly will not have an explicit form for general f .

Now, the final form of the (A3) condition (associated with the general cost-function $f(d(x, y))$) is able to be written. Taking two arbitrary, unit vectors ξ and η such that $\xi \perp \eta$,

one has that

$$\begin{aligned}
 D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = & \left(\frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} - \frac{1}{R^2 f'' \sin^2(R^{-1}d)} \right) + \frac{(p \cdot \xi)^2}{|p|^2} \left(\frac{f'''}{f' f''} - 2 \frac{f''}{f'^2} \right. \\
 & + \frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} + \frac{1}{R^2 f'' \sin^2(R^{-1}d)} \Bigg) + \frac{(p \cdot \eta)^2}{|p|^2} \left(-\frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} \right. \\
 & - \frac{1}{R^2 f'' \sin^2(R^{-1}d)} + 2 \frac{f' \cos(R^{-1}d)}{R^3 f''^2 \sin^3(R^{-1}d)} + \frac{f' f'''}{R^2 f''^3 \sin^2(R^{-1}d)} \Bigg) \\
 & + \frac{(p \cdot \xi)^2 (p \cdot \eta)^2}{|p|^4} \left(\frac{f''''}{f''^2} - \frac{f' f'''}{R^2 f''^3 \sin^2(R^{-1}d)} - \frac{f'''^2}{f''^2} - 5 \frac{f'''}{f' f''} + 8 \frac{f''}{f'^2} \right. \\
 & \left. \left. - 2 \frac{f' \cos(R^{-1}d)}{R^3 f''^2 \sin^3(R^{-1}d)} - 3 \frac{\cos(R^{-1}d)}{R f' \sin(R^{-1}d)} - 3 \frac{1}{R^2 f'' \sin^2(R^{-1}d)} \right) \right). \quad (4.14)
 \end{aligned}$$

For convenience, (4.14) is used to define four functions of d : $P_1(d)$, $P_2(d)$, $P_3(d)$ and $P_4(d)$, such that one may write

$$D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = P_1(d) + \frac{(p \cdot \xi)^2}{|p|^2} P_2(d) + \frac{(p \cdot \eta)^2}{|p|^2} P_3(d) + \frac{(p \cdot \xi)^2 (p \cdot \eta)^2}{|p|^4} P_4(d). \quad (4.15)$$

To analyse (4.15), various orientations of the vector p relative to ξ and η must now be considered. These calculations can effectively be reduced down to analysing four cases.

Case I: If $p \perp \text{span } \xi, \eta$, then

$$D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = P_1(d) =: O_1(d). \quad (4.16a)$$

Case II: If $p \parallel \xi$, then

$$D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = P_1(d) + P_2(d) =: O_2(d). \quad (4.16b)$$

Case III: If $p \parallel \eta$, then

$$D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = P_1(d) + P_3(d) =: O_3(d) \quad (4.16c)$$

Case IV: If $p \in \text{span } \xi, \eta$ and $p \cdot \xi = p \cdot \eta$, then

$$D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = P_1(d) + \frac{P_2(d)}{2} + \frac{P_3(d)}{2} + \frac{P_4(d)}{4} =: O_4(d). \quad (4.16d)$$

Here, four new functions of d are again defined for clarity. Subsequently, the combined negativity of O_1 , O_2 , O_3 and O_4 is tantamount to the (A3) condition being satisfied. This is formally stated in the following lemma:

Lemma 4.3.1: *Given a cost-function of the form $c(x, y) = f(d(x, y))$ and $\mathbb{S}^n$ an embedded sphere in $\mathbb{R}^{n+1}$ with arbitrary radius R , equipped with the round Riemannian metric with an associated geodesic distance d , then the following statements are true:*

- (i) *If $O_i(d) \leq C < 0$ for $i \in \{2, 3, 4\}$ for all $d \in (0, R\pi)$ and $n > 2$, then the strong (A3) condition is satisfied with constant C , for the cost-function.*
- (ii) *If $O_i(d) < 0$ for $i \in \{2, 3, 4\}$ for all $d \in (0, R\pi)$ and $n > 2$, then the (A3w) condition is satisfied for the cost-function.*
- (iii) *If $O_i(d) \leq C < 0$ for $i \in \{1, 2, 3, 4\}$ for all $d \in (0, R\pi)$ and $n \geq 2$, then the strong (A3) condition is satisfied with constant C for the cost-function.*
- (iv) *If $O_i(d) < 0$ for $i \in \{1, 2, 3, 4\}$ for all $d \in (0, R\pi)$ and $n \geq 2$, then the (A3w) condition is satisfied for the cost-function for $n \geq 2$.*

With the above calculations, verification of the (A3w) condition is now reduced to verifying that (4.16a)–(4.16d) are non-positive for $d \in (0, \frac{R\pi}{2})$. Correspondingly, (4.16a)–(4.16d) being strictly negative indicate that the strong (A3) condition is satisfied. The primary benefit gained by the use of (4.16a)–(4.16d), is that one may use these relations to easily generate explicit, analytic expressions that allow for straight-forward verification of the (A3) criterion for a general class of f . Such calculations will be the focus of the next section.

4.4 Examples

In this section, various examples of cost-functions will be analysed using (4.16a)–(4.16d) to see if they do indeed satisfy the (A3w) or the (A3) condition. These examples include the some of the cost-functions analysed in [TW06] and [MTW05] for the Euclidean case and are encompassed by the general cost-function

$$c(x, y) = f(d(x, y)), \quad (4.17)$$

where $d(x, y)$ represents the geodesic distance between x and y with respect to the underlying Riemannian manifold.

Remark: *The forthcoming results will be conveyed with the understanding that they hold for all $d \in (0, \frac{R\pi}{2})$, unless otherwise indicated.*

4.4.1 $c(x, y) = \frac{1}{2}d^2(x, y)$

The first example consider is the cost-function that has been the most studied out of all the possibilities encompassed by (4.17): $c(x, y) = \frac{1}{2}d^2(x, y)$. Indeed, this is the only cost-function for which the stereographic formulation may be applied without any extraneous geometric conditions, that serve to validate analysing the Optimal Transportation Problem

in only one chart. This is due to the gradient estimate first proved in [McC01] for compact Riemannian manifolds, which was then improved upon in the case of round sphere in [DL06]. To the author's knowledge, there exists no such gradient bounds for the general cost-function depicted in (4.17). This estimate will now be briefly reviewed, as it will also motivate points of the discussion at the end of this chapter.

As was stated above, the scenario where $c(x, y) = \frac{1}{2}d^2(x, y)$ has been studied previously in [McC01] in the context of Optimal Transportation on general Riemannian manifolds; and it has been specifically analysed on the round sphere in both [Loe05] and [DL06]. In [McC01], McCann studies *gradient mappings* defined on Riemannian manifolds, which are mappings of the form

$$G_\phi(m) := \exp_m(\nabla_m \phi),$$

where ϕ is the associated *gradient-potential* of the mapping. McCann shows that such maps are indeed optimal in the transportation of measures on Riemannian manifolds for the cost-function $c(x, y) = \frac{1}{2}d^2(x, y)$. In addition to this, McCann also shows that if ϕ is c -convex, then the length of its gradient can not exceed the diameter of the manifold. Translating into the context of stereographic projections of round spheres, this means that the transport vector defined in (4.8) is not necessarily bounded in the projected local chart, but it is well-defined there. Delanoë and Loeper improved this result on $\mathbb{S}^n$ by proving the following gradient bound in [DL06]:

Theorem 4.4.1 [DL06]: *Given $\phi : \mathbb{S}^n \mapsto \mathbb{R}$ a c -convex function, such that*

$$\int_{G_\phi^{-1}(\mathbb{S}^n)} (h \circ G_\phi) \, d\text{Vol} = \int_{\mathbb{S}^n} (h \cdot \rho) \, d\text{Vol} \quad (4.18)$$

for some $\rho \in L^\infty(\mathbb{S}^n, d\text{Vol})$ with any $h \in C^0(\mathbb{S}^n)$, where $d\text{Vol}$ stands for the canonical Lebesgue measure on $\mathbb{S}^n$, then the following estimate holds a.e.

$$|d\phi| \leq \pi - \frac{1}{2\pi} \left\{ \frac{1}{\|\rho\|_{L^\infty(\mathbb{S}^n)}} \left[\frac{n \text{Vol}(\mathbb{S}^n)}{2 \text{Vol}(\mathbb{S}^{n-1})} \right]^2 \right\}^{1/n}. \quad (4.19)$$

Remark: *Recalling that the notation remarked upon in Section 1.2, the expression in (4.18) is often denoted as*

$$G_{\phi\#} d\text{Vol} = \rho d\text{Vol}.$$

Theorem 4.4.1 correlates to the transportation vector p , as defined in (4.8), being strictly bounded in the stereographically projected local chart. Thus, the assumption that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$ is not needed in the case where $c(x, y) = \frac{1}{2}d^2(x, y)$. The possibility of extending (4.19) to the general case of $c(x, y) = f(d(x, y))$, will be discussed at the end of this chapter. The (A3) condition will now be analysed for the case when $c(x, y) = \frac{1}{2}d^2(x, y)$, using the results of Section 4.3.

Using (4.14), it is calculated that

$$P_1(d) = \frac{\cos(R^{-1}d)}{Rd \sin(R^{-1}d)} - \frac{1}{R^2 \sin^2(R^{-1}d)} \quad (4.20a)$$

$$P_2(d) = -\frac{2}{d^2} + \frac{\cos(R^{-1}d)}{Rd \sin(R^{-1}d)} + \frac{1}{R^2 \sin^2(R^{-1}d)} \quad (4.20b)$$

$$P_3(d) = -\frac{\cos(R^{-1}d)}{Rd \sin(R^{-1}d)} - \frac{1}{R^2 \sin^2(R^{-1}d)} + 2\frac{d \cos(R^{-1}d)}{R^3 \sin^3(R^{-1}d)} \quad (4.20c)$$

$$P_4(d) = \frac{8}{d^2} - 2\frac{d \cos(R^{-1}d)}{R^3 \sin^3(R^{-1}d)} - 3\frac{\cos(R^{-1}d)}{Rd \sin(R^{-1}d)} - 3\frac{1}{R^2 \sin^2(R^{-1}d)}. \quad (4.20d)$$

As stated in Subsection 4.3.1, (4.20a)–(4.20d) are considered for $d \in (0, \frac{R\pi}{2})$. First, the limits as $d \rightarrow 0$ are calculated to be

$$\lim_{d \rightarrow 0} P_1(d) = -\frac{2}{3R^2}, \quad \lim_{d \rightarrow 0} P_2(d) = 0, \quad \lim_{d \rightarrow 0} P_3(d) = 0, \quad \lim_{d \rightarrow 0} P_4(d) = 0.$$

Using these limits to analyse $O_1(d)$, $O_2(d)$, $O_3(d)$ and $O_4(d)$, it follows that

$$\lim_{d \rightarrow 0} D_{p_l p_k}^2 c_{\hat{x}_i \hat{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = -\frac{2}{3R^2}, \quad \forall p \in \mathbb{R}^n.$$

This confirms the result of Loeper presented at the end of [Loe05] for the case where $R = 1$. Using (4.20a)–(4.20d), it is also calculated that

$$O'_i(d) < 0, \quad \forall d \in \left(0, \frac{R\pi}{2}\right), \quad \forall R > 0,$$

where $i \in \{1, 2, 3, 4\}$. Figure 4.2 below contains plots of both sets of functions $P_i(d)$ and $O_i(d)$, in the case when $R = 1$.

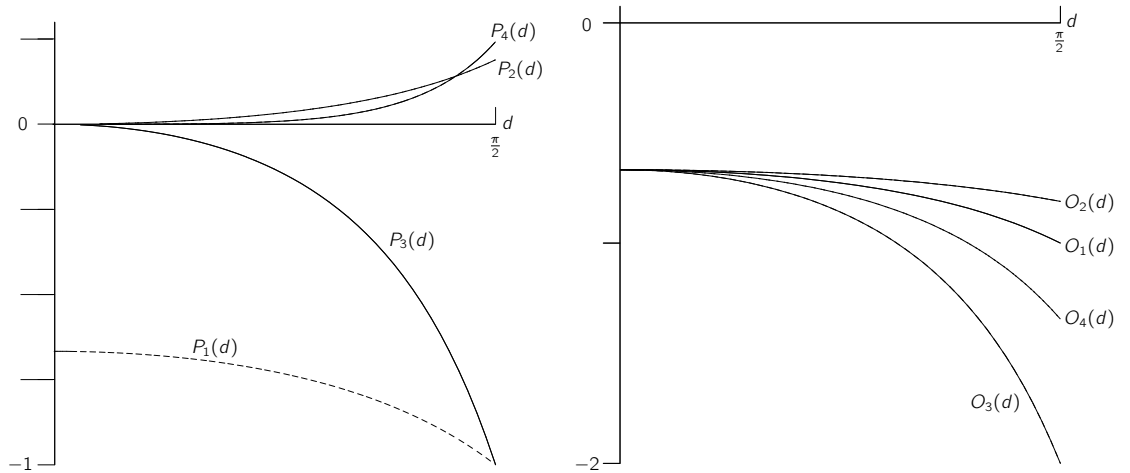


Figure 4.2: $P_i(d)$ and $O_i(d)$ corresponding to $c(x, y) = \frac{1}{2}d^2(x, y)$ with $R = 1$

With the above calculations, it has been shown that the (A3) condition is indeed satisfied on the round sphere of any radius, for the cost of $c(x, y) = \frac{1}{2}d^2(x, y)$. As noted earlier, this is in contrast to the Euclidean case where the cost-function $c(x, y) = \frac{1}{2}|x - y|^2$ only satisfies the (A3w) condition, as the (A3) term goes to 0 as $y \rightarrow x$ (see [MTW05, TW06]). This contrast between the Euclidean and spherical cases indicates that the underlying geometry (manifested through the cost-function) does indeed affect the local regularity of the Optimal Transportation Problem.

4.4.2 $c(x, y) = 2R^2 \sin^2\left(\frac{d(x, y)}{2R}\right)$

If one were to consider the round sphere embedded in $\mathbb{R}^{n+1}$ equipped with a Euclidean metric, the cost-function of $2R^2 \sin^2\left(\frac{1}{2R}d(x, y)\right)$ is equivalent to the example studied in Subsection 4.4.1 with $d(x, y)$ taken to being the geodesic distance of the Euclidean space into which the sphere is embedded. This situation is depicted in Figure 4.3 on the next page.

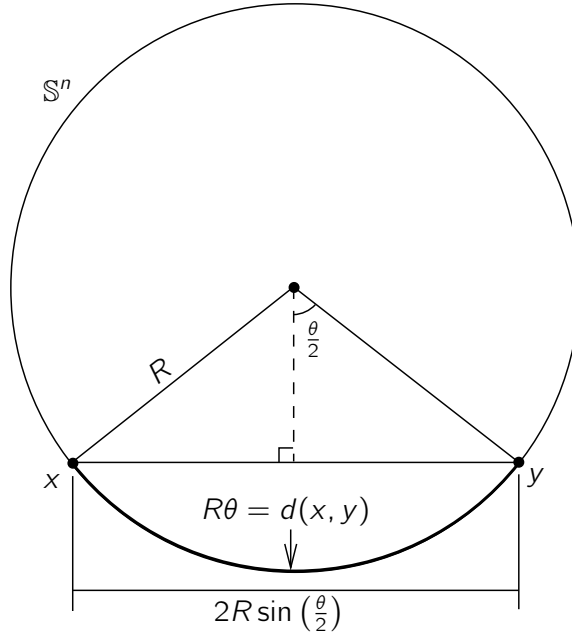


Figure 4.3: Chordal distance between points

Again, using the results of Subsubsection 4.3.1.2, it is calculated that

$$P_1(d) = \frac{1}{R^2 \left[1 - 2 \cos^2\left(\frac{d}{2R}\right)\right]} \quad (4.21a)$$

$$P_2(d) = \frac{4 \sin^2\left(\frac{d}{2R}\right) \cos^2\left(\frac{d}{2R}\right)}{R^2 \left[1 - 8 \cos^6\left(\frac{d}{2R}\right) + 12 \cos^4\left(\frac{d}{2R}\right) - 6 \cos^2\left(\frac{d}{2R}\right)\right]} \quad (4.21b)$$

$$P_3(d) = 0 \quad (4.21c)$$

$$P_4(d) = \frac{2 \sin^4\left(\frac{d}{2R}\right) \cos^2\left(\frac{d}{2R}\right)}{R^2 \left[8 \cos^6\left(\frac{d}{2R}\right) - 12 \cos^4\left(\frac{d}{2R}\right) + 6 \cos^2\left(\frac{d}{2R}\right) - 1\right]}. \quad (4.21d)$$

From these relations, the following limits are calculated:

$$\lim_{d \rightarrow 0} P_1(d) = -\frac{1}{R^2}, \quad \lim_{d \rightarrow 0} P_2(d) = 0, \quad \lim_{d \rightarrow 0} P_3(d) = 0, \quad \lim_{d \rightarrow 0} P_4(d) = 0;$$

thus,

$$\lim_{d \rightarrow 0} D_{p_l p_k}^2 c_{\tilde{x}_i \tilde{x}_j}(d) \xi_i \xi_j \eta_k \eta_l = -\frac{1}{R^2}, \quad \forall p \in \mathbb{R}^n. \quad (4.22)$$

Finally, an elementary calculation using (4.21a)–(4.21d) in the definitions of $O_i(d)$ yields

$$O'_i(d) < 0 \quad \forall d \in \left(0, \frac{R\pi}{2}\right) \quad \forall R > 0. \quad (4.23)$$

Given the properties depicted in (4.22) and (4.23), it follows that this cost-function does indeed satisfy the strong (A3) condition on the half-sphere. This is readily confirmed by the plots of $P_i(d)$ and $O_i(d)$ in the case where $R = 1$, shown in Figure 4.4 below.

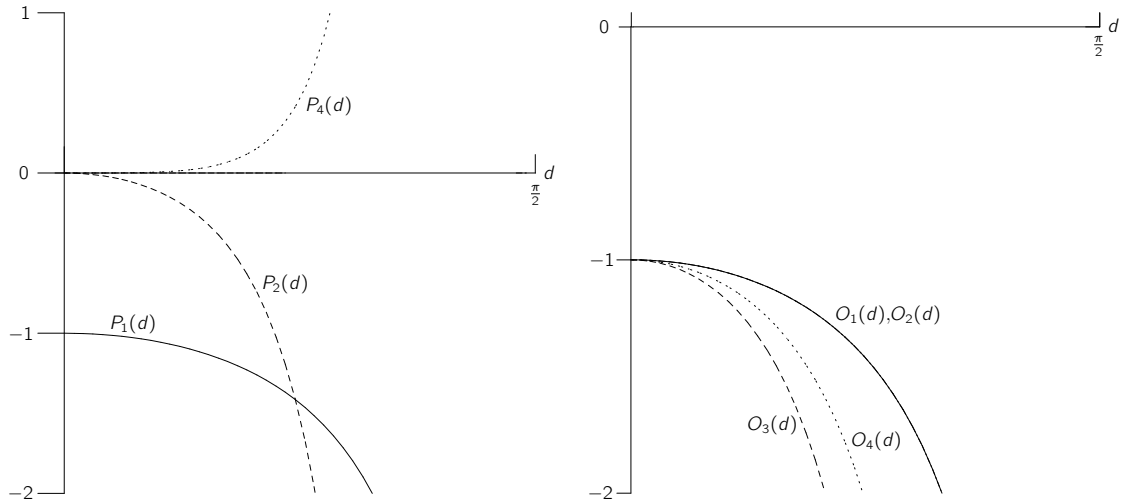


Figure 4.4: $P_i(d)$ and $O_i(d)$ corresponding to $c(x, y) = 2R^2 \sin^2\left(\frac{d(x, y)}{2R}\right)$ with $R = 1$

4.4.3 $c(x, y) = \sqrt{1 - d^2(x, y)}$

This example gives a demonstration as to just how important R scaling can be in affecting the (A3) condition. This radial scaling dependence manifests itself through the limiting behaviour of the orientation terms as $d \rightarrow 0$. Using the general expression for $O_i(d)$, one readily calculates that

$$\lim_{d \rightarrow 0} O_i(d) = \frac{2}{3R^2} - 1 \quad \text{for } i \in \{1, 2, 3, 4\}.$$

Further calculation shows that for $R > \sqrt{2/3}$, and $\text{diam}(\Omega^- \cup \Omega^+) < 1$, the (A3) condition is satisfied. For $R = \sqrt{2/3}$, the (A3) condition is satisfied if $\text{dist}(\Omega^-, \Omega^+) > 0$ and $\text{diam}(\Omega^- \cup \Omega^+) < 1$. If one only has that $\text{dist}(\Omega^-, \Omega^+) \geq 0$ and $\text{diam}(\Omega^- \cup \Omega^+) \leq 1$, then

only the (A3w) condition is satisfied. In other cases where $R \geq \sqrt{2/3}$, the (A3) condition will be violated.

If $R < \sqrt{2/3}$ and $\text{diam}(\Omega^- \cup \Omega^+) < 1$ with $\text{dist}(\Omega^-, \Omega^+) > h^*$, where $h^* > 0$ solves $P_1(h^*) = 0$, then $c(x, y) = \sqrt{1 - d^2}$ satisfies the strong (A3) condition. This cost-function will violate the (A3) condition in all other cases where $R < \sqrt{2/3}$.

$$4.4.4 \quad c(x, y) = -\sqrt{1 - d^2(x, y)}$$

This example is closely related to the previous example, in that it also has a dependence on the radial scaling of the round sphere. In particular, one has that

$$\lim_{d \rightarrow 0} O_i(d) = 1 - \frac{2}{3R^2} \quad \text{for } i \in \{1, 2, 3, 4\}.$$

From this, it is observed that for $R \geq \sqrt{2/3}$, the (A3) condition will be violated at least for some small values of d . Indeed, further calculation indicates that

$$O_i(d) \geq 0 \quad \text{for } i \in \{1, 2, 3\}, \quad \forall d \in [0, 1],$$

for any $R \geq \sqrt{2/3}$, thus proving the (A3) condition will be violated in this case.

If $R < \sqrt{2/3}$ and $\text{diam}(\Omega^- \cup \Omega^+) < \min\{h^*, R, 1\}$, where $h^* > 0$ solves $P_2(h^*) = 0$, then the strong (A3) condition will be satisfied. In all other cases where $R < \sqrt{2/3}$, the (A3) condition will be violated.

$$4.4.5 \quad c(x, y) = \sqrt{1 + d^2(x, y)}$$

This example bears some resemblance to the previous one; but in this scenario, there is no sign-symmetry broken by a variance in the radial scaling. Indeed, it is calculated that

$$\lim_{d \rightarrow 0} O_i(d) = -\frac{2}{3R^2} - 1, \quad \text{for } i \in \{1, 2, 3, 4\},$$

along with

$$O'_i(d) \leq 0, \quad i \in \{1, 2, 3, 4\}.$$

In this scenario, the (A3) condition will be strongly satisfied for all values of $R > 0$.

$$4.4.6 \quad c(x, y) = -\sqrt{1 + d^2(x, y)}$$

In this example, it is calculated that

$$O_i(d) > 0, \quad i \in \{1, 2, 3, 4\},$$

for any $R > 0$. Thus, the (A3) condition will always be violated for this cost-function.

$$4.4.7 \quad c(x, y) = \pm \frac{1}{m} d^m(x, y)$$

In this subsection, a generalisation of the example in Subsection 4.4.1 is considered:

$$c(x, y) = \pm \frac{d^m}{m}(x, y) \quad m \neq 0, \quad c(x, y) = \pm \log(d(x, y)) \quad m = 0. \quad (4.24)$$

Calculating as before, one finds the following:

$$O_1(d) = \pm \frac{(m-1)R \sin(R^{-1}d) \cos(R^{-1}d) - d}{(m-1)R^2 d^{m-1} \sin^2(R^{-1}d)} \quad (4.25a)$$

$$O_2(d) = \mp \frac{mR \sin(R^{-1}d) - 2d \cos(R^{-1}d)}{R d^m \sin(R^{-1}d)} \quad (4.25b)$$

$$O_3(d) = \mp \frac{mR \sin(R^{-1}d) - 2d \cos(R^{-1}d)}{(m-1)^2 R^3 d^{m-2} \sin^3(R^{-1}d)} \quad (4.25c)$$

$$O_4(d) = \pm \frac{\cos(R^{-1}d)}{2(m-1)^2 R^3 d^{m-3} \sin^3(R^{-1}d)} \pm \frac{\cos(R^{-1}d)}{4R d^{m-1} \sin(R^{-1}d)} \\ \mp \frac{6m-5}{4(m-1)^2 R^2 d^{m-2} \sin^2(R^{-1}d)} \pm \frac{m^2-2m+2}{2(m-1)d^m} - \frac{(m-2)^2}{4d^2}. \quad (4.25d)$$

From (4.25a)–(4.25d), the following series of observations can be made for various values of m and sign on the cost-function depicted in (4.24).

4.4.7.1 (+), $m < 0$:

In this scenario, the R scaling has no effect on the (A3) results. From (4.25a)–(4.25d), the following is calculated:

$$\begin{aligned} P_i(d) &\geq 0, \quad \text{for } i \in \{1, 2\}, \\ P_j(d) &\leq 0, \quad \text{for } j \in \{3, 4\}, \\ |P_1(d)| &\geq |P_3(d)|, \end{aligned} \quad (4.26)$$

for $d \in (0, \frac{R\pi}{2})$. Analysing limits as $y \rightarrow x$, it is observed that

$$\begin{aligned} \lim_{d \rightarrow 0} P_k(d) &= 0, \quad \text{for } k \in \{1, 2, 3\}, \\ \lim_{d \rightarrow 0} P_4(d) &= -\infty. \end{aligned} \quad (4.27)$$

From (4.26) and (4.27), it is thus concluded

$$O_i(d) \geq 0, \quad \text{for } i \in \{1, 2, 3\},$$

for $d \in (0, \frac{R\pi}{2})$; that is, the (A3) condition does not hold in this scenario.

4.4.7.2 $(-)$, $m < 0$:

Although this case bears many similarities to the scenario depicted in Subsubsection 4.4.7.1, it also has some notable differences. In particular, the radial scaling of the sphere does affect whether or not the (A3) condition is satisfied. However, results that are true for all $R > 0$ will first be conveyed. Analysing the $P_i(d)$ expressions, it is calculated that

$$\begin{aligned} P_i(d) &\leq 0, \quad \text{for } i \in \{1, 2\}, \\ P_3(d) &\geq 0, \\ |P_1(d)| &\geq |P_3(d)|, \end{aligned} \quad (4.28)$$

for $d \in (0, \frac{R\pi}{2})$ and $\forall R > 0$. To proceed, the limits as $y \rightarrow x$ are calculated:

$$\begin{aligned} \lim_{d \rightarrow 0} P_k(d) &= 0, \quad \text{for } k \in \{1, 2, 3\}, \\ \lim_{d \rightarrow 0} P_4(d) &= -\infty. \end{aligned} \quad (4.29)$$

From (4.28) and (4.29), one has the following:

$$O_i(d) \leq 0, \quad \text{for } i \in \{1, 2, 3\},$$

for $d \in (0, \frac{R\pi}{2})$. So far, no mention has been made of the $O_4(d)$ expression in the current analysis. This is due to the fact that this term can switch signs on the interval $d \in (0, \frac{R\pi}{2})$, which is a result of $P_4(d)$ becoming positive for some $d > 0$. (4.29) already shows that $O_4(d) \rightarrow -\infty$ as $y \rightarrow x$; and thus, it is known that there exists some constant M such that $d < M$ implies $O_4(d) < 0$ by the continuity of $O_4(d)$. Thus, the analysis of the $O_4(d)$ expression is reduced to analysing the equation

$$\begin{aligned} 0 &= -2d^5 \cos\left(\frac{d}{R}\right) + (6m - 5)Rd^4 \sin\left(\frac{d}{R}\right) - (m - 1)^2 R^2 d^3 \sin^2\left(\frac{d}{R}\right) \cos\left(\frac{d}{R}\right) \\ &\quad - 2(m - 1)(m^2 - 2m + 2)R^3 d^2 \sin^3\left(\frac{d}{R}\right) \\ &\quad - (m - 1)^2(m - 2)^2 R^3 d^m \sin^3\left(\frac{d}{R}\right); \end{aligned} \quad (4.30)$$

the right-hand side is simply the numerator of (4.25d). For $m < 0$ and $d \in (0, \frac{R\pi}{2})$, the fourth term on the right-hand side of (4.30) is the only positive term. If we consider $d \in (0, 1]$ and $m < 0$, it is observed that

$$\left| 2(m - 1)(m^2 - 2m + 2)R^3 d^2 \sin^3\left(\frac{d}{R}\right) \right| \leq \left| (m - 1)^2(m - 2)^2 R^3 d^m \sin^3\left(\frac{d}{R}\right) \right|, \quad (4.31)$$

as $(m - 1)^2(m - 2)^2 > 2(m - 1)(m^2 - 2m + 2)$ for all values of m . Thus, by (4.29) and (4.30) it is seen that $O_4(d) \leq 0$ for $d \in [0, 1]$. The actual value d^* where $O_4(d^*) = 0$ can not be explicitly represented as (4.30) is a transcendental equation. This value of d^* depends

on both R and m . Even though an analytic representation of d^* does not exist, it can be calculated from (4.30) that

$$\lim_{m \rightarrow -\infty} d^* = 1, \quad \forall R > 0. \quad (4.32)$$

This can be seen from (4.30) by noticing the fourth term on the right-hand side dominates all terms for large negative values of m and $d > 1$ and some $R > 0$ fixed. However, (4.31) holds for all $m < 0$, $R > 0$ and $d \leq 1$; thus, one can intuitively reconcile (4.32) from (4.31) without resorting to a limit calculation. (4.32) represents the manifestation of the R scaling dependency for this specific scenario. In particular, if $R \leq \frac{2}{\pi}$, the (A3w) condition will be satisfied. This is true in the full-sphere case if $R \leq \frac{1}{\pi}$. Of course, this is not a sharp estimate; but such restrictions on the radius of the sphere will ensure the (A3w) condition is satisfied for any $m < -2$. Instead of restricting the radius of the sphere, an analogous restriction may be employed on the source and target domains: $\text{diam}(\Omega^- \cup \Omega^+) \leq 1$. This will also ensure that the (A3w) criterion is satisfied. In order for the strong (A3) condition to be satisfied, one of the aforementioned restrictions is required plus the criterion that $\text{dist}(\Omega^-, \Omega^+) > 0$, as

$$\lim_{d \rightarrow 0} O_i(d) = 0, \quad \text{for } i \in \{1, 2, 3\},$$

which is a straight-forward consequence of (4.29).

As was already mentioned, the limit (4.32) only corresponds to a sufficient condition on R for the (A3w) condition to be satisfied. The corresponding necessary condition on R can indeed be calculated numerically for given values of m ; but such a condition can not be explicitly stated due to the transcendental nature of (4.30). However for a fixed R , this scenario undergoes a bifurcation in the m parameter whereupon the (A3) condition will be at least weakly satisfied for all $R > 0$. The value for m , that corresponds to this bifurcation, depends on R and is again only able to be calculated numerically. One may analyse this bifurcation point as $R \rightarrow \infty$. Assuming $R > 0$ and $d > 1$ is such that $\frac{d}{R} \ll 1$, the right-hand side of (4.30) is approximated by

$$(-2m^3 + 5m^2 - 4)d^5 + o(d^7).$$

By this approximation, it is seen that if $m^* < m < 0$, where m^* is defined as the the negative root of the polynomial $-2m^3 + 5m^2 - 4$, then for an arbitrary fixed $d \in (0, \frac{R\pi}{2})$

$$\begin{aligned} 0 > \lim_{R \rightarrow \infty} \left[-2d^5 \cos\left(\frac{d}{R}\right) + (6m - 5)Rd^4 \sin\left(\frac{d}{R}\right) \right. \\ \quad \left. - (m - 1)^2 R^2 d^3 \sin^2\left(\frac{d}{R}\right) \cos\left(\frac{d}{R}\right) \right. \\ \quad \left. - 2(m - 1)(m^2 - 2m + 2)R^3 d^2 \sin^3\left(\frac{d}{R}\right) \right]. \end{aligned}$$

It can be calculated that

$$m^* \approx -.7807764064, \quad (4.33)$$

which is a numerical limit for the (A3), m -parameter bifurcation as R tends to infinity. This is shown using elementary methods of calculus to conclude that for $m > m^*$, the first three terms of (4.30) dominate the single positive fourth term on the right-hand side of (4.30).

Remark: *It indeed does require the sum of all three of these terms to dominate this one positive term in the $O_4(d)$ expression; thus, there is no simple asymptotic statement to justify this behaviour outside of analysing the sign of the right-hand side of (4.30) and it's derivatives.*

Again, (4.33) correlates to a limit of R tending to infinity. As R decreases, this bifurcation will happen for a value of $m < m^*$. If R falls below the previously discussed bifurcation point dependent on m (we recall this has to be greater than $\frac{2}{\pi}$), then the (A3) condition is at least weakly satisfied. This combined with the previous statements regarding the bifurcation point in the R parameter fills out the current spectrum of results regarding this scenario. The results for this particular case are summarised as follows:

- If $R > \frac{2}{\pi}$ and $m > m^*$, the (A3w) condition is satisfied for any Ω^- and Ω^+ such that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$.
- If $R > \frac{2}{\pi}$ and $m < m^*$, one must numerically check to see if (4.30) has real zeros for $d \in (0, \frac{R\pi}{2})$. If this is the case, the smallest positive root of (4.30) represents an upper bound for the diameter of $\Omega^- \cup \Omega^+$ for the (A3w) condition to hold.
- If $R \leq \frac{2}{\pi}$ the (A3w) condition will hold for any Ω^- and Ω^+ such that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$.

All these conclusions can be strengthened to having the (A3) condition satisfied, provided $\text{dist}(\Omega^-, \Omega^+) > 0$.

4.4.7.3 (+), $m = 0$:

This, scenario is exactly the same as the case studied in Subsubsection 4.4.7.1, except that

$$\lim_{d \rightarrow 0} P_1(d) = 2, \quad \forall R > 0,$$

which is calculated from (4.25a). Thus, the (A3) condition will not be satisfied in this case.

4.4.7.4 (−), $m = 0$:

Straight-forward calculations from (4.25a)–(4.25d) indicate that

$$\begin{aligned} O_i(d) &< 0, & \text{for } i \in \{1, 4\}, \\ O_j(d) &\leq 0, & \text{for } j \in \{2, 3\}, \end{aligned}$$

for $d \in (0, \frac{R\pi}{2})$. Indeed, analysing limits, it can be ascertained that that

$$\begin{aligned}\lim_{d \rightarrow 0} O_i(d) &= -2, \quad \text{for } i \in \{1, 2, 3\}, \\ \lim_{d \rightarrow 0} O_4(d) &= -\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{d \rightarrow \frac{R\pi}{2}} O_i(d) &= 0, \quad \text{for } i \in \{2, 3\} \\ \lim_{d \rightarrow \frac{R\pi}{2}} O_j(d) &< 0, \quad \text{for } i \in \{1, 4\}.\end{aligned}$$

These results are true for all values of $R > 0$. Thus the (A3w) condition is satisfied independent of the radial scaling of the round sphere. To have the (A3) condition be satisfied, it is required that $\text{dist}(\text{cut } \Omega^-, \Omega^+) > 0$.

4.4.7.5 (+), $0 < m < 1$

As with the cases where $m < 0$, the scenarios for when $m > 0$ also have a complex structure in that both the varying of R and m have bifurcations in all of the terms $O_i(d)$ for $i \in \{1, 2, 3, 4\}$. The following exposition will be less explicit as compared to the $m < 0$ cases, as the analysis is extremely similar to examples already presented in this section.

In this case the (A3) condition will be violated for all values of $R > 0$, except in a very special situation. First, one can easily verify that $O_1(d) > 0$ for $d \in [0, \frac{R\pi}{2})$; but there are values of $0 < m < 1$, $R > 0$ and d such that $O_j(d) < 0$. Specifically, for $m > m^*$ and any value of $R > 0$, one has the (A3) condition being satisfied on $\mathbb{S}^2$ provided $\text{dist}(\Omega^-, \Omega^+) > h^*$, where h^* represents the second positive root of the equation $O_4(d) = 0$. m^* is the root of the equation

$$\lim_{R \rightarrow \infty} O_4\left(\frac{R\pi}{2}\right) = 0, \quad (4.35)$$

and is approximately .806. If $R \lesssim .071$, then h^* is determined by the smallest positive root of $O_2(d) = 0$ which is equivalent to the equation $O_3(d) = 0$.

Remarks:

- (1) *It should be noted that $O_4(d)$ has a parameter dependence on m for cost-functions of the form $c(x, y) = \pm \frac{1}{m} d^m(x, y)$. With this, it is clear what is meant by m^* being the root of (4.35).*
- (2) *This is one of the special cases where the dimensionality of $\mathbb{S}^n$ has an affect on the (A3) condition. Indeed, for $n > 2$, one must have $O_1(d)$ less than zero, which is not the case in the current scenario. However, for $n = 2$ the $O_1(d)$ is not considered, as it is clearly not possible to have a transport vector orthogonal to both arbitrary vectors η and ξ with $\eta \perp \xi$ in a two dimensional space.*

4.4.7.6 (+), $1 < m < 2$

In this case,

$$O_i(d) < 0, \quad \text{for } i \in \{1, 4\};$$

and $O_2(d)$ and $O_3(d)$ are monotone decreasing with

$$\lim_{d \rightarrow 0} O_j(d) = \infty, \quad \text{for } j \in \{2, 3\}.$$

It can be calculated that for this scenario there exists $h^* \in (0, \frac{R\pi}{2})$ such that $O_2(h^*) = O_3(h^*) = 0$. Thus, h^* represents the minimal distance separation between Ω^- and Ω^+ for the strong (A3) condition to be satisfied in this particular case. This conclusion is independent of radial scaling.

4.4.7.7 (+), $2 < m < \infty$

In this case, one has that

$$O_i(d) < 0, \quad \text{for } i \in \{2, 3\},$$

and $O_2(d)$ and $O_3(d)$ are monotone decreasing with

$$\lim_{d \rightarrow 0} O_j(d) = \infty, \quad \text{for } j \in \{1, 4\}.$$

The strong (A3) condition will be satisfied in this case if $\text{dist}(\Omega^-, \Omega^+) > h^*$, where h^* is defined by the equation $O_4(h^*) = 0$ (there is only one root of this equation in the interval $(0, \frac{R\pi}{2})$). It is a straight-forward calculation to see that $h^* \in (0, \frac{R\pi}{2})$. The results of this scenario are invariant under radial scaling.

4.4.7.8 (−), $0 < m < 1$

Unlike the case for $m < 0$, a change in sign is not necessarily tantamount to violation of the (A3) condition. Here one has that

$$O_i(d) < 0, \quad \text{for } i \in \{1, 4\}$$

and

$$O_j(0) < 0, \quad O'_j(d) > 0, \quad \text{for } j \in \{2, 3\}.$$

It is calculated that there exists a $h^* \in (0, \frac{R\pi}{2})$ such that $O_2(h^*) = O_3(h^*) = 0$. Thus, if $\text{diam}(\Omega^- \cup \Omega^+) < h^*$, then the (A3) condition holds.

4.4.7.9 $(-)$, $1 < m < 2$

In this case, it is calculated that

$$\begin{aligned} O_1(d) &> 0, \\ O_j(0) &< 0, \quad O'_j(d) > 0, \quad \text{for } j \in \{2, 3, 4\}, \end{aligned}$$

for all $d \in (0, \frac{R\pi}{2})$ and $R > 0$. Thus, the (A3) condition will not hold for this cost-function on $\mathbb{S}^n$ for $n > 2$. On $\mathbb{S}^2$ however, there exists $h^* \in (0, \frac{R\pi}{2})$ such that $O_4(h^*) = 0$. Moreover, if $\text{diam}(\Omega^- \cup \Omega^+) < h^*$, then the (A3) condition will be strongly satisfied on $\mathbb{S}^2$.

4.4.7.10 $(-)$, $2 \leq m < \infty$

Here the (A3) condition can not be satisfied for any $R > 0$ with $2 \leq m < \infty$, as it is calculated that

$$O_i(d) > 0, \quad \text{for } i \in \{2, 3\},$$

for all $d \in (0, \frac{R\pi}{2})$.

4.4.7.11 Remarks on the $c(\mathbf{x}, \mathbf{y}) = \pm \frac{1}{m} d^m(\mathbf{x}, \mathbf{y})$ Example

This concludes a particularly long example that demonstrates the use of Lemma 4.3.1. The complex structure that correlates to the above set of scenarios is due to the fact that one must analyse four different expressions for various orientations of $\xi \perp \eta$ relative to the transport vector p , while varying two different parameters: the exponent m and the radius of the round sphere R . It has been demonstrated in this example that bifurcations in the (A3) behaviour happen relative to both these parameters for different orientation vectors, thus resulting in a large group of scenarios and corresponding results.

4.4.8 Final Remarks on Examples

Through the presented examples, it has been shown that the tie between regularity of Optimal Transportation potentials and geometry is complex indeed. Loeper showed in [Loe05] how the (A3) condition related to cross-sectional curvatures on a manifold; specifically, curvature can have implications toward the (A3) criterion in the limit as $d \rightarrow 0$. In the examples of this section, it has been shown that curvature can also affect the result of the (A3) criterion as $d \rightarrow \text{diam}(\mathbb{S}^n)$ in a very explicit, albeit non-intrinsic way. It has also been shown that the underlying topology can also affect the outcome of the (A3) criterion of cost-functions; that is, some costs satisfy the (A3) condition on $\mathbb{S}^2$ but not on spheres of higher dimension. This link between geometry and regularity in Optimal Transportation is becoming the focus of intense research efforts by many researchers in both geometry and partial differential equations.

4.5 Conclusions

4.5.1 Geometric Implications

In this chapter, results have been presented correlating to the verification of the (A3) condition on round spheres with cost-functions of the form

$$c(x, y) = f(d(x, y)). \quad (4.36)$$

The full generality of (4.36) has been able to be studied at the expense of utilising an inherently non-intrinsic approach in the current set of calculations. However, the use of the stereographic projection as the centrepiece of the analysis presented in this chapter is uniquely powerful in the context of round spheres for two main reasons. First, the stereographic formulation is rotationally invariant on $\mathbb{S}^n$; indeed, the analysis in this chapter was carried out on an arbitrary fixed $x \in \mathbb{S}^n$ against a variable Optimal Transportation target y . The second (and most important) simplification the stereographic formulation affords is the ability to explicitly represent the geodesic distance between two points on a sphere in terms of our projected coordinates; that is, (4.4) is valid as a representation of the geodesic distance between points $x, y \in \mathbb{S}^n$ in terms of our projected coordinates $\hat{x}, \hat{y} \in \mathbb{R}^n$. Such an explicit and technically manageable representation of geodesic distance on a general Riemannian manifold is rare in any coordinate system one may choose for a chart on that manifold. Indeed, to the author's knowledge, representations of geodesic distances on even an ellipsoid result in the necessity to use highly esoteric special functions based on implicit or integral representations.

As the (A3) analysis in this chapter reduces to the analysis of covariant derivatives of $f(d(\hat{x}, \hat{y}))$, nothing can be calculated without an explicit representation of geodesic distance. Thus, given the above comments, it is computationally difficult to extend the methods in this chapter to other Riemannian manifolds beyond that of the round sphere.

As eluded to in Section 4.1 and Subsection 4.4.1, verification of the (A3) criterion is only one part to proving the regularity of potential functions associated with certain costs. The other part of proving regularity lies in the existence of gradient estimates analogous to those presented in Theorem 4.4.1, for cost-functions other than $\frac{1}{2}d^2(x, y)$. To circumvent this, the assumption that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$ has been made throughout the chapter. Without this assumption or a gradient estimate, the stereographic formulation becomes invalid as a point may be mapped to its cut-locus on the sphere; and thus, move outside of the chart where the analysis was performed. In the case where $f'(d) < 0$, one can not escape the assumption that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$ in the stereographic formulation. In this situation, one only need to consider a case where $\text{cut}(\Omega^-) = \Omega^+$ to observe that the optimal mapping correlates to a mapping that takes every point in $x \in \Omega^-$ and maps it to that point's particular cut-locus. Thus, to analyse cases where $f'(d) < 0$ free from the assumption that $\text{cut}(\Omega^-) \cap \Omega^+ = \emptyset$, requires the use of geometrically intrinsic methods. As for the cases where $f'(d) > 0$, it is

possible that analogies to Theorem 4.4.1 exist given the variational nature of the Optimal Transportation Problem. Indeed, the author suspects that such gradient bounds exist, given the close representation of optimal solutions presented in [McC01] for costs of the form

$$c(x, y) = \int_0^{d(x, y)} \lambda(t) dt,$$

to the *gradient maps* discussed earlier in this chapter. It may be possible to modify the intrinsic arguments presented in [DL06] to these slight generalisations of gradient mappings under the restriction that $f'(d) > 0$. This research is one possibility to further the results presented here.

4.5.2 Further Studies

Outside of the remarks of the previous subsection, there are other avenues of research that are possible to pursue stemming from this thesis.

First, there is the possibility of analogising the stereographic analysis of this chapter to other manifolds beyond the construction of a round sphere. The application of the calculations in this chapter to a completely general classes of manifolds will not be possible, as the stereographic analysis presented is dependent on the explicit representation of geodesic distance on the sphere. However, going off the importance that symmetry and the explicit geodesic distance representation both played in these calculations, it seems that Lie Groups would be the next place to look in efforts to extend the stereographic formulation of the (A3) condition. Not only do Lie Groups have strong symmetry properties, they also have the right curvature conditions to indicate that the (A3) condition should be at least satisfied for $c(x, y) = \frac{1}{2}d^2(x, y)$. Indeed, the sectional curvature of a Lie Group can be represented as

$$R(X, Y) = \frac{1}{4} \|[X, Y]\|^2, \quad (4.37)$$

where $[X, Y]$ is the Lie Bracket of left-invariant vector fields X and Y on the Lie Group (see [dC92, Ch. 4, Exercise 1]). This combined with the theory presented by Loeper in [Loe05], makes it reasonable to think that at least the (A3w) condition will be satisfied (in at least some bounded domain) given the results known for $c(x, y) = \frac{1}{2}d^2(x, y)$ on both the round sphere and the Euclidean case. If attention is further restricted to Lie Groups where

$$C < \|[X, Y]\|^2 \leq 4C, \quad (4.38)$$

holds everywhere for some arbitrary constant C , it is then ascertained, via the Sphere Theorem (see [dC92, Chapter 13]), that the Lie Group is then homeomorphic to a sphere and thus compact. Thus, the gradient estimates of McCann presented in [McC01] can be applied in this scenario. Of course, strong gradient bounds can possibly be proven for certain Lie Groups; but this argument gives a strong indication that Lie Group structures would

be a good place to look to apply the non-intrinsic techniques presented here. Beyond Lie Groups, it is difficult to see applying the current techniques outside the context of very specific manifolds due to few explicit representations of geodesic distances corresponding to arbitrary metrics.

Lastly, the variation of parameters for certain classes of cost-functions lead to complex behaviour through bifurcations of the orientation terms defined earlier in Subsubsection 4.3.1.2. Given the transcendental nature of these bifurcation points, it would be interesting to run numerical analyses on various families of costs using the stereographic formulation to ascertain visualisations of the various parameter bifurcations that occur.



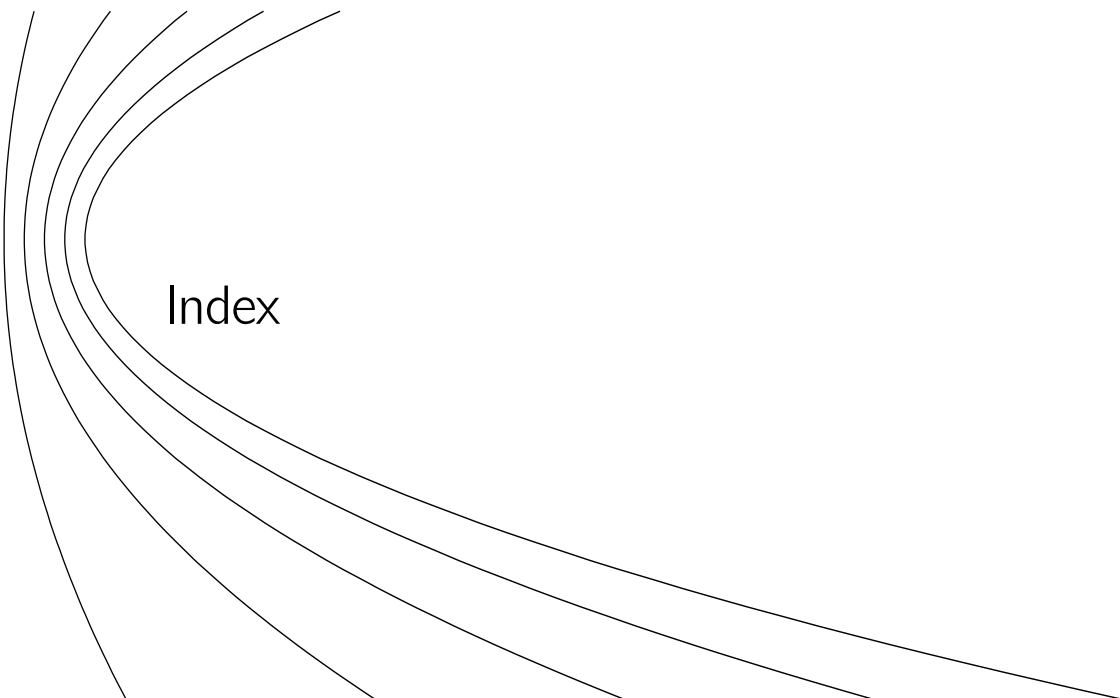
Bibliography

- [Bre91] Yann Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), no. 4, 375–417. MR MR1100809 (92d:46088)
- [Caf96] Luis A. Caffarelli, *Allocation maps with general cost functions*, Partial differential equations and applications, Lecture Notes in Pure and Appl. Math., vol. 177, Dekker, New York, 1996, pp. 29–35. MR MR1371577 (97f:49055)
- [CNS88] Luis Caffarelli, Louis Nirenberg, and Joel Spruck, *Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces*, Comm. Pure Appl. Math. **41** (1988), no. 1, 47–70. MR MR917124 (90b:35093)
- [dC92] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty. MR MR1138207 (92i:53001)
- [DL06] Philippe Delanoë and Grégoire Loeper, *Gradient estimates for potentials of invertible gradient-mappings on the sphere*, Calc. Var. Partial Differential Equations **26** (2006), no. 3, 297–311. MR MR2232207 (2007b:35108)
- [Eva98] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR MR1625845 (99e:35001)
- [Eva01] ———, *Partial differential equations and Monge-Kantorovich mass transfer*, On-line notes found at http://math.berkeley.edu/~evans/Monge-Kantorovich_survey.pdf, September 2001.

- [Fed69] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR MR0257325 (41 #1976)
- [Gan94] Wilfrid Gangbo, *An elementary proof of the polar factorization of vector-valued functions*, Arch. Rational Mech. Anal. **128** (1994), no. 4, 381–399. MR MR1308860 (96a:49048)
- [Ger96] Claus Gerhardt, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Differential Geom. **43** (1996), no. 3, 612–641. MR MR1412678 (97g:53067)
- [GF63] I. M. Gelfand and S. V. Fomin, *Calculus of variations*, Revised English edition translated and edited by Richard A. Silverman, Prentice-Hall Inc., Englewood Cliffs, N.J., 1963. MR MR0160139 (28 #3353)
- [GM95] Wilfrid Gangbo and Robert J. McCann, *Optimal maps in Monge's mass transport problem*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 12, 1653–1658. MR MR1367824 (96i:49004)
- [GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, REPRINT of the 1998 edition. MR MR1814364 (2001k:35004)
- [Kan42] L. Kantorovich, *On the translocation of masses*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **37** (1942), 199–201. MR MR0009619 (5,174d)
- [Kan48] L. V. Kantorovich, *On a problem of Monge (in Russian)*, Uspekhi Mat. Nauk. **3** (1948), 225–226.
- [Lev99] Vladimir Levin, *Abstract cyclical monotonicity and Monge solutions for the general Monge-Kantorovich problem*, Set-Valued Anal. **7** (1999), no. 1, 7–32. MR MR1699061 (2000j:90075)
- [Lev04] V. L. Levin, *Optimality conditions and exact solutions of the two-dimensional Monge-Kantorovich problem*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **312** (2004), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 11, 150–164, 314. MR MR2117888 (2005k:49123)
- [Loe05] G. Loeper, *On the regularity of maps solutions of optimal transportation problems*, arXiv Mathematics e-prints (2005), <http://arxiv.org/abs/math/0504137> arXiv: math/0504137.
- [LT86] Gary M. Lieberman and Neil S. Trudinger, *Nonlinear oblique boundary value problems for nonlinear elliptic equations*, Trans. Amer. Math. Soc. **295** (1986), no. 2, 509–546. MR MR833695 (87h:35114)

- [LTU86] P.-L. Lions, N. S. Trudinger, and J. I. E. Urbas, *The Neumann problem for equations of Monge-Ampère type*, Comm. Pure Appl. Math. **39** (1986), no. 4, 539–563. MR MR840340 (87j:35114)
- [LV04] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, arXiv Mathematics e-prints (2004), <http://arxiv.org/abs/math/0412127> arXiv:math/0412127.
- [McC01] Robert J. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. **11** (2001), no. 3, 589–608. MR MR1844080 (2002g:58017)
- [Mit70] D. S. Mitrinović, *Analytic inequalities*, In cooperation with P. M. Vasić. Die Grundlehren der mathematischen Wissenschaften, Band 1965, Springer-Verlag, New York, 1970. MR MR0274686 (43 #448)
- [Mon81] G. Monge, *Mémoire sur la théorie des déblais et des remblais*, Histoire de l'Académie Royale des Sciences de Paris (1781), 666–704.
- [MTW05] Xi-Nan Ma, Neil S. Trudinger, and Xu-Jia Wang, *Regularity of potential functions of the optimal transportation problem*, Arch. Ration. Mech. Anal. **177** (2005), no. 2, 151–183. MR MR2188047 (2006m:35105)
- [Pog64] A. V. Pogorelov, *Monge-Ampère equations of elliptic type*, Translated from the first Russian edition by Leo F. Boron with the assistance of Albert L. Rabenstein and Richard C. Bollinger, P. Noordhoff Ltd., Groningen, 1964. MR MR0180763 (31 #4993)
- [Roc66] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–510. MR MR0193549 (33 #1769)
- [SF04] A. Sobolevskii and U. Frish, *Application of optimal transportation theory to the reconstruction of the early Universe*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **312** (2004), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 11, 303–309, 317. MR MR2117895 (2005h:85003)
- [Stu06] Karl-Theodor Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131. MR MR2237206
- [SUW04] Weimin Sheng, John Urbas, and Xu-Jia Wang, *Interior curvature bounds for a class of curvature equations*, Duke Math. J. **123** (2004), no. 2, 235–264. MR MR2066938 (2005d:35087)
- [Tru90] Neil S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, Arch. Rational Mech. Anal. **111** (1990), no. 2, 153–179. MR MR1057653 (91g:35118)

- [Tru95] ———, *On the Dirichlet problem for Hessian equations*, Acta Math. **175** (1995), no. 2, 151–164. MR MR1368245 (96m:35113)
- [Tru07] ———, *Optimal transportation and nonlinear partial differential equations*, 26th Brazilian Mathematical Colloquium, 2007, Slides from conference talk.
- [TW97] Neil S. Trudinger and Xu-Jia Wang, *Hessian measures. I*, Topol. Methods Non-linear Anal. **10** (1997), no. 2, 225–239, Dedicated to Olga Ladyzhenskaya. MR MR1634570 (2000a:35061)
- [TW06] ———, *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, arXiv Mathematics e-prints (2006), [arXiv:math/0601086](https://arxiv.org/abs/math/0601086).
- [TWar] ———, *On strict convexity and continuous differentiability of potential functions in optimal transportation*, Arch. Ration. Mech. Anal. (to appear).
- [Urb95] John Urbas, *Nonlinear oblique boundary value problems for Hessian equations in two dimensions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), no. 5, 507–575. MR MR1353259 (96h:35071)
- [Urb97] ———, *On the second boundary value problem for equations of Monge-Ampère type*, J. Reine Angew. Math. **487** (1997), 115–124. MR MR1454261 (98f:35057)
- [Urb98a] ———, *Mass transfer problems*, Lecture notes given at the University of Bonn, 1998.
- [Urb98b] ———, *Oblique boundary value problems for equations of Monge-Ampère type*, Calc. Var. Partial Differential Equations **7** (1998), no. 1, 19–39. MR MR1624426 (99h:35068)
- [Urb01] ———, *The second boundary value problem for a class of Hessian equations*, Comm. Partial Differential Equations **26** (2001), no. 5-6, 859–882. MR MR1843287 (2002g:35071)



Index

- (A1) condition, 15
- (A2) condition, 15
- (A3) condition, 15
- (A3w) condition, 18
- admissible solution, 34
- Augmented Monge Cost Functional, 11
- boundary condition
 - natural, 27, 33
 - oblique, 46
 - second type, 27
- boundary point lemma
 - Hopf, 69
- c -concave function, 19
- c -convex function, 19
- c -convex set, 20
- c -segment, 20
- c -support function, 19
- condition
 - (A1), 15
 - (A2), 15
 - (A3), 15
 - (A3w), 18
 - mass-balance, 4
- cost-function, 4
- c^* -concave function, 19
- c^* -convex function, 19
- c^* -convex set, 20
- c^* -segment, 20
- c^* -support function, 19
- dual formulation
 - Kantorovich, 14
- dual-convex function, 22
- equation
 - Optimal Transportation, 28
 - Quotient Transportation, 39
- formula
 - Urbas type, 47
- function
 - c^* -convex, 19
 - admissible, 34
 - c -concave, 19
 - c -convex, 19
 - c -support, 19
 - c^* -concave, 19
 - c^* -support, 19
 - dual-convex, 22
 - semi-concave, 19
 - uniformly c -convex, 54

- functional
 - Augmented Monge Cost, 11
 - Kantorovich Dual, 14
 - Kantorovich Relaxed, 14
 - Monge Cost, 4
- gradient map, 82, 95
- gradient-potential, 82
- half-sphere stereographic projection, 75
- Hopf boundary point lemma, 69
- inequality
 - Newton, 43
- Kantorovich Dual Formulation, 14
- Kantorovich Dual Functional, 14
- Kantorovich Relaxed Formulation, 13
- Kantorovich Relaxed Functional, 14
- Lagrange multiplier, 11
- Lebesgue dominated convergence theorem, 6, 26
- Leray-Schauder fixed point theorem, 38, 69
- map
 - gradient, 82, 95
 - measure-preserving, 3
- mass-balance condition, 4
- matrix
 - modified-Hessian, 33
- measure
 - projection, 13
 - push-forward, 4
- measure-preserving map, 3
- method of continuity, 66
- modified-Hessian matrix, 33
- Monge Cost Functional, 4
- natural boundary condition, 27, 33
- Newton inequality, 43
- oblique boundary condition, 46
- operator
 - trace of, 42
- Optimal Transportation Equation, 28
- projection of measure, 13
- push-forward (measure), 4
- Quotient Transportation Equation, 39
- relaxed formulation
 - Kantorovich, 13
- second type boundary condition, 27
- semi-concave function, 19
- set
 - c -convex, 20
 - c^* -convex, 20
- solution
 - viscosity, 34
- stereographic projection
 - half-sphere, 75
- theorem
 - Lebesgue dominated convergence, 6, 26
 - Leray-Schauder fixed point, 38, 69
- trace (operator), 42
- uniformly c -convex function, 54
- Urbas type formula, 47
- viscosity solution, 34