

STATEMENT

ON HOMOMORPHISMS OF AN ORTHOGONALLY  
DECOMPOSABLE HILBERT SPACE

by

Trien Bang Dang

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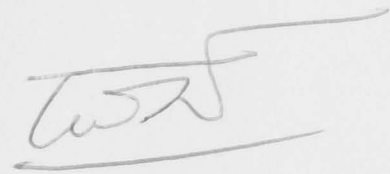
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## STATEMENT

Unless it is stated in the text to the contrary, the material contained in this thesis is the product of my own research.



T.B. Dang

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## INTRODUCTION

Let  $\mathcal{M}$  be a von Neumann algebra on the Hilbert space  $H$ .  $\xi \in H$  is called a cyclic vector for  $\mathcal{M}$  if

$$H = \overline{\{x\xi : x \in \mathcal{M}\}};$$

$\xi$  is said to be separating for  $\mathcal{M}$  if, for each  $a \in \mathcal{M}$  satisfying  $a\xi = 0$ ,  $a = 0$ . Assume that  $\mathcal{M}$  has cyclic separating vector  $\xi_0$ . Then

$$x\xi_0 \mapsto x^*\xi_0, \quad x \in \mathcal{M}$$

is a preclosed antilinear operator defined on the dense subset  $\mathcal{M}\xi_0$  of  $H$ . The closure of this operator is denoted by  $S$  and its polar decomposition by

$$S = J_{\xi_0} \Delta_{\xi_0}^{\frac{1}{2}}.$$

Araki [1] and Connes [1] showed that the set

$$H_{\xi_0}^+ = \overline{\{\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0 : x \in \mathcal{M}^+\}}$$

is a self-dual cone in  $H$ ; and, for each  $\xi \in H$  satisfying  $J_{\xi_0} \xi = \xi$ ,  $\xi$  has the unique orthogonal decomposition, that is, there are  $\xi^+ \in H_{\xi_0}^+$  and  $\xi^- \in H_{\xi_0}^+$  such that  $\xi = \xi^+ - \xi^-$  and  $(\xi^+, \xi^-) = 0$ .

It was also shown that

$$H_{\xi_0}^+ = \overline{\{xJ_{\xi_0} xJ_{\xi_0} \xi_0 : x \in \mathcal{M}\}}.$$

Connes [1] also characterized order isomorphisms on  $H$  whose generating positive cone is  $H_{\xi_0}^+$ .

In this thesis we shall study those order homomorphisms preserving orthogonal decomposition. For the sake of convenience, we shall call these o.d. homomorphisms.

The thesis consists of twelve sections. Sections 1 and 2 contain preliminaries in-

cluding Tomita-Takesaki's theory and various results of Araki and Connes on the order structure of  $H$ . Section 3 gives an example of a Hilbert space ordered by a simple cone  $P$  such that there are no von Neumann algebra  $\mathcal{M}$  on  $H$  and cyclic separating vector  $\xi_0$  satisfying

$$P = \overline{\{\Delta_{\xi_0}^{\frac{1}{2}} x \xi_0 : x \in \mathcal{M}^+\}}.$$

In Section 4 we find some basic properties of o.d. homomorphisms and show that there is a correspondence between Jordan isomorphisms on  $\mathcal{M}$  and o.d. isomorphisms on  $H$ ; and when  $\Delta_{\xi_0} = 1$  there is a correspondence between  $\tau$ -bounded unital Jordan homomorphisms on  $\mathcal{M}$  and  $\xi_0$ -preserving o.d. homomorphisms on  $H$ .

Section 5 deals with characterizations of o.d. homomorphisms. The main result of this section is that  $\phi$  is an o.d. homomorphism if and only if  $\phi(H_{\xi_0}^+) \subset H_{\xi_0}^+$  and  $\phi^* \phi$  belongs to the centre of  $\mathcal{M}$ . In Section 6, we show that if  $\phi$  is an o.d. homomorphism and  $\phi = u|\phi|$  is the polar decomposition of  $\phi$ , then both  $u$  and  $|\phi|$  are o.d. homomorphisms.

In Section 7, we characterize the generators of  $C_0$ -semigroups and uniform semigroups of o.d. homomorphisms.

In Section 8, we consider the differentiability of the absolute-valued function  $\theta : H^J \rightarrow H_{\xi_0}^+$  defined by

$$\theta(\xi) = |\xi|, \quad \xi \in H^J,$$

where

$$H^J = \{\xi \in H : J\xi = \xi\}.$$

We have only been able to prove the differentiability of  $\theta$  for  $\xi \in H_{\xi_0}^+$ .

In Section 9, we find equivalent conditions for all o.d. isomorphisms being normal. Also, it is shown that if  $\mathcal{M}$  is finite-dimensional then all o.d. homomorphisms are normal if and only if  $\mathcal{M}$  is a factor.

Let  $\eta_0 \in H_{\xi_0}^+$  be another cyclic and separating vector for  $\mathcal{M}$ . We set

$$L(\xi_o, \eta_o, H_{\xi_o}^+) = \{\phi \in L(H) : \phi(H_{\xi_o}^+) \subset H_{\xi_o}^+, \phi\xi_o = \eta_o\}.$$

We show in Section 10 that all o.d. homomorphisms in  $L(\xi_o, \eta_o, H_{\xi_o}^+)$  are extreme points; and  $\mathcal{M}$  is commutative if and only if for any positive cyclic separating vectors  $\xi$  and  $\eta$ , every extreme point of  $L(\xi, \eta, H_{\xi_o}^+)$  is an o.d. homomorphism.

In Section 11, we show various results concerning o.d. homomorphisms satisfying  $\phi\xi_o = \xi_o$ .

A continuous linear operator  $\delta : H \rightarrow H$  is a derivation if it satisfies the following conditions:

$$(i) J_{\xi_o} \delta = \delta J_{\xi_o}.$$

$$(ii) \text{ If } \xi \in H_{\xi_o}^+, \eta \in H_{\xi_o}^+ \text{ and } (\xi, \eta) = 0, \text{ then } (\delta\xi, \eta) = 0.$$

$D(H^+)$  denotes the set of all derivations. A continuous linear operator  $\phi$  is said to be derivation-preserving if  $J_{\xi_o} \phi = \phi J_{\xi_o}$  and  $\phi^* D(H^+) \phi \subset D(H^+)$ .

In Section 12, we show that a continuous linear operator  $\delta$  is an order homomorphic derivation if and only if  $\delta \in (\mathcal{M} \cap \mathcal{M}')^+$ . We characterize the generators of uniform semigroups of derivations. We also characterize the derivation-preserving operators to derive another characterization of o.d. homomorphisms.

## §1. TOMITA-TAKESAKI THEORY

Let  $\mathcal{M}$  be a von Neumann algebra acting on a Hilbert space  $H$ . We assume that there exists a cyclic and separating vector  $\xi_0 \in H$  for  $\mathcal{M}$ . Then, the operator

$$x\xi_0 \mapsto x^*\xi_0 \quad \text{for all } x \in \mathcal{M}$$

is a preclosed antilinear operator defined on the dense subset  $\mathcal{M}\xi_0$  of  $H$ . The closure of this operator is denoted by  $S$  and its polar decomposition by

$$S = J_{\xi_0} \Delta_{\xi_0}^{\frac{1}{2}}.$$

The operators  $J_{\xi_0}$  and  $\Delta_{\xi_0}$  depend on  $\xi_0$ . They are characterized as follows:

(1.1) (Araki [1]). The operator  $J_{\xi_0}$  has the following properties:

(i)  $(J_{\xi_0}\xi, J_{\xi_0}\eta) = (\eta, \xi)$  for all  $\xi, \eta \in H$ .

(ii)  $J_{\xi_0}^2 = 1$ .

(iii)  $j(\mathcal{M}) = \mathcal{M}'$ , where  $j(x) = J_{\xi_0} x J_{\xi_0}$  for  $x \in \mathcal{M}$ .

(iv)  $J_{\xi_0}\xi_0 = \xi_0$ .

(v)  $(xj(x)\xi_0, \xi_0) \geq 0$  for all  $x \in \mathcal{M}$ .

Conversely, if a linear operator  $J$  on  $H$  satisfies these conditions, then  $J = J_{\xi_0}$ .

This operator is called the *conjugation operator* associated with  $(\mathcal{M}, H, \xi_0)$ .

1.2 (Takesaki [1]). A densely defined operator  $\Delta$  satisfies  $\Delta = \Delta_{\xi_0}$  if and only if

(i)  $\Delta^* = \Delta$ .

(ii)  $\Delta$  is positive definite.

(iii)  $\sigma_t(x) = \Delta^{it} x \Delta^{-it} \in \mathcal{M}$  for every  $x \in \mathcal{M}$  and  $t \in \mathbb{R}$ .

(iv)  $\omega_{\xi_0}$  satisfies the KMS-conditions with respect to  $\{\sigma_t\}$ .

(v)  $\Delta^{it}\xi_0 = \xi_0$  for all  $t \in \mathcal{R}$ .

The operator  $\Delta_{\xi_0}$  is called the *modular operator* associated with  $(\mathcal{M}, H, \xi_0)$ .

Note. The condition (iv) above means that, for any  $x \in \mathcal{M}$  and  $y \in \mathcal{M}$ , there exists a bounded continuous function  $F$  on  $\{z \in \mathcal{C} : 0 \leq \text{Im}(z) \leq 1\}$  such that  $F$  is regular inside this set and satisfies

$$F(t) = \omega_{\xi_0}(\sigma_t(x)y)$$

and

$$F(t+i) = \omega_{\xi_0}(y\sigma_t(x))$$

for all  $t \in \mathcal{R}$ , where  $\omega_{\xi_0}$  is a linear functional defined on  $\mathcal{M}$  by

$$\omega_{\xi_0}(a) = (a\xi_0, \xi_0), \quad a \in \mathcal{M}.$$



## §2. THE NATURAL CONES

Let  $(\mathcal{M}, H, \xi_0)$  denote the triple consisting of von Neumann algebra  $\mathcal{M}$  acting on Hilbert space  $H$ , with  $\xi_0 \in H$  being a cyclic separating vector for  $\mathcal{M}$ . The *natural cone*  $H_{\xi_0}^+$  associated with  $(\mathcal{M}, H, \xi_0)$  is defined as

$$H_{\xi_0}^+ = \overline{\{\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0 : x \in \mathcal{M}^+\}}.$$

The theory of such cone was developed by Araki [1] and Connes [1].

When  $(\mathcal{M}, H, \xi_0)$  is fixed and there is no confusion, we shall denote  $J_{\xi_0}$ ,  $\Delta_{\xi_0}$  and  $H_{\xi_0}^+$  by  $J$ ,  $\Delta$  and  $H^+$  respectively. We set

$$H^J = \{\xi \in H : J\xi = \xi\}.$$

The natural cone has the following properties (Bratteli-Robinson [1], Propositions 2.5.26 and 2.5.28).

(2.1) (Araki [1], Theorem 4). Let  $H^+$  be the natural cone associated with  $(\mathcal{M}, H, \xi_0)$ . It follows that

$$(i) \quad H^+ = \overline{\{xj(x)\xi_0 : x \in \mathcal{M}\}}, \text{ where } j(x) = JxJ.$$

$$(ii) \quad \Delta^{it}(H^+) = H^+ \text{ for all } t \in \mathcal{R}.$$

$$(iii) \quad \text{If } f \text{ is a positive definite function, then } f(\log \Delta)(H^+) \subset H^+.$$

$$(iv) \quad H^+ \subset H^J.$$

$$(v) \quad xj(x)(H^+) \subset H^+ \text{ for all } x \in \mathcal{M}.$$

$$(vi) \quad H^+ \text{ is self-dual: } \xi \in H^+ \text{ if and only if } (\xi, \eta) \geq 0 \text{ for every } \eta \in H^+.$$

$$(vii) \quad H^+ \cap (-H^+) = \{0\}.$$

$$(viii) \quad \text{For any } \xi \in H^J, \text{ there exist } \xi^+ \in H^+ \text{ and } \xi^- \in H^+ \text{ such that}$$



$\xi = \xi^+ - \xi^-$  and  $(\xi^+, \xi^-) = 0$ . This "orthogonal decomposition" is unique, and the "positive part"  $\xi^+$  is determined by

$$\|\xi^+\| = \inf\{\|\xi + \eta\| : \eta \in H^+\}.$$

(ix)  $H$  is linearly spanned by  $H^+$ .

With respect to the order introduced by the natural cone  $H^+$ ,  $H^J$  is a real ordered Hilbert space and  $H = H^J + iH^J$  is a complex ordered Hilbert space. This order structure of  $H$  has been investigated by Araki [1] and Connes [1]. Their results are listed below. Most of them can be found in the book by Bratteli and Robinson [1], which we shall refer to as BR.

(2.2) (Araki [1], Theorem 4; BR, Proposition 2.5.30). Let  $\eta_o$  be a cyclic and separating vector for  $\mathcal{M}$  and  $H_{\eta_o}^+$  be the natural cone associated with  $(\mathcal{M}, H, \eta_o)$ , that is,

$$H_{\eta_o}^+ = \overline{\{xJ_{\eta_o}xJ_{\eta_o}\eta_o : x \in \mathcal{M}\}}.$$

Then, if  $\eta_o \in H^+$  ( $= H_{\xi_o}^+$ ), we have  $H_{\eta_o}^+ = H^+$  and  $J_{\eta_o} = J_{\xi_o}$ .

(2.3) (BR, Lemma 2.5.40). The map  $b : \mathcal{M} \rightarrow H$  defined by

$$b(x) = \Delta^{\frac{1}{4}} x \xi_o$$

is an order isomorphism of  $\mathcal{M}^h$ , the set of all self-adjoint operators in  $\mathcal{M}$ , onto the subset

$$F_{\xi_o} = \{\xi \in H^J : -\alpha\xi_o \leq \xi \leq \alpha\xi_o \text{ for some } \alpha > 0\}.$$

(2.4) (Connes [1], Lemma 4.3). For  $\xi \in H^+$ , the following conditions are equivalent

(i)  $\xi$  is cyclic for  $\mathcal{M}$ .

(ii)  $\xi$  is separating for  $\mathcal{M}$ .

(iii) If  $(\xi, \eta) = 0$  and  $\eta \in H^+$ , then  $\eta = 0$ .

(2.5) (Araki [1], Theorem 4; BR, Theorem 2.5.31). The map  $\xi \mapsto \omega_\xi$  is a norm-topological isomorphism from  $H^+$  onto  $(\mathcal{M}_*)^+$ , and

$$\|\xi - \eta\|^2 \leq \|\omega_\xi - \omega_\eta\| \leq \|\xi - \eta\| \cdot \|\xi + \eta\| \quad \text{for } \xi, \eta \in H^+.$$

(2.6) (BR, Corollary 2.5.32). For any automorphism  $\alpha$  of  $\mathcal{M}$ , there is a unique unitary operator  $u$  on  $H$  such that  $u(H^+) = H^+$  and

$$\alpha(x) = u x u^* \quad \text{for all } x \in \mathcal{M}.$$

(2.7) (BR, Theorem 3.2.15).

(i) For every Jordan automorphism  $\alpha$  of  $\mathcal{M}$ , there is a unique unitary operator  $u$  on  $H$  such that  $u(H^+) = H^+$  and

$$(2.7.ia) \quad (\alpha(x), \xi, \xi) = (u x u^* \xi, \xi) \quad \text{for all } x \in \mathcal{M} \text{ and } \xi \in H^+.$$

(ii) For every unitary operator  $u$  such that  $u(H^+) = H^+$ , there is a unique Jordan automorphism  $\alpha$  of  $\mathcal{M}$  which satisfies (2.7.i,a).

(iii) The pair  $(\alpha, u)$  satisfies

$$(a) \quad \Delta_{u\xi_0}^{\frac{1}{4}} \alpha(x) u \xi_0 = u(\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0) \quad \text{for all } x \in \mathcal{M}.$$

(b) There is a projection  $e \in \mathcal{M} \cap \mathcal{M}'$  such that  $u x u^* = \alpha(x)e + J\alpha(x^*)J(1 - e)$  for all  $x \in \mathcal{M}$ .

(2.8) (Connes [1], Theorem 3.9). Let  $T: H \rightarrow H$  be a positive definite and invertible operator such that  $T(H^+) = H^+$ . Then,  $T = aj(a)$  for some  $a \in \mathcal{M}$ .

(2.9) (Connes [1], Theorem 3.3). Let  $\phi: H \rightarrow H$  be a continuous linear bijection such that  $\phi(H^+) = H^+$ , and let  $\phi = u|\phi|$  be the polar decomposition of  $\phi$ . Then,

(i)  $u(H^+) = H^+$ .

(ii)  $|\phi| = aj(a)$  for some  $a \in \mathcal{M}$ .

(2.10) (Connes [1], Theorem 3.4 and Lemma 5.3). For a continuous linear operator  $\delta$  on  $H$ , the following conditions are equivalent:

(i)  $e^{t\delta}(H^+) = H^+$  for all  $t \in \mathcal{R}$ .

(ii)  $J\delta = \delta J$  and  $(\delta\xi, \eta) = 0$  whenever  $\xi, \eta \in H^+$  and  $(\xi, \eta) = 0$ .

(iii)  $\delta = x + j(x)$  for some  $x \in \mathcal{M}$ .

(iv)  $J\delta = \delta J$  and  $[\delta, \mathcal{M}] \subset \mathcal{M}$ , where  $[\delta, x] = \delta x - x\delta$ .

The set of all such  $\delta$  will be denoted by  $D(H^+)$ .

A subset  $F$  of  $H^+$  is called a *subcone* if  $\xi \in F$ ,  $\eta \in F$  and  $\lambda \in \mathcal{R}^+$  imply  $\xi + \eta \in F$  and  $\lambda\xi \in F$ . It is said to be *hereditary* if  $0 \leq \xi \leq \eta$  and  $\eta \in F$  imply  $\xi \in F$ . A hereditary subcone of  $H^+$  is called a *face* of  $H^+$ . The set

$$F_\xi = \{\eta \in H^+ : 0 \leq \eta \leq \lambda\xi \text{ for some } \lambda > 0\}$$

is a face of  $H^+$ , which is called the face generated by  $\xi$ .

For any  $\xi \in H^+$ , the cyclic projection  $p_\xi$  is defined by  $p_\xi = [\mathcal{M}'\xi]$ .

(2.11) (Connes [1], Theorem 4.2). (i) For every  $\xi \in H^+$ ,  $\bar{F}_\xi = p_\xi j(p_\xi)H^+$ .

(ii) For every face  $F$  of  $H^+$ , there exists a projection  $e \in \mathcal{M}$  such that  $\bar{F} = ej(e)H^+$ . Furthermore,  $ej(e) = P_F$  is the projection to the subspace generated by  $F$ .

For every  $\xi \in H^+$ , the cyclic projection  $p_\xi = [\mathcal{M}'\xi]$  is an element of  $\mathcal{M}$ .  $p_\xi$  coincides with the support of the linear functional  $\omega_\xi$ , that is,  $p_\xi = s(\omega_\xi)$ . The following result is a direct consequence of (2.5).

(2.12) (Araki [1], Theorem 4; Strătilă-Zsidó [1], 10.24, Corollary 1). Let  $\xi \in H^+$  and  $\eta \in H^+$ .

(i) If  $(\xi, \eta) = 0$ ,  $p_\xi p_\eta = 0$ .

(ii) If  $(\xi, \rho) = 0$  and  $\rho \in H^+$  imply that  $(\eta, \rho) = 0$ , then  $p_\eta \leq p_\xi$ .

(iii) If  $\eta \leq \xi$ ,  $p_\eta \leq p_\xi$ .

### 13. AN ORDERED HILBERT SPACE

The ordered Hilbert space  $H$  ordered by the cone  $H^+$  is called a *positive ordered Hilbert space*. It is known that  $H$  is a positive ordered Hilbert space if and only if  $H$  is commutative, and  $p$  is an order unit in  $H$ . It also has the property that  $H^+$  is a lattice in  $H$ . The following theorem characterizes positive ordered Hilbert spaces. The following result is an immediate consequence of Corollary 1, Theorem 2.2, Definition 3.1 and Theorem 3.2.

Let  $H$  be a separable Hilbert space equipped with a proper cone  $H^+$ . Then, there exist a von Neumann algebra  $M$  on  $H$  and a cyclic separation vector  $\xi$  for  $M$  such that  $H^+$  is equal to the normal cone  $H^+$  associated with  $(M, H, \xi)$  if and only if  $H$  is self-dual, homogeneous and isotropic.

The definition of self-duality is given in (2.1)(v).

$H$  is said to be homogeneous if for every  $\lambda \in H^+$ ,  $\lambda \neq 0$ ,

$$UP_\lambda = P_\lambda U \quad \text{for all } U \in \mathcal{U}(H).$$

where  $P_\lambda$  and  $P_{\lambda^2}$  are the projections on the closed subspaces generated by  $\lambda$  and  $\lambda^2$  respectively.  $H$  is said to be isotropic if  $\lambda \in H^+$  implies  $\lambda^2 \in H^+$  for all  $\lambda \in H^+$ .

We denote by  $D(H)$  the set of all continuous linear operators  $T$  on  $H$  such that

$$T(H^+) \subseteq H^+ \quad \text{for all } T \in D(H).$$

$D(H)$  is an involutive Lie algebra. Let  $\mathcal{D}(H)$  be the quotient of  $D(H)$  by its center.

Then, an operator  $T$  on  $\mathcal{D}(H)$  is called an *abstract derivation* if it satisfies the following conditions

$$(i) \quad T^2 = 0,$$

$$(ii) \quad T(A_1 A_2) = (T A_1) A_2 + A_1 (T A_2) \quad \text{for all } A_1, A_2 \in \mathcal{D}(H).$$

### §3. AN ORDERED HILBERT SPACE

The ordered Hilbert space  $H$ , ordered by the natural cone  $H^+$ , is lattice-ordered if and only if  $\mathcal{M}$  is commutative, and it is an antilattice if and only if  $\mathcal{M}$  is a factor (Kadison [2] and Bös [1]). Among various order structures between these two extreme cases, only some are defined by natural cones associated with some  $(\mathcal{M}, H, \xi_0)$ . The characterization of such order structures was given by Connes [1]. The following result is an immediate consequence of Connes ([1], Theorem 5.8, Definition 5.6 and Theorem 5.2).

*Let  $H$  be a separable Hilbert space equipped with a proper and closed cone  $P$ . Then, there exist a von Neumann algebra  $\mathcal{M}$  on  $H$  and a cyclic separating vector  $\xi_0$  for  $\mathcal{M}$  such that  $P$  is equal to the natural cone  $H^+$  associated with  $(\mathcal{M}, H, \xi_0)$  if and only if  $P$  is self-dual, homogeneous and orientable.*

The definition of self-duality was given in (2.1)(vi).

$P$  is said to be *homogeneous* if, for every face  $F$  of  $P$ ,

$$e^{t(P_F - P_{F^\perp})}(P) = P \quad \text{for all } t \in \mathcal{R},$$

where  $P_F$  and  $P_{F^\perp}$  are the projections on the closed subspaces generated by  $F$  and  $F^\perp = \{\xi \in P : (\xi, \eta) = 0 \text{ for all } \eta \in F\}$  respectively.

We denote by  $D(P)$  the set of all continuous linear operators  $\delta$  on  $H$  such that

$$e^{t\delta}(P) = P \quad \text{for all } t \in \mathcal{R}.$$

$D(P)$  is an involutive Lie algebra. Let  $\underline{D}(P)$  be the quotient of  $D(P)$  by its centre. Then, an operator  $I$  on  $\underline{D}(P)$  is called an *orientation* if it satisfies the following three conditions:

$$(i) \quad I^2 = 1.$$

$$(ii) \quad [I\delta_1, \delta_2] = [\delta_1, I\delta_2] = I[\delta_1, \delta_2] \quad \text{for all } \delta_1, \delta_2 \in \underline{D}(P).$$



(iii)  $I(\delta^*) = -I(\delta)^*$  for all  $\delta \in \underline{D}(P)$ .

$P$  is said to be *orientable* if  $\underline{D}(P)$  admits an orientation.

In this section, we show an example of a Hilbert space, ordered by a simple cone, which is not orientable. It is a kind of "spin factor" considered in the book by Hanche-Olsen and Størmer [1].

Let  $H$  be a real Hilbert space and suppose that there is a complete orthonormal sequence  $\theta_0, \theta_1, \dots, \theta_N$ , where  $1 \leq N \leq \infty$ . Then, every element  $\xi \in H$  is expressed as

$$\xi = \sum_{n=0}^N \xi_n \theta_n,$$

where

$$\xi_n = (\xi, \theta_n).$$

We set

$$H^+ = \left\{ \xi \in H : \xi_0 \geq \left( \sum_{n=1}^N \xi_n^2 \right)^{\frac{1}{2}} \right\}.$$

This is a closed and proper cone, and it makes  $H$  an ordered Hilbert space.

We shall prove the following properties of this cone.

(3.1).  $H$  is a vector lattice when  $N = 1$ . For all  $N \geq 2$ ,  $H$  is an antilattice.

(3.2). For each  $N$ ,  $H^+$  is a self-dual cone.

(3.3). For each  $N$ ,  $H^+$  is a homogeneous cone.

(3.4). There is a von Neumann algebra  $\mathcal{M}$  on the complexification  $H^c$  of  $H^+$  and a cyclic separating vector  $\mu \in H^c$  for  $\mathcal{M}$  such that  $H^+$  is the associated natural cone  $H_\mu^+$  if and only if  $N = 1$  or  $N = 3$ .

For  $\xi, \eta \in H$ , we write  $\xi \geq \eta$  if  $\xi - \eta \in H^+$ .

(3.1).  $H$  is a vector lattice when  $N = 1$ . For all  $N \geq 2$ ,  $H$  is an antilattice,

**Proof.** Let  $N = 1$  ; then  $H^+ = \{(\xi_0, \xi_1) : \xi_0 \geq |\xi_1|\}$ . Let  $\mathcal{R}^2$  be the two-dimensional Euclidean space equipped with pointwise order. Then  $\mathcal{R}^2$  is a Hilbert lattice and the map

$$(\xi_0, \xi_1) \mapsto (\xi_0 + \xi_1, \xi_0 - \xi_1)$$

is an order isomorphism of  $H$  onto  $\mathcal{R}^2$ . Hence,  $H$ , ordered by  $H^+$ , is lattice-ordered.

We next prove that  $H$  is an antilattice when  $N \geq 2$ . By definition,  $H$  is an antilattice if  $\xi \wedge \eta = 0$  implies  $\xi = 0$  or  $\eta = 0$  (Kadison [2], Luxemburg [1]). Suppose, therefore, there are  $\xi = (\xi_n)$ ,  $\eta = (\eta_n)$  in  $H^+$  such that  $\xi \wedge \eta = 0$ . We assume that  $\xi \neq 0$  and  $\eta \neq 0$ . Then,  $\xi_0 \neq 0$  and  $\eta_0 \neq 0$ . Since  $(\alpha\xi) \wedge \eta = 0$  for any positive number  $\alpha$ , we may assume that  $\xi_0 = \eta_0 = 1$ . For convenience, we shall use the notation

$$\|\xi\|_0 = \left( \sum_{n=1}^N \xi_n^2 \right)^{\frac{1}{2}}.$$

Now, consider an element  $\rho$  of  $H$  defined by

$$\rho = \left( 0, \frac{\xi_1 + \eta_1}{2}, \frac{\xi_2 + \eta_2}{2}, \dots \right).$$

Then, since  $\|\xi\|_0 \leq 1$  and  $\|\eta\|_0 \leq 1$ , we have  $\xi \geq \rho$  and  $\eta \geq \rho$ . Hence, by the assumption,  $\rho \leq \xi \wedge \eta = 0$ , and this implies  $\eta_n = -\xi_n$  for  $n \geq 1$ . Thus,

$$\xi = (1, \xi_1, \xi_2, \dots)$$

and

$$\eta = (1, -\xi_1, -\xi_2, \dots).$$

We now consider

$$\nu = (1 - \|\xi\|_0, 0, 0, \dots).$$

It is obvious that  $\xi \geq \nu$  and  $\eta \geq \nu$ . Then,  $\nu \leq 0$  and this implies  $\|\xi\|_0 = 1$  and  $\|\eta\|_0 = 1$ . We may assume that  $\xi_1 > 0$ ; otherwise, we take  $-\xi_1$ .

If  $\xi_1 = 1$ , we have  $\xi_2 = \xi_3 = \dots = 0$  because  $\|\xi\|_0 = 1$ . Then

$$\xi = (1, 1, 0, 0, \dots)$$

and

$$\eta = (1, -1, 0, 0, \dots).$$

In this case, take a number  $t$  such that  $1 < t < \frac{\sqrt{7}}{2}$  and set

$$\lambda = (-1, \frac{1}{2}, t, 0, 0, \dots).$$

Then,  $\xi \geq \lambda$  and  $\eta \geq \lambda$ . Hence,  $\lambda \leq 0$ , which is a contradiction because

$$(\frac{1}{2})^2 + t^2 > 1.$$

Therefore,  $0 < \xi_1 < 1$ . We set

$$t = -\xi_1 + \sqrt{\xi_1^2 + 3}$$

and

$$\mu = (-1, t, 0, 0, \dots).$$

Then  $\xi \geq \mu$  and  $\eta \geq \mu$ . Hence,  $\mu \leq 0$ , which is a contradiction because  $t > 1$ .

Therefore, we have  $\xi = 0$  or  $\eta = 0$ . □

(3.2). For every  $N$ ,  $H^-$  is self-dual.

**Proof.** If  $\xi, \eta \in H^+$ ,

$$\begin{aligned} (\xi, \eta) &= \sum_{n=0}^N \xi_n \eta_n = \xi_0 \eta_0 + \sum_{n=1}^N \xi_n \eta_n \\ &\geq \xi_0 \eta_0 - \|\xi\|_0 \|\eta\|_0 \\ &\geq 0. \end{aligned}$$

Suppose  $(\xi, \eta) \geq 0$  for every  $\eta \in H^+$ . Take

$$\eta = \left(1, \frac{-\xi_1}{\|\xi\|_o}, \frac{-\xi_2}{\|\xi\|_o}, \dots\right).$$

Then,

$$0 \leq (\xi, \eta) = \xi_o - \|\xi\|_o,$$

that is,  $\xi \in H^+$ . □

The following three results are needed to prove (3.3).

(3.4). Let  $\xi, \eta \in H^+ \setminus \{0\}$ . Then,  $(\xi, \eta) = 0$  if and only if there exists a positive number  $\alpha$  such that  $\xi_o = \|\xi\|_o$ ,  $\eta_o = \|\eta\|_o$ ,  $\eta_n = -\alpha\xi_n$ ,  $\eta_o = \alpha\xi_o$ .

**Proof.** Suppose that  $(\xi, \eta) = 0$ . Then,

$$\xi_o \eta_o = - \sum_{n=1}^N \xi_n \eta_n \leq \|\xi\|_o \|\eta\|_o \leq \xi_o \eta_o.$$

This implies that  $\xi_o = \|\xi\|_o$  and  $\eta_o = \|\eta\|_o$ ; and the equality

$$\|\xi\|_o \|\eta\|_o = \sum_{n=1}^N (-\xi_n) \eta_n$$

implies that  $\eta_n = -\alpha\xi_n$  for some positive number  $\alpha$ . Consequently,  $\eta_o = \alpha\xi_o$ .

The converse is obvious. □

(3.5). Let  $F$  be a face in  $H^+$ .

(i) If there exists  $\xi \in F$  such that  $\xi_o > \|\xi\|_o$ , then  $F = H^+$ .

(ii) If  $\xi_o = \|\xi\|_o$  for every  $\xi \in F$ , then for an arbitrary non-zero  $\xi \in F$ ,  $F = \{\alpha\xi : \alpha \geq 0\}$  and  $F^\perp = \{\alpha\eta : \alpha \geq 0\}$ , where  $\eta = (\xi_o, -\xi_1, -\xi_2, \dots)$ .

**Proof.** (i) Let  $\rho \in H^+$  and  $\alpha$  a positive number such that

$$\alpha^2(\xi_o^2 - \|\xi\|_o^2) + 2\alpha \left( \sum_{n=1}^N \xi_n \rho_n - \xi_o \rho_o \right) + (\rho_o^2 - \|\rho\|_o^2) \geq 0.$$

This inequality is equivalent to  $\alpha\xi \geq \rho$ . Hence,  $\rho \in F$ .

(ii) Let  $\rho$  and  $\xi$  be arbitrary elements of  $F$ . Since  $F$  is a cone,  $\xi + \rho \in F$ . By the assumption,  $\xi_0 + \rho_0 = \|\xi + \rho\|_0$ ,  $\xi_0 = \|\xi\|_0$  and  $\rho_0 = \|\rho\|_0$ ; and hence there is a real number  $\alpha$  such that  $\rho_n = \alpha\xi_n$  for all  $n \geq 1$ . Again by the assumption,  $\rho_0 = |\alpha|\xi_0$ .  $\alpha$  must be non-negative; otherwise,  $\rho + |\alpha|\xi \in F$  and  $\rho + |\alpha|\xi = (2|\alpha|\xi_0, 0, 0, \dots)$  would be a contradiction. Hence, there exists a positive number  $\alpha$  such that  $\rho = \alpha\xi$ . Therefore,

$$F = \{\alpha\xi : \alpha \geq 0\}.$$

It follows now from (3.4) that

$$F^\perp = \{\alpha\eta : \alpha \geq 0\},$$

where  $\eta = (\xi_0, -\xi_1, -\xi_2, \dots)$ . □

We note that self-dual cones in a real Hilbert space are generating (Bellissard-Iochum [1], Lemma 10), that is,  $H = H^+ - H^+$ , and  $H^c = (H^+ - H^+) + i(H^+ - H^+)$ . Hence, (3.5)(i) states that  $F$  spans  $H^c$  and (3.4) implies  $F^\perp = \{0\}$ . But it follows from (3.5)(ii) that  $F$  and  $F^\perp$  span one-dimensional subspaces. Therefore, (3.5) implies the following.

(3.6) Let  $F$  be a face in  $H^+$ .

(i) If there exists  $\xi \in F$  such that  $\xi_0 > \|\xi\|_0$ , then  $P_F = 1$  and  $P_{F^\perp} = 0$ .

(ii) If  $\xi_0 = \|\xi\|_0$  for all  $\xi \in F$ , choose  $\xi \in F$  such that  $\|\xi\| = 1$ . Then,  $P_F$  and  $P_{F^\perp}$  are expressed as  $N \times N$  matrices of the following form:

$$P_F = \begin{pmatrix} \xi_0^2 & \xi_0\xi_1 & \xi_0\xi_2 & \dots \\ \xi_0\xi_1 & \xi_1^2 & \xi_1\xi_2 & \dots \\ \xi_0\xi_2 & \xi_1\xi_2 & \xi_2^2 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad P_{F^\perp} = \begin{pmatrix} \xi_0^2 & -\xi_0\xi_1 & -\xi_0\xi_2 & \dots \\ -\xi_0\xi_1 & \xi_1^2 & \xi_1\xi_2 & \dots \\ -\xi_0\xi_2 & \xi_1\xi_2 & \xi_2^2 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$



and so

$$P_F - P_{F^\perp} = 2 \begin{pmatrix} 0 & \xi_0 \xi_1 & \xi_0 \xi_2 & \dots \\ \xi_0 \xi_1 & 0 & 0 & \dots \\ \xi_0 \xi_2 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We are now ready to prove (3.3). Recall that  $H^+$  is said to be homogeneous if for every face  $F$  of  $H^+$ ,

$$e^{t(P_F - P_{F^\perp})}(H^+) = H^+ \quad \text{for all } t \in \mathcal{R}.$$

It follows from Evans [Hanche-Olsen [1], Theorem 1] that this condition is equivalent to  $((P_F - P_{F^\perp})\xi, \eta) = 0$  for all  $\xi, \eta \in H^+$  satisfying  $(\xi, \eta) = 0$ .

(3.3) For each  $N$ ,  $H^+$  is a homogeneous cone.

**Proof.** Let  $F$  be an arbitrary face of  $H^+$ . Suppose  $\xi, \eta \in H^+$  and  $(\xi, \eta) = 0$ .

The case when  $P_F$  and  $P_{F^\perp}$  satisfy (3.6)(i) is trivial. Now assume, as in (3.6)(ii),

$$P_F - P_{F^\perp} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \dots \\ \beta_1 & 0 & 0 & \dots \\ \beta_2 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then, using the same  $\alpha$ ,  $\xi$ , and  $\eta$  as in (3.4), we have

$$\begin{aligned} ((P_F - P_{F^\perp})\xi, \eta) &= \left( \sum_{n=1}^N \beta_n \xi_n \right) \eta_0 + \sum_{n=1}^N (\beta_n \xi_0) \eta_n \\ &= \alpha \left( \sum_{n=1}^N \beta_n \xi_n \right) \xi_0 - \alpha \left( \sum_{n=1}^N \beta_n \xi_n \xi_0 \right) \\ &= 0. \end{aligned}$$

□

Recall that  $H$  is a real Hilbert space which possesses a complete orthonormal

sequence  $\theta_0, \theta_1, \theta_2, \dots, \theta_N$ , where  $1 \leq N \leq \infty$ ; and

$$H^+ = \{\xi \in H : \xi_0 \geq \|\xi\|_0\}.$$

Obviously,  $\theta_0 \in H^+$  and, furthermore,  $\theta_0$  satisfies the following condition: if  $\xi \in H^+$  and  $(\xi, \theta_0) = 0$ , then  $\xi = 0$ . Therefore, if there is a von Neumann algebra  $\mathcal{M}$  on  $H^c$ , the complexification of  $H$ , and a cyclic separating vector  $\mu \in H$  for  $\mathcal{M}$  such that  $H^+$  is the natural cone associated with  $(\mathcal{M}, H^c, \mu)$ , then by (2.4)  $\theta_0$  is a positive cyclic separating vector for  $\mathcal{M}$  and hence,  $H^+$  is the natural cone associated with  $(\mathcal{M}, H^c, \theta_0)$  by (2.2).

(3.7). Suppose that there is a von Neumann algebra  $\mathcal{M}$  on  $H^c$  such that  $H^+$  is the natural cone associated with  $(\mathcal{M}, H^c, \theta_0)$ . Then, the number of mutually orthogonal (non-zero) projections in  $\mathcal{M}$  is at most 2.

**Proof.** Let  $p, q$  and  $r$  be three non-zero orthogonal projections in  $\mathcal{M}$  such that  $p \perp q$  and  $p \perp r$ . Let  $J$  denote the conjugation operator associated with  $(\mathcal{M}, H^c, \theta_0)$ . We consider  $\xi = pJp\theta_0$ ,  $\eta = qJq\theta_0$  and  $\rho = rJr\theta_0$ . Then,  $\xi \neq 0$  because otherwise (see Connes' proof of [1, Lemma 4.4])

$$\begin{aligned} (\Delta^{\frac{1}{2}}p\theta_0, p\theta_0) &= (Jp\theta_0, J\Delta^{\frac{1}{2}}p\theta_0) \\ &= (Jp\theta_0, p\theta_0) \\ &= 0 \end{aligned}$$

would imply  $p = 0$ . Similarly,  $\eta \neq 0$  and  $\rho \neq 0$ . Hence,  $\xi, \eta$  and  $\rho$  are mutually orthogonal non-zero elements in  $H^+$ . Then, by (3.4), there are positive numbers  $\alpha$  and  $\beta$  such that  $\eta_0 = \alpha\xi_0$ ,  $\eta_n = -\alpha\xi_n$ ,  $\rho_0 = \beta\xi_0$  and  $\rho_n = -\beta\xi_n$ . Hence  $\eta$  and  $\rho$  cannot be orthogonal.  $\square$

We are now ready to prove (3.4).

(3.4). There is a von Neumann algebra  $\mathcal{M}$  on the complexification  $H^c$  of  $H$  and a cyclic separating vector  $\mu \in H^c$  for  $\mathcal{M}$  such that  $H^+$  is the associated natural cone if and only if  $N = 1$  or  $N = 3$ .

**Proof.** When  $N = 1$ , let  $\mathcal{M}$  be the algebra of all  $2 \times 2$  matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Then,  $H^+$  is the natural cone associated with  $(\mathcal{M}, H^c, \theta_0)$ .

When  $N \geq 2$ ,  $H$  is an antilattice by (3.2). Hence, such an algebra  $\mathcal{M}$  is a factor (Bös [1]). Therefore,  $\mathcal{M}$  is the algebra of  $n \times n$  matrices for some  $n$ . By (3.7), we have  $n = 2$ , and hence  $N = 3$ . Conversely, when  $N = 3$ ,  $H^+$  is the natural cone associated with  $M_2(\mathbb{C})$  with the Pauli matrices as  $\{\theta_n\}$  (BR, Example 4.2.7).  $\square$

#### §4. O.D. HOMOMORPHISMS

The studies of those operators preserving orthogonal decompositions in general orthogonally decomposable ordered Banach spaces have been done in Yamamuro ([1], [2]). We shall investigate the properties of such operators defined on an ordered Hilbert space associated with a von Neumann algebra with a cyclic separating vector.

Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $H$  and  $\xi_0 \in H$  a cyclic and separating vector for  $\mathcal{M}$ . Let  $H^+$  denote the natural cone associated with  $(\mathcal{M}, H, \xi_0)$ ,  $J$  the conjugation operator and

$$H^J = \{\xi \in H : J\xi = \xi\}.$$

Then, by (2.1)(viii), every element  $\xi \in H^J$  admits a unique orthogonal decomposition  $\xi = \xi^+ - \xi^-$ , where  $\xi^+ \in H^+$ ,  $\xi^- \in H^+$  and  $(\xi^+, \xi^-) = 0$ . We define the absolute-value mapping  $|\cdot| : H^J \rightarrow H^J$  by

$$|\xi| = \xi^+ + \xi^-, \quad \xi \in H^J.$$

With the foregoing notations, a continuous linear operator  $\phi : H \rightarrow H$  is called an o.d. (orthogonally decomposable) homomorphism if, for every  $\xi \in H^J$ ,  $\phi\xi \in H^J$  and  $\phi\xi = \phi\xi^+ - \phi\xi^-$  is the orthogonal decomposition of  $\phi\xi$ .

It follows immediately from this definition that

(4.1). *The following conditions are equivalent:*

(i)  $\phi$  is an o.d. homomorphism.

(ii)  $\phi(H^+) \subset H^+$ , and  $(\phi\xi, \phi\eta) = 0$  whenever  $\xi, \eta \in H^+$  and  $(\xi, \eta) = 0$ .

(iii)  $|\phi\xi| = \phi|\xi|$  for  $\xi \in H^J$ .

**Proof.** (i)  $\Rightarrow$  (ii). For any  $\xi \in H^+$ ,  $\xi - 0$  is the orthogonal decomposition of  $\xi$ , so  $\phi(\xi) - \phi(0)$  is the orthogonal decomposition of  $\phi\xi$ , that is,  $\phi\xi \in H^+$ . Now

for any  $\xi, \eta \in H^+$  such that  $(\xi, \eta) = 0$ ,  $\xi - \eta$  is the orthogonal decomposition of  $\zeta = \xi - \eta \in H^J$ , so  $\phi\xi - \phi\eta$  is the orthogonal decomposition of  $\phi(\zeta)$ , that is,  $(\phi\xi, \phi\eta) = 0$ .

(ii)  $\Rightarrow$  (iii). For any  $\xi \in H^J$  with its orthogonal decomposition  $\xi = \xi^+ - \xi^-$ , by hypothesis,  $\phi\xi^+ \in H^+$  and  $\phi\xi^- \in H^+$ , so

$$\begin{aligned} J\phi\xi &= J\phi\xi^+ - J\phi\xi^- \\ &= \phi\xi^+ - \phi\xi^- \\ &= \phi\xi. \end{aligned}$$

Hence  $\phi\xi$  has the unique orthogonal decomposition  $(\phi\xi)^+ - (\phi\xi)^-$ . On the other hand, by hypothesis,  $\phi\xi = \phi\xi^+ - \phi\xi^-$ , where  $\phi\xi^+ \in H^+$ ,  $\phi\xi^- \in H^+$  and  $(\phi\xi^+, \phi\xi^-) = 0$ , that is,  $\phi\xi^+ - \phi\xi^-$  is an orthogonal decomposition of  $\phi\xi$ . Hence, by the uniqueness of orthogonal decomposition in  $H^J$ ,  $(\phi\xi)^+ = \phi\xi^+$  and  $(\phi\xi)^- = \phi\xi^-$ . Thus, we have

$$\begin{aligned} |\phi\xi| &= (\phi\xi)^+ + (\phi\xi)^- \\ &= \phi\xi^+ + \phi\xi^- \\ &= \phi|\xi|. \end{aligned}$$

(iii)  $\Rightarrow$  (i). For any  $\xi \in H^J$  with its orthogonal decomposition  $\xi = \xi^+ - \xi^-$ , by hypothesis  $\phi\xi^+ = \phi|\xi^+| = |\phi\xi^+|$  and hence,  $\phi\xi^+ \in H^+$ . Similarly,  $\phi\xi^- \in H^+$ . Also, by hypothesis,  $\phi\xi^+ + \phi\xi^- = \phi|\xi| = |\phi\xi| = (\phi\xi)^+ + (\phi\xi)^-$ . On the other hand,  $\phi\xi^+ - \phi\xi^- = (\phi\xi)^+ - (\phi\xi)^-$ . Hence,  $\phi\xi^+ = (\phi\xi)^+$  and  $\phi\xi^- = (\phi\xi)^-$ . Thus,  $\phi\xi = \phi\xi^+ - \phi\xi^-$  is the orthogonal decomposition of  $\phi\xi$ .  $\square$

#### (4.2). Examples of o.d. homomorphisms

**Example (4.2.i).** A unitary operator  $u$  satisfying  $u(H^+) = H^+$  is an o.d. homomorphism.

**Example (4.2.ii).** Every central projection is an o.d. homomorphism. Let  $p$  be an arbitrary central projection.  $p = p^2 = pJpJ$ , so  $p(H^+) \subset H^+$  by (2.1)(v).



Furthermore, for any  $\xi \in H^+$ ,  $\eta \in H^+$  such that  $(\xi, \eta) = 0$ , since  $p\xi \in [\mathcal{M}'\xi]$  and  $p\eta \in [\mathcal{M}'\eta]$  we have  $(p\xi, p\eta) = 0$  by (2.12). So  $p$  is an o.d. homomorphism.

**Example (4.2.iii).** (Yamamuro [2] (3.4)). For  $a \in \mathcal{M}$ , the operator  $aJaJ$  is an o.d. homomorphism if and only if  $a^*a \in \mathcal{M} \cap \mathcal{M}'$ . Indeed, suppose  $a \in \mathcal{M}$  and  $aj(a)$  is an o.d. homomorphism. Let  $p$  be any projection in  $\mathcal{M}$ . Then  $pj(p)\xi_o \in H^+$ ,  $(1-p)j(1-p)\xi_o \in H^+$  and  $(pj(p)\xi_o, (1-p)j(1-p)\xi_o) = 0$ , so  $(aj(a)pj(p)\xi_o, aj(a)(1-p)j(1-p)\xi_o) = 0$ , that is,

$$((1-p)a^*ap\xi_o, Jpa^*a(1-p)J\xi_o) = 0.$$

Thus,

$$\begin{aligned} (\Delta^{\frac{1}{2}}pa^*a(1-p)\xi_o, pa^*a(1-p)\xi_o) &= (Jpa^*a(1-p)\xi_o, J\Delta^{\frac{1}{2}}pa^*a(1-p)\xi_o) \\ &= (Jpa^*a(1-p)\xi_o, (1-p)a^*ap\xi_o) \\ &= 0, \end{aligned}$$

and hence,  $pa^*a(1-p) = 0$ , that is,  $pa^*a = pa^*ap$ . Since  $pa^*ap$  is self-adjoint,  $pa^*a = (pa^*a)^* = a^*ap$ . Therefore,  $a^*a \in \mathcal{M} \cap \mathcal{M}'$ .

Conversely, suppose that  $a \in \mathcal{M}$ ,  $a^*a \in \mathcal{M} \cap \mathcal{M}'$ ,  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ . Then, by (2.12) we have

$$\begin{aligned} (aj(a)\xi, aj(a)\eta) &= (a^*aj(a^*a)\xi, \eta) \\ &= (a^*a a^*a\xi, \eta) \\ &= 0. \end{aligned}$$

**Example (4.2.iv).** As mentioned in the beginning of §3,  $H^J$  is lattice-ordered by  $H^+$  if and only if  $\mathcal{M}$  is commutative. Then, o.d. homomorphisms are exactly lattice homomorphisms. This follows immediately from the following.

**(4.3).** Suppose that  $H$  is lattice-ordered. Then, for  $\xi \in H^+$  and  $\eta \in H^+$ ,  $(\xi, \eta) = 0$  if and only if  $\xi \wedge \eta = 0$ .

**Proof.** Suppose that  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ . Since  $0 \leq \xi \wedge \eta \leq \xi$  and  $0 \leq \xi \wedge \eta \leq \eta$ , by the self-duality of  $H^+$  we have

$$0 \leq (\xi \wedge \eta, \xi \wedge \eta) \leq (\xi, \eta) = 0 ,$$

that is,  $\xi \wedge \eta = 0$ .

Conversely, suppose that  $\xi \in H^+$ ,  $\eta \in H^+$  and  $\xi \wedge \eta = 0$ . Then,  $|\xi - \eta| = \xi + \eta$ . Consequently,  $\|\xi - \eta\|^2 = \|\xi + \eta\|^2$ . But

$$\begin{aligned} \|\xi - \eta\|^2 &= (\xi - \eta, \xi - \eta) \\ &= \|\xi\|^2 - 2(\xi, \eta) + \|\eta\|^2, \end{aligned}$$

and

$$\|\xi + \eta\|^2 = \|\xi\|^2 + 2(\xi, \eta) + \|\eta\|^2.$$

Therefore,  $(\xi, \eta) = 0$ . □

The von Neumann algebra  $\mathcal{M}$  itself is an ordered Banach space. Let  $\mathcal{M}^h$  be the hermitian part of  $\mathcal{M}$ . Then, the positive cone  $\mathcal{M}^+$  consists of all positive operators, and, for every  $x \in \mathcal{M}^h$ , there exist  $x^+ \in \mathcal{M}^+$  and  $x^- \in \mathcal{M}^+$  such that  $x = x^+ - x^-$  and  $x^+x^- = 0$ . The following theorem of Kadison [3] shows that the o.d. homomorphisms with respect to this orthogonal decomposition are exactly the Jordan homomorphisms, when restricted to unital operators.

(4.4) (Kadison [3]). Let  $\alpha : \mathcal{M} \rightarrow \mathcal{M}$  be a linear operator such that  $\alpha(1) = 1$ . Then,  $\alpha$  is a Jordan homomorphism if and only if  $|\alpha(x)| = \alpha(|x|)$  for every  $x \in \mathcal{M}^h$ .

Theorem (2.7) of Connes shows that there is a correspondence between Jordan isomorphisms of  $\mathcal{M}$  and o.d. isomorphisms on  $H$ . The unitary operator  $u$  such that  $u(H^+) = H^+$ , which appears in (2.7), is obviously an o.d. isomorphism.

There is also a correspondence between unital Jordan homomorphisms on  $\mathcal{M}$  and  $\xi_0$ -preserving o.d. homomorphisms on  $H$  in a special case when  $\Delta_{\xi_0} = 1$ . This condition, as shown below, is equivalent to that  $\xi_0$  is a trace vector for  $\mathcal{M}$ . A vector  $\xi \in H$  is called a *trace vector* for the von Neumann algebra  $\mathcal{M}$  on  $H$  if, for any  $x, y \in \mathcal{M}$ ,

$$(xy\xi, \xi) = (yx\xi, \xi).$$

(4.5). Let  $\mathcal{M}$  be a von Neumann algebra on  $H$ ,  $\xi_0$  a cyclic and separating vector for  $\mathcal{M}$ . It follows that

(i)  $\Delta_{\xi_0} = 1$  if and only if  $\xi_0$  is a trace vector.

(ii)  $\mathcal{M}$  is finite if and only if there exists a cyclic separating vector  $\eta \in H$  such that  $\Delta_\eta = 1$ .

**Proof.** (i) Suppose that  $\Delta_{\xi_0} = 1$ . Then  $Jx\xi_0 = x^*\xi_0$  for all  $x \in \mathcal{M}$ , and hence, for any  $x, y \in \mathcal{M}$ ,

$$\begin{aligned} (xy\xi_0, \xi_0) &= (y\xi_0, x^*\xi_0) \\ &= (Jy^*\xi_0, Jx\xi_0) \\ &= (x\xi_0, y^*\xi_0) \\ &= (yx\xi_0, \xi_0). \end{aligned}$$

Conversely, suppose that  $\xi_0$  is a trace vector. Then, the mapping

$$\mathcal{M}\xi_0 \ni a\xi_0 \mapsto a^*\xi_0 \in \mathcal{M}\xi_0$$

is isometric because  $\xi_0$  is a trace vector. Furthermore, as  $\xi_0$  is cyclic and separating, this mapping extends to a well-defined antiunitary operator  $S$ . Hence,  $\Delta_{\xi_0} = 1$ .

(ii) If  $\mathcal{M}$  is finite, there is a positive faithful normal central linear functional  $\phi$  on  $\mathcal{M}$  (Strătilă-Zsidó [1], Theorems 7.10 and 7.11). So there is a cyclic separating vector  $\eta \in H^+$  such that

$$\phi(x) = (x\eta, \eta)$$

for  $x \in \mathcal{M}$  by (2.5). Hence  $\eta$  is a cyclic separating trace vector for  $\mathcal{M}$ . It now follows from (4.5)(i) that  $\Delta_\eta = 1$ .

Conversely, let  $\eta \in H$  be a cyclic and separating vector for  $\mathcal{M}$  such that  $\Delta_\eta = 1$ . Then by (4.5)(i),  $\eta$  is also a trace vector for  $\mathcal{M}$ . Hence the finite trace

$$\tau(x) = (x\eta, \eta)$$

on  $\mathcal{M}$  satisfies  $\tau(x) \neq 0$  if  $x \in \mathcal{M}^+$  and  $x \neq 0$ . Thus by (Strătilă-Zsidó [1], Corollary 7.14),  $\mathcal{M}$  is finite.  $\square$

Therefore, we can find a cyclic and separating vector  $\xi_0 \in H$  such that  $\Delta_{\xi_0} = 1$  if and only if  $\mathcal{M}$  is a finite von Neumann algebra of countable type. When  $\mathcal{M}$  is such an algebra and  $\xi_0$  is a cyclic trace vector, define a trace  $\tau$  by

$$\tau(x) = (x\xi_0, \xi_0), \quad x \in \mathcal{M}$$

and  $\|\cdot\|_\tau: \mathcal{M} \rightarrow \mathcal{R}$  by

$$\|a\|_\tau = \tau(a^*a)^{\frac{1}{2}}, \quad a \in \mathcal{M}.$$

Then, since  $\xi_0$  is a cyclic trace vector,  $\|\cdot\|_\tau$  is a norm in  $\mathcal{M}$ .

We shall call a linear operator  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$   $\tau$ -bounded if it is bounded with respect to the  $\|\cdot\|_\tau$ -norm, that is, if there exists a positive number  $k$  such that  $\|\alpha(x)\|_\tau \leq k \cdot \|x\|_\tau$  for all  $x \in \mathcal{M}$ .

(4.6). Suppose that  $\Delta_{\xi_0} = 1$ .

(i) If  $\phi$  is an o.d. homomorphism on  $H$  such that  $\phi\xi_0 = \xi_0$ , there exists a  $\tau$ -bounded unital Jordan homomorphism  $\alpha$  on  $\mathcal{M}$  such that

$$\alpha(x)\xi_0 = \phi(x\xi_0), \quad x \in \mathcal{M}.$$

(ii) If  $\alpha$  is a  $\tau$ -bounded unital Jordan homomorphism, there exists an o.d. homomorphism  $\phi$  on  $H$  such that  $\phi\xi_0 = \xi_0$  and

$$\phi(x\xi_0) = \alpha(x)\xi_0, \quad x \in \mathcal{M}.$$

**Proof.** (i) Let  $\phi$  denote an arbitrary o.d. homomorphism satisfying  $\phi\xi_0 = \xi_0$ . Since  $\Delta_{\xi_0} = 1$ , for any  $a \in \mathcal{M}^h$ ,  $-\|a\|\xi_0 \leq a\xi_0 \leq \|a\|\xi_0$ , and hence

$$-\|a\|\xi_0 = \phi(-\|a\|\xi_0) \leq \phi(a\xi_0) \leq \phi(\|a\|\xi_0) = \|a\|\xi_0,$$

that is,  $\phi(a\xi_0) \in F_{\xi_0}$ . Hence, by (2.3) there is a unique  $b \in \mathcal{M}^h$  such that



$b\xi_o = \phi(a\xi_o)$ . We now define  $\alpha_o: \mathcal{M}^h \rightarrow \mathcal{M}^h$  by

$$\alpha_o(a) = b,$$

where  $b\xi_o = \phi(a\xi_o)$ . Then,  $\alpha_o$  is obviously well-defined, (real) linear and  $\alpha_o(1) = 1$ .

Furthermore, because for each  $a \in \mathcal{M}$ , there are two unique operators  $a_1, a_2 \in \mathcal{M}^h$  satisfying  $a = a_1 + ia_2$ , we may define the linear extension  $\alpha$  of  $\alpha_o$  on  $\mathcal{M}$  by

$$\alpha(a) = \alpha_o(a_1) + i\alpha_o(a_2), \quad a \in \mathcal{M},$$

where  $a = a_1 + ia_2$  and  $a_1, a_2 \in \mathcal{M}^h$ . Then,  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  satisfies  $\alpha(1) = 1$  and  $\alpha(a)\xi_o = \phi(a\xi_o)$  for  $a \in \mathcal{M}$ .

Note that when  $\Delta_{\xi_o} = 1$ , for all  $x \in \mathcal{M}^h$ ,

$$x\xi_o = x^+\xi_o - x^-\xi_o$$

is the orthogonal decomposition of  $x\xi_o$ , and hence,

$$\begin{aligned} |\alpha(x)|\xi_o &= |\alpha(x)\xi_o| = |\phi(x\xi_o)| \\ &= \phi(|x\xi_o|) = \phi(|x|\xi_o) \\ &= \alpha(|x|)\xi_o. \end{aligned}$$

Since  $\xi_o$  is separating for  $\mathcal{M}$ , we have, for every  $x \in \mathcal{M}^h$ ,

$$|\alpha(x)| = \alpha(|x|).$$

It now follows from (4.4) of Kadison that  $\alpha$  is a Jordan homomorphism. Moreover, if

$\|a_n - a\|_\tau = ((a_n - a)\xi_o, (a_n - a)\xi_o) \rightarrow 0$ , then

$$\begin{aligned} \|\alpha(a_n) - \alpha(a)\|_\tau &= (\alpha(a_n - a)\xi_o, \alpha(a_n - a)\xi_o) \\ &= (\phi(a_n - a)\xi_o, \phi(a_n - a)\xi_o) \\ &\leq \|\phi\|^2 \|(a_n - a)\xi_o\|^2 \rightarrow 0. \end{aligned}$$

Hence,  $\alpha$  is  $\tau$ -bounded.

(ii) Let  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  be an arbitrary  $\tau$ -bounded Jordan homomorphism satisfying  $\alpha(1) = 1$ . We define a mapping  $\phi_o: \mathcal{M}\xi_o \rightarrow \mathcal{M}\xi_o$  by

$$\phi_o(a\xi_o) = \alpha(a)\xi_o, \quad a \in \mathcal{M}.$$

Then because  $\alpha$  is  $\tau$ -bounded, the closure  $\phi$  of  $\phi_o$  is a bounded linear operator on



$H = \overline{\mathcal{M}\xi_0}$  satisfying  $\phi(H^+) \subset H^+$ ,  $\phi\xi_0 = \xi_0$  and  $\phi(a\xi_0) = \alpha(a)\xi_0$  for  $a \in \mathcal{M}$ .

Now for any  $a \in \mathcal{M}^h$ , since  $\Delta_{\xi_0} = 1$  we have

$$|\phi(a\xi_0)| = |\alpha(a)\xi_0| = |\alpha(a)|\xi_0.$$

Hence by (4.4) of Kadison we have

$$|\phi(a\xi_0)| = |\alpha(a)|\xi_0 = \alpha(|a|)\xi_0 = \phi(|a|\xi_0) = \phi(|a\xi_0|).$$

Furthermore, for any  $\xi \in H^J = H^+ - H^+$ , there exist  $(a_n) \in \mathcal{M}^h$  such that

$$a_n\xi_0 \rightarrow \xi.$$

Since  $H^+$  is self-dual, for any  $\xi \in H^J$  and  $\eta \in H^J$  we have

$$\begin{aligned} (\xi, \eta) &= (\xi^+, \eta^+) - (\xi^+, \eta^-) - (\xi^-, \eta^+) + (\xi^-, \eta^-) \\ &\leq (\xi^+, \eta^+) + (\xi^+, \eta^-) + (\xi^-, \eta^+) + (\xi^-, \eta^-) \\ &= (|\xi|, |\eta|). \end{aligned}$$

and hence

$$\begin{aligned} |||\xi| - |\eta|||^2 &= |||\xi|||^2 - 2(|\xi|, |\eta|) + |||\eta|||^2 \\ &= \|\xi\|^2 - 2(|\xi|, |\eta|) + \|\eta\|^2 \\ &\leq \|\xi\|^2 - 2(\xi, \eta) + \|\eta\|^2 \\ &= \|\xi - \eta\|^2. \end{aligned}$$

Consequently  $|a_n\xi_0| \rightarrow |\xi|$ . As  $\phi$  is continuous we have

$$\phi(a_n\xi_0) \rightarrow \phi\xi$$

and hence

$$|\phi(a_n\xi_0)| \rightarrow |\phi\xi|.$$

But for each  $n$ ,  $|\phi(a_n\xi_0)| = \phi(|a_n\xi_0|)$ , so

$$|\phi(a_n\xi_0)| = \phi(|a_n\xi_0|) \rightarrow \phi(|\xi|),$$

that is,  $|\phi\xi| = \phi|\xi|$ . Therefore  $\phi$  is an o.d. homomorphism.  $\square$

This correspondence between  $\alpha$  and  $\phi$  is unknown when  $\Delta_{\xi_0} \neq 1$ . The main difficulty is in finding the orthogonal decomposition of  $\Delta_{\xi_0}^{\frac{1}{2}}x\xi_0$  when  $x \in \mathcal{M}^h$ . For

example, for  $x \in \mathcal{M}^h$ ,

$$\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0 = \Delta_{\xi_0}^{\frac{1}{4}} x^+ \xi_0 - \Delta_{\xi_0}^{\frac{1}{4}} x^- \xi_0$$

is not the orthogonal decomposition of  $\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0$ . In fact, we have the following.

(4.7) *The following conditions are equivalent:*

$$(i) (\Delta_{\xi_0}^{\frac{1}{4}} x^+ \xi_0, \Delta_{\xi_0}^{\frac{1}{4}} x^- \xi_0) = 0 \text{ for every } x \in \mathcal{M}^h.$$

$$(ii) |\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0| = \Delta_{\xi_0}^{\frac{1}{4}} |x| \xi_0 \text{ for every } x \in \mathcal{M}^h.$$

$$(iii) \Delta_{\xi_0} = 1.$$

**Proof.** The equivalence of (i) and (ii) is obvious. It is also obvious that (iii) implies (i).

(i)  $\Rightarrow$  (ii). Condition (i) is equivalent to

$$x, y \in \mathcal{M}^+, xy = 0 \Rightarrow (\Delta_{\xi_0}^{\frac{1}{4}} x \xi_0, \Delta_{\xi_0}^{\frac{1}{4}} y \xi_0) = 0.$$

So for any projection  $p \in \mathcal{M}$ , we have

$$\begin{aligned} (J(1-p)\xi_0, p\xi_0) &= (\Delta_{\xi_0}^{\frac{1}{4}} p\xi_0, (1-p)\xi_0) \\ &= (\Delta_{\xi_0}^{\frac{1}{4}} p\xi_0, \Delta_{\xi_0}^{\frac{1}{4}} (1-p)\xi_0) \\ &= 0, \end{aligned}$$

that is,  $(\xi_0, p\xi_0) = (Jp\xi_0, p\xi_0)$ . Consequently,

$$\begin{aligned} \|Jp\xi_0 - p\xi_0\|^2 &= (Jp\xi_0 - p\xi_0, Jp\xi_0 - p\xi_0) \\ &= (Jp\xi_0, Jp\xi_0) - (Jp\xi_0, p\xi_0) - (p\xi_0, Jp\xi_0) + (p\xi_0, p\xi_0) \\ &= 0. \end{aligned}$$

Hence  $Jp\xi_0 = p\xi_0$  for any projection  $p \in \mathcal{M}$ . Then by spectral theory we have  $Ja\xi_0 = a\xi_0$  for any  $a \in \mathcal{M}^h$ . Now for any  $x \in \mathcal{M}$ , there are  $x_1 \in \mathcal{M}^h$ ,  $x_2 \in \mathcal{M}^h$  such that  $a = a_1 + ia_2$ , hence

$$\begin{aligned}
 Ja\xi_o &= Ja_1\xi_o + Jia_2\xi_o \\
 &= a_1\xi_o - ia_2\xi_o \\
 &= a^*\xi_o.
 \end{aligned}$$

Therefore,  $\Delta_{\xi_o} = 1$ . □

In conclusion, an o.d. homomorphism on  $H$  is a lattice homomorphism when  $H$  is lattice-ordered. Generally, there is a correspondence between Jordan isomorphisms on  $\mathcal{M}$  and o.d. isomorphisms on  $H$ . When  $\Delta_{\xi_o} = 1$ , there is a correspondence between  $\tau$ -bounded unital Jordan homomorphisms on  $\mathcal{M}$  and  $\xi_o$ -preserving o.d. homomorphisms on  $H$ . Therefore at least in these two cases, considering o.d. homomorphisms is equivalent to considering  $\tau$ -bounded Jordan homomorphisms on  $\mathcal{M}$ . Since the order structure of  $H$ , being a Hilbert space, is simpler than that of  $\mathcal{M}$ , there are some advantages in considering o.d. homomorphisms on  $H$ , one of which is the characterization of these maps in the next section.

## §5. CHARACTERIZATIONS OF O.D. HOMOMORPHISMS

Throughout the rest of this thesis, unless otherwise stated, we always assume that  $(\mathcal{M}, H, \xi_0)$  is given, that is, given  $\mathcal{M}$  a von Neumann algebra on Hilbert space  $H$  and  $\xi_0 \in H$  cyclic and separating for  $\mathcal{M}$ . The natural cone associated with  $(\mathcal{M}, H, \xi_0)$  will be denoted by  $H^+$ .

We shall start this section with a characterization of bijective o.d. homomorphisms. To do this, we need the following result.

(5.1). *Let  $\phi$  be an o.d. homomorphism. If  $\phi$  is bijective,  $\phi$  is an order isomorphism, that is,  $\phi(H^+) = H^+$ .*

**Proof.** Let  $\xi$  be an arbitrary vector in  $H^J$  such that  $\phi\xi \in H^+$ . Let  $\xi = \xi^+ - \xi^-$  be the orthogonal decomposition of  $\xi$ . Then we have

$$\phi\xi^+ \geq \phi\xi^- \geq 0.$$

Since  $H^+$  is self dual,

$$0 \leq (\phi\xi^-, \phi\xi^-) \leq (\phi\xi^+, \phi\xi^-) = 0.$$

Hence  $\phi\xi^- = 0$ ; and since  $\phi$  is bijective,  $\xi^- = 0$ . Therefore,  $\xi \in H^+$ . This shows that  $\phi(H^+) = H^+$ .  $\square$

(5.2) *Let  $\phi$  be a bijective continuous linear operator on  $H$ . Then  $\phi$  is an o.d. homomorphism if and only if, for the polar decomposition  $\phi = u|\phi|$ ,  $u$  is a unitary operator such that  $u(H^+) = H^+$  and  $|\phi| \in \mathcal{M} \cap \mathcal{M}'$ .*

**Proof.** It follows from (5.1) that  $\phi$  is an order isomorphism. Then by theorem (2.9) of Connes, there is an operator  $a \in \mathcal{M}^+$  such that  $aj(a) = |\phi| = u^*\phi$ . Since  $\phi$  and  $u^*$  are o.d. homomorphisms,  $aj(a)$  is also an o.d. homomorphism. It follows from Example (4.2.iii) that  $a^2 = a^*a \in \mathcal{M} \cap \mathcal{M}'$ . Therefore  $a \in \mathcal{M} \cap \mathcal{M}'$  and

$$|\phi| = aj(a) = a^2 \in \mathcal{M} \cap \mathcal{M}'.$$

The converse is obvious.  $\square$

The following result is needed in characterizing o.d. homomorphisms.

(5.3). Let  $a \in \mathcal{M}$  and  $\delta = a + j(a)$ . Then,  $\delta(H^+) \subset H^+$  if and only if  $\delta$  is an o.d. homomorphism.

**Proof.** Suppose that  $\delta(H^+) \subset H^+$ . It follows from (2.10) that  $\delta$  satisfies  $(\xi, \delta\eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ . Since  $\delta\eta \in H^+$  by the assumption, we can apply this condition again to obtain  $(\delta\xi, \delta\eta) = 0$ . Hence  $\delta$  is an o.d. homomorphism.

The converse follows immediately from (4.1).  $\square$

(5.4). The following conditions are equivalent:

(i)  $\phi$  is an o.d. homomorphism.

(ii)  $\phi(H^+) \subset H^+$  and  $\phi^*\phi \in \mathcal{M} \cap \mathcal{M}'$ .

**Proof.** (i)  $\Rightarrow$  (ii). Let  $\phi$  be an o.d. homomorphism and set  $\delta = \phi^*\phi$ . It follows from (4.1) that  $\phi(H^+) \subset H^+$ . Since  $H^+$  is self-dual, for each  $\xi \in H^+$ , for all  $\eta \in H^+$ ,

$$(\phi^*\xi, \eta) = (\xi, \phi\eta) \geq 0,$$

that is,  $\phi^*\xi \in H^+$ . So  $\phi^*(H^+) \subset H^+$ , and hence  $\delta(H^+) = \phi^*\phi(H^+) \subset H^+$ . Since  $(\xi, \delta\eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ , by (2.10)

$$\delta = a + j(a),$$

for some  $a \in \mathcal{M}$ . It follows from (5.3) that  $\delta$  is also an o.d. homomorphism. We now set

$$\psi = 1 + \alpha\delta,$$

where  $\alpha$  is a positive number such that  $\alpha\|\delta\| < 1$ . Then  $\psi$  is bijective. Furthermore,

$\psi$  is an o.d. homomorphism because if  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ , then



$$\begin{aligned}
(\psi\xi, \psi\eta) &= (\xi + \alpha\delta\xi, \eta + \alpha\delta\eta) \\
&= (\xi, \eta) + \alpha(\xi, \delta\eta) + \alpha(\delta\xi, \eta) + \alpha^2(\delta\xi, \delta\eta) \\
&= 0.
\end{aligned}$$

Since  $\psi = |\psi|$ , we have  $\psi = |\psi| \in \mathcal{M} \cap \mathcal{M}'$  by (5.2). Hence,  $\delta \in \mathcal{M} \cap \mathcal{M}'$ .

(ii)  $\Rightarrow$  (i). For  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ , we have, by (2.12),

$$(\phi\xi, \phi\eta) = (\phi^*\phi\xi, \eta) = 0.$$

It follows immediately from (4.1) that  $\phi$  is an o.d. homomorphism.  $\square$

Note that (5.4) implies all elements of  $(\mathcal{M} \cap \mathcal{M}')^+$  are o.d. homomorphisms.

A bijective continuous linear operator  $\phi$  is called an *o.d. isomorphism* if  $\phi$  and  $\phi^{-1}$  are o.d. homomorphisms.

(5.4) has two immediate corollaries. The first one generalizes (5.1).

(5.5). *If  $\phi$  is a bijective o.d. homomorphism,  $\phi$  is an o.d. isomorphism.*

**Proof.** Let  $\phi = u|\phi|$  be the polar decomposition of  $\phi$ . It follows from (5.4) that  $u$  is a unitary satisfying  $u(H^+) = H^+$  and  $|\phi|$  is a bijective operator in  $(\mathcal{M} \cap \mathcal{M}')^+$ . So  $u^*(H^+) = H^+$  and  $|\phi|^{-1} \in (\mathcal{M} \cap \mathcal{M}')^+$ ; that is both  $u^*$  and  $|\phi|^{-1}$  are o.d. homomorphisms. Therefore  $\phi^{-1} = |\phi|^{-1}u^*$  is also an o.d. homomorphism.  $\square$

In a Banach lattice, all order isomorphisms are lattice isomorphisms. When  $\mathcal{M}$  is not commutative, this is no longer true. In fact, Yamamuro [2] has proved that every order isomorphism is an o.d. isomorphism if and only if  $\mathcal{M}$  is commutative. Furthermore, we have the following.

(5.6). *The following conditions are equivalent:*

(i) *If  $\phi(H^+) \subset H^+$  and  $\phi$  is bijective, then  $\phi(H^+) = H^+$ .*

(ii) *If  $\phi(H^+) \subset H^+$ ,  $\phi\xi_0 = \xi_0$  and  $\phi$  is bijective, then  $\phi(H^+) = H^+$ .*

(iii)  $H^J$  is isomorphic to the one-dimensional ordered space of all real numbers.

**Proof.** (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (i) are obvious.

(ii)  $\Rightarrow$  (iii). Without loss of generality, we assume  $\|\xi_0\| = 1$ , and set

$$\phi\xi = \frac{1}{2}(\xi + (\xi, \xi_0)\xi_0), \quad \xi \in H.$$

As  $H^+$  is self-dual and  $\xi_0$  is in  $H^+$ , we have  $\phi(H^+) \subset H^+$ . It is also obvious that  $\phi\xi_0 = \xi_0$ ,  $\phi$  is bijective and

$$\phi^{-1}\xi = 2\xi - (\xi, \xi_0)\xi_0.$$

Since  $\phi^{-1}(H^+) \subset H^+$ , we have, for all  $\xi \in H^+$ ,

$$2\xi \geq (\xi, \xi_0)\xi_0.$$

Hence for an arbitrary  $\xi \in H^J$  with the orthogonal decomposition  $\xi = \xi^+ - \xi^-$ , we have  $2\xi^+ \geq (\xi^+, \xi_0)\xi_0$  and  $2\xi^- \geq (\xi^-, \xi_0)\xi_0$ . Thus, by the self-duality of  $H^+$ ,

$$0 = 4(\xi^+, \xi^-) \geq (\xi^+, \xi_0)(\xi^-, \xi_0),$$

and hence  $\xi^+ = 0$  or  $\xi^- = 0$ , that is,  $\xi \in H^+$  or  $-\xi \in H^+$ . This means that  $H^J$  is totally ordered. Hence  $H^J$  is isomorphic to the set of real numbers equipped with the usual order.  $\square$

The second corollary of (5.4) is due to Yamamuro [4], which offers another characterization of o.d. homomorphisms.

(5.7). A continuous linear operator  $\phi: H \rightarrow H$  is an o.d. homomorphism if and only if  $\phi(H^+) \subset H^+$  and  $\phi^*x\phi \in \mathcal{M} \cap \mathcal{M}'$  whenever  $x \in \mathcal{M} \cap \mathcal{M}'$ .

**Proof.** Suppose that  $\phi$  is an o.d. homomorphism and  $y \in (\mathcal{M} \cap \mathcal{M}')^+$ . Since  $y^{\frac{1}{2}} \in (\mathcal{M} \cap \mathcal{M}')^+$ ,  $y^{\frac{1}{2}}$  is an o.d. homomorphism; and hence  $y^{\frac{1}{2}}\phi$  is also an o.d. homomorphism. So, by (5.4), we have  $\phi^*y\phi = (y^{\frac{1}{2}}\phi)^*(y^{\frac{1}{2}}\phi) \in \mathcal{M} \cap \mathcal{M}'$ . Now for  $a \in (\mathcal{M} \cap \mathcal{M}')^h$ ,  $1 + \alpha a \in (\mathcal{M} \cap \mathcal{M}')^+$  with  $\alpha$  being a positive number satisfying

$\alpha\|a\| < 1$ . Then  $\phi^*(1 + \alpha a)\phi \in \mathcal{M} \cap \mathcal{M}'$ , and hence  $\phi^*a\phi \in \mathcal{M} \cap \mathcal{M}'$ . Since for any  $x \in \mathcal{M} \cap \mathcal{M}'$ ,  $x$  is a linear combination of two self-adjoint operators in  $\mathcal{M} \cap \mathcal{M}'$ , we have  $\phi^*x\phi \in \mathcal{M} \cap \mathcal{M}'$ .

The converse follows when we set  $x = 1$ . □

As a consequence, an order homomorphism  $\phi$  is an o.d. homomorphism if  $\phi^*\mathcal{M}\phi \subset \mathcal{M}$ , because then, since  $J\phi = \phi J$ , (1.1)(iii) implies

$$\phi^*j(\mathcal{M})\phi \subset j(\mathcal{M}),$$

and hence  $\phi^*(\mathcal{M} \cap \mathcal{M}')\phi \subset \mathcal{M} \cap \mathcal{M}'$ . This type of operator, "implemented" operators, on  $\mathcal{M}$  has been studied by Demoen and Vanheuverzwijn [1], in which they clarified the relation between the implementability and another kind of orthogonality preservation. We shall give a brief account of their results.

Let  $\mathcal{M}_*$  be the predual of  $\mathcal{M}$  and  $\mathcal{M}_*^h$  be the hermitian part of  $\mathcal{M}_*$ . Then every element  $\omega \in \mathcal{M}_*^h$  admits a unique orthogonal decomposition (the Jordan decomposition)

$$\omega = \omega^+ - \omega^-, \quad \omega^+ \in \mathcal{M}_*^+, \quad \omega^- \in \mathcal{M}_*^+, \quad \omega^+ \perp \omega^-,$$

where  $\mathcal{M}_*^+$  is the positive part of  $\mathcal{M}_*$ , and the orthogonality,  $\omega^+ \perp \omega^-$ , is defined by  $s(\omega^+)s(\omega^-) = 0$  ( $s(\cdot)$  denotes the support of  $\cdot$ ). Naturally, we can consider o.d. homomorphisms on this orthogonally decomposable space  $\mathcal{M}_*$ .

When a linear operator  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  is normal, the dual  $\alpha_*: \mathcal{M}_* \rightarrow \mathcal{M}_*$  is defined. We shall say that  $\alpha_*$  is an o.d. homomorphism if  $\alpha_*(\mathcal{M}_*^+) \subset \mathcal{M}_*^+$  and  $\alpha_*(\omega_1) \perp \alpha_*(\omega_2)$  whenever  $\omega_1 \in \mathcal{M}_*^+$ ,  $\omega_2 \in \mathcal{M}_*^+$  and  $\omega_1 \perp \omega_2$ .

Demoen and Vanheuverzwijn [1] gave the following definition:

A unital, normal and completely positive map  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  is said to be (isometrically) implementable if there exists an isometry  $v: H \rightarrow H$  such that  $\alpha(x) = v^*xv$  for all  $x \in \mathcal{M}$  and  $v(H^+) \subset H^+$ ;

and they have proved the following theorem.

[DV]. Let  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  be a unital and normal Schwartz map; then  $\alpha$  is implementable if and only if  $\alpha_*$  is an o.d. homomorphism.

(A Schwartz map is a linear map  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  such that  $\alpha(x^*x) \geq \alpha(x^*)\alpha(x)$  for all  $x \in \mathcal{M}$ .)

It immediately follows from the remark after (5.7) that the operator  $v$  in the definition above of implementability is an o.d. homomorphism. To make clear the relation between the two orthogonality preservations of  $\phi$  on  $H$  and  $\alpha_*$  on  $\mathcal{M}_*$ , we give the following definition.

A linear operator  $\alpha: \mathcal{M} \rightarrow \mathcal{M}$  is said to be *implementable* if there exists a continuous linear operator  $\phi: H \rightarrow H$  such that  $\alpha(x) = \phi^*x\phi$  for all  $x \in \mathcal{M}$ .

It is easy to see that implementable operators are normal. Hence if  $\alpha$  is implementable,  $\alpha_*: \mathcal{M}_* \rightarrow \mathcal{M}_*$  is well-defined. Under this circumstance, we have the following result.

(5.8). The following conditions are equivalent:

- (i)  $\phi$  is an o.d. homomorphism.
- (ii)  $\alpha_*$  is an o.d. homomorphism.

**Proof.** By (2.5), for every  $\omega \in \mathcal{M}_*^+$ , there is a unique  $\xi \in H^+$  such that  $\omega = \omega_\xi$ , and for  $\xi \in H^+$  and  $\eta \in H^+$ ,  $(\xi, \eta) = 0$  if and only if  $\omega_\xi \perp \omega_\eta$ .

(i)  $\Rightarrow$  (ii). Since  $\alpha(\mathcal{M}^+) \subset \mathcal{M}^+$ ,  $\alpha_*(\mathcal{M}_*^+) \subset \mathcal{M}_*^+$  is obvious. Suppose that  $\omega_i \in \mathcal{M}_*^+$  ( $i = 1, 2$ ) and  $\omega_1 \perp \omega_2$ . Then  $\omega_1 = \omega_\xi$  and  $\omega_2 = \omega_\eta$  for some  $\xi \in H^+$  and  $\eta \in H^+$  such that  $(\xi, \eta) = 0$ . Since  $\phi$  is an o.d. homomorphism,  $(\phi\xi, \phi\eta) = 0$ . Consequently,  $\omega_{\phi\xi} \perp \omega_{\phi\eta}$ . On the other hand,  $\omega_{\phi\xi} = \alpha_*(\omega_\xi)$  and  $\omega_{\phi\eta} = \alpha_*(\omega_\eta)$ . Therefore  $\alpha_*$  is an o.d. homomorphism.

(ii)  $\Rightarrow$  (i). If  $\xi \in H^+$ , since  $\alpha_*(\omega_\xi) \in \mathcal{M}_*^+$ ,  $\alpha_*(\omega_\xi) = \omega_\eta$  for some  $\eta \in H^+$ ,



that is  $\omega_{\phi\xi} = \omega_{\eta}$  and  $\eta \in H^+$ . Thus,  $\phi(H^+) \subset H^+$ . So by the remark after (5.7),  $\phi$  is an o.d. homomorphism.  $\square$

The following result follows immediately from (5.7).

(5.9). If  $\alpha$  is implementable and  $\phi(H^+) \subset H^+$ , then  $\alpha(\mathcal{M} \cap \mathcal{M}') \subset \mathcal{M} \cap \mathcal{M}'$ .

**Proof.** It follows from the assumption that  $\phi$  is an o.d. homomorphism. So  $\alpha(\mathcal{M} \cap \mathcal{M}') = \phi^*(\mathcal{M} \cap \mathcal{M}')\phi \subset \mathcal{M} \cap \mathcal{M}'$  by (5.7).  $\square$



## §6. POLAR DECOMPOSITIONS OF O.D. HOMOMORPHISMS

Every continuous linear operator  $\phi : H \rightarrow H$  admits the polar decomposition

$$\phi = u|\phi| ,$$

where  $|\phi| = (\phi^* \phi)^{\frac{1}{2}}$  and  $u$  is a partial isometry such that  $u^* u = s(|\phi|)$ , the support of  $|\phi|$ . If  $\phi \in \mathcal{M}$ , then  $u \in \mathcal{M}$  and  $|\phi| \in \mathcal{M}$ .

The theorem (2.9) of Connes states that if  $\phi(H^+) = H^+$ , then  $u(H^+) = H^+$  and  $|\phi|(H^+) = H^+$ , that is, when  $\phi$  is an order isomorphism,  $u$  and  $|\phi|$  are also order isomorphisms. However, this character is not enjoyed by order homomorphisms, that is, there are operators  $\phi$  on  $H$  satisfying  $\phi(H^+) \subset H^+$ , with the polar decomposition  $\phi = u|\phi|$ , where either  $u$  is not an order homomorphism, or both  $u$  and  $|\phi|$  are not order homomorphisms.

(6.1). *An example of an order homomorphism  $\phi$  for which  $|\phi|(H^+) \subset H^+$  and  $u(H^+) \not\subset H^+$ .*

We consider the space of  $2 \times 2$  matrices

$$H_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; \quad a, b, c, d \in \mathcal{C} .$$

Then,

$$\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} u & x \\ y & z \end{bmatrix} \right) = \text{trace} \begin{bmatrix} \bar{u} & \bar{y} \\ \bar{x} & \bar{z} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

defines an inner product on  $H_2$ ; and  $H_2$  is a Hilbert space with respect to this inner product. Let  $M_2(\mathcal{C})$  be the set of all operators of the form

$$\phi = \begin{bmatrix} \alpha & \beta \\ \chi & \delta \end{bmatrix} , \quad \alpha, \beta, \chi, \delta \in \mathcal{C} ,$$

defined on  $H_2$  such that if  $A \in H_2$ ,  $\phi(A)$  is the usual matrices product of  $\phi A$ . It

follows easily that  $M_2(\mathcal{C})$  is a von Neumann algebra on  $H_2$ . Furthermore, it is simple to check that

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H_2$$

is a cyclic and separating trace vector for  $M_2(\mathcal{C})$ . So by (4.5)(i),

$$\Delta_I = 1,$$

and hence

$$\begin{aligned} J_I \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \\ &= \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}. \end{aligned}$$

Since  $H_2^+ = \overline{\{J_I \phi J_I \phi I : \phi \in M_2(\mathcal{C})\}}$ ,  $H_2^+$  is the set of all matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & a\bar{c} + b\bar{d} \\ c\bar{a} + d\bar{b} & |c|^2 + |d|^2 \end{pmatrix}.$$

In other words,

$$\begin{pmatrix} p & q \\ \bar{q} & r \end{pmatrix} \in H_2^+$$

if and only if  $p \geq 0$ ,  $r \geq 0$  and  $pr \geq |q|^2$ . Consequently,  $H_2^J$  consists of all Hermitian matrices.

We now consider the operator  $\phi : H_2 \rightarrow H_2$  defined by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+d & b \\ c & d \end{pmatrix}$$

It is easy to see that  $\phi(H^+) \subset H^+$ . Also,

$$\begin{aligned} \left( \phi^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \begin{pmatrix} x & y \\ z & u \end{pmatrix} \right) &= \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \begin{pmatrix} x+u & y \\ z & u \end{pmatrix} \right) \\ &= a\bar{x} + b\bar{y} + c\bar{z} + (a+d)\bar{u} . \end{aligned}$$

So,

$$\phi^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & a+d \end{pmatrix}$$

and hence

$$\phi^* \phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+d & b \\ c & a+2d \end{pmatrix}$$

Let  $\phi = u|\phi|$  be the polar decomposition of  $\phi$ . We have

$$|\phi| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{3a+d}{\sqrt{5}} & b \\ c & \frac{a+2d}{\sqrt{5}} \end{pmatrix}$$

and

$$u \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+2d}{\sqrt{5}} & b \\ c & \frac{2a-d}{\sqrt{5}} \end{pmatrix}$$

Then, it is easy to see that  $|\phi|(H_2^+) \subset H_2^+$ , and

$$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \in H_2^+$$

but

$$u \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} \frac{11}{\sqrt{5}} & 0 \\ 0 & -\frac{3}{\sqrt{5}} \end{pmatrix} \in H_2^+ \quad \square$$

(6.2). An example of an order homomorphism  $\phi$  for which  $|\phi|(H^+) \not\subset H^+$  and  $u(H^+) \not\subset H^+$ .

Let  $M_3$  be an algebra of all  $3 \times 3$  matrices of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \chi \\ 0 & \delta & \theta \end{pmatrix}, \quad \alpha, \beta, \chi, \delta, \theta \in \mathbb{C}.$$

The Hilbert space  $H_3$  consists of all  $3 \times 3$  matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}, \quad a, b, c, d, e \in \mathbb{C}.$$

with the underlying inner product defined in the same way as in (6.1). Then  $M_3$  is a von Neumann algebra on  $H_3$ , and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a cyclic separating trace vector. So by (4.5)(i)  $\Delta_I = 1$ , and hence  $H_3^+$  is equal to  $M_3^+$  as sets. We now consider  $\phi : H_3 \rightarrow H_3$  defined by

$$\phi \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & a+e \end{pmatrix}$$

Then  $\phi(H_3^+) \subset H_3^+$  and

$$|\phi| \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} = \begin{pmatrix} \frac{3a+e}{\sqrt{5}} & 0 & 0 \\ 0 & b & c \\ 0 & d & \frac{a+2e}{\sqrt{5}} \end{pmatrix}$$

Hence

$$|\phi| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & \frac{2}{\sqrt{5}} \end{pmatrix} \in H_3^+.$$

From  $u^*\phi = |\phi|$ , it follows that  $u(H_3^+) \subset H_3^+$ , because  $u(H_3^+) \subset H_3^+$  implies  $u^*(H_3^+) \subset H_3^+$ .  $\square$

Thus the polar decomposition is not compatible with the order structure of  $H$ . However, for every order homomorphism  $\phi$  with the polar decomposition  $\phi = u|\phi|$ , if  $\phi$  is also an o.d. homomorphism, then both  $u$  and  $|\phi|$  are order homomorphisms; in fact, as the following result shows, both  $u$  and  $|\phi|$  are o.d. homomorphisms.

**(6.3).** Let  $\phi$  be an o.d. homomorphism,  $\phi = u|\phi|$  be the polar decomposition.

It follows that

(i)  $|\phi|$  is an o.d. homomorphism.

(ii)  $s(|\phi|)$  is an o.d. homomorphism.

(iii)  $\overline{|\phi|(H)} \cap H^+ = \overline{|\phi|(H^+)}$ .

(iv)  $s(|\phi|)(H) \cap H^+ = s(|\phi|)(H^+)$ .

(v)  $u$  is an o.d. homomorphism.

**Proof.** (i) and (ii). It follows from (5.4) that  $|\phi| = (\phi^*\phi)^{\frac{1}{2}} \in (\mathcal{M} \cap \mathcal{M}')^+$ , and hence  $s(|\phi|) \in (\mathcal{M} \cap \mathcal{M}')^+$ . So by the remark after (5.4),  $|\phi|$  and  $s(|\phi|)$  are o.d.



homomorphisms.

(iii). Since  $|\phi|$  is an o.d. homomorphism,  $|\phi|(H^+) \subset |\phi|(H) \cap H^+$ . Hence  $\overline{|\phi|(H^+)} \subset \overline{|\phi|(H) \cap H^+} \subset \overline{|\phi|(H)} \cap H^+$ .

Let  $\xi$  be an arbitrary element in  $H$  such that  $|\phi|\xi \in H^+$ . By (2.1)(ix) of Connes we have

$$\xi = \xi_1 + i\xi_2,$$

where  $\xi_1 \in H^J$  and  $\xi_2 \in H^J$ . Since  $|\phi|$  is an o.d. homomorphism,  $J|\phi| = |\phi|J$  and hence  $|\phi|\xi_2 = 0$ . So  $|\phi|\xi_1 \in H^+$ . Let  $\xi_1 = \xi_1^+ - \xi_1^-$  be the orthogonal decomposition of  $\xi_1$ . Then  $|\phi|\xi_1^+ \in H^+$ ,  $|\phi|\xi_1^- \in H^+$  and  $(|\phi|\xi_1^+, |\phi|\xi_1^-) = 0$  because  $\phi$  is an o.d. homomorphism. Therefore  $|\phi|\xi^- = 0$ , otherwise

$$|\phi|\xi_1^+ - |\phi|\xi_1^- = |\phi|\xi_1 \in H^+$$

would contradict the unique orthogonal decomposition of  $|\phi|\xi_1$ . Thus  $|\phi|\xi = |\phi|\xi_1^+$ .

Consequently

$$|\phi|(H) \cap H^+ \subset |\phi|(H^+),$$

and hence

$$\overline{|\phi|(H) \cap H^+} \subset \overline{|\phi|(H) \cap H^+} \subset \overline{|\phi|(H^+)}.$$

Therefore  $\overline{|\phi|(H) \cap H^+} = \overline{|\phi|(H^+)}.$

(iv) is proved similarly as (iii) because  $s(|\phi|)$  is also an o.d. homomorphism.

(v). Since  $s(|\phi|)$  is the projection on  $\overline{|\phi|(H)}$ , it follows from (iv) and (iii)

that

$$\begin{aligned} s(|\phi|)(H^+) &= s(|\phi|)(H) \cap H^+ \\ &= \overline{|\phi|(H) \cap H^+} \\ &= \overline{|\phi|(H^+)}. \end{aligned}$$

So for each  $\xi \in H^+$ , there is a sequence  $(\xi_n) \subset H^+$  such that

$$\lim_n |\phi| \xi_n = s(|\phi|) \xi.$$

Then, as  $u$  is a partial isometry, we have

$$\begin{aligned} u\xi &= uu^*u\xi \\ &= us(|\phi|)\xi \\ &= \lim_n u|\phi|\xi_n \\ &= \lim_n \phi\xi_n, \end{aligned}$$

where, as  $\phi$  is an o.d. homomorphism,  $\phi\xi_n \in H^+$  for each  $n$ . So  $u\xi \in H^+$  because  $H^+$  is closed. We have just proved  $u(H^+) \subset H^+$ .

Now for  $\xi \in H^+$ ,  $\eta \in H^+$  satisfying  $(\xi, \eta) = 0$ , it follows from (ii) that

$$\begin{aligned} (u\xi, u\eta) &= (s(|\phi|)\xi, \eta) \\ &= (s(|\phi|)^2\xi, \eta) \\ &= (s(|\phi|)\xi, s(|\phi|)\eta) \\ &= 0. \end{aligned}$$

□

**Remark.** Since  $s(|\phi|)$  is an o.d. homomorphism, it follows from (5.4) that

$$\begin{aligned} s(|\phi|) &= s(|\phi|) s(|\phi|) \\ &= s(|\phi|)^* s(|\phi|) \in \mathcal{M} \cap \mathcal{M}'. \end{aligned}$$

Hence  $n(\phi) = 1 - s(|\phi|) \in \mathcal{M} \cap \mathcal{M}'$  ( $n(\phi)$  is the projection on the kernel of  $\phi$ ); and by Example (4.2.ii)  $n(\phi)$  is also an o.d. homomorphism.

## §7. SEMIGROUPS OF O.D. HOMOMORPHISMS

Recall that when  $H$  is lattice-ordered, or equivalently when  $\mathcal{M}$  is commutative, o.d. homomorphisms with respect to the natural orthogonal decompositions  $\xi = \xi^+ - \xi^-$ , where  $\xi^+ \in H^+$ ,  $\xi^- \in H^+$  and  $(\xi^+, \xi^-) = 0$ , are exactly lattice homomorphisms (Example (4.2.iv)). Then since  $H$  is reflexive, the underlying norm of  $H$  is an order continuous norm. Nagel and Uhlig ([1], Theorem 3.4) proved that

[N-U]. *Let  $B$  be a complex Banach lattice that has order continuous norm. The following conditions are equivalent for a  $C_0$ -semigroup  $\{T(t) : t \geq 0\}$  of continuous linear operators on  $B$  and its generator  $G$ .*

(a)  $|T(t)x| = T(t)|x|$ ,  $x \in B$ ,  $t \geq 0$ .

(b) *The generator  $G$  is a real, local operator and its domain  $D(G)$  is a vector sublattice of  $B$ .*

(c)  $x \in D(G)$  implies  $|x| \in D(G)$ ; and  $G|x| = \text{sign}(x) Gx$ .

The same result for the case when  $B$  is  $\sigma$ -complete is also due to Nagel and Uhlig. Later, Arendt [1] has proved a more general form of this theorem. Note that when  $B$  is a  $\sigma$ -complete Banach lattice, each element  $x \in B^+$  determines a projection operator  $P_x$  by

$$P_x y = \lim_{n \rightarrow \infty} (y \wedge nx) \quad \text{for all } y \in B^+,$$

and  $P_x y = P_x y^+ - P_x y^-$  for all  $y \in B$ . Then for each  $x \in B$ ,  $\text{sign}(x)$  is defined by

$$\text{sign}(x)y = P_{x^+}(y) - P_{x^-}(y), \quad y \in B.$$

Arendt [1] has proved the following equation:

$$\text{sign}(x)y = \inf \left\{ \frac{1}{t} (|x + ty| - |x|) - (1 - P_{|x|})y : t > 0 \right\},$$

for all  $x \in B$  and  $y \in B$ . The existence of the limit of the first term in the right-hand side was proved in the same paper by Arendt. Approximately, "sign" denotes the derivative of the absolute value function  $x \rightarrow |x|$ , and, in this respect, it is related to the Kato's inequality. Kato [1] has proved the following distributional inequality for the Laplacian  $\Delta$ :

$$\Delta|f| \geq \text{Re}(\text{sign}(f) \Delta f),$$

when  $f$  and  $\Delta f$  are in the  $L^1_{loc}(\mathbb{R}^n)$ . Later Simon [1] has shown that this equation is closely related to the positivity of the semigroup generated by  $\Delta$ . The Kato's inequality is in the following form:

$$G|x| \geq \text{Re}(\text{sign}(x) Gx).$$

We now return to the ordered Hilbert space  $H$  in  $(\mathcal{M}, H, \xi_0)$ . In general, whether or not  $H$  is lattice-ordered, there exist naturally  $C_0$ -semigroups of o.d. homomorphisms, for example,  $\{\Delta^{it} : t \geq 0\}$  is one. We shall generalize [N-U] to the case of  $C_0$ -semigroups of o.d. homomorphisms.

Let  $\{T(t) : t \geq 0\}$  be a  $C_0$ -semigroup on the Hilbert space  $H$  and  $G$  be its generator. We set

$$H^J = \{\xi \in H : J\xi = \xi\},$$

and

$$D(G)^J = D(G) \cap H^J,$$

where  $D(G)$  denotes the domain of  $G$ . For the orthogonal decomposition  $\xi = \xi^+ - \xi^-$  we set

$$|\xi| = \xi^+ + \xi^-.$$

(7.1). Let  $\{T(t) : t \geq 0\}$  be a  $C_0$ -semigroup on  $H$  such that  $JT(t) = T(t)J$

for all  $t \geq 0$  and  $G$  be the generator. Then the following conditions are equivalent.

(i) Each  $T(t)$  is an o.d. homomorphism.

(ii) If  $\xi \in D(G)^J$ , then  $|\xi| \in D(G)^J$  and

$$G|\xi| = \lim_{t \downarrow 0} t^{-1}(|\xi + tG\xi| - |\xi|). \quad (\text{Kato's equality})$$

**Proof.** (i)  $\Rightarrow$  (ii). Recall that in the proof of (4.6)(ii) we have shown that for any  $\xi \in H^J$ ,  $\eta \in H^J$ ,  $|||\xi| - |\eta|||^2 \leq \|\xi - \eta\|^2$ .

On the other hand, by (BR, Proposition 3.1.23),  $\xi \in D(G)$  if and only if

$$\sup\{t^{-1}\|T(t)\xi - \xi\| : 0 < t \leq 1\} < +\infty.$$

Since  $|T(t)\phi| = T(t)|\xi|$  for each  $t \geq 0$ , we have

$$\begin{aligned} |||T(t)|\xi| - |\xi||| &= |||T(t)\xi| - |\xi||| \\ &\leq \|T(t)\xi - \xi\|. \end{aligned}$$

This shows that  $\xi \in D(G)^J$  implies  $|\xi| \in D(G)^J$ .

Moreover, for  $\xi \in D(G)^J$ ,

$$\begin{aligned} G|\xi| &= \lim_{t \downarrow 0} t^{-1}(T(t)|\xi| - |\xi|) \\ &= \lim_{t \downarrow 0} t^{-1}(|\xi + (T(t)\xi - \xi)| - |\xi|), \end{aligned}$$

and since  $T(t)J = JT(t)$  for  $t \geq 0$ , we have



$$\begin{aligned}
JG\xi &= J \cdot \lim_{t \downarrow 0} \frac{T(t) - 1}{t} \xi \\
&= \lim_{t \downarrow 0} \frac{T(t) - 1}{t} \cdot J\xi \\
&= \lim_{t \downarrow 0} \frac{T(t) - 1}{t} \xi \\
&= G\xi,
\end{aligned}$$

that is,  $G\xi \in H^J$ , and so  $|\xi + tG\xi|$  appearing in the right-hand side of Kato's equality is meaningful. Now as  $t \downarrow 0$ ,

$$\begin{aligned}
&\|t^{-1}(|\xi + T(t)\xi - \xi| - |\xi|) - t^{-1}(|\xi + tG\xi| - |\xi|)\| \\
&= t^{-1}||\xi + (T(t)\xi - \xi)| - |\xi + tG\xi|| \\
&\leq \|t^{-1}(T(t)\xi - \xi) - G\xi\| \rightarrow 0.
\end{aligned}$$

Hence we have the required Kato's equality

$$G|\xi| = \lim_{t \downarrow 0} t^{-1}(|\xi + tG\xi| - |\xi|).$$

(i)  $\Rightarrow$  (ii). The proof of this part is the same as that in the case of Banach lattices (Arendt [1], Corollary 3.8). For each  $\xi \in D(G)^J$ , since  $T(t)$  commute with  $J$ ,  $T(t)\xi \in D(G)^J$  for all  $t \geq 0$ ; and hence we may define

$$\phi_\xi(t) = T(s-t)|T(t)\xi|, \quad t \in (0, s).$$

Then, for  $\epsilon > 0$ ,

$$\begin{aligned}
\frac{1}{\epsilon} (\phi_{\xi}(t+\epsilon) - \phi_{\xi}(t)) &= \frac{1}{\epsilon} (T(s-t-\epsilon)|T(t+\epsilon)\xi| - T(s-t)|T(t)\xi|) \\
&= \frac{1}{\epsilon} T(s-t-\epsilon) [|T(t+\epsilon)\xi| - T(\epsilon)|T(t)\xi|] \\
&= \frac{1}{\epsilon} T(s-t-\epsilon) [|T(t+\epsilon)\xi| - |T(t)\xi| - (T(\epsilon)-1)|T(t)\xi|] \\
&= T(s-t-\epsilon) \left[ \frac{1}{\epsilon} (|T(t)\xi| + (T(\epsilon)-1)|T(t)\xi| - |T(t)\xi|) \right. \\
&\quad \left. - T(s-t-\epsilon) \left[ \frac{1}{\epsilon} (T(\epsilon)-1)|T(t)\xi| \right] \right],
\end{aligned}$$

where

$$\begin{aligned}
&\left\| \frac{1}{\epsilon} (|T(t)\xi| + (T(\epsilon)-1)|T(t)\xi| - |T(t)\xi|) \right. \\
&\quad \left. - \frac{1}{\epsilon} (|T(t)\xi| + \epsilon GT(t)\xi| - |T(t)\xi|) \right\| \\
&= \left\| \frac{1}{\epsilon} (|T(t)\xi| + (T(\epsilon)-1)|T(t)\xi| - |T(t)\xi| + \epsilon GT(t)\xi|) \right\| \\
&\leq \left\| \frac{1}{\epsilon} (T(\epsilon)-1)|T(t)\xi| - GT(t)\xi| \right\| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.
\end{aligned}$$

Hence by the assumption (Kato's equality), we have

$$GT(t)\xi = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (|T(t)\xi| + (T(\epsilon)-1)|T(t)\xi| - |T(t)\xi|).$$

Therefore the first and second terms converge to the same limit. Thus,

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} (\phi_{\xi}(t+\epsilon) - \phi_{\xi}(t)) = 0, \quad \text{for all } t \in (0, s).$$

Since the function  $\phi_{\xi}$  is continuous for each  $\xi \in D(G)^J$ , it is constant. In particular,  $\phi_{\xi}(0) = \phi_{\xi}(s)$ , that is,  $T(s)|\xi| = |T(s)\xi|$ . This proves that each  $T(t)$  is an o.d. homomorphism.  $\square$

The condition (7.2)(ii) below says that  $C_0$ -semigroups of o.d. homomorphisms

are "local" in the sense of (Nagel and Uhlig [1]).

(7.2) Let  $\{T(t) : t \geq 0\}$  be a  $C_0$ -semigroup of o.d. homomorphisms and  $G$  be its generator. It follows that

$$(i) \quad GJ = JG.$$

$$(ii) \quad \text{If } \xi \in D(G) \cap H^+, \eta \in H^+ \text{ and } (\xi, \eta) = 0, \text{ then } (G\xi, \eta) = 0.$$

**Proof.** (i) was proved in (7.1).

(ii) Suppose  $\xi \in D(G) \cap H^+, \eta \in H^+$  and  $(\xi, \eta) = 0$ . We have

$$\begin{aligned} (G\xi, \eta) &= \lim_{t \downarrow 0} t^{-1}(T(t)\xi - \xi, \eta) \\ &= \lim_{t \downarrow 0} t^{-1}(T(t)\xi, \eta) \geq 0, \end{aligned}$$

because  $T(t)\xi \in H^+$  for all  $t \geq 0$  and  $\xi \in H^+$ . On the other hand,

$$\begin{aligned} (G\xi, \eta) &= \lim_{t \downarrow 0} t^{-1}(T(t)\xi - \xi, T(t)\eta) \\ &= \lim_{t \downarrow 0} t^{-1}(-\xi, T(t)\eta) \leq 0, \end{aligned}$$

because  $(T(t)\xi, T(t)\eta) = 0$  for all  $t \geq 0$ . Hence  $(G\xi, \eta) = 0$ .  $\square$

This locality is not shared by the  $C_0$ -semigroups of order homomorphisms as the following statement shows. In the case of uniformly continuous group  $\{e^{tG}\}$  such that  $JG = GJ$ , the locality is exactly the same as  $G \in D(H^+)$  (see (2.10)), or equivalently,  $G$  is a derivation.

(7.3). If every uniformly continuous semigroup of order homomorphisms is local,  $\mathcal{M}$  is commutative.

**Proof.** Let  $p$  be an arbitrary projection in  $\mathcal{M}$ . Since  $pj(p)(H^+) \subset H^+$ ,  $\{e^{tpj(p)} : t \geq 0\}$  is a uniformly continuous semigroup of order homomorphisms with the

generator  $pj(p)$ . Hence,  $pj(p)$  is local by the hypothesis. Then, for  $\xi \in H^+$ ,  $\eta \in H^-$  satisfying  $(\xi, \eta) = 0$ , we have

$$(pj(p)\xi, pj(p)\eta) = (pj(p)\xi, \eta) = 0.$$

So  $pj(p)$  is an o.d. homomorphism. It follows from Example (4.2.ii) that

$$p = p^*p \in \mathcal{M} \cap \mathcal{M}'.$$

□

We now characterize generators of uniformly continuous semigroups of o.d. homomorphisms.

(7.4). Let  $\{T(t) : t \geq 0\}$  be a uniformly continuous semigroup and  $G$  be its generator. The following conditions are equivalent.

(i)  $T(t)$  is an o.d. homomorphism for each  $t \geq 0$ .

(ii)(a)  $JG = GJ$  and  $(G\xi, \eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ .

(b)  $G + G^* \in \mathcal{M} \cap \mathcal{M}'$ .

(iii)(a)  $G = a + j(a)$  for some  $a \in \mathcal{M}$ .

(b)  $a + a^* \in \mathcal{M} \cap \mathcal{M}'$ .

**Proof.** (i)  $\Rightarrow$  (ii)(a) was proved in (7.2). To prove (i)  $\Rightarrow$  (ii)(b), we note that since  $t^{-1}(T(t) - 1)$  converges to  $G$  uniformly,  $t^{-1}(T^*(t) - 1)$  converges to  $G^*$  uniformly, and hence

$$\begin{aligned} \lim_{t \downarrow 0} t^{-1}(T^*(t)T(t) - 1) &= \lim_{t \downarrow 0} t^{-1}(T^*(t)\hat{T}(t) - T^*(t) + T^*(t) - 1) \\ &= \lim_{t \downarrow 0} t^{-1}[T^*(t)(T(t) - 1) + T^*(t) - 1] \\ &= G + G^* \quad (\text{uniformly}). \end{aligned}$$

But each  $T(t)$  is an o.d. homomorphism, so by (5.4)  $T^*(t)T(t) \in \mathcal{M} \cap \mathcal{M}'$  for each  $t > 0$ , and hence  $G + G^* \in \mathcal{M} \cap \mathcal{M}'$  because  $\mathcal{M} \cap \mathcal{M}'$  is strongly closed.

(ii)  $\Rightarrow$  (iii). By theorem (2.10) of Connes (ii)(a) is equivalent to (iii)(a). Now condition (ii)(b) becomes  $a + a^* + j(a + a^*) \in \mathcal{M} \cap \mathcal{M}'$  where  $a \in \mathcal{M}$  and  $a + j(a) = G$ . By (1.1)(iii),  $j(a + a^*) \in \mathcal{M}'$  and hence

$$a + a^* \in -j(a + a^*) + \mathcal{M} \cap \mathcal{M}' \subset \mathcal{M}'.$$

Consequently,  $a + a^* \in \mathcal{M} \cap \mathcal{M}'$ .

(iii)  $\Rightarrow$  (i). By theorem (2.10) of Connes, condition (iii)(a) implies  $T(t)(H^+) = H^+$  for  $t \geq 0$ . We now only need to show for each  $t > 0$ ,  $(T(t)\xi, T(t)\eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ . As  $G$  is the generator of the uniformly continuous semigroup  $\{T(t) : t \geq 0\}$ ,  $G^*$  is the generator of the uniformly continuous semigroup  $\{T^*(t) : t \geq 0\}$ . Moreover, since  $a \in \mathcal{M}$  and  $a^* + a \in \mathcal{M} \cap \mathcal{M}'$ , we have  $a(a^* + a) = (a^* + a)a$ , and hence  $aa^* = a^*a$ . Consequently,  $G$  commutes with  $G^*$ . Thus by (Yoshida [1], IX.6, Corollary 2),

$$e^{tG^*} \cdot e^{tG} = e^{t(G^*+G)},$$

and hence by (2.12) we have, for each  $t > 0$ ,

$$(T(t)\xi, T(t)\eta) = (T^*(t)T(t)\xi, \eta)$$

$$= (e^{tG^*} e^{tG} \xi, \eta)$$

$$= (e^{t(G^*+G)} \xi, \eta)$$

$$= \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} (G^*+G)^n \xi, \eta \right)$$

$$= 0.$$

□

It follows from the proof of (7.4)(iii)  $\Rightarrow$  (7.4)(i) that if  $\{T(t) : t \geq 0\}$  is a uniform semigroup of o.d. homomorphisms, then each  $T(t)$  has to be normal, that is  $T^*(t)T(t) = T(t)T^*(t)$ . We now generalize (7.4) in the following form.

(7.5). Let  $\{T(t) : t \geq 0\}$  be a  $C_0$ -semigroup such that  $T(t)(H^+) \subset H^+$  and  $G$  be its generator. Then, the following conditions are equivalent.

(i) Each  $T(t)$  is normal and an o.d. homomorphism.



(ii)(a)  $G$  is normal and the real part of its spectrum is bounded above.

(b)  $G + G^*$  is affiliated with  $\mathcal{M} \cap \mathcal{M}'$ .

**Proof.** By (Rudin [1], Theorem 13.37) the condition (ii)(a) is equivalent to that each  $T(t)$  is normal.

(i)  $\Rightarrow$  (ii). Since  $\{T(t) : t \geq 0\}$  is a  $C_0$ -semigroup of normal operators with the generator  $G$ ,  $\{T^*(t)T(t) : t \geq 0\}$  is also a  $C_0$ -semigroup with the generator  $G + G^*$ .

Let  $\xi \in D(G + G^*)$ . Then by (BR, Proposition 3.1.23),

$$\sup\{t^{-1}\|T^*(t)T(t)\xi - \xi\| : 0 < t \leq 1\} < +\infty.$$

For an arbitrary unitary operator  $u \in (\mathcal{M} \cap \mathcal{M}')'$ , because  $T^*(t)T(t) \in \mathcal{M} \cap \mathcal{M}'$  we have for each  $t > 0$ ,

$$\begin{aligned} \sup\{t^{-1}\|T^*(t)T(t)u\xi - u\xi\| : 0 < t \leq 1\} \\ &= \sup\{t^{-1}\|uT^*(t)T(t)\xi - u\xi\| : 0 < t \leq 1\} \\ &= \sup\{t^{-1}\|T^*(t)T(t)\xi - \xi\| : 0 < t \leq 1\} \\ &< +\infty. \end{aligned}$$

Hence, again by (BR, Proposition 3.1.23),

$$u\xi \in D(G + G^*).$$

Moreover, because  $T^*(t)T(t) \in \mathcal{M} \cap \mathcal{M}'$  for each  $t \geq 0$ , we have

$$\begin{aligned} u(G + G^*)\xi &= u \lim_{t \downarrow 0} t^{-1}(T^*(t)T(t)\xi - \xi) \\ &= \lim_{t \downarrow 0} t^{-1}(uT^*(t)T(t)\xi - u\xi) \\ &= \lim_{t \downarrow 0} t^{-1}(T^*(t)T(t)u\xi - u\xi) \\ &= (G + G^*)u\xi, \end{aligned}$$

that is,  $G + G^*$  commutes with every unitary operator in  $(\mathcal{M} \cap \mathcal{M}')'$ . Hence  $G + G^*$  is affiliated with  $\mathcal{M} \cap \mathcal{M}'$ .

(ii)  $\Rightarrow$  (i). Since (ii)(a) implies that  $\{T(t) : t \geq 0\}$  is a  $C_0$ -semigroup of normal

operators,  $G + G^*$  is the generator of the  $C_0$ -semigroup  $\{T^*(t)T(t) : t \geq 0\}$ . By Hille-Phillip-Yosida's theorem (Dunford and Schwartz [1]),

$$\text{Range}[\lambda - (G + G^*)] = H,$$

and  $R(\lambda, G + G^*) = (\lambda - (G + G^*))^{-1}$  exists for all  $\lambda$  large enough. Hence, for any unitary operator  $u \in (\mathcal{M} \cap \mathcal{M}')'$ , it follows from (ii)(b) that

$$\begin{aligned} uR(\lambda, G + G^*)(\lambda - (G + G^*)) &= u \\ &= R(\lambda, G + G^*)(\lambda - (G + G^*))u \\ &= R(\lambda, G + G^*)u(\lambda - (G + G^*)). \end{aligned}$$

Since  $\text{Range}[\lambda - (G + G^*)] = H$ , we have

$$uR(\lambda, G + G^*) = R(\lambda, G + G^*)u,$$

that is,  $R(\lambda, G + G^*) \in \mathcal{M} \cap \mathcal{M}'$  for all  $\lambda$  large enough.

It follows immediately from the formula

$$T^*(t)T(t)\xi = \lim_{n \rightarrow \infty} \left[ \frac{n}{t} R\left(\frac{n}{t}, G + G^*\right) \right]^n \xi$$

that  $T^*(t)T(t) \in \mathcal{M} \cap \mathcal{M}'$ , and hence each  $T(t)$  is an o.d. homomorphism.  $\square$

For the modular operator  $\Delta$ ,  $\{\Delta^{it} : t \in \mathcal{R}\}$  is a  $C_0$ -group of unitary operators and its generator is  $i\Delta$ . It is known that  $\Delta^{it}(H^+) = H^+$ . The following result gives a property of this  $C_0$ -group. Note that a unitary operator  $u$  is an o.d. homomorphism if and only if  $u(H^+) \subset H^+$ .

(7.6). Let  $\{u(t) : t \in \mathcal{R}\}$  be a  $C_0$ -group of unitary operators such that  $Ju(t) = u(t)J$  for all  $t \in \mathcal{R}$ , and let  $G$  be its generator. Then  $u(t)(H^+) = H^+$  for all  $t \in \mathcal{R}$ , if and only if the following conditions are satisfied.

(i)  $\xi \in D(G)^J$  implies  $|\xi| \in D(G)$ .

(ii)  $G$  is local.

**Proof.** Suppose that  $u(t)(H^+) = H^+$  for all  $t \in \mathcal{R}$ . Then  $\{u(t)\}$  is a

$C_0$ -group of o.d. homomorphisms. Therefore (i) follows from (7.1), and (ii) follows from (7.2). For the converse, we need to prove the "dissipativity" of  $G$  (Batty and Robinson [1], Corollary 2.2.2) in the following sense (Yamamuro [2]):

$$(\xi^+, G\xi) \leq 0 \quad \text{for every } \xi \in D(G)^J.$$

To prove this, suppose that  $\xi \in D(G)^J$ . Then by (i),  $\xi^+ \in D(G)$  and  $\xi^- \in D(G)$ . Moreover, by (ii),  $(\xi^+, G\xi^-) = 0$ , and hence, because each  $u(t)$  is a contraction,

$$\begin{aligned} (\xi^+, G\xi) &= (\xi^+, G\xi^+) - (\xi^-, G\xi^-) \\ &= (\xi^+, G\xi^+) \\ &\leq 0. \end{aligned}$$

□

From the consideration of semigroups of o.d. homomorphisms, we observe that there are two general problems. The first is the differentiability of the absolute value involved in the Kato's equality. In the case of Banach lattice, the absolute value  $x \mapsto |x|$  is differentiable in a suitable sense and it was possible to write the limit in the Kato's equality as  $\text{sign}(x)Gx$ . The second problem is whether or not an o.d. homomorphism is a normal operator as we have encountered in (7.4) and (7.5). On Hilbert spaces, the normal semigroups are much more easily handled than non-normal ones. These two problems will be considered in the following two sections.

## §8. ABSOLUTE VALUES AND ITS DIFFERENTIABILITY

Let  $E$  be a vector space,  $F$  a topological vector space for some topology  $\tau$  and  $\theta$  a function from  $E$  to  $F$ .  $\theta$  is  $\tau$ -Gâteaux differentiable if for every  $x, y \in E$ ,

$$D_x \theta(y) = \lim_{h \downarrow 0} h^{-1}(\theta(x + hy) - \theta(x))$$

exists in  $F$  for the topology  $\tau$ . When  $F$  is a Banach space, we say  $\theta$  is Gâteaux differentiable when it is norm Gâteaux differentiable.

As mentioned in §7, Arendt [1] has proved the Gâteaux differentiability of the map  $\theta$  defined by

$$\theta(x) = |x|, \quad x \in B,$$

where  $B$  is a Banach lattice. The proof essentially depended upon the inequality

$$|x + y| \leq |x| + |y|, \quad x \in B, \quad y \in B,$$

which holds in any Banach lattice.

In the ordered Hilbert space  $H$  associated with  $(\mathcal{M}, H, \xi_0)$ , the absolute value  $\theta(\xi) = |\xi|$  for  $\xi \in H^J$  is difficult to handle, and one of the reasons for the difficulty is that the above inequality does not hold in general.

(8.1). *The following conditions are equivalent.*

(i)  $|\xi + \eta| \leq |\xi| + |\eta|$  for all  $\xi \in H^J$  and  $\eta \in H^J$ .

(ii)  $\mathcal{M}$  is commutative.

**Proof.** (i)  $\Rightarrow$  (ii). Since  $|\xi| = \xi^+ + \xi^-$  for  $\xi = \xi^+ - \xi^-$ , it is obvious that  $\pm \xi \leq |\xi|$ . Now suppose that  $\pm \xi \leq \eta$  for some  $\eta \in H^J$ . Then since  $\eta \pm \xi \in H^+$ , we have

$$\begin{aligned}
|\xi| &= \frac{\xi + \eta}{2} + \frac{\xi - \eta}{2} \\
&\leq \frac{1}{2} |\xi + \eta| + \frac{1}{2} |\xi - \eta| \\
&= \frac{1}{2} (\xi + \eta) + \frac{1}{2} (\eta - \eta) \\
&= \eta.
\end{aligned}$$

This shows that  $|\xi|$  is the least upper bound of  $\xi$  and  $-\xi$ , that is,

$$|\xi| = \sup(\xi, -\xi).$$

Therefore  $H^J$  is lattice-ordered, and hence  $\mathcal{M}$  is commutative.

(ii)  $\Rightarrow$  (i). Since  $H^J$  is lattice-ordered, (i) is obvious.  $\square$

In the proof of (7.1) we have proved that the absolute-valued function is a Lipschitz map, that is,

$$|||\xi| - |\eta||| \leq \|\xi - \eta\| \quad \text{for all } \xi \in H^J \text{ and } \eta \in H^J.$$

Our next result about the absolute-valued function will show that it is additive for "absolutely orthogonal" elements.

(8.2). If  $\xi \in H^J$ ,  $\eta \in H^J$  and  $(|\xi|, |\eta|) = 0$ , then  $|\xi + \eta| = |\xi| + |\eta|$ .

**Proof.** Suppose that  $\xi \in H^J$ ,  $\eta \in H^J$  and  $(|\xi|, |\eta|) = 0$ . Then, since  $H^+$  is self-dual,  $|\xi| \geq \xi^\pm$  and  $|\eta| \geq \eta^\pm$ , we have  $(\xi^\pm, \eta^\pm) = 0$ , and hence  $(\xi, \eta) = (\xi^+ - \xi^-, \eta^+ - \eta^-) = 0$ . Then, since  $|||\rho||| = \|\rho\|$  for every  $\rho \in H^J$ , we have

$$|||\xi + \eta|||^2 = \|\xi + \eta\|^2 = \|\xi\|^2 + \|\eta\|^2$$

and

$$|||\xi| + |\eta|||^2 = |||\xi|||^2 + |||\eta|||^2 = \|\xi\|^2 + \|\eta\|^2.$$

Therefore,



$$|||\xi + \eta|||^2 = |||\xi| + |\eta|||^2.$$

Again by the self-duality of  $H^+$ ,

$$(\xi^+, |\xi + \eta|) \geq (\xi^+, \xi + \eta) = (\xi^+, \xi) = \|\xi^+\|^2$$

and

$$(\xi^-, |\xi + \eta|) \geq (\xi^-, -(\xi + \eta)) = (\xi^-, -\xi) = \|\xi^-\|^2.$$

Hence,

$$(|\xi|, |\xi + \eta|) \geq \|\xi^+\|^2 + \|\xi^-\|^2 = \|\xi\|^2.$$

Similarly,

$$(|\eta|, |\xi + \eta|) \geq \|\eta\|^2.$$

Therefore,

$$\|\xi\|^2 + \|\eta\|^2 \leq (|\xi| + |\eta|, |\xi + \eta|) \leq |||\xi| + |\eta||| \cdot \|\xi + \eta\|.$$

Then, by the equalities proved above, we arrive at

$$|||\xi| + |\eta|||^2 = (|\xi| + |\eta|, |\xi + \eta|) = \|\xi + \eta\|^2.$$

Finally,

$$\begin{aligned} ||(|\xi| + |\eta|) - |\xi + \eta|||^2 &= |||\xi| + |\eta|||^2 - 2(|\xi| + |\eta|, |\xi + \eta|) + \|\xi + \eta\|^2 \\ &= 0. \end{aligned}$$

and the required equality is proved.  $\square$

We now turn to the problem of the differentiability of  $\theta(\xi) = |\xi|$ , or, the existence of the limit

$$\lim_{t \downarrow 0} \frac{1}{t} (|\xi + t\eta| - |\xi|).$$

When  $\mathcal{M}$  is commutative,  $H^J$  is lattice-ordered, and hence the absolute-valued function  $|\cdot|$  is convex. Then by Arendt ([1], Proposition 3.3)  $|\cdot|$  is Gâteaux differentiable. The following example shows a case where  $\mathcal{M}$  is a factor and  $|\cdot|$  is

Gâteaux differentiable, and hence rules out the possibility that Gâteaux differentiability of  $|\cdot|$  decides the commutativity of  $\mathcal{M}$ .

(8.2). *Example.* We consider the triple  $(M_2, H_2, I)$  as defined in Example (6.1).

Let

$$A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in H_2^J$$

The characteristic equation of  $A$  is

$$\lambda^2 - (a + c)\lambda + ac - |b|^2 = 0.$$

Hence,

$$(a + c)^2 \lambda^2 = \lambda^4 + 2D_A \lambda^2 + D_A^2,$$

where  $D_A = ac - |b|^2$  is the determinant of  $A$ . Since

$$\|A\|^2 = \text{trace } A^2 = a^2 + 2|b|^2 + c^2,$$

we have

$$\|A\|^2 \lambda^2 = \lambda^4 + D_A^2$$

and hence,

$$\begin{aligned} (\|A\|^2 + 2|D_A|)\lambda^2 &= \lambda^4 + 2|D_A|\lambda^2 + |D_A|^2 \\ &= (\lambda^2 + |D_A|)^2. \end{aligned}$$

Now by the Cayley-Hamilton's theorem we have

$$(\|A\|^2 + 2|D_A|)A^2 = (A^2 + |D_A|)^2.$$

As  $A^2 + |D_A|$  is a positive matrix, we have

$$|A| = \frac{A^2 + |D_A|}{\sqrt{\|A\|^2 + 2|D_A|}}.$$

Hence, for

$$B = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \chi \end{bmatrix} \in H_2^J$$

with  $\alpha, \chi \in \mathfrak{R}$ , we have after tedious calculations,

$$\begin{aligned} \lim_{t \downarrow 0} \frac{|A+tB| - |A|}{t} &= \frac{AB + BA}{\sqrt{\|A\|^2 + 2|D_A|}} + \frac{2(c\alpha + a\chi - \bar{b}\beta - b\bar{\beta})}{\sqrt{\|A\|^2 + 2|D_A|}} \\ &\quad - \frac{a\alpha + c\chi + c\alpha + a\chi}{(\|A\|^2 + 2|D_A|)} \frac{(A^2 + 2|D_A|)}{\sqrt{\|A\|^2 + 2|D_A|}}, \end{aligned}$$

when  $D_A \geq 0$ ; and

$$\begin{aligned} \lim_{t \downarrow 0} \frac{|A+tB| - |A|}{t} &= \frac{AB + BA}{\sqrt{\|A\|^2 + 2|D_A|}} - \frac{2(2\alpha + a\chi - \bar{b}\beta - b\bar{\beta})}{\sqrt{\|A\|^2 + 2|D_A|}} \\ &\quad - \frac{a\alpha + c\chi + (c\alpha + a\chi) + 2(b\bar{\beta} + \bar{b}\beta)}{(\|A\|^2 + 2|D_A|)} \frac{(A^2 + 2|D_A|)}{\sqrt{\|A\|^2 + 2|D_A|}}, \end{aligned}$$

when  $D_A < 0$ . □

It remains an open problem whether or not the absolute-valued function is Gâteaux differentiable. In general, we have only been able to prove the existence of the limit when  $\xi \in H^+$ .

The following property of the positive parts will be used later.

(8.4). If  $\xi \in H^J$ ,  $\eta \in H^J$  and  $\xi \leq \eta$ , then  $\|\xi^+\| \leq \|\eta^+\|$ .

**Proof.** Since  $\eta \leq \eta^+$ , we have  $\xi \leq \eta^+$ . Hence by (2.1)(viii),

$$\begin{aligned} \|\xi^+\| &= \inf\{\|\xi + \rho\| : \rho \in H^+\} \\ &\leq \|\xi + (\eta^+ - \xi)\| = \|\eta^+\|. \end{aligned} \quad \square$$

The following result has also been found by Yamamuro.

(8.5). If  $\xi \in H^+$ , the following limit exists for every  $\eta \in H^J$ ,

$$\lim_{t \downarrow 0} \frac{1}{t} (|\xi + t\eta| + |\xi|).$$

**Proof.** We set

$$C_n = \{\eta \in H^J : \eta \leq n\xi\} \quad \text{and} \quad C_\infty = \overline{\bigcup_{n=1}^{\infty} C_n}.$$

Obviously,  $C_n$  and  $C_\infty$  are closed and convex. Therefore, we can define projections  $P_n$  and  $P_\xi$  on  $C_n$  and  $C_\infty$  respectively. In other words, for each  $\eta \in H^J$ ,  $P_n\eta$  is the unique element of  $C_n$  such that

$$\|P_n\eta - \eta\| \leq \|\rho - \eta\|, \quad \text{for all } \rho \in C_n,$$

and  $P_\xi\eta$  is the unique element of  $C_\infty$  such that

$$\|P_\xi\eta - \eta\| \leq \|\rho - \eta\|, \quad \text{for all } \rho \in C_\infty.$$

For  $\eta \in H^J$ , we temporarily set

$$\eta \wedge \xi = \eta - (\eta - \xi)^+,$$

although  $\eta \wedge \xi$  is not equal to the greatest lower bound of  $\eta$  and  $\xi$ .

$$(i) \quad P_n\eta = \eta \wedge n\xi.$$

Since

$$n\xi - (\eta \wedge n\xi) = n\xi - \eta - (\eta - n\xi)^+ = (\eta - n\xi)^- \in H^+,$$

we have  $\eta \wedge n\xi \in C_n$ . Next we prove

$$\|\eta - (\eta \wedge n\xi)\| \leq \|\eta - \rho\| \quad \text{for all } \rho \in C_n.$$

Since  $\eta - \eta \wedge n\xi = (\eta - n\xi)^+$ , we need to prove

$$\|(\eta - n\xi)^+\| \leq \|\eta - \rho\| \quad \text{for all } \rho \in C_n.$$

But because  $\rho \leq n\xi$ , it follows from (8.4) that

$$\|(\eta - n\xi)^+\| \leq \|(\eta - \rho)^+\| \leq \|\eta - \rho\|.$$

Thus,  $\eta \wedge n\xi$  coincides with  $P_n\eta$ .

$$(ii) \quad \lim_{n \rightarrow \infty} \|P_n\eta - \eta\| = \|P_\xi\eta - \eta\|.$$

Since  $\eta - P_n\eta = (\eta - n\xi)^+$  and  $\xi \in H^+$ , it follows from (8.4) that

$$\|P_n\eta - \eta\| = \|(\eta - n\xi)^+\| \geq \|\eta - (n+1)\xi^+\| = \|P_{n+1}\eta - \eta\|.$$

Furthermore, for each  $n$ , we have

$$\|P_n\eta - \eta\| = \inf\{\|\rho - \eta\| : \rho \in C_n\} \geq \inf\{\|\rho - \eta\| : \rho \in C_\infty\}.$$

Therefore, if our assertion is false, there exists  $\epsilon > 0$  such that

$$\|P_\xi\eta - \eta\| + \epsilon < \|P_n\eta - \eta\|, \quad \text{for all } n.$$

Then there exists  $\rho \in C_\infty$  such that

$$\|\rho - \eta\| < \|P_n\eta - \eta\| - \epsilon, \quad \text{for all } n.$$

Since

$$C_\infty = \overline{\bigcup_{n=1}^{\infty} C_n}$$

and  $\{C_n\}$  is increasing, there exist  $\rho_i \in C_{n_i}$  such that  $\rho_i \rightarrow \rho$ . Then there exists  $i_0$  such that if  $i \geq i_0$ ,

$$\|\rho_i - \eta\| < \|P_n\eta - \eta\| - \epsilon \quad \text{for all } n,$$

that is, for all  $i \geq i_0$ ,

$$\|P_{n_i}\eta - \eta\| \leq \|\rho_i - \eta\| < \|P_n\eta - \eta\| - \epsilon \quad \text{for all } n.$$

which is a contradiction.

(iii)  $\{P_n\eta\}$  is bounded.

Because  $\xi \in H^+$ , we have  $\eta - \eta\xi \leq \eta$ . Applying (8.4) to this inequality we have

$$\|P_n\eta\| \leq \|\eta\| + \|(\eta - n\xi)^+\| \leq 2\|\eta\|.$$

(iv)  $\lim_{n \rightarrow \infty} P_n\eta = P_\xi\eta$  for every  $\xi \in H^J$ .



By (iii) there is  $\{n_k\}$  such that  $\{P_{n_k}\eta\}$  converges weakly to some  $\rho_o$ . Since  $P_{n_k}\eta \in C_{n_k} \subset C_\infty$  and  $C_\infty$  is weakly closed, we have  $\rho_o \in C_\infty$ .

Since  $\{P_{n_k}\eta - \eta\}$  converges weakly to  $\rho_o - \eta$ , it follows from (ii) that

$$\|\rho_o - \eta\| \leq \lim_{k \rightarrow \infty} \|P_{n_k}\eta - \eta\| = \|P_\xi\eta - \eta\|.$$

Since  $\rho_o \in C_\infty$ , we arrive at

$$\|\rho_o - \eta\| = \|P_\xi\eta - \eta\|.$$

Hence, by the unicity of the projection, we have  $\rho_o = P_\xi\eta$ . Therefore,  $\{P_{n_k}\eta - \eta\}$  converges weakly to  $P_\xi\eta - \eta$ . Combining this with (ii), we now have

$$\lim_{k \rightarrow \infty} (P_{n_k}\eta - \eta) = P_\xi\eta - \eta$$

and hence,

$$\lim_{k \rightarrow \infty} P_{n_k}\eta = P_\xi\eta.$$

Thus,

$$\lim_{n \rightarrow \infty} P_n\eta = P_\xi\eta.$$

Now we have

$$\begin{aligned} n\left(\left|\xi + \frac{1}{n}\eta\right| - \xi\right) - \eta &= |n\xi + \eta| - (n\xi + \eta) \\ &= 2(n\xi + \eta)^- \\ &= -\eta - [(-\eta) \wedge n\xi] \\ &= -\eta - P_n(-\eta). \end{aligned}$$

Therefore by (iv) we have

$$\lim_{n \rightarrow \infty} n\left(\left|\xi + \frac{1}{n}\eta\right| - \xi\right) - \eta = \lim_{n \rightarrow \infty} -P_n(-\eta) = -P_\xi(-\eta).$$



## §9. NORMALCY OF O.D. ISOMORPHISMS

We shall characterize  $(\mathcal{M}, H, \xi_0)$  which has the property that every o.d. isomorphism is a normal operator. This is connected to an unexpected problem. We shall first explain this connection.

Let  $B$  be an ordered Banach space. Let  $G(B)$  be the set of all isometric order isomorphisms. This is a group. The set of all continuous linear operators  $T$  on  $B$  such that

$$-\lambda x \leq Tx \leq \lambda x \quad \text{for all } x \in B^+,$$

where  $\lambda$  is a constant depending only on  $T$ , is called the *ideal centre* of  $B$ . An element of  $G(B)$  which commutes with every member of the ideal centre is called a *symmetry*, and the set of all symmetries is denoted by  $S(B)$ . This is a subgroup of  $G(B)$ . In Yamamuro ([3]), it was proved that  $S(B)$  consists of only the identity operator when  $B$  is a Banach lattice. This means that  $S(B)$  is the smallest when  $B$  is lattice-ordered. The opposite case when  $S(B)$  is the largest, that is  $S(B) = G(B)$ , has also been given a characterization when  $B = H^J$ .

When  $B = H^J$ , it is obvious that  $G(H^J)$  is the group of all unitary operators  $u$  on  $H$  such that  $u(H^+) = H^+$ . Bös [1] has proved that the ideal centre of  $H^J$  is equal to  $(\mathcal{M} \cap \mathcal{M}')^h$ , the hermitian part of  $\mathcal{M} \cap \mathcal{M}'$ . Therefore,  $S(H^J)$  is the group of all unitary operators  $u$  such that  $u(H^+) = H^+$  and  $u \in R(\mathcal{M}, \mathcal{M}')$ , the algebra generated by  $\mathcal{M}$  and  $\mathcal{M}'$ , which is the commutant of the centre. Hence,  $S(H^J) = G(H^J)$  is equivalent to the fact that all unitary operators  $u$  such that  $u(H^+) = H^+$  belong to  $R(\mathcal{M}, \mathcal{M}')$ . As the following result shows, this property is equivalent to that all o.d. isomorphisms are normal operators.

(9.1). *The following conditions are equivalent:*

(i) *All o.d. isomorphisms are normal.*

(ii) All unitary operators  $u$  such that  $u(H^+) = H^+$  belong to  $R(\mathcal{M}, \mathcal{M}')$ .

(iii) All unital Jordan isomorphisms on  $\mathcal{M}$  are identical on the centre.

(iv) For any central projections  $p$  and  $q$ , if  $u^*pu = q$  for some unitary operator  $u$  such that  $u(H^+) = H^+$ , then  $p = q$ .

(v) For any mutually orthogonal central non-zero projections  $p$  and  $q$ , there is no Jordan isomorphism  $\beta$  of  $\mathcal{M}_p$  onto  $\mathcal{M}_q$  such that  $\beta(p) = q$ .

**Proof.** (i)  $\Rightarrow$  (ii). Let  $u$  be an arbitrary unitary operator such that  $u(H^-) = H^-$ . Then,  $u$  is an o.d. isomorphism. Let  $p$  be an arbitrary projection in the centre. Then, since  $(1+p)(1 - \frac{1}{2}p) = 1$ ,  $1+p$  is a bijective o.d. homomorphism, and hence  $1+p$  is an o.d. isomorphism. Therefore,  $u(1+p)$  is also an o.d. isomorphism. Since  $u(1+p)$  is normal by the hypothesis, we have

$$u(1+p)^2 u^* = (1+p)^2.$$

It then follows that  $upu^* = p$ , or,  $up = pu$ . This shows that  $u$  commutes with every member of the centre. Hence,  $u \in R(\mathcal{M}, \mathcal{M}')$ .

(ii)  $\Rightarrow$  (iii). Let  $\alpha$  be an arbitrary unital Jordan isomorphism. Then, among others, it satisfies  $\alpha(x^*) = \alpha(x)^*$  for  $x \in \mathcal{M}$ , and  $\alpha(\mathcal{M} \cap \mathcal{M}') \subset \mathcal{M} \cap \mathcal{M}'$ . Now, by the theorem (2.7) of Connes, there exist a central projection  $e$  and a unitary operator  $u$  such that  $u(H^+) = H^+$  and

$$uxu^* = \alpha(x)e + J\alpha(x)^*J(1-e) \quad \text{for every } x \in \mathcal{M}.$$

Suppose  $x \in \mathcal{M} \cap \mathcal{M}'$ . Since  $u \in R(\mathcal{M}, \mathcal{M}') = (\mathcal{M} \cap \mathcal{M}')'$  by the assumption, we have  $uxu^* = x$ . On the other hand, since  $\alpha(x) \in \mathcal{M} \cap \mathcal{M}'$ , we have  $J\alpha(x)^*J = \alpha(x)$ . Thus,  $x = \alpha(x)e + \alpha(x)(1-e) = \alpha(x)$ .

(iii)  $\Rightarrow$  (iv). Let  $p$  and  $q$  be two arbitrary central projections. Suppose that  $u^*pu = q$  for some unitary operator  $u$  such that  $u(H^+) = H^+$ . Then, by (2.7), there is a Jordan isomorphism  $\alpha$  and a central projection  $e$  such that

$$uxu^* = \alpha(x)e + J\alpha(x)^*J(1-e) \quad \text{for all } x \in \mathcal{M}.$$

For  $x = q$ , we have

$$p = \alpha(q)e + J\alpha(x)^*J(1-e) = q,$$

because  $\alpha(q) = q$  by the assumption.

(iv)  $\Rightarrow$  (v). Let  $p$  and  $q$  be two arbitrary mutually orthogonal central non-zero projections. Suppose that there is a Jordan isomorphism  $\beta$  of  $\mathcal{M}_p$  onto  $\mathcal{M}_q$  such that  $\beta(p) = q$ . We set

$$\alpha(x) = (1 - (p+q))x + \beta(px) + \beta^{-1}(qx) \quad \text{for all } x \in \mathcal{M}.$$

Then  $\alpha$  is a unital Jordan isomorphism on  $\mathcal{M}$  and  $\alpha(q) = p$ . Hence, for the unitary operator  $u$  determined by  $\alpha$  by (2.7), we have  $u(H^+) = H^+$  and  $u^*pu = q$ . Then by the hypothesis,  $p = q$ , which is a contradiction.

(v)  $\Rightarrow$  (i). Let  $\phi$  be an o.d. isomorphism and  $\phi = u|\phi|$  be its polar decomposition. Then,  $u$  is a unitary operator and also an o.d. homomorphism by (6.3). Hence, by (5.6),  $u$  is an o.d. isomorphism, which is equivalent to  $u(H^+) = H^+$ . Therefore it determines a unital Jordan isomorphism  $\alpha$  by (2.7), and for some central projection  $e$ ,

$$uxu^* = \alpha(x)e + J\alpha(x)^*J(1-e) \quad \text{for all } x \in \mathcal{M}.$$

Suppose that  $\alpha(f) \neq f$  for some central projection  $f$ . Note that  $g = \alpha(f)$  is also a central projection. We define  $p$  and  $q$  as follows.

$$p = f - fg \quad \text{and} \quad q = g - g\alpha(g) \quad \text{if } gf \neq f,$$

$$p = g - f \quad \text{and} \quad q = \alpha(g) - g \quad \text{if } gf = f.$$

Obviously,  $p$  and  $q$  are mutually orthogonal central projections such that  $\alpha(p) = q$ . Then the restriction  $\beta$  of  $\alpha$  on  $\mathcal{M}_p$  is a Jordan isomorphism onto  $\mathcal{M}_q$  such that  $\alpha(p) = q$ . This contradicts the hypothesis. Therefore  $\alpha$  is identical on the centre. From the equation above between  $u$  and  $\alpha$ , it follows that  $u \in R(\mathcal{M}, \mathcal{M}')$ . Since  $|\phi| \in \mathcal{M} \cap \mathcal{M}'$  by (5.4), we can conclude that  $u$  and  $|\phi|$  commute. Hence  $\phi$  is a normal operator.  $\square$

The equivalence of (i), (ii) and (iii) in (9.1) have been proved in



Dang-Yamamuro ([1]). The conditions (iv) and (v) have also been given by Yamamuro independently.

(9.2). *Every o.d. homomorphism is normal if and only if every partial isometric o.d. homomorphism is normal.*

**Proof.** Suppose that every partial isometry which is an o.d. homomorphism is normal. Let  $u$  be an arbitrary partial isometry which is an o.d. homomorphism. Then by hypothesis and by (5.4),  $uu^* = u^*u \in \mathcal{M} \cap \mathcal{M}'$ . Let  $p$  be an arbitrary central projection. Then,  $(u^*p)^*u^*p = puu^*p \in \mathcal{M} \cap \mathcal{M}'$ , and hence by (5.4)  $u^*p$  is a partial isometry which is an o.d. homomorphism, because  $u^*(H^+) \subset H^+$  and  $p(H^+) \subset H^+$ . Therefore, by hypothesis we have

$$u^*pu = u^*p(u^*p)^* = (u^*p)^*u^*p = puu^*p.$$

Since  $u^*u = uu^*$ , we have  $u^*pu = pu^*up \in \mathcal{M} \cap \mathcal{M}'$ , and hence

$$p(u^*pu) = (u^*pu)p = pu^*up.$$

Consequently, we have

$$\begin{aligned} (up - pu)^*(up - pu) &= pu^*up - pu^*pu - u^*pup + u^*pu \\ &= 0. \end{aligned}$$

Thus,  $u \in R(\mathcal{M}, \mathcal{M}')$ .

Now let  $\phi$  be an arbitrary o.d. homomorphism and  $\phi = u_\phi|\phi|$  be the polar decomposition of  $\phi$ . Then by (6.3)  $u_\phi$  is a partial isometry which is also an o.d. homomorphism. So  $u_\phi \in R(\mathcal{M}, \mathcal{M}')$  and hence  $u_\phi$  commutes with  $|\phi| \in \mathcal{M} \cap \mathcal{M}'$ . Thus,  $\phi$  is normal.

The converse is obvious. □

It follows from (5.4) that factors satisfy conditions of (9.1). In other words, the class of von Neumann algebras for which all o.d. isomorphisms are normal is somewhat larger than the one of factors. To obtain a more precise statement, let us consider finite-dimensional algebras.

Let  $\mathcal{M}$  be a finite-dimensional von Neumann algebra. Then,  $\mathcal{M}$  is the direct sum of factors  $\mathcal{M}p_i$  ( $1 \leq i \leq n$ ), where  $p_i$  are central projections and each  $\mathcal{M}p_i$  can be regarded as the  $k_i \times k_i$  matrices algebra. Let  $1_i$  be the unit matrix of  $\mathcal{M}p_i$  and set

$$\xi_0 = 1_1 \oplus 1_2 \oplus \dots \oplus 1_n,$$

which is a trace vector for  $\mathcal{M}$  and  $H$  be the corresponding Hilbert space. Then (9.1)(v) implies the following.

(9.3) (Yamamuro [3], (5.1)). *Let  $\mathcal{M}$  be a finite-dimensional von Neumann algebra as above. Then, all o.d. isomorphisms are normal if and only if  $\mathcal{M}$  is the direct sum of factors such that any two of them are not Jordan isomorphic.*

We now show that, as long as  $\mathcal{M}$  is finite-dimensional, we can reach the class of factors if we replace "isomorphism" by "homomorphism". In other words, we are going to consider the following property:

(\*) *All o.d. homomorphisms are normal operators.*

For preparations, we shall first prove two statements.

When  $p$  is a central projection,  $\mathcal{M}p$  is a von Neumann algebra on  $pH$ , and  $p\xi_0$  is a cyclic and separating vector for  $\mathcal{M}p$ . The natural cone  $(pH)^+$  associated with  $(\mathcal{M}p, pH, p\xi_0)$  is equal to  $p(H^+)$ .

The following statements (9.4), (9.5) and (9.6) have also been found by Yamamuro. I follow his formulation.

(9.4). *Suppose that  $(\mathcal{M}, H, \xi_0)$  satisfies (\*) and  $p$  is a central projection. Then  $(\mathcal{M}p, pH, p\xi_0)$  satisfies (\*).*

**Proof.** If  $\phi$  is an o.d. homomorphism on  $pH$ ,  $\psi = \phi p$  is an o.d. homomorphism on  $H$ . Hence, by the hypothesis,  $\psi$  is a normal operator, that is,

$$(\phi p)^*(\phi p)\xi = (\phi p)(\phi p)^*\xi, \quad \xi \in H.$$

Since  $\phi$  is an operator from  $pH$  to  $pH$ , the above equality can be written as

$$\phi^*\phi\eta = \phi\phi^*\eta, \quad \eta \in pH,$$

and hence  $\phi$  is a normal operator. □

In the following statement,  $M_n(\mathcal{C})$  denotes the algebra of all  $n \times n$  matrices, the unit of which is denoted by  $1_n$ , and the zero is denoted by  $0_n$ .

(9.5). Let  $n < m$  and  $M = M_n(\mathcal{C}) \oplus M_m(\mathcal{C})$ . Then, there is an element  $a$  of  $M_{n+m}(\mathcal{C})$  such that  $a^*a = 1_n \oplus 0$ ,  $aa^* \leq 0 \oplus 1_m$  and the map  $v: M \rightarrow M$  defined by  $v(x) = axa^*$  satisfies  $v(M^+) \subset M^+$ .

**Proof.**  $M$  consists of matrices of the following form:

$$x = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

for all  $A \in M_n(\mathcal{C})$  and  $B \in M_m(\mathcal{C})$ . Since  $n < m$ , we can write this matrix in the following form:

$$x = \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & B_2 \\ 0 & B_3 & B_4 \end{pmatrix},$$

where  $B_1$  is a  $n \times n$  matrix. Take an element  $a \in M_{n+m}(\mathcal{C})$  defined by

$$\begin{pmatrix} 0_n & 0 & 0 \\ 1_n & & \\ & 0_m & \end{pmatrix}$$

Then,  $a^*a = 1_n \oplus 0_m$ ,  $aa^* \leq 0_n \oplus 1_m$  and for  $x$  as above.

$$v(x) = axa^* = \begin{pmatrix} 0_n & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0_{m-n} \end{pmatrix}$$

Then it follows immediately that  $v(M^+) \subset M^+$ . □

Now we consider the case when  $M = Mp_1 \oplus Mp_2 \oplus \dots \oplus Mp_n$  is finite-

dimensional. We retain the notations given before (9.3).

(9.6). *Let  $M$  be finite-dimensional. Then,  $(\mathcal{M}, H, \xi_0)$  satisfies (\*) if and only if  $M$  is a factor.*

**Proof.** If  $M$  is a factor and  $\phi$  is a non-zero o.d. homomorphism, it follows from (5.4) that  $\phi^*\phi = \alpha 1$  for some positive number  $\alpha$ . Since  $H$  is finite-dimensional, this implies  $\phi^*\phi = \phi\phi^*$ .

Conversely, suppose that  $(\mathcal{M}, H, \xi_0)$  satisfies (\*). Then, by (9.4), we can assume that  $M = M_{p_1} \oplus M_{p_2}$ , where  $M_{p_i}$  are algebras of  $k_i \times k_i$  matrices such that  $p_1 = 1_1 \oplus 0$  and  $p_2 = 0 \oplus 1_2$ . By (9.3), we can assume that  $k_1 < k_2$ . Take the operator  $v$  defined as in (9.5). Since  $v^*v = p_1 \in \mathcal{M} \cap \mathcal{M}'$ ,  $v$  is an o.d. homomorphism. By the assumption,

$$p_1 = v^*v = vv^* \leq p_2.$$

Therefore,  $p_1 = 0$  and hence  $M$  is a factor.  $\square$

The general characterization of  $(\mathcal{M}, H, \xi_0)$  which satisfy (\*) is an open problem. Hiai and Yamamuro [1] have proved that such an algebra is finite and furthermore, when  $H$  is separable and  $M$  is hyperfinite,  $(\mathcal{M}, H, \xi_0)$  satisfies (\*) if and only if  $M$  is a factor of type  $I_n$ . Yamamuro's conjecture is that  $(\mathcal{M}, H, \xi_0)$  satisfies (\*) if and only if  $M$  is a factor of type  $I_n$ .



## §10. EXTREMAL MAPS AND O.D. HOMOMORPHISMS

In a real ordered Banach space, a positive element  $u$  is called a *quasi-interior point* if  $f(u) > 0$  for every positive continuous linear functional  $f$ . Then by (2.4), quasi-interior points of  $H^+$  are exactly cyclic and separating vectors. This fact has been observed first by Bellisard and Iochum [1], Lemma 4.2.

Let  $E$  and  $F$  be Banach lattices with respective quasi-interior elements  $u \in E_+$  and  $v \in F_+$ . Define

$$L(u, v, E_+, F_+) = \{T \in L(E, F) : T(E_+) \subset F_+, T(u) = v\},$$

where  $L(E, F)$  denotes the set of all continuous operator from  $E$  to  $F$ . Obviously,  $L(u, v, E_+, F_+)$  is a convex set.

A theorem due to Phelps, Ellis and Nagel (Schaefer [1], Chapter III, Proposition 9.2) states that:

*Let  $E$  and  $F$  be Banach lattices with respective quasi-interior elements  $u \in E_+$  and  $v \in F_+$ . Then,  $T$  is an extreme point of  $L(u, v, E_+, F_+)$  if and only if  $T$  is a lattice homomorphism.*

The main aim of this section is to find a corresponding statement for  $H^J$  associated with  $(\mathcal{M}, H, \xi_0)$ .

We set

$$L(\xi_0, \eta_0, H^+) = \{\phi \in L(H) : \phi(H^+) \subset H^+, \phi\xi_0 = \eta_0\},$$

where  $\eta_0 \in H^+$  is another cyclic and separating vector for  $\mathcal{M}$ .

**Remark.** In this definition of  $L(\xi_0, \eta_0, H^+)$ , we chose  $\xi_0$  which is in  $(\mathcal{M}, H, \xi_0)$ . This is only for the sake of convenience; in fact, it makes no difference in the following context if we replace  $\xi_0$  with another cyclic separating vector  $\rho_0 \in H^+$ , because



$$H_{\xi_0}^+ = H_{\rho_0}^+.$$

(10.1). All o.d. homomorphisms in  $L(\xi_0, \eta_0, H^+)$  are extreme points.

**Proof.** Let  $\phi$  be an arbitrary o.d. homomorphism in  $L(\xi_0, \eta_0, H^+)$ . Suppose there are  $\phi_1$  and  $\phi_2$  in  $L(\xi_0, \eta_0, H^+)$  and  $0 < \alpha < 1$  such that

$$\phi = \alpha\phi_1 + (1-\alpha)\phi_2.$$

Then, for any  $\xi \in H^+$  and  $\eta \in H^+$  satisfying  $(\xi, \eta) = 0$ , since  $\alpha\phi_1\xi \leq \phi\xi$ ,  $\alpha\phi_1\eta \leq \phi\eta$  and  $H^+$  is self-dual, we have

$$0 \leq (\alpha\phi_1\xi, \alpha\phi_1\eta) \leq (\phi\xi, \phi\eta) = 0.$$

Hence,  $\phi_1$  is an o.d. homomorphism and similarly  $\phi_2$  is also an o.d. homomorphism.

On the other hand, we have

$$\phi^*\phi = \alpha^2\phi_1^*\phi_1 + (1-\alpha)^2\phi_2^*\phi_2 + \alpha(1-\alpha)(\phi_1^*\phi_2 + \phi_2^*\phi_1).$$

It follows from (5.4) that

$$\alpha(1-\alpha)(\phi_1^*\phi_2 + \phi_2^*\phi_1) = \phi^*\phi - \alpha^2\phi_1^*\phi_1 - (1-\alpha)^2\phi_2^*\phi_2 \in \mathcal{M} \cap \mathcal{M}'.$$

But, since  $\phi_1^*\phi_2\xi_0 = \phi_1^*\eta_0 = \phi_1^*\phi_1\xi_0$  and  $\phi_2^*\phi_1\xi_0 = \phi_2^*\eta_0 = \phi_2^*\phi_2\xi_0$ , where  $\xi_0$  is a separating vector for  $\mathcal{M}$ , we have

$$\phi_1^*\phi_2 + \phi_2^*\phi_1 = \phi_1^*\phi_1 + \phi_2^*\phi_2.$$

Thus,

$$(\phi_1 - \phi_2)^*(\phi_1 - \phi_2) = \phi_1^*\phi_1 - \phi_1^*\phi_2 - \phi_2^*\phi_1 + \phi_2^*\phi_2 = 0,$$

that is,  $\phi_1 = \phi_2$ . Hence,  $\phi$  is an extreme point of  $L(\xi_0, \eta_0, H^+)$ .  $\square$

(10.2). The following conditions are equivalent:

(i) For any positive cyclic separating vectors  $\xi$  and  $\eta$ , every extreme point of  $L(\xi, \eta, H^+)$  is an o.d. homomorphism.

(ii)  $\mathcal{M}$  is commutative.

**Proof.** (i)  $\Rightarrow$  (ii). Let  $\xi$  be an arbitrary positive cyclic separating vector. For

an arbitrary bijection  $a \in \mathcal{M}^+$ ,  $aj(a)\xi$  is a positive cyclic and separating vector by (2.4). Hence  $aj(a) \in L(\xi, aj(a)\xi, H^+)$ .

We now show that  $aj(a)$  is an extreme point of  $L(\xi, aj(a)\xi, H^+)$ . Suppose that there exist  $\phi_i \in L(\xi, aj(a)\xi, H^+)$  ( $i = 1, 2$ ) and  $\alpha \in (0, 1)$  such that

$$aj(a) = \alpha\phi_1 + (1-\alpha)\phi_2.$$

Since  $(aj(a))^{-1} = a^{-1}j(a^{-1})$  and  $a^{-1}j(a^{-1})(H^+) \subset H^+$ , we have

$$1 = \alpha a^{-1}j(a^{-1})\phi_1 + (1-\alpha)a^{-1}j(a^{-1})\phi_2,$$

and  $a^{-1}j(a^{-1})\phi_i \in L(\xi, \xi, H^+)$  ( $i = 1, 2$ ). Obviously,  $1 \in L(\xi, \xi, H^+)$  and  $1$  is an o.d. homomorphism. Hence, by (10.1),  $1$  is an extreme point of  $L(\xi, \xi, H^+)$ . Therefore,  $aj(a) = \phi_1 = \phi_2$ , that is,  $aj(a)$  is an extreme point of  $L(\xi, aj(a)\xi, H^+)$ . By the hypothesis,  $aj(a)$  is an o.d. homomorphism. Hence, by Example (4.2.iii),  $a^2 \in \mathcal{M} \cap \mathcal{M}'$ . Since  $a \in \mathcal{M}^+$ , we have  $a \in \mathcal{M} \cap \mathcal{M}'$ . For any  $a \in \mathcal{M}^h$ , there is a positive number  $\alpha$  such that  $1 + \alpha a$  is bijective and belongs to  $\mathcal{M}^+$ . Hence  $\mathcal{M}^h \subset \mathcal{M} \cap \mathcal{M}'$  and it then follows that  $\mathcal{M}$  is commutative.

(ii)  $\Rightarrow$  (i). When  $\mathcal{M}$  is commutative,  $H^J$  becomes a Banach lattice. Then, by Example (4.2.iv), o.d. homomorphisms are exactly lattice homomorphisms, and hence condition (i) holds by the Phelps-Ellis-Nagel theorem quoted in the beginning of this section.  $\square$

### §11. ON THE CONDITION $\phi\xi_0 = \xi_0$

We recall that a linear operator  $\alpha : \mathcal{M} \rightarrow \mathcal{M}$  is said to be *unital* if  $\alpha(1) = 1$ . For a cyclic and separating vector  $\xi_0 \in H$ , the condition  $\phi\xi_0 = \xi_0$ , which appeared in the previous section, can be regarded as corresponding to being unital for operators on  $\mathcal{M}$ .

To see this correspondence more clearly, let  $\mathcal{M}$  be finite-dimensional and  $\xi_0$  be a cyclic and separating vector such that  $\Delta_{\xi_0} = 1$ . Then, the equation

$$\alpha(x)\xi_0 = \phi(x\xi_0), \quad x \in \mathcal{M}$$

establishes a correspondence between operators  $\phi$  on  $H$  and  $\alpha$  on  $\mathcal{M}$ .  $\phi(H^+) \subset H^+$  if and only if  $\alpha(\mathcal{M}^+) \subset \mathcal{M}^+$ , because the map  $x \mapsto x\xi_0$  is an order isomorphism between  $\mathcal{M}^+$  and  $H^+$ . Furthermore,  $\phi$  is an o.d. homomorphism if and only if  $\alpha$  is a Jordan homomorphism. Obviously,  $\alpha$  is unital if and only if  $\phi\xi_0 = \xi_0$ .

On the other hand, while the unit 1 of  $\mathcal{M}$  is unique, there are many cyclic and separating vectors in  $H$ .

(11.1). *Let  $\phi$  be a partial isometric o.d. homomorphism. Then, the following conditions are equivalent*

$$(i) \phi\xi_0 = \xi_0.$$

$$(ii) \phi^*\phi = 1 \text{ and } (\phi\xi_0, \xi_0) = (\xi_0, \xi_0).$$

**Proof.** (i)  $\Rightarrow$  (ii). Let  $p = \phi^*\phi$ .  $p$  is a projection in  $\mathcal{M}$ . Then, since  $\phi^*\xi_0 = \phi^*\phi\xi_0 = p\xi_0$ ,

$$\|p\xi_0\|^2 = (p\xi_0, \xi_0) = (\phi^*\xi_0, \xi_0) = (\xi_0, \phi\xi_0) = \|\xi_0\|^2.$$

Hence,  $(1-p)\xi_0 = 0$ . Since  $p \in \mathcal{M}$  and  $\xi_0$  is cyclic separating for  $\mathcal{M}$ , we have  $p = 1$ .

Thus,  $\phi^*\phi = 1$ .

The second equality is obvious.

(ii)  $\Rightarrow$  (i). From the assumptions, we have

$$\|\xi_o\|^2 = (\phi^* \phi \xi_o, \xi_o) = \|\phi \xi_o\|^2.$$

Hence,

$$\|\xi_o - \phi \xi_o\|^2 = \|\xi_o\|^2 - (\xi_o, \phi \xi_o) - (\phi \xi_o, \xi_o) + \|\phi \xi_o\|^2 = 0.$$

Therefore,  $\phi \xi_o = \xi_o$ . □

As an immediate consequence,

(11.2). When  $\mathcal{M}$  is finite-dimensional, every partial isometric o.d. homomorphism  $\phi$  such that  $\phi \xi_o = \xi_o$  is a unitary operator.

**Proof.** When  $\mathcal{M}$  is finite-dimensional,  $\phi^* \phi = 1$  implies  $\phi \phi^* = 1$ . □

Note that (11.2) fails if  $\mathcal{M}$  is not finite-dimensional. Let  $\mathcal{M}$  be the tensor product of countable numbers of  $M^2(\mathbb{C})$  with respect to

$$\theta_n = \theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let  $\xi_o = \theta_1 \otimes \theta_2 \otimes \dots$ , and  $\phi$  be an operator defined by the correspondence

$$\xi_1 \otimes \dots \otimes \xi_k \otimes \theta \otimes \theta \otimes \dots \mapsto \theta \otimes \xi_1 \otimes \dots \otimes \xi_k \otimes \theta \otimes \theta \otimes \dots$$

Then,  $\phi$  is an o.d. homomorphism,  $\phi \xi_o = \xi_o$ ,  $\phi^* \phi = 1$ , but  $\phi \phi^* \neq 1$ .

About (11.1), we make two remarks.

First, the assumption that  $\phi$  is partially isometric cannot be dropped except when  $\mathcal{M}$  is a factor (see (11.5)). For example, let  $\mathcal{M}$  be the algebra of all matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & u & v \end{pmatrix}$$

Let

$$\xi_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\phi \begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & u & v \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}.$$

Then,  $\phi\xi_o = \xi_o$  and

$$\phi^*\phi = 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M} \cap \mathcal{M}'.$$

Secondly, for an o.d. homomorphism  $\phi$  and its polar decomposition  $\phi = v|\phi|$ ,  $\phi\xi_o = \xi_o$  does not imply  $v\xi_o = \xi_o$ . In fact, as the following result shows, this is equivalent to  $\phi = v$ , a case which has been dealt with in (11.1).

(11.3). Let  $\phi$  be an o.d. homomorphism such that  $\phi\xi_o = \xi_o$ , and let  $\phi = v|\phi|$  be the polar decomposition of  $\phi$ . The following conditions are equivalent:

(i)  $v\xi_o = \xi_o$ .

(ii)  $|\phi| = 1$ .

(iii)  $\phi^*\xi_o = \xi_o$ .

**Proof.** (i)  $\Rightarrow$  (ii). Since  $|\phi|\xi_o = v^*\phi\xi_o = v^*\xi_o$ , we have

$$(|\phi|\xi_o, \xi_o) = (v^*\xi_o, \xi_o) = (\xi_o, v\xi_o) = (\xi_o, \xi_o).$$

Furthermore,

$$\| |\phi|\xi_o \|^2 = (\phi^*\phi\xi_o, \xi_o) = \|\phi\xi_o\|^2 = \|\xi_o\|^2.$$

Therefore,



$$\begin{aligned} \|\phi\xi_0 - \xi_0\|^2 &= \|\phi\xi_0\|^2 - 2(\phi\xi_0, \xi_0) + \|\xi_0\|^2 \\ &= 0. \end{aligned}$$

Hence,  $|\phi|\xi_0 = \xi_0$ . Since  $|\phi| \in \mathcal{M} \cap \mathcal{M}'$ , we have  $|\phi| = 1$ .

$$(ii) \Rightarrow (iii). \quad \phi^*\xi_0 = \phi^*\phi\xi_0 = |\phi|^2\xi_0 = \xi_0.$$

$$(iii) \Rightarrow (i). \quad \text{Since } \xi_0 = \phi^*\phi\xi_0 \text{ and } \phi^*\phi \in \mathcal{M} \cap \mathcal{M}', \text{ we have } |\phi| = 1.$$

Therefore,  $\phi\xi_0 = \xi_0$ . □

The fact that  $\phi$  is an o.d. homomorphism such that  $\phi\xi_0 = \xi_0$  does not imply the dual  $\phi^*$  has the same property in general. When it does,  $\phi$  becomes the most typical o.d. isomorphism, that is, a unitary operator  $u$  such that  $u(H^+) = H^+$ .

(11.4). Let  $\phi$  be an o.d. homomorphism such that  $\phi\xi_0 = \xi_0$ . Then, the following conditions are equivalent:

$$(i) \quad \phi^* \text{ is an o.d. homomorphism such that } \phi^*\xi_0 = \xi_0.$$

$$(ii) \quad \phi \text{ is normal and } \phi^*\xi_0 = \xi_0.$$

$$(iii) \quad \phi \text{ is a unitary operator such that } \phi(H^+) = H^+.$$

$$(iv) \quad \phi \text{ is an order isomorphism and } \phi^*\xi_0 = \xi_0.$$

**Proof.** (i)  $\Rightarrow$  (ii). It follows from the hypothesis and (5.4) that  $\phi^*\phi \in \mathcal{M} \cap \mathcal{M}'$  and  $\phi\phi^* \in \mathcal{M} \cap \mathcal{M}'$ . Furthermore,  $\phi^*\phi\xi_0 = \xi_0 = \phi\phi^*\xi_0$ . Hence,  $\phi^*\phi = \phi\phi^*$ .

(ii)  $\Rightarrow$  (iii). From the hypothesis,  $\phi^*\phi = \phi\phi^* \in \mathcal{M} \cap \mathcal{M}'$ . Hence, it follows from  $\phi^*\phi\xi_0 = \xi_0 = \phi\phi^*\xi_0$  that  $\phi^*\phi = \phi\phi^* = 1$ . By (5.1),  $\phi(H^+) = H^+$ .

(iii)  $\Rightarrow$  (iv). Obvious.

(iv)  $\Rightarrow$  (i). Let  $\phi = u|\phi|$  be the polar decomposition of  $\phi$ . By (2.9),  $u$  satisfies  $u(H^+) = H^+$ , and by (11.3) we have  $|\phi| = 1$ . Hence,  $\phi = u$  and  $\phi$  satisfies the condition (i). □

Some improvements of the above results can be made under additional

assumptions.

(11.5). Let  $\mathcal{M}$  be a factor. Then, if  $\phi$  is an o.d. homomorphism such that  $\phi\xi_o = \xi_o$ , we have  $\phi^*\phi = 1$ .

**Proof.** Since  $\phi^*\phi \in \mathcal{M} \cap \mathcal{M}'$  by (5.4) and  $\mathcal{M}$  is a factor,  $\phi^*\phi = \alpha 1$  for some positive number  $\alpha$ . Hence,

$$\|\xi_o\|^2 = (\phi\xi_o, \phi\xi_o) = (\phi^*\phi\xi_o, \xi_o) = \alpha\|\xi_o\|^2.$$

Hence,  $\alpha = 1$  and  $\phi^*\phi = 1$ . □

It is a result of Kadison [1] that a unital order isomorphism on  $\mathcal{M}$  is a Jordan isomorphism. The corresponding result on  $H$  is as follows.

(11.6). If  $\Delta_{\xi_o} = 1$ , every order isomorphism  $\phi$  on  $H$  such that  $\phi\xi_o = \xi_o$  is an o.d. isomorphism.

**Proof.** As in the proof of (11.4), (iv)  $\Rightarrow$  (i), with the additional assumption  $\Delta_{\xi_o} = 1$ ,  $\phi$  determines a Jordan isomorphism  $\alpha$  by

$$\alpha(x)\xi_o = \phi(x\xi_o), \quad x \in \mathcal{M}.$$

Then, since  $\alpha$  is unital,

$$|\phi(x\xi_o)| = |\alpha(x)\xi_o| = |\alpha(x)|\xi_o = \alpha(|x|)\xi_o = \phi(|x|)\xi_o = \phi(|x\xi_o|).$$

Taking the limit, we have  $|\phi\xi| = \phi|\xi|$  for all  $\xi \in H^J$ . Hence  $\phi$  is a bijective o.d. homomorphism. By (5.5),  $\phi$  is an o.d. isomorphism. □

We make two remarks about (11.6).

**Remark 1.** If  $\Delta_{\xi_o} \neq 1$ , there is an order isomorphism  $\phi$  of  $M_2(\mathbb{C})$  such that  $\phi\xi_o = \xi_o$  which is not an o.d. isomorphism. For example, we set

$$\xi_o = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$J_{\xi_0} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

Consider  $\phi = aj(a)$  where

$$a = \begin{pmatrix} 0 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = a^{-1}.$$

Then  $\phi\xi_0 = \xi_0$  and

$$a^*a = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathcal{C}) \cap M_2(\mathcal{C})'.$$

Hence by Example (4.2.iii),  $\phi$  is not an o.d. isomorphism.

**Remark 2.** Even when  $\Delta_{\xi_0} = 1$ , if  $\phi$  is an order homomorphism (not an isomorphism),  $\phi$  is not necessarily an o.d. homomorphism. Furthermore, in the assumption of (11.6), “order isomorphism  $\phi$ ” cannot be replaced by “bijective order homomorphism  $\phi$ ”. For example, in  $M_2(\mathcal{C})$  with

$$\xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let

$$\phi \begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x + \frac{1}{3}y + \frac{1}{3}z + \frac{1}{3}u & \frac{1}{3}y \\ \frac{1}{3}z & \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z + \frac{2}{3}u \end{pmatrix}.$$

Then,  $\phi$  is an order homomorphism such that  $\phi\xi_0 = \xi_0$ . However, since

$$\phi^{-1} \begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} 2x-y-z-u & 3y \\ 3z & -x-y-z-2u \end{pmatrix}$$

$\phi^{-1}$  is not an order homomorphism. Hence,  $\phi$  is not an o.d. homomorphism.

## §12. DERIVATIONS

We shall call a continuous linear operator  $\delta : H \rightarrow H$  a *derivation* if it satisfies the following two conditions:

$$(i) \quad J\delta = \delta J.$$

$$(ii) \quad \text{If } \xi \in H^+, \eta \in H^+ \text{ and } (\xi, \eta) = 0, \text{ then } (\delta\xi, \eta) = 0.$$

Denote by  $D(H^+)$  the set of all derivations.

Fundamental properties of derivations are listed in the theorem (2.10) of Connes, to which we shall refer frequently. The aim of this section is to investigate the order properties of the derivations, particularly, their relations to o.d. homomorphisms.

When  $\phi$  is an o.d. homomorphism, it follows immediately from the definition that  $\phi^*\phi$  is a derivation, and (5.4) states that  $\phi^*\phi \in \mathcal{M} \cap \mathcal{M}'$ . In this case, we obviously have  $\phi^*\phi(H^+) \subset H^+$ . Our first result generalizes this situation and states that order homomorphic derivations belong to the centre  $\mathcal{M} \cap \mathcal{M}'$ .

(12.1). *The following conditions are equivalent:*

$$(i) \quad \delta \in D(H^+) \text{ and } \delta(H^+) \subset H^+.$$

$$(ii) \quad \delta \in (\mathcal{M} \cap \mathcal{M}')^+.$$

**Proof.** (i)  $\Rightarrow$  (ii). Since  $\delta \in D(H^+)$ , for  $\xi \in H^+$ ,  $\eta \in H^+$  satisfying  $(\xi, \eta) = 0$ , we have  $(\delta\xi, \eta) = 0$ . Then, because  $\delta(H^+) \subset H^+$ , we have  $(\delta^2\xi, \eta) = (\delta(\delta\xi), \eta) = 0$ . Similarly, we have  $(\delta^m\xi, \delta^n\eta) = 0$  for  $m = 0, 1, 2, \dots$  and  $n = 0, 1, 2, \dots$  with the convention  $\delta^0 = 1$ . Hence,

$$e^{t\delta} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta^n$$

is an o.d. homomorphism for each  $t \geq 0$ . So by (7.4),  $\delta + \delta^* \in (\mathcal{M} \cap \mathcal{M})^h$ . Then, it follows from Börs ([1], Corollary 7(2)) that there is  $\alpha \geq 0$  such that



$-\alpha\xi \leq (\delta + \delta^*)\xi \leq \alpha\xi$  for all  $\xi \in H^+$ . But, since  $\delta^*(H^+) \subset H^+$ , we have

$$-\alpha\xi \leq -(\delta + \delta^*)\xi \leq \delta\xi \leq (\delta + \delta^*)\xi \leq \alpha\xi$$

for all  $\xi \in H^+$ . Hence, again by Bös ([1], Corollary 7.(2)),  $\delta \in (\mathcal{M} \cap \mathcal{M}')^h$ .

Now for any  $\xi = \xi_1 + i\xi_2$  with  $\xi_1, \xi_2 \in H^J$  and the orthogonal decomposition  $\xi_r = \xi_r^+ - \xi_r^-$  ( $r = 1, 2$ ), as  $\delta \in (\mathcal{M} \cap \mathcal{M}')^h$ , we have

$$\begin{aligned} (\delta\xi, \xi) &= (\delta\xi_1, \xi_1) + (\delta\xi_1, i\xi_2) + (i\delta\xi_2, \xi_1) + (i\delta\xi_2, i\xi_2) \\ &= (\delta\xi_1, \xi_1) + (i\delta\xi_2, i\xi_2) \\ &= (\delta\xi_1^+, \xi_1^+) + (\delta\xi_1^-, \xi_1^-) + (\delta\xi_2^+, \xi_2^+) + (\delta\xi_2^-, \xi_2^-) \\ &\geq 0. \end{aligned}$$

Hence,  $\delta \in (\mathcal{M} \cap \mathcal{M}')^+$ .

(ii)  $\Rightarrow$  (i). It follows immediately from (2.12) and  $\delta = \delta^{\frac{1}{2}}j(\delta^{\frac{1}{2}})$  for  $\delta \in (\mathcal{M} \cap \mathcal{M}')^+$ .  $\square$

Note that the last part of the proof of (i)  $\Rightarrow$  (ii) actually shows that if  $a \in (\mathcal{M} \cap \mathcal{M}')^h$  and  $a(H^+) \subset H^+$  then  $a$  is a positive operator, which is the converse of Bös ([1], Corollary 6.(2)) for the case where  $H^+$  is the natural cone associated with  $(\mathcal{M}, H, \xi_0)$ .

The following statement is an immediate consequence of (12.1).

(12.2). Let  $u$  be a unitary derivation satisfying  $u(H^+) = H^+$ . Then,  $u = 1$ .

We shall next characterize generators of uniform semigroups of derivations.

(12.3). Let  $a : H \rightarrow H$  be a continuous linear operator. The following conditions are equivalent.

(i)  $e^{ta}$  is a derivation for each  $t \geq 0$ .

(ii)  $e^{ta} \in (\mathcal{M} \cap \mathcal{M}')^+$  for each  $t \geq 0$ .

(iii)  $a \in (\mathcal{M} \cap \mathcal{M}')^h$ .

$$(iv)(a) \quad Ja = aJ.$$

$$(b) \quad (|a\xi|, \eta) = 0 \text{ for } \xi \in H^+, \eta \in H^+ \text{ and } (\xi, \eta) = 0.$$

**Proof.** (i)  $\Rightarrow$  (ii). It follows immediately from the hypothesis that for each  $t > 0$ ,

$$J\left(\frac{e^{ta} - 1}{t}\right) = \left(\frac{e^{ta} - 1}{t}\right)J,$$

and if  $\xi \in H^+, \eta \in H^+$  and  $(\xi, \eta) = 0$ , then for each  $t > 0$ ,

$$\left(\frac{e^{ta} - 1}{t}\xi, \eta\right) = 0.$$

Hence

$$a = \lim_{t \downarrow 0} \frac{e^{ta} - 1}{t}$$

satisfies  $aJ = Ja$  and  $(a\xi, \eta) = 0$  for  $\xi \in H^+, \eta \in H^+$  satisfying  $(\xi, \eta) = 0$ . Therefore  $a \in D(H^+)$ , and hence, by (2.10)  $e^{ta}(H^+) = H^+$  for each  $t > 0$ . Then, it follows from (12.1) that  $e^{ta} \in (\mathcal{M} \cap \mathcal{M}')^+$  for each  $t > 0$ .

(ii)  $\Rightarrow$  (iii). It follows immediately from  $\mathcal{M} \cap \mathcal{M}'$  being strongly closed and

$$a = \lim_{t \downarrow 0} \frac{e^{ta} - 1}{t} \quad (\text{uniformly}).$$

(iii)  $\Rightarrow$  (iv). Since  $a \in (\mathcal{M} \cap \mathcal{M}')^h$ ,  $a$  can be written as

$$a = \frac{1}{2}(|a| + a) - \frac{1}{2}(|a| - a),$$

where  $(|a| + a)$  and  $(|a| - a)$  are positive operators and  $(|a| + a)(|a| - a) = 0$ .

Furthermore, as  $a \in \mathcal{M} \cap \mathcal{M}'$ , we have  $|a| \in \mathcal{M} \cap \mathcal{M}'$ , and hence  $\frac{1}{2}(|a| + a) \in \mathcal{M} \cap \mathcal{M}'$  and  $\frac{1}{2}(|a| - a) \in \mathcal{M} \cap \mathcal{M}'$ . It then follows from (12.1) that  $\frac{1}{2}(|a| + a)(H^+) \subset H^+$  and  $\frac{1}{2}(|a| - a)(H^+) \subset H^+$ . Hence, for  $\xi \in H^+$ , by the uniqueness of orthogonal decomposition in  $H^J$ , we have

$$(a\xi)^+ = \frac{1}{2}(|a| + a)\xi$$

and

$$(a\xi)^- = \frac{1}{2}(|a| - a)\xi.$$

Hence, for  $\xi \in H^+$ ,  $\eta \in H^+$  satisfying  $(\xi, \eta) = 0$ , we have  $((a\xi)^+, \eta) = 0$  and  $((a\xi)^-, \eta) = 0$ . That is,  $(|a\xi|, \eta) = ((a\xi)^+ + (a\xi)^-, \eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ .

(iv)  $\Rightarrow$  (i). Since  $H^+$  is self-dual, condition (iv)(b) implies  $(a\xi^+, \eta) = 0$ ,  $((a\xi)^-, \eta) = 0$ , and hence  $(a^n \xi, \eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ , for  $n = 0, 1, 2, \dots$ . Therefore, if  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ , then

$$(e^{ta} \xi, \eta) = \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} a^n \xi, \eta \right) = 0.$$

Hence,  $e^{ta}$  is a derivation for each  $t \geq 0$ . □

It follows immediately from (12.3) that

(12.4). *Every uniform semigroup of derivations is automatically a uniform semigroup of positive operators contained in the centre, and hence, of order homomorphisms.*

When  $B$  is a Banach lattice, Arendt [2] has defined a *Lamperti operator* as a continuous linear operator  $T: B \rightarrow B$  which is order-bounded and  $|Tx| \wedge |Ty| = 0$  whenever  $|x| \wedge |y| = 0$ . He has proved that  $T$  is a Lamperti operator if and only if there exists a lattice homomorphism  $\psi: B \rightarrow B$  such that  $|Tx| = \psi x$  for all  $x \in B^+$ . Yamamuro [5] has adopted this latter condition, replacing lattice homomorphism with o.d. homomorphism, as the definition of Lamperti operators on  $H$ , and developed its general theory. We are interested in the first condition adapted to  $H$ , that is,  $(|\phi\xi|, |\phi\eta|) = 0$  whenever  $\xi \in H^J$ ,  $\eta \in H^J$  and  $(|\xi|, |\eta|) = 0$ . Because of difficulty in dealing with the absolute values, it is desirable to obtain an equivalent condition

without the absolute values.

We note that condition  $(|\phi\xi|, |\phi\eta|) = 0$  whenever  $\xi \in H^J$ ,  $\eta \in H^J$  and  $(|\xi|, |\eta|) = 0$  is equivalent to that  $(|\phi\rho|, |\phi\varsigma|) = 0$  whenever  $\rho \in H^+$ ,  $\varsigma \in H^+$  and  $(\rho, \varsigma) = 0$ .

The following result has also been found by Yamamuro.

(12.5). Let  $\phi$  be a continuous linear operator such that  $J\phi = \phi J$ . Then, the following conditions are equivalent:

- (i)  $(|\phi\xi|, |\phi\eta|) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ .
- (ii)  $(\mathcal{M}\phi\xi, \phi\eta) = 0$  whenever  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ .
- (iii)  $\phi^*D(H^+)\phi \subset D(H^+)$ .

**Proof.** (i)  $\Rightarrow$  (iii). Suppose  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ . By hypothesis, we have

$$((\phi\xi)^+ + (\phi\xi)^-, (\phi\eta)^+ + (\phi\eta)^-) = 0,$$

where  $\phi\xi = (\phi\xi)^+ - (\phi\xi)^-$  and  $\phi\eta = (\phi\eta)^+ - (\phi\eta)^-$  are the orthogonal decompositions of  $\phi\xi$  and  $\phi\eta$  respectively. It follows from the self-duality of  $H^+$  that  $((\phi\xi)^\pm, (\phi\eta)^\pm) = 0$ . Then, by (2.12), for each  $a \in \mathcal{M}$ ,

$$(a\phi\xi, \phi\eta) = (a(\phi\xi)^+ - a(\phi\xi)^-, (\phi\eta)^+ - (\phi\eta)^-) = 0.$$

Since  $J\phi = \phi J$ , we also have  $(JaJ\phi\xi, \phi\eta) = (\phi\eta, a\phi\eta) = 0$ . Hence, for each  $a \in \mathcal{M}$ ,

$$(\phi^*(a + j(a))\phi\xi, \eta) = (a\phi\xi, \phi\eta) + (JaJ\phi\xi, \phi\eta) = 0;$$

moreover, as  $\phi^*(a + j(a))\phi$  commutes with  $J$ , we have  $\phi^*(a + j(a))\phi \in D(H^+)$ .

Hence, by (2.10),  $\phi^*D(H^+)\phi \subset D(H^+)$ .

(iii)  $\Rightarrow$  (ii). Let  $a$  be an arbitrary operator in  $\mathcal{M}$ . For  $\xi \in H^+$ ,  $\eta \in H^+$  satisfying  $(\xi, \eta) = 0$ , by hypothesis we have

$$\begin{aligned}
\operatorname{Re}(a\phi\xi, \phi\eta) &= (a\phi\xi, \phi\eta) + (\phi\eta, a\phi\xi) \\
&= (a\phi\xi, \phi\eta) + (Ja\phi\xi, \phi\eta) \\
&= (\phi^*a\phi\xi, \eta) + (\phi^*JaJ\phi\xi, \eta) \\
&= 0.
\end{aligned}$$

Then, since  $ia \in \mathcal{M}$ , we have

$$\operatorname{Im}(a\phi\xi, \phi\eta) = -\operatorname{Re}(ia\phi\xi, \phi\eta) = 0.$$

Hence,  $(a\phi\xi, \phi\eta) = 0$  for any  $a \in \mathcal{M}$ .

(ii)  $\Rightarrow$  (i). Suppose  $\xi \in H^+$ ,  $\eta \in H^+$  and  $(\xi, \eta) = 0$ . We denote by

$$P_{(\phi\xi)^+} = \overline{[\mathcal{M}'(\phi\xi)^+]}, \quad P_{(\phi\xi)^-} = \overline{[\mathcal{M}'(\phi\xi)^-]},$$

$$P_{(\phi\eta)^+} = \overline{[\mathcal{M}'(\phi\eta)^+]}, \quad \text{and} \quad P_{(\phi\eta)^-} = \overline{[\mathcal{M}'(\phi\eta)^-]}.$$

Then,  $P_{(\phi\xi)^\pm}$  and  $P_{(\phi\eta)^\pm}$  are all projections in  $\mathcal{M}$ . Hence, by hypothesis,

$$((\phi\xi)^+, (\phi\eta)^+) = (P_{(\phi\eta)^+}P_{(\phi\xi)^+}\phi\xi, \phi\eta) = 0,$$

and similarly,  $((\phi\xi)^+, (\phi\eta)^-) = 0$ ,  $((\phi\xi)^-, (\phi\eta)^+) = 0$ , and  $((\phi\xi)^-, (\phi\eta)^-) = 0$ .

Therefore,  $(|\phi\xi|, |\phi\eta|) = 0$ . □

The following statement follows readily from (12.5), and gives us another characterization of o.d. homomorphisms.

**(12.6).** Let  $\phi: H \rightarrow H$  be a continuous linear operator such that  $\phi(H^+) \subset H^+$ . Then,  $\phi$  is an o.d. homomorphism if and only if  $\phi^*D(H^+)\phi \subset D(H^+)$ .



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