Abstracts of Lectures, Tutorials, and Talks

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Shanks Lectures

Congruence Identities

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A congruence identity for a variety of algebras is a lattice identity that holds in all congruence lattices of members. In this talk I will describe which varieties satisfy nontrivial congruence identities and how the identities affect the shapes of congruence lattices.

Four Commutators

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We describe four important commutator operations definable on congruence lattices of algebras. For each one I will describe its ideal abelian model and tell what it means for a variety if one of these commutators is trivial for each member.

Invited Talks

The variety of involutive residuated lattices is generated by its finite members

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Involutive FL-algebras (InFL-algebras), also known as involutive residuated lattices, are generalizations of Boolean algebras, lattice-ordered groups and symmetric relation algebras. In this talk we prove that the corresponding variety has a decidable equational theory; actually, it is generated by its finite members. On the way, we explain the connections of these algebras with substructural logics and show how the interplay between algebra and logic can be fruitful for both areas.

Residuated lattices are algebras with a lattice and a monoid reduct such that multiplication is residuated; the residuals of multiplication (called divisions and denoted by $\backslash$ and $/$) are included in the type. The addition of another arbitrary constant element 0 allows for considering two new operations (negations) $\sim x = x \backslash 0$ and $-x = 0/x$ and leads to the definition of an *FL-algebra*. An *InFL-algebra* is an FL-algebra that satisfies $\sim-x = -\sim x = x$.

On the other hand, substructural logics include classical logic, as well as non-classical logics like intuitionistic, many-valued, relevant, linear, and paraconsistent. More precisely, (propositional) substructural logics are defined as axiomatic extensions of the full Lambek calculus (**FL**), which is usually presented in a sequent system formulation. The system **FL** is essentially obtained from Gentzen's system **LJ** for intuitionistc logic by removing the three structural rules of weakening, contraction and exchange. It turns out that the variety of FL-algebras is an equivalent algebraic semantics (in the sense of Blok an Pigozzi) for the substructural logic **FL**.

In this talk we define a sequent calculus **InFL**, algebraic semantics for which is the variety of InFL-algebras. We obtain the decidability of the latter by proving the cut-elimination property for the former. Traditionally, cut-elimination for a sequent calculus is shown by a syntactic and proof-theoretic analysis through a long and involved induction argument. We establish cut-elimination for **InFL** by a semantic, near-algebraic method based on the notion of a residuated frame (relational semantics for substructural logics).

The system **InFL** used for the decidability of InFL-algebras is inspired by logical considerations and its cut-elimination is obtained by appeal to

algebraic ideas. We feel that ordered algebraic structures and relational semantics play an important role in bringing the connections between algebra and logic to light. We define in detail the notion of a residuated frame and show how it can be used to prove cut-elimination, decidability, the finite model property, the finite embeddability property, as well as the amalgamation and the interpolation properties for various logics and varieties.

An Abstract Algebraic Logic view on canonical extensions

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The theory of canonical extensions originated in B. Jónsson and A. Tarski's paper "Boolean algebras with operators, I" (Am. J. Math. 73, 1951) and has been developed in the last fifteen years by several authors, among them M. Gehrke, J. Harding, B. Jónsson, A, Palmigiano, H. Priestley, Y. Venema, etc. First it was extended from Boolean algebras with operators to distributive lattices and then to lattices and also very recently to posets, always with additional operations. Two perspectives can be taken on canonical extensions. One purely algebraic and another with a strong logical motivation. This second perspective has been mainly taken by algebraic modal logicians, and also recently by researchers on substructual logics.

Abstract Algebraic Logic AAL is the field that intends to develop a general theory of the relations between logic systems and their intended classes of algebras, providing general concepts and theorems relating logical properties with algebraic ones. These theorems usually explain known results found before for specific logics.

In the talk we will adopt the perspective of AAL to try to provide a unified view on the theory of canonical extensions from the logical point of view.

Canonical extensions of bounded distributive lattice expansions

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Just over sixty years ago, Alfred Tarski proved:

Theorem 1. *Every relation algebra can be embedded in a complete and atomic relation algebra.*

This theorem is an acorn from which a mighty oak has grown. I have used this metaphor once before, about a theorem by another mathematician. See [J95], p. 354. The two instances are remarkably similar. In each case the proof of a moderately interesting theorem, when viewed from the right perspective, was seen to exemplify a previously unformulated principle with broad applicability. Perhaps we should pay more attention to this phenomenon. It might teach us to look behind the details of a proof and identify fundamental ideas that make it work. This talk will be about the theory that evolved from Tarski's result, but it will also illustrate the potential rewards for careful reading, and the price we may pay for not being thorough enough. When I first read Tarski's proof, I realized that his method could be applied to other algebraic structures beside relation algebras. This led to the theory of canonical extensions of Boolean algebras with operators. Almost half a century later, Mai Gehrke and I extended this to bounded distributive lattices with operators, but we soon realized that a still more general notion of canonical extensions was needed. A patchwork of definitions was considered, but eventually Gehrke proposed one that includes all of them as special cases. This concept, which applies to arbitrary bounded distributive lattice expansions, i.e. to arbitrary algebras having a bounded distributive lattice as a reduct, was introduced and investigated in [GJ04]. In spite of its extreme generality, it has turned out to be quite manageable and powerful. This is the concept that I should have come up with when I read Tarski's proof over half a century earlier!

References

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Near-unanimity terms are decidable

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An at least ternary operation t is a *near-unanimity operation* if it satisfies the identities

$$t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y) \approx x.$$

Near-unanimity operations come up naturally in the study of algebras. For example, if an algebra of finite signature has a near-unanimity term operation, then it has a finite base of equations; a finite algebra generating a congruence-distributive variety admits a natural duality if and only if it has a near-unanimity term operation; and if a set of relations has a compatible near-unanimity operation then the corresponding constraint satisfaction problem is solvable in polynomial time.

We prove that it is decidable of a finite algebra whether it has a near-unanimity term operation, which settles a more than ten-year-old problem.

Order and Algebra in Natural Logic

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By 'natural logic' in this work I mean logical systems whose syntax is closer to that of natural language than more standard systems. One way to state the goal of the work would be to ask for logical systems which can formalize interesting parts of natural language semantics, and which at the same time are decidable (unlike first-order logic) and which have interesting complete proof systems. I am interested in these for reasons coming from linguistic semantics, cognitive science, and other areas. However, none of this motivation informs the mathematics.

Practically all the systems in the talk will be new to the audience. The main results are a series of completeness theorems for them. There are many open problems and areas, and part of the reason for presenting the work at OAL is to advertise the subject.

Ideas from algebra and ordered structures are important tools in the area. The use of order is already clear from the prominence of monotonicity facts in semantics. As for algebra, boolean algebra is obviously important, but there is much of interest in systems which do not even have boolean connectives to start with. For example, consider standard syllogisms with a (noun phrase

level) complement operation. Completeness can be shown fairly easily using a representation result for orthoposets. This is an example of the kind of result to be presented.

The Haar theorem for finitely presented MV-algebras and unital ℓ-groups

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Every rational polyhedron P is equipped with invariants $\lambda_0(P), \lambda_1(P), \ldots$, with each $\lambda_i(P)$ proportional to the volume of the i-dimensional part of P. Using these invariants we prove that every finitely presented unital lattice-ordered abelian group G (resp., every finitely presented MV-algebra A), has a faithful invariant state s. For every element $x \in G$ (resp., $x \in A$), $s(g)$ is the integral of g over the maximal spectrum of G (resp., the maximal spectrum of A), the latter being canonically identified with a rational polyhedron P. Volume elements are measured by the $\lambda_i(P)$'s. The proof uses Geometry of Numbers, together with the Włodarczyk-Morelli theorem on decompositions of birational toric maps in blow-ups and blow-downs.

From lifting objects to lifting diagrams with respect to a given functor

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For categories $\mathcal{I}$ and $\mathcal{S}$, an *$\mathcal{I}$-indexed $\mathcal{S}$-valued diagram* is a functor $\mathbf{D} \colon \mathcal{I} \to \mathcal{S}$. For another category $\mathcal{V}$ and a functor $\mathbf{F} \colon \mathcal{V} \to \mathcal{S}$, a *lifting* of $\mathbf{D}$ with respect to $\mathbf{F}$ is a functor $\mathbf{E} \colon \mathcal{I} \to \mathcal{V}$ such that the composition $\mathbf{FE}$ is equivalent to $\mathbf{D}$.

During the few past years, many results have been proved involving, on the one hand, lifting results of *objects* and lifting objects of *diagrams*. For example, Bill Lampe proved in 1982 that every bounded semilattice is isomorphic to the compact congruence semilattice $\mathrm{Con}_c\, G$, for some *groupoid G*. He extended his result to lifting *morphisms* in 2005.

Our program is to give formal settings relating the "objects" results and the "diagrams" results. One of our results makes it possible to extend Lampe's results to many other situations, including the following.

Theorem 1. *If a variety $\mathcal{V}$ lifts all $(\vee, 0)$-semilattices, with respect to the* $\mathrm{Con_c}$ *functor, then it also lifts every diagram of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms indexed by a finite lattice. Furthermore, in the presence of large cardinals (e.g., enough Erdős cardinals), then this result can be extended to diagrams indexed by arbitrary, possibly infinite, $(\vee, 0)$-semilattices.*

For varieties $\mathcal{A}$ and $\mathcal{B}$ of algebras, the *critical point* $\mathrm{crit}(\mathcal{A}, \mathcal{B})$ is the least possible cardinality of a semilattice of the form $\mathrm{Con_c}\, A$, for some $A \in \mathcal{A}$, but not of the form $\mathrm{Con_c}\, B$, for any $B \in \mathcal{B}$, and ∞ if there is no such cardinal.

Theorem 2. *For any finitely generated lattice varieties $\mathcal{A}$ and $\mathcal{B}$, $\mathrm{crit}(\mathcal{A}, \mathcal{B})$ is either finite, or of the form $\aleph_n$ for some natural number n, or ∞.*

Four unsolved problems in congruence permutable varieties

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In general algebra, one usually considers congruence permutable varieties and the algebras they contain as being a mere first step away from the classical algebraic structures of groups and rings. Many "big" problems in the field have long been solved for congruence permutable (and more general) varieties; the focus of current research is in the outer reaches of nonstandard algebraic structures.

By contrast, in this lecture I will describe four general problems which are solved (or nearly solved) for groups and rings, but are wide open for congruence permutable varieties.

Invited Tutorials

Equations and algebras: A tutorial on algorithmic problems

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The problems at issue here have two aspects: a specification concerning inputs and a property to be tested on each input. Each problem has a computational character. Loosely speaking, each input must be amenable to entering on a computer keyboard using finitely many keystrokes. The

properties to be tested here will all arise from algebraic or logical consideration. The focus of the tutorial is the compuational difficulty of each of the problems.

The first session of the tutorial will focus on those problems where the specified inputs are equations or, more generally, finite sets of equations. An easily solved example is the problem of determining of an equation in the language of lattices whether it is true in some nontrivial lattice. A much more challenging example is the problem of determining of an equation whether it is true is some algebra whose universe is $\mathbb{R}$ and whose basic opertions are all continuous.

The second session of the tutorial will focus on those problems where the specified inputs are finite algebras. A problem of this sort which is not too hard is to determine whether the variety generated by the algebra is a minimal variety. There turns out to be an algorithm solving this problem. However, the computational complexity of this problem has not been determined.

While the main thrust of the tutorial is to summarize the considerable discoveries made over the last 60 years, some of the methods used will be sketched and several open problems will be posed.

Interplay between algebra and logic

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In the present talk, we will discuss recent development of research on residuated lattices and substructural logics. Our main aim is to show that there are fundamental and complementary interplays between algebra and logic through *algebraic logic*, and also to explain recent discoveries of unexpectedly closer interlinkages between *algebraic* methods and *proof-theoretic* methods. These observation will provide us a deeper understanding of the subject. The present talk is based on my book [GJKO] jointly written by Galatos, Jipsen and Kowalski, where these topics are extensively discussed. Here is a brief outline of my talk.

Part 1
- Formalizing logics of residuated lattices
- Provability, deducibility and equational consequence
- Substructural logics and subvarieties of the variety of residuated lattices
- Algebraization a la Blok-Pigozzi
- Parameterized local deduction theorem

Part 2
- Decidability
- Cut elimination
- Gentzen matrices and algebraic proof of cut elimination
- Finite model property and finite embeddability property
- Interpolation, Robinson property and amalgamation property

[GJKO] N. Galatos, P. Jipsen. T. Kowalski and H. Ono, *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logic and the Foundations of Mathematics, vol. 151, Elsevier, 2007.

Commutator Theory

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Commutator theory is a powerful tool for investigating varieties (= equationally definable classes) of algebras. A commutator operation on congruences, which generalizes the commutator operation on normal subgroups of groups, was first introduced for congruence modular varieties. The modular commutator has been successfully applied in numerous results, e.g. in determining which finitely generated modular varieties are residually small, or which locally finite modular varieties have decidable first order theories. Generalizations of the modular commutator to wider classes of varieties have proved equally important in extending many of these results beyond the realms of congruence modular varieties.

The aim of this tutorial is to give an introduction to commutator theory and to illustrate its power by discussing some of its applications.

Modal Logic and Universal Algebra

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Algebraic Logic aims at studying all kinds of logics using tools and techniques from Universal Algebra. Modal logics are mirrored algebraically by classes of so-called Boolean algebras with operators, and many results in Universal Algebra have been applied in this area. Recent developments

have also witnessed a significant flow of information in the opposite direction. In the tutorial I intend to give an introduction and overview of some developments on the interface of modal logic and universal algebra.

Lattice expansions, or lattice-ordered algebras, will comprise the main topic of the tutorial. I will first introduce the mirroring of modal logics by algebras in terms of two categorical dualities: a discrete and a topological one. Many issues that naturally arise in this modal logic setting carry over to the much wider setting of lattice expansions. As a key example I will sketch how a classic result from modal logic, Sahlqvist's theorem, may transform into the cornerstone of an emerging uniform and partly *automatic* representation and duality theory for lattice expansions.

Finally, I will briefly sketch the emerging field of Universal Coalgebra as a general framework for studying the behaviour of state-based evolving systems. I will try to motivate and substantiate the slogan that modal logic is to coalgebra what equational logic is to algebra.

The tutorial does not presuppose any previous exposure to the theory of modal logic.

Contributed Talks

A Minimal Q-universal variety of Heyting algebras

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We construct a variety **V** of Heyting algebras having the following three properties: **V** is Q-universal, no variety properly contained in **V** is Q-universal, and **V** is the join of its proper subvarieties. This refutes a plausible conjecture that every minimal Q-universal variety of Heyting algebras must be join irreducible.

Realization of abstract convex geometries by point configurations

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A closure space $(J, -)$ is called *a convex geometry*, if it satisfies the anti-exchange axiom. Our work is aiming toward a solution of the problem by P. Edelman and R. Jamison: *Given a finite convex geometry, decide whether it can be realized as relatively convex subsets of some finite configuration of points on a plane.*

Among several directions in this study, we discuss a connection between the Edelman-Jamison Problem and the Order Type Problem. The latter asks whether any given function $t : J[3] \to \{1, -1\}$ defined on the triples of different elements of a finite set J might be realizable as orientation of triples of points of some point configuration on a plane. This problem is known to be NP-hard.

We formulate the modification of Edelman-Jamison problem as follows: given a finite convex geometry and a circular order of its extreme elements, decide whether this convex geometry is realizable via a point configuration, with the outside layer of this configuration having the given circular order. We prove that such modification of the Edelman-Jamison problem is polynomially equivalent to the Order Type problem, thus, it is also NP-hard.

Finitely presented MV-algebras with finite automorphism group

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We address the question, which MV-algebras have finite automorphism group. The automorphism group of the free MV-algebra on 1 generator is just the group of order 2. In contrast, it is known that the automorphism group of the free MV-algebra on 2 generators is not even locally finite.

If M is a finitely generated MV-algebra, its maximal spectrum is a compact Hausdorff space of finite dimension. When the dimension is ≤ 1, we say M is a monodimensional MV-algebra. We establish: for any finitely presented MV algebra M, the automorphism group of M is finite whenever M is monodimensional. Our proof yields as a byproduct a complete isomorphism

invariant for finitely presented monodimensional MV-algebras, namely, a weighted graph.

We give examples to show that our main result does not extend to finitely generated monodimensional MV-algebras. Specifically, there exists a two-generated semisimple MV-algebra with zero-dimensional maximal spectrum whose automorphism group is not even locally finite.

We obtain enough information on extensions of automorphisms to afford a partial converse to our main result. Whether the actual converse holds is an open problem.

On a special case of the flexible atom conjecture

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An atom a in a relation algebra is said to be *flexible* if there are no forbidden cycles of diversity atoms involving the atom a. Every finite integral relation algebra with a flexible atom is known to be representable on a countable set. It is conjectured that every such relation algebra is representable on a finite set. In this talk, we will give a proof of a special case, namely that every finite integral symmetric relation algebra with exactly one flexible atom in which all mandatory cycles of diversity atoms involve the flexible atom is representable on a finite set.

We begin by constructing arbitrarily large representations of the algebra with three symmetric atoms 1', r, b, where bbb is the only forbidden cycle. This yields a complete graph with edges colored in colors r (red) and b (blue) with no monochromatic blue triangles. The special case is proved by reassigning blue edges to n shades of blue uniformly at random. By taking the number of vertices to be sufficiently large such a random coloring yields a representation of the desired algebra with positive probability.

Syntactical derivation of protoalgebraicity condition

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Abstract Gentzen-style propositional deductive systems can be treated as regular (Hilbert-style) deductive systems, usually considered in Algebraic Logic. Thus a Gentzen-style system can be identified with a family of its

theories, but now each of this theories is a set of sequents — finite sequences of formulas. The family of theories, as well as in the Hilbert-style case, must be closed under arbitrary intersections, upward directed unions and inverse substitutions, the latter taken componentwise. But Gentzen-style structures, that can be associated with a Hilbert-style deductive system, usually fall short of being deductive systems. The most important cases are families of, so called, full and axiomatic closure relations. Although they are not deductive systems, they have amenable enough properties to express some aspects of algebraizability of Hilbert-style deductive systems. Namely, it will be shown that the interrelation between full and axiomatic closure relations can be used to derive the protoagebraicity and weak algebraizability syntactical conditions. The novelty of the proof is that it deals only with the formula algebra and closure relations on it, without explicit reference to semantics.

Dualities in full homomorphisms

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We study dualities of graphs and, more generally, relational structures, with respect to full homomorphisms, that is, mappings that are both edge- and non-edge-preserving. (The research was motivated by, among other things, results from logic concerning first order definability, and by Constraint Satisfaction Problems.) We prove that for any finite set of objects $\mathcal{B}$ (finite relational structures) there is a finite duality with $\mathcal{B}$ to the left. There appears to be a suprising rich assortment of such dualities, some leading to interesting problems of Ramsey type. We explicitly analyze these concepts in the simplest case of graphs.

Set functors determined by values on objects

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The talk is devoted to DVO set functors, i.e. endofunctors of the category of sets which are uniquely determined by their object part. Known results, recent progress and connections with universal algebra problems will be mentioned.

A solution of the following problem seems to be crucial for a full characterization of DVO set functors.

Problem 1. Let $t(x_0, \ldots, x_{n-1})$ be a term in an idempotent nontrivial variety $\mathbf{V}$, let G be the stabilizer of t in $\mathbf{V}$:

$$G = \{g \in S_n \mid t(x_0, x_1, \ldots, x_{n-1}) \approx_{\mathbf{V}} t(x_{g(0)}, x_{g(1)}, \ldots, x_{g(n-1)})\}$$

and let k be an odd natural number less than the cardinality of the smallest orbit of G. Must there be a mapping $r : n \to k$ such that the term $t(x_{r(0)}, x_{r(1)}, \ldots, x_{r(n-1)})$ depends on all of the variables $x_0, \ldots x_{k-1}$?

Priestley Order-Compactifications

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We generalize the notion of a 0-dimensional compactification of a topological space to that of a Priestley order-compactification of an ordered topological space. We also generalize the notion of a Boolean basis to that of a Priestley basis, and show that there is a 1-1 correspondence between Priestley order-compactifications and Priestley bases of an ordered topological space. This generalizes a 1961 result of Dwinger that 0-dimensional compactifications of a topological space are in a 1-1 correspondence with its Boolean bases.

There are no Tarski-style axiomatizations for most logics over FL$_{\text{ew}}$

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It is well-known that intuitionistic (and also classical) propositional logic can be obtained as the smallest consequence relation satisfying some meta-rules involving, each of them, only one connective: deduction-detachment theorem, property of conjunction, property of disjunction,... In the book of Wojcicki [1] the term Tarski-style condition is used in a naive way to talk about this kind of rules. We will say that a logic has a *Tarski-style*

axiomatization whenever it can be obtained as the smallest consequence relation satisfying certain Tarski-style conditions. We point out that we do not worry whether these conditions are recursively enumerable or not, we only care about their existence.

In the talk, first of all we will present a formal definition of the notion of Tarski-style axiomatization of a logic. For the sake of generality we consider in the previous definition consequence relations that are more general than the standard ones (and are closer to sequent systems). The main result we will explain in the talk is that *all substructural logics above* $\mathsf{FL_{ew}}$ *that are not superintuitionistic logics do not have a Tarski-style axiomatization.* In other words, the logics above $\mathsf{FL_{ew}}$ such that the fusion connective does not coincide with the meet connective do not have a Tarski-style axiomatization.

References

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Study of canonical extensions of BL-algebras

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If $\mathbf{A} = (A, \{f_i, i \in I\})$ is a distributive lattice with operations, the canonical extension A^σ of the lattice $(A, \wedge, \vee)$ is a doubly algebraic distributive lattice that contains A as separating and compact sublattice. There are two different ways to extend an operation f to A^σ: the canonical extension f^σ and the dual canonical extension f^π. This gives us two possible candidates for the canonical extension of $\mathbf{A}$: the canonical extension $\mathbf{A}^\sigma$ and the dual canonical $\mathbf{A}^\pi$. A class of algebras is called σ-canonical (π-canonical) if it is closed under canonical (dual canonical) extensions.

BL-algebras are the algebraic counterpart of basic logic. Since BL-algebras can be viewed as distributive lattices with additional operations, we will analyze the σ and π-canonicity of some subvarieties of BL-algebras. The main results are the following ones:

1) A subvariety of BL-algebras is σ-canonical iff it is finitely generated by finite algebras;

2) There are at least countably many subvarieties of BL-algebras that are π-canonical.

Logics with Fusion and Implication

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In this work we introduce and study three semilattice based logics relative to the varieties of distributive lattices with fusion and \ or implication, **DLF**, **DLI** and **DLFI**-algebras, $\mathcal{S}$ (**DLF**), $\mathcal{S}$ (**DLI**) and $\mathcal{S}$ (**DLFI**) respectively.

We introduce relational semantics for these logics based on posets and ternary relations defined on them. Using these semantics, we show that the three deductive systems mentioned are not Fregean and not protoalgebraic.

Finally, we study some extensions of these logics like subintuitionistic logic, the semilattice based logics relative to Residuated Lattices, Integral Commutative Residuated Lattices, and **MTL**-algebras. For these deductive systems we describe the family of its frames and we prove that they are correct and complete with respect to them.

Compatible operations on commutative residuated lattices

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Let L be a commutative residuated lattice, and $f : L \to L$ a function. We determine a necessary and sufficient condition for f to be compatible with respect to every congruence on L. We use this characterization of compatible functions in order to prove that the variety of commutative residuated lattices is locally affine complete. Then, we find a condition on a polynomial $P(x, y)$ in L that implies that the function $x \mapsto min\{y \in L \mid P(x, y) \leq y\}$ is compatible when defined. In particular, $P_n(x, y) = y^n \to x$, for natural n, defines a family of compatible functions on some commutative residuated lattices. We show through examples that S_1 and S_2 defined respectively from P_1 and P_2 are independent as operations over this variety; i.e. neither S_1 is definable as a polynomial in the language of L enriched with S_2 nor S_2 in that enriched with S_1.

Many-valued quantum algebras

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An MV-algebra as an algebraic axiomatization of the Łukasiewicz many-valued propositional logic. It is a bounded distributive lattice with an antitone involution, where the set of those elements that have a complement form a subalgebra which is a Boolean algebra.

The logic of quantum mechanics is axiomatized by orthomodular lattices. It is natural to ask which structures generalize orthomodular lattices in the way that MV-algebras extend Boolean algebras. The structures in question are bounded lattices with antitone involutions, such that the set consisting of those x for which x^* is a complement is an orthomodular sublattice of the initial lattice.

Our goal is to derive "MV-like" algebras $\mathbf{A} = (A, \oplus, \neg, 0)$ of type $\langle 2, 1, 0 \rangle$ that generalize both MV-algebras and orthomodular lattices, and have the following properties:

  (I) A is a bounded lattice with respect to the induced natural order $\leq$,
  (II) the set of complemented elements is a sublattice which is an orthomodular lattice,
 (III) every subalgebra of $\mathbf{A}$ contained in a block is an MV-algebra.

In a stronger version we ask that

 (III') every block of $\mathbf{A}$ is an MV-algebra.

Homomorphisms for topological coalgebras

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There is a natural way to turn topological spaces into coalgebras for the filter functor. However, it was proved by Gumm that naive coalgebra homomorphisms correspond to open continuous maps. In this paper, we show that a suitably weakened homomorphism concept ensures that coalgebra homomorphisms agree with the correct homomorphisms of topological spaces, namely continuous maps. Based on this appropriate relaxation of the concept of coalgebra homomorphism, we prove the equivalence between the

usual category of topological spaces and the category of coalgebras obtained from topological spaces.

Algebraic characterization of completeness properties for core fuzzy logics II: predicate logics

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This contribution is a follow-up to *Algebraic characterization of completeness properties for core fuzzy logics I: propositional logics* presented by Carles Noguera. Here we continue our study of completeness properties of core and Δ-core fuzzy logics, only this time we move to the first-order level. We present three major (groups of) results.

First, we deal with general completeness results (w.r.t. arbitrary classes of algebras). Our main result here is an equivalent characterization of the strong completeness by model-theoretic (based on the notion of elementary embedding) and by algebraic means (based on the notion of σ-embedding). Then we present a constructive procedure that translates a contraexample for the *strong* completeness of a *propositional* (core fuzzy) logic to a contraexample for the *finite strong* completeness of its corresponding *first-order* variant. Finally, we concentrate on results for particular distinguished semantics (real-, rational-, finite-, and hyperreal-chains based ones), present tables summarizing known results for particular fuzzy logics, and show a list of open problems.

Existence of states on fuzzy structures

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In this paper we prove that every perfect residuated lattice admits at least a Bosbach state and that the Bosbach and Riečan states on a good pseudo-BCK algebra with pseudo-double negation coincide. Additionally, we will compare some results regarding the existence of states on pseudo-BCK algebras with the those on pseudo-BL algebras.

Theorem 1. *Any perfect residuated lattice admits at least a Bosbach state.*

Theorem 2. *Any perfect pseudo-BL algebra admits a unique state-morphism.*

Proposition 3. *There exist linearly ordered pseudo-BCK algebras having no Bosbach states.*

Proposition 4. *There exist ordered pseudo-BCK algebras having maximal and normal deductive systems, but having no Bosbach states.*

Theorem 5. *The Bosbach and Riečan states on a good pseudo-BCK algebra with pseudo-double negation coincide.*

Corollary 6. *If $\mathcal{BS}(A)$ is the set of all Bosbach states and $\mathcal{RS}(A)$ the set of all Riečan states on a perfect residuated lattice with pseudo-double negation A, then $\mid \mathcal{BS}(A) \mid = \mid \mathcal{RS}(A) \mid \neq 0$.*

Commutator Equations, Day Implication and the Commutator

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The theory of the commutator from the perspective of Abstract Algebraic Logic (AAL) is investigated. AAL offers a very general notion of the commutator, defined for each n-dimensional logical system. An account of the commutator associated with two-dimensional logical systems is presented. In this context an emphasis is put on logical aspects of the commutator for the equational systems associated with quasivarieties of algebras.

The talk is concerned with the notion of a commutator equation for a class of algebras. Commutator equations give rise to a new concept of the commutator, which turns out to be equivalent to the standard one introduced by Kearnes and McKenzie for relatively congruence-modular (RCM) quasivarieties. The importance of quaternary commutator equations (with parameters) in the context of additivity and other properties of the commutator is underlined. This is a consequence of the fact that for any quasivariety **Q**, the commutator (in the sense of commutator equations of **Q**) is additive iff it is determined by a set of quaternary commutator equations for **Q** (with parameters). In particular, the commutator of any RCM quasivariety is determined by a set of *quaternary* commutator equations. Observations concerning the commutator for finitely generated quasivarieties are also provided.

The focus of the talk is also on the idea of applying a general notion of an implication viewed as a set of quaternary equations that jointly possess the property of detachment relative to the equational system associated with a class of algebras. In particular, the significance of Day implication systems in commutator theory is shown.

Superspecial-Valued Lattice-Ordered Groups

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Let α be an ordinal. Within a lattice-ordered group, a value V of an element g is α-*special* if V is an α-point in the Cantor-Bendixon derivative sequence of the Yosida space $\mathcal{Y}(g)$ of g. Contrary to an assertion of the author and Martinez, this definition is shown to be independent of the choice of element, and it is also shown that an element g has an α-special value V if and only if g has an α-special component at V. Generalizing earlier results due to Conrad and this author, it is shown that α-special elements are recognizable within the underlying lattice. Conrad's results on torsion classes of finite-valued lattice-ordered groups and quasitorsion classes of special-valued lattice-ordered groups are then generalized to these new definitions.

Semigroup Properties of Operations and Cooperations defined on Finite Sets

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Let $O^n(A)$ be the set of all n-ary operations and let $cO^n(A)$ be the set of all n-ary cooperations defined on the nonempty finite set A with $|A| \geq 2$. If $f, g_1, \ldots, g_n : A^n \to A$ are n-ary operations defined on A, then by $S^n(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) := f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n))$ for all $a_1, \ldots, a_n \in A$ an $n+1$-ary operation on the set $O^n(A)$ of all n-ary operations can be defined. From this operation one can derive a binary operation $+$ by $f + g := S^n(f, g, \ldots, g)$ and obtains a semigroup $(O^n(A); +)$. Analogously, if f is an n-ary cooperation and if $g_1, \ldots, g_n$ are m-ary cooperations on A, then one can define a composition $comp_m^n(f, g_1, \ldots, g_n)$ and obtains a new m-ary cooperation defined on A. By $f + g := comp_n^n(f, g, \ldots, g)$ one defines a binary operation on the set of all n-ary cooperations. Then the set of all n-ary cooperations forms a semigroup $(cO^n(A); +)$ with respect to this binary operation. In this paper we study semigroup properties of these semigroups.

Localization in abelian ℓ-groups with strong unit via MV-algebras

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The class of MV-algebras arises as algebraic counterpart of the infinite valued Łukasiewciz sentential calculus, as Boolean algebras did with respect to the classical propositional logic. Due to the non-idempotency of the MV-algebraic conjunction, unlike Boolean algebras, MV-algebras can be non-archimedean and can contain elements x such that x^n is always greater than zero, for any $n > 0$. Perfect MV-algebras are those algebras generated by such *infinitesimal* elements.

As it is well known, MV-algebras form a category which is equivalent to the category of abelian lattice ordered groups with strong unit. This makes the interest in MV-algebras relevant outside the realm of logic. Let us denote by Γ the functor implementing this equivalence. So, via Γ^{-1} each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit (ℓ_u-group). Also it has been proved that the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups. Let us denote by Δ the functor implementing this equivalence.

We are here interested in characterizing the class of ℓ_u-groups obtained by the restriction of Γ^{-1} to the class of perfect MV-algebras. This will display the interplay between the two functors Δ and Γ. The use of such functors in developing localization issues helps in revealing non trivial properties of localization in both categories of MV-algebras and ℓ_u-groups.

Cantor–Bernstein property

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The classical Cantor–Bernstein theorem says that two sets X, Y which admit injective mappings $X \to Y$ and $Y \to X$ have the same cardinality. We obtain a different problem when the sets are equipped with an additional structure which should be preserved by the mappings (isomorphisms). It has been generalized to boolean algebras by Sikorski and Tarski: For any two σ-complete boolean algebras A and B and elements $a \in A$ and $b \in B$, if

$B \cong [0, a]_A$ and $A \cong [0, b]_B$, then $A \cong B$. This result has been generalized to MV-algebras, to orthomodular lattices, and to more general structures, e.g. effect algebras and pseudo-BCK-algebras.

We pose another question: Which algebras satisfy the Cantor–Bernstein theorem in its original form by Sikorski and Tarski, without any additional condition? We summarize some results for orthomodular lattices and compare them to the situation in MV-algebras.

Automorphisms and endomorphisms of ordered sets

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We obtain new results on this conjecture of Rival and Rutkowski [1991]. For an ordered set X, $e(X)$ $[a(X)]$ is the number of endomorphisms [respectively, automorphisms] of X.

Conjecture 1. For any sequence (P_n) of pairwise nonisomorphic, finite ordered sets

$$\lim_{n\to\infty} e(P_n)/a(P_n) \ = \ \infty.$$

In fact, we are inclined to believe that the following problem, first suggested to us by Trotter [1995], has a positive solution, which would verify a much stronger statement than the conjecture.

Problem 2. Show that there exists $c > 1$ such that for all ordered sets X of size $n \geq 2$

$$e(X)/a(X) \ \geq \ c^n.$$

We have this exponential bound for a "majority" of bipartite ordered sets. But for lattices and "most" other bipartite orders, the best we can do at this point is subexponential lower bound of the form $c^{n/\log n}$ for a fixed constant $c > 1$.

On perfect GMV-algebras and covers of MV-algebras

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GMV-algebras are a generalization of MV-algebras, dropping the commutativity condition in MV-algebras, but having two negations, left- and right one, in general. The first author proved a fundamental representation showing that every GMV-algebra is exactly an interval $\Gamma(G, u) = [0, u]$ in an ℓ-group G (not necessarily Abelian) with a strong unit u.

This result enables us to study perfect and n-perfect (or perfect if $n = 1$) GMV-algebras, i.e., ones can be split into $n + 1$ comparable slices. We are able to represent symmetric (having only one negation) perfect GMV-algebras via $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.

We show that each symmetric cover is either derived from a variety of perfect GMV-algebras generated by $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, where G varies in a cover variety of ℓ-groups of the variety of Abelian ℓ-groups, or from a p^k-perfect GMV-algebra derived from the Scrimger ℓ-group S_p, p prime, $k \geq 1$.

Properties defined by Maltsev conditions

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Let $\mathcal{V} = V(\mathbf{A})$ be the variety generated by an algebra $\mathbf{A}$. We show, for example, that if $\mathbf{A}$ is idempotent and $\mathcal{V}$ is not congruence modular then there is a four-generated subalgebra of $\mathbf{A}^2$ with a nonmodular congruence lattice. Similar results hold for other properties such as distributivity, semidistributivity, permutability, and having a majority term. These results lead to polynomial time algorithms for testing these properties. On the other hand we show that deciding if $\mathcal{V}$ is distributive (and various other properties) without the assumption that $\mathbf{A}$ is idempotent is EXPTIME complete.

We also show that if $\mathcal{V}$ is congruence distributive then a minimal sequence of Jónsson terms will have at most $2m - 1$ terms, where m is the cardinality of the 2-generated free algebra and give examples showing this is the best possible. Similar results are proved for other Maltsev properties.

A discriminator variety of linear Heyting algebras with operators arising in quantum computation

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Unlike classical computation, quantum computation [4] allows one to encode two atomic information bits in parallel. Here, in fact, the appropriate counterpart of a classical bit is the *qubit*, defined as a unit vector in $\mathbb{C}^2$, i.e. $|\psi\rangle = a|0\rangle + b|1\rangle$, where a, b are complex numbers s.t. $|a|^2 + |b|^2 = 1$, and $\{|0\rangle, |1\rangle\}$ is the canonical base. It can be shown [1] that - from a logical viewpoint - it is unnecessary to consider information quantities in Hilbert spaces other than $\mathbb{C}^2$: in fact, the algebra whose universe is the set of all qumixes of $\mathbb{C}^2$ and whose operations correspond, in an appropriate sense, to the main quantum logical gates generates the same (weak or strong) consequence relation as the algebra over the set of all density operators of arbitrary n-fold tensor products of $\mathbb{C}^2$. In order to investigate the algebraic counterpart of the strong consequence relation in a logical system associated to quantum logical gates we introduced quasi MV-algebras and $\sqrt{'}$ quasi MV-algebras [3], [2]. Successively we investigated an expansion of lattice ordered $\sqrt{'}$ QMV algebras whereby: i) lattice meet and join are promoted to the rank of fundamental operations; ii) lattice meet is residuated by a Gödel-like implication. We called these structures *linear Heyting quantum computational algebras* (shortly, *LHQC-algebras*). In this talk we outline a *representation* and a *standard completeness* theorem for this structure. Finally we show that LHQC-algebras form a discriminator variety.

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Boolean Skeleton and Pierce representation of Bounded BCK-algebras

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We define the Boolean skeleton of a bounded BCK-algebra, and we use it to obtain a representation of bounded BCK-algebras, called (weak) Pierce bBCK-representation, as (weak) Boolean products of bounded BCK-algebras. Since the class $b\mathbb{BCK}$ of bounded BCK-algebras is relative congruence distributive we show that it satisfies the relative Boolean Factor Congruence property. Moreover given a bounded BCK-algebra $\mathbf{A}$, we prove that its BCK-factor congruences can be identified with elements of the algebra, called *factor elements*, and the set of all factor elements forms a Boolean subalgebra of $\mathbf{A}$, $\mathbf{B_F(A)}$, called the *Boolean skeleton*. This fact allow us to prove that *any bounded BCK-algebra $\mathbf{A}$ admits a weak Boolean representation of bounded BCK-algebras for each subalgebra of $\mathbf{B_F(A)}$ over its associated Boolean space. Moreover every representation as a weak Boolean product of bounded BCK-algebras is equivalent to one of the above representations.* Next we analyze these representations and we show that its stalks need not to be directly indecomposable. Our main result characterizes relative subvarieties well weak Pierce representable, i.e. in its representation the stalks are directly indecomposable: *Given $\mathbb{Q}$ a relative subvariety of $b\mathbb{BCK}$. Then $\mathbb{Q}$ is well weak Pierce representable if and only if for any $\mathbf{A} \in \mathbb{Q}$, $B_F(\mathbf{A}) = \{a \in A : a \to \neg a = \neg a$ and $\neg a \to a = a\}$.* We show some examples of relative subvarieties well weak Pierce representable and we analyze some special cases in which the stalks are simple.

Is the clone lattice on uncountable cardinals dually atomic?

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A clone C on a set X is a set of finitary operations $f : X^n \to X$ which contains all the projections and is closed under composition. The family of all clones forms a complete algebraic lattice $Cl(X)$ with greatest element $\mathcal{O}$.

The coatoms (dual atoms) of this lattice $Cl(X)$ are called "precomplete clones" or "maximal clones" on X.

For finite X every clone $C \neq \mathcal{O}$ is contained in a precomplete clone, that is: the clone lattice $Cl(X)$ is *dually atomic*. The question whether this holds also for infinite X had been open for many years.

Theorem 1. *If κ is a regular cardinal satisfying $2^\kappa = \kappa^+$, then the clone lattice on a set of size κ is not dually atomic.*

(In 2002 we had proved this theorem for $\kappa = \aleph_0$.)

The method behind our proof is "forcing with large creatures". However, no advanced set theory (beyond a basic understanding of ordinals, cardinals and transfinite induction) will be required from the audience.

In my talk I will highlight a few ideas from the proof.

Behavioral algebraization

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The theory of Abstract Algebraic Logic (AAL) aims at drawing a strong bridge between logic and algebra, by generalizing the well known Lindenbaum-Tarski method. Despite of its success, the scope of applicability of the current theory of AAL is still quite limited. To wit, logics with a many-sorted language or with non-truth-functional connectives simply fall out of its scope.

The subject of our work is precisely the quest for a more general AAL framework. We present a generalization of AAL obtained by substituting the role of unsorted equational logic with many-sorted behavioral logic. While the step towards many-sortedness seems to be the obvious way to bring the possible many-sorted character of a given logic to its algebraic counterpart, the incorporation of behavioral reasoning in the algebraization process allows one to cope also with connectives that fail to be congruent.

We illustrate these ideas by exploring the examples of paraconsistent logic C1 of da Costa and the exogenous Global and Probabilistic propositional logics.

Complexity of term equation problem

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We study the computational complexity of the problem of satisfiability of equation between terms over a finite algebra (TermSat). We describe many classes of algebras where the complexity of TermSat is determined by the clone of term operations. We classify the complexity for algebras generating the maximal clones. Using this classification we describe a lot of algebras where TermSat is NP-complete. We classify the situation for clones generated by an order or a permutation relation. We introduce the concept of semiaffine algebras and show polynomial time algorithms solving the satisfiability problem for them.

Monadic BL-algebras

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BL-algebra was introduced by P. Hajek as an algebraic counterpart of Basic logic $\mathcal{BL}$. The predicate Basic logic $\mathcal{QBL}$ is defined in standard way: having some universe and complete BL-algebra, which generate the variety of all BL-algebras, and defining existential (universal) quantifier as supremum (infimum), then the predicate calculus is defined as all formulas having value 1 for any assignment. Monadic BL-algebras is an algebraic model for the monadic predicate calculus $q\mathcal{BL}$, in which only a single fixed individual variable occurs. We follow P. R. Halmos' study of monadic Boolean algebras.

We define and study monadic BL-algebras as a pair (A, A_0) of BL-algebras one of which - A_0 is a special case of relatively complete subalgebra - m-relatively complete subalgebra.

A subalgebra A_0 of BL-algebra $(A, \vee, \wedge, \rightarrow, *, \exists, \forall, 1, 0)$ is said to be relatively complete if and only if for every $a \in A$ the sets $\{b \in A_0 : a \leq b\}, \{b \in A_0 : a \geq b\}$ have the least and the greatest elements respectively.

A subalgebra A_0 of BL-algebra A is said to be m-relatively complete if A_0 is relatively complete and in addition holds

(#) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a * a \Rightarrow v \geq a \& v * v \leq x)$

Primal and functionally complete algebras of logic

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Suppose A is a finite universe with $|A| > 1$, 0 and 1 are two distinct fixed elements of A and $f : A^2 \to A$ satisfies for all $x \in A$

$$f(0, x) = f(x, 1) = f(x, x) = 1, \quad f(1, x) = x.$$

We often write $x \to y$ instead of $f(x, y)$.

Denote by c_0 and c_1 the constant unary operations on A with values 0 and 1. The question we are addressing here is:

Let X be a set of operations on A such that $f, c_0 \in X$. When is the (nonindexed) algebra $\langle A, X \rangle$ primal or equivalently X complete? Applying Rosenberg's classification of maximal clones, we answer this question in terms of three types of maximal clones.

As an application we obtain simple criteria for pocrims with a weak vt-operator to be functionally complete.

Completions and a variety of Heyting algebras

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There has been considerable recent progress on the topic of completions of ordered algebraic structures. The canonical completion, introduced by Jónsson and Tarski for Boolean algebras with operators, has been extended to the setting of distributive lattice with operators, and even to the setting of general lattices with operators. The well-known technique of MacNeille completions, often studied in detail in particular settings, has begun to receive a more systematic treatment. Recent results have for instance shown that any variety of lattices with operators that is closed under MacNeille completions is also closed under canonical extensions; and a systematic study of identities preserved by MacNeille completions has been initiated. There are also results showing that various types of completions, such as the canonical completion, the MacNeille completion, and the ideal lattice completion, all are instances of a more general type of construction. In sum, results from the past 10 years are leading to a more coherent view of the topic of completions of ordered algebraic structures.

The main result to be presented here represents a small contribution to this program. It is a specialized result about completions for a certain variety of Heyting algebras. This result has immediate application to give the completeness, with respect to the algebraic semantics, of a natural intuitionistic logic. But perhaps the main reason for interest in this result, certainly the one that motivated my study of it, is its relationship to the larger setting of completions of ordered structures. This result provides an example of an elusive phenomenon, and naturally points to a number of interesting questions.

Canonical extensions of Stone algebras: the natural way

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A construction of canonical extensions of Stone algebras will be presented that uses the natural duality based on the three-element generating algebra **3** rather than the Priestley duality based on **2** that is traditionally used to build the canonical extension. The new approach has the advantage that the canonical extension so constructed inherits its algebra structure pointwise from a power of the generator, so that the extension of the fundamental operations and closure of the variety under the formation of canonical extensions occur in a transparent way. The method has the potential to be applied similarly to other classes of bounded distributive lattice expansions.

We present the origins of canonical extensions going back to B. Jónsson and A. Tarski (1951-52), and fifty years later, for bounded distributive lattice expansions and lattice expansions, to M. Gehrke in collaboration with B. Jónsson, J. Harding, Y. Venema and others. We show how the Priestley and Banaschewski dualities can be used in tandem in an alternative construction of the canonical extensions of bounded distributive lattices. We compare the traditional approach to the canonical extensions of the bounded distributive lattice expansions with a proposed natural way based on the natural dualities and illustrate the latter by presenting a construction of the canonical extensions of Stone algebras via the new approach.

Ordered monoids which are epimorphic images of submonoids of ordered groups

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Our talk has two aims. The first one is to bring the notion of a formally integral totally ordered commutative monoid into the structure theory of commutative residuated chains. The reason for this is that there is a reasonable connection between this class of monoids and totally ordered Abelian groups. The second goal of this talk is to present solutions to three problems which arose in the theory of formally integral totally ordered commutative monoids.

Geometric description of residuated semigroups

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A geometric characterization of residuated semigroups is presented in this talk based on the notions of the rotation-invariance property, and the pseudo-inverse of monotone functions. This provides a deeper understanding of associativity via geometric insight. It is then demonstrated how the intuition, which is gained from the geometric description may help in conjecturing and proving results in the fields of algebra, logic and functional equations. It is interesting to notice that when conjecturing algebraic results based on geometric motivations, one can first heuristically prove them using geometry; then the steps of the geometric hint can be translated step by step into an algebraic proof.

Representability of small relation algebras and involutive FL-algebras

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This talk reports on two related topics. The first part is joint work with P. Hertzel, R. Kramer and R. Maddux, and comprises a survey of representability results about relation algebras of size 16.

The variety RA of relation algebras was originally defined by Tarski as the class of algebras $(A, +, 0, \cdot, 1, ^-, ;, e, ^\smile)$ such that $(A, +, 0, \cdot, 1, ^-)$ is a Boolean algebra, $(A, ;, e, ^\smile)$ is an involutive monoid (i.e. ; is associative with unit e, $x^{\smile\smile} = x$ and $(x;y)^\smile = y^\smile;x^\smile$), ; and $\smile$ distribute over $+$, and $x^\smile;(x;y)^- \leq y^-$. A relation algebra is *integral* if e is an atom, and *symmetric* if it satisfies the identity $x^\smile = x$.

The variety RRA of representable relation algebras is generated by the *square relation algebras* $\mathrm{Re}(U) = (\mathcal{P}(U^2), \cup, \emptyset, \cap, U^2, ^-, \circ, id_U, ^{-1})$ where U is any set. D. Monk [1964] proved that RRA is a nonfinitely axiomatizable subvariety of RA, and R. Hirsch and I. Hodkinson [2001] proved that it is undecidable whether a finite relation algebra is a member of RRA. Never-the-less this decision has been made for many specific finite relation algebras. In particular we show that exactly 71 of the 102 integral relation algebras of size 16 are representable, and we outline what is known for the integral relation algebras of size 32.

The second part considers symmetric relation algebras as a subvariety of involutive FL-algebras [i.e. residuated lattices with a constant d satisfying the identity $d/(x\backslash d) = x = (d/x)\backslash d$]. We discuss how atom structures for symmetric relation algebras are related to the recently introduced involutive frames. This connection allows several relation algebraic constructions to be applied to residuated lattices in general.

Semiprime ideals and separation theorem in posets

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Krull in 1929 introduced the concept of semiprime ideal. Accordingly, if R is a commutative ring with unity, then an ideal I of R is semiprime whenever $a^n \in I$, n is positive integer implies that $a \in I$. Krull also proved, using well ordering theorem, that every semiprime ideal is the intersection of all the prime ideals which contain it.

M.H. Stone in 1936 proved the famous prime separation theorem for distributive lattices. Y. Rav in 1989 studied the lattice of semiprime ideals and proved the analogue of the prime separation theorem for semiprime ideals.

In this paper, the concept of a *semiprime* ideal in a poset is introduced as an ideal I of a poset P if $(a, b)^l \subseteq I$ and $(a, c)^l \subseteq I$ together imply $\{a, (b, c)^u\}^l \subseteq I$. We have studied the relations between semiprime ideals of a poset P and ideals of the ideal lattice of $Id(P)$ of all ideals of P and also the relation between prime ideals of P and prime ideals of $Id(P)$. We extend separation theorem for semiprime ideals to a finite poset with 1 and

also extend the separation for prime ideals to the case of finite posets. Then we prove the prime separation theorem for finite posets P for which $Id(P)$ is distributive. Some counterexamples are also given.

Failure of amalgamation in Hilbert lattices

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We show that amalgamation property fails for Hilbert lattices (i.e., lattices of closed subspaces of Hilbert spaces). The argument proceeds by reconstructing a counterexample to amalgamation in orthomodular lattices, due to Bruns and Harding [1], within a certain V-formation of Hilbert lattices. The construction of that V-formation employs some linear operator theory, as well as an example of a non-modular Hilbert lattice devised by von Neumann in a letter to Birkhoff.

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A finite algebra with an EXPTIME complete clone membership problem

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In 1977 Kozen proved (comp. [Koz77]) that for a finite family of unary functions (over a finite set) with one distinguished member, it is a PSPACE complete problem to decide whether the distinguished function can be obtained as a composition of the others. An unpublished result of Friedman shows that allowing non-unary functions causes the problem to become EXPTIME complete.

This problem splits into subproblems – for a fixed, finite algebra **A** we define

$$\begin{array}{ll} \text{INPUT} & \text{a function } f : A^k \to A \\ \text{PROBLEM} & \text{decide if } f \in \text{Clo}\mathbf{A}. \end{array}$$

We construct a finite algebra with binary operations only such that the membership problem for the clone it generates is EXPTIME complete. This shows that the clone membership problem for a fixed finite algebra can be as difficult as the clone membership problem with an algebra as a part of the input.

This question follows the line of research devoted to describing the computational complexity of natural algebraic problems generated by fixed, finite structures. Such problems are studied extensively and a number of results is already known (for example [Szé02], [JM06]).

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The lattice of quasivarieties of differential groupoids

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A *differential groupoid* is an algebra with one fundamental binary operation satisfying the identities $x \cdot x = x$, $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$, and $x \cdot (x \cdot y) = x$.

Let **Dm** denote the variety of differential groupoids. As is known, each proper subvariety $\mathbf{D}_{i,n}$ of **Dm** is defined by one additional identity $xy^{i+n} = xy^i$, where $i \geqslant 0$ and $n > 0$, which leads to an explicit description of the lattice $\mathrm{L_v}(\mathbf{Dm})$ of subvarieties of **Dm**.

We prove that **Dm** is a $\mathcal{Q}$-universal variety, i.e., there is no convenient description for the lattice $\mathrm{L_q}(\mathbf{Dm})$ of subquasivarieties of **Dm**. The method of the proof does not allow us to prove that some subvariety of the form $\mathbf{D}_{i,n}$ is $\mathcal{Q}$-universal. This leads to the following natural question: Which proper subvarieties of differential groupoids are $\mathcal{Q}$-universal?

For varieties of the form $\mathbf{D}_{i,1}$, we completely solve this question. Namely, the lattice $\mathrm{L_q}(\mathbf{D}_{i,1})$ is either finite (if $i \leqslant 1$) or $\mathcal{Q}$-universal.

We also show that, for every $n > 1$, the cardinality of $\mathrm{L_q}(\mathbf{D}_{0,n})$ is 2^ω.

Representable integral residuated lattices and their subreducts

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A residuated lattice is said to be representable if it is a subdirect product of linearly ordered residuated lattices. C. J. van Alten proved that an integral residuated lattice is representable if and only if it satisfies the identity

$$(x\backslash y) \vee (w/(w/[((y\backslash x)\backslash z)\backslash z])) = 1.$$

This identity also characterizes the representable algebras in the classes of all C-subreducts of integral residuated lattices as long as $\{\vee, \backslash, /, 1\} \subseteq C$, but it has been an open question to find an axiomatization when $\vee \notin C$. We prove van Alten's conjecture that the identity

$$((x\backslash y)\backslash u)\backslash [([w/(w/[((y\backslash x)\backslash z)\backslash z])]\backslash u)\backslash u] = 1$$

can be used to characterize representable members of the class of all $\{\backslash, /, 1\}$-subreducts of integral residuated lattices (which are known as pseudo-BCK-algebras or biresiduation algebras), as well as representable integral residuated lattices themselves.

Algebraization of Logics Defined by Literal–Paraconsistent or Literal–Paracomplete Matrices

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Literal paraconsistent–paracomplete matrices, or LPP–matrices, were introduced by Lewin and Mikenberg in an attempt to give a meaningful semantics, via the classic method of matrices, to a large family of paraconsistent and paracomplete logics. The main characteristic of these logics is that paraconsistency and/or paracompleteness occurs only in the most elementary levels, that of propositional letters and their (iterated) negations, called literals, but more complex formulas behave classically with respect to negations. Needless to say many of these logics have appeared in the literature under different names and formalizations.

We study the algebraizability of the logics constructed using literal–paraconsistent and literal–paracomplete matrices described by Lewin and Mikenberg, proving that they are all algebraizable but not finitely algebraizable

(in the sense of Blok and Pigozzi). A characterization of the finitely algebraizable logics defined by LPP–matrices is given.

Relevance logic and the calculus of relations

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The relevant model structures introduced by Routley and Meyer as semantics of relevance logic are closely related to the atom structures of relation algebras. By combining this connection with the notion of representability for relation algebras we arrive at new semantics for relevance logic in which formulas are interpreted as binary relations, similar to the manner in which formulas may be interpreted as sets to obtain sound and complete semantics for the classical propositional calculus. We make some elementary observations and raise some questions about the interpretation of formulas as binary relations and about the connections betweeen relevant model structures and atom structures of relation algebras.

On Decidability and Complexity of Semilinear Varieties

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The real $\mathbb{Q}$-linear space $\mathcal{R}_{\mathbb{Q}lin}$ is the vector space $\mathbb{R}$ over $\mathbb{Q}$ equipped with the natural ordering. The theory of $\mathcal{R}_{\mathbb{Q}lin}$ admits quantifier elimination in the language of ordered groups equipped with a set of unary function symbols $\lambda \in \mathbb{Q}$, i.e.: $\{+, -, 0, <\} \cup \{\lambda : \lambda \in \mathbb{Q}\}$. This means that the theory of $\mathcal{R}_{\mathbb{Q}lin}$ is decidable and the sets definable in it exactly are the semilinear sets in $\mathbb{R}^n$, i.e. subsets of $\mathbb{R}^n$ given by Boolean combinations of linear polynomial equations and inequalities with rational coefficients.

It is easy to see that the existential theory of $\mathcal{R}_{\mathbb{Q}lin}$ is NP-complete. We will exploit this result in order to characterize decidability and complexity of certain varieties of algebras.

A function is *semilinear* if its graph is a semilinear set. We call *semilinear* an algebra of the form $\langle S, f_1, \ldots, f_k \rangle$ such that $S \subseteq \mathbb{R}$ is a semilinear set, and all its basic operations f_i are semilinear functions over S. We call a variety *weakly semilinear* if it is generated by a class of semilinear algebras (over the same S). We call a variety *strongly semilinear* if it is generated (up to isomorphism) by one semilinear algebra.

Theorem 1. (*i*) *Let* $\mathbb{V}$ *be a strongly semilinear variety. Then the equational theory of* $\mathbb{V}$ *can be faithfully translated in polynomial time into the existential theory of* $\mathcal{R}_{\mathbb{Q}_{lin}}$. *Then, the satisfiability problem for the equational theory of* $\mathbb{V}$ *is in* NP. (*ii*) *Let* $\mathbb{V}$ *be a finitely axiomatizable weakly semilinear variety. Then, the equational theory of* $\mathbb{V}$ *is decidable.*

To see this method at work, we will study of decidability and complexity of varieties based on uninorms and triangular norms.

Epicompletion in Frames with Skeletal Maps

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This research concerns the category $\mathfrak{Frm}\mathfrak{S}$ of frames with frame homomorphisms which map dense elements to dense elements, called *skeletal maps*. In an earlier article the authors show that the formation of the so-called absolute $\varepsilon : A \to \varepsilon A$ is a monoreflection on the subcategory $\mathfrak{KR\mathfrak{eg}}\mathfrak{S}$ of all compact regular frames with skeletal maps in the subcategory of $\mathfrak{KR\mathfrak{eg}}\mathfrak{S}$, $\mathfrak{SPR\mathfrak{eg}}\mathfrak{S}$, consisting of all the strongly projectable frames – i.e., those in which every polar is complemented. In fact, they have shown that ε is the maximum monoreflection on $\mathfrak{KR\mathfrak{eg}}\mathfrak{S}$.

In the current research, ε is extended to an epireflection ψ on $\mathfrak{KN\mathfrak{Ar}}\mathfrak{S}$, the category of all compact, normal joinfit frames (with skeletal maps), in the subcategory $\mathfrak{SP\mathfrak{Ar}}\mathfrak{S}$ consisting of all the strongly projectable frames. The techniques used to define ψ involve coproducts and pushouts in frames with skeletal maps.

It is unknown whether ψ is a monoreflection. We consider restrictions of ψ to categories of algebraic frames (with skeletal coherent maps), and show that on the category $\mathfrak{KN\mathfrak{Y}}\mathfrak{S}$ of all Yosida frames ψ is monoreflective.

Polynomial clones on squarefree groups

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A clone on a set A is a collection of finitary functions on A that contains all projections and is closed under all compositions. The clone of polynomial

functions on an algebra $\mathbf{A} := \langle A, F \rangle$ is the smallest clone on A that contains all fundamental operations F of $\mathbf{A}$ and all constant functions on A.

We prove that the polynomial clone on a squarefree group has only finitely many extensions. This confirms a conjecture by Paweł Idziak. We also show that such an extension is uniquely determined by its binary functions but not by the congruence lattice and the commutator operation of the corresponding algebra.

An Investigation of the Lattice *Ext*S4

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This research is a continuation of that which was started in [A. Muravitsky, *Notre Dame Journal of Formal logic*, vol. 47, 2006, no. 4, 525–540]. There we were engaged mostly in investigation of the lattice structure of the interval $[\mathbf{Grz} \cap \mathbf{S5}, \mathbf{S4} + p \rightarrow \Box p]$. In this talk we focus on the interval $[\mathbf{S4}, \mathbf{Grz}]$ and show interconnect between the two intervals, as well as their relation to the lattice of the logics $\{\tau(L)|L \in Ext\mathbf{Int}\}$, where $\tau : Ext\mathbf{Int} \longrightarrow Ext\mathbf{S4}$ is a well-known mapping defined through *Tarski translation*. Also, we define two classes of intervals so that the first class determines a congruence on $Ext\mathbf{S4}$ such that the quotient lattice is isomorphic to $Ext\mathbf{Int}$. On the other hand, the second class being arranged by set inclusion forms a dual of $Ext\mathbf{Int}$.

Lattices of Quasivarieties as Congruence Lattices of Semilattices with Operators

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Our objective is to provide, for the lattice of quasivarieties contained in a qiven quasivariety, a description similar to the one that characterizes the lattice of subvarieties of a given variety as the lattice of fully invariant congruences on a countably generated free algebra.

Given a variety V, let $\mathbf{F}$ be the V-free algebra on ω generators, and let $\mathbf{T}$ denote the join semilattice of compact congruences of $\mathbf{F}$. The endomorphisms of $\mathbf{F}$ act naturally on $\mathbf{T}$. Let $\widehat{E} = \{\widehat{\varepsilon} : \varepsilon \in \text{End } \mathbf{F}\}$, and consider the

algebra $\mathbf{S}_V = \langle \mathbf{T}, \vee, 0, \widehat{E} \rangle$. Note that the operations of $\widehat{E}$ are *operators*, i.e., $(\vee, 0)$-homomorphisms. With this setup, we have our main result.

Theorem 1. *For a variety V, the lattice of quasivarieties contained in V is dually isomorphic to* Con $\mathbf{S}_V$.

We can use our construction to devise a new class of Q-lattices, that are congruence lattices semilattices with operators in a tractable way.

Theorem 2. *Let $\mathbf{S}$ be a join semilattice with 0 and 1, and let G be a group of operators acting on $\mathbf{S}$. Then there is a quasivariety B such that the lattice of quasivarieties contained in B is dually isomorphic to* $\mathrm{Con}(\mathbf{S}, +, 0, G)$.

Algebraic characterization of completeness properties for core fuzzy logics I: propositional logics

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In his famous book *Metamathematics of Fuzzy Logic*, Hájek considers the problem of finding a basic fuzzy logic which is a common fragment of the most important fuzzy logics, namely Łukasiewicz, Gödel and product logics. There, he introduced a logic, named BL, and he proposed it for the role of basic fuzzy logic. It was shown that BL is the logic of all continuous t-norms and their residua. However, Esteva and Godo observed that the minimal condition for a t-norm to have a residuum, and therefore to determine a logic, is left-continuity (continuity is not necessary). Therefore, they proposed a weaker logic, called MTL (monoidal t-norm based logic), which was proved to be (by Jenei and Montagna) the logic of left-continuous t-norms and their residua. This talk is a contribution to the algebraic study of t-norm based fuzzy logics. In the general framework of propositional core and Δ-core fuzzy logics (algebraizable expansions of MTL) we consider three properties of completeness with respect to any semantics of linearly ordered algebras. Useful algebraic characterizations of these completeness properties are obtained and their relations are studied. Moreover, we propose four kinds of distinguished semantics for these logics –namely the class of algebras defined over the real unit interval, the rational unit interval, the hyperreals and finite chains, respectively– and we survey the known completeness methods and results for prominent logics. We also obtain new interesting relations between the real, rational and hyperreal semantics, and

good characterizations for the completeness with respect to the semantics of finite chains.

Positive Sugihara monoids

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An algebra $\mathbf{A} = \langle A; \cdot, \rightarrow, \wedge, \vee, e \rangle$ which has lattice and commutative monoid reducts, and which satisfies $x \cdot y \leq z \iff y \leq x \rightarrow z$, is called a *commutative residuated lattice* (briefly, *CRL*). An *involutive* CRL has an additional unary operation $\neg$ which satisfies $\neg\neg x \approx x$ and $x \rightarrow \neg y \approx y \rightarrow \neg x$. An involutive CRL is called a *Sugihara monoid* if its lattice reduct is distributive and its monoid operation is idempotent. Sugihara monoids algebraize the logical system $\mathbf{RM}^t$, which is derived from '$\mathbf{R}$–mingle' by adding the sentential constant t. The $\{\neg\}$–free subreducts of Sugihara monoids are called *positive Sugihara monoids*, and they correspond to the full $\{\neg\}$–free fragment of $\mathbf{RM}^t$.

We examine the structure of positive Sugihara monoids with the purpose of showing that they form a *primitive* variety, i.e., a variety in which every subquasivariety is already a variety. This stands in contrast to the case of Sugihara monoids, the class of which is generated as a variety by a proper subquasivariety. We actually prove a stronger result, that every finite subdirectly irreducible positive Sugihara monoid is a retract of a free algebra.

The result also has a logical interpretation. A logical consequence relation is *structurally complete* if each of its proper extensions contains new theorems, in addition to new derivable rules. Our result can be used to show that the negation-free fragment of $\mathbf{RM}^t$ is structurally complete, as are all of its extensions.

Topological relational quantales

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We introduce and present results about a class of quantales, the *topological relational quantales*, that can be associated with tuples (X, R) such that X is a topological space and R is a lower-semicontinuous equivalence relation on X. Our motivating case studies and examples include concrete quantales such as those of type $\mathcal{P}(R)$, R being an equivalence relation, and the quantale $\overline{Pen}$ defined by Mulvey and Resende to classify Penrose tilings. In this setting, we focus in particular on the question of the representability of quantales into quantales of binary relations on a set, which has already been studied by some authors in the literature.

We present sufficient conditions for the representability of unital involutive quantales into quantales of relations, so that joins are represented as unions, non-commutative products as compositions of relations, and involutions as taking inverses. One of the most significant conditions for our representation theorem is that the quantale Q is join-generated by its *functional elements*, i.e. those $a \in Q$ s.t. $a^* \cdot a \le e$. This condition is analogous to that given by Jónsson and Tarski (1952) for representability of relation algebras, and indeed our methodology is similar in that is based on notions derived from the theory of canonical extensions. The connection with the theory of canonical extensions is an element of novelty, for, as far as we know, the theorems of representation for quantales that appear in the literature rely on techniques inspired only to functional analysis and the theory of C^*-algebras.

Algebras from quantum computation and Abelian lattice-ordered groups

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Quasi-MV algebras were introduced in [3] in an attempt to describe an appropriate abstract counterpart of the algebra of all density operators of $\mathbb{C}^2$, endowed with operations corresponding to a few significant quantum

logical gates. This variety of algebras, apart from its original motivation related to quantum computation, has an intrinsic interest as a generalisation of MV algebras to the semisubtractive but not point regular case. Quasi-MV algebras were later expanded in [2] by an operation of square root of the inverse ($\sqrt{'}$) whose quantum computational significance is especially noteworthy.

In this talk we discuss a connection between $\sqrt{'}$ quasi-MV algebras and a class of ℓ-groups with additional operators. Daniele Mundici established a well-known equivalence between the categories of MV algebras and Abelian ℓ-groups with strong unit via an invertible functor (the *Gamma functor:* [1]). A partial analogue of Mundici's Gamma functor turns out to be available in the present framework too. We show, in fact, that:

- "Well-behaved" $\sqrt{'}$ quasi-MV algebras (*cartesian* algebras, in the terminology of [2]) are subalgebras of intervals in appropriate Abelian ℓ-groups with projection and rotation operators;
- "Very well-behaved" $\sqrt{'}$ quasi-MV algebras (*pair* algebras, in the terminology of [2]) are isomorphic copies of intervals in such Abelian ℓ-groups.

A categorical equivalence between pair algebras and MV algebras will be also briefly discussed.

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The reducts of $(\mathbb{N}, =)$

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In Model Theory, reducts of a relational structure Γ are usually considered up to *first-order interdefinability*. We initiate the systematic study of reducts up to *primitive positive interdefinability*. Classifications of the latter type are harder to obtain, since there are more reducts to distinguish. The simplest structure where such a classification can be studied is the structure $(\mathbb{N}, =)$.

To obtain our classification we use the concept of a *clone*, exploiting the fact that for ω-categorical structures Γ, there is a one-to-one correspondence between the reducts of Γ and the locally closed clones that contain the automorphisms of Γ. Therefore, to classify the reducts of $(\mathbb{N}, =)$, we classify the local clones on $\mathbb{N}$ that additionally contain all permutations of $\mathbb{N}$. Seen from the Universal Algebraic side, our work can thus be understood as a contribution to the revealing of the lattice of local clones on $\mathbb{N}$.

Utilizing the Galois-correspondence $Inv - Pol$ from the theory of clones, we describe the reducts of $(\mathbb{N}, =)$ sometimes by generating sets of operations, sometimes by generating sets of relations, and sometimes by syntactical restrictions on the formulas in the language of equality which define their relations.

Quantified Propositional Gödel Logics – A Survey

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In 1932, Gödel introduced a family of finite-valued propositional logics to show that intuitionistic logic does not have a characteristic finite matrix. The propositional Gödel logics are well understood: Any infinite set of truth-values characterizes the same set of tautologies.

Propositional Gödel logic can be extended by quantifiers in different ways, in particular by first-order quantifiers (universal and existential quantification over object variables) and propositional quantifiers (universal and existential quantification over propositions).

While there is only one infinite-valued propositional Gödel logic, uncountably many different quantified propositional Gödel logics are induced by different infinite subsets of truth-values over $[0, 1]$.

In contrast to classical propositional logic, propositional quantification may increase the expressive power of Gödel logics. More precisely, statements about the topological structure of the set of truth-values (taken as infinite subsets of the real interval $[0, 1]$) can be expressed using propositional quantifiers.

The purpose of this paper is to summarize and relate recent (and not so recent) results dealing with decidability, axiomatisability, and quantifier elimination for certain quantified propositional Gödel logics. We present the different results obtained and propose further research directions.

Completing distributive lattice expansions

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An algebra $\mathbf{A} = \langle A; F \rangle$ is a *distributive lattice expansion* if there are terms $\wedge, \vee$ in the term clone of $\mathbf{A}$, such that $\langle A; \wedge, \vee \rangle$ is a distributive lattice. How do you embed $\mathbf{A}$ into an algebra $\mathbf{B}$ such that $\langle B; \wedge, \vee \rangle$ is a complete distributive lattice? This turns out to be a very difficult problem which is very much the subject of active investigation.

Two common approaches are to take B to be the Dedekind-MacNeille completion of A or to be the Priestley completion of A (take B to be the lattice of all isotone maps from the partially ordered set of all prime ideals of A into 2). If $f : A \to A$ is an operation of $\mathbf{A}$, then there are two standard ways to extend f to B: $f^l(b) := \bigvee \{f(a) \mid b \geq a \in A\}$ and $f^u(b) := \bigwedge \{f(a) \mid b \leq a \in A\}$.

This approach mimics Dedekind's construction of the reals from the rationals: first construct the lattice completion, then extend the other operations to the completion. In this paper, I take a more algebraic approach. Robinson's algebraic construction of the reals from the rationals via ultrapowers is generalized, inspired by using Jónsson's Varietal Structure Theorem for congruence distributive varieties.

Monadic bounded commutative residuated ℓ-monoids

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An algebra $\mathcal{M} = (M; \odot, \vee, \wedge, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is called a *bounded commutative $R\ell$-monoid* iff (i) $(M; \odot, 1)$ is a commutative monoid, (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice, and (iii) $x \odot y \leq z \Longleftrightarrow x \leq y \to z$, (iv) $x \odot (x \to y) = x \wedge y$, for each $x, y, z \in M$. Bounded commutative residuated ℓ-monoids are a generalization of algebras of propositional logics such as BL-algebras, i.e. algebraic counterparts of the basic fuzzy logic (and hence consequently MV-algebras, i.e. algebras of the Łukasiewicz infinite valued logic) and Heyting algebras, i.e. algebras of the intuitionistic logic. Therefore, bounded $R\ell$-monoids could be taken as algebras of a more general propositional logic than the basic and intuitionistic logic. Monadic MV-algebras are an algebraic model of the predicate calculus of the Łukasiewicz

infinite valued logic in which only a single individual variable occurs. We introduce and study monadic residuated ℓ-monoids as a generalization of monadic MV-algebras.

The Equational Definability of Truth Predicates

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An *algebraizable logic* is a substitution-invariant consequence relation over formulas that is both 'equivalential' and 'truth-equational'. In this characterization, the former demand is well understood, but not the latter. In the talk, we provide the first intrinsic characterization of truth-equationality that is readily falsifiable in logics with enough small models.

The *Leibniz operator* of a logic maps a theory T to its 'Leibniz congruence' $\Omega\, T$, that is, the largest congruence relation on formulas that makes T a union of congruence classes. The operator has a similar action on the logical filters of all algebras over the same signature. The main result is:

Theorem 1. *The following conditions on a logic $\mathcal{S}$ are equivalent:*

(1) $\mathcal{S}$ *is* truth-equational, *i.e., the truth predicate of the class of reduced matrix models of $\mathcal{S}$ is explicitly definable by some fixed set of unary equations.*

(2) *The Leibniz operator Ω of $\mathcal{S}$ is* completely order reflecting *on all algebras, i.e., for any set of $\mathcal{S}$–filters $\mathcal{F} \cup \{G\}$ of an algebra, if $\bigcap \Omega\,[\mathcal{F}] \subseteq \Omega\, G$ then $\bigcap \mathcal{F} \subseteq G$.*

(3) *The Leibniz operator is completely order reflecting on the theories of $\mathcal{S}$.*

(4) *The Suszko operator of $\mathcal{S}$ is injective on all algebras.*

The *Suszko operator* maps a logical filter F to its 'Suszko congruence', which is the intersection of all Leibniz congruences of filters containing F. The *Suszko-reduced* matrix models of $\mathcal{S}$ are those whose Suszko congruence is the identity relation. The implication $(4) \Rightarrow (1)$ gives

Corollary 2. *Over the Suszko-reduced models of $\mathcal{S}$, the* implicit *definability of the truth predicate entails its* equational *definability.*

Previously, this was known only for *protoalgebraic* systems. The analogue of the corollary for the *Leibniz*-reduced models is shown to be false, i.e., global *injectivity* of the Leibniz operator does not entail truth-equationality.

An order-theoretic approach to image processing

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The aim of this work is to propose a general order-theoretic framework for the study of compression and reconstruction of digital images. Our approach places under a common umbrella a number of existing results and provides a new direction for future research in this field. The central mathematical structure used in this work is that of a module over a quantale. The compression and reconstruction of digital images is achieved by means of adjoint pairs of operators between such modules.

Algebra and Topology in Lambda Calculus

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This contribution concerns the study of lambda calculus by universal algebraic means. Lambda calculus is first and foremost a proof theoretic topic and the use of a semantic approach is non-standard. In particular, the algebraic semantic approach originates to a great extent with the introduction of the variety of lambda abstraction algebras by D. Pigozzi and the author in nineties. The intention of the speaker is to give a survey talk about the use of universal algebraic and topological methods in the study of the lattice of lambda theories (i.e., congruences extending lambda calculus), and of the model theory of lambda calculus. The starting point for this study is a recent result by the author that, for every variety of lambda abstraction algebras, there exists exactly one lambda theory whose term algebra generates the variety. Thus, the properties of a lambda theory can be studied by means of the variety of lambda abstraction algebras generated by its term algebra. Many longstanding open problems of lambda calculus can be restated in terms of algebraic properties of varieties of lambda abstraction algebras.

Derived Semidistributive Lattices

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Let $\mathcal{C}(L)$ denote the set of covers of a lattice L: $\gamma \in \mathcal{C}(L)$ iff $\gamma = (\gamma_0, \gamma_1) \in L^2$ and $\gamma_0 \prec \gamma_1$. For $\gamma, \delta \in \mathcal{C}(L)$ say $\gamma \leq \delta$ iff $\gamma_0 \leq \delta_0$, $\gamma_1 \not\leq \delta_0$, and $\gamma_1 \leq \delta_1$. For $\alpha \in \mathcal{C}(L)$ let $\mathcal{C}(L, \alpha)$ be the connected component of α in the poset $\langle \mathcal{C}(L), \leq \rangle$. We prove:

Theorem 1. *If L is a finite semidistributive lattice and $\alpha \in \mathcal{C}(L)$, then $\mathcal{C}(L, \alpha)$ is a semidistributive lattice. If L is bounded, then $\mathcal{C}(L, \alpha)$ is bounded.*

We call $\mathcal{C}(L, \alpha)$ the semidistributive lattice derived from L and α. For example, let $\mathcal{S}_n$ be the permutohedron on n letters and $\mathcal{T}_n$ be the associahedron on $n + 1$ letters. Let also $\alpha = (\bot, \alpha_1)$ be an atom. Then $\mathcal{C}(\mathcal{S}_n, \alpha) = \mathcal{S}_{n-1}$ and $\mathcal{C}(\mathcal{T}_n, \alpha) = \mathcal{T}_{n-1}$, up to isomorphism. Theorem 1 is a consequence of new characterizations of join semidistributive and lower bounded lattices:

Proposition 2. *A finite lattice is join semidistributive if and only if the projection $(\,\cdot\,)_0 : \mathcal{C}(L) \longrightarrow L$ creates pullbacks.*

Following the work by Barbut et al. on lattices of Coxeter groups, we say that $f : \mathcal{C}(L) \longrightarrow N$ is a *strict facet labeling* if (i) it is constant on each component $\mathcal{C}(L, \alpha)$, (ii) $f(\gamma) < f(\delta)$ whenever $u \neq \gamma_0$ and $u \wedge \gamma_0 < \delta_0 \prec \delta_1 \leq u \prec \gamma_1$, (iii) the dual property of (ii) holds.

Proposition 3. *A finite join semidistributive lattice is lower bounded if and only if it has a strict facet labeling.*

Advances in monadic $n \times m-$valued Łukasiewicz algebras with negation

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In [2] (see also [3]), monadic $n \times m$–valued Łukasiewicz algebras with negation (or $MNS_{n \times m}$–algebras) were introduced as $n \times m$–valued Łukasiewicz algebras with negation (see [1, 2, 4]) endowed with a quantifier. These algebras coincide with monadic n–valued Łukasiewicz algebras in $m = 2$ case. In this note, the congruences on these algebras are determined and

subdirectly irreducible algebras are characterized. From this last result it is proved that the variety of $MNS_{n \times m}$–algebras is a discriminator variety and as a consequence, the principal congruences are obtained. Furthermore, the number of congruences of finite $MNS_{n \times m}$–algebras is computed. Finally, a topological duality for $MNS_{n \times m}$–algebras is described.

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Small monoids $=\mathbf{EDA} \cap \overline{\mathbf{G_{nil}}}$?

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Recall that the free spectrum of $\mathbf{A}$ (or just $\mathbf{A}$ itself) is *large* (or *log-exponential*) if there exists a positive real number c such that for all n high enough $f_n^{\mathbf{A}} \geq 2^{2^{cn}}$; otherwise, $\mathbf{A}$ is said to be *small* (or *sub-log-exponential*). The pseudovariety $\mathbf{EDA}$ consists of finite monoids that satisfy the pseudoidentity $(x^\omega y^\omega)(y^\omega x^\omega)(x^\omega y^\omega) \approx x^\omega y^\omega$. The pseudovariety $\overline{\mathbf{G_{nil}}}$ consists of monoids with nilpotent subgroups.

Conjecture 1. A finite monoid $\mathbf{M}$ is small if and only if it is in $\mathbf{EDA} \cap \overline{\mathbf{G_{nil}}}$.

Tbe forward direction of Conjecture 1 has been proven. Let $\mathbf{B_2^1}$ denote the Perkins monoid, a monoid in $\mathbf{EDA} \cap \overline{\mathbf{G_{nil}}}$ and the source of significant counterexamples. Perhaps not surprisingly it is non-trivial to determine the asymptotic growth of its free spectrum. But at least we know that $\mathbf{B_2^1}$ is not a counterexample to Conjecture 1.

Theorem 2. *The free spectrum of $\mathbf{B_2^1}$ is small; indeed, it is at least $2^{O(n^2)}$ and no greater than $2^{O(n^3)}$.*

Problem 3. Determine precisely the asymptotic class $O(q(n))$, where the growth of the free spectrum of $\mathbf{B_2^1}$ is $2^{O(q(n))}$.

Problem 4. True or false? If **M** is a small non-commutative monoid, then there exists a positive integer k such that the growth of the free spectrum of **M** is log-n^k or log-$(n^k log(n))$.

Perhaps $\mathbf{B_2^1}$ provides a "no" to the above question.

Interestingly, it turns out that the pseudovariety $\mathbf{EDA} \cap \overline{\mathbf{G_{nil}}}$ shows up in a significant dichotomy conjecture of theoretical computer science.

Conjecture 5. A finite monoid **M** has **polynomial-length programs** (PLP) if and only if it is in $\mathbf{EDA} \cap \overline{\mathbf{G_{nil}}}$; otherwise, **M** is **universal**.

Definitions can be found at www2.ift.ulaval.ca/$\sim$ tesson, "Questions Related to Finite Monoids and Semigroups", P. Tesson. Connections between the two conjectures will be discussed briefly.

Sublattices of subsemigroup lattices. A survey

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Given a class $\mathcal{K}$ of semigroups, one might get interested in the class $\mathbf{S}(\mathrm{Sub}\mathcal{K})$ of lattices embeddable into subsemigroup lattices of semigroups from $\mathcal{K}$. We review results concerning dscription of classes $\mathbf{S}(\mathrm{Sub}\mathcal{K})$ for different $\mathcal{K}$. From those results, it follows, in particular, that for some $\mathcal{K}$, the class $\mathbf{S}(\mathrm{Sub}\mathcal{K})$ appears to be first-order.

Clones of 2-step Nilpotent Groups

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We are interested in the following problem: Is it true that for every finite group G there exists a k such that $\mathrm{Clo}(G)$ is determined by the k-ary algebraic relations of G?

The 2nd and 3rd authors have shown that the 3-ary algebraic relations determine the clone of a finite group G with abelian Sylow subgroups. They have also shown that if G is a finite nonabelian nilpotent group, then the k-ary algebraic relations of G for

$$(1) \qquad\qquad k = |G|^{[G:Z(G)]-1}$$

will determine $\mathrm{Clo}(G)$.

The main objective of this talk is to present the following result, which shows that for 2-step nilpotent groups the bound in (1) can be considerably improved.

Theorem 1. *If G is a finite 2-step nilpotent group, then $\mathrm{Clo}(G)$ is determined by the 4-ary algebraic relations of G.*

Primitive positive clones of groupoids

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Kearnes and Szendrei have proven that if a finite group G has Abelian Sylow subgroups, then the term operations of G are precisely those operations on G which preserve the subgroups of G^3. They have also demonstrated that this is not the case for every finite group. While ternary relations may not determine the clone of a finite group among all clones on the universe, they may be enough to distinguish between groups. Kearnes and Szendrei pose the following problem: *Suppose that G and H are groups defined on the same set. Show that $\mathrm{Sub}(G^3) = \mathrm{Sub}(H^3)$ implies $\mathrm{Clo}(G) = \mathrm{Clo}(H)$.*

We prove that if $\mathbf{A}$ is a finite groupoid with an identity element, then all of the homomorphisms in $\mathrm{P_{fin}HS}\mathbf{A}$ are completely determined by $\mathrm{Sub}(\mathbf{A}^3)$. This means that $\mathrm{Sub}(\mathbf{A}^3)$ determines the centralizer clone of $\mathbf{A}$ and, hence, the primitive positive clone generated by the operations of $\mathbf{A}$. It follows that the groups G and H in the Kearnes-Szendrei problem have the same primitive positive clones.

A representation theorem for quantale algebras

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We prove a representation theorem for quantale algebras generalizing the result of [1] stating that for every quantale Q there exists a semigroup S and a quantic nucleus j on the power-set of S such that Q is isomorphic to the range of j. We begin by introducing the category Q-**Alg** of algebras over a given unital commutative quantale Q (shortly Q-algebras). By analogy with monoid rings of [3] we construct a free Q-algebra from a given semigroup. Lastly we introduce the notion of quantale algebra nucleus which generalizes

the notions of quantic nucleus [1] and module nucleus [2]. The concept gives rise to the following representation theorem for Q-algebras.

Theorem 1. *For every Q-algebra A there exists a semigroup S and a quantale algebra nucleus j on the free Q-algebra over S such that A is isomorphic to the range of j.*

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On Binary Discriminator Varieties I, II, and III

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Binary **0**-discriminator varieties generalise (pointed) ternary discriminator varieties to the **0**-arithmetical case. Examples of such varieties abound in the literature and include pseudocomplemented semilattices, fixedpoint **0**-discriminator varieties (in the sense of Blok and Pigozzi), and residually finite varieties of basic logic algebras.

The study of binary discriminator varieties is facilitated by restricting attention to a certain class of algebras generalising implicative BCK- or Tarski algebras. We discuss this class, and the role it plays in the theory of binary discriminator varieties. Some connections with Agliano and Ursini's theory of subtractive varieties with EDPI are established, and a sheaf representation result for a certain naturally arising subclass of binary discriminator varieties is presented. We also describe some relationships between binary **0**-discriminator varieties, fixedpoint **0**-discriminator varieties, and ternary discriminator varieties. We conclude with some brief remarks on the role of the binary discriminator in the study of pointed discriminator logics, *viz.*, those algebraisable deductive systems whose equivalent quasivarieties are pointed discriminator varieties.

Embedding algebras into semimodules

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We study algebras that embed as subreducts into semimodules (modules) over commutative semirings (rings).

The problem is solved only partly. We will sketch our proof that idempotent subreducts of such semimodules are precisely Szendrei modes, and tell about related results, namely Stronkowski's non-idempotent generalization of this theorem, and several attempts to determine (idempotent) subreducts of modules over commutative rings.

Nilpotent subsemigroups of the semigroup of order-decreasing transformations of a power set

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A transformation α of a poset Ω is called order-decreasing, if $\alpha(x) \leqslant x$ for any x of Ω. Set $D(\Omega)$ of all order-decreasing transformations is a semigroup under composition. Let B_n stand for the power set of the set $\{1, ..., n\}$, partially ordered by inclusion. For $k \leqslant n$ we consider a set $\Lambda(B_n, k)$ of all ordered partitions of the set B_n into k blocks $Q_1, \ldots, Q_k$ satisfying following terms:

(1) for all i, $1 \leqslant i < k$, $A \in Q_i$ there exists B of Q_{i+1} such that $B < A$;
(2) for all i, $1 < i \leqslant k$, $A, B \in Q_i$, $B < A$, there exist $B_1 \in Q_1, \ldots, B_{i-1} \in Q_{i-1}$, such that $B_1 > B_2 > \ldots > B_{i-1} > A$.

Theorem 1. *There exists one-to-one correspondence between sets $\Lambda(B_n, k)$ and $Nil(B_n, k)$, the set of subsemigroups from $D(B_n)$, which are maximal among all nilpotent subsemigroups of $D(B_n)$ of nilpotency class k.*

Theorem 2. *Let $T_1 \in Nil(B_n, k)$, $T_2 \in Nil(B_m, k)$, $n \neq m$. Then T_1 and T_2 are non-isomorphic.*

Clones of algebras with parallelogram terms

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The critical algebraic relations of an algebra $\mathbf{A}$ are the subalgebras of finite powers of $\mathbf{A}$ that are completely $\cap$-irreducible and directly indecomposable. It is well known that if $\mathbf{A}$ is finite, then the critical algebraic relations of $\mathbf{A}$ determine the clone of $\mathbf{A}$. Our main results are

- a structure theorem for the critical algebraic relations of an algebra with a parallelogram term, and
- the theorem that an algebra has a parallelogram term if and only if it has an edge term.

Edge terms, a common generalization of Maltsev terms and near unanimity terms, were discovered recently by Berman, Idziak, Marković, McKenzie, Valeriote, and Willard while investigating finite algebras with 'few algebraic relations'. As an application of our structure theorem we prove that

- if A is a finite set, then for any fixed $k \geq 3$ there are only finitely many clones $\mathcal{C}$ on A such that $\mathcal{C}$ contains a k-ary parallelogram term and $(A; \mathcal{C})$ generates a residually small variety.

Algebraic aspects of some modal $n + 1$-valued Łukasiewicz logics

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In this lecture, we briefly introduce an $n + 1$-valued modal system which is complete with respect to the $n + 1$-valued KRIPKE models (i.e. the truth values of formulas range in the subset $\mathbb{L}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0, 1]$, the connectors $\neg$ and $\rightarrow$ are interpreted by the ŁUKASIEWICZ implication and negation and the truth value of the formula $\Box\phi$ in an world u is defined as the infimum of the truth values of ϕ in the worlds accessible from u. See HANSOUL G. and TEHEUX B., *Completeness results for many-valued Łukasiewicz modal systems and relational semantics*, `arXiv:math/0612542v1` for details).

We then present two different ways to go from models to structures. The first class of structures considered is the class of frames and the second is the class of $n + 1$-frames. The latters are frames in which we associate to each

world u a divisor m of n. This number represents the set $Ł_m$ of truth values that are allowed for a formula in the world u when we add a valuation to the $n+1$-frame. A logic that is complete with respect to a class of frames is called KRIPKE *complete* and a logic that is complete with respect to a class of $n+1$-frames is $n+1$-KRIPKE *complete*.

The core of the lecture is to show how these two problems of completeness can be considered with the help of preservation results under two types of extensions of $n+1$-valued modal algebras: canonical extension to obtain $n+1$-KRIPKE complete logics and *extension by idempotents* to obtain KRIPKE complete logics (the extension by idempotents of an $n+1$-valued modal algebra A is the largest extension that has, for Boolean algebra of idempotents, the canonical extension of the algebra of idempotents of A).

Connecting the profinite completion, the canonical extension and the MacNeille completion using duality

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We consider completions of bounded distributive lattice expansions in two ways. Firstly, we show that there is a natural connection between a lattice expansion, its canonical extension and its profinite completion if we view them as cones for a diagram of finite quotients. Secondly, we look for differences and similarities between the canonical extension, the profinite completion and the MacNeille completion of modal algebras. We provide criteria for the three constructions to coincide based on an analysis using the Jonsson-Tarski duality for modal algebras. This leads to sufficient conditions under which the lower MacNeille completion and the profinite completion of a modal algebra are the same.

Mereotopology and Boolean contact algebras

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In this talk we are going to present a nonclassical approach to the notion of space and topology - sometimes called 'pointless' topology. In contrast to the classical approach in which the notion of point is primitive and geometric figures are sets of points, this approach adopts as its primitive the notion

of region as an abstraction of a 'solid' spatial body. Consequently, in this theory regions are the first-order entities whereas points can be described as certain sets of regions, i.e., they become second-order entities. Together with some basic relations and operations these regions form an algebraic structure called a (Boolean) contact algebra. Such a contact algebra is a Boolean algebra extended by a binary relation C, called a contact relation, which satisfies certain axioms. The lattice operations provide the order structure, while the contact relation corresponds to the topological part of the formalism. The topological component becomes apparent in the notion of a standard model. Such a model consists of regular closed sets of a given topological space. Two region are said to be in contact, i.e. in the relation C, if they share a common point. Three major questions arise:

(1) Is it possible to weaken the algebraic structure to, say, distributive lattices and retain reasonable spatial expressivity?

(2) Which lattices/Boolean algebras can serve as basic sets of regions, e.g. does every Boolean algebra admit a sufficiently expressive contact relation?

(3) Can every contact algebra be represented as substructure of a standard model?

In this talk we are going to answer Question 1 and 3 and present some progress towards a solution of Question 2.

PMV-algebras of matrices

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MV-algebras of various kinds have been heavily investigated in the recent times. The isomorphism theorems between the MV-algebras and the interval MV-algebras in the corresponding lattice-ordered algebraic structures support research that utilizes the established properties of these structures in order to obtain specific information about the initial MV-algebras. In this talk we will discuss a concrete shape of any product MV-algebra that naturally embeds in a real algebra of matrices. We will use the structure theorem about lattice-ordered real algebras of matrices. In particular we establish that PMV-algebras in concern are precisely the intervals between the zero matrix and a conjugate of a certain positive matrix.

The representation of continuous lattices and a new Cartesian closed category

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Deng Zike in [1] introduced quasi-atoms in completely distributive lattices which generated a base and gave the representation of completely distributive lattices by using the base. Since completely distributive lattices are a kind of special continuous lattices, it is natural to discuss whether there is a representation in continuous lattices. Cartesian closedness is the basic categorical property to support effectively various fundamental operations of functional programming languages. There have been lots of discussions on this topic(see [3,4]). Thus it is interesting to seek the new cartesian closed full subcategory of the category **POSET**.

In this paper we introduce quasi-compact elements in continuous lattices, which are different with meet-irreducible elements and join-irreducible elements. And then we give quasi-bases of continuous lattices and a procedure for generating continuous lattices by using the quasi-bases. From the definition of quasi-compact elements, we introduce ST-lattices which is different with continuous lattices and discuss some properties of ST-lattices. Lastly, we introduce ST-lattice category which is a full subcategory of **POSET** and prove that the ST-lattice category is Cartesian closed.

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Negation from the perspective of neighborhood semantics

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We generalize the correspondence between the properties of negations and the conditions on the negation-neighborhood frames $\langle F, \leqslant, N \rangle$, where F is a non-empty set, $\leqslant$ a partial order on F and $N : F \to \mathcal{P}(\mathcal{U}(F))$ satisfying the following conditions: (1) for any $s_1, s_2 \in F$, if $s_1 \leqslant s_2$, then $N(s_1) \subseteq N(s_2)$; (2) for any $s \in S$, and $X, Y \in \mathcal{U}(F)$, if $X \subseteq Y$ and $Y \in N(s)$, then $X \in N(s)$.

Theorem 1. *The first six sequents (see the table below) are canonical. On correspondence we have the following table:*

sequents	corresponding property
N1 $\top \vdash \neg\bot$	*for any $s \in F, \emptyset \in N(s)$*
N2 $\neg\top \vdash \bot$	*for any $s \in F, F \notin N(s)$*
N3 $\neg x \wedge \neg y \vdash \neg(x \vee y)$	*N is reduced to the binary relation R_1 if $N(s) \neq \emptyset$ for each $s \in F$*
N4 $\neg(x \wedge y) \vdash \neg x \vee \neg y$	*N is reduced to binary relation R_2 if $N(s) \neq Full$ for each $s \in F$*
N5 $x \wedge \neg x \vdash \bot$	*if $s \in X$, then $X \notin N(s)$*
N6 $x \vdash \neg\neg x$	*$\{t\mid \uparrow s \in N(t)\} \in N(s)$*
N7 $\neg\neg x \vdash x$	*$\{t \mid \downarrow s \in N(t)\} \notin N(s)$*

Conjecture 2. $\neg\neg x \vdash x$ is not canonical.

Comparing our results with those in the literatures shows that the way here is more general in the sense that it can capture some additional single property of the negation with antitony.

Solvability of systems of polynomial equations over finite algebras

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Let SysPol(A) denote the solvability problem of a system of polynomial equations over a finite algebra A. In the talk the algorithmic complexity of SysPol(A) is considered. It turns out that the problem has a dichotomy in the class of finite groupoids with an identity element. In other words, SysPol(A) is either in the complexity class **P** or **NP**-complete, if A is a finite grupoid with an identity element. Developing the underlying idea further yields the following dichotomy theorem:

Theorem 1. *Let A be a finite algebra of finite signature that admits a non-trivial idempotent Maltsev condition. Then SysPol(A) is in $\mathbf{P}$ whenever A has a compatible idempotent ternary operation that extends to the idempotent term operation $xy^{n-1}z$ of a finite semilattice of Abelian semigroups, and SysPol(A) is $\mathbf{NP}$-complete otherwise.*

This result encompasses the cases of semilattices, lattices, rings, modules, quasigroups and is a substantial extension of most of the earlier results on the topic.

A subvariety of Ockham - Nelson algebras

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In [2], A. V. Figallo introduced a new class of algebras as a generalization of N–lattices ([5]) or Nelson algebras, following A. Monteiro ([1]). These algebras, which from now on, we shall call De Morgan–Nelson algebras were obtained from the latter by omitting the Kleene law: $(x \wedge \sim x) \vee (y \vee \sim y) = y \vee \sim y$. In this note, we prove that in these algebras the identity $(x \to y) \to x = x$ is equivalent to the one known as Pierce law $((x \to y) \to x) \to x = 1$ whose importance can be seen in [4]. Furthermore, we introduce and investigate the subvariety of Ockham–Nelson algebras ([3]) verifying Kleene law.

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