Solidarity and Ergodic Properties

of

Semi-Markov Transition Probabilities

by

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I, Cheong Choong Kong, declare that this thesis is my own original work save and insofar as where I have expressly acknowledged my debt to other authors.

Cheong Choong Kong.

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ABSTRACT

The chief purpose of this thesis is to establish for semi-Markov processes the same type of behaviour that is characteristic of the better-known Markov chains; this is achieved mainly through the use of Laplace and Laplace-Stieltjes transforms and a frequent appeal to renewal theory. The mathematical tools needed for the task are developed in the first chapter. In the second chapter the solidarity nature of geometric ergodicity within an irreducible class is examined, and necessary and sufficient conditions are derived for geometric ergodicity in the particular case of a process with a finite state space. In chapter three it is shown that the Laplace transforms of the transition probabilities pertaining to an irreducible class all have the same abscissa of convergence, a fact that permits the definition of α-recurrence and leads to a result for α-recurrent processes that generalizes the familiar ergodic theorem of Markov chain theory; quasi-stationary distributions are also studied in the same chapter. Chapter four is devoted to some general ratio-limit theorems involving a parameter λ (λ equals zero in the usual ratio-limit theorems), and the last chapter applies the results obtained in the earlier part of the thesis to the study of an inventory model and a continuous-time Markov branching process.

Preface

This thesis is the culmination of my three years of research in the Department of Statistics of the Institute of Advanced Studies, Australian National University. The gist of chapter two appears in my paper (reference [3]) in Zeitschrift für Wahrscheinlichkeitstheorie; the greater portion of chapters three and four has been accepted for publication in the same journal (see [4]), and the last sections of chapters one, three and five are now being put into paper form with a view to publication in the very near future.

Not all the research that I did between March 1965 and January 1968 is accounted for in this thesis. Approximately twelve months were spent in collaboration with Dr. D. Vere-Jones in the investigation of Markov processes defined on a topological (more specifically, a locally compact) state space and having continuous transition probabilities. Although our efforts in this direction were fairly fruitful, the subject matter bears too tenuous a relationship to the theme of this thesis to justify its inclusion here. Dr. Vere-Jones and I hope eventually to publish a joint paper based on our work.

I should like to thank Dr. Vere-Jones, my supervisor, for his patient guidance, and his many invaluable comments and suggestions; and Professor P.A.P. Moran for his advice and encouragement. I should like also to express my thanks and gratitude to the Australian Government, whose generous Colombo Plan scholarship gave me the opportunity of extending my education beyond the confines of the strictly academic, and contributed to making my eight years' sojourn in this country the pleasant and highly rewarding one that it has been.

C.K. Cheong.

Australian National University,
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CHAPTER I

INTRODUCTION

§1. Definitions and notation.

Semi-Markov processes were first defined and studied by Levy [26], [27] and Smith [37] in 1954, and a similar type of process was introduced in the same year in a paper by Takács [43]. The theory of such processes has since been extensively developed, mainly by Pyke and Schaufele. In 1961 Pyke [30], [31] gave a definition of a semi-Markov process and studied it in the special case of a finite state space; included in the first paper was a bibliography of works on the subject that have been published between 1954 and 1961. The definition of Pyke in [30] was a 'constructive' one which did not admit instantaneous states and defined the process only up to its first explosion; a more general definition allowing both instantaneous states and explosions was given by Pyke and Schaufele [32] in 1964. A general definition has also been given by Feller [15]. Roughly, a semi-Markov process is a generalization of a Markov process and differs from it in that the holding time in a given state has an arbitrary distribution and is not necessarily independent of the state immediately following the next transition.

Let us write I^+ for the state space of the process, and denote by Z_t both the process and the state of the process at time t; it is unlikely that ambiguity will arise over the meaning of Z_t in a given context. Before we proceed to describe the process, we make the following definitions. $U_t = t - \sup\{u < t : Z_u \neq Z_t\}$, $V_t = \inf\{u > t : Z_u \neq Z_t\} - t$, $X_t = U_t + V_t$ and $Z_t^+ = Z_{t+V_t}$.

The interpretation of the various quantities is as follows: U_t is the time that has expired since the last transition before time t, V_t the remaining time to the next transition, X_t the length of time spent in Z_t, and Z_t^+ the state immediately following the next transition. Let $N_i(t) = \text{card}\{0 < u \leq t : Z_{u-} \neq Z_u = i\}$ and $N_t = \sum_{i \varepsilon I^+} N_i(t)$; these are respectively the number of transitions into state i, and the total number of transitions, in $(0,t]$. (Pyke called the process $\{N_i(t) : i \varepsilon I^+, t \geq 0\}$ a Markov Renewal Process.) Z_t is said to be regular if $\Pr[N_i(t) < \infty] = 1$ for all $i \varepsilon I^+$ and all $t \geq 0$, and strongly regular if $\Pr[N(t) < \infty] = 1$ for all $t \geq 0$. A regular process obviously need not be strongly regular (see, for example, remark 2 of section 4, [33]). We observe that in [30], strong regularity was called regularity.

The process that we shall study in this thesis is the process defined by Pyke and Schaufele [32]; our notation is consistent with theirs, and we refer the reader to their paper for the complete definition and detailed description of the process. We shall assume that Z_t satisfies hypothesis A of [32]; for the sake of completeness, we note below the salient points of this hypothesis.

Hypothesis A. Z_t is a separable stochastic process defined on a complete probability space having a denumerable state space I^+, compactified by the addition of ∞. Almost all sample functions are right continuous, have left limits on $[0,\infty)$, and are such that if $Z_t \to \infty$ as $t \to \tau$ from one side, then $Z_t \to \infty$ as $t \to \tau$ from both sides. The stochastic process $\{(Z_t,U_t) : t \geq 0\}$ is a Markov process having the strong Markov property for all stopping times which are almost surely finite. Evolution of Z_t is determined by a matrix of one-step transition probabilities Q_{ij} which satisfy

$\sum_{j\varepsilon I^+} Q_{ij}(\infty) = 1$, $\sum_{j\varepsilon I^+} Q_{ij}(0) < 1$, and

$Q_{ij}(x) = \Pr[Z_t^+ = j, X_t \leq x | Z_t = i, U_t = 0,(Z_v,U_v);\ 0 \leq v < t]$. The transition probabilities, $P_{ij}(t) = \Pr[Z_t = j | Z_0 = i, U_0 = 0]$, are related to the Q_{ij}'s by the equation

$$P_{ij}(t) = \delta_{ij}\{1-H_j(t)\} + \sum_{k\varepsilon I^+} \int_0^t P_{kj}(t-u)dQ_{ik}(u) \qquad (1.1),$$

where $H_i(t) = \sum_{k\varepsilon E} Q_{ik}(t)$ is the unconditional distribution of the holding time in state i. H_i is clearly an honest distribution function.

The strong Markov property assumed in hypothesis A is an important one for it enables us later to establish the transition identities that are used throughout the thesis. Other important implications of the hypothesis are that the process is regular and every state in I^+ is stable (stable and instantaneous states were defined in [32]). A significant feature of Pyke's definition is that it includes all denumerable Markov processes; on account of this, the results derived in this thesis for a general semi-Markov process quite often yield, as special cases, new results in the theory of denumerable, continuous-time Markov processes.

In this thesis we are mainly concerned with the properties of quantities connected with the transition probabilities $P_{ij}(t)$. The behaviour of $P_{ij}(t)$ is closely associated with that of the first passage distributions

$$G_{ij}(t) = \Pr[N_j(t) > 0 | Z_0 = i, U_0 = 0]$$

and the means

$$M_{ij}(t) = E[N_j(t) | Z_0 = i, U_0 = 0] + \delta_{ij}.$$

Playing a vital role in linking these are the 'taboo' quantities defined below.

Let D be a sub-set of I^+, possibly empty but strictly contained in I^+; then

$${}_DP_{ij}(t) = \Pr[Z_t = j;\ N_k(t) = 0 \text{ for all } k \in D | Z_0 = i, U_0 = 0],$$

$${}_DG_{ij}(t) = \Pr[\text{for some } u \leq t,\ N_j(u) > 0,\ N_k(u) = 0 \text{ for all } k \in D | Z_0 = i, U_0 = 0],$$

and

$${}_DM_{ij}(t) = \{1 - \delta_{jD}\}E[N_j(s_k) | Z_0 = i, U_0 = 0] + \delta_{ij},$$

where $\delta_{jD} = 1$ if $j \in D$ and zero otherwise, $s_k = \min(t,t_k)$ and $t_k = \inf\{t > V_0 : Z_t \in D\}$.

Note that in all cases the 'taboo' is imposed only after the first exit from the initial state. Clearly, if $j \in D$, ${}_DP_{ij}(t) = \delta_{ij}\{1-H_j(t)\}$.

The expectations of H_j and G_{ij} will be denoted by η_j and μ_{ij} respectively i.e. $\eta_j = \int_0^\infty\{1-H_j(t)\}dt$ and $\mu_{ij} = \int_0^\infty\{1-G_{ij}(t)\}dt$.

States i and j are said to communicate if either $i = j$ or $G_{ij}(\infty)G_{ji}(\infty) > 0$. Communication is an equivalence relation, and a set of states is said to be irreducible if it is an equivalence class under this relation. Z_t is said to be irreducible if there is only one irreducible class.

We shall write $U_c(t)$ for the function which is one for $t \geq c$ and zero for $t < c$, and because of the frequent appearance of transforms and convolutions, we shall use the following abbreviations:

$$A^*(s) = \int_{0-}^{\infty} e^{-st}dA(t), \qquad A^o(s) = \int_{0-}^{\infty} e^{-st}A(t)dt,$$

$$(1-A)^o(s) = \int_{0-}^{\infty} e^{-st}\{1-A(t)\}dt, \qquad A^+(s) = \int_{0-}^{\infty} te^{-st}dA(t)$$

and

$$A * B(t) = \int_{0-}^{t} B(t-u)dA(u).$$

Important results in the thesis will be numbered according to the chapters in which they occur; thus theorem (equation) 2.3 will refer to the third theorem (equation) of chapter two.

§2. Structure of thesis and its relationship to relevant literature.

Chapter I: In §3 we list the main identities governing transitions between states. These are merely extensions of familiar results in the theory of Markov processes (see [5] for example), and most of them have already been quoted and used by Pyke [31], and Pyke and Schaufele [32]. The appearance of convolutions in all the identities makes the problems posed in this thesis specially amenable to solution by the method of Laplace and Laplace-Stieltjes transforms. It may in fact be said that the elementary lemmas on transforms and convolutions in §4 are the basic mathematical tools used in this thesis, and we therefore feel justified in devoting an entire section to them, their elementary nature notwithstanding. Lemmas 1.1 - 1.4 are due to Widder [47], and the rest, though apparently new, are easily proved by imitating Widder's methods.

The result in §5 gives us a simple criterion for the validity of the key-renewal theorem (for a discussion on the key-renewal theorem, see [38]). Let A be a renewal function on $[0,\infty)$ i.e. $A(t) = \sum_{n=0}^{\infty} B^{(n)}(t)$, where $B^{(0)}(t) = U_0(t)$, $B^{(1)}(t)$ is some distribution function (possibly dishonest), and $B^{(n)}(t)$ is the n-fold convolution of $B^{(1)}(t)$ with itself $(n > 1)$. If $B^{(1)}(t)$ is non-lattice, $\nu = \int_0^{\infty}\{1-B^{(1)}(t)\}dt$ and ν^{-1} is interpreted as zero if ν is infinite, then the key-renewal theorem states that $\lim_{t\to\infty} \int_0^t g(t-u)dR(u) = \nu^{-1} \int_0^\infty g(u)du$ provided g satisfies certain conditions. The conditions stipulated by Smith in his 1954 paper [36] are too stringent and exclude a large class of functions. A later set of conditions of his (1961, [40] : page 469), and Feller's condition of direct Riemann integrability (see [16]), are more easily satisfied; in fact, lemma 1.8 of §5 asserts that all functions which are Lebesgue integrable and of bounded total variation in $[0,\infty)$ are directly Riemann integrable. This result is used at a crucial stage in the proof of the main ergodic theorem of Chapter III where, in the case of an α-null recurrent process, the relevant function satisfies none of Smith's earlier conditions.

Chapter II: We restrict our attention in this chapter to an irreducible class i.e. an equivalence class of mutually communicating states. By a solidarity property (of an irreducible class E) we shall mean a property possessed by either every state in E, or none. Well-known examples of such properties are those of recurrence and positive recurrence in the theory of Markov processes.

In §1 we assume the existence of $\lim_{t\to\infty} P_{ij}(t)$ and denote it by Π_{ij}. (The existence of the limit when j is stable and $\eta_j < \infty$ was proved by Smith [37], who showed that $\Pi_{ij} = G_{ij}(\infty)\eta_j\mu_{jj}^{-1}$ if $\mu_{jj} < \infty$ and $\Pi_{ij} = 0$ if $\mu_{jj} = \infty$ or Z_t is transient. Recently in 1966, Yackel [48] proved the existence of the limit when j is an instantaneous state, and discussed the case of a process in which $\eta_j = \infty$ for every j in I^+.) A transition $i \to j$ is said to be geometrically ergodic if there exists a positive λ_{ij} such that $P_{ij}(t) - \Pi_{ij} = o(e^{-\lambda_{ij}t})$ as $t \to \infty$. In 1959 Kendall [19] proved the following solidarity theorem for irreducible, discrete-time Markov chains: if for some i, the transition $i \to i$ is geometrically ergodic, then for all j,k in I^+, the transition $j \to k$ is geometrically ergodic. This was improved upon by Vere-Jones [45] in 1962: Vere-Jones showed that there exists a parameter of convergence $\lambda > 0$ which is common to all transitions $i \to j$; in the case of a transient chain, a best possible parameter is obtainable. In [20], Kingman extended Vere-Jones' results to continuous-time, transient Markov processes; and finally, in [21], to both transient and recurrent processes by applying Vere-Jones' results to the discrete skeleton of the continuous-time process. We ask in the first section of this chapter if the results of Kingman would continue to hold for a general semi-Markov process. An unqualified, affirmative answer is obtained if the process is transient; if the process is recurrent, the solidarity theorem holds provided certain regularity conditions are imposed on the holding time distributions $H_i(t)$. These conditions are trivially satisfied if the process is Markov, and the holding time distributions consequently negative exponential. We are aided in the proof of our theorem by results of

Leadbetter [25] and Stone [42]. We conclude the section by establishing a necessary condition for geometric ergodicity.

In §2 we persist with the assumption of irreducibility but consider only irreducible classes with a finite number of states. It has been shown by Pyke ([30] : lemma 4.1) that when the state space is finite, the process is necessarily strongly regular. There is thus only a finite number of transitions in any finite interval of time and we can, when considering transitions between two given states in a finite time, enumerate all possible paths of the process. This strong regularity is made use of in this section to establish necessary and sufficient conditions for the finite process to be geometrically ergodic.

<u>Chapter III</u>. We consider again an irreducible class E. In §1 and §2 we show that the transforms $P_{ij}{}^{o}(s)$ and $G_{ij}{}^{+}(s)$ have abscissae of convergence that are common to all transitions, $i \to j$, within E. Denoting the common abscissa of convergence of $P_{ij}{}^{o}(s)$ by $-\alpha$ and calling α the convergence parameter of E, we are able to define α-transience, α-positive recurrence and α-null recurrence $(\alpha \geq 0)$, and demonstrate that each is a solidarity property. Our theorems yield as special cases results of Kingman [20] on continuous-time Markov processes and Vere-Jones [45] on discrete-time Markov chains. When $\alpha = 0$, our results show that ordinary transience, recurrence and positive recurrence are solidarity properties; this result is not

entirely new, for the solidarity property of ordinary recurrence follows immediately from theorem 5.1 of Pyke [30].

In §3 we prove the existence of, and evaluate, the limits as $t \to \infty$ of $e^{\alpha t}P_{ij}(t)$ $(i,j \in E)$ under the following conditions: $\alpha > 0$, or $\alpha = 0$ and $\eta_j < \infty$; $e^{\alpha t}\{1-H_j(t)\}$ is of bounded, total variation in $[0,\infty)$; and at least one non-zero $Q_{k\ell}(t)$ $(k,\ell \in E)$ is non-lattice. The last condition eliminates periodicity, and the regularity condition on $H_j(t)$ prevents extremely erratic behaviour of the function $e^{\alpha t}P_{ij}(t)$. If Z_t is Markovian (whence $H_j(t) = 1 - e^{-\frac{t}{\eta_j}}$), all the conditions are trivially satisfied; if Z_t is semi-Markov and $\alpha = 0$, then $e^{\alpha t}\{1-H_j(t)\}$ has total variation in $[0,\infty)$ equal to one. Our theorem thus generalizes the results of Vere-Jones [45] and Kingman [20] for Markov processes, and the result of Smith ([37] : theorem 5) for semi-Markov processes with $\alpha = 0$ and $\eta_j < \infty$. In an analogous manner to the Markovian case, the limits L_{ij} of the quantities $e^{\alpha t}P_{ij}(t)$ are positive if E is α-positive recurrent, and zero otherwise.

When E is α-positive recurrent, we cannot extend the results of Vere-Jones and Kingman further by expressing as they did the limits L_{ij} as the product $x_i m_j$, where x_i and m_j are respectively the unique α-invariant vector and α-invariant measure. Even in the case $\alpha = 0$ an invariant measure does not exist in general, and an example will be produced in §4 to show this. However, when Z_t is a Markov process, a little more may be added to known results. For a continuous-time Markov process, Kingman [20] has proved the existence of a unique α-invariant vector x_i, and a unique α-invariant measure m_j, satisfying respectively the two sets of equations

$$\sum_{k\varepsilon E} P_{ik}(t)x_k = e^{-\alpha t}x_i ,$$

$$\sum_{k\varepsilon E} m_k P_{kj}(t) = e^{-\alpha t}m_j, \quad \text{for all } i,j \varepsilon E, \quad t \geq 0.$$

He showed furthermore that E is α-positive recurrent if and only if $\sum_{k\varepsilon E} m_k x_k$ converges, and then $L_{ij} = \dfrac{x_i m_j}{\sum_{k\varepsilon E} x_k m_k}$. In §4 we complement Kingman's results by proving that the unique α-invariant vector and measure are respectively given by $x_i = G_{i1}^*(-\alpha)$ and $m_j = {}_1P_{1j}^{\,o}(-\alpha)$, where 1 is some fixed state in E; moreover, $L_{ij} = (q_1-\alpha)L_{11}x_i m_j$ and $\sum_{k\varepsilon E} L_{kk} = 1$, where $L_{11} = x_1 m_1 [\sum_{k\varepsilon E} x_k m_k]^{-1}$ and q_1^{-1} is the mean of the holding time in state 1.

We consider in §5 the sum $R_j(\lambda,t) = \sum_{k\varepsilon E} u_k P_{kj}(t)e^{\lambda t}$ and $S_i(\lambda,t) = \sum_{k\varepsilon E} P_{ik}(t)e^{\lambda t}v_k$; and establish conditions under which they are finite for every t, and converge to finite limits as $t \to \infty$. Our results yield as a special case a theorem of Vere-Jones ([46] : theorem 6.1) for discrete-time Markov chains. The result concerning the convergence of $S_i(\lambda,t)$ to a finite limit, as $t \to \infty$, is used at the end of the section to study the existence of 'quasi-stationary distributions'; these are the limits (when they exist) of the quantities

$$V_{ij}(t) = \frac{P_{ij}(t)}{\sum_{k\varepsilon E} P_{ik}(t)} .$$

$V_{ij}(t)$ is of particular interest when the irreducible class E is not closed i.e. there is a positive probability of a transition from some state in E to some state outside E. When this is so, $\sum_{k\varepsilon E} Q_{ik}(\infty) < 1$ for some i in E, and E is easily shown to be transient (once the process escapes from E, it cannot return since E is irreducible); $V_{ij}(t)$ may then be interpreted as the probability that the process goes from i to j in time t, conditional on its being in E up to time t. Let $1 - r_i = \lim_{t\to\infty} \sum_{k\varepsilon E} P_{ik}(t)$ $(i \varepsilon E)$ be the probability that the process starting from i remains forever in E. If $1 - r_i > 0$, $V_{ij}(t) \to 0$; we then study the more interesting quantity

$$W_{ij}(t) = \frac{P_{ij}(t)r_j}{\sum_{k\varepsilon E} P_{ik}(t)r_k},$$

whose interpretation is the probability that at time t the process is in state j, given that it started from i, was in E till time t, and will eventually escape from E. We shall derive sufficient conditions under which $W_{ij}(t)$ tends to an honest distribution (over E) as $t \to \infty$.

For finite, discrete-time Markov chains, quasi-stationary distributions were first studied by Seneta and Darroch [9] in 1965, and a discussion of the continuous time case was given by the two authors in [10]; in 1966 Seneta and Vere-Jones [34], using the concept of α-positive recurrence, obtained results for the denumerable, discrete-time Markov chain. A portion of the results of Seneta and Vere-Jones emerge as a special case of our theorem 3.8 (for general semi-Markov processes) and corollary 3.5 (for continuous-time Markov processes).

Chapter IV: We dispense with the assumption of irreducibility in this chapter. Let $\overline{P}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} P_{ij}(x)dx$ and $\overline{M}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} dM_{ij}(x)$. After some preliminary lemmas in §1, we proceed in §2 to evaluate, under certain conditions, the limits of the ratios

$$\frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{k\ell}(\lambda,t)} \quad \text{and} \quad \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{k\ell}(\lambda,t)}, \quad i,j,k,\ell \in I^+.$$

Results for the case $\lambda = 0$ have already been proved by Pyke and Schaufele [32], and it is in extending their results to cases $\lambda \geq 0$ that the usefulness of the elementary lemma in Chapter I viz lemma 1.5, is exemplified. In the case of an irreducible class E with convergence parameter α, our results yield the limits of

$$\frac{\overline{P}_{ij}(\alpha,t)}{\overline{P}_{k\ell}(\alpha,t)} \quad \text{and} \quad \frac{\overline{M}_{ij}(\alpha,t)}{\overline{M}_{k\ell}(\alpha,t)},$$

and generalize a result of Vere-Jones [45] for discrete-time Markov chains.

Chapter V: In §1 we use the analytical technique and results of earlier chapters to study an inventory model in which the restocking policy is a special case of the familiar (S,s) type, while in §2 we apply corollary 3.5 to the problem of deriving quasi-stationary distributions for continuous-time Markov branching processes.

§3. Transition Identities.

The identities in this section are straightforward generalizations of well-known identities in the theory of Markov processes. Most of them have been quoted, usually without proof, and used in the paper by Pyke and Schaufele [32].

Let $G_{jj}{}^{o}(t) = U_0(t)$, $G_{jj}{}^{(1)}(t) = G_{jj}(t)$ and, for $n > 1$, $G_{jj}{}^{(n)}(t)$ = the n-fold Laplace-Stieltjes convolution of $G_{jj}(t)$ with itself. For $i \neq j$, define $G_{ij}{}^{(1)}(t) = G_{ij}(t)$ and $G_{ij}{}^{(n)}(t) = G_{ij}{*}G_{jj}{}^{(n-1)}(t)$, $(n > 1)$. The iterates of the taboo distributions, ${}_kG_{ij}{}^{(n)}(t)$, are similarly defined by replacing in the above definitions $G_{jj}{}^{(0)}(t)$ with ${}_kG_{jj}{}^{(0)}(t)$ and $G_{ij}{}^{(n)}(t)$ with ${}_kG_{ij}{}^{(n)}(t)$.

An attractive feature of semi-Markov processes, which makes its theory so tractable, is that its points of transition are regeneration points which erase all memory of the past. Moreover, the points of entry into a given fixed state are points of regeneration of a recurrent (or regenerative) event. (See Smith [37] for a definition and description of a regenerative process. A recent, thorough development of the theory of regenerative events is given by Kingman in [22], [23] and [24].) $M_{jj}(t)$ is therefore the renewal function associated with the recurrent event "Z_t enters state j". The length of the time interval between two successive occurrences of this event has the distribution $G_{jj}(t)$, so

$$M_{jj}(t) = \sum_{n=0}^{\infty} G_{jj}^{(n)}(t),$$

and

$$M_{jj}(t) = 1 + G_{jj}*M_{jj}(t) \qquad (1.2).$$

A similar renewal theory type of argument yields

$${}_kM_{jj}(t) = 1 + {}_kG_{jj}*{}_kM_{jj}(t), \quad k \neq j \qquad (1.2a).$$

When $i \neq j$, $M_{ij}(t)$ is the renewal function associated with a 'delayed' recurrent event (see, for instance [14] : chapter XIII), so we have $M_{ij}(t) = \sum_{n=1}^{\infty} G_{ij}^{(n)}(t)$, and

$$M_{ij}(t) = M_{jj}*G_{ij}(t) \qquad (1.3).$$

Similarly, $${}_kM_{ij}(t) = {}_kM_{jj}*{}_kG_{ij}(t) \qquad (1.3a).$$

Let $\rho = \inf\{t : t \geq 0, Z_t \neq Z_0\}$ and $\theta_j = \inf\{t : t > \rho, Z_t = j\}$. These are random variables representing respectively the time spent in the initial state and the first entrance time into state j. Let $\hat{\theta}_j = \theta_j - \rho$ and $\hat{G}_{ij}(t) = \Pr[\hat{\theta}_j \leq t | Z_0 = i, U_0 = 0]$. Then, since $\hat{\theta}_j$ and ρ are independent random variables because of the strong Markov property,

$$G_{ij}(t) = \Pr[\theta_j \leq t | Z_0 = i, U_0 = 0] = \hat{G}_{ij}*H_i(t) \qquad (1.4).$$

The next group of equations are forward equations derived by considering the last transition. Since Z_t is regular, the number of transitions into a fixed state j in any time interval $[0,t)$, $t < \infty$, is finite with probability one; hence by the theorem of total probability,

$$P_{ij}(t) = \delta_{ij}\{1-H_j(t)\} + \sum_{n=1}^{\infty} G_{ij}{}^{(n)}*(1-H_j)(t),$$

and

$$P_{ij}(t) = M_{ij}*(1-H_j)(t) \tag{1.5}$$

($(1-H_j)(t)$ denotes the function $1 - H_j(t)$).

Similarly,

$${}_kP_{ij}(t) = {}_kM_{ij}*(1-H_j)(t) \tag{1.5a}$$

Furthermore, since

$$G_{ij}(t) = {}_iG_{ij}(t) + \sum_{n=1}^{\infty} {}_jG_{ii}{}^{(n)}*{}_iG_{ij}(t)$$

and

$$P_{ij}(t) = {}_iP_{ij}(t) + \sum_{n=1}^{\infty} G_{ii}{}^{(n)}*{}_iP_{ij}(t),$$

we have

$$G_{ij}(t) = {}_jM_{ii}*{}_iG_{ij}(t) \tag{1.6}$$

and

$$P_{ij}(t) = M_{ii}*{}_iP_{ij}(t) \tag{1.7}.$$

By considering the first transition, we obtain the backward equation

$$G_{ij}(t) = Q_{ij}(t) + \sum_{\substack{k\in I^+ \\ k\neq j}} Q_{ik}*G_{kj}(t) \tag{1.8}$$

The last three identities of this section are first entrance formulas which may be proved by means of the strong Markov property and familiar probabilistic arguments (see, for example, Chung's proof in [5], chapter II.11).

$$G_{ij}(t) = {}_kG_{ij}(t) + {}_jG_{ik}*G_{kj}(t) \tag{1.9}$$

$$P_{ij}(t) = {}_kP_{ij}(t) + G_{ik}*P_{kj}(t) \tag{1.10}$$

$${}_kP_{ij}(t) = {}_{h,k}P_{ij}(t) + {}_kG_{ih}*{}_kP_{hj}(t), \quad h \neq k \tag{1.11}$$

Identies (1.1) - (1.11) hold for all i,j,k,h in I^+, except where expressly stated otherwise. It should be borne in mind that ${}_DP_{ij}(t) = \delta_{ij}\{1-H_j(t)\}$ if $j \in D$.

§4. Properties of convolutions and Laplace-Stieltjes transforms.

From now on, t,x and u will denote real variables, s a complex one, and λ, δ, γ, α and β real parameters; the real part of s will be written Rs. The first four lemmas of this section are due to Widder [47]; lemmas 1.5 and 1.6 appear to be new, but are very easily proved by the standard methods.

All functions appearing in the lemmas of this section are assumed non-negative in $[0,\infty)$ and identically zero in $(-\infty,0)$.

<u>Lemma 1.1.</u> ([47] : page 40). If $A^*(s)$ converges for $s = -\lambda + i\delta$, with $\lambda > 0$, then $A(\infty)$ exists and $A(t) - A(\infty) = o(e^{-\lambda t})$ as $t \to \infty$.

<u>Lemma 1.2.</u> ([47] : page 39). If $A(\infty)$ exists and $A(t) - A(\infty) = o(e^{-\lambda t})$ as $t \to \infty$, for some λ, then $A^*(s)$ converges for all $Rs > -\lambda$.

<u>Lemma 1.3.</u> ([47] : page 58). If $A(t)$ is monotonic, then the real point of the axis of convergence of $A^*(s)$ is a singularity of $A^*(s)$.

<u>Lemma 1.4.</u> ([47] : page 89). If $C(t) = A*B(t)$, and $A^*(s_0)$ and $B^*(s_0)$ converge, one of them absolutely, then $C^*(s_0)$ converges and $C^*(s_0) = A^*(s_0)B^*(s_0)$.

Note that unlike corollary 1.2 below, lemma 1.4 makes no assumption of monotonicity. With the help of lemmas 1.1 and 1.2, the condition of absolute convergence in lemma 1.4 may be removed to obtain the following slightly weaker result:

<u>Lemma 1.4.a.</u> If $C(t) = A*B(t)$ and if both $A^*(-\lambda)$ and $B^*(-\lambda)$ converge for some $\lambda > 0$, then for all $Rs > \frac{-\lambda}{2}$, $C^*(s)$ converges and $C^*(s) = A^*(s)B^*(s)$.

<u>Proof</u>: By lemma 1.1, $A - A(t) = o(e^{-\lambda t})$ and $B - B(t) = o(e^{-\lambda t})$ as $t \to \infty$. Since

$$AB - C(t) = B\{A-A(t)\} + \int_0^{\frac{t}{2}} \{B - B(t-u)\}dA(u) + \int_{\frac{t}{2}}^{t} \{B - B(t-u)\}dA(u),$$

$AB - C(t) = o(e^{\frac{-\lambda t}{2}})$ as $t \to \infty$, so by lemma 1.2, $C^*(s)$ converges for all $Rs > \frac{-\lambda}{2}$. We can now appeal to yet another result of Widder's ([47] : page 91) to assert that $C^*(s) = A^*(s)B^*(s)$ for $Rs > \frac{-\lambda}{2}$.

<u>Lemma 1.5.</u> Suppose $C(t) = A*B(t)$. Then for every λ,

(i) $$\int_0^t e^{\lambda x}dC(x) = \int_0^t e^{\lambda x} \int_0^{t-x} e^{\lambda u}dB(u)dA(x),$$

(ii) $$\int_0^t e^{\lambda x}C(x)dx = \int_0^t e^{\lambda x} \int_0^{t-x} e^{\lambda u}B(u)dudA(x)$$

and

(iii) $$\int_0^t xe^{\lambda x}dC(x) = \int_0^t e^{\lambda x}\int_0^{t-x} ue^{\lambda u}dB(u)dA(x) + \int_0^t xe^{\lambda x}\int_0^{t-x} e^{\lambda u}dB(u)dA(x).$$

<u>Proof</u>: The proofs of all three equations are very similar, and we shall concern ourselves with (iii) only. On integrating by parts, the left hand side of (iii) becomes

$$te^{\lambda t}\int_0^t B(t-u)dA(u) - \int_0^t\{\int_0^x B(x-u)dA(u)\}\{e^{\lambda x} + \lambda xe^{\lambda x}\}dx$$

$$= \int_0^t te^{\lambda t}B(t-u)dA(u) - \int_0^t\int_u^t B(x-u)e^{\lambda x}dxdA(u) - \lambda\int_0^t\int_u^t B(x-u)xe^{\lambda x}dxdA(u)$$

$$= \int_0^t te^{\lambda t}B(t-u)dA(u) - \int_0^t e^{\lambda u}\int_u^t B(x-u)e^{\lambda(x-u)}dxdA(u) - \lambda\int_0^t e^{\lambda u}\int_u^t B(x-u)xe^{\lambda(x-u)}dxdA(u)$$

$$= \int_0^t te^{\lambda t}B(t-u)dA(u) - \int_0^t e^{\lambda u}\int_0^{t-u} B(x)e^{\lambda x}dxdA(u) - \lambda\int_0^t e^{\lambda u}\int_0^{t-u} xe^{\lambda x}B(x)dxdA(u)$$

$$-\lambda\int_0^t ue^{\lambda u}\int_0^{t-u} e^{\lambda x}B(x)dxdA(u)$$

$$= \int_0^t e^{\lambda u}\{(t-u)e^{\lambda(t-u)}B(t-u) - \int_0^{t-u} e^{\lambda x}B(x)dx - \lambda\int_0^{t-u} xe^{\lambda x}B(x)dx\}dA(u)$$

$$+ \int_0^t ue^{\lambda u}\{e^{\lambda(t-u)}B(t-u) - \lambda\int_0^{t-u} e^{\lambda x}B(x)dx\}dA(u)$$

$$= \int_0^t e^{\lambda u}\int_0^{t-u} xe^{\lambda x}dB(x)dA(u) + \int_0^t ue^{\lambda u}\int_0^{t-u} e^{\lambda x}dB(x)dA(u).$$

If (i) is established, (iii) may of course be easily obtained from it by differentiating both sides of the equation with respect to λ.

To emphasize the convolution structure present in the equations of Lemma 1.5, we rephrase the lemma thus:

Lemma 1.5a. Suppose $C(t) = A*B(t)$ and let $F^*(\lambda,t) = \int_0^t e^{\lambda x} dF(x)$, $F^o(\lambda,t) = \int_0^t e^{\lambda x} F(x)dx$ and $F^+(\lambda,t) = \int_0^t xe^{\lambda x} dF(x)$, where F may be any of the functions A,B or C. Then for every λ,

(i) $$C^*(\lambda,t) = B^*(\lambda,.) * A^*(\lambda,t),$$

(ii) $$C^o(\lambda,t) = B^o(\lambda,.) * A^*(\lambda,t)$$

and

(iii) $$C^+(\lambda,t) = B^+(\lambda,.) * A^*(\lambda,t) + B^*(\lambda,.) * A^+(\lambda,t).$$

Corollary 1.1. Suppose $A(t)$ is monotonic increasing. $C^o(-\lambda)$ converges if and only $A^*(-\lambda)$ and $B^o(-\lambda)$ converge. If $C^o(-\lambda)$ converges, then $C^o(-\lambda) = A^*(-\lambda)B^o(-\lambda)$.

Corollary 1.2. Suppose $A(t)$ and $B(t)$ are monotonic increasing. $C^*(-\lambda)$ converges if and only if $A^*(-\lambda)$ and $B^*(-\lambda)$ converge. If $C^*(-\lambda)$ converges, then $C^*(-\lambda) = A^*(-\lambda)B^*(-\lambda)$.

Proofs: Corollary 1.1 follows from (ii) and corollary 1.2 from (i).

<u>Corollary 1.3</u>. Suppose $A(t)$ and $B(t)$ are monotonic increasing. $C^+(-\lambda)$ converges if and only if $A^+(-\lambda)$ and $B^+(-\lambda)$ converge. If $C^+(-\lambda)$ converges, then $C^+(-\lambda) = A^+(-\lambda)B^*(-\lambda) + A^*(-\lambda)B^+(-\lambda)$.

<u>Proof</u>: The 'if' part follows easily from (iii). Suppose $C^+(-\lambda)$ converges. Then

$$\begin{aligned}\infty > C^+(-\lambda) &\geq \int_0^{\frac{t}{2}} e^{\lambda x} \int_0^{t-x} ue^{\lambda u} dB(u)dA(x) \\ &\geq \int_0^{\frac{t}{2}} e^{\lambda x} dA(x) \int_0^{\frac{t}{2}} ue^{\lambda u} dB(u) \\ &\geq \int_0^{T} e^{\lambda x} dA(x) \int_0^{\frac{t}{2}} ue^{\lambda u} dB(u) \quad \text{for all } t \geq 2T.\end{aligned}$$

Hence $B^+(-\lambda)$ converges; similarly, $A^+(-\lambda)$ converges. Clearly from (iii), $C^+(-\lambda) = A^+(-\lambda)B^*(-\lambda) + A^*(-\lambda)B^+(-\lambda)$.

<u>Corollary 1.4</u>. Suppose $F(t) = G(t) + H*F(t)$, $H(t)$ is monotonic increasing, $G^o(-\lambda)$ and $H^*(-\lambda)$ converge, and $H^*(-\lambda) < 1$.

Then $F^o(-\lambda)$ converges and $F^o(-\lambda) = \dfrac{G^o(-\lambda)}{1-H^*(-\lambda)}$.

<u>Proof</u>: The proof of this corollary is similar to that of corollary 1.5 below. Alternatively, it may be regarded as a direct consequence of lemma 1.3.

Corollary 1.5. Suppose $F(t) = G(t) + H*F(t)$, $F(t)$, $G(t)$ and $H(t)$ are all monotonic increasing, $F^*(-\lambda)$, $G^+(-\lambda)$ and $H^+(-\lambda)$ converge, and $H^*(-\lambda) < 1$. Then $F^+(-\lambda)$ converges, and

$$F^+(-\lambda) = \frac{G^+(-\lambda) + H^+(-\lambda)F^*(-\lambda)}{1 - H^*(-\lambda)} .$$

Proof: From (iii),

$$\int_0^t xe^{\lambda x}dF(x) = \int_0^t xe^{\lambda x}dG(x) + \int_0^t e^{\lambda x}\int_0^{t-x} ue^{\lambda u}dF(u)dH(x) + \int_0^t xe^{\lambda x}\int_0^{t-x} e^{\lambda u}dF(u)dH(x)$$

$$\leq G^+(-\lambda) + \int_0^t xe^{\lambda x}dF(x)H^*(-\lambda) + F^*(-\lambda)H^+(-\lambda) .$$

Hence

$$\int_0^t xe^{\lambda x}dF(x) \leq \frac{G^+(-\lambda) + H^+(-\lambda)F^*(-\lambda)}{1 - H^*(-\lambda)}$$

$$< \infty,$$

and $F^+(-\lambda)$ converges. The rest of the lemma now follows easily.

Lemma 1.6. Suppose $A(t)$ is monotonic increasing, $A(\infty)$ exists and $\lambda > 0$. Then $A^*(-\lambda)$ converges if and only if $\int_0^\infty e^{\lambda t}\{A(\infty) - A(t)\}dt$ converges.

Note: The lemma obviously cannot be extended to include the case $\lambda = 0$, for otherwise any honest distribution function with an infinite first moment will provide us with a contradiction.

Proof: Let $R(t) = \int_0^t e^{\lambda u} dA(u)$ and $S(t) = \lambda \int_0^t e^{\lambda u}\{A(\infty) - A(u)\}du$; then $S(t) = R(t) + e^{\lambda t}\{A(\infty) - A(t)\} + A(0) - A(\infty)$. If $A^*(-\lambda)$ converges, $e^{\lambda t}\{A(\infty) - A(t)\} \to 0$ by lemma 1.1, so $S(\infty)$ converges. If $\int_0^\infty e^{\lambda t}\{A(\infty) - A(t)\}dt$ converges, then

$$\infty > \int_0^\infty e^{\lambda t}\{A(\infty) - A(t)\}dt = \int_0^\infty e^{\lambda t} d\bar{A}(t)$$

where $\bar{A}(t) = \int_0^t \{A(\infty) - A(u)\}du$. By lemma 1.1, $\bar{A}(\infty) - \bar{A}(t) = \int_t^\infty \{A(\infty) - A(u)\}du = o(e^{-\lambda t})$. Now $e^{\lambda t}\int_t^\infty \{A(\infty) - A(u)\}du \geq e^{\lambda t}\int_t^{t+1} \{A(\infty)-A(u)\}du \geq e^{\lambda t}\{A(\infty) - A(t+1)\}$, so $e^{\lambda t}\{A(\infty) - A(t)\} \to 0$ and $R(\infty) = A^*(-\lambda)$ converges.

In succeeding chapters, we wish to apply Widder's lemmas to the Laplace-Stieltjes transorms of $P_{ij}(t)$, $G_{ij}(t)$ and the corresponding taboo quantities. The next lemma shows that $\int_0^T e^{-st} dP_{ij}(t)$ exists for every finite T, so we may define the Laplace-Stieltjes transform of $P_{ij}(t)$ by

$$P_{ij}^*(s) = \int_0^\infty e^{-st} dP_{ij}(t) = \lim_{T\to\infty} \int_0^T e^{-st} dP_{ij}(t).$$

$G_{ij}(t)$, being a monotonic function, presents no difficulty.

Lemma 1.7. $P_{ij}(t)$ and ${}_kP_{ij}(t)$ are functions of bounded variation in $[0,T]$ for every finite T.

Proof: Let $\mathrm{Var}_T(A)$ denote the variation of the function A in the interval $[0,T]$. We have $\mathrm{Var}_T(1-H_j) \leq 1$, and $\mathrm{Var}_T(M_{ij}) = M_{ij}(t)$; $M_{ij}(t)$ is finite by virtue of the regularity of the process. Hence by (1.5) and theorem 11.2b of Widder,

$$\begin{aligned}\mathrm{Var}_T(P_{ij}) &\leq \mathrm{Var}_T(1-H_j)\mathrm{Var}_T(M_{ij}) \\ &\leq M_{ij}(t) \\ &< \infty.\end{aligned}$$

Similarly, from (1.5a), $\mathrm{Var}_T({}_kP_{ij}) \leq {}_kM_{ij}(t) < \infty$.

§5. Comments on the key-renewal theorem.

Suppose $A(t)$ is a renewal function i.e. $A(t) = \sum_{n=0}^{\infty} B^{(n)}(t)$, where $B^{(0)}(t) = U_0(t)$, $B^{(1)}(t)$ is a distribution function (possibly dishonest) and $B^{(n)}(t)$, $n > 1$, is the n-fold convolution of $B^{(1)}(t)$ with itself. Let $\nu = \int_0^{\infty}\{1-B^{(1)}(t)\}dt$ and interpret ν^{-1} as zero if the integral diverges. The key renewal theorem states that

$$\lim_{t\to\infty} A * g(t) = \nu^{-1}\int_0^{\infty} g(x)dx$$

if one of the following conditions holds:

(A) g is bounded, Lebesgue integrable and non-increasing in $[0,\infty)$;

(B) g is Riemann integrable in every finite interval and

$$\sum_{n=0}^{\infty} \sup_{n\le x<n+1} |g(x)| < \infty;$$

(C) g is directly Riemann integrable.

(g is directly Riemann integrable if for sufficiently small h, the sums $\overline{\sigma}(h) = h\sum_{n=1}^{\infty} \sup_{(n-1)h\le x<nk} g(x)$ and $\underline{\sigma}(h) = h\sum_{n=1}^{\infty} \inf_{(n-1)h\le x<nh} g(x)$ converge absolutely, and $\lim_{h\to 0}\{\overline{\sigma}(h) - \underline{\sigma}(h)\} = 0$.)

If in addition $\nu < \infty$, then the key-renewal theorem holds if $B^{(k)}$ possesses an absolutely continuous component for some k, g is bounded and Lebesgue integrable in $[0,\infty)$, and $\lim_{t\to\infty} g(t) = 0$ (Smith, [36]).

Conditions A and B are due to Smith (1954, [36]; 1961, [40]), and condition C to Feller (1964, [16]). It is easy to see that A implies C, which in turn implies B, so that B is satisfied by the largest class of functions. (The class of functions satisfying B is strictly larger than the class of functions satisfying C: it requires little effort to construct an example of a function which satisfies B but not C.)

In applications of the key-renewal theorem, it may happen that A does not hold, as when g is not non-increasing, and B and C are not easily verifiable directly. One such situation occurs in the proof of theorem 3.4 of this thesis; there, the relevant function satisfies instead the condition

(D) g is non-negative, Lebesgue integrable and of bounded total variation in $[0,\infty)$.

The chief object of this section is to prove that C follows from D. Let $M_n(h)$ and $m_n(h)$ denote respectively the supremum and infimum of g in the interval $[(n-1)h,nh)$, and write $\mathrm{Var}(g)$ for the total variation of g in $[0,\infty)$.

<u>Lemma 1.8.</u> If g is non-negative, Lebesgue integrable and of bounded total variation in $[0,\infty)$, then g is directly Riemann integrable (whence $A * g(t) \to \nu^{-1} \int_0^\infty g(t)dt$).

<u>Proof</u>: For every fixed $h > 0$, since g is Lebesgue integrable, $\infty > \int_0^\infty g(t)dt \geq h \sum_{n=1}^{N} m_n(h)$ for every N; hence $\underline{\sigma}(h) < \infty$. For every N, $h \sum_{n=1}^{\infty N} M_n(h) - h \sum_{n=1}^{\infty N} m_n(h) \leq h\mathrm{Var}(g)$, so $\overline{\sigma}(h) < \infty$ and $\lim_{h\to 0}\{\overline{\sigma}(h) - \underline{\sigma}(h)\} = 0$.

Another useful criterion for direct Riemann integrability was given by Feller ([16] : page 349, example a); this we state as our next lemma.

<u>Lemma 1.9</u>. Suppose g is non-negative and continuous. Then g is directly Riemann integrable if and only if $\overline{\sigma}(1) < \infty$.

<u>Proof</u>: Suppose g is directly Riemann integrable. We can then select a number h, $0 < h < 1$, so that $\overline{\sigma}(h) = h \sum_{n=1}^{\infty} M_n(h) < \infty$. For every r, there exists n such that $M_n(h) = M_r(1)$. Hence

$h \sum_{r=1}^{\infty} M_r(1) \leq h \sum_{n=1}^{\infty} M_n(h) < \infty$, so $\overline{\sigma}(1) < \infty$.

Suppose now that $\overline{\sigma}(1) < \infty$, and h is any positive number smaller than one. For every n, there exists r such that $M_n(h) \leq M_r(1)$; hence $\overline{\sigma}(h) = h \sum_{n=1}^{\infty} M_n(h) \leq h\{[h^{-1}] + 1\} \sum_{r=1}^{\infty} M_r(1) < \infty$, where $[h^{-1}]$ denotes the integral part of h^{-1}. Now choose N so that $\sum_{n=N+1}^{\infty} M_n(1) < \frac{\varepsilon}{3}$. Since g is uniformly continuous in $[0,N+1]$, we can choose $h < 1$ so that $M_n(h) - m_n(h) < \frac{\varepsilon}{3(N+1)}$ for all $n \le [\frac{N+1}{h}]$. Then

$$\overline{\sigma}(h) - \underline{\sigma}(h) = h \sum_{n=1}^{[\frac{N+1}{h}]} \{M_n(h) - m_n(h)\} + h \sum_{n=[\frac{N+1}{h}]+1}^{\infty} \{M_n(h) - m_n(h)\}$$

$$\le h[\frac{N+1}{h}] \frac{\varepsilon}{3(N+1)} + h \sum_{n=[\frac{N+1}{h}]+1}^{\infty} M_n(h)$$

$$\le \frac{\varepsilon}{3} + h\{[h^{-1}] + 1\} \sum_{n=N+1}^{\infty} M_n(1)$$

$$< \varepsilon.$$

The converse of lemma 1.8 is false; while a directly Riemann integrable function is necessarily Lebesgue integrable, it is not true that it must also be of bounded total variation. As a counter example, consider the continuous, non-negative function g defined as follows: for every non-negative integer n,

$$g(n) = g(n + \frac{r}{(n+1)^3}) = \frac{1}{(n+1)^2},$$

$$g(n + \frac{r}{(n+1)^3} + \frac{1}{2(n+1)^3}) = 0, \qquad r = 0,1,\dots,(n+1)^3 - 1.$$

In the interval $[n + \frac{r}{(n+1)^3}, n + \frac{r}{(n+1)^3} + \frac{1}{2(n+1)^3}]$ (and the interval $[n + \frac{r}{(n+1)^3} + \frac{1}{2(n+1)^3}, n + \frac{r+1}{(n+1)^3}])$, g is a straight line joining its values at the end points of the interval. For this function g, $M_n(1) = \frac{1}{n^2}$ and $\bar{\sigma}(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so by lemma 1.9, g is directly Riemann integrable. However, the variation of g in $[0,N]$ is

$$\cancel{\sum_{n=1}^{N} 2\{(n+1)^3 - 1\}\frac{1}{n^2} + \frac{1}{N^2} + \frac{1}{(N+1)^2}}$$ which is unbounded in N.

$$1 \quad + 2\sum_{n=1}^{N}(n+1) \quad + \frac{1}{(N+1)^2}$$

CHAPTER II

Geometric Ergodicity

§1. A solidarity theorem.

Let E be an irreducible class i.e. for every i in E, i communicates with a state j if, and only if, $j \in E$. In this and the subsequent chapter, we restrict our attention to only those transitions which occur between the states of E. In the present chapter, we assume that $0 < n_j < \infty$ for every j in E; under this assumption, the limits Π_j of the transition probabilities $P_{ij}(t)$ exist, are independent of the original states i, and are positive or zero according as E is positive recurrent or not. (See theorem 5 of Smith [37], or theorem 3.4 of this thesis.) We shall call a property a 'solidarity property' if it is possessed by either every state of E, or none; or by either every transition within E, or none. A transition from i to j $(i,j \in E)$ is said to be geometrically ergodic if there exists a positive number λ_{ij} such that $P_{ij}(t) - \Pi_j = o(e^{-\lambda_{ij}t})$ as $t \to \infty$; a state i is geometrically ergodic if the transition from i to i is geometrically ergodic.

If Z_t is Markovian, Kendall [19], Kingman [20] and [21], and Vere-Jones [45] have shown that all transitions in E are geometrically ergodic, given that some state in E is (see §2 of chapter I for a fuller discussion); the main purpose of this section is to examine whether this type of solidarity behaviour persists in a general semi-Markov process.

Transcience and recurrence will be comprehensively treated in the next chapter. It suffices here to say that a state i is transient if $G_{ii}(\infty) < 1$, positive - recurrent if $G_{ii}(\infty) = 1$ and $\mu_{ii} < \infty$, and null-recurrent if $G_{ii}(\infty) = 1$ and $\mu_{ii} = \infty$; and that the states of an irreducible class are either all transient, all positive - recurrent, or all null-recurrent. E is transient, positive-recurrent or null-recurrent according as whether its states are transient, positive-recurrent or null-recurrent.

We show firstly that a null-recurrent state cannot be geometrically ergodic, so that we need consider the transient and positive-recurrent cases only. Suppose to the contrary that a state i is null-recurrent and geometrically ergodic. Then the first passage distribution $G_{ii}(t)$ is honest and has an infinite mean, and there exists a positive number λ such that $e^{\lambda t}P_{ii}(t) \to o$ as $t \to \infty$. Since $P_{ii}(t) \geq 1 - H_i(t)$, it follows that $e^{\lambda t}\{1-H_i(t)\} \to o$. Hence there exists a positive δ, $\delta < \lambda$, for which both the integrals $P_{ii}{}^{o}(-\delta)$ and $(1-H_i)^{o}(-\delta)$ converge . Putting $i = j = k$ in (1.10) and using corollary 1.1, we see that $G_{ii}{}^{*}(-\delta)$ converges. But this is a contradiction, since the convergence of $G_{ii}{}^{*}(-\delta)$, $\delta > o$, implies the existence of all moments of $G_{ii}(t)$.

Suppose a state b in E is geometrically ergodic. If E is transient, the results of Kingman and Vere-Jones carry over without change to the general semi-Markov process, and we can then assert that all transitions within E are geometrically ergodic. However, if E is positive recurrent, difficulties arise and conditions must be imposed on the holding time distribution $H_b(t)$.

These conditions are of course satisfied when Z_t is Markovian, and $H_b(t)$ consequently negative exponential.

The first condition is rather obvious: we want the tail $1 - H_b(t)$ to tend to zero exponentially fast as $t \to \infty$. We shall see later that this is also a necessary condition for every transition to be geometrically ergodic.

Denote the first two moments of $H_b(t)$ by η_b and $\eta_{b,2}$ and write $N_b(t) = 1 + \sum_{n=1}^{\infty} H_b^{(n)}(t)$, where $H_b^{(n)}(t)$ is the n-fold convolution of $H_b(t)$ with itself. $N_b(t)$ is a renewal function, and if $1 - H_b(t) = O(e^{-\lambda t})$ for some positive λ, all the moments of $H_b(t)$ exist and the corresponding renewal process is positive recurrent. By a well-known renewal theorem (see [38]), $N_b(t) - \frac{t}{\eta_b} = \frac{\eta_{b,2}}{2\eta_b^2} + R(t)$, where $R(t)$ is a remainder function that tends to zero. The other condition that we have to impose on $H_b(t)$ is one that ensures that $R(t)$ tends to zero exponentially fast. Sufficient conditions for this to happen have been obtained by Leadbetter [25] and Stone [42], and we now quote their results without proof.

<u>Leadbetter's theorem</u>. If $H_b(t)$ is absolutely continuous with density function $h_b(t)$; and for some $\lambda > o$, $H_b^*(-\lambda)$ converges and $\frac{H_b^*(-\lambda+i\gamma)}{-\lambda+i\gamma} \in L_1(-\infty,\infty)$; then there exists $\delta > o$ such that

$$N_b(t) - \frac{t}{\eta_b} = \frac{\eta_{b,2}}{2\eta_b^2} + o(e^{-\delta t}) \quad \text{as} \quad t \to \infty.$$

<u>Stone's theorem</u>. If $H_b^*(-\lambda)$ converges for some $\lambda > 0$, and $H_b(t)$ is lattice or strongly non-lattice, then there exists $\delta > o$ such that

$$N_b(t) - \frac{t}{\eta_b} = \frac{\eta_{b,2}}{2\eta_b^2} + o(e^{-\delta t}) \quad \text{as} \quad t \to \infty.$$

(A function $A(t)$ is said to be lattice if there exists $\gamma \neq o$ such that $A^*(i\gamma) = 1$, and strongly non-lattice if $\liminf_{|\gamma|\to\infty} \left|1-A^*(-i\gamma)\right| > o$.) A simple proof of Stone's result was obtained by Teugels [44].

Note that by Stone's theorem, the condition ' $\frac{H_b^*(-\lambda+i\gamma)}{-\lambda+i\gamma} \in L_1(-\infty,\infty)$ ' in Leadbetter's theorem is superfluous; for if $H_b(t)$ is absolutely continuous and $H_b^*(-\lambda)$ converges, then $H_b(t)$ must be strongly non-lattice (see [25]).

We shall need also a modified version of Stone's theorem to treat the case of a 'delayed' renewal process; in such a process, the time to the first renewal has a different distribution, $H_k(t)$ say. If $\tilde{N}_b(t)$ is the renewal function associated with the delayed process, then

$$\tilde{N}_b(t) = 1 + H_k(t) + H_k * \sum_{n=1}^{\infty} H_b^{(n)}(t) = 1 + H_k * N_b(t).$$

The following modified form of Stone's theorem is easily proved.

<u>Stone's modified theorem</u>. If $H_b^*(-\lambda)$ and $H_k^*(-\theta)$ converge, $\lambda > o$, $\theta > o$, $H_b(t)$ is lattice or strongly non-lattice, and $\tilde{N}_b^*(s) = 1 + \frac{H_k^*(s)}{1-H_b^*(s)}$; then there exists $\gamma > o$ such that

$$\tilde{N}_b(t) - \frac{t}{\eta_b} = \frac{\eta_{b,2}}{2\eta_b^2} + \frac{\eta_b - \eta_k}{\eta_b} + o(e^{-\gamma t}) \quad \text{as} \quad t \to \infty,$$

where η_k is the mean of H_k. The equation is satisfied by $\gamma = \frac{1}{2}\min\{\delta,\theta\}$, where δ is the parameter of convergence in the enunciation of Stone's original theorem.

Proof: Let $\beta = \min\{\delta,\theta\}$. From the definition of $\tilde{N}_b(t)$,

$$\tilde{N}_b(t) - \frac{t}{\eta_b} = 1 + \int_0^t \{N_b(t-u) - \frac{t-u}{\eta_b}\} dH_k(u) - \frac{1}{\eta_b}\int_0^t \{1-H_k(u)\} du \ ,$$

and

$$\left|\tilde{N}_b(t) - \frac{t}{\eta_b} - \frac{\eta_{b,2}}{2\eta_b^2} - \frac{\eta_b - \eta_k}{\eta_b}\right| \leq \int_0^t \left|N_b(t-u) - \frac{t-u}{\eta_b} - \frac{\eta_{b,2}}{2\eta_b^2}\right| dH_k(u)$$

$$+ \frac{1}{\eta_b}\int_t^\infty \{1-H_k(u)\} du + \frac{\eta_{b,2}}{2\eta_b^2}\{1-H_k(t)\}$$

$$= \left(e^{-\frac{\beta t}{2}}\right) \qquad \text{as} \quad t \to \infty.$$

Note that Stone's original theorem is a special case of Stone's modified theorem with $H_k = H_b$.

In proving the solidarity theorem for geometric ergodicidy, we shall be relying heavily on the transition identities of §3, chapter I; by means of Laplace-Stieltjes transforms and their properties, we shall make inferences regarding a general transition, j to k, from what is known of a given transition, b to b. To facilitate this task, we now list explicitly the transition identities which we shall be using. Taking Laplace-Stieltjes transforms in (1.9) - (1-11), we have, for every b,j,k in E, with $b \neq j$, $b \neq k$, $j \neq k$:

$$P_{bb}^{\;*}(s) = \frac{1-H_b^*(s)}{1-G_{bb}^{\;*}(s)} \tag{2.1}$$

$$G_{bb}^{\;*}(s) = {}_kG_{bb}^{\;*}(s) + {}_bG_{bk}^{\;*}(s)G_{kb}^{\;*}(s) \tag{2.2}$$

$$G_{kb}^{\;*}(s) = {}_kG_{kb}^{\;*}(s) + {}_bG_{kk}^{\;*}(s)G_{kb}^{\;*}(s) \tag{2.3}$$

$$P_{kb}^{\;*}(s) = G_{kb}^{\;*}(s)P_{bb}^{\;*}(s) \tag{2.4}$$

$${}_bP_{kk}^{\;*}(s) = \frac{1-H_k^*(s)}{1-{}_bG_{kk}^{\;*}(s)} \tag{2.5}$$

$${}_bP_{bk}^{\;*}(s) = {}_bG_{bk}^{\;*}(s)\,{}_bP_{kk}^{\;*}(s) \tag{2.6}$$

$$P_{bk}^{\;*}(s) = \frac{{}_bP_{bk}^{\;*}(s)}{1-G_{bb}^{\;*}(s)} \tag{2.7}$$

$$P_{kk}^{\;*}(s) = {}_bP_{kk}^{\;*}(s) + G_{kb}^{\;*}(s)P_{bk}^{\;*}(s) \tag{2.8}$$

$$G_{jb}^{\;*}(s) = {}_kG_{jb}^{\;*}(s) + {}_bG_{jk}^{\;*}(s)G_{kb}^{\;*}(s) \tag{2.9}$$

$${}_bP_{jk}^{\;*}(s) = {}_bG_{jk}^{\;*}(s)\,{}_bP_{kk}^{\;*}(s) \tag{2.10}$$

$$P_{jk}^{\;*}(s) = {}_bP_{jk}^{\;*}(s) + G_{jb}^{\;*}(s)P_{bk}^{\;*}(s) \tag{2.11}$$

Equations (2.1) - (2.11) hold for all $Rs > 0$; if one side of an equation has an analytic continuation in some half-plane, then so has the other side, and the equation continues to hold in the half-plane.

THEOREM 2.1. Suppose there exists $\lambda > 0$ and a state b in E satisfying $P_{bb}(t) - \Pi_b = O(e^{-\lambda t})$ as $t \to \infty$; and furthermore, $H_b(t)$ is lattice or strongly non-lattice, and $1 - H_b(t) = O(e^{-\delta t})$ as $t \to \infty$, for some $\delta > o$. Then there exists $\gamma > o$ such that for all j,k in E,

$$P_{jk}(t) - \Pi_k = o(e^{-\gamma t}) \quad \text{as} \quad t \to \infty.$$

Proof: Let $j,k \in E$, $j \neq k$, $j \neq b$, $k \neq b$. We shall show that for some $\gamma > o$, all the transforms $P_{kb}^*(-\gamma)$, $P_{bk}^*(-\gamma)$, $P_{kk}^*(-\gamma)$ and $P_{jk}^*(-\gamma)$ converge; whence the desired result follows by lemma 1.1.

Let $\delta' = \min(\lambda,\delta)$. We show firstly that $G_{bb}^*(-\lambda')$ converges for some $\lambda' > 0$. If $\Pi_b = 0$, then $P_{bb}^o(-\theta)$ and $(1-H_b)^o(-\theta)$ converge for all $\theta < \delta'$, and as in the discussion in the fourth paragraph of this section, $G_{bb}^*(-\theta)$ converges for all $\theta < \delta'$. If $\Pi_b > 0$, then, by lemma 1.2, and since $P_{bb}^*(s)$ is analytic in its region of convergence ([47]: theorem 5a), we can choose a positive number $\lambda' < \delta'$ such that $P_{bb}^*(s)$ and $H_b^*(s)$ are analytic for all $Rs \geq -\lambda'$, and $P_{bb}^*(s)$ is non-vanishing for real values of s in the interval $[-\lambda', 0]$. By (2.1), $G_{bb}^*(s)$ has no singularity on the segment, $[-\lambda',0]$, of the real line; so by lemma 1.3, $G_{bb}^*(-\lambda')$ converges.

Applying corollary 1.2 to (2.2) and (2.3), we see that all the transforms in those two equations are convergent in the half-plane $\Omega' = Rs \geq -\lambda'$. In particular, $G_{kb}^{*}(-\lambda')$ converges, hence by (1.4), $H_k^{*}(-\lambda')$ converges.

By (2.4) and lemma 1.4, $P_{kb}^{*}(-\lambda')$ converges. By (2.3) and the irreducibility of E, ${}_bG_{kk}^{*}(s) \neq 1$ anywhere in Ω'. (For suppose otherwise: ${}_bG_{kk}^{*}(s_o) = 1$ for some $s_o = x_o + it_o$, $x_o \geq -\lambda'$, whence ${}_bG_{kk}^{*}(x_o) \geq 1$ and, by (2.3), ${}_kG_{kb}^{*}(x_o) = 0$; by (1.6), $G_{kb}^{*}(x_o) = 0$, so we have a contradiction.) From (1.2a), ${}_bM_{kk}^{*}(s) = [1 - {}_bG_{kk}^{*}(s)]^{-1}$ is analytic in Ω', and as it is the transform of a monotonic function, it is also convergent there. By lemma 1.4 and (2.5), ${}_bP_{kk}^{*}(-\lambda') = \{1 - H_k^{*}(-\lambda')\}{}_bM_{kk}^{*}(-\lambda')$ converges. The convergence of ${}_bP_{bk}^{*}(-\lambda')$ now follows from (2.6).

By (2.5) and (2.6), we may write (2.7) as

$$P_{bk}^{*}(s) = \frac{{}_bG_{bk}^{*}(s)\{1-H_k^{*}(s)\}}{\{1-G_{bb}^{*}(s)\}\{1-{}_bG_{kk}^{*}(s)\}}.$$ Using (2.1), we may further re-write

this as

$$P_{bk}^{*}(s) = \frac{{}_bG_{bk}^{*}(s)}{1-{}_bG_{kk}^{*}(s)} \cdot P_{bb}^{*}(s)U_k^{*}(s) \qquad (2.12),$$

where $U_k^{*}(s) = \dfrac{1-H_k^{*}(s)}{1-H_b^{*}(s)}$. It is at this juncture that we now apply Stone's result. Since $U_k^{*}(s) = 1 + \dfrac{1}{1-H_b^{*}(s)} - \{1 + \dfrac{H_k^{*}(s)}{1-H_b^{*}(s)}\}$, there exists, by Stone's modified theorem, $\lambda'' > o$ such that $U_k(t) - \dfrac{\eta_k}{\eta_b} = o\left(e^{-\lambda''t}\right)$

as $t \to \infty$. Note that λ'' is independent of k. By lemma 1.2, $U_k^*(s)$ is convergent for all $Rs > -\lambda''$. We have already seen that ${}_bG_{bk}{}^*(s)$, $[1-{}_bG_{kk}{}^*(s)]^{-1}$ and $P_{bb}{}^*(s)$ converge in Ω'; so by (2.12) and lemma 1.4a, there exists a positive number γ, $\gamma < \min(\lambda',\lambda'')$, such that $P_{bk}{}^*(-\gamma)$ converges. By (2.8), $P_{kk}{}^*(-\gamma)$ converges; the convergence of $P_{jk}{}^*(-\gamma)$ now follows easily from identities (2.9) - (2.11), thus completing the proof.

Note that if Z_t is a Markov process, the conditions on $H_b(t)$ are trivially satisfied.

Theorem 2.1 cannot be improved upon to assert the existence of a best possible parameter of convergence, γ, for all transitions; an example is given in [45] of a discrete-time, positive recurrent Markov chain, for which the best common parameter of convergence is not the best possible parameter for all transitions.

However, if E is transient, there is a best possible parameter for all transitions, and a stronger solidarity result holds (see theorems 3.1 and 3.4).

We take the opportunity here to comment briefly on an earlier solidarity theorem of the author's ([3]: theorem 3.1), which is the same as theorem 2.1 except that the conditions on $H_b(t)$ are replaced by the condition:

$$1 - H_j(t) = 0(e^{-\lambda t}) \quad \text{as} \quad t \to \infty \quad \text{and} \quad \frac{H_j^*(-\lambda+i\gamma)}{-\lambda+i\gamma} \in L_1(-\infty,\infty) \quad \text{for}$$

every $j \in E$. An error exists in the proof of that theorem; this can be rectified by including in the theorem the hypothesis that $H_j(t)$ is absolutely continuous for every j in E. The additional hypothesis is required for the validity of Leadbetter's theorem, which was invoked in the proof and wrongly quoted (as theorem D in [3]). Theorem 2.1 thus represents a significant improvement on theorem 3.1 of [3]: at the time of writing [3], the author was unaware of Stone's result; by the use of this result and equation (1.4), we have been able to replace the clumsy conditions of the earlier result by the more pleasing condition that $H_b(t)$ is lattice or strongly non-lattice, and $1 - H_b(t) = 0(e^{-\lambda t})$. Using the terminology of [3], we remark also that in the light of theorem 2.1, every geometrically ergodic class is strongly geometrically ergodic. (A geometrically ergodic class was defined in [3] to be a class E such that for every i,j in E, there exists $\lambda_{ij} > 0$ satisfying $P_{ij}(t) - \Pi_j = o\left(e^{-\lambda_{ij}t}\right)$; a geometrically ergodic class was said to be strongly geometrically ergodic if all the λ_{ij}'s could be replaced by a single parameter common to all transitions.)

We conclude this section with a necessary condition for geometric ergodicity. As expected, a stronger result is obtained if E is transient. In the transient case, we have seen that if $P_{bb}(t) = o(e^{-\lambda t})$ for some state b and some positive λ, then $G_{bb}^{*}(-\theta)$ converges for all $\theta < \lambda$. By (2.2) and corollary 1.2, $G_{kb}^{*}(-\theta)$ converges for all $\theta < \lambda$ and $k \in E$; finally, by the same corollary and (1.4), $H_k^*(-\theta)$ converges and $1 - H_k(t) = o(e^{-\theta t})$ for all $\theta < \lambda$. Let us call E geometrically ergodic if there exists $\lambda > o$ such that $P_{jk}(t) - \Pi_k = o(e^{-\lambda t})$ for every j,k in E. If E is positive recurrent and geometrically ergodic, we can select a state b and a positive number λ_b such that $P_{jk}^{*}(s)$ is analytic for all $Rs \geq -\lambda_b$ and j,k in E, and $P_{bb}^{*}(s) \neq 0$ for all real s in the interval $[-\lambda_b, 0]$. By (2.4), $G_{kb}^{*}(-\lambda_b)$ converges for every k, and by (1.4), $H_k^*(-\lambda_b)$ converges and $1 - H_k(t) = o\left(e^{-\lambda_b t}\right)$. We summarize the preceeding remarks in the following:

<u>THEOREM 2.2</u>. (i) If E is transient, and there exist a state b and $\lambda > o$ satisfying $P_{bb}(t) = o(e^{-\lambda t})$; then for all j in E and every $\theta < \lambda$, $1 - H_j(t) = o(e^{-\theta t})$.

(ii) If E is positive-recurrent and geometrically ergodic, then there exists $\gamma > o$ such that for all j in E, $1 - H_j(t) = o(e^{-\gamma t})$.

§2. Geometric ergodicity in a finite, irreducible class.

We assume in this section that the irreducible class, E, is the whole state space. However, this is only for simplicity of exposition, and it can be easily checked that the assumption results in no loss of generality; our results remain valid even if E is a proper sub-set of the state space.

Our object here is to establish a sufficient condition for geometric ergodicity when E has a finite number, m, of states. (Recall that E is geometrically ergodic if all the transitions within it are geometrically ergodic.) That we succeed is due, to a large extent, to lemma 4.1 of [30]. Pyke showed by the lemma that if $m < \infty$, then there can be at most a finite number of transitions in any finite period of time; it is this strong regularity property that allows us to enumerate all possible paths taken by Z_t in moving from one given state to another in finite time, and admits the method of induction used in the proof of theorem 2.3.

In the sequel, we shall say that a Laplace-Stieltjes transform, $A^*(s)$, is 'analytic at the origin', if there exists a positive δ such that $A^*(-\delta)$ converges. Note that by lemmas 1.1 and 1.2 $A^*(s)$ is analytic at the origin if and only if $A(t)$ tends to its limit exponentially fast.

Lemma 2.1. $G_{ij}^*(s)$ is analytic at the origin for all i,j in E if and only if $Q_{ij}^*(s)$ is analytic at the origin for all i,j in E.

Proof: The necessity part follows at once from (1.8) and corollary 1.2.

The proof of the sufficiency part proceeds by induction. The lemma is trivially true for $m = 1$; suppose it holds for $m = N$. We now consider a semi-Markov process Z_t of $N + 1$ states, with one-step transition probabilities $Q_{ij}(t)$ and first passage distributions $G_{ij}(t)$. Suppose $Q_{ij}^*(s)$ is analytic at the origin for all i,j, and let k and ℓ be two arbitrarily chosen states, $k \neq \ell$, $k \neq N + 1$, $\ell \neq N + 1$.

We wish to show firstly that ${}_kG_{k\ell}^*(s)$ is analytic at the origin. To pave the way for the induction argument, we define a new semi-Markov process $\overline{Z}_t$, with N states and the following one-step transition probabilities: for $i,j = 1,2,\ldots,N$, $i \neq k$, $j \neq k$, and some fixed state j_o, $j_o \neq k$, $j_o \neq N + 1$,

$$\overline{Q}_{ik}(t) = \begin{cases} \left[1 + \dfrac{p_{ik}}{p_{i,N+1}}\right] Q_{i,N+1}(t) & \text{if } p_{i,N+1} \neq 0, \\[2ex] Q_{ik}(t) & \text{if } p_{i,N+1} = 0; \end{cases}$$

$$\overline{Q}_{kj}(t) = \begin{cases} \left[1 + \dfrac{p_{N+1,k}}{\sum\limits_{\substack{h=1 \\ h\neq k}}^{N} p_{N+1,h}}\right] Q_{N+1,j}(t) & \text{if } \sum\limits_{\substack{h=1 \\ h\neq k}}^{N} p_{N+1,h} \neq 0, \\[4ex] \delta_{jj_o} & \text{if } \sum\limits_{\substack{h=1 \\ h\neq k}}^{N} p_{N+1,h} = 0; \end{cases}$$

$\overline{Q}_{ii}(t) = \overline{Q}_{kk}(t) = 0$; and $\overline{Q}_{ij}(t) = Q_{ij}(t)$. The notation here is obvious: all quantities related to $\overline{Z}_t$ will be marked by a horizontal bar; thus $\overline{G}_{ij}$ is the first passage distribution for $\overline{Z}_t$. It is easily checked that $\sum_{j=1}^{N} \overline{p}_{ij} = 1$ for all $i = 1,\ldots,N$. The new process may not be irreducible, but this will not affect the proof. Now, ${}_kG_{k\ell}^*(s)$ is determined by the Q_{ij}'s according to the equation

$$
\begin{aligned}
{}_kG_{k\ell}^*(s) = Q_{k\ell}^*(s) + \sum_{\substack{j=1 \\ j\neq k,\ell}}^{N+1} Q_{kj}^*(s) Q_{j\ell}^*(s) \; + \\
+ \sum_{\substack{j=1 \\ j\neq k,\ell}}^{N+1} Q_{kj}^*(s) \sum_{n=1}^{\infty} \sum_{\underline{\alpha}_n \in S_n} Q_{j\alpha_1}^*(s) Q_{\alpha_1\alpha_2}^*(s) \ldots \\
\ldots Q_{\alpha_n\ell}^*(s) \qquad (2.13),
\end{aligned}
$$

where $S_n = \{\underline{\alpha}_n = (\alpha_1,\ldots,\alpha_n) : \alpha_i = 1,2,\ldots,N+1,\ \alpha_i \neq k,\ell\}$ is the set of all paths of n states not containing k or ℓ. By the definition of $\overline{Q}_{ij}(t)$, (2.13) may be re-written as

$$
\begin{aligned}
{}_kG_{k\ell}^*(s) = Q_{k\ell}^*(s) + \sum_{\substack{j=1 \\ j\neq k,\ell}}^{N+1} Q_{kj}^*(s) Q_{j\ell}^*(s) \\
+ \sum_{\substack{j=1 \\ j\neq k,\ell}}^{N} Q_{kj}^*(s) \sum_{n=1}^{\infty} \sum_{\underline{\alpha}_n \in \overline{S}_n} a_{\underline{\alpha}_n} \overline{Q}_{j\alpha_1}^*(s) \ldots \overline{Q}_{\alpha_n\ell}^*(s) \\
+ Q_{k,N+1}^*(s) \sum_{n=1}^{\infty} \sum_{\underline{\alpha}_n \in \overline{S}_n} a_{\underline{\alpha}_n} \overline{Q}_{k\alpha_1}^*(s) \ldots \overline{Q}_{\alpha_n\ell}^*(s) \qquad (2.14),
\end{aligned}
$$

where $a_{\underline{\alpha}_n} \leq 1$ is a constant, and

$\bar{S}_n = \{\underline{\alpha}_n = (\alpha_1,\ldots,\alpha_n);\ \alpha_i = 1,\ldots,N,\ \alpha_i \neq \ell\}$. For each $j = 1,\ldots,N$,

$$\bar{G}_{j\ell}^{*}(s) = \bar{Q}_{j\ell}^{*}(s) + \sum_{n=1}^{\infty} \sum_{\underline{\alpha}_n \varepsilon S_n}^{\infty} \bar{Q}_{j\alpha_1}^{*}(s) \ldots \bar{Q}_{\alpha_n \ell}^{*}(s) \qquad (2.15),$$

$$\bar{G}_{k\ell}^{*}(s) = \bar{Q}_{k\ell}^{*}(s) + \sum_{n=1}^{\infty} \sum_{\underline{\alpha}_n \varepsilon S_n} \bar{Q}_{k\alpha_1}^{*}(s) \ldots \bar{Q}_{\alpha_n \ell}^{*}(s) \qquad (2.16).$$

By the induction hypothesis, $\bar{G}_{j\ell}^{*}(s)$ and $\bar{G}_{k\ell}^{*}(s)$ are analytic at the origin. A comparison of (2.14) with (2.15) and (2.16) shows that ${}_kG_{k\ell}^{*}(s)$ is also analytic at the origin.

It can be similarly shown, by reversing the roles of k and ℓ in the definition of $\bar{Z}_t$, that ${}_\ell G_{kk}^{*}(s)$ is also analytic at the origin. From (1.9),

$$G_{k\ell}^{*}(s) = \frac{{}_kG_{k\ell}^{*}(s)}{1 - {}_\ell G_{kk}^{*}(s)} \qquad (2.17),$$

$$G_{kk}^{*}(s) = {}_\ell G_{kk}^{*}(s) + {}_kG_{k\ell}^{*}(s) G_{\ell k}^{*}(s) \qquad (2.18).$$

By (2.17) and lemma 1.3, $G_{k\ell}^{*}(s)$ is analytic at the origin. Similarly, $G_{\ell k}^{*}(s)$ is analytic at the origin, and by (2.18), so is $G_{kk}^{*}(s)$.

The proof for the remaining cases when either one or both of k and ℓ is $N+1$ will be omitted. This proof can be easily obtained by modifying slightly the definition of $\overline{Z}_t$ and proceeding in exactly the same way as before.

After lemma 2.1, the proof of the main result below presents little difficulty.

THEOREM 2.3. If $H_i^*(s)$ is analytic at the origin for every i, and $\limsup_{|\gamma|\to\infty} \left|H_b^*(-i\gamma)\right| < 1$ for some b, then E is geometrically ergodic.

Proof: We first observe that $G_{ij}^*(s)$ is analytic at the origin for every i,j by lemma 2.1. Also, by the hypothesis of the theorem and (1.4), $\limsup_{|\gamma|\to\infty} \left|G_{bb}^*(-i\gamma)\right| < 1$, and this implies that $\liminf_{|\gamma|\to\infty} \left|1-G_{bb}^*(-i\gamma)\right| > 0$. From (2.1),

$$P_{bb}^*(s) = 1 + \frac{1}{1-G_{bb}^*(s)} - \{1 + \frac{H_b^*(s)}{1-G_{bb}^*(s)}\} ;$$

since both $H_b^*(s)$ and $G_{bb}^*(s)$ are analytic at the origin, and all moments of $H_b(t)$ and $G_{bb}(t)$ consequently exist, there is a $\lambda > 0$ satisfying $P_{bb}(t) - \frac{\eta_b}{\mu_{bb}} = o(e^{-\lambda t})$, by Stone's modified theorem. An appeal to theorem 2.1 completes the proof.

Note that theorem 2.3 improves theorem 4.1 of Cheong [3]. An immediate corollary to theorem 2.3 is that every finite, irreducible Markov process is necessarily geometrically ergodic. The theorem is best possible in that it no longer holds if E has infinitely many states; this must be so, or else every non-geometrically ergodic, irreducible Markov process would produce a contradiction.

CHAPTER III

Ergodic Theorems

§1. Convergence of $P^{o}_{ij}(s)$; α-recurrence.

We continue to assume in this chapter that E is irreducible. In the first two sections solidarity type theorems will be proved for the transitions within E: these will lead to the main ergodic theorem in §3; and to a discussion on invariant measures and vectors in §4; and the convergence of certain sums, and quasi-stationary distributions, in §5.

THEOREM 3.1. If for some $\lambda > 0$ and a pair of states a,b in E (a,b possibly identical), $P_{ab}{}^{o}(-\lambda)$ converges, then for all states i,j in E, $P_{ij}{}^{o}(-\lambda)$ converges. The theorem holds for $\lambda = 0$, provided $\eta_j < \infty$ for every j in E.

Remark: The proof of theorem 3.1 resembles closely that of theorem 2.1: the underlying method of both proofs consists of operating on the transition identities by transforms (either Laplace or Laplace-Stieltjes), and drawing conclusions regarding a general transition, given information on some specific one. Theorem 3.1 is more easily proved because we are considering here the Laplace transforms $P_{ij}{}^{o}(\lambda)$, which, unlike the Laplace-Stieltjes transforms $P_{ij}{}^{*}(\lambda)$, are positive. As an example of the usefulness of this positivity, consider equation (3.1) below: because every term in the equation is positive, the convergence of $P_{bb}{}^{o}(-\lambda)$ allows us to deduce that both $(1-H_b)^{o}(-\lambda)$ and $G_{bb}{}^{*}(-\lambda)$ converge, and that $G_{bb}{}^{*}(-\lambda) < 1$.

<u>Proof</u>: Let i and j be any two states of E, $i \neq j$, $i \neq b$, $j \neq b$. We wish to show firstly that $P_{bb}{}^{o}(-\lambda)$ and $G_{ij}{}^{*}(-\lambda)$ converge, and $G_{bb}{}^{*}(-\lambda) < 1$. Applying corollary 1.1 to equation (1.10) (with b replacing j and k, and a replacing i), we see that $P_{bb}{}^{o}(-\lambda)$ and $G_{ab}{}^{*}(-\lambda)$ converge, and

$$\infty > P_{ab}{}^{o}(-\lambda) = \delta_{ab}(1-H_b)^{o}(-\lambda) + G_{ab}{}^{*}(-\lambda)P_{bb}{}^{o}(-\lambda).$$

Similarly, $G_{bb}{}^{*}(-\lambda)$ and $(1-H_b)^{o}(-\lambda)$ converge and

$$\infty > P_{bb}{}^{o}(-\lambda) = (1-H_b)^{o}(-\lambda) + G_{bb}{}^{*}(-\lambda)P_{bb}{}^{o}(-\lambda) \qquad (3.1).$$

Since every state is assumed stable, $(1-H_b)^{o}(-\lambda) > 0$, so $G_{bb}{}^{*}(-\lambda) < 1$. By corollary 1.2 and equation (1.9),

$$1 > G_{bb}{}^{*}(-\lambda) = {}_jG_{bb}{}^{*}(-\lambda) + {}_bG_{bj}{}^{*}(-\lambda)G_{jb}{}^{*}(-\lambda)$$

$$\infty > G_{jb}{}^{*}(-\lambda) = {}_jG_{jb}{}^{*}(-\lambda) + {}_bG_{jj}{}^{*}(-\lambda)G_{jb}{}^{*}(-\lambda) \qquad (3.2),$$

and all the transforms in the two equations converge. From the first equation, ${}_jG_{bb}{}^{*}(-\lambda) < 1$; hence ${}_jG_{bb}{}^{*}(-\lambda) \neq 1$ anywhere in the half-plane $\Omega = Rs \geq -\lambda$, and $G_{bj}{}^{*}(s) = \dfrac{{}_bG_{bj}{}^{*}(s)}{1 - {}_jG_{bb}{}^{*}(s)}$ is convergent in Ω.

By corollary 1.2, $G_{jj}{}^{*}(-\lambda)$ converges and

$G_{jj}{}^{*}(-\lambda) = {}_bG_{jj}{}^{*}(-\lambda) + {}_jG_{jb}{}^{*}(-\lambda)G_{bj}{}^{*}(-\lambda) < \infty$. Finally, we obtain the convergence of $G_{ij}{}^*(-\lambda)$ from $G_{jj}{}^*(-\lambda) = {}_iG_{jj}{}^*(-\lambda) + {}_jG_{ji}{}^*(-\lambda)G_{ij}{}^*(-\lambda)$.

By the convergence of $G_{jj}{}^*(-\lambda)$ and (1.4), $H_j{}^*(-\lambda)$ converges. If $\lambda > 0$, an appeal to lemma 1.6 yields the convergence of $(1-H_j)^o(-\lambda)$. If $\lambda = 0$, $(1-H_j)^o(-\lambda) = \eta_j < \infty$ by the hypotheses of the theorem.

Since $G_{bb}{}^*(-\lambda) < 1$, $1 - G_{bb}{}^*(-\lambda)$ has no root in Ω; by irreducibility and (3.2), neither has $1 - {}_bG_{jj}{}^*(-\lambda)$. Thus $M_{bb}{}^*(-\lambda) = [1 - G_{bb}{}^*(-\lambda)]^{-1}$ and ${}_bM_{jj}{}^*(-\lambda) = [1 - {}_bG_{jj}{}^*(-\lambda)]^{-1}$ are both convergent. Applying corollary 1.1 to equations (1.10) and (1.11), we obtain the following sequences of equations and inequalities:

$$P_{jb}{}^o(-\lambda) = G_{jb}{}^*(-\lambda)P_{bb}{}^o(-\lambda) < \infty$$

$${}_bP_{jj}{}^o(-\lambda) = (1-H_j)^o(-\lambda)[1 - {}_bG_{jj}{}^*(-\lambda)]^{-1} < \infty$$

$${}_bP_{bj}{}^o(-\lambda) = {}_bG_{bj}{}^*(-\lambda)\,{}_bP_{jj}{}^o(-\lambda) < \infty$$

$$P_{bj}{}^o(-\lambda) = {}_bP_{bj}{}^o(-\lambda)[1-G_{bb}{}^*(-\lambda)]^{-1} < \infty$$

$$P_{jj}{}^o(-\lambda) = {}_bP_{jj}{}^o(-\lambda) + G_{jb}{}^*(-\lambda)P_{bj}{}^o(-\lambda) < \infty$$

$$P_{ij}{}^o(-\lambda) = G_{ij}{}^*(-\lambda)P_{jj}{}^o(-\lambda) < \infty.$$

The last inequality completes the proof.

It is worth noting that in theorem 3.1, the additional condition '$\eta_j < \infty$ for every j in E' has to be imposed when $\lambda = 0$. The reason for this is evident from the proof of the theorem. A necessary condition for the convergence of $P_{jj}{}^o(-\lambda)$ is the convergence of $(1-H_j)^o(-\lambda)$. When $\lambda > 0$, the convergence of $(1-H_j)^o(-\lambda)$ follows from the implications $G_{jj}^{*}(-\lambda) < \infty \to H_j^{*}(-\lambda) < \infty \to (1-H_j)^{o}(-\lambda) < \infty$. The last implication stems from lemma 1.6, which is false if $\lambda = 0$.

Theorem 3.1 is essential to the development of our theory in that it guarantees the existence of a common, best possible abscissa of convergence, $-\alpha$ say, of all the $P_{ij}^{o}(s)$, $i,j \in E$. It ensures furthermore that if $\alpha > 0$, or $\alpha = 0$ and $\eta_j < \infty$ for every j, then $P_{ij}^{o}(-\alpha)$ either all diverge or all converge. We shall call α the convergence parameter of E.

In the special case when Z_t is a Markov process, theorem 3.1 has been obtained by Kingman [20]. Kingman showed that α, which he called the 'decay parameter', is the limit of $-t^{-1}\log P_{ij}(t)$ as $t \to \infty$. His arguments are based on the Markovian property of the transition probabilities, and cannot be adapted for the general semi-Markov process.

A state i is called recurrent (transient) if $G_{ii}(\infty) = 1$ ($G_{ii}(\infty) < 1$); if it is recurrent, it is called positive-recurrent if $\mu_{ii} = G_{ii}^{+}(0) < \infty$, and null-recurrent otherwise. To avoid confusion, we will designate these usual definitions by the terms 'ordinary recurrence' 'ordinary transience', and so on.

We now follow Vere-Jones [45] and Kingman [20] in defining an α-recurrent (α-transient) state i as one for which $P_{ii}^{o}(-\alpha)$ diverges (converges); if i is α-recurrent, we call it α-positive recurrent if $G_{ii}^{+}(-\alpha) < \infty$, and α-null recurrent otherwise. We shall see after the proof of lemma 3.2 that if $\eta_i < \infty$, then 0-recurrence of the state i is equivalent to ordinary recurrence.

The next solidarity result, which we state as a corollary, follows immediately from theorem 3.1.

Corollary 3.1. If $\alpha > 0$, or $\alpha = 0$ and $\eta_j < \infty$ for every j, then the states of E are either all α-transient, or all α-recurrent.

Henceforth, E will be called α-transient if its states are α-transient, and α-recurrent otherwise. If Z_t itself is irreducible, then Z_t will be called α-transient or α-recurrent according as its states are α-transient or α-recurrent.

We now proceed to investigate what happens in a class E with convergence parameter α.

Lemma 3.1. For all i,j,k in E and all $\lambda \leq \alpha$, $G_{ij}^{*}(-\lambda)$ and ${}_kM_{ij}^{*}(-\lambda)$ converge, $G_{jj}^{*}(-\lambda) \leq 1$, and

$\lim_{t\to\infty} e^{\lambda t}\{1-H_j(t)\} = \lim_{t\to\infty} e^{\lambda t}{}_kP_{ij}(t) = 0$. If, moreover, $\alpha > 0$ or $\alpha = 0$ and $\eta_j < \infty$, then $(1-H_j)^{o}(-\lambda)$ and ${}_kP_{ij}^{o}(-\lambda)$ converge.

Proof: It is clearly sufficient to prove the lemma for $\lambda = \alpha$. The equation $P_{jj}^{o}(-\gamma) = (1-H_j)^{o}(-\gamma) + G_{jj}^{*}(-\gamma)P_{jj}^{o}(-\gamma)$ shows that for all $\gamma < \alpha$, $G_{jj}^{*}(-\gamma)$ converges and $G_{jj}^{*}(-\gamma) < 1$. Since $G_{jj}^{*}(-\gamma)$ is a non-decreasing function of γ and bounded by one for all $\gamma < \alpha$, $\lim_{\gamma\to\alpha} G_{jj}^{*}(-\gamma)$ exists, and the limit is less than or equal to one. By the monotone convergence theorem, $1 \geq \lim_{\gamma\to\alpha} G_{jj}^{*}(-\gamma) = \lim_{\gamma\to\alpha}\int_0^\infty e^{\gamma t}dG_{jj}(t) = \int_0^\infty e^{\alpha t}dG_{jj}(t)$; thus, $G_{jj}^{*}(-\alpha)$ converges and $G_{jj}^{*}(-\alpha) \leq 1$. The convergence of $G_{kj}^{*}(-\alpha)$ follows from the equation $1 \geq G_{jj}^{*}(-\alpha) = {}_kG_{jj}^{*}(-\alpha) + {}_jG_{jk}^{*}(-\alpha)G_{kj}^{*}(-\alpha)$, and the convergence of $H_j^{*}(-\alpha)$ from (1.4). When $\alpha > 0$, lemma 1.6 yields the convergence of $(1-H_j)^{o}(-\alpha)$. By irreducibility, ${}_kG_{jj}^{*}(-\alpha) < 1$, so ${}_kM_{jj}^{*}(-\alpha) = [1 - {}_kG_{jj}^{*}(-\alpha)]^{-1} < \infty$; and from (1.3a), ${}_kM_{ij}^{*}(-\alpha) = {}_kM_{jj}^{*}(-\alpha){}_kG_{ij}^{*}(-\alpha) < \infty$. Next, if $\alpha > 0$ or $\alpha = 0$ and $\eta_j < \infty$, ${}_kP_{jj}^{o}(-\alpha) = (1-H_j)^{o}(-\alpha)[1 - {}_kG_{jj}^{*}(-\alpha)]^{-1} < \infty$, and ${}_kP_{ij}^{o}(-\alpha) = {}_kG_{ij}^{*}(-\alpha){}_kP_{jj}^{o}(-\alpha) < \infty$. When $\alpha = 0$, $e^{\alpha t}\{1-H_j(t)\} \to 0$ trivially. For $\alpha > 0$, since $(1-H_j)^{o}(-\alpha)$ converges and $e^{\alpha t}\{1-H_j(t+1)\} \leq \int_t^{t+1} e^{\alpha u}\{1-H_j(u)\}du$, we have $\lim_{t\to\infty} e^{\alpha t}\{1-H_j(t)\} = 0$. Finally, from (1.5a), $e^{\alpha t}{}_kP_{ij}(t) = \int_o^t e^{\alpha(t-u)}\{1-H_j(t-u)\}d_k\overline{M}_{ij}(\alpha,u)$, where ${}_k\overline{M}_{ij}(\alpha,t) = \int_o^t e^{\alpha u}d_kM_{ij}(u)$; since ${}_k\overline{M}_{ij}(\alpha,\infty) = {}_kM_{ij}^{*}(-\alpha) < \infty$ and $\lim_{t\to\infty} e^{\alpha t}\{1-H_j(t)\} = 0$, we have $\lim_{t\to\infty} e^{\alpha t}{}_kP_{ij}(t) = 0$.

($\lim_{t\to\infty} A * B(t) = A(\infty)B(\infty)$ if both the limits exist; see [47], page 86)

<u>Lemma 3.2.</u> If $\alpha > 0$, or $\alpha = 0$ and $\eta_j < \infty$, then j is α-recurrent if and only if $G_{jj}^{*}(-\alpha) = 1$. If E is α-recurrent then $G_{ij}^{*}(-\alpha)G_{jk}^{*}(-\alpha) = G_{ik}^{*}(-\alpha)$ (whence $G_{ij}^{*}(-\alpha)G_{ji}^{*}(-\alpha) = 1$) for all i,j,k in E.

Proof: $P_{jj}{}^{o}(-\lambda) = (1-H_j)^o(-\lambda)M_{jj}{}^{*}(-\lambda)$ (equation (1.5)), so from the previous lemma and the hypotheses of the present one, $P_{jj}{}^{o}(-\alpha)$ diverges if and only if $M_{jj}{}^{*}(-\alpha)$ diverges i.e. if and only if $G_{jj}{}^{*}(-\alpha) = 1$, since $M_{jj}{}^{*}(-\alpha) = [1 - G_{jj}{}^{*}(-\alpha)]^{-1}$. This proves the first statement of the lemma. Suppose E is α-recurrent; then

$$1 = G_{ii}{}^{*}(-\alpha) = {}_jG_{ii}{}^{*}(-\alpha) + {}_iG_{ij}{}^{*}(-\alpha)G_{ji}{}^{*}(-\alpha)$$

$$G_{ij}{}^{*}(-\alpha) = {}_iG_{ij}{}^{*}(-\alpha) + {}_jG_{ii}{}^{*}(-\alpha)G_{ij}{}^{*}(-\alpha).$$

Eliminating ${}_iG_{ij}{}^{*}(-\alpha)$ from the last two equations, we get

$$1 - {}_jG_{ii}{}^{*}(-\alpha) = G_{ij}{}^{*}(-\alpha)G_{ji}{}^{*}(-\alpha)\{1-{}_jG_{ii}{}^{*}(-\alpha)\}.$$

Since ${}_jG_{ii}{}^{*}(-\alpha) < 1$ by irreducibility, $G_{ij}{}^{*}(-\alpha)G_{ji}{}^{*}(-\alpha) = 1$. Now, on eliminating ${}_kG_{ij}{}^{*}(-\alpha)$ from $G_{ik}{}^{*}(-\alpha) = {}_jG_{ik}{}^{*}(-\alpha) + {}_kG_{ij}{}^{*}(-\alpha)G_{jk}{}^{*}(-\alpha)$ and $G_{ij}{}^{*}(-\alpha) = {}_kG_{ij}{}^{*}(-\alpha) + {}_jG_{ik}{}^{*}(-\alpha)G_{kj}{}^{*}(-\alpha)$, we get $G_{ik}{}^{*}(-\alpha) = G_{ij}{}^{*}(-\alpha)G_{jk}{}^{*}(-\alpha)$.

Corollary 3.2. If $\eta_j < \infty$, then the definitions of 0-recurrence and ordinary recurrence, with respect to the state j, are equivalent.

Corollary 3.3. If $\alpha > 0$, or $\alpha = 0$ and $\eta_j < \infty$, j is α-recurrent if and only if $M_{jj}{}^{*}(-\alpha)$ diverges.

Proof: The corollary follows at once from the lemma and (1.2).

Lemma 3.2 yields as a special case a result of Vere-Jones [45] for discrete-time Markov chains.

§2. Convergence of $G_{ij}^{+}(s)$; α-positive recurrence.

We now use for the first time, in the proof of the next theorem, corollaries 1.3 and 1.5.

THEOREM 3.2. If for some state b and some $\lambda \geq 0$, $G_{bb}^{*}(-\lambda) = 1$, then $G_{kk}^{*}(-\lambda) = 1$ for every state k. If for some b and some λ, $G_{bb}^{+}(-\lambda)$ converges and $G_{bb}^{*}(-\lambda) \leq 1$, then $G_{jk}^{+}(-\lambda)$ converges for every j and k.

Proof: Making the obvious substitution of indices in (1.9), we get, for $k \neq b$, the following:

$$G_{bb}(t) = {}_kG_{bb}(t) + {}_bG_{bk} * G_{kb}(t) \tag{3.3}$$

$$G_{kb}(t) = {}_kG_{kb}(t) + {}_bG_{kk} * G_{kb}(t) \tag{3.4}$$

$$G_{bk}(t) = {}_bG_{bk}(t) + {}_kG_{bb} * G_{bk}(t) \tag{3.5}$$

$$G_{kk}(t) = {}_bG_{kk}(t) + {}_kG_{kb} * G_{bk}(t) \tag{3.6}$$

Suppose firstly that $G_{bb}^{*}(-\lambda) = 1$. By corollary 1.2, the Laplace-Stieltjes transforms of all the distributions in (3.3) and (3.4) are convergent in the half-plane $Rs \geq -\lambda$. By irreducibility and (3.3), ${}_kG_{bb}^{*}(-\lambda) < 1$, so from (3.5), $G_{bk}^{*}(-\lambda) = {}_bG_{bk}^{*}(-\lambda)[1-{}_kG_{bb}^{*}(-\lambda)]^{-1}$ converges; the convergence of $G_{kk}^{*}(-\lambda)$ follows from (3.6). Taking transforms and eliminating ${}_bG_{bk}^{*}(-\lambda)$ from (3.3) and (3.5), we get $G_{bk}^{*}(-\lambda)G_{kb}^{*}(-\lambda) = 1$; and on eliminating ${}_bG_{kk}^{*}(-\lambda)$ from (3.4) and (3.6), we have

$$G_{kk}^{*}(-\lambda) = \frac{G_{kb}^{*}(-\lambda) - {}_kG_{kb}^{*}(-\lambda)}{G_{kb}^{*}(-\lambda)} + {}_kG_{kb}^{*}(-\lambda)G_{bk}^{*}(-\lambda) = 1.$$

Assume now that $G_{bb}^{+}(-\lambda)$ converges and $G_{bb}^{*}(-\lambda) \leq 1$. Corollary 1.3 applied to (3.3) and (3.4) gives the convergence of ${}_kG_{bb}^{+}(-\lambda)$, ${}_bG_{bk}^{+}(-\lambda)$, $G_{kb}^{+}(-\lambda)$, ${}_kG_{kb}^{+}(-\lambda)$ and ${}_bG_{kk}^{+}(-\lambda)$. Since ${}_kG_{bb}^{*}(-\lambda) < 1$, $G_{bk}^{+}(-\lambda)$ converges by (3.5) and corollary 1.5. A repetition of the arguments yields the convergence of $G_{jk}^{+}(-\lambda)$ for every j.

With $\lambda = 0$, theorem 3.2 says that ordinary recurrence and ordinary positive-recurrence are solidarity properties. Suppose $\alpha > 0$: by lemma 3.2, a state i is α-recurrent if, and only if, $G_{ii}^{*}(-\alpha) = 1$; thus, theorem 3.2 says also that α-recurrence and α-positive recurrence are solidarity properties. We summarize these remarks in the next solidarity theorem.

<u>THEOREM 3.3</u>. Ordinary transience, when $\eta_j < \infty$ ordinary positive-recurrence and ordinary null-recurrence are solidarity properties. So, too, are α-transience, α-positive recurrence and α-null recurrence.

The first half of theorem 3.3 is not entirely new. Pyke in [30] proved ([30]: theorem 5.1) that i is ordinarilly recurrent if and only if it is ordinarilly recurrent in the imbedded Markov chain $\{J_n = Z_{N_t} : t \geq 0\}$. Since ordinary recurrence is a solidarity property of irreducible Markov chains, it follows immediately that ordinary recurrence in an irreducible class of a semi-Markov process is a solidarity property.

§3. The main ergodic theorem.

We suppose in this section that E is an irreducible class with convergence parameter α.

Let $\overline{G}_{ij}(\alpha,t) = \int_0^t e^{\alpha u} dG_{ij}(u)$; $\overline{G}_{jj}(\alpha,t)$ is a distribution function by lemma 3.1, and honest if $G_{jj}^*(-\alpha) = 1$ i.e. E is α-recurrent. Define the iterates by $\overline{G}_{jj}^{(0)}(\alpha,t) = U_0(t)$, $\overline{G}_{ij}^{(1)}(\alpha,t) = \overline{G}_{ij}(\alpha,t)$ and, for $n > 1$,

$\overline{G}_{ij}^{(n)}(\alpha,t) = \overline{G}_{ij}(\alpha,\cdot) * \overline{G}_{jj}^{(n-1)}(\alpha,t)$. By lemma (1.5(i)),

$$\overline{G}_{jj}^{(n)}(\alpha,t) = \int_0^t e^{\alpha u} dG_{jj}^{(n)}(u) \qquad (3.7).$$

We shall be considering the main ergodic theorem only in the aperiodic case; a corresponding result can be obtained for the periodic case without much additional difficulty. Let us assume that at least one non-zero one-step transition probability, $Q_{ik}(t)$ say, is non-lattice; and call this <u>hypothesis B</u>. It is easy to see that if A and B are distributions, then $A * B(t)$ is non-lattice if either A or B is non-lattice (see, for example, Lukacs [28] : lemma 8.5.2). Thus, if $Q_{ik}(t)$ is non-lattice, then by (1.8) and (1.9),

$$G_{ij}(t) = Q_{ij}(t) + \sum_{\substack{h\in E \\ h\neq k,j}} Q_{ih} * G_{hj}(t) + Q_{ik} * G_{kj}(t)$$

$$G_{jj}(t) = {}_iG_{jj}(t) + {}_jG_{ji} * G_{ij}(t),$$

so $G_{jj}(t)$ is non-lattice for every j in E. $G_{jj}(\alpha,t)$ is clearly non-lattice, and this allows us to apply the key-renewal theorem to derive the main ergodic theorem in the aperiodic case.

Denote by $\mathrm{Var}(f)$ the total variation of f in $[0,\infty)$.

<u>THEOREM 3.4.</u> Suppose hypothesis B is satisfied. For every i,j in E, if $\alpha > 0$, or $\alpha = 0$ and $n_j < \infty$, and $\mathrm{Var}[e^{\alpha t}\{1-H_j(t)\}] < \infty$, then

$$\lim_{t\to\infty} e^{\alpha t}P_{ij}(t) = L_{ij} = \begin{cases} 0 \text{ if } E \text{ is } \alpha\text{-transient or} \\ \qquad \alpha\text{-null recurrent,} \\ \\ G_{ij}{}^{*}(-\alpha)L_j \text{ if } E \text{ is} \\ \qquad \alpha\text{-positive recurrent,} \end{cases}$$

where $L_j = \lim_{t\to\infty} e^{\alpha t}P_{jj}(t) = \dfrac{(1-H_j)^o(-\alpha)}{G_{jj}{}^{+}(-\alpha)} > 0.$

<u>Proof</u>: For every i,j in E, define $\overline{M}_{ij}(\alpha,t) = \int_0^t e^{\alpha u}dM_{ij}(u)$.

Then $\overline{M}_{jj}(\alpha,t) = \sum_{n=0}^{\infty} \int_0^t e^{\alpha u}dG_{jj}{}^{(n)}(u) = \sum_{n=0}^{\infty} \overline{G}_{jj}{}^{(n)}(\alpha,t)$ by (3.7).

Suppose first that E is α-transient. By corollary 3.3 and equation (1.3), $M_{ij}^{*}(-\alpha) = \overline{M}_{ij}(\alpha,\infty)$ converges. Equation (1.5) gives rise to $e^{\alpha t}P_{ij}(t) = \int_0^t e^{\alpha(t-u)}\{1-H_j(t-u)\}d\overline{M}_{ij}(\alpha,u)$, and the theorem follows for the α-transient case from lemma 3.1.

Suppose now that E is α-recurrent. Under the conditions of the theorem $G_{jj}^{*}(-\alpha) = 1$; hence $\overline{G}_{jj}(\alpha,t)$ is an honest distribution function and $\overline{M}_{jj}(\alpha,t)$ is a renewal function.
From (1.7),

$$e^{\alpha t}P_{jj}(t) = \int_0^t e^{\alpha(t-u)}\{1-H_j(t-u)\}d\overline{M}_{jj}(\alpha,u) \qquad (3.8).$$

Since $\overline{G}_{jj}(\alpha,t)$ is non-lattice, $e^{\alpha t}\{1-H_j(t)\}$ is Lebesgue integrable (lemma 3.1) and $\mathrm{Var}[e^{\alpha t}\{1-H_j(t)\}] < \infty$, we may apply the key-renewal theorem (see lemma 1.8) to (3.8) to obtain

$$L_j = \lim_{t\to\infty} e^{\alpha t}P_{jj}(t) = \frac{(1-H_j)^{o}(-\alpha)}{\overline{\mu}_{jj}} \quad ,$$

where $\overline{\mu}_{jj} = \int_0^\infty t\, d\,\overline{G}_{jj}(\alpha,t) = G_{jj}^{+}(-\alpha)$ if E is α-positive recurrent, and $\overline{\mu}_{jj}^{-1} = 0$ if E is α-null recurrent. To complete the proof we note (by (1.10)) that

$$e^{\alpha t}P_{ij}(t) = \int_0^t e^{\alpha(t-u)}P_{jj}(t-u)d\overline{G}_{ij}(\alpha,u),$$

and on taking limits, obtain

$$L_{ij} = \lim_{t\to\infty} e^{\alpha t}P_{ij}(t) = G_{ij}{}^{*}(-\alpha)L_{j}.$$

For $\alpha = 0$, theorem 3.4 is simply theorem 5 of Smith [37]. In the special case of a Markov process, theorem 3.4 is well-known for $\alpha = 0$. For $\alpha > 0$ Kingman [20] has shown that the limits exist, and are positive or zero according as E is α-positive recurrent or not. Our results give the actual values of the limits in terms of the distributions H_j and G_{ij}; these agree with those obtained by Vere-Jones [45] for discrete-time Markov chains.

It is interesting to note that theorem 3.4 cannot be improved by removing the condition '$\eta_j < \infty$' when $\alpha = 0$: when $\alpha = 0$ and $\eta_j = \infty$, the transition probability $P_{ij}(t)$ may have an ordinary limit, a (C,1) limit, or no limit at all at $t \to \infty$; this situation cannot, of course, occur in irreducible Markov processes. Several 'peculiar' examples of semi-Markov processes have been constructed by Smith [41] (see also Derman [11] and Yackel [48]).

§4. Invariant measures and vectors.

When Z_t is a Markov process and E is α-recurrent, Vere-Jones [45] and Kingman [20] proved the existence of a unique, α-invariant measure and a unique, α-invariant vector (unique up to a constant multiple). Thus in the continuous time case, Kingman showed that there exist unique positive numbers m_j and x_i satisfying, for all i,j in E and $t > 0$,

$$\sum_{k\epsilon E} m_k P_{kj}(t) = e^{-\alpha t} m_j \, ,$$

$$\sum_{k\epsilon E} P_{ik}(t) x_k = e^{-\alpha t} x_i \, .$$

Kingman also showed that E is α-positive recurrent if and only if $\sum_{k\epsilon E} m_k x_k$ converges and when this is so $\lim_{t\to\infty} e^{\alpha t} P_{ij}(t) = L_{ij} = \frac{x_i m_j}{\sum_k x_k m_k}$.

Unfortunately, these results no longer hold when Z_t is a general semi-Markov process, for even in the simple case $\alpha = 0$, an invariant measure usually does not exist. Consider, for example, the alternating process (see Cox, [7] : chapter 7) in which one state has a negative-exponentially distributed holding time with unit mean, and the other a fixed holding time of unit length. Thus $E = \{1,2\}$, $p_{12} = p_{21} = 1$, $Q_{12}(t) = H_1(t) = 1 - e^{-t}$ and $Q_{21}(t) = H_2(t) = U_1(t)$. Z_t is semi-Markov and E obviously irreducible and positive recurrent. The ergodic limits are $\lim_{t\to\infty} P_{ij}(t) = \Pi_j = \frac{\eta_j}{\eta_1+\eta_2} = \frac{1}{2}$ $(i,j \in E)$. Let X_j be the random variable representing the holding time in state j, and suppose $1 < t < 2$.

Then

$$P_{12}(t) = \Pr[X_1 < t \le X_1 + 1] + \cancel{P[2X_1 + 1 < t \le 2X_1 + 2]} \quad \Pr[Y+1 < t \le Y+2]$$

$$P_{22}(t) = \Pr[1 + X_1 < t < 2 + X_1], \quad \text{where } Y = X_1 + X_2$$

In particular,

$$P_{12}\left(\frac{3}{2}\right) = \cancel{1 + e^{-\frac{1}{2}} - \frac{7}{2}e^{-\frac{3}{2}}} \quad \cancel{\text{and}} \quad 1 - e^{-\frac{3}{2}} - \frac{1}{2}e^{-\frac{1}{2}}$$

$$P_{22}\left(\frac{3}{2}\right) = 1 - e^{-\frac{1}{2}} .$$

If an invariant measure m_j exists, $m_j = \sum_{k=1}^{2} m_k P_{kj}(t)$ for every t and every j; so on letting $t \to \infty$, we must have $m_1 = m_2 = \frac{1}{2}$ or some constant multiple of $\frac{1}{2}$. But $\frac{1}{2} \neq \cancel{\frac{1}{2}\{1 + e^{-\frac{1}{2}} - \frac{7}{2}e^{-\frac{3}{2}} + 1 - e^{-\frac{1}{2}}\}}$, $\frac{1}{2}\left\{1 - e^{-\frac{3}{2}} - \frac{1}{2}e^{-\frac{1}{2}} + 1\right\}$ so there cannot be an invariant measure.

Let us assume for the remainder of this section that Z_t is a Markov process and E is α-recurrent. Under this assumption, a little more may be added to Kingman's results.

For an α-positive recurrent E, theorem 3.4 gives the actual values of the limits L_{ij} in terms of the distributions $H_i(t)$ and $G_{ij}(t)$; one consequence of this is that given these distributions, the α-invariant measure and α-invariant vector may be found. From theorem 3.4 and the results of Kingman, $\dfrac{(1-H_j)^o(-\alpha)}{G_{jj}^{\ +}(-\alpha)} = \dfrac{x_j m_j}{\sum_k x_k m_k}$ and

$$\frac{G_{ij}^{*}(-\alpha)(1-H_j)^{o}(-\alpha)}{G_{jj}^{+}(-\alpha)} = \frac{x_i m_j}{\sum_k x_k m_k}\text{, whence } x_i = x_j G_{ij}^{*}(-\alpha).$$ Suppose

that state 1 belongs to E and put $x_1 = 1$ and $x_i = G_{i1}^{*}(-\alpha)$; then

$$m_j = \frac{G_{1j}^{*}(-\alpha)(1-H_j)^{o}(-\alpha)\sum_k x_k m_k}{G_{jj}^{+}(-\alpha)} = L_{1j}\sum_k x_k m_k$$

$$= \frac{{}_1P_{1j}^{\ o}(-\alpha)\sum_k x_k m_k}{G_{11}^{+}(-\alpha)}$$

(The last equality follows from the equation $P_{1j}(t) = M_{11} * {}_1P_{1j}(t)$.) Thus $m_j' = {}_1P_{1j}^{\ o}(-\alpha)$ and $x_i = G_{i1}^{*}(-\alpha)$ are respectively the α-invariant measure and α-invariant vector. If $H_1(t) = 1 - e^{-q_1 t}$, we also have

$$L_{ij} = \frac{x_i m_j}{\sum_k x_k m_k}$$

$$= (q_1-\alpha)L_{11}G_{i1}^{*}(-\alpha)\,{}_1P_{1j}^{\ o}(-\alpha) .$$

The arguments we have just used depend on the positivity of the limits L_{ij}; when E is α-null recurrent, these limits are no longer positive, but the quantities ${}_1P_{1j}^{\ o}(-\alpha)$ and $G_{i1}^{*}(-\alpha)$ remain the respective invariant measure and vector. To prove this we shall have to anticipate a ratio-limit theorem which will be proved in the next chapter. With this theorem, we shall be able to treat both the α-positive recurrent and α-null recurrent cases simultaneously, and expand Kingman's results for continuous-time Markov processes. By virtue of the Markov property of $P_{ij}(t)$, we may write

$$P_{ij}(t+x) = \sum_{k\epsilon E} P_{ik}(t)P_{kj}(x) \qquad (3.9)$$

for every i,j in E and every t and x. If $1 \in E$, then from (3.9),

$$\frac{\int_o^T e^{\alpha(t+x)} P_{i1}(t+x)dx}{\int_o^T e^{\alpha x} P_{11}(x)dx} = e^{\alpha t} \sum_{k\epsilon E} P_{ik}(t) \frac{\int_o^T e^{\alpha x} P_{k1}(x)dx}{\int_o^T e^{\alpha x} P_{11}(x)dx} \tag{3.10}$$

$$\frac{\int_o^T e^{\alpha(t+x)} P_{1j}(t+x)dt}{\int_o^T e^{\alpha t} P_{11}(t)dt} = e^{\alpha x} \sum_{k\epsilon E} \frac{\int_o^T e^{\alpha t} P_{1k}(t)dt}{\int_o^T e^{\alpha t} P_{11}(t)dt} P_{kj}(x) \tag{3.11}$$

The left hand side of (3.10) equals

$$\frac{\int_o^T e^{\alpha x} P_{i1}(x)dx + \int_T^{T+t} e^{\alpha x} P_{i1}(x)dx - \int_o^t e^{\alpha x} P_{i1}(x)dx}{\int_o^T e^{\alpha x} P_{11}(x)dx} \quad : \quad \text{for a fixed } t,$$

the last two terms of the numerator are bounded in T, while the denominator diverges to infinity as $T \to \infty$; since

$$\lim_{T\to\infty} \frac{\int_o^T e^{\alpha x} P_{i1}(x)dx}{\int_o^T e^{\alpha x} P_{11}(x)dx} = G_{i1}^{\,*}(-\alpha)$$ according to theorem 4.1 of chapter four,

the left hand side of (3.10) tends to $G_{i1}^{\,*}(-\alpha)$, so by Fatou's lemma,

$$\infty > G_{i1}^{\,*}(-\alpha) = \lim_{T\to\infty} e^{\alpha t} \sum_{k\epsilon E} P_{ik}(t) \frac{\int_o^T e^{\alpha x} P_{k1}(x)dx}{\int_o^T e^{\alpha x} P_{11}(x)dx}$$

$$\geq e^{\alpha t} \sum_{k\epsilon E} P_{ik}(t) G_{k1}^{\,*}(-\alpha) .$$

Similarly, using (3.11) and the result $\lim_{T\to\infty} \frac{\int_o^T e^{\alpha x} P_{1j}(x)dx}{\int_o^T e^{\alpha x} P_{11}(x)dx} = {}_1P_{1j}^{\,o}(-\alpha)$

(theorem 4.1), we obtain

$$\infty > {}_1P_{1j}{}^{o}(-\alpha) \geq e^{\alpha t} \sum_{k \in E} {}_1P_{1k}{}^{o}(-\alpha)P_{kj}(t).$$

$m_j = {}_1P_{1j}{}^{o}(-\alpha)$ and $x_i = G_{i1}{}^{*}(-\alpha)$ are thus respectively an α-subinvariant measure and an α-subinvariant vector; by theorem 4 of Kingman [20], they are respectively the unique α-invariant measure and vector, and $\sum_{k \in E} m_k x_k$ converges if and only if E is α-positive recurrent. We have already seen that if E is α-positive recurrent, then $L_{ij} = (q_1-\alpha)L_{11}x_i m_j$ and $\sum_k L_{kk} = 1$, where q_1^{-1} is the mean of the holding time in state 1 and $L_{11} = \dfrac{x_1 m_1}{\sum_k x_k m_k}$. We now summarize our results in the following

THEOREM 3.5. If Z_t is a Markov process and E is α-recurrent, then $m_j = {}_1P_{1j}{}^{o}(-\alpha)$ and $x_i = G_{i1}{}^{*}(-\alpha)$ are respectively the unique (apart from constant multiples) α-invariant measure and α-invariant vector. $\sum_{k \in E} m_k x_k$ converges if and only if E is α-positive recurrent, and then, as $t \to \infty$,

$$L_{ij} = \lim_{t \to \infty} e^{\alpha t} P_{ij}(t) = (q_1-\alpha)L_{11}x_i m_j,$$

$$\sum_{k \in E} L_{kk} = 1,$$

where q_1^{-1} is the mean of the holding time distribution of state 1 and

$$L_{11} = x_1 m_1 [\sum_k x_k m_k]^{-1}.$$

Theorem 3.5 complements the result of Kingman ([20] : theorem 4) on continuous-time Markov processes, and yields as a special case Vere-Jones' result on discrete-time Markov chains ([45] : theorem 2).

§5. Convergence of $\sum_i u_i e^{\lambda t} P_{ij}(t)$ and $\sum_j e^{\lambda t} P_{ij}(t) v_j$.

In this section we investigate the conditions under which the sums $R_j(\lambda,t) = \sum_{i\varepsilon E} u_i e^{\lambda t} P_{ij}(t)$ and $S_i(\lambda,t) = \sum_{j\varepsilon E} e^{\lambda t} P_{ij}(t) v_j$ tend to finite limits as $t \to \infty$, where u_i and v_j are numerical, possibly complex, vectors, and $\lambda \leq \alpha$, the convergence parameter of the irreducible class E. A study of these sums was made for discrete-time Markov chains by Vere-Jones [46], who also obtained sufficient conditions for the convergence of the double sum $\sum_{i,j\varepsilon E} u_i e^{\lambda t} P_{ij}(t) v_j$ to a finite limit. In trying to extend his results to the general semi-Markov process, we are handicapped by our inability to assert the existence of invariant measures and vectors; however, while we can gather no information on the double sum that is worthy of comment here, we succeed in obtaining an almost complete generalization of Vere-Jones' results for $R_j(\lambda,t)$ and $S_i(\lambda,t)$. Our result on the convergence of $S_i(\lambda,t)$ will be used towards the end of this section in a discussion of 'quasi-stationary distributions'.

The sum $R_j(\lambda,t)$ may be treated in exactly the same manner as its counterpart in the theory of discrete-time Markov chains; $S_i(\lambda,t)$, however, poses problems which we cannot overcome except by imposing conditions to ensure regular behaviour of the holding time distributions $H_k(t)$. One is that the sums $\sum_{k\varepsilon E} {}_iP_{ik}{}^{o}(-\alpha)|v_k|$ and $\sum_{k\varepsilon E} {}_iM_{ik}{}^{*}(-\alpha)|v_k|$ converge (as usual, α denotes the convergence parameter). We shall comment later on how this relates to a similar condition of Vere-Jones for discrete-time Markov chains. The other condition involves a uniform bounded variation property which we shall call <u>hypothesis C</u>: there exists a number d such that $\mathrm{Var}[e^{\alpha t}\{1-H_k(t)\}] \leq d$ for every k. ($\mathrm{Var}(f)$ denotes the total variation of the function f in $[0,\infty)$.) Note that hypothesis C is trivially satisfied if Z_t is Markovian, in which case $\mathrm{Var}[e^{\alpha t}\{1-H_k(t)\}] = 1$ for every k.

To eliminate periodic behaviour and so simplify the description, we assume <u>hypothesis B</u>: at least one non-zero one-step transition probability is non-lattice. Then by theorem 3.4, provided $\mathrm{Var}[e^{\alpha t}\{1-H_k(t)\}]$ is finite, $e^{\alpha t}P_{ij}(t)$ tends to a limit as $t \to \infty$, which is positive if and only if E is α-positive recurrent and $\lambda = \alpha$.

We study $R_j(\lambda,t)$ first: suppose $\mathrm{Var}[e^{\lambda t}\{1-H_j(t)\}]$ is finite and $\sum_{k\epsilon E}|u_k|G_{kj}^{\ *}(-\lambda)$ converges; we shall show that $R_j(\lambda,t)$ is finite for every (fixed) t, and $\lim_{t\to\infty} R_j(\lambda,t)$ exists. From equation (1.10)

$$\sum_{k=1}^{N}|u_k|e^{\lambda t}P_{kj}(t) = \sum_{k=1}^{N}|u_k|\int_0^t e^{\lambda(t-u)}P_{jj}(t-u)d\bar{G}_{kj}(\lambda,u) \qquad (3.12)$$

$$\leq e^{\lambda t}\sum_{k=1}^{N}|u_k|\bar{G}_{kj}(\lambda,t)$$

$$\leq e^{\lambda t}\sum_{k\epsilon E}|u_k|G_{kj}^{\ *}(-\lambda) \quad .$$

Since the upper bound is finite and independent of N, $\sum_{k\varepsilon E} u_k e^{\lambda t} P_{kj}(t)$ is finite for every t. Now, replacing N by ∞ in (3.12) and interchanging the order of summation and integration on the right hand side, we have

$$R_j(\lambda,t) = \int_0^t e^{\lambda(t-u)} P_{jj}(t-u) d\{\sum_{k\varepsilon E} u_k \overline{G}_{kj}(\lambda,u)\} \tag{3.13}$$

$$\to L_j' \sum_{k\varepsilon E} u_k G_{kj}{}^*(-\lambda) \qquad \text{(by theorem 3.4)},$$

where $L_j' = \dfrac{(1-H_j)^o(-\alpha)}{G_{jj}{}^+(-\alpha)}$ if E is α-positive recurrent and $\lambda = \alpha$, and $L_j' = 0$ otherwise. If E is α-positive recurrent and $\lambda = \alpha$, and u_k is a real and non-negative vector, then the convergence of $\sum_{k\varepsilon E} u_k G_{kj}{}^*(-\alpha)$ is a necessary condition as well for the existence of $\lim_{t\to\infty} R_j(\alpha,t)$; for clearly the right hand side of (3.13) diverges if $\sum_{k\varepsilon E} u_k G_{kj}{}^*(-\alpha)$ diverges. We have thus proved:

<u>THEOREM 3.6.</u> Suppose the hypotheses of theorem 3.4 hold, and $\lambda \le \alpha$. If $\sum_{k\varepsilon E} |u_k| G_{kj}{}^*(-\lambda)$ converges, then $R_j(\lambda,t)$ is finite for every t, and

$$\lim_{t\to\infty} R_j(\lambda,t) = \sum_{k\varepsilon E} u_k \{\lim_{t\to\infty} e^{\lambda t} P_{kj}(t)\} = L_j' \sum_{k\varepsilon E} u_k G_{kj}{}^*(-\lambda),$$

where

$$L_j' = \frac{(1-H_j)^o(-\alpha)}{G_{jj}{}^+(-\alpha)} > 0 \text{ if } E \text{ is } \alpha\text{-positive recurrent}$$

and $\lambda = \alpha$, and $L_j' = 0$ otherwise. If E is α-positive recurrent and $\lambda=\alpha$ and u_k is a real, non-negative vector, then the convergence of $\sum u_k G_{kj}^{*}(-\alpha)$ is a necessary condition for the existence of $\lim_{t\to\infty} R_j(\lambda,t)$.

For the sum $S_i(\lambda,t)$, we prove the following weaker result.

THEOREM 3.7. Suppose the hypotheses of theorem 3.4 hold, and $\lambda \leq \alpha$. If $\sum_{k\varepsilon E} {}_iM_{ik}^{*}(-\lambda)|v_k|$ converges, then $S_i(\lambda,t)$ is finite for every t. If in addition, $\sum_{k\varepsilon E} {}_iP_{ik}^{o}(-\lambda)|v_k|$ converges and hypothesis C holds (i.e. $\mathrm{Var}[e^{\lambda t}\{1-H_k(t)\}] \leq d$ for some d and all k), then

$$\lim_{t\to\infty} S_i(\lambda,t) = \sum_{k\varepsilon E} \lim_{t\to\infty}\{e^{\lambda t}P_{ik}(t)\}v_k = \sum_{k\varepsilon E} G_{ik}^{*}(-\lambda)L_k' v_k,$$

where

$$L_k' = \frac{(1-H_k)^{o}(-\alpha)}{G_{kk}^{+}(-\alpha)} > 0 \text{ if } E \text{ is } \alpha\text{-positive recurrent}$$

and $\lambda = \alpha$, and $L_k' = 0$ otherwise.

Proof: As Vere-Jones has commented in [46], we may separate v_k into its real and imaginary parts, and separate these again into positive and negative parts; since $\sum_{k\varepsilon E} e^{\lambda t}P_{ik}(t)|v_k|$ converges if and only if each of the corresponding sums involving its component parts converges, it suffices in what follows to consider only real and non-negative vectors v_k.

Suppose $\sum_{k\varepsilon E} {}_iM_{ik}^{*}(-\lambda)v_k$ converges. From (1.7) and (1.5a), we have

$$\sum_{k=1}^{N} e^{\lambda t} P_{ik}(t)v_k = \int_0^t \sum_{k=1}^{N} e^{\lambda(t-u)} {}_iP_{ik}(t-u)v_k d\overline{M}_{ii}(\lambda,u) \qquad (3.14)$$

$$\sum_{k=1}^{N} e^{\lambda t} {}_iP_{ik}(t)v_k = \sum_{k=1}^{N} \int_0^t e^{\lambda(t-u)}\{1-H_k(t-u)\} d_i\overline{M}_{ik}(\lambda,u)v_k \qquad (3.15)$$

$$\leq e^{\lambda t} \sum_{k\varepsilon E} {}_iM_{ik}^{*}(-\lambda)v_k \quad .$$

From (3.15) $\sum_{k\varepsilon E} e^{\lambda t} {}_iP_{ik}(t)v_k$ is finite; from (3.16), $S_i(\lambda,t)$ is finite for every t, and

$$S_i(\lambda,t) = \int_0^t \sum_{k\varepsilon E} e^{\lambda(t-u)} {}_iP_{ik}(t-u)v_k d\overline{M}_{ii}(\lambda,u) \qquad (3.16).$$

We would like now to apply the key-renewal theorm to (3.14) in order to assert the existence of $\lim_{t\to\infty} S_i(\lambda,t)$. Before doing this, we have to be certain that the integrand, $\sum_{k\varepsilon E} e^{\lambda t} {}_iP_{ik}(t)v_k$, does not behave too erratically for large t; in fact, regularity of $\sum_{k\varepsilon E} e^{\lambda t} {}_iP_{ik}(t)v_k$ follows from the regularity of $\sum_{k\varepsilon E} e^{\lambda t}\{1-H_k(t)\}v_k$, which in turn is ensured by hypothesis C. Since hypothesis C holds,

$$\mathrm{Var}[\sum_{k\epsilon E} e^{\lambda t}\,{}_iP_{ik}(t)v_k] \le \sum_{k\epsilon E} \mathrm{Var}[e^{\lambda t}\,{}_iP_{ik}(t)v_k]$$

$$\le \sum_{k\epsilon E} \mathrm{Var}[e^{\lambda t}\{1-H_k(t)\}]\mathrm{Var}[{}_iM_{ik}(\lambda,t)]v_k$$

$$\le d \sum_{k\epsilon E} {}_iM_{ik}^{*}(-\lambda)v_k$$

$$< \infty$$

($\mathrm{Var}[A*B(t)] \le \mathrm{Var}(A)\mathrm{Var}(B)$: see Widder [47], theorem 11.2b). Since $\sum_{k\epsilon E} {}_iP_{ik}^{o}(-\lambda)v_k$ converges, we have

$$\infty > \sum_{k\epsilon E} \int_0^{\infty} e^{\lambda t}\,{}_iP_{ik}(t)dtv_k = \int_0^{\infty} \sum_{k\epsilon E} e^{\lambda t}\,{}_iP_{ik}(t)v_k dt.$$

The integrand in (3.16) is thus Lebesgue integrable and of total bounded variation in $[0,\infty)$; application of the key-renewal theorem now yields

$$\lim_{t\to\infty} S_i(\lambda,t) = \frac{\int_0^{\infty} \sum_{k\epsilon E} e^{\lambda t}\,{}_iP_{ik}(t)v_k dt}{G_{ii}^{+}(-\lambda)} \tag{3.17}$$

where $[G_{ii}^{+}(-\lambda)]^{-1}$ is zero unless E is α-positive recurrent and $\lambda = \alpha$. If E is α-positive recurrent and $\lambda = \alpha$, then

$\lim_{t\to\infty} S_i(\alpha,t) = \dfrac{\sum_{k\epsilon E} {}_iP_{ik}^{o}(-\alpha)v_k}{G_{ii}^{+}(-\alpha)} > 0.$ Appealing to theorem 3.4 and applying the key-renewal theorem to the equation

$$e^{\alpha t}P_{ik}(t) = \int_0^t e^{\alpha(t-u)}\,{}_iP_{ik}(t-u)\,d\bar{M}_{ii}(\alpha,u),$$

we obtain

$$\lim_{t\to\infty} e^{\alpha t}P_{ik}(t) = G_{ik}{}^*(-\alpha)L_k' = \frac{{}_iP_{ik}{}^o(-\alpha)}{G_{ii}{}^+(-\alpha)} \quad ;$$

the proof is now complete.

We devote the next few paragraphs to examining the hypotheses of theorems 3.6 and 3.7 when Z_t is Markovian. If time is discrete and the chain aperiodic, then by theorem 6.1 of Vere-Jones [46], the respective sufficient conditions for the finiteness of $R_j(\lambda,t)$ and $S_i(\lambda,t)$, and for the convergence of these sums to finite limits as $t \to \infty$, are (in our terminology) the convergence of

$$\sum_{k\epsilon E} |u_k|\; G_{kj}{}^*(-\lambda) \quad \text{and} \quad \sum_{k\epsilon E} {}_iP_{ik}{}^o(-\lambda)\; |v_k|.$$

This result emerges as a special case of our theorems: as we have already observed, hypothesis C is trivially satisfied when Z_t is Markovian; hypothesis B is also trivially satisfied if the chain is aperiodic; moreover,

$$\sum_{k\epsilon E} {}_iM_{ik}{}^*(-\lambda)|v_k| = \sum_{k\epsilon E} {}_iP_{ik}{}^o(-\lambda)|v_k|$$

if time is discrete.

When Z_t is a continuous-time Markov process, theorems 3.6 and 3.7 yield new results, and for this reason we shall discuss this special case in greater detail. The existence of λ-subinvariant measures and vectors may be demonstrated as in the discussion preceeding theorem 3.5; adopting the arguments used there, but replacing the convergence parameter, α, by λ $(\leq \alpha)$, we see that ${}_iP_{ij}{}^{o}(-\lambda)$ (i fixed) and $G_{ij}{}^{*}(-\lambda)$ (j fixed) are the respective solutions to the inequalities $e^{\lambda t} \sum_{k\epsilon E} y_k P_{kj}(t) \leq y_j$ and $e^{\lambda t} \sum_{k\epsilon E} P_{ik}(t) z_k \leq z_i$.

If $\lambda = \alpha$ and E is α-recurrent, these solutions are unique up to constant multiples. If $\lambda < \alpha$, or $\lambda = \alpha$ and E is α-transient, they are no longer unique, but we shall show in the next paragraph that they possess the following minimality property: if ξ_k (ζ_k) is any λ-subinvariant measure (vector), then

$$\frac{\xi_j}{\xi_i} \geq \frac{{}_iP_{ij}{}^{o}(-\lambda)}{(1-H_i)^{o}(-\lambda)} \qquad \left(\frac{\zeta_i}{\zeta_j} \geq G_{ij}{}^{*}(-\lambda)\right).$$

The minimality property may be established by considering the discrete skeletons of Z_t and applying to these the known results concerning discrete-time chains. (The discrete skeleton on scale h of the Markov process Z_t is the Markov chain, $Z_n(h)$, defined by the transition matrix $P_{ij}(h)$.) Suppose ξ_j is a λ-subinvariant measure; then for every $h > 0$ and every j in E,

$$\xi_j \geq e^{\lambda h} \sum_{k\epsilon E} \xi_k P_{kj}(h).$$

Hence ξ_j is a $e^{\lambda h}$-subinvariant measure of the discrete-time Markov chain $Z_n(h)$. If we write $\ell_{ij}^{(n)}(h)$ for the probability of the event $\{Z_n(h) = j;\ Z_m(h) \neq i,\ 1 \leq m < n | Z_0(h) = i\}$, and $L_{ij}(h,r)$ for the generating function $\sum_{n=0}^{\infty} r^n \ell_{ij}^{(n)}(h)$ (we define $\ell_{ij}^{(0)}(h)$ as being equal to one or zero according as whether i and j are identical or not), then it is known ([46] : §4) that $L_{ij}(h,r)$ is a r-subinvariant measure; furthermore, if y_k is an r-subinvariant measure, then ([46] : lemma 4.1) $\frac{y_j}{y_i} \geq L_{ij}(h,r)$. Since

$$L_{ij}(h,r) = \sum_{h=0}^{\infty} r^n P_{ij}(nh)[\sum_{n=0}^{\infty} r^n P_{ii}(nh)]^{-1}$$

([46] : equation (4)), we have

$$\frac{\xi_j}{\xi_i} \geq \frac{\sum_{n=0}^{\infty} (e^{\lambda h})^n P_{ij}(nh)}{\sum_{n=0}^{\infty} (e^{\lambda h})^n P_{ii}(nh)} .$$

Because the inequality holds for every h, we may allow h to approach zero, and so obtain

$$\frac{\xi_j}{\xi_i} \geq \frac{P_{ij}{}^{o}(-\lambda)}{P_{ii}{}^{o}(-\lambda)}$$

$$= {}_iP_{ij}{}^{o}(-\lambda)[(1-H_i)^{o}(-\lambda)]^{-1},$$

by (1.7). (Note that all the transforms used in the above argument are finite because $\lambda < \alpha$ or E is α-transient.) It may similarly be shown that if ζ_i is any λ-subinvariant vector, then $\frac{\zeta_i}{\zeta_j} \geq G_{ij}^*(-\lambda)$.

By virtue of the uniqueness property of the λ-subinvariant measures and vectors when E is α-recurrent and $\lambda = \alpha$, and the minimality property when E is α-transient or $\lambda < \alpha$, we have the following result:

$$\sum_{k\varepsilon E} {}_iP_{ik}{}^o(-\lambda)|v_k| \qquad (\sum_{k\varepsilon E} |u_k| G_{kj}{}^*(-\lambda))$$

converges if

$$\sum_{k\varepsilon E} \xi_k|v_k| \qquad (\sum_{k\varepsilon E} |u_k|\zeta_k)$$

converges for some λ-subinvariant measure ξ_k (λ-subinvariant vector ζ_k). Let us write $H_k(t) = 1 - e^{-q_k t}$; we may now formulate theorems 3.6 and 3.7 in the following way for continuous-time Markov processes.

<u>Corollary 3.4</u>. Suppose $\lambda \leq \alpha$; and $L_j' = \dfrac{(1-H_j)^o(-\alpha)}{G_{jj}{}^+(-\alpha)} > 0$

if E is α-positive recurrent and $\lambda = \alpha$, $L_j' = 0$ otherwise.

(i) If $\sum_{k\varepsilon E} |u_k|\zeta_k$ converges for some λ-subinvariant vector ζ_k, then $R_j(\lambda,t)$ is finite for every t, and

$$\lim_{t\to\infty} R_j(\lambda,t) = \sum_{k\varepsilon E} u_k\{\lim_{t\to\infty} e^{\lambda t}P_{kj}(t)\} = L_j' \sum_{k\varepsilon E} u_k G_{kj}{}^*(-\lambda).$$

If E is α-positive recurrent and $\lambda = \alpha$, and u_k is real and non-negative, then the convergence of $\sum_{k\epsilon E} u_k \zeta_k$, where ζ_k is the unique λ-invariant vector, is necessary for the existence of $\lim_{t\to\infty} R_j(\lambda,t)$.

(ii) If $\sum_{k\epsilon E} (q_k-\lambda)\xi_k|v_k|$ converges for some λ-subinvariant measure ξ_k, then $S_i(\lambda,t)$ is finite for every t. If in addition $\sum_{k\epsilon E} \xi_k|v_k|$ converges, then

$$\lim_{t\to\infty} S_i(\lambda,t) = \sum_{k\epsilon E} \{\lim_{t\to\infty} e^{\lambda t} p_{ik}(t)\} v_k = \sum_{k\epsilon E} G_{ik}{}^*(-\lambda) L_k' v_k \ .$$

The corollary follows from the preceeding remarks and theorems 3.6 and 3.7; we note that

$$\sum_{k\epsilon E} (q_k-\lambda)\xi_k|v_k| = \sum_{k\epsilon E} [(1-H_k)^o(-\lambda)]^{-1}\xi_k|v_k|$$

$$= \sum_{k\epsilon E} {}_iM_{ik}{}^*(-\lambda)[{}_iP_{ik}{}^o(-\lambda)]^{-1}\xi_k|v_k|$$

(by 1.5a)), which converges only if $\sum_{k\epsilon E} {}_iM_{ik}{}^*(-\lambda)|v_k|$ converges. We note also that conditional on $\sum_{k\epsilon E} \xi_k|v_k|$ converging, $\sum_{k\epsilon E} (q_k-\lambda)\xi_k|v_k|$ converges if the q_k's are bounded, uniformly in k i.e. the means of the holding time distributions are bounded away from zero, uniformly in k.

Our result concerning the convergence of $S_i(\lambda, t)$ finds a use in the theory of 'quasi-stationary distributions'. Suppose E is irreducible but not closed i.e. from some state in E, it is possible to go to some state outside E (and never to return to E, since E is irreducible). This means that for some i in E,

$\sum_{k \in E} Q_{ik}(\infty) = \sum_{k \in E} p_{ik} < 1$; when this is so, an elementary argument shows that E is transient. In many applications, we are interested in the ratio

$$V_{ij}(t) = \frac{P_{ij}(t)}{\sum_{k \in E} P_{ik}(t)}, \quad i, j \in E,$$

which represents the probability of Z_t being in state j at time t, given that it began in state i and is still in the class E at time t. The probability that the process, starting from i in E, is still in E at time t is given by $\sum_{k \in E} P_{ik}(t)$. This probability is obviously a decreasing function of t, and therefore tends to a limit as $t \to \infty$; the limit, which we shall denote by $1-r_i$, may be interpreted as the probability of Z_t remaining for ever in E, given that it started in i. Using the relationship $P_{ik}(t) = {}_jP_{ik}(t) + G_{ij} * P_{jk}(t)$ and the fact that E is irreducible, we see at once that the r_i's are either all equal to one or all less than one. If $r_i < 1$, $\dfrac{P_{ij}(t)}{\sum_{k \in E} P_{ik}(t)}$ tends to zero;

a more interesting quantity for study then is $W_{ij}(t) = \dfrac{P_{ij}(t)r_j}{\sum_{k\varepsilon E} P_{ik}(t)r_k}$

$i,j \varepsilon E$, which is the probability that Z_t is in j at time t, given that it started in i, is still in E at t, AND will eventually escape from E. Note that $V_{ij}(t)$ and $W_{ij}(t)$ coincide when $r_i = 1$ i.e. when with probability one, Z_t eventually escapes from E.

When time is discrete and Z_t Markovian, it has been shown by various authors (Darroch and Seneta [9], Seneta and Vere-Jones [34]) that under certain conditions $W_{ij}(t)$ tends to an honest distribution (the 'quasi-stationary distribution') as $t \to \infty$. Our next theorem extends this result to the general semi-Markov process.

<u>THEOREM 3.8</u>. Suppose the hypotheses of theorem 3.4 hold, and E is not closed. If E is α-positive recurrent, $\sum_{k\varepsilon E} {}_iM_{ik}^*(-\alpha)r_k$ and $\sum_{k\varepsilon E} {}_iP_{ik}^o(-\alpha)r_k$ converge, and hypothesis C holds, then $W_{ij}(t)$ tends to an honest distribution which is independent of the initial state i.

<u>Proof</u>: By theorem 3.4, $e^{\alpha t}P_{ij}(t) \to G_{ij}^*(-\alpha)L_j$, where $L_j = \dfrac{(1-H_j)^o(-\alpha)}{G_{jj}^+(-\alpha)} > 0$;

by theorem 3.7, $\sum_{k\varepsilon E} e^{\alpha t}P_{ik}(t)r_k \to \sum_{k\varepsilon E} G_{ik}^*(-\alpha)L_kr_k$. Thus

$$\lim_{t\to\infty} W_{ij}(t) = \lim_{t\to\infty} \frac{e^{\alpha t} P_{ij}(t) r_j}{\sum_{k\epsilon E} e^{\alpha t} P_{ik}(t) r_k}$$

$$= \frac{G_{ij}^{\ *}(-\alpha) L_j r_j}{\sum_{k\epsilon E} G_{ik}^{\ *}(-\alpha) L_k r_k}$$

$$> 0.$$

The form of the limit shows that it is an honest distribution. To show its independence of i, we multiply both numerator and denominator by $G_{ji}^{\ *}(-\alpha)$; by lemma 3.2, the limit is then equal to $\frac{L_j r_j}{\sum_{k\epsilon E} G_{jk}^{\ *}(-\alpha) L_k r_k}$.

Corollary 3.5. Suppose Z_t is a continuous-time Markov process and E is not closed. If E is α-positive recurrent, and $\sum_{k\epsilon E} (q_k-\alpha) m_k r_k$ and $\sum_{k\epsilon E} m_k r_k$ converge, where m_k is the unique α-invariant measure, then $W_{ij}(t)$ tends to an honest distribution which is independent of the initial state i.

Proof: By theorem 3.5 and corollary 3.4,

$$\lim W_{ij}(t) = \frac{\lim e^{\alpha t} P_{ij}(t) r_j}{\lim \sum e^{\alpha t} P_{ik}(t) r_k} = \frac{m_j r_j}{\sum m_k r_k} \quad .$$

CHAPTER IV

Ratio-Limit Theorems

§1. Preliminary lemmas.

For every j in I^+, let $\alpha(j)$ denote the convergence parameter of the irreducible class containing j, and for all i,j,k in I^+, let

$$\overline{P}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} P_{ij}(x)dx, \qquad {}_k\overline{P}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} {}_kP_{ij}(x)dx,$$

$$\overline{G}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} dG_{ij}(x), \qquad {}_k\overline{G}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} d_kG_{ij}(x),$$

$$\overline{M}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} dM_{ij}(x), \qquad {}_k\overline{M}_{ij}(\lambda,t) = \int_0^t e^{\lambda x} d_kM_{ij}(x).$$

In this chapter we are interested in the asymptotic behaviour of the ratios

$$\frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{k\ell}(\lambda,t)} \quad \text{and} \quad \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{k\ell}(\lambda,t)} \quad \text{under the assumption } G_{k\ell}(\infty) > 0.$$

When $\lambda = 0$, our results have already been obtained by Pyke and Schaufele [32] by means of an Abelian lemma ([32] : lemma 3.1) which utilizes the convolution relationship existing between the quantities $P_{ij}(t)$, $G_{ij}(t)$ and $M_{ij}(t)$. Because of lemma 1.5a, this convolution relationship is preserved when the quantities are replaced by the respective integrals $\overline{P}_{ij}(\lambda,t)$, $\overline{G}_{ij}(\lambda,t)$ and $\overline{M}_{ij}(\lambda,t)$.

Under the assumption $\lambda \leq \alpha(j)$, $\overline{M}_{jj}(\lambda,t)$ is a renewal function, so the boundedness hypothesis of the Abelian lemma (viz. $\overline{M}_{jj}(\lambda,t) - \overline{M}_{jj}(\lambda,t-1) \leq d$ for some $d > 0$ and all t) is satisfied. We may therefore, if we so wish, continue to quote the Abelian lemma, and retain with slight modification the arguments of Pyke and Schaufele, to derive the desired results for $\lambda \geq 0$. But we prefer instead to set down our arguments in full, and illustrate how very simply but effectively the convolution nature of the transition identities may be exploited, by lemma 1.5 and an elementary renewal theoretic result, to yield what we want.

When Z_t is Markovian, our results are partly known. (See Chung [5] for $\lambda = 0$, and Vere-Jones [45] for a discrete-time, irreducible Markov chain with $\lambda = R$, the 'radius of convergence' of the chain.)

<u>Lemma 4.1</u>. For all i,j in I^+, $i \neq j$, if $0 \leq \lambda \leq \alpha(j)$, then

$$\lim_{t\to\infty} \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = G_{ij}^*(-\lambda);$$

where the equation is interpreted to mean $\limsup\limits_{t \to \infty} \dfrac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = \infty$

if $G_{ij}^*(-\lambda)$ diverges.

<u>Proof</u>: Since the lemma is trivially true if $G_{ij}^*(-\lambda) = 0$, we shall assume that $G_{ij}^*(-\lambda) > 0$. Applying lemma 1.5 to equation (1.3), we obtain

$$\int_o^t e^{\lambda x} dM_{ij}(x) = \int_o^t e^{\lambda x} \int_o^{t-x} e^{\lambda u} dM_{jj}(u) dG_{ij}(x) \tag{4.1}$$

and

$$\frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = \int_o^t e^{\lambda x} dG_{ij}(x) - K(t) \tag{4.2},$$

where

$$K(t) = \frac{\int_o^t e^{\lambda x} \int_{t-x}^t e^{\lambda u} dM_{jj}(u) dG_{ij}(x)}{\int_o^t e^{\lambda u} dM_{jj}(u)} \tag{4.3}$$

Suppose firstly that $G_{ij}^*(-\lambda)$ converges (this will be the case if i and j belong to the same irreducible class). It obviously suffices to show that $\lim_{t\to\infty} K(t) = 0$. If $M_{jj}^*(-\lambda)$ converges, then given $\varepsilon > 0$, we can choose T such that

$\int_T^\infty e^{\lambda x} dG_{ij}(x) < \frac{\varepsilon}{2}$, and $T' > T$ such that for all $t \geq T'$,

$\int_{t-T}^\infty e^{\lambda u} dM_{jj}(u) < \frac{\varepsilon}{2} \cdot \frac{\int_o^T e^{\lambda u} dM_{jj}(u)}{G_{ij}^*(-\lambda)}$. Thus for all $t \geq T'$,

$$K(t) = \frac{\int_o^T e^{\lambda x} \int_{t-x}^t e^{\lambda u} dM_{jj}(u) dG_{ij}(x)}{\int_o^t e^{\lambda u} dM_{jj}(u)} + \frac{\int_T^t e^{\lambda x} \int_{t-x}^t e^{\lambda u} dM_{jj}(u) dG_{ij}(x)}{\int_o^t e^{\lambda u} dM_{jj}(u)}$$

$$\leq \frac{G_{ij}^*(-\lambda) \int_{t-T}^t e^{\lambda u} dM_{jj}(u)}{\int_o^t e^{\lambda u} dM_{jj}(u)} + \int_T^t e^{\lambda x} dG_{ij}(x)$$

$$< \varepsilon \; .$$

If $M_{jj}^*(-\lambda)$ diverges, we resort to a renewal theoretic argument. Since $\lambda \leq \alpha(j)$, $\overline{G}_{jj}(\lambda,t)$ is a distribution function (possibly dishonest) by lemma 3.1, so $\overline{M}_{jj}(\lambda,t) = \sum_{n=0}^{\infty} \int_{o}^{t} e^{\lambda x} dG_{jj}^{(n)}(x) = \sum_{n=0}^{\infty} \overline{G}_{jj}^{(n)}(\lambda,t)$ is a renewal function. From a well-known result in renewal theory (See [1] for example), $\overline{M}_{jj}(\lambda,t) - \overline{M}_{jj}(\lambda,t-T)$ is bounded in t for every fixed T, by $C(T)$ say. From (4.3),

$$K(t) \leq \frac{G_{ij}^*(-\lambda)C(T)}{\int_o^t e^{\lambda u} dM_{jj}(u)} + \int_T^t e^{\lambda u} dG_{ij}(u), \quad \text{and}$$

clearly $K(t) \to 0$ as $t \to \infty$.

Suppose now that $G_{ij}^*(-\lambda)$ diverges. From (4.1),

$$\frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} \geq \int_o^T e^{\lambda x} dG_{ij}(x) \frac{\int_o^{t-T} e^{\lambda x} dM_{jj}(x)}{\int_o^t e^{\lambda x} dM_{jj}(x)} \tag{4.4}$$

If $M_{jj}^*(-\lambda)$ converges, then

$$\frac{\int_o^{t-T} e^{\lambda x} dM_{jj}(x)}{\int_o^t e^{\lambda x} dM_{jj}(x)} \to 1; \text{ if } M_{jj}^*(-\lambda) \text{ diverges, then}$$

$$\frac{\int_o^{t-T} e^{\lambda x} dM_{jj}(x)}{\int_o^t e^{\lambda x} dM_{jj}(x)} = 1 - \frac{\int_{t-T}^{t} e^{\lambda x} dM_{jj}(x)}{\int_o^t e^{\lambda x} dM_{jj}(x)} \to 1. \quad \text{Hence, from (4.4),}$$

$$\limsup_{t\to\infty} \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} \geq \int_0^T e^{\lambda x} dG_{ij}(x) \quad \text{for every } T.$$

Lemma 4.1 is false if the condition $\lambda \leq \alpha(j)$ is omitted. As a counter-example, consider once again an alternating process with states 1 and 2, $p_{11} = p_{22} = 0$ and $p_{12} = p_{21} = 1$. Let the holding time in each state have the same negative exponential distribution, $1 - e^{-t}$. The process is Markovian and obviously irreducible and positive recurrent with convergence parameter $\alpha = 0$. Both the first passage times have the same Erlangian distribution i.e. $G_{11}(t) = G_{22}(t) = 1 - e^{-t} - te^{-t}$. Now

$$M_{jj}^*(s) = [1 - G_{11}^*(s)]^{-1} = [1 - (s+1)^{-2}]^{-1} = 1 - [2(s+2)]^{-1} + [2s]^{-1},$$

and consequently $M_{jj}(t) = \frac{t}{2} + \frac{3}{4} + \frac{1}{4}.e^{-2t}$. Let $0 < \lambda < 1$; from (4.3),

$$K(t) \geq \frac{\int_1^t e^{\lambda x} \int_{t-1}^t e^{\lambda u} dM_{jj}(u) dG_{ij}(x)}{\int_0^t e^{\lambda u} dM_{jj}(u)}$$

$$\sim (1 - e^{-\lambda}) \int_1^t e^{\lambda x} dG_{ij}(x).$$

Hence $K(t)$ does not tend to zero, and from (4.2), it is not true that

$$\lim_{t\to\infty} \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = G_{ij}^*(-\lambda).$$ (Note that because $\lambda > \alpha$ in this example, $\overline{M}_{jj}(\lambda,t)$ may not be a renewal function, so renewal theoretic arguments are no longer available to us.)

Lemma 4.2. Under the conditions of lemma 4.1,

$$\lim_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = G_{ij}{}^*(-\lambda)$$

if $G_{ij}{}^*(-\lambda)$ converges, and $\limsup_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = \infty$ if $G_{ij}{}^*(-\lambda)$ diverges.

Proof: From (1.10) and lemma 1.5,

$$\frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = \int_0^t e^{\lambda x} dG_{ij}(x) - K(t),$$

where

$$K(t) = \frac{\int_0^t e^{\lambda x} \int_{t-x}^t e^{\lambda u} P_{jj}(u)\,du\,dG_{ij}(x)}{\int_0^t e^{\lambda u} P_{jj}(u)\,du} .$$

By theorem 3.4, $e^{\lambda t} P_{jj}(t)$ is bounded, and this in turn implies the boundedness of $\int_{t-T}^t e^{\lambda u} P_{jj}(u)du$ for every fixed T. By exactly the same line of argument as was used in the proof of lemma 4.1, $K(t) \to 0$.

Lemma 4.3. For all i,j in I^+ (i,j possibly identical), if $0 \le \lambda \le \alpha(i)$, then

$$\lim_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = {}_iP_{ij}{}^o(-\lambda)$$

if ${}_iP_{ij}{}^o(-\lambda)$ converges, and $\limsup_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = \infty$ if ${}_iP_{ij}{}^o(-\lambda)$ diverges.

Proof: From (1.7) and lemma ~~5.1~~, 1.5,

$$\frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = \frac{\int_o^t e^{\lambda x}\int_o^{t-x} e^{\lambda u}\,{}_iP_{ij}(u)du\,dM_{ii}(x)}{\int_o^t e^{\lambda u}dM_{ii}(u)} .$$

If ${}_iP_{ij}{}^o(-\lambda)$ diverges, it can be easily shown as before that

$\limsup_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = \infty.$ Otherwise,

$$\frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = \int_o^t e^{\lambda x}\,{}_iP_{ij}(x)dx - K(t),$$

where

$$K(t) = \frac{\int_o^t e^{\lambda x}\int_{t-x}^t e^{\lambda u}\,{}_iP_{ij}(u)du\,dM_{ii}(x)}{\int_o^t e^{\lambda x}dM_{ii}(x)}$$

$$\leq \int_T^\infty e^{\lambda u}\,{}_iP_{ij}(u)du + \frac{{}_iP_{ij}{}^o(-\lambda)\int_{t-T}^t e^{\lambda u}dM_{ii}(u)}{\int_o^t e^{\lambda x}dM_{ii}(x)} .$$

The rest of the proof is now obvious.

We end this section by noting that ${}_iP_{ii}{}^o(-\lambda) = (1-H_i)^o(-\lambda)$; and that, by lemma 3.1, the limits in lemmas 4.1 and 4.2 are finite if i and j happen to be in the same irreducible class, while the limit in lemma 4.3 is finite if i and j are in the same irreducible class, and $\lambda > 0$ or $\eta_j < \infty$.

§2. Ratio-limit theorems.

THEOREM 4.1. Suppose i and j are two arbitrary states in I^+, $i \neq j$.

(i) If $\lambda \leq \alpha(j)$, then $\lim_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = G_{ij}^*(-\lambda)$.

(ii) If $\lambda \leq \alpha(i)$, and $\lambda > 0$ or $\eta_i < \infty$, then

$$\lim_{t\to\infty} \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)} = \frac{{}_iP_{ij}{}^o(-\lambda)}{(1-H_i)^o(-\lambda)}.$$

(iii) If $\lambda \leq \min\{\alpha(i),\alpha(j)\}$, $G_{ij}(\infty) > 0$, $G_{ij}^*(-\lambda)$ converges and $\lambda > 0$ or $\eta_i < \infty$, then

$$0 \leq \lim_{t\to\infty} \frac{\overline{P}_{ii}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = \frac{G_{ij}^*(-\lambda)(1-H_i)^o(-\lambda)}{{}_iP_{ij}{}^o(-\lambda)} < \infty.$$

Remarks: (1) The equations in (i) and (ii) are interpreted to mean that the limit superior of the left hand side is infinite if the quantity on the right hand side diverges; in (iii), the limit is zero if ${}_iP_{ij}{}^o(-\lambda)$ diverges.

(2) Since $P_{ij}(t) = G_{ij}*P_{jj}(t) = M_{ii}*{}_iP_{ij}(t)$ (equations (1.10) and (1.7)), $G_{ij}(\infty) > 0$ if and only if ${}_iP_{ij}{}^o(-\lambda) > 0$; furthermore, because every state is assumed stable, $(1-H_i)^o(-\lambda) > 0$. Thus the denominators in the limits in (ii) and (iii) are positive.

(3) The situation is considerably simpler if i and j belong to the same irreducible class. In this case, if $\lambda \le \alpha(i) = \alpha(j)$, and $\lambda > 0$ or $\eta_i < \infty$ and $\eta_j < \infty$, then $G_{ij}(\infty) > 0$, ${}_iP_{ij}{}^o(-\lambda)$, $G_{ij}{}^*(-\lambda)$ and $(1-H_i)^o(-\lambda)$ converge (by lemma 3.1), and all three limits of the theorem are finite and positive.

<u>Proof</u>: (i) is lemma 4.2.

(ii) $$\lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)} = \lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} \cdot \frac{\overline{M}_{ii}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)} \quad .$$

Since by lemma 4.3 $\lim \dfrac{\overline{P}_{ii}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = (1-H_i)^o(-\lambda) > 0$, $\lim\limits_{t\to\infty} \dfrac{\overline{M}_{ii}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)}$ exists

and equals $[(1-H_i)^o(-\lambda)]^{-1} < \infty$; under the stipulated conditions, $[(1-H_i)^o(-\lambda)]^{-1} > 0$. Appealing once again to lemma 4.3, we obtain

$$\lim \frac{\overline{P}_{ij}(\lambda,t)}{P_{ii}(\lambda,t)} = \lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} \lim \frac{\overline{M}_{ii}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)}$$

$$= \frac{{}_iP_{ij}{}^o(-\lambda)}{(1-H_i)^o(-\lambda)}$$

(iii) follows easily from (i) and (ii) by means of the following equations:

$$\lim \frac{\overline{P}_{ii}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = \lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} \left[\lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)}\right]^{-1}$$

$$= \frac{G_{ij}^{*}(-\lambda)(1-H_i)^{o}(-\lambda)}{{}_iP_{ij}^{\;o}(-\lambda)} \quad .$$

For the case $G_{ji}(\infty) > 0$, we have the following result, which we state as a corollary.

<u>Corollary 4.1</u>. If $\lambda \leq \min\{\alpha(i),\alpha(j)\}$, $G_{ji}(\infty) > 0$, $G_{ji}^{*}(-\lambda)$ and ${}_jP_{ji}^{\;o}(-\lambda)$ converge, and $\lambda > 0$ or $\eta_j < \infty$, then

$$0 < \lim_{t\to\infty} \frac{\overline{P}_{ii}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)} = \frac{{}_jP_{ji}^{\;o}(-\lambda)}{G_{ji}^{*}(-\lambda)(1-H_j)^{o}(-\lambda)} < \infty.$$

<u>Proof</u>: From the last part of theorem 4.1, with i and j interchanged,

$$\infty > \lim_{t\to\infty} \frac{\overline{P}_{jj}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)} = \frac{G_{ji}^{*}(-\lambda)(1-H_j)^{o}(-\lambda)}{{}_jP_{ji}^{\;o}(-\lambda)} > 0.$$

The corollary follows at once by taking reciprocals in the above.

It is evident that with some appropriate bound on λ, and assuming finiteness of the relevant quantities, theorem 4.1 enables one to evaluate the limits of all ratios of the form

$\dfrac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{k\ell}(\lambda,t)}$, where i,j,k,ℓ are arbitrary states of I^{+}, $G_{k\ell}(\infty) > 0$,

and it is possible to travel between the sets of states $\{i,j\}$ and

$\{k,\ell\}$ in at least one direction. For example, suppose $i \neq j$, $i \neq k$, $k \neq \ell$, $0 < G_{ik}^{*}(-\lambda) < \infty$, $G_{k\ell}(\infty) > 0$, ${}_iP_{ij}^{o}(-\lambda)$, ${}_iP_{ik}^{o}(-\lambda)$ and ${}_kP_{k\ell}^{o}(-\lambda)$ converge, and $0 < \lambda \leq \min\{\alpha(i),\alpha(k)\}$. Then, by theorem 4.1,

$$\lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{k\ell}(\lambda,t)} = \lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{ii}(\lambda,t)} \lim \frac{\overline{P}_{ii}(\lambda,t)}{\overline{P}_{kk}(\lambda,t)}\left[\lim \frac{\overline{P}_{k\ell}(\lambda,t)}{\overline{P}_{kk}(\lambda,t)}\right]^{-1}$$

$$= \frac{{}_iP_{ij}^{o}(-\lambda)G_{ik}^{*}(-\lambda)(1-H_k)^{o}(-\lambda)}{{}_iP_{ik}^{o}(-\lambda)\,{}_kP_{k\ell}^{o}(-\lambda)}$$

$$< \infty.$$

<u>THEOREM 4.2.</u> Suppose i and j are two arbitrary states in I^{+}, $i \neq j$.

(i) If $\lambda \leq \alpha(j)$, then $\lim\limits_{t\to\infty} \dfrac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = G_{ij}^{*}(-\lambda)$.

(ii) If $\lambda \leq \min\{\alpha(i),\alpha(j)\}$, $G_{ij}^{*}(-\lambda)$ converges, and $\lambda > 0$ or $\eta_j < \infty$, then

$$\lim_{t\to\infty} \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = {}_iM_{ij}^{*}(-\lambda)\ .$$

(iii) If $\lambda \leq \min\{\alpha(i),\alpha(j)\}$, $G_{ji}(\infty) > 0$, $G_{ji}^{*}(-\lambda)$ converges and $\lambda > 0$ or $\eta_i < \infty$, then

$$\lim_{t\to\infty} \frac{\overline{M}_{ii}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = \frac{{}_jM_{ji}^{*}(-\lambda)}{G_{ji}^{*}(-\lambda)}\ .$$

The equations in (i), (ii), (iii) are all given the usual interpretation if $G_{ij}{}^*(-\lambda)$, ${}_iM_{ij}{}^*(-\lambda)$ or ${}_jM_{ji}{}^*(-\lambda)$ diverges.

<u>Proof</u>: (i) is lemma 4.1.

(ii) is trivially true if $G_{ij}(\infty) = 0$, so assume $G_{ij}(\infty) > 0$. Then

$$\lim \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)} = \lim \frac{\overline{M}_{ij}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} \left[\lim \frac{\overline{P}_{jj}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)}\right]^{-1} \left[\lim \frac{\overline{P}_{ij}(\lambda,t)}{\overline{P}_{jj}(\lambda,t)}\right]^{-1} \lim \frac{P_{ij}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)}$$

$$= G_{ij}{}^*(-\lambda)[(1-H_j)^o(-\lambda)]^{-1}[G_{ij}{}^*(-\lambda)]^{-1}{}_iP_{ij}{}^o(-\lambda)$$

(by the preliminary lemmas in §1)

$$= {}_iM_{ij}{}^*(-\lambda)$$

(by (1.5a))

(iii) $$\lim \frac{\overline{M}_{ii}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} = \lim \frac{\overline{M}_{ji}(\lambda,t)}{\overline{M}_{jj}(\lambda,t)} \left[\lim \frac{\overline{M}_{ji}(\lambda,t)}{\overline{M}_{ii}(\lambda,t)}\right]^{-1}$$

$$= {}_jM_{ji}{}^*(-\lambda)[G_{ji}{}^*(-\lambda)]^{-1}$$

(by (i) and (ii).

When $\lambda = 0$ all the conditions in theorems 4.1 and 4.2, excepting those involving the finiteness of the means η_i and η_j, are trivially satisfied. In this special case, lemma 4.3, the first two parts of theorem 4.1, and theorem 4.2 have been proved by Pyke and Schaufele [32]. They obtained different expressions for theorems 4.1(ii) and 4.2(iii), but it is easy to see that these are equivalent to ours. For instance, in theorem 4.1(ii),

$$\lim \frac{\overline{P}_{ij}(0,t)}{\overline{P}_{ii}(0,t)} = \lim \frac{\int_o^t P_{ij}(x)dx}{\int_o^t P_{ii}(x)dx} = \frac{{}_iP_{ij}{}^o(0)}{(1-H_i)^o(0)} = \frac{(1-H_i)^o(0)\,{}_iM_{ij}{}^*(0)}{(1-H_i)^o(0)} = \frac{\eta_j}{\eta_i}\,{}_iM_{ij}{}^*(0),$$

which agrees with Pyke and Schaufele's corollary 3.1.

When Z_t is Markovian and t discrete, it is again easy to show, and we discuss below a simple example to illustrate our assertion, that our results yield a theorem of Vere-Jones ([45] : theorem III). In the discrete time case, summation takes the place of integration, and instead of a convergence parameter α, we have a 'radius of convergence' R, which is of course common to all transitions if the chain is irreducible. Vere-Jones proved that for all states i,j,k,ℓ in an irreducible, aperiodic, R-recurrent chain,

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} p_{ij}^{(n)}R^n}{\sum_{n=0}^{N} p_{k\ell}^{(n)}R^n} = \frac{x_i m_j}{x_k m_\ell} ,$$

where $p_{ij}^{(n)}$ is the n-step transition probability, and x_i and m_j are respectively the R-invariant vector and R-invariant measure discussed in §4 of the previous chapter. Thus

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} p_{ij}^{(n)} R^n}{\sum_{n=0}^{N} p_{jj}^{(n)} R^n} = \frac{x_i}{x_j}$$

$$= \frac{G_{i1}^*(-R)}{G_{j1}^*(-R)} \qquad \text{(by theorem 3.5)}$$

$$= \frac{G_{i1}^*(-R)G_{1j}^*(-R)}{G_{j1}^*(-R)G_{1j}^*(-R)}$$

$$= G_{ij}^*(-R) \qquad \text{(by lemma 3.2)},$$

which is in agreement with our theorem 4.1.

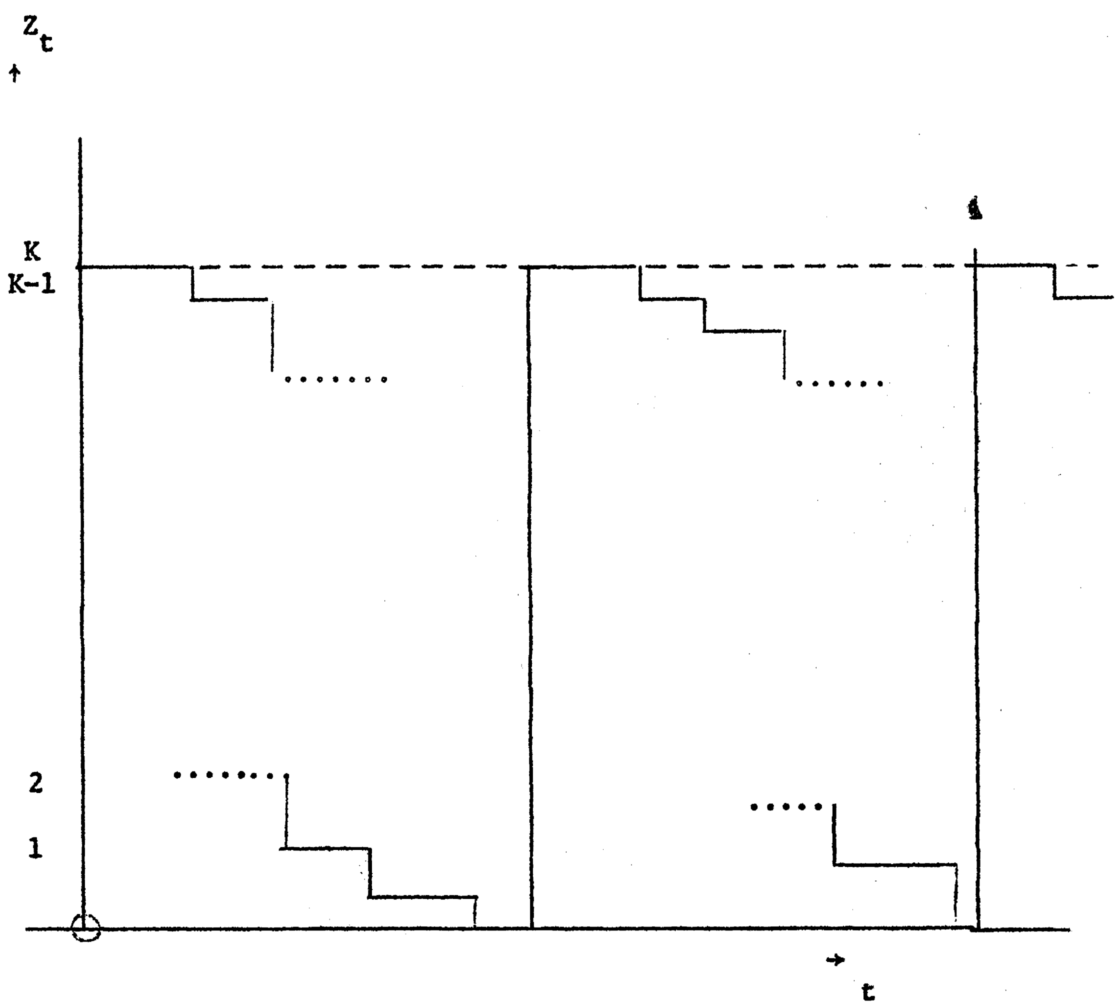
CHAPTER V

Applications

§1. An inventory model.

In this section we apply the familiar analytical arguments used throughout this thesis, and the theorems proved in the previous chapters, to examine an inventory model which is a special case of the (S,s) model (see [29]) with $s = 0$. The model which we shall consider is the following. A store contains material in discrete units, has a finite capacity K, and receives demands for goods at moments which are the regeneration points of a renewal process $\{X_n : n = 1,2,\ldots\}$. We denote the random variable representing the time between the ith. and (i+1)th. demands by X_i and write $\Pr[X_i \leq t] = X(t)$. Let $S_0 = 0$, $S_n = \sum_{j=1}^{N} X_j$ $(n > 1)$, $X^{(n)}(t) = \Pr[S_n \leq t]$, and let $M(t) = \sum_{n=0}^{\infty} X^{(n)}(t)$ be the renewal function of the process $\{X_n\}$. Every demand is for exactly one item; as soon as the store becomes empty it replenishes itself by ordering K items from the supplier. No backlog is kept, and demands made when the store is empty are rejected. A time-lag, Y, exists between the ordering and the arrival of supplies, and Y is a random variable with the distribution $Y(t)$. It will be assumed that $X(t)$ and $Y(t)$ are honest distribution functions.

Let Z_t be the amount of stock at time t; Z_t is a stochastic process whose state space is the finite set $E = \{0,1,2,\ldots,K\}$, and our first aim is to see if Z_t is semi-Markov. The cyclic manner in which Z_t evolves is very clear: from every state $i > 0$, Z_t proceeds to $i-1$ in one transition; when it gets down to zero, the only possible state after the next transition is K. If Z_t is in



A possible path of Z_t

state 0, it obviously remains in it for as long as the supplies take to arrive, so the holding time in the state is distributed as Y and is independent of the past and future states of the process. This independence of past and future states is easily seen to apply to any other state $i < K$, the holding time of which must have the same distribution as X_1. The holding time in state K is the time that elapses between the arrival of fresh supplies for the store and the receipt of the next demand; if we consider the ordering renewal process $\{X_n\}$ as having begun at the last time point at which the store became empty, then the holding time in state K is simply the forward recurrence-time (see Cox [7]) of $\{X_n\}$ at time u, given that $Y = u$. Denote the forward recurrence-time of X_n at time u by ξ_u; then

$$\Pr[\xi_u \leq t] = \sum_{n=1}^{\infty} \Pr[u < S_n \leq u+t < S_{n+1}]$$

$$= \int_u^{u+t} \{1-X(u+t-v)\} dM(v),$$

and the unconditional holding time in K is given by

$$H_K(t) = \int_0^{\infty} \Pr[\xi_u \leq t] dY(u) \qquad (5.1).$$

Since $\Pr[\xi_u \leq t]$ is an honest distribution function (the easiest way to show this is to use the key-renewal theorem), so obviously is $H_K(t)$. Note that if we are told that the time spent in the 0th. state immediately preceding the Kth. state is of length y, then the

<u>conditional</u> distribution of the holding time in the state K is $\Pr[\xi_y \leq t]$. The only situation in which ξ_y is independent of y is that which occurs when $\{X_n\}$ is a Poisson process. Thus Z_t cannot be semi-Markov unless $\{X_n\}$ is Poisson i.e. unless $X(t)$ is a negative-exponential distribution.

We make the following assumption about the model: from the moment, τ say, at which supplies arrive, the time to the next demand is independent of what has happened prior to τ, and its distribution is given by (5.1). From the preceeding discussion it is now clear that Z_t is a semi-Markov process with the following one-step transition probabilities and unconditional holding time distributions:

$$Q_{K,K-1}(t) = H_K(t) = \int_0^\infty \int_u^{u+t} \{1-X(u+t-v)\}dM(v)dY(u) \tag{5.2}$$

$$Q_{i,i-1}(t) = H_i(t) = X(t) \qquad (0 < i < K) \tag{5.3}$$

$$Q_{0K}(t) = H_0(t) = Y(t) \tag{5.4}$$

and $\qquad Q_{ij}(t) = 0$ otherwise.

Since $Q_{0K}(\infty) = Q_{i,i-1}(\infty) = 1 \quad (i > 0)$, E is irreducible.

(It is of incidental interest to observe that Z_t is a special case of the 'cyclic-k-state process' defined and discussed in [2].)

We proceed now to study the transition probabilities $P_{ij}(t)$, and to do this we require a knowledge of the first passage distributions. The first entrance time from $K-1$ to K is the sum of two independent random variables: one (the time taken to go from $K-1$ to 0 for the first time) is distributed as S_{K-1}, and the other (the time-lag of the supplying process) has the distribution $Y(t)$. Thus

$$G_{K-1,K}(t) = X^{(K-1)} * Y(t) \tag{5.5}$$

In the same way we obtain, for every $i < K$:

$$G_{KK}(t) = H_K * G_{K-1,K}(t) \tag{5.6}$$

$$G_{iK}(t) = X^{(i)} * Y(t) \tag{5.7}$$

$$G_{Ki}(t) = H_K * X^{(K-1-i)}_{\cancel{K-1-i}}(t) \tag{5.8}$$

$$G_{ii}(t) = G_{iK} * G_{Ki}(t) \tag{5.9}$$

(In the equations above $X_0(t)$ is defined to be identically one for $t \geq 0$.) By (5.5) and (5.6) $G_{KK}(\infty) = 1$, so by irreducibility and solidarity (theorem 3.3), E is recurrent. To obtain the means of G_{ii} we need to know the means of X_1, Y and H_K. Suppose X_1 and Y have means η and θ respectively, and denote the mean of H_i by η_i as usual. From (5.4) and (5.3) $\eta_0 = \theta$ and $\eta_i = \eta$ for $0 < i < K$; and from (5.1) $\eta_K = \int_0^\infty E[\xi_t]dY(t)$. We know also that $\eta_K \leq \eta$; this must be so since the interval of time spent in K is contained in a time interval between two successive demands. The means of $G_{ii}(t)$ can now be obtained from (5.5) and (5.6):

$$\mu_{ii} = \mu_{KK} = \eta_K + \mu_{K-1,K} = \eta_K + (K-1)\eta + \theta. \tag{5.10}$$

We digress here briefly to discuss an interesting aspect of semi-Markov processes that distinguishes it from Markov processes. We see from (5.10) that Z_t is positive-recurrent if and only if both η and θ are finite; thus a semi-Markov process, unlike a Markov one, may not be positive-recurrent even when the state space is finite and irreducible. This cannot happen, however, if every η_j is finite; for then, by a theorem of Pyke ([30] : theorem 5.1), a state in a finite class E is positive-recurrent if and only if it is positive-recurrent in the imbedded Markov chain $\{Z_{N_t} : t > 0\}$.

With the knowledge of η_i and μ_{ii}, theorem 3.4 yields at once the limits of the transition probabilities $P_{ij}(t)$.

<u>THEOREM 5.1</u>. Z_t is positive-recurrent if and only if η and θ are both finite. If Z_t is positive-recurrent, and either $X(t)$ or $Y(t)$ is non-lattice, then

$$\pi_j = \lim_{t\to\infty} P_{ij}(t) = \begin{cases} \theta\tau & \text{if } j = 0 \\ \eta\tau & \text{if } 0 < j < K \\ \eta_K\tau & \text{if } j = K, \end{cases}$$

where $\tau = [\eta_K + (K-1)\eta + \theta]^{-1}$.

Note that because the convergence parameter is zero, the bounded variation condition of theorem 3.4 is trivially satisfied. The limiting distribution may alternatively be obtained by using theorem 1, chapter two, of [2].

Suppose at $t = 0$ the store has just been restocked. Quantities of interest are $G_{K0}(t)$, $P_{Kj}(t)$ ($P_{K0}(t)$ in particular) and $P_{Kj}(t)[\sum_{i=1}^{K} P_{Ki}(t)]^{-1}$, which give respectively the probabilities that the store will not be empty till time t at least, the store contains j items at time t, and the store contains j items at time t given that it has not been empty till time t. $G_{K0}(t)$ may be obtained from (5.8); to obtain $P_{ij}(t)$ we use Laplace-Stieltjes transforms and enlist the aid of the transition identities of §3, chapter one.

For $0 < j < K$

$$P_{jj}^{*}(s) = \frac{1-H_j^*(s)}{1-G_{jj}^{*}(s)}$$

$$= \frac{1-X^*(s)}{1-H_K^*(s)Y^*(s)[X^*(s)]^{K-1}}$$

by (5.7), (5.8) and (5.9); similarly,

$$P_{00}^{*}(s) = \frac{1-Y^*(s)}{1-H_K^*(s)Y^*(s)[X^*(s)]^{K-1}}$$

and

$$P_{KK}^{*}(s) = \frac{1-H_K^*(s)}{1-H_K^*(s)Y^*(s)[X^*(s)]^{K-1}} \quad .$$

Since $P_{Kj}^*(s) = G_{Kj}^*(s)P_{jj}^*(s)$ for $j \neq K$, we have by (5.8),

$$P_{K0}^*(s) = \frac{H_K^*(s)[X^*(s)]^{K-1}[1-Y^*(s)]}{1-H_K^*(s)Y^*(s)[X^*(s)]^{K-1}}$$

and

$$P_{Kj}^*(s) = \frac{H_K^*(s)[X^*(s)]^{K-1-j}[1-X^*(s)]}{1-H_K^*(s)Y^*(s)[X^*(s)]^{K-1}}$$

$(0 < j < K)$. Inverting the transforms gives us the desired probabilities.

Having derived the time-dependent transition probabilities $P_{ij}(t)$ and evaluated their limits Π_j, we now wish to know the rate at which $P_{ij}(t)$ converges to Π_j. An examination of the form of the individual P_{ij}'s will provide the answer, but a less exact answer can be given more easily. By theorem 2.2 we know that the process cannot be geometrically ergodic unless $X(t)$ and $Y(t)$ are such that their tails tend to zero exponentially fast. Conversely, if both the tails approach zero exponentially fast, and either

$\limsup_{|\gamma|\to\infty}|X^*(-i\gamma)| < 1$ or $\limsup_{|\gamma|\to\infty}|Y^*(-i\gamma)| < 1$, then by theorem 2.3 the process is geometrically ergodic. (Note that $1 - H_K(t) \leq 1 - X(t)$.)

Let us now consider a particular model in which the time-lag has a constant value, one say, and $X(t)$ is negative-exponential with parameter λ. Thus $Y(t) = U_1(t)$, $X(t) = 1 - e^{-\lambda t}$, $\theta = 1$ and $\eta = \frac{1}{\lambda}$. Even without using (5.2) we know at once, from the peculiar nature of the negative-exponential distribution, that $H_K(t) = 1 - e^{-\lambda t}$. The conditions of theorem 5.1 are easily satisfied, so

$$\lim_{t\to\infty} P_{ij}(t) = \begin{cases} \frac{\lambda}{K+\lambda} & \text{if } j = 0 \\ \frac{1}{K+\lambda} & \text{if } j > 0 \, . \end{cases}$$

The probability $P_{K0}(t)$ is given by

$$P_{K0}{}^{*}(s) = \frac{(\frac{\lambda}{\lambda+s})^K [1-e^{-s}]}{1 - (\frac{\lambda}{\lambda+s})^K e^{-s}} \, .$$

If both $X(t)$ and $Y(t)$ are negative-exponential, Z_t is a Markov process and the problem simplifies considerably.

A slightly different and more difficult model than ours has been studied by Fabens [12] (see also [13]). In Faben's model $X(t)$ is a gamma distribution, an order is sent out for $K-k$ $(k < K)$ items whenever the stock reaches a fixed level k, and a backlog is kept, so that the state space of Z_t is the infinite set of integers in $(-\infty,K]$. By means of semi-Markov analysis, Fabens obtained the limiting probability distribution of the stock level.

§2. Quasi-stationary distributions in continuous-time Markov branching processes.

Let $\{Z_t;\ t \geq 0\}$ be the size of the population at time t of a continuous-time, time-homogeneous Markov branching process; denote by f_k the probability that a dying particle is replaced by k objects $(k = 0,1,2,\ldots;\ f_1 = 0)$, and by $\lambda\,\Delta t$ the probability that a particle existing at time t will die in the interval $(t, t+\Delta t)$. Z_t is a Markov process which has 0 as an absorbing state, and one-step transition probabilities and unconditional holding time distributions given, for $i > 0$, by: $Q_{i,i-1}(t) = f_0(1-e^{-i\lambda t})$, $Q_{i,i+k}(t) = f_{k+1}(1-e^{-i\lambda t})$ for $k \geq 1$, $Q_{ij}(t) = 0$ otherwise, and $H_i(t) = 1 - e^{-i\lambda t}$.

Let $f(s) = \sum_{k=0}^{\infty} f_k s^k$ and $F_i(s,t) = \sum_{k=0}^{\infty} P_{ik}(t)s^k$, $|s| \leq 1$; it is well-known that $F_i(s,t) = \{F_1(s,t)\}^i$ and $F_1(s,t_1+t_2) = F_1[F_1(s,t_1),t_2]$ for every $t,t_1,t_2 \geq 0$ (see [17], for example). $P_{i0}(t)$ is the probability that the process with an initial population of i individuals will become extinct by time t; it is obviously an increasing function of t, and its limit, which we shall denote by r_i, gives the probability of eventual extinction; putting $s = 0$ in the equation $F_i(s,t) = \{F_1(s,t)\}^i$ and taking limits, we get $r_i = r^i$, where $r = r_1$. If we write

$f'(s) = \frac{df(s)}{ds}$, $F_i'(s,t) = \frac{\partial F_i(s,t)}{\partial s}$, and $m = f'(1)$ for the mean number of off-spring per individual, then it is again well-known that $r = 1$ (i.e. extinction is certain) if and only if $m \leq 1$.

We assume that $0 < f_0 < 1$ and $f_j > 0$ for some $j > 1$; under these conditions, $1 \geq r_i > 0$ for every $i > 0$, and the class $E = \{i : i > 0\}$ is irreducible. The main purpose of this section is to use corollary 3.5 to investigate the size of the population size at time t, conditional on extinction not having occured till then. Specifically, we wish to prove

<u>THEOREM 5.2</u>. If $m > 1$, $W_{ij}(t)$ tends to an honest distribution (over E) as $t \to \infty$, for all i,j in E; if $m < 1$, $V_{ij}(t)$ tends to an honest distribution if $\sum_{k=2}^{\infty} (k \log k) f_k$ converges. The limit distributions are independent of the initial states i.

(Recall that $W_{ij}(t) = \dfrac{P_{ij}(t) r_j}{\sum_{k=1}^{\infty} P_{ik}(t) r_k}$ and $V_{ij}(t) = \dfrac{P_{ij}(t)}{\sum_{k=1}^{\infty} P_{ik}(t)}$.)

For the discrete-time Galton-Watson process, theorem 5.2 was proved by Seneta and Vere-Jones [34]; their result was an improvement on Yaglom's [49], which required the more restrictive condition $f''(1) < \infty$. In another paper ([18]) the two authors, in collaboration with Heathcote, showed that when $m < 1$, the quasi-stationary distribution in fact exists without additional conditions. When time is continuous theorem 5.2 is a step forward from the corresponding result of Sevast'yanov [35] which, like Yaglom's, needed the finiteness of $f''(1)$. Our result, however, is weaker than that of Zolotarev [50], who obtained the limiting form of the generating function of $V_{ij}(t)$ under the condition $1 < m < \infty$ or $m < 1$. Theorem 5.2 is therefore not new, and the main reason for its inclusion in this thesis is that it illustrates very well the usefulness of corollary 3.5.

The proof of theorem 5.2 will rely on lemma A below, which we shall apply to Z_t by means of discrete skeleton arguments and the following result of Conner [6].

<u>Conner's theorem.</u> For every positive integer n and every positive number h, $\sum_{k=2}^{\infty} (k^n \log k) f_k$ converges if and only if $\sum_{k=2}^{\infty} (k^n \log k) P_{1k}(h)$ converges.

<u>Lemma A.</u> ([34] : section 5, lemma (iii)): Let $\{Z_n;\ n \geq 0\}$ be a Galton-Watson process and suppose the class $E = \{i : i > 0\}$ is irreducible. The convergence parameter R is equal to m^{-1} if $m \leq 1$, and to $[f'(r)]^{-1}$ if $m > 1$. When $m < 1$, E is R-positive recurrent if and only if $\sum_{k=2}^{\infty} (k \log k) f_k$ converges; when $m > 1$, E is always R-positive recurrent.

For every $h > 0$ let us denote the discrete skeleton (on scale h) of Z_t by $Z_n(h)$. $Z_n(h)$ is a discrete-time Markov chain governed by the transition matrix $P_{ij}(h)$; and its n-step transition probabilities, which we shall denote by $P_{ij}^{(n)}(h)$, satisfy $P_{ij}^{(n)}(h) = P_{ij}(nh)$. For every $h > 0$ E is obviously irreducible in Z_t if and only if it is irreducible in $Z_n(h)$; to avoid confusion, the irreducible class E will henceforth be written E(h) if the process under consideration is $Z_n(h)$.

Lemma 5.1. E is α-positive recurrent if and only if E(h) is $e^{\alpha h}$-positive recurrent for every $h > 0$.

Proof: Because of solidarity (theorem 3.3) we need consider only one state in E, j say.

If E is α-positive recurrent $e^{\alpha t}P_{jj}(t) \to L_j > 0$ as $t \to \infty$ (theorem 3.4), whence $e^{\alpha nh}P_{jj}(nh) = (e^{\alpha h})^n P_{jj}^{(n)}(h) \to L_j$ as $n \to \infty$, for every $h > 0$.

To prove the converse we require a theorem of Kingman ([21] : theorem 2) which states that if g is a continuous function from the positive reals to the reals, and for every $h > 0$ $\lim_{n\to\infty} g(nh)$ exists and is finite, then this limit is independent of h and $g(t) \to \lim_{n\to\infty} g(nh)$ as $t \to \infty$. (Kingman's theorem is based on a result of Croft [8].)

Now suppose that for every $h > 0$ $E(h)$ is $e^{\alpha h}$-positive recurrent. Then for every $h > 0$ $e^{\alpha nh}P_{jj}(nh)$ tends to a positive, finite limit $L(h)$; since $e^{\alpha t}P_{jj}(t)$ is continuous (see [5]), Kingman's theorem allows us to conclude that $L(h)$ is independent of h and $e^{\alpha t}P_{jj}(t) \to$ $\to L(h) > 0$ as $t \to \infty$.

Let $a = -\lambda(m-1)$ and $b = -\lambda(f'(r)-1)$.

Lemma 5.2. Let α be the convergence parameter of E. Then $\alpha = a$ if $m \leq 1$ and $\alpha = b$ if $m > 1$.

Proof: Since $a = b$ when $m \leq 1$ and $f'(r)$ consequently equals m, it suffices to show that $\alpha = b$ in the general case $m \lesseqgtr 1$.

From $F_1(s,t_1+t_2) = F_1[F_1(s,t_1),t_2]$ we have $P_{11}(t_1+t_2) = F_1'(0,t_1+t_2) = F_1'[F_1(0,t_1),t_2]F_1'(0,t_1)$ and

$$\frac{P_{11}(t_1+t_2)}{P_{11}(t_1)} = F_1'[F_1(0,t_1)t_2] \qquad (5.11).$$

As $F_1(0,t) = P_{10}(t)$ increases monotonically to r,

$$P_{11}(t+1) \leq P_{11}(1)F_1'(r,t) \qquad (5.12)$$

$F_1'(r,t)$ is given by the equation

$$F_1'(r,t) = \exp \lambda\int_0^t\{f'(F_1(r,x)) - 1\}dx \qquad (5.13)$$

(see Harris [17] : page 121, equation (2)). For every t the monotone convergence theorem applied to $P_{10}(t+\tau) = \sum_{k=0}^{\infty} P_{1k}(t)P_{k0}(\tau)$ yields $r = \sum_{k=0}^{\infty} P_{1k}(t)r^k$. Hence $f'(F_1(r,t)) = f'(r)$ for all t, and from (5.12) and (5.13),

$$F_1'(r,t) = e^{-bt} \tag{5.14}$$

and $P_{11}(t+1) \leq P_{11}(1)e^{-bt}$; it is now clear that the convergence parameter $\geq b$. (Note that we have obtained $P_{11}(t) = 0(e^{-bt})$ without any condition cf. equation (21) and theorem 10 of Zolotarev [50].)

We now have to show that the convergence parameter $\leq b$. Since $F_1(0,t) \to r$ as $t \to \infty$,

$$\lim_{n\to\infty} \frac{P_{11}(n+1)}{P_{11}(n)} = F_1'(r,1) = e^{-b}$$

by (5.11), (5.13) and the monotone convergence theorem; hence given any $\varepsilon > 0$, there exists M such that

$P_{11}((n+1)M) \geq P_{11}(M)(e^{-b}-\varepsilon)^{nM}$, which shows that $P_{11}{}^{o}(-\lambda)$ diverges whenever $\lambda > b$.

<u>Proof of theorem 5.</u> For every $h > 0$ let $m(h)$ be the mean number of off-spring per individual for the discrete skeletonic chain $Z_n(h)$; then $m(h) = \sum_{k=1}^{\infty} kP_{1k}(h) = e^{-ah}$ (see [17]).

Consider firstly the case $m > 1$. We then have $a < 0$ and $m(h) > 1$; by lemma A $E(h)$ is $\alpha(h)$-positive recurrent, where

$$\alpha(h) = [\sum_{k=1}^{\infty} kP_{1k}(h)r^{k-1}(h)]^{-1} = [F_1'(r(h),h)]^{-1}$$ and $r(h)$ is the probability of extinction of the $Z_n(h)$ chain. As $r(h) = \lim_{n\to\infty} P_{10}^{(n)}(h) = r$, we have $\alpha(h) = e^{bh}$ by (5.14); so by lemma 5.1, E is b-positive recurrent. Now suppose $m < 1$ and $\sum_{k=2}^{\infty}(k\log k)f_k$ converges. By Conner's theorem $\sum_{k=2}^{\infty}(k\log k)P_{1k}(h)$ converges for every h, whence by lemma A $E(h)$ is $\alpha(h)$-positive recurrent for every h, where $\alpha(h) = [m(h)]^{-1} = e^{ah}$. Finally, by lemma 5.1, E is a-positive recurrent.

We have thus shown that, under the conditions of the theorem, E is a-positive recurrent if $m < 1$, and b-positive recurrent if $m > 1$. Hence, if m_k and x_k are respectively the unique α-invariant measure and vector, then $\sum_{k=1}^{\infty} m_k x_k$ converges (theorem 3.5). If $m < 1$ (and $r = 1$) $\sum_{k=1}^{\infty} kP_{ik}(t) = ie^{-at}$, so $x_k = k$ is the invariant vector; we then have $\sum_{k=1}^{\infty}(q_k-a)m_k = \sum_{k=1}^{\infty}(k\lambda-a)m_k<\infty$ and $\sum_{k=1}^{\infty} m_k < \infty$. By corollary 3.5 $V_{ij}(t)$ tends to an honest distribution which is independent of the initial state i. If $m > 1$ (and $0 < r < 1$), then $F_1'(r,t) = e^{-bt}$ by (5.14). Differentiating the relation $F_i(s,t) = [F_1(s,t)]^i$ and using induction, we can easily show that $F_i'(r,t) = ir^{i-1}F_1'(r,t)$, from which it follows that $ir^{i-1}e^{-bt} = \sum_{k=1}^{\infty} kP_{ik}(t)r^{k-1}$. Hence $x_i = ir^{i-1}$ is the unique b-invariant vector and as before, $\sum_{k=1}^{\infty}(q_k-b)m_kr^k = \sum_{k=1}^{\infty}(k\lambda-b)m_kr^k < \infty$ and $\sum_{k=1}^{\infty} m_kr^k < \infty$. The proof can now be completed by appealing once again to corollary 3.5.

When $m = 1$ the convergence parameter is zero by lemma 5.2, so E must be transient and corollary 3.5 cannot be used. However, results for this case have been obtained through other means by Zolotarev [50].

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Key to References

AMS:	The Annals of Mathematical Statistics
Berk. Symp.:	Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability
Bio.:	Biometrika
Doklady:	Doklady Akademii Nauk SSSR
JAP:	Journal of Applied Probability
JMAA:	Journal of Mathematical Analysis and Applications
JRSS:	Journal of the Royal Statistical Scoiety
PAMS:	Proceedings of the American Mathematical Society
PICM:	Proceedings of the International Congress of Mathematicians
PJM:	Pacific Journal of Mathematics
PLMS:	Proceedings of the London Mathematical Society
PNAS:	Proceedings of the National Academy of Sciences, U.S.A.
PRS:	Proceedings of the Royal Society
PRSE:	Proceedings of the Royal Society of Edinburgh
QJM:	The Quarterly Journal of Mathematics (Oxford.)
TAMS:	Transactions of the American Mathematical Society
TV:	Teoriya Veroyatnostei i ee Primeneniya
Z. Wahr.:	Zeitschrift für Wahrscheinlichkeitstheorie