H^∞ Control of Linear Quantum Stochastic Systems

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***Abstract*—The purpose of this paper is to formulate and solve a H^∞ controller synthesis problem for a class of noncommutative linear stochastic systems which includes many examples of interest in quantum technology. The paper includes results on the class of such systems for which the quantum commutation relations are preserved (such a requirement must be satisfied in a physical quantum system). A quantum version of standard (classical) dissipativity results are presented and from this a quantum version of the Strict Bounded Real Lemma is derived. This enables a quantum version of the two Riccati solution to the H^∞ control problem to be presented. This result leads to controllers which may be realized using purely quantum, purely classical or a mixture of quantum and classical elements. This issue of physical realizability of the controller is examined in detail, and necessary and sufficient conditions are given. Our results are constructive in the sense that we provide explicit formulas for the Hamiltonian function and coupling operator corresponding to the controller.**

***Index Terms*—H^∞ robust control, dissipativity, quantum controller realization, quantum feedback control, quantum optics, strict bounded real lemma.**

I. Introduction

RECENT developments in quantum and nano technology have provided a great impetus for research in the area of quantum feedback control systems; e.g., see [1], [4], [11], [16], [27], [29], and [30]. In particular, it is now being realized that robustness is a critical issue in quantum feedback control systems, as it is in classical (i.e., nonquantum) feedback control systems; e.g., see [9], [10], and [31]. However, the majority of feedback control results for quantum systems do not address the issue of robustness directly. The aim of this paper is to address the problem of systematic robust control system design for quantum systems via a H^∞ approach.

We present a H^∞ controller synthesis result for a class of noncommutative linear stochastic systems which includes many examples of interest in quantum technology. The synthesis objective is to find a disturbance attenuating controller which bounds the influence of certain signals, called the disturbance input signals, on another set of signals, called the performance output signals. In this way, the undesirable effects of disturbances on performance is reduced in a systematic and quantifiable way. This follows from a quantum version of the small gain theorem [10]; indeed, the controller will be robustly stabilizing against certain kinds of uncertainties, which in principle could include parameter uncertainties, modelling errors, etc. To illustrate the results, we consider some design examples in quantum optics.

A feature of our approach is that the control designer can choose to synthesize a controller which may be quantum, classical or a mixed quantum-classical controller for the plant. The majority of the available results in quantum feedback control consider the controller to be a classical (i.e., nonquantum) system, which may be implemented using analog or digital electronics. Classical controllers process measurement data obtained by monitoring the quantum system to determine control actions which influence the dynamics of the quantum system in a feedback loop. In contrast, quantum controllers are themselves quantum systems, and the closed loop is fully quantum; e.g., see [5], [10], [18], and [28]–[31]. In [29] and [30], a transfer function approach to quantum control based on the chain scattering approach to H^∞ control has been proposed. However, the plants and controllers considered therein are single input single output (SISO) systems having only quantum degrees of freedoms. Moreover, no systematic treatment is given of the physical realizability of the resulting controllers. On the other hand, our approach is developed for a fairly general class of multiple input multiple output (MIMO) quantum linear stochastic systems with possibly mixed quantum and classical degrees of freedom and addresses the physical realizability issue.

Our approach involves deriving a quantum version of the Strict Bounded Real Lemma (e.g., see [21]). We begin by considering a general problem of dissipativity for quantum systems in a manner that generalizes Willems' theory of dissipative systems (see [26]), originally developed for nonlinear deterministic classical systems. The paper characterizes this dissipation property in algebraic terms. This then leads to a quantum version of the Strict Bounded Real Lemma. This lemma is then applied to the closed-loop system formed from the interconnection between the quantum plant and the controller. By following an algebraic approach to the H^∞ control problem such as in [21], this enables us to derive a quantum version of the celebrated two Riccati solution to the H^∞ control problem; e.g., see [15] and [32].

The two Riccati quantum H^∞ result which is derived leads to formulas for some, but not all, of the controller state space matrices. Controller noise sources (needed for physical realizability, as discussed shortly) are not determined by these Riccati equations. If the designer chooses to synthesize a classical controller, then the standard classical H^∞ controller suffices, and no further matrices nor noise sources need be determined. However, if the designer chooses to synthesize a controller that is itself a quantum system, or contains a component that is a quantum system, then the controller design must be completed

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by selecting the undetermined matrices and noise sources to ensure that the controller is physically meaningful. For example, in a quantum controller, quantum mechanics dictates that the time evolution of a closed system preserve certain commutation relations. This requirement constrains the possible controller matrices and noise sources for a physically realizable controller. To address this issue, the paper considers the question of physical realizability. Starting with a standard parameterization of purely quantum linear systems in terms of a quadratic Hamiltonian and a linear coupling operator (e.g., see [12]), we then derive necessary and sufficient conditions for given controller state space matrices to be physically realizable. These conditions are constructive in that if a set of controller state space matrices are physically realizable, then we can construct the required Hamiltonian function and coupling operator.

We begin in Section II by presenting the class of models under consideration and we present a result describing the condition such systems must satisfy in order to correspond to a physical quantum system in that the quantum commutation relations are preserved. In Section IV, we consider the question of dissipation for quantum systems and derive a quantum version of the Strict Bounded Real Lemma. In Section V, we set up the H^∞ problem to be solved and present our main result which is a two-Riccati solution to this quantum H^∞ control problem. This section also considers the question of physical realizability. In Section VI, we consider the application of our quantum H^∞ control results to the question of stability robustness and we establish a version of the small gain theorem for quantum systems subject to parameter uncertainty. In Section VII, we present some examples from quantum optics to illustrate the theory which has been developed. In particular, we consider the control of quantum optical plants using quantum, classical, and quantum-classical controllers. We also consider the design of a purely quantum controller which leads to robustness against uncertainty in one of the physical parameters of the cavity system. The paper is organized so that the Appendix contains all of the proofs of the results which are presented.

II. Linear Quantum Stochastic Models

In this paper, we are interested in physical systems that contain one or more components that are quantum in nature. It is helpful to have in mind an interconnection of components, some of which are "classical", meaning that nonquantum descriptions suffice, and some for which "quantum" descriptions are required. Such systems are common in quantum optics laboratories, and may occur, for instance, in schemes for implementing quantum computing and information processing algorithms. We use noncommutative or quantum probability theory (e.g., see [7] and the references therein) to describe the systems of interest. This framework is quite general and encompasses quantum and classical mechanical systems. Quantum noise, which may arise from measurements or interactions between subsystems and the environment, is central.

We consider linear noncommutative stochastic systems of the form

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t); \quad x(0) = x_0 \\ dy(t) &= Cx(t)dt + Ddw(t) \end{aligned} \tag{1}$$

where $A, B, C,$ and D are, respectively, real $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n_w}, \mathbb{R}^{n_y\times n}$ and $\mathbb{R}^{n_y\times n_w}$ matrices (n, n_w, n_y are positive integers), and $x(t) = [x_1(t) \ \dots \ x_n(t)]^T$ is a vector of self-adjoint possibly noncommutative system variables.

The initial system variables $x(0) = x_0$ consist of operators (on an appropriate Hilbert space) satisfying the *commutation relations*[1]

$$[x_j(0), x_k(0)] = 2i\Theta_{jk}, \quad j,k = 1,\dots,n \tag{2}$$

where Θ is a real antisymmetric matrix with components Θ_{jk}, and $i = \sqrt{-1}$. Here, the commutator is defined by $[A,B] = AB - BA$. To simplify matters without loss of generality, we take the matrix Θ to be of one of the following forms:

- *Canonical* if $\Theta = \mathrm{diag}(J, J, \dots, J)$ or
- *Degenerate canonical* if $\Theta = \mathrm{diag}(0_{n'\times n'}, J, \dots, J)$, where $0 < n' \le n$.

Here, J denotes the real skew-symmetric 2×2 matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the "diag" notation indicates a block diagonal matrix assembled from the given entries. To illustrate, the case of a system with one classical variable and two conjugate quantum variables is characterized by $\Theta = \mathrm{diag}(0, J)$, which is degenerate canonical. It is assumed that x_0 is Gaussian, with density operator ρ.

The vector quantity w describes the input signals and is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t) \tag{3}$$

where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is a self adjoint, adapted process (see, e.g., [7], [19], [20] for a discussion of adapted processes). The noise $\tilde{w}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt \tag{4}$$

where $F_{\tilde{w}}$ is a nonnegative Hermitian matrix; e.g., see [6], [20]. This determines the following commutation relations for the noise components:

$$\begin{aligned} [d\tilde{w}(t), d\tilde{w}^T(t)] &= d\tilde{w}(t)d\tilde{w}^T(t) - (d\tilde{w}(t)d\tilde{w}^T(t))^T \\ &= 2T_{\tilde{w}}dt \end{aligned} \tag{5}$$

where we use the notation $S_{\tilde{w}} = (1/2)(F_{\tilde{w}} + F_{\tilde{w}}^T), T_{\tilde{w}} = (1/2)(F_{\tilde{w}} - F_{\tilde{w}}^T)$ so that $F_{\tilde{w}} = S_{\tilde{w}} + T_{\tilde{w}}$. For instance, $F_{\tilde{w}} = \mathrm{diag}(1, I + iJ)$ describes a noise vector with one classical component and a pair of conjugate quantum noises (here I is the 2×2 identity matrix). The noise processes can be represented as operators on an appropriate Fock space (a particular, yet important, type of Hilbert space); e.g., see [6] and [20].

The process $\beta_w(t)$ serves to represent variables of other systems which may be passed to the system (1) via an interaction. Therefore, we require that $\beta_w(0)$ is an operator on a Hilbert

[1]In the case of a single degree of freedom quantum particle, $x = (x_1, x_2)^T$ where $x_1 = q$ is the position operator, and $x_2 = p$ is the momentum operator. The annihilation operator is $a = (q + ip)/2$. The commutation relations are $[a, a^*] = 1$, or $[q, p] = 2i$.

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space distinct from that of x_0 and the noise processes. We also assume $\beta_w(t)$ commutes with $x(t)$ for all $t \geq 0$ (two vectors x, y of operators are said to commute if $xy^T - (yx^T)^T = 0$); this will simplify matters for the present work. Moreover, since we had earlier specified that $\beta_w(t)$ should be an adapted process, we make note that $\beta_w(t)$ also commutes with $d\tilde{w}(t)$ for all $t \geq 0$.

To simplify the exposition, we now set up some conventions to put the system (1) into a standard form. First, note that there will be no change to the dynamics of $x(t)$ and $y(t)$ if we enlarge $w(t)$, by adding additional dummy noise components, and at the same time enlarging B by inserting suitable columns of zeros. Secondly, we may add dummy components to y by enlarging C and D by inserting additional dummy rows to each of these matrices. Our original output can be recovered by discarding or "disconnecting" the dummy components/entries. Therefore, we make the following assumptions on the system (1): (i) n_y is even, and (ii) $n_w \geq n_y$. We also make an assumption that $F_{\tilde{w}}$ is of the *canonical* form $F_{\tilde{w}} = I + i\mathrm{diag}(J, \ldots, J)$. Hence n_w has to be even. Note that if $F_{\tilde{w}}$ is not canonical but of the form $F_{\tilde{w}} = I + i\mathrm{diag}(0_{n' \times n'}, \mathrm{diag}(J, \ldots, J))$ with $n' \geq 1$, we may enlarge $w(t)$ (and hence also $\tilde{w}(t)$) and B as before such that the enlarged noise vector, say $\tilde{w}'$, can be taken to have an Ito matrix $F_{\tilde{w}'}$ which is canonical.

Equation (1) is a linear quantum stochastic differential equation. General quantum stochastic differential equations of this type are described in [19], [20], [6]. In (1) the integral with respect to $dw(t)$ is taken to be a quantum stochastic integral. The solution $x(t)$ depends only on the past noise $w(s)$, for $0 \leq s \leq t$; i.e., it is adapted, and a property of the Ito increments is that $d\tilde{w}(t)$ commutes with $x(t)$.

Equation (1) describes a noncommutative linear stochastic system, which need not necessarily correspond to a physical system. This issue does not normally arise in physical modeling, but as we shall see it is of considerable importance when we come to synthesizing physically realizable controllers below in Section III and Subsection V-D. The following theorem provides an algebraic characterization of precisely when the linear system (1) preserves the commutation relations as time evolves, a property enjoyed by open physical systems undergoing an overall unitary evolution, [14]. The proof is given in the Appendix.

Theorem 2.1: Under the assumptions discussed above for the system (1), we have $[x_i(0), x_j(0)] = 2i\Theta_{ij}$ implies $[x_i(t), x_j(t)] = 2i\Theta_{ij}$ for all $t \geq 0$ if and only if

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0 \tag{6}$$

III. Physical Realizability of Linear QSDES

Unlike classical systems, which we may regard here as always being physically realizable (for the purpose of controller synthesis), at least approximately via classical analog or digital electronics, a quantum system represented by the linear QSDE (1) need not necessarily represent the dynamics of a meaningful physical system. An example of a meaningful physical system here could be a system made up of interconnection of various quantum optical devices such as optical cavities, beam splitters, optical amplifiers. In particular, we have already seen from the previous section that in physical devices, the canonical commutation relations need to be preserved for all positive times leading to the requirement that the constraint (6) be satisfied by the matrices A and B of (1). However, as we shall shortly see, there is another constraint related to the output signal $y(t)$ which is required for (1) to be physically realizable.

A. Open Quantum Harmonic Oscillator

In order to formally present a definition of an open quantum harmonic oscillator, we will require the following notation. For a square matrix T, $\mathrm{diag}_m(T)$ denotes the block diagonal matrix $\mathrm{diag}(T, \ldots, T)$, where T appears m times as a diagonal block. The symbol P_m denotes a $2m \times 2m$ *permutation matrix* defined so that if we consider a column vector $a = [a_1 \ a_2 \ \ldots \ a_{2m}]^T$, then $P_m a = [a_1 \ a_3 \ \ldots \ a_{2m-1} \ a_2 \ a_4 \ \ldots \ a_{2m}]^T$. An $m \times m$ permutation matrix is a full-rank real matrix whose columns (or, equivalently, rows) consist of standard basis vectors for $\mathbb{R}^m$; i.e., vectors in $\mathbb{R}^m$ whose elements are all 0 except for one element which has the value 1. A permutation matrix P has the unitary property $PP^T = P^T P = I$. Note that $P_m^T[a_1 \ a_2 \ \ldots \ a_{2m}]^T = [a_1 \ a_{m+1} \ a_2 \ a_{m+2} \ \ldots \ a_m \ a_{2m}]^T$.

Let us also further introduce the notation $N_w = (n_w/2)$ and $N_y = (n_y/2)$,

$$M = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

and $\Gamma = P_{N_w}\mathrm{diag}_{N_w}(M)$. Moreover, let * denote the adjoint of a Hilbert space operator (by this we mean that the operator is a map from one Hilbert space to another), and let $X^\#$ denote the operation of taking the adjoint of each element of X, where X is a matrix/array of Hilbert space operators. Also, let $X^\dagger = (X^\#)^T$.

Then we have the following definition of an open quantum harmonic oscillator by generalizing slightly the linear model given in [12, Sect. 4].

Definition 3.1: The system (1) (with $\beta_w = 0$) is said to be an *open quantum harmonic oscillator* if Θ is canonical and there exist a quadratic Hamiltonian $H = x(0)^T R x(0)$, with a real and symmetric Hamiltonian matrix R of dimension $n \times n$, and a coupling operator $L = \Lambda x(0)$, with complex-valued coupling matrix Λ of dimension $n_w \times n$, such that

$$x_k(t) = U(t)^* x_k(0) U(t), \quad k = 1, \ldots, n$$
$$y_l(t) = U(t)^* w_l(t) U(t), \quad l = 1, \ldots, n_y$$

where $\{U(t); t \geq 0\}$ is an adapted process of unitary operators satisfying the following QSDE ([12, Sect. 2.5]):

$$dU(t) = \left(-iH dt - \frac{1}{2}L^\dagger L dt + [-L^\dagger \quad L^T]\Gamma dw(t)\right) U(t)$$
$$U(0) = I.$$

In this case, the matrices A, B, C, D are given by

$$A = 2\Theta(R + \Im(\Lambda^\dagger \Lambda)) \tag{7}$$
$$B = 2i\Theta[-\Lambda^\dagger \Lambda^T]\Gamma \tag{8}$$
$$C = P_{N_y}^T \begin{bmatrix} \Sigma_{N_y} & 0_{N_y \times N_w} \\ 0_{N_y \times N_w} & \Sigma_{N_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix} \tag{9}$$

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$$D = P_{N_y}^T \begin{bmatrix} \Sigma_{N_y} & 0_{N_y \times N_w} \\ 0_{N_y \times N_w} & \Sigma_{N_y} \end{bmatrix} P_{N_w} = [\, I_{n_y \times n_y} \quad 0_{n_y \times (n_w - n_y)} \,] \tag{10}$$

where $\Sigma_{N_y} = [I_{N_y \times N_y} \;\; 0_{N_y \times (N_w - N_y)}]$.

B. Augmentation of a Linear QSDE

If Θ is degenerate canonical, then we may perform an augmentation in which Θ is embedded into a larger skew symmetric matrix $\tilde{\Theta}$, which is canonical *up to permutation* (this means $\tilde{\Theta}$ becomes canonical after permutation of appropriate rows and columns). To do this, let $\theta = [\Theta_{ij}]_{i,j=n'+1,\ldots,n} = \mathrm{diag}_{(n-n')/2(}(J)$ if $n' < n$. Here $\mathrm{diag}_m(J)$ denotes a $m \times m$ block diagonal matrix with m matrices J on the diagonal. Define

$$\tilde{\Theta} = \begin{bmatrix} 0_{n' \times n'} & 0_{n' \times (n-n')} & I_{n' \times n'} \\ 0_{(n-n') \times n'} & \theta & 0_{(n-n') \times n'} \\ -I_{n' \times n'} & 0_{n' \times (n-n')} & 0_{n' \times n'} \end{bmatrix}$$

where the middle block of rows is dropped whenever $n = n'$. Then by definition $\tilde{\Theta}$ is canonical up to permutation and contains Θ as a sub-matrix by removing appropriate rows and columns of $\tilde{\Theta}$. Let $\tilde{n} = n + n'$, the dimension of the rows and columns of $\tilde{\Theta}$. Define the vector $\tilde{x}(t) = [x_1(t) \; x_2(t) \; \ldots \; x_n(t) \; z_1(t) \; z_2(t) \; \ldots \; z_{n'}(t)]^T$ of variables. We now define the following linear QSDE:

$$\begin{aligned} d\tilde{x}(t) &= \begin{bmatrix} A & 0_{n \times n_c} \\ A' & A'' \end{bmatrix} \tilde{x}(t)dt + \begin{bmatrix} B \\ B' \end{bmatrix} dw(t) \\ \tilde{y}(t) &= [\, C \quad C' \,] \tilde{x}(t)dt + D dw(t) \end{aligned} \tag{11}$$

where A', A'', B', and C' are, respectively, some real $n' \times n$, $n' \times n'$, $n' \times n_w$, and $n_y \times n'$ matrices, and the initial variables $\tilde{x}(0)$ satisfy the commutation relations $\tilde{x}_0 \tilde{x}(0)^T - (\tilde{x}(0)\tilde{x}(0)^T)^T = 2i\tilde{\Theta}$. We shall refer to the system (11) as an *augmentation* of (1).

Remark 3.2: In the Proof of Theorem 3.4 it is shown that the augmentation can be chosen to preserve commutation relations whenever the original system does.

C. Formal Definition of Physical Realizability

With open quantum harmonic oscillators and augmentations having been defined, we are now ready to introduce a formal definition of physical realizability of the QSDE (1). A discussion regarding the definition follows after Theorem 3.4 in which necessary and sufficient conditions for physical realizability are given.

Definition 3.3: The system (1) is said to be physically realizable if one of the following holds.

1) Θ is canonical and (1) represents the dynamics of an open quantum harmonic oscillator.
2) Θ is degenerate canonical and there exists an augmentation (11) which, after a suitable relabelling of the components $\tilde{x}_1(t), \ldots, \tilde{x}_{\tilde{n}}(t)$ of $\tilde{x}(t)$, represents the dynamics of an open quantum harmonic oscillator.

The following theorem, whose proof is given in the Appendix, provides necessary and sufficient conditions for physical realizability.

Theorem 3.4: The system (1) is physically realizable if and only if:

$$iA\Theta + i\Theta A^T + BT_w B^T = 0, \tag{12}$$

$$\begin{aligned} B \begin{bmatrix} I_{n_y \times n_y} \\ 0_{(n_w - n_y) \times n_y} \end{bmatrix} &= \Theta C^T P_{N_y}^T \\ &\quad \times \begin{bmatrix} 0_{N_y \times N_y} & I_{N_y \times N_y} \\ -I_{N_y \times N_y} & 0_{N_y \times N_y} \end{bmatrix} P_{N_y} \\ &= \Theta C^T \mathrm{diag}_{N_y}(J) \end{aligned} \tag{13}$$

and D satisfies (10). Moreover, for canonical Θ, the Hamiltonian and coupling matrices have explicit expressions as follows. The Hamiltonian matrix R is uniquely given by $R = (1/4)(-\Theta A + A^T \Theta)$, and the coupling matrix Λ is given uniquely by

$$\Lambda = -\frac{1}{2} i [\, 0_{N_w \times N_w} \quad I_{N_w \times N_w} \,] (\Gamma^{-1})^T B^T \Theta. \tag{14}$$

In the case that Θ is degenerate canonical, a physically realizable augmentation of the system can be constructed to determine the associated Hamiltonian and coupling operators using the above explicit formulas.

Remark 3.5: Note that the Hamiltonian and coupling operators are determined by (12), while conditions (10) and (13) relate to the required form of the output equation.

Remark 3.6: It is possible to consider the problem of realization more broadly than discussed above by including additional components, such as beam splitters and phase shifts that commonly occur in quantum optics. While Theorem 3.4 characterizes the existence of physically realizable controllers, detailed development of an efficient realization methodology is beyond the scope of the present paper.

IV. Dissipation Properties

In this section, we describe various dissipation properties frequently used in control engineering, suitably adapted to the quantum context. These properties concern the influence of disturbance inputs on energy transfers and stability. In particular, we give a quantum version of the Strict Bounded Real Lemma (Corollary 4.5) which will be employed in Section V for quantum H^∞ controller synthesis. In this section, we consider the following quantum system of the form (1):

$$\begin{aligned} dx(t) &= Ax(t)dt + [\, B \quad G \,][\, dw(t)^T \quad dv(t)^T \,]^T \\ dz(t) &= Cx(t)dt + [\, D \quad H \,][\, dw(t)^T \quad dv(t)^T \,]^T. \end{aligned} \tag{15}$$

In this quantum system, the input channel has two components, $dw = \beta_w dt + d\tilde{w}$ which represents disturbance signals, and dv, which represents additional noise sources.

Definition 4.1: Given an operator valued quadratic form

$$r(x, \beta_w) = [x^T \beta_w^T] R \begin{bmatrix} x \\ \beta_w \end{bmatrix}$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}$$

is a given real symmetric matrix, we say the system (15) is *dissipative with supply rate* $r(x, \beta_w)$ if there exists a positive operator valued quadratic form $V(x) = x^T X x$ (where X is a real

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positive definite symmetric matrix) and a constant $\lambda > 0$ such that

$$\langle V(x(t))\rangle + \int_0^t \langle r(x(s), \beta_w(s))\rangle ds \leq \langle V(x(0))\rangle + \lambda t \quad \forall t > 0 \quad (16)$$

for all Gaussian states ρ for the initial variables $x(0)$. Here we use the shorthand notation $\langle\cdot\rangle$ for expectation over all initial variables and noises.

We say that the system (15) is *strictly dissipative* if there exists a constant $\epsilon > 0$ such that inequality (16) holds with the matrix R replaced by the matrix $R + \epsilon I$.

The term $\langle V(x(t))\rangle$ serves as a generalization to quantum stochastic systems (15) of the notion of an abstract internal energy for the system at time t. On the other hand, the term $\langle r(x(t), \beta_w(t))\rangle$ is a quantum generalization of the notion of abstract power flow into and out of the system at time t. Both of these are notions which are widely used in the stability analysis of linear and nonlinear deterministic systems [26], [24]. The dissipation inequality (16) is a generalization of the corresponding inequality that was introduced for classical stochastic systems in [23], see [10].

The following theorem, whose proof is given in the Appendix, relates the property of dissipativeness to certain linear matrix inequalities.

Theorem 4.2: Given a quadratic form $r(x, \beta_w)$ defined as above, then the quantum stochastic system (15) is dissipative with supply rate $r(x, \beta_w)$ if and only if there exists a real positive definite symmetric matrix X such that the following matrix inequality is satisfied:

$$\begin{pmatrix} A^T X + XA + R_{11} & R_{12} + XB \\ B^T X + R_{12}^T & R_{22} \end{pmatrix} \leq 0. \quad (17)$$

Furthermore, the system is strictly dissipative if and only if there exists a real positive definite symmetric matrix X such that the following matrix inequality is satisfied:

$$\begin{pmatrix} A^T X + XA + R_{11} & R_{12} + XB \\ B^T X + R_{12}^T & R_{22} \end{pmatrix} < 0. \quad (18)$$

Moreover, if either of (17) or (18) holds, then the required constant $\lambda \geq 0$ can be chosen as $\lambda = \lambda_0$, where

$$\lambda_0 = \operatorname{tr}\left[\begin{bmatrix} B^T \\ G^T \end{bmatrix} X [B \quad G]\right] F]. \quad (19)$$

Here, the matrix F is defined by the following relation:

$$F dt = \begin{bmatrix} dw \\ dv \end{bmatrix} [\, dw^T \quad dv^T \,]. \quad (20)$$

We now present some corollaries to the above theorem corresponding to special cases of the matrix R defined in terms of the error output operator $\beta_z(t) = Cx(t) + D\beta_w(t)$.

Definition 4.3: The quantum stochastic system (15) is said to be *Bounded Real with disturbance attenuation* g if the system (15) is dissipative with supply rate

$$r(x, \beta_w) = \beta_z^T \beta_z - g^2 \beta_w^T \beta_w = [x^T \beta_w^T] \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - g^2 I \end{bmatrix} \begin{bmatrix} x \\ \beta_w \end{bmatrix}.$$

Also, the quantum stochastic system (15) is said to be *Strictly Bounded Real with disturbance attenuation* g if the system (15) is strictly dissipative with this supply rate.

Using the above definition of a bounded real system, we obtain the following corollary from Theorem 4.2. (e.g., see also [8] for the corresponding classical result.)

Corollary 4.4: The quantum stochastic system (15) is bounded real with disturbance attenuation g if and only if there exists a positive definite symmetric matrix $X \in \mathbb{R}^{n\times n}$ such that the following matrix inequality is satisfied:

$$\begin{pmatrix} A^T X + XA + C^T C & C^T D + XB \\ B^T X + D^T C & D^T D - g^2 I \end{pmatrix} \leq 0.$$

Furthermore, the quantum stochastic system is strictly bounded real with disturbance attenuation g if and only if there exists a positive definite symmetric matrix $X \in \mathbb{R}^{n\times n}$ such that the following matrix inequality is satisfied:

$$\begin{pmatrix} A^T X + XA + C^T C & C^T D + XB \\ B^T X + D^T C & D^T D - g^2 I \end{pmatrix} < 0.$$

Moreover, in both cases the required constant $\lambda \geq 0$ can be chosen as $\lambda = \lambda_0$, where λ_0 is defined by (19).

Now combining this corollary with the standard Strict Bounded Real Lemma (e.g., see [21], [33]) we obtain the following Corollary.

Corollary 4.5: The following statements are equivalent.

i) The quantum stochastic system (15) is strictly bounded real with disturbance attenuation g.

ii) A is a stable matrix and $\|C(sI - A)^{-1}B + D\|_\infty < g$.[2]

iii) $g^2 I - D^T D > 0$ and there exists a positive definite matrix $\tilde{X} > 0$ such that

$$A^T \tilde{X} + \tilde{X} A + C^T C + (\tilde{X} B + C^T D) \times (g^2 I - D^T D)^{-1} (B^T \tilde{X} + D^T C) < 0.$$

iv) $g^2 I - D^T D > 0$ and the algebraic Riccati equation

$$A^T X + XA + C^T C + (XB + C^T D) \times (g^2 I - D^T D)^{-1} (B^T X + D^T C) = 0$$

has a stabilizing solution $X \geq 0$.

Furthermore, if these statements hold then $X < \tilde{X}$.

V. H^∞ Controller Synthesis

In this section, we consider the problem of H^∞ controller design for quantum systems. As we shall see, we do not restrict ourselves to classical controllers. The closed-loop plant-controller system is defined in Subsection V-A, and then in Subsection V-C we apply the Strict Bounded Real Lemma to the closed-loop system to obtain our main results. In Subsection V-D, we provide conditions under which a controller is physically realizable.

[2]The H^∞ norm notation used here is standard [32], and applies to the classical transfer function $C(sI - A)^{-1}B + D$, *not* the quantum system (15). In this paper we do not define nor use transfer functions for quantum systems.

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A. The Closed-Loop Plant-Controller System

The general linear model (1) described above is the prototype for the interconnection of components which will make up the quantum control system. In control system design, we prescribe a system called the *plant*, and seek to find another system, called a *controller*, in such a way that desired closed-loop behavior is achieved. We now introduce our plant and controller models, and the resulting closed loop.

We consider *plants* described by noncommutative stochastic models of the following form defined in an analogous way to the quantum system (1):

$$\begin{aligned} dx(t) &= Ax(t)dt + [B_0 \quad B_1 \quad B_2] \\ &\quad \times [dv(t)^T \quad dw(t)^T \quad du(t)^T]^T; \quad x(0) = x_0 \\ dz(t) &= C_1 x(t)dt + D_{12}du(t) \\ dy(t) &= C_2 x(t)dt + [D_{20} \quad D_{21} \quad 0_{n_y \times n_u}] \\ &\quad \times [dv(t)^T \quad dw(t)^T \quad du(t)^T]^T. \end{aligned} \tag{21}$$

Here $x(t)$ is a vector of plant variables. The input $w(t)$ represents a *disturbance* signal of the form (3). The signal $u(t)$ is a *control* input of the form

$$du(t) = \beta_u(t)dt + d\tilde{u}(t) \tag{22}$$

where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is the adapted, self-adjoint part of $u(t)$. Also, $dv(t)$ represents any additional quantum noise in the plant. The vectors $v(t)$, $\tilde{w}(t)$, and $\tilde{u}(t)$ are quantum noises with Ito matrices F_v, $F_{\tilde{w}}$, and $F_{\tilde{u}}$ which are all nonnegative Hermitian.

Controllers are assumed to be noncommutative stochastic systems of the form

$$\begin{aligned} d\xi(t) &= A_K\xi(t)dt + [B_{K1} \quad B_K] \\ &\quad \times [dv_K(t)^T \quad dy(t)^T]^T; \quad \xi(0) = \xi_0 \\ du(t) &= C_K\xi(t)dt + [\, B_{K0} \quad 0_{n_u \times n_y}\,] \\ &\quad \times [\, dv_K(t)^T \quad dy(t)^T\,]^T \end{aligned} \tag{23}$$

where $\xi(t) = [\xi_1(t) \ \ldots \ \xi_{n_K}(t)]^T$ is a vector of self-adjoint controller variables. The noise $v_K(t) = [v_{K1}(t) \ \ldots \ v_{KK_v}(t)]^T$ is a vector of noncommutative Wiener processes (in vacuum states) with nonzero Ito products as in (4) and with canonical Hermitian Ito matrix F_{v_K}.

At time $t = 0$, we also assume that $x(0)$ commutes with $\xi(0)$. The *closed-loop system* is obtained by making the identification $\beta_u(t) = C_K\xi(t)$ and interconnecting (21) and (23) to give

$$\begin{aligned} d\eta(t) &= \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}\eta(t)dt \\ &\quad + \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix}\begin{bmatrix} dv(t) \\ dv_K(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} B_1 \\ B_KD_{21} \end{bmatrix} dw(t) \\ dz(t) &= [C_1 \quad D_{12}C_K]\eta(t)dt \\ &\quad + [0 \quad D_{12}B_{K0}]\begin{bmatrix} dv(t) \\ dv_K(t) \end{bmatrix} \end{aligned} \tag{24}$$

where $\eta(t) = [x(t)^T \ \xi(t)^T]^T$. That is, we can write

$$\begin{aligned} d\eta(t) &= \tilde{A}\eta(t)dt + \tilde{B}dw(t) + \tilde{G}d\tilde{v}(t) \\ &= \tilde{A}\eta(t)dt + [\tilde{B} \quad \tilde{G}]\begin{bmatrix} dw(t) \\ d\tilde{v}(t) \end{bmatrix} \\ dz(t) &= \tilde{C}\eta(t)dt + \tilde{H}d\tilde{v}(t) \\ &= \tilde{C}\eta(t)dt + [0 \quad \tilde{H}]\begin{bmatrix} dw(t) \\ d\tilde{v}(t) \end{bmatrix} \end{aligned} \tag{25}$$

where

$$\begin{aligned} \tilde{v}(t) &= \begin{bmatrix} v(t) \\ v_K(t) \end{bmatrix}; \quad \tilde{A} = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \\ \tilde{B} &= \begin{bmatrix} B_1 \\ B_KD_{21} \end{bmatrix}; \quad \tilde{G} = \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix} \\ \tilde{C} &= [C_1 \quad D_{12}C_K]; \quad \tilde{H} = [0 \quad D_{12}B_{K0}]. \end{aligned}$$

Note that the closed-loop system (25) is a system of the form (1).

B. H^∞ Control Objective

The goal of the H^∞ controller synthesis problem is to find a controller (23) for a *given* disturbance attenuation parameter $g > 0$ such that the closed-loop system (25) satisfies

$$\begin{aligned} &\int_0^t \langle \beta_z(s)^T\beta_z(s) + \epsilon\eta(s)^T\eta(s)\rangle ds \\ &\quad \le (g^2 - \epsilon)\int_0^t \langle \beta_w(s)^T\beta_w(s)\rangle ds + \mu_1 + \mu_2 t, \quad \forall t > 0 \end{aligned} \tag{26}$$

for some real constants $\epsilon, \mu_1, \mu_2 > 0$. Here, $\beta_z(t)$ is error output operator $\beta_z(t) = \tilde{C}\eta(t)$ corresponding to the closed-loop system (25). Thus the controller bounds the effect of the "energy" in the signal $\beta_w(t)$ on the "energy" in the error signal $\beta_z(t)$.

Remark 5.1: It should be noted that the closed-loop system (25) will meet the objective (26) if it is strictly bounded real with disturbance attenuation g. Indeed, it follows from Definition 4.3 that the closed-loop system (25) will be strictly bounded real with disturbance attenuation g if and only if there exists a real positive definite symmetric matrix X and a constant $\lambda > 0$ such that

$$\begin{aligned} &\langle \eta(t)^T X\eta(t)\rangle + \int_0^t \langle \beta_z(s)^T\beta_z(s) - g^2\beta_w(s)^T\beta_w(s) \\ &\quad + \epsilon\eta(s)^T\eta(s) + \epsilon\beta_w(s)^T\beta_w(s)\rangle ds \\ &\quad \le \langle \eta(0)^T X\eta(0)\rangle + \lambda t, \quad \forall t > 0. \end{aligned} \tag{27}$$

From this, (26) follows with $\mu_1 = \langle \eta(0)^T X\eta(0)\rangle$ and $\mu_2 = \lambda$.

Necessary and sufficient conditions for the existence of a specific type of controller which achieves this goal for a given g are given in the next section, as well as explicit formulas for A_K, B_K, and C_K. The results parallel the corresponding well-known results for classical linear systems (see, e.g., [3], [21]).

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C. Necessary and Sufficient Conditions

In order to present our results on quantum H^∞ control, we will require that the plant system (21) satisfies the following assumptions.

Assumption 5.2:

1) $D_{12}^T D_{12} = E_1 > 0$.
2) $D_{21} D_{21}^T = E_2 > 0$.
3) The matrix $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ is full rank for all $\omega \geq 0$.
4) The matrix $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ is full rank for all $\omega \geq 0$.

Our results will be stated in terms of the following pair of algebraic Riccati equations:

$$\begin{aligned} &\left(A - B_2 E_1^{-1} D_{12}^T C_1\right)^T X + X\left(A - B_2 E_1^{-1} D_{12}^T C_1\right) \\ &\quad + X\left(B_1 B_1^T - g^2 B_2 E_1^{-1} B_2'\right) X \\ &\quad + g^{-2} C_1^T \left(I - D_{12} E_1^{-1} D_{12}^T\right) C_1 = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &\left(A - B_1 D_{21}^T E_2^{-1} C_2\right) Y + Y\left(A - B_1 D_{21}^T E_2^{-1} C_2\right) \\ &\quad + Y\left(g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2\right) Y \\ &\quad + B_1\left(I - D_{21}^T E_2^{-1} D_{21}\right) B_1^T = 0. \end{aligned} \tag{29}$$

The solutions to these Riccati equations will be required to satisfy the following assumption.

Assumption 5.3:

i) $A - B_2 E_1^{-1} D_{12}^T C_1 + (B_1 B_1^T - g^2 B_2 E_1^{-1} B_2') X$ is a stability matrix.
ii) $A - B_1 D_{21}^T E_2^{-1} C_2 + Y(g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2)$ is a stability matrix.
iii) The matrix XY has a spectral radius strictly less than one.

Our results will show that if the Riccati equations (28), (29) have solutions satisfying Assumption 5.3, then a controller of the form (23) will solve the H^∞ control problem under consideration if its system matrices are constructed from the Riccati solutions as follows:

$$\begin{aligned} A_K &= A + B_2 C_K - B_K C_2 + (B_1 - B_K D_{21}) B_1^T X \\ B_K &= (I - YX)^{-1}\left(Y C_2^T + B_1 D_{21}^T\right) E_2^{-1} \\ C_K &= -E_1^{-1}\left(g^2 B_2^T X + D_{12}^T C_1\right). \end{aligned} \tag{30}$$

We are now in a position to present our main result concerning H^∞ controller synthesis.

Theorem 5.4: Necessity. Consider the system (21) and suppose that Assumption 5.2 is satisfied. If there exists a controller of the form (23) such that the resulting closed-loop system (25) is strictly bounded real with disturbance attenuation g, then the Riccati equations (28) and (29) will have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying Assumption 5.3.

Sufficiency. Suppose the Riccati equations (28) and (29) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying Assumption 5.3. If the controller (23) is such that the matrices A_K, B_K, C_K are as defined in (30), then the resulting closed-loop system (25) will be strictly bounded real with disturbance attenuation g. Also the constant $\lambda \geq 0$ in Definition 4.1 can be chosen as in (19), $\lambda = \lambda_0$ with B, G, F replaced by the corresponding matrices of the closed-loop system.

The controller parameters B_{K0}, B_{K1}, and the controller noise v_K are not given in the construction described in the sufficiency part of Theorem 5.4. They are free as far as the H^∞ objective is concerned. In the next subsection, we show how they may be chosen to give a controller that is physically realizable.

D. Physical Realization of Controllers

We now show that given an *arbitrary* choice of commutation matrix Θ_K for the controller, it is always possible to find a physically realizable controller in the sense of Definition 3.3. This means that the controller can be chosen to be purely quantum, purely classical, or a combination of quantum and classical components.

Theorem 5.5: Assume $F_y = D_{20} F_v D_{20}^T + D_{21} F_w D_{21}^T$ is canonical. Let $\{A_K, B_K, C_K\}$ be an arbitrary triple (such as given by (30)), and select the controller commutation matrix Θ_K to be canonical or degenerate canonical, as desired. Then there exists controller parameters B_{K0}, B_{K1}, and the controller noise v_K such that the controller (23) is physically realizable. In particular, $2i\Theta_K = \xi(t)\xi(t)^T - (\xi(t)\xi(t)^T)^T$ for all $t \geq 0$ whenever $2i\Theta_K = \xi(0)\xi(0)^T - (\xi(0)\xi(0)^T)^T$.

The proof of this theorem depends on the following lemma for the case in which Θ_K is canonical. For the degenerate canonical case, this lemma can be applied to an augmentation of the controller. We shall use the notation of Section III-A, and as in the discussion in Section II, we may take B_K to have an even number of columns and C_K to have an even number of rows.

Lemma 5.6: Let F_y be canonical and $\{A_K, B_K, C_K\}$ be such that $A_K \in \mathbb{R}^{n_K \times n_K}$, $B_K \in \mathbb{R}^{n_K \times m_K}$, $C_K \in \mathbb{R}^{l_k \times n_K}$, $n_K = 2N_\xi$, $m_K = 2N_y$ and $l_K = 2N_u$ for positive integers N_ξ, N_y and N_u, and $\Theta_K = \mathrm{diag}_{N_\xi}(J)$ is canonical. Then there exists an integer $N_{v_K} \geq N_u$ and $B_{K1} \in \mathbb{R}^{n_K \times 2N_{v_K}}$ such that the system (23) is physically realizable with

$$\begin{aligned} \Sigma_{N_u} &= \begin{bmatrix} I_{N_u \times N_u} & 0_{N_u \times (N_{v_K} - N_u)} \end{bmatrix} \\ B_{K0} &= P_{N_u}^T \begin{bmatrix} \Sigma_{N_u} & 0_{N_u \times N_{v_K}} \\ 0_{N_u \times N_{v_K}} & \Sigma_{N_u} \end{bmatrix} P_{N_{v_K}} \\ &= \begin{bmatrix} I_{l_K \times l_K} & 0_{l_K \times (2N_{v_K} - l_K)} \end{bmatrix} \\ R &= \frac{1}{2}(Z + Z^T) \end{aligned} \tag{31}$$

$$B_{K1} = \begin{bmatrix} B_{K1,1} & B_{K1,2} \end{bmatrix} \tag{32}$$

$$\Lambda = \begin{bmatrix} \frac{1}{2} C_K^T P_{N_u}^T \begin{bmatrix} I \\ iI \end{bmatrix} & \Lambda_{b1}^T & \Lambda_{b2}^T \end{bmatrix}^T \tag{33}$$

$$B_{K1,1} = -i\Theta_K C_K^T \mathrm{diag}_{N_u}(iJ) \tag{34}$$

$$\begin{aligned} \Lambda_{b2} &= -i\begin{bmatrix} I_{N_y \times N_y} & 0_{N_y \times N_y} \end{bmatrix} P_{N_y} \mathrm{diag}_{N_y}(M) \\ &\quad \times B_K^T \Theta_K \end{aligned} \tag{35}$$

$$\begin{aligned} B_{K1,2} &= 2i\Theta_K \begin{bmatrix} -\Lambda_{b1}^\dagger & \Lambda_{b1}^T \end{bmatrix} P_{N_{v_K} - N_u} \\ &\quad \times \mathrm{diag}_{N_{v_K} - N_u}(M) \end{aligned} \tag{36}$$

where $Z = -(1/2)\Theta A_K$ and $N_{v_K} \geq N_u + 1$. Here, Λ_{b1} is any complex $(N_{v_K} - N_u) \times n_K$ matrix such that

$$\begin{aligned} \Lambda_{b1}^\dagger \Lambda_{b1} = \Xi + i\Bigg(&\frac{1}{2}(Z - Z^T) - \frac{1}{4} C_K^T P_{N_u}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \\ &\times P_{N_u} C_K - \Im(\Lambda_{b2}^\dagger \Lambda_{b2})\Bigg) \end{aligned} \tag{37}$$

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where Ξ is any real symmetric $n_K \times n_K$ matrix such that the right hand side of (37) is nonnegative definite.

The Proof of Lemma 5.6 is given in the Appendix.

Remark 5.7: Note that the condition $N_{v_K} \geq N_u$ is significant since it implies that there is no direct feedthrough of the signal $y(t)$ to $u(t)$ as required for (23). For compatibility between the (23) and (21), it is necessary that the corresponding Ito matrices satisfy the following condition:

$$F_u = B_{K0} F_{v_K} B_{K0}^T. \tag{38}$$

However, since F_{v_K} and F_u are, by convention, in canonical form, (38) is always satisfied. To see this, we simply note that the $2N_u$ elements of $B_{K0}v_K$ are a subset of pairs of conjugate real and imaginary quadratures in v_K. Hence it follows that if F_{v_K} is canonical then F_u must also be canonical and (38) is automatically satisfied.

VI. Robust Stability

The H^∞ control approach of Section V leads to a closed-loop quantum system of the form (25) which is strictly bounded real with disturbance attenuation g. We now show that this property can be used to guarantee stability robustness against real parameter uncertainties. Indeed, we will suppose that the true closed-loop quantum system corresponding to the system (25) is described by the equations

$$d\eta(t) = \bar{A}\eta(t)dt + \tilde{G}d\tilde{v}(t) \tag{39}$$

where $\bar{A} = \tilde{A} + \tilde{B}\Delta\tilde{C}$ and Δ is a constant but unknown uncertainty matrix satisfying

$$\Delta^T\Delta \leq \frac{1}{g^2} I. \tag{40}$$

Definition 6.1: The closed-loop quantum system (39) is said to be *mean square stable* if there exists a real positive definite matrix $X > 0$ and a constant $\lambda > 0$ such that

$$\langle \eta(t)^T X \eta(t)\rangle + \int_0^t \langle \eta(s)^T \eta(s)\rangle ds \leq \langle \eta(0)^T X \eta(0)\rangle + \lambda t \quad \forall t > 0$$

for all Gaussian states ρ.

The following lemma and theorem relates the robust stability of the above system to its H^∞ properties. The proofs of this lemma and this theorem can be found in the Appendix.

Lemma 6.2: The quantum system (39) is mean square stable if and only if the matrix $\bar{A}$ is a stable matrix.

Theorem 6.3: If the closed-loop quantum system (25) is strictly bounded real with disturbance attenuation g, then the true closed-loop system (39) is mean-square stable for all Δ satisfying (40).

VII. H^∞ Synthesis in Quantum Optics

Quantum optics is an important area in quantum physics and quantum technology and provides a promising means of implementing quantum information and computing devices; e.g., see [17]. In this section we give some examples of controller design for simple quantum optics plants based on optical cavities and optical amplifiers coupled to optical fields; e.g., see [2] and [14]. We give explicit realizations of controllers which are fully quantum, fully classical, and mixed quantum-classical using standard quantum optical components and electronics.

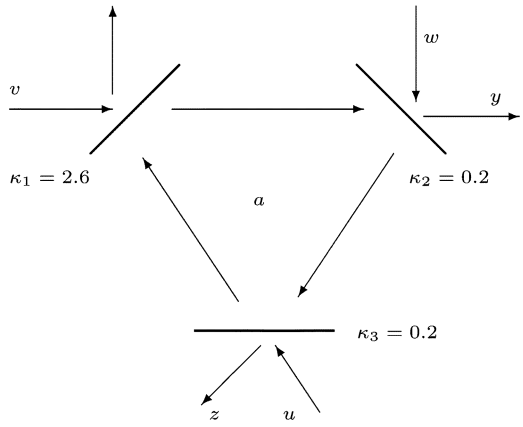


Fig. 1. Optical cavity (plant).

A. Quantum Controller Synthesis

We consider an optical cavity resonantly coupled to three optical channels v, w, u as in Fig. 1. The control objective is to attenuate the effect of the disturbance signal w on the output z—physically this means to dim the light emerging from z resulting from light shone in at w.

The dynamics of this optical cavity system is described by the evolution of its annihilation operator a and its creation operator a^* (the adjoint of a)

$$\begin{aligned}
da(t) &= -\frac{\gamma}{2}a(t)dt - \sqrt{\kappa_1}dA_1(t) - \sqrt{\kappa_2}dA_2(t) \\
&\quad - \sqrt{\kappa_3}dA_3(t); \quad a(0) = a \\
da^*(t) &= -\frac{\gamma}{2}a^*(t)dt - \sqrt{\kappa_1}dA_1^*(t) - \sqrt{\kappa_2}dA_2^*(t) \\
&\quad - \sqrt{\kappa_3}dA_3^*(t); \quad a^*(0) = a^* \\
&\qquad\qquad (\gamma = \kappa_1 + \kappa_2 + \kappa_3) \\
dB_3(t) &= \sqrt{\kappa_3}a(t)dt + dA_3(t) \\
dB_2(t) &= \sqrt{\kappa_2}a(t)dt + dA_2(t)
\end{aligned} \tag{41}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$ are respectively the annihilation process of the input fields in channels v, w, u (in the vacuum state), while $B_3(t)$, $B_2(t)$ are respectively the output fields of channels u and w. In this model, the boson commutation relation $[a(t), a^*(t)] = [a, a^*] = 1$ holds. Derivation of the above dynamics for a cavity can be found in, for example, [13], [14], [29].

The operators in the dynamics (41) are not self-adjoint and in general the system coefficients may be complex-valued. However, it is convenient to work with self-adjoint operators and real-valued coefficients (since the latter is the usual setting for the formulation and solution of control problems). Therefore, we rewrite (41) in the quadrature notation of (21), with $x_1(t) = q(t) = a(t) + a^*(t)$, $x_2(t) = p(t) = (a(t) - a^*(t)/i$, $v(t) =$

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$(v_1(t) = A_1(t) + A_1^*(t), v_2(t) = (A_1(t) - A_1^*(t))/i)^T$, $w(t) = (w_1(t) = A_2(t) + A_2^*(t), w_2(t) = (A_2(t) - A_2^*(t))/i)^T$, $u(t) = (u_1(t) = A_3(t) + A_3^*(t), u_2(t) = (A_3(t) - A_3^*(t))/i)^T$, $z(t) = (z_1(t) = B_3(t) + B_3^*(t), z_2(t) = (B_3(t) - B_3^*(t))/i)^T$, and $y(t) = (y_1(t) = B_2(t) + B_2^*(t), y_2(t) = (B_2(t) - B_2^*(t))/i)^T$. The quantum noises v, $\tilde{w}$ have Hermitian Ito matrices $F_v = F_{\tilde{w}} = I + iJ$. This leads to a system of the form (21) with the following system matrices:

$$\begin{aligned} A &= -\frac{\gamma}{2}I; \quad B_0 = -\sqrt{\kappa_1}I \\ B_1 &= -\sqrt{\kappa_2}I; \quad B_2 = -\sqrt{\kappa_3}I \\ C_1 &= \sqrt{\kappa_3}I; \quad D_{12} = I \\ C_2 &= \sqrt{\kappa_2}I; \quad D_{21} = I. \end{aligned}$$

Moreover, it follows from the definition of x_1 and x_2 and the commutation relation $[a, a^*] = 1$ that the commutation matrix for this plant is $\Theta_P = J$.

In our example, we will choose the total cavity decay rate $\kappa = 3$ and the coupling coefficients $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$. With a disturbance attenuation constant of $g = 0.1$, it was found that the Riccati equations (28) and (29) have stabilizing solutions satisfying Assumption 5.3. These Riccati solutions were as follows: $X = Y = 0_{2\times 2}$. Then, it follows from Theorem 4.2 that if a controller of the form (23) is applied to this system with matrices A_K, B_K, C_K defined as in (30) then the resulting closed-loop system will be strictly bounded real with disturbance attenuation g. In our case, these matrices are given by

$$A_K = -1.1I, B_K = -0.447I, C_K = -0.447I.$$

In this case, the controller (23) can be implemented with another optical cavity with annihilation operator a_K (with quadratures $\xi_1 = q_K = a_K + a_K^*$, $\xi_2 = p_K = (a_K - a_K^*)/i$, $\xi = (q_K, p_K)^T$), corresponding to $\Theta_K = J$, connected at the output with a 180° phase shifter (see Remark 3.6). The controller cavity has coupling coefficients $\kappa_{K1} = 0.2$, $\kappa_{K2} = 1.8$, $\kappa_{K3} = 0.2$, and $\kappa_K = 2.2$ and is a physically realizable system with dynamics

$$\begin{aligned} d\xi(t) &= A_K\xi(t)dt + [\,B_{K1} \quad B_K\,][\,dv_K^T \quad dy^T\,]^T \\ d\tilde{u}(t) &= -C_K\xi(t)dt + [\,I_{2\times 2} \quad 0_{2\times 4}][\,dv_K^T \quad dy^T\,]^T \end{aligned}$$

where $B_{K1} = [-0.447I \quad -1.342I]$, $v_K(t) = (v_{K11}(t), v_{K12}(t), v_{K21}(t), v_{K22}(t))^T$ are the quadratures of two independent canonical quantum noise sources, and $\tilde{u}(t)$ is the output of the cavity. The overall output of the controller is $u(t)$, given by $u(t) = K_s\tilde{u}(t)$, where $K_s = -I_{2\times 2}$. Here K_s models the 180° phase shift at the output of the cavity. Thus, the overall controller (an optical cavity cascaded with a 180° phase shifter) is of the form (23) with $B_{K0} = [-I \ 0]$ and B_{K1} as given before. This controller is illustrated in Fig. 2. An experiment based on this example has recently been completed [34].

B. Robust Stability in Quantum Optics

We now modify the above example to allow for uncertainty in one of the optical cavity parameters using the results of Section VI. Indeed, we consider the same set up as in Fig. 1 and assume that there is uncertainty in the value of the coupling

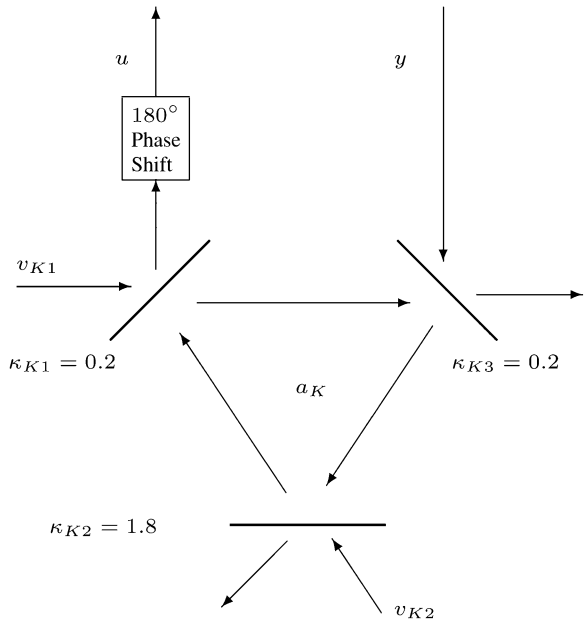


Fig. 2. Optical cavity quantum realization of the controller ($\Theta_K = J$) for the plant shown in Fig. 1.

coefficient κ_1 corresponding to the optical channel v. In this case, the (21) describing the optical cavity now have matrices

$$\begin{aligned} &A = -\frac{\gamma+\delta}{2}I; \quad B_0 = \sqrt{\kappa_1+\delta}I; \quad B_1 = -\sqrt{\kappa_2}I \\ &B_2 = -\sqrt{\kappa_3}I; \quad C_1 = \sqrt{\kappa_3}I; \quad D_{12} = I \\ &C_2 = \sqrt{\kappa_2}I; \quad D_{21} = I. \end{aligned} \tag{42}$$

This is our true system which depends on the unknown parameter δ.

In order to apply our H^∞ theory together with the results of Section VI to this system, we must overbound the uncertainty in the matrix A. Indeed, let S be any nonsingular matrix. If $|\delta| \le \mu$, then we can write $-(\delta/2)I = \tilde{B}_1\Delta\tilde{C}_1$ where $\tilde{B}_1 = (\mu/2)S$, $\tilde{C}_1 = S^{-1}$ and $\Delta = -(\delta/\mu)I$ satisfies $\Delta^T\Delta \le I$. Hence, if we consider a family of systems of the form (21) with the system matrices

$$\begin{aligned} &A = -\frac{\gamma}{2}I + \tilde{B}_1\Delta\tilde{C}_1; \quad B_0 = \sqrt{\kappa_1+\delta}I \\ &B_1 = -\sqrt{\kappa_2}I; \quad B_2 = -\sqrt{\kappa_3}I; \quad C_1 = \sqrt{\kappa_3}I \\ &C_2 = \sqrt{\kappa_2}I; \quad D_{12} = I; \quad D_{21} = I \end{aligned} \tag{43}$$

where $\Delta^T\Delta \le I$, this will include the true system. Now, in order to apply the result of Section VI to this problem, we consider the H^∞ problem defined by a system of the form (21) where

$$\begin{aligned} &A = -\frac{\gamma}{2}I; \quad B_0 = \sqrt{\kappa_1+\delta}I; \quad B_{10} = -\sqrt{\kappa_2}I \\ &B_1 = [B_{10} \quad \tilde{B}_1]; \quad B_2 = -\sqrt{\kappa_3}I; \quad C_{10} = \sqrt{\kappa_3}I \\ &C_1 = \begin{bmatrix} C_{10} \\ g\tilde{C}_1 \end{bmatrix}; \quad D_{120} = I; \quad D_{12} = \begin{bmatrix} D_{120} \\ 0 \end{bmatrix} \\ &C_2 = \sqrt{\kappa_2}I; \quad D_{210} = I; \quad D_{21} = [D_{210} \quad 0]. \end{aligned} \tag{44}$$

Here, g is the disturbance attenuation parameter in the H^∞ control problem to be considered. Note that the matrix B_0 depends

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on the unknown parameter δ. However, this matrix is not involved in the calculation of the H^∞ controller.

As in the original example, we will choose the nominal cavity decay rate $\gamma = 3$ and the nominal coupling coefficients of $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$. Also, we let $\mu = 0.1$. That is, we are considering a 10% variation in the coupling coefficient. With a disturbance attenuation constant of $g = 0.1$ and $S = 1.5I$, it was found that the Riccati equations (28) and (29) have stabilizing solutions satisfying Assumption 5.3. These Riccati solutions were as follows: $X = 0.1733I$, $Y = 0.0022I$. Also, the corresponding controller matrices were given by

$$A_K = -1.0997I, \quad B_K = -0.4464I$$
$$C_K = -0.4464I. \tag{45}$$

Now as in the original example, the controller defined by the matrices (45) can be implemented by another optical cavity. In this case, $\kappa_{K1} = 0.1993$, $\kappa_{K2} = 1.8008$, $\kappa_{K3} = 0.1993$, and $\gamma_K = 2.1993$. As in the original example, the controller is illustrated in Fig. 2.

It follows from Theorem 4.2 that the resulting closed-loop system satisfies the strict bounded real condition with disturbance attenuation g. Indeed, this closed-loop system will be described by the (25) where

$$\tilde{A} = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}$$
$$\tilde{B} = \begin{bmatrix} B_1 \\ B_KD_{21} \end{bmatrix} = \begin{bmatrix} B_{10} & \tilde{B}_1 \\ B_KD_{210} & 0 \end{bmatrix}$$
$$\tilde{G} = \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix}$$
$$\tilde{C} = [C_1 \quad D_{12}C_K] = \begin{bmatrix} C_{10} & D_{120}C_K \\ g\tilde{C}_1 & 0 \end{bmatrix}$$
$$\tilde{D} = [0 \quad D_{12}B_{K0}].$$

Now, since this system is strictly bounded real with disturbance attenuation g, it follows from Corollary 4.5 that $\|\tilde{C}(sI - \tilde{A})^{-1}\tilde{B}\|_\infty < g$. From this, we can conclude that

$$\left\| [C_{10} \quad D_{120}C_K](sI - \tilde{A})^{-1} \begin{bmatrix} B_{10} \\ B_KD_{210} \end{bmatrix} \right\|_\infty < g \tag{46}$$

and

$$\left\| [\tilde{C}_1 \quad 0](sI - \tilde{A})^{-1} \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \right\|_\infty < 1. \tag{47}$$

Using Corollary 4.5, (46) implies that the nominal closed-loop system strictly bounded real with disturbance attenuation g. Also, (47) implies that the closed-loop system

$$d\eta(t) = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \eta(t)dt + \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix} \begin{bmatrix} dv(t) \\ dv_K(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} dw(t)$$
$$dz(t) = [\tilde{C}_1 \quad 0]\eta(t)dt$$

is strictly bounded real with unity disturbance attenuation. From this, it follows from Theorem 6.3 that the closed-loop uncertain system

$$d\eta(t) = \begin{bmatrix} A + \tilde{B}_1\Delta\tilde{C}_1 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \eta(t)dt + \begin{bmatrix} B_0 & B_2B_{K0} \\ B_KD_{20} & B_{K1} \end{bmatrix} \begin{bmatrix} dv(t) \\ dv_K(t) \end{bmatrix}$$

is mean square stable for all matrices Δ such that $\Delta^T\Delta \leq I$. Hence, we can conclude that the true closed-loop system is mean square stable.

Note that for this example, it is also possible to verify that the true closed-loop system must not only be mean square stable but must also be strictly bounded real with disturbance attenuation g.

C. Classical Controller Synthesis

In Section VII-A, we obtained a quantum controller corresponding to the choice $\Theta_K = J$. We now show that if we instead choose $\Theta_K = 0$, the controller that is realized is classical, with appropriate transitions to and from the quantum plant.

Now, suppose we choose v_K to be the quadratures of two independent noise channels (i.e., $F_{v_K} = I_{4\times4} + i\mathrm{diag}(J, J)$). Setting $\Theta_K = 0_{2\times2}$, (12) and the compatibility requirement (38) in this context results in the following pair of equations:

$$B_KJB_K^T + B_{K1}\mathrm{diag}(J, J)B_{K1}^T = 0 \tag{48}$$
$$B_{K0}(I_{4\times4} + i\mathrm{diag}(J, J))B_{K0}^T = I + iJ. \tag{49}$$

In order to find B_{K0} and B_{K1} solving (48) and (49), we assume the following forms for B_{K0} and B_{K1}:

$$B_{K0} = [\tilde{B}_{K0} \quad 0_{2\times2}]; \quad B_{K1} = [0_{2\times2} \quad \tilde{B}_{K1}].$$

Since $B_K = -0.447I$, substitution of these forms into (48) and (49) gives

$$\tilde{B}_{K0}(I + iJ)\tilde{B}_{K0}^T = I + iJ; \quad 0.447^2J + \tilde{B}_{K1}J\tilde{B}_{K1}^T = 0.$$

It can be readily checked, by direct substitution, that these equations are solved by $\tilde{B}_{K0} = I_{2\times2}$ and $\tilde{B}_{K1} = -0.447\tilde{I}$, where $\tilde{I} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. This completely specifies the classical realization of the controller, illustrated in Fig. 3. The quantum signal y is converted to a classical signal $y_c = (y_{c1}, y_{c2})^T = (y_1 - v_{K21}, v_{K22} + y_2)^T$ by imperfect continuous measurement of the real and imaginary quadratures of the optical beam, implemented in Fig. 3 by a beam splitter and two homodyne detectors [2]. The classical signal y_c is processed by a classical linear system $(A_K, B_K, C_K, 0)$ to produce a classical control signal u_c, which then modulates (displaces) a field v_{K1} to produce the optical control signal $du = u_cdt + dv_{K1}$. This classical controller achieves exactly the same H^∞ performance as the quantum controller of Subsection VII-A.

This classical controller has access to the full quantum signal y, and the quantum measurement occurs in the controller. The algebra based on the commutation relations enforces the quantum measurement, and also the modulation. If we were to include measurement as part of the plant specification, then in general a different classical controller will result, with different

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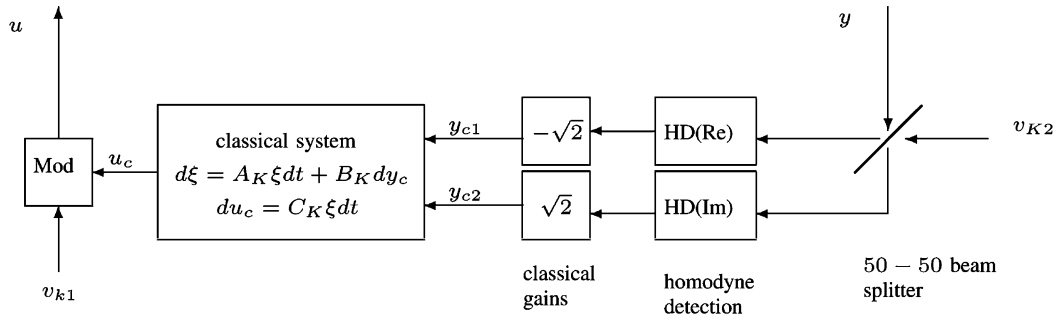


Fig. 3. Classical realization of the controller ($\Theta_K = 0$) for the plant shown in Fig. 1. The controller includes quantum measurement and classical modulation of optical fields.

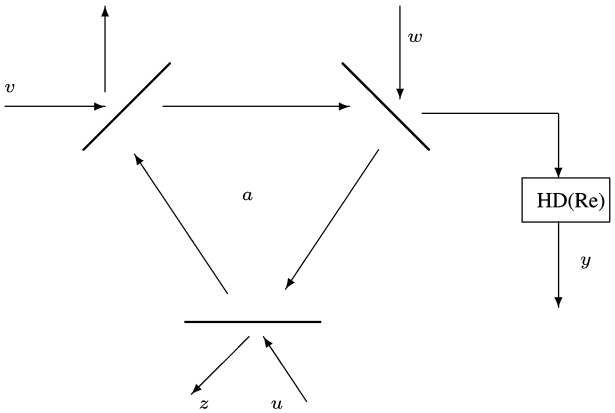


Fig. 4. Optical cavity (plant) with classical output. The (real) quadrature measurement is achieved by homodyne photodetection (HD(Re)).

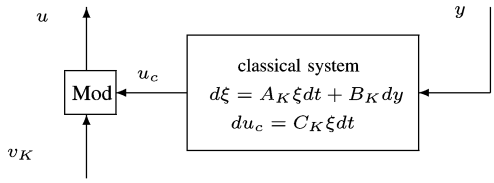


Fig. 5. Classical controller ($\Theta_K = 0$) for the plant of Fig. 4.

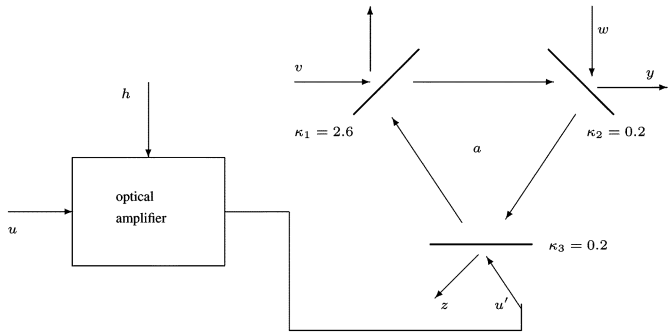


Fig. 6. Optical amplifier-cavity system (plant).

H^∞ performance. To see this, suppose that y is replaced by its real quadrature in the plant specification; this situation is described by the matrices

$$\begin{aligned} &A = -\frac{\gamma}{2}I; \quad B_0 = -\sqrt{\kappa_1}I; \quad B_1 = -\sqrt{\kappa_2}I \\ &B_2 = -\sqrt{\kappa_3}I; \quad C_1 = \sqrt{\kappa_3}I; \quad D_{12} = I \\ &C_2 = \sqrt{\kappa_2}[1 \quad 0]; \quad D_{21} = [1 \quad 0] \end{aligned} \tag{50}$$

and is illustrated in Fig. 4. Thus the output of the plant is a classical single-variable signal.

With a disturbance attenuation constant of $g = 0.134$, it was found that the Riccati equations (28) and (29) have the following stabilizing solutions satisfying Assumption 5.3:

$$X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 0.121 \end{bmatrix}.$$

It now follows from Theorem 4.2 that if a controller of the form (23) is applied to this system with the following matrices A_K, B_K, C_K defined as in (30), then the resulting closed-loop system will be strictly bounded real with disturbance attenuation $g = 0.667$

$$\begin{aligned} &A_K = \begin{bmatrix} -1.1 & 0 \\ 0 & -1.3 \end{bmatrix}; \quad B_K = \begin{bmatrix} -0.447 \\ 0 \end{bmatrix} \\ &C_K = \begin{bmatrix} -0.447 & 0 \\ 0 & -0.447 \end{bmatrix}. \end{aligned}$$

In this case, the controller (23), (51) is a classical system which can be implemented using standard electronic devices. This second classical controller is illustrated in Fig. 5, and is different to the previous one. Here we have chosen $B_{K0} = I, B_{K1} = 0$, and the quantum noise is canonical. The control signal is $du = u_c dt + dv_K$, a coherent optical field.

D. Classical-Quantum Controller Synthesis

As a final example, we illustrate the synthesis of a controller with both classical and quantum components. The plant has two degrees of freedom, and is formed as a cascade of an optical amplifier [14] and the cavity discussed above. This plant is illustrated in Fig. 6.

The optical amplifier has an auxiliary input h, which is an inverted heat bath with Ito matrix $F_h = (2N+1)I + iJ$, where $N > 0$ is a positive thermal parameter. The complete system shown in Fig. 6 is of the form (21) with matrices

$$\begin{aligned} &A = \begin{bmatrix} -\frac{\gamma}{2}I & -\sqrt{\kappa_3\alpha}\,I \\ 0 & -\frac{\alpha-\beta}{2}I \end{bmatrix} \\ &B_0 = \begin{bmatrix} -\sqrt{\kappa_1}I & 0 \\ 0 & \sqrt{\beta}I \end{bmatrix} \\ &B_1 = \begin{bmatrix} -\sqrt{\kappa_2}I \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} -\sqrt{\kappa_3}I \\ -\sqrt{\alpha}I \end{bmatrix} \\ &C_1 = [\sqrt{\kappa_3}I \quad 0]; \quad D_{12} = I \\ &C_2 = [\sqrt{\kappa_2}I \quad 0]; \quad D_{20} = 0; \quad D_{21} = I. \end{aligned} \tag{51}$$

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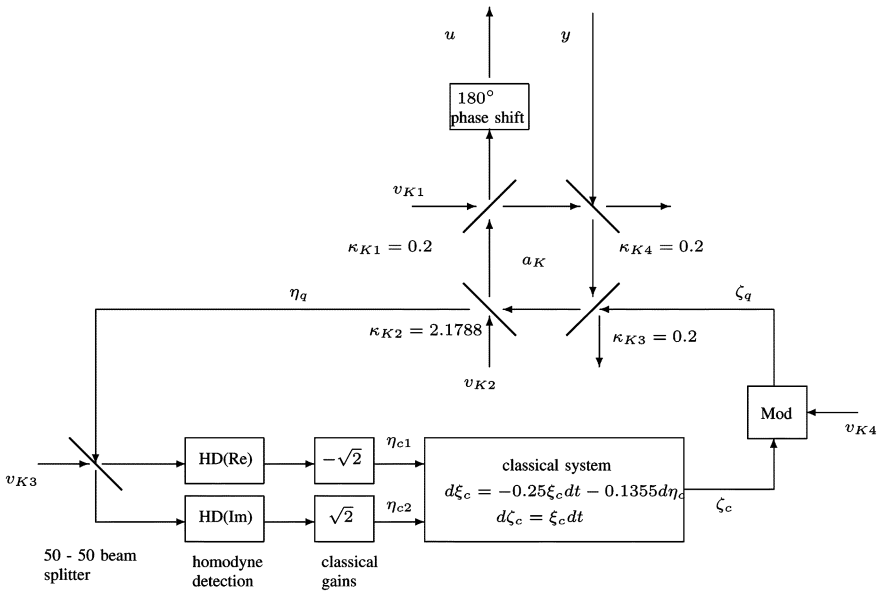


Fig. 7. Quantum-classical controller ($\Theta_K = \mathrm{diag}(J, 0_{2\times 2})$) for the plant of Fig. 6.

Here, α and β are parameters of the optical amplifier. The signals have Ito matrices $F_u = F_{\tilde{w}} = I + iJ$ and $F_v = \mathrm{diag}(I + iJ, (2N+1)I + iJ)$, and the parameters values are $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$, $\alpha = 1$ and $\beta = 0.5$.

With a H^∞ gain $g = 0.1$, the Riccati equations (28) and (29) have stabilizing solutions satisfying Assumption 5.3: $X = Y = 0_{2\times 2}$. Using (30), the controller matrices A_K, B_K, C_K are

$$A_K = \begin{bmatrix} -1.3894I & -0.4472I \\ -0.2I & -0.25I \end{bmatrix}$$
$$B_K = \begin{bmatrix} -0.4472I \\ 0_{2\times 2} \end{bmatrix}; \quad C_K = [-0.4472I \quad 0_{2\times 2}].$$

We choose $\Theta_K = \mathrm{diag}(J, 0_{2\times 2})$ in order to implement a degenerate canonical controller, with both classical and quantum degrees of freedom. We write $\xi = (\xi_q, \xi_c)^T$, where $\xi_q = (\xi_1, \xi_2)^T$ are classical and $\xi_c = (\xi_3, \xi_4)^T$ are quantum variables. A realization is shown in Fig. 7, which consists of a four-mirror optical cavity, a classical system, and homodyne detection and modulation for interfacing the classical and quantum components. The quantum noises in Fig. 7 are all canonical. The cavity has coupling coefficients $\kappa_{K1} = \kappa_{K3} = \kappa_{K4} = 0.2$ and $\kappa_{K2} = 1.33$. The interconnection fields are given by $d\eta_q = \xi_q dt + dv_{K2}$, and $d\zeta_q = \zeta_c dt + dv_{K4}$, where $\eta_c = (\eta_{c1}, \eta_{c2})^T = (\eta_{q1} - v_{K31}, \eta_{q2} + v_{K32})^T$. For this realization, we have

$$B_{K1} = \begin{bmatrix} -0.4472I & -1.4761I & 0_{2\times 2} & -0.4472I \\ 0_{2\times 2} & -0.1355I & 0.1355\tilde{I} & 0_{2\times 2} \end{bmatrix}$$
$$\tilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$B_{K0} = [-I \quad 0_{2\times 2} \quad 0_{2\times 2}].$$

VIII. Conclusion

In this paper, we have formulated and solved an H^∞ synthesis problem for a class of noncommutative stochastic models. Models important to quantum technology, such as those arising in quantum optics, are included in this class. We have provided results for the physical realization of the controllers. Our results are illustrated with examples from quantum optics, which demonstrate the synthesis of quantum, classical and quantum-classical controllers. Future work will include further development of the approach initiated here, and application of the synthesis methods to particular problems in quantum technology.

Appendix A
Proofs

Proof of Theorem 2.1: To preserve the commutation relations for all $i, j = 1, \ldots, n$ and all $t \geq 0$, we must have $d[x_i, x_j] = 0$ for all $i, j = 1, \ldots, n$. We now develop a general expression for $d[x_i, x_j]$. Indeed, let $e_k = [0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0]^T$, where the 1 is in the k-th row. It is easy to see that for any $i, j \in \{1, \ldots, n\}$, $[x_i, x_j] = e_i^T x x^T e_j - e_j^T x x^T e_i$. Therefore, $d[x_i, x_j] = e_i^T d(xx^T) e_j - e_j^T d(xx^T) e_i$. Now, we expand $d(xx^T)$ using the quantum Ito rule (e.g., see [20]) as follows:

$$\begin{aligned} &d(xx^T) \\ &\quad = (dx)x^T + x d(x^T) + dx d(x^T) \\ &\quad = Axx^T dt + B dw x^T + x x^T A^T dt + x d(w^T) B^T \\ &\qquad + Axx^T A^T dt^2 + Ax d(w^T) dt B^T + B dw dt x^T A^T \\ &\qquad + B dw (dw)^T B^T \\ &\quad = Axx^T dt + B dw x^T + x x^T A^T dt + x d(w^T) B^T \\ &\qquad + B (dw)(dw)^T B^T. \end{aligned}$$

Substituting $dw = \beta_w dt + d\tilde{w}$ into the above and noting that $\beta_w \beta_w^T dt^2$ and $\beta_w d\tilde{w}^T dt$ vanish to order dt gives

$$\begin{aligned} &d(xx^T) \\ &\quad = Axx^T dt + B\beta_w x^T dt + B d\tilde{w} x^T + x x^T A^T dt \\ &\qquad + x\beta_w^T B^T dt + x d\tilde{w}^T B^T + B d\tilde{w} d\tilde{w}^T B^T. \end{aligned}$$

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We now write $A = [A_1^T \ A_2^T \ \dots \ A_n^T]^T$ and $B = [B_1^T \ B_2^T \ \dots \ B_n^T]^T$, where the vectors A_k and B_k denote the k-th row of matrices A and B, respectively. Then we have

$$\begin{aligned} e_i^T d(xx^T)e_j &= e_i^T Axx^T e_j dt + e_i^T B\beta_w x^T e_j dt + e_i^T Bd\tilde{w}x^T e_j \\ &\quad + e_i^T xx^T A^T e_j dt + e_i x\beta_w^T B^T e_j dt + e_i xd\tilde{w}^T B^T e_j \\ &\quad + e_i^T Bd\tilde{w}(d\tilde{w})^T B^T e_j \\ &= A_i xx_j dt + B_i\beta_w x_j dt + B_i d\tilde{w}x_j + x_i A_j x dt \\ &\quad + x_i B_j \beta_w dt + x_i B_j d\tilde{w} + (B_i d\tilde{w})(B_j d\tilde{w}). \end{aligned} \tag{52}$$

Also, we have

$$\begin{aligned} e_j^T d(xx^T)e_i = A_j xx_i dt + B_j\beta_w x_i dt + B_j d\tilde{w}x_i \\ + x_j A_i x dt + x_j B_i\beta_w dt + x_j B_i d\tilde{w} \\ + (B_j d\tilde{w})(B_i d\tilde{w}). \end{aligned} \tag{53}$$

Subtracting (53) from (52) gives us

$$\begin{aligned} &e_i^T d(xx^T)e_j - e_j^T d(xx^T)e_i \\ &= ((A_i x)x_j - x_j(A_i x))dt + ((B_i\beta_w)x_j - x_j(B_i\beta_w))dt \\ &\quad + (B_i d\tilde{w})x_j - x_j(B_i d\tilde{w}) + (x_i(A_j x) - (A_j x)x_i)dt \\ &\quad + (x_i(B_j\beta_w) - (B_j\beta_w)x_i)dt \\ &\quad + (x_i(B_j d\tilde{w}) - (B_j d\tilde{w})x_i) \\ &\quad + ((B_i d\tilde{w})(B_j d\tilde{w}) - (B_j d\tilde{w})(B_i d\tilde{w})) \\ &= ((A_i x)x_j - x_j(A_i x))dt + ((B_i\beta_w)x_j - x_j(B_i\beta_w))dt \\ &\quad + (x_i(A_j x) - (A_j x)x_i)dt \\ &\quad + (x_i(B_j\beta_w) - (B_j\beta_w)x_i)dt \\ &\quad + ((B_i d\tilde{w})(B_j d\tilde{w}) - (B_j d\tilde{w})(B_i d\tilde{w})). \end{aligned} \tag{54}$$

Here we are using the fact that elements of $d\tilde{w}$ commute with those of x and β_w due to the adaptedness of x and β_w. Hence,

$$\begin{aligned} &e_i^T d(xx^T)e_j - e_j^T d(xx^T)e_i \\ &= [A_i x, x_j]dt - [x_j, B_i\beta_w]dt \\ &\quad + [x_i, A_j x]dt + [x_i, B_j\beta_w]dt + [B_i d\tilde{w}, B_j d\tilde{w}] \\ &= \sum_{k=1}^n A_{ik}[x_k, x_j]dt - \sum_{k=1}^n B_{ik}[x_j, \beta_{wk}]dt \\ &\quad + \sum_{k=1}^n A_{jk}[x_i, x_k]dt + \sum_{k=1}^n B_{jk}[x_i, \beta_{wk}]dt \\ &\quad + \sum_{k=1}^n \sum_{l=1}^n B_{ik}B_{jl}[d\tilde{w}_k, d\tilde{w}_l] \\ &= \left(2i\sum_{k=1}^n A_{ik}\Theta_{kj} + 2i\sum_{k=1}^n A_{jk}\Theta_{ik} \right. \\ &\quad - \sum_{k=1}^n B_{ik}C_{jk}^{x\beta_w} + \sum_{k=1}^n B_{jk}C_{ik}^{x\beta_w} \\ &\quad \left. + \sum_{k=1}^n \sum_{l=1}^n B_{ik}B_{jl}(F_{\tilde{w},kl} - F_{\tilde{w},lk}) \right) dt \end{aligned} \tag{55}$$

where $C_{ij}^{x\beta_w} = [x_i, \beta_{wj}]$.

Since $C^{x\beta_w} = [C_{ij}^{x\beta_w}]_{i=1,\dots,n,j=1\dots,n_{\beta_w}} = 0$ (by assumption) and $F_{\tilde{w}} - F_{\tilde{w}}^T = 2T_{\tilde{w}}$, (55) takes the form

$$d(xx^T - (xx^T)^T) = 2(iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T)dt \tag{56}$$

from which the result follows. □

Proof of Theorem 3.4: Let us first consider the case where Θ is canonical. If the system is realizable then (7)–(10) holds. Since $U(t)$ is unitary for each $t \geq 0$, we have that $d\left(x(t)x(t)^T - (x(t)x(t)^T)^T\right) = 0$; i.e., the canonical commutation relations are preserved. By Theorem 2.1 this is equivalent to (12). Let $M_1, M_2, \dots, M_{N_y}$ be column vectors such that $[M_1 \ M_2 \ \dots \ M_{N_y}] = \Lambda^T[I_{N_y\times N_y} \ 0]^T$. Then using (8) and (9), we obtain the following after some algebraic manipulations:

$$\begin{aligned} &B\left[I_{n_y\times n_y} \quad 0_{n_y\times(n_w-n_y)}\right]^T \\ &= 2i\Theta[-\Lambda^\dagger \quad \Lambda^T]\Gamma\left[I_{n_y\times N_y} \quad 0_{n_y\times(n_w-N_y)}\right]^T \\ &= 2\Theta\left[-\Im(M_1) \quad \Re(M_1) \quad \dots \quad -\Im\left(M_{N_y}\right) \quad \Re\left(M_{N_y}\right)\right] \\ &= \Theta\left(P_{N_y}^T\begin{bmatrix} 0_{N_y\times N_y} & -I_{N_y\times N_y} \\ I_{N_y\times n_y} & 0_{N_y\times n_y}\end{bmatrix}P_{N_y}C\right)^T \\ &= \Theta C^T P_{N_y}^T\begin{bmatrix} 0 & I_{N_y\times N_y} \\ -I_{N_y\times N_y} & 0\end{bmatrix}P_{N_y} \\ &= \Theta C^T \mathrm{diag}_{N_y}(J). \end{aligned}$$

Therefore, we conclude that (13), (12), and (10) are necessary for realizability.

Conversely, now suppose that (13), (12) and (10) hold. We will argue that these conditions are sufficient for realizability by showing that they imply the existence a symmetric matrix R and a coupling matrix Λ such that (7)-(9) are satisfied. First we note that after some simple algebraic manipulation $-i\Theta^{-1}B\Gamma^{-1} = i\Theta B\Gamma^{-1} = [-Z^\# \ Z]$, for some complex matrix Z. Hence, $B = i\Theta[-Z^\# \ Z]\Gamma$. Substituting the last expression into (12) and after further manipulations we get:

$$iA\Theta + i\Theta A^T - \frac{1}{2}\Theta(Z^\# Z^T - ZZ^\dagger)\Theta = 0.$$

Writing $Z^\# Z^T - ZZ^\dagger = 2i\Im(Z^\# Z^T)$, we may rewrite the last expression as follows:

$$\begin{aligned} &iA\Theta + i\Theta A^T - \frac{1}{2}\Theta(Z^\# Z^T - ZZ^\dagger)\Theta \\ &\quad = iA\Theta + i\Theta A^T - i\Theta\Im(Z^\# Z^T)\Theta \\ &\quad = i\Theta(\Theta^{-1}A + A^T\Theta^{-1} - \Im(Z^\# Z^T))\Theta \\ &\quad = i\Theta(\Theta^{-1}A - (\Theta^{-1}A)^T - \Im(Z^\# Z^T))\Theta \\ &\quad = 0 \end{aligned}$$

implying that $\Theta^{-1}A - (\Theta^{-1}A)^T - \Im(Z^\# Z^T) = 0$. Since $\Theta^{-1}A$ is real, we have the decomposition $\Theta^{-1}A = -\Theta A = V + W$ for a unique pair of real symmetric matrix V and real skew symmetric matrix W and obtain the condition $2W - \Im(Z^\# Z^T) = 0$. Hence, $W = (1/2)\Im(Z^\# Z^T)$. Setting $R = (1/2)V$ and $\Lambda = 2Z^T$, we get $A = 2\Theta(R + \Im(\Lambda^\dagger\Lambda))$ and $B = 2i\Theta[-\Lambda^\dagger \ \Lambda^T]$ as desired, and also prove the second statement of the theorem. After substituting the expression, just obtained for B (in terms of Λ, Θ, and Γ) into (13) and more algebraic manipulations

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we then get (9). Since the expression for D has been hypothesized as (10), we conclude that (13), (12) along with (10) gives matrices A, B, C, D which are the coefficients of a realizable system.

Now, we consider the case where Θ is degenerate canonical, i.e., $\Theta = \mathrm{diag}(0_{n'\times n'}, \mathrm{diag}_{(n-n')/2)}(J))$. Let us write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = [B_1 \quad B_2] \quad C = [C_1 \quad C_2]$$

with $A_{11} \in \mathbb{R}^{n'\times n'}$, $A_{12} \in \mathbb{R}^{n'\times(n-n')}$, $A_{21} \in \mathbb{R}^{(n-n')\times n'}$, $A_{22} \in \mathbb{R}^{(n-n')\times(n-n')}$, $B_1 \in \mathbb{R}^{n\times n_y}$, $B_2 \in \mathbb{R}^{n\times(n_w-n_y)}$, $C_1 \in \mathbb{R}^{n_y\times n'}$ and $C_2 \in \mathbb{R}^{n_y\times(n-n')}$. Consider the following augmentation:

$$\begin{aligned} d\tilde{x}(t) &= \begin{bmatrix} A_{11} & A_{12} & 0_{n'\times n'} \\ A_{21} & A_{22} & 0_{(n-n')\times n'} \\ A_1' & A_2' & A'' \end{bmatrix} \tilde{x}(t)dt \\ &\quad + \begin{bmatrix} B_1 & B_2 \\ B_1' & 0 \end{bmatrix} dw(t) \\ d\tilde{y}(t) &= [\,C \quad 0_{n_y\times n'}\,]\tilde{x}(t)dt + Ddw(t) \end{aligned}$$

where $B_1' = -C_1^T P_{N_y}^T [\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}] P_{N_y}$, and A_1', A_2' and A'' satisfy the following:

$$\begin{aligned} A_1' - (A_1')^T &= i[B_1' \quad 0]T_w \begin{bmatrix} (B_1')^T \\ 0 \end{bmatrix} \\ \begin{bmatrix} A'' & -A_2'\mathrm{diag}_{\frac{n-n'}{2}}(J) \end{bmatrix} &= -\begin{bmatrix} A_{11}^T & A_{21}^T \end{bmatrix} \\ &\quad - i[B_1' \quad 0]T_w B^T. \end{aligned}$$

It follows by inspection that such matrices A_1', A_2', and A'' exist. Let $A' = [A_1' \;\; A_2']$ and define

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A & 0_{n\times n'} \\ A' & A'' \end{bmatrix}; \tilde{B} = \begin{bmatrix} B_1 & B_2 \\ B_1' & 0 \end{bmatrix} \\ \tilde{C} &= [C \quad 0_{n\times n'}]. \end{aligned}$$

If (12) holds then it can be verified, by direct substitution, that the matrices $\tilde{A}$ and $\tilde{B}$ satisfy:

$$i\tilde{A}\tilde{\Theta} + i\tilde{\Theta}\tilde{A}^T + \tilde{B}T_w\tilde{B}^T = 0. \tag{57}$$

Recalling that $\tilde{\Theta}$ is only canonical up to permutation, we now need to transform it into canonical form. To do this, introduce the variable $z = P\tilde{x}$ where P is a permutation matrix such that $P\tilde{\Theta}P^T = \mathrm{diag}_{(\tilde{n}/2)}(J)$. Then the components of z are a relabelling of the components of $\tilde{x}$. This gives us the following dynamics for z:

$$\begin{aligned} dz(t) &= P\tilde{A}P^T z(t) + P\tilde{B}dw(t) \\ dy(t) &= \tilde{C}P^T z(t)dt + Ddw(t). \end{aligned}$$

Denoting $\hat{A} = P\tilde{A}P^T$, $\hat{B} = P\tilde{B}$, $\hat{C} = \tilde{C}P^T$, and $\hat{\Theta} = \mathrm{diag}_{(\tilde{n}/2)}(J)$, we see that (57) implies that

$$i\hat{A}\hat{\Theta} + i\hat{\Theta}\hat{A}^T + \hat{B}T_w\hat{B}^T = 0. \tag{58}$$

Continuing further using (13), we have the following:

$$\begin{aligned} &\hat{B}\begin{bmatrix} I_{n_y\times n_y} \\ 0_{(n_w-n_y)\times n_y} \end{bmatrix} \\ &= P\begin{bmatrix} B \\ B_1' \quad 0 \end{bmatrix}\begin{bmatrix} I_{n_y\times n_y} \\ 0_{(n_w-n_y)\times n_y} \end{bmatrix} \\ &= P\begin{bmatrix} \Theta C^T \\ -C_1^T \end{bmatrix} P_{N_y}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{N_y} \\ &= P\tilde{\Theta}\begin{bmatrix} C^T \\ 0 \end{bmatrix} P_{N_y}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{N_y} \\ &= (P\tilde{\Theta}P^T)P\begin{bmatrix} C^T \\ 0 \end{bmatrix} P_{N_y}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{N_y} \\ &= \hat{\Theta}\hat{C}^T P_{N_y}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{N_y} = \hat{\Theta}\hat{C}^T \mathrm{diag}_{N_y}(J). \end{aligned} \tag{59}$$

If D is given by (10) then (58) and (59) implies, as we have already shown for the case of canonical Θ, the system defined by the matrices $(\hat{A}, \hat{B}, \hat{C}, D)$ is realizable in the sense of Point 1 of the theorem. Hence, the original system defined by the matrices (A, B, C, D) is then realizable in the sense of Point 2 of the theorem.

Finally, suppose conversely that (1) is realizable and let $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ be a suitable augmentation. Then $(P\tilde{A}P^T, P\tilde{B}, \tilde{C}P^T, D)$ is a quantum harmonic oscillator, with P as defined before. Hence, $P\tilde{A}P^T$, $P\tilde{B}$, $\tilde{C}P^T$, and D are given by the right hand sides of (7)–(10) for a canonical Θ and some R and Λ. It follows that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are given by the same set of equations by replacing $\tilde{\Theta}$, R and Λ by $\tilde{\Theta} = P^T\Theta P$, $\tilde{R} = P^T R P$ and $\tilde{\Lambda} = \Lambda P$, respectively. We then have, from the same line of arguments given for the case of canonical Θ, that

$$\begin{aligned} &\tilde{B}\begin{bmatrix} I_{n_y\times n_y} \\ 0_{(n_w-n_y)\times n_y} \end{bmatrix} \\ &= \tilde{\Theta}\tilde{C}^T \times P_{N_y}^T \begin{bmatrix} 0 & I_{N_y\times N_y} \\ -I_{N_y\times N_y} & 0 \end{bmatrix} P_{N_y} \\ &= \tilde{\Theta}\tilde{C}^T \mathrm{diag}_{N_y}(J) \end{aligned} \tag{60}$$

(57) holds, and D satisfies (10). Reading off the first n rows of both sides of (60) then gives us (13), while reading of the first n rows and columns of both sides of (57) gives us (12), as required. This completes the proof. □

The Proof of Theorem 4.2 will use the following lemma.

Lemma A.1: Consider a real symmetric matrix X and corresponding operator valued quadratic form $x^T X x$ for the system (15). Then the following statements are equivalent.

i) There exists a constant $\lambda \geq 0$ such that $\langle \rho, x^T X x \rangle \leq \lambda$ for all Gaussian states ρ.[3]
ii) The matrix X is negative semidefinite.

[3]Here $\langle \rho, \cdot \rangle$ denotes the expectation with respect to the Gaussian state ρ.

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Proof: $(i) \Rightarrow (ii)$. To establish this part of the lemma, consider a Gaussian state ρ which has mean $\bar{x}$ and covariance matrix $Y \geq 0$. Then, we can write

$$\begin{aligned}\langle \rho, x^T X x\rangle &= \sum_{i=1}^n \sum_{j=1}^n X_{ij}\langle \rho, x_i x_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n X_{ij}[Y_{ij} + \bar{x}_i \bar{x}_j] \\ &= \bar{x}^T X \bar{x} + \mathrm{tr}[XY]. \end{aligned} \tag{61}$$

Now for any constant $\alpha > 0$, consider the inequality of part (i) where ρ is a Gaussian state with mean $\alpha\bar{x}$ and covariance matrix Y. Then it follows from this bound and (61) that $\alpha^2 \bar{x}^T X \bar{x} + \mathrm{tr}[XY] \leq \lambda$ for all $\alpha > 0$. From this, it immediately follows that $\bar{x}^T X \bar{x} \leq 0$. However $\bar{x}$, the mean of the Gaussian state ρ was arbitrary. Hence, we can conclude that condition (ii) of the lemma is satisfied. $(ii) \Rightarrow (i)$. Suppose that the matrix X is negative semidefinite and let ρ be any Gaussian state and suppose that ρ has mean $\bar{x}$ and covariance matrix $Y \geq 0$. Then, it follows from (61) that $\langle \rho, x^T X x\rangle = \bar{x}^T X \bar{x} + \mathrm{tr}[XY]$. However, $X \leq 0$ and $Y \geq 0$ implies $\bar{x}^T X \bar{x} \leq 0$ and $\mathrm{tr}[XY] \leq 0$. Hence, $\langle \rho, x^T X x\rangle \leq 0$ and condition (ii) is satisfied with $\lambda = 0$. □

Proof of Theorem 4.2: Let the system be dissipative with $V(x) = x^T X x$. By Ito's rule, the table (20) and the quantum stochastic differential equation (15), we have

$$\begin{aligned} &d\langle V(x(t))\rangle \\ &\quad = \langle dx^T(t) X x(t) + x^T(t) X dx(t) + dx^T(t) X dx(t)\rangle \\ &\quad = \langle x^T(t)(A^T X + XA)x(t) + \beta_w^T(t) B^T X x(t) \\ &\quad + x^T(t) X B \beta_w(t) + \lambda_0\rangle dt, \end{aligned} \tag{62}$$

where λ_0 is given by (19). We now note that (e.g., see [20, p. 215]) $\langle V(x(t)\rangle = \langle \rho, E_0[V(x(t))]\rangle$, where E_0 denotes expectation with respect to ϕ, and ρ is an initial Gaussian state. Combining this with the integral of (62) and (16), we find that

$$\begin{aligned}\Big\langle \rho, \int_0^t E_0[x^T(s)(A^T X + XA)x(s) + \beta_w^T(s) B^T X x(s) \\ + x^T(s) X B \beta_w(s) + \lambda_0 + r(x(s), \beta_w(s))] ds \Big\rangle \leq \lambda t.\end{aligned}$$

Let $t \to 0$ to obtain

$$\begin{aligned}\Big\langle \rho, x^T(A^T X + XA)x + \beta_w^T B^T X + x^T X B \beta_w + \lambda_0 \\ + [x^T \beta_w^T] R \begin{bmatrix} x \\ \beta_w \end{bmatrix} \Big\rangle \leq \lambda.\end{aligned}$$

Here, x and β_w denote the initial conditions. An application of Lemma A.1 implies (17). Also, (18) is a straightforward consequence of this inequality when R is replaced by $R + \epsilon I$ where $\epsilon > 0$.

To establish the converse part of the theorem, we first assume that (17) is satisfied. Then with $V(x) = x^T X x$, it follows from (62) that

$$\langle V(x(t))\rangle - \langle V(x(0))\rangle + \int_o^t \langle r(x(s), \beta_w(s))\rangle ds \leq \lambda_0 t$$

for all $t > 0$ and all $\beta_w(t)$. Hence, inequality (16) is satisfied with λ given by (19).

If matrix inequality (18) is satisfied, then it follows by similar reasoning that there exists an $\epsilon > 0$ such that

$$\begin{aligned}\langle V(x(t))\rangle - \langle V(x(0))\rangle + \int_0^t \langle r(x(s), \beta_w(s)) \\ + \epsilon(x(s)^T x(s) + \beta_w(s)^T \beta_w(s))\rangle ds \leq \lambda_0 t.\end{aligned}$$

Hence, inequality (16) is satisfied with $\lambda = \lambda_0$ given by (19) and with R replaced by by $R + \epsilon I$. □

Proof of Theorem 5.4: Using the Strict Bounded Real Lemma Corollary 4.5, the theorem follows directly from the corresponding classical H^∞ result; e.g., see [15], [21], and [32]. □

The Proof of Theorem 5.5 will use the following lemma.

Lemma A.2: If S is a Hermitian matrix then there is a real constant α_0 such that $\alpha I + S \geq 0$ for all $\alpha \geq \alpha_0$.

Proof: Since S is Hermitian it has real eigenvalues and is diagonalizable. Hence, $S = V^\dagger E V$ for some real diagonal matrix E and orthogonal matrix V. Now let $\alpha_0 = -\lambda$, where λ is the smallest eigenvalue of S. The result follows since $\alpha I + S = V^\dagger(\alpha I + E)V$ while $\alpha I + E \geq 0$ for all $\alpha \geq \alpha_0$. □

Proof of Lemma 5.6: The main idea is to explicitly construct matrices $R \in \mathbb{R}^{n_K \times n_K}$, $\Lambda \in \mathbb{C}^{(N_{v_K}+N_y)\times n_K}$, $B_{K1} \in \mathbb{R}^{n_K \times 2N_{v_K}}$ and $B_{K0} \in \mathbb{R}^{l_K \times 2N_{v_K}}$, with $N_{v_K} \geq N_u$, such that (7)–(10) are satisfied by identifying A_K, $[B_{K1}\ B_K]$, C_K, $[B_{K0}\ \ 0_{l_K \times m_K}]$, ξ, $w_K = (v_K^T, y^T)^T$ and u with A, B, C, D, x, w and y, respectively. To this end, let $Z = (1/2)\Theta_K^{-1} A = -(1/2)\Theta_K A$, with $\Theta_K = \mathrm{diag}_{N_\xi}(J)$. We first construct matrices $\Lambda_{b2}, \Lambda_{b1}, B_{K1,1}$, and $B_{K1,2}$ according to the following procedure:

1) Construct the matrix Λ_{b2} according to (35).
2) Construct a real symmetric $n_K \times n_K$ matrix Ξ_1 such that the matrix

$$\begin{aligned}\Xi_2 = \Xi_1 + i\Big(\frac{Z - Z^T}{2} - \frac{1}{4} C_K^T P_{N_u}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \\ \times P_{N_u} C_K - \Im(\Lambda_{b2}^\dagger \Lambda_{b2})\Big)\end{aligned}$$

is nonnegative definite. It follows from Lemma A.2 that such a matrix Ξ_1 always exists.
3) Construct a matrix Λ_{b1} such that $\Lambda_{b1}^\dagger \Lambda_{b1} = \Xi_2$, where Λ_{b1} has *at least* 1 row. This can be done, for example, using the singular value decomposition of Ξ_2 (in this case Λ_{b1} will have n_K rows).
4) Construct the matrices $B_{K1,1}$ and $B_{K1,2}$ according to (34) and (36), respectively.

Let $R = (1/2)(Z + Z^T)$. We now show that there exists an integer $N_{v_K} \geq N_u$ such conditions (7)–(10) are satisfied with the matrix R as defined and with $B_{K1} = [B_{K1,1}\ B_{K1,2}]$ and

$$\Lambda = \begin{bmatrix} \frac{1}{2}[I \quad iI] P_{N_u} C_k \\ \Lambda_{b1} \\ \Lambda_{b2} \end{bmatrix}. \tag{63}$$

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First note that necessarily $N_{v_K} \geq N_u + 1 > N_u$ since B_{K1} has at least $2N_u + 2$ columns. Also, by virtue of our choice of Λ_{b1}, we have

$$\begin{aligned}\Im(\Lambda_{b1}^\dagger \Lambda_{b1}) &= \Im(\Xi_2) \\ &= \frac{1}{2}(Z - Z^T) - \frac{1}{4} C_K^T P_{N_u}^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{N_u} \\ &\quad \times C_K - \Im(\Lambda_{b2}^\dagger \Lambda_{b2})\end{aligned}$$

and hence

$$\begin{aligned}\Im(\Lambda^\dagger \Lambda) = \Im(\Lambda_{b1}^\dagger \Lambda_{b1}) + \Im(\Lambda_{b2}^\dagger \Lambda_{b2}) + \frac{1}{4} C_K^T P_{N_u}^T \\ \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P_{N_u} C_K = \frac{1}{2}(Z - Z^T).\end{aligned}$$

Since $R = (Z + Z^T/2)$, we have $R + \Im(\Lambda^\dagger \Lambda) = Z$. Therefore, (7) is satisfied.

Now, as in the Proof of Theorem 3.4, observe that $i\Theta_K B_K \mathrm{diag}_{N_y}(M^\dagger) P_{N_y}^T = [T \quad -T^\#]$ for some $n_K \times N_y$ complex matrix T. But by taking the conjugate transpose of both sides of (35) which defined Λ_{b2}, we conclude that $T = -\Lambda_{b2}^\dagger$. Hence

$$B_K = 2i\Theta_K \begin{bmatrix} -\Lambda_{b2}^\dagger & \Lambda_{b2}^T \end{bmatrix} P_{N_y} \mathrm{diag}_{N_y}(M). \tag{64}$$

From (34) which defined $B_{K1,1}$, we obtain

$$\begin{aligned}B_{K1,1} &= -i\Theta_K C_K^T \mathrm{diag}_{N_u}(iJ) \\ &= -i\Theta_K C_K^T \mathrm{diag}_{N_u}(iJ)(2\mathrm{diag}_{N_u}(M^\dagger))\mathrm{diag}_{N_u}(M) \\ &= i\Theta_K C_K^T \mathrm{diag}_{N_u}\left(\begin{bmatrix} -1 & 1 \\ i & i \end{bmatrix}\right) \mathrm{diag}_{N_u}(M) \\ &= i\Theta_K C_K^T P_{N_u}^T \begin{bmatrix} -I & I \\ iI & iI \end{bmatrix} P_{N_u} \mathrm{diag}_{N_u}(M). \end{aligned} \tag{65}$$

Combining (36), (64), and (65) gives us

$$\begin{aligned}&[B_{K1,1} \quad B_{K1,2} \quad B_K] \\ &= 2i\Theta_K \left[\frac{1}{2} C_K^T P_{N_u}^T \begin{bmatrix} -I & I \\ iI & iI \end{bmatrix} P_{N_u} \right. \\ &\quad \left. \times [-\Lambda_{b1}^\dagger \quad \Lambda_{b1}^T] P_{N_{v_K} - N_u} \quad [-\Lambda_{b2}^\dagger \quad \Lambda_{b2}^T] P_{N_y}\right] \\ &\quad \times P_{N_{w_K}}^T P_{N_{w_K}} \mathrm{diag}_{n_{w_K}}(M) \\ &= 2i\Theta_K \left[-\frac{1}{2} C_K^T P_{N_u}^T \begin{bmatrix} I \\ -iI \end{bmatrix} \quad -\Lambda_{b1}^\dagger \quad -\Lambda_{b2}^\dagger \right. \\ &\quad \left. \frac{1}{2} C_K^T P_{N_u}^T \begin{bmatrix} I \\ iI \end{bmatrix} \quad \Lambda_{b1}^T \quad \Lambda_{b2}^T \right] P_{N_{w_K}} \mathrm{diag}_{N_{w_K}}(M) \\ &= 2i\Theta_K \left[-\frac{1}{2} C_K^T P_{N_u}^T \begin{bmatrix} I \\ -iI \end{bmatrix} \quad -\Lambda_{b1}^\dagger \quad -\Lambda_{b2}^\dagger \right. \\ &\quad \left. \frac{1}{2} C_K^T P_{N_u}^T \begin{bmatrix} I \\ iI \end{bmatrix} \quad \Lambda_{b1}^T \quad \Lambda_{b2}^T \right] \Gamma \\ &= 2i\Theta_K [-\Lambda^\dagger \quad \Lambda^T] \Gamma.\end{aligned}$$

Therefore, (8) is also satisfied. Moreover, it is straightforward to verify (9) by substituting Λ as defined by (63) into the right hand side of (9). Finally, since $N_{v_K} \geq N_u$, it follows that $[B_{K0} \quad 0_{l_K \times m_k}]$ is precisely the right hand side of (10). This completes the Proof of Theorem 5.5. □

Proof of Lemma 6.2: We first observe that the system (39) is mean square stable if and only if it is dissipative with a supply rate defined by the matrix $R = \mathrm{diag}(I, 0)$. Hence, it follows from Theorem 4.2 that the system (39) is mean square stable if and only if there exists a real positive definite symmetric matrix X such that $\bar{A}^T X + X\bar{A} + I \leq 0$. Hence, using a standard Lyapunov result (e.g., see [32]), it follows that the system (39) is mean square stable if and only if the matrix $\bar{A}$ is asymptotically stable. □

Proof of Theorem 6.3: It follows from Corollary 4.5 that the closed-loop quantum system (25) is strictly bounded real with disturbance attenuation g, then $\tilde{A}$ is a stable matrix and $\|\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\|_\infty < g$. From this, it follows using the standard small gain theorem (e.g., see Theorem 9.1 on page 218 of [32]) that the matrix $\bar{A} = \tilde{A} + \tilde{B}\Delta\tilde{C}$ is stable for all Δ satisfying (40). Hence using Lemma 6.2, it follows that the true closed-loop system (39) is mean square stable for all Δ satisfying (40). □

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