Aspects of overdetermined systems of partial differential equations in projective and conformal geometry

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis discusses aspects of overdetermined systems of partial differential equations (PDEs) in projective and conformal geometry. The first part deals with projective differential geometry. A projective surface is a 2-dimensional smooth manifold equipped with a projective structure i.e. a class of torsion-free affine connections that have the same geodesics as unparameterised curves. Given any projective surface we can ask whether it admits a torsion-free affine connection (in its projective class) that has skew-symmetric Ricci tensor. This is equivalent to solving a particular overdetermined system of semi-linear partial differential equations. It turns out that there are local obstructions to solving the system of PDEs in two dimensions. These obstructions are constructed out of local invariants of the projective structure. We give examples of projective surfaces that admit skew-symmetric Ricci tensor and examples that do not because of non-vanishing obstructions. We relate projective surfaces admitting skew-symmetric Ricci tensor to 3-webs in 2 dimensions. We also give examples of projective structures in higher dimensions that admit skew-symmetric Ricci tensor. The second part of the thesis deals with conformal differential geometry. On Möbius surfaces introduced in [5], we can define an analogous overdetermined system of semi-linear PDEs as in the projective case. This is called the scalar-flat Möbius Einstein-Weyl equation and is conformally invariant. We derive local algebraic constraints for Möbius surfaces to admit a solution to this equation and give local obstructions. These obstructions are similarly constructed out of local conformal invariants of the Möbius structure. Again we provide examples of Möbius surfaces that admit a solution and examples that do not because of non-vanishing obstructions. Finally, we also investigate the conformally Einstein equation on Möbius surfaces and derive obstructions. In contrast to the previous two equations, the conformally Einstein equation is linear.

Contents

Chapter 1

Introduction

This thesis is divided accordingly as follows: Chapter 1 starts with a review of smooth projective structures, and then introduces the projective Einstein-Weyl (pEW) equation which will occupy most part of the thesis. We will prolong the pEW equation and derive a closed system, from which the integrability condition can be determined. Chapter 2 is concerned with extracting local obstructions for the existence of solutions of the pEW equation in 2-dimensions. Chapter 3 reformulates the pEW equation using tractors, and this chapter can be skipped for readers not familiar with tractor calculus. The local obstructions derived in Chapter 2 are recalculated in Chapter 3 with a two-fold purpose: 1) to verify accuracy and 2) to verify the projective invariance of the obstructions obtained. Chapter 4 deals with projective structures in 2-dimensions that admit skew-symmetric Ricci tensor and relates them to the theory of second order ordinary differential equations (ODEs) and 3-webs. Chapter 5 gives examples of projective structures that admit a solution to pEW and examples that do not. In Chapters 6 and 7 we turn to conformal geometry. In Chapter 6, we discuss a conformally invariant equation on Möbius surfaces called the scalar-flat Möbius Einstein-Weyl (sf-MEW) equation. We proceed to find local obstructions for the existence of solutions to the sf-MEW equation and give examples. In Chapter 7, we discuss the conformally Einstein equation on Möbius surfaces and also derive local obstructions. The reader is assumed to have some knowledge of affine connections, and the notation for tensorial objects as found in [26]. In particular, an affine connection is usually denoted with a covariant index like so: ∇_a. Round brackets over the indices of the tensorial objects denote symmetrisation, e.g. $T_{(ab)} = \frac{1}{2}T_{ab} + \frac{1}{2}T_{ba}$, while square brackets denote skew-symmetrisation like so $T_{[ab]} = \frac{1}{2}T_{ab} - \frac{1}{2}T_{ba}$.

1.1 Preliminaries

We shall work in the smooth category of manifolds. A projective structure $(M, [\nabla])$ on a smooth oriented manifold M is a class of torsion-free affine connections $[\nabla]$ that have the same unparameterised curves as geodesics. Equivalently, two torsion-free affine connections ∇ and $\widehat{\nabla}$ are projectively equivalent (and belong to $[\nabla]$) if and only if

$$\widehat{\nabla}_a \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a \tag{1.1}$$

for some 1-form Υ_a. Details can be found in [15]. The projective transformation formula (1.1) for connections acting on 1-forms extends to connections acting on n-forms, where $n = \dim M$ by Leibniz rule; as a result, we have

$$\widehat{\nabla}_a \omega_{bc\ldots d} = \nabla_a \omega_{bc\ldots d} - (n+1)\Upsilon_a \omega_{bc\ldots d} + (n+1)\Upsilon_{[a}\omega_{bc\ldots d]} \tag{1.2}$$

for any volume form $\omega_{bc\ldots d} \in \Gamma(\Lambda^n)$. However, the last term $\Upsilon_{[a}\omega_{bc\ldots d]}$ is identically zero and hence we have

$$\widehat{\nabla}_a \omega_{bc\ldots d} = \nabla_a \omega_{bc\ldots d} - (n+1)\Upsilon_a \omega_{bc\ldots d}.$$

For oriented projective structures we can find a connection in $[\nabla]$ to preserve any given volume form locally, i.e. such that $\nabla_a \omega_{bc\ldots d} = 0$ (this connection is necessarily flat on the bundle Λ^n). To see this, given any connection $\nabla \in [\nabla]$, we have $\nabla_a \omega_{bc\ldots d} = \mu_a \omega_{bc\ldots d}$ for some 1-form μ_a. This is because the bundle of volume forms is 1-dimensional, so that its covariant derivative is proportional to the volume form. Now take the projective transformation to be given by $\Upsilon_a = \frac{1}{n+1}\mu_a$, and this normalises the connection so that $\widehat{\nabla}_a \omega_{bc\ldots d} = 0$, where $\widehat{\nabla}$ is projectively related to the ∇ we started with. We then relabel $\widehat{\nabla}$ as ∇ to obtain a connection for which the volume form is parallel. Since M is oriented, we can take roots of the volume form and we define the density line bundle $\mathcal{E}(w)$ to be the $-\frac{w}{n+1}$-th root of Λ^n (see [15] for more details). Sections of $\mathcal{E}(w)$ are projective densities of weight w. Given σ a section of $\mathcal{E}(w)$, we have under a projective change of connection that

$$\widehat{\nabla}_a \sigma = \nabla_a \sigma + w \Upsilon_a \sigma.$$

By definition, $\mathcal{E}(-(n+1)) = \Lambda^n = \mathcal{E}_{[bc\ldots d]}$ and the line bundle $\mathcal{E}_{[bc\ldots d]}(n+1)$ is trivial. Let $\epsilon_{bc\ldots d}$ be the tautological section of this bundle, which satisfies $\nabla_a \epsilon_{bc\ldots d} = 0$. The inverse of the tautological form $\epsilon^{bc\ldots d}$ identifies the bundle of

n-forms Λ^n with the density line bundle $\mathcal{E}(-(n+1))$. For a chosen volume form $\omega_{bc\ldots d}$ the associated connection will be called special. A special connection has the property that its Ricci tensor is symmetric. Given any projective structure we can always locally restrict to this class of special connections and the projective changes will be restricted to closed (and hence locally exact) 1-forms Υ_a. A choice of volume form $\omega_{bc\ldots d}$ is often referred to as a choice of scale. Any other volume form consistent with the orientation is of the form $\Omega^{n+1}\omega_{bc\ldots d}$ for some positive smooth function Ω and then by (1.2) the transformation rule (1.1) holds with $\Upsilon_a = \nabla_a \log \Omega$.

1.1.1 Curvature of a projective structure

Let us now look at the curvature tensor associated to an arbitary torsion-free affine connection in the general projective class. The curvature of any affine connection $\nabla \in [\nabla]$ decomposes as follows:

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} - 2\mathrm{P}_{[ab]}\delta_d{}^c$$

where $\mathrm{P}_{ab} = \frac{n}{n^2-1}R_{ab} + \frac{1}{n^2-1}R_{ba}$ is called the projective Schouten tensor or projective rho tensor and $R_{ab} = R_{ca}{}^c{}_b$ is called the Ricci tensor. The totally tracefree part of $R_{ab}{}^c{}_d$ is called the projective Weyl tensor, denoted as $W_{ab}{}^c{}_d$. It has the same symmetry properties as the curvature tensor: $W_{[ab}{}^c{}_{d]} = 0$ and $W_{ab}{}^c{}_d = W_{[ab]}{}^c{}_d$. The projective Weyl tensor is projectively invariant, that is to say it is a universal polynomial in the jets of the projective structure that is invariantly defined as a projectively weighted tensor. A projectively invariant quantity does not change under a projective change of connection. In 3 or more dimensions, the quantity $Y_{abc} = \frac{1}{n-2}\nabla_d W_{ab}{}^d{}_c$ is called the Cotton-York tensor. It has the symmetry properties $Y_{[abc]} = 0$ and $Y_{abc} = Y_{[ab]c}$. By the second Bianchi identity, $Y_{abc} = \nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac}$. In 2 dimensions the projective Weyl tensor vanishes identically due to symmetry considerations. However we can define the quantity $Y_{abc} = \nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac}$ just as well in 2 dimensions. It turns out to be projectively invariant. This quantity is known as early as 1889 by Roger Liouville (see [4]), but as tradition decrees it is more befitting to again call it the Cotton-York tensor. The skew-symmetric part of the projective rho tensor $\mathrm{P}_{[ab]}$ is called the Faraday 2-form F_{ab}. It is always exact, i.e. $F_{ab} = \nabla_{[a}\alpha_{b]}$ for some 1-form α_a, and hence it is always closed, i.e. $\nabla_{[a}F_{bc]} = 0$. If we restrict to a class of special connections in $[\nabla]$ this forces $F_{ab} = 0$, and hence the Ricci tensor (and the projective rho tensor) for any special connection is symmetric. Given any projective structure $[\nabla]$, we ask is it always possible to find a connection in $[\nabla]$ so that the Ricci tensor is skew, i.e. $R_{(ab)} = 0$? The answer turns out to be no, as we see in the following section.

1.2 An analogue of Einstein-Weyl equation on projective manifolds

The condition of a projective structure admitting a torsion-free affine connection with skew-symmetric Ricci tensor is equivalent to finding a connection $D \in [\nabla]$ such that the symmetric part of its projective rho tensor can be made to vanish, i.e. $\mathrm{P}_{(ab)} = 0$. By definition, this condition depends only on the projective class and is therefore projectively invariant. In this case the projective rho tensor $\mathrm{P}_{ab} = \mathrm{P}_{[ab]} = F_{ab}$ is precisely the Faraday 2-form and the curvature decomposes into

$$R_{ab}{}^{c}{}_{d} = W_{ab}{}^{c}{}_{d} + \delta_a{}^c F_{bd} - \delta_b{}^c F_{ad} - 2F_{ab}\delta_d{}^c.$$

Choosing any ∇ from $[\nabla]$, we have

$$D_a\omega_b = \nabla_a\omega_b + \alpha_a\omega_b + \alpha_b\omega_a \tag{1.3}$$

for some 1-form α_a by the projective transformation formula (1.1). Here the connection D_a has skew-symmetric Ricci tensor. We can use the background connection ∇_a and its associated curvature to write down conditions on α_a to admit skew-symmetric Ricci tensor. It is well known that the projective rho tensor transforms accordingly as $\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b$ under a projective change of connection, where Υ_a is some 1-form. Hence using (1.3) the skew-symmetric Ricci condition is equivalent to asking whether

$$0 = \widehat{\mathrm{P}}_{(ab)} = \nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{(ab)}$$

for some 1-form α_a. This is the pEW equation written in terms of an arbitary connection $\nabla \in [\nabla]$. However, the preferred background connection ∇_a we will work instead with is the special one so that $\nabla_a\omega_{bc\ldots e} = 0$. Note however that the connection D_a we are looking for with skew-symmetric Ricci tensor does not lie in the special class of connections. Normalising the projective structure by taking the class of special connections in $[\nabla]$, the projective rho tensor $\mathrm{P}_{ab} = \mathrm{P}_{(ab)}$ becomes symmetric. Taking ∇_a as the background representative for the special class of connection we obtain the projective Einstein-Weyl equation:

$$\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} = 0. \tag{1.4}$$

Observe that α_a is not necessarily exact. In fact $\widehat{\mathrm{P}}_{ab} = F_{ab} = \nabla_{[a}\alpha_{b]}$ and we can rewrite (1.4) as

$$\nabla_a \alpha_b + \alpha_a \alpha_b + \mathrm{P}_{ab} = F_{ab}. \tag{1.5}$$

Exactness of α_a implies $F_{ab} = 0$. In this case we can write α_a as a gradient of some function f, i.e. $\alpha_a = \nabla_a f$. Now let $\sigma = e^f$. Then $\nabla_a \sigma = \nabla_a(e^f) = e^f \nabla_a f = \sigma \nabla_a f$, and therefore

$$\begin{aligned}
\nabla_a \nabla_b \sigma + \mathrm{P}_{(ab)}\sigma &= \nabla_a(\sigma \nabla_b f) + \mathrm{P}_{(ab)}\sigma \\
&= (\nabla_a \sigma)(\nabla_b f) + \sigma(\nabla_a \nabla_b f) + \mathrm{P}_{(ab)}\sigma \\
&= \sigma(\nabla_a f)(\nabla_b f) + \sigma(\nabla_a \nabla_b f) + \mathrm{P}_{(ab)}\sigma \\
&= \sigma(\nabla_a \alpha_b + \alpha_a \alpha_b + \mathrm{P}_{(ab)}) \\
&= 0
\end{aligned}$$

provided that (1.5) holds. But the left hand-side of this expression is none other than projectively invariant Hessian operator [7] which determines whether a projective structure admits a representative in its connection class that has Ricci curvature zero. The projective Einstein-Weyl equation can therefore be seen as a generalisation of the projective to Ricci-flat (or projectively Einstein condition), recovering the latter equation when the 1-form α_a is exact (since we are working locally, we can also make do with α_a being closed, i.e $F_{ab} = 0$). This is the main reason for justifying the nomenclature Einstein-Weyl: The Einstein-Weyl equation in conformal geometry (see [12] and [13]) generalises the conformal-to-Einstein equation exactly in the same manner. The solution to (1.4) is the 1-form α_a determined up gauge equivalence: a projective transformation changes α_a by a locally exact 1-form. While the projective invariance of (1.4) is automatic from the projectively invariant geometric condition the equation determines, it can also be verified by direct calculation.

Proposition 1.1. *The equation (1.5) is projectively invariant.*

Proof. Under a change of projective scale,

$$\begin{aligned}
\alpha_b \mapsto \widehat{\alpha}_b &= \alpha_b + \Upsilon_b, \\
\mathrm{P}_{ab} \mapsto \widehat{\mathrm{P}}_{ab} &= \mathrm{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b, \\
F_{ab} \mapsto \widehat{F}_{ab} &= F_{ab} + \nabla_{[a}\Upsilon_{b]} = F_{ab},
\end{aligned}$$

where Υ_b is a closed 1-form. Hence we obtain

$$\begin{aligned}
\widehat{\nabla}_a \widehat{\alpha}_b &= \nabla_a \widehat{\alpha}_b - \Upsilon_a \widehat{\alpha}_b - \Upsilon_b \widehat{\alpha}_a \\
&= \nabla_a \alpha_b + \nabla_a \Upsilon_b - \Upsilon_a \alpha_b - \Upsilon_a \Upsilon_b - \Upsilon_b \alpha_a - \Upsilon_a \Upsilon_b,
\end{aligned}$$

and so

$$\begin{aligned}
&\widehat{\nabla}_a\widehat{\alpha}_b + \widehat{\alpha}_a\widehat{\alpha}_b + \widehat{\mathrm{P}}_{ab} - \widehat{F}_{ab}\\
=&\nabla_a\alpha_b + \nabla_a\Upsilon_b - \Upsilon_a\alpha_b - \Upsilon_a\Upsilon_b - \Upsilon_b\alpha_a - \Upsilon_a\Upsilon_b\\
&+ \alpha_a\alpha_b + \Upsilon_a\alpha_b + \Upsilon_b\alpha_a + \Upsilon_a\Upsilon_b + \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - F_{ab}\\
=&\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} - F_{ab},
\end{aligned}$$

verifying the projective invariance of (1.5). □

1.3 Prolongation and integrability conditions

Applying the theory of prolongation developed in [3] and [14] to equation (1.4) gives a closed system. Prolongation involves introducing new unknowns into the system and trying to solve the derivatives of the new unknowns in terms of known variables. If this can be done, the system is said to be closed. As an example, the prolongation of the Einstein-Weyl equation in conformal geometry is carried out in [12], giving rise to a closed system. The general aim of the prolongation procedure is three-fold: 1) to get algebraic constraints on the system, and in the process hopefully derive obstructions to existence of solutions to the original equation, 2) to get explicit bounds on the dimension of the solution space of the system, and 3) to read off directly the symbol of the differential operator associated to the equation. Most of the equations that are prolonged by this process are overdetermined, that is to say they have more equations than unknowns. Most of them are also linear in their variables, so that obstructions and bounds can be readily obtained, e.g. in [4]. However, equation (1.4) and the Einstein-Weyl equation are semi-linear (linear in the symbol and non-linear in the lower order terms) and so there is no systematic algorithm developed to find obstructions for these equations. They also do not share the properties that the linear equations have. We shall now derive integrability conditions for equation (1.4). Differentiating

$$\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} = F_{ab}$$

gives

$$\begin{aligned}
\nabla_c F_{ab} &= \nabla_c\nabla_a\alpha_b + (\nabla_c\alpha_a)\alpha_b + \alpha_a(\nabla_c\alpha_b) + \nabla_c\mathrm{P}_{ab}\\
&= \nabla_c\nabla_a\alpha_b + (F_{ca} - \mathrm{P}_{ca} - \alpha_c\alpha_a)\alpha_b + \alpha_a(F_{cb} - \mathrm{P}_{cb} - \alpha_c\alpha_b) + \nabla_c\mathrm{P}_{ab}\\
&= \nabla_c\nabla_a\alpha_b + F_{ca}\alpha_b - \mathrm{P}_{ca}\alpha_b + \alpha_a F_{cb} - \alpha_a\mathrm{P}_{cb} - 2\alpha_a\alpha_b\alpha_c + \nabla_c\mathrm{P}_{ab}.
\end{aligned}$$

Exchanging the a and c indices gives

$$\nabla_a F_{cb} = \nabla_a \nabla_c \alpha_b + F_{ac}\alpha_b - \mathrm{P}_{ac}\alpha_b + \alpha_c F_{ab} - \alpha_c \mathrm{P}_{ab} - 2\alpha_a\alpha_b\alpha_c + \nabla_a \mathrm{P}_{cb},$$

and so taking the difference, we obtain

$$\begin{aligned}\nabla_c F_{ab} - \nabla_a F_{cb} =& \nabla_c \nabla_a \alpha_b + F_{ca}\alpha_b - \mathrm{P}_{ca}\alpha_b + \alpha_a F_{cb} - \alpha_a \mathrm{P}_{cb} - 2\alpha_a\alpha_b\alpha_c + \nabla_c \mathrm{P}_{ab}\\ &- \nabla_a \nabla_c \alpha_b - F_{ac}\alpha_b + \mathrm{P}_{ac}\alpha_b - \alpha_c F_{ab} + \alpha_c \mathrm{P}_{ab} + 2\alpha_a\alpha_b\alpha_c - \nabla_a \mathrm{P}_{cb}\\ =& R_{ac}{}^d{}_b\alpha_d + F_{ca}\alpha_b - \mathrm{P}_{ca}\alpha_b + \alpha_a F_{cb} - \alpha_a \mathrm{P}_{cb} + Y_{cab}\\ &- F_{ac}\alpha_b + \mathrm{P}_{ac}\alpha_b - \alpha_c F_{ab} + \alpha_c \mathrm{P}_{ab}\\ =& W_{ac}{}^d{}_b\alpha_d + \alpha_a \mathrm{P}_{bc} - \alpha_c \mathrm{P}_{ab} + F_{ca}\alpha_b - \mathrm{P}_{ca}\alpha_b + \alpha_a F_{cb} - \alpha_a \mathrm{P}_{cb} + Y_{cab}\\ &- F_{ac}\alpha_b + \mathrm{P}_{ac}\alpha_b - \alpha_c F_{ab} + \alpha_c \mathrm{P}_{ab}\\ =& W_{ac}{}^d{}_b\alpha_d + 2F_{ca}\alpha_b + \alpha_a F_{cb} - \alpha_c F_{ab} + Y_{cab}.\end{aligned}$$

Since $\nabla_{[a}F_{bc]} = 0$,

$$\nabla_c F_{ab} - \nabla_a F_{cb} = \nabla_c F_{ab} + \nabla_a F_{bc} = -\nabla_b F_{ca}$$

and so

$$\nabla_b F_{ac} = -\nabla_b F_{ca} = W_{ac}{}^d{}_b\alpha_d + 2F_{ca}\alpha_b + \alpha_a F_{cb} - \alpha_c F_{ab} + Y_{cab}$$

or that

$$\nabla_c F_{ab} = W_{ab}{}^d{}_c\alpha_d + 2F_{ba}\alpha_c + \alpha_a F_{bc} - \alpha_b F_{ac} - Y_{abc}. \tag{1.6}$$

This expresses the derivative of F_{ab} in terms of known quantities of the system and curvature of the system, namely $W_{ab}{}^d{}_c$, Y_{abc}, F_{ab}, and α_a. The system is now closed with the 2 equations of the system given by:

$$\begin{aligned}\nabla_c \alpha_b =& F_{cb} - \alpha_c\alpha_b - \mathrm{P}_{cb}\\ \nabla_c F_{ab} =& W_{ab}{}^d{}_c\alpha_d - Y_{abc} + 2F_{ba}\alpha_c + \alpha_a F_{bc} - \alpha_b F_{ac}.\end{aligned}$$

Taking the derivative of (1.6) once more and skewing the indices gives the equation

$$\begin{aligned}W_{cd}{}^e{}_b F_{ae} - W_{cd}{}^e{}_a F_{be} =& (\nabla_d W_{ab}{}^f{}_c)\alpha_f - (\nabla_c W_{ab}{}^f{}_d)\alpha_f + W_{ab}{}^f{}_c F_{df} - W_{ab}{}^f{}_d F_{cf}\\ &+ W_{ab}{}^f{}_c\alpha_d\alpha_f - W_{ab}{}^f{}_d\alpha_c\alpha_f - W_{ab}{}^f{}_c\mathrm{P}_{df} + W_{ab}{}^f{}_d\mathrm{P}_{cf}\\ &+ \nabla_c Y_{abd} - \nabla_d Y_{abc} + \alpha_b W_{cd}{}^e{}_a\alpha_e - \alpha_a W_{cd}{}^e{}_b\alpha_e\\ &+ Y_{cdb}\alpha_a + 2Y_{bac}\alpha_d - 2Y_{bad}\alpha_c - Y_{cda}\alpha_b\\ &+ 4F_{ab}F_{cd} + 2F_{ac}F_{bd} - 2F_{bc}F_{ad}.\end{aligned} \tag{1.7}$$

This is the first integrability condition that must be satisfied for (1.4) to hold. The algebraic constraint (1.7) has symmetries of the Riemann curvature tensor and is projectively invariant.

1.4 The pEW equation in flat space

A smooth manifold $(M^n, [\nabla])$ with a projective structure is called projectively flat iff $W_{ab}{}^c{}_d = 0$ when $n \geq 3$, and $Y_{abd} = 0$ when $n = 2$. In this section we investigate equation (1.4) on projectively flat manifolds. This is in the same spirit of [12] where the authors investigate the Einstein-Weyl equation on conformally flat manifolds. The results we obtain are similar to those obtained in [12], namely that the only solutions to (1.4) on projectively flat structures are local projective rescalings of Ricci-flat structures.

Proposition 1.2. *On projectively flat manifolds M^n with $n \geq 3$, the Cotton tensor $Y_{abc} = 0$.*

Proof. This follows from the vanishing of the projective Weyl tensor. Alternatively, from [15], the curvature tensor on projectively flat manifolds decomposes into $R_{ab}{}^c{}_d = \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad}$, where $\mathrm{P}_{ab} = \frac{1}{n-1} R_{ca}{}^c{}_b$. Differentiating this equation gives

$$\nabla_e R_{ab}{}^c{}_d = \delta_a{}^c \nabla_e \mathrm{P}_{bd} - \delta_b{}^c \nabla_e \mathrm{P}_{ad}$$

and so

$$\nabla_e R_{ab}{}^e{}_d = \nabla_a \mathrm{P}_{bd} - \nabla_b \mathrm{P}_{ad} = Y_{abd}.$$

By the Bianchi identity,

$$\nabla_e R_{ab}{}^c{}_d + \nabla_b R_{ea}{}^c{}_d + \nabla_a R_{be}{}^c{}_d = 0,$$

and so

$$\begin{aligned} 0 =& \nabla_e R_{ab}{}^e{}_d + \nabla_b R_{ea}{}^e{}_d + \nabla_a R_{be}{}^e{}_d \\ =& Y_{abd} + (n-1)\nabla_b \mathrm{P}_{ad} - (n-1)\nabla_a \mathrm{P}_{bd} \\ =& Y_{abd} + (n-1) Y_{bad} \\ =& (2-n) Y_{abd}, \end{aligned}$$

which implies that $Y_{abc} = 0$ for $n \geq 3$. □

The closed system for (1.4) in projectively flat space for $n \geq 2$ is given by

$$\begin{aligned} \nabla_c \alpha_b =& F_{cb} - \alpha_c \alpha_b - \mathrm{P}_{cb}, \\ \nabla_c F_{ab} =& 2F_{ba}\alpha_c + \alpha_a F_{bc} - \alpha_b F_{ac}, \end{aligned}$$

and the integrability condition (1.7) reduces to

$$0 = 4F_{ab}F_{cd} + 2F_{ac}F_{bd} - 2F_{bc}F_{ad}. \tag{1.8}$$

We claim that (1.8) forces $F_{ab} = 0$. This conclusion is entirely algebraic and has nothing to do with the form of F_{ab}. Rewriting (1.8) as

$$2F_{ab}F_{cd} = F_{bc}F_{ad} - F_{ac}F_{bd} \tag{1.9}$$

and relabelling indices gives

$$2F_{ca}F_{bd} = F_{ab}F_{cd} - F_{cb}F_{ad} = F_{ab}F_{cd} + F_{bc}F_{ad},$$

and so

$$4F_{ca}F_{bd} = 2F_{ab}F_{cd} + 2F_{bc}F_{ad} = F_{bc}F_{ad} - F_{ac}F_{bd} + 2F_{bc}F_{ad},$$

which simplifies to

$$F_{ca}F_{bd} = F_{bc}F_{ad}.$$

Substituting back into (1.9), we have

$$F_{ab}F_{cd} = F_{ca}F_{bd} = F_{bc}F_{ad}.$$

Assume now that $F_{ab} \neq 0$. Let X^a, Y^b be two linearly independent non-zero vectors generically chosen so that they are not eigenvectors of F_{ab} (viewed as an $n \times n$ skew-symmetric matrix) with eigenvalues 0 and also that

$$F_{ab}X^b = m_a \neq 0, \qquad F_{ab}Y^b = \ell_a \neq 0.$$

Then

$$\begin{aligned}(m_aY^a)F_{cd} =& (F_{ab}F_{cd})X^bY^a \\ =& (F_{bc}F_{ad})X^bY^a \\ =& m_c\ell_d\end{aligned}$$

and we have

$$F_{cd} = fm_c\ell_d,$$

where $f = \frac{1}{m_aY^a}$. If $m_aY^a = 0$, then either m_c or ℓ_d is zero, which is a contradiction. Transvecting once more with Y^c gives

$$-\ell_d = F_{cd}Y^c = fm_cY^c\ell_d = \ell_d,$$

forcing ℓ_d to be zero, which again is a contradiction. Hence (1.8) forces $F_{ab} = 0$, and (1.4) reduces to the projective Ricci-flat equation $\nabla_a\nabla_b\sigma + \mathrm{P}_{ab}\sigma = 0$, with $\alpha_a = \nabla_a \log\sigma$. We have the following:

Theorem 1.3. *For $n \geq 3$, the only local solutions of the projective Einstein-Weyl equation (1.4) in flat space are projective rescalings of solutions to the projectively Ricci-flat equation.*

Chapter 2

Deriving local obstructions to the pEW equation for projective surfaces

A projective surface is a smooth 2-dimensional manifold equipped with a projective structure. In this chapter we derive local obstructions to existence of solutions to the pEW equation on projective surfaces. In higher dimensions, no local obstructions to (1.4) have been found. The obstructions we obtain are the resultants of polynomials with coefficients given by invariants of the projective structure. Computing these obstructions and checking that they do not vanish tell us that the projective surface does not admit skew-symmetric Ricci tensor, thus giving us a means to distinguish projective surfaces that admit skew-symmetric Ricci tensor from those that do not. That being said, in Chapter 5 we give an example of a projective surface that does not admit a solution to (1.4) but the obstructions vanish. Parts of Chapters 1, 2 and 5 appear in [28]. In 2 dimensions, affine structures with skew-symmetric Ricci-tensor have been investigated in [8]. The relationship between surfaces with skew-symmetric Ricci tensor and affine Osserman structures is also investigated in [16]. We first set up the closed system for the pEW equation in 2-dimensions, then derive the algebraic constraints that give rise to the obstructions. Previous work devoted to finding local obstructions for various other geometries include [23] and [1] for 4-dimensional manifolds to be conformal to Einstein, and [19] and [20] for metrics in general dimensions to be conformal to Einstein. Local obstructions to whether a fixed projective structure is metrisable are given in [4] and [25].

2.1 The closed system for the pEW equation in 2 dimensions

We shall rederive the closed system for the pEW equation in 2 dimensions, this time using the weighted volume form to dualise F_{ab}. In 2 dimensions, choose an inverse weighted volume form ϵ^{ab} to satsify $\epsilon^{ac}\epsilon_{bc} = \delta_b{}^a$ so that $\epsilon^{ab}\epsilon_{ab} = 2! = 2$ (we have $\nabla_a \epsilon_{bc} = 0$). We shall use the term weighted volume form to denote ϵ^{ab} and its inverse ϵ_{ab} interchangeably. Choose a special connection $\nabla \in [\nabla]$ such that the projective rho tensor $\mathrm{P}_{ab} = \mathrm{P}_{(ab)}$ is symmetric. Then $F_{ab} = \nabla_{[a}\alpha_{b]}$ is determined by a scalar density $F = \epsilon^{ab}F_{ab}$ and we can reexpress

$$F_{ab} = \frac{1}{2}\epsilon_{ab}F$$

and

$$\epsilon^{ab}F_{bc} = \frac{1}{2}\epsilon^{ab}\epsilon_{bc}F = -\frac{1}{2}\delta_c{}^a F.$$

We can rewrite (1.4) in 2 dimensions as

$$\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} = \frac{1}{2}\epsilon_{ab}F. \tag{2.1}$$

Differentiating (2.1) gives

$$\begin{aligned}
\frac{1}{2}\epsilon_{ab}\nabla_c F &= \nabla_c\nabla_a\alpha_b + (\nabla_c\alpha_a)\alpha_b + \alpha_a(\nabla_c\alpha_b) + \nabla_c\mathrm{P}_{ab}\\
&= \nabla_c\nabla_a\alpha_b + (\frac{1}{2}\epsilon_{ca}F - \mathrm{P}_{ca} - \alpha_c\alpha_a)\alpha_b + \alpha_a(\frac{1}{2}\epsilon_{cb}F - \mathrm{P}_{cb} - \alpha_c\alpha_b) + \nabla_c\mathrm{P}_{ab}\\
&= \nabla_c\nabla_a\alpha_b + \frac{1}{2}\epsilon_{ca}F\alpha_b - \mathrm{P}_{ca}\alpha_b + \frac{1}{2}\epsilon_{cb}F\alpha_a - \alpha_a\mathrm{P}_{cb} - 2\alpha_a\alpha_b\alpha_c + \nabla_c\mathrm{P}_{ab}.
\end{aligned}$$

Skewing gives

$$\begin{aligned}
\frac{1}{2}\epsilon_{ab}\nabla_c F - \frac{1}{2}\epsilon_{cb}\nabla_a F =& R_{ac}{}^d{}_b\alpha_d + \frac{1}{2}\epsilon_{ca}F\alpha_b - \mathrm{P}_{ca}\alpha_b + \frac{1}{2}\epsilon_{cb}F\alpha_a - \alpha_a\mathrm{P}_{cb} + Y_{cab}\\
&- \frac{1}{2}\epsilon_{ac}F\alpha_b + \mathrm{P}_{ac}\alpha_b - \frac{1}{2}\epsilon_{ab}F\alpha_c + \alpha_c\mathrm{P}_{ab}\\
=& \epsilon_{ca}\alpha_b F + \frac{1}{2}\epsilon_{cb}\alpha_a F - \frac{1}{2}\epsilon_{ab}\alpha_c F + Y_{cab}.
\end{aligned}$$

We have made use of the fact that in 2 dimensions, the Weyl tensor $W_{ab}{}^c{}_d$ vanishes by symmetry considerations and the projective Cotton-York tensor $Y_{abc} = \nabla_a\mathrm{P}_{bc} - \nabla_b\mathrm{P}_{ac}$ is projectively invariant. Since $\epsilon_{[ab}\nabla_{c]}F = 0$, we have

$$\frac{1}{2}\epsilon_{ac}\nabla_b F = \epsilon_{ca}\alpha_b F + \frac{1}{2}\epsilon_{cb}\alpha_a F - \frac{1}{2}\epsilon_{ab}\alpha_c F + Y_{cab} \tag{2.2}$$

completing the pEW closed system. Contracting equation (2.2) with the weighted volume form ϵ^{ac} gives

$$\begin{aligned}\nabla_b F =& -2F\alpha_b - \frac{1}{2}F\alpha_b - \frac{1}{2}F\alpha_b - Y_b \\ =& -3F\alpha_b - Y_b,\end{aligned} \tag{2.3}$$

where we have used the weighted volume form ϵ^{ac} to dualise $Y_b = \epsilon^{ac}Y_{acb}$. Equation (2.3) is precisely equation (1.6) in 2 dimensions. It is projectively invariant because ϵ^{ab} has projective weight -3, which means that both F and Y_b have weight -3 as well. One observes from equation (2.3) that for $F \neq 0$, we can express

$$\alpha_a = -\frac{1}{3F}\nabla_a F - \frac{1}{3F}Y_a \tag{2.4}$$

so that α_a is determined by the curvature terms F and Y_a. We shall subsequently assume that $F \neq 0$, since $F = 0$ implies that α_a is locally a gradient and the solution to pEW is then a projective rescaling of the solution to the projective to Ricci-flat equation. In this case the manifold is locally projectively flat. Also from (2.3) we can deduce

Proposition 2.1. *Suppose* $(M^2, [\nabla])$ *is not (locally) projectively flat, i.e.* $Y_{abc} \neq 0$ *and admits a solution to (1.4). Then* F *cannot vanish on an open set.*

Proof. If F vanishes on an open neighbourhood U_p around some point $p \in M$, then by (2.3) the Cotton-York 1-form $Y_a = 0$ on U_p. This is a contradiction since M^2 is not (locally) projectively flat. $\square$

2.2 Constraints of the pEW system in 2 dimensions

Differentiating (2.3) one more time gives a first constraint of the pEW system. We first obtain

$$\begin{aligned}\nabla_a\nabla_b F =& -3(\nabla_a F)\alpha_b - 3F(\nabla_a\alpha_b) - \nabla_a Y_b \\ =& -3(-3F\alpha_a - Y_a)\alpha_b - 3F(\frac{1}{2}\epsilon_{ab}F - \alpha_a\alpha_b - \mathrm{P}_{ab}) - \nabla_a Y_b \\ =& 9F\alpha_a\alpha_b + 3Y_a\alpha_b - \frac{3}{2}\epsilon_{ab}F^2 + 3\alpha_a\alpha_b F + 3\mathrm{P}_{ab}F - \nabla_a Y_b\end{aligned} \tag{2.5}$$

and skewing gives

$$0 = 3Y_a\alpha_b - 3Y_b\alpha_a - 3\epsilon_{ab}F^2 - \nabla_a Y_b + \nabla_b Y_a.$$

We again dualise with the weighted volume form ϵ^{ab} to obtain

$$0 = -6\alpha_a Y^a - 6F^2 - 2\nabla_a Y^a,$$

which we rewrite as

$$0 = 3\alpha_a Y^a + 3F^2 + \nabla_a Y^a, \tag{2.6}$$

where $Y^a = \epsilon^{ab} Y_b$. It can be verified that (2.6) is projectively invariant, taking into account of the weight -6 of Y^a. In the flat case when $Y_a = 0$, we necessarily have $F = 0$ by (2.6). The 1-form α_a is therefore locally exact, and equation (1.4) specialises to the projective Ricci-flat equation. We shall therefore restrict our attention to non-flat projective surfaces, that is one with Y_a non-zero. This ensures that $F \neq 0$. In higher dimensions constraint (2.6) is precisely (1.7) and has symmetries of the Riemann curvature tensor. However in 2 dimensions the space of such tensors is 1-dimensional and so the constraint reduces to a single scalar equation. Let

$$\phi = \epsilon^{ac}(\nabla_a Y_c - \nabla_c Y_a) = 2\nabla_a Y^a$$

and we can rewrite equation (2.6) as

$$\alpha_a Y^a = -F^2 - \frac{\phi}{6}. \tag{2.7}$$

This gives us the first constraint of the pEW system. We shall obtain a second constraint as follows. Differentiating (2.7) gives

$$\alpha_a(\nabla_c Y^a) + (\nabla_c \alpha_a) Y^a = -2F(\nabla_a F) - \frac{\nabla_a \phi}{6}.$$

Using (2.1) and (2.3), we obtain

$$\alpha_a(\nabla_c Y^a) + (\frac{1}{2}\epsilon_{ca} F - \alpha_c \alpha_a - \mathrm{P}_{ca}) Y^a = -2F(-3\alpha_a F - Y_a) - \frac{\nabla_a \phi}{6}.$$

Contracting the equation with Y^c, we obtain

$$\alpha_a(Y^c \nabla_c Y^a) - (\alpha_c Y^c)^2 - \mathrm{P}_{ca} Y^a Y^c = 6(\alpha_a Y^a) F^2 - \frac{Y^a \nabla_a \phi}{6}. \tag{2.8}$$

Using equation (2.7) again we obtain

$$\begin{aligned}
\alpha_a(Y^c \nabla_c Y^a) =& (\alpha_c Y^c)^2 + \mathrm{P}_{ca} Y^a Y^c + 6(\alpha_a Y^a) F^2 - \frac{Y^a \nabla_a \phi}{6} \\
=& (-F^2 - \frac{\phi}{6})^2 + \mathrm{P}_{ca} Y^a Y^c + 6(-F^2 - \frac{\phi}{6}) F^2 - \frac{Y^a \nabla_a \phi}{6} \\
=& F^4 + \frac{\phi}{3} F^2 + \frac{\phi^2}{36} + \mathrm{P}_{ca} Y^a Y^c - 6F^4 - \phi F^2 - \frac{Y^a \nabla_a \phi}{6} \\
=& -5F^4 - \frac{2\phi}{3} F^2 + \frac{\phi^2}{36} + \mathrm{P}_{ca} Y^a Y^c - \frac{Y^a \nabla_a \phi}{6}
\end{aligned}$$

We shall now introduce the vector $W^a := Y^b\nabla_b Y^a - \frac{2\phi}{3}Y^a$. It is projectively invariant with weight -12. To see this invariance, since the Cotton-York vector Y^a is projectively invariant and has a weight of -6, under a projective change,

$$\begin{aligned}
\widehat{\nabla}_a\widehat{Y}^b = \widehat{\nabla}_a Y^b =& \nabla_a Y^b - 5\Upsilon_a Y^b + \delta_a{}^b\Upsilon_c Y^c,\\
\widehat{\nabla}_a\widehat{Y}^a = \widehat{\nabla}_a Y^a =& \nabla_a Y^a - 3\Upsilon_a Y^a.
\end{aligned}$$

Together these give

$$\begin{aligned}
\widehat{\phi} =& \phi - 6\Upsilon_a Y^a,\\
\widehat{Y^a\nabla_a Y^b} =& Y^a\nabla_a Y^b - 4(\Upsilon_c Y^c)Y^b,
\end{aligned}$$

so that

$$\begin{aligned}
\widehat{W}^b :=& \widehat{Y^a\nabla_a Y^b} - \frac{2}{3}\widehat{\phi}\widehat{Y}^b\\
=& Y^a\nabla_a Y^b - 4(\Upsilon_c Y^c)Y^b - \frac{2}{3}(\phi - 6\Upsilon_a Y^a)Y^b\\
=& Y^a\nabla_a Y^b - 4(\Upsilon_c Y^c)Y^b - \frac{2}{3}\phi Y^b + 4\Upsilon_a Y^a Y^b\\
=& Y^a\nabla_a Y^b - \frac{2}{3}\phi Y^b\\
=& W^b.
\end{aligned}$$

Hence $W^b = Y^a\nabla_a Y^b - \frac{2}{3}\phi Y^b$ is projectively invariant. Now consider

$$\begin{aligned}
\alpha_a W^a =& \alpha_b(Y^a\nabla_a Y^b - \frac{2}{3}\phi Y^b)\\
=& \alpha_b(Y^a\nabla_a Y^b) - \frac{2}{3}\phi(\alpha_b Y^b)\\
=& -5F^4 - \frac{2\phi}{3}F^2 + \frac{\phi^2}{36} + \mathrm{P}_{ca}Y^aY^c - \frac{Y^a\nabla_a\phi}{6} - \frac{2}{3}\phi(-F^2 - \frac{\phi}{6})\\
=& -5F^4 - \frac{2\phi}{3}F^2 + \frac{\phi^2}{36} + \mathrm{P}_{ca}Y^aY^c - \frac{Y^a\nabla_a\phi}{6} + \frac{2}{3}\phi F^2 + \frac{\phi^2}{9}\\
=& -5F^4 + \frac{5\phi^2}{36} + \mathrm{P}_{ca}Y^aY^c - \frac{Y^a\nabla_a\phi}{6}.
\end{aligned} \tag{2.9}$$

Let

$$\ell = \frac{5\phi^2}{12} + 3\mathrm{P}_{ac}Y^aY^c - \frac{Y^a\nabla_a\phi}{2}.$$

This quantity has projective weight -12 and projectively transforms as follows:

$$\ell \mapsto \widehat{\ell} = \ell + 3\Upsilon_a W^a.$$

We can write (2.9) as

$$\alpha_a W^a = -\,5F^4 + \frac{\ell}{3}, \tag{2.10}$$

giving us the second constraint of the pEW system. Together (2.7) and (2.10) allows us to solve for α_a, under the assumption that the projective invariant $\rho := Y_a W^a$ is non-zero. Since both Y_a and W^a are projectively invariant, the scalar density $\rho = Y_a W^a = Y_b Y^a \nabla_a Y^b = -Y^b Y^a \nabla_a Y_b$ is also projectively invariant and has weight -15. The case where $\rho = 0$ will be dealt in Section 2.4.

2.3 Local obstructions to the pEW system for $\rho \neq 0$

We can use the weighted volume form ϵ^{ab} to raise indices and its inverse ϵ_{ab} to lower indices so that $W^a = \epsilon^{ab} W_b$ and $W_b = W^a \epsilon_{ab}$. We shall now assume that the manifold $(M^2, [\nabla])$ has a projective structure with $\rho = Y_a W^a \neq 0$. This is a local generic condition. It suffices to assume that $\rho \neq 0$ at a point since by continuity, $\rho \neq 0$ in an open neighbourhood of that point. Under this assumption, the basis vectors Y^a, W^a span a 2-dimensional vector space. The case where Y^a and W^a are linearly dependent at a point (i.e. $\rho = 0$) will be dealt with in Section 2.4. Together the two constraint equations (2.7) and (2.10) imply

$$\alpha_a = \frac{1}{3\rho}\left(\frac{\phi}{2} + 3F^2\right) W_a - \frac{1}{3\rho}\left(15F^4 - \ell\right) Y_a. \tag{2.11}$$

We shall show that α_a given by (2.11) rescales correctly under a projective rescaling. We first need the spinor identity lemma

Lemma 2.2. *In 2 dimensions, the following identity is valid:*

$$\begin{aligned} Y_{[a} W_b \Upsilon_{c]} =& \frac{1}{6} Y_a W_b \Upsilon_c + \frac{1}{6} Y_c W_a \Upsilon_b + \frac{1}{6} Y_b W_c \Upsilon_a \\ &- \frac{1}{6} Y_a W_c \Upsilon_b - \frac{1}{6} Y_c W_b \Upsilon_a - \frac{1}{6} Y_b W_a \Upsilon_c \\ =& 0. \end{aligned} \tag{2.12}$$

Contracting (2.12) with ϵ^{ab} gives

$$Y_a W^a \Upsilon_c - Y_c \Upsilon_a W^a + Y^a \Upsilon_a W_c = 0,$$

or

$$\Upsilon_a = -\frac{Y^b \Upsilon_b}{\rho} W_a + \frac{W^b \Upsilon_b}{\rho} Y_a \tag{2.13}$$

upon rearranging.

Proposition 2.3. *Under a projective change of connection, α_a given by (2.11) transforms accordingly as $\alpha_a \mapsto \widehat{\alpha}_a = \alpha_a + \Upsilon_a$.*

Proof. Under a projective change of connection, we have

$$\begin{aligned}
\widehat{\alpha}_a =& \frac{1}{3\rho}\left(\frac{\widehat{\phi}}{2} + 3F^2\right) W_a - \frac{1}{3\rho}\left(15F^4 - \widehat{\ell}\right) Y_a \\
=& \frac{1}{3\rho}\left(\frac{(\phi - 6\Upsilon_a Y^a)}{2} + 3F^2\right) W_a - \frac{1}{3\rho}\left(15F^4 - \ell - 3\Upsilon_a W^a\right) Y_a \\
=& \frac{1}{3\rho}\left(\frac{\phi}{2} + 3F^2\right) W_a - \frac{1}{3\rho}\left(15F^4 - \ell\right) Y_a - \frac{\Upsilon_b Y^b}{\rho} W_a + \frac{\Upsilon_b W^b}{\rho} Y_a \\
=& \alpha_a - \frac{\Upsilon_b Y^b}{\rho} W_a + \frac{\Upsilon_b W^b}{\rho} Y_a \\
=& \alpha_a + \Upsilon_a,
\end{aligned}$$

where in the last step we made use of (2.13). □

Substituting (2.11) back into the pEW equation (1.4) yields further constraints on F. The first polynomial constraint $P_1(F) = 0$ comes from computing $F = \nabla_a \epsilon^{ab} \alpha_b = \nabla_a \alpha^a$ for the formula of α_a in (2.11). A computation gives

$$\begin{aligned}
P_1(F) =& -90F^6 + 15\left(\frac{Y^a \nabla_a \rho}{\rho} - \frac{5\phi}{2}\right) F^4 - \left(\frac{3W^a \nabla_a \rho}{\rho} + 6\ell - 3\nabla_a W^a\right) F^2 \\
& - 9\rho F + \left(\frac{W^a \nabla_a \phi}{2} + \frac{\phi}{2}\nabla_a W^a + Y^a \nabla_a \ell + \frac{\phi\ell}{2} - \frac{\phi W^a \nabla_a \rho}{2\rho} - \ell \frac{Y^a \nabla_a \rho}{\rho}\right) \\
=& 0.
\end{aligned}$$

The second and third polynomial constraints in F come from contracting (1.4) with $W^a W^b$ and $W^a Y^b$ respectively and again using (2.7) and (2.10) to eliminate $\alpha_a Y^a$ and $\alpha_a W^a$. We note that another possible contraction $Y^a Y^b \nabla_a \alpha_b + \alpha_a Y^a \alpha_b Y^b + \mathrm{P}_{ab} Y^a Y^b$ is identically zero and yields no new information. Evaluating $W^a W^b \nabla_a \alpha_b + \alpha_a \alpha_b W^a W^b + \mathrm{P}_{ab} W^a W^b = 0$ for α_a in (2.11) gives

$$\begin{aligned}
P_2(F) =& -275F^8 + \left(\frac{-5W^e Y^d \nabla_e W_d}{\rho} + \frac{50\ell}{3}\right) F^4 + 20\rho F^3 + \left(\frac{W^e W^a \nabla_e W_a}{\rho}\right) F^2 \\
& + \frac{\phi W^e W^a \nabla_e W_a}{6\rho} + \mathrm{P}_{ea} W^e W^a + \frac{\ell^2}{9} + \frac{\ell W^e Y^d \nabla_e W_d}{3\rho} + \frac{W^e \nabla_e \ell}{3} \\
=& 0,
\end{aligned}$$

while evaluating $W^aY^b\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b W^aY^b + \mathrm{P}_{ab}W^aY^b = 0$ gives

$$\begin{aligned}
P_3(F) = &-40F^6 + \left(\frac{-5Y^a(W^e\nabla_e Y_a + Y^e\nabla_a W_e)}{2\rho} - \frac{25\phi}{6}\right)F^4\\
&+\left(\frac{2\ell}{3} + \frac{W^e(W^d\nabla_e Y_d + Y^d\nabla_d W_e)}{2\rho}\right)F^2\\
&+\rho F - \frac{W^e\nabla_e\phi}{12} + \frac{\phi W^b W^e\nabla_e Y_b}{12\rho} + \frac{\ell W^e Y^a\nabla_e Y_a}{6\rho}\\
&+\mathrm{P}_{ae}W^eY^a + \frac{Y^e\nabla_e\ell}{6} + \frac{\phi W^eY^a\nabla_a W_e}{12\rho} - \frac{\ell\phi}{18} + \frac{\ell Y^bY^a\nabla_a W_b}{6\rho}\\
=&0.
\end{aligned}$$

The polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ obtained by replacing F with the indeterminate t turn out to have coefficients which are scalar invariants of the projective structure. For example, under projective rescalings, the coefficient of the degree 4 term in $P_1(t)$ transforms as follows:

$$\begin{aligned}
15\left(\widehat{\frac{Y^a\nabla_a\rho}{\rho} - \frac{5\phi}{2}}\right) =&15\left(\frac{Y^a\widehat{\nabla}_a\rho}{\rho} - \frac{5\widehat{\phi}}{2}\right)\\
=&15\left(\frac{Y^a\nabla_a\rho - 15\Upsilon_aY^a\rho}{\rho} - \frac{5(\phi - 6\Upsilon_aY^a)}{2}\right)\\
=&15\left(\frac{Y^a\nabla_a\rho}{\rho} - 15\Upsilon_aY^a - \frac{5\phi}{2} + 15\Upsilon_aY^a\right)\\
=&15\left(\frac{Y^a\nabla_a\rho}{\rho} - \frac{5\phi}{2}\right).
\end{aligned}$$

It is therefore projectively invariant. We have therefore established the following:

Theorem 2.4. *Let $(M^2, [\nabla])$ be a projective surface with $\rho \neq 0$. Suppose M^2 admits a solution to (1.4). Then there exists polynomials $P_1(t)$, $P_2(t)$, $P_3(t)$ in the single variable t with coefficients given by the invariants of the projective structure such that when $t = F$,*

$$P_1(F) = P_2(F) = P_3(F) = 0$$

must hold.

Let $\mathrm{Res}(P(t), Q(t))$ be the resultant of polynomials $P(t)$ and $Q(t)$ in the single variable t. $\mathrm{Res}(P(t), Q(t)) = 0$ is a necessary and sufficient condition for $P(t)$ and $Q(t)$ to share a common root. The desired local obstructions for there to be solutions of (1.4) are therefore given by the resultants of any two of the three polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$. The resultants are constructed from the coefficients of $P_1(t)$, $P_2(t)$ and $P_3(t)$ and are therefore invariants of the projective structure. We have

Corollary 2.5. *Let $(M^2, [\nabla])$ be a projective surface with $\rho \neq 0$. Suppose M^2 admits a solution to (1.4). Then*

$$\mathit{Res}(P_1(t), P_2(t)) = \mathit{Res}(P_2(t), P_3(t)) = \mathit{Res}(P_1(t), P_3(t)) = 0$$

must hold.

In principle one can find these resultants explicitly, but that involves more lengthy computations. In practice, checking that the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ associated to a given projective structure at a particular point do not share a common root will suffice to show that the projective surface admits no solution to (1.4) (since the obstructions do not vanish identically), so explicit computations of the resultant polynomials are not necessary. We conclude this section with a couple of remarks. Firstly, it is not clear that the three polynomial equations $P_1(F) = 0$, $P_2(F) = 0$ and $P_3(F) = 0$ altogether give a sufficient condition for projective structures with $\rho \neq 0$ to admit a solution to (1.4). We have an example of a projective structure with $\rho = 0$ given in Section 5.3 where the obstructions vanish but it does not admit a solution to (1.4). We suspect it is possible to construct a similar example with $\rho \neq 0$. Secondly, more polynomial constraints can also be found by differentiating $P_1(F) = 0$, $P_2(F) = 0$ and $P_3(F) = 0$ repeatedly, contracting with Y^a or W^a, and again using the relations in (2.7) and (2.10) to generate more polynomial constraint equations in F with projectively invariant coefficients, so we suspect that the resultants given in Corollary 2.5 do not form a complete set of obstructions.

2.3.1 Concise way of extracting obstructions

We can eliminate the single odd degree terms so that even degree terms remain in $P_1(t)$, $P_2(t)$, $P_3(t)$. Namely, define

$$\begin{aligned}
Q_1(t^2) &= -\,20t^2 P_3(t) + P_2(t),\\
Q_2(t^2) &= -\,9P_3(t) - P_1(t),\\
Q_3(t^2) &= -\,\frac{20}{9}t^2 P_1(t) - P_2(t).
\end{aligned}$$

Then the three polynomials $Q_1(t^2)$, $Q_2(t^2)$, $Q_3(t^2)$ will be quartic, cubic and quartic polynomials of t^2 respectively since only even powers of t remain. This allows the obstructions to be extracted easily since now the resultant of any of these 2 polynomials will at most be the determinant of an 8 by 8 matrix. Supposing that

$$\begin{aligned}
P_1(t) =& -90t^6 + a_1 t^4 + a_2 t^2 - 9\rho t + a_3,\\
P_2(t) =& -275t^8 + b_1 t^4 + 20\rho t^3 + b_2 t^2 + b_3,\\
P_3(t) =& -40t^6 + c_1 t^4 + c_2 t^2 + \rho t + c_3,
\end{aligned}$$

where

$$\begin{aligned}
a_1 =& 15\left(\frac{Y^a\nabla_a\rho}{\rho} - \frac{5\phi}{2}\right)\\
a_2 =& -\left(\frac{3W^a\nabla_a\rho}{\rho} + 6\ell - 3\nabla_a W^a\right)\\
a_3 =& \left(\frac{W^a\nabla_a\phi}{2} + \frac{\phi}{2}\nabla_a W^a + Y^a\nabla_a\ell + \frac{\phi\ell}{2}\right) - \frac{\phi W^a\nabla_a\rho}{2\rho} - \ell\frac{Y^a\nabla_a\rho}{\rho}\\
b_1 =& \frac{-5W^eY^d\nabla_e W_d}{\rho} + \frac{50\ell}{3}\\
b_2 =& \frac{W^eW^a\nabla_e W_a}{\rho}\\
b_3 =& \frac{\phi W^eW^a\nabla_e W_a}{6\rho} + \mathrm{P}_{ea}W^eW^a + \frac{\ell^2}{9} + \frac{\ell W^eY^d\nabla_e W_d}{3\rho} + \frac{W^e\nabla_e\ell}{3}\\
c_1 =& \frac{-5Y^a(W^e\nabla_e Y_a + Y^e\nabla_a W_e)}{2\rho} - \frac{25\phi}{6}\\
c_2 =& \frac{2\ell}{3} + \frac{W^e(W^d\nabla_e Y_d + Y^d\nabla_d W_e)}{2\rho}\\
c_3 =& -\frac{W^e\nabla_e\phi}{12} + \frac{\phi W^bW^e\nabla_e Y_b}{12\rho} + \frac{\ell W^eY^a\nabla_e Y_a}{6\rho} + \mathrm{P}_{ae}W^eY^a + \frac{Y^e\nabla_e\ell}{6}\\
&+ \frac{\phi W^eY^a\nabla_a W_e}{12\rho} - \frac{\ell\phi}{18} + \frac{\ell Y^bY^a\nabla_a W_b}{6\rho},
\end{aligned}$$

then a computation gives

$$\begin{aligned}
Q_1(X) =& 525X^4 - 20c_1X^3 + (b_1 - 20c_2)X^2 + (b_2 - 20c_3)X + b_3,\\
Q_2(X) =& 450X^3 - (9c_1 + a_1)X^2 - (9c_2 + a_2)X - (9c_3 + a_3),\\
Q_3(X) =& 475X^4 - \frac{20}{9}a_1X^3 - \left(\frac{20}{9}a_2 + b_1\right)X^2 - \left(\frac{20}{9}a_3 + b_2\right)X - b_3,
\end{aligned}$$

where $X = t^2$. The local obstructions are therefore

$$Q_{12} = \mathrm{Res}(Q_1(X), Q_2(X)) =$$
$$\begin{vmatrix} 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 & 0 & 0 \\ 0 & 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 & 0 \\ 0 & 0 & 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 \\ 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) & 0 & 0 & 0 \\ 0 & 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) & 0 & 0 \\ 0 & 0 & 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) & 0 \\ 0 & 0 & 0 & 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) \end{vmatrix},$$

$$Q_{23} = \mathrm{Res}(Q_2(X), Q_3(X)) =$$
$$\begin{vmatrix} 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) & 0 & 0 & 0 \\ 0 & 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) & 0 & 0 \\ 0 & 0 & 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) & 0 \\ 0 & 0 & 0 & 450 & -(9c_1 + a_1) & -(9c_2 + a_2) & -(9c_3 + a_3) \\ 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 & 0 & 0 \\ 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 & 0 \\ 0 & 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 \end{vmatrix},$$

and

$$Q_{13} = \mathrm{Res}(Q_1(X), Q_3(X)) =$$
$$\begin{vmatrix} 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 & 0 & 0 & 0 \\ 0 & 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 & 0 & 0 \\ 0 & 0 & 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 & 0 \\ 0 & 0 & 0 & 525 & -20c_1 & (b_1 - 20c_2) & (b_2 - 20c_3) & b_3 \\ 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 & 0 & 0 & 0 \\ 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 & 0 & 0 \\ 0 & 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 & 0 \\ 0 & 0 & 0 & 475 & -\frac{20}{9}a_1 & -\left(\frac{20}{9}a_2 + b_1\right) & -\left(\frac{20}{9}a_3 + b_2\right) & -b_3 \end{vmatrix},$$

and the knowledge of the values of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ at a point will let us determine the values of the obstruction at that point.

2.3.2 Solving for X

Now the polynomials

$$\begin{aligned} Q_1(X) =& 525X^4 - 20c_1X^3 + (b_1 - 20c_2)X^2 + (b_2 - 20c_3)X + b_3, \\ Q_2(X) =& 450X^3 - (9c_1 + a_1)X^2 - (9c_2 + a_2)X - (9c_3 + a_3), \\ Q_3(X) =& 475X^4 - \frac{20}{9}a_1X^3 - \left(\frac{20}{9}a_2 + b_1\right)X^2 - \left(\frac{20}{9}a_3 + b_2\right)X - b_3, \end{aligned}$$

all have to satisfy $Q_1(F^2) = Q_2(F^2) = Q_3(F^2) = 0$ for there to admit a solution to (1.4). We can rescale these polynomials so that they become monic:

$$\begin{aligned}
\tilde{Q}_1(X) =& X^4 - \frac{20c_1}{525}X^3 + \frac{(b_1 - 20c_2)}{525}X^2 + \frac{(b_2 - 20c_3)}{525}X + \frac{b_3}{525}, \\
\tilde{Q}_2(X) =& X^3 - \frac{(9c_1 + a_1)}{450}X^2 - \frac{(9c_2 + a_2)}{450}X - \frac{(9c_3 + a_3)}{450}, \\
\tilde{Q}_3(X) =& X^4 - \frac{4a_1}{855}X^3 - \left(\frac{4a_2}{855} + \frac{b_1}{475}\right)X^2 - \left(\frac{4a_3}{855} + \frac{b_2}{475}\right)X - \frac{b_3}{475}.
\end{aligned}$$

Now we find that

$$\begin{aligned}
\tilde{Q}_3(X) - \tilde{Q}_1(X) = & \left(\frac{20c_1}{525} - \frac{4a_1}{855}\right)X^3 - \left(\frac{4a_2}{855} + \frac{b_1}{475} + \frac{(b_1 - 20c_2)}{525}\right)X^2 \\
& - \left(\frac{4a_3}{855} + \frac{b_2}{475} + \frac{(b_2 - 20c_3)}{525}\right)X - \frac{b_3}{475} - \frac{b_3}{525} \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\tilde{Q}_2(X) \qquad (2.15) \\
&= \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)X^3 - \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_1 + a_1)}{450}X^2 \\
&\quad - \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_2 + a_2)}{450}X - \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_3 + a_3)}{450}. \qquad (2.16)
\end{aligned}$$

Taking the difference between equations (2.14) and (2.15) allows us to eliminate the cubic term to obtain a polynomial at most quadratic in X:

$$\begin{aligned}
&\tilde{Q}_3(X) - \tilde{Q}_1(X) - \left(\frac{4c_1}{105} - \frac{4}{855}a_1\right)\tilde{Q}_2(X) \\
&= \left(\frac{20c_1}{525} - \frac{4a_1}{855}\right)X^3 - \left(\frac{4a_2}{855} + \frac{b_1}{475} + \frac{(b_1 - 20c_2)}{525}\right)X^2 \\
&\quad - \left(\frac{4a_3}{855} + \frac{b_2}{475} + \frac{(b_2 - 20c_3)}{525}\right)X - \frac{b_3}{475} - \frac{b_3}{525} \\
&\quad - \left[\left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)X^3 - \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_1 + a_1)}{450}X^2 \right. \\
&\quad \left. - \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_2 + a_2)}{450}X - \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_3 + a_3)}{450}\right] \\
&= \left(\left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_1 + a_1)}{450} - \left(\frac{4a_2}{855} + \frac{b_1}{475} + \frac{(b_1 - 20c_2)}{525}\right)\right)X^2 \\
&\quad \left(\left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_2 + a_2)}{450} - \left(\frac{4a_3}{855} + \frac{b_2}{475} + \frac{(b_2 - 20c_3)}{525}\right)\right)X - \frac{b_3}{475} - \frac{b_3}{525} \\
&\quad + \left(\frac{4c_1}{105} - \frac{4a_1}{855}\right)\frac{(9c_3 + a_3)}{450}.
\end{aligned}$$

Multiplying throughout by $\frac{1}{2} \cdot 15 \cdot 225 \cdot 7 \cdot 57$, we clear denominators and obtain the polynomial

$$\begin{aligned}
&\left(513c_1^2 - 120c_1a_1 - 7a_1^2 - 3150a_2 - 2700b_1 + 25650c_2\right) X^2 \\
&(513c_1c_2 + 57a_2c_1 - 63a_1c_2 - 7a_1a_2 - 3150a_3 - 2700b_2 + 25650c_3)\, X \\
&+ 513c_1c_3 + 57c_1a_3 - 63a_1c_3 - 7a_1a_3 - 2700b_3.
\end{aligned}$$

Let

$$\begin{aligned}
A =&513c_1^2 - 120c_1a_1 - 7a_1^2 - 3150a_2 - 2700b_1 + 25650c_2, \\
B =&513c_1c_2 + 57a_2c_1 - 63a_1c_2 - 7a_1a_2 - 3150a_3 - 2700b_2 + 25650c_3, \\
C =&513c_1c_3 + 57c_1a_3 - 63a_1c_3 - 7a_1a_3 - 2700b_3.
\end{aligned}$$

We find that the polynomial constraint $AX^2 + BX + C = 0$ must hold for there to admit a solution to (1.4). A criterion for there to be real solutions to X is then $B^2 - 4AC \geq 0$. The quadratic formula allows us to solve for X, then F and finally α_a using (2.11).

2.3.3 On projective surfaces with $\rho \neq 0$ admitting more than one solution to pEW

The aim of this section is to investigate whether it is possible for a given projective structure to admit more than one solution to pEW, and derive constraints if possible for more than one solution to exist. From the example given by the second order ODE (5.1) in Section 5.2 in Chapter 5, it is possible for projective structures with $\rho = 0$ to admit 2 distinct solutions to pEW. We would like to understand if the same phenomenon exists when $\rho \neq 0$, however we are not able to proceed quite far with the results. In the case where $\rho \neq 0$, this would imply that the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ share more than one distinct common root. Now suppose there exist (α_a, F) and (β_a, J) that are two solutions to (1.4) for a fixed projective structure $[\nabla]$, with $F \neq J$. This means that the integrability conditions have to be satisfied for (α_a, F) and (β_a, J) separately. Let $\gamma_a = \alpha_a - \beta_a$. Then

$$3\alpha_a Y^a + 3F^2 = 3\beta_a Y^a + 3J^2,$$

so that

$$\gamma_a Y^a = J^2 - F^2.$$

Differentiating once more, we obtain

$$\begin{aligned}\alpha_a W^a &= -\,5F^4 + \frac{\ell}{3},\\ \beta_a W^a &= -\,5J^4 + \frac{\ell}{3},\end{aligned}$$

which gives

$$\gamma_a W^a = 5J^4 - 5F^4,$$

so that for $\rho = Y_a W^a \neq 0$, we have

$$\gamma_a = \frac{5(J^4 - F^4)}{\rho} Y_a + \frac{F^2 - J^2}{\rho} W_a.$$

We therefore have the proposition

Proposition 2.6. *Let $(M^2, [\nabla])$ be a smooth projective structure with $\rho \neq 0$, admitting a solution (α_a, F) to (1.4), with F non-zero. Then there does not exist another solution (β_a, J) such that $F = -J$.*

Proof. Suppose there exists β_a such that $J = -F$. Then $\gamma_a = \alpha_a - \beta_a = 0$, so that $\alpha_a = \beta_a$. But this means that $J = F = -J$, or that $J = F = 0$, which is a contradiction. □

Hence using Proposition 2.6, we conclude that for projective structures $(M^2, [\nabla])$ with $\rho \neq 0$ that admit two different solutions to pEW, that $J^2 - F^2 \neq 0$ for any pair of distinct solutions (α_a, F) and (β_a, J) to (1.4). This means that $J \neq \pm F$, as $J^2 = F^2$ implies $\alpha_a = \beta_a$. Then the two solutions (α_a, F) and (β_a, J) have to satisfy

$$AX^2 + BX + C = AY^2 + BY + C = 0,$$

where $X = F^2$, $Y = J^2$, and hence taking difference,

$$0 = (X - Y)(A(X + Y) + B).$$

Since $X \neq Y$, we obtain

$$F^2 + J^2 = X + Y = -\frac{B}{A} > 0.$$

Hence we have

Proposition 2.7. *Let $(M^2, [\nabla])$ be a projective structure with $\rho \neq 0$ that admits at least one solution (α_a, F) to (1.4), with F non-zero. If in addition $A = 0$, then either 1) $B = 0$, in which case $C = 0$ as well, or*
2) $B \neq 0$, in which case $X = F^2 = -\frac{C}{B} > 0$, so that $C \neq 0$ and (α_a, F) is the only solution to (1.4).

2.4 Local obstructions to the pEW system for $\rho = 0$

In this section we shall further investigate the case where $\rho = 0$. This means that the vectors W^a and Y^a are linearly dependent everywhere on M^2 and we can express $W^a = fY^a$ for some density f of weight -6. Under the assumption that the projective structure has $Y_a \neq 0$ and $\rho = 0$, the quantity f is projectively invariant. It is however not invariant in the classical sense since it is rational in the jets of projective structure. Constraint (2.10) becomes

$$f\alpha_a Y^a = -5F^4 + \frac{\ell}{3},$$

upon which multiplying (2.7) throughout by f and eliminating $\alpha_a Y^a$ gives the quartic equation

$$15F^4 - 3fF^2 - \left(\ell + \frac{f\phi}{2}\right) = 0. \tag{2.17}$$

Let $h = \ell + \frac{f\phi}{2}$, a projective invariant density of weight -12. The term h is projectively invariant under the assumption that $W^a = fY^a$ as

$$\begin{aligned} \widehat{h} = \widehat{\ell + \frac{f\phi}{2}} =& \ell + 3\Upsilon_a W^a + \frac{f\phi}{2} - 3\Upsilon_a Y^a f \\ =& \ell + \frac{f\phi}{2} = h. \end{aligned}$$

Differentiating equation (2.17) gives

$$60F^3(\nabla_a F) - 6fF(\nabla_a F) - 3(\nabla_a f)F^2 - \nabla_a h = 0,$$

which we contract with Y^a to get

$$\begin{aligned} 0 =& 60F^3(Y^a\nabla_a F) - 6fF(Y^a\nabla_a F) - 3(Y^a\nabla_a f)F^2 - Y^a\nabla_a h \\ =& 60F^3(\frac{\phi F}{2} + 3F^3) - 6fF(\frac{\phi F}{2} + 3F^3) - 3(Y^a\nabla_a f)F^2 - Y^a\nabla_a h \\ =& 180F^6 + (30\phi - 18f)F^4 - 3(\phi f + Y^a\nabla_a f)F^2 - Y^a\nabla_a h. \end{aligned} \tag{2.18}$$

We have used the fact that $Y^a\nabla_a F = \frac{\phi F}{2} + 3F^3$ to eliminate all derivatives of F. Equation (2.18) projective transforms like so:

$$
\begin{aligned}
&180F^6 + \widehat{(30\phi - 18f)}F^4 - 3\widehat{(\phi f + Y^a\nabla_a f)}F^2 - \widehat{Y^a\nabla_a h} \\
=&180F^6 + (30\phi - 18f)F^4 - 3(\phi f + Y^a\nabla_a f)F^2 - Y^a\nabla_a h \\
&- 180\Upsilon_a Y^a F^4 + 18\Upsilon_a Y^a f F^2 + 18\Upsilon_a Y^a f F^2 + 12\Upsilon_a Y^a h \\
=&180F^6 + (30\phi - 18f)F^4 - 3(\phi f + Y^a\nabla_a f)F^2 - Y^a\nabla_a h \\
&- 12\Upsilon_a Y^a(15F^4 - 3fF^2 - h) \\
=&180F^6 + (30\phi - 18f)F^4 - 3(\phi f + Y^a\nabla_a f)F^2 - Y^a\nabla_a h
\end{aligned}
$$

where in the last step we have used (2.17). Equation (2.18) is therefore projectively invariant. Multiplying (2.17) by $12F^2$ gives

$$180F^6 - 36fF^4 - 12hF^2 = 0,$$

which can be used to eliminate the degree 6 term in (2.18) to obtain

$$(30\phi + 18f)F^4 - 3(\phi f + Y^a\nabla_a f - 4h)F^2 - Y^a\nabla_a h = 0. \tag{2.19}$$

Substituting

$$F^4 = \frac{3fF^2 + h}{15}$$

into (2.19) gives the quadratic equation

$$
\begin{aligned}
0 =&(30\phi + 18f)\left(\frac{3fF^2 + h}{15}\right) - 3(\phi f + Y^a\nabla_a f - 4h)F^2 - Y^a\nabla_a h \\
=&6\phi f F^2 + 2\phi h - 3(\phi f + Y^a\nabla_a f - 4h)F^2 + \frac{18}{5}f^2F^2 + \frac{6hf}{5} - Y^a\nabla_a h \\
=&\left(3\phi f - 3Y^a\nabla_a f + 12h + \frac{18}{5}f^2\right)F^2 + \frac{6hf}{5} - Y^a\nabla_a h + 2\phi h.
\end{aligned}
$$

Let us call

$$
\begin{aligned}
k =&3\phi f - 3Y^a\nabla_a f + 12h + \frac{18}{5}f^2, \\
m =&\frac{6hf}{5} - Y^a\nabla_a h + 2\phi h.
\end{aligned}
$$

We shall show that k, which has weight -12 and m, which has weight -18 are projectively invariant. We have

$$\begin{aligned}
\widehat{k} =& 3\widehat{\phi}f - 3Y^a\widehat{\nabla}_a f + 12h + \frac{18}{5}f^2\\
=& 3\phi f - 18\Upsilon_a Y^a f - 3Y^a\nabla_a f + 18\Upsilon_a Y^a f + 12h + \frac{18}{5}f^2\\
=& 3\phi f - 3Y^a\nabla_a f + 12h + \frac{18}{5}f^2\\
=& k,\\
\widehat{m} =& \frac{6hf}{5} - Y^a\widehat{\nabla}_a h + 2\widehat{\phi}h\\
=& \frac{6hf}{5} - Y^a\nabla_a h + 12\Upsilon_a Y^a h + 2\phi h - 12\Upsilon_a Y^a h\\
=& \frac{6hf}{5} - Y^a\nabla_a h + 2\phi h\\
=& m.
\end{aligned}$$

Hence we have

$$kF^2 + m = 0. \tag{2.20}$$

Substituting $F^2 = -\frac{m}{k}$ given by (2.20) into (2.17) gives

$$15\left(\frac{m}{k}\right)^2 + 3f\left(\frac{m}{k}\right) - h = 0, \tag{2.21}$$

and we clear the denominator k^2 to obtain the vanishing of the first obstruction of weight -36 in Theorem 2.8. To obtain a second obstruction, differentiating (2.20) gives

$$(\nabla_a k)F^2 + 2Fk\nabla_a F + \nabla_a m = 0,$$

which contracted into Y^a gives

$$\begin{aligned}
(Y^a\nabla_a k)F^2 + 2Fk(Y^a\nabla_a F) + Y^a\nabla_a m =& (Y^a\nabla_a k)F^2 + 2Fk(\frac{\phi F}{2} + 3F^3) + Y^a\nabla_a m\\
=& (Y^a\nabla_a k)F^2 + k\phi F^2 + 6kF^4 + Y^a\nabla_a m\\
=& 6kF^4 + (k\phi + Y^a\nabla_a k)F^2 + Y^a\nabla_a m\\
=& 0.
\end{aligned}$$

Now substituting

$$6kF^4 = -6mF^2,$$

which is a consequence of (2.20), we obtain

$$(k\phi + Y^a\nabla_a k)F^2 + Y^a\nabla_a m - 6mF^2 = 0, \tag{2.22}$$

which we multiply by k to get

$$(k\phi + Y^a\nabla_a k)kF^2 + kY^a\nabla_a m - 6mkF^2 = 0.$$

Again we make use of (2.20) to get

$$-m(k\phi + Y^a\nabla_a k) + kY^a\nabla_a m + 6m^2 = kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m) = 0.$$

The quantity $kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m)$ is a projective invariant of weight -36 and is our desired second obstruction. We therefore have the following:

Theorem 2.8. *Let $(M^2, [\nabla])$ be a projective structure with $\rho = 0$. Suppose M^2 admits a solution to (1.4). Then both*

$$15m^2 + 3fmk - hk^2 = 0, \tag{2.23}$$

$$kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m) = 0 \tag{2.24}$$

must hold.

We now attempt to solve for α_a in the case where $\rho = 0$ and (1.4) both hold. The formula for α_a is given by

$$\alpha_a = -\frac{k}{6m}\nabla_a\left(\frac{m}{k}\right) + \frac{k}{3m}Y_a F. \tag{2.25}$$

To obtain this formula, observe that differentiating $F^2 = -\frac{m}{k}$ gives

$$2F\nabla_a F = 2F(-3\alpha_a F - Y_a) = -\nabla_a\left(\frac{m}{k}\right),$$

and so

$$-6\alpha_a\left(-\frac{m}{k}\right) - 2Y_a F = -6\alpha_a F^2 - 2Y_a F = -\nabla_a\left(\frac{m}{k}\right),$$

which rearranges to give

$$\alpha_a = -\frac{k}{6m}\nabla_a\left(\frac{m}{k}\right) + \frac{k}{3m}Y_a F.$$

Now contracting both sides of (2.25) with W^a gives

$$f\left(\frac{m}{k} - \frac{\phi}{6}\right) = f(-F^2 - \frac{\phi}{6}) = f\alpha_a Y^a = \alpha_a W^a = -\frac{k}{6m}W^a\nabla_a\left(\frac{m}{k}\right),$$

which recovers (2.24) upon expanding the terms and equating them.

Remark 2.9. Are there projective structures that admit a solution to pEW and have $\rho = 0$? The answer is yes, and an example is given in Section 5.2 of Chapter 5.

Chapter 3

Tractor calculus formalism

In this chapter we shall proceed to derive the same polynomial constraints in Theorem 2.4 and obstructions in Theorem 2.8 by tractor calculus. This makes the projective invariance of the coefficients of the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ explicit, since the objects constructed out of tractors will be projectively invariant. For readers unfamiliar with the formalism, this section can be skipped on first reading. Tractor calculus is not used in the other sections of the thesis except Chapter 7, which deals with the conformal version.

3.1 Preliminaries: the cotractor bundle

In this section we review the tractor bundle construction in projective geometry. On a smooth projective manifold $(M, [\nabla])$ we can associate a co-tractor bundle $\mathcal{T}^*$ as defined in [2]. This is the first jet prolongation of $\mathcal{E}(1)$. Consider the direct sum bundle $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$ and a section given by

$$\begin{pmatrix} \sigma \\ \nabla_a \sigma \end{pmatrix}$$

for $\sigma \in \Gamma(\mathcal{E}(1))$. The covariant derivative acting on a density σ, denoted $\nabla_a \sigma$, is determined by a projective scale τ and is given by $\nabla^\tau \sigma = \tau d(\tau^{-1}\sigma)$. If another projective scale $\widehat{\tau} = \Omega^{-1}\tau$ is chosen, this gives another formula for the derivative. In particular,

$$\nabla^{\hat{\tau}} \sigma = \nabla^\tau \sigma + \Upsilon \sigma \tag{3.1}$$

where Υ is a 1-form given by $d \log \Omega$. In abstract index notation, let $\widehat{\nabla}_a \sigma$ denote $\nabla^{\hat{\tau}} \sigma$ and the equation (3.1) reads

$$\widehat{\nabla}_a \sigma = \nabla_a \sigma + \Upsilon_a \sigma.$$

A different choice of projective scale therefore determines a new section of the direct sum bundle $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$ given by

$$\begin{pmatrix} \sigma \\ \widehat{\nabla}_a \sigma \end{pmatrix} = \begin{pmatrix} \sigma \\ \nabla_a \sigma + \Upsilon_a \sigma \end{pmatrix}.$$

It is natural to identify all sections of $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$ that are related in this way by a projective change of scale. Note that an arbitrary section μ_a of $\mathcal{E}_a(1)$ may not be locally exact. Declare two sections of $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$ to be projectively equivalent iff

$$\begin{pmatrix} \sigma \\ \widehat{\mu}_a \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \end{pmatrix}$$

for some 1-form Υ_a. The projectively equivalent class of sections

$$\left[\begin{pmatrix} \sigma \\ \mu_a \end{pmatrix} \right]$$

related in this way is called a co-tractor, denoted by $\mathbb{T}_A$. It is a section of the bundle $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$ modulo the equivalence relation defined above. This bundle is called the co-tractor bundle and let us denote this bundle by $\mathcal{T}^*$. Without a splitting, $\mathcal{T}^*$ is a composition series. An arbitrary projective scale τ determines a splitting of the tractor bundle into the direct sum bundle $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$ and we can write

$$\mathbb{T}_A \stackrel{\tau}{=} \begin{pmatrix} \sigma \\ \mu_a \end{pmatrix}$$

to denote the choice of projective scale used in this splitting. A projective change of scale from τ to $\hat{\tau} = \Omega^{-1}\tau$ determines a new splitting of the co-tractor into components related to the splitting by τ by the following:

$$\mathbb{T}_A \stackrel{\tau}{=} \begin{pmatrix} \sigma \\ \mu_a \end{pmatrix} \sim \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \end{pmatrix} \stackrel{\hat{\tau}}{=} \mathbb{T}_A.$$

The projective scale also determines a splitting of tensor products of co-tractor bundles. There is a canonical projectively invariant connection $\nabla^{\mathcal{T}^*}$ on the co-tractor bundle that acts on sections of the co-tractor bundle. By definition, the tractor connection is a linear map

$$\nabla^{\mathcal{T}^*} : \Gamma(\mathcal{T}^*) \to \Gamma(\Lambda^1 \otimes \mathcal{T}^*)$$

satisfying Leibniz rule. A choice of projective scale τ determines a splitting of $\Lambda^1 \otimes \mathcal{T}^*$ into a direct sum of vector bundles $\Lambda^1 \otimes \mathcal{E}(1) \oplus \Lambda^1 \otimes \mathcal{E}_a(1)$. This same scale τ determines a splitting of $\mathcal{T}^*$ into $\mathcal{E}(1) \oplus \mathcal{E}_a(1)$. Hence we obtain two linear connections on the direct sum bundle: the first is given by

$$\nabla^\tau : \mathcal{E}(1) \to \Lambda^1 \otimes \mathcal{E}(1)$$

which is the connection on the density bundle determined by τ and the second is an induced affine connection

$$\nabla : \mathcal{E}_a(1) \to \Lambda^1 \otimes \mathcal{E}_a(1)$$

which is given by some choice of $\nabla \in [\nabla]$. For any splitting of the co-tractor, the covariant derivative is given by the following formula:

$$\nabla_b^{\mathcal{T}^*} \mathbb{T}_A = \nabla_b^{\mathcal{T}^*} \begin{pmatrix} \sigma \\ \mu_a \end{pmatrix} = \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_a + \mathrm{P}_{ab} \sigma \end{pmatrix} \tag{3.2}$$

for some choice of torsion-free affine connection $\nabla \in [\nabla]$. The above formula defines a connection since it is linear and Leibniz rule holds: for any function f and co-tractor $\mathbb{T}_A$,

$$\begin{aligned}
\nabla_b^{\mathcal{T}^*}(f\mathbb{T}_A) = \nabla_b^{\mathcal{T}^*} \begin{pmatrix} f\sigma \\ f\mu_a \end{pmatrix} &= \begin{pmatrix} \nabla_b(f\sigma) - f\mu_b \\ \nabla_b(f\mu_a) + \mathrm{P}_{ab} f\sigma \end{pmatrix} \\
&= \begin{pmatrix} f\nabla_b\sigma + \sigma\nabla_b f - f\mu_b \\ (\nabla_b f)\mu_a + f(\nabla_b\mu_a) + \mathrm{P}_{ab} f\sigma \end{pmatrix} \\
&= f \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_a + \mathrm{P}_{ab}\sigma \end{pmatrix} + \begin{pmatrix} \sigma\nabla_b f \\ (\nabla_b f)\mu_a \end{pmatrix} \\
&= f \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_a + \mathrm{P}_{ab}\sigma \end{pmatrix} + \nabla_b f \begin{pmatrix} \sigma \\ \mu_a \end{pmatrix} \\
&= f\nabla_b^{\mathcal{T}^*}\mathbb{T}_A + (\nabla_b f)\mathbb{T}_A.
\end{aligned}$$

In fact the definition of the tractor connection is independent of the choice of torsion free affine connection.

Proposition 3.1. *The formula for the tractor connection $\nabla^{\mathcal{T}^*}$ acting on sections of the co-tractor bundle is independent of any choice of affine connections in* $[\nabla]$.

Proof. Suppose we pick another torsion-free connection $\widehat{\nabla}$ projectively related to ∇. Let

$$\mathbb{T}_A = \begin{pmatrix} \sigma \\ \mu_a \end{pmatrix}$$

denote the splitting of a co-tractor by a projective scale τ and

$$\widehat{\mathbb{T}}_A = \begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}_a \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a \sigma \end{pmatrix}$$

denote the splitting of a co-tractor by another projective scale $\Omega^{-1}\tau$. Then

$$\begin{aligned}
\widehat{\nabla}_b^{\mathcal{T}^*} \widehat{\mathbb{T}}_A &= \begin{pmatrix} \widehat{\nabla}_b \widehat{\sigma} - \widehat{\mu}_b \\ \widehat{\nabla}_b \widehat{\mu}_a + \widehat{\mathrm{P}_{ab}} \widehat{\sigma} \end{pmatrix} \\
&= \begin{pmatrix} \nabla_b \sigma + \Upsilon_b \sigma - \mu_b - \Upsilon_b \sigma \\ \nabla_b(\mu_a + \Upsilon_a \sigma) - \Upsilon_a(\mu_b + \Upsilon_b \sigma) + \mathrm{P}_{ab}\sigma - (\nabla_a \Upsilon_b)\sigma + \Upsilon_a \Upsilon_b \sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_a + \Upsilon_a(\nabla_b \sigma) + \mathrm{P}_{ab}\sigma - \Upsilon_a \mu_b \end{pmatrix} \\
&= \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_a + \mathrm{P}_{ab}\sigma + \Upsilon_a(\nabla_b \sigma - \mu_b) \end{pmatrix} \\
&= \begin{pmatrix} \nabla_b \widehat{\sigma - \mu_b} \\ \nabla_b \widehat{\mu_a + \mathrm{P}}_{ab}\sigma \end{pmatrix} \\
&= \widehat{\nabla_b^{\mathcal{T}^*} \mathbb{T}_A}.
\end{aligned}$$

$\square$

Let $\mathcal{T}$ denote the tractor bundle which is dual to the cotractor bundle. Under a choice of projective scale the tractor bundle splits into the direct sum of vector bundles $\mathcal{T} = \mathcal{E}(-1) \oplus \mathcal{E}^a(-1)$. Sections of the tractor bundle are called tractors and are denoted by $\mathbb{T}^A$. Sections $\mathbb{T}^A$ of the tractor bundle change under rescaling by $(\widehat{\rho} \quad \widehat{\nu}^a) = (\rho - \Upsilon_a \nu^a \qquad \nu^a)$. A splitting of a tractor using some scale allows us to write $\mathbb{T}^A = (\rho \quad \nu^a)$. There is a canonical connection on the tractor bundle called the tractor connection on $\mathcal{T}$ given by

$$\nabla_b^{\mathcal{T}}(\rho \quad \nu^a) = (\nabla_b \rho - \mathrm{P}_{cb}\nu^c \quad \nabla_b \nu^a + \rho \delta_b{}^a). \tag{3.3}$$

The following can be easily verified:

Proposition 3.2. *The tractor connection $\nabla^{\mathcal{T}}\mathbb{T}^A$ is projectively invariant.*

The Thomas' D-operator $D_A : \mathcal{E}(w) \to \mathcal{E}_A(w-1)$ acts on a section $\sigma \in \Gamma(\mathcal{E}(w))$ by

$$D_A \sigma = \begin{pmatrix} w\sigma \\ \nabla_a \sigma \end{pmatrix}. \tag{3.4}$$

This is projectively invariant because

$$\widehat{D}_A\sigma = \begin{pmatrix} w\sigma \\ \widehat{\nabla}_a\sigma \end{pmatrix} = \begin{pmatrix} w\sigma \\ \nabla_a\sigma + w\Upsilon_a\sigma \end{pmatrix} = \begin{pmatrix} \widehat{w\sigma} \\ \widehat{\nabla_a\sigma} \end{pmatrix} = D_A\sigma. \tag{3.5}$$

When $w = 1$, we have a canonical map into the co-tractor bundle:

$$D_A : \mathcal{E}(1) \to \mathcal{E}_A(0) = \mathcal{T}^*. \tag{3.6}$$

Define $\mathbb{X}^A : \mathcal{T}^* \to \mathcal{E}(1)$ by $\mathbb{X}^A\mathbb{T}_A = \sigma$ to be the projecting part of a co-tractor field into the top slot in $\mathcal{E}(1)$. So $\mathbb{X}^A D_A\sigma = \sigma$ on $\mathcal{E}(1)$. $\mathbb{X}^A$ satisfies

$$\mathbb{X}^B D_A\sigma = \delta_A{}^B\sigma.$$

We subsequently use drop $\mathcal{T}$ in $\nabla^{\mathcal{T}}$ and use ∇_a to denote the connection acting on tractors or co-tractors. We can expand a tractor in terms of tractor basis under a choice of scale. The tractor bases are obtain from considering the projections which are invariant:

$$\mathbb{X}^A : \mathcal{T}^* \to \mathcal{E}(1), \qquad \mathbb{X}^b_A : \mathcal{T} \to \mathcal{E}^b(-1),$$

and the splitting maps of the short exact sequence which are not invariant:

$$\mathbb{Y}_A : \mathcal{E}(1) \to \mathcal{T}^*, \qquad \mathbb{Y}^A_b : \mathcal{E}^b(-1) \to \mathcal{T}.$$

In a scale, the elements of the tractor bases that are invariant are given by

$$\mathbb{X}^A = (1 \quad 0), \qquad \mathbb{X}^b_A = \begin{pmatrix} 0 \\ \delta_a{}^b \end{pmatrix},$$

and the elements of the tractor bases that are not invariant are given by:

$$\mathbb{Y}_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbb{Y}^A_b = (0 \quad \delta_b{}^a).$$

We can see this as follows:

$$\mathbb{X}^A\mathbb{T}_A = \sigma,$$

and since $\mathbb{T}_A$ and σ are projectively invariant, $\mathbb{X}^A$ has to be invariant as well. Similarly,

$$\mathbb{X}^a_A\mathbb{T}^A = v^a,$$

and since $\mathbb{T}^A$ and v^a are both projectively invariant, $\mathbb{X}^a_A$ has to be projectively invariant as well. Together we have the identities

$$\mathbb{X}^A\mathbb{Y}_A = 1, \qquad \mathbb{X}^b_A\mathbb{Y}^A_a = \delta_a{}^b, \qquad \mathbb{X}^A\mathbb{X}^b_A = \mathbb{Y}_A\mathbb{Y}^A_b = 0.$$

A cotractor $\mathbb{T}_A$ can be written as

$$\mathbb{T}_A = \sigma \mathbb{Y}_A + \mu_a \mathbb{X}^a_A.$$

Under a projective transformation,

$$\sigma \widehat{\mathbb{Y}}_A + (\mu_a + \Upsilon_a \sigma)\widehat{\mathbb{X}}^a_A = \widehat{\mathbb{T}}_A = \mathbb{T}_A = \sigma \mathbb{Y}_A + \mu_a \mathbb{X}^a_A,$$

but

$$\begin{aligned} \sigma \widehat{\mathbb{Y}}_A + (\mu_a + \Upsilon_a \sigma)\widehat{\mathbb{X}}^a_A &= \sigma \widehat{\mathbb{Y}}_A + \mu_a \mathbb{X}^a_A + \Upsilon_a \sigma \mathbb{X}^a_A \\ &= \sigma \mathbb{Y}_A + \mu_a \mathbb{X}^a_A \end{aligned}$$

and so

$$\widehat{\mathbb{Y}}_A = \mathbb{Y}_A - \Upsilon_a \mathbb{X}^a_A.$$

A tractor $\mathbb{T}^A$ can be written as

$$\mathbb{T}^A = \rho \mathbb{X}^A + v^a \mathbb{Y}^A_a.$$

Under a projective transformation,

$$(\rho - \Upsilon_a v^a)\widehat{\mathbb{X}}^A + v^a \hat{\mathbb{Y}}^A_a = \widehat{\mathbb{T}}^A = \mathbb{T}^A = \rho \mathbb{X}^A + v^a \mathbb{Y}^A_a,$$

and since

$$\rho \mathbb{X}^A + v^a \mathbb{Y}^A_a = \rho \mathbb{X}^A - \Upsilon_a v^a \mathbb{X}^A + v^a \widehat{\mathbb{Y}}^A_a,$$

we have

$$\begin{aligned} -\Upsilon_a v^a \mathbb{X}^A + v^a \widehat{\mathbb{Y}}^A_a &= v^a \mathbb{Y}^A_a \\ \widehat{\mathbb{Y}}^A_a &= \mathbb{Y}^A_a + \Upsilon_a \mathbb{X}^A. \end{aligned}$$

The tractor connection acts on the tractor bases as follows:

$$\begin{aligned} \nabla_a \mathbb{X}^B =& \mathbb{Y}^B_a, \\ \nabla_c \mathbb{Y}^B_a =& - \mathrm{P}_{ac} \mathbb{X}^B \end{aligned}$$

The tractor connection acts on the cotractor bases as follows:

$$\begin{aligned} \nabla_a \mathbb{X}^b_B =& - \delta_a{}^b \mathbb{Y}_B, \\ \nabla_a \mathbb{Y}_B =& \mathrm{P}_{ab} \mathbb{X}^b_B. \end{aligned}$$

With the cotractor bases, we can write out explicitly how Thomas' D-operator acts on co-tractors. For a general co-tractor $\mathbb{T}_{BC...D} \in \Gamma\mathcal{E}_{BC...D}(w)$, the D-operator is given by

$$D_A \mathbb{T}_{BC...D} = w\mathbb{Y}_A \mathbb{T}_{BC...D} + \mathbb{X}_A^a \nabla_a \mathbb{T}_{BC...D}$$

where ∇_a is the connection on $\mathcal{E}_{BC...D}(w)$ obtained by coupling the tractor connection $\nabla_a^{\mathcal{T}^*}$ on $\mathcal{E}_{BC...D}$ with the connection on $\mathcal{E}(w)$ corresponding to the trivialisation of the density line bundle. We remark that

$$\widehat{\nabla}_a \widehat{\mathbb{T}}_{BC...D} = \widehat{\nabla}_a \mathbb{T}_{BC...D} = \nabla_a \mathbb{T}_{BC...D} + w\Upsilon_a \mathbb{T}_{BC...D}$$

because of the weight of the co-tractor, so that this definition of the D-operator is also projectively invariant.

3.2 Tractor formalism for pEW

Consider the weight -1 cotractor α_A (equivalently, a section of $\mathcal{E}_A(-1)$) given in some scale by

$$\alpha_A = \begin{pmatrix} 1 \\ \alpha_a \end{pmatrix} = \mathbb{Y}_A + \alpha_a \mathbb{X}_A^a,$$

where α_a is a 1-form.

Proposition 3.3. *Solutions α_a to (1.4) are in 1-1 correspondences with cotractors α_A satisfying the tractor equation $D_{(A}\alpha_{B)} + \alpha_A\alpha_B = 0$.*

Proof. A computation using the standard tractor bases gives

$$\begin{aligned}
D_{(A}\alpha_{B)} =& \frac{1}{2}D_A\alpha_B + \frac{1}{2}D_B\alpha_A \\
=& \frac{1}{2}[\mathbb{X}_A^a\nabla_a\alpha_B - \mathbb{Y}_A\alpha_B] + \frac{1}{2}[\mathbb{X}_B^b\nabla_b\alpha_A - \mathbb{Y}_B\alpha_A] \\
=& \frac{1}{2}\mathbb{X}_A^a\nabla_a(\mathbb{Y}_B + \alpha_b\mathbb{X}_B^b) + \frac{1}{2}\mathbb{X}_B^b\nabla_b(\mathbb{Y}_A + \alpha_a\mathbb{X}_A^a) \\
& - \frac{1}{2}\mathbb{Y}_A(\mathbb{Y}_B + \alpha_b\mathbb{X}_B^b) - \frac{1}{2}\mathbb{Y}_B(\mathbb{Y}_A + \alpha_a\mathbb{X}_A^a) \\
=& \frac{1}{2}\mathbb{X}_A^a[-\alpha_a\mathbb{Y}_B + (\nabla_a\alpha_b + \mathrm{P}_{ab})\mathbb{X}_B^b] \\
& + \frac{1}{2}\mathbb{X}_B^b[-\alpha_b\mathbb{Y}_A + (\nabla_b\alpha_a + \mathrm{P}_{ab})\mathbb{X}_A^a] \\
& - \frac{1}{2}\mathbb{Y}_A\mathbb{Y}_B - \frac{1}{2}\mathbb{Y}_B\mathbb{Y}_A - \frac{1}{2}\alpha_b\mathbb{Y}_A\mathbb{X}_B^b - \frac{1}{2}\alpha_a\mathbb{X}_A^a\mathbb{Y}_B \\
=& -\mathbb{Y}_{AB} - 2\alpha_a\mathbb{Z}_{AB}^a + (\nabla_{(a}\alpha_{b)} + \mathrm{P}_{ab})\mathbb{X}_{AB}^{ab}
\end{aligned}$$

where $\mathbb{Y}_{AB} = \mathbb{Y}_{(A}\mathbb{Y}_{B)}$, $\mathbb{Z}^a_{AB} = \mathbb{Y}_{(A}\mathbb{X}^a_{B)}$ and $\mathbb{X}^{ab}_{AB} = \mathbb{X}^a_{(A}\mathbb{X}^b_{B)}$. In the matrix representation, this is equivalent to

$$D_{(A}\alpha_{B)} = \begin{pmatrix} -1 \\ -\alpha_a \\ \nabla_{(a}\alpha_{b)} + \mathrm{P}_{ab} \end{pmatrix} \tag{3.7}$$

in some choice of projective scale (we have normalised the middle component by $\frac{1}{2}$). The symmetric cotractor given by (3.7) is a section of $\mathcal{E}_{(AB)}(-2)$. Taking the tensor product of α_A with itself, we have

$$\begin{aligned} \alpha_A\alpha_B = \alpha_{(A}\alpha_{B)} &= (\mathbb{Y}_A + \alpha_a\mathbb{X}^a_A)(\mathbb{Y}_B + \alpha_b\mathbb{X}^b_B) \\ &= \mathbb{Y}_{AB} + 2\alpha_a\mathbb{Z}^a_{AB} + \alpha_a\alpha_b\mathbb{X}^{ab}_{AB}, \end{aligned}$$

which is a section of $\mathcal{E}_{(AB)}(-2)$. In the matrix representation, this is given by

$$\alpha_A\alpha_B = \begin{pmatrix} 1 \\ \alpha_a \\ \alpha_a\alpha_b \end{pmatrix},$$

where we have again normalised the middle component by $\frac{1}{2}$. It is then evident that the projective Einstein-Weyl equation is satisfied if and only if

$$D_{(A}\alpha_{B)} + \alpha_A\alpha_B = (\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab})\mathbb{X}^{ab}_{AB} = 0. \tag{3.8}$$

$\square$

3.3 Prolongation of the tractor pEW equation

Again we can apply the theory of prolongation to the tractor pEW equation (3.8). We can rewrite (3.8) as

$$D_A\alpha_B + \alpha_A\alpha_B = F_{AB},$$

where $F_{AB} = D_{[A}\alpha_{B]}$. Differentiating this gives

$$D_C D_A\alpha_B + (D_C\alpha_A)\alpha_B + \alpha_A(D_C\alpha_B) = D_C F_{AB}$$

and upon skewing, we obtain

$$\Omega_{AC}{}^D{}_B\alpha_D + 2F_{CA}\alpha_B + \alpha_A(D_C\alpha_B) - \alpha_C(D_A\alpha_B) = D_C F_{AB} - D_A F_{CB} = D_B F_{AC},$$

since the identity $D_{[A}F_{BC]} = 0$ holds. Here $\Omega_{AC}{}^{D}{}_{B}\alpha_D = (D_C D_A - D_A D_C)\alpha_B$. We can thus rewrite as

$$D_B F_{AC} = \Omega_{AC}{}^{D}{}_{B}\alpha_D - 2F_{AC}\alpha_B + \alpha_A F_{CB} - \alpha_C F_{AB}.$$

Differentiating once more gives

$$\begin{aligned} D_A D_B F_{CD} =& (D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E + \Omega_{CD}{}^{E}{}_{B}(D_A\alpha_E) - 2(D_A F_{CD})\alpha_B - 2F_{CD}(D_A\alpha_B) \\ & + (D_A\alpha_C)F_{DB} + \alpha_C(D_A F_{DB}) - (D_A\alpha_D)F_{CB} - \alpha_D(D_A F_{CB}) \\ =& (D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E + \Omega_{CD}{}^{E}{}_{B}(F_{AE} - \alpha_A\alpha_E) - 2(D_A F_{CD})\alpha_B \\ & - 2F_{CD}(F_{AB} - \alpha_A\alpha_B) + (F_{AC} - \alpha_A\alpha_C)F_{DB} + \alpha_C(D_A F_{DB}) \\ & - (F_{AD} - \alpha_A\alpha_D)F_{CB} - \alpha_D(D_A F_{CB}). \end{aligned}$$

Upon skewing, we obtain the constraint

$$\begin{aligned} & D_A D_B F_{CD} - D_B D_A F_{CD} \\ =& \Omega_{BA}{}^{E}{}_{C}F_{ED} + \Omega_{BA}{}^{E}{}_{D}F_{CE} \\ =& (D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E + \Omega_{CD}{}^{E}{}_{B}(F_{AE} - \alpha_A\alpha_E) - 2(D_A F_{CD})\alpha_B - 2F_{CD}(F_{AB} - \alpha_A\alpha_B) \\ & + (F_{AC} - \alpha_A\alpha_C)F_{DB} + \alpha_C(D_A F_{DB}) - (F_{AD} - \alpha_A\alpha_D)F_{CB} - \alpha_D(D_A F_{CB}) \\ & - (D_B\Omega_{CD}{}^{E}{}_{A})\alpha_E - \Omega_{CD}{}^{E}{}_{A}(F_{BE} - \alpha_B\alpha_E) + 2(D_B F_{CD})\alpha_A \\ & + 2F_{CD}(F_{BA} - \alpha_B\alpha_A) - (F_{BC} - \alpha_B\alpha_C)F_{DA} - \alpha_C(D_B F_{DA}) \\ & + (F_{BD} - \alpha_B\alpha_D)F_{CA} + \alpha_D(D_B F_{CA}) \\ =& (D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E + \Omega_{CD}{}^{E}{}_{B}(F_{AE} - \alpha_A\alpha_E) - 2(D_A F_{CD})\alpha_B - 2F_{CD}(F_{AB} - \alpha_A\alpha_B) \\ & + (F_{AC} - \alpha_A\alpha_C)F_{DB} - (F_{AD} - \alpha_A\alpha_D)F_{CB} \\ & - (D_B\Omega_{CD}{}^{E}{}_{A})\alpha_E - \Omega_{CD}{}^{E}{}_{A}(F_{BE} - \alpha_B\alpha_E) + 2(D_B F_{CD})\alpha_A \\ & + 2F_{CD}(F_{BA} - \alpha_B\alpha_A) - (F_{BC} - \alpha_B\alpha_C)F_{DA} + \alpha_C(D_D F_{AB}) \\ & + (F_{BD} - \alpha_B\alpha_D)F_{CA} + \alpha_D(D_C F_{BA}) \end{aligned}$$

and upon substituting

$$\begin{aligned} D_D F_{AB} =& \Omega_{AB}{}^{E}{}_{D}\alpha_E - 2F_{AB}\alpha_D + \alpha_A F_{BD} - \alpha_B F_{AD}, \\ D_B F_{CD} =& \Omega_{CD}{}^{E}{}_{B}\alpha_E - 2F_{CD}\alpha_B + \alpha_C F_{DB} - \alpha_D F_{CB}, \\ D_C F_{BA} =& \Omega_{BA}{}^{E}{}_{C}\alpha_E - 2F_{BA}\alpha_C + \alpha_B F_{AC} - \alpha_A F_{BC}, \\ D_A F_{CD} =& \Omega_{CD}{}^{E}{}_{A}\alpha_E - 2F_{CD}\alpha_A + \alpha_C F_{DA} - \alpha_D F_{CA}, \end{aligned}$$

we obtain

$$
\begin{aligned}
&\Omega_{BA}{}^{E}{}_{C}F_{ED}+\Omega_{BA}{}^{E}{}_{D}F_{CE}\\
=&(D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E+\Omega_{CD}{}^{E}{}_{B}(F_{AE}-\alpha_A\alpha_E)\\
&-2(\Omega_{CD}{}^{E}{}_{A}\alpha_E-2F_{CD}\alpha_A+\alpha_C F_{DA}-\alpha_D F_{CA})\alpha_B\\
&-2F_{CD}(F_{AB}-\alpha_A\alpha_B)+(F_{AC}-\alpha_A\alpha_C)F_{DB}-(F_{AD}-\alpha_A\alpha_D)F_{CB}\\
&-(D_B\Omega_{CD}{}^{E}{}_{A})\alpha_E-\Omega_{CD}{}^{E}{}_{A}(F_{BE}-\alpha_B\alpha_E)\\
&+2(\Omega_{CD}{}^{E}{}_{B}\alpha_E-2F_{CD}\alpha_B+\alpha_C F_{DB}-\alpha_D F_{CB})\alpha_A\\
&+2F_{CD}(F_{BA}-\alpha_B\alpha_A)-(F_{BC}-\alpha_B\alpha_C)F_{DA}+(F_{BD}-\alpha_B\alpha_D)F_{CA}\\
&+\alpha_C(\Omega_{AB}{}^{E}{}_{D}\alpha_E-2F_{AB}\alpha_D+\alpha_A F_{BD}-\alpha_B F_{AD})\\
&+\alpha_D(\Omega_{BA}{}^{E}{}_{C}\alpha_E-2F_{BA}\alpha_C+\alpha_B F_{AC}-\alpha_A F_{BC})\\
=&(D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E+\Omega_{CD}{}^{E}{}_{B}F_{AE}-\Omega_{CD}{}^{E}{}_{B}\alpha_A\alpha_E-2\Omega_{CD}{}^{E}{}_{A}\alpha_E\alpha_B\\
&+4F_{CD}\alpha_A\alpha_B-2\alpha_B\alpha_C F_{DA}+2\alpha_B\alpha_D F_{CA}\\
&-2F_{CD}F_{AB}+2F_{CD}\alpha_A\alpha_B+F_{AC}F_{DB}-\alpha_A\alpha_C F_{DB}-F_{AD}F_{CB}+\alpha_A\alpha_D F_{CB}\\
&-(D_B\Omega_{CD}{}^{E}{}_{A})\alpha_E-\Omega_{CD}{}^{E}{}_{A}F_{BE}+\Omega_{CD}{}^{E}{}_{A}\alpha_B\alpha_E+2\Omega_{CD}{}^{E}{}_{B}\alpha_E\alpha_A\\
&-4F_{CD}\alpha_B\alpha_A+2\alpha_A\alpha_C F_{DB}-2\alpha_A\alpha_D F_{CB}\\
&+2F_{CD}F_{BA}-2F_{CD}\alpha_B\alpha_A+F_{AD}F_{BC}-F_{AD}\alpha_B\alpha_C\\
&+\Omega_{AB}{}^{E}{}_{D}\alpha_E\alpha_C-2F_{AB}\alpha_C\alpha_D+\alpha_A\alpha_C F_{BD}-\alpha_B\alpha_C F_{AD}\\
&+F_{BD}F_{CA}-\alpha_B\alpha_D F_{CA}+\Omega_{BA}{}^{E}{}_{C}\alpha_E\alpha_D-2F_{BA}\alpha_C\alpha_D+\alpha_B F_{AC}\alpha_D-\alpha_A F_{BC}\alpha_D\\
=&(D_A\Omega_{CD}{}^{E}{}_{B})\alpha_E+\Omega_{CD}{}^{E}{}_{B}F_{AE}-\Omega_{CD}{}^{E}{}_{A}\alpha_E\alpha_B-4F_{CD}F_{AB}+2F_{AC}F_{DB}\\
&-(D_B\Omega_{CD}{}^{E}{}_{A})\alpha_E-\Omega_{CD}{}^{E}{}_{A}F_{BE}+\Omega_{CD}{}^{E}{}_{B}\alpha_E\alpha_A\\
&+2F_{AD}F_{BC}+\Omega_{AB}{}^{E}{}_{D}\alpha_E\alpha_C+\Omega_{BA}{}^{E}{}_{C}\alpha_E\alpha_D.
\end{aligned}
$$

Expanding this expression out in terms of tractor bases, we get constraint (1.7).

3.4 Tractor pEW equation in 2 dimensions

In this section D_A is Thomas' D-operator which is related to the standard tractor connection and satisfies Leibniz rule. Consider an invariant of the projective structure the tractor $\mathbb{V}^A$ of weight -5 given in some scale by

$$\begin{pmatrix} \frac{\phi}{6} & Y^a \end{pmatrix}$$

where $Y^a = \epsilon^{ab}Y_b$ and $\phi = \epsilon^{ab}(\nabla_a Y_b - \nabla_b Y_a) = 2\nabla_a Y^a$. The Cotton-York vector Y^a is projectively invariant and its divergence ϕ transforms according to $\widehat{\phi} =$

$\phi - 6\Upsilon_a Y^a$ as one changes the projective connection. The tractor expression is given by $\mathbb{V}^A = \frac{\phi}{6}\mathbb{X}^A + Y^a \mathbb{Y}^A_a$. Let $\mathbb{V}_A$ denote the cotractor of weight -4 given in a projective scale by

$$\mathbb{V}_A = \begin{pmatrix} 0 \\ Y_a \end{pmatrix}.$$

The cotractor $\mathbb{V}_A = Y_a \mathbb{X}^a_A$ is clearly projectively invariant and satisfies $\mathbb{V}_A \mathbb{V}^A = 0$. However it is not dual to $\mathbb{V}^A$ (there is no canonical way to get $\mathbb{V}_A$ from $\mathbb{V}^A$). We find that equation (2.3) in the tractor formalism is

$$\alpha_A = -\frac{1}{3F} D_A F - \frac{1}{3F} \mathbb{V}_A. \tag{3.9}$$

Proposition 3.4. *Suppose $(M^2, [\nabla])$ admits a solution to (1.4). Then the following algebraic constraints*

$$\alpha_A \mathbb{V}^A = -F^2 \tag{3.10}$$

and

$$(\mathbb{V}^A D_A \mathbb{V}^B)\alpha_B = -5F^4 \tag{3.11}$$

must hold.

Proof. Expanding $\alpha_A \mathbb{V}^A$ gives

$$\begin{pmatrix} 1 \\ \alpha_a \end{pmatrix} \begin{pmatrix} \frac{\phi}{6} & Y^a \end{pmatrix} = \alpha_a Y^a + \frac{\phi}{6} = -F^2$$

from constraint equation (2.6). We can obtain equation (3.11) from (3.10) by differentiating with respect to D_A, using (3.9) to eliminate $D_A F$ and finally contracting with $\mathbb{V}^A$. Differentiating equation (3.10) gives

$$(D_A \alpha_B)\mathbb{V}^B + \alpha_B (D_A \mathbb{V}^B) = -2F D_A F.$$

Substituting $D_A F = -3F\alpha_A - \mathbb{V}_A$ from (3.9) gives

$$(D_A \alpha_B)\mathbb{V}^B + \alpha_B (D_A \mathbb{V}^B) = 6F^2 \alpha_A + 2F\mathbb{V}_A.$$

Contracting with $\mathbb{V}^A$ gives

$$(\mathbb{V}^A D_A \alpha_B)\mathbb{V}^B + \alpha_B (\mathbb{V}^A D_A \mathbb{V}^B) = 6F^2 \alpha_A \mathbb{V}^A + 2F\mathbb{V}_A \mathbb{V}^A = 6F^2(-F^2) = -6F^4.$$

Since pEW holds,

$$\mathbb{V}^A \mathbb{V}^B D_A \alpha_B = -\mathbb{V}^A \mathbb{V}^B \alpha_A \alpha_B = -(-F^2)^2 = -F^4.$$

Hence we have

$$-F^4 + \alpha_B (\mathbb{V}^A D_A \mathbb{V}^B) = -6F^4$$

which gives (3.11). ∎

Corollary 3.5. *Let $(M^2, [\nabla])$ be a projective surface with $\mathbb{V}^A \neq 0$ satisfying the tractor geodesic equation $\mathbb{V}^A D_A \mathbb{V}^B = 0$. Then $(M^2, [\nabla])$ cannot admit any solution to the projective Einstein-Weyl equation.*

Proof. Suppose (1.4) holds on M^2. Since $\mathbb{V}^A D_A \mathbb{V}^B = 0$, this forces $F = 0$ by (3.11). But $F = 0$ if and only if M^2 is projectively flat, i.e. $\mathbb{V}^A = 0$, which is a contradiction. □

Introduce $\mathbb{W}^A = \mathbb{V}^B D_B \mathbb{V}^A$, a tractor of weight -11 and the cotractor $\mathbb{W}_A = \mathbb{V}^A D_A \mathbb{V}_B$ of weight -10. In a projective scale,

$$\begin{aligned}\mathbb{W}^A =& \left(\frac{Y^a \nabla_a \phi}{6} - \mathrm{P}_{ab} Y^a Y^b - \frac{5\phi^2}{36}\right) \mathbb{X}^A + (Y^a \nabla_a Y^d - \frac{2\phi}{3} Y^d) \mathbb{Y}_d^A \\ =& -\frac{\ell}{3} \mathbb{X}^A + W^d \mathbb{Y}_d^A,\end{aligned}$$

while $\mathbb{W}_A$ is given by $\mathbb{W}_A = (Y^c \nabla_c Y_d - \frac{2\phi}{3} Y_d) \mathbb{X}_A^d = W_d \mathbb{X}_A^d$. Equations (3.10) and (3.11) can be rewritten as

$$\mathbb{V}^A D_A F = 3F^3, \tag{3.12}$$

$$\mathbb{W}^A D_A F = 15F^5 - \mathbb{V}_A \mathbb{W}^A. \tag{3.13}$$

Since the tractor bundle has rank 3, we need another basis in order to solve for $D_A F$. The canonical one to use is the projector $\mathbb{X}^A$. We have

$$\mathbb{X}^A D_A F = -3F,$$

where -3 is the weight of F. In matrix form, where we let Y^1 (resp. Y_1) denote the first component of the vector Y^a (resp. covector Y_a) and Y^2 (resp. Y_2) denote the second component of the vector Y^a (resp. covector Y_a) and so on, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\phi}{6} & Y^1 & Y^2 \\ -\frac{\ell}{3} & W^1 & W^2 \end{pmatrix} \begin{pmatrix} -3F \\ \nabla_1 F \\ \nabla_2 F \end{pmatrix} = \begin{pmatrix} -3F \\ 3F^3 \\ 15F^5 - \mathbb{V}_A \mathbb{W}^A \end{pmatrix}.$$

The matrix on the left has determinant $\rho = Y_a W^a = \mathbb{V}_A \mathbb{W}^A$. Inverting it is particularly difficult, but if we introduce a new set of basis vectors

$$\mathbb{V}'^A = \mathbb{V}^A - \frac{\phi}{6} \mathbb{X}^A, \qquad \mathbb{W}'^A = \mathbb{W}^A + \frac{\ell}{3} \mathbb{X}^A,$$

we can reexpress constraints (3.12) and (3.13) as

$$\begin{aligned}\mathbb{V}'^A D_A F =& \mathbb{V}^A D_A F - \frac{\phi}{6} \mathbb{X}^A D_A F = 3F^3 + \frac{\phi}{2} F, \\ \mathbb{W}'^A D_A F =& \mathbb{W}^A D_A F + \frac{\ell}{3} \mathbb{X}^A D_A F = 15F^5 - \rho - \ell F.\end{aligned}$$

These are the tractor version of constraints (2.7) and (2.9). The matrix form now becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & Y^1 & Y^2 \\ 0 & W^1 & W^2 \end{pmatrix} \begin{pmatrix} -3F \\ \nabla_1 F \\ \nabla_2 F \end{pmatrix} = \begin{pmatrix} -3F \\ 3F^3 + \frac{\phi F}{2} \\ 15F^5 - \rho - \ell F \end{pmatrix}.$$

This can be easily solved to obtain

$$D_A F = \left(\frac{15F^5 - \rho - \ell F}{\rho} \right) \mathbb{V}_A - \left(\frac{3F^3 + \frac{\phi}{2} F}{\rho} \right) \mathbb{W}_A - 3F \mathbb{Y}_A.$$

and α_A is given by

$$\alpha_A = -\frac{1}{3F} D_A F - \frac{1}{3F} \mathbb{V}_A = -\frac{5F^4}{\rho} \mathbb{V}_A + \frac{\ell}{3\rho} \mathbb{V}_A + \frac{F^2}{\rho} \mathbb{W}_A + \frac{\phi}{6\rho} \mathbb{W}_A + \mathbb{Y}_A.$$

We check that under a projective transformation,

$$\widehat{\alpha}_A = \alpha_A + \frac{\Upsilon_c W^c}{\rho} \mathbb{V}_A - \frac{\Upsilon_c Y^c}{\rho} \mathbb{W}_A - \Upsilon_c \mathbb{X}^c_A = \alpha_A,$$

since

$$\Upsilon_c \mathbb{X}^c_A = \left(\frac{\Upsilon_a W^a Y_c}{\rho} - \frac{\Upsilon_a Y^a W_c}{\rho} \right) \mathbb{X}^c_A = \frac{\Upsilon_a W^a}{\rho} \mathbb{V}_A - \frac{\Upsilon_a Y^a}{\rho} \mathbb{W}_A$$

follows from (2.13). Computing $D_A \alpha_B$ gives

$$\begin{aligned} D_A \alpha_B = & \left(\frac{2F D_A F}{\rho} - \frac{F^2 D_A \rho}{\rho^2} \right) \mathbb{W}_B + \frac{F^2}{\rho} D_A \mathbb{W}_B - \left(\frac{20F^3 D_A F}{\rho} - \frac{5F^4 D_A \rho}{\rho^2} \right) \mathbb{V}_B \\ & - \frac{5F^4}{\rho} D_A \mathbb{V}_B + \frac{\ell}{3\rho} D_A \mathbb{V}_B + \left(\frac{D_A \ell}{3\rho} - \frac{\ell D_A \rho}{3\rho^2} \right) \mathbb{V}_B + \left(\frac{D_A \phi}{6\rho} - \frac{\phi D_A \rho}{6\rho^2} \right) \mathbb{W}_B \\ & + \frac{\phi}{6\rho} D_A \mathbb{W}_B + D_A \mathbb{Y}_B. \end{aligned}$$

Symmetrising the indices gives

$$\begin{aligned}
&D_{(A}\alpha_{B)}\\
=&\frac{1}{2}\left(\frac{2FD_AF}{\rho}-\frac{F^2D_A\rho}{\rho^2}\right)\mathbb{W}_B+\frac{F^2}{\rho}D_{(A}\mathbb{W}_{B)}-\frac{1}{2}\left(\frac{20F^3D_AF}{\rho}-\frac{5F^4D_A\rho}{\rho^2}\right)\mathbb{V}_B\\
&-\frac{5F^4}{\rho}D_{(A}\mathbb{V}_{B)}+\frac{1}{2}\left(\frac{2FD_BF}{\rho}-\frac{F^2D_B\rho}{\rho^2}\right)\mathbb{W}_A-\frac{1}{2}\left(\frac{20F^3D_BF}{\rho}-\frac{5F^4D_B\rho}{\rho^2}\right)\mathbb{V}_A\\
&+\frac{\ell}{3\rho}D_{(A}\mathbb{V}_{B)}+\frac{1}{2}\left(\frac{D_A\ell}{3\rho}-\frac{\ell D_A\rho}{3\rho^2}+\right)\mathbb{V}_B+\frac{1}{2}\left(\frac{D_A\phi}{6\rho}-\frac{\phi D_A\rho}{6\rho^2}\right)\mathbb{W}_B+\frac{\phi}{6\rho}D_{(A}\mathbb{W}_{B)}\\
&+\frac{1}{2}\left(\frac{D_B\ell}{3\rho}-\frac{\ell D_B\rho}{3\rho^2}\right)\mathbb{V}_A+\frac{1}{2}\left(\frac{D_B\phi}{6\rho}-\frac{\phi D_B\rho}{6\rho^2}\right)\mathbb{W}_A+D_{(A}\mathbb{Y}_{B)}\\
=&\frac{2F\mathbb{W}_{(A}D_{B)}F}{\rho}-\frac{F^2\mathbb{W}_{(A}D_{B)}\rho}{\rho^2}+\frac{F^2}{\rho}D_{(A}\mathbb{W}_{B)}-\frac{5F^4}{\rho}D_{(A}\mathbb{V}_{B)}-\frac{20F^3\mathbb{V}_{(A}D_{B)}F}{\rho}\\
&+\frac{5F^4\mathbb{V}_{(A}D_{B)}\rho}{\rho^2}+\frac{\ell}{3\rho}D_{(A}\mathbb{V}_{B)}+\frac{\phi}{6\rho}D_{(A}\mathbb{W}_{B)}+\frac{\mathbb{V}_{(A}D_{B)}\ell}{3\rho}-\frac{\ell\mathbb{V}_{(A}D_{B)}\rho}{3\rho^2}\\
&+\frac{\mathbb{W}_{(A}D_{B)}\phi}{6\rho}-\frac{\phi\mathbb{W}_{(A}D_{B)}\rho}{6\rho^2}+D_{(A}\mathbb{Y}_{B)},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&D_{(A}\alpha_{B)}+\alpha_A\alpha_B\\
=&\frac{2F\mathbb{W}_{(A}D_{B)}F}{\rho}-\frac{F^2\mathbb{W}_{(A}D_{B)}\rho}{\rho^2}+\frac{F^2}{\rho}D_{(A}\mathbb{W}_{B)}-\frac{5F^4}{\rho}D_{(A}\mathbb{V}_{B)}-\frac{20F^3\mathbb{V}_{(A}D_{B)}F}{\rho}\\
&+\frac{5F^4\mathbb{V}_{(A}D_{B)}\rho}{\rho^2}+\frac{\ell}{3\rho}D_{(A}\mathbb{V}_{B)}+\frac{\phi}{6\rho}D_{(A}\mathbb{W}_{B)}+\frac{\mathbb{V}_{(A}D_{B)}\ell}{3\rho}-\frac{\ell\mathbb{V}_{(A}D_{B)}\rho}{3\rho^2}+\frac{\mathbb{W}_{(A}D_{B)}\phi}{6\rho}\\
&-\frac{\phi\mathbb{W}_{(A}D_{B)}\rho}{6\rho^2}+D_{(A}\mathbb{Y}_{B)}+\left(-\frac{5F^4}{\rho}\mathbb{V}_A+\frac{\ell}{3\rho}\mathbb{V}_A+\frac{F^2}{\rho}\mathbb{W}_A+\frac{\phi}{6\rho}\mathbb{W}_A+\mathbb{Y}_A\right)\\
&\times\left(-\frac{5F^4}{\rho}\mathbb{V}_B+\frac{\ell}{3\rho}\mathbb{V}_B+\frac{F^2}{\rho}\mathbb{W}_B+\frac{\phi}{6\rho}\mathbb{W}_B+\mathbb{Y}_B\right)\\
=&0.
\end{aligned}$$

We remark that

$$D_A\mathbb{Y}_B=D_{(A}\mathbb{Y}_{B)}=-\mathbb{Y}_A\mathbb{Y}_B+\mathbb{X}^a_A\nabla_A\mathbb{Y}_B=\mathrm{P}_{ab}\mathbb{X}^a_A\mathbb{X}^b_B-\mathbb{Y}_A\mathbb{Y}_B.$$

Contracting with $\mathbb{V}^A\mathbb{V}^B$ is identically zero, as we shall show. Since $\mathbb{V}_A\mathbb{V}^A=0$, we obtain

$$\begin{aligned}
&\mathbb{V}^A\mathbb{V}^BD_{(A}\alpha_{B)}+\alpha_A\mathbb{V}^A\alpha_B\mathbb{V}^B\\
=&\frac{2F\mathbb{V}^A\mathbb{V}^B\mathbb{W}_{(A}D_{B)}F}{\rho}-\frac{F^2\mathbb{V}^A\mathbb{V}^B\mathbb{W}_{(A}D_{B)}\rho}{\rho^2}+\frac{F^2}{\rho}\mathbb{V}^A\mathbb{V}^BD_{(A}\mathbb{W}_{B)}-\frac{5F^4}{\rho}\mathbb{V}^A\mathbb{V}^BD_{(A}\mathbb{V}_{B)}\\
&+\frac{\ell}{3\rho}\mathbb{V}^A\mathbb{V}^BD_{(A}\mathbb{V}_{B)}+\frac{\phi}{6\rho}\mathbb{V}^A\mathbb{V}^BD_{(A}\mathbb{W}_{B)}+\frac{\mathbb{V}^A\mathbb{V}^B\mathbb{W}_{(A}D_{B)}\phi}{6\rho}-\frac{\phi\mathbb{V}^A\mathbb{V}^B\mathbb{W}_{(A}D_{B)}\rho}{6\rho^2}\\
&+\mathbb{V}^A\mathbb{V}^BD_{(A}\mathbb{Y}_{B)}+\left(\frac{F^2}{\rho}\mathbb{W}_A\mathbb{V}^A+\frac{\phi}{6\rho}\mathbb{W}_A\mathbb{V}^A+\mathbb{Y}_A\mathbb{V}^A\right)\left(\frac{F^2}{\rho}\mathbb{W}_B\mathbb{V}^B+\frac{\phi}{6\rho}\mathbb{W}_B\mathbb{V}^B+\mathbb{Y}_B\mathbb{V}^B\right).
\end{aligned}$$

Substituting $\mathbb{V}_A\mathbb{W}^A = \rho$, $\mathbb{V}^A\mathbb{V}^B D_A\mathbb{V}_B = -\mathbb{V}_A\mathbb{W}^A = -\rho$, $\mathbb{V}^A D_A\rho = -\mathbb{V}^A\mathbb{V}^B D_A\mathbb{W}_B$, $\mathbb{Y}_A\mathbb{V}^A = \frac{\phi}{6}$, $\mathbb{V}^A\mathbb{V}^B D_A\mathbb{Y}_B = \mathrm{P}_{ab}Y^aY^b - \frac{\phi^2}{36}$, and $\mathbb{V}^A D_A\phi = Y^a\nabla_a\phi - \phi^2$, we get

$$
\begin{aligned}
&\mathbb{V}^A\mathbb{V}^B D_{(A}\alpha_{B)} + \alpha_A\mathbb{V}^A\alpha_B\mathbb{V}^B \\
&\quad= -2F\mathbb{V}^B D_B F + F^2\frac{\mathbb{V}^B D_B\rho}{\rho} + \frac{F^2}{\rho}\mathbb{V}^A\mathbb{V}^B D_{(A}\mathbb{W}_{B)} - \frac{5F^4}{\rho}\mathbb{V}^A\mathbb{V}^B D_{(A}\mathbb{V}_{B)} \\
&\qquad + \frac{\ell}{3\rho}\mathbb{V}^A\mathbb{V}^B D_{(A}\mathbb{V}_{B)} + \frac{\phi}{6\rho}\mathbb{V}^A\mathbb{V}^B D_{(A}\mathbb{W}_{B)} - \frac{\mathbb{V}^B D_B\phi}{6} + \frac{\phi\mathbb{V}^B D_B\rho}{6\rho} + \mathbb{V}^A\mathbb{V}^B D_{(A}\mathbb{Y}_{B)} \\
&\qquad + \left(-F^2 - \frac{\phi}{6} + \frac{\phi}{6}\right)\left(-F^2 - \frac{\phi}{6} + \frac{\phi}{6}\right) \\
&\quad= -6F^4 + 5F^4 - \frac{\ell}{3} - \frac{\mathbb{V}^B D_B\phi}{6} + \mathrm{P}_{ab}Y^aY^b - \frac{\phi^2}{36} + F^4 \\
&\quad= -\frac{\ell}{3} - \frac{Y^a\nabla_a\phi}{6} + \frac{\phi^2}{6} + \mathrm{P}_{ab}Y^aY^b - \frac{\phi^2}{36} \\
&\quad= \frac{\ell}{3} - \frac{\ell}{3} \\
&\quad\equiv 0.
\end{aligned}
$$

The polynomial constraint $P_2(F) = 0$ is given by contracting with $\mathbb{W}^A\mathbb{W}^B$:

$$
\begin{aligned}
&\mathbb{W}^A\mathbb{W}^B D_{(A}\alpha_{B)} + \alpha_A\mathbb{W}^A\alpha_B\mathbb{W}^B \\
&\quad= \frac{F^2}{\rho}\mathbb{W}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} - \frac{5F^4}{\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{V}_B - 20F^3\mathbb{W}^A D_A F + \frac{5F^4\mathbb{W}^A D_A\rho}{\rho} \\
&\qquad + \frac{\ell}{3\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{V}_B + \frac{\phi}{6\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{W}_B + \frac{\mathbb{W}^B D_B\ell}{3} - \frac{\ell W^B D_B\rho}{3\rho} \\
&\qquad + \mathbb{W}^A\mathbb{W}^B D_A\mathbb{Y}_B + \left(-5F^4 + \frac{\ell}{3} + \mathbb{Y}_A\mathbb{W}^A\right)\left(-5F^4 + \frac{\ell}{3} + \mathbb{Y}_B\mathbb{W}^B\right),
\end{aligned}
$$

but $\mathbb{Y}_A\mathbb{W}^A = -\frac{\ell}{3}$, and using (3.13) we have

$$
\begin{aligned}
&\mathbb{W}^A\mathbb{W}^B D_{(A}\alpha_{B)} + \alpha_A\mathbb{W}^A\alpha_B\mathbb{W}^B \\
&= \frac{F^2}{\rho}\mathbb{W}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} - \frac{5F^4}{\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{V}_B - 300F^8 + 20\rho F^3 + \frac{5F^4\mathbb{W}^A D_A\rho}{\rho} \\
&\quad + \frac{\ell}{3\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{V}_B + \frac{\phi}{6\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{W}_B + \frac{\mathbb{W}^B D_B\ell}{3} - \frac{\ell W^B D_B\rho}{3\rho} \\
&\quad + \mathbb{W}^A\mathbb{W}^B D_A\mathbb{Y}_B + 25F^8 \\
&= -275F^8 + \left(\frac{5\mathbb{W}^A D_A\rho}{\rho} - \frac{5\mathbb{W}^A\mathbb{W}^B D_A\mathbb{V}_B}{\rho}\right)F^4 + 20\rho F^3 + \frac{\mathbb{W}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)}}{\rho}F^2 \\
&\quad + \mathbb{W}^A\mathbb{W}^B D_A\mathbb{Y}_B + \frac{\mathbb{W}^B D_B\ell}{3} - \frac{\ell\mathbb{W}^B D_B\rho}{3\rho} + \frac{\phi}{6\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{W}_B + \frac{\ell}{3\rho}\mathbb{W}^A\mathbb{W}^B D_A\mathbb{V}_B \\
&= 0,
\end{aligned}
$$

which gives $P_2(F) = 0$. The polynomial constraint $P_3(F) = 0$ is given by contracting with $\mathbb{V}^A\mathbb{W}^B$:

$$
\begin{aligned}
&\mathbb{V}^A\mathbb{W}^B D_{(A}\alpha_{B)} + \alpha_A\mathbb{V}^A\alpha_B\mathbb{W}^B \\
=&\frac{F^2}{\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} - \frac{1}{2}\left(\frac{20F^3\mathbb{V}^A D_A F}{\rho} - \frac{5F^4\mathbb{V}^A D_A\rho}{\rho^2}\right)\mathbb{V}_B\mathbb{W}^B \\
&- \frac{5F^4}{\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} + \frac{1}{2}\left(\frac{2F\mathbb{W}^B D_B F}{\rho} - \frac{F^2\mathbb{W}^B D_B\rho}{\rho^2}\right)\mathbb{W}_A\mathbb{V}^A \\
&- \frac{5F^6}{\rho^2}\mathbb{V}_B\mathbb{W}^B\mathbb{W}_A\mathbb{V}^A + \frac{\phi}{6}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} + \frac{\ell}{3\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} \\
&+ \frac{1}{2}\left(\frac{\mathbb{V}^A D_A\ell}{3\rho} - \frac{\ell\mathbb{V}^A D_A\rho}{3\rho^2}\right)\mathbb{V}_B\mathbb{W}^B + \frac{1}{2}\left(\frac{\mathbb{W}^B D_B\ell}{6\rho} - \frac{\phi\mathbb{W}^B D_B\rho}{6\rho^2}\right)\mathbb{W}_A\mathbb{V}^A \\
&+ \mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{Y}_{B)} \\
=&\frac{F^2}{\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} - 10F^3\mathbb{V}^A D_A F + \frac{5F^4\mathbb{V}^A D_A\rho}{2\rho} - \frac{5F^4}{\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} \\
&- F\mathbb{W}^B D_B F + \frac{F^2\mathbb{W}^B D_B\rho}{2\rho} + 5F^6 + \frac{\phi}{6}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} + \frac{\ell}{3\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} \\
&+ \frac{\mathbb{V}^A D_A\ell}{6} - \frac{\ell\mathbb{V}^A D_A\rho}{6\rho} - \frac{\mathbb{W}^B D_B\ell}{12} + \frac{\phi\mathbb{W}^B D_B\rho}{12\rho} + \mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{Y}_{B)} \\
=&\frac{F^2}{\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} - 30F^6 + \frac{5F^4\mathbb{V}^A D_A\rho}{2\rho} - \frac{5F^4}{\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} - 15F^6 \\
&+ \rho F + \frac{F^2\mathbb{W}^B D_B\rho}{2\rho} + 5F^6 + \frac{\phi}{6}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} + \frac{\ell}{3\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} + \frac{\mathbb{V}^A D_A\ell}{6} \\
&- \frac{\ell\mathbb{V}^A D_A\rho}{6\rho} - \frac{\mathbb{W}^B D_B\ell}{12} + \frac{\phi\mathbb{W}^B D_B\rho}{12\rho} + \mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{Y}_{B)} \\
=&-40F^6 + \left(\frac{5\mathbb{V}^A D_A\rho}{2\rho} - \frac{5\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)}}{\rho}\right)F^4 + \left(\frac{\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)}}{\rho} + \frac{\mathbb{W}^B D_B\rho}{2\rho}\right)F^2 \\
&+ \rho F + \frac{\phi}{6}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{W}_{B)} + \frac{\ell}{3\rho}\mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{V}_{B)} + \frac{\mathbb{V}^A D_A\ell}{6} - \frac{\ell\mathbb{V}^A D_A\rho}{6\rho} - \frac{\mathbb{W}^B D_B\ell}{12} \\
&+ \frac{\phi\mathbb{W}^B D_B\rho}{12\rho} + \mathbb{V}^A\mathbb{W}^B D_{(A}\mathbb{Y}_{B)} \\
=&0.
\end{aligned}
$$

Finally to get $P_1(F)=0$, we compute

$$\begin{aligned}
F =& D_A\alpha^A \\
=& \left(\frac{2F\mathbb{W}^AD_AF}{\rho}-\frac{F^2\mathbb{W}^AD_A\rho}{\rho^2}\right)+\frac{F^2}{\rho}D_A\mathbb{W}^A-\left(\frac{20F^3\mathbb{V}^AD_AF}{\rho}-\frac{5F^4\mathbb{V}^AD_A\rho}{\rho^2}\right) \\
&-\frac{5F^4}{\rho}D_A\mathbb{V}^A+\left(\frac{\mathbb{V}^AD_A\ell}{3\rho}-\frac{\ell\mathbb{V}^AD_A\rho}{3\rho^2}\right)+\left(\frac{\mathbb{W}^AD_A\rho}{6\rho}-\frac{\phi\mathbb{W}^AD_A\rho}{6\rho^2}\right)+\frac{\phi}{6\rho}D_A\mathbb{W}^A \\
=& \frac{30F^6-2F\rho}{\rho}-\frac{F^2\mathbb{W}^AD_A\rho}{\rho^2}+\frac{F^2}{\rho}D_A\mathbb{W}^A-\frac{60F^6}{\rho}+\frac{5F^4\mathbb{V}^AD_A\rho}{\rho^2} \\
&+\frac{\mathbb{V}^AD_A\ell}{3\rho}-\frac{\ell\mathbb{V}^AD_A\rho}{3\rho^2}+\frac{\mathbb{W}^AD_A\rho}{6\rho}-\frac{\phi\mathbb{W}^AD_A\rho}{6\rho^2}+\frac{\phi}{6\rho}D_A\mathbb{W}^A \\
=& -\frac{30F^6}{\rho}+\frac{5F^4\mathbb{V}^AD_A\rho}{\rho^2}+\left(\frac{D_A\mathbb{W}^A}{\rho}-\frac{\mathbb{W}^AD_A\rho}{\rho^2}\right)F^2-2F \\
&+\frac{\mathbb{V}^AD_A\ell}{3\rho}-\frac{\ell\mathbb{V}^AD_A\rho}{3\rho^2}+\frac{\mathbb{W}^AD_A\rho}{6\rho}-\frac{\phi\mathbb{W}^AD_A\rho}{6\rho^2}+\frac{\phi}{6\rho}D_A\mathbb{W}^A
\end{aligned}$$

making use of the fact that $D_A\mathbb{V}^A=0$ and $D_A\mathbb{Y}^A=0$. We then rearrange and multiply by ρ to get

$$\begin{aligned}
&-30F^6+\frac{5\mathbb{V}^AD_A\rho}{\rho}F^4+\left(D_A\mathbb{W}^A-\frac{\mathbb{W}^AD_A\rho}{\rho}\right)F^2-3\rho F \\
&+\left(\frac{\mathbb{V}^AD_A\ell}{3}-\frac{\ell\mathbb{V}^AD_A\rho}{3\rho}\right)+\frac{\mathbb{W}^AD_A\rho}{6}-\frac{\phi\mathbb{W}^AD_A\rho}{6\rho}+\frac{\phi}{6}D_A\mathbb{W}^A=0.
\end{aligned}$$

The case where $\mathbb{V}_A\mathbb{W}^A=0$ can also be worked out in the tractor formalism. We take note however that the condition $W^a=fY^a$ is equivalent to $\mathbb{X}^a_A\mathbb{W}^A=f\mathbb{X}^a_A\mathbb{V}^A$ for a -6 density f and not $\mathbb{W}^A=f\mathbb{V}^A$, which is a stronger invariant condition. In the case where $\rho=0$, we have

$$\begin{aligned}
\mathbb{W}^A=-\frac{\ell}{3}\mathbb{X}^A+W^d\mathbb{Y}^A_d=-\frac{\ell}{3}\mathbb{X}^A+fY^d\mathbb{Y}^A_d=&-\frac{\ell}{3}\mathbb{X}^A+f(\mathbb{V}^A-\frac{\phi}{6}\mathbb{X}^A) \\
=& f\mathbb{V}^A-\left(\frac{\ell}{3}+\frac{f\phi}{6}\right)\mathbb{X}^A \\
=& f\mathbb{V}^A-\frac{h}{3}\mathbb{X}^A.
\end{aligned}$$

Hence $-5F^4=\alpha_A\mathbb{W}^A=f\alpha_A\mathbb{V}^A-\frac{h}{3}=-fF^2-\frac{h}{3}$, which gives us

$$5F^4=fF^2+\frac{h}{3},$$

or equation (2.17). Applying the tractor D-operator throughout gives

$$20F^3(D_AF)=(D_Af)F^2+2F(D_AF)f+\frac{D_Ah}{3}$$

and contracting with $\mathbb{V}^A$ gives

$$\begin{aligned}20F^3(\mathbb{V}^A D_A F) =& (\mathbb{V}^A D_A f)F^2 + 2F(\mathbb{V}^A D_A F)f + \frac{\mathbb{V}^A D_A h}{3}\\ 20F^3(3F^3) =& (\mathbb{V}^A D_A f)F^2 + 2F(3F^3)f + \frac{\mathbb{V}^A D_A h}{3}\\ 60F^6 =& (\mathbb{V}^A D_A f)F^2 + 6F^4 f + \frac{\mathbb{V}^A D_A h}{3}.\end{aligned}$$

Now multiplying (2.17) by $12F^2$ throughout gives

$$60F^6 = 12fF^4 + 4hF^2,$$

and so we obtain

$$12fF^4 + 4hF^2 = (\mathbb{V}^A D_A f)F^2 + 6F^4 f + \frac{\mathbb{V}^A D_A h}{3},$$

or that

$$6fF^4 + (4h - \mathbb{V}^A D_A f)F^2 - \frac{\mathbb{V}^A D_A h}{3} = 0.$$

Again using

$$6F^4 f = \frac{6}{5}f^2F^2 + \frac{2}{5}hf,$$

obtained from (2.17), we get

$$\left(\frac{6}{5}f^2 + 4h - \mathbb{V}^A D_A f\right)F^2 + \left(\frac{2}{5}hf - \frac{\mathbb{V}^A D_A h}{3}\right) = 0.$$

Multiplying throughout by 3, we obtain

$$\left(\frac{18}{5}f^2 + 12h - 3\mathbb{V}^A D_A f\right)F^2 + \left(\frac{6}{5}hf - \mathbb{V}^A D_A h\right) = 0. \tag{3.14}$$

A computation shows that

$$\begin{aligned}\mathbb{V}^A D_A f &= Y^a \nabla_a f - \phi f,\\ \mathbb{V}^A D_A h &= Y^a \nabla_a h - 2\phi h.\end{aligned}$$

Hence

$$\begin{aligned}\frac{18}{5}f^2 + 12h - 3\mathbb{V}^A D_A f = \frac{18}{5}f^2 + 12h - 3(Y^a\nabla_a f - \phi f) &= \frac{18}{5}f^2 + 12h - 3Y^a\nabla_a f + 3\phi f\\ &= k,\\ \frac{6}{5}hf - \mathbb{V}^A D_A h = \frac{6}{5}hf - (Y^a\nabla_a h - 2\phi h) &= \frac{6}{5}hf - Y^a\nabla_a h + 2\phi h\\ &= m,\end{aligned}$$

and (3.14) becomes (2.20). The desired obstructions (2.23) and (2.24) can be obtained analogously.

Chapter 4

Projective surfaces admitting pEW and web geometry

In this chapter we discuss in greater detail the relationship between projective surfaces with skew-symmetric Ricci tensor and its characterisation in terms of second order ordinary differential equations (ODEs) using ideas in [8]. We also discuss the relationship between solutions to (1.4) and 3-webs (and its extension to Veronese webs) on the surface. Most of the material in this chapter will overlap with [22] and existing literature on 3-webs such as [17]. Let us first recall the relationship between 2-dimensional projective structures and second order ODEs. Details can be found for instance in [4].

4.0.1 Deriving the second order ODE from the geodesic equation

Given a projective structure $[\nabla]$ on M^2, we can associate to it a second order ODE as follows. Let t be an affine parameter defining the geodesic curve $\gamma(t)$ on M^2 via

$$t \mapsto \gamma(t) = (x(t), y(t)).$$

We can write the geodesic equations as

$$\begin{aligned} \ddot{x} + \Gamma^1_{11}\dot{x}\dot{x} + 2\Gamma^1_{12}\dot{x}\dot{y} + \Gamma^1_{22}\dot{y}\dot{y} = 0, \\ \ddot{y} + \Gamma^2_{11}\dot{x}\dot{x} + 2\Gamma^2_{12}\dot{x}\dot{y} + \Gamma^2_{22}\dot{y}\dot{y} = 0, \end{aligned}$$

from which we can eliminate the parameter t to obtain a second order ODE at most cubic in its first derivatives associated to a projective structure. It is given by

$$\frac{d^2y}{dx^2} = \Gamma^1_{22}\left(\frac{dy}{dx}\right)^3 + (2\Gamma^1_{12} - \Gamma^2_{22})\left(\frac{dy}{dx}\right)^2 + (\Gamma^1_{11} - 2\Gamma^2_{12})\left(\frac{dy}{dx}\right) - \Gamma^2_{11}.$$

We can label

$$A_3 = \Gamma^1_{22}, \qquad A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, \qquad A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, \qquad A_0 = -\Gamma^2_{11}$$

to obtain

$$\frac{d^2y}{dx^2} = A_3\left(\frac{dy}{dx}\right)^3 + A_2\left(\frac{dy}{dx}\right)^2 + A_1\left(\frac{dy}{dx}\right) + A_0. \tag{4.1}$$

Conversely, let $y = y(x)$. If we further normalise the connection coefficients to be trace-free in this coordinate system, then a second order ODE of the form (4.1) where $A_0 = A_0(x,y)$, $A_1 = A_1(x,y)$, $A_2 = A_2(x,y)$, $A_3 = A_3(x,y)$ defines a projective structure with the 6 components of the trace-free symmetric connection coefficients Π^c_{ab} given by

$$\Pi^1_{11} = \frac{1}{3}A_1, \;\; \Pi^1_{12} = \frac{1}{3}A_2, \;\; \Pi^1_{22} = A_3, \;\; \Pi^2_{11} = -A_0, \;\; \Pi^2_{21} = -\frac{1}{3}A_1, \;\; \Pi^2_{22} = -\frac{1}{3}A_2. \tag{4.2}$$

The connection coefficients Π^c_{ab} depend on the chosen coordinate system (in this case (x,y)) but have the property of being independent of the choice of connection in the projective class. Furthermore we have chosen the normalisation so that $\Pi^a_{ab} = 0$ is satisfied.

Remark 4.1. If we choose a volume form ϵ_{bc} to be constant in the chosen coordinates and impose the normalisation condition that $\nabla_a \epsilon_{bc} = 0$ for $\nabla \in [\nabla]$, then the connection coefficients Γ^c_{ab} are obliged to be trace-free, since

$$0 = \nabla_a \epsilon_{bc} = \partial_a \epsilon_{bc} - \Gamma^d_{ab}\epsilon_{dc} - \Gamma^d_{ac}\epsilon_{bd} = -\Gamma^d_{ab}\epsilon_{dc} - \Gamma^d_{ac}\epsilon_{bd}$$

and so the connection coefficients have to satisfy

$$\Gamma^1_{11} = -\Gamma^2_{12}, \qquad \Gamma^1_{21} = -\Gamma^2_{22}.$$

We use Π^c_{ab} to denote the trace-free part of Γ^c_{ab}; we can check that

$$\Pi^c_{ab} = \Gamma^c_{ab} - \frac{1}{3}\Gamma^d_{ad}\delta^c_b - \frac{1}{3}\Gamma^d_{bd}\delta^c_a.$$

The connection coefficients Γ^c_{ab} are related to Π^c_{ab} by a projective transformation.

Remark 4.2. In [21] it was mentioned that given a holomorphic projective structure in 2 dimensions, we can choose $x = c$, where c is constant, to be geodesic, to get rid of A_3. Further, one can use geodesic polar coordinates to get rid of A_2. However, we are unable to reproduce this aforementioned normalisation procedure. Instead, choosing the coordinate functions given by $x = c$ and $y = d$ to be constant, so that the level sets are geodesics, gives $\dot{x} = \dot{y} = 0$, which forces $\Gamma^1_{22} = A_3 = 0$ and $\Gamma^2_{11} = -A_0 = 0$ in the geodesic equations.

4.1 The pEW equation in 2 dimensions

In this section we shall fix a coordinate system (x, y) and give the pEW equation (1.4) as a system of 3 equations on 3 unknowns involving quantites from the projective structure, such as the Ricci tensor, in this coordinate system. The Ricci tensor, which is identically the projective Schouten tensor, can be explicitly computed from the connection coefficients using that

$$\mathrm{P}_{ab} = \partial_c \Pi^c_{ab} - \partial_a \Pi^c_{cb} + \Pi^c_{ce}\Pi^e_{ab} - \Pi^c_{ae}\Pi^e_{cb} = \partial_c \Pi^c_{ab} - \Pi^c_{ae}\Pi^e_{cb}.$$

This gives us

$$\begin{aligned}
\mathrm{P}_{11} &= -\frac{\partial A_0}{\partial y} + \frac{1}{3}\frac{\partial A_1}{\partial x} - \frac{2}{9}A_1^2 + \frac{2}{3}A_0 A_2,\\
\mathrm{P}_{22} &= \frac{\partial A_3}{\partial x} - \frac{1}{3}\frac{\partial A_2}{\partial y} - \frac{2}{9}A_2^2 + \frac{2}{3}A_1 A_3,\\
\mathrm{P}_{12} = \mathrm{P}_{21} &= \frac{1}{3}\frac{\partial A_2}{\partial x} - \frac{1}{3}\frac{\partial A_1}{\partial y} - \frac{1}{9}A_1 A_2 + A_0 A_3.
\end{aligned}$$

To obtain (1.4), let $\alpha = Pdx + Qdy$ where $P = P(x, y)$ and $Q = Q(x, y)$. Following conventions in [4], we have

$$\nabla^{\Pi}_a \alpha_b = \partial_a \alpha_b - \Pi^c_{ab}\alpha_c.$$

The projective Einstein-Weyl equation becomes

$$\nabla^{\Pi}_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} = \partial_{(a}\alpha_{b)} - \Pi^c_{ab}\alpha_c + \alpha_a\alpha_b + \mathrm{P}_{ab} = 0,$$

which is equivalent to the following system of equations:

$$\begin{aligned}
\frac{\partial P}{\partial x} - \Pi^1_{11}P - \Pi^2_{11}Q + P^2 + \mathrm{P}_{11} &= 0,\\
\frac{\partial Q}{\partial y} - \Pi^1_{22}P - \Pi^2_{22}Q + Q^2 + \mathrm{P}_{22} &= 0,\\
\frac{1}{2}\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial P}{\partial y} - \Pi^1_{12}P - \Pi^2_{12}Q + QP + \mathrm{P}_{12} &= 0.
\end{aligned}$$

Introducing

$$F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and substituting the quantities from (4.2) we find that (1.4) becomes

$$\begin{aligned}
\frac{\partial P}{\partial x} - \frac{1}{3}A_1 P + A_0 Q + P^2 - \frac{\partial A_0}{\partial y} + \frac{1}{3}\frac{\partial A_1}{\partial x} - \frac{2}{9}A_1^2 + \frac{2}{3}A_0 A_2 = 0,\\
\frac{\partial Q}{\partial y} - A_3 P + \frac{1}{3}A_2 Q + Q^2 + \frac{\partial A_3}{\partial x} - \frac{1}{3}\frac{\partial A_2}{\partial y} - \frac{2}{9}A_2^2 + \frac{2}{3}A_1 A_3 = 0,\\
\frac{\partial Q}{\partial x} - \frac{1}{3}A_2 P + \frac{1}{3}A_1 Q + QP + \frac{1}{3}\frac{\partial A_2}{\partial x} - \frac{1}{3}\frac{\partial A_1}{\partial y} - \frac{1}{9}A_1 A_2 + A_0 A_3 = \frac{1}{2}F.
\end{aligned}$$

This system of equations is overdetermined and generically admits no non-trivial solution.

Proposition 4.3. *The second order ODE of the form*

$$\frac{d^2 y}{dx^2} = -\frac{\partial \phi}{\partial y}\left(\frac{dy}{dx}\right)^2 - \frac{\partial \phi}{\partial x}\left(\frac{dy}{dx}\right) \tag{4.3}$$

where $\phi = \phi(x, y)$ is smooth and $\frac{\partial^2 \phi}{\partial x \partial y} \neq 0$ defines a projective structure $(M^2, [\nabla])$ that has skew-symmetric Ricci tensor. In particular, $(M^2, [\nabla])$ admits a solution α to the projective Einstein-Weyl equation with α given locally by

$$\alpha = \frac{1}{3}\frac{\partial \phi}{\partial x}dx - \frac{1}{3}\frac{\partial \phi}{\partial y}dy.$$

Proof. For projective structures with $A_0 = A_3 = 0$, $A_1 = -\frac{\partial \phi}{\partial x}$ and $A_2 = -\frac{\partial \phi}{\partial y}$, we have the symmetric connections given by

$$\Pi_{11}^1 = -\frac{1}{3}\frac{\partial \phi}{\partial x}, \quad \Pi_{12}^1 = -\frac{1}{3}\frac{\partial \phi}{\partial y}, \quad \Pi_{21}^2 = \frac{1}{3}\frac{\partial \phi}{\partial x}, \quad \Pi_{22}^2 = \frac{1}{3}\frac{\partial \phi}{\partial y}, \quad \Pi_{22}^1 = \Pi_{11}^2 = 0,$$

and the components of the Ricci tensor are given by

$$\begin{aligned}
\mathrm{P}_{11} &= -\frac{1}{3}\frac{\partial^2 \phi}{\partial x^2} - \frac{2}{9}\left(\frac{\partial \phi}{\partial x}\right)^2,\\
\mathrm{P}_{22} &= \frac{1}{3}\frac{\partial^2 \phi}{\partial y^2} - \frac{2}{9}\left(\frac{\partial \phi}{\partial y}\right)^2,\\
\mathrm{P}_{12} = \mathrm{P}_{21} &= -\frac{1}{3}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{1}{3}\frac{\partial^2 \phi}{\partial y \partial x} - \frac{1}{9}\left(-\frac{\partial \phi}{\partial x}\right)\left(-\frac{\partial \phi}{\partial y}\right)\\
&= -\frac{1}{9}\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right).
\end{aligned}$$

For $P = \frac{1}{3}\frac{\partial\phi}{\partial x}$ and $Q = -\frac{1}{3}\frac{\partial\phi}{\partial y}$, it can be verified easily that

$$\begin{aligned}\frac{\partial P}{\partial x} - \Pi^1_{11}P - \Pi^2_{11}Q + P^2 + \mathrm{P}_{11} =& \frac{1}{3}\frac{\partial^2\phi}{\partial x^2} - \left(-\frac{1}{3}\frac{\partial\phi}{\partial x}\right)\left(\frac{1}{3}\frac{\partial\phi}{\partial x}\right) + \left(\frac{1}{3}\frac{\partial\phi}{\partial x}\right)^2 + \mathrm{P}_{11}\\ =& \frac{1}{3}\frac{\partial^2\phi}{\partial x^2} + \frac{2}{9}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{3}\frac{\partial^2\phi}{\partial x^2} - \frac{2}{9}\left(\frac{\partial\phi}{\partial x^2}\right)^2\\ =& 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial Q}{\partial y} - \Pi^1_{22}P - \Pi^2_{22}Q + Q^2 + \mathrm{P}_{22} =& -\frac{1}{3}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{1}{3}\frac{\partial\phi}{\partial y}\right)\left(-\frac{1}{3}\frac{\partial\phi}{\partial y}\right) + \left(-\frac{1}{3}\frac{\partial\phi}{\partial y}\right)^2 + \mathrm{P}_{22}\\ =& -\frac{1}{3}\frac{\partial^2\phi}{\partial y^2} + \frac{2}{9}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{3}\frac{\partial^2\phi}{\partial y^2} - \frac{2}{9}\left(\frac{\partial\phi}{\partial y}\right)^2\\ =& 0,\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial P}{\partial y} - \Pi^1_{12}P - \Pi^2_{12}Q + QP + \mathrm{P}_{12} =& -\frac{1}{6}\frac{\partial^2\phi}{\partial x\partial y} + \frac{1}{6}\frac{\partial^2\phi}{\partial y\partial x} - \left(-\frac{1}{3}\frac{\partial\phi}{\partial y}\right)\left(\frac{1}{3}\frac{\partial\phi}{\partial x}\right)\\ &- \left(\frac{1}{3}\frac{\partial\phi}{\partial x}\right)\left(-\frac{1}{3}\frac{\partial\phi}{\partial y}\right) + \left(-\frac{1}{3}\frac{\partial\phi}{\partial y}\right)\left(\frac{1}{3}\frac{\partial\phi}{\partial x}\right)\\ &+ \mathrm{P}_{12}\\ =& \frac{1}{9}\left(\frac{\partial\phi}{\partial y}\right)\left(\frac{\partial\phi}{\partial x}\right) - \frac{1}{9}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\\ =& 0,\end{aligned}$$

or that the projective Einstein-Weyl equations hold. □

Proposition 4.4. *Let $(M^2, [\nabla])$ be a projective surface that admits a solution to (1.4), with F nowhere vanishing. Then there exists local coordinates such that the projective structure is given by a second order ODE of the form*

$$\frac{d^2y}{dx^2} = -\frac{\partial\phi}{\partial y}\left(\frac{dy}{dx}\right)^2 - \frac{\partial\phi}{\partial x}\left(\frac{dy}{dx}\right)$$

where $\phi = \phi(x, y)$ is smooth and $\frac{\partial^2\phi}{\partial x\partial y} \neq 0$.

Proof. By assumption, since M^2 admits a solution to (1.4), there exists an affine connection $D \in [\nabla]$ such that the curvature of D is given by

$$\begin{aligned}
R^{D\,c}_{ab\;\;d} =& -2F_{ab}\delta_d{}^c - F_{ad}\delta_b{}^c + F_{bd}\delta_a{}^c \\
=& -F\epsilon_{ab}\delta_d{}^c + \frac{1}{2}F\epsilon_{bd}\delta_a{}^c - \frac{1}{2}F\epsilon_{ad}\delta_b{}^c \\
=& F\left(\frac{1}{2}\epsilon_{bd}\delta_a{}^c - \frac{1}{2}\epsilon_{ad}\delta_b{}^c - \epsilon_{ab}\delta_d{}^c\right) \\
=& -\frac{3}{2}F\epsilon_{ab}\delta_d{}^c.
\end{aligned}$$

The Ricci tensor of this connection D is given by

$$R^D_{ab} = R^{D\,c}_{ca\;\;b} = -\frac{3}{2}F\epsilon_{ca}\delta_b{}^c = -\frac{3}{2}F\epsilon_{ba},$$

which is skew. By Theorem 6.1 of [8], at any point $p \in M$, there exists local coordinates around p such that the non-vanishing connection symbols of D are given by $\Gamma^1_{11} = -\frac{\partial\phi}{\partial x}$ and $\Gamma^2_{22} = -\frac{\partial\phi}{\partial y}$ for some ϕ with $\frac{\partial^2\phi}{\partial x\partial y} \neq 0$. Using these coordinates we can write down the ODE

$$\frac{d^2y}{dx^2} = -\frac{\partial\phi}{\partial y}\left(\frac{dy}{dx}\right)^2 - \frac{\partial\phi}{\partial x}\left(\frac{dy}{dx}\right)$$

which gives rise to a projective structure on M^2. Since $D \in [\nabla]$ is a representative of the projective structure, this second order ODE has to be the projective structure of M^2, written in special local coordinates. $\square$

4.1.1 The Cotton-York tensor in 2 dimensions and constraints to pEW

In two dimensions, the Cotton-York tensor in the chosen local coordinate system is given by

$$Y_{abc} = \nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac} = \partial_a \mathrm{P}_{bc} - \Pi^e_{ab}\mathrm{P}_{ec} - \Pi^e_{ac}\mathrm{P}_{be} - \partial_b \mathrm{P}_{ac} + \Pi^e_{ba}\mathrm{P}_{ec} + \Pi^e_{bc}\mathrm{P}_{ae}.$$

In particular,

$$Y_{121} = \partial_1 \mathrm{P}_{21} - \Pi^1_{11}\mathrm{P}_{21} - \Pi^2_{11}\mathrm{P}_{22} - \partial_2 \mathrm{P}_{11} + \Pi^1_{21}\mathrm{P}_{11} + \Pi^2_{21}\mathrm{P}_{21}$$

and

$$Y_{122} = \partial_1 \mathrm{P}_{22} - \Pi^1_{12}\mathrm{P}_{21} - \Pi^2_{12}\mathrm{P}_{22} - \partial_2 \mathrm{P}_{12} + \Pi^1_{22}\mathrm{P}_{11} + \Pi^2_{22}\mathrm{P}_{21}.$$

We shall relate the Cotton-York tensor to the relative invariant $Y = (L_1 dx + L_2 dy) \otimes (dx \wedge dy)$ found in [4] (due to Liouville and Tresse). A computation gives

$$
\begin{aligned}
&Y_{121}\\
=&\frac{\partial}{\partial x}\left(\frac{1}{3}\frac{\partial A_2}{\partial x}-\frac{1}{3}\frac{\partial A_1}{\partial y}-\frac{1}{9}A_1A_2+A_0A_3\right)-\frac{1}{3}A_1\left(\frac{1}{3}\frac{\partial A_2}{\partial x}-\frac{1}{3}\frac{\partial A_1}{\partial y}-\frac{1}{9}A_1A_2+A_0A_3\right)\\
&+A_0\left(\frac{\partial A_3}{\partial x}-\frac{1}{3}\frac{\partial A_2}{\partial y}-\frac{2}{9}A_2^2+\frac{2}{3}A_1A_3\right)-\frac{\partial}{\partial y}\left(-\frac{\partial A_0}{\partial y}+\frac{1}{3}\frac{\partial A_1}{\partial x}-\frac{2}{9}A_1^2+\frac{2}{3}A_0A_2\right)\\
&+\frac{1}{3}A_2\left(-\frac{\partial A_0}{\partial y}+\frac{1}{3}\frac{\partial A_1}{\partial x}-\frac{2}{9}A_1^2+\frac{2}{3}A_0A_2\right)-\frac{1}{3}A_1\left(\frac{1}{3}\frac{\partial A_2}{\partial x}-\frac{1}{3}\frac{\partial A_1}{\partial y}-\frac{1}{9}A_1A_2+A_0A_3\right)\\
=&-\frac{2}{3}\frac{\partial^2 A_1}{\partial x\partial y}+\frac{1}{3}\frac{\partial^2 A_2}{\partial x^2}+\frac{\partial^2 A_0}{\partial y^2}-A_0\frac{\partial A_2}{\partial y}-A_2\frac{\partial A_0}{\partial y}\\
&+A_3\frac{\partial A_0}{\partial x}+2A_0\frac{\partial A_3}{\partial x}+\frac{2}{3}A_1\frac{\partial A_1}{\partial y}-\frac{1}{3}A_1\frac{\partial A_2}{\partial x}\\
=&-L_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&Y_{122}\\
=&\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial x}-\frac{1}{3}\frac{\partial A_2}{\partial y}-\frac{2}{9}A_2^2+\frac{2}{3}A_1A_3\right)-\frac{1}{3}A_2\left(\frac{1}{3}\frac{\partial A_2}{\partial x}-\frac{1}{3}\frac{\partial A_1}{\partial y}-\frac{1}{9}A_1A_2+A_0A_3\right)\\
&+A_3\left(-\frac{\partial A_0}{\partial y}+\frac{1}{3}\frac{\partial A_1}{\partial x}-\frac{2}{9}A_1^2+\frac{2}{3}A_0A_2\right)-\frac{\partial}{\partial y}\left(\frac{1}{3}\frac{\partial A_2}{\partial x}-\frac{1}{3}\frac{\partial A_1}{\partial y}-\frac{1}{9}A_1A_2+A_0A_3\right)\\
&-\frac{1}{3}A_2\left(\frac{1}{3}\frac{\partial A_2}{\partial x}-\frac{1}{3}\frac{\partial A_1}{\partial y}-\frac{1}{9}A_1A_2+A_0A_3\right)+\frac{1}{3}A_1\left(\frac{\partial A_3}{\partial x}-\frac{1}{3}\frac{\partial A_2}{\partial y}-\frac{2}{9}A_2^2+\frac{2}{3}A_1A_3\right)\\
=&-\frac{2}{3}\frac{\partial^2 A_2}{\partial x\partial y}+\frac{1}{3}\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_3}{\partial x^2}+A_3\frac{\partial A_1}{\partial x}+A_1\frac{\partial A_3}{\partial x}\\
&-A_0\frac{\partial A_3}{\partial y}-2A_3\frac{\partial A_0}{\partial y}-\frac{2}{3}A_2\frac{\partial A_2}{\partial x}+\frac{1}{3}A_2\frac{\partial A_1}{\partial y}\\
=&-L_2.
\end{aligned}
$$

Using the convention

$$Y_a = \epsilon^{bc} Y_{bca},$$

we have

$$
\begin{aligned}
Y_1 &= 2Y_{121} = -2L_1,\\
Y_2 &= 2Y_{122} = -2L_2.
\end{aligned}
$$

The differential consequence (2.3) reads

$$\frac{\partial F}{\partial x} = -3FP + 2L_1, \tag{4.4}$$

$$\frac{\partial F}{\partial y} = -3FQ + 2L_2, \tag{4.5}$$

and the constraint equation (2.6) is

$$-2\frac{\partial L_2}{\partial x}+2\frac{\partial L_1}{\partial y}+6L_1Q-6L_2P+3F^2=0. \tag{4.6}$$

It can be verified that the solution α to pEW from Proposition 4.3 satisfies (4.4), (4.5) and (4.6).

4.2 Projective version of Derdzinski's correspondence

In this section we give an analogue of Derdzinski's correspondence in Corollary 4.2 from [8] which highlights studying affine connections on surfaces with skew symmetric Ricci in the projective setting. We remark that projective flatness is used in a different sense in [8] compared to this thesis. Recall Corollary 4.2 from [8]:

Corollary 4.5. *Given a smooth surface Σ, the assignment $(\nabla,\alpha_a)\mapsto(D,\alpha_a)$, where $D_a v^b = \nabla_a v^b + \frac{3}{2}\alpha_a v^b$ defines a 1-1 correspondence between the set of pairs (∇,α_a) consisting of a torsion-free affine connection ∇ on Σ along with a 1-form α_a such that $\nabla_{[a}\alpha_{b]} = F_{ab}$ is the Ricci tensor of ∇, and the set of pairs (D,α_a) consisting of any flat connection on Σ with torsion tensor given by $T_{ab}{}^c = -3\alpha_{[a}\delta_{b]}{}^c$.*

The projective version of Derdzinski's correspondence can be formulated as follows:

Lemma 4.6. *On projective surfaces $(M,[\nabla])$, the assignment*

$$\alpha_a \mapsto D_aV^b = \nabla_aV^b+\frac{1}{2}\alpha_aV^b-\delta_a{}^b\alpha_cV^c,$$

for an arbitrary vector field $V^a\in\mathcal{E}^a(-\frac{3}{2})$ defines a 1-1 correspondence between solutions of the projective Einstein-Weyl (pEW) equation, and the set of flat connections on the bundle $\mathcal{E}^a(-\frac{3}{2})$.

Proof. Fix a connection $\nabla\in[\nabla]$. We would like to show

$$\left\{\begin{matrix}\text{Solutions of pEW equation}\\ \text{up to projective change of connection}\end{matrix}\right\}\xleftrightarrow{1-1}\left\{\text{flat connections on }\mathcal{E}^a\left(-\frac{3}{2}\right)\right\}$$

$\Rightarrow$. Suppose the surface M admits a solution α_a to the pEW equation given by

$$\nabla_a \alpha_b + \alpha_a \alpha_b + \mathrm{P}_{ab} = F_{ab}.$$

This means that the projectively related connection on TM given by

$$\widehat{\nabla}_a v^b = \nabla_a v^b - \alpha_a v^b - \delta_a{}^b \alpha_c v^c$$

for $v^b \in \Gamma(TM)$ has skew-symmetric Ricci tensor. Define a connection D_a on the bundle $\mathcal{E}^a(-\frac{3}{2}) = \mathcal{E}(-\frac{3}{2}) \otimes TM$ by

$$D_a V^b = \nabla_a V^b + \frac{1}{2}\alpha_a V^b - \delta_a{}^b \alpha_c V^c$$

for $V^a \in \Gamma(\mathcal{E}^a(-\frac{3}{2}))$. This is precisely the connection induced by $\widehat{\nabla}$ on $\mathcal{E}^a(-\frac{3}{2})$. A computation shows that

$$\begin{aligned}
D_a D_b V^c =& \nabla_a(D_b V^c) + \frac{3}{2}\alpha_a(D_b V^c) - \delta_a{}^c(D_b V^d)\alpha_d + \alpha_b(D_a V^c)\\
=& \nabla_a(\nabla_b V^c + \frac{1}{2}\alpha_b V^c - \delta_b{}^c \alpha_d V^d) + \frac{3}{2}\alpha_a(\nabla_b V^c + \frac{1}{2}\alpha_b V^c - \delta_b{}^c \alpha_d V^d)\\
&- \delta_a{}^c(\nabla_b V^d + \frac{1}{2}\alpha_b V^d - \delta_b{}^d \alpha_e V^e)\alpha_d + \alpha_b(\nabla_a V^c + \frac{1}{2}\alpha_a V^c - \delta_a{}^c \alpha_d V^d)\\
=& \nabla_a \nabla_b V^c + \frac{1}{2}(\nabla_a \alpha_b)V^c + \frac{1}{2}\alpha_b(\nabla_a V^c) - \delta_b{}^c(\nabla_a \alpha_d)V^d - \delta_b{}^c \alpha_d(\nabla_a V^d)\\
&+ \frac{3}{2}\alpha_a \nabla_b V^c + \frac{3}{4}\alpha_a \alpha_b V^c - \frac{3}{2}\alpha_a \delta_b{}^c \alpha_d V^d\\
&- \delta_a{}^c \alpha_d \nabla_b V^d - \delta_a{}^c \frac{1}{2}\alpha_b \alpha_d V^d + \delta_a{}^c \alpha_b \alpha_e V^e\\
&+ \alpha_b \nabla_a V^c + \frac{1}{2}\alpha_a \alpha_b V^c - \delta_a{}^c \alpha_b \alpha_d V^d\\
=& \nabla_a \nabla_b V^c + \frac{1}{2}(\nabla_a \alpha_b)V^c + \frac{3}{2}\alpha_b(\nabla_a V^c) - \delta_b{}^c(\nabla_a \alpha_d)V^d - \delta_b{}^c \alpha_d(\nabla_a V^d)\\
&+ \frac{3}{2}\alpha_a \nabla_b V^c + \frac{5}{4}\alpha_a \alpha_b V^c - \frac{3}{2}\alpha_a \delta_b{}^c \alpha_d V^d - \delta_a{}^c \alpha_d \nabla_b V^d - \delta_a{}^c \frac{1}{2}\alpha_b \alpha_d V^d.
\end{aligned}$$

Therefore,

$$\begin{aligned}
R^D{}_{ab}{}^c{}_dV^d =&(D_aD_b - D_bD_a)V^c \\
=&\nabla_a\nabla_bV^c + \frac{1}{2}(\nabla_a\alpha_b)V^c + \frac{3}{2}\alpha_b(\nabla_aV^c) - \delta_b{}^c(\nabla_a\alpha_d)V^d - \delta_b{}^c\alpha_d(\nabla_aV^d) \\
&+ \frac{3}{2}\alpha_a\nabla_bV^c + \frac{5}{4}\alpha_a\alpha_bV^c - \frac{3}{2}\alpha_a\delta_b{}^c\alpha_dV^d - \delta_a{}^c\alpha_d\nabla_bV^d - \delta_a{}^c\frac{1}{2}\alpha_b\alpha_dV^d \\
&- \nabla_b\nabla_aV^c - \frac{1}{2}(\nabla_b\alpha_a)V^c - \frac{3}{2}\alpha_a(\nabla_bV^c) + \delta_a{}^c(\nabla_b\alpha_d)V^d + \delta_a{}^c\alpha_d(\nabla_bV^d) \\
&- \frac{3}{2}\alpha_b\nabla_aV^c - \frac{5}{4}\alpha_b\alpha_aV^c + \frac{3}{2}\alpha_b\delta_a{}^c\alpha_dV^d + \delta_b{}^c\alpha_d\nabla_aV^d + \delta_b{}^c\frac{1}{2}\alpha_a\alpha_dV^d \\
=&R_{ab}{}^c{}_dV^d + \frac{1}{2}(\nabla_a\alpha_b)V^c - \delta_b{}^c(\nabla_a\alpha_d)V^d - \delta_b{}^c\alpha_a\alpha_dV^d - \frac{1}{2}(\nabla_b\alpha_a)V^c \\
&+ \delta_a{}^c(\nabla_b\alpha_d)V^d + \delta_a{}^c\alpha_b\alpha_dV^d \\
=&F_{ab}V^c - \delta_b{}^c(\nabla_a\alpha_d + \alpha_a\alpha_d - \mathrm{P}_{ad})V^d + \delta_a{}^c(\nabla_b\alpha_d + \alpha_b\alpha_d + \mathrm{P}_{bd})V^d.
\end{aligned}$$

Since pEW holds, we have

$$R^D{}_{ab}{}^c{}_dV^d = F_{ab}V^c - \delta_b{}^cF_{ad}V^d + \delta_a{}^cF_{bd}V^d \equiv 0,$$

so that the connection is flat on the bundle $\mathcal{E}^a(-\frac{3}{2})$. $\Leftarrow$. Conversely, given any torsion-free affine connection ∇ in the projective class, it induces a connection D on the bundle $\mathcal{E}^a(-\frac{3}{2})$. Suppose we have a flat connection D_a on the bundle $\mathcal{E}^a(-\frac{3}{2})$. Then

$$0 = R^D{}_{ab}{}^c{}_dV^d = F_{ab}V^c - \delta_b{}^c(\nabla_a\alpha_d + \alpha_a\alpha_d - \mathrm{P}_{ad})V^d + \delta_a{}^c(\nabla_b\alpha_d + \alpha_b\alpha_d + \mathrm{P}_{bd})V^d$$

must hold for any section V^a of $\mathcal{E}^a(-\frac{3}{2})$. Tracing the indices a and c gives

$$(\nabla_b\alpha_d + \alpha_b\alpha_d + \mathrm{P}_{bd} - F_{bd})V^d = 0.$$

Since V^a is arbitrary, the pEW equation

$$\nabla_b\alpha_d + \alpha_b\alpha_d + \mathrm{P}_{bd} - F_{bd} = 0$$

must hold. □

Remark 4.7. The flat connection induced by the projective structure on the bundle $\mathcal{E}^a(-\frac{3}{2})$ can be seen as an alternative route to establish Theorem 6.1 in [8], which we now proceed to show.

4.2.1 Obtaining Derdzinski's normal form for solutions to pEW equation

We now apply Derdzinski's argument to the projective setting of Lemma 4.6 to write down his normal form for solutions of the pEW equation. Choose linearly independent sections of $\mathcal{E}^a(-\frac{3}{2})$ written locally as $U^a = (U^1, U^2) = (e^\chi, 0)$ and $V^a = (V^1, V^2) = (0, e^\psi)$, where $\partial_a = (\partial_1, \partial_2) = (\partial_x, \partial_y) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ are local coordinates for the tangent vector fields adapted to the linearly independent sections and $\chi = \chi(x, y)$ and $\psi = \psi(x, y)$ are smooth functions. This gives $U^a\partial_a = e^\chi\partial_x$ and $V^a\partial_a = e^\psi\partial_y$. We now impose the condition that they are covariantly constant with respect to some affine connection D in the projective class:

$$D_aV^b = D_aU^b = 0.$$

This means that the system of 4 equations have to be satisfied for V^a (and U^a respectively)

$$\begin{aligned}
D_1V^1 =& \partial_1V^1 + \Gamma^1_{11}V^1 + \Gamma^1_{12}V^2 + \frac{1}{2}\alpha_1V^1 - \alpha_1V^1 - \alpha_2V^2 = 0,\\
D_1V^2 =& \partial_1V^2 + \Gamma^2_{11}V^1 + \Gamma^2_{12}V^2 + \frac{1}{2}\alpha_1V^2 = 0,\\
D_2V^1 =& \partial_2V^1 + \Gamma^1_{21}V^1 + \Gamma^1_{22}V^2 + \frac{1}{2}\alpha_2V^1 = 0,\\
D_2V^2 =& \partial_2V^2 + \Gamma^2_{21}V^1 + \Gamma^2_{22}V^2 + \frac{1}{2}\alpha_2V^2 - \alpha_1V^1 - \alpha_2V^2 = 0.
\end{aligned}$$

For our choice of local coordinates $(V^1, V^2) = (0, e^\psi\partial_y)$, this reduces to the equations

$$\begin{aligned}
\Gamma^1_{12} - \alpha_2 &= 0,\\
\partial_x\psi + \Gamma^2_{12} + \frac{1}{2}\alpha_1 &= 0,\\
\Gamma^1_{22} &= 0,\\
\partial_y\psi + \Gamma^2_{22} - \frac{1}{2}\alpha_2 &= 0.
\end{aligned}$$

Similarly, our choice of local coordinates for $(U^1, U^2) = (e^\chi\partial_x, 0)$ gives

$$\begin{aligned}
\partial_x\chi + \Gamma^1_{11} - \frac{1}{2}\alpha_1 &= 0,\\
\Gamma^2_{11} &= 0,\\
\partial_y\chi + \Gamma^1_{21} + \frac{1}{2}\alpha_2 &= 0,\\
\Gamma^2_{21} &= \alpha_1.
\end{aligned}$$

Let $\phi = \chi - \psi$. The projective structure then is given by

$$\begin{aligned}
\Gamma^2_{11} &= 0, \\
\Gamma^1_{22} &= 0, \\
\partial_x\phi = \partial_x\chi - \partial_x\psi = \alpha_1 - \Gamma^1_{11} + \Gamma^2_{12} &= 2\Gamma^2_{12} - \Gamma^1_{11}, \\
\partial_y\phi = \partial_y\chi - \partial_y\psi = -\Gamma^1_{21} - \alpha_2 + \Gamma^2_{22} &= \Gamma^2_{22} - 2\Gamma^1_{21}.
\end{aligned}$$

This gives rise to the second order ODE (4.3). In particular, from Proposition 4.3 we have α_a given by $\alpha = \alpha_1 dx + \alpha_2 dy = \Gamma^2_{21}dx + \Gamma^1_{12}dy$ a solution to pEW equation.

4.3 Derdzinski's example from first order ODE

In this section we show that Derdzinski's example (4.3) arises from the derivative of a first order ODE. We can use Lemma 3.2 in [22] to show that it is point equivalent to the derivative of a first order ODE. Consider the following first order differential equation

$$\frac{dy}{dx} = e^{-\phi(x,y)}. \tag{4.7}$$

Differentiating with respect to x gives

$$\begin{aligned}
\frac{d^2y}{dx^2} =& e^{-\phi(x,y)}\left(-\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y}\left(\frac{dy}{dx}\right)\right) \\
=& -\frac{\partial\phi}{\partial x}\left(\frac{dy}{dx}\right) - \frac{\partial\phi}{\partial y}\left(\frac{dy}{dx}\right)^2,
\end{aligned} \tag{4.8}$$

where in the last step we made the substitution given by (4.7). This is Derdzinski's normal form (4.3) and we can read off the projective structure given by

$$A_0 = 0, \qquad A_1 = -\partial_x\phi, \qquad A_2 = -\partial_y\phi, \qquad A_3 = 0$$

and this admits a solution to the pEW equation following Proposition 4.3. The solution is given by

$$\alpha = \frac{1}{3}\partial_x\phi dx - \frac{1}{3}\partial_y\phi dy$$

and

$$\begin{aligned}
d\alpha = \frac{1}{3}\partial_{yx}\phi dy \wedge dx - \frac{1}{3}\partial_{xy}\phi dx \wedge dy &= -\frac{2}{3}\partial_{xy}\phi dx \wedge dy \\
&= F dx \wedge dy,
\end{aligned}$$

where

$$F = -\frac{2}{3}\frac{\partial^2\phi}{\partial y \partial x}.$$

Remark 4.8. If we instead substituted

$$\frac{dy}{dx} = e^{-\phi(x,y)}$$

into (4.8), we obtain

$$\frac{d^2y}{dx^2} = -\frac{\partial\phi}{\partial x}e^{-\phi} - \frac{\partial\phi}{\partial y}e^{-2\phi}.$$

which we can regard as a projective structure with connection coefficients given by

$$A_0 = -\frac{\partial\phi}{\partial x}e^{-\phi} - \frac{\partial\phi}{\partial y}e^{-2\phi}, \qquad A_1 = 0, \qquad A_2 = 0, \qquad A_3 = 0.$$

This projective structure is not point equivalent to the projective structure given by (4.8) as they have different Liouville invariants in general. This observation shows that the same second order ODE can define different projective structures.

4.3.1 Different normal form example

In this section we show that the any second order ODE given by the derivative of a first order ODE admits a solution to pEW. Consider the following first order differential equation

$$\frac{dy}{dx} = G(x, y).$$

Differentiating with respect to x gives

$$\frac{d^2y}{dx^2} = \frac{\partial G}{\partial y}\left(\frac{dy}{dx}\right) + \frac{\partial G}{\partial x}. \tag{4.9}$$

We can read off the projective structure given by

$$A_0 = \partial_x G, \qquad A_1 = \partial_y G, \qquad A_2 = 0, \qquad A_3 = 0$$

and this admits a solution to the pEW equation. This is quite different from the normal form given by (4.3). The solution is given by

$$\alpha = \frac{2}{3}\partial_y G dx$$

and

$$d\alpha = \frac{2}{3}\partial_{yy} G dy \wedge dx = -\frac{2}{3}\partial_{yy} G dx \wedge dy = F dx \wedge dy,$$

where

$$F = -\frac{2}{3}\frac{\partial^2 G}{\partial y \partial y}.$$

Remark 4.9. If we make the further substitution

$$\frac{dy}{dx} = G(x,y)$$

back into (4.9), we obtain

$$\frac{d^2y}{dx^2} = G\partial_y G + \partial_x G$$

which we can regard as a projective structure with connection coefficients given by

$$A_0 = G\partial_y G + \partial_x G, \qquad A_1 = 0, \qquad A_2 = 0, \qquad A_3 = 0.$$

This projective structure is not point equivalent to the projective structure given by (4.9) and have different Liouville invariants in general. However, they share a 1-parameter family of solutions, namely the integral curves given by solutions of $\frac{dy}{dx} = G(x,y)$.

Conjecture A. *Suppose we have a projective structure given in local coordinates as*

$$\frac{d^2y}{dx^2} = A_1\left(\frac{dy}{dx}\right) + A_0.$$

The condition that there admit a solution to pEW for the given projective structure is that

$$\frac{\partial A_1}{\partial x} = \frac{\partial A_0}{\partial y}$$

must hold. As a consequence, we have

$$A_1 = \frac{\partial G}{\partial y} \qquad \text{and} \qquad A_0 = \frac{\partial G}{\partial x}$$

for some $G = G(x,y)$.

For this projective structure the equations we are trying to solve are

$$\begin{aligned} \frac{\partial P}{\partial x} - \frac{1}{3}A_1 P + A_0 Q + P^2 - \frac{\partial A_0}{\partial y} + \frac{1}{3}\frac{\partial A_1}{\partial x} - \frac{2}{9}A_1^2 =& 0, \\ \frac{\partial Q}{\partial y} + Q^2 =& 0, \\ \frac{\partial Q}{\partial x} + \frac{1}{3}A_1 Q + QP - \frac{1}{3}\frac{\partial A_1}{\partial y} =& \frac{1}{2}F. \end{aligned}$$

The evidence supporting this conjecture comes from taking $\alpha = \frac{2}{3}A_1 dx$ so that setting $P = \frac{2}{3}A_1$ and $Q = 0$, the pEW equation is satisfied iff

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_0}{\partial y} = 0$$

holds. We suspect that we can use the constraints from the pEW system to eliminate all other possible solutions of α.

Remark 4.10. Given a projective structure that admits skew-symmetric Ricci tensor, we can use Lemma 4.6 to write down locally the connection coefficients of the projective structure in Derdzinski's normal form (4.3). The second order ODE corresponding to the projective structure turns out to be point equivalent to the derivative of a first order ODE. See also Theorem 3.3 of [22].

4.4 Complex structure on the pEW surface

Fix $U^a, V^a \in \Gamma(\mathcal{E}^a(-\frac{3}{2}))$, so that $D_a U^b = D_a V^b = 0$. Introduce $\epsilon^{ab} = 2U^{[a}V^{b]}$ and observe that $D_a \epsilon^{bc} = 0$. Its inverse is given by ϵ_{ab} and we raise and lower indices according to $V^a = \epsilon^{ab} V_b$, $V_b = \epsilon_{ab} V^a$. Normalise so that

$$1 = \epsilon_{ab} U^a V^b.$$

We can write $\epsilon_{ab} = 2U_{[a}V_{b]}$.

Proposition 4.11. *The tensor $g_{ab} = 2U_{(a}V_{b)}$ defines a Lorentzian metric of signature $(1,1)$ on $\mathcal{E}^a(-\frac{3}{2})$, and is covariantly constant with respect to D_a.*

Proposition 4.12. *The tensor*

$$J_a{}^b = U_a U^b + V_a V^b$$

defines an almost complex structure on M.

We check that

$$J_a{}^b J_b{}^c = (U_a U^b + V_a V^b)(U_b U^c + V_b V^c) = -U_a V^c + V_a U^c = -\delta_a{}^c.$$

Proposition 4.13. *A pEW surface admits a compatible complex structure, i.e. there exists a projective connection D_a in its projective class such that*

$$D_a J_b{}^c = 0.$$

4.5 Twistor theory for holomorphic projective surfaces with skew-symmetric Ricci tensor

In the real-analytic setting, there exists a holomorphic projective structure on M^2 for which the real-analytic projective structure is the real slice. In particular, 2-dimensional projective structures in the holomorphic category admit a twistor theory. Mini-mini-twistor space, denoted $\mathcal{Z}$ is a complex surface with a 2-parameter family rational curves having normal bundle $\mathcal{O}(1)$. In [21], Hitchin showed that mini-mini-twistor spaces are in 1-1 correspondence with holomorphic 2-dimensional projective structures. The twistor theory for holomorphic projective structures with skew-symmetric Ricci tensor have been dealt with in the appendix of [6] and in [10] separately. We would like to give a brief summary of known results. Recall the following proposition of Dunajski-West in [10]:

Proposition 4.14. *There is a one to one correspondence between holomorphic* 2 *dimensional projective structures such that the corresponding second order ODE is point equivalent to the derivative of a first order ODE, and complex surfaces which contain a holomorphic rational curve with normal bundle* $\mathcal{O}(1)$ *and fiber holomorphically over* $\mathbb{CP}^1$.

Together with the observation that any projective surface admitting a skew-symmetric Ricci tensor has a projective structure whose corresponding second order ODE is point equivalent to the derivative of a first order ODE, we have the result, also found in [6]:

Proposition 4.15. *There is a one to one correspondence between holomorphic* 2 *dimensional projective structures admitting skew-symmetric Ricci tensor and complex surfaces which contain a holomorphic curve with normal bundle* $\mathcal{O}(1)$ *and fiber holomorphically over* $\mathbb{CP}^1$.

4.6 3-webs in 2 dimensions

We shall follow [18]. A 3-web W_3 on a smooth manifold M^2 is 3 distinct pairwise transverse foliations of smooth curves on M^2 in generic position, i.e. such that the tangent lines to the curves of different foliations through any point $p \in M$ are distinct and locally any 2 curves of transverse foliations have at most 1 common point. It can be encoded by three differential 1-forms ω_1, ω_2, ω_3 such that any 2 of them are linearly independent.

Proposition 4.16. *[18] There is a normalisation such that*

$$\omega_1 + \omega_2 + \omega_3 = 0. \tag{4.10}$$

In view of Lemma 4.6, the correct projective weight for each of the three 1-forms is $\frac{3}{2}$, so that $\omega_1, \omega_2, \omega_3$ should be regarded as sections of $\mathcal{E}_a(\frac{3}{2})$.

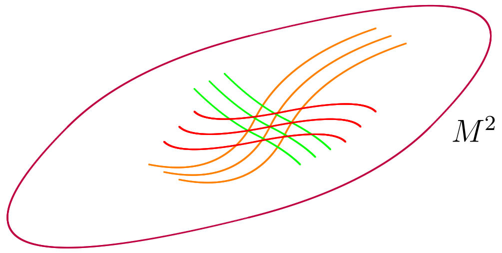

Figure 4.1: Example of a 3-web on surfaces

Proposition 4.17. *[18] Two normalisations* ω_1, ω_2, ω_3 *and* $\widehat{\omega}_1$, $\widehat{\omega}_2$, $\widehat{\omega}_3$ *determine the same* 3*-web if and only if*

$$\widehat{\omega}_1 = \Omega^{\frac{3}{2}}\omega_1, \quad \widehat{\omega}_2 = \Omega^{\frac{3}{2}}\omega_2, \quad \widehat{\omega}_3 = \Omega^{\frac{3}{2}}\omega_3.$$

For some non-zero smooth function $\Omega \in \mathcal{E}$*.*

We shall now follow [17] to derive the structure equations associated to a 3-web. Since the 1-forms ω_1, ω_2 and ω_3 are pairwise linearly independent on M^2, we have

$$d\omega_1 = \frac{3}{2}\,\alpha_1 \wedge \omega_1, \qquad d\omega_2 = \frac{3}{2}\alpha_2 \wedge \omega_2, \qquad d\omega_3 = \frac{3}{2}\alpha_3 \wedge \omega_3,$$

and we deduce from (4.10) that

$$\begin{aligned} 0 = \alpha_1 \wedge \omega_1 + \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3 =& \alpha_1 \wedge \omega_1 + \alpha_2 \wedge \omega_2 + \alpha_3 \wedge (-\omega_1 - \omega_2) \\ =& (\alpha_1 - \alpha_3) \wedge \omega_1 + (\alpha_2 - \alpha_3) \wedge \omega_2. \end{aligned}$$

We can therefore write

$$\begin{aligned} \alpha_1 - \alpha_3 =& a\omega_1 + b\omega_2, \\ \alpha_2 - \alpha_3 =& b\omega_1 + c\omega_2 \end{aligned}$$

for some functions a, b and c. We find that

$$\alpha_1 - (a-b)\omega_1 = \alpha_2 - (c-b)\omega_2 = \alpha_3 + b(\omega_1 + \omega_2)$$

and taking

$$\alpha := \alpha_1 - (a-b)\omega_1 = \alpha_2 - (c-b)\omega_2 = \alpha_3 + b(\omega_1 + \omega_2)$$

we obtain

$$d\omega_1 = \frac{3}{2}\,\alpha \wedge \omega_1, \tag{4.11a}$$

$$d\omega_2 = \frac{3}{2}\alpha \wedge \omega_2, \tag{4.11b}$$

$$d\omega_3 = \frac{3}{2}\alpha \wedge \omega_3. \tag{4.11c}$$

The 1-form α is the connection form for the web structure equations (4.11). Under a rescaling of the web as given by Proposition 4.17, equations (4.11) transform as

$$\begin{aligned} d\widehat{\omega}_1 = d(\Omega^{\frac{3}{2}}\omega_1) = \frac{3}{2}\Omega^{\frac{1}{2}}(d\Omega)\omega_1 + \Omega^{\frac{3}{2}}(d\omega_1) &= \frac{3}{2}\Omega^{\frac{3}{2}}(\Omega^{-1}d\Omega + \alpha)\wedge\omega_1 \\ &= \frac{3}{2}\widehat{\alpha}\wedge\widehat{\omega}_1, \\ d\widehat{\omega}_2 = d(\Omega^{\frac{3}{2}}\omega_2) = \frac{3}{2}\Omega^{\frac{1}{2}}(d\Omega)\omega_2 + \Omega^{\frac{3}{2}}(d\omega_2) &= \frac{3}{2}\Omega^{\frac{3}{2}}(\Omega^{-1}d\Omega + \alpha)\wedge\omega_2 \\ &= \frac{3}{2}\widehat{\alpha}\wedge\widehat{\omega}_2, \end{aligned}$$

where $\widehat{\alpha} = \alpha + \Upsilon$, where $\Upsilon = \Omega^{-1}d\Omega$. Now

$$\frac{3}{2}d\alpha = K\omega_1 \wedge \omega_2,$$

where K is the curvature of the web (or Chern) connection and is also known as the Blaschke curvature. Now

$$\frac{3}{2}d\alpha = \frac{3}{2}d\widehat{\alpha} = \widehat{K}\widehat{\omega}_1 \wedge \widehat{\omega}_2 = \Omega^3\widehat{K}\omega_1 \wedge \omega_2,$$

so that $\widehat{K} = \Omega^{-3}K$ and K has projective weight -3. Later on we shall relate K to the projective density F of weight -3, thus justifying our earlier assertion that the 1-forms ω_1, ω_2 and ω_3 are best viewed as having projective weight $\frac{3}{2}$. Observe that given a projective structure $[\nabla]$ on M, for the coframe (θ^a), and connection symbols Γ^a_{bc}, we have the connection 1-forms $\Gamma^a_b = \Gamma^a_{bc}\theta^c$. Let

$$\theta^1 \wedge \theta^2 = \frac{1}{2}\epsilon_{ij}\theta^i \wedge \theta^j$$

denote the web surface element, where $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The connection 1-forms (Γ^i_j) for a particular representative (torsion-free affine connection) in $[\nabla]$ give the structure equations

$$d\theta^a + \Gamma^a_b \wedge \theta^b = 0,$$

and curvature

$$\Omega^a{}_b = d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b$$

which defines the curvature tensor

$$\Omega^a{}_b = \frac{1}{2} R_{cd}{}^a{}_b \theta^c \wedge \theta^d.$$

Now take

$$\theta^1 = \omega_1, \theta^2 = \omega_2,$$

and compare it with the web structure equations (4.11). We obtain

$$\Gamma^a_b = -\frac{3}{2}\delta^a{}_b\alpha,$$

and

$$\Omega^a{}_b = -\frac{3}{2}\delta^a{}_b d\alpha = -K\theta^1 \wedge \theta^2 \delta^a{}_b = -\frac{K}{2}\epsilon_{cd}\theta^c \wedge \theta^d \delta^a{}_b,$$

so that

$$\frac{1}{2} R_{cd}{}^a{}_b = -\frac{K}{2}\epsilon_{cd}\delta^a{}_b,$$

or $K = \frac{3}{2}F$ in our convention. This gives

$$d\alpha = F\theta^1 \wedge \theta^2 = \frac{1}{2}F\epsilon_{ab}\theta^a \wedge \theta^b.$$

We see that any 3-web W_3 on M^2 defines a torsion-free affine (web or Chern) connection with skew-symmetric Ricci. This is due to Goldberg in [17]. It has also been shown in [17] and [18] that

Proposition 4.18. *The leaves of the foliations of the* 3*-web structure given by* $\{\omega_1 = 0\}$, $\{\omega_2 = 0\}$ *and* $\{\omega_3 = -\omega_1 - \omega_2 = 0\}$ *are unparameterised curves for the geodesics on* M^2*.*

Hence it is natural to consider the projective structure induced by the web connection. We have

Proposition 4.19. *Let* W_3 *be a smooth 3-Web structure on the surface* M*, with the web connection given by* ∇*. Then* $(M, [\nabla])$ *is a projective surface admitting a torsion-free affine connection* ∇ *with skew-symmetric Ricci.*

The projective structure $[\nabla]$ is the class of torsion-free affine connections projectively related to the web or Chern connection ∇. Conversely given an affine connection on M^2 with skew-symmetric Ricci, one can use Derdzinski's correspondence in Corollary 4.2 in [8] to obtain linearly independent 1-forms ω_1 and

ω_2 parallel with respect to D satisfying the web structure equations. The foliations given by $\{\omega_1 = 0\}$, $\{\omega_2 = 0\}$ and $\{\omega_3 = -\omega_1 - \omega_2 = 0\}$ then define a (local) 3-web on M^2. This generalises straightforwardly to the projective setting We have

Proposition 4.20. *Let $(M, [\nabla])$ be a projective surface admitting a torsion-free affine connection ∇ with skew-symmetric Ricci. Then there exists a 3-web structure on the surface, with the web connection given by ∇.*

Proof. Suppose M^2 admits a connection with skew-symmetric Ricci. Then by Lemma 4.6, there is a flat connection on the dual bundle $\mathcal{E}_a(\frac{3}{2})$ and in particular, there exist linearly independent non-zero sections μ_a, ν_b of $\mathcal{E}_a(\frac{3}{2})$ satisfying $D_a\mu_b = 0$ and $D_a\nu_b = 0$. In particular this is equivalent to μ_a, ν_a satisfying the equations

$$\nabla_a\mu_b = \frac{1}{2}\alpha_a\mu_b - \alpha_b\mu_a,$$
$$\nabla_a\nu_b = \frac{1}{2}\alpha_a\nu_b - \alpha_b\nu_a,$$

for some 1-form α_a yet to be determined. Skewing gives

$$\nabla_{[a}\mu_{b]} = \frac{3}{2}\alpha_{[a}\mu_{b]},$$
$$\nabla_{[a}\nu_{b]} = \frac{3}{2}\alpha_{[a}\nu_{b]},$$

which are the web structure equations (4.11) and so taking $\omega_1 = \mu_a$, $\omega_2 = \nu_a$, $\omega_3 = -\mu_a - \nu_a$ gives the desired three 1-forms, and the foliations $\{\omega_1 = 0\}$, $\{\omega_2 = 0\}$, $\{\omega_3 = 0\}$ define a 3-web on M^2. The web structure equations also give rise to α_a which determines the web connection ∇. $\square$

Note that with Propositions 4.19 and 4.20 we are not establishing a 1-1 correspondence between 3-webs in 2 dimensions and projective surfaces with skew-symmetric Ricci tensor. Certainly the map from 3-webs to projective surfaces with skew Ricci is surjective, however it is not clear whether two 3-web structures with the same web curvature are necessarily web diffeomorphic to each other. In view of Theorem 2.3 in [22], we may establish a 1-1 correspondence between Veronese webs in 2 dimensions up to a Möbius transformation with projective surfaces with skew-symmetric Ricci tensor.

4.7 3-webs and first order ODEs

In this section we start with a 3-web given locally by a web function f and write out the projective structure in the associated coordinate system. We then give a solution α_a to the pEW equation in this coordinate system. We formulate this section following Section 2.2 of [18], with ω_3 chosen so that the normalisation $d\omega_3 = 0$ holds. The results in this section can easily be generalised to Veronese webs, in view of Proposition 2.1 of [22]. In particular, we get the connection coefficients (4) in [22] associated a 3-web (where we denote f for the web function instead of w). Assuming that M^2 is a simply connected domain of $\mathbb{R}^2$, there exists a smooth function f such that ω_3 is proportional to df. i.e. $\omega_3 \wedge df = 0$. The function f is called a web function, defined up to renormalisation. Choose f such that

$$\omega_3 = df,$$

and similarly we find functions x, y for ω_1 and ω_2 respectively such that

$$\omega_1 = adx, \quad \omega_2 = bdy,$$

for some functions a, b to be determined. The functions x and y are linearly independent and can be viewed as local coordinates. In these coordinates, and with the normalization $\omega_1 + \omega_2 + \omega_3 = 0$, we have $a = -f_x$ and $b = -f_y$, so that

$$\omega_1 = -f_x dx, \quad \omega_2 = -f_y dy, \quad \omega_3 = df,$$

where $f_x = \frac{\partial f}{\partial x}$. Because of the linear independence criterion, $f_x f_y$ is nowhere vanishing. Foliations given by $\{\omega_1 = 0\}$, $\{\omega_2 = 0\}$, $\{\omega_3 = 0\}$ are geodesic with respect to the web (or Chern) connection, with the integral curves of $\{\omega_1 = 0\}$ and $\{\omega_2 = 0\}$ given locally by $x = c$, $y = d$ where c, d are constants. Recall that this choice of coordinates forces $A_3 = A_0 = 0$ in (4.1). The leaves of the foliations of $\{\omega_3 = 0\}$ are integral curves of the first order ODE

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Now

$$d\omega_3 = 0,$$

so that α is proportional to ω_3. Write $\alpha = \frac{2}{3} H \omega_3$ for some function H to be determined. Let ∂_1 and ∂_2 be the basis of vector fields dual to ω_1, ω_2, so that $\omega_i(\partial_j) = \delta_{ij}$. We have

$$\partial_1 = -\frac{1}{f_x}\frac{\partial}{\partial x}, \quad \partial_2 = -\frac{1}{f_y}\frac{\partial}{\partial y}.$$

Dually,

$$\omega_1 = -f_x dx, \quad \omega_2 = -f_y dy.$$

Let $X_1 = \partial_1$, $X_2 = \partial_2$, so that the vector fields $(X_a) = (X_1, X_2) = (\partial_1, \partial_2)$ form a frame on M, and $(\theta^a) = (\theta^1, \theta^2) = (\omega_1, \omega_2)$ form a coframe dual to (X_a), so that

$$\theta^a(X_b) = \delta_b{}^a.$$

The structure equations defining the connection coefficients are given by

$$\begin{aligned} \nabla_{X_a} X_b =& \Gamma^c_{ba} X_c, \\ \nabla_{X_a} \theta^b =& -\Gamma^b_{ca} \theta^c. \end{aligned}$$

The torsion tensor is given by

$$T(\partial_1, \partial_2) = \nabla_{\partial_1} \partial_2 - \nabla_{\partial_2} \partial_1 - [\partial_1, \partial_2].$$

The web structure equations define a torsion-free affine connection called a web connection, so that

$$[\partial_1, \partial_2] = \nabla_{\partial_1} \partial_2 - \nabla_{\partial_2} \partial_1.$$

We have

$$\begin{aligned} \theta^c([X_a, X_b]) = \theta^c(\nabla_{X_a} X_b - \nabla_{X_b} X_a) =& \theta^c(\nabla_{X_a} X_b) - \theta^c(\nabla_{X_b} X_a) \\ =& \theta^c(\Gamma^d_{ba} X_d) - \theta^c(\Gamma^d_{ab} X_d) \\ =& \Gamma^c_{ba} - \Gamma^c_{ab}. \end{aligned}$$

Since

$$\begin{aligned} \theta^1([\partial_1, \partial_2]) =& \omega_1([\partial_1, \partial_2]) \\ =& -d\omega_1(\partial_1, \partial_2) = -\frac{3}{2}(\alpha \wedge \omega_1)(\partial_1, \partial_2) = \frac{3}{2}\alpha(\partial_2), \\ \theta^2([\partial_1, \partial_2]) =& \omega_2([\partial_1, \partial_2]) \\ =& -d\omega_2(\partial_1, \partial_2) = -\frac{3}{2}(\alpha \wedge \omega_2)(\partial_1, \partial_2) = -\frac{3}{2}\alpha(\partial_1), \end{aligned}$$

we therefore obtain

$$\begin{aligned} \Gamma^1_{21} - \Gamma^1_{12} =& \frac{3}{2}\alpha(\partial_2), \\ \Gamma^2_{21} - \Gamma^2_{12} =& -\frac{3}{2}\alpha(\partial_1). \end{aligned}$$

Now to determine the connection coefficients, we compute

$$\begin{aligned} \nabla_{\partial_1} \theta^1 =& -\Gamma^1_{11}\theta^1 - \Gamma^1_{21}\theta^2, \\ \nabla_{\partial_2} \theta^2 =& -\Gamma^2_{12}\theta^1 - \Gamma^2_{22}\theta^2, \\ \nabla_{\partial_2} \theta^1 =& -\Gamma^1_{12}\theta^1 - \Gamma^1_{22}\theta^2, \\ \nabla_{\partial_1} \theta^2 =& -\Gamma^2_{11}\theta^1 - \Gamma^2_{21}\theta^2. \end{aligned}$$

Since the connection forms are given by $\Gamma^i_j = -\frac{3}{2}\delta_j{}^i\alpha$, we have

$$\Gamma^2_{11} = \Gamma^1_{22} = 0,$$

and using $\alpha = \frac{2}{3}H\omega_3 = -\frac{2}{3}H\omega_1 - \frac{2}{3}H\omega_2$ gives

$$\Gamma^j_{jk}\theta^k = \Gamma^i_j = \delta_j{}^i(H\omega_1 + H\omega_2),$$

and so

$$\begin{aligned}
\Gamma^1_{11}\theta^1 + \Gamma^1_{12}\theta^2 =& H\theta^1 + H\theta^2,\\
\Gamma^2_{21}\theta^1 + \Gamma^2_{22}\theta^2 =& H\theta^1 + H\theta^2.
\end{aligned}$$

Moreover, since the connection is torsion free,

$$\Gamma^1_{21} = \Gamma^1_{12} + \frac{3}{2}\alpha(\partial_2) = H - H = 0, \qquad \Gamma^2_{12} = \Gamma^2_{21} + \frac{3}{2}\alpha(\partial_1) = H - H = 0.$$

Hence the connection coefficients are given by

$$\Gamma^2_{11} = \Gamma^1_{22} = 0, \qquad \Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{21} = \Gamma^2_{22} = H, \qquad \Gamma^1_{21} = \Gamma^2_{12} = 0.$$

from which we obtain

$$\begin{aligned}
\nabla_{\partial_1}\theta^1 =& -\Gamma^1_{11}\theta^1 - \Gamma^1_{12}\theta^2 = -H\theta^1,\\
\nabla_{\partial_2}\theta^2 =& -\Gamma^2_{21}\theta^1 - \Gamma^2_{22}\theta^2 = -H\theta^2,\\
\nabla_{\partial_2}\theta^1 =& -H\theta^1,\\
\nabla_{\partial_1}\theta^2 =& -H\theta^2.
\end{aligned}$$

We would like to find the connection coefficients of the web connection associated to a 3-web in terms of the chosen coordinate system (x, y) and web function f. With respect to the coordinate basis ∂_x and ∂_y, since

$$\nabla_{X_a}X_b = \Gamma^c_{ba}X_c,$$

we obtain

$$\begin{aligned}
\nabla_{\partial_1}\partial_1 =& -\frac{1}{f_x}\nabla_{\partial_x}\left(-\frac{1}{f_x}\partial_x\right) = \frac{1}{f_x^2}\nabla_{\partial_x}\partial_x - \frac{f_{xx}}{f_x^3}\partial_x,\\
\nabla_{\partial_2}\partial_2 =& -\frac{1}{f_y}\nabla_{\partial_y}\left(-\frac{1}{f_y}\partial_y\right) = \frac{1}{f_y^2}\nabla_{\partial_y}\partial_y - \frac{f_{yy}}{f_y^3}\partial_y,\\
\nabla_{\partial_1}\partial_2 =& -\frac{1}{f_x}\nabla_{\partial_x}\left(-\frac{1}{f_y}\partial_y\right) = \frac{1}{f_xf_y}\nabla_{\partial_x}\partial_y - \frac{f_{yx}}{f_y^2f_x}\partial_y,\\
\nabla_{\partial_2}\partial_1 =& -\frac{1}{f_y}\nabla_{\partial_y}\left(-\frac{1}{f_x}\partial_x\right) = \frac{1}{f_xf_y}\nabla_{\partial_y}\partial_x - \frac{f_{yx}}{f_x^2f_y}\partial_x,
\end{aligned}$$

from which we get the structure equations

$$
\begin{aligned}
\nabla_{\partial_x}\partial_x =& f_x^2\nabla_{\partial_1}\partial_1 + \frac{f_{xx}}{f_x}\partial_x = f_x^2\left(-\frac{H}{f_x}\partial_x\right) + \frac{f_{xx}}{f_x}\partial_x = \left(\frac{f_{xx}}{f_x} - Hf_x\right)\partial_x,\\
\nabla_{\partial_y}\partial_y =& f_y^2\nabla_{\partial_2}\partial_2 + \frac{f_{yy}}{f_y}\partial_y = f_y^2\left(-\frac{H}{f_y}\partial_y\right) + \frac{f_{yy}}{f_y}\partial_y = \left(\frac{f_{yy}}{f_y} - Hf_y\right)\partial_y,\\
\nabla_{\partial_y}\partial_x =& f_xf_y\nabla_{\partial_2}\partial_1 + \frac{f_{yx}}{f_x}\partial_x = f_xf_y\left(-\frac{H}{f_x}\partial_x\right) + \frac{f_{yx}}{f_x}\partial_x = \left(-Hf_y + \frac{f_{yx}}{f_x}\right)\partial_x,\\
\nabla_{\partial_x}\partial_y =& f_xf_y\nabla_{\partial_1}\partial_2 + \frac{f_{yx}}{f_y}\partial_y = f_xf_y\left(-\frac{H}{f_y}\partial_y\right) + \frac{f_{yx}}{f_y}\partial_y = \left(-Hf_x + \frac{f_{xy}}{f_y}\right)\partial_y,
\end{aligned}
$$

which gives rise to the connection coefficients

$$
\Gamma^y_{yx} = \Gamma^y_{xy} = \frac{f_{xy}}{f_y} - Hf_x = 0, \quad \Gamma^x_{xy} = \Gamma^x_{yx} = \frac{f_{xy}}{f_x} - Hf_y = 0, \quad \Gamma^y_{xx} = \Gamma^x_{yy} = 0,
$$
$$
\Gamma^x_{xx} = \frac{f_{xx}}{f_x} - Hf_y, \quad \Gamma^y_{yy} = \frac{f_{yy}}{f_y} - Hf_x,
$$

from which we deduce that $H = \frac{f_{xy}}{f_xf_y}$ and the connection coefficients are

$$
\Gamma^y_{yx} = \Gamma^y_{xy} = 0, \quad \Gamma^x_{xy} = \Gamma^x_{yx} = 0, \quad \Gamma^y_{xx} = \Gamma^x_{yy} = 0,
$$
$$
\Gamma^x_{xx} = \frac{f_{xx}}{f_x} - \frac{f_{xy}}{f_y}, \quad \Gamma^y_{yy} = \frac{f_{yy}}{f_y} - \frac{f_{xy}}{f_x}.
$$

These are the connection coefficients (4) obtained in [22]. We therefore have

$$
\begin{aligned}
\alpha = \frac{2}{3}H\omega_3 = \frac{2}{3}\frac{f_{xy}}{f_xf_y}\omega_3 = \frac{2}{3}\frac{f_{xy}}{f_xf_y}(-\omega_1 - \omega_2) =& \frac{2}{3}\frac{f_{xy}}{f_xf_y}(f_xdx + f_ydy)\\
=& \frac{2}{3}\frac{f_{xy}}{f_y}dx + \frac{2}{3}\frac{f_{xy}}{f_x}dy
\end{aligned}
\tag{4.12}
$$

and it can be verified that α given by (4.12) is a solution to pEW with the projective structure in this coordinate system given by the connection coeffcients above. A computation shows that

$$
\begin{aligned}
K =& \frac{1}{f_xf_y}\left(\frac{f_{xyx}}{f_x} - \frac{f_{xy}f_{xx}}{(f_x)^2} - \frac{f_{xyy}}{f_y} + \frac{f_{xy}f_{yy}}{(f_y)^2}\right)\\
=& \frac{f_{xyx}}{f_y(f_x)^2} - \frac{f_{xy}f_{xx}}{(f_x)^3f_y} - \frac{f_{xyy}}{(f_x)(f_y)^2} + \frac{f_{xy}f_{yy}}{(f_y)^3f_x}.
\end{aligned}
$$

Remark 4.21. We avoid writing

$$
K = \frac{1}{f_xf_y}\left(\log\left(\frac{f_x}{f_y}\right)\right)_{xy}
$$

because of the ambiguity in the domain for which $\log\left(\frac{f_x}{f_y}\right)$ is defined. Following [22], because the sign of $\frac{f_x}{f_y}$ is alway fixed, we can take $K = \frac{1}{f_x f_y}\left(\log\left(\frac{f_x}{f_y}\right)\right)_{xy}$ if $\frac{f_x}{f_y} > 0$ and $K = \frac{1}{f_x f_y}\left(\log\left(-\frac{f_x}{f_y}\right)\right)_{xy}$ if $\frac{f_x}{f_y} < 0$. Also the quantity K defined here is negative of K defined in [18], because of the conventions chosen.

We can verify the accuracy of our computations by computing the connection coefficients dually. Dually, using the equations

$$\nabla_{X_a}\theta^b = -\Gamma^b_{ca}\theta^c,$$

and the fact that $H = \frac{f_{xy}}{f_x f_y}$, we obtain

$$\begin{aligned}
\nabla_{\partial_1}\theta^1 &= -\frac{1}{f_x}\nabla_{\partial_x}(-f_x dx) = \nabla_{\partial_x}dx + \frac{f_{xx}}{f_x}dx,\\
\nabla_{\partial_2}\theta^2 &= -\frac{1}{f_y}\nabla_{\partial_y}(-f_y dy) = \nabla_{\partial_y}dy + \frac{f_{yy}}{f_y}dy,\\
\nabla_{\partial_1}\theta^2 &= -\frac{1}{f_x}\nabla_{\partial_x}(-f_y dy) = \frac{f_y}{f_x}\nabla_{\partial_x}dy + \frac{f_{yx}}{f_x}dy,\\
\nabla_{\partial_2}\theta^1 &= -\frac{1}{f_y}\nabla_{\partial_y}(-f_x dx) = \frac{f_x}{f_y}\nabla_{\partial_y}dx + \frac{f_{yx}}{f_y}dx,
\end{aligned}$$

so that

$$\begin{aligned}
\nabla_{\partial_x}dx =& \nabla_{\partial_1}\theta^1 - \frac{f_{xx}}{f_x}dx = \left(-\frac{f_{xx}}{f_x} + Hf_x\right)dx = \left(-\frac{f_{xx}}{f_x} + \frac{f_{xy}}{f_y}\right)dx,\\
\nabla_{\partial_y}dy =& \nabla_{\partial_2}\theta^2 - \frac{f_{yy}}{f_y}dy = \left(-\frac{f_{yy}}{f_y} + Hf_y\right)dy = \left(-\frac{f_{yy}}{f_y} + \frac{f_{xy}}{f_x}\right)dy,\\
\nabla_{\partial_x}dy =& \frac{f_x}{f_y}(\nabla_{\partial_1}\theta^2) - \frac{f_{yx}}{f_y}dy = \frac{f_x}{f_y}(-H\theta^2) - \frac{f_{yx}}{f_y}dy = \frac{f_x}{f_y}\left(\frac{f_{xy}}{f_x}\right)dy - \frac{f_{yx}}{f_y}dy\\
=& 0,\\
\nabla_{\partial_y}dx =& \frac{f_y}{f_x}(\nabla_{\partial_2}\theta^1) - \frac{f_{yx}}{f_x}dx = \frac{f_y}{f_x}(-H\theta^1) - \frac{f_{yx}}{f_x}dx = \frac{f_y}{f_x}\left(\frac{f_{xy}}{f_y}\right)dx - \frac{f_{yx}}{f_x}dx\\
=& 0.
\end{aligned}$$

This gives the connection coefficients in terms of the coordinate functions x and y as

$$\begin{aligned}
\Gamma^y_{yx} = \Gamma^y_{xy} = 0, \quad \Gamma^x_{xy} = \Gamma^x_{yx} = 0, \quad \Gamma^y_{xx} = \Gamma^x_{yy} = 0,\\
\Gamma^x_{xx} = \frac{f_{xx}}{f_x} - \frac{f_{xy}}{f_y}, \quad \Gamma^y_{yy} = \frac{f_{yy}}{f_y} - \frac{f_{xy}}{f_x}.
\end{aligned}$$

This gives rise to the second order ODE

$$\begin{aligned}\frac{d^2y}{dx^2} =&\Gamma^x_{yy}\left(\frac{dy}{dx}\right)^3 + (2\Gamma^x_{xy} - \Gamma^y_{yy})\left(\frac{dy}{dx}\right)^2 + (\Gamma^x_{xx} - 2\Gamma^y_{xy})\left(\frac{dy}{dx}\right) - \Gamma^y_{xx}\\ =&\left(\frac{f_{xy}}{f_x} - \frac{f_{yy}}{f_y}\right)\left(\frac{dy}{dx}\right)^2 + \left(\frac{f_{xx}}{f_x} - \frac{f_{xy}}{f_y}\right)\left(\frac{dy}{dx}\right) \end{aligned} \tag{4.13}$$

associated to the projective structure, which is precisely equation (8) in Proposition 3.1 of [22]. Moreover, by Lemma 3.2 of [22], the second order ODE (4.13) is point equivalent to the derivative of a first order ODE. We can read off solutions to pEW from (4.13). We obtain

$$\widehat{\alpha} = -\frac{1}{3}\left(-\frac{f_{xy}}{f_y} + \frac{f_{xx}}{f_x}\right) dx + \frac{1}{3}\left(-\frac{f_{yy}}{f_y} + \frac{f_{xy}}{f_x}\right) dy$$

and a computation shows that

$$\begin{aligned} d\widehat{\alpha} =& -\frac{1}{3}\partial_y\left(-\frac{f_{xy}}{f_y} + \frac{f_{xx}}{f_x}\right) dy \wedge dx + \frac{1}{3}\partial_x\left(-\frac{f_{yy}}{f_y} + \frac{f_{xy}}{f_x}\right) dx \wedge dy\\ =&\frac{2}{3}\left(\frac{f_{xy}f_{yy}}{f_y^2} - \frac{f_{xy}f_{xx}}{f_x^2} + \frac{f_{xxy}}{f_x} - \frac{f_{yyx}}{f_y}\right) dx \wedge dy\\ =&\frac{2}{3}f_xf_yK dx \wedge dy\\ =&\frac{2}{3}K\omega_1 \wedge \omega_2\\ =&F\omega_1 \wedge \omega_2 \qquad \text{(Recall } K = \tfrac{3}{2}F.)\\ =&d\alpha, \end{aligned}$$

where $\alpha = \frac{2}{3}\frac{f_{xy}}{f_y}dx + \frac{2}{3}\frac{f_{xy}}{f_x}dy$ is given by (4.12). Thus α differs from $\widehat{\alpha}$ by a locally exact 1-form. We have

$$\Upsilon = \widehat{\alpha} - \alpha = -\frac{1}{3}\left(\frac{f_{xx}}{f_x} + \frac{f_{yx}}{f_y}\right) dx - \frac{1}{3}\left(\frac{f_{yy}}{f_y} + \frac{f_{xy}}{f_x}\right) dy,$$

and a computation shows that

$$d\Upsilon = 0.$$

Hence given a 3-web, we have found α in terms of the web function (and the chosen coordinate system) that satisfies pEW.

4.8 Global obstructions to pEW

Derdzinski has shown in [8] that the only oriented closed (compact without boundary) surface admitting a globally defined connection with skew-symmetric Ricci tensor is diffeomorphic to the 2-torus $\mathbb{T}^2$. To prove this one has to to use a result of Milnor [24] which says that a surface with genus $g \geq 2$ does not posses any flat affine connection (with or without torsion). We provide an alternative proof as follows. Suppose $(M^2, [\nabla])$ is a closed oriented smooth surface with a globally defined projective structure that admits a skew-symmetric Ricci. Now by Lemma 4.6, there is a globally defined flat connection on the weighted tangent bundle. In particular, it admits a globally defined parallel section V which is a vector field tensored with some non-vanishing density. Now V cannot have any zeroes, or otherwise parallel transport would mean V is identically zero. But by the theorem of Poincare-Hopf, a nowhere vanishing vector field exists only on surfaces with vanishing Euler characteristic. Hence M^2 has to be diffeomorphic to the 2-torus, and the Euler characteristic can be seen as a global obstruction to closed smooth surfaces admitting skew-symmetric Ricci.

Chapter 5

Examples

This chapter presents various examples of projective structures that either admit or do not admit skew-symmetric Ricci tensor. The first two examples in Section 5.1 are projective surfaces with non-vanishing obstructions, and hence do not admit any skew-symmetric Ricci tensor in its projective class of connections. The next 2 examples in Section 5.2 are projective structures that do admit skew-symmetric tensor, and hence have vanishing obstructions. We also give an example of a projective surface for which the obstructions vanish, but yet does not admit a real solution to pEW in Section 5.3. Following this in Section 5.4 we give 2 examples of three-dimensional projective structures that do not admit skew-symmetric Ricci tensor, namely the projective geometry induced by the metric structure on 'sol', and the Berger sphere geometries. In Section 5.5 we then give 2 different examples of a three dimensional projective structure that do admit a solution to pEW, the first of which can be seen as a generalisation of Derdzinski's normal form in [8]. In the final Section 5.6, this normal form again is generalised to higher dimensions to obtain affine structures with skew-symmetric Ricci tensor, thus yielding solutions to pEW in higher dimensions by taking the projectively related class of connections.

5.1 Projective surfaces with non-vanishing obstructions

5.1.1 An example with $\rho \neq 0$

Ingredients required for computing the polynomial constraints

We shall first specify the objects needed in order to compute the polynomials $P_1(t)$, $P_2(t)$, $P_3(t)$ for a projective structure with $\rho = Y_a W^a \neq 0$. In practice, to show an example of a projective structure which does not admit any solution to pEW, one only needs to evaluate these polynomials at a generically chosen point and show that there are no common solutions at that point. The curvature terms Y^a, W^a, etc. needed to compute the coefficients of the polynomials get out of hand quickly and we suggest a program like MAPLE or MATLAB to do general computations. The difficulty of performing the computations by hand is illustrated in the example. The data from the projective structure that is needed to compute the coefficients of the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ come from the following objects: Y^a, W^a, ϕ, ρ, $\nabla_a W^a$, $Y^d W^e \nabla_e Y_d$, $W^e W^d \nabla_e Y_d$, $W^e Y^d \nabla_d W_e$, $W^d W^e \nabla_d W_e$, $Y^a Y^b \nabla_a W_b$, $Y^b W^a \nabla_a W_b$, $W^a \nabla_a \rho$, $Y^a \nabla_a \rho$, $\mathrm{P}_{ab} W^a W^b$, $\mathrm{P}_{ab} Y^a Y^b$, $\mathrm{P}_{ab} W^a Y^b$, ℓ, $W^a \nabla_a \phi$, $Y^a \nabla_a \phi$, $W^a \nabla_a \ell$ and $Y^a \nabla_a \ell$. Explicitly, working in local coordinates (x, y) and denoting $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$, and using $Y^1 = Y_2$, $Y^2 = -Y_1$, we have

$$
\begin{aligned}
W^1 &= Y^1 \partial_1 Y^1 + Y^2 \partial_2 Y^1 + \Pi^1_{11} Y^1 Y^1 + 2\Pi^1_{12} Y^1 Y^2 + \Pi^1_{22} Y^2 Y^2 - \frac{2\phi}{3} Y^1, \\
W^2 &= Y^1 \partial_1 Y^2 + Y^2 \partial_2 Y^2 + \Pi^2_{11} Y^1 Y^1 + 2\Pi^2_{12} Y^1 Y^2 + \Pi^2_{22} Y^2 Y^2 - \frac{2\phi}{3} Y^2, \\
\phi &= 2\partial_1 Y^1 + 2\partial_2 Y^2.
\end{aligned}
$$

We check that $\rho = Y_a W^a = Y_1 W^1 + Y_2 W^2 \neq 0$ on an open dense set and proceed to compute $\nabla_a W^a = \partial_1 W^1 + \partial_2 W^2$. We also find

$$
\begin{aligned}
\nabla_1 Y_1 &= \partial_1 Y_1 - \Pi^1_{11} Y_1 - \Pi^2_{11} Y_2, & \nabla_1 Y_2 &= \partial_1 Y_2 - \Pi^1_{12} Y_1 - \Pi^2_{12} Y_2, \\
\nabla_2 Y_1 &= \partial_2 Y_1 - \Pi^1_{21} Y_1 - \Pi^2_{21} Y_2, & \nabla_2 Y_2 &= \partial_2 Y_2 - \Pi^1_{22} Y_1 - \Pi^2_{22} Y_2, \\
\nabla_1 W_1 &= \partial_1 W_1 - \Pi^1_{11} W_1 - \Pi^2_{11} W_2, & \nabla_1 W_2 &= \partial_1 W_2 - \Pi^1_{12} W_1 - \Pi^2_{12} W_2, \\
\nabla_2 W_1 &= \partial_2 W_1 - \Pi^1_{21} W_1 - \Pi^2_{21} W_2, & \nabla_2 W_2 &= \partial_2 W_2 - \Pi^1_{22} W_1 - \Pi^2_{22} W_2,
\end{aligned}
$$

so that

$$
\begin{aligned}
Y^aW^b\nabla_bY_a &= Y^1W^1\nabla_1Y_1 + Y^1W^2\nabla_2Y_1 + Y^2W^1\nabla_1Y_2 + Y^2W^2\nabla_2Y_2,\\
W^aW^b\nabla_bY_a &= W^1W^1\nabla_1Y_1 + W^1W^2\nabla_2Y_1 + W^2W^1\nabla_1Y_2 + W^2W^2\nabla_2Y_2,\\
W^aY^b\nabla_bW_a &= W^1Y^1\nabla_1W_1 + W^1Y^2\nabla_2W_1 + W^2Y^1\nabla_1W_2 + W^2Y^2\nabla_2W_2,\\
W^aW^b\nabla_bW_a &= W^1W^1\nabla_1W_1 + W^1W^2\nabla_2W_1 + W^2W^1\nabla_1W_2 + W^2W^2\nabla_2W_2,\\
Y^aY^b\nabla_bW_a &= Y^1Y^1\nabla_1W_1 + Y^1Y^2\nabla_2W_1 + Y^2Y^1\nabla_1W_2 + Y^2Y^2\nabla_2W_2,\\
Y^aW^b\nabla_bW_a &= Y^1W^1\nabla_1W_1 + Y^1W^2\nabla_2W_1 + Y^2W^1\nabla_1W_2 + Y^2W^2\nabla_2W_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
Y^a\nabla_a\rho &= W^bY^a\nabla_aY_b - Y^aY^b\nabla_aW_b,\\
W^a\nabla_a\rho &= W^aW^b\nabla_aY_b - W^aY^b\nabla_aW_b,
\end{aligned}
$$

but a simpler method is to compute

$$
\begin{aligned}
Y^a\nabla_a\rho &= Y^1\partial_1\rho + Y^2\partial_2\rho,\\
W^a\nabla_a\rho &= W^1\partial_1\rho + W^2\partial_2\rho.
\end{aligned}
$$

Also computing

$$
\begin{aligned}
\mathrm{P}_{ab}Y^aY^b &= \mathrm{P}_{11}Y^1Y^1 + 2\mathrm{P}_{12}Y^1Y^2 + \mathrm{P}_{22}Y^2Y^2,\\
\mathrm{P}_{ab}Y^aW^b &= \mathrm{P}_{11}Y^1W^1 + \mathrm{P}_{12}Y^1W^2 + \mathrm{P}_{12}Y^2W^1 + \mathrm{P}_{22}Y^2W^2,\\
\mathrm{P}_{ab}W^aW^b &= \mathrm{P}_{11}W^1W^1 + 2\mathrm{P}_{12}W^1W^2 + \mathrm{P}_{22}W^2W^2,
\end{aligned}
$$

and

$$
\begin{aligned}
Y^a\nabla_a\phi &= Y^1\partial_1\phi + Y^2\partial_2\phi,\\
W^a\nabla_a\phi &= W^1\partial_1\phi + W^2\partial_2\phi,
\end{aligned}
$$

allows us to find

$$\ell = \frac{5\phi^2}{12} + 3\mathrm{P}_{ac}Y^aY^c - \frac{Y^a\nabla_a\phi}{2},$$

which gives us

$$
\begin{aligned}
Y^a\nabla_a\ell &= Y^1\partial_1\ell + Y^2\partial_2\ell,\\
W^a\nabla_a\ell &= W^1\partial_1\ell + W^2\partial_2\ell.
\end{aligned}
$$

We shall now compute the ingredients for the following example with $\rho \neq 0$.

Computing the ingredients for an example with $\rho \neq 0$

This example is a projective structure with $A_1 = A_2 = 0$, $A_0 = y$, $A_3 = xy$ on $\mathbb{R}^2$. This gives rise to the connection coefficients

$$\Pi^1_{22} = xy, \quad \Pi^2_{11} = -y, \quad \Pi^1_{11} = \Pi^1_{21} = \Pi^2_{12} = \Pi^2_{22} = 0.$$

We compute

$$\mathrm{P}_{11} = -1, \quad \mathrm{P}_{22} = y, \quad \mathrm{P}_{12} = xy^2,$$

for this projective structure, and the relative invariants are

$$L_1 = -2y^2, \quad L_2 = 3xy,$$

so that

$$Y_1 = -2L_1 = 4y^2, \quad Y_2 = -2L_2 = -6xy.$$

Alternatively,

$$Y^1 = Y_2 = -6xy, \quad Y^2 = -Y_1 = -4y^2.$$

We have

$$\partial_2 Y^1 = -6x, \quad \partial_1 Y^1 = -6y, \quad \partial_2 Y^2 = -8y, \quad \partial_1 Y^2 = 0,$$

which gives

$$\phi = 2\partial_1 Y^1 + 2\partial_2 Y^2 = -28y.$$

We get

$$W^1 = -52xy^2 + 16xy^5, \quad W^2 = 32y^3 - 36x^2y^3 - \frac{224}{3}y^3,$$

so that

$$\rho = Y_1 W^1 + Y_2 W^2 = 48xy^4 + 216x^3y^4 + 64xy^7 = 8xy^4(6 + 27x^2 + 8y^3).$$

Observe that $\rho = 0$ on $\{x = 0\} \cup \{y = 0\} \cup \{6 + 27x^2 + 8y^3 = 0\}$ but $\rho \neq 0$ on an open dense set. We have

$$\begin{aligned} \partial_2 W^1 &= -104xy + 80xy^4, \partial_1 W^1 = -52y^2 + 16y^5, \\ \partial_2 W^2 &= -128y^2 - 108x^2y^2, \quad \partial_1 W^2 = -72xy^3, \end{aligned}$$

which gives

$$\nabla_a W^a = \partial_1 W^1 + \partial_2 W^2 = -180y^2 + 16y^5 - 108x^2y^2.$$

Now we compute

$$\begin{aligned}
\nabla_1 W_1 &= 20xy^3 + 16xy^6,\\
\nabla_1 W_2 &= -52y^2 + 16y^5,\\
\nabla_2 W_1 &= 108x^2y^2 + 128y^2,\\
\nabla_2 W_2 &= -104xy - 36x^3y^4 + \frac{112}{3}xy^4,
\end{aligned}$$

and

$$\begin{aligned}
\nabla_1 Y_1 &= -6xy^2,\\
\nabla_1 Y_2 &= -6y,\\
\nabla_2 Y_1 &= 8y,\\
\nabla_2 Y_2 &= -6x - 4xy^3.
\end{aligned}$$

Altogether this gives

$$\begin{aligned}
Y^a W^b \nabla_b Y_a =& (-6xy)(-52xy^2 + 16xy^5)(-6xy^2)\\
&+ (-4y^2)(-52xy^2 + 16xy^5)(-6y)\\
&+ (-6xy)(32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(8y)\\
&+ (-4y^2)(32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(-6x - 4xy^3),
\end{aligned}$$

$$\begin{aligned}
W^a W^b \nabla_b Y_a =& (-52xy^2 + 16xy^5)(-52xy^2 + 16xy^5)(-6xy^2)\\
&+ (32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(-52xy^2 + 16xy^5)(-6y)\\
&+ (-52xy^2 + 16xy^5)(32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(8y)\\
&+ (32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(-6x - 4xy^3),
\end{aligned}$$

$$\begin{aligned}
W^a Y^b \nabla_b W_a =& (-52xy^2 + 16xy^5)(-6xy)(20xy^3 + 16xy^6)\\
&+ (32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(-4y^2)(-104xy - 36x^3y^4 + \frac{112}{3}xy^4)\\
&+ (-6xy)(32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(-52y^2 + 16y^5)\\
&+ (-52xy^2 + 16xy^5)(-4y^2)(108x^2y^2 + 128y^2),
\end{aligned}$$

$$
\begin{aligned}
W^aW^b\nabla_bW_a =&(-52xy^2+16xy^5)(-52xy^2+16xy^5)(20xy^3+16xy^6)\\
&+(32y^3-36x^2y^3-\frac{224}{3}y^3)(32y^3-36x^2y^3-\frac{224}{3}y^3)\\
&\times(-104xy-36x^3y^4+\frac{112}{3}xy^4)\\
&+(-52xy^2+16xy^5)(32y^3-36x^2y^3-\frac{224}{3}y^3)(-52y^2+16y^5)\\
&+(-52xy^2+16xy^5)(32y^3-36x^2y^3-\frac{224}{3}y^3)(108x^2y^2+128y^2),
\end{aligned}
$$

$$
\begin{aligned}
Y^aY^b\nabla_bW_a =&(-6xy)(-6xy)(20xy^3+16xy^6)\\
&+(-4y^2)(-4y^2)(-104xy-36x^3y^4+\frac{112}{3}xy^4)\\
&+(-6xy)(-4y^2)(-52y^2+16y^5)\\
&+(-6xy)(-4y^2)(108x^2y^2+128y^2),
\end{aligned}
$$

$$
\begin{aligned}
W^aY^b\nabla_aW_b =&(-52xy^2+16xy^5)(-6xy)(20xy^3+16xy^6)\\
&+(32y^3-36x^2y^3-\frac{224}{3}y^3)(-4y^2)(-104xy-36x^3y^4+\frac{112}{3}xy^4)\\
&+(-52xy^2+16xy^5)(-4y^2)(-52y^2+16y^5)\\
&+(-6xy)(32y^3-36x^2y^3-\frac{224}{3}y^3)(108x^2y^2+128y^2).
\end{aligned}
$$

We also have

$$
\begin{aligned}
Y^a\nabla_a\rho =&(-6xy)(48y^4+648x^2y^4+64y^7)+(-4y^2)(192xy^3+864x^3y^3448+xy^6),\\
W^a\nabla_a\rho =&(-52xy^2+16xy^5)(48y^4+648x^2y^4+64y^7)\\
&+(32y^3-36x^2y^3-\frac{224}{3}y^3)(192xy^3+864x^3y^3448+xy^6).
\end{aligned}
$$

We still have to find

$$
\begin{aligned}
\mathrm{P}_{ab}W^aY^b =&-(-52xy^2+16xy^5)(-6xy)+y(32y^3-36x^2y^3-\frac{224}{3}y^3)(-4y^2)\\
&+xy^2(-52xy^2+16xy^5)(-4y^2)+xy^2(32y^3-36x^2y^3-\frac{224}{3}y^3)(-6xy),\\
\mathrm{P}_{ab}W^aW^b =&-(-52xy^2+16xy^5)(-52xy^2+16xy^5)\\
&+y(32y^3-36x^2y^3-\frac{224}{3}y^3)(32y^3-36x^2y^3-\frac{224}{3}y^3)\\
&+2xy^2(32y^3-36x^2y^3-\frac{224}{3}y^3)(-52xy^2+16xy^5),\\
\mathrm{P}_{ab}Y^aY^b =&-(-6xy)(-6xy)+y(-4y^2)(-4y^2)+2xy^2(-4y^2)(-6xy),
\end{aligned}
$$

the last of which enables us to compute

$$\ell = \frac{980}{3}y^2 - 108x^2y^2 + 144x^2y^5 + 48y^5 - 56y^2,$$

which gives

$$\begin{aligned}\partial_1\ell &= -216xy^2 + 288xy^5,\\ \partial_2\ell &= \frac{1960}{3}y - 216x^2y + 720x^2y^4 + 240y^4 - 112y,\end{aligned}$$

enabling us to find

$$\begin{aligned}Y^a\nabla_a\ell =&Y^1\partial_1\ell + Y^2\partial_2\ell\\ =&(-6xy)(-216xy^2 + 288xy^5)\\ &+ (-4y^2)(\frac{1960}{3}y - 216x^2y + 720x^2y^4 + 240y^4 - 112y)\end{aligned}$$

and

$$\begin{aligned}W^a\nabla_a\ell =&W^1\partial_1\ell + W^2\partial_2\ell\\ =&(-52xy^2 + 16xy^5)(-216xy^2 + 288xy^5)\\ &+ (32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(\frac{1960}{3}y - 216x^2y + 720x^2y^4 + 240y^4 - 112y).\end{aligned}$$

Finally,

$$W^a\nabla_a\phi = (32y^3 - 36x^2y^3 - \frac{224}{3}y^3)(-28).$$

Having all the ingredients, we are ready to cook up the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ at any point $p \in M$ where $\rho(p) \neq 0$. We shall do this at an arbitrarily chosen point $(x, y) = (1, 1)$.

The polynomials for this example at the point $(1, 1)$

At the point $(1, 1)$, the ingredients for the example from the previous section take the following values:

$$\begin{aligned}Y^1 = -6, \quad Y^2 = -4,\\ W^1 = -36, \quad W^2 = -\frac{236}{3},\\ \phi = -28, \quad \rho = 328, \quad \nabla_aW^a = -272,\end{aligned}$$

$$Y^aW^b\nabla_bY_a = -\frac{4592}{3}, \qquad W^aW^b\nabla_aY_b = -\frac{575968}{9},$$
$$W^aY^b\nabla_bW_a = -\frac{67840}{9}, \qquad W^aW^b\nabla_aW_b = -\frac{601856}{27},$$
$$Y^aY^b\nabla_aW_b = \frac{13360}{3}, \qquad Y^bW^a\nabla_aW_b = \frac{735104}{9},$$
$$Y^a\nabla_a\rho = -10576, \qquad W^a\nabla_a\rho = -\frac{437024}{3},$$

$$\mathrm{P}_{ab}Y^aY^b = 28, \qquad \mathrm{P}_{ab}W^aY^b = \frac{2144}{3}, \qquad \mathrm{P}_{ab}W^aW^b = \frac{95008}{9},$$
$$W^a\nabla_a\phi = \frac{6608}{3}, \qquad \ell = \frac{1064}{3},$$
$$Y^a\nabla_a\ell = -\frac{16720}{3}, \qquad W^a\nabla_a\ell = -\frac{933344}{9}.$$

We can now substitute the data into $P_1(t)$, $P_2(t)$ and $P_3(t)$. At the point $(1,1)$, for the pEW to hold on this example, the following polynomial equations must hold:

$$\begin{aligned}
P_1(F) &= -90F^6 + \frac{185760}{328}F^4 - \frac{528608}{328}F^2 - 2952F - \frac{134912}{328} = 0,\\
P_2(F) &= -275F^8 + \frac{13774080}{2952}F^4 + 6560F^3 - \frac{601856}{8856}F^2 + \frac{523957248}{26568} = 0\\
P_3(F) &= -40F^6 + \frac{30960}{328}F^4 + \frac{125360}{984}F^2 + 328F + \frac{31603200}{26568} = 0.
\end{aligned}$$

If we run a program through MATLAB to solve these polynomials, we find that they do not share a common root, i.e. there exists no F (real or complex) such that $P_1(F) = P_2(F) = P_3(F) = 0$. We can also compute the resultants of the polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$, and even Q_{12}, Q_{13} and Q_{23}. A computation using MAPLE shows that the obstructions at the point $p = (1,1)$ take on the values

$$\begin{aligned}
Q_{12}(p) &= -\frac{1457890459574161592339200000}{1681} \approx -8.6728 \times 10^{23},\\
Q_{13}(p) &= -\frac{188610437798501965389961756672000000000}{452190681} \approx -4.1710 \times 10^{29},\\
Q_{23}(p) &= \frac{1457890459574161592339200000}{1681} \approx 8.6728 \times 10^{23}.
\end{aligned}$$

The three polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ therefore do not share a common root, since the obstructions $Q_{12}(p)$, $Q_{13}(p)$ and $Q_{23}(p)$ are non-zero. In particular the obstructions do not vanish at p and on an open set near p. Moreover, since $Q_{12}(p)$, $Q_{13}(p)$ and $Q_{23}(p)$ are polynomials in p we can conclude that there is no solution to pEW anywhere on $\mathbb{R}^2$, as the set where these polynomials Q_{12}, Q_{13} and Q_{23} are non-zero is Zariski open and hence dense in $\mathbb{R}^2$.

5.1.2 An example with $\rho = 0$

Using the obstructions obtained in the case where $\rho = 0$, we shall now show that the projective structure on $\mathbb{R}^2$ with connection coefficients given by

$$\Pi^1_{11} = -\frac{x^2}{6}, \quad \Pi^1_{22} = -\frac{x^2}{2}, \quad \Pi^2_{21} = \frac{x^2}{6}, \quad \Pi^2_{11} = \Pi^1_{21} = \Pi^2_{22} = 0$$

does not admit a solution to pEW. We shall show that equations (2.23) and (2.24) do not hold (for any y on the y-axis). This projective structure in fact has a metric connection in its projective class. We find that

$$\mathrm{P}_{11} = -\frac{x}{3} - \frac{x^4}{18}, \quad \mathrm{P}_{12} = 0, \quad \mathrm{P}_{22} = \frac{x^4}{6} - x,$$

and the projective invariants are

$$Y^a = \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 2x^3 - 2 \\ 0 \end{pmatrix}, \qquad W^a = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}x^2(x^3-1)(x^3+5) \\ 0 \end{pmatrix}.$$

We therefore find that

$$\phi = 12x^2, \quad \rho = Y_a W^a = 0, \quad f = -\frac{x^2}{3}(x^3+5) = -\frac{x^5}{3} - \frac{5x^2}{3}.$$

We also have

$$\begin{aligned} \ell = \frac{5\phi^2}{12} + 3\mathrm{P}_{ab}Y^aY^b - \frac{Y^a\nabla_a\phi}{2} &= \frac{5(12x^2)^2}{12} + 3\mathrm{P}_{11}Y^1Y^1 - \frac{Y^1\partial_1\phi}{2} \\ &= 60x^4 + 3\left(-\frac{x}{3} - \frac{x^4}{18}\right)(2x^3-2)^2 - \frac{(2x^3-2)(24x)}{2} \\ &= -\frac{2}{3}x^{10} - \frac{8}{3}x^7 + \frac{130}{3}x^4 + 20x, \end{aligned}$$

which gives

$$\begin{aligned} h = \ell + \frac{\phi f}{2} &= -\frac{2}{3}x^{10} - \frac{8}{3}x^7 + \frac{130}{3}x^4 + 20x + 6x^2\left(-\frac{x^5}{3} - \frac{5x^2}{3}\right) \\ &= -\frac{2}{3}x^{10} - \frac{14}{3}x^7 + \frac{100}{3}x^4 + 20x. \end{aligned}$$

We now compute

$$\begin{aligned}
k =&\frac{18}{5}f^2 + 3\phi f + 12h - 3Y^e\nabla_e f \\
=&\frac{18}{5}\left(-\frac{x^5}{3} - \frac{5x^2}{3}\right)^2 + 3(12x^2)\left(-\frac{x^5}{3} - \frac{5x^2}{3}\right) \\
&+ 12\left(-\frac{2}{3}x^{10} - \frac{8}{3}x^7 + \frac{130}{3}x^4 + 20x + 6x^2\left(-\frac{x^5}{3} - \frac{5x^2}{3}\right)\right) \\
&- 3(2x^3 - 2)\left(-\frac{5x^4}{3} - \frac{10x}{3}\right) \\
=&-\frac{38}{5}x^{10} - 54x^7 + 360x^4 + 220x
\end{aligned}$$

and

$$\begin{aligned}
m =&\frac{6hf}{5} - Y^a\nabla_a h + 2\phi h \\
=&\frac{4}{15}x^{15} + \frac{8}{15}x^{12} - 64x^9 + \frac{1180}{3}x^6 + \frac{2000}{3}x^3 + 40.
\end{aligned}$$

Using MATLAB, we obtain that the obstructions are

$$\begin{aligned}
&15m^2 + 3fmk - hk^2 \\
=&15(\frac{4}{15}x^{15} + \frac{8}{15}x^{12} - 64x^9 + \frac{1180}{3}x^6 + \frac{2000}{3}x^3 + 40)^2 \\
&+ 3\left(-\frac{x^5}{3} - \frac{5x^2}{3}\right)(\frac{4}{15}x^{15} + \frac{8}{15}x^{12} - 64x^9 + \frac{1180}{3}x^6 + \frac{2000}{3}x^3 + 40) \\
&(-\frac{38}{5}x^{10} - 54x^7 + 360x^4 + 220x) \\
&- (-\frac{2}{3}x^{10} - \frac{14}{3}x^7 + \frac{100}{3}x^4 + 20x)(-\frac{38}{5}x^{10} - 54x^7 + 360x^4 + 220x)^2 \\
=&\frac{8}{5}(x^3 - 1)^2(26x^{24} + 583x^{21} + 660x^{18} - 42985x^{15} \\
&- 48500x^{12} + 864000x^9 + 744500x^6 - 102500x^3 + 15000) \\
=&\frac{208}{5}x^{30} + \frac{4248}{5}x^{27} - 768x^{24} - \frac{349776}{5}x^{21} + 61008x^{18} + 1468824x^{15} \\
&- 1651200x^{12} - 1164000x^9 + 1543200x^6 - 212000x^3 + 24000, \\
&k(Y^a\nabla_a m) - m(Y^a\nabla_a k + k\phi - 6m) \\
=&\frac{32}{75}x^{30} + \frac{144}{25}x^{27} - 208x^{24} - \frac{117408}{25}x^{21} + \frac{311104}{5}x^{18} \\
&+ 136208x^{15} + 71536x^{12} - 996160x^9 + 978560x^6 - \frac{824000}{3}x^3 + 27200
\end{aligned}$$

and we can conclude that this projective structure does not admit any solution to pEW locally.

5.2 Projective surfaces admitting skew-symmetric Ricci tensor (and hence with vanishing obstructions)

5.2.1 An example with $\rho \neq 0$

A second order ODE that is the derivative of a first order ODE will define a smooth projective structure that has a solution to (1.4) and generically has $\rho \neq 0$. We refer to example (4.9) in Section 4.3.1 for more details.

5.2.2 An example with $\rho = 0$

Consider the second order ODE given by

$$\frac{d^2y}{dx^2} = -\frac{1}{4y^3}. \tag{5.1}$$

The solutions are $y = \pm\sqrt{ax^2 + bx + c}$ where a, b, c are constants such that $4ac - b^2 = -1$. This is well-defined away from $y = 0$. We can regard (5.1) as giving rise to a projective structure with connection coefficients given by

$$A_0 = -\frac{1}{4y^3}, \quad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0.$$

Using MATLAB, this projective structure have the invariants Y^a and W^a given by

$$Y^a = \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{6}{y^5} \end{pmatrix}, \qquad W^a = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{60}{y^{11}} \end{pmatrix}.$$

We find that $\rho = W^1Y^2 - W^2Y^1 = 0$ and

$$f = \frac{10}{y^6}, \qquad \ell = \frac{420}{y^{12}}, \qquad h = \frac{120}{y^{12}}, \qquad k = \frac{1080}{y^{12}}, \qquad m = -\frac{4320}{y^{18}}.$$

A computation shows that

$$15m^2 + 3fmk - hk^2 = 0,$$
$$kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m) = 0,$$

and so the obstructions for this example vanish. This projective structure in fact admits two distinct real solutions to pEW. We find that

$$F = \pm\frac{2}{y^3},$$

and

$$\alpha = \pm \frac{1}{y^2}dx + \frac{1}{y}dy.$$

5.3 Projective surface with vanishing obstructions but not admitting a solution to pEW

The example of Hitchin [21] is a second order ODE given by

$$\frac{d^2y}{dx^2} = \frac{1}{4y^3}. \tag{5.2}$$

The solutions are $y = \pm\sqrt{ax^2 + bx + c}$ where a, b, c are constants such that $4ac - b^2 = 1$ (together with $y = \infty$ according to [21]). Since $b^2 - 4ac < 0$, $y \neq 0$ and (5.2) is well-defined. We can regard (5.2) as giving rise to a projective structure with connection coefficients given by

$$A_0 = \frac{1}{4y^3}, \quad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0.$$

Using MATLAB, this projective structure have the invariants Y^a and W^a given by

$$Y^a = \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{6}{y^5} \end{pmatrix}, \qquad W^a = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{60}{y^{11}} \end{pmatrix}.$$

We find that $\rho = W^1Y^2 - W^2Y^1 = 0$ and

$$f = -\frac{10}{y^6}, \qquad \ell = \frac{420}{y^{12}}, \qquad h = \frac{120}{y^{12}}, \qquad k = \frac{1080}{y^{12}}, \qquad m = \frac{4320}{y^{18}}.$$

A computation shows that

$$\begin{aligned} 15m^2 + 3fmk - hk^2 =& 15\left(\frac{4320}{y^{18}}\right)^2 + 3\left(-\frac{10}{y^6}\right)\left(\frac{4320}{y^{18}}\right)\left(\frac{1080}{y^{12}}\right) \\ &- \left(\frac{120}{y^{12}}\right)\left(\frac{1080}{y^{12}}\right)^2 \\ =& 0, \end{aligned}$$

and

$$kY^a\nabla_a m - m(Y^a\nabla_a k + k\phi - 6m) = 0,$$

and so the obstructions for this example vanish. We claim that this projective structure does not admit any real solution to pEW. The obstructions hence do not distinguish real solutions from complex ones. Equation (2.20) for this example is given by

$$\left(\frac{1080}{y^{12}}\right) F^2 + \frac{4320}{y^{18}} = 0,$$

or that

$$F^2 + \frac{4}{y^6} = 0, \tag{5.3}$$

and this equation has no real solutions. However, this projective structure admits a complex solution which we can determine as follows. From (5.3), F is necessarily given by

$$F = \pm\frac{2i}{y^3}.$$

Using the formula for α_a given by (2.25), we find that

$$\alpha_a = -\frac{y^6}{6}\nabla_a\left(\frac{1}{y^6}\right) \pm \frac{iy^3}{6}Y_a,$$

so that in local coordinates

$$\alpha = P dx + Q dy = \pm\frac{iy^3}{6}\left(\frac{6}{y^5}\right) dx + \frac{1}{y}dy = \pm\frac{i}{y^2}dx + \frac{1}{y}dy. \tag{5.4}$$

We conclude that this example admits no real solutions to pEW.

Remark 5.1. We can try to solve the pEW equation directly for this projective structure, however in this approach it is not clear whether we are finding all possible solutions. The pEW equation for this projective structure is given by the following system of equations:

$$\frac{\partial P}{\partial x} + \frac{Q}{4y^3} + P^2 + \frac{3}{4y^4} = 0, \quad \frac{\partial Q}{\partial y} + Q^2 = 0, \quad \frac{1}{2}\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial P}{\partial y} + QP = 0,$$

and completing to form the closed system gives two extra equations

$$\frac{\partial F}{\partial x} = -3PF - \frac{6}{y^5}, \quad \frac{\partial F}{\partial y} = -3QF,$$

where $F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Hence $Q = \frac{1}{y+s(x)}$ where s is a function of x or $Q = 0$ is the most general solution. If $Q = 0$, then $\frac{\partial P}{\partial y} = 0$ so that $F = 0$, which we eliminate. Substituting $Q = \frac{1}{y+s(x)}$ into the last equation gives

$$\frac{1}{2}\frac{\partial P}{\partial y} = -\frac{P}{y + s(x)} + \frac{s'(x)}{2(y + s(x))^2},$$

which solves to

$$P = \frac{a(x) + s'(x)y}{(y + s(x))^2}$$

This gives

$$F = \frac{a(x) - s'(x)s(x)}{(y + s(x))^3}.$$

Now set $s(x) = 0$, although we still have to check whether there other possible solutions where $s(x) \neq 0$. Then

$$P = \frac{a(x)}{y^2}, \qquad Q = \frac{1}{y},$$

and the first equation now becomes

$$\frac{a'(x)}{y^2} + \frac{1}{4y^4} + \frac{a(x)^2}{y^4} + \frac{3}{4y^4} = \frac{a'(x)y^2 + a(x)^2 + 1}{y^4} = 0.$$

Setting $a(x) = d$, where d is constant, (and again there might be other possible solutions, such as $a(x) = -\tan(\frac{x+c}{y^2})$, where c is constant) gives

$$\frac{1 + d^2}{y^4} = 0,$$

so that there are no real solutions. Over the complex field this recovers $\alpha = \pm\frac{i}{y^2}dx + \frac{1}{y}dy$. The method of computing α using the constraint equations (2.25) tell us definiteively that α has to be of the form (5.4) so that there are no possible real solutions.

5.4 Three dimensional non-projectively Ricci-skew examples

In this section we show that sol and Berger sphere geometries do not locally admit a solution to pEW. In the higher dimensional setting no local obstructions to pEW are known to the author. To set up the projective Einstein-Weyl equation on sol and the Berger sphere geometries (the conformal case is done in [13]), we consider the class of connections projectively related to the Levi-Civita connection ∇^g on M^3. The curvature tensor associated to ∇^g is given by

$$R_{ab}{}^c{}_d = \tilde{P}_a{}^c g_{bd} - \tilde{P}_b{}^c g_{ad} + \tilde{P}_{bd}\delta_a{}^c - \tilde{P}_{ad}\delta_b{}^c$$

where $\tilde{P}_{ab}$ is the conformal Schouten tensor on M. The projective Schouten tensor is given by

$$\mathrm{P}_{ab} = \frac{1}{2}R_{ca}{}^c{}_b = \frac{1}{2}(Jg_{ab} + \tilde{P}_{ab}).$$

Taking the trace free part of $R_{ab}{}^c{}_d$ gives

$$\begin{aligned}
W_{ab}{}^c{}_d =& R_{ab}{}^c{}_d - \frac{1}{2}\delta_a{}^c R_{bd} + \frac{1}{2}\delta_b{}^c R_{ad} \\
=& R_{ab}{}^c{}_d - \frac{1}{2}\delta_a{}^c (J g_{bd} + \tilde{P}_{bd}) + \frac{1}{2}\delta_b{}^c (J g_{bd} + \tilde{P}_{bd}) \\
=& \left(\tilde{P}_a{}^c - \frac{1}{2}\delta_a{}^c J\right) g_{bd} - \left(\tilde{P}_b{}^c - \frac{1}{2}\delta_b{}^c J\right) g_{ad} + \left(\frac{1}{2}\delta_b{}^c - \delta_b{}^c\right) \tilde{P}_{ad} \\
&+ \left(\delta_a{}^c - \frac{1}{2}\delta_a{}^c\right) \tilde{P}_{bd} \\
=& \left(\tilde{P}_a{}^c - \frac{1}{2}\delta_a{}^c J\right) g_{bd} - \left(\tilde{P}_b{}^c - \frac{1}{2}\delta_b{}^c J\right) g_{ad} - \frac{1}{2}\delta_b{}^c \tilde{P}_{ad} + \frac{1}{2}\delta_a{}^c \tilde{P}_{bd}.
\end{aligned}$$

The two examples fall under the Class A classification of homogeneous 3-geometries (see [13]). For background on the geometries of 3 dimensional Lie groups see [29]. The structural constants are given by

$$[e_1, e_2] = -(A + B)e_3 \qquad [e_3, e_1] = -(C + A)e_2 \qquad [e_2, e_3] = -(B + C)e_1$$

where A, B and C are constants. With respect to the chosen orthonormal frame (e_1, e_2, e_3), the projective Schouten tensor is diagonal and given by

$$\mathrm{P}_{11} = BC, \quad \mathrm{P}_{22} = AC, \quad \mathrm{P}_{33} = AB$$

and all remaining terms equal 0. Let the smooth 1-form α have components X, Y, Z. That is

$$\alpha = X\theta^1 + Y\theta^2 + Z\theta^3$$

where the basis of 1-forms θ^1, θ^2, θ^3 is dual to e_1, e_2, e_3. The projective Einstein-Weyl equation can now be expressed in terms of the variables X, Y, Z. They are given by the following system of 6 equations

$$\begin{aligned}
e_1X + X^2 + BC &= 0, & e_2X + e_1Y + 2XY + (A - B)Z &= 0, \\
e_2Y + Y^2 + AC &= 0, & e_3X + e_1Z + 2XZ + (C - A)Y &= 0, \\
e_3Z + Z^2 + AB &= 0, & e_3Y + e_2Z + 2YZ + (B - C)X &= 0.
\end{aligned}$$

We introduce $F_{ab} = \nabla_{[a}\alpha_{b]}$. Using the connection coefficients specified by tableau (2.9) in [13], this has components

$$\begin{aligned}
F_{12} &= \frac{1}{2}(\nabla_{e_1}\alpha)(e_2) - \frac{1}{2}(\nabla_{e_2}\alpha)(e_1) = \frac{1}{2}[e_1\alpha(e_2) - \alpha(\nabla_{e_1}e_2)] - \frac{1}{2}[e_2\alpha(e_1) - \alpha(\nabla_{e_2}e_1)] \\
&= \frac{1}{2}e_1Y - \frac{1}{2}e_2X + \left(\frac{A+B}{2}\right)Z, \\
F_{13} &= \frac{1}{2}(\nabla_{e_1}\alpha)(e_3) - \frac{1}{2}(\nabla_{e_3}\alpha)(e_1) = \frac{1}{2}[e_1Z - \alpha(\nabla_{e_1}e_3)] - \frac{1}{2}[e_3X - \alpha(\nabla_{e_3}e_1)] \\
&= \frac{1}{2}e_1Z - \frac{1}{2}e_3X - \left(\frac{A+C}{2}\right)Y, \\
F_{23} &= \frac{1}{2}(\nabla_{e_2}\alpha)(e_3) - \frac{1}{2}(\nabla_{e_3}\alpha)(e_2) = \frac{1}{2}[e_2Z - \alpha(\nabla_{e_2}e_3)] - \frac{1}{2}[e_3Y - \alpha(\nabla_{e_3}e_2)] \\
&= \frac{1}{2}e_2Z - \frac{1}{2}e_3Y + \left(\frac{B+C}{2}\right)X.
\end{aligned}$$

Let us call $P = F_{23}$, $Q = F_{31} = -F_{13}$, $R = F_{12}$. We can introduce these auxiliary variables to rewrite the pEW system as the following set of equations:

$$\begin{array}{lll}
e_1X = -BC - X^2 & e_2X = -R - XY + BZ & e_3X = -CY + Q - XZ \\
e_1Y = R - XY - AZ & e_2Y = -AC - Y^2 & e_3Y = CX - P - YZ \\
e_1Z = AY - Q - XZ & e_2Z = -BX + P - YZ & e_3Z = -AB - Z^2.
\end{array} \tag{5.5}$$

We can in turn determine the derivatives of P, Q and R by applying the commutation relations. This gives

$$\begin{aligned}
e_1P =&(A-B)C^2 + B^2(A-C) - 2PX + QY + RZ \\
e_2P =&BR + B(C-A)Z - 3PY \\
e_3P =&-CQ + C(A-B)Y - 3PZ \\
e_1Q =&-AR + A(B-C)Z - 3QX \\
e_2Q =&(B-C)A^2 + C^2(B-A) - 2QY + RZ + PX \\
e_3Q =&CP + C(A-B)X - 3QZ \\
e_1R =&AQ + A(B-C)Y - 3RX \\
e_2R =&-BP + B(C-A)X - 3RY \\
e_3R =&(C-A)B^2 + A^2(C-B) - 2RZ + PX + QY.
\end{aligned}$$

Together we have 6 constraints obtained from computing the commutation relations on P, Q and R. Three of them are given by

$$\begin{aligned}
-3RQ + A(C-B)P + X(AB^2 + A^2B + BC^2 - AC^2 - B^2C - A^2C) = 0, \\
-3QP + C(B-A)R + Z(CA^2 + C^2A + AB^2 - CB^2 - A^2B - C^2B) = 0, \\
-3PR + B(A-C)Q + Y(BC^2 + B^2C + CA^2 - BA^2 - C^2A - B^2A) = 0,
\end{aligned}$$

while the other three are given by

$$\begin{aligned}
&2Y^2C(A-B) - 2B(C-A)Z^2 - 6P^2\\
&+2(ABC^2 - B^2C^2 + AC^3 - A^2BC - BC^3 + AB^2C + AB^3 - B^3C) = 0,\\
&2X^2B(C-A) - 2A(B-C)Y^2 - 6R^2\\
&+2(CAB^2 - A^2B^2 + CB^3 - C^2AB - AB^3 + CA^2B + CA^3 - A^3B) = 0,\\
&2Z^2A(B-C) - 2C(A-B)X^2 - 6Q^2\\
&+2(BCA^2 - C^2A^2 + BA^3 - B^2CA - CA^3 + BC^2A + BC^3 - C^3A) = 0.
\end{aligned}$$

These are precisely the integrability conditions (1.7) for pEW to hold.

5.4.1 sol

Following [13], the structure constants for 'sol' are given by taking $A = 0$, $B = 1$, $C = -1$, so that the orthonormal basis e_1, e_2, e_3 commute according to

$$[e_1, e_2] = -e_3, \qquad [e_3, e_1] = e_2, \qquad [e_2, e_3] = 0.$$

With respect to this orthonormal frame, the Ricci tensor on sol is diagonal with components

$$R_{11} = -2, \qquad R_{22} = 0, \qquad R_{33} = 0,$$

and the conformal Schouten tensor is diagonal with components

$$\tilde{P}_{11} = -\frac{3}{2}, \qquad \tilde{P}_{22} = \frac{1}{2}, \qquad \tilde{P}_{33} = \frac{1}{2}.$$

The metric trace of the conformal Schouten tensor is

$$J = g^{ab}\tilde{P}_{ab} = P_{11} + P_{22} + P_{33} = -\frac{3}{2} + \frac{1}{2} + \frac{1}{2} = -\frac{1}{2}.$$

The projective Schouten tensor is easily computed and is also diagonal with components

$$\mathrm{P}_{11} = -1, \qquad \mathrm{P}_{22} = 0, \qquad \mathrm{P}_{33} = 0.$$

The projective Weyl tensor has components given by

$$\begin{aligned}
W_{12}{}^{1}{}_{2} &= W_{e_1e_2}{}^{e_1}{}_{e_2} = -1, & W_{32}{}^{3}{}_{2} &= W_{e_3e_2}{}^{e_3}{}_{e_2} = 1,\\
W_{23}{}^{2}{}_{3} &= W_{e_2e_3}{}^{e_2}{}_{e_3} = 1, & W_{13}{}^{1}{}_{3} &= W_{e_1e_3}{}^{e_1}{}_{e_3} = -1,
\end{aligned}$$

and all other components vanish. The projective Cotton-York tensor is non-zero and has components

$$Y_{123} = 1, \quad Y_{231} = 0, \quad Y_{312} = -1,$$

and all other components vanish. To see this, we can compute explicitly. For instance,

$$\begin{aligned}
Y(e_1,e_2,e_3) =&(\nabla_{e_1}\mathrm{P})(e_2,e_3)-(\nabla_{e_2}\mathrm{P})(e_1,e_3)\\
=&e_1\mathrm{P}(e_2,e_3)-\mathrm{P}(\nabla_{e_1}e_2,e_3)-\mathrm{P}(e_1,\nabla_{e_2}e_3)-e_2\mathrm{P}(e_1,e_3)\\
&+\mathrm{P}(\nabla_{e_2}e_1,e_3)+\mathrm{P}(e_1,\nabla_{e_2}e_3)\\
=&0-0-0-0+0+\mathrm{P}(e_1,-e_1)\\
=&-\mathrm{P}(e_1,e_1)\\
=&1.
\end{aligned}$$

The pEW equation for sol is given by

$$\begin{aligned}
e_1X+X^2-1&=0, & e_2X+e_1Y-Z+2XY&=0,\\
e_2Y+Y^2&=0, & e_3X+e_1Z-Y+2XZ&=0,\\
e_3Z+Z^2&=0, & e_3Y+e_2Z+2YZ+2X&=0,
\end{aligned}$$

and the associated closed system is given by

$$\begin{array}{lll}
e_1X=-X^2+1 & e_2X=Z-R-XY & e_3X=Y+Q-XZ\\
e_1Y=R-XY & e_2Y=-Y^2 & e_3Y=-X-P-YZ\\
e_1Z=-Q-XZ & e_2Z=-X+P-YZ & e_3Z=-Z^2.
\end{array}\tag{5.6}$$

We can in turn determine the derivatives of P, Q, R by applying the commutation relations. This gives

$$\begin{array}{lll}
e_1P=-2PX+QY+RZ & e_2P=R-Z-3PY & e_3P=Q+Y-3PZ\\
e_2Q=-2QY+RZ+PX+1 & e_3Q=X-P-3QZ & e_1Q=-3QX\\
e_3R=-2RZ+PX+QY-1 & e_1R=-3RX & e_2R=-X-3RY-P.
\end{array}\tag{5.7}$$

A computation shows that the commutation relations are satisfied on X,Y and Z, but give immediate constraints when applied to P, Q, R. For example, one expects

$$[e_2,e_3]P=0.$$

Expanding e_2e_3P gives

$$-Y^2-2RZ+3Z^2+12PYZ+4PX-3P^2-2QY+1$$

while expanding e_3e_2P gives

$$4PX+12PZY-2QY-1+Z^2-3Y^2+3P^2-2RZ.$$

Taking the difference, we obtain the first constraint that

$$[e_2, e_3]P = e_2e_3P - e_3e_2P = 2Z^2 - 6P^2 + 2 + 2Y^2 = 0,$$

or

$$3P^2 = Z^2 + Y^2 + 1.$$

Together we have 6 constraints obtained from computing the commutation relations on P,Q,R. The other five are

$$\begin{aligned} &Q = 3PR, \qquad && R = -3QP, \qquad && 2X = 3QR, \\ &-3Q^2 = 1 + X^2, \qquad && -3R^2 = 1 + X^2. && \end{aligned}$$

These constraints can be computed from the equation (1.8) and they form an inconsistent system. Squaring $2X = 3QR$ gives

$$4X^2 = 9Q^2R^2 = (-3Q^2)(-3R^2) = (1 + X^2)(1 + X^2) = 1 + 2X^2 + X^4,$$

which gives

$$X^4 - 2X^2 + 1 = (X^2 - 1)^2 = 0.$$

Solving for X, we have $X^2 = 1$, or $X = \pm 1$. This implies that

$$Q^2 = R^2 = \frac{-1 - X^2}{3} = -\frac{2}{3},$$

and so Q, R have no real solutions. If complex solutions are allowed, with $Q = \pm\sqrt{\frac{2}{3}}i$, $R = \pm\sqrt{\frac{2}{3}}i$, we can compute

$$Q^2 = (3PR)^2 = 9P^2R^2.$$

Since $Q^2 = R^2$, cancellation on both sides gives

$$1 = 9P^2,$$

or $P = \pm\frac{1}{3}$. Now multiplying $Q = 3PR$ on both sides by $-3P$ gives

$$R = -3QP = -9P^2R = -R,$$

or $R = 0$, which is an inconsistency.

5.4.2 Berger sphere

Substituting $A = 1$, $B = 1$, $C = t$ into the 6 constraint equations give

$$\begin{aligned}
0 =& -3RQ + (t-1)P + X(2-2t) \\
0 =& -3QP \\
0 =& -3PR + (1-t)Q + Y(2t-2) \\
0 =& -2(t-1)Z^2 - 6P^2 + 2(1-t) \\
0 =& 2X^2(t-1) - 2(1-t)Y^2 - 6R^2 + 2(4t-3-t^2) \\
0 =& 2Z^2(1-t) - 6Q^2 + 2(1-t).
\end{aligned}$$

For $t = 1$, we recover the standard sphere and this forces $P = Q = R = 0$, i.e. $F_{ab} = 0$, so that solutions are projective rescalings of Ricci-flat connections. For $t \neq 1$, $3PQ = 0$ forces either P or $Q = 0$. Assume $P = 0$. Then from $0 = -2(t-1)Z^2 - 6P^2 + 2(1-t)$ we get

$$Z^2 = -1,$$

so that $Z = \pm i$. Substituting this into the equation $0 = 2Z^2(1-t) - 6Q^2 + 2(1-t)$ we obtain

$$Q = 0,$$

but from the equation $0 = -3PR + (1-t)Q + Y(2t-2)$ we get

$$Q = 2Y,$$

so $Y = 0$, and consequently $X = 0$. Also, $3R^2 = -(t-1)(t-3)$. Similarly for $Q = 0$, we have $\alpha = \pm i\theta^3$. We conclude that over the reals, the Berger sphere does not admit a solution to (1.4).

5.5 Three dimensional examples

Here we give examples of projective structures in 3 dimensions that admit a solution to pEW.

5.5.1 3 functions of 2 variables

In (x, y, t) space consider 3 functions of 2 variables $P = P(x, y)$, $Q = Q(y, t)$, $R = R(x, t)$, i.e. P is independent of t, Q is independent of x, R is independent of y. We claim that the affine connection ∇ with connection symbols given by

$$\Gamma^1_{11} = \Gamma^x_{xx} = \partial_x P - \partial_x R,$$
$$\Gamma^2_{22} = \Gamma^y_{yy} = -\partial_y P + \partial_y Q,$$
$$\Gamma^0_{00} = \Gamma^t_{tt} = \partial_t R - \partial_t Q,$$

and all remaining connection symbols equaling zero has skew-symmetric Ricci. A computation using

$$R_{ca}{}^c{}_b = \partial_c \Gamma^c_{ab} - \partial_a \Gamma^c_{cb} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^c_{ad}\Gamma^d_{cb} \tag{5.8}$$

gives

$$\begin{aligned}
&R_{11} = R_{22} = R_{00} = 0, && \\
&R_{12} = -\partial_x \Gamma^y_{yy} = \partial_x \partial_y P, && R_{21} = -\partial_y \Gamma^x_{xx} = -\partial_y \partial_x P, \\
&R_{10} = -\partial_x \Gamma^t_{tt} = -\partial_x \partial_t R, && R_{01} = -\partial_t \Gamma^x_{xx} = \partial_t \partial_x R, \\
&R_{20} = -\partial_y \Gamma^t_{tt} = \partial_y \partial_t Q, && R_{02} = -\partial_t \Gamma^y_{yy} = -\partial_y \partial_t Q,
\end{aligned}$$

thus verifying the skew-symmetry of the Ricci tensor. Now take the projective class related to this affine connection. Using that $\alpha_a = -\frac{1}{n+1}\Gamma^c_{ac}$, we compute that in a local coordinate system $\{dx, dy, dt\}$, α is given by $\alpha_1 dx + \alpha_2 dy + \alpha_0 dt = -\frac{1}{4}(\partial_x P - \partial_x R)dx - \frac{1}{4}(-\partial_y P + \partial_y Q)dy - \frac{1}{4}(\partial_t R - \partial_t Q)dt$. We have under a projective change of connections of the form $\Gamma^c_{ab} \to \hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta^c_a \alpha_b + \delta^c_b \alpha_a$, that the connections symbols get modified as follows:

$$\begin{aligned}
&\hat{\Gamma}^1_{12} = \Gamma^1_{12} + \alpha_2 = \frac{1}{4}P_y - \frac{1}{4}Q_y, && \hat{\Gamma}^0_{02} = \Gamma^0_{02} + \alpha_2 = \frac{1}{4}P_y - \frac{1}{4}Q_y, \\
&\hat{\Gamma}^0_{01} = \Gamma^0_{01} + \alpha_1 = \frac{1}{4}R_x - \frac{1}{4}P_x, && \hat{\Gamma}^2_{21} = \Gamma^2_{21} + \alpha_1 = \frac{1}{4}R_x - \frac{1}{4}P_x, \\
&\hat{\Gamma}^1_{10} = \Gamma^1_{10} + \alpha_0 = \frac{1}{4}Q_t - \frac{1}{4}R_t, && \hat{\Gamma}^2_{20} = \Gamma^2_{20} + \alpha_0 = \frac{1}{4}Q_t - \frac{1}{4}R_t, \\
&\hat{\Gamma}^0_{00} = \Gamma^0_{00} + 2\alpha_0 = \frac{1}{2}R_t - \frac{1}{2}Q_t, && \hat{\Gamma}^1_{11} = \Gamma^1_{11} + 2\alpha_1 = \frac{1}{2}P_x - \frac{1}{2}R_x, \\
&\hat{\Gamma}^2_{22} = \Gamma^2_{22} + 2\alpha_2 = \frac{1}{2}Q_y - \frac{1}{2}P_y. &&
\end{aligned}$$

The Ricci tensor is now symmetric, and we check that α_a satisfies the pEW equation $\nabla_{(a}\alpha_{b)} + \alpha_a \alpha_b + \frac{1}{2}R_{ab} = 0$.

5.5.2 2 functions of 3 variables

This example is found by experimenting with MAPLE. In (x, y, t) space consider 2 functions of 3 variables $\phi = \phi(x, y, t)$, $\psi = \psi(x, y, t)$. We claim that the torsion-free affine connection ∇ with connection symbols given by

$$
\begin{aligned}
&\Gamma^1_{10} = \partial_x\phi, \qquad \Gamma^1_{20} = \partial_y\phi, && \Gamma^2_{10} = \partial_x\psi, \qquad \Gamma^2_{20} = \partial_y\psi,\\
&\Gamma^1_{00} = (2+c)\psi\partial_y\phi + c\phi\partial_y\psi + \partial_t\phi, && \Gamma^2_{00} = -(2+c)\psi\partial_x\phi - c\phi\partial_x\psi + \partial_t\psi,\\
&\Gamma^0_{00} = \partial_x\phi + \partial_y\psi,
\end{aligned}
$$

where c is an arbitrary constant, and all remaining connection symbols equal 0 has skew-symmetric Ricci. Again using (5.8), we obtain

$$
\begin{aligned}
&R_{11} = R_{22} = R_{00} = R_{12} = R_{21} = 0,\\
&R_{10} = \phi_{xx} + \psi_{xy}, && R_{01} = -\phi_{xx} - \psi_{xy},\\
&R_{20} = \phi_{xy} + \psi_{yy}, && R_{02} = -\phi_{xy} - \psi_{yy}.
\end{aligned}
$$

Now take the projective class related to this affine connection. Using that $\alpha_a = -\frac{1}{n+1}\Gamma^c_{ac}$, we compute that the only non-vanishing component of α_a is given by $\alpha_0 = -\frac{1}{2}(\partial_x\phi + \partial_y\psi)$. We have under a projective change of connections of the form $\Gamma^c_{ab} \to \hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta^c_a\alpha_b + \delta^c_b\alpha_a$, that the connections symbols get modified as follows:

$$
\begin{aligned}
&\hat{\Gamma}^1_{10} = \Gamma^1_{10} + \alpha_0 = \frac{1}{2}\partial_x\phi - \frac{1}{2}\partial_y\psi, && \hat{\Gamma}^1_{00} = \Gamma^1_{00} = (2+c)\psi\partial_y\phi + c\phi\partial_y\psi + \partial_t\phi,\\
&\hat{\Gamma}^2_{20} = \Gamma^2_{20} + \alpha_0 = \frac{1}{2}\partial_x\phi - \frac{1}{2}\partial_y\psi, && \hat{\Gamma}^2_{00} = \Gamma^2_{00} = -(2+c)\psi\partial_x\phi - c\phi\partial_x\psi + \partial_t\psi,\\
&\hat{\Gamma}^1_{20} = \Gamma^1_{20} = \partial_y\phi, && \hat{\Gamma}^2_{10} = \Gamma^2_{10} = \partial_x\psi,\\
&\hat{\Gamma}^0_{00} = \Gamma^0_{00} + 2\alpha_0 = 0.
\end{aligned}
$$

The Ricci tensor is now symmetric, with components

$$
\begin{aligned}
&R_{11} = R_{22} = R_{12} = R_{21} = 0, && R_{00} = \phi_{zx} + \psi_{zy} - \frac{1}{2}(\phi_x + \psi_y)^2,\\
&R_{10} = R_{01} = \frac{1}{2}\psi_{xy} + \frac{1}{2}\phi_{xx}, && R_{20} = R_{02} = \frac{1}{2}\psi_{yy} + \frac{1}{2}\phi_{xy},
\end{aligned}
$$

and we check that α_a satisfies the pEW equation $\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \frac{1}{2}R_{ab} = 0$.

5.6 Higher dimensional examples

We generalise the 3 functions of 2 variables case in 3-dimensions to $\binom{n}{2}$ functions of 2 variables in n dimensions. This is also an appropriate generalization of Derdzinski's normal form [8] to higher dimensions. In $(x_1, x_2, \ldots x_n)$ space consider $\binom{n}{2}$ functions of 2 variables denoted by $P_{ij} = -P_{ji}$, $i \neq j$, where P_{ij} depend only on x_i and x_j. We claim that the affine connection ∇ with connection symbols given by

$$\Gamma^i_{ii} = \sum_j \partial_i P_{ij}$$

for $1 \leq i \leq n$ and all remaining connection symbols equal 0 gives rise to an affine structure with skew-symmetric Ricci. A computation using (5.8) gives

$$R_{ci}{}^c{}_i = \partial_i \Gamma^i_{ii} - \partial_i \Gamma^i_{ii} + \Gamma^i_{ii}\Gamma^i_{ii} - \Gamma^i_{ii}\Gamma^i_{ii} = 0,$$

and

$$R_{ci}{}^c{}_j = \partial_c \Gamma^c_{ij} - \partial_i \Gamma^c_{cj} + \Gamma^c_{cd}\Gamma^d_{ij} - \Gamma^c_{id}\Gamma^d_{cj} = -\, \partial_i \Gamma^j_{jj} = -\partial_i \partial_j P_{ji},$$

since all other terms in $\Gamma^j_{jj} = \sum_k \partial_j P_{jk}$ that do not depend on x_i will be annihilated by ∂_i. But now

$$R_{ci}{}^c{}_j = -\partial_i \partial_j P_{ji} = \partial_i \partial_j P_{ij} = \partial_j \partial_i P_{ij} = -R_{cj}{}^c{}_i,$$

thus verifying the skew-symmetry of the Ricci tensor for this given affine structure.

Chapter 6

Local obstructions to a conformally invariant equation on Möbius surfaces

6.1 Introduction

In this chapter and the next we discuss conformal geometry, and it will be clear from context that tensor quantities in conformal geometry are different from the ones derived in projective geometry even though the script used to denote them are similar. Let $(M^2, [g])$ be a Riemann surface. This is a smooth 2-dimensional oriented manifold equipped with a conformal structure $[g]$, which is an equivalence class of smooth Riemannian metrics under the equivalence relation $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$ for any smooth positive nowhere vanishing function Ω. Since every metric g_{ab} in dimension 2 is locally a conformal rescaling of the flat metric δ_{ab}, conformal geometry in dimension 2 carries no local information. To remedy this, one can impose on Riemann surfaces additional local structure present in conformal manifolds of dimension $n > 2$. This is the motivation behind Möbius structures [5]. A Riemann surface with a Möbius structure will henceforth be called a Möbius surface. We can study on Möbius surfaces a well-defined conformally invariant equation, which we call the scalar-flat Möbius Einstein-Weyl (sf-MEW) equation. This equation specialises the Einstein-Weyl equation in conformal geometry in higher dimensions to the 2-dimensional setting. We derive algebraic constraints for a given Möbius surface to admit a solution to sf-MEW, and from there derive obstructions to existence of solutions to the equation. Checking that the obstructions do not vanish tells us definitively that the

Möbius surface cannot admit any solution to the sf-MEW equation locally. Most of the material in this chapter appears in [27]. Again abstract indices [26] will be used throughout Chapters 6 and 7 to describe tensors on the conformal manifold. We have already used g_{ab} to denote the metric tensor. For another instance, if we write ω_a to denote a smooth 1-form ω, then the 2-form $d\omega$ can be written as $\nabla_{[a}\omega_{b]} = \frac{1}{2}(\nabla_a\omega_b - \nabla_b\omega_a)$.

6.2 Conformal geometry and Möbius structures

A Möbius surface is a Riemann surface $(M^2, [g])$ equipped with a smooth Möbius structure as defined in 2.1 of [5]. Taking the Weyl derivative D_a in the definition to be the Levi-Civita connection ∇_a for a particular representative metric g_{ab} in the conformal class $[g]$, a Möbius structure on $(M^2, [g])$ is a smooth second order linear differential operator $D_{(ab)_\circ} : \mathcal{E}[1] \to \mathcal{E}_{(ab)_\circ}[1]$ such that $D_{(ab)_\circ} - \nabla_{(a}\nabla_{b)_\circ}$ is a zero order operator acting on sections of the density line bundle of weight 1, denoted by $\mathcal{E}[1]$. The definition of a density line bundle in the conformal setting can be found in [2] for instance. Let $\mathrm{P}_{(ab)_\circ}$ be the symmetric trace-free tensor denoting the difference, i.e.

$$\mathrm{P}_{(ab)_\circ}\sigma := (D_{(ab)_\circ} - \nabla_{(a}\nabla_{b)_\circ})\sigma,$$

where σ is a section of $\mathcal{E}[1]$. Since the operator $D_{(ab)_\circ}$ is invariantly defined, under a conformal rescaling of the metric $\widehat{g}_{ab} = \Omega^2 g_{ab}$ we find that

$$\widehat{\mathrm{P}}_{(ab)_\circ} = \mathrm{P}_{(ab)_\circ} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c + \frac{1}{2}g_{ab}\nabla_c\Upsilon^c$$

where $\Upsilon_a = \nabla_a \log\Omega$. Let K denote the Gauss curvature of g_{ab}, i.e. $K = \frac{R}{2}$, where R is the scalar curvature of g_{ab}. Define the Rho tensor by $\mathrm{P}_{ab} := \mathrm{P}_{(ab)_\circ} + \frac{K}{2}g_{ab}$. Under a conformal rescaling, K transforms as $\widehat{K} = K - \nabla_a\Upsilon^a$ and therefore

$$\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c.$$

Hence for any representative metric g_{ab} in the conformal class $[g]$ with its associated Levi-Civita connection ∇_a, a Möbius structure in the sense of [5] determines a symmetric tensor P_{ab} satisfying the following two properties:
1) The metric trace of P_{ab} is the Gauss curvature K of g_{ab};
2) Under a conformal rescaling of the metric $\hat{g}_{ab} = \Omega^2 g_{ab}$, the tensor P_{ab} transforms accordingly as

$$\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c, \tag{6.1}$$

where $\Upsilon_a = \nabla_a \log \Omega$. This is the definition of a Möbius structure given in [11] which we shall subsequently use. A Möbius surface will be given by $(M^2, [g], [\mathrm{P}])$, where $[\mathrm{P}]$ denotes the Möbius structure on $(M^2, [g])$, which is the equivalence class of smooth symmetric tensors related to a representative Rho tensor P_{ab} by formula (6.1) under conformal rescalings of the metric. In 2 dimensions, the Schouten tensor P_{ab} is not well-defined and a Möbius structure remedies that by equipping the manifold with a Rho tensor that behaves like the Schouten tensor under conformal rescalings. In 2 dimensions, a Rho tensor P_{ab} allows us to write

$$R_{abcd} = K(g_{ac}g_{bd} - g_{bc}g_{ad}) \equiv \mathrm{P}_{ac}g_{bd} - \mathrm{P}_{bc}g_{ad} + \mathrm{P}_{bd}g_{ac} - \mathrm{P}_{ad}g_{bc}, \tag{6.2}$$

even though the tensor P_{ab} cannot be recovered from the Riemannian curvature tensor alone, in contrast to the higher dimensional setting. A fixed representative metric g_{ab} from the conformal class $[g]$ can be viewed as having conformal weight 2 and induces a volume form $\epsilon_{ab} = \epsilon_{[ab]}$ of conformal weight 2. We set our convention so that $\epsilon^{ab}\epsilon_{cb} = \delta_c{}^a$ and we raise and lower indices using the metric. The Cotton-York tensor given by

$$Y_{abc} = \nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac}$$

is a conformal invariant of the Möbius structure. This means that it is a universal polynomial in the jets of the Möbius structure that is invariantly defined as a conformally weighted tensor. Under conformal rescalings of the metric, the quantity Y_{abc} remains unchanged. We can use the volume form ϵ_{ab} to dualise, so that

$$Y_{abc} = \frac{1}{2}\epsilon_{ab}Y_c,$$

where $Y_c = \epsilon^{ab}Y_{abc}$ is now a 1-form of conformal weight -2. Observe that

$$\frac{1}{2}\epsilon_{ab}Y^b = Y_{ab}{}^b = \nabla_a K - \nabla^b \mathrm{P}_{ab}.$$

The vanishing of Y_a characterises flat Möbius surfaces. In [5] it is shown that flat Möbius surfaces are precisely those which are integrable.

6.3 The MEW equation on Möbius surfaces

A Weyl connection D_a on a Riemann surface $(M^2, [g])$ is a torsion-free connection that preserves the conformal class $[g]$, or equivalently $D_a g_{bc} = 2\alpha_a g_{bc}$ for some 1-form α_a. The 1-form α_a is determined up to a gauge freedom; under a conformal rescaling of the metric $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$, we have $\alpha_a \mapsto \widehat{\alpha}_a = \alpha_a + \Upsilon_a$, where again $\Upsilon_a = \nabla_a \log \Omega$. The Möbius Einstein-Weyl (MEW) equation asks for a fixed Möbius surface $(M^2, [g], [\mathrm{P}])$ whether there is a compatible Weyl connection D_a such that the second order linear differential operator $D_{(ab)_\circ}$ is given by the trace-free symmetric Hessian of the Weyl derivative, i.e. $D_{(ab)_\circ} = D_{(a}D_{b)_\circ}$. On Möbius surfaces $(M^2, [g], [\mathrm{P}])$, the MEW equation for a representative metric $g_{ab} \in [g]$ and its associated Levi-Civita connection ∇_a is given by

$$\text{Trace-free part of } (\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab}) = 0.$$

Let Λ denote the trace term, so that we have

$$\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} = \Lambda g_{ab}. \tag{6.3}$$

The nomenclature MEW is chosen because (6.3) is the Einstein-Weyl equation in conformal geometry in higher dimensions. In higher dimensions, the Einstein-Weyl equation asks for the symmetric part of the Ricci tensor of the Weyl connection D_a to be pure trace, see for instance [12] and [13]. This is a conformally invariant system of PDEs. Analogous to the pEW equation, a common procedure to treat equations such as (6.3) is through prolongation [3]. This involves expressing first derivatives of the dependent variables in terms of the variables themselves. We get a closed system if the overdetermined system is finite type. However, in 2 dimensions (6.3) is not finite type in the sense of [30] because of its symbol. (We can compare this to the conformal Killing equation also has the same symbol and is also not finite type in 2 dimensions). To see this explicitly, we can attempt to prolong equation (6.3) and see where it leads to. Introducing $s = \alpha^a\alpha_a - 2\Lambda$, we can rewrite (6.3) as

$$\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha_c\alpha^c}{2}g_{ab} = -\frac{s}{2}g_{ab}. \tag{6.4}$$

The reason for doing so is that s is the scalar curvature of the Weyl connection D_a and is conformally invariant of weight -2. It makes the conformal invariance of the left-hand side term also more apparent. Introducing $F_{ab} = \frac{1}{2}\epsilon_{ab}F = \nabla_{[a}\alpha_{b]}$ as the extra dependent variable where $F = \epsilon^{ab}F_{ab}$, we can rewrite (6.4) as

$$\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha^c\alpha_c}{2}g_{ab} = \frac{1}{2}\epsilon_{ab}F - \frac{s}{2}g_{ab}. \tag{6.5}$$

We now attempt to write a closed system for this equation. Differentiating and skewing (6.4), we find

$$\nabla_a F = -2\alpha_a F + 2\epsilon_{ab}M^b, \tag{6.6}$$

where tautologically

$$M_a = \nabla^b F_{ba} + 2\alpha^b F_{ba} = \frac{1}{2}\epsilon_{ba}\nabla^b F + \epsilon_{ba}\alpha^a F.$$

Also, we have

$$\nabla_a s = 2M_a - 2\alpha_a s - \epsilon_{ac}Y^c \tag{6.7}$$

as a consequence of (6.4). However, we cannot express $\nabla_a M_b$ in terms of the other known quantities to close up the system. We have the following

Proposition 6.1. *Prolongation of (6.4) does not form a closed system in* 2 *dimensions.*

In dimensions $n > 2$, the EW equation (6.3) is finite type. Prolongation of (6.3) in general gives us $\frac{n-2}{n-1}\nabla_a M_b$ in terms of the other known variables of the system and (6.3) can be prolonged to form a closed system. In 2 dimensions the expression $\frac{n-2}{n-1}\nabla_a M_b$ vanishes and the only piece of information that remains is the divergence of M_a:

$$\nabla_a M^a = -2\alpha_a M^a - F^2.$$

Since the equation cannot be prolonged to form a closed system, it makes no sense to derive local obstructions as we cannot derive integrability constraints on the solutions. Equation (6.7) is precisely Calderbank's definition of Einstein-Weyl when $Y_a = 0$ (see the first equation of [5]). When the Möbius structure is flat and (6.4) holds, we have

$$D_a s - 2D^b F_{ba} = \nabla_a s + 2\alpha_a s - 2\nabla^b F_{ba} - 4\alpha^b F_{ba} = 2M_a - 2M_a = 0.$$

Note however that [5] does not mention (6.7) is a differential consequence of (6.4). While Calderbank's EW equation is defined on flat Möbius surfaces, our MEW equation is defined on general non-flat Möbius surfaces. The MEW equation becomes finite type if we consider further restrictions on the Weyl structure. By taking $s = 0$ in (6.4), we obtain the sf-MEW equation. This will be discussed in the next section. By taking $F = 0$ instead, we obtain the conformally Einstein equation. This will be discussed in Chapter 7.

6.4 The sf-MEW equation on Möbius surfaces

If we impose the additional condition that the scalar curvature s of the Weyl connection D_a is 0, the equation becomes finite type and forms a closed system.

This is the sf-MEW equation and is given by

$$\nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha_c\alpha^c}{2}g_{ab} = 0. \tag{6.8}$$

The 'scalar-flat' in the terminology of sf-MEW hence refers to the vanishing of the scalar curvature of D_a. The sf-MEW equation is a conformally invariant, finite type, overdetermined system of semi-linear partial differential equations. Taking $s = 0$ in (6.5) gives

$$\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha^c\alpha_c}{2}g_{ab} = \frac{1}{2}\epsilon_{ab}F. \tag{6.9}$$

Observe that tracing the equation gives $\nabla_a\alpha^a + K = 0$ as a necessary constraint of the system. Differentiating (6.9), we find that

$$\nabla_a F = -2\alpha_a F - Y_a$$

is a consequence of the original equation. The derivative of the extra dependent variable F is now given by known quantities α_a, F and Y_a of the system. The prolonged system is therefore given by

$$\nabla_a\alpha_b = \frac{1}{2}\epsilon_{ab}F + \frac{\alpha^c\alpha_c}{2}g_{ab} - \alpha_a\alpha_b - \mathrm{P}_{ab}, \tag{6.10}$$

$$\nabla_a F = -2\alpha_a F - Y_a. \tag{6.11}$$

Equation (6.11) can similarly be obtained by taking $s = 0$ in (6.7) and substituting the formula for $M_a = \frac{1}{2}\epsilon_{ab}Y^b$ into (6.6). We can use the prolonged system to derive algebraic constraints for there to be a solution to (6.8). We obtain as a necessary condition for there to be solutions of (6.8) three polynomial equations in a single variable t with coefficients given by conformal invariants of the Möbius structure, under the assumption that a certain conformal invariant M_{ab} defined in (6.22) is non-zero. This is the result of Theorem 6.4. The resultants of any two of these 3 polynomials will then have to vanish for there to be a common root t, and since the resultants are given only by the invariants of the Möbius structure, we obtain obstructions for there to be solutions of (6.8) in Corollary 6.5. We also discuss the case where $M_{ab} = 0$ in Theorem 6.6. We conclude the chapter in Section 6.8 by giving three examples of Möbius structures on $\mathbb{R}^2$ with the flat metric δ_{ab} for which one admits a solution to (6.8), one does not because of non-vanishing obstructions and one with vanishing obstructions but does not admit any real solution to (6.8).

Remark 6.2. We remark on the case of global solutions to (6.8) on closed Möbius surfaces $(M^2, [g], [\mathrm{P}])$. Since $\nabla_a \alpha^a + K = 0$ is a consequence, we find that integrating this equation over M^2, the integral of the divergence term vanishes, so that $0 = -\int_{M^2} K = -2\pi\chi(M^2)$ by the Gauss-Bonnet formula and this implies that M^2 has to be diffeomorphic to the torus.

6.5 Deriving algebraic constraints for sf-MEW to hold

In this section we derive the algebraic constraints that have to be satisfied for equation (6.9) to hold. In the flat case when $Y_a = 0$, we necessarily have $F = 0$ by differentiating (6.11) and skewing. The 1-form α_a is therefore exact, and equation (6.9) specialises to the conformally Einstein equation for flat Möbius surfaces. We therefore restrict our attention to non-flat Möbius structures, that is one with Y_a non-zero. By (6.11) this ensures that $F \neq 0$. Introduce the vector $U^a = \epsilon^{ab} Y_b$. Locally this is obtained by rotating the vector Y^a 90° clockwise on the plane. We have $Y_b = U^a \epsilon_{ab}$. The 1-form U_a has conformal weight -2. Differentiating (6.11) and skewing with ϵ^{ac} gives the first constraint of the system, namely that

$$-2F^2 = \epsilon^{ac} \nabla_a Y_c + 2\epsilon^{ac} Y_c \alpha_a = \nabla_a U^a + 2\alpha_a U^a. \tag{6.12}$$

Define the quantities

$$\mu := \frac{\nabla_c Y^c}{2}, \qquad \phi := \frac{\nabla_c U^c}{2}.$$

The scalar μ has weight -4 and transforms as $\widehat{\mu} = \mu - \Upsilon_a Y^a$ under conformal rescaling. The scalar ϕ also has weight -4 and transforms as $\widehat{\phi} = \phi - \Upsilon_a U^a$ under conformal rescaling. We can rewrite equation (6.12) as

$$\alpha_a U^a = -F^2 - \phi. \tag{6.13}$$

Set $W_a = Y^c \nabla_c U_a + \phi Y_a - 3\mu U_a$. The 1-form W_a can be verified to be conformally invariant of weight -6. To see this, we have

$$\begin{aligned}
\widehat{W}_a =& Y^c \widehat{\nabla}_c U_a + \widehat{\phi} Y_a - 3\widehat{\mu} U_a \\
=& Y^c(\nabla_c U_a - 3\Upsilon_c U_a - \Upsilon_a U_c + g_{ca} \Upsilon^e U_e) + (\phi - \Upsilon_c U^c) Y_a - 3(\mu - \Upsilon_c Y^c) U_a \\
=& Y^c \nabla_c U_a - 3\Upsilon_c Y^c U_a + Y_a \Upsilon^e U_e + \phi Y_a - \Upsilon_c U^c Y_a - 3\mu U_a + 3\Upsilon_c Y^c U_a \\
=& W_a.
\end{aligned}$$

Differentiating (6.13) and using (6.10), we find that

$$\alpha_a W^a = 3(\alpha_b Y^b)F^2 + \frac{5}{2}\rho F - Y^b \nabla_b \phi + \mathrm{P}_{ba} U^a Y^b + 3\mu F^2 + 3\mu\phi,$$

where $\rho := U_a U^a = Y_a Y^a$ and has weight -6. To simplify notation, set

$$\ell = 3\mu\phi + \mathrm{P}_{ab} U^a Y^b - Y^c \nabla_c \phi.$$

The scalar ℓ has conformal weight -8. We have

Proposition 6.3. *Under a conformal rescaling,* $\widehat{\ell} = \ell + \Upsilon_a W^a$.

Proof. We have

$$\begin{aligned}
\mathrm{P}_{ab}\widehat{U^a Y^b} &= \mathrm{P}_{ab} U^a Y^b - (\nabla_a \Upsilon_b) U^a Y^b + \Upsilon_a \Upsilon_b U^a Y^b,\\
3\widehat{\mu\phi} &= 3\mu\phi - 3\phi(\Upsilon_b Y^b) - 3\mu(\Upsilon_a U^a) + 3(\Upsilon_a U^a)(\Upsilon_b Y^b),\\
-\widehat{Y^c \nabla_c \phi} &= -Y^c \nabla_c \phi + Y^c U^e (\nabla_c \Upsilon_e) + \Upsilon_e (Y^c \nabla_c U^e) + 2\Upsilon_c Y^c (\nabla_b U^b) - 4(\Upsilon_c Y^c)(\Upsilon_b U^b).
\end{aligned}$$

A computation shows

$$\widehat{\ell} = \ell + \Upsilon_a (Y^c \nabla_c U^a + \phi Y^a - 3\mu U^a) = \ell + \Upsilon_a W^a.$$

□

We find that

$$\alpha_a W^a = \ell + \frac{5}{2}\rho F + (3\mu + 3\alpha_a Y^a)F^2. \tag{6.14}$$

We can now solve for α_a assuming the scalar density σ of conformal weight -10 given by $\sigma := Y_a W^a \neq 0$. It is given by

$$\alpha_a = \frac{Y_a}{\sigma}\left(\ell + \frac{5}{2}\rho F + (3\mu + 3\alpha_c Y^c)F^2\right) - \frac{\epsilon_{ab} W^b}{\sigma}\left(F^2 + \phi\right). \tag{6.15}$$

Contracting with Y^a on both sides gives

$$\alpha_a Y^a = \frac{\rho}{\sigma}\left(\ell + \frac{5}{2}\rho F + (3\mu + 3\alpha_c Y^c)F^2\right) + \frac{\tau}{\sigma}\left(F^2 + \phi\right),$$

where $\tau := U_a W^a$ has conformal weight -10. Hence

$$\left(1 - \frac{3\rho F^2}{\sigma}\right)\alpha_a Y^a = \frac{\rho}{\sigma}\left(\ell + \frac{5}{2}\rho F + 3\mu F^2\right) + \frac{\tau}{\sigma}\left(F^2 + \phi\right), \tag{6.16}$$

and for $P_0(F) := \sigma - 3\rho F^2 \neq 0$, we obtain

$$\alpha_a Y^a = \frac{\rho}{\sigma - 3\rho F^2}\left(\ell + \frac{5}{2}\rho F + 3\mu F^2\right) + \frac{\tau}{\sigma - 3\rho F^2}\left(F^2 + \phi\right).$$

Substituting this expression into (6.15) gives

$$\begin{aligned}\alpha_a =&\frac{Y_a}{\sigma}\left(\ell+\frac{5}{2}\rho F+3\mu F^2\right)-\frac{\epsilon_{ab}W^b}{\sigma}\left(F^2+\phi\right)\\&+\frac{3Y_a\rho F^2}{\sigma(\sigma-3\rho F^2)}\left(\ell+\frac{5}{2}\rho F+3\mu F^2\right)+\frac{3Y_a\tau F^2}{\sigma(\sigma-3\rho F^2)}\left(F^2+\phi\right),\end{aligned}$$

which simplifies to

$$\alpha_a =\frac{Y_a}{\sigma-3\rho F^2}\left(\ell+\frac{5}{2}\rho F+3\mu F^2\right)-\frac{\epsilon_{ab}W^b-3F^2U_a}{\sigma-3\rho F^2}\left(F^2+\phi\right).$$

We can rewrite this expression to obtain

$$(\sigma-3\rho F^2)\alpha_a=\left(Y_a\ell-\epsilon_{ab}W^b\phi\right)+\frac{5}{2}Y_a\rho F+\left(\frac{\nabla_a\rho}{2}\right)F^2+3F^4U_a.$$

The case where $P_0(F)=\sigma-3\rho F^2=0$ will be discussed in Section 6.6. Call $L_a=Y_a\ell-\epsilon_{ab}W^b\phi$. The 1-form L_a has conformal weight -10, and we have

$$(\sigma-3\rho F^2)\alpha_a=L_a+\frac{5}{2}Y_a\rho F+\frac{\nabla_a\rho}{2}F^2+3F^4U_a. \tag{6.17}$$

The derivation of (6.17) still holds when $\sigma=0$. We shall now derive polynomial constraints for there to admit a solution of (6.9). This involves differentiating equation (6.17) and using the equations (6.10) and (6.11) in the prolonged system to substitute. We then contract by the quantities U^aU^b, Y^aY^b and ϵ^{ab} to produce three polynomial constraint equations

$$P_1(F)=0,\qquad P_2(F)=0,\qquad P_3(F)=0,$$

in the variable F with coefficients given by conformal invariants of the Möbius structure. The three polynomial constraint equations have to be satisfied for there to be solutions to (6.9). After a routine computation we find that the first polynomial constraint is given by

$$\begin{aligned}P_1(F)=&\frac{63}{2}\rho^2F^8-12\rho\sigma F^6+\begin{pmatrix}12\rho\sigma\phi-63\rho^2\phi^2+3\rho U^a\nabla_a\sigma+\frac{1}{2}(\tau+3\mu\rho)^2\\+\frac{1}{2}(3\rho\phi-\sigma)^2+\frac{3}{2}\rho U^aU^b\nabla_a\nabla_b\rho-9\rho^2\mathrm{P}_{ab}U^aU^b\end{pmatrix}F^4\\&+\left(\frac{15}{2}\rho^3\mu+\frac{5}{2}\tau\rho^2+\frac{15}{2}\rho^2(U^aU^b\nabla_bY_a)\right)F^3\\&+\begin{pmatrix}(3\rho\phi-\sigma)(U^a\nabla_a\sigma)+21\rho\phi^2\sigma-3\phi\sigma^2+(\rho\ell+\phi\tau)(3\mu\rho+\tau)\\+\frac{25}{8}\rho^4+3\rho U^aU^b\nabla_bL_a+6\rho\sigma\mathrm{P}_{ab}U^aU^b-\sigma\frac{U^aU^b\nabla_a\nabla_b\rho}{2}\end{pmatrix}F^2\\&+\left(\frac{5}{2}\rho^2(\rho\ell+\phi\tau)-\frac{5}{2}(U^aU^b\nabla_bY_a)\sigma\rho\right)F\\&-\sigma\phi(U^a\nabla_a\sigma)+\frac{1}{2}(\rho\ell+\phi\tau)^2-\frac{1}{2}\phi^2\sigma^2-\sigma(U^aU^b\nabla_bL_a+\sigma\mathrm{P}_{ab}U^aU^b)\\=&0.\end{aligned}$$

The second constraint is given by

$$
\begin{aligned}
\tilde{P}_2(F) = & \left(\frac{\sigma - 15\rho F^2}{\sigma - 3\rho F^2}\right)\left((\rho\ell + \tau\phi) + \frac{5}{2}\rho^2 F + (\tau + 3\mu\rho)F^2\right)^2 - \frac{9}{2}\rho^2 F^8 \\
& - (9(Y^b Y^a \nabla_b U_a)\rho + 3\rho(3\phi\rho - \sigma))F^6 - 25\rho^2(\tau + 3\mu\rho)F^3 \\
& + \begin{pmatrix} 3(Y^a Y^b \nabla_b U_a)\sigma - \frac{3}{2}\rho Y^a Y^b \nabla_a \nabla_b \rho + \frac{3}{2}(\tau + 3\mu\rho)^2 \\ +9\rho^2 \mathrm{P}_{ab} Y^a Y^b + 3\phi\sigma\rho - \frac{1}{2}(3\phi\rho - \sigma)^2 \end{pmatrix} F^4 \\
& + \begin{pmatrix} \frac{1}{2} Y^a Y^b \nabla_a \nabla_b \rho\sigma - \frac{185}{8}\rho^4 - 3(Y^a Y^b \nabla_b L_a)\rho - 6\rho\sigma \mathrm{P}_{ab} Y^a Y^b \\ +\phi\sigma(3\phi\rho - \sigma) + (3\mu\rho + \tau)(\rho\ell + \phi\tau) - (3\mu\rho + \tau)(Y^a \nabla_a \sigma) \end{pmatrix} F^2 \\
& + \left(\frac{11}{2}\rho\sigma(\tau + 3\rho\mu) - \frac{27}{2}\rho^2(\rho\ell + \phi\tau) - \frac{5}{2}\rho^2(Y^a \nabla_a \sigma)\right) F \\
& + (Y^a Y^b \nabla_b L_a)\sigma - \frac{5}{2}\sigma\rho^3 + \mathrm{P}_{ab} Y^a Y^b \sigma^2 - \frac{1}{2}(\rho\ell + \phi\tau)^2 \\
& - \frac{1}{2}\phi^2\sigma^2 - (\rho\ell + \tau\phi)(Y^a \nabla_a \sigma) \\
= & 0,
\end{aligned}
$$

and we clear denominators to obtain $P_2(F) := P_0(F)\tilde{P}_2(F) = 0$. Finally the third constraint is given by

$$
\begin{aligned}
P_3(F) = & -6\tau F^6 + 18\rho^2 F^5 + (3Y^b \nabla_b \sigma + 24(\rho\ell + \tau\phi) - 6\sigma\mu)F^4 + 13\sigma\rho F^3 \\
& + \begin{pmatrix} (3\phi - \frac{\sigma}{\rho})Y^b \nabla_b \sigma + 30\mu\phi\sigma + 30\phi\rho\ell + 30\phi^2\tau \\ -(3\mu + \frac{\tau}{\rho})(U^b \nabla_b \sigma) - 10\sigma\ell + 3\rho\epsilon^{ab}\nabla_b L_a \end{pmatrix} F^2 \\
& + \left(25\phi\sigma\rho - \frac{5\rho}{2}(U^b \nabla_b \sigma) - 8\sigma^2\right) F \\
& - \frac{\phi\sigma}{\rho} Y^b \nabla_b \sigma - (U^b \nabla_b \sigma)(\ell + \frac{\phi\tau}{\rho}) - (\epsilon^{ab}\nabla_b L_a)\sigma \\
= & 0.
\end{aligned}
$$

We have derived the polynomial constraints $P_1(F) = P_2(F) = P_3(F) = 0$ for (6.9) to hold under the assumption that the generic condition $P_0(F) \neq 0$ holds on M^2. The polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ in Theorem 6.4 are obtained by replacing F with the indeterminate t (the polynomial $P_2(t)$ is given by $P_0(t)\tilde{P}_2(t)$).

6.6 The case where $P_0(F) = 0$

In this section, we examine the case where $P_0(F) = \sigma - 3\rho F^2 = 0$ and (6.9) both hold on the Möbius surface $(M^2, [g], [\mathrm{P}])$ and show that they imply the vanishing of a conformally invariant symmetric tensor M_{ab} of weight 0 constructed out of invariants of the Möbius structure. Under the assumption $P_0(F) = 0$, we obtain from (6.16) that

$$0 = \rho\ell + \frac{5}{2}\rho^2 F + 3\rho\mu F^2 + \tau F^2 + \tau\phi.$$

Substituting $F^2 = \frac{\sigma}{3\rho}$ gives

$$0 = \rho\ell + \frac{5}{2}\rho^2 F + \mu\sigma + \frac{\tau\sigma}{3\rho} + \tau\phi,$$

which upon rearranging gives

$$F = -\frac{2}{5}\left(\frac{\rho\ell + \mu\sigma + \frac{\tau\sigma}{3\rho} + \tau\phi}{\rho^2}\right). \tag{6.18}$$

Also, the first constraint (6.13) becomes

$$\alpha_a U^a = -(F^2 + \phi) = -\left(\frac{\sigma}{3\rho} + \phi\right) = -m,$$

where

$$m := \frac{\sigma}{3\rho} + \phi.$$

Differentiating once more and using (6.10) we obtain

$$\begin{aligned}\alpha_a W^a =& \alpha_a(Y^c\nabla_c U^a + \phi Y^a - 3\mu U^a)\\ =& -Y^c\nabla_c m + \frac{1}{2}\rho F + \mathrm{P}_{ca}U^cY^a + (\phi - m)(\alpha_c Y^c) + 3\mu m.\end{aligned}$$

Let

$$\psi = 3\mu m + \mathrm{P}_{ca}U^cY^a - Y^c\nabla_c m.$$

We obtain the following expression for α_a:

$$\alpha_a = \frac{Y_a}{\sigma}\left(\frac{1}{2}\rho F + \psi + (\phi - m)\alpha_c Y^c\right) - \frac{\epsilon_{ab}W^b}{\sigma}m. \tag{6.19}$$

Contracting with Y^a on both sides, we obtain

$$\begin{aligned}\alpha_a Y^a =& \frac{\rho}{\sigma}\left(\frac{1}{2}\rho F + \psi + (\phi - m)\alpha_c Y^c\right) + \frac{\tau m}{\sigma}\\ =& \frac{\rho^2}{2\sigma}F + \frac{\psi\rho}{\sigma} - \frac{1}{3}\alpha_c Y^c + \frac{\tau m}{\sigma},\end{aligned}$$

from which we obtain

$$\alpha_a Y^a = \frac{3\rho^2}{8\sigma}F + \frac{3(\psi\rho + \tau m)}{4\sigma}. \tag{6.20}$$

Substituting (6.20) into (6.19) now gives

$$\alpha_a = \left(\frac{3\rho}{8\sigma}F + \frac{3(\psi\rho + \tau m)}{4\sigma\rho}\right)Y_a - \frac{mU_a}{\rho}$$

and a further substitution of (6.18) gives

$$\alpha_a = \left(-\frac{3}{20}\left(\frac{\ell}{\sigma} + \frac{\mu}{\rho} + \frac{\tau}{3\rho^2} + \frac{\tau\phi}{\rho\sigma}\right) + \frac{3(\psi\rho + \tau m)}{4\sigma\rho}\right)Y_a - \frac{mU_a}{\rho}.$$

Defining the quantity k by

$$k := -\frac{3\rho}{20}\left(\frac{\ell}{\sigma} + \frac{\mu}{\rho} + \frac{\tau}{3\rho^2} + \frac{\tau\phi}{\rho\sigma}\right) + \frac{3(\psi\rho + \tau m)}{4\sigma}$$

then gives

$$\alpha_a = \frac{kY_a}{\rho} - \frac{mU_a}{\rho}. \tag{6.21}$$

Let

$$M_{ab} := \nabla_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha_c\alpha^c}{2}g_{ab} \tag{6.22}$$

for α_a given by (6.21). Hence for Möbius structures with $P_0(F) = 0$ and (6.9) both holding, we necessarily must have α_a given by (6.21), and the tensor M_{ab} given by (6.22) automatically vanishes. Conversely for Möbius structures with $M_{ab} = 0$, taking α_a to be as given in (6.21), we obtain a solution to (6.9). This is the result of Theorem 6.6.

6.7 Statement of results

The symmetric tensor M_{ab} is an invariant of the Möbius structure that can be used to distinguish sf-MEW Möbius surfaces with $P_0(F) = 0$ from those with $P_0(F) \neq 0$. For the formulation of Theorem 6.4, we therefore assume that $M_{ab} \neq 0$, which allows us to work locally in an open set $U \subset M^2$ where $P_0(F) = \sigma - 3\rho F^2 \neq 0$.

Theorem 6.4. *Let $(M^2, [g], [\mathrm{P}])$ be a Möbius surface, with $M_{ab} \neq 0$. Suppose the surface locally admits a solution to (6.8). Then there exist polynomials $P_1(t)$, $P_2(t)$, $P_3(t)$ in a single variable t with coefficients given by conformal invariants of the Möbius structure such that when $t = F$, where $F = \epsilon^{ab}F_{ab}$,*

$$P_1(F) = P_2(F) = P_3(F) = 0$$

must hold.

We have explicitly computed the polynomial constraints $P_1(F) = P_2(F) = P_3(F) = 0$ in Section 6.5. For any polynomials $P(t)$, $Q(t)$ in a single variable t, let $\mathrm{Res}(P(t), Q(t))$ denote the resultant of $P(t)$ and $Q(t)$. $\mathrm{Res}(P(t), Q(t)) = 0$ is necessary and sufficient for $P(t)$ and $Q(t)$ to share a common root. As a corollary, we obtain local obstructions for there to be solutions of (6.8).

Corollary 6.5. *Suppose the Möbius surface* $(M^2, [g], [\mathrm{P}])$ *has* $M_{ab} \neq 0$ *and locally admits a solution to (6.8). Then the following conformal invariants of the Möbius structure have to vanish:*

$$Res(P_1(t), P_2(t)) = Res(P_1(t), P_3(t)) = Res(P_2(t), P_3(t)) = 0.$$

For the case when $M_{ab} = 0$, we have

Theorem 6.6. *Let* $(M^2, [g], [\mathrm{P}])$ *be a Möbius surface with* $M_{ab} = 0$*. Then the surface locally admits a solution to (6.8), with* α_a *given by (6.21).*

However, the author does not know of any examples of non-flat Möbius surfaces for which $M_{ab} = 0$.

6.8 Examples

In this section we give examples of three different Möbius structures on the Euclidean plane $\mathbb{R}^2$. The first example has non-vanishing obstruction, while the second and third have vanishing obstructions. In the third example we show that the Möbius structure does not admit a real solution to (6.8) despite having vanishing obstructions. Since $\rho = Y_a Y^a > 0$ for non-flat Möbius surfaces, computing the discriminant of $P_0(F)$ gives $12\rho\sigma > 0$, or $\sigma > 0$ for there to be solutions of $P_0(F) = 0$. (For $\sigma = 0$, it would mean $-3\rho F^2 = 0$ which cannot happen if $\rho > 0$ and $F \neq 0$.) In the first 2 examples we have $\sigma \leq 0$, so that $P_0(F) = \sigma - 3\rho F^2 \neq 0$ and we do not need to compute the Möbius invariant M_{ab}. The last example has $\sigma > 0$, but a simple argument eliminates the possibility that $P_0(F) = 0$ can hold.

6.8.1 Example with non-vanishing obstruction

The first example will be the Möbius structure given by $\mathrm{P}_{ab} = x_{(a}\epsilon_{b)c}x^c$ on $\mathbb{R}^2$ with the flat metric δ_{ab}. On $\mathbb{R}^2$ we have $K = 0$. Here x^a are standard local coordinates in $\mathbb{R}^2$ so that $\partial_a x_b = \delta_{ab}$. Let $r = x_a x^a$. A computation shows that for this example,

$$\partial_a \mathrm{P}_{bc} = \delta_{a(b}\epsilon_{c)d}x^d + x_{(b}\epsilon_{c)a}.$$

This gives

$$Y_c = 2\epsilon^{ab}\partial_a \mathrm{P}_{bc} = -4x_c, \quad U_c = -4\epsilon_{ca}x^a,$$

from which we obtain

$$\begin{aligned}
\partial_a U_c &= -4\epsilon_{ca}, \qquad & \phi &= \frac{\partial_a U^a}{2} = 0,\\
\partial_a Y_c &= -4\delta_{ac}, \qquad & \mu &= \frac{\partial_a Y^a}{2} = -4.
\end{aligned}$$

We therefore have

$$\begin{aligned}
W_a = Y^c\partial_c U_a + \phi Y_a - 3\mu U_a =& Y^c(-4\epsilon_{ac}) - 3(-4)U_a\\
=& 16\epsilon_{ac}x^c + 12(-4)\epsilon_{ac}x^c\\
=& -32\epsilon_{ac}x^c\\
=& 8U_a,
\end{aligned}$$

and so deduce that the Möbius structure has $\sigma = 0$. For this Möbius structure, we have

$$\begin{aligned}
P_1(t) =& 256r^2\left(\frac{63}{2}t^8 + 56t^4 - 640rt^3 + 672r^2t^2 + 320r^3t + 32r^4\right),\\
P_2(t) =& 256r^2\left(-\frac{9}{2}t^8 + 56t^4 - 7392r^2t^2 + 1472r^3t + 288r^4\right),\\
P_3(t) =& -768rt^4(t^2 - 6rt - 4r^2).
\end{aligned}$$

Using MAPLE, we find

$$\mathrm{Res}(P_1(t), P_3(t)) = 2^{142}\cdot 3^{10}r^{44}(2^4\cdot 3^2\cdot 7^2r^8 + 2^2\cdot 7^2\cdot 59\cdot 251r^4 - 3^2\cdot 131)$$

which is non-zero for general r. A similar computation shows that the local obstructions given by $\mathrm{Res}(P_2(t), P_3(t))$ and $\mathrm{Res}(P_1(t), P_2(t))$ do not vanish on any open set. We conclude that the Möbius structure for this example admits no local solution to (6.9).

6.8.2 Example with vanishing obstruction (and solution to sf-MEW)

Consider the Möbius structure given by $\mathrm{P}_{ab} = x_a x_b - \frac{1}{2}\delta_{ab}x_c x^c$ on $\mathbb{R}^2$ with the flat metric δ_{ab}. Again we have $K = 0$. It can be verified that $\alpha_a = \pm\epsilon_{ab}x^b$ is a solution to (6.9), since

$$\partial_a\alpha_b = \partial_a(\pm\epsilon_{bc}x^c) = \pm\epsilon_{ba} = \mp\epsilon_{ab},$$

and therefore, using $\epsilon_{ac}\epsilon_{bd} = \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{cb}$, we find

$$\begin{aligned}\partial_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha_c\alpha^c}{2}\delta_{ab} =& 0 + \epsilon_{ac}x^c\epsilon_{bd}x^d + (x_a x_b - \frac{1}{2}\delta_{ab}x_c x^c) - \frac{x_c x^c}{2}\delta_{ab} \\ =& \delta_{ab}x_c x^c - x_a x_b + x_a x_b - \frac{1}{2}\delta_{ab}x_c x^c - \frac{x_c x^c}{2}\delta_{ab} \\ =& 0.\end{aligned}$$

We have

$$F_{ab} = \partial_{[a}\alpha_{b]} = \pm\epsilon_{[ba]} = \mp\epsilon_{ab},$$

so that $F = \mp 2$ depending on the sign of α_a chosen. For this example, we have

$$\partial_a \mathrm{P}_{bc} = \delta_{ab}x_c + \delta_{ac}x_b - \delta_{bc}x_a,$$

and so

$$Y_c = 2\epsilon^{ab}\partial_a \mathrm{P}_{bc} = 4\epsilon_{cb}x^b, \quad U_c = -4x_c,$$

from which we obtain

$$\begin{aligned}\partial_a U_c = -4\delta_{ca}, \qquad & \phi = \frac{\partial_a U^a}{2} = -4, \\ \partial_a Y_c = 4\epsilon_{ca}, \qquad & \mu = \frac{\partial_a Y^a}{2} = 0.\end{aligned}$$

We therefore have

$$\begin{aligned}W_a = Y^c\partial_c U_a + \phi Y_a - 3\mu U_a =& Y^c(-4\delta_{ac}) + (-4)Y_a \\ =& -32\epsilon_{ab}x^b \\ =& -8Y_a,\end{aligned}$$

so that the Möbius structure has $\tau = 0$. We also find that $\ell = 0$. Computing $P_1(t)$ gives

$$P_1(t) = \rho^2(t^2-4)\left(\frac{63}{2}t^6 + 222t^4 + (512 - \frac{9\rho^2}{32})t^2 + (384 + \frac{\rho^2}{2})\right),$$

where $\rho = 16x_a x^a$. Computing $\tilde{P}_2(t)$ gives

$$\begin{aligned}\tilde{P}_2(t) =& \left(\frac{8+15t^2}{8+3t^2}\right)\left(\frac{25}{4}\rho^4 t^2\right) - \frac{9}{2}\rho^2 t^8 + 48\rho^2 t^6 + (136\rho^2 - \frac{9\rho^4}{32})t^4 \\ &+ (-640\rho^2 - \frac{197}{8}\rho^4)t^2 - 1536\rho^2 + 18\rho^4.\end{aligned}$$

Multiplying throughout by $P_0(t) = -\rho(8+3t^2)$ and expanding the terms on the right, we obtain

$$
\begin{aligned}
P_2(t) =& -\rho(8+3t^2)\tilde{P}_2(t)\\
=& \frac{27}{2}\rho^3(t^2-4)\left(t^8 - 4t^6 + (\frac{\rho^2}{16} - \frac{224}{3})t^4 - (\frac{6400}{27} + \frac{19}{18}\rho^2)t^2 + (\frac{8}{3}\rho^2 - \frac{2048}{9})\right).
\end{aligned}
$$

The third polynomial $P_3(t)$ for this example is

$$P_3(t) = 2\rho^2 t(9t^2-16)(t^2-4).$$

It therefore can be seen that $P_1(t)$, $P_2(t)$ and $P_3(t)$ share a common root t^2-4, attained when $t = F$. The local obstructions vanish for this example.

6.8.3 Example with vanishing obstruction but does not admit a real solution to sf-MEW

Consider the Möbius structure given by $\mathrm{P}_{ab} = \frac{1}{2}\delta_{ab}x_c x^c - x_a x_b$ on $\mathbb{R}^2$ with the flat metric δ_{ab}. Again we have $K = 0$. We shall claim that obstructions vanish for this example but it does not admit a solution to (6.9). As we shall see below, for this Möbius structure, we have $\sigma = 8\rho$, and $\mu = 0$, $\ell = 0$, $\tau = 0$. The polynomial constraint $P_0(F) = 0$ cannot hold because equation (6.18) yields $F = 0$, which cannot happen if the Möbius structure is not flat. A computation shows that

$$\partial_a \mathrm{P}_{bc} = \delta_{bc}x_a - \delta_{ab}x_c - \delta_{ac}x_b.$$

This gives

$$Y_c = 2\epsilon^{ab}\partial_a \mathrm{P}_{bc} = 2\epsilon^{ab}(\delta_{bc}x_a - \delta_{ac}x_b) = -4\epsilon_{cb}x^b, \qquad U_c = 4x_c,$$

from which we obtain

$$
\begin{aligned}
\partial_a U_c &= 4\delta_{ca}, \qquad \phi = \frac{\partial_a U^a}{2} = 4,\\
\partial_a Y_c &= -4\epsilon_{ca}, \qquad \mu = \frac{\partial_a Y^a}{2} = 0.
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
W_a = Y^c\partial_c U_a + \phi Y_a - 3\mu U_a =& Y^c(4\delta_{ac}) + 4Y_a\\
=& 8Y_a\\
=& -32\epsilon_{ab}x^b
\end{aligned}
$$

and the Möbius structure has $\tau = 0$. Computing $P_1(t)$ for this example gives

$$P_1(t) = \rho^2(t^2+4)\left(\frac{63}{2}t^6 - 222t^4 + (512 + \frac{9\rho^2}{32})t^2 + (\frac{\rho^2}{2} - 384)\right),$$

where again $\rho = 16x_a x^a$. The second polynomial $\tilde{P}_2(t)$ is given by

$$\begin{aligned}\tilde{P}_2(t) = &\left(\frac{8-15t^2}{8-3t^2}\right)\left(\frac{25}{4}\rho^4 t^2\right) - \frac{9}{2}\rho^2 t^8 - 48\rho^2 t^6 + (136\rho^2 + \frac{9\rho^4}{32})t^4 \\ &+ (640\rho^2 - \frac{197}{8}\rho^4)t^2 - 1536\rho^2 - 18\rho^4,\end{aligned}$$

so that clearing denominators we obtain

$$\begin{aligned}P_2(t) =& (8-3t^2)\tilde{P}_2(t) \\ =& \frac{27}{2}\rho^2(t^2+4)\left(t^8 + 4t^6 - (\frac{\rho^2}{16} + \frac{224}{3})t^4 + (\frac{6400}{27} - \frac{19}{18}\rho^2)t^2 - (\frac{8}{3}\rho^2 + \frac{2048}{9})\right).\end{aligned}$$

The third polynomial $P_3(t)$ for our example is given by

$$P_3(t) = 2\rho^2 t(9t^2+16)(t^2+4).$$

It can be seen that the three polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ all share a common factor, namely t^2+4. The local obstructions all vanish for this example. Dividing $P_1(t)$, $P_2(t)$ and $P_3(t)$ by their common factors $\rho^2(t^2+4)$, we obtain

$$\begin{aligned}S_1(t) =& \frac{63}{2}t^6 - 222t^4 + (512 + \frac{9\rho^2}{32})t^2 + (\frac{\rho^2}{2} - 384), \\ S_2(t) =& \frac{27}{2}\left(t^8 + 4t^6 - (\frac{\rho^2}{16} + \frac{224}{3})t^4 + (\frac{6400}{27} - \frac{19}{18}\rho^2)t^2 - (\frac{8}{3}\rho^2 + \frac{2048}{9})\right), \\ S_3(t) =& 2t(9t^2+16),\end{aligned}$$

and we find that

$$\begin{aligned}&\mathrm{Res}(S_1(t), S_2(t)) \\ &\quad = \frac{1076168025}{67108864}\rho^8(243\rho^6 + 12704256\rho^4 + 131135897600\rho^2 + 251658240000)^2, \\ &\mathrm{Res}(S_1(t), S_3(t)) = 80289792000000\rho^2 - 61662560256000000, \\ &\mathrm{Res}(S_2(t), S_3(t)) = -3583180800(73600 + 81\rho^2)^2(3\rho^2 + 256).\end{aligned}$$

Hence the three polynomials $P_1(t)$, $P_2(t)$ and $P_3(t)$ share no other common factors besides t^2+4 and the Möbius structure does not admit any real solution to (6.9).

However, the Möbius structure does admit a complex solution to (6.9). It can be verified that $\alpha_a = \pm i\epsilon_{ab}x^b$ is a solution to (6.9), since

$$\partial_a\alpha_b = \partial_a(\pm i\epsilon_{bc}x^c) = \pm i\epsilon_{ba} = \mp i\epsilon_{ab},$$

and using $\epsilon_{ac}\epsilon_{bd} = \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{cb}$, we find

$$\begin{aligned}\partial_{(a}\alpha_{b)} + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha_c\alpha^c}{2}\delta_{ab} =& 0 - \epsilon_{ac}x^c\epsilon_{bd}x^d + (\frac{1}{2}\delta_{ab}x_cx^c - x_ax_b) + \frac{x_cx^c}{2}\delta_{ab}\\ =& -\delta_{ab}x_cx^c + x_ax_b - x_ax_b + \frac{1}{2}\delta_{ab}x_cx^c + \frac{x_cx^c}{2}\delta_{ab}\\ =& 0.\end{aligned}$$

We have

$$F_{ab} = \partial_{[a}\alpha_{b]} = \pm i\epsilon_{[ba]} = \mp i\epsilon_{ab},$$

so that $F = \mp 2i$ depending on the sign of α_a chosen.

Chapter 7

The conformally Einstein equation on Möbius surfaces

On Möbius surfaces $(M^2, [g], [\mathrm{P}])$, the conformally Einstein equation is a well defined overdetermined system of linear PDEs. In this chapter we first give a geometric description of the conformally Einstein equation and apply the theory of prolongation to the overdetermined system on Möbius surfaces in Section 7.1. We find local obstructions to the existence of conformally Einstein metrics on the Möbius surface and discuss this further in Sections 7.3 and 7.4. We give examples where the obstructions vanish and examples where they do not vanish in Sections 7.2 and 7.5. On a conformal manifold of dimension $n > 2$, it is well known that parallel sections of the standard tractor bundle with nowhere vanishing scale are in 1-1 correspondence with solutions of the conformally Einstein equation. It turns out however that an additional term involving curvature in the form of the Cotton-York tensor appears in the prolongation of the conformally Einstein equation in 2-dimensions, in contrast to the higher dimensional setting. In Section 7.6 establish 1-1 correspondence between solutions of the conformally Einstein equation on Möbius surfaces with nowhere vanishing scale and parallel sections of the modified standard tractor connection called the prolongation connection. We conclude the chapter with a summary table in Section 7.7.

7.1 Conformal-to-Einstein equation on Möbius surfaces

The conformally Einstein equation on conformal manifolds has been studied extensively, for instance in [19] and [20]. Algebraic obstructions to the existence of conformally Einstein metrics have also been found, for example in dimension 4 the Bach tensor is an obstruction to the existence of conformally Einstein metrics. Other obstructions are found for self-dual manifolds in [1] and a complete set of obstructions is found recently in [9]. In this section we apply the prolongation proceedure to the linear system of PDEs associated to the conformally Einstein condition on Möbius surfaces and see that the closed system differs from that obtained in the higher dimensional setting. Prolongation of the conformally Einstein equation in higher dimensions is done in [2] and [20]. The closed system has consequences in deriving algebraic obstructions for the existence of conformally Einstein metric on general non-flat Möbius surfaces. Let $(M^2, [g], [\mathrm{P}])$ be a Möbius surface. A Weyl connection D_a is called closed if the 1-form α_a determined by the Weyl derivative is closed, i.e $\nabla_{[a}\alpha_{b]} = F_{ab} = 0$. The conformal-to-Einstein equation asks for a fixed Möbius surface, whether there is a compatible Weyl connection D_a that is closed, such that the second order linear differential operator is given by the symmetric trace-free Hessian of the Weyl derivative, i.e. $D_{(ab)_\circ} = D_{(a}D_{b)_\circ}$. Choosing a representative metric g_{ab} in the conformal class $[g]$ with its associated Levi-Civita connection ∇_a, we can write the conformally Einstein equation as

$$\nabla_a\alpha_b + \alpha_a\alpha_b + \mathrm{P}_{ab} - \frac{\alpha_c\alpha^c}{2}g_{ab} = -\frac{s}{2}g_{ab}. \tag{7.1}$$

This is obtained from the MEW equation precisely when $F = 0$. Substituting $\alpha_a = \frac{\nabla_a\sigma}{\sigma}$ into (7.1) gives us the conformally Einstein equation

$$(\nabla_{(a}\nabla_{b)_\circ} + \mathrm{P}_{(ab)_\circ})\sigma = 0, \tag{7.2}$$

where $(\ldots)_\circ$ denotes symmetric trace-free part. This equation is conformally invariant when σ has conformal weight 1, or equivalently σ is a section of the density line bundle $\mathcal{E}[1]$. Rewriting (7.2), we obtain

$$\nabla_a\nabla_b\sigma + \mathrm{P}_{ab}\sigma + \Lambda g_{ab} = 0. \tag{7.3}$$

Note that we have used the same notation Λ even though the trace quantity in (7.3) is not the same as the trace term appearing in (6.3). We shall prolong the conformal-to-Einstein equation and find remarkably that it differs from the prolongation in higher dimensions with the addition of a curvature term! Let $\mu_a = \nabla_a \sigma$. Then (7.3) is

$$\nabla_a \mu_b = -\mathrm{P}_{ab}\sigma - \Lambda g_{ab}. \tag{7.4}$$

Differentiating we obtain

$$\nabla_c \nabla_a \mu_b + (\nabla_c \mathrm{P}_{ab})\sigma + \mathrm{P}_{ab}\mu_c + \nabla_c \Lambda g_{ab} = 0,$$

so that skewing a and c indices, we get

$$R_{ac}{}^{d}{}_{b}\mu_d + Y_{cab}\sigma + \mathrm{P}_{ab}\mu_c - \mathrm{P}_{cb}\mu_a + \nabla_c \Lambda g_{ab} - \nabla_a \Lambda g_{cb} = 0.$$

Since on Möbius surfaces we have (6.2) holding, we obtain

$$\begin{aligned}
&(\delta_a{}^d \mathrm{P}_{cb} - \delta_c{}^d \mathrm{P}_{ab} + \mathrm{P}_a{}^d g_{cb} - \mathrm{P}_c{}^d g_{ab})\mu_d + Y_{cab}\sigma + \mathrm{P}_{ab}\mu_c - \mathrm{P}_{cb}\mu_a + \nabla_c\Lambda g_{ab} - \nabla_a \Lambda g_{cb}\\
=&\mu_a \mathrm{P}_{cb} - \mu_c \mathrm{P}_{ab} + \mathrm{P}_a{}^d \mu_d g_{cb} - \mathrm{P}_c{}^d \mu_d g_{ab} + Y_{cab}\sigma + \mathrm{P}_{ab}\mu_c - \mathrm{P}_{cb}\mu_a + \nabla_c \Lambda g_{ab} - \nabla_a \Lambda g_{cb}\\
=&\mathrm{P}_a{}^d \mu_d g_{cb} - \mathrm{P}_c{}^d \mu_d g_{ab} + Y_{cab}\sigma + \nabla_c \Lambda g_{ab} - \nabla_a \Lambda g_{cb}\\
=&0.
\end{aligned}$$

Tracing a and b indices, we get

$$\nabla_a \Lambda = -Y_{ac}{}^{c}\sigma + \mathrm{P}_a{}^d \mu_d.$$

Now in dimensions $n > 2$, $Y_{ac}{}^{c} = 0$ because the Cotton-York tensor is totally trace-free. However, in 2 dimensions,

$$Y_{abc} = \frac{1}{2}\epsilon_{ab} Y_c,$$

where $Y_c = \epsilon^{ab} Y_{abc}$, so that

$$Y_{ac}{}^{c} = \frac{1}{2}\epsilon_{ac} Y^c = \frac{1}{2} U_a.$$

Hence we obtain

$$\nabla_a \Lambda = -\frac{1}{2} U_a \sigma + \mathrm{P}_a{}^d \mu_d. \tag{7.5}$$

We now use the closed system to derive algebraic obstructions for there to admit a solution to (7.3). Differentiating equation (7.5) once more gives

$$\nabla_b \nabla_a \Lambda = -\frac{1}{2}(\nabla_b U_a)\sigma - \frac{1}{2} U_a \mu_b + (\nabla_b \mathrm{P}_a{}^d)\mu_d + \mathrm{P}_a{}^d(-\Lambda \delta_{bd} - \mathrm{P}_{bd}\sigma),$$

so that skewing gives

$$0 = \frac{1}{2}(\nabla_a U_b - \nabla_b U_a)\sigma - \frac{1}{2}U_a\mu_b + \frac{1}{2}U_b\mu_a + Y_{ba}{}^d\mu_d,$$

or contracting with the volume form ϵ^{ba}, we get

$$0 = (\nabla_a Y^a)\sigma + 2Y_d\mu^d.$$

In our notation, we have

$$Y_a\mu^a + \mu\sigma = 0. \tag{7.6}$$

This is the first constraint equation we obtain for (7.3) to hold. It is one equation in 3 unknowns. Differentiating this equation once more gives

$$\begin{aligned}(\nabla_e Y_d)\mu^d + (\nabla_e\mu)\sigma + &\mu(\nabla_e\sigma) + Y_d(-\Lambda\delta_e{}^d - \mathrm{P}_e{}^d\sigma) \\ &= (\nabla_e Y_d)\mu^d + (\nabla_e\mu)\sigma + \mu\mu_e - \Lambda Y_e - \mathrm{P}_e{}^d Y_d\sigma = 0.\end{aligned} \tag{7.7}$$

This is the second constraint equation and is given by 2 equations in 4 unknowns. Recalling that both Y_a and U_a have weight -2, we have under conformal transformations

$$\begin{aligned}\widehat{\nabla_e Y_d} &= \nabla_e Y_d - 3\Upsilon_e Y_d - \Upsilon_d Y_e + g_{ed}\Upsilon^c Y_c,\\ \widehat{U^e\nabla_e Y_d} &= U^e\nabla_e Y_d - 3\Upsilon_e U^e Y_d + U_d\Upsilon^c Y_c,\end{aligned}$$

so that

$$V_d := U^e\nabla_e Y_d + \mu U_d - 3\phi Y_d$$

is conformally invariant of weight -6. In turns out that $V_a = -W_a$, where W_a is defined in Section 6.5 of Chapter 6. Contracting (7.7) with U^a gives

$$(U^e\nabla_e Y_d)\mu^d + (U^e\nabla_e\mu)\sigma + \mu(\mu_e U^e) - \mathrm{P}_{ed}U^e Y^d\sigma = 0$$

and so

$$\begin{aligned}V_d\mu^d = (U^e\nabla_e Y_d + \mu U_d - 3\phi Y_d)\mu^d &= -(U^e\nabla_e\mu)\sigma + \mathrm{P}_{ed}U^e Y^d\sigma - 3\phi(Y_d\mu^d)\\ &= -(U^e\nabla_e\mu)\sigma + \mathrm{P}_{ed}U^e Y^d\sigma + 3\phi\mu\sigma.\end{aligned}$$

Let us call

$$k := \mathrm{P}_{ed}U^e Y^d - U^e\nabla_e\mu + 3\phi\mu.$$

The quantity k has conformal weight -8 and under conformal rescaling, $\widehat{k} = k + \Upsilon_c V^c$, and we have

$$V_d\mu^d = k\sigma.$$

Combining the 2 constraints

$$
\begin{aligned}
Y_d \mu^d &= -\mu\sigma, \\
V_d \mu^d &= k\sigma,
\end{aligned}
$$

and assuming that $U_d V^d \neq 0$, we can solve for μ^d to obtain

$$
\mu^d = -\frac{\mu \epsilon^{ad} V_a \sigma}{U_c V^c} + \frac{k\sigma U^d}{U_c V^c} = \frac{1}{U_c V^c}\left(\mu \epsilon^{da} V_a + kU^d\right)\sigma. \tag{7.8}
$$

We remark here that since $V_a = -W_a$, $U_a V^a = -U_a W^a = -\tau$ from Chapter 6. The case where $U_d V^d = 0$ will be mentioned in Section 7.4. Let us call

$$
K_a := \frac{1}{U_c V^c}\left(\mu \epsilon_a{}^b V_b + kU_a\right). \tag{7.9}
$$

The 1-form K_a has conformal weight 0 and under conformal rescalings, $\widehat{K}_a = K_a + \Upsilon_a$. Then from (7.8) we get

$$
\mu_a = K_a \sigma,
$$

and substituting this back into (7.4) we find that

$$
\nabla_a \mu_b = (\nabla_a K_b)\sigma + K_b \mu_a = -\Lambda g_{ab} - \mathrm{P}_{ab}\sigma,
$$

which gives

$$
(\nabla_a K_b)\sigma + K_b K_a \sigma + \mathrm{P}_{ab}\sigma = -\Lambda g_{ab}.
$$

This implies that

$$
\Lambda = -\frac{1}{2}(\nabla_c K^c + K_c K^c + K)\sigma,
$$

so that

$$
(\nabla_a K_b)\sigma + K_b K_a \sigma + \mathrm{P}_{ab}\sigma - \frac{1}{2}(\nabla_c K^c + K_c K^c + K)\sigma g_{ab} = 0
$$

must hold and assuming $\sigma \neq 0$, we obtain

$$
\nabla_a K_b + K_b K_a + \mathrm{P}_{ab} - \frac{1}{2}(\nabla_c K^c + K_c K^c + K) g_{ab} = 0.
$$

Equivalently, we have the consequence that

$$
\text{trace-free part of } (\nabla_a K_b + K_a K_b + \mathrm{P}_{ab}) = 0
$$

must be satisfied for (7.3) to hold. We have the following proposition

Proposition 7.1. *Let $(M^2, [g], [\mathrm{P}])$ be a Möbius surface with $U_aV^a \neq 0$. Suppose it admits a solution to the conformal-to-Einstein equation (7.3). Then the following tensorial obstruction E_{ab} of conformal weight 0 given by*

$$E_{ab} := \nabla_a K_b + K_b K_a + \mathrm{P}_{ab} - \frac{1}{2}(\nabla_c K^c + K_c K^c + K)g_{ab} \tag{7.10}$$

must vanish, where K_a is given by (7.9). Conversely, suppose the tensor E_{ab} defined in (7.10) vanishes for some non-zero 1-form K_a given by (7.9). Then $\nabla_{[a}K_{b]} = 0$, and taking $\alpha_a = K_a = \nabla_a \log \sigma$, we find that σ satisfies the conformal-to-Einstein equation (7.3) on M^2.

7.2 Examples

We now give 2 examples of Möbius structures on the Euclidean plane $\mathbb{R}^2$, one with vanishing obstruction and one without. In both examples $U_aV^a \neq 0$. The first example will be the Möbius structure on $\mathbb{R}^2$ given by

$$\begin{aligned} \mathrm{P} =& \mathrm{P}_{11}dxdx + 2\mathrm{P}_{12}dxdy + \mathrm{P}_{22}dydy \\ =& \left(y - x + \frac{1}{2}y^4 - \frac{1}{2}x^4\right) dxdx - 2x^2y^2dxdy + \left(x - y + \frac{1}{2}x^4 - \frac{1}{2}y^4\right) dydy. \end{aligned}$$

For this Möbius structure, we obtain

$$\begin{aligned} Y =& Y_1dx + Y_2dy = (-4xy^2 - 4y^3 - 2)dx + (4x^3 + 4x^2y + 2)dy, \\ \phi =& 6x^2 + 8xy + 6y^2, \\ \mu =& 2x^2 - 2y^2, \\ V =& V_1dx + V_2dy \\ =& (48x^3y^2 + 112x^2y^3 + 88xy^4 + 24y^5 + 32xy + 8x^5 + 40x^2 + 8yx^4)dx \\ &+ (-24x^5 - 88yx^4 - 32xy - 112x^3y^2 - 48x^2y^3 - 8xy^4 - 8y^5 - 40y^2)dy, \\ U_aV^a =& 80x^2 - 128xy^4 - 128y^5 + 128yx^4 + 128x^5 + 192x^5y^3 - 64y^7x \\ &- 192y^5x^3 + 64x^7y - 80y^2 - 128y^6x^2 - 32y^8 + 128y^2x^6 + 32x^8, \\ k =& 40x^4 - 40y^4 + 24x^5y^2 + 24x^4y^3 - 24y^4x^3 - 24y^5x^2 \\ &+ 8x^6y - 32y^3x - 8y^6x + 32x^3y + 8x^7 - 8y^7, \end{aligned}$$

and K_a remarkably simplifies to give

$$K = K_1dx + K_2dy = x^2dx + y^2dy.$$

We find that the following holds:

$$\partial_a K_b + K_a K_b + \mathrm{P}_{ab} - \frac{1}{2}(\partial_c K^c + K_c K^c) g_{ab} = 0,$$

so that the obstruction tensor E_{ab} vanishes and taking $\sigma = e^{\frac{x^3+y^3}{3}}$ gives us a solution to the conformal-to-Einstein equation. The second example will be the Möbius structure given by $\mathrm{P}_{ab} = x_{(a}\epsilon_{b)c}x^c$ on $\mathbb{R}^2$. We shall show that this Möbius structure admits no solution to (7.3) by showing that the tensor obstruction does not vanish. On $\mathbb{R}^2$, we have $K = 0$. Let x^a be standard local coordinates in $\mathbb{R}^2$ so that $\partial_a x_b = \delta_{ab}$. A computation shows that

$$\partial_a \mathrm{P}_{bc} = \delta_{a(b}\epsilon_{c)d}x^d + x_{(b}\epsilon_{c)a},$$

from which we obtain

$$Y_c = 2\epsilon^{ab}\partial_a \mathrm{P}_{bc} = -4x_c, \qquad U_c = -4\epsilon_{ca}x^a.$$

This gives

$$\partial_a Y_c = -4\delta_{ac}, \qquad \partial_a U_c = -4\epsilon_{ca},$$

and so

$$\mu = \frac{\partial_a Y^a}{2} = -4, \qquad \phi = \frac{\partial_a U^a}{2} = 0.$$

We also obtain

$$\begin{aligned} V_a = U^c\partial_c Y_a + \mu U_a - 3\phi Y_a =& U^c(-4\delta_{ca}) - 4U_a \\ =& -8U_a, \end{aligned}$$

and so $U_a V^a = -8U_a U^a = -128x_c x^c$. Further computations of the various quantities yield

$$\begin{aligned} k = 3\mu\phi + \mathrm{P}_{ab}U^a Y^b - U^c \nabla_c \mu =& \mathrm{P}_{ab}U^a Y^b \\ =& \left(\frac{1}{2}x_a\epsilon_{bc}x^c + \frac{1}{2}x_b\epsilon_{ac}x^c\right)(-4\epsilon^{ad}x_d)(-4x^b) \\ =& 8(x_b x^b)^2, \\ K_a = -\frac{1}{128x_c x^c}\left(kU_a + \mu\epsilon_{ac}V^c\right) =& -\frac{1}{128x_c x^c}\left(kU_a - 8(-4)\epsilon_{ac}U^c\right) \\ =& -\frac{1}{128x_c x^c}\left(kU_a - 32Y_a\right) \\ =& -\frac{1}{128x_c x^c}\left(-32(x_c x^c)^2\epsilon_{ab}x^b - 32(-4x_a)\right) \\ =& \frac{1}{4x_c x^c}\left((x_c x^c)^2\epsilon_{ab}x^b - 4x_a\right). \end{aligned}$$

We then find (with the aid of MAPLE) that

$$\partial_a K_b + K_a K_b + \mathrm{P}_{ab} - \frac{1}{2}(\partial_c K^c + K_c K^c) g_{ab} \neq 0,$$

and so we conclude that this Möbius structure admits no solution to the conformal-to-Einstein equation. Also observe that $\partial_{[a} K_{b]} \neq 0$ for this Möbius structure.

7.3 Proposal to derive obstructions via the determinant method

Following the theory developed in [4], we would like to relate the obstructions obtained in the previous section to the determinant of some 4 by 4 matrix. However, the computational complexity increases when we take more derivatives, and so here we only give an overview of the steps involved. Together constraint equations (7.6) and (7.7) give us 3 equations in 4 unknowns. We can write this as in matrix form as

$$\begin{pmatrix} \mu & Y_1 & Y_2 & 0 \\ \nabla_1 \mu - \mathrm{P}_{11} Y^1 - \mathrm{P}_{12} Y^2 & \nabla_1 Y_1 + g_{11}\mu & \nabla_1 Y_2 + g_{12}\mu & -Y_1 \\ \nabla_2 \mu - \mathrm{P}_{21} Y^1 - \mathrm{P}_{22} Y^2 & \nabla_2 Y_1 + g_{21}\mu & \nabla_2 Y_2 + g_{22}\mu & -Y_2 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^1 \\ \mu^2 \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this system of equations it is not clear which additional constraint equation is the best thing to augment to get a 4 by 4 matrix with non-zero determinant. We can differentiate (7.7) one more time and obtain 3 more equations (note that skewing yields identically zero). They are given by

$$
\begin{aligned}
&(\nabla_f \nabla_e \mu - (\nabla_f \mathrm{P}_{ed})Y^d - \mathrm{P}_{ed}\nabla_f Y^d)\sigma + (\nabla_e \mu - \mathrm{P}_{ed}Y^d)\mu_f + (\nabla_f \nabla_e Y_d + g_{ed}\nabla_f \mu)\mu^d \\
&- (\nabla_f \Lambda)Y_e + (\nabla_e Y_d + g_{ed}\mu)(\nabla_f \mu^d) - \Lambda(\nabla_f Y_e) \\
=&(\nabla_f \nabla_e \mu - (\nabla_f \mathrm{P}_{ed})Y^d - \mathrm{P}_{ed}\nabla_f Y^d)\sigma + (\nabla_e \mu - \mathrm{P}_{ed}Y^d)\mu_f + (\nabla_f \nabla_e Y_d + g_{ed}\nabla_f \mu)\mu^d \\
&+ (\frac{1}{2}U_f\sigma - \mathrm{P}_{fd}\mu^d)Y_e + (\nabla_e Y_d + g_{ed}\mu)(-\delta_f{}^d\Lambda - \mathrm{P}_f{}^d\sigma) - \Lambda(\nabla_f Y_e) \\
=&(\nabla_f \nabla_e \mu - (\nabla_f \mathrm{P}_{ed})Y^d - \mathrm{P}_{ed}\nabla_f Y^d)\sigma + (\nabla_e \mu - \mathrm{P}_{ed}Y^d)\mu_f + (\nabla_f \nabla_e Y_d + g_{ed}\nabla_f \mu)\mu^d \\
&+ (\frac{1}{2}U_f\sigma - \mathrm{P}_{fd}\mu^d)Y_e - \Lambda(\nabla_e Y_f) - g_{ef}\mu\Lambda - \mathrm{P}_f{}^d\sigma\nabla_e Y_d - \mathrm{P}_{ef}\mu\sigma - \Lambda(\nabla_f Y_e) \\
=&(\nabla_f \nabla_e \mu - (\nabla_f \mathrm{P}_{ed})Y^d - \mathrm{P}_{ed}\nabla_f Y^d - \mathrm{P}_f{}^d\nabla_e Y_d - \mathrm{P}_{ef}\mu + \frac{1}{2}U_f Y_e)\sigma \\
&+ (\nabla_e \mu - \mathrm{P}_{ed}Y^d)\mu_f + (\nabla_f \nabla_e Y_d + g_{ed}\nabla_f \mu - \mathrm{P}_{fd}Y_e)\mu^d \\
&- \Lambda(\nabla_f Y_e + \nabla_e Y_f + g_{ef}\mu) \\
=&(\nabla_f \nabla_e \mu - (\nabla_f \mathrm{P}_{ed})Y^d - \mathrm{P}_{ed}\nabla_f Y^d - \mathrm{P}_f{}^d\nabla_e Y_d - \mathrm{P}_{ef}\mu + \frac{1}{2}U_f Y_e)\sigma \\
&+ (\nabla_f \nabla_e Y_d + g_{ed}\nabla_f \mu + g_{fd}\nabla_e \mu - \mathrm{P}_{fd}Y_e - g_{fd}\mathrm{P}_{ec}Y^c)\mu^d \\
&- \Lambda(\nabla_f Y_e + \nabla_e Y_f + g_{ef}\mu) \\
=&(\nabla_f \nabla_e \mu - (\nabla_{(e}\mathrm{P}_{f)d})Y^d - 2\mathrm{P}_{d(e}\nabla_{f)}Y^d - \mathrm{P}_{ef}\mu + \frac{1}{2}U_{(e}Y_{f)})\sigma \\
&+ (\nabla_{(e}\nabla_{f)}Y_d + 2g_{d(e}\nabla_{f)}\mu - Y_{(e}\mathrm{P}_{f)d} - g_{d(e}\mathrm{P}_{f)c}Y^c)\mu^d \\
&- \Lambda(2\nabla_{(e}Y_{f)} + g_{ef}\mu) \\
=&0.
\end{aligned}
$$

From this we obtain 6 equations in 4 unknowns and the determinants of the 4 by 4 minors associated to the 6 by 4 matrix will give us the obstructions. However, it is quite difficult to compute these determinants explicitly. In the next section we discuss a similar method to compute the obstruction when $U_aV^a = 0$ and the method can be applied even when $U_aV^a \neq 0$. The obstruction is given by the determinant of a 4 by 4 matrix.

7.4 Deriving obstructions when $U_aV^a = 0$

In this section we discuss the case where $U_aV^a = 0$ and proceed to derive obstructions. One obstruction can be computed quite readily; since $U_aV^a = 0$, $V^a = fY^a$ for some scalar density f of weight -4. Note here that f is not an invariant in the classical sense since it can be rational in the jets of the Möbius structure.

However assuming that $U_aV^a = 0$ holds on the non-flat Möbius surface, f is well-defined and is conformally invariant. Then

$$k\sigma = V_a\mu^a = fY_a\mu^a = -f\mu\sigma,$$

so that $k + f\mu$ is an obstruction to conformally Einstein on Möbius surfaces with $U_aV^a = 0$. We now would like to find more obstructions in the form of solving for μ_a or in the form of a non-zero determinant of a matrix associated to some linear system of equations. Contracting equation (7.7) by Y^a gives

$$\begin{aligned}(Y^e\nabla_eY_d)\mu^d + (Y^e\nabla_e\mu)\sigma + \mu(-\mu\sigma) - \Lambda\rho - \mathrm{P}_{ed}Y^eY^d\sigma = \\ (Y^e\nabla_eY_d)\mu^d + (Y^e\nabla_e\mu - \mu^2 - \mathrm{P}_{ed}Y^eY^d)\sigma - \Lambda\rho = 0.\end{aligned} \tag{7.11}$$

Let

$$X_d = Y^e\nabla_eY_d - 2\mu Y_d - \frac{\nabla_d\rho}{6}.$$

The 1-form X_d is conformally invariant of weight -6. Let

$$M_c = X_c + \frac{\nabla_c\rho}{6}.$$

Let

$$Q := \mathrm{P}_{ab}Y^aY^b - Y^a\nabla_a\mu + 3\mu^2.$$

This has weight -4. Then $\widehat{M_c} = M_c - \Upsilon_c\rho$, $\widehat{Q} = Q + \Upsilon_cM^c - \frac{1}{2}\Upsilon_c\Upsilon^c\rho$ and we have from (7.11)

$$M_c\mu^c = Q\sigma + \Lambda\rho. \tag{7.12}$$

Differentiating this equation gives

$$(\nabla_aM_b)\mu^b + M_b(-\Lambda\delta_a{}^b - \mathrm{P}_a{}^b\sigma) = (\nabla_aQ)\sigma + Q\mu_a + \left(-\frac{1}{2}U_a\sigma + \mathrm{P}_{ab}\mu^b\right)\rho + \Lambda(\nabla_a\rho),$$

so that we obtain

$$(\nabla_aQ + \mathrm{P}_{ab}M^b - \frac{1}{2}U_a\rho)\sigma + (-\nabla_aM_b + Qg_{ab} + \mathrm{P}_{ab}\rho)\mu^b + \Lambda(\nabla_a\rho + M_a) = 0. \tag{7.13}$$

Let $C_a = \nabla_aQ + \mathrm{P}_{ab}M^b - \frac{1}{2}U_a\rho$, $D_{ab} = -\nabla_aM_b + Qg_{ab} + \mathrm{P}_{ab}\rho$ and $E_a = \nabla_a\rho + M_a$. From equation (7.13), we have

$$C_a\sigma + D_{ab}\mu^b + E_a\Lambda = 0.$$

This gives 2 extra equations on 4 unknowns, so together with equations (7.6) and (7.12) we obtain a linear system of 4 equations in 4 unknowns:

$$\begin{pmatrix} \mu & Y_1 & Y_2 & 0 \\ Q & -M_1 & -M_2 & \rho \\ C_1 & D_{11} & D_{12} & E_1 \\ C_2 & D_{21} & D_{22} & E_2 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^1 \\ \mu^2 \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{7.14}$$

where

$$\begin{aligned} C_1 &= \nabla_1 Q + \mathrm{P}_{11} M^1 + \mathrm{P}_{12} M^2 - \frac{\rho}{2} U_1, & E_1 &= \nabla_1 \rho + M_1, \\ C_2 &= \nabla_2 Q + \mathrm{P}_{21} M^1 + \mathrm{P}_{22} M^2 - \frac{\rho}{2} U_2, & E_2 &= \nabla_2 \rho + M_2, \\ D_{11} &= -\nabla_1 M_1 + Q g_{11} + \mathrm{P}_{11}\rho, & D_{12} &= -\nabla_1 M_2 + Q g_{12} + \mathrm{P}_{12}\rho, \\ D_{21} &= -\nabla_2 M_1 + Q g_{12} + \mathrm{P}_{12}\rho, & D_{22} &= -\nabla_2 M_2 + Q g_{22} + \mathrm{P}_{22}\rho. \end{aligned}$$

The obstruction is then given by the determinant of the 4 by 4 matrix in (7.14). The next example in Section 7.5 shows that it is generically non-zero. However, it is quite difficult to check the conformal invariance of the determinant and invert the matrix to solve for μ_a. We note that under conformal rescalings,

$$\begin{aligned} \widehat{C}_a &= C_a - 7\Upsilon_a Q - 7\Upsilon_a \Upsilon_b M^b - \frac{1}{2}\Upsilon_b \Upsilon^b E_a - D_{ab}\Upsilon^b + \frac{7}{2}\Upsilon_a \Upsilon_b \Upsilon^b \rho, \\ \widehat{D}_{ab} &= D_{ab} + \Upsilon_b E_a + 7\Upsilon_a M_b - 7\Upsilon_a \Upsilon_b \rho, \\ \widehat{E}_a &= E_a - 7\Upsilon_a \rho. \end{aligned}$$

A possible future project will be to relate the determinant obstruction obtained in (7.14) to the tensor E_{ab} given by (7.10), and to check the conformal invariance of the determinant. Also we would like to see whether we can solve for μ_a in terms of invariants of the Möbius structure.

7.5 Example with $U_a V^a = 0$

In this example the Möbius structure on $\mathbb{R}^2$ has $U_a V^a = 0$ but the obstructions are non-vanishing, and so it does not admit a solution to (7.3). Take the Möbius structure given by

$$\mathrm{P} = \mathrm{P}_{11} dx^2 + 2\mathrm{P}_{12} dx dy + \mathrm{P}_{22} dy^2 = \left(\frac{x^2}{2} - \frac{y^2}{2}\right) dx^2 + \left(\frac{y^2}{2} - \frac{x^2}{2}\right) dy^2.$$

We compute and find,

$$Y^a = \begin{pmatrix} 2y \\ -2x \end{pmatrix}, \quad U^a = \begin{pmatrix} -2x \\ -2y \end{pmatrix}, \quad V^a = \begin{pmatrix} 8y \\ -8x \end{pmatrix}.$$

Here $f = 4$, $\mu = 0$, $k = 4xy(y^2 - x^2)$, so that $f\mu + k = 4xy^3 - 4x^3y \neq 0$ and we can conclude that the Möbius structure does not admit a conformally Einstein scale. Moreover, we find

$$M_a = \begin{pmatrix} 12x \\ 12y \end{pmatrix}, \qquad E_a = \begin{pmatrix} 20x \\ 20y \end{pmatrix}, \qquad Q = 4y^2x^2 - 2y^4 - 2x^4,$$
$$C_a = \begin{pmatrix} 6y^2x + 2x^3 \\ 6yx^2 + 2y^3 \end{pmatrix}, \qquad D_{ab} = \begin{pmatrix} 12 + 4y^2x^2 - 4y^4 & 0 \\ 0 & 12 + 4y^2x^2 - 4x^4 \end{pmatrix},$$

so that the 4 by 4 matrix in (7.14) is given by

$$\begin{pmatrix} 0 & 2y & -2x & 0 \\ 4y^2x^2 - 2y^4 - 2x^4 & -12x & -12y & 4y^2 + 4x^2 \\ 6y^2x + 2x^3 & 12 + 4y^2x^2 - 4y^4 & 0 & 20x \\ 6yx^2 + 2y^3 & 0 & 12 + 4y^2x^2 - 4x^4 & 20y \end{pmatrix}.$$

The determinant of the matrix is non-zero and is given by $-256y^3x^7 + 256y^7x^3 - 384y^9x + 384x^9y - 2304y^5x + 2304yx^5$.

7.6 Conformal tractor calculus on Möbius surfaces

Because of the way we defined Möbius surfaces, the conformal class $[g]$ on M^2 is assumed to be Riemannian and so of signature $(2, 0)$. However, the tractor construction in this section essentially goes through without change for conformal classes with metrics of signature $(1, 1)$ or $(0, 2)$. The conformally Einstein equation (7.3) is finite type and prolongs to form a closed system. Details can be found in [19] and [20]. In this section we shall discuss how to construct the conformal standard tractor bundle $\mathcal{E}_A$ on Möbius surfaces. We have used the tractor bundle here to really mean its dual, the co-tractor bundle, under the identification via the tractor metric. This essentially follows the construction given in [2]. The jet exact sequence at the 2-jets of the density line bundle $\mathcal{E}[1]$ gives

$$0 \to \mathcal{E}_{(ab)}[1] \to J^2(\mathcal{E}[1]) \to J^1(\mathcal{E}[1]) \to 0, \tag{7.15}$$

and the conformal structure further decomposes $\mathcal{E}_{(ab)}[1]$ into the direct sum $\mathcal{E}_{(ab)_\circ}[1]\oplus\mathcal{E}[-1]$. The symmetric trace-free bundle $\mathcal{E}_{(ab)_\circ}[1]$ is a smooth subbundle of $J^2(\mathcal{E}[1])$, and the tractor bundle $\mathcal{E}_A$ is simply the quotient bundle of $J^2(\mathcal{E}[1])$ by $\mathcal{E}_{(ab)_\circ}[1]$, defined by the exact sequence

$$0 \to \mathcal{E}_{(ab)_\circ}[1] \to J^2(\mathcal{E}[1]) \to \mathcal{E}_A \to 0. \tag{7.16}$$

The short exact sequence (7.15) at the 2-jets level and the short exact sequence

$$0 \to \mathcal{E}_a[1] \to J^1(\mathcal{E}[1]) \to \mathcal{E}[1] \to 0$$

at the 1-jet level determine a composition series for $\mathcal{E}_A$ described by $\mathcal{E}_A = \mathcal{E}[-1] \mathbin{+\!\!\!\!\supset} \mathcal{E}_a[1] \mathbin{+\!\!\!\!\supset} \mathcal{E}[1]$ where $\mathbin{+\!\!\!\!\supset}$ is the semi-direct sum. A choice of metric $g_{ab} \in [g]$ determines a splitting of the exact sequence, and identifies the standard tractor bundle $\mathcal{E}_A$ with the direct sum $\mathcal{E}[-1]\oplus\mathcal{E}_a[1]\oplus\mathcal{E}[1]$. On Möbius surfaces, the conformal cotractor bundle has an invariant metric h_{AB} of signature $(2,1)$ called the tractor metric, and an invariant connection ∇_a preserving h_{AB} called the tractor connection. On a Möbius surface $(M^2, [g], [\mathrm{P}])$, we have a section of the standard tractor bundle $\mathbb{T}_A \in \Gamma\mathcal{E}_A$ given in a conformal scale obtained by choosing a representative metric $g \in [g]$ by

$$\mathbb{T}_A \overset{g}{=} \begin{pmatrix} \sigma \\ \mu_b \\ \Lambda \end{pmatrix} \in \begin{matrix} \mathcal{E}[1] \\ \oplus \\ \mathcal{E}_a[1] \\ \oplus \\ \mathcal{E}[-1] \end{matrix},$$

and under conformal rescalings of the metric,

$$\widehat{\mathbb{T}}_A \overset{\widehat{g}}{=} \begin{pmatrix} \sigma \\ \mu_b + \Upsilon_b\sigma \\ \Lambda - \Upsilon^b\mu_b - \frac{1}{2}\Upsilon_b\Upsilon^b\sigma \end{pmatrix}.$$

Using the tractor bases convention as used in [20] for instance, we have

$$\mathbb{T}_A \overset{g}{=} \sigma\mathbb{Y}_A + \mu_a\mathbb{Z}_A^a + \Lambda\mathbb{X}_A,$$

where under conformal recalings, the bases transform according to

$$\begin{aligned}
\widehat{\mathbb{Y}}_A =& \mathbb{Y}_A - \Upsilon_a\mathbb{Z}_A^a - \frac{1}{2}\Upsilon_a\Upsilon^a\mathbb{X}_A, \\
\widehat{\mathbb{Z}}_A^a =& \mathbb{Z}_A^a + \Upsilon^a\mathbb{X}_A, \\
\widehat{\mathbb{X}}_A =& \mathbb{X}_A,
\end{aligned}$$

and the tractor connection acts on the bases according to

$$\begin{aligned}\nabla_a \mathbb{Y}_A =& \mathrm{P}_{ab}\mathbb{Z}^b_A,\\ \nabla_a \mathbb{Z}_{bA} =& -\mathrm{P}_{ab}\mathbb{X}_A - g_{ab}\mathbb{Y}_A,\\ \nabla_a \mathbb{X}_A =& \mathbb{Z}_{aA}.\end{aligned}$$

We have the standard tractor connection acting on sections of the tractor bundle by

$$\nabla_a \mathbb{T}_B = \nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \Lambda \end{pmatrix} = \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \mathrm{P}_{ab}\sigma + \Lambda g_{ab} \\ \nabla_a\Lambda - \mathrm{P}_{ad}\mu^d \end{pmatrix}.$$

Recall that in dimensions $n > 2$, there is a 1-1 correspondence between solutions of the conformally Einstein equation and parallel sections of the standard tractor bundle with nowhere vanishing scale σ (see for instance [19]). We would like to establish a similar correspondence on Möbius surfaces. However, we have seen from the prolongation of the conformally Einstein equation (7.3) that an additional term involving U_a appears in the closed system, so the standard tractor connection is not the right object in the correspondence. We have however in the flat case, where $U_a = 0$, that

Proposition 7.2. *There is a 1-1 correspondence between solutions of the conformally Einstein equation on flat Möbius surfaces $(M^2, [g], [\mathrm{P}])$ and parallel sections $\mathbb{I}_A$ of the standard tractor bundle with $\sigma = \mathbb{X}^A\mathbb{I}_A$ nowhere vanishing.*

Proof. Suppose the flat Möbius surface admits a solution to (7.3). In a particular conformal scale, a parallel section of the tractor bundle $\mathcal{E}_A$ is given by

$$\mathbb{I}_A = \begin{pmatrix} \sigma \\ \nabla_a\sigma \\ -\frac{1}{2}(\Delta\sigma + K\sigma) \end{pmatrix}$$

and we find from the prolonged system of (7.3) that

$$\begin{aligned}\nabla_a \mathbb{I}_B = \nabla_a \begin{pmatrix} \sigma \\ \nabla_b\sigma \\ -\frac{1}{2}(\Delta\sigma + K\sigma) \end{pmatrix} =& \begin{pmatrix} \nabla_a\sigma - \nabla_a\sigma \\ \nabla_a\nabla_b\sigma + \mathrm{P}_{ab}\sigma - \frac{1}{2}(\Delta\sigma + K\sigma)g_{ab} \\ \nabla_a(-\frac{1}{2}(\Delta\sigma + K\sigma)) - \mathrm{P}_{ad}\nabla^d\sigma \end{pmatrix}\\ =& 0,\end{aligned}$$

where the equation on the bottom slot holds as a differential consequence of (7.3). It is precisely (7.5). Conversely given a parallel section of the standard tractor connection with nowhere vanishing scale, $\sigma = \mathbb{X}^A \mathbb{I}_A$ defines a solution to (7.3). $\square$

We define the prolongation connection on general Möbius surfaces by the following formula:

$$
\begin{aligned}
D_a \mathbb{T}_B = D_a \begin{pmatrix} \sigma \\ \mu_b \\ \Lambda \end{pmatrix} &= \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma + \Lambda g_{ab} \\ \nabla_a \Lambda - \mathrm{P}_{ad}\mu^d + \frac{1}{2} U_a \sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma + \Lambda g_{ab} \\ \nabla_a \Lambda - \mathrm{P}_{ad}\mu^d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} U_a \sigma \end{pmatrix} \\
&= \nabla_a \mathbb{T}_B + \frac{1}{2} U_a \mathbb{T}^A \mathbb{X}_A \mathbb{X}_B.
\end{aligned}
\tag{7.17}
$$

Proposition 7.3. *There is a 1-1 correspondence between solutions of the conformally Einstein equation on Möbius surfaces and sections of the standard tractor bundle with $\sigma = \mathbb{X}^A \mathbb{I}_A$ non-vanishing and parallel with respect to the prolongation connection defined in (7.17).*

Proof. The proof is similar to the proof in Proposition 7.2. $\square$

7.6.1 Almost Möbius Einstein scales

We shall work with the prolongation connection on the standard tractor bundle. Let $(M^2, [g], [\mathrm{P}])$ be a Möbius surface equipped with a parallel section $\mathbb{I}_A$ (with respect to the prolongation connection) of the standard tractor bundle. Such a surface is called almost Möbius Einstein. We can look the scale singularity of σ. If σ is nowhere vanishing, then by Proposition 7.3 the surface locally admits a conformally Einstein metric everywhere. Otherwise the scale singularity set $\Sigma = \{p \in M^2 | \sigma(p) = 0\}$ could be either codimension 1, in which case Σ is a curve on the Riemann surface, or codimension 2 in which case Σ is a collection of isolated singularity points. Such Möbius surfaces are conformally Einstein away from the scale singularity set of σ.

Proposition 7.4. *Let $(M^2, [g], [\mathrm{P}], \mathbb{I}_A)$ be an almost Möbius Einstein structure with a codimension 1 scale singularity set Σ. Then on Σ, which is a curve in M^2, we have the normal cotractor be given by*

$$
\mathbb{N}_A := \mathbb{I}_A|_\Sigma = \begin{pmatrix} 0 \\ n_a \\ -H \end{pmatrix},
$$

where $n_a := \nabla_a \sigma|_\Sigma \neq 0$ is the normal covector of Σ in M^2, and $H := \nabla_a n^a$ is the mean curvature of Σ in M^2. The normal vector field $n^a := \nabla^a \sigma|_\Sigma$ is a multiple of $U^a|_\Sigma$.

Proof. The last statement follows from the constraint equation (7.6). Since $\sigma = 0$ on Σ, $Y_a \mu^a = 0$ on Σ and so $\mu^a|_\Sigma =: n^a$ is some multiple of U^a. □

This tells us that Y^a is tangent to Σ.

7.7 Summary of equations studied on Möbius surfaces

We can summarise the equations studied on Möbius surfaces in Chapters 6 and 7 in the table here:

	$F = 0$	$F \neq 0$
$s = 0$	Conformally Einstein on flat surfaces	sf-MEW
$s \neq 0$	Conformally Einstein	MEW

The most general equation (6.4) called MEW is not finite type and cannot be prolonged to form a closed system. Taking $s = 0$, $F \neq 0$ yields the sf-MEW equation (6.8). Taking $s \neq 0$, $F = 0$ gives us the conformal-to-Einstein equation (7.3). Requiring both $s = 0$ and $F = 0$ to hold force $Y_a = 0$, so that the Möbius structure is flat. In this setting we obtain the conformally Einstein equation on flat Möbius surfaces.

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