Convergence Results for Variational Regularization and Landweber Iteration Under Heuristic Rules

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

We present general convergence results on the variational regularization and the Landweber iteration for inverse problems. First, we consider the variational regularization for inverse problems in a general form. We propose a heuristic parameter choice rule for choosing the regularization parameter which does not require the information on the noise level and is therefore purely data driven. Under variational source conditions, we obtain *a posteriori* error estimates. A variant of the same heuristic rule is also proposed for the Landweber iteration for solving linear and as well as nonlinear inverse problems in Banach spaces. We present a new convergence result which requires neither the Gâteaux differentiability of the forward operator nor the reflexivity of the image space. To address the slow convergence, we also present a new convergence result under the heuristic rule for an accelerated Landweber iteration for linear inverse problems in Hilbert spaces. According to the Bakushinskii veto, convergence in the worst case scenario can not be expected in general. However, by imposing certain conditions on the random noise, we establish convergence results for the heuristic rule. Applications of the results are addressed and numerical simulations are reported.

Contents

Chapter 1

Introduction

Inverse problems are all over science and engineering. They deal with searching for a certain cause based on an observed or desired effect. A common example is electrical impedance tomography: finding the electrical conductivity of a medium by creating voltage and current measurements at the boundary of the medium. This problem has been applied in searching petroleum underground and generating medical images from inside the human body. Mathematically, inverse problems can be formulated in this way:

find the solution q of an equation $F(q) = u$ for some u.

Usually we think of q as the parameter, while u as the data. This is how inverse problems are typically formulated. These problems deal with determining causes for a desired or observed effect. They occur in many areas, such as imaging, physics, and finances (c.f. [16],[77]).

First, we focus on a general classic formulation. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces equipped with the usual norm $\|\cdot\|$ and duality mapping $\langle \cdot, \cdot \rangle_{\mathcal{X}^*, \mathcal{Y}^*}$, where $\mathcal{X}^*$ and $\mathcal{Y}^*$ are the corresponding dual spaces. We focus on inverse problems of the form

$$F(x) = y, \tag{1.0.1}$$

where $F : \mathcal{D}(F) \subseteq \mathcal{X} \to \mathcal{Y}$ is any operator, linear or nonlinear, between $\mathcal{X}$ and $\mathcal{Y}$. Inherently, inverse problems are ill-posed, leading to stability issues in solving (1.0.1). Instead of exact data y, a noisy data $y^\delta \in \mathcal{Y}$ is available satisfying

$$\left\|y^\delta - y\right\| \leq \delta$$

with noise level $\delta > 0$. To overcome ill-posedness, we use regularization, which involves solving a sequence of related well-posed problems to obtain stable approximate solutions of (1.0.1), called regularized solutions. Doing this requires rules for choosing the regularization parameter. Such rules can be *a-priori* (depending only on the noise level δ) and *a-posteriori* (depending on the noise level δ and the noisy data y^δ). Given a certain parameter choice rule, along with other assumptions, we need to prove the regularization property, which simply tells us that regularized solutions converge to the exact solution of (1.0.1) as the noise level decays.

In regularization theory, one aim is to obtain convergence results under suitable assumptions. Later on, the assumptions used may prove to be too rigid and may fail to satisfy in some problems, hence they are relaxed as much as possible. Also, some rules in choosing the regularization parameter require knowledge of the noise level δ, but in real applications such knowledge may be unavailable. Also, an inaccurate estimate of the noise level may lead to unreliable estimates. Luckily, parameter choice rules not requiring the noise level are available. Using noise level-free parameter choice rules and relaxed assumptions make regularization theory cover a wider range of inverse problems.

The main aim of this dissertation is to obtain more general convergence results for both variational and iterative regularization schemes. This involves using heuristic parameter choice rules and using more relaxed assumptions on the problem's formulation, involving its parameter and data spaces. This is to generalize regularization theory and to make regularization methods more practical to solving inverse problems.

First, we study a general framework for variational regularization. Let $(\mathcal{Q}, \tau_\mathcal{Q})$ and $(\mathcal{U}, \tau_\mathcal{U})$ be two topological spaces with topologies $\tau_\mathcal{Q}$ and $\tau_\mathcal{U}$, respectively (e.g. Banach, Hilbert spaces). We consider a more general inverse problem of the form

$$\mathcal{S}(q, u^\dagger) = 0 \tag{1.0.2}$$

for exact data $u^\dagger \in \mathcal{U}$ and parameter $q \in \mathcal{Q}$ to be determined, and $\mathcal{S} : \mathcal{Q} \times \mathcal{U} \to [0, \infty]$ is a proper data misfit functional. Same as in (1.0.1), problem (1.0.2) can be ill-posed. However, (1.0.2) is more general, since it only shows the relationship between the parameter and data variables without an explicit forward operator. For instance, parameter estimation problems expressed as a weak formulation may fall in this category.

Given a noisy data $\tilde{u}$, the variational regularization for solving (1.0.2) is to solve the minimization problem

$$\tilde{q}_\alpha \in \arg\min_{q \in \mathcal{Q}} \{\mathcal{S}(q, \tilde{u}) + \alpha \mathcal{R}(q)\} \tag{1.0.3}$$

to approximate an $\mathcal{R}$-minimizing solution $q^\dagger$ with $\mathcal{S}(q^\dagger, u^\dagger) = 0$. Here $\alpha > 0$ is the regularization parameter. We propose the Hanke-Raus rule for choosing the parameter $\alpha > 0$ in (1.0.3): let $A \geq 0$, $\alpha_0 > 0$ and $0 < \gamma < 1$ be given numbers, and define a discrete exponential grid given by

$$\Delta_\gamma = \left\{\alpha_0 \gamma^j : j = 0, 1, \ldots\right\}.$$

Define $\alpha_* := \alpha_*(\tilde{u}) \in \Delta_\gamma$ such that

$$\alpha_* \in \arg\min_{\alpha \in \Delta_\gamma} \left\{\Theta(\alpha, \tilde{u}) := \left(\frac{1}{\alpha} + A\right) \mathcal{S}(\tilde{q}_\alpha, \tilde{u})\right\} \tag{1.0.4}$$

and use $\tilde{q}_{\alpha_*}$ as an approximate solution. The framework (1.0.3) generalizes the one in [40] for variational regularization for solving ill-posed problems (1.0.1) in Banach spaces. To extend this result, we provide a convergence analysis for the variational regularization of the form (1.0.3) for solving (1.0.2) under the Hanke-Raus rule (1.0.4).

Second, we consider the simplest iterative scheme, the Landweber method. It solves an approximate solution of ill-posed problems of the form (1.0.1) using a gradient scheme for solving a corresponding least-squares problem. For this method, we focus on its analog for Banach spaces with a nonsmooth p-convex penalty functional with $p \geq 2$ [48]. Given an initial guess $\xi_0 \in \mathcal{X}^*$ and $x_0 := \operatorname{argmin}_{x \in \mathcal{X}} \{\mathcal{R}(x) - \langle \xi_0, x\rangle\}$, set $x_0^\delta = x_0$ and $\xi_0^\delta = \xi_0$ define the iterates of the Landweber method by

$$\begin{aligned} \xi_{n+1}^\delta &= \xi_n^\delta - \mu_n^\delta L(x_n^\delta)^* j_r^{\mathcal{Y}}(F(x_n^\delta) - y^\delta), \\ x_{n+1}^\delta &= \operatorname{argmin}_{x \in \mathcal{X}} \left\{\mathcal{R}(x) - \left\langle \xi_{n+1}^\delta, x\right\rangle\right\}, \end{aligned} \tag{1.0.5}$$

where $\{L(x) : \mathcal{X} \to \mathcal{Y}\}$ is a family of bounded linear operator, $j_r^{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}^*$ is the duality mapping with respect to the gauge function $t \mapsto t^{r-1}$ for $r \geq 1$, and μ_n^δ is an appropriately chosen variable step size. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, $L(x)$ is the Fréchet derivative of F, and $\mathcal{R}(x) = \frac{1}{2}\|x\|^2$ the duality mapping becomes the identity map, and (1.0.5) is the classic Landweber method. For Hilbert spaces, this method is well-studied for both linear and nonlinear problems [16, Chap. 6], [23, 24].

The iteration (1.0.5) can approximate the solutions of (1.0.1) with special properties, by using an appropriate choice of the penalty functional $\mathcal{R}$. Hence, the convergence of (1.0.5) is studied in terms of an $\mathcal{R}$-minimizing solution of (1.0.1), denoted by $x^\dagger$. For this method, the number of iterations serve as the regularization parameter, so (1.0.5) should be terminated appropriately in order to become a regularization method. A well-studied parameter choice rule (also a stopping rule in this case) is the discrepancy principle, which says terminate (1.0.5) after $n_* = n_*(\delta, y^\delta)$ iterations when

$$\left\|F(x^\delta_{n_*}) - y^\delta\right\| \le \tau\delta < \left\|F(x^\delta_n) - y^\delta\right\|, \quad 0 \le n < n_*,$$

and use $x^\delta_{n_*}$ as an approximation to $x^\dagger$ (c.f. [48].) The performance of discrepancy principle relies on a good estimate of the noise level δ. To avoid using the noise level, this study focuses on a specific heuristic or noise level-free parameter rule known as the Hanke-Raus rule [24], which selects the regularization parameter $n_* = n_*(y^\delta)$ that minimizes a factor of the residual, i.e., for a fixed constants $a \ge 0$ and $p \ge 2$, choose $n_* = n_*(y^\delta)$ such that

$$n_* \in \operatorname{argmin}\left\{(n+a)\left\|F(x^\delta_n) - y^\delta\right\|^p, n = 0, 1, \ldots\right\}, \tag{1.0.6}$$

and then use $x^\delta_{n_*}$ as an approximation to $x^\dagger$. Despite Bakushinskii's veto [3] over the non-convergence of noise level-free rules (hence the term heuristic), convergence can still be obtained without considering the worst-case convergence. This can be done by using additional assumptions on the noise (c.f. [24, 56]).

Moreover, for the Landweber iteration for Banach spaces (1.0.5), the key assumptions (c.f. [48]) used for the convergence analysis are the following:

(a) the data space $\mathcal{Y}$ is uniformly smooth;

(b) the mapping $x \mapsto L(x)$ from $\mathcal{X}$ to $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ is continuous. Here, $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of bounded linear operators between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$.

Another issue this study tackles deals with using those two. Assumptions (a) excludes data contaminated by non-Gaussian noise, such as impulse noise. Moreover, when the forward operator F is smooth, an obvious choice for L satisfying assumption (b) is the Fréchet derivative. However, in some instance inverse problems are expressed as non-smooth PDEs with a nonsmooth forward operator, violating assumption (b).

As a consequence, in this study we develop a convergence analysis of the Landweber iteration (1.0.5) under the heuristic rule (1.0.6) by relaxing assumptions (a) and (b).

The Landweber iteration is known to be computationally cheap, a quality preferred for large-scale problems. However, it requires many iterations to obtain a reliable solution. To address this issue, acceleration strategies are considered. One well-studied strategy is using Nesterov acceleration, known for achieving a fast rate of $\mathcal{O}(1/n^2)$ for gradient method of minimizing smooth convex functionals. In this study we considered the method for solving (1.0.1) on

Hilbert spaces, with $F = T$ a bounded linear operator. Given the two most recent approximates x_n^δ and x_{n-1}^δ, the next approximate x_{n+1}^δ is computed by

$$\begin{aligned} z_n^\delta &= x_n^\delta + \frac{n-1}{n+\beta}(x_n^\delta - x_{n-1}^\delta), \\ x_{n+1}^\delta &= x_n^\delta + \omega T^*(y^\delta - T z_n^\delta), \end{aligned} \tag{1.0.7}$$

with $0 < \omega \|T\|^2 \leq 1$ and $\beta \geq 1$. The convergence of (1.0.7) under a-priori and a-posteriori stopping rules has been established [65]. To generalize existing theory, we establish the convergence of (1.0.7) of convergence under the Hanke-Raus rule (1.0.6).

All sets of convergence analyses are done theoretically using standard and non-standard assumptions on the problem, and they all build up towards the main result, the regularization property. Numerical results from examples in parameter estimation, image deblurring, and plasma physics are also shown to verify the theoretical results.

Outline of the dissertation

The dissertation is organized as follows, specifying the original contributions.

- Some relevant mathematical background material needed for this dissertation are reviewed in Chapter 2.

- Chapter 3 contains the dissertation's first original contribution: a general convergence analysis of variational regularization under the Hanke-Raus rule. Instead of an explicit forward operator, the inverse problem is expressed as an equation in terms of a data misfit functional satisfying only a quasi-triangle inequality. Such form covers a broader range of inverse problems, such as electrical impedance tomography, as show in the numerical examples. The contents in this chapter are already published in [63].

- Chapter 4 covers the Landweber iteration. The second original contribution of this dissertation is in Section 4.2. A new convergence analysis on the Landweber method in Banach spaces with a convex penalty functional under the Hanke-Raus rule and the discrepancy principle was established. The convergence analysis involves weaker assumptions on the forward operator and the data space to cover non-smooth inverse problems whose measurement data is contaminated with non-Gaussian noise. The results involving the discrepancy principle were already published in [72]. For completeness, we also present a new convergence analysis of the Hanke-Raus rule for the classical Landweber method for nonlinear inverse problems in Hilbert spaces in Section 4.3.

- Finally, an original contribution in Section 4.4 deals with Nesterov acceleration for Landweber method to solve linear problems in Hilbert spaces. We also prove its convergence under the Hanke-Raus rule. This is the first known result on integrating an acceleration scheme and a heuristic rule to Landweber iteration.

1.1 The Hanke-Raus rule

This section gives a brief background on the Hanke-Raus rule, which has been gradually extended to more general inverse problems. We start in its basic formulation for linear problems. Next, we touch on the noise condition, an important assumption for establishing convergence under these rules. We discuss how this abstract condition could make sense towards making heuristic rules applicable to real problems. Finally we briefly discuss some of its progress towards nonlinear problems and Banach spaces.

1.1.1 Motivation

There are two instances in using an unreliable noise level to parameter choice rules that pose severe loss of accuracy. For one, suppose when we have a discrete vector with a finite number of measurements, with each measurement having an underlying error, and we may not know the standard deviation and/or the worst-case error bound. Another one is when using discrete data towards an interpolation or approximation process. Then we have to estimate the L^2-norm of the difference between the constructed function and the true data function based on the discrete noise information, and the *a-priori* assumed smoothness properties of the data [16].

One of the most well-studied heuristic rules was developed by Hanke and Raus [24] for linear problems in Hilbert spaces of the form (4.1.1). To describe this rule, consider (4.1.1) for Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ with F being a bounded linear operator between $\mathcal{X}$ and $\mathcal{Y}$. Given an exact solution (4.1.1) denoted by $x^\dagger$, the paper [24] compared the values of the exact error $\left\|x_\alpha^\delta - x^\dagger\right\|^2$ and a multiple of the residual $\left\|Fx_\alpha^\delta - y^\delta\right\|^2/\alpha$ with respect to the regularization parameter α. Preferably, a suitable choice for a regularization parameter is one that minimizes $\left\|x_\alpha^\delta - x^\dagger\right\|^2$. Despite that, in reality the exact solution $x^\dagger$ is not known, so the exact error is not computable. Using a-posteriori error bounds for Tikhonov regularization and Landweber iteration, it can be derived that

$$\left\|x_\alpha^\delta - x^\dagger\right\| \sim \frac{1}{\sqrt{\alpha}} \left\|Fx_\alpha^\delta - y^\delta\right\|.$$

implying that the term on the right can serve as a surrogate for the actual error. From this result, it was proposed to choose a regularization parameter

$$\alpha_* = \alpha(y^\delta) \in \mathrm{argmin}_{\alpha>0}\psi_{HR}(\alpha, y^\delta), \quad \text{where } \psi_{HR}(\alpha, y^\delta) := \left\|Fx_\alpha^\delta - y^\delta\right\| / \sqrt{\alpha}, \tag{1.1.1}$$

and use $x_{\alpha_*}^\delta$ as an approximate solution for (4.1.1). For Landweber iteration, this can be used by setting $\alpha = 1/(n+1)$ for an iteration number n. Moreover, using a source condition of the form $x^\dagger - x_0 \in R((F^*F)^\mu)$ for some $\mu > 0$, this rule admits error estimates of the form

$$\left\|x_{\alpha_*}^\delta - x^\dagger\right\| \le C\left(1 + \frac{\delta}{\delta^*}\right) \max\left\{\delta_*, \delta\right\}^{2\mu/(2\mu+1)} \tag{1.1.2}$$

for a uniform constant C, $\delta_* := \left\|Fx_{\alpha_*}^\delta - y^\delta\right\|$ and $\delta := \left\|y^\delta - y\right\|$. The error estimate in (1.1.2) involves two quantities: δ_* and δ. Knowing the value of δ_* give the following check on this parameter choice rule: if $\delta_* \ll \delta$ (or what is believed to be the noise level δ), then one should be careful about the chosen parameter $\alpha(y^\delta)$ since the factor δ/δ_* is large, causing (1.1.2) to blow up. On the contrary, if $\delta_* \gg \delta$, then this situation is not critical, as the magnitude of δ_* determines the noise level (c.f. [16, Chap. 4]). Apparently, with additional conditions, the factor δ/δ_* can be removed, and we get the optimal-like convergence rates of the form

$$\left\|x_{\alpha_*}^\delta - x^\dagger\right\| \le C \max\left\{\delta_*, \delta\right\}^{2\mu/(2\mu+1)}. \tag{1.1.3}$$

1.1.2 Overcoming Bakushinskii's veto

However, despite some advantages, heuristic rules suffer from the following drawback.

Theorem 1.1.1 (Bakushinskii's veto)**.** [3] *Consider ill-posed problems of the form* (4.1.1) *for Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ with F being a bounded linear operator between $\mathcal{X}$ and $\mathcal{Y}$. Then any noise level-free parameter choice rule of the form $\alpha = \alpha(y^\delta)$ cannot give rise to convergence in the worst case scenario.*

Theorem 1.1.1 somehow discourages the use of noise level-free rules. Despite that, convergence and convergence rates can still be obtained without considering this worst-case convergence. This can be done using a so-called noise condition, which restricts the noise allowed in the given data [55, 56, 57].

The authors in [56] provided a comprehensive numerical study about noise level-free parameter choicer rules for Tikhonov regularization of linear inverse problems. Aside from the Hanke-Raus rule, it also includes the quasi-optimality criterion [66, 57]. It uses the difference between successive approximates for different values of the regularization parameter. When the noise condition holds, the heuristic parameter choice rule $\alpha_* = \alpha(y^\delta)$ is a good estimator to the optimal α_{opt} with the minimum total error given by $\left\|x^\delta_{\alpha_{\text{opt}}} - x^\dagger\right\|$. On the other hand, violating it causes α_* to be too small and terribly underestimate α_{opt}.

Moreover, Stefan Kindermann analyzed a more general version of the Hanke-Raus minimizing functional. When the noise $y^\delta - y$ is contained in a set of restricted noisy data, then the Landweber iteration (4.1.5) under the Hanke-Raus rule converges. Using spectral calculus, the restricted noisy data condition is of the form $y^\delta - y \in \mathcal{N}_p$, where

$$\mathcal{N}_p := \{e \in \mathcal{Y} \,:\, \text{(1.1.4) holds}\,\},$$

with

$$t^p \int_t^\infty \lambda^{-1} dE_\lambda \|e\|^2 \leq \nu \int_0^t \lambda^{p-1} dE_\lambda \|e\|^2, \quad \forall 0 < t \leq t_1, \tag{1.1.4}$$

for $p \geq 1$ and fixed $t, \nu > 0$, and $\{F_\lambda\}$ is a spectral family of FF^*. By using a more general version of the Hanke-Raus minimizing functional, given by

$$\psi_{HR,\tau}(\alpha, y^\delta) := \sqrt{\frac{1}{\alpha} \int |r_\alpha(\lambda)|^{2+\frac{1}{\tau}} dE_\lambda \|Py^\delta\|^2}$$

for a parameter τ usually set as the qualification index, and P is the orthogonal projector onto $\overline{R(F)}$. Using spectral calculus, it is easy to see that if $\tau = \infty$, then $\psi_{HR,\infty} = \psi_{HR}$ in (1.1.1). In addition, the heuristic rule (1.1.1) can still give optimal convergence depending on the smoothness of the exact solution $x^\dagger$, even though in general heuristic parameter choice rules only get sub-optimal rates. Also, none among the heuristic parameter choice rules (Hanke-Raus rule, quasi-optimality criterion, L-curve method) is better, and their performances depend on the smoothness of the exact solution $x^\dagger$ [55].

1.1.3 More on the noise condition

We first consider linear inverse problems of the form (4.1.1) for Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ with F being a bounded linear operator. Convergence analysis for Landweber iteration and

Tikhonov regularization were obtained under the assumption that the exact data is not completely dominated by the noise, i.e., $\|y\| \geq \|y - y^\delta\|$. In addition, they also used a noise condition of the form

$$\left\|Q(y^\delta - y)\right\| \geq \varepsilon \left\|y^\delta - y\right\| \tag{1.1.5}$$

where $\varepsilon > 0$ is a constant prescribing the randomness of the noise, and Q denotes the orthogonal projector onto the orthogonal complement of the range of F. With P denoting the orthogonal complement onto the range of F, condition (1.1.5) is equivalent to

$$\left\|P(y^\delta - y)\right\| \leq \varepsilon' \left\|y^\delta - y\right\| \tag{1.1.6}$$

for some $0 \leq \varepsilon' < 1$. For inverse problems in Banach spaces, the noise conditions (1.1.5)–(1.1.6) can also be generalized towards problems of the form (4.1.1), with F a linear or nonlinear operator. This comes while assuming Bakushinskii's veto (Theorem 1.1.1) also holds under such spaces. The paper in [38], focusing on heuristic rules for convex variational regularization for linear inverse problems, weakened condition (1.1.6) into the following form: there exists a constant $0 < \sigma < 1$ such that

$$\left\langle y^\delta - y, v \right\rangle \leq (1 - \sigma) \left\|y^\delta - y\right\| \|v\| \tag{1.1.7}$$

for all $v \in \{F(x) - y : x \in \mathcal{D}(F)\}$. On the other hand, a noise condition of the form

$$\left\|y^\delta - y - v\right\| \geq \kappa \left\|y^\delta - y\right\| \tag{1.1.8}$$

for all $v \in \{F(x) - y : x \in \mathcal{D}(F)\}$, was also used for regularization in Banach spaces under heuristic rules, including the Hanke-Raus rule. It can be showed that condition (1.1.7) implies condition (1.1.8) (c.f. [40]). Meanwhile, condition (1.1.8) can be interpreted in the following way. Usually the forward operator F has a smoothing effect, so $F(x)$ admits a certain regularity. On the contrary, the noise $y^\delta - y$ in general may contain severe irregularity due to the random nature of the noise. Subtracting a regular function of the form $F(x) - y$ from the noise $y^\delta - y$ cannot significantly remove the noise [40]. The noise condition is independent of the regularization method. There is no way to verify such conditions in reality, unless the exact data y is accessible. Despite that, heuristic parameter choice rules are useful as long as there is a good understanding of the noise in a given problem. To add, a random noise is a desirable instance to use heuristic parameter choice rules. The least desirable instance happens with a mildly ill-posed forward operator and a smooth data noise. This is because the noise condition tells us that the noise should be sufficiently "nonsmooth" (c.f. [55, 56] for some illustrations).

1.1.4 Extending the Hanke-Raus rule

The Hanke-Raus rule, among other heuristic rules, is very well-understood for linear problems in Hilbert space, thanks to spectral calculus of bounded linear operators. For nonlinear problems, it is less developed, and spectral calculus only comes in handy when dealing with norms of Fréchet derivative. Despite that, heuristic rules have been extended to various regularization methods. We first present the extensions of the rule in variational regularization, then we move to the extensions for iterative regularization.

Variational regularization

For variational regularization of the form (3.0.4), the authors in [36] analyzed Hanke-Raus rule similar to (1.1.11), with Hilbert spaces $\mathcal{Q}$ and $\mathcal{U}$, using a nonlinear forward operator F, with the solution solved in a constraint set $C \subset \mathcal{Q}$. Aside from that, they also analyzed its convergence under another heuristic rule, the balancing principle. The obtained convergence estimates similar to (1.1.2) relied on a source condition based on Lagrange multipliers. Moreover, the authors in [38] considered the Hanke-Raus rule for convex variational regularization for linear inverse problems, with a Banach space $\mathcal{Q}$ and a Hilbert space $\mathcal{U}$. Remarkably, they derived error estimates in terms of Bregman distances induced by the convex penalty functional. Such estimates are also similar to (1.1.2).

More recently, the convergence of (3.0.4) for Banach spaces $\mathcal{Q}$ and $\mathcal{U}$ with the misfit term $\mathcal{S}(q, \tilde{u}) := \|F(q) - \tilde{u}\|_{\mathcal{Y}}^p$ for $1 < p < \infty$ under the heuristic rule has been established. Here, the parameter is chosen over an exponential grid $\Delta_\gamma := \{\alpha_0 \gamma^j : j = 0, 1, \ldots\}$ for a preassigned number $\alpha_0 > 0$. Then $\alpha_* := \alpha_*(\tilde{u}) \in \Delta_\gamma$ is chosen such that

$$\alpha_* \in \arg\min_{\alpha \in \Delta_\gamma} \frac{\|F(q_\alpha) - \tilde{u}\|_{\mathcal{Y}}^p}{\alpha}.$$

Convergence analysis was also done using the noise condition (1.1.8), with convergence expressed in terms of $\mathcal{R}$-minimizing solution of (3.0.1). Moreover, convergence rates were expressed in terms of Bregman distances [40].

Iterative regularization

We focus again on inverse problems of the form (4.1.1) for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with F a nonlinear operator. A convergence analysis of another iterative method, the iteratively regularized Gauss-Newton method (IRGNM), under the Hanke-Raus rule has been established [47]. The method is of the form

$$x_{n+1}^\delta = x_n^\delta - (\alpha_n I + F'(x_n^\delta)^* F'(x_n^\delta))^{-1}(F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta) + \alpha_n(x_n^\delta - x_0)) \tag{1.1.9}$$

where $x_0^\delta = x_0 \in \mathcal{D}(F)$ is an initial guess and $\{\alpha_n\}$ is a sequence of positive numbers such that

$$\alpha_n > 0, \quad q \leq \frac{\alpha_{n+1}}{\alpha_n} \leq 1 \text{ and } \quad \lim_{n\to\infty} \alpha_n = 0 \tag{1.1.10}$$

for some constants $\alpha_0 > 0$ and $0 < q < 1$ (e.g. $\alpha_n = 0.1 \times 2^{-n}$). Convergence was established using the noise condition (1.1.8) which suits nonlinear problems. The Hanke-Raus rule for (1.1.9) chooses $n_* = n_*(y^\delta)$ such that

$$n_* \in \text{argmin} \left\{ \frac{\|F(x_n^\delta) - y^\delta\|}{\alpha_n}, \quad n = 0, 1, \dots \right\}, \tag{1.1.11}$$

and $x_{n_*}^\delta$ is used as an approximation to $x^\dagger$. The key method in the proof of convergence involves the following:

1. defining an auxiliary integer $\tilde{n}_*$ similar to discrepancy principle to ensure that (1.1.11) is well-defined; and

2. general source conditions to derive a-posteriori error estimates.

Notice that the heuristic rule (1.1.11) re-uses the same relaxation parameter $\{\alpha_n\}$ in the iterative method.

The Hanke-Raus rule for regularization of inverse problems of the form(4.1.1) using the augmented Lagrangian method was also analyzed [43]. Here, F is a bounded linear operator between a Banach space $\mathcal{X}$ and a Hilbert space $\mathcal{Y}$. Starting from an initial guess $\lambda_0 = 0$, the method is of the form

$$\begin{aligned} x_n^\delta &\in \arg\min_{x\in\mathcal{X}} \left\{ \frac{t_n}{2} \left\|Fx - y^\delta\right\|^2 + \mathcal{R}(x) - \left\langle \lambda_{n-1}^\delta, Fx - y^\delta \right\rangle \right\}, \\ \lambda_n^\delta &= \lambda_{n-1}^\delta - t_n(Fx_n^\delta - y^\delta), \end{aligned} \tag{1.1.12}$$

with $\{t_n\}$ a given sequence of positive numbers such that $s_n := \sum_{j=1}^{n} t_j \to \infty$ as $n \to \infty$, and a proper convex function $\mathcal{R} : \mathcal{X} \to [0, \infty]$. The Hanke-Raus rule for (1.1.12) chooses an integer $n_* = n_*(y^\delta)$ such that

$$n_* \in \operatorname{argmin} \left\{ s_n \left\| F x_n^\delta - y^\delta \right\|^2, \quad n = 1, 2, \ldots \right\}, \tag{1.1.13}$$

and use $x_{n_*}^\delta$ as an $\mathcal{R}$-minimizing solution of (4.1.1). Convergence holds while assuming the noise condition of the form (1.1.8) with $x \in \mathcal{D}(F) \cap \mathcal{D}(\mathcal{R})$. Using the penalty function $\mathcal{R}$, obtained error estimates are in terms of Bregman distances. Remarkably, the numerical simulations also used a slightly modified version from (1.1.13), given by

$$n_* \in \operatorname{argmin} \left\{ (s_n + a) \left\| F x_n^\delta - y^\delta \right\|^2, \quad n = 1, 2, \ldots \right\}, \tag{1.1.14}$$

for a given constant $a \geq 0$. This fixed constant has a similar role as the constant $A \geq 0$ in Rule 3.1 for variational regularization in Chapter 3. It prevents the regularization parameter index n_* from becoming too small. The simulations in [43] show that a suitably large fixed constant $a > 0$ in the Hanke-Raus rule for (1.1.12) gives more accurate solutions. The same convergence analysis used for (1.1.13) should hold when using (1.1.14). Introducing this fixed constant has also further used for other regularization methods, as we discuss below.

A more general result involves problems (4.1.1) with F a linear/nonlinear operator between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with a Fréchet derivative F'. More recently the authors in [81] established the convergence of the Hanke-Raus rule for the non-stationary iterated Tikhonov regularization in Banach spaces, which takes the form

$$\begin{aligned} x_n^\delta &\in \arg\min_{x \in \mathcal{D}(F)} \left\{ \frac{1}{r} \left\| F(x) - y^\delta \right\|^r + \alpha_n D_{\xi_{n-1}^\delta} \mathcal{R}(x, x_{n-1}^\delta) \right\}, \\ \xi_n^\delta &= \xi_{n-1}^\delta - \frac{1}{\alpha_n} F'(x_n^\delta)^* J_r(F(x_n^\delta) - y^\delta), \end{aligned} \tag{1.1.15}$$

where $1 \leq r < \infty$ and α_n is a sequence of positive numbers such that $s_n := \sum_{l=1}^{n} \frac{1}{\alpha_l} \to \infty$ as $n \to \infty$, and $F'(x)^* : \mathcal{Y}^* \to \mathcal{X}^*$ denotes the adjoint of $F'(x)$. In addition J_r denotes the duality mapping as in (1.0.5) and $D_\xi R(\tilde{x}, x)$ denotes the Bregman distance induced by $\mathcal{R}$ at x in the direction of $\xi \in \partial \mathcal{R}(x)$ (Definition 2.26). The corresponding Hanke-Raus rule for (1.1.15) chooses an integer $n_* = n_*(y^\delta)$ such that

$$n_* \in \operatorname{argmin} \left\{ (s_n + a) \left\| F(x_n^\delta) - y^\delta \right\|^r, \quad n = 1, 2, \ldots \right\}, \tag{1.1.16}$$

given a fixed number $a \geq 0$ similar in (1.1.14). We note that when F is a bounded linear operator and $\mathcal{Y}$ is a Hilbert spaces, (1.1.15) with $r = 2$, $\alpha_n = 1/t_n$ and $\xi_n^\delta = F^*\lambda_n^\delta$ is simply the augmented Lagrangian method in (1.1.12) .

Still in Banach spaces, the authors in [21] applied the Hanke-Raus rule for the IRGNM in Banach spaces with convex penalizing term, generalizing the result in [43]. Given a nonlinear forward operator F between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, the IRGNM for solving (4.1.1) then defines a sequence $\{x_n^\delta\}$ iteratively computed by

$$x_{n+1}^\delta \in \arg\min_{x \in \mathcal{D}(F)} \left\{ \left\| y^\delta - F(x_n^\delta) - L(x_n^\delta)(x - x_n^\delta) \right\|^r + \alpha_n \mathcal{R}(x) \right\} \tag{1.1.17}$$

with an initial guess $x_0^\delta := x_0 \in \mathcal{D}(F)$, where $r \geq 1$, $\{L(x) : x \in \mathcal{D}(F)\}$ is a family of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, $\mathcal{R} : \mathcal{X} \to [0, \infty)$ is the penalty functional, and $\{\alpha_n\}$ is a sequence of positive numbers satisfying (1.1.10). When F is Fréçhet differentiable, the operator L can be simply the Fréchet derivative F' of F. The Hanke-Raus rule for the iteration (1.1.17) chooses $n_* = n_*(y^\delta) \geq 1$ such that

$$n_* \in \operatorname{argmin} \left\{ \frac{1}{\alpha_n} \max \left\{ \left\| F(x_n^\delta) - y^\delta \right\|^r, \left\| F(x_{n-1}^\delta) - y^\delta \right\|^r \right\}, \quad n = 1, 2, \ldots \right\}, \tag{1.1.18}$$

and use $x_{n_*}^\delta$ as the approximate to the sought solution of (4.1.1). This is a slightly modified version compared to (1.1.14) and (1.1.16), since (1.1.18) is based on the modified discrepancy principle previously applied to IRGNM [49, 50]. Remarkably, *a posteriori* error estimates for the heuristic rule were obtained by using the variational source condition on the sought solution, similar to the assumption 3.2.1 for variational regularization in Chapter 3. Moreover, when $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, $L = F'$, $r = 2$, and $\mathcal{R}(x) = \|x\|^2$, iteration (1.1.17) is simply the IRGNM for Hilbert spaces in (1.1.9).

Chapter 2

Mathematical Background

2.1 Basic functional analysis

This dissertation presents results made from general topological spaces down to Hilbert spaces. We gather all the necessary basic functional analysis background. Most of the results here can be found on almost any analysis textbook, so we omit their proofs. We present some citations in case the reader want to read the proofs.

2.1.1 General topology

Denote by $\mathcal{P}(x)$ the power set of a set $\mathcal{X}$.

Definition 2.1. A *topological space* is a pair $(\mathcal{X}, \tau_{\mathcal{X}})$ consisting of a non-empty set $\mathcal{X}$ and a nonempty family $\tau_{\mathcal{X}} \in \mathcal{P}(\mathcal{X})$ of subsets of $\mathcal{X}$, where $\tau_{\mathcal{X}}$ satisfies the following properties:

- $\emptyset \in \tau_{\mathcal{X}}$ and $\mathcal{X} \in \tau_{\mathcal{X}}$;
- given an arbitrary index set I, $B_i \in \tau_{\mathcal{X}}$ for all $i \in I$ implies $\cup_{i \in I} B_i \in \tau_{\mathcal{X}}$;
- if $B_1, B_2 \in \tau_{\mathcal{X}}$, then $B_1 \cap B_2 \in \tau_{\mathcal{X}}$.

Definition 2.2. Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be a topological space. A set $A \subset \mathcal{X}$ is called *open* if $A \in \tau_{\mathcal{X}}$, and *closed* if $\mathcal{X} \setminus A \in \tau_{\mathcal{X}}$. The union of all open sets contained in $A \subset \mathcal{X}$ is called the *interior* of A, denoted by $\text{int}A$. The *closure* of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Definition 2.3. Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be a topological space. A net $(x_i)_{i \in I}$ *converges* to x with respect to the $\tau_{\mathcal{X}}$-topology if for every neighbourhood $N \in \mathcal{N}(x)$ there exists an index set $i_0 \in I$ such that $x_i \in N$ for all $i \succeq i_0$. In this instance, we write $x_i \xrightarrow{\tau_{\mathcal{X}}} x$.

Definition 2.4. Let $T : \mathcal{D}(T) \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be an operator. We say T is *closed* with respect to topologies $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{Y}}$ if for any net $(x_i)_{i \in I} \subset \mathcal{D}(T)$ such that $x_i \xrightarrow{\tau_{\mathcal{X}}}$ and $Tx_i \xrightarrow{\tau_{\mathcal{Y}}} Tx$, it follows that $x \in \mathcal{D}(T)$ and $Tx = y$.

Definition 2.5. A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is a *Hausdorff space* if for any two distinct x_1, x_2 there are open sets $B_1, B_2 \in \tau_{\mathcal{X}}$ with $x_1 \in B_1$, $x_2 \in B_2$, and $B_1 \cap B_2 \neq \emptyset$.

Proposition 2.1. *A topological space is a Hausdorff space if and only if each convergent net converges to exactly one element.*

By using the index set $I = \mathbb{N}$, an example of nets are *sequences* $(x_k)_{k \in \mathbb{N}}$ with the usual ordering $\leq$ on $\mathbb{N}$.

Definition 2.6. Let $\mathcal{X}$ be a Hausdorff space, let $C \subseteq \mathcal{X}$. Then C is *$\tau_{\mathcal{X}}$-sequentially closed* if the limit of every $\tau_{\mathcal{X}}$-convergent sequence $(x_n)_{n \in \mathbb{N}}$ that lies in C is also in C.

Definition 2.7. A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is $\tau_{\mathcal{X}}$- *sequentially compact* if $\mathcal{X}$ is a Hausdorff space and every sequence in $\mathcal{X}$ has a convergent subsequence with respect to $\tau_{\mathcal{X}}$-topology.

Remark 2.1. In general topology, a closed set is sequentially closed, but the converse is false.

2.1.2 Metric and normed spaces

Here, let $\mathcal{X}$ be a non-empty set.

Definition 2.8. A *metric* on $\mathcal{X}$ is a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfying the following properties:

(a) $d(x, y) \geq 0$ for all $x, y \in \mathcal{X}$, and $d(x, y) = 0$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$.

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{X}$.

A *metric space* $(\mathcal{X}, d_{\mathcal{X}})$ is a set $\mathcal{X}$ equipped with a metric $d_{\mathcal{X}}$.

We assume the reader is familiar with the properties of linear spaces, also known as vector space.

Definition 2.9. A *norm* on a linear space X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

(a) $\|x\| \geq 0$ for all $x \in \mathcal{X}$;

(b) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in \mathcal{X}$ and $\lambda \in \mathbb{R}$;

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$ (triangle inequality);

(d) $\|x\| = 0$ implies that $x = 0$.

A *normed linear space* (or simply *normed space*) $(\mathcal{X}, \|\cdot\|)$ is a linear space $\mathcal{X}$ equipped with a norm $\cdot$.

From the previous definitions, we see that a normed linear space is a metric space equipped with the metric

$$d(x, y) = \|x - y\| \tag{2.1.1}$$

Definition 2.10. Let $(\mathcal{X}, d)$ be a metric space. Given $\rho > 0$ and $x \in \mathcal{X}$, the *open ball* of radius ρ centred at $x \in \mathcal{X}$ is the set $\mathcal{B}_\rho(x) := \{y \in \mathcal{X} \,:\, d(x, y) < \rho\}$ while the *closed ball* is defined as $\overline{\mathcal{B}}_\rho(x) := \{y \in \mathcal{X} \,:\, d(x, y) \leq \rho\}$.

Remark 2.2. From the definition, some topological spaces are not metric spaces. However, Every metric space is a topological space, with the topology of all the open balls in the metric space.

The normed space has a canonical topology: a set V is a neighborhood of a point x if there exists $\rho > 0$ such that $\mathcal{B}_\rho(x) \subset V$, called the *strong topology*. If no modifiers in use, topological notions (closedness, openness, neighborhood, continuity, convergence, compactness, etc.) used in this paper are understood to be with respect to strong topology.

We describe two different notions of convergent sequences in terms of metric spaces. In normed spaces, these notions can be correspondingly obtained using (2.1.1).

Definition 2.11. Let $(\mathcal{X}, d)$ be a metric space. A sequence (x_n) in $\mathcal{X}$ converges to $x \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. Moreover, it is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

Definition 2.12. A metric space $(\mathcal{X}, d)$ is *complete* if every Cauchy sequence in $\mathcal{X}$ converges to a limit in $\mathcal{X}$. A subset C of $\mathcal{X}$ is *complete* if the metric subspace (C, d_C) is complete.

With respect to the strong topology, we also say that a sequence (x_n) in a normed space $\mathcal{X}$ *converges (strongly)* to $x \in \mathcal{X}$, written as $x_n \to x$ as $n \to \infty$, if $\lim_{n\to\infty} \|x_n - x\| = 0$, and we call x the limit of the sequence. On the other hand, (x_n) is a *Cauchy sequence* if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$.

2.1.3 Bounded operators on normed spaces

Here, let $\mathcal{X}, \mathcal{Y}$ be normed spaces and $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator from $\mathcal{X}$ to $\mathcal{Y}$. Denote by $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ the set of continuous linear operators $T : \mathcal{X} \to \mathcal{Y}$.

Proposition 2.2. [14, Prop. III.2.1] *If $T : \mathcal{X} \to \mathcal{Y}$ is a linear operator, the following statements are equivalent:*

(i) $T \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$.

(ii) T is continuous at 0.

(iii) T is continuous at some point.

(iv) There is a positive constant c such that $\|Ax\| \leq c\|x\|$ for all $x \in \mathcal{X}$. If $T \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$ and

$$\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\},$$

then

$$\begin{aligned}\|A\| &= \sup\{\|Ax\| : \|x\| = 1\} \\ &= \sup\{\|Ax\| / \|x\| : x \neq 0\} \\ &= \inf\{c > 0 : \|Ax\| \leq c\|x\| \ \textit{for}\ x \in \mathcal{X}\}.\end{aligned}$$

We call $\|A\|$ the *norm* of A. A continuous linear operator is also called a *bounded linear operator.*

Definition 2.13. A linear operator $T : \mathcal{X} \to \mathcal{Y}$ is called a *compact linear operator* if for every bounded sequence (x_n) in $\mathcal{X}$, the sequence (Tx_n) has a convergent subsequence.

Remark 2.3. Note that a compact linear operator T is bounded and, by linearity, is also continuous. A continuous linear operator is also called a *bounded linear operator.* In other words, a linear operator $T : \mathcal{X} \to \mathcal{Y}$ is *bounded* if

$$\|T\|_{\mathscr{L}(\mathcal{X},\mathcal{Y})} := \sup_{\|x\|=1} \|Ax\| < \infty. \tag{2.1.2}$$

Proposition 2.3. [68, Prop. 1.7] If $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a normed space, and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space, then $\left(\mathscr{L}(\mathcal{X}, \mathcal{Y}), \|\cdot\|_{\mathscr{L}(\mathcal{X},\mathcal{Y})}\right)$ is a Banach space.

2.1.4 Banach space

Definition 2.14. A normed linear space that is complete with respect to the metric (2.1.1) is called a *Banach space*.

Example 2.1.1. Let $x = (x_n)_{n=1}^N \in \mathbb{R}^N$. For $1 \leq p < \infty$, define the p-norm of $\mathbb{R}^N$ (or $\mathbb{C}^n$) by

$$\|x\|_p = \left(\sum_{n=1}^{N} |x_n|^p \right)^{1/p}.$$

For $p = \infty$, define the infinity norm (or maximum norm) by

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_N|\}.$$

Then the space $\mathbb{R}^N$ equipped with the p-norm for $1 \leq p \leq \infty$ is a finite-dimensional Banach space.

Example 2.1.2. For $1 \leq p < \infty$, the sequence space $\ell^p(\mathbb{N})$ consists of all infinite sequences $x = (x_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

with the p-norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

If $p = \infty$, the sequence space $\ell^\infty(\mathbb{N})$ consists of all bounded sequences with the infinity norm

$$\|x\|_\infty = \sup\{|x_n| \ : \ n = 1, 2, \ldots\}.$$

Then $\ell^p(\mathbb{N})$ is an infinite-dimensional Banach space for $1 \leq p \leq \infty$.

Example 2.1.3. Let $1 \leq p < \infty$, and $\Omega \subseteq \mathbb{R}^n$ a measurable set. Then $L^p(\Omega)$ is the set of Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) whose p-th power is Lebesgue integrable, i.e.,

$$\int_\Omega |f(x)|^p \, dx < \infty,$$

equipped with the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, the space $L^\infty(\Omega)$ is the space of essentially bounded Lebesgue measurable functions, with

$$\|f\|_\infty = \inf\{M \ : \ |f(x)| \le M \text{ a.e. in } \Omega\}$$

as its norm. The space $L^p(\Omega)$ for $1 \le p \le \infty$ is a Banach space.

Definition 2.15. The *topological dual* (or *dual space*) of a normed space $(\mathcal{X}, \|\cdot\|)$ is the normed space $(\mathcal{X}^*, \|\cdot\|_*)$, where $\mathcal{X}^* = \mathscr{L}(X, \mathbb{R})$, the Banach space of all bounded (and continuous) linear functionals $\xi : \mathcal{X} \to \mathbb{R}$ equipped with the norm $\|\cdot\|_* = \|\cdot\|_{\mathscr{L}(X,\mathbb{R})}$, where $\|\xi\|_{\mathscr{L}(X,\mathbb{R})} = \sup_{\|\xi\|=1} |\xi(x)|$.

Note 2.1. The topological dual $(\mathcal{X}^*, \|\cdot\|_*)$ is also a Banach space via Proposition 2.3.

Theorem 2.1.1 (Cauchy's inequality)**.** Let $\mathcal{X}$ be a Banach space and let $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. Then we have

$$\left|\langle x^*, x\rangle_{\mathcal{X}^*\times\mathcal{X}}\right| \le \|x^*\|_{\mathcal{X}^*} \|x\|_{\mathcal{X}}.$$

2.1.5 Weak topology

We consider a particular topology in $\mathcal{X}$ where every element of the dual space $\mathcal{X}^*$ is continuous. The coarsest of such topologies (the one with fewer open sets) is called the *weak topology*, denoted by $\sigma(\mathcal{X})$. In fact, $(\mathcal{X}, \sigma)$ is a Hausdorff space. If $\mathcal{X}$ is finite dimensional, the weak topology coincides with the strong topology.

Definition 2.16. We say that a sequence (x_n) in $\mathcal{X}$ *converges weakly* to x, written as $x_n \rightharpoonup x$, as $n \to \infty$, if $\lim_{n\to\infty} \langle L, x_n - x\rangle = 0$ for each $L \in \mathcal{X}^*$. In this instance, we call x the *weak limit* of the sequence.

Remark 2.4. By Cauchy's inequality, $|\langle L, x_n - x\rangle| \le \|L\|_* \|x_n - x\|$. This shows that if (x_n) is strongly convergent, then it is also weakly convergent, and their limits coincide.

Proposition 2.4. [68, Prop. 1.22]. Let (x_n) converge weakly to x as $n \to \infty$. Then (x_n) is bounded.

2.1.6 Reflexivity and weak topology

Definition 2.17. The topological *bidual* of a normed space $(\mathcal{X}, \|\cdot\|)$ is the topological dual of $(\mathcal{X}^*, \|\cdot\|_*)$, denoted by $(\mathcal{X}^{**}, \|\cdot\|_{**})$. To describe its elements, each $x \in \mathcal{X}$ defines a linear functional $\mu_x : \mathcal{X} \to \mathbb{R}$, where

$$\mu_x(\xi) = \langle \xi, x\rangle_{\mathcal{X}^*,\mathcal{X}}, \quad (\forall \xi \in \mathcal{X}^*).$$

By Cauchy's inequality, $|\langle \xi, x\rangle| \leq \|\xi\| \cdot \|x\|$, for each $x \in \mathcal{X}$ and each $\xi \in \mathcal{X}^*$. Then we have $\|\mu_x\|_{**} \leq \|x\|$, implying that $\mu_x \in \mathcal{X}^{**}$.

The function $\mathcal{J} : \mathcal{X} \to \mathcal{X}^{**}$ defined by $\mathcal{J}(x) = \mu_x$ is the *canonical embedding* of $\mathcal{X}$ into $\mathcal{X}^{**}$. Clearly $\mathcal{J}$ is linear and continuous. In addition, $\mathcal{J}$ is an *isometry*, i.e., $\|\mu_x\|_{**} = \|x\| \quad (\forall x \in \mathcal{X})$, and this implies $\mathcal{J}$ is injective.

Definition 2.18. The normed space $(\mathcal{X}, \|\cdot\|)$ is *reflexive* if $\mathcal{J}$ is also surjective, i.e., every element μ of $\mathcal{X}^{**}$ is of the form $\mu = \mu_{\mathcal{X}}$ for some $x \in \mathcal{X}$.

Note 2.2. Since the normed space $(\mathcal{X}, \|\cdot\|)$ is homeomorphic to $(\mathcal{X}^{**}, \|\cdot\|_{**})$, then $(\mathcal{X}, \|\cdot\|)$ is necessarily Banach space.

Theorem 2.1.2. [68, Thm. 1.24] Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. Then the following are equivalent:

(i) $\mathcal{X}$ is reflexive.

(ii) Every bounded sequence in $\mathcal{X}$ has a weakly convergent subsequence.

2.1.7 Differentiability

Definition 2.19. Let $f : \mathcal{D}(f) \subseteq \mathcal{X} \to \mathcal{Y}$ be a mapping between normed spaces $\mathcal{X}$ and $\mathcal{Y}$ on the domain $\mathcal{D}(f)$. We say f is *Gâteaux differentiable* at $x \in \mathcal{D}(f)$ if there exists a continuous linear mapping $Df(x) : \mathcal{X} \to \mathcal{Y}$ such that for all $h \in \mathcal{X}$,

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = Df(x)h.$$

Definition 2.20. Let $f : \mathcal{D}(f) \subseteq \mathcal{X} \to \mathcal{Y}$ be a mapping between normed spaces $\mathcal{X}$ and $\mathcal{Y}$ on the domain $\mathcal{D}(f)$. We call f *Fréchet differentiable* in $x \in \mathcal{D}(f)$ if there is a continuous linear mapping $Df(x) : \mathcal{X} \to \mathcal{Y}$ such that for $z \in \mathcal{X}$,

$$\lim_{\|z\|\to 0} \frac{\|f(x+z) - f(x) - Df(x)z\|}{\|z\|} = 0.$$

If f is Fréchet differentiable, then it is also Gâteaux differentiable in x. The two derivatives coincide when x belongs to the core of the $\mathcal{D}(f)$, which usually coincides with the interior of $\mathcal{D}(f)$. Also, f cannot be Gâteaux or Fréchet differentiable if x is a boundary point of $\mathcal{D}(f)$ [77].

2.1.8 Hilbert spaces

Definition 2.21. An *inner product* in a real vector space $\mathcal{H}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ such that

1. $\langle x, x \rangle > 0$ for every $0 \neq x \in \mathcal{H}$;

2. $\langle x, y \rangle = \langle y, x \rangle$ for each $x, y \in \mathcal{H}$;

3. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$, $\forall \alpha \in \mathbb{R}$, $\forall x, y \in \mathcal{H}$

Definition 2.22. The function $\|\cdot\| : \mathcal{H} \to \mathbb{R}$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$ is a *norm* in $\mathcal{H}$

Clearly, for every $x \neq 0$, we have $\|x\| > 0$. In addition, for every $\alpha \in \mathbb{R}$ and $x \in \mathcal{H}$, we have $\|\alpha x\| = |\alpha| \, \|x\|$.

Proposition 2.5. *For each $x, y \in \mathcal{H}$ the following hold:*

1. *Cauchy-Schwarz inequality:* $|\langle x, y \rangle| \leq \|x\| \, \|y\|$.

2. *triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.23. A *Hilbert space* is a Banach space $\mathcal{X}$ whose norm is associated to an inner product, i.e. $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathcal{X}$.

Example 2.1.4. Briefly, we mention some examples of Hilbert spaces.

1. The Euclidean space $\mathbb{R}^n$, described in Example 2.1.1, is a Hilbert space with the inner product $\langle x, y \rangle = x \cdot y$.

2. The sequence space $\ell^2(\mathbb{N})$ in Example 2.1.2 is a Hilbert space equipped with the inner product $\langle x, y \rangle = \sum_{n \in \mathbb{N}} x_n y_n$.

3. The space $L^2(\Omega)$ via Example 2.1.3 is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_\Omega f(x) g(x) dx, \quad \forall f, g \in L^2(\Omega).$$

Given $y \in \mathcal{H}$, we define a function $L_y : \mathcal{H} \to \mathbb{R}$ by $L_y(x) = \langle y, h \rangle$ for $h \in \mathcal{H}$. It is easy to see that L_y is linear and continuous via Cauchy-Schwarz inequality. Moreover $\|L_y\|_* = \|y\|$ (see [68]). The converse result is given by the following, whose proof can be read in standard analysis textbooks.

Theorem 2.1.3 (Riesz Representation Theorem)**.** [68, Theorem 1.41] *Let $L : \mathcal{H} \to \mathbb{R}$ be a continuous linear functional on $\mathcal{H}$. Then there exists a unique $y_L \in \mathcal{H}$ such that*

$$L(h) = \langle y_L, h \rangle$$

for each $h \in \mathcal{H}$. Moreover, the function $L \mapsto y_L$ is a linear isometry.

As a consequence of Theorem 2.1.3, if $\mathcal{H}^*$ is the dual of $\mathcal{H}$, then the inner product $\langle \cdot, \cdot \rangle_* :$ $H^* \times \mathcal{H}^* \to \mathbb{R}$ defined by

$$\langle L_1, L_2 \rangle_* = L_1(y_{L_2}) = \langle y_{L_1}, y_{L_2} \rangle$$

turns $\mathcal{H}^*$ into a Hilbert space, which is isometrically isomorphic to $\mathcal{H}$. Consequently, the norm associated with $\langle \cdot, \cdot \rangle_*$ is precisely $\|\cdot\|_*$ [68]. Thus Hilbert spaces are *self-dual.*

The following result is important in describing the geometry of a Hilbert space.

Corollary 2.1.1. [68, Corollary 1.42] *Hilbert spaces are reflexive.*

2.2 Banach space and convex analysis

In this chapter, we focus on the theoretical background concerning Banach spaces. The first parts focus on convex analysis. After that, we gather important results on the geometry of a Banach space with respect to its norm. We end this chapter with a result relating the geometric property of a Banach space with its respective duality mapping. Most of the results here can be found in any convex analysis or Banach space theory textbook, so we omit their proofs.

2.2.1 Basic concepts

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$. Often the norm may be written as $\|\cdot\|_{\mathcal{X}}$ to indicate its respective space. Denote by $\mathcal{X}^*$ the dual space of $\mathcal{X}$. For any $\xi \in \mathcal{X}^*$ and $x \in \mathcal{X}$ denote by $\langle \xi, x \rangle = \xi(x)$ the duality pairing. If $\mathcal{Y}$ is another Banach space and $A : \mathcal{X} \to \mathcal{Y}$ a bounded linear operator, then the bounded (continuous) linear operator $A^* : \mathcal{Y}^* \to \mathcal{X}^*$ defined as

$$\langle A^*\xi, x \rangle_{\mathcal{X}^*,\mathcal{X}} = \langle \xi, Ax \rangle_{\mathcal{Y}^*,\mathcal{Y}}, \forall x \in \mathcal{X}, \xi \in \mathcal{Y}^*$$

is called the *adjoint* operator (or simply adjoint) of A. Denote by $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

Denote by $\mathcal{N}(A) = \{x \in \mathcal{X} : Ax = 0\}$ the *null space* of A and

$$\mathcal{N}(A)^{\perp} := \{\xi \in \mathcal{X}^* : \langle \xi, x \rangle = 0, \, \forall x \in \mathcal{N}(A)\}$$

be the *annihilator* of A.

Definition 2.24. For $p > 1$, denote by $p^* > 1$ the *conjugate exponent* of p satisfying

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Theorem 2.2.1 (Young's inequality)**.** Let $a, b \in \mathbb{R}$ and $p, p^* > 1$ be conjugate exponents. Then

$$a \cdot b \leq \frac{1}{p} |a|^p + \frac{1}{p^*} |b|^{p^*}.$$

Proposition 2.6. [77, Lemma 2.55] Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $A \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$. If $\mathcal{X}$ is uniformly convex (and consequently, reflexive), then there holds

$$\mathcal{N}(A)^{\perp} = \overline{\mathcal{R}(A^*)},$$

where $\overline{\mathcal{R}(A^*)}$ denotes the closure of $\mathcal{R}(A^*)$, the range space of A^* in $\mathcal{X}^*$.

2.2.2 Convex analysis

In this section let $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$ be a *convex* function. Define $\mathcal{D}(\mathcal{R}) := \{x \in \mathcal{X} : \mathcal{R}(x) < +\infty\}$ its *effective domain*. We call $\mathcal{R}$ *proper* if $\mathcal{D}(\mathcal{R}) \neq \emptyset$.

Definition 2.25. The *subgradient* of $\mathcal{R}$ at $x \in \mathcal{X}$ is given by

$$\partial \mathcal{R}(x) := \{\xi \in \mathcal{X}^* : \mathcal{R}(z) \geq \mathcal{R}(x) + \langle \xi, z - x \rangle, \forall z \in \mathcal{X}\}.$$

The multi-valued mapping $\partial : \mathcal{X} \to 2^{\mathcal{X}^*}$ is called the *subdifferential* of $\mathcal{R}$. The set $\partial \mathcal{R}(x)$ denotes the subdifferential (i.e. the set of all subgradients) of $\mathcal{R}$ at x. We call

$$\mathcal{D}(\partial \mathcal{R}) := \{x \in \mathcal{D}(\mathcal{R}) : \partial \mathcal{R}(x) \neq \emptyset\}$$

Definition 2.26. [7] For $x \in \mathcal{D}(\partial \mathcal{R})$ and $\xi \in \partial \mathcal{R}(x)$, the *Bregman distance* in x and ξ induced by $\mathcal{R}$ is given by

$$D_{\xi} \mathcal{R}(\hat{x}, x) := \mathcal{R}(\hat{x}) - \mathcal{R}(x) - \langle \xi, \hat{x} - x \rangle, \quad \forall \hat{x} \in \mathcal{X}.$$

Remark 2.5. Clearly, $D_\xi\mathcal{R}(\hat{x}, x) \geq 0$ from Definition 2.25.

Proposition 2.7. [77, Lemma 2.62] For all $x, x_1 \in \mathcal{D}(\partial\mathcal{R})$, $\xi \in \partial\mathcal{R}(x)$, $\xi_1 \in \partial\mathcal{R}(x_1)$, $x_2 \in \mathcal{X}$ there holds

$$D_\xi\mathcal{R}(x_2, x) - D_\xi\mathcal{R}(x_1, x) = D_{\xi_1}\mathcal{R}(x_2, x_1) + \langle \xi_1 - \xi, x_2 - x_1 \rangle$$

Definition 2.27. A proper convex function $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$ is called *uniformly convex* if there is a continuous increasing function $h : [0, \infty) \to [0, \infty)$ with the property $h(t) = 0$ implies $t = 0$, such that

$$\mathcal{R}(\lambda\bar{x} + (1-\lambda)x) + \lambda(1-\lambda)h(\|\bar{x} - x\|) \leq \lambda\mathcal{R}(\bar{x}) + (1-\lambda)\mathcal{R}(x)$$

for all $x, \bar{x} \in \mathcal{X}$ and for all $\lambda \in (0, 1)$. If $h(t) = c_0 t^p$ for some $c_0 > 0$ and $p \geq 2$, then $\mathcal{R}$ is called *p-convex.*

Remark 2.6. It follows that

$$\text{strong convexity} \Rightarrow \text{uniform convexity} \Rightarrow \text{strict convexity} \Rightarrow \text{convexity}.$$

Theorem 2.2.2. [85, Theorem 3.5.10] *If $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$ is proper, lower semi-continuous, and uniformly convex, then*

$$D_\xi\mathcal{R}(\bar{x}, x) \geq h(\|\bar{x} - x\|), \quad \forall \bar{x} \in \mathcal{X}, \forall x \in D(\partial\mathcal{R}), \forall \xi \in \partial\mathcal{R}(x).$$

In particular, if $\mathcal{R}$ is p-convex, then

$$D_\xi\mathcal{R}(\bar{x}, x) \geq c_0 \|\bar{x} - x\|^p, \quad \forall \bar{x} \in \mathcal{X}, \forall x \in D(\partial\mathcal{R}), \forall \xi \in \partial\mathcal{R}(x). \tag{2.2.1}$$

Definition 2.28. Let $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$. Its Legendre-Fenchel *conjugate* (or simply conjugate) is defined by

$$\mathcal{R}^*(\xi) := \sup_{x \in \mathcal{X}} \{\langle \xi, x \rangle - \mathcal{R}(x)\}, \quad \forall \xi \in \mathcal{X}^*.$$

Proposition 2.8. [68, Prop. 3.56] If $\mathcal{R}$ is proper, lower semi-continuous, and convex, then so is $\mathcal{R}^*$.

Proposition 2.9. ([10, Prop. I.2.14],[68, Prop. 3.59, Rem. 3.60]) If $\mathcal{X}$ is reflexive and $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$ is proper, lower semi-continuous, and convex, then

$$\xi \in \partial\mathcal{R}(x) \Leftrightarrow x \in \partial\mathcal{R}^*(\xi) \Leftrightarrow \mathcal{R}(x) + \mathcal{R}^*(\xi) = \langle \xi, x \rangle.$$

Proposition 2.10. [85, Corollary 3.5.11] *If $\mathcal{R}$ is p-convex satisfying* (2.2.1) *for $p \geq 2$, then $\mathcal{D}(\mathcal{R}^*) = \mathcal{X}^*$, $\mathcal{R}^*$ is Fréchet differentiable, and its gradient $\nabla : \mathcal{X}^* \to \mathcal{X}$ satisfies*

$$\|\nabla\mathcal{R}^*(\xi_1) - \nabla\mathcal{R}^*(\xi_2)\| \leq \left(\frac{\|\xi_1 - \xi_2\|}{2c_0}\right)^{\frac{1}{p-1}}, \forall \xi_1, \xi_2 \in \mathcal{X}^*.$$

Moreover,

$$\mathcal{R}^*(\xi_2) - \mathcal{R}^*(\xi_1) - \langle \xi_2 - \xi_1, \nabla\mathcal{R}^*(\xi_1)\rangle \leq \frac{1}{p^*(2c_0)^{p^*-1}} \|\xi_2 - \xi_1\|^{p^*}$$

for any $\xi_1, \xi_2 \in \mathcal{X}^*$.

2.2.3 Duality mappings

Definition 2.29. A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies $\varphi(0) = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$ is called a *weight function* (or *gauge function*).

Definition 2.30. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a weight function. The (set-valued) mapping $J^{\mathcal{X}}_{\varphi} : \mathcal{X} \to 2^{\mathcal{X}^*}$ defined by

$$J^{\mathcal{X}}_{\varphi}(x) := \{\xi \in \mathcal{X}^* \ : \ \langle \xi, x\rangle = \|\xi\| \, \|x\| \, , \ \|\xi\| = \varphi(\|x\|)\}$$

is called the *duality mapping* of $\mathcal{X}$ with weight function φ. The duality mapping corresponding to the weight function $\varphi(t) = t$ is called *normalized duality mapping.*

If φ is a weight function, then $\psi(t) := \int_0^t \varphi(s)ds$ is a convex function [10, Lemma 4.3]. We use ψ to present the following important result (see the proof in ([1], [10, Theorem 4.4]).

Theorem 2.2.3 (Asplund's theorem)**.** *If $J^{\mathcal{X}}_{\varphi}$ is a duality mapping of weight function φ, then $J^{\mathcal{X}}_{\varphi}(x) = \partial\psi(\|x\|)$, for all $x \in \mathcal{X}$*

Example 2.2.1. [10, Prop. 4.8(i)] If $\mathcal{X}$ is a normed space, then $J^{\mathcal{X}}_2(x) = x$, and therefore $J^{\mathcal{X}}_p(x) = \|x\|^{p-2} \cdot x$ for $p \geq 1$.

2.2.4 Geometry of Banach space norms

We recall some notions of how a Banach space is geometrically "nice". To do this, we start from the subdifferential of the $\frac{1}{p}\|\cdot\|^p$ for $p \geq 1$, as presented in [77].

Definition 2.31. We call a Banach space $\mathcal{X}$ *convex of power type p* or *p-convex* if there exists a constant $c_p > 0$ such that

$$\tfrac{1}{p}\|x - y\|^p \geq \tfrac{1}{p}\|x\|^p - \left\langle j^{\mathcal{X}}_p(x), y\right\rangle + \tfrac{c_p}{p}\|y\|^p$$

for all $x, y \in \mathcal{X}$, and all $j_p^{\mathcal{X}} \in J_p^{\mathcal{X}}$.

Definition 2.32. We call a Banach space $\mathcal{X}$ *smooth of power type* p or *p-smooth* if there exists a constant $G_p > 0$ such that

$$\tfrac{1}{p}\left\|x-y\right\|^p \leq \tfrac{1}{p}\left\|x\right\|^p - \left\langle j_p^{\mathcal{X}}(x), y\right\rangle + \tfrac{G_p}{p}\left\|y\right\|^p$$

for all $x, y \in \mathcal{X}$, and all $j_p^{\mathcal{X}} \in J_p^{\mathcal{X}}$.

Definition 2.33. We call a Banach space $\mathcal{X}$ *strictly convex* if for all $x, y \in \mathcal{X}$ such that $x \neq y$ and $\|x\| = \|y\| = 1$, it follows that $\|\alpha x + (1-\alpha)y\| < 1$, $\forall \alpha \in (0,1)$.

Definition 2.34. We call $\mathcal{X}$ *uniformly convex* if, for the *modulus of convexity* $\delta_{\mathcal{X}} : [0,2] \to [0,1]$, defined as

$$\delta_{\mathcal{X}}(\varepsilon) := \inf\left\{1 - \left\|\tfrac{1}{2}(x+y)\right\|,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\},$$

there holds $\delta_{\mathcal{X}}(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Definition 2.35. We call $\mathcal{X}$ *smooth* if for every $x \neq 0$, there exists a unique $\xi \in \mathcal{X}^*$ such that $\|\xi\| = 1$ and $\langle \xi, x\rangle = \|x\|$.

Definition 2.36. We call $\mathcal{X}$ *uniformly smooth* if for the *modulus of smoothness* $\rho_{\mathcal{X}} : [0,\infty) \to [0,\infty)$ defined as

$$\rho_{\mathcal{X}}(\tau) := \frac{1}{2}\sup\left\{\|x+y\| + \|x-y\| - 2\ \ \|x\| = 1, \|y\| \leq \tau\right\}$$

we have the limit condition

$$\lim_{\tau \to 0} \frac{\rho_{\mathcal{X}}(\tau)}{\tau} = 0.$$

Example 2.2.2. [77, Ex. 2.46] The polarization identity

$$\tfrac{1}{2}\left\|x-y\right\|^2 = \tfrac{1}{2}\left\|x\right\|^2 - \langle x, y\rangle + \tfrac{1}{2}\left\|y\right\|^2,$$

guarantees that Hilbert spaces are 2-convex and 2-smooth.

Example 2.2.3. ([10, Thm. 4.9] , [76]). The space $L^p(\Omega)$ described in Example 2.1.3 is uniformly convex for $1 < p < \infty$. Conversely, the spaces $L^\infty(\Omega)$ and $L^1(\Omega)$ are neither smooth nor strictly convex.

2.3 Spectral theory

We present some background on spectral theory. We also include some estimates involving norms of self-adjoint linear operators. These are important tools in obtaining convergence results in Hilbert spaces.

Consider a compact linear operator $K : \mathcal{X} \to \mathcal{Y}$ between Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$. Let $(\sigma_n; v_n, u_n)$ be a singular system for K. Since $(\sigma_n^2; v_n)$ is an eigensystem for the self-adjoint compact operator K^*K,

$$K^*Kx = \sum_{n=1}^{\infty} \sigma_n^2 \langle x, v_n \rangle v_n$$

holds. Alternatively, for $\lambda \in \mathbb{R}$ and $x \in \mathcal{X}$, we write

$$K^*Kx = \int \lambda dE_\lambda x,$$

where E_λ is called the *spectral family*, denoted by

$$E_\lambda x := \sum_{\substack{n=1 \\ \sigma_n^2 < \lambda}}^{\infty} \langle x, v_n \rangle v_n \quad (+Px), \tag{2.3.1}$$

where P is an orthogonal projector onto $N(K^*K)$, only appearing in (2.3.1) only for $\lambda > 0$. Given a piecewise continuous function f and a self-adjoint compact operator K^*K, define

$$f(K^*K) := \int f(\lambda) dE_\lambda := \sum_{n=1}^{\infty} f(\sigma_n^2) \langle \cdot, v_n \rangle v_n. \tag{2.3.2}$$

Also, $f(K^*K)$ is a bounded self-adjoint linear operator on $\mathcal{X}$. Moreover, for all (piecewise) continuous functions f, we have

$$f(K^*K)K^* = K^*f(KK^*), \tag{2.3.3}$$

where $f(KK^*)$ is analogously defined by the spectral family F_λ defined by

$$F_\lambda y :=:= \sum_{\substack{n=1 \\ \sigma_n^2 < \lambda}}^{\infty} \langle y, u_n \rangle u_n \quad + (I - Q),$$

where $I - Q$ is an orthogonal projector onto $N(KK^*) = R(KK^*)^\perp$. Also, given $f(K^*K)$ via

(2.3.2), and for all $x, y \in \mathcal{X}$, it follows that

$$\langle f(K^*K)x, y\rangle = \int f(\lambda) d\,\langle E_\lambda x, y\rangle\,, \tag{2.3.4}$$

and

$$\|f(K^*K)x\|^2 = \int f^2(\lambda) d\,\|E_\lambda x\|^2\,. \tag{2.3.5}$$

From these formulas, it follows that

$$\|f(K^*K)\| \leq \sup |f| \tag{2.3.6}$$

and

$$\|f(K^*K)\| \leq \sup\left\{\sqrt{\lambda}\,|f|\right\}, \tag{2.3.7}$$

where both supremum are taken over the smallest interval containing the spectrum of K^*K, i.e., over $\left[0, \|K\|^2\right]$.

Proposition 2.11 (Interpolation inequality)**.** Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. Then for all $q > r \geq 0$, there holds

$$\|(T^*T)^r x\| \leq \|(T^*T)^q x\|^{\frac{r}{q}}\, \|x\|^{1-\frac{r}{q}}\,.$$

Proposition 2.12. [16, Proposition 2.18] *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. Then* $R(T^*) = R((T^*T)^{1/2})$.

Proposition 2.13. Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. Then $\|Tz\|^2 = \left\|(T^*T)^{1/2}z\right\|^2$ for all $z \in \mathcal{X}$.

Proof. This is an auxiliary result in the proof of [16, Theorem 4.3]. For completeness we show the proof. For any $z \in \mathcal{X}$,

$$\|Tz\|^2 = \langle Tz, Tz\rangle = \langle T^*Tz, z\rangle = \left\langle (T^*T)^{1/2}z, (T^*T)^{1/2}z\right\rangle = \left\|(T^*T)^{1/2}\right\|^2.$$

□

Chapter 3

Variational Regularization

Inverse problems occur across sciences and engineering, and they deal with reconstructing parameters based on some measurement data. Let $(\mathcal{Q}, \tau_{\mathcal{Q}})$ and $(\mathcal{U}, \tau_{\mathcal{U}})$ be topological spaces $\mathcal{Q}$ and $\mathcal{U}$ with respective topologies $\tau_{\mathcal{Q}}$ and $\tau_{\mathcal{U}}$. The most extensively studied form of inverse problems has to do with solving an equation of the form

$$F(q) = u^\dagger, \tag{3.0.1}$$

where $F : \mathcal{Q} \to \mathcal{U}$ is a mapping between $\mathcal{Q}$ and $\mathcal{U}$ and $u^\dagger \in \mathcal{U}$ is the exact data. Often, the space $\mathcal{Q}$ is referred to as the parameter or solution space, while $\mathcal{U}$ is the measurement or data space. Equivalently, a solution to the operator equation (3.0.1) is also a solution to

$$\mathcal{J}\left(F(q), u^\dagger\right) = 0, \tag{3.0.2}$$

where $\mathcal{J} : \mathcal{Q} \times \mathcal{U} \to [0, +\infty]$ is a proper data *misfit functional*. As (3.0.1) can have many solutions, we may take a proper penalty functional or regularization functional $\mathcal{R} : \mathcal{Q} \to [0, \infty]$ to capture the feature by finding a particular solution $q^\dagger \in \mathcal{Q}$ that minimizes $\mathcal{R}$, i.e.,

$$\mathcal{R}(q^\dagger) = \min\left\{\mathcal{R}(q) : q \in \mathcal{Q} \text{ and } \mathcal{J}\left(F(q), u^\dagger\right) = 0\right\},$$

and we call $q^\dagger$ an *$\mathcal{R}$-minimizing solution.*

Inherently, inverse problems are not well-posed in Hadaramd's sense. Topologically speaking, this means if a sequence of right-hand side vectors in (3.0.1) converge with respect to $\tau_{\mathcal{U}}$, then the sequences of solutions of the corresponding equations, if exist, may not converge with respect to $\tau_{\mathcal{Q}}$. A common scenario for inverse problems is as follows. Rather than the exact data $u^\dagger$, a noisy measurement $\tilde{u} \in \mathcal{U}$ is available, with $\delta := \mathcal{J}(F(q^\dagger), \tilde{u})$ as the noise level. This makes (3.0.1) ill-posed, and as an alternative to solving (3.0.1), we solve the minimization problem

$$\min_{q\in\mathcal{Q}} \mathcal{J}\left(F(q), \tilde{u}\right). \tag{3.0.3}$$

Since (3.0.1) is already ill-posed, equation (3.0.3) is also ill-posed. To stabilize (3.0.3), we may consider solving the minimization problem

$$\min_{q\in\mathcal{Q}} \left\{\mathcal{J}\left(F(q), \tilde{u}\right) + \alpha\mathcal{R}(q)\right\}, \tag{3.0.4}$$

where $\alpha > 0$ is called the *regularization parameter*. The functional $T_{\alpha,\tilde{u}}(q) := \mathcal{J}\left(F(q), \tilde{u}\right) + \alpha\mathcal{R}(q)$ is called the Tikhonov functional, and the strategy of solving (3.0.4) is called *Tikhonov regularization*.

For a given $\tilde{u} \in \mathcal{U}$ and $\alpha > 0$, we take the solution to (3.0.4) as an approximate solution to (3.0.1). In Tikhonov regularization, the following fundamental questions must be answered:

- **Existence**: given the noisy data $\tilde{u}$ and $\alpha > 0$, there exists a solution to (3.0.2);
- **Stability**: small perturbations on the noisy data $\tilde{u}$ should not change the minimizers of $T_{\alpha,\tilde{u}}$ too much. This is important because (3.0.4) is solved numerically.
- **Convergence**: the more exact the measured data $\tilde{u}$ is, the closer is the minimizer of $T_{\alpha,\tilde{u}}$ to the exact solution of (3.0.2), given a rule for choosing the regularization parameter α.
- **Convergence rates**: the bound of the error between the minimizer of $T_{\alpha,\tilde{u}}$ and the exact solution of (3.0.2) expressed in terms of the misfit between $\tilde{u}$ and $u^\dagger$ under a suitable rule for choosing the regularization parameter.

Normally, convergence rates are attainable for variational regularization using source conditions. For inverse problems in Hilbert spaces, there is a useful abstract tool for expressing the smoothness of the $\mathcal{R}$-minimizing solution $q^\dagger$. A more powerful tool, variational source condition, combines the source condition and the nonlinearity of the problem (3.0.1).

The performance of Tikhonov regularization depends crucially on the choice of the regularization parameter. There are several different types of parameter choice rules.

1. *A-priori* parameter choice rules $\alpha = \alpha(\delta)$ do not depend on the noisy data $\tilde{u}$ but depend on the noise level δ and possibly on the smoothness condition on the sought solution.

For (3.0.4) with power-type misfit norms, this rule can be formulated to satisfy the limit conditions

$$\alpha(\delta) \to 0 \text{ and } \frac{\delta^p}{\alpha(\delta)} \to 0 \text{ as } \delta \to 0 \tag{3.0.5}$$

for $1 < p < \infty$. For instance in (3.0.4) for Hilbert spaces, we can set $\alpha \sim \delta^{2/(2\mu+1)}$ for some $\mu > 0$ dependent on the source condition. Normally such strategies have limited use since they depend on the smoothness condition, which are hard to verify in applications.

2. *A-posteriori* parameter choice rules depend on both the noise level and noisy data $\tilde{u}$. The most well-studied is the *discrepancy principle*: choose $\alpha = \alpha(\delta, \tilde{u})$ such that

$$\tau_1 \delta \leq \left\| F(q_\alpha^\delta) - \tilde{u} \right\| \leq \tau_2 \delta$$

for $1 \leq \tau_1 \leq \tau_2$. In some applications, a weaker rule with only the constant τ_2 is used, especially for iterative regularization schemes (c.f. [53]).

3. *Heuristic rules* $\alpha = \alpha(\tilde{u})$ depend only on the noisy data $\tilde{u}$. This will be more thoroughly discussed in this chapter.

We briefly present an overview on important developments of Tikhonov regularization in the following.

1. If $\mathcal{Q}$ and $\mathcal{U}$ are Hilbert spaces with the usual norm topology, then (3.0.4) with $\mathcal{J}(q, \tilde{u}) := \|F(q) - \tilde{u}\|_{\mathcal{Y}}^2$ and $\mathcal{R}(q) = \|q - \bar{q}\|^2$ for a given $\bar{q} \in \mathcal{Q}$ is the Tikhonov regularization for inverse problems in Hilbert spaces. Here $F : \mathcal{D}(F) \subseteq \mathcal{Q} \to \mathcal{U}$ is a Fréchet differentiable forward operator with Fréchet derivative F'. Convergence and optimal convergence rates with respect to the $\bar{q}$-minimum norm solution $x^\dagger$ can be achieved under a-priori parameter choice rules [16, 17], using a source condition of the form

$$q^\dagger - \bar{q} \in R((F'(q^\dagger)^* F'(q^\dagger))^\mu)$$

for some $\mu \in [1/2, 1]$. The same thing under discrepancy principle was first shown [75] under very restrictive conditions and then enhanced in [45, 39, 46] using weaker conditions.

2. If $\mathcal{Q}$ and $\mathcal{U}$ are Banach spaces with norm topology, then (3.0.4) with the power-type misfit norm $\mathcal{J}(q, \tilde{u}) := \frac{1}{p} \|F(q) - \tilde{u}\|_{\mathcal{Y}}^p$ for $1 < p < \infty$ was studied with a general penalty functional $\mathcal{R}$. Convergence under *a-priori* and *a-posteriori* stopping criterion has been

studied (c.f. [77, Chap. 4] and the references therein). Convergence rate is obtained under a *variational source condition* of the form

$$\beta E(q,q^\dagger) \le \mathcal{R}(q) - \mathcal{R}(q^\dagger) + \varphi(\|F(q) - F(q^\dagger)\|) \tag{3.0.6}$$

for all $x \in M \subseteq \mathcal{D}(F) \cup \mathcal{D}(\mathcal{R})$, where M is normally a sublevel set, φ is some index function, and $E(q,q^\dagger)$ represents an error measure between $q \in \mathcal{Q}$ and the $\mathcal{R}$-minimizing solution $q^\dagger$. In terms of heuristic rules, the convergence of (3.0.4) for Banach spaces $\mathcal{Q}$ and $\mathcal{U}$ with the misfit term $\mathcal{J}(q,\tilde{u}) := \|F(q) - \tilde{u}\|_{\mathcal{Y}}^p$ for $1 < p < \infty$ was also established in [40].

3. The dissertation [69] considered operator equations (3.0.1) with $\mathcal{Q}$, $\mathcal{U}$ as general topological spaces. This leads to (3.0.4) in terms of a generalized misfit functional $\mathcal{J}$. This formulation extends towards problems with more general misfit functionals beyond the usual norm. These includes Bregman distances, probability metrics, and information divergence measures. The obtained convergence rates are in terms of Bregman distance, and also uses source conditions.

4. The dissertation [19], which builds on the results of [69], considered (3.0.4) by considering the operator equation (3.0.1) with $\mathcal{Q}$ and $\mathcal{U}$ as Hausdorff spaces. Remarkably, this assumed that the exact data $u^\dagger$ and the noisy data $\tilde{u}$ belong to different spaces, e.g. setting a third space $\tilde{U}$ such as $\tilde{u} \in \tilde{U}$. This approach leads to new way of proving the convergence, instead of simply replacing the norm by misfit functionals. It also assumed a variation source condition of the form (3.0.6) with a general misfit functional. In turn, the obtained convergence rates were in terms of the general error functional under both *a-priori* and *a posteriori* parameter choice rules.

3.1 A general form of variational regularization

Now we consider variational regularization for inverse problems in a more general form. Let $(\mathcal{Q}, \tau_{\mathcal{Q}})$ and $(\mathcal{U}, \tau_{\mathcal{U}})$ be two topological spaces with topology $\tau_{\mathcal{Q}}$ and $\tau_{\mathcal{U}}$ respectively. Instead of (3.0.2), we consider inverse problems of the form

$$\mathcal{S}(q, u^\dagger) = 0 \tag{3.1.1}$$

where $\mathcal{S} : \mathcal{Q}\times\mathcal{U} \to [0,\infty]$ is a proper data misfit functional. Here the properness means $\mathcal{S}$ takes a finite value at some point in $\mathcal{Q}\times\mathcal{U}$. Equation (3.1.1) may have more than one solutions, given one exists. In order to pick one with the desired properties, we use a proper penalty functional $\mathcal{R} : \mathcal{Q} \to (-\infty,\infty]$ to determine a solution $q^\dagger$ such that

$$\mathcal{R}(q^\dagger) = \min\left\{\mathcal{R}(q) : q \in \mathcal{Q} \text{ and } \mathcal{S}(q,u^\dagger) = 0\right\}, \tag{3.1.2}$$

and we call it an $\mathcal{R}$-minimizing solution when it exists. Here, we consider the instance where problem (3.1.1) is ill-posed, in the sense that given $u \in \mathcal{U}$, the equation $\mathcal{S}(q,u) = 0$ may not have a solution, or the solution may not continuously depend on u. Now, let $\tilde{u}$ be a noisy data. The variational regularization for solving (3.1.1) is to use the solution of the minimization problem

$$\tilde{q}_\alpha \in \arg\min_{q\in\mathcal{Q}} \{\mathcal{S}(q,\tilde{u}) + \alpha\mathcal{R}(q)\} \tag{3.1.3}$$

to approximate an $\mathcal{R}$-minimizing solution $q^\dagger$, where $\alpha > 0$ is the regularization parameter.

The framework (3.1.1) covers a broader range of inverse problems not covered by (3.0.2), while (3.1.3) provides a more general form of variational regularization than (3.0.4). For instance, consider the problem of estimating the parameter $q \in \mathcal{Q}$ in the elliptic equation

$$-\operatorname{div}(q\nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{3.1.4}$$

given measurements $u \in H_0^1(\Omega)$, where $\mathcal{Q} := \{q \in L^\infty(\Omega) : \gamma_0 \le q \le \gamma_1\}$ for some positive constants γ_0 and γ_1. Let $u(q)$ denote the solution of (3.1.4). In [84] problem (3.1.4) was formulated in the form (3.1.1) with

$$\mathcal{S}(q,u) := \int_\Omega q\,|\nabla(u(q) - u)|^2 \tag{3.1.5}$$

which has the advantage that the functional $q \mapsto \mathcal{S}(q,u)$ is convex [26]. Clearly, such formulation cannot be covered by (3.0.2). In section 3.3, we will formulate a general convex framework to regularize a family of elliptic parameter estimation problems although the problems themselves are nonlinear. As the second example, consider the electrical impedance tomography which will be discussed in detail in Section 3.4. When formulating this inverse problem in terms of the Kohn-Vogelius functional ([58, 59, 60]), it takes the form (3.1.1) and it does not take the form (3.0.2).

Throughout this chapter we will assume that (3.1.1) has a solution in $\operatorname{dom}(\mathcal{R})$, the domain of $\mathcal{R}$. We will use $\xrightarrow{\tau_{\mathcal{Q}}}$ and $\xrightarrow{\tau_{\mathcal{U}}}$ to denote the convergence with respect to the topologies $\tau_{\mathcal{Q}}$ and $\tau_{\mathcal{U}}$ respectively. We will consider using the variational regularization (3.1.3) to solve (3.1.1). Our analysis is based on the following conditions on the penalty functional $\mathcal{R} : \mathcal{Q} \to [0, \infty]$ and the data misfit term $\mathcal{S} : \mathcal{Q} \times \mathcal{U} \to [0, \infty]$.

Assumption 3.1.1.

(i) *For each $u \in \mathcal{U}$ there is a $q \in \mathcal{Q}$ such that $\mathcal{R}(q) < \infty$ and $\mathcal{S}(q, u) < \infty$.*

(ii) *$\mathcal{R} : \mathcal{Q} \to [0, \infty]$ is proper, nonnegative and $\tau_{\mathcal{Q}}$-lower semi-continuous over $\mathcal{Q}$. That is, $\{q_n\} \subset \mathcal{Q}$ with $q_n \xrightarrow{\tau_{\mathcal{Q}}} q \in \mathcal{Q}$ implies that*

$$\mathcal{R}(q) \le \liminf_{n\to\infty} \mathcal{R}(q_n).$$

(iii) *For every $\rho > 0$, the sublevel set*

$$\mathcal{M}_\rho := \{q \in \mathcal{Q} : \mathcal{R}(q) \le \rho\}$$

is $\tau_{\mathcal{Q}}$-sequentially compact. That is, every sequence $\{q_n\}$ in $\mathcal{M}_\rho$ has a $\tau_{\mathcal{Q}}$-convergent subsequence in $\mathcal{Q}$.

(iv) *The mapping $(q, u) \to \mathcal{S}(q, u)$ is $(\tau_{\mathcal{Q}} \times \tau_{\mathcal{U}})$-lower semi-continuous over $\mathcal{Q} \times \mathcal{U}$, i.e. $\{(q_n, u_n)\} \subset \mathcal{Q} \times \mathcal{U}$ with $q_n \xrightarrow{\tau_{\mathcal{Q}}} q \in \mathcal{Q}$ and $u_n \xrightarrow{\tau_{\mathcal{U}}} u \in \mathcal{U}$ implies that*

$$\mathcal{S}(q, u) \le \liminf_{n\to\infty} \mathcal{S}(q_n, u_n).$$

(v) *For each $q \in \mathcal{Q}$ the function $u \to \mathcal{S}(q, u)$ is $\tau_{\mathcal{U}}$-continuous.*

Under Assumption 3.1.1, by a standard argument one can show that (3.1.1) has an $\mathcal{R}$-minimizing solution $q^\dagger$ and the minimization problem (3.1.3) has a solution $\tilde{q}_\alpha$ for each $\alpha > 0$. The proof follows from that of [77, Prop. 4.1] and [69].

Theorem 3.1.1. *Let Assumption 3.1.1 hold.*

(i) *If (3.1.1) has a solution, then it has an $\mathcal{R}$-minimizing solution.*

(ii) *Let $\tilde{u} \in \mathcal{U}$ and $\alpha > 0$ and define $\mathcal{T}_\alpha(q) := \mathcal{S}(q, \tilde{u}) + \alpha\mathcal{R}(q)$. Then there exists a minimizer of $\mathcal{T}_\alpha(q)$.*

Proof. We will only prove (ii) since (i) can be proved similarly. According to Assumption 3.1.1 (i), there is at least one $\hat{q} \in \mathcal{Q}$ such that $\mathcal{T}(\hat{q}) < \infty$. Thus

$$0 \le c := \inf \{\mathcal{T}_\alpha(q) : q \in \mathcal{D}\} < \infty.$$

Let $\{q_k\} \in \mathcal{Q}$ be a minimizing sequence such that

$$\lim_{k \to \infty} \mathcal{T}_\alpha(q_k) = c.$$

Let $\rho = c+1$. Then there is an N such that $\mathcal{T}_\alpha(q_k) \le \rho$ for all $k \ge N$. Since $\mathcal{R}(q_k) \le T_\alpha(q_k)/\alpha$, it follows that $\{q_k\} \subseteq \mathcal{M}_{\rho/\alpha}$. Hence, by Assumption 3.1.1 (iii), $\{q_k\}$ has a $\tau_{\mathcal{Q}}$-convergent subsequence in $\mathcal{Q}$, which we denote again by $\{q_k\}$ for ease of notation. Let $\tilde{q}$ be its associated limit. Then Assumption 3.1.1 (ii) shows that

$$\mathcal{R}(\tilde{q}) \le \liminf_{k \to \infty} \mathcal{R}(q_k),$$

In addition, using Assumption 3.1.1 (iv), there holds

$$\mathcal{S}(\tilde{q}, \tilde{u}) \le \liminf_{k \to \infty} \mathcal{S}(q_k, \tilde{u}).$$

These imply that

$$\begin{aligned} \mathcal{S}(\tilde{q}, \tilde{u}) + \alpha \mathcal{R}(\tilde{q}) &\le \liminf_{k \to \infty} \mathcal{S}(q_k, \tilde{u}) + \alpha \liminf_{k \to \infty} \mathcal{R}(q_k) \\ &\le \liminf_{k \to \infty} \left(\mathcal{S}(q_k, \tilde{u}) + \alpha \mathcal{R}(q_k)\right) = c. \end{aligned}$$

Along with the definition of c, this shows that $\tilde{q}$ minimizes $\mathcal{T}_\alpha(q)$. □

In order to use the minimizer $\tilde{q}_\alpha$ of (3.1.3) to approximate the sought solution, the choice of the regularization parameter α is very crucial. The most natural choice rule is the discrepancy principle which selects α such that

$$\tau_1 \delta \le \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \tau_2 \delta \tag{3.1.6}$$

for given constants $1 \le \tau_1 \le \tau_2 < \infty$, where δ denotes an upper bound on the noise level: $\mathcal{S}(q^\dagger, \tilde{u}) \le \delta$. Under Assumption 3.1.1, one can use a standard argument to obtain convergence results with α chosen by (3.1.6). We note that the performance of the discrepancy principle

(3.1.6) depends crucially on the estimate of noise level. Unfortunately, an upper bound on $\mathcal{S}(q^\dagger, \tilde{u})$ is usually obtained by estimation and a good estimate may be difficult to obtain. For instance, consider (3.1.3) with $\mathcal{S}$ given by (3.1.5), in order to obtain a good upper bound on noise level $\mathcal{S}(q^\dagger, \tilde{u})$ one has to rely on a bound on the sought solution $q^\dagger$ which might lead $\mathcal{S}(q^\dagger, \tilde{u})$ to be overestimated or underestimated and thus it could result in a poor approximate solution. Therefore, it is necessary to develop parameter choice rules which depend only on the data $\tilde{u}$.

In this chapter, we consider the Hanke-Raus heuristic rule for the variational regularization (3.1.3) using the most general set of conditions possible for general topological spaces. We present a general error estimate using a generalized source condition, as well as a convergence result. The material in the following sections is taken from our joint paper [63].

3.2 Hanke-Raus heuristic rule

Now we formulate the Hanke-Raus heuristic rule for the variational regularization (3.1.3), as motivated by [40], [16, Chapter 4], and [24].

Rule 3.1. *Let $A \geq 0$, $\alpha_0 > 0$ and $0 < \gamma < 1$ be given numbers, and let*

$$\Delta_\gamma = \left\{\alpha_0 \gamma^j : j = 0, 1, \ldots\right\}.$$

Define $\alpha_* := \alpha_*(\tilde{u}) \in \Delta_\gamma$ such that

$$\alpha_* \in \arg\min_{\alpha \in \Delta_\gamma} \left\{\Theta(\alpha, \tilde{u}) := \left(\frac{1}{\alpha} + A\right) \mathcal{S}(\tilde{q}_\alpha, \tilde{u})\right\}$$

and use $\tilde{q}_{\alpha_*}$ as an approximate solution.

Using Rule 3.1 requires preassigned choices for constants A and α_0. In [40], variational regularization in Banach spaces under Rule 3.1 was studied using $A = 0$. When using Rule 3.1, we should avoid to choose $\alpha_0 > 0$ too small since otherwise it may produce a small regularization parameter α_* and hence result in a bad approximate solution that is oscillatory. On the other hand, by the definition of $\tilde{q}_\alpha$ we have

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) + \alpha \mathcal{R}(\tilde{q}_\alpha) \leq \mathcal{S}(q_m, \tilde{u}) + \alpha \mathcal{R}(q_m),$$

where $q_m \in \mathcal{Q}$ denotes an element such that $\mathcal{R}(q_m) = \min_{q \in \mathcal{Q}} \mathcal{R}(q)$ whose existence is guaranteed by Assumption 3.1.1. Thus $\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \leq \mathcal{S}(q_m, \tilde{u})$ and so

$$\frac{1}{\alpha}\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \to 0 \quad \text{as } \alpha \to \infty.$$

Consequently, if α_0 is too large and $A = 0$, Rule 3.1 may output a large parameter α_* which may result in an approximate solution with less accuracy. Therefore, for Rule 3.1 to generate an accurate approximate solution, we suggest to choose $A > 0$ to be suitably large or $\alpha_0 > 0$ to be suitably small. The choice of $\alpha_0 > 0$ can be based on a rough guess of the optimal regularization parameter.

3.2.1 A posteriori error estimates

We first derive *a posteriori* error estimates on $\tilde{q}_{\alpha_*}$ with α_* chosen by Rule 3.1. To achieve this, we need conditions on the sought $\mathcal{R}$-minimizing solution $q^\dagger$. We will assume the following variational source condition on $q^\dagger$.

Assumption 3.2.1. *There is an error function $\mathcal{E} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ with $\mathcal{E}(q, q) = 0$ whenever $q \in \mathcal{Q}$ such that*

$$\mathcal{E}(q, q^\dagger) \leq \mathcal{R}(q) - \mathcal{R}(q^\dagger) + \varphi(\mathcal{S}(q, u^\dagger)) \tag{3.2.1}$$

for all $q \in \mathcal{M}_\rho := \{q \in \mathcal{Q} : \mathcal{R}(q) \leq \rho\}$ with $\rho > \mathcal{R}(q^\dagger)$, where $\varphi : [0, \infty) \to [0, \infty)$ is a concave index function. Here φ is called an index function if it is continuous and strictly increasing with $\varphi(0) = 0$.

After discussing the error estimates we will give a detailed discussion on Assumption 3.2.1 in Proposition 3.1. For now, in order to carry out the derivation of the error estimate, we need the following "quasi-triangle inequality" on the data misfit term $\mathcal{S}$.

Assumption 3.2.2. *There is a constant $C_S \geq 1$ such that for any $u^\dagger \in \mathcal{U}$ and $q^\dagger \in \mathcal{Q}$ satisfying $\mathcal{S}(q^\dagger, u^\dagger) = 0$ and any $u \in \mathcal{U}$ and $q \in \mathcal{Q}$ there holds*

$$\mathcal{S}(q, u^\dagger) \leq C_{\mathcal{S}} \left(\mathcal{S}(q^\dagger, u) + \mathcal{S}(q, u)\right).$$

Now we are ready to prove the following *a posteriori* error estimate.

Theorem 3.2.1. *Let $\mathcal{R}$ and $\mathcal{S}$ satisfy Assumption 3.1.1 and Assumption 3.2.2. Assume that $q^\dagger$ is an $\mathcal{R}$-minimizing solution of* (3.1.1) *satisfying Assumption 3.2.1 with $\Phi(t) := t/\varphi(t)$ being strictly increasing. Let α_* be determined by Rule 3.1. If $\tilde{q}_{\alpha_*} \in \mathcal{M}_\rho$ and $\delta_* := \mathcal{S}(\tilde{q}_{\alpha_*}, \tilde{u}) \neq 0$,* *then*

$$\mathcal{E}(\tilde{q}_{\alpha_*}, q^\dagger) \le C\left(1 + \frac{\delta}{\delta_*}\right)(\delta + \varphi(\delta + \delta_*)) . \tag{3.2.2}$$

where $\delta := \mathcal{S}(q^\dagger, \tilde{u})$ *denotes the noise level and* C *is a constant depending only on* α_0, γ, A *and* $C_{\mathcal{S}}$.

Proof. First, we show that if $\tilde{q}_\alpha \in \mathcal{M}_\rho$, then

$$\mathcal{E}(\tilde{q}_\alpha, q^\dagger) \le \frac{\delta}{\alpha} + C_{\mathcal{S}}\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})), \tag{3.2.3}$$

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le 2\delta + \Phi^{-1}(3C_{\mathcal{S}}\alpha). \tag{3.2.4}$$

To see this, by using the minimality of $\tilde{q}_\alpha$ we have

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) + \alpha\mathcal{R}(\tilde{q}_\alpha) \le \mathcal{S}(q^\dagger, \tilde{u}) + \alpha\mathcal{R}(q^\dagger) \tag{3.2.5}$$

and since $\mathcal{S}(q^\dagger, \tilde{u}) = \delta$, (3.2.5) leads us to

$$\alpha\left(\mathcal{R}(\tilde{q}_\alpha) - \mathcal{R}(q^\dagger)\right) + \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \delta.$$

From Assumption 3.2.1 it then follows that

$$\alpha\mathcal{E}(\tilde{q}_\alpha, q^\dagger) + \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \delta + \alpha\varphi(\mathcal{S}(\tilde{q}_\alpha, u^\dagger)). \tag{3.2.6}$$

Note that φ is a concave index function, so there holds $\varphi(bt) \le b\varphi(t)$ for any $t \ge 0$ and $b \ge 1$. By virtue of Assumption 3.2.2 (note $C_S \ge 1$), we then have

$$\begin{aligned}\varphi(\mathcal{S}(\tilde{q}_\alpha, u^\dagger)) &\le \varphi(C_{\mathcal{S}}(\mathcal{S}(q^\dagger, \tilde{u}) + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})) \\ &\le C_{\mathcal{S}}\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})).\end{aligned}$$

Combining this with (3.2.6) yields

$$\alpha\mathcal{E}(\tilde{q}_\alpha, q^\dagger) + \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \delta + C_{\mathcal{S}}\alpha\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})). \tag{3.2.7}$$

Since $\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \ge 0$, by dropping this term on the left hand side of (3.2.7) we can obtain (3.2.3). Next, by dropping the nonnegative term $\alpha\mathcal{E}(\tilde{q}_\alpha, q^\dagger)$ on the left hand side of (3.2.7) we have

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \delta + C_{\mathcal{S}}\alpha\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})). \tag{3.2.8}$$

Notice that (3.2.8) is implicit in $\mathcal{S}(\tilde{q}_\alpha, \tilde{u})$, which we want to estimate in the following manner. If $C_{\mathcal{S}}\alpha\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})) \le \delta$, then

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le 2\delta. \tag{3.2.9}$$

If $C_{\mathcal{S}}\alpha\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})) > \delta$, we have from (3.2.8) that

$$\begin{aligned}\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) &\le 2\delta + C_{\mathcal{S}}\alpha\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u}))\\ &\le 3C_{\mathcal{S}}\alpha\varphi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})).\end{aligned}$$

Consequently $\Phi(\delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u})) \le 3C_{\mathcal{S}}\alpha$ and hence

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \delta + \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le \Phi^{-1}(3C_{\mathcal{S}}\alpha)$$

which together with (3.2.9) shows (3.2.4).

Now we are ready to derive the a posteriori error estimate (3.2.2). Since $\tilde{q}_{\alpha_*}$ is assumed to be in $\mathcal{M}_\rho$, we may use (3.2.3) to derive that

$$\begin{aligned}\mathcal{E}(\tilde{q}_{\alpha_*}, q^\dagger) &\le \frac{\delta}{\alpha_*} + C_{\mathcal{S}}\varphi(\delta + \mathcal{S}(\tilde{q}_{\alpha_*}, \tilde{u}))\\ &\le \frac{\delta}{\delta_*}\Theta(\alpha_*, \tilde{u}) + C_{\mathcal{S}}\varphi(\delta + \delta_*),\end{aligned} \tag{3.2.10}$$

where we used

$$\frac{\delta}{\alpha_*} = \frac{\delta}{\delta_*}\left(\frac{1}{\alpha_*}\right)\delta_* \le \frac{\delta}{\delta_*}\left(\frac{1}{\alpha_*} + A\right)\delta_* = \frac{\delta}{\delta_*}\Theta(\alpha_*, \tilde{u}).$$

We need to estimate $\Theta(\alpha_*, \tilde{u})$. If $\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \le 3\delta$ for all $\alpha \in \Delta_\gamma$, we have

$$\begin{aligned}\Theta(\alpha_*, \tilde{u}) \le \Theta(\alpha_0, \tilde{u}) &\le \left(\frac{1}{\alpha_0} + A\right)\mathcal{S}(\tilde{q}_{\alpha_0}, \tilde{u})\\ &\le 3\left(\frac{1}{\alpha_0} + A\right)\delta.\end{aligned} \tag{3.2.11}$$

Next we will assume that $\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) > 3\delta$ for some $\alpha \in \Delta_\gamma$. By (3.2.5) and the nonnegativity of $\mathcal{R}$ we have

$$\begin{aligned}\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) &\le \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) + \alpha\mathcal{R}(\tilde{q}_\alpha)\\ &\le \alpha\mathcal{R}(q^\dagger) + \mathcal{S}(q^\dagger, \tilde{u})\\ &= \alpha\mathcal{R}(q^\dagger) + \delta.\end{aligned}$$

This implies that $\lim_{\alpha\searrow 0} \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \leq \delta$. Therefore there must exist a largest number $\hat{\alpha}$ in Δ_γ such that

$$\mathcal{S}(\tilde{q}_{\gamma\hat{\alpha}}, \tilde{u}) \leq 3\delta < \mathcal{S}(\tilde{q}_{\hat{\alpha}}, \tilde{u}).$$

We need a lower bound for $\hat{\alpha}$. By (3.2.5) with $\tilde{q}_{\hat{\alpha}}$, we have

$$\begin{aligned} 3\delta + \hat{\alpha}\mathcal{R}(\tilde{q}_{\hat{\alpha}}) &\leq \mathcal{S}(\tilde{q}_{\hat{\alpha}}, \tilde{u}) + \hat{\alpha}\mathcal{R}(\tilde{q}_{\hat{\alpha}}) \\ &\leq \mathcal{S}(q^\dagger, \tilde{u}) + \hat{\alpha}\mathcal{R}(q^\dagger) \\ &= \delta + \hat{\alpha}\mathcal{R}(q^\dagger). \end{aligned}$$

This implies that $\mathcal{R}(\tilde{q}_{\hat{\alpha}}) \leq \mathcal{R}(q^\dagger) < \rho$ and thus $\tilde{q}_{\hat{\alpha}} \in \mathcal{M}_\rho$. Consequently we may use (3.2.4) to obtain

$$3\delta < \mathcal{S}(\tilde{q}_{\hat{\alpha}}, \tilde{u}) \leq 2\delta + \Phi^{-1}(3C_{\mathcal{S}}\hat{\alpha}).$$

Thus $\delta \leq \Phi^{-1}(3C_{\mathcal{S}}\hat{\alpha})$. This together with the monotonicity of Φ gives

$$\hat{\alpha} \geq \frac{1}{3C_{\mathcal{S}}}\Phi(\delta).$$

We may now use this lower bound of $\hat{\alpha}$ and the minimality of $\Theta(\alpha_*, \tilde{u})$ to derive that

$$\begin{aligned} \Theta(\alpha_*, \tilde{u}) &\leq \Theta(\gamma\hat{\alpha}, \tilde{u}) = \left(\frac{1}{\gamma\hat{\alpha}} + A\right)\mathcal{S}(\tilde{q}_{\gamma\hat{\alpha}}, \tilde{u}) \\ &\leq 3\left(\frac{1}{\gamma\hat{\alpha}} + A\right)\delta \leq 3A\delta + \frac{9C_{\mathcal{S}}}{\gamma}\frac{\delta}{\Phi(\delta)} \\ &= 3A\delta + \frac{9C_{\mathcal{S}}}{\gamma}\varphi(\delta). \end{aligned} \tag{3.2.12}$$

Combining (3.2.11) and (3.2.12)

$$\begin{aligned} \Theta(\alpha_*, \tilde{u}) &\leq \left(3A + 3\left(\frac{1}{\alpha_0} + A\right)\right)\delta + \frac{9C_{\mathcal{S}}}{\gamma}\varphi(\delta) \\ &\leq C\left(\delta + \varphi(\delta + \delta_*)\right) \end{aligned}$$

where C is a constant in terms only of A, α_0, γ, and C_S. In the previous inequality we used the fact that φ is strictly increasing. Combining that and (3.2.10) we obtain the desired estimate (3.2.2). □

The a posteriori estimate in Theorem 3.2.1 involves the quantity δ_*. If δ_* is about the order of δ, it gives convergence rate $O(\varphi(\delta))$. If δ_* is much larger than δ, only weaker convergence rates are available. If δ_* is significantly smaller than δ, the factor δ/δ_* blows up and the approximation may diverge. Therefore, the quantity δ_* provides an a posteriori check of Rule 3.1, its value should always be monitored and the computed approximation should be discarded if δ_* is presumably too small.

In Theorem 3.2.1 we have assumed that Rule 3.1 defines a positive number α_* with $\tilde{q}_{\alpha_*} \in \mathcal{M}_\rho$. This requirement can be guaranteed if the noisy data $\tilde{u}$ satisfies the following condition.

Assumption 3.2.3. *There is a constant $\kappa > 0$ such that*

$$\mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \geq \kappa \mathcal{S}(q^\dagger, \tilde{u}) \tag{3.2.13}$$

for any minimizer $\tilde{q}_\alpha$ of (3.1.3) *with any $\alpha \in \Delta_\gamma$.*

This assumption is motivated from the work [24] for linear ill-posed inverse problems $Fq = u^\dagger$ in Hilbert spaces. The analysis of Hanke and Raus on their heuristic rule in [24] for Tikhonov regularization

$$\min_{q \in \mathcal{Q}} \left\{ \|Fq - \tilde{u}\|_{\mathcal{U}}^2 + \alpha \|q\|_{\mathcal{Q}}^2 \right\} \tag{3.2.14}$$

is based on the condition

$$\|\mathcal{P}(\tilde{u} - u^\dagger)\|_{\mathcal{U}} \geq \kappa \|\tilde{u} - u^\dagger\|_{\mathcal{U}}$$

on the noisy data $\tilde{u}$, where $0 < \kappa < 1$ and $\mathcal{P}$ denotes the orthogonal projection onto the orthogonal complement of the range of F. It turns out that this condition implies that

$$\|Fq - \tilde{u}\|_{\mathcal{U}} \geq \kappa \|u^\dagger - \tilde{u}\|_{\mathcal{U}}$$

for all $q \in \mathcal{Q}$. Note that $\|Fq - \tilde{u}\|_{\mathcal{U}}$ and $\|u^\dagger - \tilde{u}\|_{\mathcal{U}}$ correspond to the discrepancy and the noise level in Tikhonov regularization (3.2.14). By comparing with the variational regularization (3.1.3), this leads us to make the above assumption. Intuitively, Assumption 3.2.3 means that there does not exist overfitting for the noisy data. It would be interesting to explore a statistical interpretation on Assumption 3.2.3 which however remains open.

Lemma 3.2.1. *Let $\mathcal{R}$ and $\mathcal{S}$ satisfy Assumption 3.1.1 and Assumption 3.2.2. Let the noisy data $\tilde{u}$ satisfy Assumption 3.2.3. If*

$$\mathcal{S}(q^\dagger, \tilde{u}) \leq \alpha_0 \mathcal{R}(q^\dagger) \quad \textit{and} \quad \left(1 + \frac{2(1 + A\alpha_0)}{\kappa}\right) \mathcal{R}(q^\dagger) \leq \rho,$$

then Rule 3.1 determines a positive parameter $\alpha_* \in \Delta_\gamma$ with $\tilde{q}_{\alpha_*} \in \mathcal{M}_\rho$.

Proof. Let $\delta := \mathcal{S}(q^\dagger, \tilde{u})$. From Assumption 3.2.3 it follows that

$$\Theta(\alpha, \tilde{u}) = \left(\frac{1}{\alpha} + A\right) \mathcal{S}(\tilde{q}_\alpha, \tilde{u}) \geq \left(\frac{1}{\alpha} + A\right) \kappa \mathcal{S}(q^\dagger, \tilde{u}) = \left(\frac{1}{\alpha} + A\right) \kappa\delta$$

for all $\alpha > 0$. This implies that $\Theta(\alpha, \tilde{u}) \to \infty$ as $\alpha \to 0$. Therefore there must exist an $\alpha_* \in \Delta_\gamma$ that gives the minimum of $\Theta(\alpha, \tilde{u})$ over Δ_γ. By using Assumption 3.2.3 and the minimizing property of $\tilde{q}_{\alpha_0}$ we have

$$\begin{aligned}
\left(\frac{1}{\alpha_*} + A\right) \kappa\delta \leq \Theta(\alpha_*, \tilde{u}) \leq \Theta(\alpha_0, \tilde{u}) &= \left(\frac{1}{\alpha_0} + A\right) \mathcal{S}(\tilde{q}_{\alpha_0}, \tilde{u}) \\
&\leq \left(\frac{1}{\alpha_0} + A\right) \left(\delta + \alpha_0 \mathcal{R}(q^\dagger)\right) \\
&= (1 + A\alpha_0) \left(\frac{\delta}{\alpha_0} + \mathcal{R}(q^\dagger)\right).
\end{aligned}$$

In view of the condition $\delta = \mathcal{S}(q^\dagger, \tilde{u}) \leq \alpha_0 \mathcal{R}(q^\dagger)$ we can obtain

$$\frac{\kappa\delta}{\alpha_*} \leq \left(\frac{1}{\alpha_*} + A\right) \kappa\delta \leq 2(1 + A\alpha_0)\mathcal{R}(q^\dagger)$$

which shows that

$$\alpha_* \geq \frac{\kappa\delta}{2(1 + A\alpha_0)\mathcal{R}(q^\dagger)}.$$

Consequently, it follows from the minimizing property of $\tilde{q}_{\alpha_*}$ and the other condition that

$$\mathcal{R}(\tilde{q}_{\alpha_*}) \leq \frac{\delta}{\alpha_*} + \mathcal{R}(q^\dagger) \leq \left(\frac{2(1 + A\alpha_0)}{\kappa} + 1\right) \mathcal{R}(q^\dagger) \leq \rho.$$

Thus $\tilde{q}_{\alpha_*} \in \mathcal{M}_\rho$ and the proof is complete. □

We remark that the *a posteriori* error estimate in Theorem 3.2.1 is based on the variational source condition on $q^\dagger$ specified in Assumption 3.2.1. Variational source conditions were first introduced in [32], as a generalization of the standard source conditions in Hilbert spaces, to derive convergence rates of Tikhonov regularization in Banach spaces. This kind of source conditions was further generalized and refined later, see [6, 22, 33, 77] for more information. Recently it has been shown in [18] that, for inverse problems of the form (3.0.1) in Banach spaces, under certain natural conditions, a variational source condition is always satisfied with a suitable concave index function φ. The error function $\mathcal{E}$ in Assumption 3.2.1 is used to measure the speed of convergence. In the original version of variational source conditions, Bregman distance induced by $\mathcal{R}$ is used as an error function. Use of a general error function has the advantage of covering a wider range of applications, see [18, 20]. In the following result we extend the work in [18] for inverse problems of the form (3.1.1).

Proposition 3.1. *Let Assumption 3.1.1 hold and let $\rho > \mathcal{R}(q^\dagger)$. Furthermore, assume that there is an error function $\mathcal{E} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ with $\mathcal{E}(q, q) = 0$ whenever $q \in \mathcal{Q}$ such that the following conditions hold:*

(i) $\mathcal{E}(q^*, q^\dagger) \leq \mathcal{R}(q^*) - \mathcal{R}(q^\dagger)$ for all $q^* \in \mathcal{M}_\rho$ with $\mathcal{S}(q^*, u^\dagger) = 0$;

(ii) The mapping $q \to -\mathcal{E}(q, q^\dagger) + \mathcal{R}(q)$ is $\tau_\mathcal{Q}$ lower semi-continuous on $\mathcal{M}_\rho$;

(iii) There exist $\beta > 1$ and $c \in \mathbb{R}$ such that $\beta\mathcal{E}(q, q^\dagger) - \mathcal{R}(q) \leq c$ for all $q \in \mathcal{M}_\rho$.

Then there exists a concave strictly increasing index function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\mathcal{E}(q, q^\dagger) \leq \mathcal{R}(q) - \mathcal{R}(q^\dagger) + \varphi(\mathcal{S}(q, u^\dagger)), \quad \forall q \in \mathcal{M}_\rho.$$

Proof. The proof essentially follows [18] with refinements. For $r \geq 0$ we define

$$D(r) := \sup_{q \in \mathcal{M}_\rho} \left\{ \mathcal{E}(q, q^\dagger) - \mathcal{R}(q) + \mathcal{R}(q^\dagger) - r\mathcal{S}(q, u^\dagger) \right\}.$$

It is clear that $D(r)$ is monotonically decreasing with respect to $r \in [0, \infty)$. This is clear because for $0 \leq r_1 < r_2 < \infty$,

$$D(r_2) - D(r_1) = \sup_{q \in \mathcal{M}_\rho} \left\{ (r_1 - r_2)\mathcal{S}(q, u^\dagger) \right\} < 0.$$

Furthermore, by (iii) we have

$$D(r) \leq D(0) < \sup_{q \in \mathcal{M}_\rho} \left\{ \beta\mathcal{E}(q, q^\dagger) - \mathcal{R}(q) + \mathcal{R}(q^\dagger) \right\} \leq c + \mathcal{R}(q^\dagger) < \infty$$

for all $r \in [0, \infty)$. We claim that

$$\lim_{r \to \infty} D(r) \leq 0.$$

If there is an $\hat{r} \geq 0$ such that $D(\hat{r}) \leq 0$, then this claim holds obviously by the monotonicity of $D(r)$. So we may assume that $D(r) > 0$ for all $r \geq 0$. Let $\varepsilon > 0$ be an arbitrarily small number. For each $r \geq 1$ we may choose $q_r \in \mathcal{M}_\rho$ such that

$$\mathcal{E}(q_r, q^\dagger) - \mathcal{R}(q_r) + \mathcal{R}(q^\dagger) - r\mathcal{S}(q_r, u^\dagger) \geq D(r) - \varepsilon. \tag{3.2.15}$$

According to (iii) we have

$$\begin{aligned}\mathcal{E}(q_r, q^\dagger) - \mathcal{R}(q_r) &= \frac{1}{\beta}\left(\beta\mathcal{E}(q_r, q^\dagger) - \mathcal{R}(q_r)\right) - \left(1 - \frac{1}{\beta}\right)\mathcal{R}(q_r)\\ &\leq \frac{c}{\beta} - \left(1 - \frac{1}{\beta}\right)\mathcal{R}(q_r).\end{aligned}$$

From the definition of q_r,

$$D(r) - \varepsilon + r\mathcal{S}(q_r, u^\dagger) - \mathcal{R}(q^\dagger) \leq \mathcal{E}(q_r, q^\dagger) - \mathcal{R}(q_r),$$

so by virtue of the previous two inequalities, we get

$$\begin{aligned}\left(1 - \frac{1}{\beta}\right)\mathcal{R}(q_r) + r\mathcal{S}(q_r, u^\dagger) &\leq \frac{c}{\beta} + \mathcal{R}(q^\dagger) + \varepsilon - \underbrace{D(r)}_{>0}\\ &\leq \frac{c}{\beta} + \mathcal{R}(q^\dagger) + \varepsilon \end{aligned} \tag{3.2.16}$$

This implies that $\{\mathcal{R}(q_r)\}$ is bounded. Thus, by Assumption 3.1.1 (ii), there exist $\hat{q} \in \mathcal{Q}$ and a sequence $\{r_n\}$ with $r_n \to \infty$ such that $q_{r_n} \xrightarrow{\tau_{\mathcal{Q}}} \hat{q}$ as $n \to \infty$. According to Assumption 3.1.1 (i) we have

$$\mathcal{R}(\hat{q}) \leq \liminf_{n\to\infty} \mathcal{R}(q_{r_n}) \leq \rho$$

where ρ is a positive constant taken from (3.2.16). Then $\hat{q} \in \mathcal{M}_\rho$. From (3.2.16) with $q_r = q_{r_n}$ we have $\mathcal{S}(q_{r_n}, u^\dagger) \to 0$ as $n \to \infty$. In view of Assumption 3.1.1 (iii),

$$\mathcal{S}(\hat{q}, u^\dagger) \leq \liminf_{r_n\to\infty} \mathcal{S}(q_{r_n}, u^\dagger) \to 0.$$

Therefore $\mathcal{S}(\hat{q}, u^\dagger) = 0$. Thus it follows from (3.2.15), the nonnegativity of $\mathcal{S}$, (ii) and (i) that

$$\begin{aligned}\lim_{r\to\infty} D(r) = \lim_{n\to\infty} D(r_n) &= \limsup_{n\to\infty} D(r_n)\\ &\leq \limsup_{n\to\infty}\left\{\mathcal{E}(q_{r_n}, q^\dagger) - \mathcal{R}(q_{r_n}) + \mathcal{R}(q^\dagger) - r_n\mathcal{S}(q_{r_n}, u^\dagger) + \varepsilon\right\}\\ &\leq \limsup_{n\to\infty}\left\{\mathcal{E}(q_{r_n}, q^\dagger) - \mathcal{R}(q_{r_n})\right\} + \mathcal{R}(q^\dagger) + \varepsilon\\ &\leq \mathcal{E}(\hat{q}, q^\dagger) - \mathcal{R}(\hat{q}) + \mathcal{R}(q^\dagger) + \varepsilon\\ &\leq \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we must have $\lim_{r\to\infty} D(r) = 0$.

It is now ready to show the validity of the variational source condition. If $D(r) \leq 0$ for some $r \geq 0$, then using the definition of $D(r)$, there holds for all $q \in \mathcal{M}_\rho$ that

$$E(q, q^\dagger) \leq \mathcal{R}(q) - \mathcal{R}(q^\dagger) + r\mathcal{S}(q, u^\dagger),$$

and by taking $\varphi(t) = rt$, indeed the variational source condition is valid. Hence we may assume $D(r) > 0$ for all $r \geq 0$. By the definition of $D(r)$ we have

$$\begin{aligned}\mathcal{E}(q, q^\dagger) - \mathcal{R}(q) + \mathcal{R}(q^\dagger) &= \mathcal{E}(q, q^\dagger) - R(q) + \mathcal{R}(q^\dagger) - r\mathcal{S}(q, u^\dagger) + r\mathcal{S}(q, u^\dagger)\\ &\leq D(r) + r\mathcal{S}(q, u^\dagger)\end{aligned}$$

for all $r \geq 0$ and $q \in \mathcal{M}_\rho$. Let

$$\varphi(t) := \inf_{r\geq 0} \{D(r) + rt\}.$$

Then taking the infimum on both sides, we arrive at

$$\begin{aligned}\mathcal{E}(q, q^\dagger) - \mathcal{R}(q) + \mathcal{R}(q^\dagger) &\leq \inf_{r\geq 0} \left\{D(r) + r\mathcal{S}(q, u^\dagger)\right\}\\ &= \varphi(\mathcal{S}(q, u^\dagger)).\end{aligned}$$

By definition φ is a nonnegative, concave, upper semi-continuous, increasing function on $[0, \infty)$.Note φ takes the infimum of affine functions which are concave. Hence φ is concave and upper semicontinous. Moreover, because $D(r) \to 0$ as $r \to \infty$, we have $\varphi(0) = 0$. Moreover, it follows from (iii) that

$$\varphi(t) \leq D(0) \leq \sup_{q\in\mathcal{M}_\rho} \left\{\beta\mathcal{E}(q, q^\dagger) - \mathcal{R}(q) + \mathcal{R}(q^\dagger)\right\} \leq c + \mathcal{R}(q^\dagger) < \infty$$

for all $t \geq 0$. Thus φ is continuous on $[0, \infty)$. Moreover, by replacing $\varphi(t)$ by $\varphi(t) + \sqrt{t}$ if necessary, we may guarantee the validity of the variational source condition with a concave strictly increasing index function. □

3.2.2 Convergence

For our heuristic Rule 3.1, it is natural to ask, for a family of noisy data $\{u^\delta\}$ satisfying $u^\delta \xrightarrow{\tau_\mathcal{U}} u^\dagger$ as $\delta \to 0$, if q_α^δ is defined as

$$q_\alpha^\delta \in \arg\min_{q\in\mathcal{Q}} \left\{\mathcal{S}(q, u^\delta) + \alpha\mathcal{R}(q)\right\}, \tag{3.2.17}$$

and $\alpha_* := \alpha_*(u^\delta)$ is chosen by Rule 3.1 with $\Theta(\alpha, \tilde{u})$ replaced by $\Theta(\alpha, u^\delta)$, i.e.

$$\alpha_* \in \arg\min_{\alpha\in\Delta_\gamma} \left\{ \Theta(\alpha, u^\delta) := \left(\frac{1}{\alpha} + A\right) \mathcal{S}(q_\alpha^\delta, u^\delta) \right\}, \tag{3.2.18}$$

is it possible to guarantee the convergence of $q_{\alpha_*}^\delta$ to $q^\dagger$ as $\delta \to 0$? In Theorem 3.2.1, we have derived *a posteriori* error estimates under variational source conditions on the sought solution $q^\dagger$ for the regularization parameter α_* chosen by Rule 3.1. This result may not be able to answer the convergence issue in a general situation. On one hand, although Proposition 3.1 provides a result concerning the validity of variational source conditions, the three conditions stipulated in Proposition 3.1 need to be verified for its use. For some examples of convex $\mathcal{R}$ these three conditions have been verified in [10]; however, there exist examples of $\mathcal{R}$ which may not satisfy these three conditions. On the other hand, the *a posteriori* error estimates given in Theorem 3.2.1 rely on the quasi-triangle inequality specified in Assumption 3.2.2. For the electrical impedance tomography given in Section 3.4, it is not clear if the quasi-triangle inequality holds for the Kohn-Vogelius functional. Therefore, Theorem 3.2.1 may not be applicable for this example. Furthermore, in order to derive a convergence result from the *a posteriori* error estimate, one still needs to show that $\delta_* \to 0$ as $\delta \to 0$ which requires some additional work.

In this subsection we will provide a general convergence result without using any source conditions nor Assumption 3.2.2. Therefore, the result in this subsection can be used in a most general setting and thus is significant.

Bakushinskii's veto ([3]) states that heuristic rules can not lead to convergence in the sense of worst case scenario for any regularisation method. In order to obtain a convergence result, certain conditions on the noisy data should be imposed. We will use the following condition which means that $\{u^\delta\}$ satisfies Assumption 3.2.3 in a uniform sense.

Assumption 3.2.4. $\{u^\delta\}$ is a family of noisy data with $u^\delta \xrightarrow{\mathcal{T}_\mathcal{U}} u^\dagger$ as $\delta \to 0$, and there is a constant $0 < \kappa < 1$ such that

$$\mathcal{S}(q_\alpha^\delta, u^\delta) \geq \kappa \mathcal{S}(q^\dagger, u^\delta)$$

for every u^δ and every solution q_α^δ of (3.2.17) with all $\alpha \in \Delta_\gamma$.

Now we are ready to prove the following convergence result on Rule 3.1.

Theorem 3.2.2. *Let $\mathcal{R}$ and $\mathcal{S}$ satisfy Assumption 3.1.1, let $\{u^\delta\}$ be a family of noisy data satisfying Assumption 3.2.4, and let $\alpha_* := \alpha_*(u^\delta) \in \Delta_\gamma$ be chosen by (3.2.18). Then*

$$\mathcal{S}(q_{\alpha_*}^\delta, u^\delta) \to 0 \quad \textit{and} \quad \mathcal{R}(q_{\alpha_*}^\delta) \to \mathcal{R}^\dagger \quad \textit{as } \delta \to 0,$$

where

$$\mathcal{R}^\dagger := \min\{\mathcal{R}(q) : q \in \mathcal{Q} \text{ and } \mathcal{S}(q, u^\dagger) = 0\}.$$

Moreover, any sequence from $\{q^\delta_{\alpha_*}\}$ *contains a* $\tau_{\mathcal{Q}}$*-convergent subsequence whose limit is an* $\mathcal{R}$*-minimizing solution of* (3.1.1). *If, in addition,* (3.1.1) *has a unique* $\mathcal{R}$*-minimizing solution* $q^\dagger$*, then* $q^\delta_{\alpha_*} \xrightarrow{\tau_{\mathcal{Q}}} q^\dagger$ *as* $\delta \to 0$.

Proof. We first show that

$$\Theta(\alpha_*(u^\delta), u^\delta) \to 0 \quad \text{as } \delta \to 0. \tag{3.2.19}$$

To this end, let $q^\dagger \in \mathcal{Q}$ be an $\mathcal{R}$-minimizing solution of (3.1.1). We take a constant $\tau > 1$ and define $\hat{\alpha} := \hat{\alpha}(u^\delta) \in \Delta_\gamma$ to be the largest number such that

$$\hat{\alpha}^2 + \mathcal{S}(q^\delta_{\hat{\alpha}}, u^\delta) \le \tau \mathcal{S}(q^\dagger, u^\delta). \tag{3.2.20}$$

This $\hat{\alpha}$ is well-defined. In fact, for any $\alpha > 0$ we have from the minimality of q^δ_α that

$$\mathcal{S}(q^\delta_\alpha, u^\delta) + \alpha \mathcal{R}(q^\delta_\alpha) \le \mathcal{S}(q^\dagger, u^\delta) + \alpha \mathcal{R}(q^\dagger)$$

which implies that

$$\limsup_{\alpha \to 0} \mathcal{S}(q^\delta_\alpha, u^\delta) \le \mathcal{S}(q^\dagger, u^\delta) < \tau \mathcal{S}(q^\dagger, u^\delta). \tag{3.2.21}$$

Therefore, for $\tau > 1$, there must exist a finite number $\hat{\alpha} \in \Delta_q$ satisfying (3.2.20). Since $u^\delta \xrightarrow{\tau_{\mathcal{U}}} u^\dagger$, by Assumption 3.1.1 (iv) we have $\mathcal{S}(q^\dagger, u^\delta) \to \mathcal{S}(q^\dagger, u^\dagger) = 0$ as $\delta \to 0$. Thus, by the definition of $\hat{\alpha}(u^\delta)$ we can see that $\hat{\alpha} := \hat{\alpha}(u^\delta) \to 0$ as $\delta \to 0$. We claim that

$$\frac{\mathcal{S}(q^\dagger, u^\delta)}{\hat{\alpha}} \to 0 \quad \text{as } \delta \to 0. \tag{3.2.22}$$

To see this, we use the minimizing property of $q^\delta_{\hat{\alpha}/\gamma}$ to derive that

$$\mathcal{S}(q^\delta_{\hat{\alpha}/\gamma}, u^\delta) + \frac{\hat{\alpha}}{\gamma} \mathcal{R}(q^\delta_{\hat{\alpha}/\gamma}) \le \mathcal{S}(q^\dagger, u^\delta) + \frac{\hat{\alpha}}{\gamma} \mathcal{R}(q^\dagger). \tag{3.2.23}$$

Since $\hat{\alpha} < \hat{\alpha}/\gamma$, we have

$$\left(\frac{\hat{\alpha}}{\gamma}\right)^2 + \mathcal{S}(q^\delta_{\hat{\alpha}/\gamma}, u^\delta) > \tau \mathcal{S}(q^\dagger, u^\delta)$$

which together with (3.2.23) implies that

$$\tau\mathcal{S}(q^\dagger, u^\delta) + \frac{\hat\alpha}{\gamma}\mathcal{R}(q^\delta_{\hat\alpha/\gamma}) \le \mathcal{S}(q^\delta_{\hat\alpha/\gamma}, u^\delta) + \frac{\hat\alpha}{\gamma}\mathcal{R}(q^\delta_{\hat\alpha/\gamma}) + \left(\frac{\hat\alpha}{\gamma}\right)^2$$
$$\le \mathcal{S}(q^\dagger, u^\delta) + \frac{\hat\alpha}{\gamma}\mathcal{R}(q^\dagger) + \left(\frac{\hat\alpha}{\gamma}\right)^2. \tag{3.2.24}$$

Since $\tau > 1$, we obtain

$$\mathcal{R}(q^\delta_{\hat\alpha/\gamma}) \le \frac{\hat\alpha}{\gamma} + \mathcal{R}(q^\dagger).$$

In view of $\hat\alpha \to 0$ as $\delta \to 0$, we can derive that

$$\limsup_{\delta\to 0} \mathcal{R}(q^\delta_{\hat\alpha/\gamma}) \le \mathcal{R}(q^\dagger). \tag{3.2.25}$$

This shows that $\{q^\delta_{\hat\alpha/\gamma}\} \subseteq \mathcal{M}_\rho$ with $\rho = \mathcal{R}(q^\dagger) + 1$ for all small $\delta > 0$. By Assumption 3.1.1 (ii), $\{q^\delta_{\hat\alpha/\gamma}\}$ has a $\tau_\mathcal{Q}$-convergent subsequence, and, for simplicity of exposition, we denote this subsequence again by $\{q^\delta_{\hat\alpha/\gamma}\}$. Let $\hat q \in \mathcal{Q}$ be the $\tau_\mathcal{Q}$-limit of this subsequence. By (3.2.25) and the $\tau_\mathcal{Q}$-lower semi-continuity of $\mathcal{R}$, we have

$$\mathcal{R}(\hat q) \le \liminf_{\delta\to 0} \mathcal{R}(q^\delta_{\hat\alpha/\gamma}) \le \limsup_{\delta\to 0} \mathcal{R}(q^\delta_{\hat\alpha/\gamma}) \le \mathcal{R}(q^\dagger). \tag{3.2.26}$$

Since $u^\delta \xrightarrow{\tau_\mathcal{U}} u^\dagger$ and $q^\delta_{\hat\alpha/\gamma} \xrightarrow{\tau_\mathcal{Q}} \hat q$ as $\delta \to 0$, we may use Assumption 3.1.1 (iii) and (3.2.23) to obtain

$$\mathcal{S}(\hat q, u^\dagger) \le \liminf_{\delta\to 0} \mathcal{S}(q^\delta_{\hat\alpha/\gamma}, u^\delta) \le \liminf_{\delta\to 0} \left\{\mathcal{S}(q^\dagger, u^\delta) + \frac{\hat\alpha}{\gamma}\mathcal{R}(q^\dagger)\right\} = 0.$$

Therefore, it follows from the nonnegativity of $\mathcal{S}$ that $\mathcal{S}(\hat q, u^\dagger) = 0$. Consequently, $\hat q$ is a solution of $S(q, u^\dagger) = 0$ with $\mathcal{R}(\hat q) \le \mathcal{R}(q^\dagger)$. By the $\mathcal{R}$-minimality of $q^\dagger$, we must have $\mathcal{R}(\hat q) = \mathcal{R}(q^\dagger)$ which together with (3.2.26) shows that

$$\lim_{\delta\to 0} \mathcal{R}(q^\delta_{\hat\alpha/\gamma}) = \mathcal{R}(\hat q) = \mathcal{R}(q^\dagger). \tag{3.2.27}$$

In view of (3.2.24), we have

$$(\tau - 1)\frac{\mathcal{S}(q^\dagger, u^\delta)}{\hat\alpha} \le \frac{\hat\alpha}{\gamma^2} + \frac{1}{\gamma}\left(\mathcal{R}(q^\dagger) - \mathcal{R}(q^\delta_{\hat\alpha/\gamma})\right).$$

Since $\tau > 1$, we may use $\hat\alpha \to 0$ and (3.2.27) to conclude that $\mathcal{S}(q^\dagger, u^\delta)/\hat\alpha \to 0$ as $\delta \to 0$ which shows (3.2.22).

Now we are ready to show (3.2.19). In fact, by the minimality of $\alpha_*(u^\delta)$, the choice of $\hat{\alpha}$, and (3.2.22) we have

$$
\begin{aligned}
0 \le \Theta(\alpha_*, u^\delta) \le \Theta(\hat{\alpha}, u^\delta) &= \left(\frac{1}{\hat{\alpha}} + A\right) \mathcal{S}(q^\delta_{\hat{\alpha}}, u^\delta) \\
&\le \tau \left(\frac{1}{\hat{\alpha}} + A\right) \mathcal{S}(q^\dagger, u^\delta) \to 0
\end{aligned}
$$

as $\delta \to 0$ which shows (3.2.19). Note that

$$
\Theta(\alpha_*, u^\delta) = \left(\frac{1}{\alpha_*} + A\right) \mathcal{S}(q^\delta_{\alpha_*}, u^\delta) \ge \frac{\mathcal{S}(q^\delta_{\alpha_*}, u^\delta)}{\alpha_*},
$$

we may use (3.2.19), $\alpha_* \le \alpha_0$ and $\mathcal{S}(q^\delta_{\alpha_*}, u^\delta) \ge \kappa \mathcal{S}(q^\dagger, u^\dagger)$ from Assumption 3.2.4 to derive that

$$
\mathcal{S}(q^\delta_{\alpha_*}, u^\delta) \to 0 \quad \text{and} \quad \frac{\mathcal{S}(q^\dagger, u^\delta)}{\alpha_*} \to 0 \quad \text{as } \delta \to 0. \tag{3.2.28}
$$

Next we show the convergence result. By using the minimality of $q^\delta_{\alpha_*}$, the nonnegativity of $\mathcal{S}$ and (3.2.28) we have

$$
\limsup_{\delta \to 0} \mathcal{R}(q^\delta_{\alpha_*}) \le \limsup_{\delta \to 0} \left\{ \frac{\mathcal{S}(q^\dagger, u^\delta)}{\alpha_*} + \mathcal{R}(q^\dagger) \right\} = \mathcal{R}(q^\dagger).
$$

Thus $q^\delta_{\alpha_*} \in \mathcal{M}_\rho$ with $\rho = \mathcal{R}(q^\dagger) + 1$ for all small $\delta > 0$. Therefore, we can use a similar argument for dealing with the convergence of $\{q^\delta_{\hat{\alpha}/\gamma}\}$ to conclude that, by choosing a subsequence if necessary, there exists an $\mathcal{R}$-minimizing solution q_* of (3.1.1) such that $q^\delta_{\alpha_*} \xrightarrow{\tau_Q} q_*$ and $\mathcal{R}(q^\delta_{\alpha_*}) \to \mathcal{R}(q_*) = \mathcal{R}(q^\dagger)$ as $\delta \to 0$.

If $q^\dagger$ is the unique $\mathcal{R}$-minimizing solution (3.1.1), then the above argument shows that every sequence from $\{q^\delta_{\alpha_*}\}$ has a subsequence converging to $q^\dagger$. Using a subsequence-subsequence argument, the whole sequence $\{q^\delta_{\alpha_*}\}_{\delta>0}$ must converge to $q^\dagger$. The proof is complete. $\square$

3.3 Application: a convex framework for elliptic parameter estimation

We will provide a unified framework for treating parameter estimation in a class of elliptic problems which have been considered in [26, 27, 25, 31, 70, 84] individually. To this end, we consider the elliptic problems whose weak formulation takes the form

$$a(u, v; q) + b(u, v) = \ell(v), \quad \forall v \in V, \tag{3.3.1}$$

where V is a Hilbert space, $\ell \in V^*$, i.e. ℓ is a bounded linear functional on V, q is a parameter belonging to a Banach space Z, and $a(\cdot, \cdot; \cdot)$ and $b(\cdot, \cdot)$ satisfy the following conditions:

(A1) $a(\cdot, \cdot; \cdot)$ is trilinear over $V \times V \times Z$, and $b(\cdot, \cdot)$ is bilinear over $V \times V$. Both a and b are symmetric with respect to the first two arguments.

(A2) There exists a constant C_0 such that

$$|a(u, v; q)| \leq C_0 \|u\|_V \|v\|_V \|q\|_Z \quad \text{and} \quad |b(u, v)| \leq C_0 \|u\|_V \|v\|_V$$

for all $u, v \in V$ and $q \in Z$.

(A3) There is a bounded convex set $\mathcal{A} \subset Z$ and a constant $c_0 > 0$ such that

$$a(u, u; q) + b(u, u) \geq c_0 \|u\|_V^2$$

for all $u \in V$ and $q \in \mathcal{A}$.

(A4) There is a reflexive Banach space B such that $Z \hookrightarrow B$ and $\mathcal{A}$ is closed in B; moreover for any sequence $\{q_n\} \subset \mathcal{A}$ and $q \in \mathcal{A}$ satisfying $\|q_n - q\|_B \to 0$ there holds

$$\sup_{\|u\|_V \leq \beta} |a(u, v; q_n - q)| \to 0 \qquad \text{as } n \to \infty$$

for each $v \in V$ and each constant $\beta > 0$.

We are interested in estimating the parameter $q \in \mathcal{A}$ from a measurement $\tilde{u}$ of u in V. We will use the minimizer

$$\tilde{q}_\alpha = \arg\min_{q \in \mathcal{A}} \{a(u(q) - \tilde{u}, u(q) - \tilde{u}; q) + b(u(q) - \tilde{u}, u(q) - \tilde{u}) + \alpha \mathcal{R}(q)\} \tag{3.3.2}$$

to approximate the sought parameter $q^\dagger \in \mathcal{A}$, where $\mathcal{R}(q)$ is a suitably chosen regularization functional, and $u(q) \in V$ denotes the unique solution of (3.3.1) for $q \in \mathcal{A}$, whose existence, according to the conditions (A1)–(A3), is guaranteed by the Lax-Milgram theorem; moreover

$$\|u(q)\|_V \leq \frac{1}{c_0} \|\ell\|_{V^*}. \tag{3.3.3}$$

In order to apply the theory developed in Section 2, we take $\mathcal{U} = V$, $\mathcal{Q} = \mathcal{A}$, $\tau_{\mathcal{U}} =$ strong topology on V, $\tau_{\mathcal{Q}} =$ weak topology on B, and define $\mathcal{S} : \mathcal{Q} \to \mathcal{U}$ by

$$\mathcal{S}(q,u) := a(u(q)-u, u(q)-u; q) + b(u(q)-u, u(q)-u),$$

then the identification of $q^\dagger$ becomes the equation (3.1.1) and (3.3.2) becomes the form (3.1.3). Our heuristic rule then choose the regularization parameter $\alpha_* \in \Delta_\gamma$ by Rule 3.1 with

$$\Theta(\alpha,\tilde{u}) = \left(\frac{1}{\alpha} + A\right)\left[a(u(\tilde{q}_\alpha)-\tilde{u}, u(\tilde{q}_\alpha)-\tilde{u}; \tilde{q}_\alpha) + b(u(\tilde{q}_\alpha)-\tilde{u}, u(\tilde{q}_\alpha)-\tilde{u})\right]. \tag{3.3.4}$$

We need to check that $\mathcal{S}$ satisfies Assumption 3.1.1 (iv) and (v) and Assumption 3.2.2. The verification of Assumption 3.2.2 can be proceeded as follows. By using (A1), (A2) and the boundedness of $\mathcal{A}$ in Z there is a constant $C_1 \geq 1$ such that

$$\frac{1}{C_1}\|u(q)-u\|_V^2 \leq \mathcal{S}(q,u) \leq C_1\|u(q)-u\|_V^2$$

for all $q \in \mathcal{A}$ and $u \in V$. Therefore, for any $q \in \mathcal{A}$ and $u \in V$ we have

$$\begin{aligned} S(q,u^\dagger) &\leq C_1\|u(q)-u^\dagger\|_V^2 = C_1\|u(q)-u(q^\dagger)\|_V^2 \\ &\leq 2C_1\left(\|u(q^\dagger)-u\|_V^2 + \|u(q)-u\|_V^2\right) \\ &\leq 2C_1^2\left(\mathcal{S}(q^\dagger,u) + \mathcal{S}(q,u)\right) \end{aligned}$$

which shows Assumption 3.2.2.

To check Assumption 3.1.1 (v) for $\mathcal{S}$, we note that

$$\begin{aligned} \mathcal{S}(q,\bar{u}) - \mathcal{S}(q,u) &= a(u-\bar{u}, u-\bar{u}; q) + b(u-\bar{u}, u-\bar{u}) + 2a(u-\bar{u}, u(q)-u; q) \\ &\quad + 2b(u-\bar{u}, u(q)-u) \end{aligned}$$

for any $q \in \mathcal{A}$ and $u, \bar{u} \in V$. Therefore, it follows from (A2) that

$$\begin{aligned} &|\mathcal{S}(q,\bar{u}) - \mathcal{S}(q,u)| \\ &\leq C_0(\|q\|_Z+1)\|u-\bar{u}\|_V^2 + 2C_0(\|q\|_Z+1)\|u-\bar{u}\|_V\|u(q)-u\|_V \end{aligned} \tag{3.3.5}$$

which immediately implies that, for each fixed $q \in \mathcal{A}$, the function $u \to \mathcal{S}(q,u)$ is $\tau_{\mathcal{U}}$-continuous, and hence Assumption 3.1.1 (v) is verified.

The verification of Assumption 3.1.1 (iv) is more subtle. We will achieve it by showing that for each fixed $u \in V$, the function $q \to \mathcal{S}(q, u)$ on $\mathcal{A}$ is convex and continuous with respect to the strong topology on B. To this end, we need a series of lemmas.

Given $q \in \mathcal{A}$ we can define a mapping $u'(q) : Z \to V$ such that, for each $h \in Z$, $\eta := u'(q)h$ is a solution of the problem

$$a(\eta, v; q) + b(\eta, v) = -a(u(q), v; h), \qquad \forall v \in V. \tag{3.3.6}$$

According to the given conditions (A1)–(A3), this $\eta := u'(q)h$ is uniquely defined with the property

$$\|u'(q)h\|_V \leq \frac{C_0}{c_0}\|u(q)\|_V\|h\|_Z \leq \frac{C_0}{c_0^2}\|\ell\|_{V^*}\|h\|_Z.$$

Therefore $u'(q) : Z \to V$ is a bounded linear map.

Lemma 3.3.1. *For any $p, q \in \mathcal{A}$ there holds*

$$\|u(p) - u(q) - u'(q)(p - q)\|_V \leq \frac{C_0}{c_0^3}\|\ell\|_{V^*}\|p - q\|_Z^2.$$

Proof. To see this, we recall that

$$\begin{aligned} a(u(q), v; q) + b(u(q), v) &= \ell(v), \qquad \forall v \in V, \\ a(u(p), v; p) + b(u(p), v) &= \ell(v), \qquad \forall v \in V. \end{aligned}$$

From these two equations we can derive that

$$a(u(p) - u(q), v; p) + b(u(p) - u(q), v) = -a(u(q), v; p - q).$$

In view of the definition of $u'(q)(p - q)$ we then have

$$\begin{aligned} &a(u(p) - u(q) - u'(q)(p - q), v; p) + b(u(p) - u(q) - u'(q)(p - q), v) \\ &= -a(u'(q)(p - q), v; p - q). \end{aligned}$$

By taking $v = u(q + h) - u(q) - u'(q)(p - q)$ we can obtain

$$\begin{aligned} c_0\|u(q + h) - u(q) - u'(q)(p - q)\|_V &\leq C_0\|u'(q)(p - q)\|_V\|p - q\|_Z \\ &\leq \frac{C_0}{c_0^2}\|\ell\|_{V^*}\|p - q\|_Z^2. \end{aligned}$$

The proof is thus complete. □

Lemma 3.3.2. *For each $u \in V$, the function $q \to \mathcal{S}(q,u)$ is a convex function on the convex set $\mathcal{A}$.*

Proof. For any two points $q_0, q_1 \in \mathcal{A}$ let $q_t := (1-t)q_0 + tq_1$ for $0 \le t \le 1$. It follows from Lemma 3.3.1 that

$$\frac{d}{dt}u(q_t) = u'(q_t)h, \tag{3.3.7}$$

where $h := q_1 - q_0$. Therefore, by using the definition of $\mathcal{S}(q_t,u)$ and (3.3.6) we have

$$\begin{aligned}\frac{d}{dt}\mathcal{S}(q_t,u) &= a(u(q_t)-u, u(q_t)-u; h) + 2a(u'(q_t)h, u(q_t)-u; q_t)\\ &\quad + 2b(u'(q_t)h, u(q_t)-u)\\ &= a(u(q_t)-u, u(q_t)-u; h) - 2a(u(q_t), u(q_t)-u; h)\\ &= a(u,u;h) - a(u(q_t), u(q_t); h). \end{aligned}\tag{3.3.8}$$

Consequently, by using (3.3.7), (3.3.6) and (A3) we further have

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{S}(q_t,u) &= -2a(u(q_t), u'(q_t)h; h)\\ &= 2\left[a(u'(q_t)h, u'(q_t)h; h) + b(u'(q_t)h, u'(q_t)h)\right]\\ &\ge 2c_0\|u'(q_t)h\|_V^2 \ge 0.\end{aligned}$$

This implies that $\mathcal{S}((1-t)q_0 + tq_1, u) \le (1-t)\mathcal{S}(q_0,u)) + t\mathcal{S}(q_1,u)$ for any $q_0, q_1 \in \mathcal{A}$ and $0 \le t \le 1$. Therefore $q \to \mathcal{S}(q,u)$ is convex over $\mathcal{A}$. □

Lemma 3.3.3. *For each $u \in V$ the function $q \to \mathcal{S}(q,u)$ is continuous on the set $\mathcal{A}$ with respect to the strong topology in B.*

Proof. Let $\{q_n\} \subset \mathcal{A}$ and $q \in \mathcal{A}$ be such that $\|q_n - q\|_B \to 0$ as $n \to \infty$. Since $\mathcal{A}$ is a bounded set in Z, we may use (3.3.3) to conclude that $\{\|u(q_n)\|_V\}$ is a bounded sequence. Note that (A2) and (A3) imply that

$$|||v||| := \sqrt{a(v,v;q) + b(v,v)}, \qquad v \in V$$

is an equivalent norm on V under which $(V, |||\cdot|||)$ becomes a Hilbert space. Thus, by taking a subsequence if necessary, we may assume that $\{u(q_n)\}$ converges weakly to some u in $(V, |||\cdot|||)$, i.e.

$$a(u(q_n), v; q) + b(u(q_n), v) \to a(u, v; q) + b(u, v)$$

as $n \to \infty$ for all $v \in V$. In view of the definition of $u(q_n)$ and the linearity of $q \to a(\cdot, \cdot; q)$ we have

$$\begin{aligned}\ell(v) = a(u(q_n), v; q_n) + b(u(q_n), v) = a(u(q_n), v; q) + b(u(q_n), v)\\ + a(u(q_n), v; q_n - q).\end{aligned}$$

By taking $n \to \infty$ and using (A4) we can obtain

$$\ell(v) = a(u, v; q) + b(u, v), \qquad \forall v \in V.$$

This shows that $u = u(q)$ and thus $\{u(q_n)\}$ converges weakly to $u(q)$ in $(V, |||\cdot|||)$, i.e.

$$a(u(q_n), v; q) + b(u(q_n), v) \to a(u(q), v; q) + b(u(q), v), \quad \forall v \in V.$$

Moreover, since $\ell \in V^*$, it is also a bounded linear functional on $(V, |||\cdot|||)$ and hence $\ell(u(q_n)) \to \ell(u(q))$. Using these facts, (A1) and (A4) we can derive that

$$\begin{aligned}\mathcal{S}(q_n, u) &= a(u(q_n) - u, u(q_n) - u; q_n) + b(u(q_n) - u, u(q_n) - u)\\ &= a(u(q_n), u(q_n) - u; q_n) + b(u(q_n), u(q_n) - u)\\ &\quad - a(u, u(q_n) - u; q) - b(u, u(q_n) - u)\\ &\quad - a(u, u(q_n) - u; q_n - q)\\ &= \ell(u(q_n) - u) - a(u(q_n) - u, u; q) - b(u(q_n) - u, u)\\ &\quad - a(u(q_n) - u, u; q_n - q)\\ &\to \ell(u(q) - u) - a(u(q) - u, u; q) - b(u(q) - u, u)\\ &= \ell(u(q) - u) - a(u, u(q) - u; q) - b(u, u(q) - u).\end{aligned} \tag{3.3.9}$$

According to the definition of $u(q)$ we have

$$\ell(u(q) - u) = a(u(q), u(q) - u; q) + b(u(q), u(q) - u).$$

Therefore we can obtain from (3.3.9) that $\mathcal{S}(q_n, u) \to \mathcal{S}(q, u)$ as $n \to \infty$. Thus $q \to \mathcal{S}(q, u)$ is continuous on $\mathcal{A}$ with respect to the strong topology in B. $\square$

Lemma 3.3.4. *For each $u \in V$ the function $q \to \mathcal{S}(q,u)$ is τ_Q lower semi-continuous on $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is convex and closed in B and since, by Lemma 3.3.2 and Lemma 3.3.3, $q \to \mathcal{S}(q,u)$ is convex and continuous on $\mathcal{A}$ with respect to the strong topology in B, the τ_Q lower semi-continuity of $q \to \mathcal{S}(q,u)$ then follows from a well-known fact in functional analysis. □

Now we are ready to verify Assumption 3.1.1 (iv) for $\mathcal{S}$. Let $\{q_n\} \subset \mathcal{A}$ and $\{u_n\} \subset V$ be such that $q_n \xrightarrow{\tau_Q} q \in \mathcal{A}$ and $u_n \xrightarrow{\tau_U} u \in V$. We write

$$\mathcal{S}(q_n,u_n) - \mathcal{S}(q,u) = [\mathcal{S}(q_n,u_n) - \mathcal{S}(q_n,u)] + [\mathcal{S}(q_n,u) - \mathcal{S}(q,u)]\,. \tag{3.3.10}$$

According to (3.3.5) we have

$$|\mathcal{S}(q_n,u_n) - \mathcal{S}(q_n,u)| \le C_0(\|q_n\|_Z + 1)\left(\|u_n - u\|_V + \|u(q_n) - u\|_V\right)\|u_n - u\|_V.$$

Since $\{q_n\} \subset \mathcal{A}$ is a bounded, $\{\|u(q_n)\|_V\}$ is also bounded by virtue of (3.3.3). Thus, by using $u_n \xrightarrow{\tau_U} u$ we can conclude that

$$\lim_{n\to\infty} (\mathcal{S}(q_n,u_n) - \mathcal{S}(q_n,u)) = 0.$$

Consequently, it follows from (3.3.10) and Lemma 3.3.4 that

$$\begin{aligned}
&\liminf_{n\to\infty} (\mathcal{S}(q_n,u_n) - \mathcal{S}(q,u)) \\
&\quad = \lim_{n\to\infty} (S(q_n,u_n) - \mathcal{S}(q_n,u)) + \liminf_{n\to\infty} (S(q_n,u) - \mathcal{S}(q,u)) \\
&\quad = \liminf_{n\to\infty} (S(q_n,u) - \mathcal{S}(q,u)) \ge 0,
\end{aligned}$$

i.e. $\mathcal{S}(q,u) \le \liminf_{n\to\infty} \mathcal{S}(q_n,u_n)$ and thus Assumption 3.1.1 (iv) is verified.

Example 3.3.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the determination of q in the diffusion problem

$$\begin{aligned}
-\mathrm{div}(q\nabla u) = f \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \partial\Omega
\end{aligned} \tag{3.3.11}$$

from an measurement $\tilde{u}$ of u in $H_0^1(\Omega)$. If we take $V = H_0^1(\Omega)$, $Z = L^\infty(\Omega)$, $B = L^2(\Omega)$, and define $\mathcal{A} = \{q \in L^\infty(\Omega) : \gamma_0 \le q \le \gamma_1 \text{ a.e. }\}$ for some positive constants γ_0 and γ_1, then the elliptic problem (3.3.11) can be formulated into the form (3.3.1) with

$$a(u, v; q) = \int_\Omega q\nabla u \cdot \nabla v dx, \quad b(u, v) = 0, \quad \ell(v) = \int_\Omega f v dx,$$

and the inverse problem takes the form (3.1.1) with

$$S(q, u) = \int_\Omega q|\nabla(u(q) - u)|^2.$$

It is easy to see that (A1)–(A3) hold. For verifying (A4), we first use the Cauchy-Schwarz inequality and the Poincaré inequality to obtain

$$|a(u, v; q_n - q)| \leq C\|u\|_V \left(\int_\Omega |q_n - q|^2 |\nabla v|^2\right)^{1/2}$$

for some universal constant C. Thus

$$\sup_{\|u\|_V \leq \beta} |a(u, v; q_n - q)| \leq C\beta \left(\int_\Omega |q_n - q|^2 |\nabla v|^2\right)^{1/2}.$$

By virtue of $\|q_n - q\|_{L^2(\Omega)} \to 0$ and $\{q_n\} \subset \mathcal{A}$, we may use the Lebesgue dominated convergence theorem to conclude

$$\int_\Omega |q_n - q|^2 |\nabla v|^2 \to 0 \quad \text{as } n \to \infty.$$

We thus verify (A4).

Example 3.3.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the estimation of the parameters $q = (q_1, q_2)$ in the problem

$$-\text{div}(q_1 \nabla u) + q_2 u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3.3.12}$$

from a measured data $\tilde{u}$ of u in $H^1(\Omega)$. If we take $V = H_0^1(\Omega)$, $Z = L^\infty(\Omega) \times L^\infty(\Omega)$, $B = L^2(\Omega) \times L^2(\Omega)$ and define

$$\mathcal{A} = \{(q_1, q_2) \in L^\infty(\Omega) \times L^\infty(\Omega) : \gamma_0 \leq q_1 \leq \gamma_1 \text{ and } \gamma_2 \leq q_2 \leq \gamma_3 \text{ a. e. }\}$$

for some positive constants $\gamma_0, \gamma_1, \gamma_2$ and γ_3, then the elliptic problem has the weak formulation (3.3.1) with

$$a(u, v; (q_1, q_2)) = \int_\Omega (q_1 \nabla u \cdot \nabla v + q_2 uv) dx, \quad b(u, v) = 0, \quad \ell(v) = \int_\Omega f v dx$$

and the inverse problem takes the form (3.1.1) with

$$\mathcal{S}((q_1, q_2), u) = \int_\Omega \left(q_1 |\nabla (u(q_1, q_2) - u)|^2 + q_2 |u(q_1, q_2) - u|^2 \right),$$

where $u(q_1, q_2) \in H_0^1(\Omega)$ denotes the unique solution of (3.3.12) for given $(q_1, q_2) \in \mathcal{A}$. It is easy to see that (A1)–(A4) hold.

In the following we will provide numerical results to validate the developed theory. For the inverse problems discussed above, the regularized solutions are defined by (3.3.2) and our heuristic rule chooses the regularization parameter $\alpha_* \in \Delta_\gamma$ by Rule 3.1 with $\Theta(\alpha, \tilde{u})$ given by (3.3.4). Therefore we need to solve for $\alpha \in \Delta_\gamma$ the minimization problem (3.3.2). When B is a Hilbert space and the regularization functional $\mathcal{R}(q) = \phi(q, q)$ for a bounded symmetric bilinear form ϕ, we may use a Newton-type method to solve it. To be more precise, according to (3.3.8) and the convexity of $\mathcal{S}$, solving (3.3.2) is equivalent to finding $(u, q) \in V \times \mathcal{A}$ such that

$$\begin{aligned} & a(u, v; q) + b(u, v) = \ell(v), \qquad \forall v \in V, \\ & a(\tilde{u}, \tilde{u}; p - q) - a(u, u; p - q) + 2\alpha\phi(q, p - q) \geq 0, \qquad \forall p \in \mathcal{A}. \end{aligned} \tag{3.3.13}$$

We then solve (3.3.13) by applying the projected Newton method or the semi-smooth Newton method [8, 29, 37, 78]) which enables us to consider (3.3.13) first by replacing the inequality by equality and then by projecting the computational result for q back onto $\mathcal{A}$ after each iteration. This thus leads to Algorithm 1. In case $\mathcal{R}(q)$ is nonsmooth, e.g. $\mathcal{R}(q)$ is the total variation of q, we will use an iteratively reweighted least-squares (IRLS) algorithm [15, 67] and then apply the projected Newton method.

Algorithm 1 Projected Newton method for (3.3.13)

1: Given initial guess u_0, q_0.
2: **for** $k = 0, 1, 2, \ldots$ **do**
3: Obtain $u_{k+1}, \tilde{q}_{k+1}$ by solving

$$\begin{aligned} & a(u_{k+1}, v; q_k) + b(u_{k+1}, v) + a(u_k, v; \tilde{q}_{k+1}) = \ell(v) + a(u_k, v; q_k), \ \forall\ v \in V, \\ & 2\alpha\phi(\tilde{q}_{k+1}, p) - 2a(u_k, u_{k+1}, p) = -a(\tilde{u}, \tilde{u}; p) - a(u_k, u_k, p), \ \forall p \in B. \end{aligned}$$

4: Project $\tilde{q}_{k+1}$ onto $\mathcal{A}$ to obtain q_{k+1}.
5: Check stop condition.
6: **end for**

We remark that the projected Newton method is robust and efficient and enjoys a local superlinear convergence. Since our heuristic rule requires to compute $\tilde{q}_\alpha$ for a number of $\alpha \in \Delta_\gamma$, we adopt a path-following strategy: let $\alpha_n = \alpha_0\gamma^n$ and use Algorithm 1 on the decreasing sequences $\{\alpha_n\}$ with the solution $\tilde{q}_{\alpha_n}$ as an initial guess for computing $\tilde{q}_{\alpha_{n+1}}$. In practical applications the measurement data is usually obtained in the L^2-space while our regularization methods require H^1 data. Therefore, we need to obtain an H^1-measurement data from an L^2-measurement data by a smoothing procedure. In the following computations, we will always use the Clément interpolation ([13]) to achieve this goal. It should be warned that using this smoothing procedure may result in an H^1-noisy data with large noise level; for instance, from an L^2-noisy data $\bar{u}$ with noise level $\delta := \|\bar{u} - u^\dagger\|_{L^2}$, the Clément interpolation produces an H^1-noisy data $\tilde{u}$ with $\|\tilde{u} - u^\dagger\|_{H^1} = O(\sqrt{\delta})$. Consequently, we may expect a slower rate of convergence at least in theory. We refer the readers to [52, Section 2.2] for a detailed account.

In order to implement Algorithm 1 numerically, we need to discretize the variational equations. We may replace V and B by suitably chosen finite-dimensional spaces V_h and B_h and replace $\mathcal{A}$ by $\mathcal{A}_h := \mathcal{A} \cap B_h$ in Algorithm 1. When V and B are spaces consisting of functions defined on a bounded domain $\Omega \subset \mathbb{R}^n$, we may construct V_h and B_h by finite elements. Let $\mathcal{T}_h$ be a regular triangulation of Ω with maximum mesh size $h := \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$ and suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$. We may take

$$\begin{aligned} V_h &:= \{v_h : v_h \in C(\bar{\Omega}), v_h|_T \in P_2(T)\ \forall\ T \in \mathcal{T}_h\},\\ B_h &:= \{v_h : v_h \in C(\bar{\Omega}), v_h|_T \in P_1(T)\ \forall\ T \in \mathcal{T}_h\},\\ \mathcal{A}_h &:= \mathcal{A} \cap B_h, \end{aligned} \tag{3.3.14}$$

where $P_l(T)$ denotes the spaces of polynomials of degree l defined on T.

In the following we will provide some numerical results for which the experiments are performed using FreeFem++ [28] with 128 grid points on $\partial\Omega$.

Example 3.3.3. We consider the inverse problem given in Example 3.3.1, where

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

is the unit disk in $\mathbb{R}^2$. We first consider the case that the sought solution is smooth given by

$$q^\dagger(x) = 1 + x_1^2 + x_2^2.$$

Assume that $u^\dagger(x) = \sin(\pi(x_1^2 + x_2^2))$. Then f can be obtained via the elliptic equation. The noisy data $\tilde{u}$ is produced from $u^\dagger$ by adding random Gaussian noise with relative noise level $\delta_{rel} := \|\tilde{u} - u^\dagger\|_{L^2(\Omega)}/\|u^\dagger\|_{L^2(\Omega)}$. We then use $\tilde{u}$ to reconstruct $q^\dagger$. Since the sought solution is smooth, we take

$$\mathcal{R}(q) = \int_\Omega |\nabla q|^2 dx.$$

Then the corresponding minimization problem (3.3.2) can be solved by Algorithm 1.

In our computation, we take $\gamma_0 = 0.1$ and $\gamma_1 = 20$ in the definition of $\mathcal{A}$. In order to determine the regularization parameter by our heuristic Rule 3.1, we use the set Δ_γ of grid points $\{\alpha_n := \alpha_0\gamma^n : n = 0, 1, \cdots\}$ with $\alpha_0 = 1$ and $\gamma = 0.6$. Note that Rule 3.1 involves a nonnegative constant A. To test the performance of Rule 3.1, we will report numerical results corresponding to various values of the relative noise levels δ_{rel} and various values of A.

Table 3.1: Relative error with respect to A values and relative noise levels for Example 4.1.

err \ A / δ_{rel}	0	1e1	1e2	1e3	1e4	1e5	1e6	1e7
1e-2	1.946e-2	1.946e-2	1.293e-2	8.597e-3	3.788e-3	1.054e-3	6.577e-4	1.820e-4
1e-3	2.585e-3	2.585e-3	2.585e-3	1.728e-3	1.154e-3	5.111e-4	2.191e-4	4.189e-5
1e-4	3.395e-4	3.395e-4	3.395e-4	3.395e-4	2.239e-4	1.462e-4	9.423e-5	2.152e-5
1e-5	3.657e-5	3.657e-5	3.657e-5	3.657e-5	2.409e-5	2.409e-5	1.477e-5	8.903e-6
1e-6	3.375e-6	3.375e-6	3.375e-6	3.375e-6	3.375e-6	3.375e-6	2.021e-6	1.200e-6
1e-7	2.651e-7	2.651e-7	2.651e-7	2.651e-7	2.651e-7	2.651e-7	2.651e-7	2.651e-7
1e-8	3.451e-8	3.451e-8	3.451e-8	3.451e-8	3.451e-8	3.451e-8	3.451e-8	3.451e-8

In Table 3.1 we report the relative errors err $:= \|\tilde{q}_{\alpha_*} - q^\dagger\|_{L^2}/\|q^\dagger\|_{L^2}$ of the reconstructed solutions $\tilde{q}_{\alpha_*}$ with respect to various values of A and δ_{rel}. From Table 3.1 we can see that Rule 2.1 with $A = 0$ can produce reasonable good approximations. When the noise level is suitably small, the computational results with $A = 0$ can compete with the ones obtained with large values of A. When the noise level is somewhat large, the computational results with $A = 0$ are somewhat worse than the ones obtained with large values of A; however, the results are still acceptable. For a given noise level, Table 3.1 also indicates that increasing A from 0 to 10^7 can improve the reconstruction results gradually. However, we have to warn that if we continue to increase A, the reconstructed solutions may be deteriorated eventually because Rule 3.1 can produce a very small regularization parameter. It seems that there is a value A which produces the most satisfactory reconstruction result; how to choose this A is still a puzzle. If we do not intend to produce a most satisfactory result, Rule 3.1 with $A = 0$ usually suffices for reconstruction purpose.

In order to visualize the performance of Rule 3.1, we plot in Figure 3.1 the reconstruction results and the Θ-curve concerning the relationship between $\Theta(\alpha_n, \tilde{u})$ and n for $A = 0, 10^3$ and

10^6 using a noisy data with $\delta_{rel} = 1\%$. From Figure 3.1 we can see that if $A = 0$, then the Θ-curve may start from a small value, and the value of Θ may increase to a local maximum and then decrease to a local minimum, and eventually goes up. If α_0 is large, then the minimum value of Θ may achieve at α_0 which clearly can not be used as a correct regularization parameter. For this situation, if we insist on using $A = 0$, we may find a regularization parameter by looking at the α value achieving the next local minimum. This usually produce a regularization parameter which can give a reasonable good approximate solution. Otherwise, we may use a large value of A to prevent Θ from being too small at the beginning.

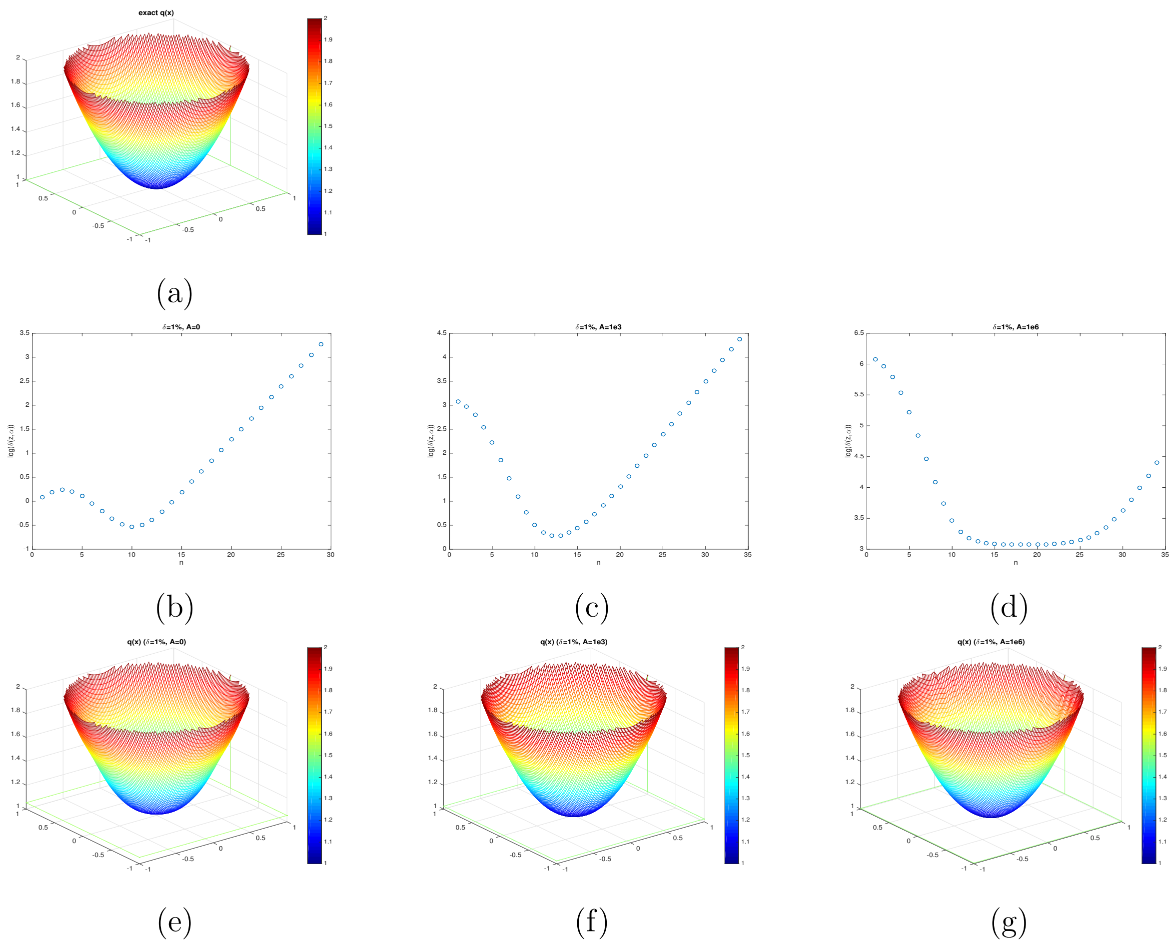

Figure 3.1: The reconstruction for Example 3.3.3 with using noisy data with $\delta_{rel} = 1\%$: (a) represents exact parameter; (b), (c) and (d) plot the relationship between $\Theta(\alpha_n, \tilde{u})$ and n for $A = 0$, 10^3 and 10^6 respectively; (e), (f) and (g) plot reconstruction results for $A = 0$, 10^3 and 10^6

Example 3.3.4. We consider again the inverse problem given in Example 3.3.1 with Ω being the unit disk in $\mathbb{R}^2$. In this example we assume the sought solution is piecewise constant given by

$$q^\dagger(x) = \begin{cases} 5, & x \in [-0.7, -0.2] \times [-0.2, 0.2], \\ 2, & x \in [0.2, 0.5] \times [-0.5, 0.1], \\ 1, & \text{otherwise}. \end{cases}$$

Assuming $f(x) = x_1 + x_2$, we may solve the elliptic problem to obtain the exact data $u^\dagger$. We then add random Gaussian noise to $u^\dagger$ to produce a noisy data $\tilde{u}$ with relative noise level $\delta_{rel} := \|\tilde{u} - u^\dagger\|_{L^2(\Omega)}/\|u^\dagger\|_{L^2(\Omega)}$. We will use $\tilde{u}$ to reconstruct $q^\dagger$. Due to the a priori information on $q^\dagger$, we take the regularization functional $\mathcal{R}(q)$ to be the total variation of q over Ω, i.e.

$$\mathcal{R}(q) = \int_\Omega |Dq|.$$

Now the corresponding minimization problem (3.3.2) can not be solved by Algorithm 1 directly. Instead we use an IRLS strategy to reduce (3.3.2) to a sequence of minimization problems in which each subproblem can be solved by Algorithm 1. More precisely, we take a sequence of small positive numbers $\{\varepsilon_j\}$; with an initial guess q_0, we then define

$$q_{j+1} \in \arg\min_{q \in \mathcal{A}} \left\{ \int_\Omega q|\nabla(u(q) - \tilde{u})|^2 dx + \alpha \int_\Omega \omega_j |\nabla q|^2 dx \right\}, \tag{3.3.15}$$

where $\omega_j := (|\nabla q_j|^2 + \varepsilon_j^2)^{-1/2}$. In summary, we have the following algorithm.

Algorithm 2 The IRLS algorithm for Example 3.3.4

1: Given an initial guess q_0, let $\omega_0 = (|\nabla q_0|^2 + \varepsilon_0^2)^{-\frac{1}{2}}$.
2: **for** $j = 0, 1, ...$ **do**
3: Find q_{j+1} from (3.3.15) by Algorithm 1;
4: Let $\omega_{j+1} = (|\nabla q_{j+1}|^2 + \varepsilon_{j+1}^2)^{-\frac{1}{2}}$;
5: **end for**

When using our heuristic rule to determine the regularization parameter, we use the set $\Delta_\gamma := \{\alpha_0 \gamma^n : n = 0, 1, \cdots\}$ with $\alpha_0 = 0.01$ and $\gamma = 0.6$. The constants γ_0 and γ_1 appearing in $\mathcal{A}$ are taken to be 0.1 and 20 respectively. For each $\alpha \in \Delta_\gamma$ the regularized solution $\tilde{q}_\alpha$ is computed by Algorithm 2 with $q_0 \equiv 1$ and $\varepsilon_j = 0.01$ for all j. We remark that Algorithm 2 consists of two components: the outer IRLS iteration and the inner projected Newton iteration. Due to the superlinear convergence of the projected Newton method, in our numerical experiments we only need to take a few IRLS iterations and Newton iterations to find $\tilde{q}_\alpha$ for each $\alpha \in \Delta_\gamma$.

In Table 3.2 we report the relative errors err $:= \|\tilde{q}_{\alpha_*} - q^\dagger\|_{L^2}/\|q^\dagger\|_{L^2}$ of the reconstructed solutions $\tilde{q}_{\alpha_*}$ with respect to various values of A and δ_{rel}. From Table 3.2 we can see that for a fixed value A, the reconstruction error decreases as the noise level decreases which is consistent

Table 3.2: Relative error with respect to A values and relative noise levels for Example 4.2.

err \ A; δ_{rel}	0	1e1	1e2	1e3	1e4	1e5	1e6	1e7
1e-2	1.359e-2	1.359e-2	1.359e-2	1.359e-2	1.359e-2	1.359e-2	8.374e-3	3.111e-3
1e-3	1.147e-3	1.147e-3	1.147e-3	1.147e-3	1.147e-3	1.147e-3	1.147e-3	7.062e-4
1e-4	1.743e-4	1.743e-4	1.743e-4	1.743e-4	1.743e-4	1.743e-4	1.743e-4	1.114e-4
1e-5	1.980e-5	1.980e-5	1.980e-5	1.980e-5	1.980e-5	1.980e-5	1.980e-5	1.980e-5
1e-6	1.125e-6	1.125e-6	1.125e-6	1.125e-6	1.125e-6	1.125e-6	1.125e-6	1.125e-6
1e-7	1.305e-7	1.305e-7	1.305e-7	1.305e-7	1.305e-7	1.305e-7	1.305e-7	1.305e-7
1e-8	7.728e-8	7.728e-8	7.728e-8	7.728e-8	7.728e-8	7.728e-8	7.728e-8	7.728e-8

with Theorem 3.2.2. For small noise level, the reconstruction results by Rule 3.1 with $A = 0$ can compete with the ones obtained with large values of A. For large noise level, the reconstruction results by Rule 3.1 with $A = 0$ are somewhat worse than the ones obtained with large values of A; however, the results can still be accepted as reasonable good approximate solutions.

In Figure 3.2 we provide further information by plotting the reconstruction results and the Θ-curve for $A = 0$, 10^3 and 10^6 using a noisy data with $\delta_{rel} = 0.1\%$. From Figure 3.2 we can observe the same phenomenon which has been commented in Example 3.3.3. This observation suggests that, for a successful use of Rule 3.1, we may use a suitably small α_0, or a suitably large A; or otherwise, we may have a successful use of Rule 3.1 with $A = 0$ by looking at the Θ-curve, ignoring the potential minimum at α_0, and taking the regularization parameter to be the α achieving the next local minimum.

3.4 Application: electrical impedance tomography

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary $\partial\Omega$, let ν denote the unit outward normal on $\partial\Omega$, and let $\rho \in H^{-1}(\Omega)$ be given. We consider the electrical impedance tomography which consists in determining the conductivity q in the elliptic equation

$$-\text{div}(q\nabla u) = \rho \quad \text{in } \Omega \tag{3.4.1}$$

by virtue of a family of Cauchy data

$$(f_\ell, g_\ell), \qquad \ell = 1, \cdots, L,$$

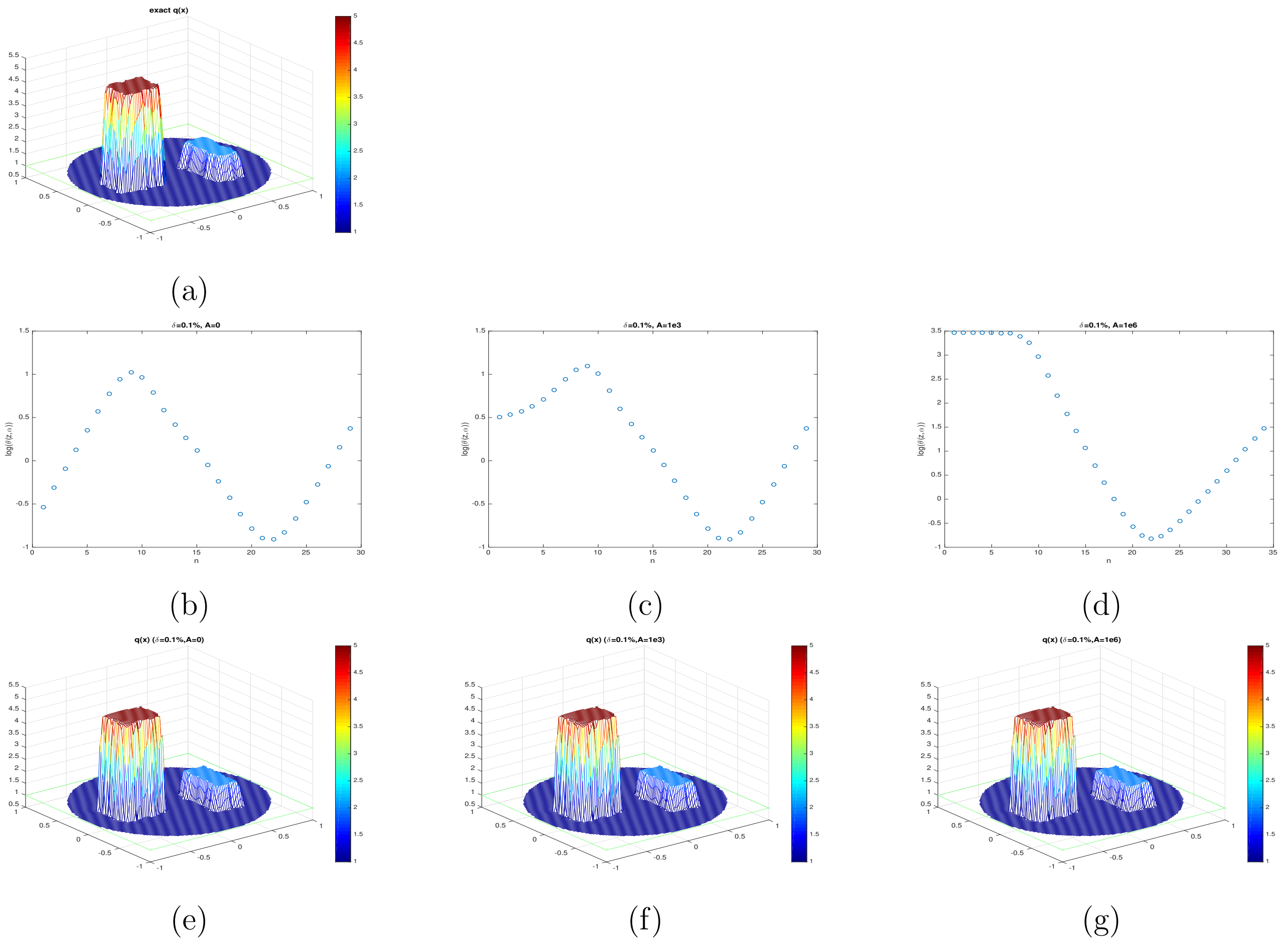

Figure 3.2: The reconstruction for Example 3.3.4 with using noisy data with $\delta_{rel} = 0.1\%$: (a) represents exact parameter; (b), (c) and (d) plot the relationship between $\Theta(\alpha_n, \tilde{u})$ and n for $A = 0$, 10^3 and 10^6 respectively. (e), (f) and (g) plot reconstruction results for $A = 0$, 10^3 and 10^6

where $f_\ell = u|_{\partial\Omega}$ and $g_\ell = q\frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ for some $u \in H^1(\Omega)$ satisfying (3.4.1). We assume that

$$q \in \mathcal{Q} := \{q \in L^\infty(\Omega) : \underline{q} \le q(x) \le \bar{q} \text{ a.e. in } \Omega\}$$

with known positive constants $\underline{q}$ and $\bar{q}$.

We will adopt the variational approach of Kohn and Vogelius [58, 59, 60] to identify q. To this end, for a given $f \in H^{1/2}(\partial\Omega)$ we use $u_D[q, f] \in H^1(\Omega)$ to denote the unique weak solution of

$$-\mathrm{div}(q\nabla u) = \rho \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega, \tag{3.4.2}$$

i.e. $u = f$ on $\partial\Omega$ and

$$\int_\Omega q\nabla u \cdot \nabla\phi = \int_\Omega \rho\phi, \quad \forall \phi \in H_0^1(\Omega),$$

where

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

For a given $g \in H^{-1/2}(\partial\Omega)$ we also use $u_N[q, g] \in H_\diamond^1(\Omega)$ to denote the unique weak solution of

$$-\mathrm{div}(q\nabla v) = \rho \text{ in } \Omega, \quad q\frac{\partial v}{\partial \nu} = g \text{ on } \partial\Omega \tag{3.4.3}$$

with vanishing boundary mean, i.e.

$$\int_\Omega q\nabla v \cdot \nabla\psi = \int_\Omega \rho\psi + \int_{\partial\Omega} g\psi, \quad \forall \psi \in H_\diamond^1(\Omega), \tag{3.4.4}$$

where

$$H_\diamond^1(\Omega) := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u = 0\right\}.$$

According to the theory of elliptic equations, $u_D[q, f]$ and $u_N[q, g]$ are well-defined and

$$\begin{aligned} \|u_D[q, f]\|_{H^1(\Omega)} &\le C_D \left(\|\rho\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}\right), \\ \|u_N[q, g]\|_{H^1(\Omega)} &\le C_N \left(\|\rho\|_{H^{-1}(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}\right) \end{aligned} \tag{3.4.5}$$

for some universal constants C_D and C_N depending only on $\underline{q}$, $\bar{q}$ and Ω.

Now we define the mapping $\mathcal{S} : \mathcal{Q} \times (H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))^L \to [0, \infty)$ by

$$\mathcal{S}(q, \{(f_\ell, g_\ell)\}_{\ell=1}^L) := \sum_{\ell=1}^L \int_\Omega q|\nabla(u_D[q, f_\ell] - u_N[q, g_\ell])|^2$$

for $q \in \mathcal{Q}$ and $\{(f_\ell, g_\ell)\}_{\ell=1}^L \in (H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))^L$. Then for the exactly given Cauchy data $\{(f_\ell^\dagger, g_\ell^\dagger)\}_{\ell=1}^L \in (H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))^L$, the electrical impedance tomography amounts to solving

$$\mathcal{S}(q, \{(f_\ell^\dagger, g_\ell^\dagger)\}_{\ell=1}^L) = 0 \quad \text{in } \mathcal{Q}.$$

In case only measurement data $\{(\tilde{f}_\ell, \tilde{g}_\ell)\}_{\ell=1}^L \in (H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))^L$ is available, we may reconstruct the conductivity $q \in \mathcal{Q}$ by considering the minimization problem

$$\tilde{q}_\alpha \in \arg\min_{q \in \mathcal{Q}} \left\{ \sum_{\ell=1}^L \int_\Omega q|\nabla(u_D[q, \tilde{f}_\ell] - u_N[q, \tilde{g}_\ell])|^2 + \alpha \mathcal{R}(q) \right\} \tag{3.4.6}$$

with a suitably chosen regularization functional $\mathcal{R} : \mathcal{Q} \to [0, \infty)$. This minimization problem clearly takes the form (3.1.3). The regularization property of (3.4.6) has been analyzed recently in [30] under a priori choice of the regularization parameter. Our heuristic rule chooses the regularization parameter as

$$\alpha_* \in \arg\min_{\alpha \in \Delta_\gamma} \left\{ \left(\frac{1}{\alpha} + A \right) \sum_{\ell=1}^L \int_\Omega \tilde{q}_\alpha |\nabla(u_D[\tilde{q}_\alpha, \tilde{f}_\ell] - u_N[\tilde{q}_\alpha, \tilde{g}_\ell])|^2 \right\}.$$

In order to apply the convergence result for Rule 3.1, we are going to show that if $\{q^{(n)}\} \subset \mathcal{Q}$ and $\{(f_\ell^{(n)}, g_\ell^{(n)})\} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ satisfying $\|q^{(n)} - q\|_{L^1(\Omega)} \to 0$ and $\|f_\ell^{(n)} - f_\ell\|_{H^{1/2}(\partial\Omega)} + \|g_\ell^{(n)} - g_\ell\|_{H^{-1/2}(\partial\Omega)} \to 0$ as $n \to \infty$ for $\ell = 1, \cdots, L$, then

$$\lim_{n\to\infty} \mathcal{S}(q^{(n)}, \{(f_\ell^{(n)}, g_\ell^{(n)})\}_{\ell=1}^L) = \mathcal{S}(q, \{(f_\ell, g_\ell)\}_{\ell=1}^L)$$

which implies that Assumption 3.1.1 (iv) and (v) hold with $\mathcal{U} := (H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))^L$, $\tau_{\mathcal{Q}} :=$ strong topology on $L^1(\Omega)$ and $\tau_{\mathcal{U}} :=$ strong topology on $\mathcal{U}$. To see this, it suffices to show that if $\{q^{(n)}\} \subset \mathcal{Q}$ and $\{(f^{(n)}, g^{(n)})\} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ satisfy $\|q^{(n)} - q\|_{L^1(\Omega)} \to 0$ and $\|f^{(n)} - f\|_{H^{1/2}(\partial\Omega)} + \|g^{(n)} - g\|_{H^{-1/2}(\partial\Omega)} \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} \Gamma(q^{(n)}, (f^{(n)}, g^{(n)})) = \Gamma(q, (f, g)), \tag{3.4.7}$$

where

$$\Gamma(q,(f,g)) := \int_\Omega q|\nabla(u_D[q,f]-u_N[q,g])|^2.$$

We first show that

$$\|u_D[q^{(n)},f]-u_D[q,f]\|_{H^1(\Omega)}\to 0, \quad \|u_N[q^{(n)},g]-u_N[q,g]\|_{H^1(\Omega)}\to 0 \tag{3.4.8}$$

as $n\to\infty$. By the definition of $u_D[q^{(n)},f]$ and $u_D[q,f]$ we have $u_D[q^{(n)},f]-u_D[q,f]\in H_0^1(\Omega)$ and

$$\int_\Omega q^{(n)}\nabla u_D[q^{(n)},f]\cdot\nabla\phi = \int_\Omega \rho\phi = \int_\Omega q\nabla u_D[q,f]\cdot\nabla\phi$$

for any $\phi\in H_0^1(\Omega)$. Therefore

$$\begin{aligned} 0 &= \int_\Omega q^{(n)}\nabla u_D[q^{(n)},f]\cdot\nabla\phi - \int_\Omega q\nabla u_D[q,f]\cdot\nabla\phi \\ &= \int_\Omega q^{(n)}\nabla\left(u_D[q^{(n)},f]-u_D[q,f]\right)\cdot\nabla\phi + \int_\Omega (q^{(n)}-q)\nabla u_D[q,f]\cdot\nabla\phi. \end{aligned}$$

By taking $\phi = u_D[q^{(n)},f]-u_D[q,f]$ in the above equation, using the condition $q^{(n)}\geq \underline{q}$, and the Cauchy-Schwartz inequality we have

$$\begin{aligned} \underline{q}\int_\Omega |\nabla(u_D[q^{(n)},f]-u_D[q,f])|^2 &\leq \int_\Omega q^{(n)}|\nabla(u_D[q^{(n)},f]-u_D[q,f])|^2 \\ &= \int_\Omega (q-q^{(n)})\nabla u_D[q,f]\cdot\nabla(u_D[q^{(n)},f]-u_D[q,f]) \\ &\leq \left(\int_\Omega |\nabla(u_D[q^{(n)},f]-u_D[q,f])|^2\right)^{1/2}\left(\int_\Omega |q^{(n)}-q|^2|\nabla u_D[q,f]|^2\right)^{1/2} \end{aligned}$$

Consequently

$$\underline{q}^2\int_\Omega |\nabla(u_D[q^{(n)},f]-u_D[q,f])|^2 \leq \int_\Omega |q^{(n)}-q|^2|\nabla u_D[q,f]|^2.$$

Since $\{q^{(n)}\}\subset\mathcal{Q}$ and $\|q^{(n)}-q\|_{L^1(\Omega)}\to 0$, by the dominated convergence theorem we can show that the right hand side converges to zero as $n\to\infty$. Therefore, we may use the Poincaré inequality to conclude the first result in (3.4.8). The second result in (3.4.8) can be proved in a similar way.

Next we will show that

$$\lim_{n\to\infty}\Gamma(q^{(n)},(f,g)) = \Gamma(q,(f,g)). \tag{3.4.9}$$

We first write

$$\begin{aligned}
&\Gamma(q^{(n)},(f,g))-\Gamma(q,(f,g)) \\
&=\int_\Omega q^{(n)}|\nabla(u_D[q^{(n)},f]-u_N[q^{(n)},g])|^2-\int_\Omega q|\nabla(u_D[q,f]-u_N[q,g])|^2 \\
&=I_1^{(n)}+I_2^{(n)}+I_3^{(n)},
\end{aligned} \tag{3.4.10}$$

where

$$\begin{aligned}
I_1^{(n)} &= \int_\Omega q^{(n)}|\nabla u_D[q^{(n)},f]|^2-\int_\Omega q|\nabla u_D[q,f]|^2, \\
I_2^{(n)} &= -2\int_\Omega q^{(n)}\nabla u_D[q^{(n)},f]\cdot\nabla u_N[q^{(n)},g]+2\int_\Omega q\nabla u_D[q,f]\cdot\nabla u_N[q,g], \\
I_3^{(n)} &= \int_\Omega q^{(n)}|\nabla u_N[q^{(n)},g]|^2-\int_\Omega q|\nabla u_N[q,g]|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
I_1^{(n)} = &\int_\Omega (q^{(n)}-q)|\nabla u_D[q,f]|^2 \\
&+\int_\Omega q^{(n)}\nabla(u_D[q^{(n)},f]-u_D[q,f])\cdot\nabla(u_D[q^{(n)},f]+u_D[q,f]),
\end{aligned}$$

we have

$$\begin{aligned}
|I_1^{(n)}| \le &\int_\Omega |q^{(n)}-q||\nabla u_D[q,f]|^2 \\
&+\bar{q}\left(\|u_D[q^{(n)},f]\|_{H^1(\Omega)}+\|u_D[q,f]\|_{H^1(\Omega)}\right)\|u_D[q^{(n)},f]-u_D[q,f]\|_{H^1(\Omega)}.
\end{aligned}$$

The first term converges to zero by the dominated convergence theorem and the second term converges to zero by the first result in (3.4.8). Hence $I_1^{(n)} \to 0$ as $n \to \infty$. By a similar argument we can also show that $I_3^{(n)} \to 0$ as $n \to \infty$. For the term $I_2^{(n)}$ we can write

$$\begin{aligned}
I_2^{(n)} = 2&\int_\Omega q^{(n)}\nabla(u_D[q,f]-u_D[q^{(n)},f])\cdot\nabla u_N[q^{(n)},g] \\
&+2\int_\Omega q^{(n)}\nabla u_D[q,f]\cdot\nabla(u_N[q,g]-u_N[q^{(n)},g]) \\
&+2\int_\Omega (q-q^{(n)})\nabla u_D[q,f]\cdot\nabla u_N[q,g].
\end{aligned}$$

Therefore

$$
\begin{aligned}
|I_2^{(n)}| \le\ & 2\bar{q}\|u_N[q^{(n)},g]\|_{H^1(\Omega)}\|u_D[q,f]-u_D[q^{(n)},f]\|_{H^1(\Omega)} \\
&+ 2\bar{q}\|u_D[q,f]\|_{H^1(\Omega)}\|u_N[q,g]-u_N[q^{(n)},g]\|_{H^1(\Omega)} \\
&+ 2\int_\Omega |q-q^{(n)}||\nabla u_D[q,f]||\nabla u_N[q,g]|.
\end{aligned}
$$

By using (3.4.8), the first two terms on the right hand side converge to zero, and, by the dominated convergence theorem, the last term on the right hand side also converges to zero. Therefore $I_2^{(n)} \to 0$ as $n \to \infty$. Combining the above results with (3.4.10) we thus obtain (3.4.9).

Finally we show (3.4.7). According to (3.4.9), it remains only to show that

$$
\lim_{n\to\infty}\left(\Gamma(q^{(n)},(f^{(n)},g^{(n)})) - \Gamma(q^{(n)},(f,g))\right) = 0. \tag{3.4.11}
$$

Let $v^{(n)} := u_D[q^{(n)},f^{(n)}] - u_D[q^{(n)},f]$ and $w^{(n)} := u_N[q^{(n)},g^{(n)}] - u_N[q^{(n)},g]$. Then $v^{(n)} \in H^1(\Omega)$ and $w^{(n)} \in H^1_\diamond(\Omega)$ are weak solutions of

$$
\begin{cases}
-\mathrm{div}(q^{(n)}\nabla v^{(n)}) = 0 & \text{in } \Omega \\
v^{(n)} = f^{(n)} - f & \text{on } \partial\Omega
\end{cases}
\quad \text{and} \quad
\begin{cases}
-\mathrm{div}(q^{(n)}\nabla w^{(n)}) = 0 & \text{in } \Omega \\
q^{(n)}\frac{\partial w^{(n)}}{\partial \nu} = g^{(n)} - g & \text{on } \partial\Omega
\end{cases}
$$

respectively. Thus, it follows from (3.4.5) that

$$
\begin{aligned}
\|v^{(n)}\|_{H^1(\Omega)} &\le C_D\|f^{(n)}-f\|_{H^{1/2}(\partial\Omega)}, \\
\|w^{(n)}\|_{H^1(\Omega)} &\le C_N\|g^{(n)}-g\|_{H^{-1/2}(\partial\Omega)}.
\end{aligned} \tag{3.4.12}
$$

Note that

$$
\begin{aligned}
\Gamma(q^{(n)},(f^{(n)},g^{(n)})) - \Gamma(q^{(n)},(f,g)) &= \int_\Omega q^{(n)}\left|\nabla(u_D[q^{(n)},f^{(n)}] - u_N[q^{(n)},g^{(n)}])\right|^2 \\
&\quad - \int_\Omega q^{(n)}\left|\nabla(u_D[q^{(n)},f] - u_N[q^{(n)},g])\right|^2 \\
&= \int_\Omega q^{(n)}\nabla(v^{(n)} - w^{(n)})\cdot\nabla\eta^{(n)},
\end{aligned}
$$

where

$$
\eta^{(n)} := u_D[q^{(n)},f^{(n)}] - u_N[q^{(n)},g^{(n)}] + u_D[q^{(n)},f] - u_N[q^{(n)},g].
$$

By using (3.4.5) it is easy to see that $\|\eta^{(n)}\|_{H^1(\Omega)} \leq C$ for a universal constant C independent of n. Therefore, we can use (3.4.12) to conclude that

$$
\begin{aligned}
&|\Gamma(q^{(n)}, (f^{(n)}, g^{(n)})) - \Gamma(q^{(n)}, (f, g))| \\
&\leq C\|\eta^{(n)}\|_{H^1(\Omega)}\|v^{(n)} - w^{(n)}\|_{H^1(\Omega)} \leq C \left(\|f^{(n)} - f\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|g^{(n)} - g\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right).
\end{aligned}
$$

This shows (3.4.11).

Example 3.4.1. We next consider the electrical impedance tomography discussed in Section 3.2 with $\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and $\rho(x) = \frac{3}{2}\chi_D - \frac{1}{2}\chi_{\Omega\setminus D}$, where

$$D := \{(x_1, x_2) \in \Omega : |x_1| \leq 0.5 \text{ and } |x_2| \leq 0.5\}$$

and χ_D denotes the characteristic function of D. We assume that the sought parameter is given by $q^\dagger = 3\chi_{\Omega_1} + 2\chi_{\Omega_2} + \chi_{\Omega\setminus(\Omega_1\cup\Omega_2)}$, where

$$
\begin{aligned}
\Omega_1 &:= \{(x_1, x_2) \in \Omega : 9(x_1 + 0.5)^2 + 16(x_2 - 0.5)^2 \leq 1\}, \\
\Omega_2 &:= \{(x_1, x_2) \in \Omega : (x_1 - 0.5)^2 + (x_2 + 0.5)^2 \leq 1/16\}.
\end{aligned}
$$

We will use multiple measurements of Cauchy data $(\tilde{f}_\ell, \tilde{g}_\ell), \ell = 1, \cdots, L$, to reconstruct $q^\dagger$. Since $q^\dagger$ is piecewise constant, we take $\mathcal{R}(q) = \int_\Omega |Dq|$, the total variation of q over Ω, as the regularization term. Thus the regularized solution $\tilde{q}_\alpha$ is determined by the minimizer of the functional

$$q \to \sum_{\ell=1}^{L} \int_\Omega q(x)|\nabla(u_D[q, \tilde{f}_\ell] - u_N[q, \tilde{g}_\ell])|^2 dx + \alpha \int_\Omega |Dq|$$

over $\mathcal{Q}$. Due to the nonsmoothness of $\mathcal{R}(q)$, we use an IRLS scheme and the projected Newton method to find an approximation of the regularized solution, i.e. by taking a sequence of small positive numbers $\{\varepsilon_j\}$ and an initial guess q_0, we produce an approximate solution by solving the minimisation problem

$$q_{j+1} \in \arg\min_{q\in\mathcal{Q}} \left\{ \sum_{\ell=1}^{L} \int_\Omega q(x)|\nabla(u_D[q, \tilde{f}_\ell] - u_N[q, \tilde{g}_\ell])|^2 dx + \alpha \int_\Omega \omega_j |\nabla q|^2 \right\} \tag{3.4.13}$$

iteratively, where $\omega_j = (|\nabla q_j|^2 + \varepsilon_j^2)^{-1/2}$. According to the Karush-Kuhn-Tucker theory, for

(3.4.13) we have the KKT conditions

$$\begin{aligned}
&\int_\Omega q\nabla u_\ell\cdot\nabla\phi_\ell=\int_\Omega \rho\phi_\ell,\quad \forall\phi_\ell\in H_0^1(\Omega),\ \ \ell=1,\cdots,L;\\
&\int_\Omega q\nabla v_\ell\cdot\nabla\psi_\ell=\int_{\partial\Omega} g\psi_\ell+\int_\Omega \rho\psi_\ell,\quad \forall\psi_\ell\in H_\diamond^1(\Omega),\ \ \ell=1,\cdots,L;\\
&\int_\Omega\left[(\tilde q-q)\sum_{\ell=1}^L(|\nabla u_\ell|^2-|\nabla v_\ell|^2)+2\alpha\omega_j\nabla q\cdot\nabla(\tilde q-q)\right]\ge 0,\quad \forall\tilde q\in\mathcal{Q}.
\end{aligned}\tag{3.4.14}$$

We may use the projected/semismooth Newton method to solve (3.4.14) which leads to Algorithm 3.

Algorithm 3 Projected Newton algorithm for (3.4.13)

1: Given initial guess q^0 and u_ℓ^0, v_ℓ^0 for $\ell=1,\cdots,L$.
2: **for** $k=0,1,2,...$ **do**
3: Obtain the increments $\xi_\ell^k,\eta_\ell^k,\sigma^k$ of u_ℓ^k,v_ℓ^k,q^k by solving

$$\begin{cases}
\displaystyle\int_\Omega q^k\nabla\xi_\ell\cdot\nabla\phi_\ell+\int_\Omega\sigma\nabla u_\ell^k\cdot\nabla\phi_\ell\\
\displaystyle\qquad=\int_\Omega q^k\nabla u_\ell^k\cdot\nabla\phi_\ell-\rho\phi_\ell,\quad\forall\phi_\ell\in H_0^1(\Omega),\ \ \ell=1,\cdots,L\\
\displaystyle\int_\Omega q^k\nabla\eta_\ell\cdot\nabla\psi_\ell+\int_\Omega\sigma_\ell\nabla v_\ell^k\cdot\nabla\psi_\ell\\
\displaystyle\qquad=\int_\Omega q^k\nabla v_\ell^k\cdot\nabla\psi_\ell-\rho\psi_\ell-\int_{\partial\Omega}\tilde g_\ell\psi_\ell,\quad\forall\psi_\ell\in H_\diamond^1(\Omega),\ \ \ell=1,\cdots,L\\
\displaystyle\int_\Omega 2\mu\sum_{\ell=1}^L(\nabla u_\ell^k\cdot\nabla\xi_\ell-\nabla v_\ell^k\cdot\nabla\eta_\ell)+2\alpha\int_\Omega\omega_j\nabla\sigma\cdot\nabla\mu\\
\displaystyle\qquad=\int_\Omega\mu\sum_{\ell=1}^L(|\nabla u_\ell^k|^2-|\nabla v_\ell^k|^2)+2\alpha\int_\Omega\omega_j\nabla q^k\cdot\nabla\mu,\quad\forall\mu\in\mathcal{Q}
\end{cases}$$

4: Update u_ℓ^k,v_ℓ^k,q^k by $u_\ell^{k+1}=u_\ell^k-\xi_\ell^k$, $v_\ell^{k+1}=v_\ell^k-\eta_\ell^k$ and $q^{k+1}=q^k-\sigma^k$.
5: Projection q^{k+1} onto $\mathcal{Q}$ to get new q^{k+1}.
6: Check stop condition.
7: **end for**

In our numerical simulations we use four groups of Cauchy data $(f_\ell^\dagger,g_\ell^\dagger)$, $\ell=1,\cdots,4$, where

$$\begin{aligned}
g_1^\dagger&=\chi_{(0,1]\times\{-1\}}-\chi_{[-1,0]\times\{1\}}+2\chi_{(0,1]\times\{1\}}-2\chi_{[-1,0]\times\{-1\}}\\
&\quad+3\chi_{\{-1\}\times(-1,0]}-3\chi_{\{1\}\times(0,1)}+4\chi_{\{1\}\times(-1,0]}-4\chi_{\{-1\}\times(0,1)},\\
g_2^\dagger&=x_1+x_2,\qquad g_3^\dagger=x_1-x_2,\qquad g_4^\dagger=x_1^2-x_2^2,
\end{aligned}$$

and each $f_\ell^\dagger$ is produced by solving (3.4.4) with $q=q^\dagger$ and $g=g_\ell^\dagger$ and then taking the trace of the solution on $\partial\Omega$. Now we add random Gaussian noise on $(f_\ell^\dagger,g_\ell^\dagger)$ to produce noisy data

$(\tilde{f}_\ell, \tilde{g}_\ell)$ with relative noise level

$$\delta_{rel} := \max\left\{ \frac{\|\tilde{f}_\ell - f_\ell^\dagger\|_{L^2(\partial\Omega)}}{\|f_\ell^\dagger\|_{L^2(\partial\Omega)}}, \frac{\|\tilde{g}_\ell - g_\ell^\dagger\|_{L^2(\partial\Omega)}}{\|g_\ell^\dagger\|_{L^2(\partial\Omega)}} \right\}$$

for $\ell = 1, \cdots, 4$. We then use $(\tilde{f}_\ell, \tilde{g}_\ell)$, $\ell = 1, \cdots, 4$, to reconstruct $q^\dagger$. In our numerical computation, we take $\varepsilon_j = 0.01$ for all j and solve the variational problems in Algorithm 3 by FreeFem++ [28]; in the definition of $\mathcal{Q}$ we take $\underline{q} = 0.1$ and $\bar{q} = 20$. When using our heuristic Rule 3.1, we use $\Delta_\gamma = \{\alpha_0\gamma^n : n = 0, 1, \cdots\}$ with $\alpha_0 = 0.05$ and $\gamma = 0.6$.

Table 3.3: Relative err with respect to A values and relative noise levels for Example 4.3.

err \ A ; δ_{rel}	0	1e1	1e2	1e3	1e4	1e5	1e6	1e7
1e-2	4.163e-2	4.163e-2	4.163e-2	3.336e-2	2.194e-2	1.389e-2	8.526e-3	3.809e-3
1e-3	1.200e-2	1.200e-2	1.200e-2	1.200e-2	1.031e-2	7.386e-3	4.718e-3	2.247e-3
1e-4	2.015e-3	2.015e-3	2.015e-3	2.015e-3	2.015e-3	2.015e-3	1.761e-3	1.625e-3
1e-5	5.316e-4	5.316e-4	5.316e-4	5.316e-4	5.316e-4	5.316e-4	5.316e-4	5.316e-4
1e-6	3.288e-4	3.288e-4	3.288e-4	3.288e-4	3.288e-4	3.288e-4	3.288e-4	3.288e-4
1e-7	3.278e-4	3.278e-4	3.278e-4	3.278e-4	3.278e-4	3.278e-4	3.278e-4	3.278e-4
1e-8	3.276e-4	3.276e-4	3.276e-4	3.276e-4	3.276e-4	3.276e-4	3.276e-4	3.276e-4

In Table 3.3 we report the relative errors err $:= \|\tilde{q}_{\alpha_*} - q^\dagger\|_{L^2}/\|q^\dagger\|_{L^2}$ of the reconstructed solutions $\tilde{q}_{\alpha_*}$ with respect to various values of A and δ_{rel}. In Figure 3.3 we provide further information by plotting the reconstruction results and the Θ-curve for $A = 0, 10^3$ and 10^6 using a group of noisy data with $\delta_{rel} = 0.1\%$. Table 3.3 and Figure 3.3 demonstrate the same facts concerning Rule 3.1 that have been observed in Example 3.3.3 and Example 3.3.4. In general, Rule 3.1 with $A = 0$ can produce reasonable good reconstruction results although better results can be obtained using suitably large A.

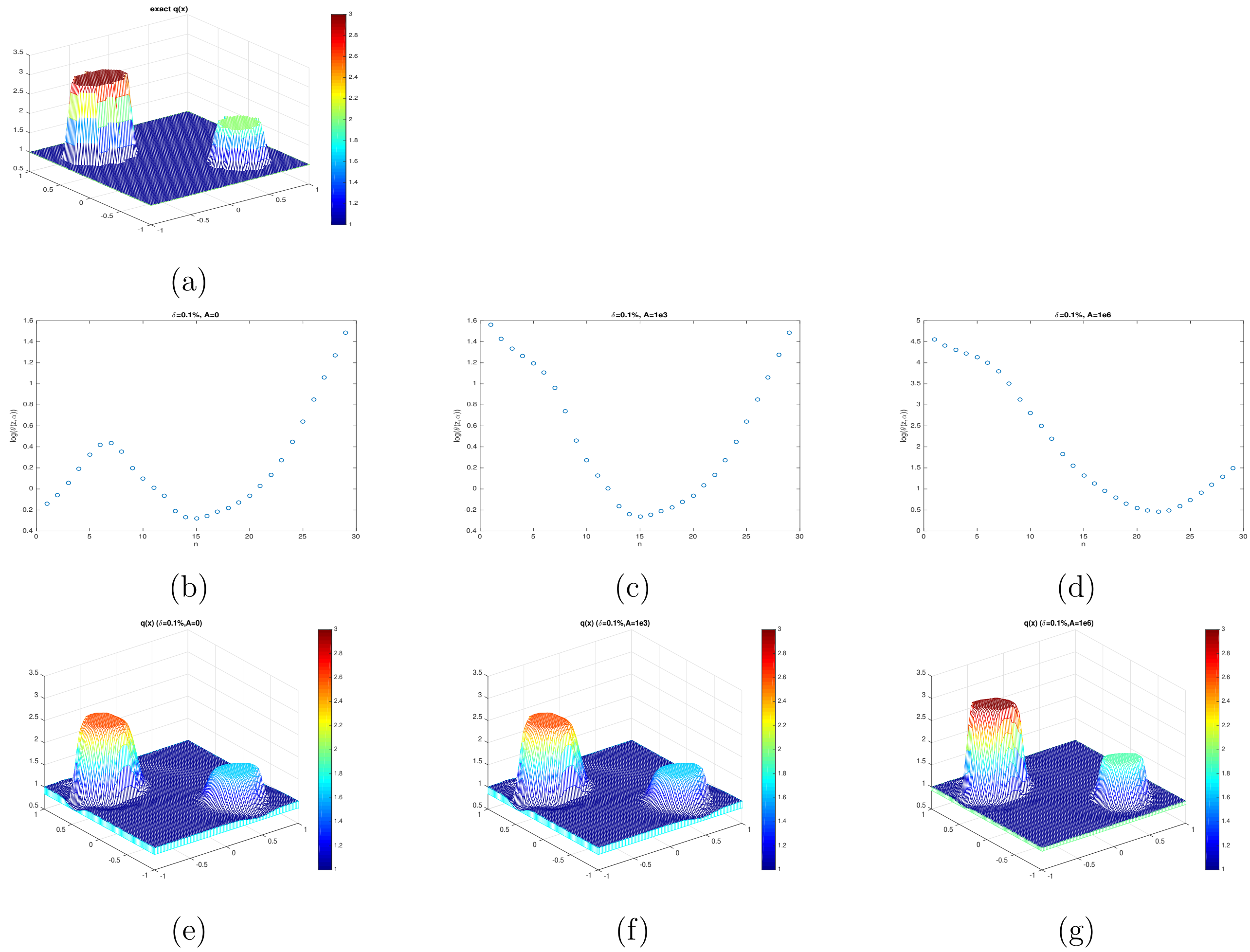

Figure 3.3: The reconstruction for Example 3.4.1 with using noisy data with $\delta_{rel} = 0.1\%$: (a) represents exact parameter; (b), (c) and (d) plot the relationship between $\Theta(\alpha_n, \tilde{u})$ and n for $A = 0$, 10^3 and 10^6 respectively. (e), (f) and (g) plot reconstruction results for $A = 0$, 10^3 and 10^6.

Chapter 4

The Landweber iteration and its variants

Variational regularization requires finding candidate regularized solutions for every candidate value of the regularization parameter $\alpha > 0$. This means for every value of $\alpha > 0$ a full numerical optimization scheme must be run to solve the problem (3.0.4). Multiple values for $\alpha > 0$ may have to be considered before finding the appropriate one, and this means a lot of demanding computational effort.

On the other hand, iterative regularization has been getting attention to solve inverse problems, due to their cheaper computational effort than Tikhonov regularization. These methods start with an initial guess and use certain input/output mechanisms to produce an iterative sequence which is intended to approximate the sought solution. Due to the ill-posedness of the underlying problem, these methods usually show the semi-convergence property, i.e. the iterate becomes close to the sought solution at the beginning, however, after a critical number of iterations, the iterate leaves the sought solution far away as the iteration proceeds. Therefore, the number of iteration plays the role of regularization parameter and the iterative method must be properly terminated according to certain stopping rules in order to render it into a regularization method.

In this chapter we will consider the prominent Landweber iteration which has the advantage of simplicity for implementation and relatively small complexity per iteration. We first give an overview on the development of Landweber iteration, we then prove new convergence results on Landweber iteration in Banach spaces terminated by the discrepancy principle which expand the applied range of the method to cover non-smooth ill-posed inverse problems and to handle the situation that the data is contaminated by various types of noise. We will also consider terminating Landweber iteration by the Hanke-Raus rule which is a heuristic rule using only the noisy data and is independent of noise level. Finally we will consider the Nesterov acceleration strategy on Landweber iteration for linear inverse problems in Hilbert spaces.

4.1 Overview

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We consider ill-posed problems of the form

$$F(x) = y \tag{4.1.1}$$

where $F : \mathcal{D}(F) \subset \mathcal{X} \to \mathcal{Y}$, with domain $\mathcal{D}(F)$, is either a linear or a nonlinear operator. In applications the exact data y is either inaccessible or not precisely available. Instead, a noisy measurement $y^\delta \in \mathcal{Y}$ is available that satisfies

$$\|y^\delta - y\| \le \delta, \tag{4.1.2}$$

where δ is the *noise level*.

When both $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces and only noisy data y^δ is available, instead of (4.1.1) it seems natural to consider the minimization problem

$$\min_{x\in\mathcal{X}} \frac{1}{2}\|F(x) - y^\delta\|^2. \tag{4.1.3}$$

Assume F is Fréchet differentiable with Fréchet derivative $F'(x)$, we may consider to use the gradient method to solve (4.1.3). Note that

$$\nabla\left(\frac{1}{2}\|F(x) - y^\delta\|^2\right) = F'(x)^*(F(x) - y^\delta).$$

The corresponding gradient method thus takes the form

$$x_{n+1}^\delta = x_n^\delta - \mu_n^\delta F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta),$$

where $x_0^\delta := x_0$ is an initial guess and $\mu_n^\delta \ge 0$ is the step-size. Here the superscript δ is used to illustrate the dependence of objects on y^δ. By taking μ_n^δ to be constant, in particular, by taking $\mu_n^\delta \equiv 1$, it leads to the Landweber iteration

$$x_{n+1}^\delta = x_n^\delta - F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta) \tag{4.1.4}$$

which was introduced and thoroughly investigated in the seminar paper [23]. When F is a linear operator, the method (4.1.4) becomes the classical Landweber iteration

$$x_{n+1}^\delta = x_n^\delta - F^*(Fx_n^\delta - y^\delta) \tag{4.1.5}$$

for solving linear ill-posed problems in Hilbert spaces which was first introduced in [61]. For extensive discussion on (4.1.5), one may refer to [16].

In order to render the Landweber iteration (4.1.4) into a regularization method for solving (4.1.1), the paper [23] considered terminating the method by the discrepancy principle, i.e., the iteration (4.1.4) is terminated after $n_* = n_*(\delta, y^\delta)$ steps with

$$\left\|y^\delta - F(x_{n_*}^\delta)\right\| \le \tau\delta < \left\|y^\delta - F(x_n^\delta)\right\|, \quad 0 \le n < n_*, \tag{4.1.6}$$

where τ is an appropriately chosen constant. Essentially, the discrepancy principle tells us to terminate the iteration when the residual is approximately equal the noise level, which means it requires the knowledge of the noisy data and the noise level, hence the term $a-posteriori$. Assuming (4.1.1) has a solution in $B_\rho(x_0) := \{x \in \mathcal{X} : \|x - x_0\| \le \rho\}$, under the scaling condition

$$\|F'(x)\| \le 1, \quad x \in \mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)$$

and the tangential cone condition

$$\|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| \le \eta \|F(x) - F(\tilde{x})\| \tag{4.1.7}$$

for $x, \tilde{x} \in \mathcal{B}_\rho(x_0) \subset \mathcal{D}(F)$ and $\eta < 1/2$, it has been shown in [23] that $x_{n(\delta,y^\delta)}^\delta$ converges to a solution $x^\dagger$ of (4.1.1) as $\delta \to 0$. Furthermore, if in addition $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ for all $x \in \mathcal{B}_\rho(x_0)$, then $x^\dagger$ is an x_0-minimum-norm solution, i.e.

$$F(x^\dagger) = y \quad \text{and} \quad \|x^\dagger - x_0\| = \min\{\|x - x_0\| : F(x) = y\}.$$

To obtain convergence rates, further conditions on F and source conditions on $x^\dagger$ should be imposed. Assume that $F'(x)$ has the decomposition

$$F'(x) = R_x F'(x^\dagger) \quad \text{and} \quad \|R_x - I\| \le C\|x - x^\dagger\|, \quad x \in B_\rho(x_0)$$

for some positive constant C, where $\{R_x : x \in B_\rho(x_0)\}$ is a family of bounded linear operators from $\mathcal{Y}$ to $\mathcal{Y}$. If the source condition

$$x^\dagger - x_0 = \left(F'(x^\dagger)^* F'(x^\dagger)\right)^\nu f \tag{4.1.8}$$

holds for some $0 < \nu \leq 1/2$ and $f \in \mathcal{X}$ with $\|f\|$ sufficiently small, it has been shown in [23] that there holds the order-optimal convergence rate

$$\left\|x_{n_*}^\delta - x^\dagger\right\| = \mathcal{O}\left(\delta^{\frac{2\mu}{2\mu+1}}\right). \tag{4.1.9}$$

One of the basic conditions required in [23] is the differentiability of the forward operator F. In some applications this may not hold. To address this issue, a modified Landweber method of the form

$$x_{n+1}^\delta = x_n^\delta + L(x_n^\delta)^*(y^\delta - F(x_n^\delta)) \tag{4.1.10}$$

was studied in [74]. Here $x \mapsto L(x)$ is a continuous mapping that inherits the scaling and the tangential cone condition imposed originally on the forward operator F. The same convergence results can be obtained under the discrepancy principle, using the same proof in [23]. On the other hand, the authors of [12] went a step further by relaxing the continuity of $x \mapsto L(x)$. Instead they used a compactness condition which can be stated as follows:

Assumption 4.1.1. There exists a Banach space Z such that $\cup_{x\in\mathcal{X}} \mathrm{R}(L(x)^*) \subset Z$ with Z compactly embedded in the Hilbert space $\mathcal{X}$. Moreover, there exists a constant $\hat{C}$ such that $\|L(x)^*\|_{L(\mathcal{Y},Z)} \leq \hat{C}$ for all $x \in \overline{\mathcal{B}}_\rho(x^\dagger)$.

Examples of non-smooth ill-posed problems are given in [12] which demonstrates that Assumption 4.1.1 is satisfied even though the continuity of $x \to L(x)$ does not hold. Therefore, the work in [12] further extended the applied range of Landweber iteration.

There is a natural need to extend the Landweber iteration in Banach spaces. Its classical framework in Hilbert spaces has a tendency to oversmooth solutions. Accordingly, when the sought solution is sparse or piecewise constant, reconstructions under Hilbert spaces may be accurate but may fail to capture the features of the sought solution.

Consider the equation (4.1.1) for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with F being a bounded linear operator between $\mathcal{X}$ and $\mathcal{Y}$. A version of Landweber iteration for linear problems in Banach spaces was developed in [76] using duality mappings. Later on this method was generalized towards nonlinear problems in [77], and is given by

$$\begin{aligned} \xi_{n+1}^{\delta} &= \xi_n^{\delta} - \mu_n^{\delta} F'(x_n^{\delta})^* j_p^{\mathcal{Y}}(F(x_n^{\delta}) - y^{\delta}), \\ x_{n+1}^{\delta} &= J_{q^*}^{\mathcal{X}^*}(\xi_{n+1}^{\delta}) \end{aligned} \tag{4.1.11}$$

where μ_n^{δ} is an appropriately chosen step size. In the first line of (4.1.11), a gradient-like step method is done to get a new approximation in the dual space $\mathcal{X}^*$, and then transported back to the solution space $\mathcal{X}$ by the second line of (4.1.11). In fact, if $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then the duality mappings in (4.1.11) become identity maps, recovering the basic formulation in (4.1.4). To prove convergence of the iterative method (4.1.11), the following assumptions are needed:

Assumption 4.1.2.

(A) The operators F and F' are continuous, and F' is bounded on the set $\overline{\mathcal{B}}_{\rho}(x^{\dagger}) \in \mathcal{D}(F)$.

(B) the operator F satisfies the tangential cone condition (4.1.7) *for some* $0 < \eta < 1$ *sufficiently small;*

(C) The solution space $\mathcal{X}$ is smooth and q-convex

(D) The data space $\mathcal{Y}$ is uniformly smooth.

Assumption 4.1.2(C) allows the use of solution spaces that are smooth but different from L^2-space. It implies that $\mathcal{X}$ is reflexive and the dual $\mathcal{X}^*$ is uniformly smooth. In addition, it also guarantees the existence of a minimum-norm solution, denoted by $x^{\dagger}$. As mentioned in Chapter 2.2, assumption 4.1.2(D) implies that the duality mapping $j_p^{\mathcal{Y}}$ is single-valued and uniformly continuous on bounded sets.

Using Assumption 4.1.2, convergence under the discrepancy principle (4.1.6) was done by extending the proof of the analogous result in [23]. The main difference is the use of Bregman distances, which is more suitable than the usual norms in Banach spaces. As proven in (c.f [77, Sec. 7.2.1]), the following hold for the method (4.1.11) :

1. for τ sufficiently large and a suitable step size μ_n,

$$D_{j_q}(x^{\dagger}, x_{n+1}^{\delta}) \leq D_{j_q}(x^{\dagger}, x_n^{\delta}),$$

   and $x_{n+1}^{\delta} \in \mathcal{D}(F)$ hold for $n \leq n_*$, where $n_* = k_*(\delta, y^{\delta})$ is chosen via (4.1.6).

2. the iterates $x_{n_*}^\delta$ converge to a solution of (4.1.1) as $\delta \to 0$. Using the similar assumption in [23], along with assuming $J_q^{\mathcal{X}}(x_0) \in \overline{R(F'(x^\dagger))}$, then $x_{n_*}^\delta$ converges to $x^\dagger$ as $\delta \to 0$.

More recently, another variant of Landweber method for linear problems in Banach spaces was developed in [5]. Its main difference from the one in [76] is the use of a p-convex penalty functional $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$ with $p > 1$ in the variant, which favours solutions with the sought properties. One advantage in their version is that it can accommodate L^1-penalty and total variation penalty functionals, which are very important in problems dealing with sparse and piecewise constant solutions, respectively. Considering (4.1.1) with F being a bounded linear operator, the algorithm in [5] can be formulated as

$$\begin{aligned} \xi_{n+1}^\delta &= \xi_n^\delta - \mu_n^\delta F^* j_r^{\mathcal{Y}}(F x_n^\delta - y^\delta), \\ x_{n+1}^\delta &= \operatorname{argmin}_{x\in\mathcal{X}} \left\{\mathcal{R}(x) - \left\langle \xi_{n+1}^\delta, x\right\rangle\right\} \end{aligned} \tag{4.1.12}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{X}^*,\mathcal{X}}$ is the usual duality pairing. Let $\mathcal{R}^*$ denote the Legendre-Fenchel conjugate of $\mathcal{R}$. Proposition 2.9 and Definition 2.25 show that for a convex function $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$,

$$x_{n+1}^\delta = \nabla\mathcal{R}^*(\xi_{n+1}^\delta).$$

Hence, the minimization step in (4.1.12) of Algorithm 4 can expressed as a minimization problem in terms of the p-convex function $\mathcal{R}$,

If we set $\mathcal{R}(x) := \frac{1}{q}\left\|x\right\|_{\mathcal{X}}^q$, then $\mathcal{R}^*(x) = \frac{1}{q^*}\left\|x\right\|_{\mathcal{X}^*}^{q^*}$, where q^* is the conjugate of q. Using the gauge function $\varphi^*(t) = t^{q^*-1}$, Asplund's theorem (Theorem 2.2.3) shows that

$$J_{q_*}^{\mathcal{X}^*} = \partial\left(\tfrac{1}{q^*}\left\|\cdot\right\|_{\mathcal{X}^*}^{q^*}\right),$$

where $J_{q_*}^{\mathcal{X}^*} = J_{\varphi^*}^{\mathcal{X}^*}$. Here $J_{q_*}^{\mathcal{X}^*} : \mathcal{X}^* \to 2^{\mathcal{X}}$ is the subdifferential of the convex function $\|x^*\| \mapsto \left\|x^*\right\|_{\mathcal{X}^*}/q^*$. Since $\nabla\mathcal{R}^* \in \partial(\mathcal{R}^*)$, it follows that $x_{k+1}^\delta = J_{q^*}^{\mathcal{X}^*}(\xi_{k+1}^\delta)$. Thus the method (4.1.12) becomes (4.1.11). The convergence analysis of (4.1.12) given in [5] requires the condition that the interior of the domain of $\mathcal{R}$ be non-empty, which may be too restrictive and thus exclude the use of TV-like functionals. This restrictive condition was eventually removed in [48] in which the method (4.1.12) was even further extended for solving nonlinear ill-posed problems in Banach spaces by the following iterative procedure

$$\begin{aligned} \xi_{n+1}^\delta &= \xi_n^\delta - \mu_n^\delta F'(x_n^\delta)^* j_r^{\mathcal{Y}}(F(x_n^\delta) - y^\delta), \\ x_{n+1}^\delta &= \operatorname{argmin}_{x\in\mathcal{X}} \left\{\mathcal{R}(x) - \left\langle \xi_{n+1}^\delta, x\right\rangle\right\}, \end{aligned} \tag{4.1.13}$$

for which the step size μ_n^δ of the both forms

$$\mu_n^\delta := \mu_0 \left\|F(x_n^\delta) - y^\delta\right\|^{p-r} \quad \text{or} \quad \mu_n^\delta := \min\left\{\frac{\mu_0 \left\|F(x_n^\delta) - y^\delta\right\|^{p(r-1)}}{\|F'(x_n^\delta)^* j_r^{\mathcal{Y}}\|}, \mu_1\right\} \left\|F(x_n^\delta) - y^\delta\right\| \tag{4.1.14}$$

are considered, see also [42], where μ_0 and μ_1 are suitably chosen positive constants. The method is then terminated by the discrepancy principle (4.1.6). The analysis given in [48] on the method defined by (4.1.13), (4.1.14) and (4.1.6) requires the following standard assumptions concerning the regularization functional $\mathcal{R}$ and the forward operator F.

Assumption 4.1.3. $\mathcal{R} : \mathcal{X} \to (-\infty, \infty]$ is proper, lower semi-continuous and p-convex with $p > 1$ in the sense that there is a constant $c_0 > 0$ such that

$$\mathcal{R}(t\bar{x} + (1-t)x) + c_0 t(1-t)\|\bar{x} - x\|^p \leq t\mathcal{R}(\bar{x}) + (1-t)\mathcal{R}(x)$$

for all $\bar{x}, x \in \mathcal{X}$ and $0 \leq t \leq 1$.

Assumption 4.1.4. *(a) F is weakly closed over $\mathcal{D}(F)$.*

(b) There exist $\rho > 0$, $x_0 \in \mathcal{X}$ and $\xi_0 \in \partial\mathcal{R}(x_0)$ such that $B_{3\rho}(x_0) \subset \mathcal{D}(F)$ and (4.1.1) has a solution x^* satisfying $D_{\xi_0}\mathcal{R}(x^*, x_0) \leq c_0\rho^p$.

(c) There is a family of bounded linear operators $\{L(x) : \mathcal{X} \to \mathcal{Y}\}_{x \in B_{3\rho}(x_0)}$ such that $\|L(x)\| \leq B$ for all $x \in B_{3\rho}(x_0)$ for some constant B, and there is a constant $0 \leq \eta < 1$ such that

$$\|F(\bar{x}) - F(x) - L(x)(\bar{x} - x)\| \leq \eta\|F(\bar{x}) - F(x)\|$$

for all $\bar{x}, x \in B_{3\rho}(x_0)$.

Assumption 4.1.4 (a) means that for any sequence $\{x_n\} \in \mathcal{D}(F)$ satisfying $x_n \rightharpoonup x \in \mathcal{X}$ and $F(x_n) \to y \in \mathcal{Y}$ there hold $x \in \mathcal{D}(F)$ and $F(x) = y$ (here we denote weak convergence and strong convergence by "$\rightharpoonup$" and "$\to$" respectively). Assumption 4.1.4 (c) is a version of the tangential cone condition, widely used in convergence analysis of iterative regularization methods for nonlinear inverse problems [23]. Notice that it is expressed in terms of a bounded linear operator $L(x)$ which does not have to be the Fréchet derivative of F. It is important to note Assumption 4.1.4 (c) implies that

$$\frac{1}{1+\eta}\|L(x)(\bar{x} - x)\| \leq \|F(\bar{x}) - F(x)\| \leq \frac{1}{1-\eta}\|L(x)(\bar{x} - x)\|. \tag{4.1.15}$$

which shows that F is continuous over $B_{3\rho}(x_0)$.

By picking $x_0 \in \mathcal{D}(\partial\mathcal{R})$ and $\xi_0 \in \partial\mathcal{R}(x_0)$ as initial guesses, which may incorporate some available information on the sought solution, we define $x^\dagger$ to be a solution of (4.1.1) with the property

$$D_{\xi_0}\mathcal{R}(x^\dagger, x_0) := \min_{x\in\mathcal{D}(F)} \{D_{\xi_0}\mathcal{R}(x, x_0) \,:\, F(x) = y\}. \tag{4.1.16}$$

When $\mathcal{X}$ is a reflexive Banach space, by using the p-convexity and the weakly lower semi-continuity of $\mathcal{R}$ together with the weakly closedness of F, $x^\dagger$ exists and is uniquely defined [48, Lemma 3.2]. Moreover, assumption 4.1.4 (b) implies that

$$D_{\xi_0}\mathcal{R}(x^\dagger, x_0) \le c_0\rho^p,$$

which, together with assumption 4.1.3, implies that $\left\|x_0 - x^\dagger\right\| \le \rho$.

In order to show that the method defined by (4.1.13), (4.1.14) and (4.1.6) is a regularization method for solving (4.1.1), besides the standard conditions specified in Assumption 4.1.3 and Assumption 4.1.4, the convergence analysis in [48] requires the following additional conditions.

Assumption 4.1.5. *(a) The Banach space $\mathcal{Y}$ is uniformly smooth and $1 < r < \infty$;*

(b) The mapping $x \to L(x)$ from $\mathcal{X}$ to $\mathscr{L}(\mathcal{X}, \mathcal{Y})$ is continuous.

Here a Banach space $\mathcal{Y}$ is called *uniformly smooth* in the sense that its modulus of smoothness

$$\rho_{\mathcal{Y}}(t) := \sup\{\|\bar{y} + y\| + \|\bar{y} - y\| - 2 : \|\bar{y}\| = 1 \text{ and } \|y\| \le t\}$$

satisfies $\lim_{t\searrow 0} \rho_{\mathcal{Y}}(t)/t = 0$. The uniform smoothness of $\mathcal{Y}$ in Assumption 4.1.5 (a) guarantees that the duality mapping $J_r^{\mathcal{Y}}$, for each $1 < r < \infty$, is single-valued and uniformly continuous on bounded sets ([10]). Therefore, Assumption 4.1.5 is crucial for establishing the stability of Landweber iteration in [48]. The main result in [48] can be summarized as follows, where, for a bounded linear operator $A : \mathcal{X} \to \mathcal{Y}$, we use $\mathcal{N}(A)$ to denote its null space and we also use

$$\mathcal{N}(A)^\perp := \{\xi \in \mathcal{X}^* : \langle \xi, x\rangle_{\mathcal{X}^*,\mathcal{X}} = 0 \text{ for all } x \in \mathcal{N}(A)\}$$

to denote the annihilator of $\mathcal{N}(A)$.

Theorem 4.1.1 ([48])**.** Let Assumption 4.1.3 and Assumption 4.1.4 hold. Let $\tau > 1$ and

$\mu_0 > 0$ be chosen such that

$$1 - \eta - \frac{1+\eta}{\tau} - \frac{1}{p^*}\left(\frac{\mu_0}{2c_0}\right)^{p^*-1} > 0. \tag{4.1.17}$$

Consider the Landweber iteration (4.1.13) and (4.1.14) with $1 \le r < \infty$. Then the discrepancy principle (4.1.6) determines a finite integer n_δ and $x_n^\delta \in B_{3\rho}(x_0)$ for all $0 \le n \le n_\delta$. Moreover

$$D_{\xi_{n+1}^\delta}\mathcal{R}(\hat{x}, x_{n+1}^\delta) \le D_{\xi_n^\delta}\mathcal{R}(\hat{x}, x_n^\delta) \tag{4.1.18}$$

for all $0 \le n < n_\delta$, where $\hat{x}$ denotes any solution of (4.1.1) in $B_{3\rho}(x_0)$.

If Assumption 4.1.5 is also satisfied and $1 < r < \infty$, then there exists a solution x^* of (4.1.1) such that

$$\|x_{n_\delta}^\delta - x^*\|_X \to 0 \quad \text{and} \quad D_{\xi_{n_\delta}^\delta}\mathcal{R}(x^*, x_{n_\delta}^\delta) \to 0$$

as $\delta \to 0$. If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in B_{3\rho}(x_0)$, then $x^* = x^\dagger$.

The convergence result in Theorem 4.1.1 depends heavily on the uniform smoothness of $\mathcal{Y}$ and the continuity of the mapping $x \in \mathcal{X} \to L(x) \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$. There are situations where the noisy data is contaminated by non Gaussian noise such as the impulsive noise or the uniform distributed noise; in such situations one may choose $\mathcal{Y}$ to be the L^1-space or the L^∞-space to effectively remove the effects of noise ([2, 11]). Note that both the L^1-space and the L^∞-space are not uniformly smooth. On the other hand, there exist ill-posed inverse problems where the forward operator F is non-smooth, i.e. not necessarily Gâteaux differentiable ([12]). For such non-smooth ill-posed problems, one needs to choose $L(x)$ carefully as the replacement of the Gâteaux derivative; for instance, one may choose $L(x)$ to be a Bouligand subderivative of F at x which is defined as a limit of Fréchet derivative of F in differentiable points. The Bouligand subderivative mapping in general is not continuous unless the forward operator is Gâteaux differentiable. In this chapter we will revisit the discrepancy principle for Landweber iteration and provide a new convergence result that requires neither the reflexivity of $\mathcal{Y}$ nor the continuity of the mapping $x \to L(x)$.

We note that the performance of the discrepancy principle depends heavily on the accurate knowledge of the noise level. Unfortunately, such noise level information is not always available or reliable in real world applications. Overestimation or underestimation on noise level may lead to significant loss of reconstruction accuracy when using the discrepancy principle. Therefore, it is necessary to develop heuristic rules for Landweber iteration that do not use any knowledge of the noise level. Based on the discrepancy principle, we will propose a heuristic rule for

Landweber iteration in the spirit of Hanke and Raus ([24]). Our heuristic rule determines an integer $n_* := n_*(y^\delta)$ by minimizing

$$\Theta(n, y^\delta) := (n+a)\|F(x_n^\delta) - y^\delta\|^p$$

over the iteration, where $a \geq 1$ is a fixed number. Although the Bakushinskii's veto ([3]) states that heuristic rules can not lead to convergence in the sense of worst case scenario for any regularization methods, we are able to prove some convergence results on our heuristic rule for Landweber iteration under certain conditions on the noisy data.

4.2 Landweber iteration in Banach spaces

In this section we will consider the method defined by (4.1.13) and (4.1.14) for solving the nonlinear ill-posed equation (4.1.1) using a p-convex regularization function $\mathcal{R}$. We will assume that $\mathcal{R}$ and F satisfy Assumption 4.1.3 and Assumption 4.1.4. However we will not assume $\mathcal{Y}$ to be reflexive. The method can be reformulated as the following algorithm.

Algorithm 4 Landweber method with convex penalty functional

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1: Given initial guess $\xi_0 \in \mathcal{X}^*$, positive constants μ_0, μ_1; parameters $p > 1, r \geq 1$.
2: Set $x_0 := \text{argmin}_{x\in\mathcal{X}} \{\mathcal{R}(x) - \langle \xi_0, x\rangle\}$.
3: Let $\xi_0^\delta := \xi_0$ and $x_0^\delta := x_0$.
4: **for** $n = 0, 1, \ldots$ **do**
5: Calculate the step size μ_n^δ by

$$\mu_n^\delta := \min\left\{\frac{\mu_0 \left\|F(x_n^\delta) - y^\delta\right\|^{p(r-1)}}{\|L(x_n^\delta)^* j_r^{\mathcal{Y}}\left(F(x_n^\delta) - y^\delta\right)\|^p}, \mu_1\right\} \left\|F(x_n^\delta) - y^\delta\right\|^{p-r} \tag{4.2.1}$$

6: Obtain

$$\begin{aligned} \xi_{n+1}^\delta &= \xi_n^\delta - \mu_n^\delta L(x_n^\delta)^* j_r^{\mathcal{Y}}\left(F(x_n^\delta) - y^\delta\right), \\ x_{n+1}^\delta &= \nabla\mathcal{R}^*(\xi_{n+1}^\delta) = \arg\min_{x\in\mathcal{X}} \left\{\mathcal{R}(x) - \langle \xi_{n+1}^\delta, x\rangle\right\}. \end{aligned} \tag{4.2.2}$$

7: Check parameter choice rule for accepting x_{n+1}^δ as approximate solution to (4.1.1).
8: **end for**

In order to make the method more transparent, it is necessary to give more explanation on the mapping $j_r^{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}^*$ for $1 \leq r < \infty$. We use $J_r^{\mathcal{Y}} : \mathcal{Y} \to 2^{\mathcal{Y}^*}$ to denote the subdifferential of the convex functional $y \to \|y\|^r/r$ over $\mathcal{Y}$. By the Asplund's theorem (see Theorem 2.2.3), $J_r^{\mathcal{Y}}$ with $1 < r < \infty$ is exactly the duality mapping on $\mathcal{Y}$ with the gauge function $t \to t^{r-1}/r$, i.e.

$$J_r^{\mathcal{Y}}(y) = \{y^* \in \mathcal{Y}^* : \|y^*\| = \|y\|^{r-1} \text{ and } \langle y^*, y\rangle = \|y\|^r\}. \tag{4.2.3}$$

While for $r = 1$, one has

$$J_1(y) = \begin{cases} \{y^* \in \mathcal{Y}^* : \|y^*\| = 1 \text{ and } \langle y^*, y\rangle = \|y\|\} & \text{if } y \neq 0, \\ \{y^* \in \mathcal{Y}^* : \|y^*\| \leq 1\} & \text{if } y = 0. \end{cases}$$

Note that $J_r^{\mathcal{Y}}$ in general is multi-valued. The mapping $j_r^{\mathcal{Y}}$ in (4.2.2) denotes a selection of $J_r^{\mathcal{Y}}$ which is defined to be a single-valued mapping from $\mathcal{Y}$ to Y^* with the property $j_r^Y(y) \in J_r^Y(y)$ for each $y \in Y$. It is easily seen that

$$\langle j_r^Y(y), y\rangle = \|y\|^r \quad \text{ and } \quad \|j_r^Y(y)\| \leq \|y\|^{r-1} \tag{4.2.4}$$

for all $y \in Y$ and $1 \leq r < \infty$. Here we adopt the convention $0^0 = 1$.

Concerning the parameter choice rule in Algorithm 4, we will consider the discrepancy principle (4.1.6) and the heuristic rule in the spirit of Hanke and Raus. We will provide new convergence results for the corresponding methods.

4.2.1 Convergence under the discrepancy principle

In this subsection we will provide a new convergence result on Landweber iteration, i.e. Algorithm 4, under the discrepancy principle

$$\left\|F(x_{n_\delta}^\delta) - y^\delta\right\| \leq \tau\delta < \left\|F(x_n^\delta) - y^\delta\right\|, \quad 0 \leq n < n_\delta \tag{4.2.5}$$

with $\tau > 1$. The new result requires neither the Gâteaux differentiability of the forward operator nor the reflexivity of the image space. Therefore, we expand the applied range of the discrepancy principle for Landweber iteration to cover non-smooth ill-posed inverse problems and to handle the situation that the data is contaminated by various types of noise.

The material of this subsection is taken from our recent paper [72]. We will not use Assumption 4.1.5. Instead we will the following compactness condition as a replacement for Assumption 4.1.5.

Assumption 4.2.1. *There exists a Banach space Z such that*

$$\bigcup_{x \in B_{3\rho}(x_0)} \mathit{Range}(L(x)^*) \subset Z$$

and Z is compactly embedded into $\mathcal{X}^*$. Moreover, there is a constant $\hat{B}$ such that $\|L(x)^*\|_{\mathscr{L}(Y^*,Z)} \leq \hat{B}$ for all $x \in B_{3\rho}(x_0)$.

Assumption 4.2.1 places a condition on the smoothing properties of $L(x)^*$ for $x \in B_{3\rho}(x_0)$ in a uniform sense. This assumption is inspired by the work [12] and is an adaptation of Assumption 4.1.1 in Hilbert spaces to Banach space setting.

In order to show the convergence of $x_{n_\delta}^\delta$ to a solution of (4.1.1), it is necessary to investigate for each n the behaviour of $(\xi_n^\delta, x_n^\delta)$ as $\delta \to 0$ which is an important step. Since the mapping $x \to L(x)$ is not assumed to be continuous, and because the mapping $y \in \mathcal{Y} \to j_r^{\mathcal{Y}}(y) \in \mathcal{Y}^*$ is no longer continuous, one may not expect the convergence of $(\xi_n^\delta, x_n^\delta)$ to a single point as $\delta \to 0$. For each n, $(\xi_n^\delta, x_n^\delta)$ may have many limit points as $\delta \to 0$. By picking any one of the limit points for each n, we may form an iterative sequence $\{(\xi_n, x_n)\}$ in $\mathcal{X}^* \times \mathcal{X}$ using Algorithm 4 corresponding to the noise-free (i.e. exact) case. It turns out that all these iterative sequences obey certain properties which will be specified in the following. We will use $\Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ to denote the set of all possible sequences $\{(\xi_n, x_n)\}$ in $\mathcal{X}^* \times \mathcal{X}$ constructed from (ξ_0, x_0) by

$$\xi_{n+1} = \xi_n - \mu_n \psi_n, \qquad \text{and} \qquad x_{n+1} = \arg\min_{x \in \mathcal{X}} \left\{ \mathcal{R}(x) - \langle \xi_{n+1}, x \rangle \right\}, \tag{4.2.6}$$

where

$$\mu_n = \begin{cases} \min\left\{ \frac{\mu_0 \|F(x_n)-y\|^{p(r-1)}}{\|\psi_n\|^p}, \mu_1 \right\} \|F(x_n) - y\|^{p-r} & \text{if } F(x_n) \neq y \\ 0 & \text{if } F(x_n) = y \end{cases} \tag{4.2.7}$$

and $\psi_n \in \mathcal{X}^*$ satisfies the properties

$$\langle \psi_n, \hat{x} - x_n \rangle \leq -(1-\eta) \|F(x_n) - y\|^r, \tag{4.2.8}$$

$$|\langle \psi_n, \hat{x} - x \rangle| \leq (1+\eta) \|F(x_n) - y\|^{r-1} \left(2 \|F(x_n) - y\| + \|F(x) - y\|\right), \tag{4.2.9}$$

for any integer n, any $x \in B_{3\rho}(x_0)$ and any solution $\hat{x}$ of (4.1.1) in $B_{3\rho}(x_0)$. We note that (4.2.8) and (4.2.9) hold when $\psi_n = L(x_n)^* J_r^{\mathcal{Y}}(F(x_n) - y)$, ensuring the well-definedness of (4.2.6) and (4.2.7). Indeed, for this choice of ψ_n, one may use Assumption 4.1.3, Assumption 4.1.4 and the argument in [48] to show that $x_n \in B_{3\rho}(x_0)$ for all $n \geq 0$. Furthermore, by the property of $j_r^{\mathcal{Y}}$

and the tangential cone condition in Assumption 4.1.4 (c), we have

$$
\begin{aligned}
\langle \psi_n, \hat{x} - x_n \rangle &= \left\langle J_r^{\mathcal{Y}}(F(x_n) - y), L(x_n)(\hat{x} - x_n) \right\rangle \\
&= \left\langle J_r^{\mathcal{Y}}(F(x_n) - y), y - F(x_n) \right\rangle \\
&\quad - \left\langle J_r^{\mathcal{Y}}(F(x_n) - y), y - F(x_n) - L(x_n)(\hat{x} - x_n) \right\rangle \\
&\le - \|F(x_n) - y\|^r + \|F(x_n) - y\|^{r-1} \|y - F(x_n) - L(x_n)(\hat{x} - x_n)\| \\
&\le - \|F(x_n) - y\|^r + \eta \|F(x_n) - y\|^r \\
&= -(1 - \eta) \|F(x_n) - y\|^{r-1}
\end{aligned}
$$

and

$$
\begin{aligned}
|\langle \psi_n, \hat{x} - x \rangle| &= \left| \left\langle J_r^{\mathcal{Y}}(F(x_n) - y), L(x_n)(\hat{x} - x) \right\rangle \right| \\
&\le \|F(x_n) - y\|^{r-1} \left(\|L(x_n)(\hat{x} - x_n)\| + \|L(x_n)(x_n - x)\| \right) \\
&\le (1 + \eta) \|F(x_n) - y\|^{r-1} \left(\|F(x_n) - y\| + \|F(x_n) - F(x)\| \right) \\
&\le (1 + \eta) \|F(x_n) - y\|^{r-1} \left(2 \|F(x_n) - y\| + \|F(x) - y\| \right)
\end{aligned}
$$

for all integer $n \ge 0$, $x \in B_{3\rho}(x_0)$ and any solution $\hat{x}$ of (4.1.1) in $B_{3\rho}(x_0)$. This verifies (4.2.8) and (4.2.9). Hence the set $\Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ is non-empty.

By using Assumption 4.2.1, we can prove the following stability result.

Lemma 4.2.1. *Let Assumption 4.1.3, Assumption 4.1.4, and Assumption 4.2.1 hold. Let $\tau > 1$ and $\mu_0 > 0$ be chosen such that (4.1.17) holds. Let $\{y^\delta\}$ be a sequence of noisy data satisfying $\|y^{\delta_l} - y\| \le \delta_l$ with $\delta_l \to 0$ as $l \to \infty$, and let n_{δ_l} be the integer determined by the discrepancy principle* (4.2.5). *Then, by taking a subsequence if necessary, there is a sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ such that*

$$
x_n^{\delta_l} \to x_n \quad \textit{and} \quad \xi_n^{\delta_l} \to \xi_n \quad \textit{as } l \to \infty
$$

for all $0 \le n \le \liminf_{l\to\infty} n_{\delta_l}$. If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in B_{3\rho}(x_0)$, then there also holds $\xi_{n+1} - \xi_n \in \mathcal{N}(L(x^\dagger))^\perp$ for all n.

Proof. Let $N := \liminf_{l\to\infty} n_{\delta_l}$. Note that $\xi_0^{\delta_l} = \xi_0$ and $x_0^{\delta_l} = x_0$. Therefore, by the diagonal sequence argument, it suffices to show that if $(\xi_n, x_n) \in \mathcal{X}^* \times \mathcal{X}$ is constructed for some $n < N$ with $\xi_n^{\delta_l} \to \xi_n$ and $x_n^{\delta_l} \to x_n$ as $l \to \infty$, then, by taking a subsequence of $\{y^{\delta_l}\}$ if necessary, we can construct $(\xi_{n+1}, x_{n+1}) \in \mathcal{X}^* \times \mathcal{X}$ such that $\xi_{n+1}^{\delta_l} \to \xi_{n+1}$ and $x_{n+1}^{\delta_l} \to x_{n+1}$ as $l \to \infty$.

For the case $N < \infty$, one may use the Landweber iteration with exact data, i.e. (4.2.6) with $\psi_n = L(x_n)^* j_r^{\mathcal{Y}}(F(x_n) - y)$ to produce a whole sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ once $\{(\xi_n, x_n) : 0 \le n \le N\}$ are constructed.

To construct (ξ_{n+1}, x_{n+1}), we consider two cases. For the case $F(x_n) = y$, we have from the fact $\mu_n^{\delta_l} \le \mu_1$, the property (4.2.4) of $j_r^{\mathcal{Y}}$, and Assumption 4.1.4 (c) that

$$\begin{aligned}\mu_n^{\delta_l} \|L(x_n^{\delta_l})^* j_r^{\mathcal{Y}}(F(x_n^{\delta_l}) - y^{\delta_l})\| &\le \mu_1 B \|F(x_n^{\delta_l}) - y^{\delta_l}\|^{p-1} \\ &\to \mu_1 B \|F(x_n) - y\|^{p-1} = 0\end{aligned}$$

as $l \to \infty$. By defining ξ_{n+1} and x_{n+1} by (4.2.6) with $\psi_n = L(x_n)^* j_r^{\mathcal{Y}}(F(x_n) - y)$ and $\mu_n = 0$, it then follows from the continuity of $\nabla \mathcal{R}^*$ that $\xi_{n+1}^{\delta_l} \to \xi_{n+1}$ and $x_{n+1}^{\delta_l} \to x_{n+1}$ as $l \to \infty$. We next consider the case $F(x_n) \ne y$. By the definition of $\xi_{n+1}^{\delta_l}$ we can write

$$\xi_{n+1}^{\delta_l} = \xi_n^{\delta_l} - \mu_n^{\delta_l} \psi_n^{\delta_l}, \quad \psi_n^{\delta_l} := L(x_n^{\delta_l})^* j_r^{\mathcal{Y}}(F(x_n^{\delta_l}) - y^{\delta_l}). \tag{4.2.10}$$

By Assumption 3.2.4 and the property (4.2.4) of the mapping $j_r^{\mathcal{Y}}$ we have

$$\|\psi_n^{\delta_l}\|_Z \le \hat{B} \|j_r^{\mathcal{Y}}(F(x_n^{\delta_l}) - y^{\delta_l})\| \le \hat{B} \|F(x_n^{\delta_l}) - y^{\delta_l}\|^{r-1}.$$

Using the continuity of F and the facts that $x_n^{\delta_l} \to x_n$ and $y^{\delta_l} \to y$ as $l \to \infty$, we can see that $\{\psi_n^{\delta_l}\}_l$ is bounded in Z. Since Z can be embedded into $\mathcal{X}^*$ compactly, by taking a subsequence of $\{y^{\delta_l}\}$ if necessary, we may assume that $\psi_n^{\delta_l} \to \psi_n$ in $\mathcal{X}^*$ as $l \to \infty$ for some $\psi_n \in \mathcal{X}^*$. Now we define μ_n and (ξ_{n+1}, x_{n+1}) by (4.2.7) and (4.2.6) respectively. It then follows from the continuity of F and $\nabla \mathcal{R}^*$ that $\mu_n^{\delta_l} \to \mu_n$, $\xi_{n+1}^{\delta_l} \to \xi_{n+1}$ and $x_{n+1}^{\delta_l} \to x_{n+1}$ as $l \to \infty$. It remains to show that ψ_n satisfies (4.2.8) and (4.2.8). Let $\hat{x}$ be any solution of (4.1.1) in $B_{3\rho}(x_0)$. We have

$$\begin{aligned}\langle \psi_n, \hat{x} - x_n \rangle &= \lim_{l \to \infty} \langle \psi_n^{\delta_l}, \hat{x} - x_n^{\delta_l} \rangle \\ &= \lim_{l \to \infty} \langle j_r^{\mathcal{Y}}(F(x_n^{\delta_l}) - y^{\delta_l}), L(x_n^{\delta_l})(\hat{x} - x_n^{\delta_l}) \rangle.\end{aligned}$$

By using the property (4.2.4) of $j_r^{\mathcal{Y}}$ and the tangential cone condition in Assumption 4.1.4 (c), we have

$$\begin{aligned}&\langle j_r^{\mathcal{Y}}(F(x_n^{\delta_l}) - y^{\delta_l}), L(x_n^{\delta_l})(\hat{x} - x_n^{\delta_l}) \rangle \\ &\le -\|F(x_n^{\delta_l}) - y^{\delta_l}\|^r + \|F(x_n^{\delta_l}) - y^{\delta_l}\|^{r-1} \big(\delta_l + \eta \|F(x_n^{\delta_l}) - y\|\big).\end{aligned}$$

Therefore, by the continuity of F, we have

$$
\begin{aligned}
&\langle \psi_n, \hat{x} - x_n\rangle \\
&\leq \lim_{l\to\infty} \left(-\|F(x_n^{\delta_l}) - y^{\delta_l}\|^r + \|F(x_n^{\delta_l}) - y^{\delta_l}\|^{r-1}\left(\|\delta_l + \eta\|F(x_n^{\delta_l}) - y\|\right)\right) \\
&= -(1-\eta)\|F(x_n) - y\|^r
\end{aligned}
$$

which shows (4.2.8). By using again the property (4.2.4) of $j_r^{\mathcal{Y}}$, the tangential cone condition, and the continuity of F, we also have

$$
\begin{aligned}
\langle \psi_n, \hat{x} - x\rangle &= \lim_{l\to\infty} \langle \psi_n^{\delta_l}, \hat{x} - x\rangle = \lim_{l\to\infty} \langle j_r^{\mathcal{Y}}(F(x_n^{\delta_l}) - y^{\delta_l}), L(x_n^{\delta_l})(\hat{x} - x)\rangle \\
&\leq \limsup_{l\to\infty} \|F(x_n^{\delta_l}) - y^{\delta_l}\|^{r-1}\left(\|L(x_n^{\delta_l})(\hat{x} - x_n^{\delta_l})\| + \|L(x_n^{\delta_l}))(x_n^{\delta_l} - x)\|\right) \\
&\leq \limsup_{l\to\infty} (1+\eta)\|F(x_n^{\delta_l}) - y^{\delta_l}\|^{r-1}\left(\|F(x_n^{\delta_l}) - y\| + \|F(x_n^{\delta_l}) - F(x)\|\right) \\
&= (1+\eta)\|F(x_n) - y\|^{r-1}(\|F(x_n) - y\| + \|F(x_n) - F(x)\|) \\
&\leq (1+\eta)\|F(x_n) - y\|^{r-1}(2\|F(x_n) - y\|_{\mathcal{Y}} + \|F(x) - y\|)
\end{aligned}
$$

for any $x \in B_{3\rho}(x_0)$ which therefore shows (4.2.9).

Note that $\xi_{n+1}^{\delta_l} - \xi_n^{\delta_l} \in \text{Range}(L(x_n^{\delta_l})^*) \subset \mathcal{N}(L(x_n^{\delta_l}))^{\perp}$. If $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in B_{3\rho}(x_0)$, we may use $x_n^{\delta_l} \in B_{3\rho}(x_0)$ to conclude that $\xi_{n+1}^{\delta_l} - \xi_n^{\delta_l} \in \mathcal{N}(L(x^\dagger))^{\perp}$. By taking $l \to \infty$ we then obtain $\xi_{n+1} - \xi_n \in \mathcal{N}(L(x^\dagger))^{\perp}$. □

Next we show that for every sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$, $\{x_n\}$ converges to a solution of (4.1.1) as $n \to \infty$. The following general convergence result plays a crucial role.

Proposition 4.1. *Consider the equation* (4.1.1) *for which Assumption 4.1.4 holds. Let $\mathcal{R} : \mathcal{X} \to (-\infty, +\infty]$ be a proper, lower semi-continuous and uniformly convex function. Let $\{x_n\} \in B_{3\rho}(x_0)$ and $\{\xi_n\} \in \mathcal{X}^*$ be such that*

(i) $\xi_n \in \partial\mathcal{R}(x_n)$ *for all n;*

(ii) *for any solution $\hat{x}$ of* (4.1.1) *in $B_{3\rho}(x_0) \cap \mathcal{D}(\mathcal{R})$ the sequence $\{D_{\xi_n}\mathcal{R}(\hat{x}, x_n)\}$ is monotonically decreasing;*

(iii) $\lim_{n\to\infty} \|F(x_n) - y\| = 0$;

(iv) *there is a subsequence $\{n_k\}$ with $n_k \to \infty$ such that for all $l < k$ and any solution $\hat{x}$ of*

(4.1.1) in $B_{3\rho}(x_0)$ there holds

$$|\langle \xi_{n_k} - \xi_{n_l}, x_{n_k} - \hat{x} \rangle| \le C \left(D_{\xi_{n_l}} \mathcal{R}(\hat{x}, x_{n_l}) - D_{\xi_{n_k}} \mathcal{R}(\hat{x}, x_{n_k}) \right)$$

for some constant C.

Then there exists a solution x_* of (4.1.1) in $B_{3\rho}(x_0) \cap \mathcal{D}(\mathcal{R})$ such that

$$\lim_{n\to\infty} D_{\xi_n} \mathcal{R}(x_*, x_n) = 0.$$

If, in addition, $\xi_{n+1} - \xi_n \in \mathcal{N}(L(x^\dagger))^\perp$ for all n, then $x_* = x^\dagger$.

Proof. This is [48, Proposition 3.6]. □

Now we prove some key result regarding the noise-free algorithm (4.2.6).

Lemma 4.2.2. *Let Assumption 4.1.3 and Assumption 4.1.4 hold. Let $\mu_0 > 0$ be chosen such that*

$$c_1 := 1 - \eta - \frac{1}{p^*} \left(\frac{\mu_0}{2c_0} \right)^{p^*-1} > 0. \tag{4.2.11}$$

The for any sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ there exists a solution x_* of (4.1.1) such that $D_{\xi_n} \mathcal{R}(x_*, x_n) \to 0$ as $n \to \infty$. If, in addition, $\xi_{n+1} - \xi_n \in \mathcal{N}(L(x^\dagger))^\perp$ for all n, then $x_* = x^\dagger$.

Proof. We will carry out the proof by checking each of the conditions in Proposition 4.1. By the definition of x_n we immediately have $\xi_n \in \partial \mathcal{R}(x_n)$, so item (i) holds. In order to show item (ii), we will use a similar argument in the proof of [48, Lemma 3.4]. Suppose $x_n \in B_{3\rho}(x_0)$ for some $n \ge 0$. Let $\hat{x}$ be any solution of (4.1.1) in $B_{3\rho}(x_0)$. Using Propositions 2.7 and 2.9 and Definition 2.26, we have

$$\begin{aligned} &D_{\xi_{n+1}} \mathcal{R}(\hat{x}, x_{n+1}) - D_{\xi_n} \mathcal{R}(\hat{x}, x_n) \\ &\quad = \mathcal{R}(x_n) - \mathcal{R}(x_{n+1}) - \langle \xi_{n+1}, \hat{x} - x_{n+1} \rangle + \langle \xi_n, \hat{x} - x_n \rangle \\ &\quad = \mathcal{R}^*(\xi_{n+1}) - \mathcal{R}^*(\xi_n) - \langle \xi_{n+1} - \xi_n, \hat{x} \rangle . \end{aligned}$$

By definition of x_n and the p-convexity of $\mathcal{R}$, and Proposition 2.10, we further obtain

$$\begin{aligned} &D_{\xi_{n+1}} \mathcal{R}(\hat{x}, x_{n+1}) - D_{\xi_n} \mathcal{R}(\hat{x}, x_n) \\ &\quad = \mathcal{R}^*(\xi_{n+1}) - \mathcal{R}^*(\xi_n) - \langle \xi_{n+1} - \xi_n, \nabla \mathcal{R}^*(\xi_n) \rangle + \langle \xi_{n+1} - \xi_n, x_n - \hat{x} \rangle \\ &\quad \le \frac{1}{p^*(2c_0)^{p^*-1}} \|\xi_{n+1} - \xi_n\|^{p^*} + \langle \xi_{n+1} - \xi_n, x_n - \hat{x} \rangle . \end{aligned} \tag{4.2.12}$$

According to (4.2.6), (4.2.7) and (4.2.8), we have

$$\langle \xi_{n+1} - \xi_n, x_n - \hat{x} \rangle = \mu_n \langle \psi_n, \hat{x} - x_n \rangle \le -(1-\eta)\mu_n \|F(x_n) - y\|$$

and

$$\begin{aligned}
\|\xi_{n+1} - \xi_n\|^{p^*} &= \mu_n^{p^*} \|\psi_n\|^{p^*} = (\mu_n)^{p^*-1} \mu_n \|\psi_n\|^{p^*} \\
&\le \frac{\mu_0^{p^*-1} \|F(x_n) - y\|^{p(r-1)(p^*-1)}}{\|\psi_n\|^{(p^*-1)p}} \|F(x_n) - y\|^{(p^*-1)(p-r)} \mu_n \|\psi_n\|^{p^*} \\
&= \mu_0^{p^*-1} \mu_n \|F(x_n) - y\|^r ,
\end{aligned}$$

where we used the fact that $p = p^*/(p^*-1)$. Plugging these two obtained estimates back to (4.2.12), we obtain

$$D_{\xi_{n+1}} \mathcal{R}(\hat{x}, x_{n+1}) - D_{\xi_n} \mathcal{R}(\hat{x}, x_n) \le -c_1 \mu_n \|F(x_n) - y\|^r . \tag{4.2.13}$$

Based on (4.2.13), Assumption 4.1.3 and Assumption 4.1.4 (b) we can show by an induction argument that $x_n \in B_{3\rho}(x_0)$ for all $n \ge 0$ and $\{D_{\xi_n} \mathcal{R}(\hat{x}, x_n)\}$ is monotonically decreasing. This shows (ii) in Proposition 4.1. Moreover, by definition of ψ_n, we can apply Assumption 4.1.4 (c) and (4.2.3) to get $\|\psi_n\| \le B \|F(x_n) - y\|^{r-1}$, which further implies $\mu_0 \|F(x_n) - y\|^{p(r-1)} \|\psi_n\|^{-p} \ge \mu_0 B^{-p}$. Using the definition of μ_n in (4.2.7), we further obtain that

$$\mu_n \|F(x_n) - y\|^r \ge \min\left\{\mu_0 B^{-p}, \mu_1\right\} \|F(x_n) - y\|^p$$

Therefore, it follows from (4.2.13) that for all n there holds

$$c_3 \|F(x_n) - y\|^p \le D_{\xi_n} \mathcal{R}(\hat{x}, x_n) - D_{\xi_{n+1}} \mathcal{R}(\hat{x}, x_{n+1}) \tag{4.2.14}$$

where $c_3 := c_1 \min\{\mu_0 B^{-p}, \mu_1\}$. Since (ii) holds, the sequence $\{D_{\xi_n} \mathcal{R}(\hat{x}, x_n)\}$ is monotonically decreasing, so the left hand side of (4.2.14) must converge to zero as $n \to \infty$, implying that

$$\|F(x_n) - y\| \to 0 \text{ as } n \to \infty, \tag{4.2.15}$$

thus obtaining (iii) in Proposition 4.1. Finally we show that (iv) in Proposition 4.1 holds. Note that (4.2.15) already holds. Moreover, if there is an integer $\hat{n}$ such that $F(x_{\hat{n}}) = y$, then $\mu_{\hat{n}} = 0$

and hence $\xi_{\hat{n}+1} = \xi_{\hat{n}}$ and $x_{\hat{n}+1} = x_{\hat{n}}$. This results to $\xi_n = \xi_{\hat{n}}$ and $x_n = x_{\hat{n}}$ for all $n \geq \hat{n}$. In turn,

$$F(x_{\hat{n}}) = y \quad (\exists\, \hat{n} \in \mathbb{N}) \quad \Rightarrow \quad F(x_n) = y \quad (\forall\, n \geq \hat{n}) \tag{4.2.16}$$

Based on facts (4.2.15) and (4.2.16), we can do the following: we choose a strictly increasing sequence $\{n_k\}$ of integers by letting $n_0 = 0$ and, for each $k \geq 1$, by letting n_k be the first integer satisfying

$$n_k \geq n_{k-1} + 1 \quad \text{and} \quad \|F(x_{n_k}) - y\| \leq \left\|F(x_{n_{k-1}}) - y\right\|.$$

For such chosen strictly increasing sequence $\{n_k\}$, it is easily seen that

$$\|F(x_{n_k}) - y\| \leq \|F(x_n) - y\|, \quad 0 \leq n < n_k. \tag{4.2.17}$$

For any integers $0 \leq l < k < \infty$, we consider

$$\langle \xi_{n_k} - \xi_{n_l}, x_{n_k} - \hat{x} \rangle = \sum_{n=n_l}^{n_k-1} \langle \xi_{n+1} - \xi_n, x_{n_k} - \hat{x} \rangle = \sum_{n=n_l}^{n_k-1} \mu_n \langle \psi_n, \hat{x} - x_{n_k} \rangle$$

It follows from (4.2.9) that

$$|\langle \xi_{n_k} - \xi_{n_l}, x_{n_k} - \hat{x} \rangle| \leq (1+\eta) \sum_{n=n_l}^{n_k-1} \mu_n \|F(x_n) - y\|^{r-1} \left(2\|F(x_n) - y\| + \|F(x_{n_k}) - y\|\right).$$

Using (4.2.17) and (4.2.14), along with (4.2.7), we obtain

$$\begin{aligned}
|\langle \xi_{n_k} - \xi_{n_l}, x_{n_k} - \hat{x} \rangle| &\leq 3(1+\eta) \sum_{n=n_l}^{n_k-1} \mu_n \|F(x_n) - y\|^r \\
&\leq 3(1+\eta)\mu_1 \sum_{n=n_l}^{n_k-1} \|F(x_n) - y\|^p \\
&\leq \frac{3(1+\eta)}{c_3} \left(D_{\xi_{n_l}} \mathcal{R}(\hat{x}, x_{n_l}) - D_{\xi_{n_k}} \mathcal{R}(\hat{x}, x_{n_k}) \right).
\end{aligned}$$

Hence we have verified (iv) in Proposition 4.1. The proof is complete. $\square$

Now we are ready to prove the main convergence result for Landweber iteration, i.e. Algorithm 4 under the discrepancy principle (4.2.5).

Theorem 4.2.1. *Let Assumption 4.1.3, Assumption 4.1.4 and Assumption 4.2.1 hold. Let $\tau > 1$ and μ_0 be chosen such that* (4.1.17) *holds. Let $\{y^\delta\}$ be a sequence of noisy data with $\|y^\delta - y\| \leq \delta \to 0$, and let n_δ be the integer determined by the discrepancy principle* (4.2.5).

Let $\{y^{\delta_l}\}$, with $\delta_l \to 0$ as $l \to \infty$, be a subsequence of $\{y^\delta\}$. Then, by taking a subsequence of $\{y^{\delta_l}\}$ if necessary, there holds

$$\left\|x_{n_{\delta_l}}^{\delta_l} - x_*\right\| \to 0 \quad \textit{and} \quad D_{\xi_{n_{\delta_l}}^{\delta_l}} \mathcal{R}(x_*, x_{n_{\delta_l}}^{\delta_l}) \to 0$$

as $l \to \infty$ for some solution x_* of (4.1.1) in $\mathcal{B}_{3\rho}(x_0)$. If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in \mathcal{B}_{3\rho}(x_0)$, then there also holds

$$\left\|x_{n_\delta}^\delta - x^\dagger\right\| \to 0 \quad \textit{and} \quad D_{\xi_{n_\delta}^\delta} \mathcal{R}(x^\dagger, x_{n_\delta}^\delta) \to 0$$

as $\delta \to 0$.

Proof. Let $\{y^{\delta_l}\}$ be a sequence of noisy data with $\|y^{\delta_l} - y\| \le \delta_l \to 0$ as $l \to \infty$. Let $N := \liminf_{l\to\infty} n_{\delta_l}$. By taking a subsequence of $\{y^{\delta_l}\}$ if necessary, we may assume $N = \lim_{l\to\infty} n_{\delta_l}$, and according to Lemma 4.2.1, we can find a sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ such that

$$\xi_n^{\delta_l} \to \xi_n \quad \text{and} \quad x_n^{\delta_l} \to x_n \quad \text{as } l \to \infty \tag{4.2.18}$$

for all $0 \le n \le N$. Let x_* be a limit point of $\{x_n\}$ which is a solution of (4.1.1) by Lemma 4.2.2. We show that

$$\lim_{l\to\infty} D_{\xi_{n_{\delta_l}}^{\delta_l}} \mathcal{R}(x_*, x_{n_{\delta_l}}^{\delta_l}) = 0. \tag{4.2.19}$$

If $N < \infty$, then we have $n_{\delta_l} = N$ for large l. Since $\left\|F(x_N^{\delta_l}) - y^{\delta_l}\right\| \le \tau\delta_l$, then by taking $l \to \infty$ and using (4.2.18) and the continuity of F, we have $F(x_N) = y$. The definition of $\{(\xi_n, x_n)\}$ then implies that $x_n = x_N$ for all $n \ge N$, and thus $x_N = x_*$. As a result, it follows from (4.2.18) that $\xi_{n_{\delta_l}}^{\delta_l} \to \xi_N$ and $x_{n_{\delta_l}}^{\delta_l} \to x_*$ as $l \to \infty$. By the lower semi-continuity of $\mathcal{R}$ we have

$$\begin{aligned}
0 &\le \limsup_{l\to\infty} D_{\xi_{n_{\delta_l}}^{\delta_l}} \mathcal{R}(x_*, x_{n_{\delta_l}}^{\delta_l}) = \mathcal{R}(x_*) - \liminf_{l\to\infty} \mathcal{R}(x_{n_{\delta_l}}^{\delta_l}) - \lim_{l\to\infty} \left\langle \xi_{n_{\delta_l}}^{\delta_l}, x_* - x_{n_{\delta_l}}^{\delta_l} \right\rangle \\
&\le \mathcal{R}(x_*) - \mathcal{R}(x_*) - \langle \xi_N, x_* - x_* \rangle = 0,
\end{aligned}$$

which shows (4.2.19). If $N = \infty$, then for any fixed integer n we have $n_{\delta_l} > n$ for large l. It follows from (4.1.18), (4.2.18), and the lower semi-continuity of $\mathcal{R}$ that

$$\begin{aligned}
0 \le \limsup_{l\to\infty} D_{\xi_{n_{\delta_l}}^{\delta_l}} \mathcal{R}(x_*, x_{n_{\delta_l}}^{\delta_l}) &\le \limsup_{l\to\infty} D_{\xi_n^{\delta_l}} \mathcal{R}(x_*, x_n^{\delta_l}) \\
&\le \mathcal{R}(x_*) - \mathcal{R}(x_n) - \langle \xi_n, x_* - x_n \rangle = D_{\xi_n} \mathcal{R}(x_*, x_n).
\end{aligned}$$

Since Lemma 4.2.2 implies that $\lim_{n\to\infty} D_{\xi_n}\mathcal{R}(x_*, x_n) = 0$, by taking $n \to \infty$ in the above equation, we can also obtain (4.2.19).

If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in \mathcal{B}_{3\rho}(x_0)$, we have from Lemma 4.2.1 that $\xi_{n+1} - \xi_n \in \mathcal{N}(L(x^\dagger))^\perp$ for all n. Consequently, it follows from Lemma 4.2.2 that $x_* = x^\dagger$. Thus, the above argument shows that any subsequence $\{y^{\delta_l}\}$ of y^δ has a subsequence, denoted by the same notation, such that $D_{\xi^{\delta_l}_{n_{\delta_l}}}\mathcal{R}(x^\dagger, x^{\delta_l}_{n_{\delta_l}}) \to 0$ as $l \to \infty$. Therefore $D_{\xi^\delta_{n_\delta}}\mathcal{R}(x^\dagger, x^\delta_{n_\delta}) \to 0$ as $\delta \to 0$. □

Now, let us consider (4.1.1) with a bounded linear operator $F : \mathcal{X} \to \mathcal{Y}$. Then Assumption 4.2.1 can be replaced by the compactness of F. Indeed, Assumption 4.2.1 was only used in the proof of the stability result of Lemma 4.2.1 to show that the sequence

$$\left\{\psi_n^{\delta_l} := F^* j_r^{\mathcal{Y}}(F x_n^{\delta_l} - y^{\delta_l})\right\}_l$$

has a convergent subsequence. Since $j_r^{\mathcal{Y}}(F x_n^{\delta_l} - y^{\delta_l})_l$ is a bounded sequence in $\mathcal{Y}^*$, this can be guaranteed by the compactness of F^* which follows from the compactness of F. We therefore have the following convergence result.

Corollary 4.2.1. *Consider* (4.1.1) *where $F : \mathcal{X} \to \mathcal{Y}$ is a compact linear operator. Let $\mathcal{R}$ satisfy Assumption 4.1.3 and let $\tau > 1$ and μ_0 be chosen such that*

$$1 - \frac{1}{\tau} - \frac{1}{p^*}\left(\frac{\mu_0}{2c_0}\right)^{p^*-1} > 0.$$

Then for the Landweber iteration (4.2.2) *with n_δ determined via discrepancy principle* (4.2.5) *there hold*

$$\left\|x^\delta_{n_\delta} - x^\dagger\right\| \to 0 \quad \text{and} \quad D_{\xi^\delta_{n_\delta}}\mathcal{R}(x^\dagger, x^\delta_{n_\delta}) \to 0$$

as $\delta \to 0$.

4.2.2 Convergence under the Hanke-Raus rule

The performance of the discrepancy principle depends heavily on the accurate knowledge of noise level. Such noise level information, however, is not always available or reliable in applications. Incorrect estimation on noise level may lead to significant loss of reconstruction accuracy when using the discrepancy principle. Therefore, it is necessary to develop heuristic rules for Landweber iteration that do not use any knowledge of the noise level in case a reliable noise level information is unavailable. Based on modifying the discrepancy principle, we propose the following heuristic rule for Landweber iteration in Banach spaces in the spirit of [24].

Rule 4.1. *Let $a \geq 1$ be a fixed number and let*

$$\Theta(n, y^\delta) := (n+a)\left\|F(x_n^\delta) - y^\delta\right\|^p .$$

We define $n_* := n_*(y^\delta)$ be an integer such that

$$n_* \in \operatorname{argmin}\left\{\Theta(n, y^\delta) : 0 \leq n \leq n_\infty\right\},$$

where $n_\infty := n_\infty(y^\delta)$ is the largest integer such that $x_n^\delta \in \mathcal{D}(F)$ for all $0 \leq n \leq n_\infty$.

Rule 4.1 can be easily implemented. During the computation, the value of $\Theta(n, y^\delta)$ is recorded versus n. When the eventual increasing tendency of $\Theta(n, y^\delta)$ is seen, we will stop and choose n_* to be the first integer that minimizes $\Theta(n, y^\delta)$ among those computed results.

With the integer $n_* := n_*(y^\delta)$ determined by Rule 4.1, we will use $x^\delta_{n_*(y^\delta)}$ as an approximate solution. A natural question is, for a family of noisy data $\{y^\delta\}$ with $y^\delta \to y$ as $\delta \to 0$, if it is possible to guarantee the convergence of $x^\delta_{n_*(y^\delta)}$ to a solution of (4.1.1) as $\delta \to 0$. The answer in general is no, according to the Bakushinskii's veto ([3]) which states that heuristic rules can not lead to convergence in the sense of worst case scenario for any regularization method. Therefore, to guarantee a convergence result, one must impose certain conditions on the noisy data. We will use the following condition.

Assumption 4.2.2. *$\left\{y^\delta\right\}$ is a family of noisy data satisfying $0 < \left\|y^\delta - y\right\| \to 0$ as $\delta \to 0$ and there is a constant $\kappa > 0$ such that*

$$\left\|y^\delta - F(x)\right\| \geq \kappa \left\|y^\delta - y\right\|$$

for every y^δ and every $x \in S(y^\delta)$, where $S(y^\delta) := \left\{x_n^\delta : 0 \leq n \leq n_\infty\right\}$ generated by (4.2.2).

Assumption 4.2.2 is analogous to Assumption 3.2.3 and Assumption 3.2.4 in Chapter 3 where we have discussed the motivation of introducing such conditions on noisy data. Under Assumption 4.2.2 it is immediate to see that Rule 4.1 defines a finite integer n_*. Indeed, if n_∞ is finite, then there is nothing to prove. So we only consider $n_\infty = \infty$. It then follows from Assumption 4.2.2 that

$$\Theta(n, y^\delta) = (n+a)\left\|F(x_n^\delta) - y^\delta\right\|^p \geq (n+a)\kappa^p \left\|y - y^\delta\right\|^p \to \infty$$

as $n \to \infty$. This implies that there must exist a finite integer n_* achieving the minimum of $\Theta(n, y^\delta)$.

In this subsection our goal is to show that $x^\delta_{n_*(y^\delta)}$ converges to a solution of (4.1.1) as $\delta \to 0$. As a first step, we will show that

$$\Theta(n_*(y^\delta), y^\delta) \to 0 \quad \text{as } \delta \to 0 \tag{4.2.20}$$

To achieve this result, we introduce an auxiliary integer $\hat{n}_\delta$ defined by the stopping rule

$$\left\|F(x^\delta_{\hat{n}_\delta}) - y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(\hat{n}_\delta + a)^2} \le \tau^p \left\|y^\delta - y\right\|^p < \left\|F(x^\delta_n) - y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(n+a)^2} \tag{4.2.21}$$

for $0 \le n < \hat{n}_\delta$, where $\tau > 1$ is a large number (not to be confused with the one in (4.2.5). This is a slight modification of the discrepancy principle (4.2.5) with an extra term $\frac{(2^p-1)c_0\rho^p}{2\mu_1(n+a)^2}$. This extra term ensures that $\hat{n}_\delta \to \infty$ as $\delta \to 0$, an important step in establishing (4.2.20).

Lemma 4.2.3. *Let Assumption 4.1.3 and Assumption 4.1.4 hold. Let μ_0 be chosen such that* (4.2.11) *holds. If $\tau > (1+\eta)/c_1$, then* (4.2.21) *defines a finite integer $\hat{n}_\delta$ with $\hat{n}_\delta \to \infty$ as $\delta \to 0$. Moreover $x^\delta_n \in B_{3\rho}(x_0)$ for all $0 \le n \le \hat{n}_\delta$.*

Proof. Using induction, we show first that $x^\delta_n \in B_{3\rho}(x_0)$ for $0 \le n \le \hat{n}_\delta$. This is trivial for $n = 0$. Now, suppose $x^\delta_k \in B_{3\rho}(x_0)$ for all $0 \le k \le n$ for some $n < \hat{n}_\delta$, we will show that $x^\delta_{n+1} \in B_{3\rho}(x_0)$. Noting that we are considering the noisy case, by doing the same to get (4.2.12) in the proof of Lemma 4.2.2, we have for $0 \le k \le n$ that

$$\begin{aligned} D_{\xi^\delta_{k+1}}\mathcal{R}(x^\dagger, x^\delta_{k+1}) - D_{\xi^\delta_k}\mathcal{R}(x^\dagger, x^\delta_k) &\le \frac{1}{p^*(2c_0)^{p^*-1}} \left\|\xi^\delta_{k+1} - \xi^\delta_k\right\|^{p^*} + \left\langle \xi^\delta_{k+1} - \xi^\delta_k, x^\delta_k - x^\dagger \right\rangle \\ &\le \frac{1}{p^*}\left(\frac{\mu_0}{2c_0}\right)^{p^*-1} \mu^\delta_n \|F(x^\delta_k) - y^\delta\|^r + \left\langle \xi^\delta_{k+1} - \xi^\delta_k, x^\delta_k - x^\dagger \right\rangle. \end{aligned} \tag{4.2.22}$$

By invoking the iterative equation (4.2.2) and Assumption 4.1.4 (c), we have

$$\begin{aligned} \left\langle \xi^\delta_{k+1} - \xi^\delta_k, x^\delta_k - x^\dagger \right\rangle &= \mu^\delta_k \left\langle j^{\mathcal{Y}}_r(F(x^\delta_k) - y^\delta), L(x^\delta_k)(x^\dagger - x^\delta_k) \right\rangle \\ &= \mu^\delta_k \left\langle j^{\mathcal{Y}}_r(F(x^\delta_k) - y^\delta), y^\delta - F(x^\delta_k) \right\rangle + \mu^\delta_k \left\langle j^{\mathcal{Y}}_r(F(x^\delta_k) - y^\delta), y - y^\delta \right\rangle \\ &\quad - \mu^\delta_k \left\langle j^{\mathcal{Y}}_r(F(x^\delta_k) - y^\delta), y - F(x^\delta_k) - L(x^\delta_k)(x^\dagger - x^\delta_k) \right\rangle \\ &\le -\mu^\delta_k \left\|F(x^\delta_k) - y^\delta\right\|^r + \mu^\delta_k \left\|F(x^\delta_k) - y^\delta\right\|^{r-1} \left\|y - y^\delta\right\| \\ &\quad + \eta\mu^\delta_k \left\|F(x^\delta_k) - y^\delta\right\|^{r-1} \left\|y - F(x^\delta_k)\right\| \end{aligned}$$

$$
\begin{aligned}
&\le -\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^r + \mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^{r-1} \left\|y-y^\delta\right\| \\
&\quad + \eta\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^{r-1} \left(\left\|y-y^\delta\right\| + \left\|F(x_k^\delta)-y^\delta\right\|\right) \\
&= -(1-\eta)\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^r + (1+\eta)\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^{r-1} \left\|y-y^\delta\right\|.
\end{aligned}
\tag{4.2.23}
$$

Combining (4.2.23) with (4.2.22) and using (4.2.11), we obtain

$$
\begin{aligned}
&D_{\xi_{k+1}^\delta}\mathcal{R}(x^\dagger, x_{k+1}^\delta) - D_{\xi_k^\delta}\mathcal{R}(x^\dagger, x_k^\delta) \\
&\le -c_1\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^p + (1+\eta)\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^{p-1} \left\|y-y^\delta\right\|.
\end{aligned}
\tag{4.2.24}
$$

According to the definition of $\hat{n}_\delta$, for $k < \hat{n}_\delta$ we have

$$
\left\|F(x_k^\delta)-y^\delta\right\|^{p-1} \left\|y-y^\delta\right\| \le \frac{1}{\tau} \left\|F(x_k^\delta)-y^\delta\right\|^{p-1} \left(\left\|F(x_k^\delta)-y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(k+a)^2}\right)^{1/p},
$$

and by Young's inequality (c.f. Theorem 2.2.1), we further arrive at

$$
\begin{aligned}
&\left\|F(x_k^\delta)-y^\delta\right\|^{p-1} \left\|y-y^\delta\right\| \\
&\le \frac{1}{\tau}\left(\frac{1}{p}\left(\left\|F(x_k^\delta)-y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(k+a)^2}\right) + \frac{1}{p^*}\left\|F(x_k^\delta)-y^\delta\right\|^p\right) \\
&\le \frac{1}{\tau}\left(\frac{1}{p}\left(\left\|F(x_k^\delta)-y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(k+a)^2}\right) + \frac{1}{p^*}\left(\left\|F(x_k^\delta)-y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(k+a)^2}\right)\right) \\
&= \frac{1}{\tau}\left(\left\|F(x_k^\delta)-y^\delta\right\|^p + \frac{(2^p-1)c_0\rho^p}{2\mu_1(k+a)^2}\right).
\end{aligned}
$$

Since $\mu_k^\delta \le \mu_1$, we can conclude that

$$
D_{\xi_{k+1}^\delta}\mathcal{R}(x^\dagger, x_{k+1}^\delta) - D_{\xi_k^\delta}\mathcal{R}(x^\dagger, x_k^\delta) \le -\left(c_1 - \frac{1+\eta}{\tau}\right)\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^p + \frac{(1+\eta)(2^p-1)c_0\rho^p}{2\tau(k+a)^2}.
$$

Then by taking the sum from $k=0$ to $k=n<\hat{n}_\delta$ on both sides, we further obtain

$$
\begin{aligned}
&D_{\xi_{n+1}^\delta}\mathcal{R}(x^\dagger, x_{n+1}^\delta) + \left(c_1 - \frac{1+\eta}{\tau}\right)\sum_{k=0}^{n}\mu_k^\delta \left\|F(x_k^\delta)-y^\delta\right\|^p \\
&\quad \le D_{\xi_0}\mathcal{R}(x^\dagger, x_0) + \frac{(1+\eta)(2^p-1)c_0\rho^p}{2\tau}\sum_{k=0}^{n}\frac{1}{(k+a)^2} \\
&\quad \le c_0\rho^p + \frac{(2^p-1)c_0\rho^p}{2}\sum_{k=0}^{\infty}\frac{1}{(k+a)^2},
\end{aligned}
$$

where we used assumption 4.1.4 (b) and the condition $\tau > (1+\eta)/c_1$ which implies that

$(1+\eta)/\tau \le c_1 < 1$. Noting that $a \ge 1$, and since $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \pi^2/6 < 2$, we can obtain

$$D_{\xi_{n+1}^\delta} \mathcal{R}(x^\dagger, x_{n+1}^\delta) + \left(c_1 - \frac{1+\eta}{\tau}\right) \sum_{k=0}^{n} \mu_k^\delta \left\|F(x_k^\delta) - y^\delta\right\|^p \le c_0 (2\rho)^p. \tag{4.2.25}$$

By using the p-convexity of $\mathcal{R}$, it follows that

$$c_0 \left\|x_{n+1}^\delta - x^\dagger\right\|^p \le D_{\xi_{n+1}^\delta} \mathcal{R}(x^\dagger, x_{n+1}^\delta) \le c_0 (2\rho)^p$$

which shows that $\left\|x_{n+1}^\delta - x^\dagger\right\| \le 2\rho$. Since $\left\|x_0 - x^\dagger\right\| \le \rho$, we therefore have $x_{n+1}^\delta \in B_{3\rho}(x_0)$.

Next, we prove that $\hat{n}_\delta$ is finite. Proving by contradiction, suppose $\hat{n}_\delta$ is infinite. Then the previous argument shows that $x_n^\delta \in B_{3\rho}(x_0)$ for all $n \ge 0$. Consequently it follows from (4.2.25) that

$$\left(c_1 - \frac{1+\eta}{\tau}\right) \sum_{k=0}^{n} \mu_k^\delta \left\|F(x_k^\delta) - y^\delta\right\|^p \le c_0 (2\rho)^p.$$

for all $n \ge 0$. Note that $\tilde{\mu}_n^\delta \ge \min\{\mu_0 B^{-p}, \mu_1\}$. Then, for all $n \ge 0$ we have

$$c_3 \sum_{k=0}^{n} \left\|F(x_k^\delta) - y^\delta\right\|^p \le c_0 (2\rho)^p, \tag{4.2.26}$$

where $c_3 := \left(c_1 - \frac{1+\eta}{\tau}\right) \min\{\mu_0 B^{-p}, \mu_1\} > 0$. In addition, the definition of $\hat{n}_\delta$ tells us that

$$\left\|F(x_k^\delta) - y^\delta\right\|^p \ge \tau^p \left\|y^\delta - y\right\|^p - \frac{(2^p - 1)c_0 \rho^p}{2\mu_1 (k+a)^2}$$

for all $k \ge 0$. Therefore, we may use (4.2.26) to further obtain

$$\begin{aligned} c_3 \tau^p \left\|y^\delta - y\right\|^p (n+1) &\le 2^p c_0 \rho^p + \frac{(2^p - 1)c_0 \rho^p}{2\mu_1} \sum_{k=0}^{n} \frac{1}{(k+a)^2} \\ &\le 2^p c_0 \rho^p + \frac{\pi^2 (2^p - 1) c_0 \rho^p}{12 \mu_1} < \infty, \end{aligned}$$

and by taking $n \to \infty$, we obtain a contradiction.

Finally, from the definition of $\hat{n}_\delta$ in (4.2.21),

$$\frac{(2^p - 1)c_0 \rho^p}{2\mu_1 (\hat{n}_\delta + a)^2} \le \tau^p \left\|y^\delta - y\right\|^p \to 0 \quad \text{as } \delta \to 0.$$

Therefore we must have $\hat{n}_\delta \to \infty$ as $\delta \to 0$. □

Lemma 4.2.4. *Let Assumption 4.1.3 and Assumption 4.1.4 hold. Let $\mu_0 > 0$ be chosen such* *that* (4.2.11) *holds, and let* $\{y^\delta\}$ *be a family of noisy data satisfying Assumption 4.2.2. Let* $n_* := n_*(y^\delta)$ *be the integer defined by Rule 4.1. Then* $\Theta(n_*(y^\delta), y^\delta) \to 0$ *as* $\delta \to 0$*. Consequently,* $\left\|F(x^\delta_{n_*(y^\delta)}) - y^\delta\right\| \to 0$ *and* $n_*(y^\delta)\left\|y^\delta - y\right\|^p \to 0$ *as* $\delta \to 0$.

Proof. Let $\hat{n}_\delta$ be the integer defined by (4.2.21). From the proof of Lemma 4.2.3 we have

$$\sum_{n=0}^{\hat{n}_\delta - 1} \left\|F(x_n^\delta) - y^\delta\right\|^p \le \frac{2^p c_0 \rho^p}{c_3}$$

By the minimality of $\Theta(n_*(y^\delta), y^\delta)$, it follows that

$$\left\|F(x_n^\delta) - y^\delta\right\|^p = \frac{\Theta(n, y^\delta)}{n+a} \ge \frac{\Theta(n_*(y^\delta), y^\delta)}{n+a}.$$

Hence,

$$\Theta(n_*(y^\delta), y^\delta) \sum_{n=0}^{\hat{n}_\delta - 1} \frac{1}{n+a} \le \sum_{n=0}^{\hat{n}_\delta - 1} \left\|F(x_n^\delta) - y^\delta\right\|^p \le \frac{2^p c_0 \rho^p}{c_3}.$$

Note that

$$\sum_{n=0}^{\hat{n}_\delta - 1} \frac{1}{n+a} \ge \sum_{n=0}^{\hat{n}_\delta - 1} \int_n^{n+1} \frac{1}{t+a} dt = \int_0^{\hat{n}_\delta} \frac{1}{t+a} dt = \log \frac{\hat{n}_\delta + a}{a}.$$

Thus

$$\Theta(n_*(y^\delta), y^\delta) \le \frac{2^p c_0 \rho^p}{c_3 \log(\hat{n}_\delta + a)/a}.$$

Since $\hat{n}_\delta \to \infty$ we must have $\Theta(n_*(y^\delta), y^\delta) \to 0$ as $\delta \to 0$. Note that

$$a\left\|F(x^\delta_{n_*(y^\delta)}) - y^\delta\right\|^p \le \Theta(n_*(y^\delta), y^\delta),$$

and Assumption 4.2.2 implies that

$$\big(n_*(y^\delta) + a\big)\, \kappa^p \left\|y^\delta - y\right\|^p \le \Theta(n_*(y^\delta), y^\delta).$$

Hence, by the recent claim, $\left\|F(x^\delta_{n_*(y^\delta)}) - y^\delta\right\| \to 0$ and $n_*(y^\delta)\left\|y^\delta - y\right\|^p \to 0$ as $\delta \to 0$. □

Lemma 4.2.5. *Let Assumption 4.1.3 and Assumption 4.1.4 hold. Let* $\mu_0 > 0$ *be chosen such that* (4.2.11) *holds, and let* $\{y^\delta\}$ *be a family of noisy data satisfying Assumption 4.2.2. Let* $n_* := n_*(y^\delta)$ *be the integer defined by Rule 4.1. Then* $x_n^\delta \in B_{3\rho}(x_0)$ *for all* $0 \le n \le n_*$ *if* δ *is sufficiently small.*

Proof. We use an induction argument. The result is trivial for $n = 0$. Now we assume that

$x_n^\delta \in B_{3\rho}(x_0)$ for some $n < n_*(y^\delta)$. We will use (4.2.24) to prove that $x_{n+1}^\delta \in B_{3\rho}(x_0)$. By the Young's inequality we have

$$(1+\eta)\|F(x_n^\delta) - y^\delta\|^{p-1}\|y^\delta - y\| \leq \frac{c_1}{p^*}\|F(x_n^\delta) - y^\delta\|^p + \frac{(1+\eta)^p}{pc_1^{p-1}}\|y^\delta - y\|^p.$$

Combining this with (4.2.24) and using $\mu_n^\delta \leq \mu_1$, we have

$$D_{\xi_{n+1}^\delta}\mathcal{R}(x^\dagger, x_{n+1}^\delta) - D_{\xi_n^\delta}\mathcal{R}(x^\dagger, x_n^\delta) \leq -\frac{c_1}{p}\mu_n^\delta\|F(x_n^\delta) - y^\delta\|^p + c_4\|y^\delta - y\|^p,$$

where $c_4 := (1+\eta)^p\mu_1/(pc_1^{p-1})$. Therefore

$$\begin{aligned} D_{\xi_{n+1}^\delta}\mathcal{R}(x^\dagger, x_{n+1}^\delta) + \frac{c_1}{p}\sum_{k=0}^{n}\mu_k^\delta\|F(x_k^\delta) - y^\delta\|^p &\leq D_{\xi_0}\mathcal{R}(x^\dagger, x_0) + c_4(n+1)\|y^\delta - y\|^p \\ &\leq c_0\rho^p + c_4 n_*(y^\delta)\|y^\delta - y\|^p \end{aligned}$$

Since $n_*(y^\delta)\|y^\delta - y\|^p \to 0$ as $\delta \to 0$, we can guarantee that

$$c_4 n_*(y^\delta)\|y^\delta - y\|^p \leq (2^p - 1)c_0\rho^p$$

for sufficiently small δ. Then

$$D_{\xi_{n+1}^\delta}\mathcal{R}(x^\dagger, x_{n+1}^\delta) \leq c_0\rho^p + (2^p - 1)c_0\rho^p = c_0(2\rho)^p.$$

By the p-convexity of $\mathcal{R}$ we have $c_0\|x_{n+1}^\delta - x^\dagger\|^p \leq c_0(2\rho)^p$ which show that $\|x_{n+1}^\delta - x^\dagger\| \leq 2\rho$. Since $\|x_0 - x^\dagger\| \leq \rho$, we therefore have $x_{n+1}^\delta \in B_{3\rho}(x_0)$. □

At this point we present two stability results based on Assumption 4.1.5 and Assumption 4.2.1 respectively.

Lemma 4.2.6. *Let Assumption 4.1.3 and Assumption 4.1.4 hold. Let $\mu_0 > 0$ be chosen such that* (4.2.11) *holds, and let $\{y^\delta\}$ be a family of noisy data satisfying Assumption 4.2.2. Let $n_* := n_*(y^\delta)$ be the integer defined by Rule 4.1.*

(a) If Assumption 4.1.5 holds and $1 < r < \infty$, then there is a sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ such that

$$x_n^\delta \to x_n \quad \text{and} \quad \xi_n^\delta \to \xi_n \quad \text{as } \delta \to 0$$

for all $0 \leq n \leq \liminf_{\delta\to 0} n_*(y^\delta)$.

(b) If Assumption 4.2.1 holds and $1 \le r < \infty$, then for any subsequence $\{y^{\delta_l}\}$, with $\delta_l \to 0$ as $l \to \infty$, of $\{y^\delta\}$, by taking a subsequence if necessary, there is a sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ such that

$$x_n^{\delta_l} \to x_n \quad \text{and} \quad \xi_n^{\delta_l} \to \xi_n \quad \text{as } l \to \infty$$

for all $0 \le n \le \liminf_{l\to\infty} n_*(y^{\delta_l})$.

If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in B_{3\rho}(x_0)$, then there also holds $\xi_{n+1} - \xi_n \in \mathcal{N}(L(x^\dagger))^\perp$ for all n.

Proof. (a) We may take $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ defined in (4.2.6) with $\psi_n := L(x_n)^* j_r^{\mathcal{Y}}(F(x_n) - y)$. Since assumption 4.1.5 hold, the mappings $y \mapsto j_r^{\mathcal{Y}}(y)$ and $x \mapsto L(x)$ are continuous. Moreover, we can see from (4.2.13) and (4.2.14) that $\{D_{\xi_n}\mathcal{R}(x^\dagger, x_n)\}_{n\in\mathbb{N}}$ is monotonically decreasing and $\|F(x_n) - y\| \to 0$ as $n \to \infty$. Hence, we may use the similar argument in the proof of Lemma 4.2.2 to conclude this stability result.

(b) Using Lemma 4.2.4, we can establish this result using the same argument in the proof of Lemma 4.2.2.

□

Now we are ready to state the main convergence result for Landweber iteration (4.2.2) under Rule 4.1.

Theorem 4.2.2. *Let Assumption 4.1.3 and Assumption 4.1.4 hold, and let μ_0 be chosen such that (4.2.11) holds. Let $\{y^\delta\}$ be a sequence of noisy data satisfying Assumption 4.2.2 and let $n_*(y^\delta)$ be the integer determined by Rule 4.1.*

(a) If Assumption 4.1.5 holds and $1 < r < \infty$, then there is a solution x_* of (4.1.1) such that

$$\left\|x^\delta_{n_*(y^\delta)} - x_*\right\| \to 0 \quad \text{and} \quad D_{\xi^\delta_{n_*(y^\delta)}}\mathcal{R}(x_*, x^\delta_{n_*(y^\delta)}) \to 0$$

as $\delta \to 0$. If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in B_{3\rho}(x_0)$, then $x_* = x^\dagger$.

(b) If Assumption 4.2.1 holds and $1 \le r < \infty$, then for any subsequence $\{y^{\delta_l}\}$, with $\delta_l \to 0$ as $l \to \infty$, of $\{y^\delta\}$, by taking a subsequence of $\{y^{\delta_l}\}$ if necessary, there hold

$$\left\|x^{\delta_l}_{n_*(y^{\delta_l})} - x_*\right\| \to 0 \quad \text{and} \quad D_{\xi^{\delta_l}_{n_*(y^{\delta_l})}}\mathcal{R}(x_*, x^{\delta_l}_{n_*(y^{\delta_l})}) \to 0$$

as $l \to \infty$ for some solution x_* of (4.1.1) *in $\mathcal{B}_{3\rho}(x_0)$. If in addition $\mathcal{N}(L(x^\dagger)) \subset \mathcal{N}(L(x))$ for all $x \in B_{3\rho}(x_0)$, then*

$$\left\|x^\delta_{n_*(y^\delta)} - x^\dagger\right\| \to 0 \quad \textit{and} \quad D_{\xi^\delta_{n_*(y^\delta)}}\mathcal{R}(x^\dagger, x^\delta_{n_*(y^\delta)}) \to 0$$

as $\delta \to 0$.

Proof. We will only prove (b) since (a) can be proved similarly. Let $\{y^{\delta_l}\}$ be a subsequence of $\{y^\delta\}$, and let $N := \liminf_{l\to\infty} n_*(y^{\delta_l})$. By taking a subsequence of $\{y^{\delta_l}\}$ if necessary, we may assume $N = \liminf_{l\to\infty} n_*(y^{\delta_l})$ and, according to Lemma 4.2.6, we can find a sequence $\{(\xi_n, x_n)\} \in \Gamma_{\mu_0,\mu_1}(\xi_0, x_0)$ such that

$$\xi_n^{\delta_l} \to \xi_n \quad \text{and} \quad x_n^{\delta_l} \to x_n \quad \text{as} \quad l \to \infty \tag{4.2.27}$$

for all $0 \le n \le N$. Let x_* be the limit of $\{x_n\}$ which is a solution of (4.1.1) in $B_{3\rho}(x_0)$. We show that

$$\lim_{l\to\infty} D_{\xi^{\delta_l}_{n_*(y^{\delta_l})}}\mathcal{R}(x_*, x^{\delta_l}_{n_*(y^{\delta_l})}) = 0. \tag{4.2.28}$$

If $N < \infty$, then (4.2.28) can be similarly proven using the argument for the corresponding case in the proof of Theorem 4.2.1 since there requires only $\left\|F(x^{\delta_l}_{n_*(y^{\delta_l})}) - y^{\delta_l}\right\| \to 0$ as $l \to \infty$, which is guaranteed by Lemma 4.2.4. If $N = \infty$ then for any fixed integer $n \ge 1$ we have $n_*(y^{\delta_l}) > n$ for large l. According to the proof of Lemma 4.2.5 we have

$$D_{\xi^{\delta_l}_{k+1}}\mathcal{R}(x_*, x^{\delta_l}_{k+1}) - D_{\xi^{\delta_l}_k}\mathcal{R}(x_*, x^{\delta_l}_k) \le C\left\|y^{\delta_l} - y\right\|^p$$

for $0 \le k < n_*(y^{\delta_l})$, where C is a positive constant independent of l. Choose n such that $0 \le n < n_*(y^{\delta_l})$. By taking the sum both sides from $k = n$ to $k = n_*(y^{\delta_l}) - 1$, the previous inequality implies that

$$\begin{aligned} D_{\xi^{\delta_l}_{n_*(y^{\delta_l})}}\mathcal{R}(x_*, x^{\delta_l}_{n_*(y^{\delta_l})}) &\le D_{\xi^{\delta_l}_n}\mathcal{R}(x_*, x^{\delta_l}_n) + C(n_*(y^{\delta_l}) - n + 1)\left\|y^{\delta_l} - y\right\|^p \\ &\le D_{\xi^{\delta_l}_n}\mathcal{R}(x_*, x^{\delta_l}_n) + Cn_*(y^{\delta_l})\left\|y^{\delta_l} - y\right\|^p. \end{aligned}$$

We follow the argument for the corresponding case in the proof of Theorem 4.2.1. From the proof of Lemma 4.2.5, the monotonicity of $\{D_{\xi^\delta_n}\mathcal{R}(x_*, x^\delta_n)\}$ also holds..Using this, (4.2.27), and the lower semi-continuity of $\mathcal{R}$, we can obtain

$$0 \le \limsup_{l\to\infty} D_{\xi^{\delta_l}_{n_*(y^{\delta_l})}} \mathcal{R}(x_*, x^{\delta_l}_{n_*(y^{\delta_l})}) \le \limsup_{l\to\infty} D_{\xi^{\delta_l}_n} \mathcal{R}(x_*, x^{\delta_l}_n) \le D_{\xi_n}\mathcal{R}(x_*, x_n)$$

Since Lemma 4.2.2 implies that $D_{\xi_n}\mathcal{R}(x_*, x_n) \to 0$ as $n \to \infty$, we can conclude (4.2.28) again.

□

As mentioned after the proof of Theorem 4.2.1, for the equation (4.1.1) with a bounded linear operator $F : \mathcal{X} \to \mathcal{Y}$, Assumption 4.2.1 can be replaced by the compactness of F. This leads to the following convergence result.

Corollary 4.2.2. *Consider the equation* (4.1.1) *where $F : \mathcal{X} \to \mathcal{Y}$ is a compact bounded linear operator. Let $\mathcal{R}$ satisfy Assumption 4.1.3 and let $\mu_0 > 0$ be chosen such that*

$$1 - \frac{1}{p^*}\left(\frac{\mu_0}{2c_o}\right)^{p^*-1} > 0.$$

Then for the Landweber iteration (4.2.2) *with the integer $n_*(y^\delta)$ determined by Rule 4.1 there hold*

$$\left\| x^\delta_{n_*(y^\delta)} - x^\dagger \right\| \to 0 \quad \textit{and} \quad D_{\xi^\delta_{n_*(y^\delta)}} \mathcal{R}(x^\dagger, x^\delta_{n_*(y^\delta)}) \to 0$$

as $\delta \to 0$.

4.2.3 Numerical examples

Example 4.2.1 (Parameter identification)**.** The problem of identifying the source or coefficient term/s in partial differential equations arises in a number of applications, such medical imaging. Here we consider a known benchmark problem for regularizing nonlinear inverse problems.

Suppose we want to solve for the space-dependent source function c in the following elliptic boundary value problem

$$\begin{cases} -\Delta u + cu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{4.2.29}$$

given a measurement u in Ω, where $\Omega \subset \mathbb{R}^m$ with $m \in \mathbb{N}$ is a smooth bounded domain. Given spaces $\mathcal{X}$ and $\mathcal{Y}$, which are to be specified later, the forward operator F

$$F : \mathcal{D}(F) \subseteq \mathcal{X} \to \mathcal{Y}, \tag{4.2.30}$$

, its derivative F', and the adjoint of F', can be formally written as

$$F(c) = A(c)^{-1}f, \quad F'(c)h = -A(c)^{-1}(h \cdot F(c)), \quad F'(c)^*w = -u(c)A(c)^{-1}w$$

for $h, w \in L^2(\Omega)$, where

$$\begin{aligned} A(c) : H^2(\Omega) \cap H_0^1(\Omega) &\to L^2(\Omega) \\ u &\to -\Delta u + cu. \end{aligned}$$

To preserve ellipticity, a straightforward choice of the domain $\mathcal{D}(F)$ is

$$\mathcal{D}(F) = \{c \in \mathcal{X} \mid c \geq 0 \text{ a.e. } \|c\|_X \leq \rho\} \tag{4.2.31}$$

for some sufficiently small $\rho > 0$. For situations requiring a nonempty interior of $\mathcal{D}(F)$ in $\mathcal{X}$, the choice

$$\mathcal{D}(F) = \{c \in \mathcal{X} \mid \exists \hat{c} \in L^\infty(\Omega),\ \hat{c} \geq 0 \text{ a.e. } \|c - \hat{c}\|_X \leq \rho\} \tag{4.2.32}$$

for some sufficiently small $\rho > 0$ has been devised [23].

So far, the problem has been studied in the context of regularization in Hilbert spaces, by setting the preimage and image spaces $\mathcal{X}$ and $\mathcal{Y}$ to be $L^2(\Omega)$ [23, 44, 47]. By treating the same $\mathcal{X}$ and $\mathcal{Y}$ as Banach spaces, the problem has also been studied in a more general setting by incorporating a non-smooth convex penalty functional to recover a non-smooth solution [40, 48, 81, 82]. However, as previously argued, $\mathcal{Y} = L^\infty(\Omega)$ or $\mathcal{Y} = L^1(\Omega)$ are more suitable choices for $\mathcal{Y}$, especially for a practically relevant situation of impulsive noise. Hence, we treat this example in a more general setting by using

$$\mathcal{X} = L^p(\Omega), \quad \mathcal{Y} = L^r(\Omega)$$

for $p, r \in [1, \infty]$. The following result guarantees that in this more general setting the forward operator F and its derivative F' are still well-defined.

Proposition 4.2. [77, Proposition 1.2] *Let $\mathcal{X} = L^p(\Omega)$, $\mathcal{Y} = L^r(\Omega)$.*

(a) For any $p, r \in [1, \infty]$, $f \in L^{\max\{1,m/2\}}(\Omega)$, the operator $F : \mathcal{D}(F) \subseteq \mathcal{X} \to \mathcal{Y}$, $F(c) = A(c)^{-1}f$, is well-defined and bounded on $\mathcal{D}(F)$ as in (4.2.31) with sufficiently small $\rho > 0$.

(b) For any

(i) $p, r \in [1, \infty]$, $f \in L^1(\Omega)$, $m \in \{1, 2\}$ *or*

(ii) $q \in (m/2, \infty]$, $p \geq 1$, $r \in [1, \infty]$, $f \in L^{m/2+\epsilon}(\Omega)$, $\epsilon > 0$

and $c \in \mathcal{D}(F)$, *the operator* $F'(c) : \mathcal{X} \to \mathcal{Y}$, $F'(c)h = -A(c)^{-1}(h \cdot F(c))$ *is well-defined and bounded.*

In addition, item (c) of assumption 4.1.4 holds for small $\rho > 0$ [23]. Hence, we can consider $\mathcal{X} = L^2(\Omega)$ and $Y = L^1(\Omega)$ for numerical simulation. We use finite differences to discretize the problem. We consider the two-dimensional problem with $\Omega = [0, 1] \times [0, 1]$ divided into $N \times N$ small squares of equal size. This results to a grid with $N + 1$ grid points in both x- and y-directions with step size $h = (N + 1)^{-1}$. We also define the sought parameter $c^\dagger$ as

$$c^\dagger(x, y) = \begin{cases} 1, & \text{if } (x - 0.65)^2 + (y - 0.36)^2 \leq 0.18^2, \\ 0.5, & \text{if } (x - 0.35)^2 + 4(y - 0.75)^2 \leq 0.2^2, \\ 0, & \text{elsewhere.} \end{cases}$$

Since $c^\dagger$ is piecewise constant, we use algorithm 4 with the 2-convex TV-like functional

$$\mathcal{R}(x) := \frac{1}{2\beta} \|x\|^2_{L^2(\Omega)} + |x|_{TV} \tag{4.2.33}$$

where $\beta > 0$ and

$$|x|_{TV} := \sup \left\{ \int_\Omega f \, \mathrm{div} g d\omega \ : \ g \in C_0^1(\Omega; \mathbb{R}^m) \text{ and } \|g\|_{L^\infty(\Omega)} \leq 1 \right\}$$

denotes the total variation of f over Ω. For our chosen image and preimage spaces $\mathcal{X}$ and $\mathcal{Y}$, implementing algorithm 4 requires solving a minimization problem of the form

$$x = \arg \min_{z \in L^2(\Omega)} \left\{ \mathcal{R}(z) - \langle \xi, z \rangle_{L^2(\Omega)} \right\}$$

for any $\xi \in L^2(\Omega)$. For our choice of $\mathcal{R}$ given by (4.2.33), this minimization problem is equivalent to solving

$$x = \arg \min_{z \in L^2(\Omega)} \left\{ \frac{1}{2\beta} \|z - \beta\xi\|^2_{L^2(\Omega)} + |x|_{TV} \right\}, \tag{4.2.34}$$

which is the total variation denoising problem, also known as the ROF model [73]. Since there is no explicit formula for the minimisation step in algorithm 4, numerical solvers are used; we will use the primal-dual hybrid gradient (PDHG) algorithm [83]. The penalty functional

Table 4.1: Results for Example 4.2.1 using Landweber iteration in Algorithm 4.

	DP		Rule 4.1 for various a									
			$5e0$		$5e1$		$5e2$		$5e3$		$5e4$	
δ	n_δ	e_{n_δ}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}
5e-2	1	3.867e-1	1	3.867e-1	1	3.867e-1	3	3.219e-1	3	3.219e-1	1963	2.211e1
5e-3	48	3.145e-1	1	3.385e-1	21	3.284e-1	35	3.195e-1	117	1.647e-1	1696	6.805e-1
5e-4	521	1.034e-1	311	1.451e-1	311	1.451e-1	435	1.149e-1	545	9.890e-2	1242	8.063e-2
5e-5	5432	6.581e-2	1975	8.004e-2	1975	8.004e-2	1975	8.004e-2	3122	7.072e-2	4232	6.751e-2

(4.2.33) is discretized first before applying PDHG, as illustrated in [41]. After setting the exact data to be $u(c^\dagger) = x + y$, we add random uniform noise to produce noisy data u^δ with noise level $\delta := \left\|u^\delta - u(c^\dagger)\right\|_{L^2(\Omega)}$.

By relaxing the uniform smoothness on $\mathcal{Y}$, we use $r = 1.0$ in the duality mapping $j_r^{\mathcal{Y}}$, in contrast to the implementation in [48]. Now we apply the result in Example 2.2.1. For $r \geq 1$ the duality mapping $j_r^{\mathcal{Y}}(y)$ for each $y \in \mathcal{Y} = L^r[0,1]$ has the pointwise expression

$$\left[j_r^{\mathcal{Y}}(y)\right](t) := |y(t)|^{r-1}\operatorname{sign}(x(t)), \quad t \in [0,1]\,.$$

Next we apply the Landweber method in algorithm 4 under the discrepancy principle (4.2.5) and the Hanke-Raus rule (Rule 4.1). We used the parameters

$$\beta = 12.0, \quad r = 1.0, \quad p = 2.0, \quad \eta = 0.01, \quad \tau = 1.001 \cdot \frac{1+\eta}{1-\eta},$$

and $\xi_0 = x_0 = 0$ as initial guess. For the step size (4.2.1), we choose

$$\mu_0 = 3.9\left(1 - \eta - \frac{1+\eta}{\tau}\right) \text{ and } \mu_1 = 10000$$

for both parameter choice rules, to ensure that $\mu_0 > 0$ and both (4.1.17) and (4.2.11) hold. We picked a large μ_1 to ensure fast convergence. Since the Fréchet derivative of F satisfies assumption 4.1.4(c) for small $\rho > 0$ (see [23]), take $L = F'$.

Table 4.2.1 shows the regularization parameters and the approximation errors $e_n := \left\|c_n^\delta - c^\dagger\right\|_{L^2}$ after applying algorithm 4 with the discrepancy principle and the Hanke-Raus rule under various noise levels. Moreover, the plot in Figure 4.1(c) illustrates that the noise condition in Assumption 4.2.2 is satisfied. In Figure 4.1(d)-(f), the plot of $\Theta(n, y^\delta)$ against n becomes flatter around n_* as the parameter a in Rule 4.1 increases, resulting to a larger n_*. The Hanke-Raus rule, illustrated in Figure 4.1(g)-(i), resulted to more accurate reconstructions using a suitably

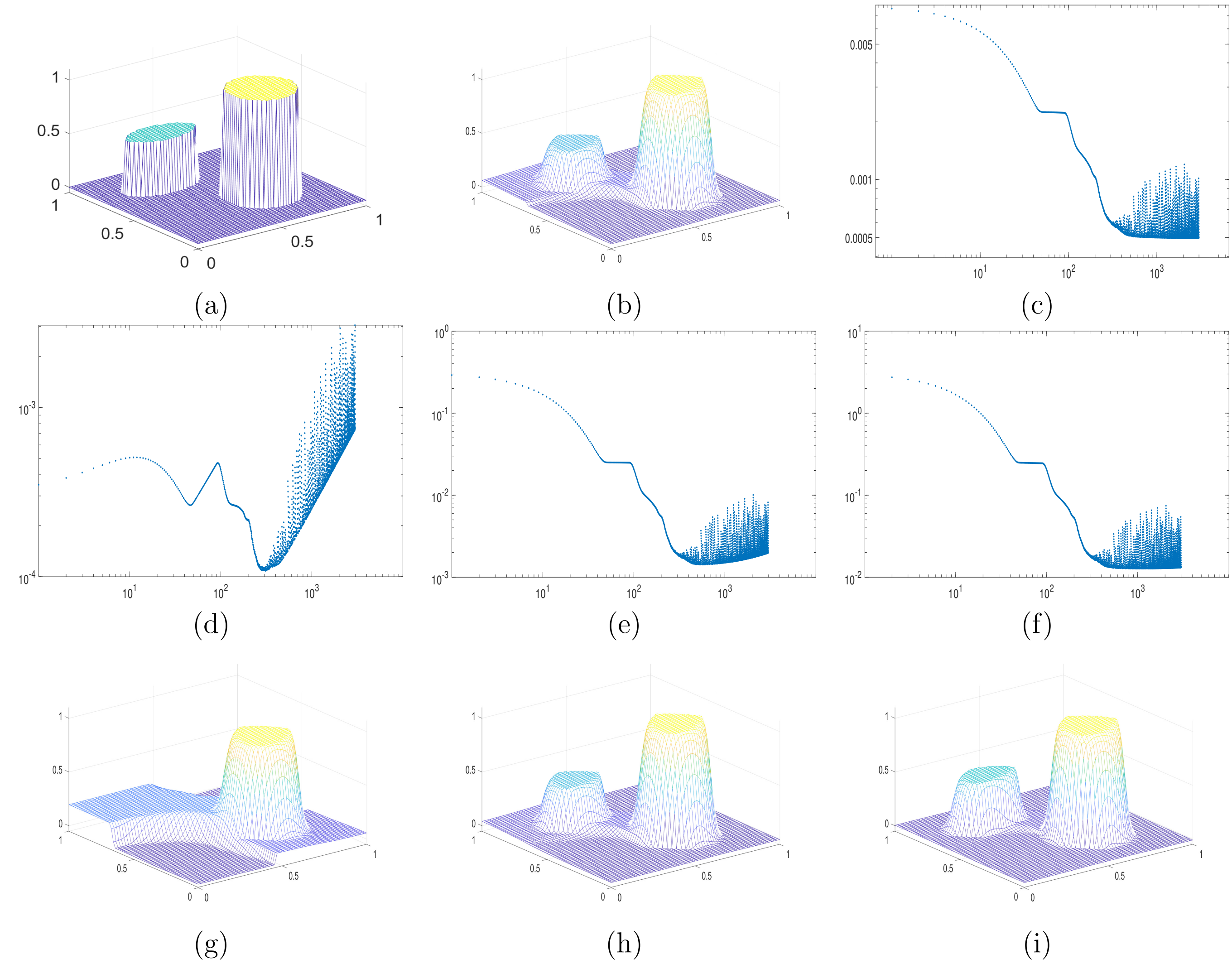

Figure 4.1: The reconstruction for Example 4.2.1 with a Gaussian noisy data with $\delta = 0.0005$: (a) plots the exact solution $c^\dagger$; (b) plots the reconstructed solution by discrepancy principle; (c) plots the relationship between the residual $\left\|F(x_n^\delta) - y^\delta\right\|$ and n; (d), (e), and (f) plot the relationship between $\Theta(n, y^\delta)$ and n for $a = 5$, 5000 and 50000, respectively; (g), (h), and (i) plot the reconstructed solutions via Hanke-Raus rule for $a = 5$, 5×10^3 and 5×10^4, respectively.

large parameter a by comparing the reconstructions to the exact one in Figure 4.1(a), and as good as the one using the discrepancy principle in (b).

On the other hand, we also apply algorithm 4 using a noisy data with impulse noise. Figure 4.2 shows the reconstruction results. The noisy data in Figure 4.2 (a) contains some outliers due to the impulse noise, making the noise level harder to estimate. The inaccurate reconstruction in Figure 4.2(c) shows that the discrepancy principle becomes ineffective with a poorly estimated noise level. However, the reconstruction in Figure 4.2(d) shows that the Hanke-Raus rule can overcome this scenario, since it does not depend on the noise level. This is despite the more erratic behavior of $\Theta(n, y^\delta)$ in Figure 4.2(b), compared to Figure 4.1 for a Gaussian noisy data, indicating a more challenging instance for choosing the regularization parameter n_*.

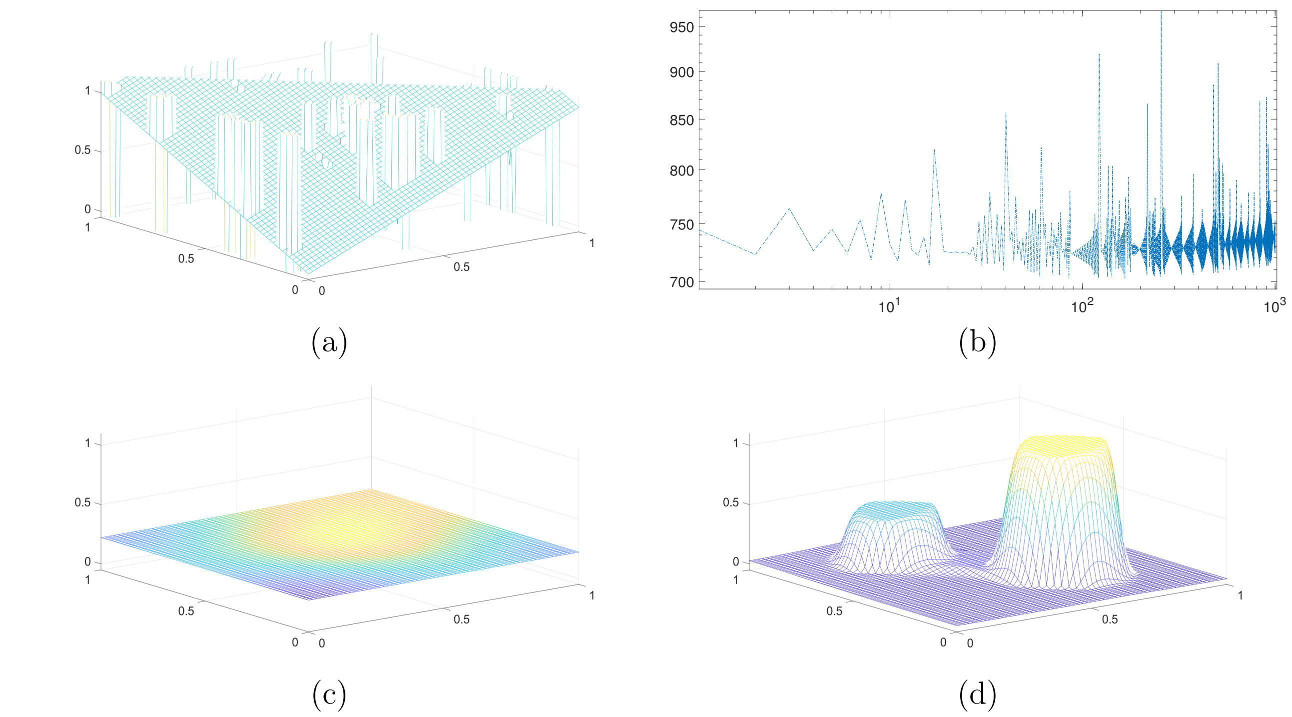

Figure 4.2: The reconstruction for Example 4.2.1 using noisy data with impulse noise: (a) plots the noisy data y^δ ; (b) plots the relationship between $\Theta(n, y^\delta)$ and n for $a = 50000$; (c) and (d) plots the reconstructions via discrepancy principle and Rule 4.1, respectively.

Table 4.2: Results for Example 4.2.1 using Nesterov acceleration for Landweber iteration in Algorithm 4.

	DP		Rule 4.1 for various a									
			$5e0$		$5e1$		$5e2$		$5e3$		$5e4$	
δ	n_δ	e_{n_δ}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}
5e-2	1	3.867e-1	1	3.867e-1	1	3.867e-1	3	1.5058e0	268	2.168e1	268	2.168e1
5e-3	23	2.896e-1	10	3.313e-1	17	3.363e-1	23	2.896e-1	919	1.328e1	919	1.328e1
5e-4	171	8.065e-2	81	1.422e-1	81	1.422e-1	134	9.296e-2	204	7.773e-2	204	7.773e-2
5e-5	-	-	218	8.440e-2	302	7.046e-2	412	6.517e-2	584	6.085e-2	584	6.085e-2

One drawback of Landweber iteration is slow convergence. Table 4.2.1 illustrates this drawback with the rapid increase in the regularization parameters n_δ and n_* as the noise level decays. To address this, the authors in [42] first proposed a Nesterov acceleration scheme to the Landweber iteration in algorithm 4. We apply their strategy here. Let $\alpha \geq 3$. Given $\xi_{-1}^\delta = \xi_0$ and $x_{-1}^\delta = x_0$, we update

$$\begin{aligned} \hat{\xi}_n &= \xi_n^\delta + \frac{n}{n+\alpha}(\xi_n^\delta - \xi_{n-1}^\delta), \\ \hat{x}_n &= x_n^\delta + \frac{n}{n+\alpha}(x_n^\delta - x_{n-1}^\delta), \\ \xi_{n+1}^\delta &= \hat{\xi}_n - \mu_n^\delta L(\hat{x}_n)^* j_r^{\mathcal{Y}}(F(\hat{x}_n) - y^\delta), \\ x_{n+1}^\delta &= \operatorname{argmin}_{x\in\mathcal{X}} \left\{ \mathcal{R}(x) - \left\langle \xi_{n+1}^\delta, x \right\rangle \right\}. \end{aligned} \tag{4.2.35}$$

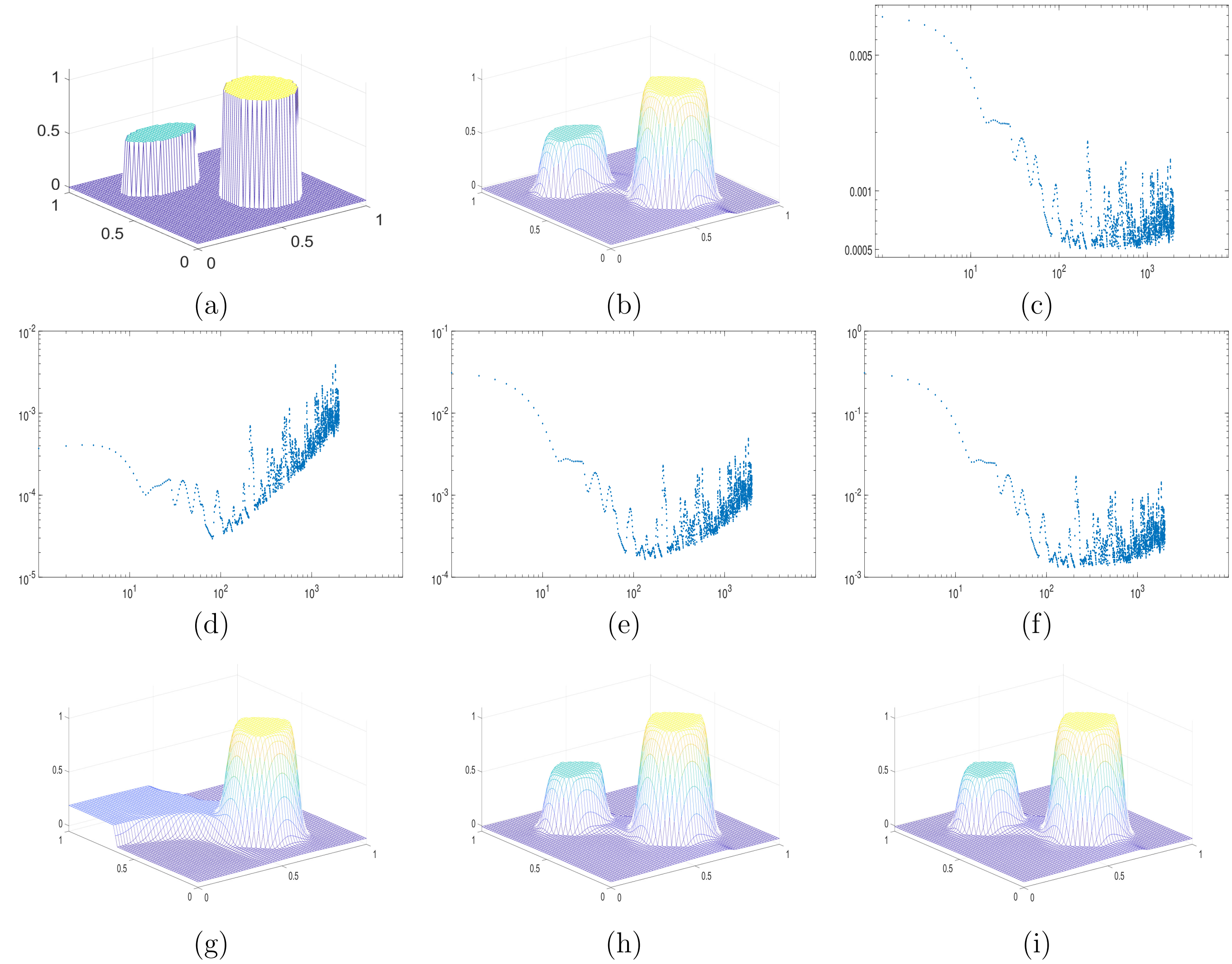

Figure 4.3: The reconstruction for Example 4.2.1 using Nesterov acceleration with a Gaussian noisy data with $\delta = 0.0005$: (a) plots the exact solution $c^\dagger$; (b) plots the reconstructed solution by discrepancy principle; (c) plots the relationship between the residual $\left\|F(x_n^\delta) - y^\delta\right\|$ and n; (d), (e), and (f) plot the relationship between $\Theta(n, y^\delta)$ and n for $a = 5$, 500 and 5000, respectively; (g), (h), and (i) plot the reconstructed solutions via Hanke-Raus rule for $a = 5$, 500 and 5000, respectively.

Although no theoretical analysis is available, numerical simulations in [42] illustrate its acceleration effect. As inspired by [42], we apply the iterative method (4.2.35) with a slightly modified discrepancy principle

$$\left\|F(\hat{x}_{n_\delta}) - y^\delta\right\| \leq \tau < \left\|F(\hat{x}_n) - y^\delta\right\|, 0 \leq n < n_\delta$$

and a slightly modified Rule 4.1 with $\Theta(n, y^\delta) := (n + a)\left\|F(\hat{x}_n) - y^\delta\right\|^p$. Table 4.2.1 shows the results using both discrepancy principle. Note that the discrepancy principle failed to converge after 10000 iterations for δ =5e-5. For both parameter choice rules, the approximation error decays, while needing much fewer regularizing iterations n_δ and n_*. Figures 4.3(d)-(f) shows the more oscillatory behavior of $\Theta(n, y^\delta)$, demonstrating the acceleration effect. Figure 4.3(c) shows that the noise condition 4.2.2 is still satisfied. The Hanke-Raus rule still provides accurate approximate solutions without any information of the noise level. As of writing, a convergence analysis of the iterative method (4.2.35) remains open.

Example 4.2.2. Now we test algorithm 4 under discrepancy principle and heuristic rule 4.1 on an image deblurring problem, where an unknown image $x^\dagger \in \mathbb{R}^{M\times N}$ is to be reconstructed from an observed image $\tilde{y} = Fx^\dagger + \nu$ downgraded by a linear convolution blurring operator F and a salt-and-pepper-noise ν.

Since the image to be reconstructed has some periodic boundary conditions, we use the total variation (TV) deblurring. The function $\mathcal{R}(x)$ is chosen as

$$\mathcal{R}(x) = |x|_{TV} + \frac{\beta}{2}\|x\|_F^2$$

with $\beta = 0.001$, which is a small perturbation of the TV function. Here $|x|_{TV}$ and $\|x\|_F$ denote the total variation of x and the Frobenius norm of x respectively. To remove the salt-and-pepper noise efficiently, we use the data fidelity term as

$$\frac{1}{r}\|Fx - \tilde{y}\|_r^r$$

with $r = 1.0$.

Next we apply algorithm 4 with $r = 1.0$, $p = 2$, and $\xi_0 = x_0 = 0$ as initial guesses. For the step size (4.2.1), we choose

$$\mu_0 = 0.008, \quad \mu_1 = 100,$$

to ensure that (4.1.17) and (4.2.11) hold. In addition, the minimization problem in (4.2.34) is solved by a primal-dual hybrid gradient method.

Figure 4.4 shows the reconstructions using the Boats (576×720) motion blur (`fspecial('motion',35,50)` in MATLAB). The noisy observed image was obtained by adding salt-and-pepper noise with noise density 0.4. To measure the quality of the reconstructed image, we show the computed PSNR (peak signal-to-noise ratio) defined by

$$\mathrm{PSNR} = 10 \cdot \log_{10} \frac{255^2}{\mathrm{MSE}} (\mathrm{dB})$$

where the MSE stands for the mean-squared error per pixel. We present the reconstruction of Landweber method via discrepancy principle and via rule 4.1. Figure 4.4(g) illustrates that

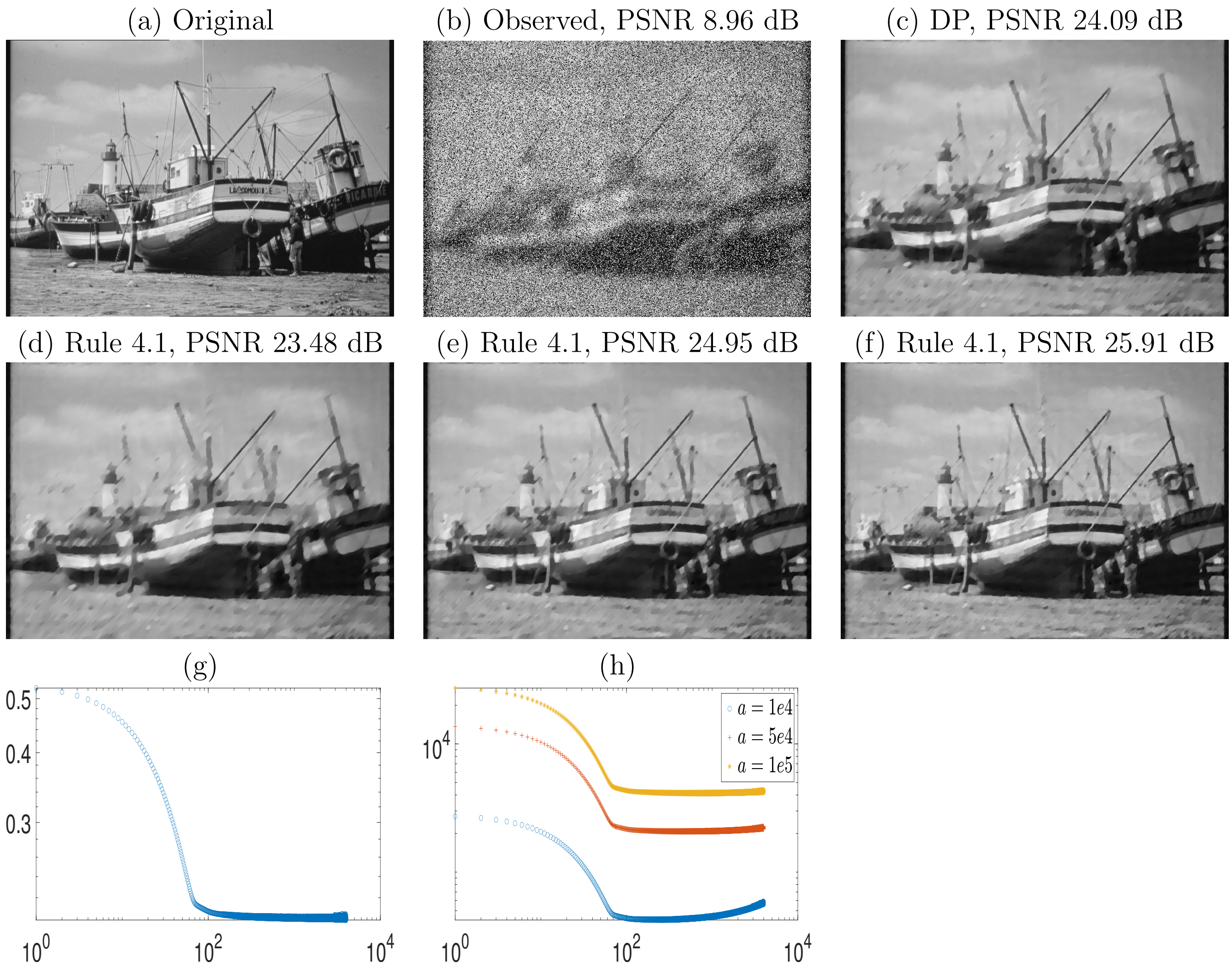

Figure 4.4: (a) The original image; (b) observed image with 40 % salt-and-pepper noise; (c) reconstruction result via discrepancy principle with $\tau = 1.01$; (d)-(f) are the reconstruction results via rule 4.1 with $a = 10000$, 50000, and 100000, respectively; (g) plots the residual $\|Fx_n^\delta - \tilde{y}\|_r^r$ versus n; (h) plots $\Theta(n, \tilde{y})$ versus n for $a = 10000$, 50000, and 100000.

assumption 4.2.2 is satisfied. Moreover, Figure 4.4(h) shows an eventual rising of $\Theta(n, y^\delta)$, illustrating the existence of n_* via rule 4.1. The graph of $\Theta(n, y^\delta)$ flattens around n_*, which increases as a increases. Both parameter choice rules give satisfactory rules, and notice that rule 4.1 gives better reconstruction for a sufficiently large fixed constant a. For this type of noisy data, the Hanke-Raus rule can indeed provide accurate reconstruction.

Example 4.2.3. Now we consider an example with a nonsmooth forward operator. To start, consider an open bounded subset $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2\}$, with a Lipschitz boundary $\partial\Omega$, and consider the nonsmooth semilinear equation

$$\begin{cases} -\Delta y + y^+ = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2.36}$$

with $u \in L^2(\Omega)$ and $y^+(x) := \max(y(x), 0)$ for almost every $x \in \Omega$. Equation (4.2.36) arises in some applications, such as in modelling the deflection of a stretched thin membrane partially covered by water (see [54]). It also appears in free boundary problems for a confined plasma; see [54, 71, 79] for some examples. For each $u \in L^2(\Omega)$, a unique solution y_u in $H_0^1(\Omega) \cap C(\Omega)$ exists [80, Theorem 4.7], so we consider the inverse problem of estimating the source term u from noisy measurements of y_u.

Define the forward operator $F : L^2 \to H_0^1(\Omega) \cap C(\Omega)$ where $y_u = F(u)$ for $u \in L^2(\Omega)$. As shown in [9, Prop. 2.1, Thm. 2.2], F is globally Lipschitz continuous, and has a directional derivative $F'(u; h)$ in the direction $h \in L^2(\Omega)$ given by $\nu \in H_0^1(\Omega)$ which solves

$$\begin{cases} -\Delta \nu + \mathbb{1}_{\{y_u = 0\}} \nu^+ + \mathbb{1}_{\{y_u > 0\}} \nu = h & \text{in } \Omega, \\ \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The operator F is Gâteaux differentiable in $u \in L^2(\Omega)$ if and only if the Lebesgue measure of the set $\{u : y_u = 0\}$ is zero [12, Prop. 3.4]. Hence, in general F not is Gâteaux differentiable. Moreover, computing directional derivative of F could be difficult, and a more convenient alternative is using the Bouligand subdifferential [9, Prop. 3.16]. Given $u \in L^2(\Omega)$, a specific Bouligand subderivative of F is given by the solution operator $G_u : L^2(\Omega) \to H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ which maps $h \in L^2(\Omega)$ to the unique solution $\nu \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta \nu + \mathbb{1}_{\{y_u > 0\}} \nu = h & \text{in } \Omega, \\ \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2.37}$$

where $y_u = F(u)$. In fact G_u is uniformly bounded for all $u \in L^2(\Omega)$ [12, Lemmas 3.7 and 3.9], and consequently satisfies the generalized tangential cone condition. Hence, assumption 4.1.4(c) holds for G_u.

Now we can use algorithm 4 to solve the inverse problem of recovering $u \in L^2(\Omega)$ from $y_u = F(u)$ satisfying (4.2.36). Since F is a mapping between Hilbert spaces, we have $r = 2$ and $J_2 \equiv I$. Since the convex quadratic penalty $\mathcal{R}(u) = \|u\|_{L^2(\Omega)}/2$ is 2-convex, set $p = 2$. By choosing $L = G_u$, the iterative equation in algorithm 4 reduces to the Bouligand-Landweber iteration [12] given by

$$u_{n+1}^\delta = u_n^\delta - \mu_n^\delta G_{u_n^\delta}^* \left(F(u_n^\delta) - y^\delta \right). \tag{4.2.38}$$

where u_n^δ solves (4.2.36) given $y = y_n^\delta$, and μ_n^δ is a given stepsize. Under the discrepancy principle (4.2.5), the authors in [12] established the convergence of (4.2.3) using a constant stepsize

$$\mu_n^\delta = \frac{2-2\eta}{\bar{L}^2}, \text{ where } \bar{L} = 5 \cdot 10^{-2}, \tag{4.2.39}$$

where $\eta = 0.1$ is a tangential constant estimate in assumption 4.1.4(c).

For the computation of the nonsmooth forward operator F, we discretize non-smooth semi-linear elliptic problem (4.2.36) using standard continuous piecewise linear finite elements, and then solve the resulting non-smooth nonlinear equation using a semi-smooth Newton method. The same discretization scheme is also done for computing the Bouligand subdifferential G_u in terms of (4.2.3). For more details, we refer the reader to [12, Section 4] and the reference therein.

We consider the two-dimensional problem with $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and use a uniform triangular Friedrichs-Keller triangulation with 128^2 vertices. The discretization scheme for computing the operator F and G_u involve standard continuous piecewise linear finite elements (see [12] for more details). We used the Python implementation available in https://github.com/clason/bouligandlandweber. The exact solution to be reconstructed is defined as

$$\begin{aligned} u^\dagger(x_1,x_2) = &\max(y^\dagger(x_1,x_2),0) \\ &+ \left[4\pi^2 y^\dagger(x_1,x_2) - 2\left((2x_1-1)^2 + 2(x_1-1+\beta)(x_1-\beta)\right)\sin(2\pi x_2)\right] \\ &\times \mathbb{1}_{(\beta,1-\beta]}(x_1) \end{aligned}$$

where

$$y^\dagger(x_1,x_2) = \left[(x_1-\beta)^2(x_1-1+\beta)^2\sin(2\pi x_2)\right]\mathbb{1}_{(\beta,1-\beta]}(x_1)$$

with $\beta = 0.005$. Figure 4.5(A) shows a plot of $u^\dagger$. It is easy to verify that $y^\dagger \in H^2(\Omega) \cup H_0^1(\Omega)$ satisfies (4.2.36) for the right-hand side $u^\dagger$ and that $y^\dagger$ vanishes on a set of measure 2β. Hence, the forward operator F is not Gâteaux differentiable at $u^\dagger$ [12, Prop. 3.4]. Hence we can use the Bouligand-Landweber iteration (4.2.3).

Given the projection of $y^\dagger$ to the finite element space, denoted by $y_h^\dagger$, random Gaussian noise is added to obtain the noisy data y_h^δ, so that $\delta = \left\|y_h^\delta - y_h^\dagger\right\|_{L^2(\Omega)}$. Figure 4.5(b) shows a noisy data with $\delta = 0.0001$.

Now we test algorithm 4 in solving the inverse problem under the discrepancy principle (4.2.5) and the Hanke-Raus rule (Rule 4.1). We used two initial solutions: the trivial point $u_0 = 0$ and the function

$$\bar{u} := u^\dagger - 10\sin(\pi x_1)\sin(2\pi x_2),$$

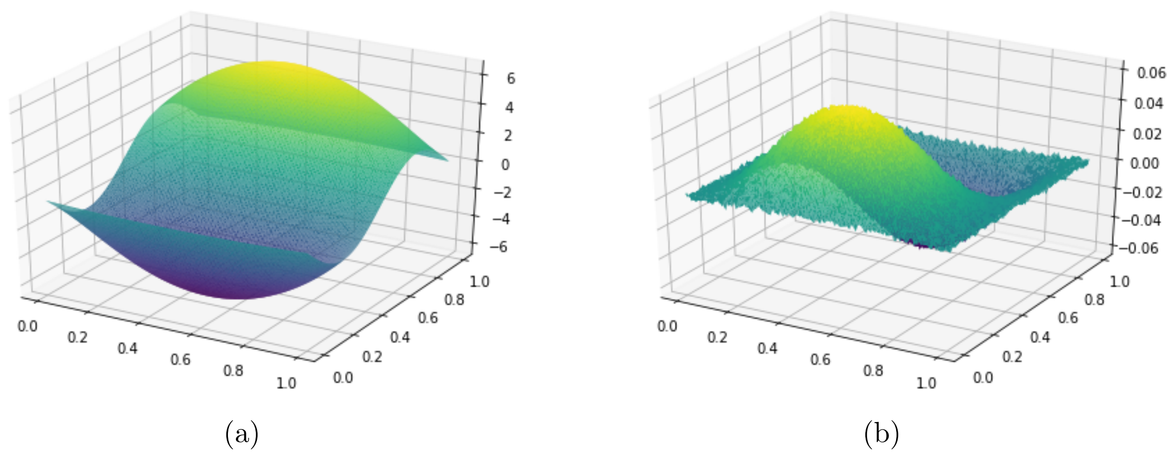

Figure 4.5: (a) plots the initial guess $u_0 = \bar{u}$; (b) plots the noisy data y_h^d with $\delta = 1.043 \times 10^{-3}$.

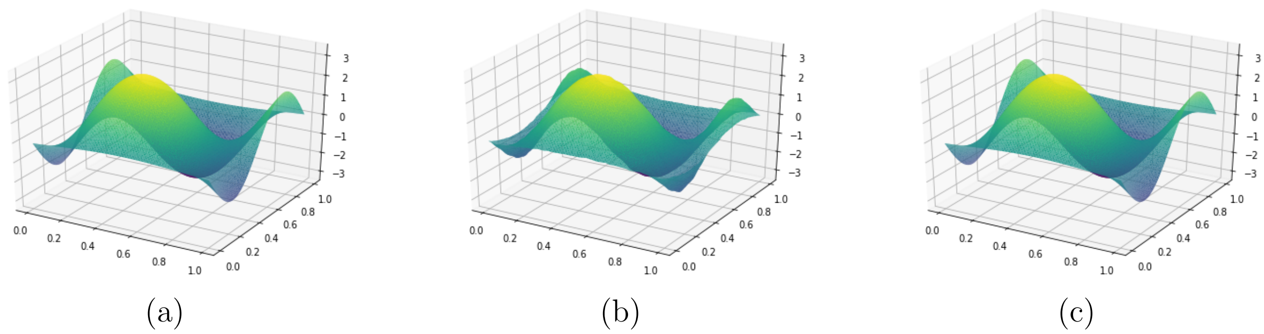

Figure 4.6: The reconstruction for Example 4.2.3 with noise level δ =1.043e-4 using the Bouligand-Landweber iteration (4.2.3) with variable stepsize (4.2.40) under the discrepancy principle: (a) plots the exact solution; (b) plots the reconstruction using $u_0 \equiv 0$; and (c) plots the reconstruction using $u_0 = \bar{u}$.

see Figure 4.5(a). Moreover, we used two different step sizes: the constant step size in (4.2.39) and the variable step size in algorithm 4 given by

$$\mu_n^\delta = \min\left\{\frac{\mu_0\|F(u_n^\delta) - y^\delta\|^2}{\|L(u_n^\delta)^*(F(u_n^\delta) - y^\delta)\|^2}, \mu_1\right\} \tag{4.2.40}$$

with $\mu_0 = 0.6$, $\mu_1 = 5.0 \times 10^5$, suitable enough to achieve convergence. Using discrepancy principle, we terminated the algorithm either when it satisfies (4.2.5) or when it exceeds 5000 iterations.

In Table 4.2.3 we report the stopping indices n_δ via discrepancy principle and the corresponding relative errors

$$E_n^\delta := \frac{\left\|u_h^\delta - u^\dagger\right\|_{L^2(\Omega)}}{\|u^\dagger\|_{L^2(\Omega)}}$$

considering the two different initial solutions and two different step sizes. The reconstructions

Table 4.3: The reconstruction for Example 4.2.3 using the Bouligand-Landweber iteration (4.2.3) using the discrepancy principle.

	Using $u_0 \equiv 0$				Using $u_0 = \bar{u}$			
	DP w/ constant step (4.2.39)		DP w/ variable step (4.2.40)		DP w/ constant step (4.2.39)		DP w/ variable step (4.2.40)	
δ	n_δ	$E^\delta_{n_\delta}$	n_δ	$E^\delta_{n_\delta}$	n_δ	$E^\delta_{n_\delta}$	n_δ	$E^\delta_{n_\delta}$
1.043e-2	7	4.007e-1	6	3.466e-1	9	1.551e-1	9	3.854e-2
5.215e-3	15	3.375e-1	19	2.374e-1	11	7.821e-2	7	3.831e-2
1.043e-3	118	2.200e-1	68	1.982e-1	16	1.401e-2	11	2.932e-3
5.215e-4	303	1.869e-1	94	1.695e-1	18	6.969e-3	13	2.555e-3
1.043e-4	3288	1.227e-1	809	1.135e-1	23	1.286e-3	19	9.823e-4
5.125e-5	-	-	1894	9.528e-2	25	8.329e-4	23	6.451e-4
1.043e-5	-	-	-	-	37	6.206e-4	29	5.589e-4
5.215e-6	-	-	-	-	55	5.510e-4	40	5.045e-4

for $\delta = 1.0432 \times 10^{-4}$ is show in in Figure 4.6. For both step sizes considered, n_δ is smaller for $\delta \leq 1.0432 \times 10^{-2}$ with $u_0 = \bar{u}$. In general, using the initial point $\bar{u}$ generated smaller relative errors and smaller termination indices. This illustrates the point in [12]: since the exact $u^\dagger$ satisfies the general source condition with $u_0 = \bar{u}$, the algorithm (4.2.3) should converge faster at an optimal rate. This optimality is also shown in the faster decay of $E^\delta_{n_\delta}$ as the noise level δ decays.

For instance, in Table 4.2.3, using the constant step (4.2.39), from $\delta = 1.043 \times 10^{-2}$ to $\delta = 1.02 \times 10^{-4}$, the relative error decreased only from 4.0×10^{-2} to 1.2×10^{-2} with $u_0 = 0$, while it decreased from 1.6×10^{-1} to 1.3×10^{-3} with $u_0 = \bar{u}$. Using the variable step size (4.2.40), the decay in the error with respect to the choice of u_0 was still observed but with fewer iterations. Correspondingly, the stopping index n_δ increases rapidly from 7 to 3288 with $u_0 \equiv 0$, while n_δ increased much slower from 6 to 809 with $u_0 = \bar{u}$.

More importantly, algorithm 4 terminated via discrepancy principle in much fewer iterations using the variable step size, often leading to better reconstruction. This strengthens the case for using a variable step for Landweber iteration. Since we established the convergence of algorithm 4 without any source condition, the method still converges with the trivial $u_0 \equiv 0$.

Tables 4.4 and 4.5 report the regularization parameters and the corresponding relative errors for the Bouligand-Landweber method with the variable step size (4.2.40) under the Hanke-Raus rule using $u_0 \equiv 0$ and $u_0 = \bar{u}$, respectively. Same as in the discrepancy principle, the decay on

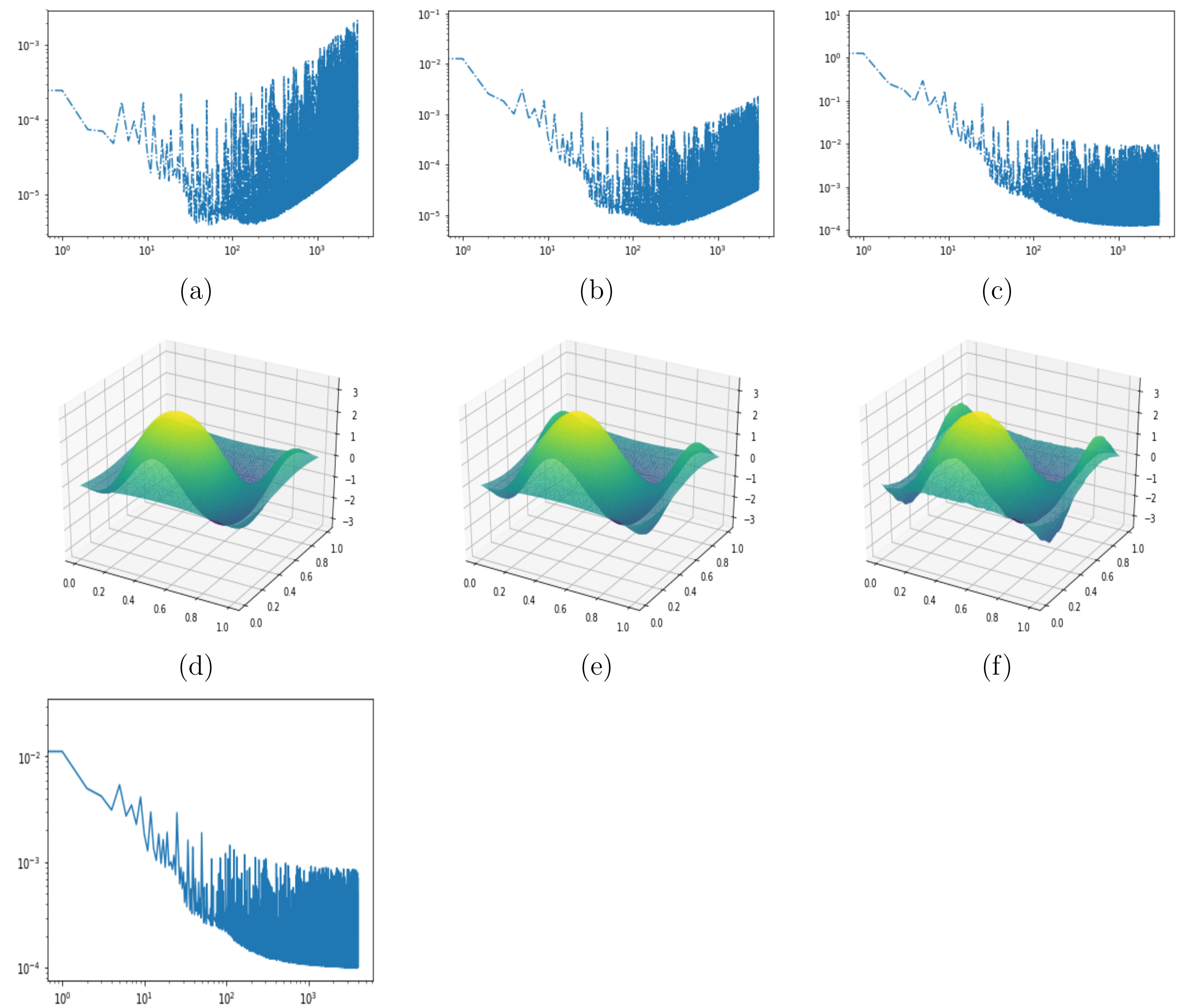

Figure 4.7: The reconstruction for Example 4.2.3 under the Hanke-Raus rule, with noise level δ =1.043e-4, using variable stepsize (4.2.40) and the trivial initial solution $u_0 \equiv 0$: (a), (b), and (c) plot the relationship between $\Theta(n, u^\delta)$ and n for $a = 1$, 100 and 10000, respectively; (d), (e), and (f) plot the reconstructed solutions via Rule 4.1 for $a = 1$, 100 and 10000, respectively; (g) plots the residual $\left\|F(u_n^\delta) - y^\delta\right\|$ versus n.

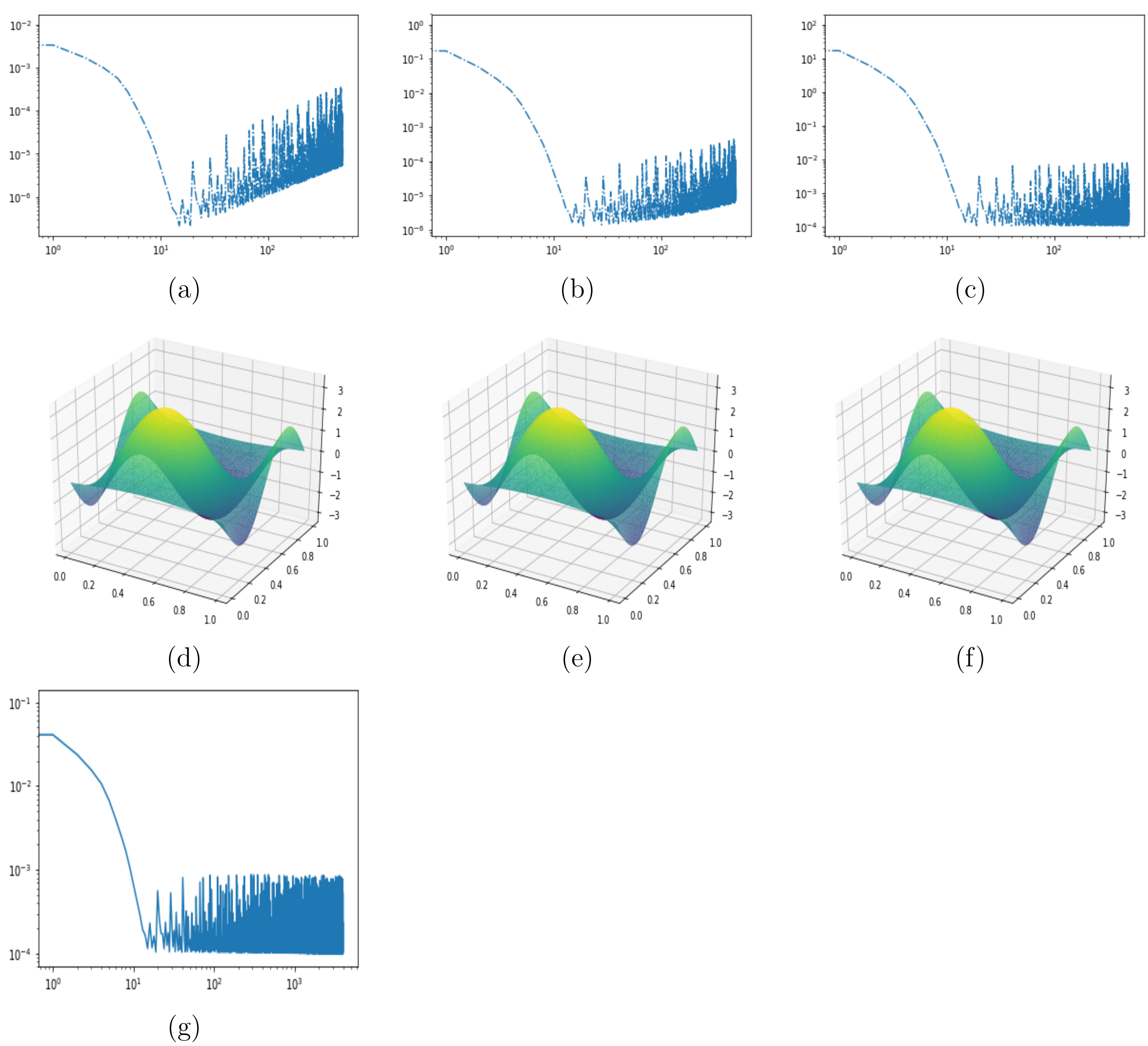

Figure 4.8: The reconstruction for Example 4.2.3 under the Hanke-Raus rule, with noise level δ =1.043e-4, using variable stepsize (4.2.40) and $u_0 = \bar{u}$: (a), (b), and (c) plot the relationship between $\Theta(n, u^\delta)$ and n for $a = 1$, 100 and 10000, respectively; (d), (e), and (f) plot the reconstructed solutions via Hanke-Raus rule for $a = 1$, 100 and 10000, respectively; (g) plots the residual $\left\|F(u_n^\delta) - y^\delta\right\|$ versus n.

Table 4.4: Results for the Bouligand-Landweber iteration with variable stepsize for Example 4.2.3 under Rule 4.1 with $u_0 \equiv 0$.

	Rule 4.1 for various a									
	1e0		1e1		1e2		1e3		1e4	
δ	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}
1.043e-2	1	5.129e-1	6	3.466e-1	6	3.466e-1	29	2.452e-1	59	2.840e-1
5.215e-3	2	4.131e-1	2	4.131e-1	19	2.374e-1	49	2.168e-1	49	2.168e-1
1.043e-3	7	3.142e-1	30	2.357e-1	54	2.108e-1	104	1.763e-1	255	1.571e-1
5.215e-4	37	2.071e-1	37	2.071e-1	70	1.811e-1	131	1.588e-1	457	1.369e-1
1.043e-4	53	1.855e-1	153	1.553e-1	192	1.479e-1	339	1.315e-1	1270	1.050e-1
5.125e-5	216	1.402e-1	216	1.402e-1	299	1.326e-1	849	1.103e-1	2502	9.052e-2

Table 4.5: Results for the Bouligand-Landweber iteration with variable stepsize for Example 4.2.3 under Rule 4.1 with $u_0 = \bar{u}$.

	Rule 4.1 for various a									
	1e0		1e1		1e2		1e3		1e4	
δ	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}	n_*	e_{n_*}
1.043e-2	5	1.627e-1	5	1.627e-2	14	3.401e-2	14	3.401e-2	64	2.203e-1
5.215e-3	7	3.831e-2	7	3.831e-2	7	3.831e-2	18	2.296e-2	47	7.396e-2
1.043e-3	11	2.932e-3	11	2.932e-3	11	2.932e-3	11	2.932e-3	90	3.087e-2
5.215e-4	13	2.555e-3	13	2.555e-3	13	2.555e-3	35	4.830e-3	83	1.014e-3
1.043e-4	15	1.590e-4	19	9.823e-4	19	9.823e-4	19	9.823e-4	117	3.580e-3
5.125e-5	17	9.381e-4	17	9.381e-4	29	6.404e-4	29	6.404e-4	132	1.679e-3
1.043e-5	29	5.589e-4	29	5.589e-4	29	5.589e-4	50	5.177e-4	110	6.060e-4
5.215e-6	27	5.504e-4	27	5.504e-4	40	5.045e-4	51	5.043e-4	128	5.396e-4

the noise level is still faster with $u_0 = \bar{u}$. Also, the increase in n_* is slower when using $u_0 = \bar{u}$. With the trivial initial point, which is more practical in many applications, a sufficiently large choice of a for Rule 4.1 can still produce relatively accurate reconstruction. Figures 4.7-4.8 shows the reconstruction plots for $\delta = 1.043 \times 10^{-4}$. To illustrate that the noise condition in assumption 4.2.2 is satisfied, we plot the residual against the iteration number in Figures 4.7(g) and 4.8(h). The reconstructions using $u_0 = \bar{u}$ appear closer to the exact solution in Figure 4.6 (a). Based on these figures, a larger choice of a also yields a larger regularization parameter n_*, making the plot of $\Theta(n, u^\delta)$ against n flatter around n_*.

4.3 Landweber iteration in Hilbert spaces

In this section we consider nonlinear ill-posed inverse problems (4.1.1) in Hilbert spaces, i.e.

$$F(x) = y, \tag{4.3.1}$$

where $F : \mathcal{D}(F) \subset \mathcal{X} \to \mathcal{Y}$ is a Fréchet differentiable nonlinear operator between two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$ with domain $\mathcal{D}(F)$. We will consider the Landweber iteration of the form

$$x_{n+1}^\delta = x_n^\delta - F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta) \tag{4.3.2}$$

to study (4.3.1), where y^δ is a noisy data of the exact data y. As usual we will assume that (4.3.1) has a solution in the ball $B_\rho(x_0)$ and F is properly scaled and satisfies the tangential cone condition in the following sense ([23]).

Assumption 4.3.1.

(a) F satisfies a tangential cone condition in a ball $B_{3\rho}(x_0) \subset \mathcal{D}(F)$ of radius 3ρ around x_0, i.e. there is a constant $0 \le \eta < 1/2$ such that

$$\|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| \le \eta \, \|F(x) - F(\tilde{x})\|$$

for all $x, \tilde{x} \in B_{3\rho}(x_0)$.

(b) The problem (4.3.1) is properly scaled, i.e. $\|F'(x)\| \le 1$ for all $x \in B_{3\rho}(x_0)$.

Under Assumption 4.3.1, the method (4.3.2) has been thoroughly studied in [23] when the iteration is terminated by the discrepancy principle as we have reviewed in Section 4.1. In Section 4.2 we have considered the heuristic rule of Hanke-Raus type for Landweber iteration in Banach spaces. This heuristic rule can be adapted to the method (4.3.2) in Hilbert spaces which takes the following form.

Rule 4.2. *Let $a \ge 1$ be a fixed integer and define $n_* := n_*(y^\delta)$ be an integer such that*

$$n_* \in \operatorname{argmin}\left\{\Theta(n, y^\delta) := (n + a)\left\|y^\delta - F(x_n^\delta)\right\|^2 \;:\; 0 \le n \le n_\infty\right\}, \tag{4.3.3}$$

where $n_\infty := n_\infty(y^\delta)$ is the largest integer such that $x_n^\delta \in \mathcal{D}(F)$ for all $0 \le n \le n_\infty$. We then use $x_{n_*}^\delta$ as an approximate solution.

Although the argument in Section 4.2 is carried out for the method with step size μ_n^δ given by (4.2.1) and the method (4.3.2) uses the step size $\mu_n^\delta \equiv 1$, the same argument can be adapted without essential changes to show a convergence result on x_{n*}^δ for the method (4.3.2) if a family of noisy data $\{y^\delta\}_{\delta>0}$ is available to satisfy Assumption 4.2.2, i.e. $0 < \delta := \|y^\delta - y\| \to 0$ as $\delta \to 0$ and there is a constant $\kappa > 0$ such that

$$\|y^\delta - F(x)\| \geq \kappa \|y^\delta - y\| \tag{4.3.4}$$

for every y^δ and every $x \in S(y^\delta)$, where $S(y^\delta) := \{x_n^\delta : 0 \leq n \leq n_\infty\}$ generated by (4.3.2). We thus have the following result.

Theorem 4.3.1. *Let Assumption 4.3.1 hold, let $\{y^\delta\}$ be a family of noisy data satisfying (4.3.4) and $0 < \delta := \|y^\delta - y\| \to 0$ as $\delta \to 0$. Let $n_* := n_*(y^\delta)$ be the integer defined by Rule 4.2. Then $x_n^\delta \in B_{3\rho}(x_0)$ for all $0 \leq n \leq n_*$ if $\delta > 0$ is sufficiently small and $\|x_{n_*(y^\delta)}^\delta - x^\dagger\| \to 0$ as $\delta \to 0$ for some solution $x^\dagger$ of (4.3.1), if in addition, $\mathcal{N}(F(x^\dagger)) \subset \mathcal{N}(F(x))$ for all $x \in B_{3\rho}(x_0)$, then $x^\dagger$ is the unique x_0-minimum-norm solution of (4.3.1).*

In this section we will derive an error estimate on $\|x_{n_*(y^\delta)}^\delta - x^\dagger\|$ for the method (4.3.2) in Hilbert spaces. To this end, we need a further condition on F and a source condition on $x^\dagger - x_0$. As in [23] we make the following assumptions.

Assumption 4.3.2. *There is a family of bounded linear operators $\{R_x : x \in B_{3\rho}(x_0)\}$ such that*

$$F'(x) = R_x F'(x^\dagger) \quad \text{and} \quad \|R_x - I\| \leq C \|x - x^\dagger\|, \quad x \in B_{3\rho}(x_0)$$

for some positive constant C.

Assumption 4.3.3. *There exists a number $\nu > 0$ and $f \in \mathcal{X}$ such that*

$$x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger))^\nu f.$$

Assumption 4.3.3 is a source condition, implying a certain "smoothness" of $x^\dagger - x_0$. Assumption 4.3.2) can be seen as a further restriction on the nonlinearity of F. It is easy to see that Assumption 4.3.2) implies Assumption 4.3.1 (a) with $\tilde{x} = x^\dagger$ for a sufficiently small ρ, since for $x \in B_{3\rho}(x_0)$ there holds

$$\|F(x) - F(x^\dagger) - F'(x)(x - x^\dagger)\| \leq \frac{3}{2} C \|F'(x^\dagger)(x - x^\dagger)\| \|x - x^\dagger\|. \tag{4.3.5}$$

Proposition 4.3. *Let Assumption 4.3.1 and Assumption 4.3.2 hold and let n_δ be the integer determined by the discrepancy principle*

$$\|F(x_{n_\delta}^\delta) - y^\delta\| \leq \tau\delta < \|F(x_n^\delta) - y^\delta\|, \quad 0 \leq n < n_\delta$$

with $\tau > 2(1+\eta)/(1-2\eta)$. Then n_δ is a finite integer and $x_n^\delta \in B_{3\rho}(x_0)$ for all $0 \leq n \leq n_\delta$. If in addition $x^\dagger - x_0$ satisfies Assumption 4.3.3 for some $0 < \nu \leq 1/2$ with sufficiently small $\|f\|$, then

$$n_\delta \leq c_1 \left(\frac{\|f\|}{\delta}\right)^{\frac{2}{2\nu+1}}$$

with some constant c_1 depending only on ν, η and τ.

Proof. This is [23, Proposition 2.2 and Theorem 3.2]. □

Lemma 4.3.1. *Let Assumption 4.3.1 and Assumption 4.3.2 hold, let y^δ be a nosy data satisfying (4.3.4), and let $n_* := n_*(y^\delta)$ be the integer defined by Rule 4.2. Assume that $\|F(x_0) - y^\delta\| > \tau\|y^\delta - y\|$ for some $\tau > 2(1+\eta)/(1-2\eta)$. If $x^\dagger - x_0$ satisfies Assumption 4.3.3 with $0 < \nu \leq 1/2$ for sufficiently small $\|f\|$, then*

$$(n_* + a)^{-\nu-1/2} \geq \frac{\delta}{c_2\|f\|},$$

where c_2 is a positive constant depending only on ν, η, τ, a and κ.

Proof. By the condition (4.3.4) and the definition of n_* and n_δ, we have

$$\begin{aligned}\kappa^2(n_* + a)\delta^2 \leq \Theta(n_*, y^\delta) \leq \Theta(n_\delta, y^\delta) &= (n_\delta + a)\|F(x_{n_\delta}^\delta) - y^\delta\|^2 \\ &\leq \tau^2(n_\delta + a)\delta^2.\end{aligned}$$

This implies that

$$\kappa^2(n_* + a) \leq \tau^2(n_\delta + a).$$

Since $\|F(x_0) - y^\delta\| > \tau\delta$, we have $n_\delta \geq 1$. Therefore

$$\kappa^2(n_* + a) \leq \tau^2(1 + a)n_\delta.$$

By virtue of Proposition 4.3, we thus complete the proof. □

In order to proceed further analysis, we need the following useful result in estimating sums.

Lemma 4.3.2. *Let p and q be nonnegative numbers. Then there holds*

$$\sum_{j=0}^{n-1}(j+1)^{-p}(n-j)^{-q} \le c_{p,q}(n+1)^{1-p-q}\begin{cases} 1, & \max\{p,q\}<1 \\ \ln(n+1), & \max\{p,q\}=1 \\ (n+1)^{\max\{p,q\}-1}, & \max\{p,q\}>1, \end{cases}$$

where $c_{p,q}$ is a positive constant depending only on p and q.

Proof. See Lemma 2.9 in [51]. □

Lemma 4.3.3. *Let all the conditions in Lemma 4.3.1 hold. If $x^\dagger - x_0$ satisfies Assumption 4.3.3 for some $0 < \nu \le 1/2$ and sufficiently small $\|f\|$, then*

$$\left\|x^\dagger - x_n^\delta\right\| \le c_\nu \|f\| (n+1)^{-\nu} \tag{4.3.6}$$

and

$$\left\|y^\delta - F(x_n^\delta)\right\| \le c_\nu \|f\| (n+1)^{-\nu-1/2} \tag{4.3.7}$$

for $0 \le n \le n_*$, where c_ν is a positive constant depending only on C, η, ν and κ.

Proof. The proof follows from that of [23, Theorem 3.1] in which the result is stated for the discrepancy principle; however, the argument can be modified slightly to obtain the desired result. For completeness we include the proof here. Let $e_n := x^\dagger - x_n^\delta$ and $K := F'(x^\dagger)$. Given $0 \le n < n_*$, according to the equation (3.10) in [23] we have

$$e_n = (I-K^*K)^n e_0 + \sum_{j=0}^{n-1}(I-K^*K)^j K^* z_{n-j-1} + \left[\sum_{j=0}^{n-1}(I-K^*K)^j K^*\right](y-y^\delta) \tag{4.3.8}$$

and

$$Ke_n = (I-KK^*)^n Ke_0 + \sum_{j=0}^{n-1}(I-KK^*)^j KK^* z_{n-j-1} + \left[I-(I-KK^*)^n\right](y-y^\delta), \tag{4.3.9}$$

where

$$z_j := (I - R^*_{x_j^\delta})(y^\delta - F(x_j^\delta)) - (F(x^\dagger) - F(x_j^\delta) - K(x^\dagger - x_j^\delta)). \tag{4.3.10}$$

We want to show that

$$\|e_j\| \le c_* \|f\| (j+1)^{-\nu}, \quad \|Ke_j\| \le c_* \|f\| (j+1)^{-\nu-1/2} \tag{4.3.11}$$

for $0 \le j \le n_*$, where

$$c_* := 2 + c_2.$$

For $j = 0$, (4.3.11) is always true due to Assumption 4.3.3 and Assumption 4.3.1 (b). By induction, assuming for a fixed $n \le n_*$ that (4.3.11) is true for $0 \le j < n$, we want to show that (4.3.11) holds for $j = n$. According to Assumption 4.3.2, it is easy to derive that

$$\left\|F(x^\dagger) - F(x_j^\delta) - K(x^\dagger - x_j^\delta)\right\| \le \frac{1}{2} C \left\|e_j\right\| \left\|K e_j\right\|$$

which together with (4.3.11) then implies that

$$\left\|F(x^\dagger) - F(x_j^\delta) - K(x^\dagger - x_j^\delta)\right\| \le \frac{1}{2} C c_*^2 \|f\|^2 (j+1)^{-2\nu-1/2}. \tag{4.3.12}$$

By using Assumption 4.3.1 (a) and noting that $\eta < 1/2$, we can see that

$$\left\|y^\delta - F(x_j^\delta)\right\| \le \delta + \frac{1}{1-\eta} \left\|K e_j\right\| \le \delta + 2\|K e_j\|.$$

By virtue of Lemma 4.3.1 we have

$$\delta \le c_2 \|f\| (n_* + a)^{-\nu-1/2} \le c_2 \|f\| (j+1)^{-\nu-1/2}.$$

Therefore

$$\left\|y^\delta - F(x_j^\delta)\right\| \le (c_2 + 2c_*) \|f\| (j+1)^{-\nu-1/2}. \tag{4.3.13}$$

Consequently, it follows from Assumption 4.3.2 that

$$\left\|(I - R_{x_j^\delta}^*)(y^\delta - F(x_j^\delta))\right\| \le C\|e_j\| \|y^\delta - F(x_j^\delta)\| \le C c_* (c_2 + 2c_*) \|f\|^2 (j+1)^{-2\nu-1/2}. \tag{4.3.14}$$

It then follows from (4.3.10), (4.3.12) and (4.3.14) that

$$\|z_j\| \le c_3 \|f\|^2 (j+1)^{-2\nu-1/2}, \tag{4.3.15}$$

with $c_3 := C c_* (c_2 + 5c_*/2)$. Since $\|K\| \le 1$, we have the following useful identities (see [51, Lemma 2.10]):

$$\left\|\sum_{j=0}^{n-1}(I-K^*K)^jK^*\right\| \le \sqrt{n}, \quad \|(I-K^*K)^n(K^*K)^\nu\| \le (n+1)^{-\nu},$$

$$\left\|(I-K^*K)^jK^*\right\| \le (j+1)^{-1/2}, \quad \left\|(I-KK^*)^jKK^*\right\| \le (j+1)^{-1}.$$

Using these bounds, Assumption 4.3.3, Lemma 4.3.2 and Lemma 4.3.1, we can obtain

$$\begin{aligned}
\|e_n\| &\le (n+1)^{-\nu}\|f\|+\sum_{j=0}^{n-1}(j+1)^{-1/2}\|z_{n-j-1}\|+\sqrt{n}\delta \\
&\le (n+1)^{-\nu}\|f\|+c_3\|f\|^2\sum_{j=0}^{n-1}(j+1)^{-1/2}(n-j)^{-2\nu-1/2}+c_2\|f\|(n+1)^{-\nu} \\
&\le (1+c_2+c_3c_4\|f\|)\|f\|(n+1)^{-\nu-1/2}
\end{aligned}$$

and

$$\begin{aligned}
\|Ke_n\| &\le (n+1)^{-\nu-1/2}\|f\|+\sum_{j=0}^{n-1}(j+1)^{-1}\|z_{n-1-j}\|+\delta \\
&\le (n+1)^{-\nu-1/2}\|f\|+c_3\|f\|^2\sum_{j=0}^{n-1}(j+1)^{-1}(n-j)^{-2\nu-1/2}+c_2(n+1)^{-\nu-1/2}\|f\| \\
&\le (1+c_2+c_3c_4\|f\|)\|f\|(n+1)^{-\nu-1/2},
\end{aligned}$$

where c_4 is a positive constant depending only on ν. Thus, if $\|f\|$ is sufficiently small so that $c_3c_4\|f\|\le 1$, then (4.3.11) is proven for $j=n$, and this verifies (4.3.6). Moreover, (4.3.7) follows from the similar derivation for (4.3.13). □

Now we are ready to prove the error estimate.

Theorem 4.3.2. *Let Assumption 4.3.1 and Assumption 4.3.2 hold, let y^δ be a nosy data satisfying (4.3.4), and let $n_* := n_*(y^\delta)$ be the integer defined by Rule 4.2. If $\|F(x_0)-y^\delta\|>\tau\delta$ for some $\tau > 2(1+\eta)/(1-2\eta)$ and $x^\dagger - x_0$ satisfies Assumption 4.3.3 with $0<\nu\le 1/2$ for sufficiently small $\|f\|$, then*

$$\|x_{n_*}^\delta - x^\dagger\|\le C_\nu\|f\|^{1/(2\nu+1)}(\delta+\delta_*)^{2\nu/(2\nu+1)}$$

for some constant C_ν depending only on C, η, ν and κ, where

$$\delta := \|y^\delta - y\| \quad \text{and} \quad \delta_* := |F(x_{n_*}^\delta)-y^\delta\|$$

denote the actual noise level and the realized noise level respectively.

Proof. By using the polar decomposition for linear operators, which implies that $K^* = (K^*K)^{1/2}U$ for some partial isometry $U : Y \to X$, we can derive, as was done in [23], from (4.3.8) that

$$e_{n_*} = (K^*K)^\nu f_{n_*} + \left[\sum_{j=0}^{n_*-1} (I - K^*K)^j K^*\right] (y - y^\delta) \tag{4.3.16}$$

and

$$Ke_{n_*} = K(K^*K)^\nu f_{n_*} + (I - (I - KK^*)^{n_*})(y - y^\delta), \tag{4.3.17}$$

where

$$f_{n_*} = (I - K^*K)^{n_*} f + \sum_{j=0}^{n_*-1} (I - K^*K)^j (K^*K)^{1/2-\nu} \tilde{z}_{n_*-j-1}$$

and $\|\tilde{z}_j\| = \|z_j\|$, $j = 0, 1, \ldots, n_* - 1$. From (4.3.16) it follows that

$$\|e_{n_*}\| \leq \|(K^*K)^\nu f_{n_*}\| + \sqrt{n_*}\delta. \tag{4.3.18}$$

By using Lemma 4.3.1 we have

$$\sqrt{n_*}\delta \leq (n_* + a)^{1/2}\delta \leq c_2^{1/(2\nu+1)} \|f\|^{1/(2\nu+1)} \delta^{2\nu/(2\nu+1)}. \tag{4.3.19}$$

We next estimate $\|(K^*K)^\nu f_{n*}\|$. By the interpolation inequality((with $q = v + 1/2$ and $r = v$ in Proposition 2.11) and Proposition 2.13, we have

$$\begin{aligned}
\|(K^*K)^\nu f_{n_*}\| &\leq \left\|(K^*K)^{\nu+1/2} f_{n_*}\right\|^{2\nu/(2\nu+1)} \|f_{n_*}\|^{1/(2\nu+1)} \\
&= \|K(K^*K)^\nu f_{n_*}\|^{2\nu/(2\nu+1)} \|f_{n_*}\|^{1/(2\nu+1)}.
\end{aligned}$$

We therefore need to estimate $\|f_{n_*}\|$ and $\|K^*(KK^*)^\nu f_{n_*}\|$. In view of Lemma 4.3.3 we can see that (4.3.15) holds for all $j = 0, \cdots, n_*$. Therefore

$$\begin{aligned}
\|f_{n_*}\| &\leq \|f\| + \sum_{j=0}^{n_*-1} (j+1)^{\nu-1/2} \|z_{n_*-j-1}\| \\
&\leq \|f\| + c_3\|f\|^2 \sum_{j=0}^{n_*-1} (j+1)^{\nu-1/2} (n_* - j)^{-2\nu-1/2} \\
&\leq \|f\| + c_3 c_5 \|f\|^2 \leq \tilde{C}\|f\|.
\end{aligned}$$

By using (4.3.17) and Assumption 4.3.1 (a) we have

$$\begin{aligned}\|K(K^*K)^\nu f_{n_*}\| &\le \|Ke_{n_*}\|+\delta \le (1+\eta)\|y - F(x_{n_*}^\delta)\|+\delta \\ &\le (1+\eta)(\delta+\delta_*)+\delta \\ &= (2+\eta)(\delta+\delta_*).\end{aligned}$$

Therefore

$$\|(K^*K)^\nu f_{n_*}\| \le \tilde{C}_\nu \|f\|^{1/(2\nu+1)} (\delta+\delta_*)^{2\nu/(2\nu+1)} \tag{4.3.20}$$

for some constant $\tilde{C}_\nu$ depending only on η, C, ν and κ. Combining (4.3.19) and (4.3.20) with (4.3.18) we thus complete the proof. □

Corollary 4.3.1. Under the same conditions in Theorem 4.3.2 there holds

$$\|x_{n_*}^\delta - x^\dagger\| = O(\delta^{4\nu^2/(2\nu+1)^2}) \quad \text{as } \delta \to 0. \tag{4.3.21}$$

Proof. By the definition of n_* and n_δ, we may use Lemma 4.3.1 to derive that

$$\begin{aligned}\delta_*^2 &= \frac{1}{n_*+a}\Theta(n_*, y^\delta) \le \frac{1}{a}\Theta(n_*, y^\delta) \le \frac{1}{a}\Theta(n_\delta, y^\delta) \\ &= \frac{1}{a}(n_\delta+a)\|F(x_{n_\delta}^\delta) - y^\delta\|^2 \le \frac{1+a}{a} n_\delta \|F(x_{n_\delta}^\delta) - y^\delta\|^2 \\ &\le \frac{(1+a)c_1\tau^2}{a}\|f\|^{2/(2\nu+1)}\delta^{4\nu/(2\nu+1)}.\end{aligned}$$

Combining this with the estimate in Theorem 4.3.2 we then obtain the desired conclusion. □

In the proof of Corollary 4.3.1 we used $n_* \ge 0$ to derive the estimate $\delta_* = O(\delta^{2\nu/(2\nu+1)})$ and hence the estimate in (4.3.21). In fact n_8 can be much larger and thus much better error estimates than (4.3.21) can be seen in numerical computation.

4.4 Nesterov acceleration for linear Landweber method

Landweber iteration, although accurate, needs so many iterations in order to obtain a reliable solution. Hence, acceleration strategies need to be introduced.

Nesterov's acceleration strategy achieves a fast rate of $\mathcal{O}(1/n^2)$ for gradient method of minimizing smooth convex functionals [64]. This strategy has been studied extensively in the optimization community, most notably in applying it to general nonsmooth convex optimization, leading to the well-known fast iterative shrinkage thresholding algorithm (FISTA) [4]. In this acceleration, instead of taking the gradient step from the current approximate, it is taken at a linear combination of the recent two approximates.

Recently, Nesterov's acceleration strategy has been introduced in [42] to accelerate Landweber iteration for ill-posed inverse problems. Although no convergence analysis could be given, the numerical results presented in [42] demonstrate remarkable acceleration effect. This work inspired the subsequently work in [65] for linear inverse problems and the works in [34, 35, 82] for nonlinear inverse problems.

In this section we will focus on the accelerated Landweber iteration by Nesterov's strategy for linear ill-posed problems

$$Ax = y \tag{4.4.1}$$

where $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $\mathcal{X}$, $\mathcal{Y}$ are Hilbert spaces, and $y \in \mathcal{R}(A)$. Here, assume that instead of the exact data y, a noisy version y^δ exists such that $\left\|y - y^\delta\right\| \leq \delta$. The method takes the form

$$\begin{aligned} z_n^\delta &= x_n^\delta + \alpha_n(x_n^\delta - x_{n-1}^\delta), \\ x_{n+1}^\delta &= z_n^\delta + \omega A^*(y^\delta - Az_n^\delta), \\ x_0^\delta &= x_{-1}^\delta = x_0, \end{aligned} \tag{4.4.2}$$

with initial points $x_{-1}^\delta = x_0^\delta = 0$ and the relaxation parameter ω satisfying

$$0 < \omega \left\|A\right\|^2 \leq 1. \tag{4.4.3}$$

The sequence α_n is chosen as

$$\alpha_n = \frac{n-1}{n+\beta}, \tag{4.4.4}$$

where $\beta \geq 1$. Using the source condition $x^\dagger \in R((T^*T)^\mu)$ for some $\mu \geq 0$, convergence rates under the discrepancy principle is given by (see [65])

$$\left\|x_{n(\delta,y^\delta)}^\delta - x^\dagger\right\| = o(\delta^{\mu/(\mu+1)}),$$

which is not optimal, unlike the one under an *a-priori* stopping criterion. This is mainly due to the suboptimality of the noise propagation error $\left\|x_{n(\delta,y^\delta)}^\delta - x_{n(\delta,y^\delta)}\right\|$. Nevertheless, algorithm (4.4.2) needs only the square root of the number of iterations required by (4.1.5) to terminate according to the discrepancy principle.

In the following we will provide a convergence analysis of the algorithm (4.4.2) under the Hanke-Raus heuristic rule.

As shown in [65], the iterates x_n^δ in (4.4.2) can be written as

$$x_n^\delta = g_n(A^*A)A^*y^\delta \tag{4.4.5}$$

where g_n is a polynomial of degree $n-1$, and the residual polynomials

$$r_n(\lambda) = 1 - \lambda g_n(\lambda) \tag{4.4.6}$$

satisfy the recurrence relation

$$r_{n+1}(\lambda) = (1-\omega\lambda)\left[r_n(\lambda) + \alpha_n(r_n(\lambda) - r_{n-1}(\lambda))\right], \quad n \in \mathbb{N},$$

with $r_0(\lambda) = r_{-1}(\lambda) = 1$. Alternatively, it can also be expressed in the form

$$r_{n+1}(\lambda) = (1-\omega\lambda)\left[(1-\theta_n)r_n(\lambda) + \theta_n\left(r_{n-1}(\lambda) + \frac{1}{\theta_{n-1}}(r_n(\lambda) - r_{n-1}(\lambda))\right)\right], \tag{4.4.7}$$

for $n \in \mathbb{N}_0$, with

$$\alpha_n = \frac{\theta_n}{\theta_{n-1}}(1-\theta_{n-1})$$

For completeness, we gather two previous results concerning the polynomial r_n in (4.4.6)-(4.4.7).

Lemma 4.4.1. *Let r_n satisfy the recurrence relation* (4.4.7) *with ω as in* (4.4.3)*. Moreover, the sequence θ_n satisfies*

$$1 \geq \theta_n \geq \frac{\theta_{n-1}}{1+\theta_{n-1}}, \quad n \in \mathbb{N}, \quad \theta_0 = 1.$$

Then

$$|r_n(\lambda)| \leq 1 \;\; \text{and} \;\; \lambda r_n^2(\lambda) \leq \frac{\theta_{n-1}^2}{\omega}(1-\omega\lambda)^{n+1} \tag{4.4.8}$$

for all $0 \leq \lambda \leq \|A\|^2$.

Proof. This is [65, Lemma 2.1]. □

Lemma 4.4.2. *Let r_n satisfy* (4.4.7) *with ω as in* (4.4.3) *and*

$$\theta_n = \frac{\beta}{n+\beta}, \quad \beta \geq 1.$$

Then for all $0 \le \lambda \le \|A\|^2$ *and* $n \in \mathbb{N}$,

$$\lambda^\mu |r_n(\lambda)| \le c_\mu \begin{cases} n^{-2\mu}, & 0 \le \mu < \frac{1}{2}, \\ n^{-(\mu+\frac{1}{2})}, & \frac{1}{2} < \mu, \end{cases}$$

where

$$c_\mu = \begin{cases} \beta^{2\mu}\omega^{-\mu}, & 0 \le \mu \le \frac{1}{2}, \\ \beta\omega^{-\mu}(2\mu-1)^{\mu-\frac{1}{2}} e^{\frac{1}{2}-\mu}, & \mu > \frac{1}{2}. \end{cases}$$

Furthermore, for any $p \ge 0$ *and* $0 \le \lambda \le \|A\|^2$,

$$\lim_{n\to\infty} n^p |r_n(\lambda)| = 0.$$

Proof. This is Proposition 2.2 of [65]. □

4.4.1 Convergence

To solve problem (4.4.1) with the iterative method (4.4.2)-(4.4.4), we propose a corresponding version of the Hanke-Raus rule:

Rule 4.3. *Let* $a \ge 1$ *be a fixed integer, and* $\Theta(n, y^\delta) := (n+a)\left\|Ax_n^\delta - y^\delta\right\|$, *for* $n \in \mathbb{N}$. *Define* $n_* := n_*(y^\delta)$ *be the integer such that*

$$n_* \in \operatorname{argmin}\left\{\Theta(n, y^\delta) := (n+a)\left\|Ax_n^\delta - y^\delta\right\|,\ n = 1, 2, \ldots\right\} \tag{4.4.9}$$

and use $x_{n_*}^\delta$ *to approximate solution* $x^\dagger$.

The following noise condition is also used.

Assumption 4.4.1. $\left\{y^\delta\right\}$ *is a family of noisy data satisfying* $0 < \left\|y^\delta - y\right\| \to 0$ *as* $\delta \to 0$ *and there is a constant* $\kappa > 0$ *such that*

$$\left\|Ax - y^\delta\right\| \ge \kappa \left\|y - y^\delta\right\|$$

for every y^δ *and every* $x \in S(y^\delta)$, *where* $S(y^\delta)$ *denotes the sequence* $\left\{x_n^\delta : 0 \le n \le n_\infty\right\}$ *generated by* (4.4.2) *using* y^δ.

The sequence $\left\{x_n^\delta\right\}$ defined by(4.4.5) is a semiiterative method with a residual formula given by (4.4.6). Then by Lemma 4.4.1, the following existing result holds.

Lemma 4.4.3. *Let $\{x_n^\delta\}$ be a semiiterative iterative method of the form (4.4.5). If its residual polynomials $\{r_n\}$ satisfy*

$$|r_n| \le 1, \quad 0 \le \lambda \le 1, \quad n \in \mathbb{N}_0,$$

and converge to zero pointwise on $(0,1]$, then the propagated data error holds:

$$\left\|x_n - x_n^\delta\right\| \le 2n\delta.$$

Proof. This is [16, Theorem 6.8]. □

We also use the result on the convergence of method (4.4.2) via discrepancy principle: terminate after $n(\delta, y^\delta)$ iterations where

$$\left\|Ax_{n(\delta,y^\delta)}^\delta - y^\delta\right\| \le \tau\delta < \left\|Ax_n^\delta - y^\delta\right\|, \quad 0 \le n < n(\delta, y^\delta) \tag{4.4.10}$$

for some $\tau > 1$. Note this result is optimal even when $n(\delta, y^\delta)$ stays bounded as $\delta \to 0$. Also, the following result involves Lemma 4.4.2.

Lemma 4.4.4. *Let $\delta > 0$ and x_n^δ be the iterates generated by (4.4.2) with α_n as in (4.4.4). Assume $x^\dagger \in R((A^*A)^\mu)$ for some $\mu \ge 0$. Let $n(\delta, y^\delta)$ be selected by discrepancy principle (4.4.10). Then it holds*

$$n(\delta, y^\delta) \le \bar{c}\delta^{-\frac{1}{\mu+1}} \tag{4.4.11}$$

where $\bar{c}$ is a constant depending on β, ω, μ, and τ. Moreover,

$$\left\|x_{n(\delta,y^\delta)}^\delta - x^\dagger\right\| = o(\delta^{\frac{\mu}{\mu+1}}).$$

Proof. This is [65, Theorem 4.1]. □

We present the main result.

Theorem 4.4.1. *Let Assumption 4.4.1 hold. Let $\delta > 0$ and x_n^δ be the iterates generated by (4.4.2)– (4.4.4). Assume $\left\|Ax_0 - y^\delta\right\| = \left\|y^\delta\right\| > \tau\left\|y^\delta - y\right\|$ for some $\tau > 1$. Assume $x^\dagger = (A^*A)^\mu v$ for some $\mu > 0$ with $\|v\| = \rho$. Let n_* be selected by Rule 4.3. Then we have*

$$n_* \le C\delta^{-\frac{1}{\mu+1}} \tag{4.4.12}$$

where C is a positive constant depending only on β, ω, μ, τ, and a. Moreover we have the error *bound*

$$\left\|x_{n_*}^\delta - x^\dagger\right\| \leq \bar{C}\,(\delta + \delta_*)^{\frac{\mu}{\mu+1}} \tag{4.4.13}$$

where

$$\delta := \left\|y^\delta - y\right\|, \quad d_* := \left\|Ax_{n_*}^\delta - y^\delta\right\|,$$

and $\bar{C}$ is a positive constant depending only on ρ, β, ω, μ, τ, and a.

Proof. First, the approximation error can be decomposed as

$$\left\|x_n^\delta - x^\dagger\right\| \leq \left\|x_n - x^\dagger\right\| + \left\|x_n - x_n^\delta\right\|. \tag{4.4.14}$$

The first expression in the right-hand side of (4.4.14) is the exact error. To estimate it, we use (4.4.5) with $\delta = 0$. This leads us to

$$\begin{aligned} x^\dagger - x_n &= x^\dagger - g_n(A^*A)A^*(Ax^\dagger) \\ &= (I - g_n(A^*A)A^*A)x^\dagger \\ &= (I - A^*Ag_n(A^*A))x^\dagger \\ &= r_n(A^*A)x^\dagger, \end{aligned} \tag{4.4.15}$$

where (4.4.15) is from (4.4.6). We follow the proof in [16, Theorem 4.34] (please also see Proposition 2.13 and the identity (2.3.3)). To estimate (4.4.15) we apply the assumption on $x^\dagger$, the interpolation inequality in Proposition 2.11 (let $r = \mu$, $q = \mu + 1/2$) , the left estimate in (4.4.8). This leads us to

$$\begin{aligned} \left\|r_n(A^*A)x^\dagger\right\| &\leq \left\|(A^*A)^{\mu+1/2} r_n(A^*A)v\right\|^{\frac{2\mu}{2\mu+1}} \left\|r_n(A^*A)v\right\|^{\frac{1}{2\mu+1}} \\ &= \left\|r_n(AA^*)Ax^\dagger\right\|^{\frac{2\mu}{2\mu+1}} \left\|r_n(A^*A)v\right\|^{\frac{1}{2\mu+1}} \\ &\leq \left(\left\|r_n(AA^*)y^\delta\right\| + \left\|r_n(AA^*)(y - y^\delta)\right\|\right)^{\frac{2\mu}{2\mu+1}} \|v\|^{\frac{1}{2\mu+1}}. \end{aligned} \tag{4.4.16}$$

where we used the left estimate in (4.4.8). Note that $r_n(AA^*)y^\delta = y^\delta - Ax_n^\delta$. We invoke again the the left estimate in (4.4.8). With $n = n_*$ in (4.4.15), we obtain

$$\left\|x^\dagger - x_{n_*}\right\| = \left\|r_{n_*}(A^*A)x^\dagger\right\| \leq c_1\,(\delta_* + \delta)^{\frac{2\mu}{2\mu+1}} \tag{4.4.17}$$

where $c_1 = \rho^{\frac{1}{2\mu+1}}$.

Now we turn to estimating the second term in the right-hand side of (4.4.14). Since Lemma

4.4.1 holds, we can use Lemma 4.4.3 with $n = n_*$, leading to

$$\left\|x_{n_*} - x_{n_*}^{\delta}\right\| \leq 2n_*\delta \tag{4.4.18}$$

We estimate the right-hand side of (4.4.18). Similarly in the proof of Lemma 4.3.1, by using assumption 4.4.1 and Lemma 4.4.4, we have

$$\begin{aligned} n_*\delta \leq (n_* + a)\delta &\leq \Theta(n_*, y^\delta) \\ &\leq \Theta(n(\delta, y^\delta), y^\delta) \\ &= (n(\delta, y^\delta) + a)\left\|Ax_{n(\delta,y^\delta)}^{\delta} - y^\delta\right\| \\ &\leq \tau(n(\delta, y^\delta) + a)\delta, \end{aligned}$$

implying that

$$n_* \leq \tau(n(\delta, y^\delta) + a).$$

Since $\left\|Ax_0^\delta - y^\delta\right\| = \left\|y^\delta\right\| > \tau\delta$ for some $\tau > 1$, it follows that $n(\delta, y^\delta) \geq 1$. Therefore

$$n_* \leq \tau(1 + a)n(\delta, y^\delta),$$

and by Lemma 4.4.4, we have

$$n_* \leq \tau(1 + a)\bar{c}\delta^{-\frac{1}{\mu+1}},$$

proving (4.4.12). Plugging the last inequality into (4.4.18), we have

$$\begin{aligned} \left\|x_{n_*} - x_{n_*}^{\delta}\right\| &\leq c_2\delta^{\frac{\mu}{\mu+1}} \\ &\leq c_2(\delta_* + \delta)^{\frac{\mu}{\mu+1}}, \end{aligned} \tag{4.4.19}$$

where $c_2 := 2\tau(1+a)\bar{c}$. Since $\mu/(\mu+1) < 2\mu/(2\mu+1)$ for all $\mu > 0$, the stability estimate (4.4.19) is bigger than (4.4.17). Plugging (4.4.19) and (4.4.17) into (4.4.14), we obtain (4.4.13). □

We also have the following result, which can be proved similarly as Corollary 4.3.1.

Corollary 4.4.1. *Under the same assumptions in Theorem 4.4.1,*

$$\left\|x_{n_*}^{\delta} - x^\dagger\right\| = \mathcal{O}(\delta^{\frac{\mu}{\mu+1}}) \text{ as } \delta \to 0. \tag{4.4.20}$$

The obtained estimates and rates in Theorem 4.4.1 and Corollary 4.4.1 are the same as in Theorem 4.4.4. The obtained convergence rate is not even optimal even for the case of $\mu \leq 1/2$. The suboptimality comes only from the stability estimate $\left\|x_{n_*} - x_{n_*}^\delta\right\|$, similar in [65]. Meanwhile, the obtained estimates on the regularization parameter n_* is much smaller than the one for the usual Landweber iteration given by $\mathcal{O}(\delta^{-2/(2\mu+1)})$ [24]. In practice the total error can still be much smaller than the obtained estimates.

4.4.2 Numerical results

In this section we present some numerical results for the linear integral equation of the first kind given by

$$Ax(s) := \int_0^1 K(s,t)x(t)dt = y(s) \tag{4.4.21}$$

with kernel

$$K(s,t) := \begin{cases} s(1-t), & s \leq t, \\ t(1-s), & s \geq t. \end{cases} \tag{4.4.22}$$

Here we choose $\mathcal{X} = \mathcal{Y} := L^2\,[0,1]$ such that the operator A is compact, self-adjoint, and injective. Then the operator equation $Ax = y$ can be rewritten as $x = -y''$ provided that $y \in H^2[0,1] \cap H_0^1\,[0,1]$. In addition A has the eigensystem $\{\sigma_j; u_j; u_j\}_{j=1}^\infty$ with $\sigma_j = (j\pi)^{-1}$ and $u_j(t) = \sqrt{2}\sin(j\pi t)$. Furthermore, interpolation theory (see [62]) shows that for $4\mu - 1/2 \notin \mathbb{N}$

$$R((A^*A)^\mu) = \left\{x \in H^{4\mu}[0,1] \,:\, x^{2l}(0) = x^{2l}(1) = 0,\ l = 0,1,\ldots,\lfloor 2\mu - 1/4 \rfloor\right\}.$$

Now, let $\mathcal{Y}_n$ be the finite element space of piecewise linear functions on a uniform grid with step size $1/(n-1)$. Denote by P_n the orthogonal projection operator from $\mathcal{Y}$ to $\mathcal{Y}_n$. Define $A_n := P_n A$ and $\mathcal{X}_n := A_n^* \mathcal{Y}_n$. Let $\{\phi_j\}_{j=1}^n$ be a basis of $\mathcal{Y}_n$. By this discretization, instead of (4.4.21)-(4.4.22), we solve a system of linear equations $A_n x_n = y_n$ in practice, where $[A_n]_{ij} = \int_0^1 (\int_0^1 K(s,t)\phi_i(t))\phi_j(t)dt$ and $[y_n]_j = \int_0^1 y(t)\phi_j(t)dt$.

The following right-hand sides for (4.4.21) are considered in the following examples, based on [65].

Example 4.4.1. Let $y(s) = s(1-s)$ and $x^\dagger(t) = 2$. Then $x^\dagger \in R((A^*A)^\mu)$ for all $\mu < 1/8$.

Example 4.4.2. Let $y(s) = s^4(1-s)^3$ and $x^\dagger(t) = -6t^2(1-t)(2-8t+7t^2)$. Then $x^\dagger \in R((A^*A)^\mu)$ for all $\mu < 5/8$.

Example 4.4.3. Let $y(s) = s^9(1-s)^{10}$ and $x^\dagger(t) = -18t^7(1-t)^8(4-18t+19t^2)$. Then $x^\dagger \in R((A^*A)^\mu)$ for all $\mu < 17/8$.

Table 4.6: Results for Landweber iteration with Nesterov acceleration under Rule 4.3 using different values of a in Example 4.4.1

δ \ a	1e1		1e2		1e3		1e4	
	n_*	L2Err	n_*	L2Err	n_*	L2Err	n_*	L2Err
1e-1	42	4.29e-1	433	4.28e-1	43	4.28e-1	385	2.92e-1
1e-2	43	4.29e-1	379	2.90e-1	928	2.38e-1	1219	1.31e0
1e-3	43	4.29e-1	1888	1.97e-1	3531	1.68e-1	5788	1.50e-1
1e-4	13159	1.20e-1	13164	1.20e-1	17659	1.11e-1	24229	1.02e-1

Table 4.7: Results for Landweber iteration under Rule 4.3 using different values of a in Example 4.4.1

δ \ a	1e1		1e2		1e3		1e4	
	n_*	L2Err	n_*	L2Err	n_*	L2Err	n_*	L2Err
1e-1	32	4.27e-1	39	4.24e-1	51	4.20e-1	824	3.10e-1
1e-2	39	4.25e-1	46	4.23e-1	1578	2.92e-1	3351	2.69e-1
1e-3	40	4.25e-1	46	4.24e-1	2001	2.85e-1	14883	2.23e-1
1e-4	40	4.25e-1	46	4.24e-1	2005	2.85e-1	15929	2.21e-1

Table 4.8: Results for Landweber iteration with Nesterov acceleration under Rule 4.3 using different values of a in Example 4.4.2

δ \ a	1e1		1e2		1e3		1e4	
	n_*	L2Err	n_*	L2Err	n_*	L2Err	n_*	L2Err
1e-1	1	1.00e0	393	3.02e-1	657	1.66e-1	699	1.57e-1
1e-2	1252	7.09e-2	1253	7.08e-2	1259	7.03e-2	1813	4.93e-2
1e-3	2710	2.59e-2	2710	2.56e-2	3800	2.02e-2	5231	1.88e-2
1e-4	10839	1.56e-2	10839	1.56e-2	10839	1.56e-2	15356	1.51e-2

All examples were tested using the grid size $m = 100$. Uniformly distributed noises with the magnitude δ' are added to the discrete exact right-hand side, so that the discrete noisy data is computed by

$$\left[y_n^\delta\right]_j := [1 + \delta' \cdot (2\mathrm{Rand}(x) - 1)] \cdot [y_n]_j\,, \quad j = 1, \ldots, m,$$

where $\mathrm{Rand}(x)$ returns a pseudo-random value drawn from a uniform distribution on $[0, 1]$. From here, the noise level is calculated as $\delta = \left\|y_n^\delta - y_n\right\|_2$, where $\|\cdot\|_2$ denotes the standard vector norm in $\mathbb{R}^n$. To assess the accuracy we compute the L^2-norm relative error for a discrete approximate solution at iteration n, denoted by x_n^δ denoted by $\mathrm{L2Err} = \left\|x_n^\delta - x^\dagger\right\|_{L^2[0,1]} / \left\|x^\dagger\right\|_{L^2[0,1]}$, where $x^\dagger$ is the exact solution to the corresponding model problem.

Table 4.9: Results for Landweber iteration under Rule 4.3 using different values of a in Example 4.4.2

$\delta \backslash a$	$1e1$		$1e2$		$1e3$		$1e4$	
	n_*	L2Err	n_*	L2Err	n_*	L2Err	n_*	L2Err
1e-1	12	9.45e-1	24	9.26e-1	1588	3.31e-1	3373	2.12e-1
1e-2	12	9.45e-1	25	9.24e-1	7887	1.24e-1	13976	8.69e-2
1e-3	12	9.45e-1	25	9.24e-1	56137	3.36e-2	74683	2.82e-2
1e-4	12	9.45e-1	379271	1.67e-2	380247	1.67e-2	389517	1.66e-2

Table 4.10: Results for Landweber iteration with Nesterov acceleration under Rule 4.3 using different values of a in Example 4.4.3

$\delta \backslash a$	$1e1$		$1e2$		$1e3$		$1e4$	
	n_*	L2Err	n_*	L2Err	n_*	L2Err	n_*	L2Err
1e-1	1	1.00e0	1006	1.71e-1	1084	1.47e-1	1842	5.02e-2
1e-2	1	1.00e0	2698	2.15e-2	2702	2.15e-2	3554	1.90e-2
1e-3	5145	3.13e-3	5145	3.13e-3	5146	3.13e-3	8310	4.34e-3
1e-4	12989	8.14e-4	12989	8.14e-4	20866	1.55e-3	20866	1.55e-3

For comparison, we also include numerical results using Landweber iteration under Rule 4.2. We also compare the performance of different values of the fixed constant a in Rule 4.3. Tables 4.6-4.7 shows the numerical results using Nesterov acceleration and the usual Landweber method for Example 4.4.1. Correspondingly, Tables 4.8-4.9 and Tables 4.10-4.11 show the results for Examples 4.4.2 and 4.4.3, respectively. Even for smaller μ values (Examples 4.4.1 and 4.4.2), using Nesterov acceleration can still yield better solutions with a sufficiently large fixed constant a for Rule 4.3. Figures 4.10(a)-(c) and Figures 4.9(a)-(c) plots the reconstruction results for Example 4.4.2 for the two methods. The acceleration effect allows more accurate reconstruction by preventing the regularization parameter n_* from being too small, even for a small choice of the fixed parameter a in Rule 4.3. This is illustrated in Figure 4.9(e) for Example 4.4.2, where for a smaller choice of a the parameter n_* by the Landweber iteration could be very small. In contrast, Figure 4.10(e) shows the oscillations on $\Theta(n, y^\delta)$ brought by the acceleration. Figures 4.9(d) and 4.10(d) illustrate that the noise condition in assumption 4.4.1 is satisfied.

Table 4.11: Results for Landweber iteration under Rule 4.3 using different values of a in Example 4.4.3

$\delta \backslash a$	$1e1$		$1e2$		$1e3$		$1e4$	
	n_*	L2Err	n_*	L2Err	n_*	L2Err	n_*	L2Err
1e-1	1	1.00e0	17	9.84e-1	1264	6.36e-1	11184	1.93e-1
1e-2	1	1.00e0	17	9.84e-1	46933	3.61e-2	52776	2.90e-2
1e-3	1	1.00e0	17	9.84e-1	139525	3.23e-3	141487	3.13e-3
1e-4	1	1.00e0	17	9.84e-1	264202	9.04e-4	266777	8.92e-4

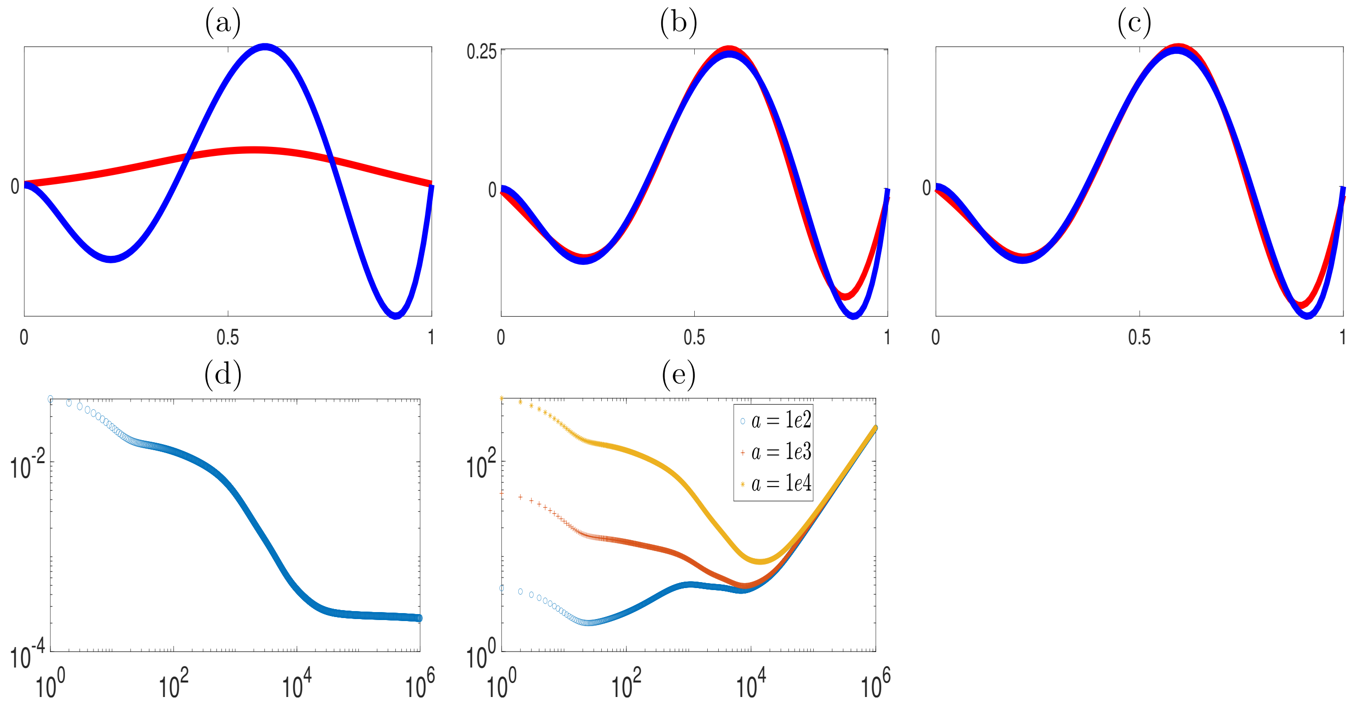

Figure 4.9: The reconstruction for Example 4.4.2 using Landweber iteration under the Hanke-Raus rule, with noise level δ =1.0e-2: (a), (b), and (c) plot the reconstructions (red) and the exact (blue) solution for $a = 100$, 1000, and 10000, respectively; (d) plots the relationship between the residual $\|Ax_n^\delta - y^\delta\|$ and n; (e) plots the relationship between $\Theta(n, y^\delta)$ and n for the aforementioned values of a.

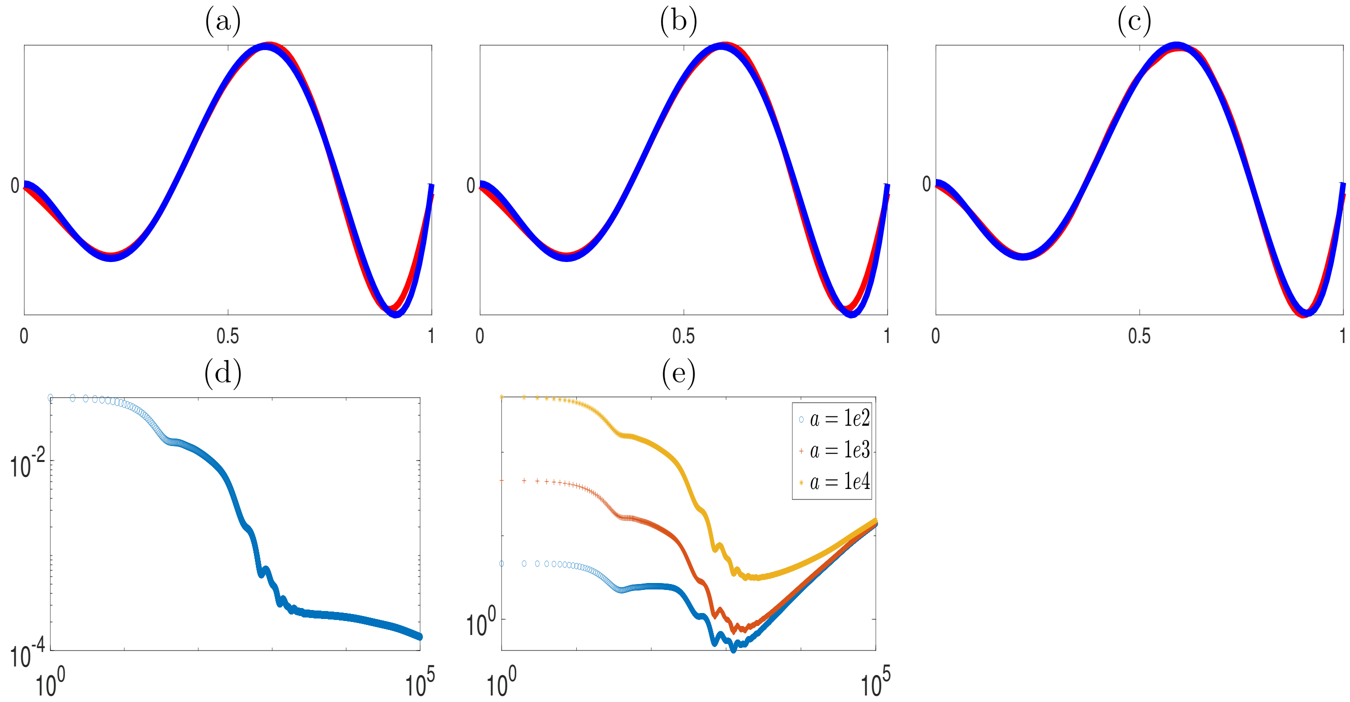

Figure 4.10: The reconstruction for Example 4.4.2 using the Nesterov acceleration for Landweber iteration under the Hanke-Raus rule, with noise level δ =1.0e-2: (a), (b), and (c) plot the reconstructions (red) and the exact (blue) solution for $a = 100$, 1000, and 10000, respectively; (d) plots the relationship between the residual $\|Ax_n^\delta - y^\delta\|$ and n; (e) plots the relationship between $\Theta(n, y^\delta)$ and n for the aforementioned values of a.

Chapter 5

Conclusion

A general convergence result for variational regularization and Landweber method for inverse problems under the Hanke-Raus heuristic was established. The Hanke-Raus is a heuristic or purely data-driven parameter choice rule that does not need any information on the noise level.

In particular, a new convergence analysis on the variational regularization for inverse problems in a general framework is performed. Such framework covers a more general type of data misfit functional satisfying only a quasi-triangle inequality. Since this new framework does not include a forward operator, it covers a broader range of problems. A heuristic parameter choice rule, known as the Hanke-Raus rule, was proposed for choosing the regularization parameter. Under the variational source condition, *a posteriori* error estimates were obtained. By imposing a certain noise condition, convergence under the Hanke-Raus rule was established. Numerical examples on elliptic parameter estimation and electrical impedance tomography were shown to demonstrate the convergence results.

One drawback of variational regularization is the relatively expensive computation needed for every candidate regularization parameter. The Landweber method is a well-studied iterative regularization scheme, and is usually attractive for its relatively low complexity per iteration. A Landweber method for solving linear or nonlinear inverse problems in Banach spaces is being studied. The method involves a nonsmooth convex penalty functional to capture solutions with specific properties. A new convergence analysis of this method which requires neither the (Gâteaux) differentiability of the forward operator nor the reflexivity of the data/image space. Both the discrepancy principle and the Hanke-Raus rule were proposed as parameter choice rules. The established convergence allows the applicability of the Landweber iteration towards non-smooth inverse problems and problems whose data is corrupted by various types of noise, especially non-Gaussian noise. Numerical examples involving a nonlinear elliptic parameter estimation, image deblurring, and a source term estimation of a non-smooth semilinear PDE were considered.

For completeness, a new convergence analysis for the classical Landweber method in Hilbert spaces with the Hanke-Raus rule was also established. Here, aside from convergence, convergence rates were also established using a source condition on the exact solution.

A disadvantage of the Landweber method is the need of a lot iterations in order to converge to a solution. To overcome this, acceleration schemes are proposed. The Landweber method with Nesterov acceleration for solving linear inverse problems in Hilbert spaces was studied. A new convergence result under the Hanke-Raus heuristic rule was established. If the exact solution satisfies a source condition, then the convergence rate under the Hanke-Raus rule is only suboptimal. Nevertheless, it is still better than the one for the classical Landweber iteration. In practice the total approximation error can still be smaller than the obtained rates. Numerical examples show, that indeed this acceleration can produce more accurate solutions in fewer iterations.

According to Bakushinskii's veto, heuristic parameter choice rules cannot be expected to converge in general under the worst-case scenario. To overcome this, a certain condition on the noisy data is being imposed in order to obtain convergence results. This condition is the so-called noise condition and was used for the new convergence results for both variational regularization and Landweber iteration under the Hanke-Raus rule. This condition is independent of the regularization method, and cannot be verified in reality unless the exact data is accessible. Nevertheless, the Hanke-Raus rule should work when the noise is understood to be random.

Our new results, both theoretical and numerical, provides more progress in the two main regularization methods. With the Hanke-Raus rule, this dissertation justifies further the use of data-driven parameter choice rules in solving a wider class of inverse problems.

Future work

We cite the following for future projects:

1. We only proved the convergence of the Landweber iteration in Banach spaces with a nonsmooth convex penalty term. Applying Nesterov acceleration to Landweber iteration is a significant improvement, but as of writing there is no sufficient theory to prove its convergence. Thus it remains an open problem.

2. Another acceleration strategy for the studied Landweber iteration in Banach spaces involves using multiple search directions. Its convergence for linear problems is already established. Proving its convergence on nonlinear inverse problems should be straightforward, since there is enough theoretical results to build up from.

3. The Landweber iteration in Banach spaces was studied in terms of a p-convex penalty functional for $2 \leq p < \infty$. An interesting issue to look at is the practical aspect when $p > 2$, and the computations involved within the iteration.

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