

Improved Doppler Positioning Techniques for Stand-Off Scenarios

Brian D. O. Anderson and Samuel Picton Drake

Abstract

In “stand-off” scenarios an estimate of the bearing to an emitter can be easily calculated from measurements of the received frequency, position and velocity of receivers. This technique is only effective if the measured frequencies can be reliably associated with a single emitter that is far away. We show how to determine if the receivers’ measurements correspond to a single emitter in a stand-off scenario and how to estimate the unknown transmit frequency.

Index Terms

Doppler Positioning, Differential Doppler, Bearing Estimation, Linearization techniques, Optimization

I. INTRODUCTION

In many civilian and defence situations it is necessary to determine the location of radio frequency (RF) emitters such as mobile phones, radio transmitters or emergency beacons. There are a number of methods for locating RF emitters such as time difference of arrival, direction finding and Doppler positioning. A summary of a variety of localisation techniques are presented in Poisel’s book [1].

In this paper we concentrate on Doppler positioning, and in particular its use to achieve direction finding of a stationary emitter. Doppler positioning is a passive localisation technique, meaning that the localization depends only on receiving signals, as opposed to transmitting them, a process which can announce the tracker’s intent and location. Passive techniques have the advantage of being more stealthy and requiring less power for operation or carriage of the associated payload such as a radar transmitter, which means that they are well suited for use in unmanned aerial vehicles (UAVs) doing continuous surveillance work, reducing the frequency required for refuelling or minimizing the broadcasting of UAV location information.

Doppler positioning has a long history. It was used to locate Russia’s first satellite SPUTNIK [2] and for surveillance by the SR-71 aircraft during the cold war [3]. Prior to the deployment of the global positioning system the TRANSIT system was used for global satellite navigation [4], and it used a Doppler positioning technique.

To determine the location of a stationary emitter using Doppler positioning it is in principle sufficient to measure the (Doppler-shifted) carrier frequency of the transmitter signal and to know the location and velocity of the sensor

B. D. O. Anderson is with the Australian National University and with Hangzhou Dianzi University, Hangzhou, China, and Data61-CSIRO (formerly NICTA Ltd.), Canberra, Australia.

S. P. Drake is with the University of Adelaide and the Defence Science and Technology Group, Australia.

at a number of different locations. Small low cost UAVs equipped with a GPS receiver and a software defined radio can easily obtain these measurements [5].

Although Doppler positioning is a well established technique there are still some advances to be made [6], [7], [8], [9]. When UAVs are used to locate an emitter it is often advantageous for the UAVs to be far enough away from the emitter to avoid being detected or subject to active interference. In this paper we describe a method for verifying, using Doppler shifted frequency measurements, that an emitter is remote (as far as the UAV positions are concerned) so that the direction vector can be assumed to be constant at all the points at which the emitter Doppler shift is measured. The emitter is assumed to be stationary, so that all Doppler shifting is attributable to the UAV's velocity only. Based on this knowledge, we explain how to determine the direction of the emitter when the emitter is sufficiently far away from the points at which the Doppler-shifted emitter frequency is received. An initial estimate of direction can be obtained by solving a least-squares problem; however, we show how to obtain an improved estimate of the emitter's direction by minimising a constrained Lagrangian. The emitter direction is determined using the Gauss-Newton method for solving the constrained Lagrangian, initializing the iterations with the least-squares solution. Simulations show that this refinement significantly improves the bearing estimate when compared with the simple least-squares estimator.

The outline of the paper is as follows: In the next section we present the fundamental Doppler positioning equations and show how they can be transformed into a linear equation with the aid of "nuisance" variables which are the quadratic terms of the emitter's location in Cartesian coordinates. We present the solutions to the linearized equations and show that if the noise on the measured frequency is sufficiently small the estimated location is in good agreement with the true location. In section III we examine "stand-off" scenarios in which the maximum distance between received signal locations is small compared with the distance from the mean receiver location to the emitter. In this section we derive a stand-off indicator which is used to determine if a set of measurements is consistent with the stand-off scenario, and separately we consider how to decide whether a new frequency measurement is from an emitter sufficiently far away for the "stand-off" approximation to continue to be valid. We demonstrate the validity of this indicator with some examples. In section III-C we show how to obtain an improved estimate of the bearing to the emitter in a stand-off scenario by using the constrained cost function with a Lagrange multiplier. We show that this estimate is a significant improvement on the least squares estimator. In the last section before the conclusions we show how to accommodate the case when the carrier frequency of the transmitter is not known precisely.

II. LINEARIZATION VIA THE INTRODUCTION OF NUISANCE VARIABLES

In this section, we will explain how the basic localization problem can be reformulated to a linear equation. This will subsequently help us in studying stand-off scenarios in which the target is remote from the UAV sensing it.

Throughout this paper, we allow the possibility of a two-dimensional or three-dimensional environment. Until Section IV, we will assume that the unshifted emitter frequency is precisely known, as opposed to a nominal value being known. One could only assume it is precisely known if, for example, there was a separate stationary receiver

providing the information, or the emitter was actually reflecting signals due to its being illuminated by a radar, with the position, velocity and precise frequency of the radar transmitter known to the receiver.

Assume that the receiver makes N measurements of the frequency of a signal associated with an emitter. In the absence of noise, the i -th frequency measurement by the receiver, call it f_i , is related to the position $\mathbf{r}_i$ and velocity $\dot{\mathbf{r}}_i$ of the receiver, the position of the emitter $\mathbf{p}$ and the unshifted emitter frequency f_0 by the Doppler effect equation:¹

$$f_i = f_0 \left(1 + \frac{\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i)}{c \|\mathbf{p} - \mathbf{r}_i\|} \right), \quad i = 1, 2, \dots, N. \quad (1)$$

(Here the over-dot indicates the derivative with respect to time $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$, the notation $\mathbf{x}^\top$ and $\|\mathbf{x}\|$ denote the transpose and Euclidean norm of a vector $\mathbf{x}$ respectively, and c is the speed of light). It is convenient to define a Doppler parameter

$$\gamma_i \equiv c \left(\frac{f_i}{f_0} - 1 \right),$$

so that Eq. (1) can be written as

$$\gamma_i^2 (\mathbf{p}^\top - \mathbf{r}_i^\top) (\mathbf{p} - \mathbf{r}_i) = \left(\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i) \right)^2. \quad (2)$$

To avoid unnecessary duplication in recording separately two- and three-dimensional cases, we shall write many of the relevant equations just for the three-dimensional case, especially where it is obvious that the corresponding two-dimensional case can be obtained by just setting certain variables to zero. The coordinates of the vectors $\mathbf{p}^\top$ and $\mathbf{r}_i^\top$ will be taken as $[x, y, z]$ and $[x_i, y_i, z_i]$ respectively.

For N independent measurements we can write Eq. (2) in linear algebra form as

$$\mathbf{A}\mathbf{w} = \mathbf{b}, \quad (3)$$

where the matrix $\mathbf{A}$ and vector $\mathbf{b}$ are known from the measurements, and the vector $\mathbf{w}$ contains entries depending on the emitter position, all being given by

$$\mathbf{A} = \begin{bmatrix} \alpha_1^x & \alpha_1^y & \alpha_1^z & -2\dot{x}_1\dot{y}_1 & -2\dot{x}_1\dot{z}_1 & -2\dot{y}_1\dot{z}_1 & 2\zeta_1^x & 2\zeta_1^y & 2\zeta_1^z \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_N^x & \alpha_N^y & \alpha_N^z & -2\dot{x}_N\dot{y}_N & -2\dot{x}_N\dot{z}_N & -2\dot{y}_N\dot{z}_N & 2\zeta_N^x & 2\zeta_N^y & 2\zeta_N^z \end{bmatrix}$$

$$\mathbf{w}^\top = \begin{bmatrix} x^2 & y^2 & z^2 & xy & xz & yz & x & y & z \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \left(\dot{\mathbf{r}}_1^\top \mathbf{r}_1 \right)^2 - \gamma_1^2 \mathbf{r}_1^\top \mathbf{r}_1 \\ \vdots \\ \left(\dot{\mathbf{r}}_N^\top \mathbf{r}_N \right)^2 - \gamma_N^2 \mathbf{r}_N^\top \mathbf{r}_N \end{bmatrix}$$

with

$$\alpha_i^x \equiv \gamma_i^2 - \dot{x}_i^2, \quad \alpha_i^y \equiv \gamma_i^2 - \dot{y}_i^2, \quad \alpha_i^z \equiv \gamma_i^2 - \dot{z}_i^2$$

¹This equation assumes that the speed of the receiver is much less than the speed of light so that relativistic effects can be ignored, and that the refractive index of air is constant and unity.

and

$$\begin{aligned}\zeta_i^x &= \dot{\mathbf{r}}_i^\top \mathbf{r}_i \dot{x}_i - \gamma_i^2 x_i \\ \zeta_i^y &= \dot{\mathbf{r}}_i^\top \mathbf{r}_i \dot{y}_i - \gamma_i^2 y_i \\ \zeta_i^z &= \dot{\mathbf{r}}_i^\top \mathbf{r}_i \dot{z}_i - \gamma_i^2 z_i \quad .\end{aligned}$$

In the two-dimensional case, columns 3,5,6 and 9 of $\mathbf{A}$ drop out, as do the corresponding entries of the vector $\mathbf{w}$. Also, in the remaining entries of $\mathbf{A}$, the quantities z_i are all zero for all i .

If there are nine or more measurements (five or more in the two dimensional case), the matrix $\mathbf{A}$ may have full column rank, and indeed simulations reveal that for generic UAV trajectories and points on those trajectories, this property holds.²

Suppose that $\mathbf{A}$ does have full column rank. If we assume all measurements are noiseless, we can find $\mathbf{w}$ and the target's location is then given by the last three (two in the two-dimensional case) entries of $\mathbf{w}$. Note that if there are more than nine (five in the two-dimensional case) measurements one can express $\mathbf{w}$ as

$$\mathbf{w} = \left(\mathbf{A}^\top \mathbf{A} \right)^{-1} \mathbf{A}^\top \mathbf{b} \quad . \quad (4)$$

If measurements are contaminated by noise and $N > 9$ (or $N > 5$ in the two-dimensional case), then Eq. (3) will not have an exact solution. However, in general Eq. (4) yields a least-squares solution of Eq. (3), and again, the last three (or two) entries of the solution can be taken as the emitter position estimate. Of course if there are nine (five) measurements exactly, then with or without noise, this equation reduces to

$$\mathbf{w} = \mathbf{A}^{-1} \mathbf{b} \quad . \quad (5)$$

When the emitter is remote from the UAV positions, one cannot expect, especially in the noisy situation, this equation to yield accurate values. However, in the non-remote case, and as illustrated by simulations below, the solution given from the last three (two) entries of Eq. (4) can be satisfactory. There is even an improvement to the solution algorithm that can be pursued but, given that our main interest in this paper is the treatment of stand-off scenarios, we will do this in detail elsewhere. This improvement arises from the fact that the only solutions, or approximations to solutions, are those of the form

$$\mathbf{w} = [x^2 \ y^2 \ z^2 \ xy \ xz \ yz \ x \ y \ z]^\top \quad (6)$$

i.e. those for which the first six entries are obtainable as quadratic expressions involving the last three. Put another way, we are interested not in the usual least-squares solution of Eq. (3) but rather a *constrained* least-squares solution, obtained using six independent quadratic constraints. (A corresponding conclusion of course holds in the two-dimensional case, there being three constraints). As the techniques for achieving this are quite different to those applying when the emitter is remote, we avoid further discussion here.

²In special cases, it need not—for example, if all measurements were taken from the same point and the receiver velocity were the same for each occasion, it is obvious that this would not be the case, and $\mathbf{A}$ would have rank 1 because its rows would be identical.

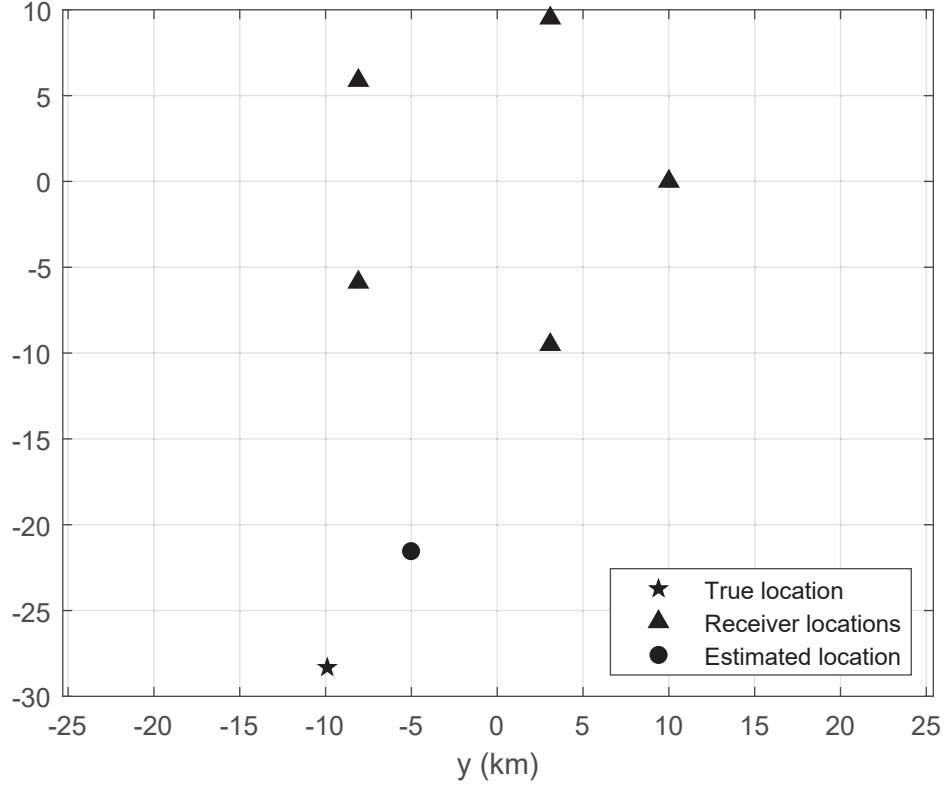


Fig. 1. Example of using Eq. (4) to determine the location of an emitter.

A. Simulated Geolocation Results

Unless otherwise stated the following parameters were used in all the simulations: both the emitter and receiver locations lie in the same two dimensional plane, the receiver moves in a perfect circle centred at zero with a radius of 10 km and a speed of 200 m/s, the receiver position and velocity measurements are noiseless, the measured frequency is calculated at five evenly-spaced intervals about a circle of radius 10 km, the transmitted frequency f_0 is 300 MHz and the measured frequency noise is zero mean Gaussian with a standard deviation of 10 Hz.

Figure 1 depicts the situation in which the frequency measurement error is zero mean Gaussian with a standard deviation of 10 Hz while UAV position and velocity measurements are assumed to be exactly known. The emitter is 30 km away.

The $\mathbf{A}$ matrix in equation (3) is a square matrix for this example, and no attempt was made to enforce the quadratic constraints on the equation solution mentioned earlier so the emitter location was obtained from Eq. (5).

III. BEARING ESTIMATION IN STAND-OFF SCENARIOS

Often we are interested in “stand-off” scenarios where the emitter is far away from the receiver locations. In these scenarios the estimate of the range to the emitter is either unreliable or unnecessary. In the next subsection, we address the question of deciding on the basis of a collection of measurements whether or not a stand-off scenario

exists. Assuming then that such a scenario has actually been identified, it becomes of interest to determine the bearing of the emitter and indeed subsection III-C describes how to estimate the bearing to the emitter when the emitter is far from the receiver locations. The final subsection considers the question of deciding whether a new measurement is consistent with the stand-off scenario already identified.

In a number of occasions in what follows, we will work with singular values of various matrices. We include in Appendix A some relevant background on singular value decomposition.

A. Testing for a stand-off scenario

Initially, we discuss the three-dimensional case. Suppose that the true position $[x, y, z]$ of the emitter is far away from each of the receiver positions $[x_i, y_i, z_i]$ at which a Doppler-shifted signal is received, and the origin of the coordinate system is located close to the receiver positions, thus implying $\sqrt{x^2 + y^2 + z^2}$ assumes a large value. It is clear intuitively that the greater the stand-off distance, the harder it will be to determine the range of the emitter, though it might well be straightforward to still obtain its direction (elevation and azimuth). Put another way, if $[x, y, z]$ is one solution to the relevant equations, then for any constant s with $s > 1$ (and maybe for some values less than 1, but this does not concern us) $[sx, sy, sz]$ should also be an approximate solution. In terms of the measurement matrix $\mathbf{A}$ and measurement vector $\mathbf{b}$, none of whose entries are expected to be large even when one at least of x, y and z is large, we have both

$$\mathbf{A} \begin{bmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \\ x \\ y \\ z \end{bmatrix} \approx \mathbf{b} \text{ and } \mathbf{A} \begin{bmatrix} s^2 x^2 \\ s^2 y^2 \\ s^2 z^2 \\ s^2 xy \\ s^2 xz \\ s^2 yz \\ sx \\ sy \\ sz \end{bmatrix} \approx \mathbf{b}$$

from which we conclude (by letting s become large) that with $\bar{\mathbf{A}}$ denoting the first six columns of $\mathbf{A}$, there must hold

$$\bar{\mathbf{A}} \begin{bmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{bmatrix} \approx \mathbf{0} \quad , \quad (7)$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} \alpha_1^x & \alpha_1^y & \alpha_1^z & -2\dot{x}_1\dot{y}_1 & -2\dot{x}_1\dot{z}_1 & -2\dot{y}_1\dot{z}_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_N^x & \alpha_N^y & \alpha_N^z & -2\dot{x}_N\dot{y}_N & -2\dot{x}_N\dot{z}_N & -2\dot{y}_N\dot{z}_N \end{bmatrix}.$$

In turn, this means that the first six columns of $\mathbf{A}$ are approximately linearly dependent, or that $\bar{\mathbf{A}}$ has a singular value very close to 0.

The changes for the two-dimensional case are trivial: in that instance, $\bar{\mathbf{A}}$ has three columns, corresponding to the first three columns of the five column matrix $\mathbf{A}$.

The stand-off emitter indicator is defined as

$$\chi \equiv 100 - 100 \frac{\sigma_{\min}}{\sigma_{\max}}, \quad (8)$$

where $\sigma_{\max}$ and $\sigma_{\min}$ denote the largest and smallest singular values of $\bar{\mathbf{A}}$. The stand-off indicator is zero if $\sigma_{\max}$ and $\sigma_{\min}$ are the same, and it is 100 if $\sigma_{\min}$ is zero.

Figure 2 shows a 2D case where the emitter is 15 km from the centre of the receiver locations. In this case the stand-off indicator is $\chi = 36$. By contrast when the emitter is 150 km from the centre of the receiver locations, as shown in figure 3, the stand-off indicator is $\chi = 95$.

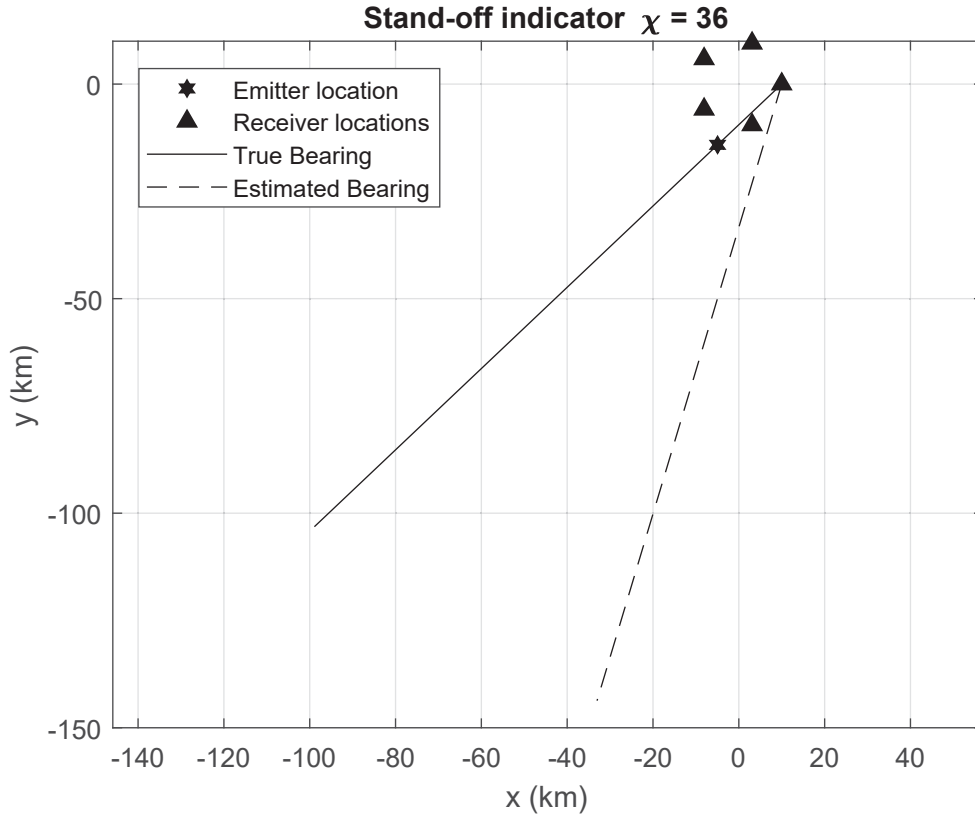


Fig. 2. Stand-off test for an emitter 15 km from the centre of the circle. The simulation parameters are given in section II-A.

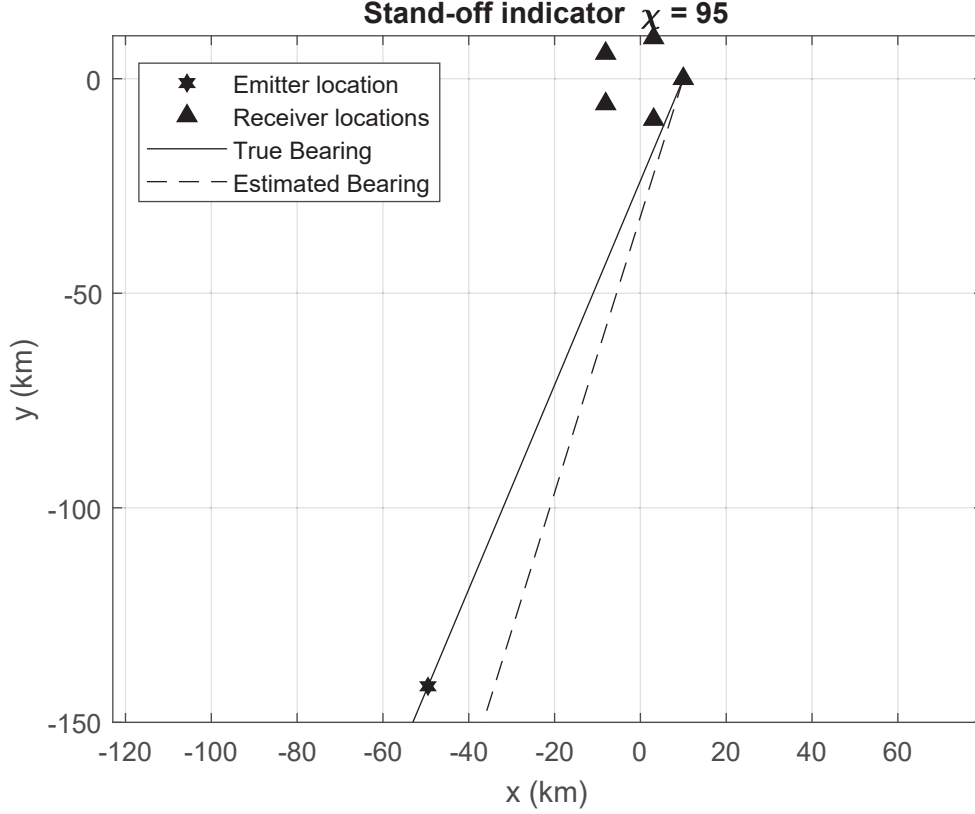


Fig. 3. Stand-off test for an emitter 150 km from the centre of the circle. The simulation parameters are given in section II-A.

B. Direction of arrival variation and singular values

In this subsection, our aim is to indicate a relation between the angular spread of directions between different positions of the receiver and a stationary target on the one hand, and the stand-off parameter χ on the other hand.

To proceed, we first note that the squared Doppler equation Eq. (2) can be written as

$$\gamma_i^2 \mathbf{u}_i^\top \mathbf{u}_i = \left(\dot{\mathbf{r}}_i^\top \mathbf{u}_i \right)^2, \quad (9)$$

if we write the displacement vector $\mathbf{p} - \mathbf{r}$ in terms of range and a direction vector

$$\mathbf{p} - \mathbf{r}_i = \rho_i \mathbf{u}_i \quad \text{where} \quad \mathbf{u}_i = \begin{bmatrix} \cos \theta_i \cos \phi_i \\ \cos \theta_i \sin \phi_i \\ \sin \theta_i \end{bmatrix}. \quad (10)$$

Equation (9) can be reformulated in terms of $\bar{\mathbf{A}}$ as

$$\bar{\mathbf{a}}_i^\top \boldsymbol{\mu}_i = 0, \quad (11)$$

1 where $\bar{\mathbf{a}}_i^\top$ is the i -th row of $\bar{\mathbf{A}}$ and

$$\boldsymbol{\mu}_i = \begin{bmatrix} \cos^2 \theta_i \cos^2 \phi_i \\ \cos^2 \theta_i \sin^2 \phi_i \\ \sin^2 \theta_i \\ \cos^2 \theta_i \cos \phi_i \sin \phi_i \\ \cos \theta_i \sin \theta_i \cos \phi_i \\ \cos \theta_i \sin \theta_i \sin \phi_i \end{bmatrix}. \quad (12)$$

2 In the noiseless case and if the emitter were infinitely far away, then the directions from the receivers to the
3 emitter would all be the same, $\boldsymbol{\mu}_i = \boldsymbol{\mu}$ and

$$\bar{\mathbf{a}}_i^\top \boldsymbol{\mu} = 0 \quad \forall \quad i.$$

4 If the emitter is far away but not infinitely far away, then for some $\boldsymbol{\mu}$ and small $\delta\boldsymbol{\mu}_i$,

$$\boldsymbol{\mu}_i = \boldsymbol{\mu} + \delta\boldsymbol{\mu}_i$$

5 and the squared Doppler equation becomes (to first order in the angular variation in θ and ϕ) away from the values
6 associated with $\boldsymbol{\mu}$:

$$\bar{\mathbf{a}}_i^\top \boldsymbol{\mu} = -\bar{\mathbf{a}}_i^\top \delta\boldsymbol{\mu}_i = -\bar{\mathbf{a}}_i^\top \mathbf{M} \delta\boldsymbol{\varphi}_i, \quad (13)$$

7 where

$$\begin{aligned} \mathbf{M} = \frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\varphi}} &= \begin{bmatrix} -\cos^2 \phi \sin 2\theta & -\cos^2 \theta \sin 2\phi \\ -\sin^2 \phi \sin 2\theta & \cos^2 \theta \sin 2\phi \\ \sin 2\theta & 0 \\ -\cos \phi \sin \phi \sin 2\theta & \cos^2 \theta \cos 2\phi \\ \cos \phi \cos 2\theta & -\frac{1}{2} \sin \phi \sin 2\theta \\ \sin \phi \cos 2\theta & \frac{1}{2} \cos \phi \sin 2\theta \end{bmatrix} \\ &= [\mathbf{m}_\theta \quad \mathbf{m}_\phi] \quad \text{and} \quad \delta\boldsymbol{\varphi} = \begin{bmatrix} \delta\theta \\ \delta\phi \end{bmatrix}. \end{aligned} \quad (14)$$

8 To assist in the following, we note that by expressing $\cos^4 \phi + \sin^4 \phi$ as $1 - 2\cos^2 \phi \sin^2 \phi$, using the Pythagorean
9 identity and the double angle formulas, a simple calculation yields the following bound on the Euclidean norm of
10 $\mathbf{m}_\theta$:

$$\|\mathbf{m}_\theta\|^2 = \sin^2 2\theta (1 - (1/4) \sin^2 2\phi) + 1 \leq 2 \quad (15)$$

11 with the upper bound attained at $\phi = 0, \theta = \pi/4$. Likewise

$$\|\mathbf{m}_\phi\|^2 = \cos^2 \theta (\cos^2 \theta (\sin 2\phi)^2 + 1) \leq 2 \quad (16)$$

12 with the upper bound achieved at $2\phi = \pi/2, \theta = 0$.

13 How is the variation in $\delta\boldsymbol{\varphi}$ reflected in the singular values of $\bar{\mathbf{A}}$?

In Appendix A we show that

$$\sigma_{\min}(\bar{\mathbf{A}})\|\boldsymbol{\mu}\| \leq \|\bar{\mathbf{A}}\boldsymbol{\mu}\| \leq \sigma_{\max}(\bar{\mathbf{A}})\|\boldsymbol{\mu}\| \quad .$$

Also, it is shown in Appendix B that the bounds on $\boldsymbol{\mu}$ (where $\boldsymbol{\mu}$ is defined by (12) with the subscripts removed) are $\sqrt{2/3} \leq \|\boldsymbol{\mu}\| \leq 1$. Accordingly, there holds

$$\sqrt{2/3}\sigma_{\min}(\bar{\mathbf{A}}) \leq \|\bar{\mathbf{A}}\boldsymbol{\mu}\| \leq \sigma_{\max}(\bar{\mathbf{A}}) \quad . \quad (17)$$

Next, let $\Delta\theta, \Delta\phi$ denote the maximum magnitudes of $\delta\theta_i, \delta\phi_i$ respectively. Then it is easily seen that (using the triangle inequality for the first inequality step)

$$\begin{aligned} \|\bar{\mathbf{A}}\boldsymbol{\mu}\| &= \left\| \begin{bmatrix} \bar{a}_1^\top M \delta\varphi_1 \\ \bar{a}_2^\top M \delta\varphi_2 \\ \vdots \\ \bar{a}_N^\top M \delta\varphi_N \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \bar{a}_1^\top m_\theta \delta\theta_1 \\ \bar{a}_2^\top m_\theta \delta\theta_2 \\ \vdots \\ \bar{a}_N^\top m_\theta \delta\theta_N \end{bmatrix} \right\| + \left\| \begin{bmatrix} \bar{a}_1^\top m_\phi \delta\phi_1 \\ \bar{a}_2^\top m_\phi \delta\phi_2 \\ \vdots \\ \bar{a}_N^\top m_\phi \delta\phi_N \end{bmatrix} \right\| \\ &\leq \left\| \begin{bmatrix} \bar{a}_1^\top m_\theta \\ \bar{a}_2^\top m_\theta \\ \vdots \\ \bar{a}_N^\top m_\theta \end{bmatrix} \right\| \Delta\theta + \left\| \begin{bmatrix} \bar{a}_1^\top m_\phi \\ \bar{a}_2^\top m_\phi \\ \vdots \\ \bar{a}_N^\top m_\phi \end{bmatrix} \right\| \Delta\phi \\ &\leq \sigma_{\max}(\bar{\mathbf{A}})(\|m_\theta\|\Delta\theta + \|m_\phi\|\Delta\phi) \quad . \end{aligned} \quad (18)$$

As the lower bound on $\|\bar{\mathbf{A}}\boldsymbol{\mu}\|$ is $\sqrt{2/3}\sigma_{\min}(\bar{\mathbf{A}})$ and the upper bound on both $\|m_\theta\|$ and $\|m_\phi\|$ is $\sqrt{2}$ it follows that $\sqrt{2/3}\sigma_{\min}(\bar{\mathbf{A}}) \leq 2\sqrt{2}\sigma_{\max}(\bar{\mathbf{A}})\max\{\Delta\theta, \Delta\phi\}$ and hence

$$\max\{\Delta\theta, \Delta\phi\} \geq \frac{1}{2\sqrt{3}} \frac{\sigma_{\min}(\bar{\mathbf{A}})}{\sigma_{\max}(\bar{\mathbf{A}})} \quad . \quad (19)$$

For example if the stand-off flag is $\chi = 95^\circ$ as in Fig. 3 the angle error is greater than 1° , however if on $\chi = 36^\circ$ as in Fig. 2 it is greater than 11° .

C. Bearing estimation with unitary constraint

Suppose that it has been determined (using the singular value test, or any other means) that a stand-off scenario exists. In this case, it is appropriate to switch from seeking to localize the target to seeking to determine its direction from the array of points at which the receiver measures Doppler data.

In a stand-off scenario

$$f_i \approx f_0 \left(1 + \frac{\dot{\mathbf{r}}_i^\top \mathbf{u}}{c} \right) \quad . \quad (20)$$

The least squares solution to Eq. (20) is

$$\mathbf{u} = \left(\mathbf{B}^\top \mathbf{B} \right)^{-1} \mathbf{B}^\top \mathbf{g} \quad , \quad (21)$$

where

$$\mathbf{B} = \begin{bmatrix} \dot{\mathbf{r}}_1^\top \\ \vdots \\ \dot{\mathbf{r}}_N^\top \end{bmatrix} \quad \mathbf{g} = c \begin{bmatrix} \frac{f_1}{f_0} - 1 \\ \vdots \\ \frac{f_N}{f_0} - 1 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_N \end{bmatrix} \quad .$$

In a noisy situation, there can be no guarantee that $\mathbf{u}$ derived by Eq. (21) is a unit vector, and it would seem prudent to seek to use this piece of information to reduce the impact of noise on the quality of the direction estimate. A straightforward way of incorporating the constraint is to reflect the constraint that $\mathbf{u}$ is a unit vector through use of a Lagrange multiplier. Finding a least squares solution of a linear equation subject to a *single* quadratic constraint is a standard problem for which there is a straightforward algorithm. The Lagrangian is

$$\mathcal{L} = (\mathbf{u}^\top \mathbf{B}^\top - \mathbf{g}^\top) (\mathbf{B}\mathbf{u} - \mathbf{g}) - \lambda (\mathbf{u}^\top \mathbf{u} - 1) \quad . \quad (22)$$

This takes an extremum when

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}^\top} = 0 \implies (\mathbf{B}^\top \mathbf{B} - \lambda \mathbf{I}) \mathbf{u} = \mathbf{B}^\top \mathbf{g} \quad (23a)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \implies \mathbf{u}^\top \mathbf{u} = 1 \quad . \quad (23b)$$

Initial estimate of $\mathbf{u}$ is obtained from (24) and updates from (25).

At once we see that, with $\mathbf{W}(\lambda) = \mathbf{B}^\top \mathbf{B} - \lambda \mathbf{I}$,

$$\mathbf{u} = \mathbf{W}^{-1}(\lambda) \mathbf{B}^\top \mathbf{g} \quad (24)$$

$$1 = \mathbf{g}^\top \mathbf{B} (\mathbf{W}^{-1}(\lambda))^\top \mathbf{W}^{-1}(\lambda) \mathbf{B}^\top \mathbf{g}$$

One way to solve the equations resulting from the Lagrange multiplier introduction is to start with an estimate of $\mathbf{u}$ obtained from the standard least squares solution (thus not necessarily of unit length) and an arbitrary estimate of λ , say zero. Then we calculate an improved estimate. Our new estimates of $\mathbf{u}$ and λ are $\mathbf{u} \mapsto \mathbf{u} + d\mathbf{u}$ and $\lambda \mapsto \lambda + d\lambda$ respectively. Using Eq. (23) and keeping only terms of first order in $d\lambda$ and $d\mathbf{u}$ we obtain

$$\begin{bmatrix} 0 & 2\mathbf{u}^\top \\ -\mathbf{u} & \mathbf{B}^\top \mathbf{B} - \lambda \mathbf{I} \end{bmatrix} \begin{bmatrix} d\lambda \\ d\mathbf{u} \end{bmatrix} = \begin{bmatrix} 1 - \mathbf{u}^\top \mathbf{u} \\ \mathbf{B}^\top \mathbf{g} - \mathbf{B}^\top \mathbf{B} \mathbf{u} + \lambda \mathbf{u} \end{bmatrix} \quad . \quad (25)$$

The iterative scheme for determining the direction vector $\mathbf{u}$ is thus as follows:

- 1) The initial estimate of λ is $\lambda_0 = 0$ and we use Eq. (23a) to determine the initial estimate of $\mathbf{u}_0 = (\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{g}$
- 2) Improved estimates of λ and $\mathbf{u}$ are $\lambda + d\lambda$ and $\mathbf{u} + d\mathbf{u}$, where $d\lambda$ and $d\mathbf{u}$ are determined by solving Eq. (25)
- 3) Determine how close the length of the direction vector is to unity:

$$t = |\mathbf{u}^\top \mathbf{u} - 1|$$

- 4) If $t > 0.01$ repeat steps 2 and 3 with $\mathbf{u} \rightarrow \mathbf{u} + d\mathbf{u}$ and $\lambda \mapsto \lambda + d\lambda$; if $t < 0.01$ the algorithm has converged to sufficient accuracy.

A comparison between the accuracy of linear estimates of the bearing using Eq. (21) and the scheme detailed above is given in Fig. 4. To generate this plot 1,000 simulations were run. The configuration parameters in these simulations were the same as previous simulations except the speed of the UAVs was normally distributed with mean zero and standard deviation 200 m/s, and their headings uniformly distributed. In this case the error of the bearing estimate has improved by approximately 20%.

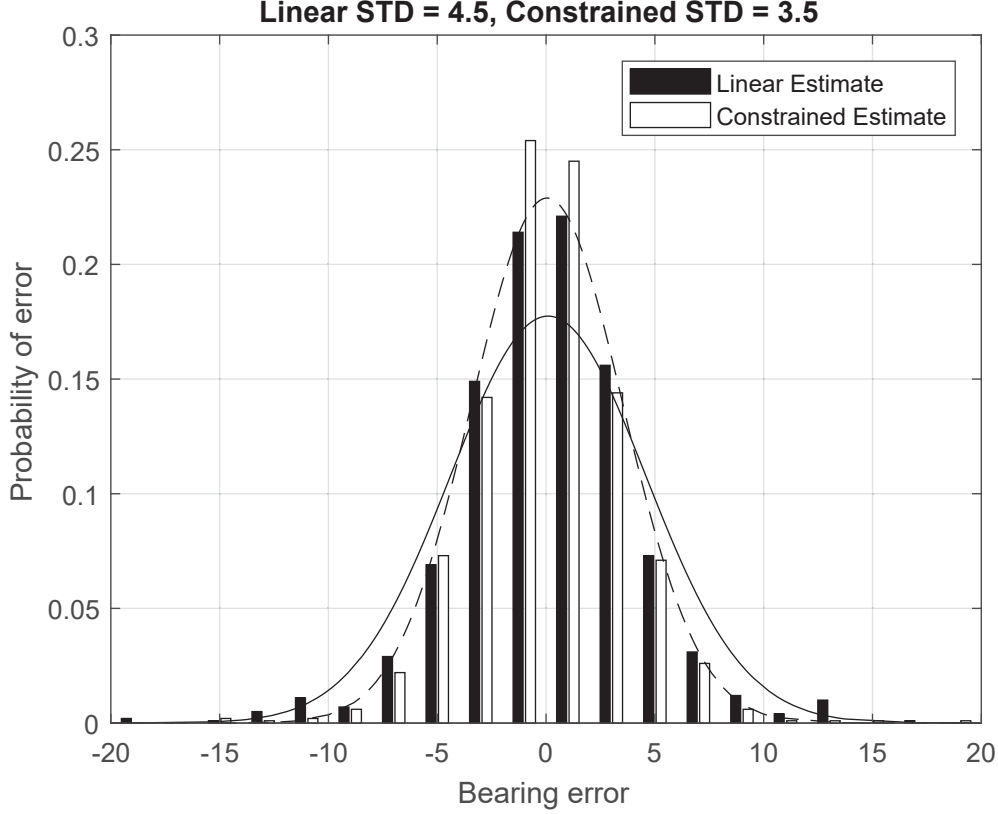


Fig. 4. Histogram of differences between true and measured bearing for linear and constrained estimates of emitter bearing. The curves are the corresponding normal distributions based on the measured mean and variance. The solid line is the linear estimate whereas the dashed line is the constrained estimate.

D. Outlier detection

Suppose that $\mathbf{u}$ is determined from N frequency measurements by solving Eq. (23), and then another measurement is made. How do we determine if this new measurement belongs to the same emitter? Again, we will use ideas of singular value decomposition to deal with this issue.

Suppose that, based on N measurements, we have concluded a target is in a stand-off situation. Suppose a further measurement is taken. Can we decide whether or not this measurement is consistent with the stand-off notion without solving the problem from the start using all $N + 1$ measurements, i.e. can we exploit what we have learnt with N measurements to simplify the task?

In linear algebra terms, we are asking under what circumstances the addition of a row to a matrix which is known to have a small singular value will cause the enlarged matrix to still have a small singular value.

Suppose that the initially given matrix $\bar{\mathbf{A}}$ is $N \times 6$ and has singular value decomposition as given by (33) in Appendix A, where now the diagonal matrix Σ and orthogonal matrix $\mathbf{Q}$ are 6×6 . Suppose also that the minimum singular value σ_6 is small relative to the other singular values. (In the two-dimensional case, $\bar{\mathbf{A}}$ is $N \times 3$ and the smallest singular value is σ_3 . We shall confine our discussion to the three-dimensional case in this section; changes

for the two-dimensional case are trivial). By the lemma in Appendix A, $\|\bar{\mathbf{A}}\mathbf{q}_6\|$ is small. Suppose an $(N+1)$ -th measurement is defined by a six-dimensional row vector

$$\bar{\mathbf{a}}_{N+1}^\top = [\bar{\mathbf{A}}_{N+1,1} \ \bar{\mathbf{A}}_{N+1,2} \ \dots \ \bar{\mathbf{A}}_{N+1,6}] \quad .$$

If $\bar{\mathbf{a}}_{N+1}^\top$ is an inlier we expect it to be confined to the space spanned by $\{\mathbf{q}_1^\top, \dots, \mathbf{q}_5^\top\}$, or equivalently we expect $\bar{\mathbf{a}}_{N+1}^\top$ to be orthogonal to $\mathbf{q}_6$, i.e. $\bar{\mathbf{a}}_{N+1}^\top \mathbf{q}_6 = 0$. In practice because the measurements are noisy and the emitter is not infinitely far away we expect the component of $\bar{\mathbf{a}}_{N+1}^\top$ parallel to $\mathbf{q}_6$ to be small, but non-zero, i.e.

$$\frac{|\bar{\mathbf{a}}_{N+1}^\top \mathbf{q}_6|}{\|\bar{\mathbf{a}}_{N+1}\|} \ll 1 \quad .$$

A good measure of acceptance of a new measurement is the percentage of $\bar{\mathbf{a}}_{N+1}$ perpendicular to $\mathbf{q}_6$:

$$\xi = 100 - 100 \times \frac{|\bar{\mathbf{a}}_{N+1}^\top \mathbf{q}_6|}{\|\bar{\mathbf{a}}_{N+1}\|} \quad . \quad (26)$$

We now consider a two-dimensional example. When the extra measurement comes from an emitter at a bearing of approximately 103° degrees different to the main emitter (bearings being measured of course at the receiver) the inlier score is $\xi = 8$, as illustrated in Fig. 5. By contrast if the bearing of the extra emitter is only 10° off that of the original emitter (as shown in Fig. 6) then the inlier score is $\xi = 91$.

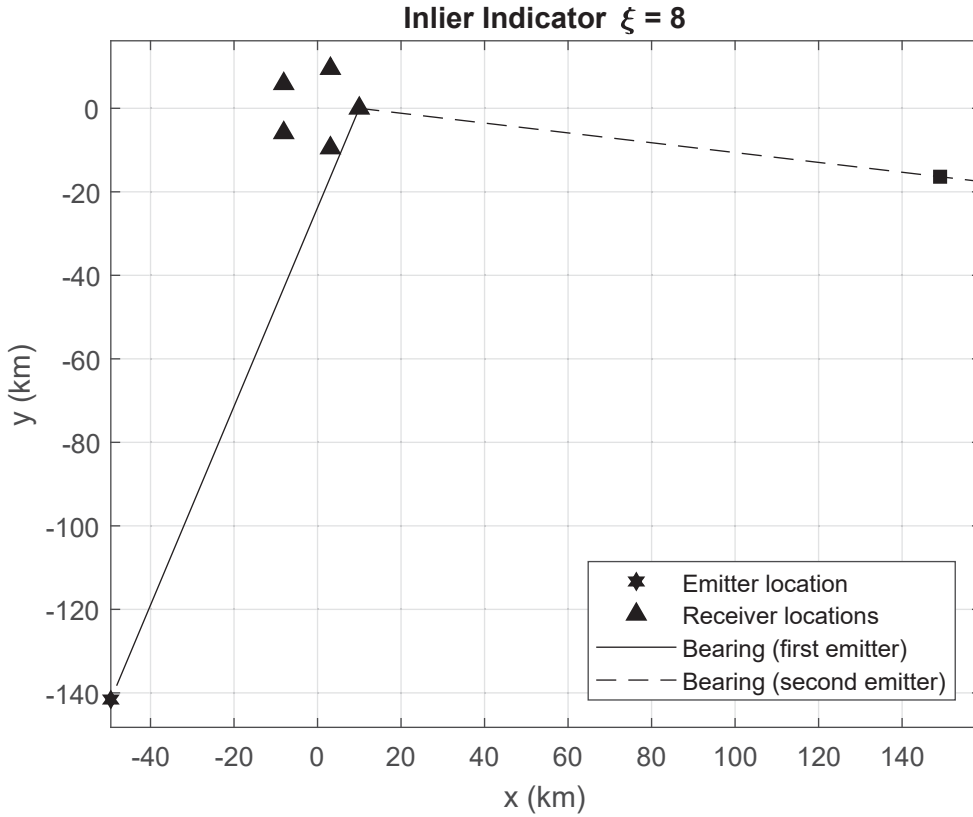


Fig. 5. An extra detection from emitter with bearing significantly different from the original emitter, giving rise to a low inlier score $\xi = 8$.

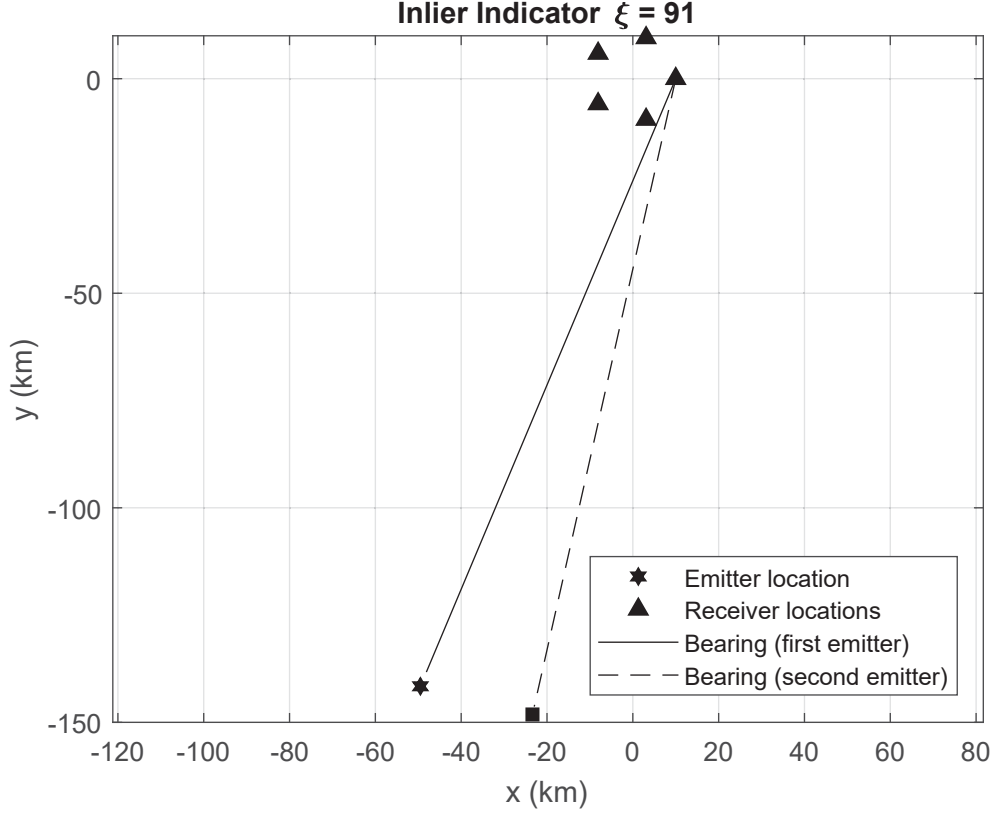


Fig. 6. An extra detection from an emitter with a bearing 10° different from that of the original emitter, giving a high inlier score $\xi = 91$.

IV. NOMINAL RATHER THAN PRECISE EMITTER FREQUENCY IS KNOWN

In this section, we shall assume that a *nominal* emitter frequency, f_0 is known to the receiver or receivers, while the true emitter frequency is $f_0(1 + \Delta)$, where Δ is a small fractional offset, and not known to the receivers.

In particular, we shall assume the validity of two approximations, namely, that Δ^2 and $\Delta(\frac{f_i}{f_0} - 1)$ are both negligible in comparison with Δ . In broad terms, the localization task is one of finding the emitter coordinates *and* Δ , or in a stand-off scenario the emitter bearing and Δ . Rather than give great detail, we shall simply summarize the key changes, which effectively make the equations more complicated.

The fundamental Doppler equation Eq. (1) is varied, to become

$$f_i = f_0 (1 + \Delta) \left(1 + \frac{\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i)}{c \|\mathbf{p} - \mathbf{r}_i\|} \right) . \quad (27)$$

The equation replacing Eq. (2) is

$$c^2 \left(\frac{f_i}{f_0(1 + \Delta)} - 1 \right)^2 \|\mathbf{p} - \mathbf{r}_i\|^2 = \left(\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i) \right)^2 . \quad (28)$$

Replacing $(1 + \Delta)^{-1}$ by $1 - \Delta$ yields

$$c^2 \left(\frac{f_i}{f_0}(1 - \Delta) - 1 \right)^2 \|\mathbf{p} - \mathbf{r}_i\|^2 \approx \left(\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i) \right)^2 .$$

Next, using the second approximation referred to above, replace $\frac{f_i}{f_0}\Delta$ by simply Δ . There results

$$c^2 \left(\left(\frac{f_i}{f_0} - 1 \right) - \Delta \right)^2 \|\mathbf{p} - \mathbf{r}_i\|^2 \approx \left(\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i) \right)^2$$

or using the Doppler parameter definition $\gamma_i = c \left(\frac{f_i}{f_0} - 1 \right)$, there results

$$c^2 (\gamma_i/c - \Delta)^2 \|\mathbf{p} - \mathbf{r}_i\|^2 \approx \left(\dot{\mathbf{r}}_i^\top (\mathbf{p} - \mathbf{r}_i) \right)^2. \quad (29)$$

From a series of such equations, one can construct a variation on the earlier linear equation, but involving a larger vector of unknowns. Previously, we had $\mathbf{A}\mathbf{w} = \mathbf{b}$ with

$$\mathbf{w} = [x^2 \ y^2 \ z^2 \ xy \ xz \ yz \ x \ y \ z]^\top.$$

It is necessary that the adjusted definition for $\mathbf{w}$ reflects the incorporation in Eq. (29) of $-2\gamma_i c \Delta \|\mathbf{p} - \mathbf{r}_i\|^2$ and of $c^2 \Delta^2 \|\mathbf{p} - \mathbf{r}_i\|^2$, with Δ an unknown. The upshot is that an additional ten entries must be included in the vector $\mathbf{w}$, viz. $\Delta(x^2 + y^2 + z^2)$, Δx , Δy , Δz , Δ , $\Delta^2(x^2 + y^2 + z^2)$, $\Delta^2 x$, $\Delta^2 y$, $\Delta^2 z$, Δ^2 . There are of course additional corresponding columns of $\mathbf{A}$, but the measurement vector $\mathbf{b}$ is unaltered.

The same rationale as previously for considering the occurrence of a stand-off scenario leads to the conclusion that a certain submatrix of $\mathbf{A}$ will have an approximately zero singular value, viz. that associated with those entries of $\mathbf{w}$ which depend quadratically on x, y, z . These are the first six columns of the $\bar{\mathbf{A}}$ matrix as previously, and the additional two columns corresponding to the new entries of $\mathbf{w}$ given by $\Delta(x^2 + y^2 + z^2)$ and $\Delta^2(x^2 + y^2 + z^2)$, this being a total of eight columns in all.

When an emitter is in a stand-off scenario, but the exact emitter frequency is unknown, the estimation of direction is straightforwardly treated. We replace Eq. (20) by

$$f_i = f_0 (1 + \Delta) \left(1 + \frac{\dot{\mathbf{r}}_i^\top \mathbf{u}}{c} \right) \approx f_0 \left(1 + \frac{\dot{\mathbf{r}}_i^\top \mathbf{u}}{c} + \Delta \right) \quad (30)$$

(Here we have neglected the second order term $\Delta(\dot{\mathbf{r}}_i^\top \mathbf{u})$ in comparison with the first order terms $\dot{\mathbf{r}}_i^\top \mathbf{u}$ and Δ .) We define a modification $\mathbf{B}_\Delta$ of the matrix $\mathbf{B}$ used earlier by

$$\mathbf{B}_\Delta = \begin{bmatrix} \dot{\mathbf{r}}_1^\top & c \\ \dot{\mathbf{r}}_2^\top & c \\ \vdots & \vdots \\ \dot{\mathbf{r}}_N^\top & c \end{bmatrix}. \quad (31)$$

The vector $\mathbf{g}$ remains as before, viz $[\gamma_1 \ \gamma_2 \ \dots \ \gamma_n]^\top$. The least-squares solution to the equation set Eq. (30) is

$$\begin{bmatrix} \mathbf{u} \\ \Delta \end{bmatrix} = \left(\mathbf{B}_\Delta^\top \mathbf{B}_\Delta \right)^{-1} \mathbf{B}_\Delta^\top \mathbf{g}. \quad (32)$$

The results for a particular example are shown in Fig. 7.

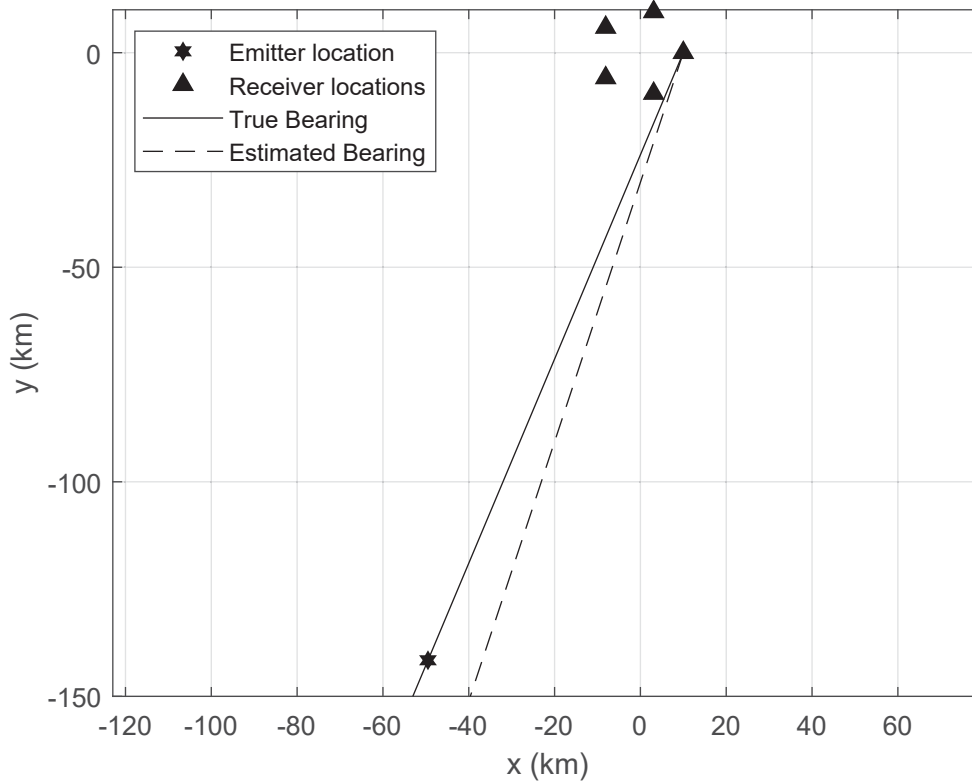


Fig. 7. True emitter frequency differs by -60 Hz from the nominal value.

V. CONCLUSIONS

This paper considers methods for determining the direction of arrival of an emitter signal based only the measured frequency of the emitter signal at the sensors and the navigation data (positions and velocities of the sensors). The methods are applicable in a two- or three-dimensional ambient space. In solving the relevant equations it is crucial to determine that two conditions are met: the emitter is at a “stand-off” distance, and that each measurement is associated with a unique emitter. To test these two conditions we create a “stand-off” indicator and an “inlier” indicator. As the values of these indicators approach 100 percent we can be confident that the conditions are met to produce a reliable estimate of the emitter’s direction based on the frequency and navigation data measurements.

In the initial sections of the paper we assumed that the emitter frequency was either known or could be measured precisely. In the last section we relaxed this condition and showed that if the emitter’s transmitter frequency was nominally, but not precisely, known then the emitter’s direction could still be estimated to a good degree of accuracy.

REFERENCES

- [1] R. Poisel, *Electronic Warfare Target Location Methods*. Artech House, 2005.
- [2] W. H. Guier and G. C. Weiffenbach, "The doppler determination of orbits," in *National Aeronautics and Space Administration Conference on Orbit and Space Trajectory Determination*, 1959, Conference Proceedings.
- [3] R. H. Graham, *Flying the SR-71 : in the Cockpit on a Secret Operational Mission*. St. Paul, MN: Zenith/MBI, 2008.
- [4] P. J. Fell, "Transit and GPS – a report on geodetic positioning activities," *Bulletin Geodesique*, vol. 60, no. 3, pp. 181–192, 1986.
- [5] L. Marsh, D. Gossink, S. P. Drake, and G. Calbert, "UAV team formation for emitter geolocation," in *2007 Information, Decision and Control*, 2007, Conference Proceedings, pp. 176–181.
- [6] D. J. Torrieri, "Statistical-theory of passive location systems," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 20, no. 2, pp. 183–198, 1984.
- [7] B. H. Lee, Y. T. Chan, F. Chan, H. J. Du, and F. A. Dilkes, "Doppler frequency geolocation of uncooperative radars," in *MILCOM 2007 - IEEE Military Communications Conference*, 2007, Conference Proceedings, pp. 1–6.
- [8] H. Witzgall, B. Pinney, and M. Tinston, "Single platform passive doppler geolocation with unknown emitter frequency," in *2010 IEEE Aerospace Conference*, 2010, Conference Proceedings, pp. 1–8.
- [9] N. H. Nguyen and K. Doğançay, "Algebraic solution for stationary emitter geolocation by a LEO satellite using doppler frequency measurements," in *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2016, Conference Proceedings, pp. 3341–3345.

APPENDIX

A. Singular value decomposition and measurement association

Let $\bar{\mathbf{A}} \in \mathbb{R}^{N \times K}$ be a matrix with more rows than columns. (In our application, K will take the value 3 or 6 according as we treat an ambient two- or three-dimensional environment, while as before, N is the number of measurements). The singular value decomposition of $\bar{\mathbf{A}}$ is of the form

$$\bar{\mathbf{A}} = \mathbf{P} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \mathbf{Q}^\top, \quad (33)$$

where $\mathbf{P}, \mathbf{Q}$ are real orthogonal matrices of size $N \times N$ and $K \times K$ respectively, and Σ is a square diagonal matrix $\text{diag}[\sigma_1, \sigma_2, \dots, \sigma_K]$ with non-negative diagonal entries decreasing down the diagonal. These are the singular values of $\bar{\mathbf{A}}$.

It is standard that for any vector $\mathbf{x}$, there holds (with the norm sign as earlier denoting a Euclidean norm)

$$\|\bar{\mathbf{A}}\mathbf{x}\| \leq \sigma_1(\bar{\mathbf{A}})\|\mathbf{x}\|. \quad (34)$$

Slightly less well known is the following inequality involving the smallest singular value:

$$\|\bar{\mathbf{A}}\mathbf{x}\| \geq \sigma_K(\bar{\mathbf{A}})\|\mathbf{x}\|. \quad (35)$$

We shall also have occasion to use a refinement of the last inequality, as follows:

Lemma 1. Let $\mathbf{q}_K$ denote the the last column of $\mathbf{Q}$. Then $\|\bar{\mathbf{A}}\mathbf{q}_K\| = \sigma_K$, i.e. $\mathbf{q}_K$ is the solution of the problem of minimizing $\|\bar{\mathbf{A}}\mathbf{x}\|$ over $\|\mathbf{x}\| = 1$.

Proof: The definition of $\mathbf{q}_K$ and the orthogonality of $\mathbf{Q}$ ensures that $\mathbf{Q}^\top \mathbf{q}_K = [0, 0, \dots, 0, 1]^\top$ and so

$$\bar{\mathbf{A}}\mathbf{q}_K = \mathbf{P} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \mathbf{Q}^\top \mathbf{q}_K = \mathbf{P} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_K \\ 0 \\ \vdots \end{bmatrix}$$

Since $\mathbf{P}$ is orthogonal, it is immediate that $\|\bar{\mathbf{A}}\mathbf{q}_K\| = \sigma_K$. However, since the singular values of $\bar{\mathbf{A}}$ are precisely the eigenvalues of $\bar{\mathbf{A}}^\top \bar{\mathbf{A}}$, the minimum of $\mathbf{x}^\top \bar{\mathbf{A}}^\top \bar{\mathbf{A}} \mathbf{x}$ over $\|\mathbf{x}\| = 1$ is σ_K^2 , and so Eq. (35) is verified $\square$.

1 B. Bounds on the magnitude of μ

The squared magnitude μ (given by (12) with the subscripts removed) is

$$\|\mu\|^2 = \cos^4 \theta (\cos^4 \phi + \sin^4 \phi) + \sin^4 \theta + \cos^4 \theta \cos^2 \phi \sin^2 \phi + \cos^2 \theta \sin^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \theta \sin^2 \phi \quad .$$

Noting that $\cos^4 \phi + \sin^4 \phi = 1 - 2 \cos^2 \phi \sin^2 \phi$ and using the Pythagorean identity $\cos^2 \phi + \sin^2 \phi = 1$ we note that the squared magnitude reduces to

$$\|\mu\|^2 = \cos^4 \theta (1 - \cos^2 \phi \sin^2 \phi) + \sin^2 \theta \quad .$$

This is clearly a maximum when $\cos \phi \sin \phi = 0$. When this occurs $\|\mu\|^2 \leq \cos^4 \theta + \sin^2 \theta$ which reduces to

$$\|\mu\|^2 \leq 1 - \cos^2 \theta \sin^2 \theta \leq 1 \quad ,$$

2 where once again we have made use of the Pythagorean identity.

To determine the minimum value of $\|\mu\|^2$ we note that

$$1 - \cos^2 \phi \sin^2 \phi = 1 - \frac{1}{4} (\sin 2\phi)^2 \geq 3/4 \quad ,$$

hence

$$\|\mu\|_{\min}^2 = \frac{3}{4} \cos^4 \theta + \sin^2 \theta \quad .$$

This is an extremum when the derivative is zero, i.e.

$$\frac{d\|\mu\|_{\min}^2}{d\theta} = \sin \theta \cos \theta (2 - 3 \cos^2 \theta) = 0 \quad .$$

This equation has three solutions $\{\sin \theta = 0, \cos \theta = 0, \cos^2 \theta = 2/3\}$ and the minimum occurs when $\cos^2 \theta = 2/3$ and the corresponding value of $\|\mu\|_{\min} = \sqrt{2/3}$. Combining this result with the one given above we determine

$$\sqrt{2/3} \leq \|\mu\| \leq 1 \quad .$$