

Localisation of Low Energy Eigenfunctions to the Schrödinger Operator

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For everyone who has helped me along the way

Declaration

The work in this thesis is my own except where otherwise stated.

Wenqi Zhang

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Abstract

We explore the phenomena where low energy eigenfunctions of the operator $L = -\Delta + V$ for $V \in L^\infty$ concentrate on a small subset of the original domain, and decay exponentially outside of this region. We consider the function u solving the equation $Lu = 1$ and show that $\frac{1}{u}$ acts as an effective potential, in the sense that one can show

$$\int |\nabla f|^2 + Vf^2 = \int u^2 |\nabla(f/u)|^2 + \frac{1}{u} f^2.$$

With this equality, it can be shown that the regions where $\frac{1}{u} < \lambda$ for some (small) eigenvalue λ approximately corresponds to where the corresponding eigenfunction is localised.

Furthermore, if the regions where $\frac{1}{u} < \lambda$ are composed of suitably disjoint subsets, we show that if a (global) eigenfunction with eigenvalue λ resides in one of these subsets, then it is well approximated in this subset by local eigenfunctions (i.e. eigenfunctions of L in this subset) with eigenvalue $\lambda \pm \delta$. For these subsets to exist, $\frac{1}{u}$ needs to vary substantially. Since u solves $Lu = 1$, this phenomena cannot occur when V is smooth, and does not vary substantially.

As a visual aid to gain intuition on how V affects the localisation, we simulate the lowest energy eigenfunction to L numerically. We generate V to take random values on a dyadic grid. We vary the maximum size of the random values and the scale of the grid. Once the grid becomes sufficiently fine, and for large values of V we observe that the lowest energy eigenfunction become concentrated in one particular region. We conjecture that the maximum size of V has a greater effect on localisation than the grid size. This is also hinted at since the theorems depend only on $\|V\|_\infty$. However, further work needs to be done for fractal potentials V , in the sense that the set of discontinuities of our V may have the same Hausdorff dimension (no matter the grid size).

Finally, we also explore briefly the possibilities of defining a Brownian motion on Ω endowed with a distance associated with $\frac{1}{u}$. The hope is to one day be able to have a microscopic (Feynman-Kac) description of the solution to $\partial_t u = Lu$. We only explore a few possibilities to define an adapted Brownian motion on Ω , we leave an explicit construction and the Feynman-Kac analysis for future work.

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Notation and terminology

In the following Ω is an open subset of $\mathbb{R}^n$

Notation

$\mathbb{R}_+^n$ the set $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$

$\Omega_+ = \Omega \cap \mathbb{R}_+^n$

$C^n(\Omega)$ space of n times continuously differentiable functions on Ω ,
 $n \geq 0$

$C^{n,\alpha}(\Omega)$ space of n times differentiable functions on Ω , $n \geq 0$ with α
Hölder continuous n^{th} order derivatives.

$C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega)$

$C_c(\Omega)$ space of continuous functions with compact support in Ω

$C_c^\infty(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Anderson localisation is a physical phenomena where the propagation of electrons becomes localised in disordered mediums [1]. Mathematically, this can be described as the low energy eigenfunctions of an elliptic operator concentrating (or localising) on a small portion of the original domain, and nearly zero on the remainder. Over the last half century, this phenomena has seen many applications in various fields such as condensed matter physics and engineering [1][2][3][4][5]. However, a mathematical treatment is lacking and to quote the 1977 Nobel lecture,

[localization] has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

—Philip W. Anderson, Nobel lecture, 8 December 1977.

In recent years the solution u to $Lu = 1$ for an elliptic operator $L = -\operatorname{div}A\nabla + V$ for some suitable matrix A and potential V (commonly referred to as the landscape function) has demonstrated remarkable accuracy in predicting the shape and location of localised low energy eigenfunctions [6]. It has been used in application successfully [7][8][9].

In 2018 a preprint containing the simplest mathematical treatment of the landscape function to date was published on arXiv [10]. In this preprint the authors demonstrated that low energy eigenfunctions of L reside in the wells of $1/u$ and decay exponentially across the barriers/extrema of $1/u$. Furthermore, if the wells of $1/u$ are disjoint and suitably far apart, we observe that the eigenfunction localises

to one of these wells, and they are well approximated by local eigenfunctions (that is, eigenfunctions of L with the domain restricted to the well).

Perhaps the simplest case for which [10] applies to is the Schrödinger operator, $L = -\Delta + V$ for some $0 < V \in L^\infty$. Even in this case, localisation can still be observed for rough potentials V , and many of the mathematical challenges simplify. Hence this thesis aims to present [10] in this simplified setting to emphasise the fundamental role u plays in localisation. As we avoid many of the issues that [10] encounter, we are able to present a self contained argument illustrating the key ideas.

We first demonstrate that the solution u to $Lu = 1$ is a unique, positive, continuous function, allowing us to safely write $\frac{1}{u}$. Afterwards, we establish the fundamental identity (Lemma 2.3), which states that for all $f \in W^{1,2}(\Omega)$ we have

$$\int_{\Omega} |\nabla f|^2 + Vf^2 = \int_{\Omega} u^2 |\nabla(f/u)|^2 + \frac{1}{u} f^2.$$

Since $u^2 |\nabla(f/u)|^2 \geq 0$ we arrive at the inequality

$$\int_{\Omega} |\nabla f|^2 + Vf^2 \geq \int_{\Omega} \frac{1}{u} f^2$$

which can be seen as replacing V with $\frac{1}{u}$ in the trivial inequality

$$\int_{\Omega} |\nabla f|^2 + Vf^2 \geq \int_{\Omega} Vf^2.$$

Hence, the landscape function can be seen as an effective (quantum) potential, in the sense that we have replaced V with a function which combines the effects of both $|\nabla f|^2$ (kinetic energy) and Vf^2 (potential energy) terms.

Without the landscape function, one can use the trivial inequality and [11] (Agmon's method) to confine eigenfunctions with (low) eigenvalue λ to regions of the form $\{x : V(x) \leq \lambda + \delta\}$. This is, however useless in cases such as piecewise constant potentials with randomly generated constants [10].

With the inequality from the fundamental lemma, Agmon's method can be used to instead confine the eigenfunction to a region of the form

$$E(\lambda + \delta) := \{x : \frac{1}{u(x)} \leq \lambda + \delta\}.$$

now since u is a continuous function, whereas V may not be, we can view $1/u$ as a continuous approximation of V . In some sense it is more powerful than just any continuous approximation as it also contains information about the kinetic term. Furthermore these regions do approximate the location of the eigenfunction well, as demonstrated in application by [FM, A+2016, L+2016].

To formulate the above observations into precise theorems, we will consider the weight

$$\omega_\lambda(x) = \max\left(\frac{1}{u(x)} - \lambda, 0\right).$$

with the associated Agmon metric given by

$$\rho(x, y) = \inf_{\gamma} \int_{\gamma} \frac{1}{\omega_\lambda(\gamma(t))^{1/2}} |\dot{\gamma}(t)| dt$$

where the infimum is taken over all absolutely continuous paths from x to y . Using the techniques of [11], presented by [10] we will develop prelocalisation estimates that show with respect to the Agmon metric, outside of the sets $E(\lambda + \delta)$ the eigenfunction (with eigenvalue λ) decays exponentially (Corollary 3.3). The existence of such a decomposition indicates that u cannot vary substantially. This cannot occur when V is smooth, or has small oscillation.

We are primarily interested in the scenario when $E(\lambda + \delta)$ decomposes into a finite collection of disjoint subsets, with a large separation between the subsets. In this case, we can show that eigenfunctions are concentrated in one of these subsets, in the sense that they are well approximated by local eigenfunctions of L . This is the main result and is presented in Theorem 4.1.

After this result is established, we shift our focus and explore the possibility that the Agmon metric is a ‘natural’ metric for our operator L . We investigate the possibilities of defining a (reflected) Brownian motion on (Ω, ρ) , inspired by [12]. We hope that in future, the heat kernel of $\exp(-\Delta + V)$ can be recovered from this Brownian by the Feynman-Kac formula.

We should mention that while we focus specifically on [11] and [10], there are more related papers by other authors [13] [and others]. While these results are different, for instance [13] only estimates a superposition of eigenfunctions over a band of frequencies, they are still related to the problem of localising eigenfunctions, and can provide a greater insight into the historical development of the ideas involved.

1.2 Preliminaries

We assume the reader is familiar with functional analysis and Hilbert space theory, for instance the Riesz-Fréchet representation theorem (see [[14], Theorem 5.5, p. 135] for a reminder), the Hilbert space L^2 , the Stone-Weierstrass theorem, and Riesz-Markov-Kakutani representation theorem. We refer the reader to [14], [15], [16] if they need a reminder. We also assume a stronger version of Rademacher's theorem, stated below

Theorem 1.1 *If u is an absolutely continuous function then u is differentiable almost everywhere. In particular, if u is Lipschitz, then u is differentiable a.e..*

This is proved in [[15], Theorem 3.11, p. 130].

We are interested in the following class of functions.

Definition 1.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The Sobolev space $W^{1,2}(\Omega)$ is given by*

$$\left\{ u \in L^2(\Omega) : \exists g_i \in L^2(\Omega) \text{ s.t. } \int_{\Omega} u \frac{\partial \varphi}{\partial x_i} = - \int_{\Omega} g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = \{1, \dots, n\} \right\}.$$

We remark that this space is a Hilbert space with inner product

$$\langle u, v \rangle_{W^{1,2}} := \langle u, v \rangle + \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. This result can be found in [[14], prop. 9.1, p.264] as well as [[17], Theorem 4.8, p. 41]. The associated norm is

$$\|u\|_{W^{1,2}}^2 := \|u\|_2^2 + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_2^2$$

where $\|\cdot\|_2$ denotes the L^2 norm.

Another fact we need is the following denseness properties of $W^{1,2}(\Omega)$.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a **bounded** open set with continuous boundary. Then*

$$\check{C}^\infty(\Omega) := \{u|_{\Omega} : u \in C_c^\infty(\mathbb{R}^n)\}$$

is dense in $W^{1,2}(\Omega)$. Furthermore $C^1(\overline{\Omega})$ is also dense in $W^{1,2}(\Omega)$.

See [[17], Theorem 7.7, p. 78] for a proof.

Since $W^{1,2}(\Omega)$ is a Hilbert space, it is of interest to study its dual space. We use $W^{-1,2}(\Omega)$ to denote the dual space of $W^{1,2}(\Omega)$. However we do not use the Riesz-Fréchet representation to identify $W^{1,2}(\Omega)$ with $W^{-1,2}(\Omega)$. Rather we identify $(L^2(\Omega), \langle \cdot, \cdot \rangle)$ to its dual, $L^2(\Omega)^*$. This gives the relationship

$$W^{1,2}(\Omega) \subset L^2(\Omega) \cong L^2(\Omega)^* \subset W^{-1,2}(\Omega).$$

where by $W^{1,2}(\Omega) \subset L^2(\Omega)$ we mean there is a bounded linear operator $T : W^{1,2}(\Omega) \rightarrow L^2(\Omega)$ such that the range of T is dense and T is injective. This fact is outlined below, for more details see [[14], Remark 3, p. 136].

Proposition 1.4 *Let Ω be a bounded open set. Then*

$$W^{1,2}(\Omega) \subset L^2(\Omega)$$

and

$$L^2(\Omega)^* \subset W^{-1,2}(\Omega)$$

PROOF: We remark that $W^{1,2}(\Omega)$ is a dense linear subspace of $L^2(\Omega)$. So we have that $W^{1,2}(\Omega) \subset L^2(\Omega)$.

To show $L^2(\Omega)^* \subset W^{-1,2}(\Omega)$ we use the restriction map $T : L^2(\Omega)^* \rightarrow W^{-1,2}(\Omega)$ which restricts continuous linear functionals on $L^2(\Omega)$ to $W^{1,2}(\Omega)$. Explicitly, for $g \in L^2(\Omega)^*$ and $\varphi \in W^{1,2}(\Omega)$ we define

$$(Tg)\varphi = \int_{\Omega} g\varphi$$

We observe that

$$\begin{aligned} |(Tg)\varphi| &\leq \int_{\Omega} |g\varphi| \\ &\leq \|g\|_2 \|\varphi\|_2 \\ &\leq \|g\|_2 \|\varphi\|_{W^{1,2}}. \end{aligned} \quad (\text{Cauchy Schwarz})$$

Hence $g \rightarrow Tg$ is a bounded linear operator. We remark that T is injective, and furthermore, because $W^{1,2}(\Omega)$ is reflexive, we note that the image of T is dense in $W^{-1,2}(\Omega)$. Hence $L^2(\Omega)^* \subset W^{-1,2}(\Omega)$.

□

We are also interested in the Sobolev space $W_0^{1,2}(\Omega)$ which we define below

Definition 1.5 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Consider*

$$W_c^{1,2}(\Omega) := \{f : f \text{ is compactly supported in } \Omega \text{ and } f \in W^{1,2}(\Omega)\}$$

we set

$$W_0^{1,2}(\Omega) := \overline{W_c^{1,2}(\Omega)}$$

where the closure is taken in the $W^{1,2}(\Omega)$ norm.

It can be shown that

Theorem 1.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, then*

$$W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}$$

where the closure is taken in the $W^{1,2}(\Omega)$ norm.

See [[17], Theorem 4.12, p. 43] for a proof. Also, we remark that $\overline{W_0^{1,2}(\Omega)} = L^2(\Omega)$ where the closure is taken in the L^2 norm.

Finally, we remark that the product rule generalises to $W^{1,2}(\Omega)$ in the following sense.

Proposition 1.7 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose $u, v \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i}v + u \frac{\partial v}{\partial x_i}$$

for $i = 1, \dots, n$.

This is proved in [[14], Proposition 9.4, p. 269].

1.2.1 Weak Formulation and Boundary Conditions

Before considering our Schrödinger operator, we begin by defining the Laplacian, Δ weakly.

Definition 1.8 *For $u, v \in W^{1,2}(\Omega)$, we define $\Delta : W^{1,2}(\Omega) \rightarrow W^{-1,2}(\Omega)$ by*

$$\langle -\Delta u, v \rangle := \int_{\Omega} \nabla u \cdot \nabla v.$$

If $\Delta u \in L^2(\Omega)$ and $f \in L^2(\Omega)$, we write $-\Delta u = f$ if for all $v \in W^{1,2}(\Omega)$ we have

$$\langle -\Delta u, v \rangle := \int_{\Omega} f v.$$

We should remark that if $\Delta u \in L^2(\Omega)$, we have

$$\int_{\Omega} -\Delta u, v = \int_{\Omega} \nabla u \cdot \nabla v$$

that is,

$$\int_{\Omega} \Delta u \, v + \nabla u \cdot \nabla v = 0.$$

This is of minor importance when we define the domain. With this weak notion in mind, we define the Neumann Laplacian Δ_N in L^2 as an unbounded operator.

Definition 1.9 *Let Ω be an open set of $\mathbb{R}^n$ and set*

$$\begin{aligned} D(\Delta_N) &:= \{u \in W^{1,2}(\Omega) : \Delta u \in L^2(\Omega)\} \\ &= \left\{ u \in W^{1,2}(\Omega) : \Delta u \in L^2(\Omega) \text{ and } \int_{\Omega} (\Delta u)v + \nabla u \cdot \nabla v = 0 \right\}. \end{aligned}$$

(by the above remark)

We define the Neumann Laplacian, $\Delta_N : D(\Delta_N) \rightarrow L^2$ by

$$\Delta_N u := \Delta u \quad (\text{when } u \in D(\Delta_N)).$$

We call this operator ‘Neumann’ because the (redundant) $\int_{\Omega} (\Delta u)v + \nabla u \cdot \nabla v = 0$ condition corresponds to Neumann boundary conditions in the classical sense. That is, for a C^1 boundary and $u, v \in C^2(\overline{\Omega})$, Green’s formula gives

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n} dS = \int_{\Omega} (\Delta u)v + \nabla u \cdot \nabla v$$

where $\frac{\partial u}{\partial n}$ is the normal derivative and dS is the surface measure. So if we impose Neumann boundary conditions, we impose

$$\frac{\partial u}{\partial n} = 0$$

giving

$$\int_{\Omega} (\Delta u)v + \nabla u \cdot \nabla v dV = 0.$$

We include the condition as a reminder of the above relationship. It also allow us to ‘integrate by parts’ for $u \in D(\Delta_N)$. That is we have

$$-\int_{\Omega} \operatorname{div}(\nabla u)v = \int_{\Omega} \nabla u \cdot \nabla v$$

for any $v \in W^{1,2}(\Omega)$.

We remark here that we can obtain the Dirichlet Laplacian in a similar fashion.

Definition 1.10 *Let Ω be an open set of $\mathbb{R}^n$ and set*

$$D(\Delta_D) := \{u \in W_0^{1,2}(\Omega) : \Delta u \in L^2(\Omega)\}.$$

We define the Dirichlet Laplacian, $\Delta_D : D(\Delta_D) \rightarrow L^2$ by

$$\Delta_D u := \Delta u \quad (\text{when } u \in D(\Delta_D)).$$

we include this example to highlight that boundary conditions are determined by the domain of our differential operator. In fact, for each closed subset $V \subseteq W^{1,2}(\Omega)$ such that $W_0^{1,2}(\Omega) \subseteq V$, we can restrict Δ_N to V , see [[17], remark 11.3, p. 123]. We use this method to obtain mixed boundary conditions, that is Dirichlet on some subset of the boundary Γ and Neumann on the remainder.

Definition 1.11 *Let Ω be an open set of $\mathbb{R}^n$ and let Γ be a closed subset of $\partial\Omega$. Consider*

$$V = \{u|_\Omega : u \in C_c^\infty(\mathbb{R}^d \setminus \Gamma)\}$$

we set

$$D(\Delta_M) := \overline{V}$$

where the closure is taken in the $W^{1,2}(\mathbb{R}^n)$ norm. When $u \in D(\Delta_M)$ we say

$$\Delta_M u := f$$

if for every $v \in D(\Delta_M)$ we have

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega -f v.$$

This domain encodes Dirichlet conditions on Γ and Neumann conditions elsewhere in the classical sense (via Green's formula) [[18], p. 101]. It should also be stated that when Γ is empty we have

$$V = \{u|_\Omega : u \in C_c^\infty(\mathbb{R}^d)\}$$

and when Ω is bounded with continuous boundary, we recall that

$$\overline{V}^{W^{1,2}(\Omega)} = W^{1,2}(\Omega)$$

from Theorem 1.3 above. This recovers the Neumann case discussed earlier.

Now that we've considered a suitable range of boundary conditions, we move onto defining our Schrödinger operator. We are primarily interested in the Neumann case, given below.

Definition 1.12 Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For $V \in L^\infty(\Omega)$ we define $L : D(\Delta_N) \rightarrow L^2$ by

$$Lu := -\Delta_N u + Vu.$$

If $u \in D(\Delta_N)$ and $f \in L^2(\Omega)$, we write $Lu = f$ if

$$\int_{\Omega} \nabla u \nabla v + Vuv = \int_{\Omega} f v$$

for all $v \in W^{1,2}(\Omega)$.

For simplicity in future sections we refer to testing $Lu = f$ against a test function g . By this we mean the following

Definition 1.13 If $u \in D(\Delta_N)$ and $f \in L^2(\Omega)$ satisfy $Lu = f$, we say testing $Lu = f$ against φ to mean

$$\begin{aligned} &\varphi \in W^{1,2}(\Omega), \text{ and} \\ &\int_{\Omega} \nabla u \nabla \varphi + V u \varphi = \int_{\Omega} f \varphi. \end{aligned}$$

We should remark here that when $V \geq 0$, L is positive, and hence self adjoint.

Finally, by replacing $-\Delta_N$ with $-\Delta_M$ and using $D(\Delta_M)$ as the domain we have the operator associated to mixed boundary conditions (and similarly with $-\Delta_D$). When we refer to L with mixed/Dirichlet boundary conditions we are referring to L with the respective domain. This helps formalise the notion of a local solution to L in later sections.

1.2.2 Existence and Uniqueness of Solutions

We now show that for $f \in L^2$, there exists a unique $u \in W^{1,2}(\Omega)$ such that $Lu = f$.

We recall that $Lu = f$ means

$$\int_{\Omega} \nabla u \nabla v + Vuv = \int_{\Omega} f v \quad (\star)$$

for every $v \in W^{1,2}(\Omega)$. For convenience, in this section we define the bilinear form $B : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{R}$ by

$$B(\varphi, \eta) := \int_{\Omega} \nabla \varphi \nabla \eta + V \varphi \eta.$$

We remark that $B : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{R}$ is a continuous bilinear form. Furthermore, if $V \geq c$ a.e. for some constant $c > 0$ we have that B is coercive in the following sense.

Proposition 1.14 *The form B defined above satisfies*

$$B(\varphi, \varphi) \geq c \|\varphi\|_{W^{1,2}}^2$$

for all $\varphi \in W^{1,2}(\Omega)$.

PROOF: We have

$$\begin{aligned} B(\varphi, \varphi) &= \int_{\Omega} |\nabla \varphi|^2 + V \varphi^2 \\ &= \int_{\Omega} \sum_{i=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 + |V^{1/2} \varphi|^2 \\ &= \sum_{i=1}^n \left\| \frac{\partial \varphi}{\partial x_i} \right\|_2^2 + \|V^{1/2} \varphi\|_2^2 \end{aligned}$$

and since $V \geq c > 0$ almost everywhere, pick $c' = \min\{1, c\} > 0$. We have

$$\sum_{i=1}^n \left\| \frac{\partial \varphi}{\partial x_i} \right\|_2^2 + \|V^{1/2} \varphi\|_2^2 \geq c' \left(\sum_{i=1}^n \left\| \frac{\partial \varphi}{\partial x_i} \right\|_2^2 + \|\varphi\|_2^2 \right).$$

combining the inequalities we obtain the desired result.

□

We now quote the Lax-Milgram theorem.

Theorem 1.15 ([14], Corollary 5.8 , p.140) *Assume that the map $[u, v] \rightarrow a(u, v)$ is a continuous coercive bilinear form on H . Then given any $\varphi \in H^*$ we have that there exists a unique element $u \in H$ such that*

$$a(u, v) = \varphi(v)$$

for all $v \in H$.

We now use this to show that for every $f \in L^2(\Omega)$ we have a solution $Lu = f$, that is $(\star)$ from above holds.

Corollary 1.16 *Suppose that $V \geq c > 0$ almost everywhere. For every $f \in L^2(\Omega)$, there exists $u \in W^{1,2}(\Omega)$ such that*

$$B(u, v) = \int_{\Omega} f v$$

for every $v \in W^{1,2}(\Omega)$.

PROOF: Identify L^2 with its dual by the Riesz representation theorem, that is, given any $f \in L^2(\Omega)$ consider the unique $g \in L^2(\Omega)^*$ such that

$$g(v) = \int_{\Omega} f v.$$

Now recall that $g \in L^2(\Omega)^* \subset W^{-1,2}(\Omega)$ (from Proposition 1.4), so by Lax-Milgram, we deduce that there exists a unique $u \in W^{1,2}(\Omega)$ such that

$$B(u, v) = \int_{\Omega} f v$$

for all $v \in W^{1,2}(\Omega)$. This means that for every $f \in L^2$, there exists a (unique) $u \in W^{1,2}(\Omega)$ satisfying $(\star)$. Hence we conclude that for every $f \in L^2(\Omega)$ there exists a unique $u \in W^{1,2}(\Omega)$ such that $Lu = f$.

□

1.2.3 Regularity of Solutions

We are primarily interested in the equations $Lu = 1$ and $(L - \mu)\varphi = 0$ (where μ is some constant) on Ω . We remark that we may apply De Giorgi, Nash, Moser theory as presented by [19] to show solutions $v \in W_0^{1,2}(\Omega)$ to $Lv = f$ for $f \in L^\infty(\Omega)$ are continuous. In this section we describe and explain how to apply it to our case, as well as provide an argument for $v \in W^{1,2}(\Omega)$ under additional assumptions on Ω . We remark that [19] treat operators of the following (more complicated) form.

Let $A = [a^{i,j}]$ be a matrix with each $a^{i,j}$ a measurable function on Ω . Let $B = [b^i]$ and $C = [c^i]$ where b^i, c^i are also measurable functions on Ω . Finally, let d be a measurable function on Ω . Also suppose that A is strictly elliptic, that is for some $\lambda > 0$ the coefficients $a^{i,j}$ satisfy

$$a^{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where ξ_j denotes the j^{th} component of ξ . Furthermore suppose that A is bounded in the sense that

$$\sum |a^{i,j}|^2 \leq \Lambda^2.$$

We also impose that B, C, d are bounded in the sense that for some $\phi^2 \geq 0$ and every $x \in \Omega$ we have

$$\lambda^{-2} \sum_{i=1}^n (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1} |d(x)| \leq \phi^2.$$

With these assumptions in mind, consider the form $\mathcal{L} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \rightarrow \mathbb{R}$ given by

$$\mathcal{L}(u, v) = \int_{\Omega} (A \nabla u + Bu) \nabla v - (C \nabla u + du)$$

and let $\tilde{L}$ denote the associated linear operator. For locally integrable functions f^i, g we say $\tilde{L}u = g + D_i f^i$ if

$$\mathcal{L}(u, v) = \int_{\Omega} f^i D_i v - gv$$

for all $v \in W_0^{1,2}(\Omega)$. In fact, [19] treat a ‘generalised Dirichlet’ case by testing against $v \in C_0^1(\Omega)$ instead, see [[19], p. 178]. However, we only state for the Dirichlet case (that is $v \in W_0^{1,2}(\Omega)$) to avoid difficulties, but to highlight the underlying assumptions.

Our bilinear form is much simpler though, with $A = Id$, $d = V \in L^\infty(\Omega)$ and $B = C = 0$. We comment that A is strictly elliptic and has integrable coefficients, and V , as well as $V - \lambda$ are bounded. We also remark that our notion of solution is simpler too. We are only be interested in solutions to $Lu = 1$ and $(L - \mu)\varphi = 0$, so we may restrict to the case where $f^i = 0$ for all i and $g \in L^\infty$. Since we are considering these simplified cases, we meet the assumptions of $f^i \in L^q(\Omega)$, and $g \in L^{q/2}(\Omega)$ for $q > n$. Rather than copy the theorems in full from [19], we state them restricted to the above assumptions (and the case that we need) to simplify the statements.

For now, we also restrict to the Dirichlet case, that is for $u \in W_0^{1,2}(\Omega)$. We can recover the $W^{1,2}(\Omega)$ case by using an even reflection, at the cost of further assumptions on our domain. The argument is given at the end of this section for the special case where Ω is a half ball in Proposition 1.25.

Interior regularity can be handled by the following theorem, which is a version of De Giorgi, Nash, Moser theorem. The (simplified) statement is

Theorem 1.17 ([19], **Theorem 8.22, p.200**) *Suppose $v \in W_0^{1,2}(\Omega)$ solves $Lv = f$ for $f \in L^\infty(\Omega)$ then on any ball $B \subset \Omega$, v is Hölder continuous on B .*

We now want an extension of Hölder regularity to the boundary. The next two theorems give bounds on subsolution and supersolutions, which can be used to bound the Hölder norm.

Theorem 1.18 ([19], **Theorem 8.25, p.202**) *Suppose $v \in W_0^{1,2}(\Omega)$ satisfies $Lv \geq f$ for $f \in L^\infty(\Omega)$, then for any $y \in \mathbb{R}^n, R > 0$ and $p > 1$ we have that*

$$\sup_{\partial\Omega \cap B(y, 2R)} v^+ = M$$

for some $M < \infty$. Furthermore, let

$$v_M^+(x) = \begin{cases} \sup\{v(x), M\}, & x \in \Omega \\ M, & x \notin \Omega \end{cases}$$

then for some finite C depending only on $\{\|f\|_\infty, n, \lambda, \Lambda, \phi, R\}$ we have

$$\sup_{B(y, R)} v_M^+ < C \|v_M^+\|_{L^p(B(y, 2R))}.$$

Theorem 1.19 ([19], **Theorem 8.26, p.203**) *Suppose $v \in W_0^{1,2}(\Omega)$ satisfies $Lv \leq f$ for $f \in L^\infty(\Omega)$. If there exists $y \in \mathbb{R}^n$ and $R > 0$ such that v is non-negative in $\Omega \cap B(y, 4R)$ then we have that*

$$\inf_{\partial\Omega \cap B(y, 4R)} v = m$$

for some $-\infty < m$. Furthermore, let

$$v_m^-(x) = \begin{cases} \inf\{v(x), m\}, & x \in \Omega \\ m, & x \notin \Omega. \end{cases}$$

then for some constant C depending on we have

$$\|v_m^-\|_{L^p(B(y, 2R))} \leq C \inf_{B(y, R)} v_m^-.$$

These two theorems require us to restrict to $f \in L^\infty(\Omega) \subset L^2(\Omega)$, but this is a large enough class of functions, since we'll be primarily interested in the solution to $Lu = 1$ and $(L - \mu)\varphi = 0$.

We should note that the above theorems do not give us Hölder continuity. However, with additional assumptions on the boundary, it can be shown that u has bounded oscillation (and hence Hölder regularity) on the intersection of any ball of radius R and Ω , with bound depending on R^α for some $\alpha > 0$ and $\sup |v|$. From this we deduce that v is α Hölder continuous.

The additional assumption on the boundary we want is the *exterior cone condition* stated below.

Definition 1.20 *A bounded open set Ω satisfies the exterior cone condition at a point $x_0 \in \partial\Omega$ if there exists a (finite) right circular cone $V = V_{x_0}$ with vertex x_0 such that $\bar{\Omega} \cap V = x_0$.*

By *finite right circular cone* we mean a cone V with vertex $x = (x_1, \dots, x_n)$ and circular base centre $y = (y_1, \dots, y_n)$, radius $r < \infty$ such that $x_i = y_i$ for some $1 \leq i \leq n$.

With this condition in mind, the (simplified) statement is

Theorem 1.21 ([19], **Theorem 8.27, p.203**) *Suppose $v \in W_0^{1,2}(\Omega)$ is a solution of $Lv = f$ for $f \in L^\infty$ in Ω and Ω satisfies the exterior cone condition at a point $x_0 \in \partial\Omega$, then v is α Hölder continuous for some $\alpha > 0$ if*

$$\sup_{\Omega \cap B(x_0, R_0)} |v| < \infty$$

and

$$\text{osc}_{\partial\Omega \cap B(x_0, R)} v < \infty$$

where $\text{osc}_{\partial\Omega \cap B(x_0, R)} v = \sup_{\partial\Omega \cap B(x_0, R)} v - \inf_{\partial\Omega \cap B(x_0, R)} v$.

We should remark here that the precise statement of this theorem involves bounding $\text{osc}_{\Omega \cap B(x, R)} v$ for any $x \in \Omega$ by theorems 1.18 and 1.19, with a constant involving R^α . This α value depends on V_{x_0} . However, we only need continuity in a qualitative sense. So we do not care about the constants involved or what $\alpha > 0$ is.

Thus, we can make the following conclusion

Corollary 1.22 *The solution $v \in W_0^{1,2}(\Omega)$ to $Lv = f$ for $f \in L^\infty(\Omega)$ are $C^{0,\alpha}(\Omega)$ for some $\alpha > 0$.*

Now to recover the above for $u \in W^{1,2}(\Omega)$, we require Ω to satisfy the *bi-Lipschitz assumption* defined below.

Definition 1.23 *A set $\Omega \subset \mathbb{R}^n$ satisfies the bi-Lipschitz assumption if for every $x \in \partial\Omega$ there exists a neighbourhood $U = U_x$ of x and a bi-Lipschitz map $F = F_x$, $F : U \rightarrow \mathbb{R}^n$ such that*

$$\begin{aligned} F(U \cap \Omega) &= H \\ F^{-1}(H) &= U \cap \Omega \end{aligned}$$

where $H \subset \{x = (x_1, \dots, x_n) : x_1 + \dots + x_n > 0\}$ is an open set.

The (simple) examples we have in mind include the half cube $(0, 1)^n$ or the half ball $B(0, 1) \cap \mathbb{R}_+^n$. More general examples include convex sets. This assumption allows us to use a reflection argument. For now we assume that Ω is a half ball, and remark that the general case can be obtained by a change of variables.

Rather than just give the reflection argument for Ω we provide it for $\bar{\Omega} \setminus K$ with suitable K . We can obtain the case for Ω by taking $K = \emptyset$. We provide this argument as we would also like to consider local solutions. We require K to satisfy the following bi-Lipschitz cone condition.

Definition 1.24 *A set $K \subset \mathbb{R}^n$, $n \neq 1$ satisfies the bi-Lipschitz cone condition if there exists $r > 0$ and $\varepsilon > 0$ such that for every point $x_0 \in \partial K$ there is a map $F : B_r(x_0) \rightarrow \mathbb{R}^n$ with $F(x_0) = 0$ satisfying the following bi-Lipschitz bounds*

$$\varepsilon|x - y| \leq |F(x) - F(y)| \leq \frac{1}{\varepsilon}|x - y|$$

and such that

$$F(K) \supset \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < \varepsilon x_1 < \varepsilon^2\}.$$

This condition is needed to ensure that connected components of $\bar{\Omega} \setminus K$ satisfy the bi-Lipschitz assumption 1.23. This allows us to guarantee Hölder continuity of local solutions.

To summarise, the above results accumulate to the following.

Proposition 1.25 ([10], Proposition 5.1, p.26) *Suppose that $\Omega \subset \mathbb{R}^n$ satisfies the bi-Lipschitz assumption above. Suppose that $K \subset \bar{\Omega}$ is compact and satisfies the bi-Lipschitz cone condition. Then there exists $\alpha > 0$ such that if $f \in L^\infty(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$ with $\varphi = 0$ on K solves $(L - \mu)\varphi = f$ in the weak sense on $\bar{\Omega} \setminus K$, then $\varphi \in C^{0,\alpha}(\Omega)$.*

PROOF: For $V \in L^\infty(\Omega)$ and any constant μ , $|V - \mu|$ is bounded, so we may replace V with $V - \mu$. To recover the case for V set $\mu = 0$. Interior regularity follows by De Giorgi, Nash, Moser theory presented above. For boundary regularity with Neumann boundary conditions we consider the special case where Ω is a half ball. We can perform a change of variables to recover the general case. Let

$$\begin{aligned} B_1 &= B(0, 1) \\ \Omega &= B_1 \cap \{x \in \mathbb{R}^n : x_1 \geq 0\} \end{aligned}$$

and suppose that K is a compact subset of $\bar{\Omega}$ satisfying the bi-Lipschitz cone condition. We let $W^{1,2}(\Omega \setminus K)$ denote the closure in the $W^{1,2}(\mathbb{R}^n)$ norm of functions in $C^1(\bar{\Omega})$ with support disjoint from K . If $\varphi \in W^{1,2}(\Omega \setminus K)$ then we can extend φ by 0 on K and so $\varphi \in W^{1,2}(\Omega)$. For $f \in L^\infty(\Omega)$ we say that φ solves $L\varphi = f$ weakly on $\Omega \setminus K$ if

$$\int_{\Omega} \nabla \varphi \nabla \eta + V \varphi \eta dx = \int_{\Omega} f \eta dx$$

for all $\eta \in C^1(\Omega)$ with support disjoint from K . We remark that η may not vanish on $x_1 = 0$ as we have Neumann boundary conditions. Let R be the reflection given by

$$R(x_1, \dots, x_n) = R(-x_1, \dots, x_n).$$

We extend V, φ, f to B_1 by the following

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \Omega \\ f(R(x)) & x \in R\Omega \end{cases}$$

and similarly for V, φ . We define $\tilde{K} = K \cup RK$. We note that $\tilde{\varphi} = 0$ on $\tilde{K}$. We extend L to B_1 by replacing V with $\tilde{V}$ etc. We claim now that $\tilde{L}\tilde{\varphi} = \tilde{f}$ weakly on $B_1 \setminus \tilde{K}$. To show this, let $\eta \in C^1(\bar{B}_1)$ with support disjoint from $\tilde{K} \cup \partial B_1$. We define

$$\eta_*(x) = \frac{1}{2}(\eta(x) + \eta(R(x))).$$

Notice that this makes η symmetrical, and by definition of $\tilde{\varphi}$ and $\tilde{f}$ we have $\tilde{\varphi}_* = \tilde{\varphi}$ and $\tilde{f}_* = \tilde{f}$. Using these properties we note that

$$\int_{B_1} \nabla \tilde{\varphi} \nabla \eta dx = \int_{B_1} \nabla \tilde{\varphi}_* \nabla \eta dx$$

if we expand out the definition for $*$ we see that

$$\begin{aligned} \int_{B_1} \nabla \tilde{\varphi}_* \nabla \eta dx &= \int_{B_1} \nabla \tilde{\varphi} \nabla \eta_* dx \\ &= 2 \int_{\Omega} \nabla \tilde{\varphi} \nabla \eta_* dx \end{aligned}$$

Now we note that $\eta_*(x) = 0$ on K . So because $L\varphi = f$ weakly on Ω we obtain

$$\begin{aligned} 2 \int_{\Omega} \nabla \varphi \nabla \eta_* dx &= 2 \int_{\Omega} (f - V\varphi) \eta_* dx \\ &= \int_{B_1} (\tilde{f} - \tilde{V}\tilde{\varphi}) \eta_* dx \\ &= \int_{B_1} (\tilde{f}_* - (\tilde{V}\tilde{\varphi})_*) \eta dx \\ &= \int_{B_1} (\tilde{f} - \tilde{V}\tilde{\varphi}) \eta dx. \end{aligned}$$

If we combine the above calculations we see that

$$\int_{B_1} \nabla \tilde{\varphi} \nabla \eta dx = \int_{B_1} (\tilde{f} - \tilde{V}\tilde{\varphi}) \eta dx$$

hence $\tilde{L}\tilde{\varphi} = \tilde{f}$ weakly on $B_1 \setminus K$.

We can now invoke the regularity theory outlined above. To actually apply Theorem 1.21, we need to find $x_0 \in \partial(\overline{\Omega} \setminus K)$ and a cone V_{x_0} to verify the exterior cone condition. We omit this step, but in the case where $K = \emptyset$ and $\Omega = B(0, 1)$ any cone exterior to Ω with vertex on $\partial\Omega$ suffices. As remarked after Theorem 1.21, the α constant depends on V_{x_0} . However, we only require Hölder continuity in a qualitative sense, so the estimates that we prove in later chapters do not depend on these constants. □

Regularity for Smooth Boundary and in Low Dimensions

For the low dimensional cases ($n \leq 4$) under the additional assumption the boundary is C^2 regular, we have a direct argument via a Sobolev embedding. We remark that if $u \in W^{1,2}(\Omega)$ solves $Lu = f$ for $f \in L^2(\Omega)$, then $u \in W^{2,2}(\Omega)$ by the following theorem.

Theorem 1.26 ([14], Theorem 9.26, p. 299) *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^2 boundary. Suppose that $u \in W^{1,2}(\Omega)$ solves $Lu = f$ in the weak sense for $f \in L^2(\Omega)$. Then $u \in W^{2,2}(\Omega)$.*

We now appeal to Morrey's inequality, stated below.

Theorem 1.27 ([14], Corollary 9.15, p. 285) *Let $m \geq 1$ be an integer and let $p \in [1, \infty)$. For $\alpha \leq [m - (n/p)]$ we have*

$$W^{m,p}(\Omega) \subset C^{0,\alpha}(\Omega).$$

This gives us $W^{2,2}(\Omega) \subset C^{0,\alpha}(\Omega)$ when $n \leq 4$, for $\alpha = \frac{1}{2}$. Hence we can also obtain $u \in C^{0,\alpha}$ for some $\alpha > 0$ by this argument.

1.2.4 Maximum Principle

This section gives a version of the maximum principle for solutions v to $Lv = f$ in the weak sense. We would like to deduce that $v \geq 0$ provided that $f \geq 0$. This allows us to deduce that the Landscape function u solving $Lu = 1$ is strictly positive, a result we need as we would like to consider $1/u$.

This next proposition is a variation of [[14], Theorem 9.30, p. 310].

Proposition 1.28 *Suppose that $f \in L^2(\Omega)$, $V \in L^\infty(\Omega)$ with $V \geq 0$ and $V > 0$ on a set of positive measure. Also suppose that $v \in H^1(\Omega)$ solves $Lv = f$ weakly, that is,*

$$\int_{\Omega} \nabla v \nabla \varphi + Vv\varphi = \int_{\Omega} f\varphi$$

for all $\varphi \in W^{1,2}(\Omega)$. Then we have

$$\inf_{\Omega} f \leq Vv(w) \leq \sup_{\Omega} f.$$

for a.e. $w \in \Omega$. Furthermore, if $f \in L^\infty$ then we have

$$\inf_{\Omega} f \leq Vv(w) \leq \sup_{\Omega} f$$

for all $w \in \Omega$.

PROOF: Suppose that f, u satisfy the above. Fix $G \in C^1(\mathbb{R})$ such that

$$|G'(s)| \leq M \quad G \text{ is strictly increasing on } (0, \infty),$$

$$G(s) = 0 \text{ for all } (s \leq 0).$$

Set $K = \sup_{\Omega} f$ and suppose that $K < \infty$. If $K = \infty$ then there is nothing to show. Set $\varphi = G(Vv - K)$, by [[14], prop. 9.5, p. 270] we note that $\varphi \in W^{1,2}(\Omega)$ and

$$\nabla (G(Vv - K)) = V(\nabla v)G'(Vv - K).$$

Recall that $Lv = f$ weakly, testing against $G(Vv - K)$ we obtain

$$\int_{\Omega} V|\nabla v|^2 G'(Vv - K) + VvG(Vv - K) = \int_{\Omega} fG(Vv - K).$$

Subtracting $KG(Vv - K)$ from both sides we see that

$$\int_{\Omega} |\nabla v|^2 G'(Vv - K) + (Vv - K)G(Vv - K) = \int_{\Omega} (f - K)G(Vv - K).$$

Now recall that $G(Vv - K)$ is nonnegative, and $K \geq f$ so

$$\int_{\Omega} (f - K)G(Vv - K) \leq 0.$$

Also note that G is increasing so $G'(s) \geq 0$ for all $s \in \mathbb{R}$. Furthermore, $V|\nabla v|^2 \geq 0$ so we have

$$\int_{\Omega} V|\nabla v|^2 G'(Vv - K) \geq 0.$$

This implies

$$\int_{\Omega} (Vv - K)G(Vv - K) \leq 0$$

and since $sG(s) \geq 0$ for all $s \in \mathbb{R}$, we have that

$$(Vv - K)G(Vv - K) = 0 \text{ a.e..}$$

Now, because $G(Vv - K) > 0$ when $Vv - K > 0$ we deduce that

$$Vv \leq K \text{ a.e..}$$

Finally, we note that

$$\begin{aligned} -Vv &\geq -K \\ -Vv &\geq \sup_{\Omega}(-f) \\ \therefore Vv &\geq \inf_{\Omega} f \end{aligned}$$

giving

$$\inf_{\Omega} f \leq Vv \leq \sup_{\Omega} f.$$

a.e. in Ω . Now if $f \in L^{\infty}(\Omega)$ then $v \in C^{0,\alpha}(\Omega)$ (by the regularity theory), so we deduce that

$$\inf_{\Omega} f \leq Vv \leq \sup_{\Omega} f.$$

in Ω .

□

An easy corollary is the following.

Corollary 1.29 *Let $V \in L^\infty(\Omega)$ such that $V \leq \bar{V}$ for some constant $\bar{V}$. Also suppose $V \geq c > 0$ for some constant c almost everywhere. Then for u solving $Lu = 1$ weakly we have $u \geq \frac{1}{\bar{V}}$. Furthermore $u \in C^{0,\alpha}(\Omega)$.*

PROOF: Suppose u solves $Lu = 1$ in the weak sense. Then, note that $1 \in L^\infty(\Omega)$ and $\inf_\Omega 1 = 1$. By the above we deduce

$$\begin{aligned} 1 &\leq Vu \\ &\leq \bar{V}u \end{aligned}$$

dividing through by $\bar{V}$ we get

$$\frac{1}{\bar{V}} \leq u.$$

Finally, $u \in C^{0,\alpha}(\Omega)$ by the regularity theory.

□

From here on, we assume that $V \in L^\infty(\Omega)$, $V \geq c$ almost everywhere for some constant $c > 0$. We can now safely define the (strictly positive) landscape function u to be the unique, continuous solution to the equation $Lu = 1$.

1.2.5 Spectral Decomposition.

This section shows that there exists an orthonormal basis of eigenfunctions of L . We begin with the following theorem.

Theorem 1.30 ([17], Theorem 7.11, p. 81) *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, open and has continuous boundary. Then the embedding*

$$j : W^{1,2}(\Omega) \rightarrow L^2(\Omega)$$

is compact.

With this embedding in mind, we have the following theorem.

Theorem 1.31 ([17], **Theorem 6.15, p. 67**) *Let $B : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{R}$ be a bounded coercive form. Let the operator $A : L^2(\Omega) \rightarrow L^2(\Omega)$ be given by the relation*

$$A = \{(x, y) \in L^2 \times L^2 : \exists u \in W^{1,2}(\Omega) \text{ s.t. } j(u) = x \text{ and } B(u, v) = \langle y, j(v) \rangle \text{ for } v \in W^{1,2}\}$$

has compact resolvent.

Recall that L was given by $\langle Lu, v \rangle := B(u, v)$ and B was a bounded coercive form. We conclude by the above statement that L has compact resolvent. This is important for the next theorem. Recall that L is self adjoint, and positive (since the potential V was nonnegative).

Theorem 1.32 ([17], **Theorem 6.17, p. 68**) *Let $A : L^2 \rightarrow L^2$ be a positive self adjoint operator with compact resolvent. Then there exists an orthonormal basis $(\psi_j)_{j \in \mathbb{N}}$ of $L^2(\Omega)$ and a sequence $(\lambda_j)_{j \in \mathbb{N}}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} \lambda_n = \infty$ such that*

$$A\psi_j = \lambda_j\psi_j \quad \forall j \in \mathbb{N}.$$

By the above theorems, we conclude that the eigenfunctions $(\psi_j)_{j \in \mathbb{N}}$ of L form an orthonormal basis of $L^2(\Omega)$, with $\psi_j \in D(L)$, as long as the boundary of Ω is continuous.

Local Eigenfunctions

In the next sections, we want to compare the above eigenfunctions to localised eigenfunctions, that is we wish to replace Ω with a suitable subset, and use mixed boundary conditions. We wish to remind the reader of 1.11, and remark that we are using mixed boundary conditions on a subset of Ω , so we modify the statements below correspondingly. Let $K \subset \bar{\Omega}$ be a compact set, and let $U \subset \bar{\Omega} \setminus K$ be a connected component. We remark here that U may not be open in $\mathbb{R}^d$ but is open in $\bar{\Omega}$. Setting $\Gamma = K \cap \partial U$ and

$$V = \{u|_U : u \in C_c^\infty(\bar{\Omega} \setminus \Gamma)\}$$

we restrict L to $\bar{V}^{W^{1,2}(\Omega)}$. As remarked earlier, this gives Dirichlet conditions on Γ and Neumann conditions on $\partial U \setminus \Gamma = \partial U \cap \partial\Omega$.

To simplify the notation, we use $W_0^{1,2}(U)$ to denote the closure of $C^1(\overline{\Omega})$ functions which are supported only in U in the $W^{1,2}(\Omega)$ norm. We remark that

$$\begin{aligned}\overline{V}^{W^{1,2}(\Omega)} &= \overline{\{u : \text{support}(u) \subset U \text{ and } u \in C_c^\infty(\overline{\Omega})\}}^{W^{1,2}(\Omega)} \\ &= W_0^{1,2}(\Omega).\end{aligned}$$

where the last line follows because $C^1(\overline{\Omega})$ is also dense in $W^{1,2}(\Omega)$ (by theorem 1.3). For convenience we take $W_0^{1,2}(\Omega)$ to be the domain for L , that is $L\varphi = f$ weakly on U if it satisfies

$$\int_{\Omega} \nabla \varphi \nabla \eta + V \eta dx = \int_{\Omega} f \eta dx$$

for all $\eta \in W_0^{1,2}(U)$. We remark that when $\partial\Omega \subset K$ then we have $\Gamma = \partial U$

$$\begin{aligned}V &= \{u|_U : u \in C_c^\infty(\overline{\Omega} \setminus \partial U)\} \\ &= \{u : u \in C_c^\infty(U)\}\end{aligned}$$

recovering the Dirichlet conditions.

Recall from the regularity theory that to ensure Hölder regularity of localised eigenfunctions we require K to satisfy the bi-Lipschitz cone condition. We remind the reader that the bi-Lipschitz cone condition is

Definition 1.33 *A set $K \subset \mathbb{R}^n$, $n \neq 1$ satisfies the bi-Lipschitz cone condition if there exists $r > 0$ and $\varepsilon > 0$ such that for every point $x_0 \in \partial K$ there is a map $F : B_r(x_0) \rightarrow \mathbb{R}^n$ with $F(x_0) = 0$ satisfying the following bi-Lipschitz bounds*

$$\varepsilon|x - y| \leq |F(x) - F(y)| \leq \frac{1}{\varepsilon}|x - y|$$

and such that

$$F[K] \supset \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < \varepsilon x_1 < \varepsilon^2\}.$$

While this restriction on K may appear restrictive, we can still pick K with a certain degree of freedom. By this, we mean that for any open set $U \subset \overline{\Omega}$ and compact set $K \subset U$ we can find a compact set K' which satisfies the bi-Lipschitz cone condition and $K \subset K' \subset U$. We use this result when we approximate eigenfunctions locally.

Lemma 1.34 ([10], Lemma 5.2, p.27) *Suppose that K is a compact subset of $\overline{\Omega}$ and let U be an open set in $\overline{\Omega}$ such that $K \subset U$. Then there is a compact set K' satisfying the bi-Lipschitz cone condition and $K \subset K' \subset U$.*

PROOF: Let $K \subset \overline{\Omega}$ be given. Since $\overline{\Omega}$ is compact we can cover $\overline{\Omega}$ with a finite number of closed cubes. For any open set U containing K set

$$d = \inf_{x \in K} d(x, \partial U).$$

For $\varepsilon = d/4 > 0$, subdivide each closed cube of the covering dyadically so that we have a finite covering by cubes of diameter smaller than ε . Call this cover C_ε . Note that the original covering of $\overline{\Omega}$ may have overlapping cubes, so these dyadic subcubes may overlap. However, C_ε is always a covering by finitely many cubes. We set

$$K' := K \cap C_\varepsilon.$$

We now note that each cube in C_ε satisfies the bi-Lipschitz cone condition, and since K' is a finite union of such cubes K' must also satisfy the bi-Lipschitz cone condition. Finally, $K' \subset U$ by our choice of ε .

□

Chapter 2

Important Identities and Key Estimates

2.1 Important Identity

Before embarking on outlining the arguments of [10], we first prove a key identity. This identity is used for the fundamental Lemma 2.3.

Lemma 2.1 *For $v, \varphi \in C^1(\Omega)$ we have*

$$\nabla v \cdot \nabla(\varphi^2 v) = |\nabla(\varphi v)|^2 - |v|^2 |\nabla \varphi|^2 \quad (2.1.1)$$

In particular, if $v = u$ and $\varphi = f/u$ satisfy the requirements then we have

$$\nabla u \cdot \nabla(f^2/u) = |\nabla f|^2 - |u|^2 |\nabla(f/u)|^2 \quad (2.1.2)$$

PROOF: Suppose v, φ are continuously differentiable. We may apply the product rule to $\nabla(\varphi^2 v)$. On the left hand side of (2.1.1) we have

$$\begin{aligned} \nabla v \cdot \nabla(\varphi^2 v) &= \nabla v \cdot (\varphi v \nabla \varphi + \varphi \nabla(\varphi v)) \\ &= \nabla v \cdot (\varphi v \nabla \varphi + \varphi(v \nabla \varphi + \varphi \nabla v)) \\ &= 2\varphi v \nabla v \cdot \nabla \varphi + \varphi^2 \nabla v \cdot \nabla v. \end{aligned} \quad (\star)$$

Now we notice that

$$\begin{aligned} |\nabla(\varphi v)|^2 &= \nabla(\varphi v) \cdot \nabla(\varphi v) \\ &= (\varphi \nabla v + v \nabla \varphi) \cdot (\phi \nabla v + v \nabla \varphi) \\ &= \varphi^2 \nabla v \cdot \nabla v + 2\varphi v \nabla v \cdot \nabla \varphi + v^2 \nabla \varphi \cdot \nabla \varphi \end{aligned}$$

a rearrangement gives

$$2\varphi v \nabla v \cdot \nabla \varphi = |\nabla(\varphi v)|^2 - \varphi^2 \nabla v \cdot \nabla v - v^2 \nabla \varphi \cdot \nabla \varphi$$

so if we substitute this into $(\star)$ we get

$$\begin{aligned} \nabla v \cdot \nabla(\varphi^2 v) &= |\nabla(\varphi v)|^2 - \varphi^2 \nabla v \cdot \nabla v - v^2 \nabla \varphi \cdot \nabla \varphi + \varphi^2 \nabla v \cdot \nabla v \\ &= |\nabla(\varphi v)|^2 - v^2 \nabla \varphi \cdot \nabla \varphi \end{aligned}$$

which is (2.1.1). To obtain (2.1.2) simply substitute $v = u$ and $\varphi = f/u$ to obtain

$$\nabla u \cdot \nabla((f/u)^2 u) = |\nabla((f/u)u)|^2 - |u|^2 |\nabla(f/u)|^2$$

which is (2.1.2) after cancellation. □

Corollary 2.2 *For $v, \varphi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ we have*

$$\int_{\Omega} \nabla v \cdot \nabla(\varphi^2 v) = \int_{\Omega} |\nabla(\varphi v)|^2 - |v|^2 |\nabla \varphi|^2 \quad (2.2.1)$$

In particular, if $v = u$ and $\varphi = f/u$ satisfy the requirements then we have

$$\int_{\Omega} \nabla u \cdot \nabla(f^2/u) = \int_{\Omega} |\nabla f|^2 - |u|^2 |\nabla(f/u)|^2 \quad (2.2.2)$$

We recall that the product rule can be generalised for functions $v, \psi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ by proposition 1.7. Thus we can generalise both (2.1.1) and (2.1.2) to $v, \psi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

2.2 Key Equalities

This section outlines the main technical tools needed for pre localisation, that is exponential decay outside of a localised region.

Fundamental Lemma 2.3 ([10], Lemma 3.1, p. 11) *Suppose that f and u belong to $W^{1,2}(\Omega)$ and V, f and $\frac{1}{u}$ belong to $L^\infty(\Omega)$. Suppose also that $Lu = 1$ weakly on Ω . Then*

$$\int_{\Omega} |\nabla f|^2 + V f^2 = \int_{\Omega} \left(u^2 \left| \nabla \left(\frac{f}{u} \right) \right|^2 + \frac{1}{u} f^2 \right).$$

PROOF: We would like to test $Lu = 1$ against $\frac{f^2}{u}$. This is a suitable test function because $1/u$ and f are bounded and since $f, u \in W^{1,2}(\Omega)$ we have $f^2/u \in W^{1,2}(\Omega)$. Testing $Lu = 1$ against $\frac{f^2}{u}$ gives

$$\int_{\Omega} \nabla u \nabla (f^2/u) + V f^2 = \int_{\Omega} \left(\frac{f^2}{u} \right).$$

Applying (2.2.2) from the corollary to the LHS we obtain

$$\int_{\Omega} \nabla u \nabla (f^2/u) + V f^2 = \int_{\Omega} |\nabla f|^2 - |u|^2 |\nabla (f/u)|^2 + V(f^2)$$

setting this equal to the RHS we obtain

$$\begin{aligned} \int_{\Omega} |\nabla f|^2 - |u|^2 |\nabla f/u|^2 + V f^2 &= \int_{\Omega} (f^2/u) \\ \int_{\Omega} |\nabla f|^2 + V f^2 &= \int_{\Omega} (|u|^2 |\nabla (f/u)|^2 + f^2/u) \end{aligned}$$

as desired. □

Before developing the consequences of the fundamental lemma, we ought to compare this to [11], where Agmon assume the inequality

$$\int_{\Omega} |\nabla f|^2 + V f^2 \geq \int_{\Omega} \lambda |f|^2.$$

for some continuous $\lambda > 0$. One can take λ to be a continuous approximation for V satisfying $\lambda \geq V$. However, this lemma shows that $\frac{1}{u}$ in a sense replaces V when conjugating by u . This suggests that $\frac{1}{u}$ is a good choice of λ . Further discussion relating to this can be found in [[10], p.11].

The next result can be viewed as a consequence of the fundamental lemma.

Lemma 2.4 ([10], Lemma 3.2, p. 12) *Suppose $\varphi \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ such that $\varphi = 0$ on a compact subset $K \subset \overline{\Omega}$ and $L\varphi = \mu\varphi$ weakly on $\overline{\Omega} \setminus K$. Let u be the landscape function and let g be a Lipschitz function on $\overline{\Omega}$. Then*

$$\int_{\Omega} \left[u^2 |\nabla (g\varphi/u)|^2 + \left(\frac{1}{u} - \mu \right) (g\varphi)^2 \right] = \int_{\Omega} |\nabla g|^2 \varphi^2. \quad (2.4.1)$$

PROOF: We would like to test $L\varphi = \mu\varphi$ against $g^2\varphi$. This is a suitable test function because g is Lipschitz and $\varphi \in W^{1,2}(\Omega)$, so the product $g^2\varphi \in W^{1,2}(\Omega)$. Testing $L\varphi = \mu\varphi$ against this function we obtain

$$\int_{\Omega} \nabla\varphi \nabla(g^2\varphi) + V\varphi g^2\varphi = \int_{\Omega} (\mu\varphi)(g^2\varphi)$$

We split the integral on the left hand side to get

$$\int_{\Omega} \nabla\varphi \nabla(g^2\varphi) + \int_{\Omega} V\varphi(g^2\varphi) = \int_{\Omega} (\mu\varphi)(g^2\varphi).$$

A rearrangement then gives

$$\int_{\Omega} (V - \mu)(g\varphi)^2 = - \int_{\Omega} \nabla\varphi \cdot \nabla(g^2\varphi). \quad (\star)$$

We use the above result to simplify $Lu - \mu u = 1 - \mu u$ in the weak sense. We now set $f = g\varphi$ and subtract $\mu(g\varphi)^2$ from both sides of Lemma 2.3. We have

$$\int_{\Omega} |\nabla(g\varphi)|^2 + (V - \mu)(g\varphi)^2 = \int_{\Omega} \left(u^2 \left| \nabla \left(\frac{(g\varphi)}{u} \right) \right|^2 + \left(\frac{1}{u} - \mu \right) (g\varphi)^2 \right)$$

If we now use $(\star)$ we obtain

$$\int_{\Omega} |\nabla(g\varphi)|^2 - (\nabla\varphi \cdot \nabla(g^2\varphi)) = \int_{\Omega} \left(u^2 \left| \nabla \left(\frac{(g\varphi)}{u} \right) \right|^2 + \left(\frac{1}{u} - \mu \right) (g\varphi)^2 \right)$$

applying the important identity (2.2.1) we get

$$\begin{aligned} \int_{\Omega} |\nabla(g\varphi)|^2 - (\nabla\varphi \cdot \nabla(g^2\varphi)) &= \int_{\Omega} |\nabla(g\varphi)|^2 - (|\nabla(\varphi g)|^2 - \varphi^2 |\nabla g|^2) \\ &= \int_{\Omega} \varphi^2 |\nabla g|^2 \end{aligned}$$

so we conclude

$$\int_{\Omega} \varphi^2 |\nabla g|^2 = \int_{\Omega} \left(u^2 \left| \nabla \left(\frac{(g\varphi)}{u} \right) \right|^2 + \left(\frac{1}{u} - \mu \right) (g\varphi)^2 \right)$$

as desired. □

We now consider the consequences of setting $g = \chi e^h$ for h, χ Lipschitz.

Lemma 2.5 *For h, χ continuously differentiable we have*

$$|\nabla g|^2 = e^{2h} \chi^2 |\nabla h|^2 + e^{2h} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2).$$

Furthermore, for h, χ Lipschitz, we have

$$\int_{\Omega} |\nabla g|^2 = \int_{\Omega} e^{2h} \chi^2 |\nabla h|^2 + e^{2h} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2).$$

PROOF: For h and χ continuously differentiable we have

$$\begin{aligned} |\nabla g|^2 &= |\nabla(\chi e^h)|^2 \\ &= (\nabla(\chi e^h)) \cdot (\nabla(\chi e^h)) \\ &= ((e^h \nabla \chi + \chi \nabla e^h)) \cdot (e^h \nabla \chi + \chi \nabla e^h) \\ &= (e^h \nabla \chi) \cdot (e^h \nabla \chi) + 2(\chi \nabla e^h) \cdot (e^h \nabla \chi) + (\chi \nabla e^h) \cdot (\chi \nabla e^h) \\ &= (e^h \nabla \chi) \cdot (e^h \nabla \chi) + 2(e^h \chi \nabla h) \cdot (e^h \nabla \chi) + (e^h \chi \nabla h) \cdot (e^h \chi \nabla h) \\ &= e^{2h} |\nabla \chi|^2 + e^{2h} \chi^2 |\nabla h|^2 + 2e^{2h} \chi (\nabla h \cdot \nabla \chi) \end{aligned}$$

Now, notice that

$$\begin{aligned} e^{2h} |\nabla \chi|^2 + 2e^{2h} \chi (\nabla h \cdot \nabla \chi) &= e^{2h} (\nabla \chi \cdot \nabla \chi + 2\chi \nabla h \cdot \nabla \chi) \\ &= e^{2h} (\nabla \chi \cdot (\nabla \chi + 2\chi \nabla h)) \\ &= e^{2h} ((\nabla \chi + \chi \nabla h - \chi \nabla h) \cdot (\nabla \chi + \chi \nabla h + \chi \nabla h)) \\ &= e^{2h} ((\chi \nabla h + \nabla \chi - \chi \nabla h) \cdot (\chi \nabla h + \nabla \chi + \chi \nabla h)) \\ &= e^{2h} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2) \end{aligned}$$

where the last line follows by a difference of squares. This implies

$$|\nabla g|^2 = e^{2h} \chi^2 |\nabla h|^2 + e^{2h} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2).$$

By Theorem 1.1 we can generalise the above expression for Lipschitz functions, giving the integrated version.

□

If we use the above identity in conjunction with Lemma 2.4 we obtain

Corollary 2.6 *With the same assumptions as Lemma 2.4 above, setting $g = \chi e^h$ with h and χ Lipschitz gives*

$$\begin{aligned} &\int_{\Omega} \left[u^2 |\nabla(\chi e^h \varphi / u)|^2 + \left(\frac{1}{u} - \mu - |\nabla h|^2 \right) (\chi e^h \varphi)^2 \right] \\ &= \int_{\Omega} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2) (e^h \varphi)^2 \end{aligned}$$

PROOF: Substituting Lemma 2.5 into (2.4.1) from Lemma 2.4, and taking $e^{2h}\chi^2|\nabla h|^2$ to the left hand side, we get

$$\begin{aligned} & \int_{\Omega} \left[u^2 |\nabla(\chi e^h \varphi / u)|^2 + \left(\frac{1}{u} - \mu - |\nabla h|^2 \right) (\chi e^h \varphi)^2 \right] \\ &= \int_{\Omega} e^{2h} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2) \varphi^2 \end{aligned}$$

and we simplify the right hand side to get

$$\begin{aligned} & \int_{\Omega} \left[u^2 |\nabla(\chi e^h \varphi / u)|^2 + \left(\frac{1}{u} - \mu - |\nabla h|^2 \right) (\chi e^h \varphi)^2 \right] \\ &= \int_{\Omega} (|\chi \nabla h + \nabla \chi|^2 - |\chi \nabla h|^2) (e^h \varphi)^2. \end{aligned} \tag{2.6.1}$$

□

Chapter 3

Prelocalisation Estimates for the Agmon Distance

3.1 Agmon Distance

To have an appropriate notion of localisation, we define a suitable distance to measure localisation. In a similar vein to [11], let w be a nonnegative continuous function on $\overline{\Omega}$. We can define the Agmon metric $\rho(x, y)$ associated with w by

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \left(w(\gamma(t)) \sum_{i=1}^n (\partial_i(\gamma(t)))^2 \right)^{\frac{1}{2}} dt.$$

where the infimum is taken over all absolutely continuous paths $\gamma : [0, 1] \rightarrow \overline{\Omega}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In a more general context, we could replace $(w(\gamma(t)) \sum_{i=1}^n (\partial_i(\gamma(t)))^2)^{\frac{1}{2}}$ with a more general function. As this is not required to understand the main theorems, we leave this for the appendix.

We should observe here that $\rho(x, y)$ is Lipschitz for continuous w , simply because

$$\rho(x, y) \leq \sup_{x \in [0, 1]} |w(x)| |x - y| \tag{3.1.1}$$

and $\sup_{x \in [0, 1]} |w(x)|$ exists and is finite by continuity of w .

3.2 Prelocalisation Estimates

For now we assume w is a positive continuous function, and ρ is the Agmon distance associated to w . We have the following lemma.

Lemma 3.1 ([10], **Lemma 3.3, p. 13**) *Let w be a nonnegative continuous function and let ρ be the Agmon distance associated with w . Suppose $h : \overline{\Omega} \rightarrow \mathbb{R}$ is such that $|h(x) - h(y)| \leq \rho(x, y)$ for all $x, y \in \overline{\Omega}$. Then h is Lipschitz and moreover*

$$|\nabla h(x)|^2 \leq w(x) \quad \text{for all } x \in \overline{\Omega}.$$

The argument below follows a similar pattern to the proof in [Agmon, 1.4, p.18], however it is simplified because of the assumption that $A = Id$.

PROOF: Let w and $h : \overline{\Omega} \rightarrow \mathbb{R}$ satisfy the conditions above. We remark that $|h(x) - h(y)| \leq \rho(x, y)$ implies that

$$|h(x) - h(y)| \leq \sup_{x \in [0,1]} |w(x)| |x - y|$$

by definition of ρ so we immediately obtain that h is Lipschitz. Hence by Theorem 1.1, we have that h is differentiable almost everywhere.

Let $x \in \overline{\Omega}$ be a point where ∇h exists. Then

$$|h(x + sy) - h(x)| \leq \inf_{\gamma} \int_0^1 \left(w(\gamma(t)) \sum_{i=1}^n (\dot{\gamma}_i(t))^2 \right)^{\frac{1}{2}} dt$$

for sufficiently small s and $y \in \mathbb{R}^n \setminus \{0\}$. We notice that $x + tsy$ is an absolutely continuous path from x to y so we have

$$\begin{aligned} \inf_{\gamma} \int_0^1 \left(w(\gamma(t)) \sum_{i=1}^n (\dot{\gamma}_i(t))^2 \right)^{\frac{1}{2}} dt &\leq \int_0^1 \left(w(x + tsy) \sum_{i=1}^n (sy_i)^2 \right)^{\frac{1}{2}} dt \\ &= \int_0^1 (w(x + tsy) |sy|^2)^{\frac{1}{2}} dt. \end{aligned}$$

So we have

$$|h(x + sy) - h(x)| \leq \int_0^1 (w(x + tsy) |sy|^2)^{\frac{1}{2}} dt \quad (\star)$$

now s is a positive scalar which does not depend on t , hence

$$\int_0^1 (w(x + tsy) |sy|^2)^{\frac{1}{2}} dt = s \int_0^1 (w(x + tsy) |y|^2)^{\frac{1}{2}} dt$$

dividing both sides of $(\star)$ by s gives

$$\frac{|h(x + sy) - h(x)|}{s} \leq \int_0^1 (w(x + tsy) \|y\|^2)^{\frac{1}{2}} dt.$$

Taking the limit as $s \rightarrow 0$ we obtain

$$\begin{aligned} (\nabla h(x)) \cdot y &\leq \lim_{s \rightarrow 0} \int_0^1 (w(x + tsy) \|y\|^2)^{\frac{1}{2}} dt \\ &= \int_0^1 \lim_{s \rightarrow 0} (w(x + tsy) \|y\|^2)^{\frac{1}{2}} dt \\ &= \int_0^1 (w(x) \|y\|^2)^{\frac{1}{2}} dt \end{aligned}$$

on the right hand side none of the variables depend on t so we have for any $y \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned} \frac{(\nabla h(x)) \cdot y}{\|y\|} &\leq w(x)^{1/2} \\ \therefore \frac{((\nabla h(x)) \cdot y)^2}{\|y\|^2} &\leq w(x) \end{aligned}$$

Taking the supremum of $\|y\| = 1$ we conclude that

$$|\nabla h(x)|^2 \leq w(x)$$

as desired. □

We now remark that the above statement (and previous lemmas) hold for continuous elliptic A by [[11], Theorem 1.4, p.18] and for w is positive. The proof is trickier in that $\sum_{i,j=1}^n a_{i,j}(x) \dot{\gamma}_i(t) \dot{\gamma}_j(t)$, where $a_{i,j}$ are the entries of A^{-1} , does not simplify as easily. In general $a_{i,j}$ may give cross terms, and so the final lines do not follow directly. However, the nonnegative case can still be deduced by letting w' be nonnegative, setting $w = w' + \varepsilon$ and taking $\varepsilon \searrow 0$.

We now have the tools needed to prove our first result on exponential decay. Recall that $V \in L^\infty(\Omega)$ and so there exists $\bar{V} > 0$ such that $V \leq \bar{V}$ a.e.. We recall that the solution u to $Lu = 1$ is positive (by the maximum principle) and continuous. This means that for a fixed $\mu \geq 0$, the weight

$$w_\mu = \left(\frac{1}{u} - \mu \right)_+$$

is non-negative and continuous. We will use this weight to define an associated Agmon distance $\rho_\mu(x, y)$ as above. Also, for any $E \subset \overline{\Omega}$ we define

$$\rho_\mu(x, E) := \inf_{y \in E} \rho_\mu(x, y).$$

The subset we will be interested in is

$$E(\nu) := \left\{ x \in \overline{\Omega} : \frac{1}{u} \leq \nu \right\}$$

where ν is a constant such that $\mu \leq \nu \leq \overline{V}$. For some compact subset K of $\overline{\Omega}$ we would like to consider

$$h_K(x) = \rho_\mu(x, E(\nu) \setminus K)$$

and using this h_K we will construct the cut off function

$$\chi(x) = \begin{cases} h_K(x) & \text{if } h_K(x) < 1 \\ 1 & \text{if } h_K(x) \geq 1. \end{cases}$$

With this notation we can state the first theorem. We can view this theorem as a minimal localisation result, localising to (approximately) $E(\nu)$.

Theorem 3.2 ([10], **Theorem 3.4, p. 14**) *Let $0 \leq \mu \leq \nu \leq \overline{V}$. Suppose $\varphi \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, $\varphi = 0$ on K and $L\varphi = \mu\varphi$ weakly on $\overline{\Omega} \setminus K$. Then for any fixed $\alpha \in (0, 1)$ we have*

$$\begin{aligned} \int_{\Omega} u^2 \left| \nabla \left(\frac{\chi e^{\alpha h_K}}{u} \right) \right|^2 + (1 - \alpha^2) \int_{\Omega} \left(\frac{1}{u} - \mu \right)_+ (\chi e^{\alpha h_K} \varphi)^2 \\ \leq (1 + 2\alpha) e^{2\alpha} (\overline{V} - \mu) \int_{0 < h_K < 1} \varphi^2. \end{aligned} \quad (3.4.1)$$

PROOF: Let $\alpha \in (0, 1)$, we begin by observing that by replacing h_K by αh_K in (2.6.1) we have

$$\begin{aligned} \int_{\Omega} \left[u^2 |\nabla(\chi e^{\alpha h_K} \varphi / u)|^2 + \left(\frac{1}{u} - \mu - |\nabla(\alpha h_K)|^2 \right) (\chi e^{\alpha h_K} \varphi)^2 \right] \\ = \int_{\Omega} (|\chi \nabla(\alpha h_K) + \nabla \chi|^2 - |\chi \nabla(\alpha h_K)|^2) (e^{\alpha h_K} \varphi)^2 \end{aligned}$$

rearranging the left hand side gives

$$LHS = \int_{\Omega} u^2 |\nabla(\chi e^{\alpha h_K} \varphi / u)|^2 + \int_{\Omega} \left(\frac{1}{u} - \mu - \alpha^2 |\nabla h_K|^2 \right) (\chi e^{\alpha h_K} \varphi)^2.$$

We note that this expression is already similar to (3.4.1). We now manipulate the second term on the left hand side. Recall that $\chi = 0$ on $E(\nu) \setminus K$, and $\varphi = 0$ on K . Hence $\chi\varphi = 0$ on $E(\nu)$, so we may safely write

$$\int_{\Omega} \left(\frac{1}{u} - \mu - \alpha^2 |\nabla h_K|^2 \right) (\chi e^{\alpha h_K} \varphi)^2 = \int_{\overline{\Omega} \setminus E(\nu)} \left(\frac{1}{u} - \mu - \alpha^2 |\nabla h_K|^2 \right) (\chi e^{\alpha h_K} \varphi)^2.$$

Now by Lemma 3.1 we have that $|\nabla h_K|^2 \leq w_\mu$. Hence $-|\nabla h_K|^2 \geq -w_\mu$, so we have

$$\int_{\overline{\Omega} \setminus E(\nu)} \left(\frac{1}{u} - \mu - \alpha^2 |\nabla h_K|^2 \right) (\chi e^{\alpha h_K} \varphi)^2 \geq \int_{\overline{\Omega} \setminus E(\nu)} \left(\frac{1}{u} - \mu - \alpha^2 w_\mu \right) (\chi e^{\alpha h_K} \varphi)^2.$$

Recall that on $\overline{\Omega} \setminus E(\nu)$ we have that $\frac{1}{u} > \nu$, and since we picked $\nu \geq \mu$ we obtain $w_\mu = \left(\frac{1}{u} - \mu \right)_+ = \left(\frac{1}{u} - \mu \right)$. Consequently,

$$\begin{aligned} \int_{\overline{\Omega} \setminus E(\nu)} \left(\frac{1}{u} - \mu - \alpha^2 w_\mu \right) (\chi e^{\alpha h_K} \varphi)^2 &= \int_{\overline{\Omega} \setminus E(\nu)} (1 - \alpha^2) w_\mu (\chi e^{\alpha h_K} \varphi)^2 \\ &= (1 - \alpha^2) \int_{\Omega} w_\mu (\chi e^{\alpha h_K} \varphi)^2. \end{aligned}$$

Which is the second term of (3.4.1). This gives us

$$\begin{aligned} \int_{\Omega} u^2 |\nabla(\chi e^{\alpha h_K} \varphi / u)|^2 + (1 - \alpha^2) \int_{\Omega} w_\mu (\chi e^{\alpha h_K} \varphi)^2 \\ \leq \int_{\Omega} (|\chi \nabla(\alpha h_K) + \nabla \chi|^2 - |\chi \nabla(\alpha h_K)|^2) (e^{\alpha h_K} \varphi)^2. \quad (\star\star) \end{aligned}$$

We now consider the right hand side. We note that when $|\nabla \chi| = 0$ then the integral is zero. Hence we may restrict to $\{0 < h_K < 1\}$. Furthermore, on this set we note that $\chi = h_K$ so we have

$$\begin{aligned} |\chi \alpha \nabla h_K + \nabla \chi|^2 - |\chi \alpha \nabla h_K|^2 &= (\chi \alpha + 1)^2 |\nabla h_K|^2 - \chi^2 \alpha^2 |\nabla h_K|^2 \\ &= ((\chi \alpha + 1)^2 - \chi^2 \alpha^2) |\nabla h_K|^2. \end{aligned}$$

Now $0 < \chi < 1$ so we deduce

$$\begin{aligned} ((\chi \alpha + 1)^2 - \chi^2 \alpha^2) &= \chi^2 \alpha^2 + 2\chi \alpha + 1 - \chi^2 \alpha^2 \\ &= 2\chi \alpha + 1 \\ &< 2\alpha + 1. \end{aligned}$$

Putting the above calculations together we get

$$\begin{aligned}
\int_{\{0 < h_K < 1\}} (|\chi \nabla(\alpha h) + \nabla \chi|^2 - |\chi \nabla(\alpha h_K)|^2) (e^{\alpha h_K} \varphi)^2 &\leq \int_{\{0 < h_K < 1\}} (2\alpha + 1) |\nabla h_K|^2 (e^{\alpha h_K} \varphi)^2 \\
&\leq \int_{\{0 < h_K < 1\}} (2\alpha + 1) w_\mu (e^{\alpha h_K} \varphi)^2 \\
&\quad \text{(by Lemma 3.1)} \\
&\leq \int_{\{0 < h_K < 1\}} (2\alpha + 1) (\bar{V} - \mu) (e^{\alpha h_K} \varphi)^2
\end{aligned}$$

The last line follows because of Corollary 1.29 gives us $\frac{1}{u} \leq \bar{V}$, and $\mu < \bar{V}$. Noting that $e^{\alpha h_K} < e^\alpha$ for $h_K < 1$, removing the constants from the integral, and combining with $(\star\star)$ we get

$$\begin{aligned}
\int_{\Omega} u^2 \left| \nabla \left(\frac{\chi e^{\alpha h_K} \varphi}{u} \right) \right|^2 + (1 - \alpha^2) \int_{\Omega} w_\mu (\chi e^{\alpha h_K} \varphi)^2 \\
\leq (2\alpha + 1) (\bar{V} - \mu) e^{2\alpha} \int_{\{0 < h_K < 1\}} \varphi^2
\end{aligned}$$

as desired. □

This theorem can now be specialised in a couple of ways to give a stronger result. To simplify the notation, for the remainder of this thesis, set

$$\bar{w}(x) = \left(\frac{1}{u(x)} - \bar{\mu} \right)_+$$

and let $\bar{\rho}$ denote the Agmon metric associated with this weight. The result we want to aim for is the following.

Corollary 3.3 ([10], Corollary 3.5, p.17) *Let K be a compact subset of $\bar{\Omega}$, Let $0 < \mu \leq \bar{\mu}$ and $0 < \delta \leq \bar{V}/10$. Set $\nu = \mu + \delta$ and suppose that $\bar{\mu} + \delta \leq \bar{V}$.*

Suppose now that $\varphi \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$, $\varphi = 0$ on K and $L\varphi = \mu\varphi$ weakly on $\bar{\Omega} \setminus K$. Then

$$\int_{\{h_K \geq 1\}} e^{h_K} (|\nabla \varphi|^2 + \bar{V} \varphi^2) \leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V} \int_{\Omega} \varphi^2 \quad (3.5.1)$$

PROOF: We will estimate $\int_{\Omega} e^{h_K} |\nabla \varphi|^2$ (kinetic energy) and $\int_{\Omega} e^{h_K} \bar{V} \varphi^2$ (potential energy) terms separately by specialising Theorem 3.2. For simplicity, we normalise φ such that

$$\int_{\Omega} \varphi^2 := 1$$

and set $f = \chi e^{\alpha h_K} \varphi$ as well as $\delta = \nu - \mu$. We are interested in the case where $\alpha = \frac{1}{2}$ as this gives us a result strong enough to prove Theorem 4.1.

We now notice that with the above simplifications in mind, Theorem 3.2 becomes

$$\int_{\Omega} u^2 \left| \nabla \left(\frac{f}{u} \right) \right|^2 + \frac{3}{4} \int_{\Omega} \omega_{\mu} f^2 \leq 2e(\bar{V} - \mu).$$

This provides a sufficient estimate for $\int_{\Omega} e^{h_K} \bar{V} \varphi^2$. Recall that $f = 0$ on $E(\nu)$ (because $\chi(x) = 0$ on $E(\nu) \setminus K$ and $\varphi = 0$ on K). Also $\omega_{\mu} \geq \nu - \mu = \delta$ on $\bar{\Omega} \setminus E(\nu)$, so we can simplify the second term on the left hand side. Furthermore, $\mu > 0$ so $\bar{V} - \mu < \bar{V}$. Altogether this gives

$$\int_{\Omega} u^2 \left| \nabla \left(\frac{f}{u} \right) \right|^2 + \frac{3}{4} \delta \int_{\bar{\Omega} \setminus E(\nu)} f^2 \leq 2e\bar{V}. \quad (3.5.2)$$

In particular, since the first term on the left hand side is strictly positive, we have

$$\int_{\bar{\Omega} \setminus E(\nu)} f^2 \leq \frac{8e\bar{V}}{3\delta}$$

This is useful because on the set $h_K \geq 1$ we have $f = e^{h_K/2} \varphi$ so

$$\begin{aligned} \bar{V} \int_{\{h_K \geq 1\}} (e^{h_K/2} \varphi)^2 &\leq \bar{V} \int_{\Omega} f^2 \\ &= \bar{V} \int_{\bar{\Omega} \setminus E(\nu)} f^2 \\ &\leq \frac{8e\bar{V}}{3\delta} \bar{V} \end{aligned} \quad (3.5.3)$$

this is the second term (potential energy) we are interested in.

For the first term, we note that

$$\begin{aligned} \int_{\{h_K \geq 1\}} e^{h_K} |\nabla \varphi|^2 &= \int_{h_K \geq 1} \left| \nabla(e^{h_K/2} \varphi) - \frac{1}{2} \nabla(h_K) e^{h_K/2} \varphi \right|^2 \\ &= \int_{h_K \geq 1} \left| \nabla f - \frac{1}{2} \nabla(h_K) f \right|^2 \\ &\leq \int_{h_K \geq 1} 2|\nabla f|^2 + \frac{f^2}{2} |\nabla h_K|^2. \end{aligned}$$

Where the last rearrangement follows by $|a - b|^2 \leq 2a^2 + 2b^2$ (as $-2ab \leq a^2 + b^2$).

We now recall that $|\nabla h_K|^2 \leq \omega_\mu$ by Lemma 3.1. Since $\mu \geq 0$ we have $|\nabla h_K|^2 \leq \bar{V}$. This gives

$$\int_{\{h_K \geq 1\}} e^{h_K} |\nabla \varphi|^2 \leq \int_{h_K \geq 1} 2|\nabla f|^2 + \frac{1}{2} \bar{V} f^2$$

Hence we just need to estimate $\int_\Omega |\nabla f|^2$. To do this, write $f = \frac{uf}{u}$, and observe that

$$\begin{aligned} \int_\Omega |\nabla f|^2 &= \int_\Omega |u \nabla(f/u) + (f/u) \nabla u|^2 \\ &= \int_\Omega u^2 |\nabla(f/u)|^2 + 2u(f/u) \nabla(f/u) \cdot \nabla u + (f/u)^2 |\nabla u|^2 \\ &\leq 2 \int_\Omega u^2 |\nabla(f/u)|^2 + (f/u)^2 |\nabla u|^2 \end{aligned}$$

where the last line follows by using $2a \cdot b \leq |a|^2 + |b|^2$. We now investigate these terms.

We begin by investigating $\int_\Omega (f/u)^2 |\nabla u|^2$. We first integrate by parts, and then apply the product rule (on the divergence term) to get

$$\begin{aligned} \int_\Omega |\nabla u|^2 (f/u)^2 &= - \int_\Omega u \operatorname{div} (\nabla u (f/u)^2) \\ &= - \int_\Omega u ((f/u)^2 \operatorname{div} \nabla u + \nabla u \cdot \nabla (f^2/u^2)) \\ &= - \int_\Omega f^2/u \operatorname{div} \nabla u - \int_\Omega u \nabla u \cdot \nabla (f^2/u^2) \end{aligned}$$

We now integrate by parts on the first term, and apply the product rule to the second term. We get

$$\begin{aligned} \int_\Omega |\nabla u|^2 (f/u)^2 &= \int_\Omega \nabla(f^2/u) \nabla u - \int_\Omega u \nabla u \cdot 2(f/u) \nabla(f/u) \\ &= \int_\Omega \nabla(f^2/u) \nabla u - \int_\Omega 2(f/u) (\nabla u) \cdot u \nabla(f/u) \end{aligned}$$

the last rearrangement is done to obtain an expression of the form $-2ab$. We note that

$$\begin{aligned} \left| \left(\frac{1}{\sqrt{2}} a + \sqrt{2} b \right) \right|^2 &= \frac{1}{2} |a|^2 + 2ab + 2|b|^2 \\ \therefore -2ab &\leq \frac{1}{2} |a|^2 + 2|b|^2 \end{aligned}$$

applying this we get

$$\int_{\Omega} |\nabla u|^2 (f/u)^2 \leq \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + 2u^2 |\nabla(f/u)|^2 + \nabla u \nabla(f^2/u) \right)$$

multiplying by 2 we get

$$\begin{aligned} 2 \int_{\Omega} |\nabla u|^2 (f/u)^2 &\leq \int_{\Omega} ((f/u)^2 |\nabla u|^2 + 4u^2 |\nabla(f/u)|^2 + 2\nabla u \nabla(f^2/u)) \\ \int_{\Omega} |\nabla u|^2 (f/u)^2 &\leq \int_{\Omega} (4u^2 |\nabla(f/u)|^2 + 2\nabla u \nabla(f^2/u)) . \end{aligned}$$

Now recall that $\int_{\Omega} u^2 |\nabla(f/u)|^2 \leq 2e\bar{V}$ by (3.5.2). As for other term, if we test $Lu = 1$ against f^2/u we obtain

$$\begin{aligned} \int_{\Omega} \nabla u \nabla(f^2/u) &= \int_{\Omega} (1 - Vu)(f^2/u) \\ &= \int_{\Omega} \left(\frac{1}{u} - V \right) f^2 \end{aligned}$$

now $\frac{1}{u} \leq \bar{V}$ and $V \geq 0$, so $\frac{1}{u} - V \leq \bar{V}$. This gives

$$\begin{aligned} \int_{\Omega} \nabla u \nabla(f^2/u) &\leq \bar{V} \int_{\Omega} f^2 \\ &\leq \frac{8e\bar{V}^2}{3\delta} . \end{aligned} \quad (\text{by (3.5.3)})$$

So putting all of this together we get

$$\int_{\Omega} |\nabla u|^2 (f/u)^2 \leq \frac{16e\bar{V}^2}{3\delta} + 8e\bar{V}$$

This now allows us to estimate $\int_{\Omega} |\nabla f|^2$. We have

$$\begin{aligned} \int_{\Omega} |\nabla f|^2 &\leq 2 \int_{\Omega} u^2 |\nabla(f/u)|^2 + (f/u)^2 |\nabla u|^2 \\ &\leq 2 \left(\int_{\Omega} u^2 |\nabla(f/u)|^2 + 8e\bar{V} + \frac{16e\bar{V}^2}{3\delta} \right) \end{aligned}$$

we now recall (3.5.2) to obtain

$$\int_{\Omega} u^2 |\nabla(f/u)|^2 \leq 2e\bar{V}$$

and hence

$$\begin{aligned} \int_{\Omega} |\nabla f|^2 &\leq 4e\bar{V} + 16e\bar{V} + \frac{32e\bar{V}^2}{3\delta} \\ &= \left(20 + \frac{32\bar{V}}{3\delta} \right) e\bar{V} . \end{aligned}$$

So we now have

$$\begin{aligned} \int_{h_K \geq 1} e^{h_K/2} |\nabla \varphi|^2 &\leq \left(40 + \frac{64\bar{V}}{3\delta}\right) e\bar{V} + \frac{4\bar{V}}{3\delta} e\bar{V} \\ &= \left(40 + \frac{68\bar{V}}{3\delta}\right) e\bar{V} \end{aligned}$$

Putting all of these calculations together

$$\begin{aligned} \int_{\{h_K \geq 1\}} e^{h_K} (|\nabla \varphi|^2 + \bar{V} \varphi^2) &\leq \left(40 + \frac{76\bar{V}}{3\delta}\right) e\bar{V} + \left(\frac{8\bar{V}}{3\delta}\right) e\bar{V} \\ &= \left(40 + \frac{76\bar{V}}{3\delta}\right) e\bar{V} \\ &= \left(\frac{120\delta + 76\bar{V}}{\delta}\right) e\bar{V} \\ &\leq \left(\frac{96\bar{V}}{3\delta}\right) e\bar{V} \\ &= 32e \left(\frac{\bar{V}}{\delta}\right) \bar{V} \int_{\Omega} \varphi^2. \end{aligned}$$

□

We now remark that the above holds in the case where $K = \emptyset$. That is, for

$$h(x) := \bar{\rho}(x, E(\bar{\mu} + \delta))$$

and ψ satisfying (in the weak sense) $L\psi = \lambda\psi$ on Ω such that $\lambda \leq \bar{\mu}$ we have

$$\int_{\{h \geq 1\}} e^h (|\nabla \psi|^2 + \bar{V} \psi^2) \leq 32e \left(\frac{\bar{V}}{\delta}\right) \bar{V} \int_{\Omega} \psi^2. \quad (3.5.4)$$

Chapter 4

Localisation Estimates Using Landscape Function

4.1 Localisation of Eigenfunctions

The main theorem (Theorem 3.2) of the previous chapter tells us that eigenfunctions ψ and localised eigenfunctions φ decay exponentially far away from where $\frac{1}{u}$ is large. This chapter now shows that an eigenfunction ψ is well approximated by localised eigenfunctions φ with a similar eigenvalue (and vice-versa).

Let $\bar{\mu}$ and δ be as defined in Corollary 3.3, that is for an eigenvalue $0 < \mu < \bar{V}$, set $\bar{\mu} \geq \mu$ and $0 < \delta \leq \bar{V}/10$ such that $\bar{\mu} + \delta \leq \bar{V}$.

Let $(E_\ell)_{1 \leq \ell \leq R}$ be any collection of subsets of $E(\bar{\mu} + \delta) = \{x \in \bar{\Omega} : \frac{1}{u} \leq \bar{\mu} + \delta\}$ satisfying the following conditions (if it exists).

1. $S = \inf\{\bar{\rho}(x, y) : x \in E_\ell, y \in E_{\ell'}, \ell \neq \ell'\} > 0$
2. $E(\bar{\mu} + \delta) = \bigcup_{\ell=1}^R E_\ell$.

We view E_ℓ as the wells of $\frac{1}{u}$, in the sense that these are the regions where $\frac{1}{u}$ is suitably small. If no such finite decomposition of $E(\bar{\mu} + \delta)$ in the above sense existed, then the region where $\frac{1}{u} \leq \bar{\mu} + \delta$ would be connected, and so the landscape function $\frac{1}{u}$ would not predict localisation. Thus we are only interested in the cases where such a decomposition exists. The existence of such wells is a geometric measure theoretic problem and we refer readers to [20], where the authors demonstrate the existence of such a decomposition by minimising a suitable functional.

We remark that the sets E_ℓ can be considered the localised regions of our eigenfunctions. There is no requirement for E_ℓ to be connected, and E_ℓ should be seen as a collection of nearby (possibly disconnected) wells. Later, we will choose suitable connected Ω_ℓ to contain E_ℓ . For the next theorems to have content, it is useful to (if possible) choose E_ℓ such that S (which we refer to as the Agmon distance between wells) is large (recall that we may merge nearby E_ℓ).

Let $S_1 < S$ be close to S (but not equal). By Lemma 1.34, we note that there exists a compact set $K_\ell \subset \bar{\Omega}$ satisfying the bi-Lipschitz cone condition and such that

$$\{x \in \bar{\Omega} : \bar{\rho}(x, E_\ell) \geq S/2\} \subset K_\ell \subset \{x \in \bar{\Omega} : \bar{\rho}(x, E_\ell) \geq S_1/2\}.$$

Define Ω_ℓ as the connected component of $\bar{\Omega} \setminus K_\ell$ containing E_ℓ . We remark that the Ω_ℓ are disjoint because the E_ℓ are at least distance S apart.

We remind the reader of how we formulated local eigenfunctions (see subsection 1.2.5 for more details). We will take the domain of L to be the closure of $C_c^\infty(\Omega_\ell)$ functions in the $W^{1,2}(\Omega)$ norm, and we use $W_0^{1,2}(\Omega_\ell)$ to denote this space. The purpose of considering Ω_ℓ is so our regularity theory (section 1.2.3) applies to $\varphi_{\ell,j}$.

For the remainder of the thesis, let $\Psi_{(a,b)}$ denote the orthogonal projection in $L^2(\Omega)$ onto the span of the eigenfunctions ψ_j with eigenvalue $\lambda_j \in (a, b)$. Also let $\Phi_{(a,b)}$ denote the span of the eigenfunctions $\varphi_{\ell,j}$ with eigenvalue $\mu_{\ell,j} \in (a, b)$. We remark that the range of $\Psi_{(0,\infty)}$ is $L^2(\Omega)$ while the range of $\Phi_{(0,\infty)}$ is $L^2(\bigcup_\ell \Omega_\ell)$.

Theorem 4.1 ([10], **Theorem 4.1, p. 19**) *Let $0 < \delta \leq \bar{V}/10$. Suppose that φ is one of the eigenfunctions $\varphi_{\ell,j}$ with eigenvalue $\mu = \mu_{\ell,j}$ such that $\mu \leq \bar{\mu} - \delta$ and $\bar{\mu} + \delta < \bar{V}$. Let S denote the Agmon distance between the wells of $E(\bar{\mu} + \delta)$, then*

$$\|\varphi - \Psi_{\mu-\delta, \mu+\delta}\varphi\|^2 \leq 1550 \left(\frac{\bar{V}}{\delta}\right)^3 e^{-S/2}$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$.

PROOF: Suppose φ solves $L\varphi = \mu\varphi$ in the weak sense on Ω_ℓ . We define the cut off function

$$f(t) = \begin{cases} 1 & t \leq \frac{S_1}{2} - 1 \\ \frac{S_1}{2} - t & \frac{S_1}{2} - 1 \leq t \leq \frac{S_1}{2} \\ 0 & \frac{S_1}{2} \leq t \end{cases}$$

and set

$$\eta(x) := f(\bar{\rho}(x, E_\ell)).$$

The function η weights the points $x \in \bar{\Omega}$ such that points suitably far away from E_ℓ are zero. Recalling (3.5.1) from Corollary 3.3, we are interested in the cases where $h_K \geq 1$. Now we can merge nearby wells so $\frac{S_1}{2} - 1 \geq 1$, so we can focus on $h_K \geq \frac{S_1}{2} - 1$.

Now notice $E_\ell \subset E(\bar{\mu} + \delta) \setminus K$. This gives

$$\begin{aligned} h_K(x) &= \bar{\rho}(x, E(\bar{\mu} + \delta) \setminus K) \\ &\leq \bar{\rho}(x, E_\ell) \end{aligned}$$

and hence restricting to the case where $h_K(x) \geq \frac{S_1}{2} - 1$ gives

$$\bar{\rho}(x, E_\ell) \geq \frac{S_1}{2} - 1.$$

With some manipulation, we can use (3.5.1) to calculate bounds on $\|\eta\varphi - \Psi_{\mu-\delta, \mu+\delta}(\eta\varphi)\|^2$ and $\|(1-\eta)\varphi - \Psi_{\mu-\delta, \mu+\delta}((1-\eta)\varphi)\|^2$.

We begin by calculating $\|\eta\varphi - \Psi_{(\mu-\delta, \mu+\delta)}(\eta\varphi)\|^2$. Let r be the distribution satisfying $r = L(\eta\varphi) - \mu\eta\varphi$ in the weak sense, that is for every $\zeta \in C_c^\infty(\bar{\Omega})$

$$r(\zeta) := \int_{\Omega} [\nabla(\eta\varphi) \nabla(\zeta) + (V - \mu)\eta\varphi\zeta].$$

We now note that for any $\zeta \in C_c^\infty(\bar{\Omega})$ we have $\eta\zeta \in W_0^{1,2}(\Omega_\ell)$. So we can test $L\varphi = \mu\varphi$ against $\eta\zeta$. This implies

$$\int_{\Omega} [\nabla\varphi \nabla(\eta\zeta) + (V - \mu)\varphi\eta\zeta] = 0.$$

Subtracting this version of 0 from r gives

$$r(\zeta) = \int_{\Omega} [\nabla(\eta\varphi) \nabla(\zeta) - \nabla\varphi \nabla(\eta\zeta)]$$

now applying the product rule we have

$$\begin{aligned} r(\zeta) &= \int_{\Omega} [(\varphi \nabla \eta + \eta \nabla \varphi) \nabla(\zeta) - \nabla \varphi (\zeta \nabla \eta + \eta \nabla \zeta)] \\ &= \int_{\Omega} [\varphi \nabla \eta \nabla \zeta - \zeta \nabla \varphi \nabla \eta]. \end{aligned}$$

We observe that the integrand is zero when $\nabla \eta = 0$, so we may restrict our attention to $\nabla \eta \neq 0$ and this occurs precisely when $\bar{\rho}(x, E_{\ell}) \in [\frac{S_1}{2} - 1, \frac{S_1}{2}]$. Notice that for $\ell \neq \ell'$, E_{ℓ} and $E_{\ell'}$ are separated by at least distance S_1 , so $\bar{\rho}(x, E_{\ell'}) \geq \frac{S_1}{2}$. Hence

$$\begin{aligned} \bar{\rho}(x, \bigcup_{\ell} E_{\ell}) &= \bar{\rho}(x, E(\bar{\mu} + \delta)) \\ &\geq \frac{S_1}{2} - 1 \end{aligned}$$

thus for any set K , we have

$$\begin{aligned} h_K(x) &= \bar{\rho}(x, E(\bar{\mu} + \delta) \setminus K) \\ &\geq \bar{\rho}(x, E(\bar{\mu} + \delta)) \\ &\geq \frac{S_1}{2} - 1. \end{aligned}$$

We set $K = \bar{\Omega} \setminus \Omega_{\ell}$, and $\bar{V}$ large enough so $|\nabla \eta|^2 \leq \bar{V}$. Now (3.5.1) implies the following bound (recall that $\|\varphi\| = 1$),

$$\begin{aligned} \int_{\nabla \eta \neq 0} e^{S_1/2-1} (|\nabla \varphi|^2 + \bar{V} \varphi^2) &\leq \int_{\nabla \eta \neq 0} e^{h_K} (|\nabla \varphi|^2 + \bar{V} \varphi^2) \\ &\leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V}. \end{aligned}$$

We now want to divide both sides by the constant $e^{S_1/2-1}$. We have

$$\begin{aligned} \int_{\nabla \eta \neq 0} (|\nabla \varphi|^2 + \bar{V} \varphi^2) &\leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V} e^{1-S_1/2} \\ &= 32e^2 \left(\frac{\bar{V}^2}{\delta e^{S_1/2}} \right) \end{aligned} \tag{4.1.1}$$

We can use these bounds to derive a bound for $|r(\zeta)|$. We begin by using the triangle inequality to obtain

$$|r(\zeta)| \leq \int_{\nabla \eta \neq 0} |\varphi \nabla \eta \nabla \zeta| + \int_{\nabla \eta \neq 0} |\zeta \nabla \varphi \nabla \eta|.$$

Extracting a factor of $\nabla\eta$ and then using Cauchy Schwarz we get

$$\begin{aligned} |r(\zeta)| &\leq \sup |\nabla\eta| \left(\int_{\nabla\eta \neq 0} |\varphi \nabla\zeta| + \int_{\nabla\eta \neq 0} |\zeta \nabla\varphi| \right) \\ &\leq \sup |\nabla\eta| \left(\|\nabla\zeta\| \left(\int_{\nabla\eta \neq 0} |\varphi|^2 \right)^{1/2} + \|\zeta\| \left(\int_{\nabla\eta \neq 0} |\nabla\varphi|^2 \right)^{1/2} \right) \end{aligned}$$

we now use that $|\nabla\eta|^2 \leq \bar{V}$

$$\begin{aligned} |r(\zeta)| &\leq \bar{V}^{1/2} \left(\|\nabla\zeta\| \left(\int_{\nabla\eta \neq 0} |\varphi|^2 \right)^{1/2} + \|\zeta\| \left(\int_{\nabla\eta \neq 0} |\nabla\varphi|^2 \right)^{1/2} \right) \\ &= \|\nabla\zeta\| \left(\int_{\nabla\eta \neq 0} \bar{V} |\varphi|^2 \right)^{1/2} + \bar{V}^{1/2} \|\zeta\| \left(\int_{\nabla\eta \neq 0} |\nabla\varphi|^2 \right)^{1/2} \end{aligned}$$

we note that the above is an expression of form $ac + bd$. If we apply Cauchy Schwarz again we get

$$\begin{aligned} |r(\zeta)| &\leq (\|\nabla\zeta\|^2 + \bar{V}\|\zeta\|^2)^{\frac{1}{2}} \left(\int_{\nabla\eta \neq 0} |\nabla\varphi|^2 + \bar{V}\varphi^2 \right)^{\frac{1}{2}} \\ &\leq (\|\nabla\zeta\|^2 + \bar{V}\|\zeta\|^2)^{\frac{1}{2}} \left(32e^2 \left(\frac{\bar{V}^2}{\delta e^{S_1/2}} \right) \right)^{\frac{1}{2}} \end{aligned}$$

If we now square both sides we get

$$|r(\zeta)|^2 \leq (\|\nabla\zeta\|^2 + \bar{V}\|\zeta\|^2) 32e^2 \frac{\bar{V}^2}{\delta e^{S_1/2}}.$$

We remark here that $r \in W^{-1,2}(\Omega)$. To simplify things, let $\varepsilon := 32e^2 \left(\frac{\bar{V}}{\delta} \right) e^{-S_1/2}$. The above simplifies to

$$(\|\nabla\zeta\|^2 + \bar{V}\|\zeta\|^2) 32e^2 \frac{\bar{V}^2}{\delta e^{S_1/2}} = \varepsilon \bar{V} (\|\nabla\zeta\|^2 + \bar{V}\|\zeta\|^2). \quad (4.1.2)$$

We will now use this upper bound on $|r(\zeta)|$ to derive a bound on $\|\eta\varphi - \Psi_{(\mu-\delta, \mu+\delta)}(\eta\varphi)\|^2$. Suppose that J denotes the set of indices where $|\lambda_j - \mu| > \delta$, and suppose that

$$\eta\varphi = \sum_{j \in J} a_j \psi_j$$

where $(\psi_j)_{j \in J}$ are eigenfunctions of L with eigenvalues $(\lambda_j)_{j \in J}$. Then

$$\|\eta\varphi - \Psi_{(\mu-\delta, \mu+\delta)}(\eta\varphi)\|^2 \leq \sum_{j \in J} a_j^2.$$

If $\eta\varphi$ was a permissible choice of ζ then we could use our bounds on $r(\zeta)$ and we'd be done. However recall that $r \in W^{-1,2}(\Omega)$ and if J were infinite then we might not be able to have $\eta\varphi \in W^{1,2}(\Omega)$.

To work around this we will consider finite linear combinations instead. Now let J be any finite set such that $j \in J$ implies $|\lambda_j - \mu| \geq \delta$, set

$$\zeta = \sum_{j \in J} \gamma_j \psi_j.$$

This is a finite linear combination of $W^{1,2}(\Omega)$ functions. Hence such a ζ is permissible.

We first observe that

$$\begin{aligned} \|\nabla \psi_j\|^2 + \bar{V} \|\psi_j\|^2 &= \int_{\Omega} |\nabla \psi_j|^2 + \bar{V} \psi_j^2 \\ &\leq \int_{\Omega} |\nabla \psi_j|^2 + (\bar{V} + V) \psi_j^2 && (\text{since } V > 0) \\ &= \int_{\Omega} |\nabla \psi_j|^2 + V \psi_j^2 + \int_{\Omega} \bar{V} \psi_j^2 \\ &= (\lambda_j + \bar{V}) \|\psi_j\|_2^2 \\ &= \lambda_j + \bar{V} \end{aligned}$$

and hence we have

$$\begin{aligned} \|\nabla \zeta\|^2 + \bar{V} \|\zeta\|^2 &\leq \int_{\Omega} |\nabla \sum_{j \in J} \gamma_j \psi_j|^2 + (V + \bar{V}) \zeta^2 \\ &= \int_{\Omega} \left| \nabla \left(\sum_{j \in J} \gamma_j \psi_j \right) \right|^2 + (V + \bar{V}) \left(\sum_{j \in J} \gamma_j \psi_j \right)^2 \\ &= \int_{\Omega} \sum_{j \in J} (|\nabla \gamma_j \psi_j|^2 + V \gamma_j \psi_j^2) + \int_{\Omega} \bar{V} \sum_{j \in J} (\gamma_j \psi_j)^2 \\ &\hspace{15em} (\text{since } \psi_j \text{ are orthonormal}) \\ &= \sum_{j \in J} \lambda_j \gamma_j^2 + \bar{V} \gamma_j^2 \\ &= \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2. \end{aligned}$$

Now if we apply this to (4.1.2) we obtain

$$\begin{aligned} r(\zeta)^2 &\leq \varepsilon \bar{V} (\|\nabla \zeta\|^2 + \bar{V} \|\zeta\|^2) \\ &\leq \varepsilon \bar{V} \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2. \end{aligned}$$

On the other hand, if we compute $r(\zeta)$ directly we have (after taking out a common factor of γ_j and using linearity),

$$\begin{aligned} r(\zeta) &= \sum_{j \in J} \gamma_j \int_{\Omega} \nabla(\eta\varphi) \nabla(\psi_j) + (V - \mu) \eta\varphi \psi_j \\ &= \sum_{j \in J} \gamma_j \left(\int_{\Omega} \nabla(\eta\varphi) \nabla(\psi_j) + (V - \lambda_j) \eta\varphi \psi_j + \int_{\Omega} (\lambda_j - \mu) \eta\varphi \psi_j \right) \end{aligned}$$

now recall that $(L - \lambda_j)\psi_j = 0$ in the weak sense, so we have

$$\begin{aligned} r(\zeta) &= \sum_{j \in J} \gamma_j \int_{\Omega} (\lambda_j - \mu) \eta\varphi \psi_j \\ &= \sum_{j \in J} \gamma_j (\lambda_j - \mu) \int_{\Omega} \eta\varphi \psi_j \\ &= \sum_{j \in J} \gamma_j (\lambda_j - \mu) \langle \eta\varphi, \psi_j \rangle \end{aligned}$$

Denoting $\beta_j := \langle \eta\varphi, \psi_j \rangle$, we note that they are the coefficients of $\eta\varphi$ in the (ψ_j) basis. We have

$$\sum_{j \in J} \gamma_j (\lambda_j - \mu) \beta_j \leq \varepsilon \bar{V} \sum_{j \in J} (\lambda_j + \bar{V}) \gamma_j^2.$$

Exploiting our freedom to choose γ_j , we set

$$\gamma_j = \beta_j \operatorname{sgn}(\lambda_j - \mu) (\lambda_j + \bar{V})^{-1/2}$$

this choice gives us

$$\left| \sum_{j \in J} |\lambda_j - \mu| \beta_j^2 (\lambda_j + \bar{V})^{-1/2} \right|^2 \leq \varepsilon \bar{V} \sum_{j \in J} \beta_j^2.$$

We are interested in controlling $\frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}}$ from below. There are two cases, either $\lambda_j \geq 2\bar{V}$ or $\lambda_j < 2\bar{V}$. We consider $\lambda_j \leq 2\bar{V}$ first, we have

$$\begin{aligned} \lambda_j + \bar{V} &\leq 3\bar{V} \\ \therefore \frac{1}{\sqrt{\lambda_j + \bar{V}}} &\geq \frac{1}{\sqrt{3\bar{V}}} \end{aligned}$$

combining this with $|\lambda_j - \mu| \geq \delta$ for $j \in J$, we obtain

$$\frac{\delta}{\sqrt{3\bar{V}}} \leq \frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}}.$$

This gives

$$\frac{\delta^2}{3\bar{V}} \left| \sum_{j \in J} \beta_j^2 \right|^2 \leq \varepsilon \bar{V} \sum_{j \in J} \beta_j^2$$

which, after rearrangement becomes

$$\left| \sum_{j \in J} \beta_j^2 \right|^2 \leq 3\varepsilon \frac{\bar{V}^2}{\delta^2} \sum_{j \in J} \beta_j^2.$$

In the other case, since $\delta^{1/2} \leq (\lambda_j - \mu)^{1/2}$ we observe

$$\begin{aligned} |\lambda_j - \mu| &= (\lambda_j - \mu)^{1/2} (\lambda_j - \mu)^{1/2} \\ &\geq \delta^{1/2} (\lambda_j - \mu)^{1/2}. \end{aligned}$$

Using $\lambda_j > 2\bar{V}$ and $\lambda_j = \frac{\lambda_j}{2} + \frac{\lambda_j}{2}$ we obtain $\lambda_j \geq \frac{\lambda_j}{2} + \bar{V}$ which implies

$$(\lambda_j - \mu)^{1/2} \geq \left(\frac{\lambda_j}{2} + \bar{V} - \mu \right)^{1/2}.$$

Noticing that $\bar{V} - \mu \geq 0$ we get

$$\left(\frac{\lambda_j}{2} + \bar{V} - \mu \right)^{1/2} \geq \left(\frac{\lambda_j}{2} \right)^{1/2}.$$

Hence we conclude that

$$|\lambda_j - \mu| \geq \delta^{1/2} \left(\frac{\lambda_j}{2} \right)^{1/2}.$$

Finally, we see that $\frac{1}{\lambda_j + \bar{V}} > \frac{2}{3\lambda_j}$ because $\frac{3\lambda_j}{2} = \lambda_j + \frac{\lambda_j}{2} > \lambda_j + \bar{V}$. Combining these estimates we see that

$$\left(\frac{\delta}{3} \right)^{1/2} \leq \frac{|\lambda_j - \mu|}{\sqrt{\lambda_j + \bar{V}}}$$

and hence

$$\frac{\delta}{3} \left| \sum_{j \in J} \beta_j^2 \right|^2 \leq \varepsilon \bar{V} \sum_{j \in J} \beta_j^2.$$

A rearrangement gives

$$\left| \sum_{j \in J} \beta_j^2 \right|^2 \leq 3\varepsilon \frac{\bar{V}}{\delta} \sum_{j \in J} \beta_j^2.$$

Recall that $\bar{V} > \delta$ so $\frac{\bar{V}}{\delta} < \frac{\bar{V}^2}{\delta^2}$. Hence in both cases we obtain

$$\left| \sum_{j \in J} \beta_j^2 \right|^2 \leq 3\varepsilon \frac{\bar{V}^2}{\delta^2} \sum_{j \in J} \beta_j^2$$

and since

$$\left| \sum_{j \in J} \beta_j^2 \right| = \sum_{j \in J} \beta_j^2$$

as the sum of positive terms, a rearrangement yields

$$\sum_{j \in J} \beta_j^2 \leq 3\varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Recall that J was an arbitrary finite subset where $j \in J$ implies $|\lambda_j - \mu| \geq \delta$. So we have

$$\begin{aligned} \|\eta\varphi - \Psi_{\mu-\delta, \mu+\delta}(\eta\varphi)\|^2 &= \sup_J \sum_{j \in J} \beta_j^2 \\ &\leq 3\varepsilon \frac{\bar{V}^2}{\delta^2}. \end{aligned}$$

We move to computing $\|(1 - \eta)\varphi - \Psi_{\mu-\delta, \mu+\delta}((1 - \eta)\varphi)\|^2$. We observe that $I - \Psi_{(\mu-\delta, \mu+\delta)}$ has operator norm at most 1 since it is a projection. Thus it suffices to find a bound for $\|(1 - \eta)\varphi\|$.

We notice that $1 - \eta = 0$ when $\bar{\rho}(x, E_\ell) \leq S_1/2 - 1$. So we have

$$\|(1 - \eta)\varphi\|^2 = \int_{\bar{\rho}(x, E_\ell) > S_1/2 - 1} |(1 - \eta)\varphi|^2.$$

We can split $\{x : \bar{\rho}(x, E_\ell) > S_1/2 - 1\}$ into $\{x : S_1/2 - 1 < \bar{\rho}(x, E_\ell) < S_1/2\}$ and $\{\bar{\rho}(x, E_\ell) \geq S_1/2\}$. On $\{x : S_1/2 - 1 < \bar{\rho}(x, E_\ell) < S_1/2\}$ we have

$$\begin{aligned} \int_{S_1/2 - 1 < \bar{\rho}(x, E_\ell) < S_1/2} |(1 - \eta)\varphi|^2 &\leq \int_{S_1/2 - 1 < \bar{\rho}(x, E_\ell) < S_1/2} |(1 - S_1/2 + S_1/2)\varphi|^2 \\ &= \int_{S_1/2 - 1 < \bar{\rho}(x, E_\ell) < S_1/2} |\varphi|^2 \end{aligned}$$

and on $\{\bar{\rho}(x, E_\ell) \geq S_1/2\}$ we have $\eta = 0$, so our expression simplifies to

$$\|(1 - \eta)\varphi\|^2 = \int_{\bar{\rho}(x, E_\ell) > S_1} |\varphi|^2.$$

If we take $K = \bar{\Omega} \setminus \Omega_\ell$, we get $E(\bar{\mu} + \delta) \setminus K \subset E_\ell$. So we have

$$h_K(x) \leq \bar{\rho}(x, E_\ell)$$

hence

$$\int_{\bar{\rho}(x, E_\ell) > S_1/2-1} |\varphi|^2 \leq \int_{h_K > S_1/2-1} |\varphi|^2.$$

Multiplying both sides by a constant gives

$$\begin{aligned} e^{S_1/2-1} \bar{V} \|(1 - \eta)\varphi\|^2 &\leq \int_{h_K > S_1/2-1} e^{S_1/2-1} \bar{V} |\varphi|^2 \\ &\leq \int_{h_K > S_1/2-1} e^{h_K} \bar{V} |\varphi|^2 \\ &\leq \int_{h_K > 1} e^{h_K} \bar{V} |\varphi|^2 \\ &\leq 32e^2 \left(\frac{\bar{V}}{\delta}\right) \bar{V} \quad (\text{using (3.5.1)}) \end{aligned}$$

If we divide both sides by the same constant, we get

$$\|(1 - \eta)\varphi\|^2 \leq 32e^2 \left(\frac{\bar{V}}{\delta}\right) e^{-S_1/2}.$$

Recall $\varepsilon = 32e^2 \left(\frac{\bar{V}}{\delta}\right) e^{-S_1/2}$. We have

$$\|(1 - \eta)\varphi - \Psi_{(\mu-\delta, \mu+\delta)}((1 - \eta)\varphi)\|^2 \leq \varepsilon.$$

If we combine the above calculations, and set $c = S - S_1$ we obtain (in the first line we use $2|ab|^2 \leq |a|^2 + |b|^2$)

$$\begin{aligned}
\|\varphi - \Psi_{(\mu-\delta, \mu+\delta)}\varphi\|^2 &\leq 2\|\eta\varphi - \Psi_{(\mu-\delta, \mu+\delta)}(\eta\varphi)\|^2 + 2\|(1-\eta)\varphi - \Psi_{(\mu-\delta, \mu+\delta)}((1-\eta)\varphi)\|^2 \\
&= 6\varepsilon \frac{\bar{V}^2}{\delta^2} + 2\varepsilon \\
&= 2\varepsilon \left(3 \frac{\bar{V}^2}{\delta^2} + 1 \right) \\
&= 64e^2 \left(\frac{\bar{V}}{\delta} \right) e^{-S_1/2} \left(\frac{3\bar{V}^2 + \delta^2}{\delta^2} \right) \\
&\leq 64e^2 \left(\frac{\bar{V}}{\delta} \right) e^{-\frac{S+c}{2}} \left(\frac{310 \bar{V}^2}{100 \delta^2} \right) \\
&\leq 64e^2 \left(\frac{\bar{V}}{\delta} \right) e^{-S/2} e^{c/2} \left(\frac{310 \bar{V}^2}{100 \delta^2} \right) \\
&\leq 1550 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-S/2} \quad (\text{for } c < 0.1)
\end{aligned}$$

as desired. □

We now turn our attention to proving the above inequality, but with the roles of φ and ψ reversed.

Theorem 4.2 *Let $0 < \delta \leq \bar{V}/10$. Suppose that ψ is one of the eigenfunctions ψ_j with eigenvalue $\lambda = \lambda_j \leq \bar{\mu} - \delta$. Let S denote the Agmon distance between the wells defined using $\bar{\mu} + \delta < \bar{V}$ (as above), then*

$$\|\psi - \Phi_{\lambda-\delta, \lambda+\delta}\psi\|^2 \leq 1550 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-S/2}$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$.

The proof is very similar to the above (but with the roles of φ and ψ reversed), except now we need to pay attention to which regions E_ℓ our eigenfunction ψ occupies.

PROOF:

We follow the same procedure as above, except now E_ℓ is no longer fixed (since ψ

isn't localised to E_ℓ). Define the cut off function (recall the definition of f from the previous proof).

$$\eta_\ell(x) = f(\bar{\rho}(x, E_\ell))$$

and set

$$\tilde{\eta}(x) = \sum_{\ell} \eta_\ell(x).$$

We again do the proof in three parts; firstly calculating the $\tilde{\eta}\psi$, secondly the $(1 - \tilde{\eta})\psi$ term, and finally combining the two terms to bound $\|\psi - \Phi\psi\|$.

Let $r(\zeta)$ be the distribution satisfying

$$\tilde{r}(\zeta) = \int_{\Omega} \nabla(\tilde{\eta}\psi) \nabla \zeta + (V - \lambda) \tilde{\eta}\psi \zeta$$

By the same reasoning as in the previous proof, $\zeta \tilde{\eta} \in W^{1,2}(\Omega)$ so we use this as a test function for $L\psi = \lambda\psi$. We have

$$\int_{\Omega} [\nabla \psi \nabla(\zeta \tilde{\eta}) + (V - \lambda)(\psi \zeta \tilde{\eta})] = 0.$$

As before, subtracting this version of zero from $\tilde{r}$ we obtain

$$\tilde{r}(\zeta) = \int_{\Omega} [\nabla(\tilde{\eta}\psi) \nabla \zeta - \nabla \psi \nabla(\zeta \tilde{\eta})].$$

By the same manipulation as before (product rule and simplification) we obtain

$$\tilde{r}(\zeta) = \int_{\Omega} [\psi \nabla \tilde{\eta} \nabla \zeta - \zeta \nabla \psi \nabla \tilde{\eta}]$$

the integrand is zero when $\nabla \tilde{\eta} = 0$, that is when $\sum_{\ell} \nabla \eta_{\ell} = 0$. Notice that $\sum_{\ell} \nabla \eta_{\ell} \neq 0$ when for any ℓ , $\bar{\rho}(x, E_{\ell}) \in [\frac{S_1}{2} - 1, \frac{S_1}{2}]$. Hence $h(x) = \bar{\rho}(x, \cup E_{\ell}) \geq \frac{S_1}{2} - 1$. Using 4.1.1 we again realise

$$\int_{\nabla \tilde{\eta} \neq 0} e^{S_1/2-1} (|\nabla \psi|^2 + \bar{V} \psi^2) \leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V}$$

so we conclude

$$\int_{\nabla \tilde{\eta} \neq 0} (|\nabla \psi|^2 + \bar{V} \psi^2) \leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V} e^{1-S_1/2}.$$

Just like we did for the previous proof, we can manipulate $|\tilde{r}(\zeta)|$ using Cauchy-Schwarz and use the above expression to obtain a bound. As before set $\varepsilon = 32e^2 \left(\frac{\bar{V}}{\delta}\right) e^{-S_1/2}$, we proceed the same way as before.

$$\begin{aligned}
|\tilde{r}(\zeta)| &\leq \int_{\nabla\tilde{\eta}\neq 0} |\psi \nabla\tilde{\eta} \nabla\zeta| + \int_{\nabla\tilde{\eta}\neq 0} |\zeta \nabla\psi \nabla\tilde{\eta}| \\
&\leq (\sup |\nabla\eta|) \left(\|\nabla\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} \psi^2 \right)^{1/2} + \|\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} |\nabla\psi|^2 \right)^{1/2} \right) \\
&\leq \bar{V}^{1/2} \left(\|\nabla\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} \psi^2 \right)^{1/2} + \|\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} |\nabla\psi|^2 \right)^{1/2} \right) \\
&\leq \|\nabla\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} \bar{V} \psi^2 \right)^{1/2} + \bar{V}^{1/2} \|\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} |\nabla\psi|^2 \right)^{1/2} \\
&\leq \|\nabla\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} \bar{V} \psi^2 \right)^{1/2} + \bar{V}^{1/2} \|\zeta\| \left(\int_{\nabla\tilde{\eta}\neq 0} |\nabla\psi|^2 \right)^{1/2} \\
&\leq (\|\nabla\zeta\|^2 + \bar{V} \|\zeta\|^2)^{\frac{1}{2}} \left(\int_{\nabla\tilde{\eta}\neq 0} |\nabla\psi|^2 + \bar{V} \psi^2 \right)^{\frac{1}{2}} \\
&\leq (\varepsilon \bar{V})^{1/2} [\|\nabla\zeta\|^2 + \bar{V} \|\zeta\|^2]^{1/2}.
\end{aligned}$$

This gives

$$\tilde{r}(\zeta)^2 \leq \varepsilon \bar{V} (\|\nabla\zeta\|^2 + \bar{V} \|\zeta\|^2). \quad (4.1.3)$$

So we have $r \in W^{-1,2}(\Omega)$ and an analogue of (4.1.2). We proceed in the same manner as the previous proof. We choose ζ as a (finite) linear combination of $\varphi_{\ell,j}$. Let $\tilde{J}$ be any finite set of indices (ℓ, j) and denote

$$\tilde{\zeta} = \sum_{(\ell,j) \in \tilde{J}} \gamma_{\ell,j} \varphi_{\ell,j}.$$

We use (4.1.3) to obtain an upper bound, we have

$$\begin{aligned}
\tilde{r}(\tilde{\zeta})^2 &\leq \varepsilon \bar{V} \left(\left\| \nabla \left(\sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right) \right\|^2 + \bar{V} \left\| \sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right\|^2 \right) \\
&\leq \varepsilon \bar{V} \left(\left\| \nabla \left(\sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right) \right\|^2 + V \left\| \sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right\|^2 \bar{V} \left\| \sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right\|^2 \right) \\
&= \varepsilon \bar{V} \left(\sum_{\tilde{j}} \|\nabla(\gamma_{\ell,j} \varphi_{\ell,j})\|^2 + V \sum_{\tilde{j}} \|\gamma_{\ell,j} \varphi_{\ell,j}\|^2 \bar{V} \sum_{\tilde{j}} \|\gamma_{\ell,j} \varphi_{\ell,j}\|^2 \right) \\
&\hspace{25em} (\text{orthogonality}) \\
&= \varepsilon \bar{V} \left(\sum_{\tilde{j}} \gamma_{\ell,j}^2 \int_{\Omega} \nabla \varphi_{\ell,j} \nabla \varphi_{\ell,j} + V \varphi_{\ell,j} \varphi_{\ell,j} + \bar{V} \sum_{\tilde{j}} \gamma_{\ell,j}^2 \right) \\
&\hspace{25em} (\text{recall } \|\varphi_{\ell,j}\|^2 = 1) \\
&= \varepsilon \bar{V} \left(\sum_{\tilde{j}} \gamma_{\ell,j}^2 \int_{\Omega} \mu_{\ell,j} \varphi_{\ell,j} \varphi_{\ell,j} + \bar{V} \sum_{\tilde{j}} \gamma_{\ell,j}^2 \right) \\
&= \varepsilon \bar{V} \left(\sum_{\tilde{j}} \gamma_{\ell,j}^2 (\mu_{\ell,j} + \bar{V}) \right)
\end{aligned}$$

We now calculate $\tilde{r}(\tilde{\zeta})$ exactly. We start by calculating $\tilde{r}(\gamma_{\ell,j} \varphi_{\ell,j})$ and noting that the expression we obtain will be helpful by linearity. For any (ℓ, j) we have

$$\tilde{r}(\gamma_{\ell,j} \varphi_{\ell,j}) = \int_{\Omega} \nabla(\tilde{\eta} \psi) \nabla(\gamma_{\ell,j} \varphi_{\ell,j}) + (V - \lambda) \tilde{\eta} \psi \gamma_{\ell,j} \varphi_{\ell,j}.$$

As before we'd like to use $(L - \mu_{\ell,j}) \varphi_{\ell,j} = 0$ on Ω_{ℓ} weakly in an analogous way to the first proof. We recall that $\varphi_{\ell,j}$ is only supported in Ω_{ℓ} but $\eta_k = 0$ on Ω_{ℓ} for $k \neq \ell$. So on Ω_{ℓ} we notice that $\tilde{\eta} = \eta_{\ell}$. We have

$$\tilde{r}(\gamma_{\ell,j} \varphi_{\ell,j}) = \int_{\Omega_{\ell}} \nabla(\eta_{\ell} \psi) \nabla(\gamma_{\ell,j} \varphi_{\ell,j}) + (V - \lambda) \eta_{\ell} \psi \gamma_{\ell,j} \varphi_{\ell,j}$$

adding and subtracting $\mu_{\ell,j}$,

$$\begin{aligned}
\tilde{r}(\gamma_{\ell,j} \varphi_{\ell,j}) &= \int_{\Omega_{\ell}} \nabla(\eta_{\ell} \psi) \nabla(\gamma_{\ell,j} \varphi_{\ell,j}) + (V - \mu_{\ell,j}) \eta_{\ell} \psi \gamma_{\ell,j} \varphi_{\ell,j} + \int_{\Omega_{\ell}} (\mu_{\ell,j} - \lambda) \eta_{\ell} \psi \gamma_{\ell,j} \varphi_{\ell,j} \\
&= \int_{\Omega_{\ell}} (\mu_{\ell,j} - \lambda) \eta_{\ell} \psi \gamma_{\ell,j} \varphi_{\ell,j} \\
&= \gamma_{\ell,j} (\mu_{\ell,j} - \lambda) \int_{\Omega} \tilde{\eta} \psi \varphi_{\ell,j}
\end{aligned}$$

Notice that $\tilde{\eta}$ is only supported on $\bigcup_{\ell} \Omega_{\ell}$. So we have that $\tilde{\eta}\psi$ is in the span of $(\varphi_{\ell,j})_{\ell,j}$. So we may let $\beta_{\ell,j}$ be the coefficients for $\tilde{\eta}\psi$ in the $(\varphi_{\ell,j})$ basis, that is

$$\beta_{\ell,j} = \int_{\Omega} \tilde{\eta}\psi \varphi_{\ell,j}.$$

Hence we have

$$\begin{aligned} \tilde{r}(\tilde{\zeta}) &= \int_{\Omega} \nabla(\tilde{\eta}\psi) \nabla \left(\sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right) + (V - \lambda) \tilde{\eta}\psi \left(\sum_{\tilde{j}} \gamma_{\ell,j} \varphi_{\ell,j} \right) \\ &= \sum_{\tilde{j}} \int_{\Omega} \nabla(\tilde{\eta}\psi) \nabla (\gamma_{\ell,j} \varphi_{\ell,j}) + (V - \lambda) \tilde{\eta}\psi (\gamma_{\ell,j} \varphi_{\ell,j}) \\ &= \sum_{\tilde{j}} \tilde{r}(\gamma_{\ell,j} \varphi_{\ell,j}) \\ &= \sum_{\tilde{j}} \gamma_{\ell,j} (\mu_{\ell,j} - \lambda) \beta_{\ell,j}. \end{aligned}$$

Using the above to rewrite (4.1.3) we obtain

$$\left| \sum_{\tilde{j}} \gamma_{\ell,j} (\mu_{\ell,j} - \lambda) \beta_{\ell,j} \right|^2 \leq \varepsilon \bar{V} \left(\sum_{j \in J} \gamma_{\ell,j}^2 (\mu_{\ell,j} + \bar{V}) \right).$$

Suppose now that $\tilde{J}$ is a finite set of (ℓ, j) such that $|\mu_{\ell,j} - \lambda| \geq \delta$. As before, set

$$\gamma_{\ell,j} = \beta_{\ell,j} (\mu_{\ell,j} + \bar{V})^{-1/2} \operatorname{sgn}(\mu_{\ell,j} - \lambda)$$

plugging this into the above we get

$$\left| \sum_{\tilde{j}} \beta_{\ell,j}^2 |\mu_{\ell,j} - \lambda| (\mu_{\ell,j} + \bar{V})^{-1/2} \right|^2 \leq \varepsilon \bar{V} \left(\sum_{\tilde{j}} \beta_{\ell,j}^2 \right).$$

As in the last proof, we want to control $\frac{|\mu_{\ell,j} - \lambda|}{\sqrt{\mu_{\ell,j} - \bar{V}}}$ from below. The two cases we have are $\mu_{\ell,j} \leq 2\bar{V}$ and $\mu_{\ell,j} > 2\bar{V}$. We consider $\mu_{\ell,j} \leq 2\bar{V}$ first, we have

$$\begin{aligned} \mu_{\ell,j} + \bar{V} &\leq 3\bar{V} \\ \therefore \frac{1}{\sqrt{\mu_{\ell,j}}} &\geq \frac{1}{\sqrt{3\bar{V}}}. \end{aligned}$$

Combining this with $|\mu_{\ell,j} - \lambda| \geq \delta$ for $(\ell, j) \in \tilde{J}$ we obtain

$$\frac{\delta}{\sqrt{3\bar{V}}} \leq \frac{|\mu_{\ell,j} - \lambda|}{\sqrt{\mu_{\ell,j} + \bar{V}}}$$

and hence

$$\frac{\delta^2}{3\bar{V}} \left| \sum_{\bar{j}} \beta_{\ell,j}^2 \right|^2 \leq \varepsilon \bar{V} \sum_{\bar{j}} \beta_{\ell,j}^2$$

which can be rearranged to give

$$\left| \sum_{\bar{j}} \beta_{\ell,j}^2 \right|^2 \leq 3\varepsilon \frac{\bar{V}^2}{\delta^2} \sum_{\bar{j}} \beta_{\ell,j}^2.$$

For the other case, since $\delta^{1/2} \leq (\mu_{\ell,j} - \lambda)^{1/2}$ we observe that

$$\begin{aligned} |\mu_{\ell,j} - \lambda| &= (\mu_{\ell,j} - \lambda)^{1/2} (\mu_{\ell,j} - \lambda)^{1/2} \\ &\geq \delta^{1/2} (\mu_{\ell,j} - \lambda)^{1/2}. \end{aligned}$$

Using $\mu_{\ell,j} > 2\bar{V}$ and $\mu_{\ell,j} = \frac{\mu_{\ell,j}}{2} + \frac{\mu_{\ell,j}}{2}$ we obtain $\mu_{\ell,j} \geq \frac{\mu_{\ell,j}}{2} + \bar{V}$, which implies

$$(\mu_{\ell,j} - \lambda)^{1/2} \geq (\mu_{\ell,j} + \bar{V} - \lambda)^{1/2}.$$

Noticing that $\bar{V} - \lambda \geq 0$ we get

$$\left(\frac{\mu_{\ell,j}}{2} + \bar{V} - \lambda \right)^{1/2} \geq \left(\frac{\mu_{\ell,j}}{2} \right)^{1/2}.$$

Hence we conclude that

$$|\mu_{\ell,j} - \lambda| \geq \delta^{1/2} \left(\frac{\lambda_j}{2} \right)^{1/2}.$$

Finally, we observe that $\frac{1}{\mu_{\ell,j} + \bar{V}} > \frac{2}{3\mu_{\ell,j}}$ because $\frac{3\mu_{\ell,j}}{2} = \mu_{\ell,j} + \frac{\mu_{\ell,j}}{2} > \mu_{\ell,j} + \bar{V}$.

Combining these estimates we see that

$$\left(\frac{\delta}{3} \right)^{1/2} \leq \frac{|\mu_{\ell,j} - \lambda|}{\sqrt{\mu_{\ell,j} + \bar{V}}}.$$

This gives

$$\frac{\delta}{3} \left| \sum_{\bar{j}} \beta_{\ell,j}^2 \right|^2 \leq \varepsilon \bar{V} \sum_{\bar{j}} \beta_{\ell,j}^2$$

which rearranges to

$$\left| \sum_{\bar{j}} \beta_{\ell,j}^2 \right|^2 \leq 3\varepsilon \frac{\bar{V}}{\delta} \sum_{\bar{j}} \beta_{\ell,j}^2.$$

Recall that $\bar{V} > \delta$ so $\frac{\bar{V}}{\delta} < \frac{\bar{V}^2}{\delta^2}$. Hence in both cases we obtain

$$\left| \sum_{\tilde{J}} \beta_{\ell,j}^2 \right|^2 \leq 3\varepsilon \frac{\bar{V}^2}{\delta^2} \sum_{\tilde{J}} \beta_{\ell,j}^2.$$

Observe that

$$\left| \sum_{\tilde{J}} \beta_{\ell,j}^2 \right| = \sum_{\tilde{J}} \beta_{\ell,j}^2$$

as a sum of positive terms. Rearranging these terms we deduce

$$\sum_{\tilde{J}} \beta_{\ell,j}^2 \leq 3\varepsilon \frac{\bar{V}^2}{\delta^2}.$$

Finally, because $\tilde{J}$ was any finite set of indices such that $|\mu_{\ell,j} - \lambda| \geq \delta$ we have

$$\begin{aligned} \|\tilde{\eta}\psi - \Phi_{(\lambda-\delta, \lambda+\delta)}(\tilde{\eta}\psi)\| &= \sup_{\tilde{J}} \sum_{\tilde{J}} \beta_{\ell,j}^2 \\ &\leq 3\varepsilon \frac{\bar{V}^2}{\delta^2}. \end{aligned}$$

We move to estimating $\|(1 - \tilde{\eta})\psi - \Phi_{\lambda-\delta, \lambda+\delta}(1 - \tilde{\eta})\psi\|$. We realise that $I - \Phi_{\lambda-\delta, \lambda+\delta}$ is a norm one projection, so it suffices to estimate $\|(1 - \tilde{\eta})\psi\|$.

Notice that $1 - \tilde{\eta} = 0$ when $\bar{\rho}(x, E_\ell) \leq \frac{s_1}{2} - 1$ for every ℓ . So our integrand is nonzero when $h(x) = \bar{\rho}(x, \cup E_\ell) > \frac{s_1}{2} - 1$. We remark that

$$\begin{aligned} \int_{h > \frac{s_1}{2} - 1} e^{S_1/2 - 1} (\bar{V}\psi^2) &\leq \int_{h > \frac{s_1}{2} - 1} e^h (\bar{V}\psi^2) \\ &\leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V}. \end{aligned}$$

Dividing both sides by $e^{\frac{s_1}{2} - 1}$ we obtain

$$\bar{V} \int_{h > \frac{s_1}{2} - 1} \psi^2 \leq 32e \left(\frac{\bar{V}}{\delta} \right) \bar{V} e^{1 - \frac{s_1}{2}}.$$

Finally, we note that $|1 - \tilde{\eta}| < 1$ on $\{x : \bar{\rho}(x, \cup E_\ell) > \frac{s_1}{2} - 1\}$, so we conclude

$$\begin{aligned} \|(1 - \tilde{\eta})\psi\|^2 &\leq 32e \left(\frac{\bar{V}}{\delta} \right) e^{-\frac{s_1}{2}} \\ &= \varepsilon. \end{aligned}$$

Combining the above calculations, and setting $c = S - S_1$ we obtain (in the first line we use $2||ab||^2 \leq ||a||^2 + ||b||^2$)

$$\begin{aligned}
||\psi - \Phi_{(\mu-\delta, \mu+\delta)}\psi||^2 &\leq 2||\tilde{\eta}\psi - \Phi_{(\mu-\delta, \mu+\delta)}(\tilde{\eta}\psi)||^2 + 2||(1 - \tilde{\eta})\psi - \Phi_{(\mu-\delta, \mu+\delta)}((1 - \tilde{\eta})\psi)||^2 \\
&= 6\varepsilon \frac{\bar{V}^2}{\delta^2} + 2\varepsilon \\
&= 2\varepsilon \left(3 \frac{\bar{V}^2}{\delta^2} + 1 \right) \\
&= 64e^2 \left(\frac{\bar{V}}{\delta} \right) e^{-S_1/2} \left(\frac{3\bar{V}^2 + \delta^2}{\delta^2} \right) \\
&\leq 64e^2 \left(\frac{\bar{V}}{\delta} \right) e^{\frac{-S+c}{2}} \left(\frac{310 \bar{V}^2}{100 \delta^2} \right) \\
&\leq 64e^2 \left(\frac{\bar{V}}{\delta} \right) e^{-S/2} e^{c/2} \left(\frac{310 \bar{V}^2}{100 \delta^2} \right) \\
&\leq 1550 \left(\frac{\bar{V}}{\delta} \right)^3 e^{-S/2} \quad (\text{for } c < 0.1)
\end{aligned}$$

as desired. □

4.2 Numerical Simulation

As a visual aid, we simulate the eigenfunctions for $L\varphi = \mu\varphi$. In this section we specialise to $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ and set V to be a random potential. We will simulate the eigenfunction with the smallest eigenvalue. We generate our random potential numerically by subdividing Ω into dyadic cubes of side length 2^{-N} , and on each sub cube we assign a value $\{10^{-5}, \bar{V}\}$ randomly. To observe the affects of the disorder of V we will vary N and $\bar{V}$ separately. We recall Corollary 3.3 which has dependency on the size of V , rather than the disorder of V . Hence we expect that increasing $\bar{V}$ would have a greater effect on the localisation of our eigenfunctions.

To help the reader visualise V we have included two examples below. For the two examples we have fixed $\bar{V}$ and varied N , but in the simulations we vary both parameters.

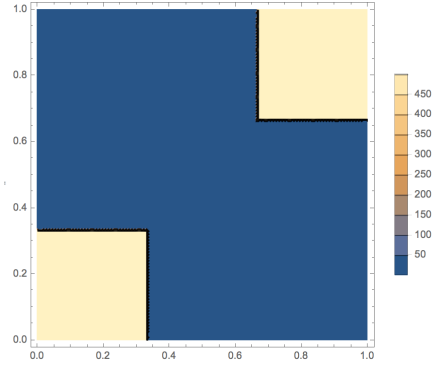


Figure 4.1: Contour plot of V for $\bar{V} = 500$ and $N = 3$.

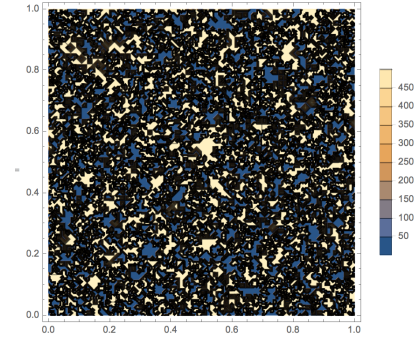


Figure 4.2: Contour plot of V for $\bar{V} = 500$ and $N = 300$.

Below are our simulations. Our first simulation is for $\bar{V} = 500$ and $N = 300$.

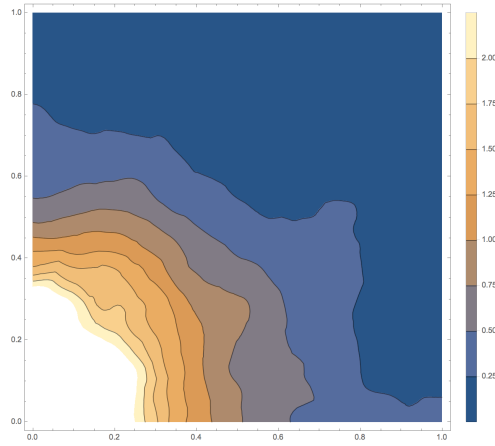


Figure 4.3: Contour plot of smallest energy eigenfunction for $N = 300$ and $\bar{V} = 500$

Figure 4.3 shows that there is some localisation when the potential and grid size are relatively small.

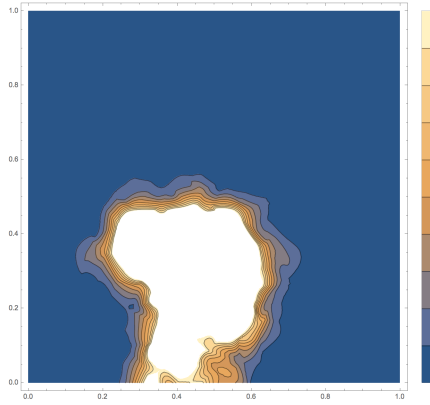


Figure 4.4: Contour plot of smallest energy eigenfunction for $N = 300$ and $\bar{V} = 5000$

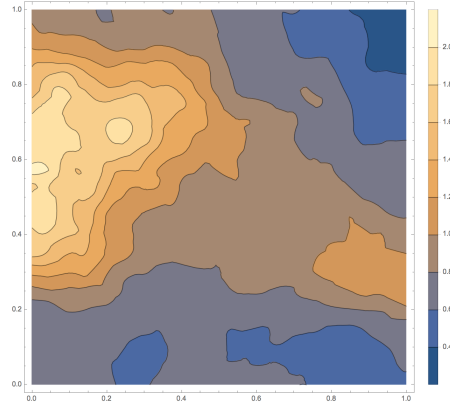


Figure 4.5: Contour plot of smallest energy eigenfunction for $N = 3000$ and $\bar{V} = 500$

Figure 4.4 shows strong localisation but figure 4.5 shows minimal localisation. This indicates that the size of $\bar{V}$ may have more effect than the size of the grid. On first inspection there appears to be less localisation than figure 4.3, but we suspect this is because the eigenfunction resides in two localised regions, rather than just a single region.

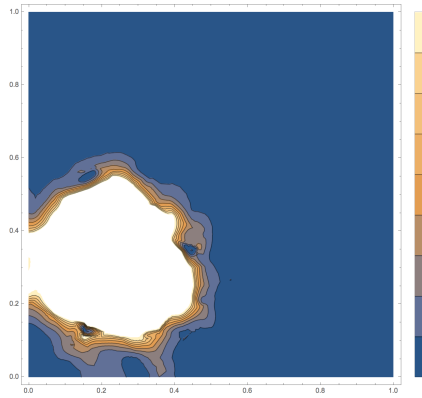


Figure 4.6: Contour plot of smallest energy eigenfunction for $N = 3000$ and $\bar{V} = 5000$

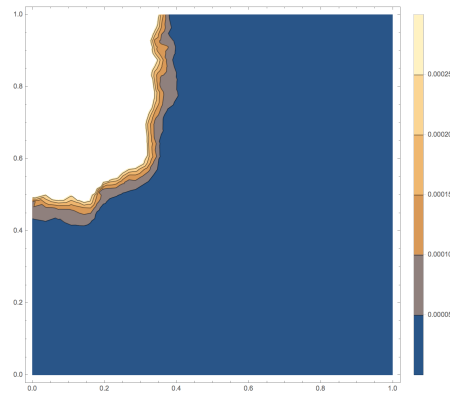


Figure 4.7: Another contour plot of smallest energy eigenfunction for $N = 3000$ and $\bar{V} = 5000$

Figures 4.6 and 4.7 show a similar localisation pattern to figure 4.4, supporting the hypothesis that the size of $\bar{V}$ has more to effect on the localisation. In figure 4.7 a different random potential $\bar{V}$ was generated, and this lead to localisation in a different region of the domain.

While our simulations in dimension $n = 2$ did show localisation of eigenfunctions, the wells of $\frac{1}{u}$ were not particularly deep. Hence these eigenfunctions did not localise to the wells of $\frac{1}{u}$ in a convincing manner. This could be because the potentials we used were not as sophisticated as [10].

However, we were able to do not only demonstrate localisation, but also that the eigenfunction resided in a well of $\frac{1}{u}$ in dimension $n = 1$. We give two examples below.

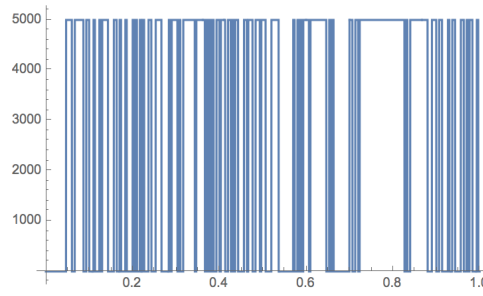


Figure 4.8: Plot of V for $N = 300$ and $\bar{V} = 5000$

Figure 4.8 is an illustration of the potential we use. We remark that there do not appear to be any obvious regions where V is small.

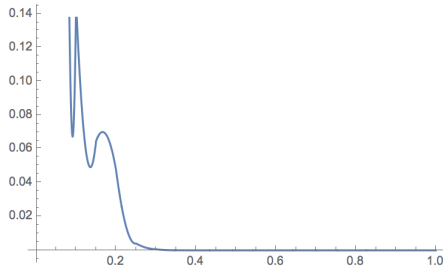


Figure 4.9: Plot of smallest energy eigenfunction for $N = 300$ and $\bar{V} = 5000$

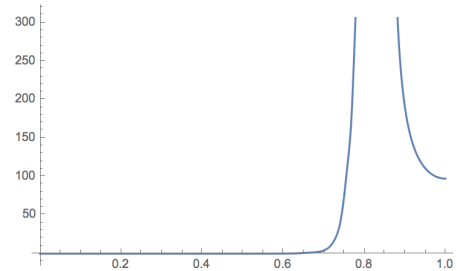


Figure 4.10: Plot of $\frac{1}{u}$ for $N = 300$ and $\bar{V} = 50000$

Figure 4.9 shows that the eigenfunction is concentrated on the left of the domain. Figure 4.10 shows that $\frac{1}{u}$ is small on the left of the domain, and becomes large on the right of our domain. This illustrates that the eigenfunction is situated away from where $\frac{1}{u}$ is large. We remark that it is easy to see where $\frac{1}{u}$ while it is not clear where V is small.

We include another example, this time with a different random V .

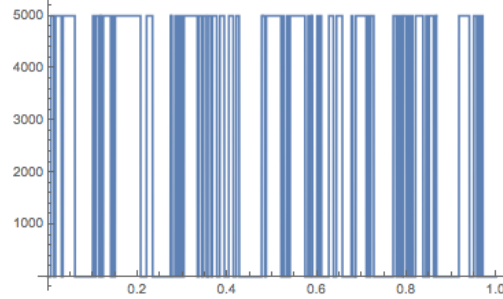


Figure 4.11: Another V with $N = 300$ and $\bar{V} = 5000$

Figure 4.11 is an illustration of the second example potential. There does not appear to be any obvious regions where V is small.

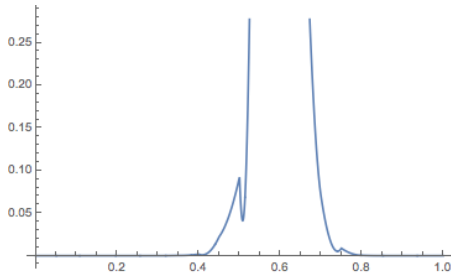


Figure 4.12: Plot of smallest energy eigenfunction

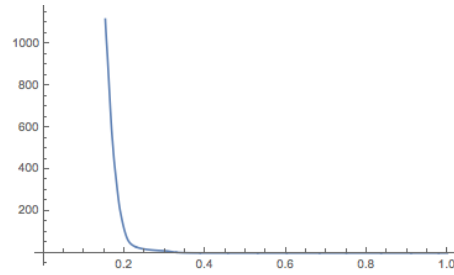


Figure 4.13: Plot of $\frac{1}{u}$

Figure 4.12 shows that the eigenfunctions are concentrated roughly about the centre of our domain. Figure 4.13 shows that $\frac{1}{u}$ is large on the right of our domain, but importantly, is small towards the centre. This is another example of where the eigenfunction is situated away from where $\frac{1}{u}$ is large.

We remark here that the set of discontinuities of V has the same Hausdorff dimension for all N . This could explain why our results are more sensitive to changes in $\bar{V}$ than they are to N . To explore this phenomena further, we should consider a fractal potential, with larger Hausdorff dimension of its set of discontinuities.

Chapter 5

Further Research

5.1 Brownian Motion

Suppose that the Agmon metric really does capture the geometry induced by L , then a reflected Brownian motion associated with the Agmon metric should give a lot of information (e.g. the corresponding heat kernel) about L via the Feynman-Kac formula. We thus aim to define a suitable Brownian motion on Ω with the Agmon metric. We were not able to properly define such a process this year, but we present some ideas which we leave for further research.

To simplify the argument, we take $\Omega = (-1, 1)$. As we're working with a connected open set in $\mathbb{R}$ we no longer have to deal with boundary regularity. We also note that the Agmon metric simplifies in the one dimensional case. Fix $\mu \geq 0$, and set

$$w = \left(\frac{1}{u(x)} - \mu \right)_+$$

Recall that this is continuous and non-negative. If $\mu < \frac{1}{V}$ this is strictly positive (recall Corollary 1.29). In general we have

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \left(w(\gamma(t)) \sum_{i=1}^n (\partial_i(\gamma(t)))^2 \right)^{\frac{1}{2}} dt.$$

In one dimension the partial derivative terms simplify to

$$\sum_{i=1}^n (\partial_i(\gamma(t)))^2 = \gamma'(t)^2$$

giving

$$\rho(x, y) = \inf_{\gamma} \int_0^1 \left(\sqrt{w_{\mu}(\gamma(t))} |\gamma'(t)| \right) dt.$$

We now use substitution to obtain

$$\begin{aligned}\rho(x, y) &= \inf_{\gamma} \operatorname{sgn}(\gamma) \int_{\gamma(0)}^{\gamma(1)} \sqrt{w_{\mu}(s)} ds \\ &= \left| \int_x^y \sqrt{w_{\mu}(s)} ds \right|.\end{aligned}$$

The last line follows by realising that γ in dimension one is simply the line segment $[x, y]$ (or the reverse). Notice that if $0 \leq \mu < \frac{1}{V}$ we have that ω is bounded below. Hence we have that

$$c|x - y| \leq \left| \int_x^y \sqrt{w_{\mu}(s)} ds \right|$$

for some constants $0 < c < \infty$. We recall that $\frac{1}{u}$ is continuous (by section 1.2.3), and hence so is $\sqrt{w_{\mu}}$. As $[-1, 1]$ is compact there is a constant $C < \infty$ for which

$$\max_{x \in [-1, 1]} \sqrt{w_{\mu}} < C.$$

This gives

$$\left| \int_x^y \sqrt{w_{\mu}(s)} ds \right| \leq C|x - y|.$$

This means that ρ is Lipschitz equivalent to the Euclidean metric.

We now move to defining a Brownian motion on Ω . Recall that Ω is a compact set with Neumann boundary conditions. This means that we are interested in a reflected Brownian. Before we consider the problem using the Agmon metric, we outline possible approaches with the Euclidean metric. There are various approaches. One can argue by defining a suitable measure (similar to the Wiener measure) on the space of continuous paths in $(-1, 1)$. Alternatively one can define our process as the weak limit of a family of processes on $\mathbb{R}^n$ with infinite drift at the boundaries (as in [12]). We are inspired by the approach in [12]. However instead of trying to understand the relevant sequence of stochastic differential equations, we just consider here the relevant sequence of infinitesimal generators of the corresponding transition semigroups.

Our goal is to show that this sequence of infinitesimal generators each define a positive contractive semigroup. Once we have these semigroups, we can use them to define the law of the sequence of processes which we would be ultimately interested in. We would like our processes to have infinite drift at the boundary

points $\{-1, 1\}$, and intuitively, this leads to all of our paths being reflected back into Ω . As in [[12], p. 471] but with ρ replaced by the Euclidean distance, for each $n \in \mathbb{N}$, consider the function

$$f_n = \exp\left(\frac{1}{n\delta(x)}\right).$$

where $\delta(x) := \inf\{\rho(x, -1), \rho(x, 1)\}$. This is a sequence of functions which explodes exponentially as x approaches $\{-1, 1\}$. Note that for fixed y , $\rho(\cdot, y)$ is a Lipschitz function. By Theorem 1.1 (and excluding the point when the infimum changes) $\delta(x)$ is differentiable a.e. and so f_n is also differentiable a.e.. Hence we may consider the Sturm-Liouville operator given by

$$L_n v = \frac{1}{2}v'' - \frac{1}{2}f'_n v'.$$

For each fixed n we would like to see if L_n generates a contractive semigroup for some domain. If we impose additional assumptions on f'_n we can use the analysis of [[22], section 2.6 p.97] to deduce more information about our semigroup acting on an appropriate L^2 space. We would like to require that f'_n is smooth, and this assumption can be realised by smoothing $\delta(x)$ (since $\exp$ is already smooth). We note that the Stein's regularised distance functions gives us a suitable smoothing of $\delta(x)$ [[12], p.464]. By imposing this additional assumption (and hence using Stein's regularised δ) we can deduce the invariant measure for L_n . By changing variables to $y = \sqrt{2}(x)$, we see that L can be written as

$$L_n v = v'' - f'_n v'$$

and by the analysis of [[22], p. 98], the invariant measure is given by

$$\begin{aligned}\omega_n(x) &= \exp\left(\int_{-1}^x -f'_n(y)dy\right) \\ &= \exp(-f_n(x)).\end{aligned}$$

So if we set

$$\gamma_n(x) = \left(\int_{-1}^1 |\exp(-f_n(x))|dx\right)^{-1}$$

then

$$\mu_n(x) = \gamma_n(x)\omega_n(x)$$

is our invariant probability measure and L_n then generates a positive contraction on $L^2(\mu_n)$. As an aside we remark that this sequence $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to

$$\mu = \frac{1}{2} \mathbf{1}_\Omega(x).$$

We also remark that [[22], 2.6.4, p. 101] states that (using the spectral decomposition) the kernel of our semigroup is given by

$$\rho_t^n(x, y) = \sum_{k \in \mathbb{N}} \exp(-\lambda_k^n t) e_k^n(x) e_k^n(y)$$

where λ_k^n is the eigenvalue associated to the eigenfunction e_k^n . We note that for Neumann conditions there are infinitely many eigenvalues for L_n which form an orthonormal basis for $L^2(\mu_n)$.

If we continue to follow the [12] approach, we would now aim to show that the associated stochastic processes converge weakly to a process with transition semigroup given by a weak limit of $\left((e^{tL_n})_{t \geq 0} \right)_{n \in \mathbb{N}}$. This approach is left as a topic for further exploration.

We remark that if we could show that $\rho_t^n \rightarrow \rho_t$ converges in a suitable sense, we could construct our reflected Brownian using a Wiener measure approach. We follow the method outlined by [23] and [24]. We would like to define a probability measure on the space of all paths $\omega : [0, \infty) \rightarrow (-1, 1)$ and show that the paths which satisfy the definition of Brownian motion occur almost surely. Since we anticipate a continuous process we begin by considering the paths indexed at countably many times. We can use continuity later to move from countably many times to continuous time. Hence we consider paths indexed by

$$\mathcal{B} = \prod_{t \in \mathbb{Q}^+} (\mathbb{R} \cup \{\infty\}).$$

Note $\mathbb{R} \cup \{\infty\}$ is the one point compactification of $\mathbb{R}$, and is Hausdorff because $\mathbb{R}$ is Hausdorff. Hence $\mathcal{B}$ is compact (by Tychonoff's theorem), and Hausdorff. We omit the proof that the product of Hausdorff sets is Hausdorff, the idea is that if x and y are distinct, we use the Hausdorff property in a factor which they differ (which exists since x and y are distinct) to construct disjoint sets. This means that we can invoke the Riesz (Markov-Kakutani) representation theorem. So to construct a positive Borel measure on $\mathcal{B}$ it suffices to construct a positive linear functional $R : C(\mathcal{B}) \rightarrow \mathbb{R}$.

We construct R by constructing $R' : U \rightarrow \mathbb{R}$ where U is some dense subset of $C(\mathcal{B})$. We take U to be the subspace of continuous functions which depend only on a finite number of time steps. We can verify that this subset is dense by Stone-Weierstrass. Because we want the time increments of our process to be independent, we define R' for $\varphi = F(\omega(t_1), \dots, \omega(t_k))$ by

$$R'(\varphi) = \int \dots \int \rho_{t_1}(x_1) \rho_{t_2-t_1}(x_2 - x_1) \dots \rho_{t_k-t_{k-1}}(x_k - x_{k-1}) F(x_1, \dots, x_k) dx_k \dots dx_1.$$

From here, the next steps would involve computing or estimating R' to show that we can extend it on all of $C(\mathcal{B})$ to R , and that the continuous paths have measure 1.

More work is required to see if such an approach works, and give a stochastic process which is a suitable analogue of the reflected Brownian motion on Ω . We leave this for future research.

Chapter 6

Appendix

6.1 Agmon's Prelocalisation Estimates

The purpose here is to contrast the above estimates obtained in [10] to earlier estimates obtained in [11]. We encourage the reader to look at this section for completeness and context. However we remark that this section isn't central to understanding [10] or the next chapters.

The estimates presented in [11] were obtained before Lemma 2.3 was discovered. Instead they are based instead on the general hypothesis 3.6.1, and a growth hypothesis 3.6.2. We should also remark that [11] treats solutions $u \in W_0^{1,2}(\Omega)$ (the Dirichlet case) to $Lu = f$ for suitable f , rather than eigenfunctions $L\varphi = \mu\varphi$.

The proofs presented earlier from [10] are simpler and have stronger results for multiple reasons. We treat the Neumann case ($W^{1,2}(\Omega)$), which means that there is no compact support hypothesis imposed on our test functions. Furthermore, we restrict to the case of eigenfunctions which allows greater freedom in choosing our test functions. Finally we are able to replace the use of assumption 3.6.1 by Lemma 2.3.

We remind the reader that we impose the additional assumption that $V \geq 0$. The results are true for $V \in L^\infty(\Omega)$ (in fact $V \in L_{loc}^1(\Omega)$). We can also generalise this to $u \in W_{loc}^{1,2}$. However these results are harder because of extra error terms needed for the negative part of V and for u_ε , a $W^{1,2}(\Omega)$ approximation for u .

Theorem 6.1 ([11], Theorem 1.5, p. 19) *Suppose $\exists \lambda > 0$ such that $\lambda \in$*

$C(\Omega)$ and

$$\int |\nabla \varphi|^2 + V|\varphi|^2 \geq \int \lambda |\varphi|^2 \quad (3.6.1)$$

for all $\varphi \in W_0^{1,2}(\Omega)$. Let $\rho_\lambda(x, y)$ be the distance given by

$$\rho_\lambda(x, y) = \inf_{\gamma} \int_0^1 [\lambda(\gamma(t)) \sum_{i=1}^n \partial_i \gamma^2]^{1/2} dt.$$

For a fixed $y^0 \in \Omega$ set $\rho_\lambda = \rho_\lambda(x, y^0)$.

Let h be Lipschitz on Ω such that $|\nabla h|^2 < \lambda$ a.e.. Suppose that u solves $Lu = f$ weakly for $f \in L^2(\Omega)$. If u also satisfies

$$\int_{\Omega} |u|^2 \lambda e^{-2(1-\delta)\rho_\lambda} < \infty \quad (3.6.2)$$

then if Ω is complete in ρ_λ we have

$$\int_{\Omega} |u|^2 (\lambda - |\nabla h|^2) e^{2h} \leq \int_{\Omega} |f|^2 (\lambda - |\nabla h|^2) e^{2h}. \quad (3.6.3)$$

On the other hand, if Ω is not complete in ρ_λ , then for some $d > 0$, set $\Omega_d = \{x : \rho_\lambda(x, \{\infty\}) > d\}$, we instead have

$$\begin{aligned} \int_{\Omega_d} |u|^2 (\lambda - |\nabla h|^2) e^{2h} &\leq \int_{\Omega} |f|^2 (\lambda - |\nabla h|^2) e^{2h} \\ &\quad + \frac{2(1+2d)}{d^2} \int_{\Omega \setminus \Omega_d} |u|^2 \lambda e^h. \end{aligned} \quad (3.6.4)$$

The proof of this theorem is quite long. Before giving a proof we will first outline the idea. Recall Lemma 2.1 which states that for two continuously differentiable functions $v, \psi : \Omega \rightarrow \mathbb{R}$ we have

$$\nabla v \cdot \nabla(\phi^2 v) = |\nabla(\phi v)|^2 - |v|^2 |\nabla \phi|^2.$$

If we let u be a weak solution to $Lu = f$ on Ω we have

$$\int_{\Omega} \nabla u \cdot \nabla \varphi + V u = \int_{\Omega} f \varphi$$

for every $\varphi \in W_0^{1,2}(\Omega)$. Suppose we could replace φ with $u\phi^2$ where ϕ has compact support. If we use Lemma 2.1 we get

$$\int_{\Omega} |\nabla(\phi u)|^2 - |u|^2 |\nabla \phi|^2 + V |u|^2 \phi^2 = \int_{\Omega} f u \phi^2.$$

If we invoke the assumption (3.6.1) we get

$$\int_{\Omega} \lambda |u\phi|^2 - |u|^2 |\nabla \phi|^2 \leq \int_{\Omega} f u \phi^2.$$

We show that the above is valid if ϕ is Lipschitz and compactly supported in (3.6.7).

However, if we ignore the compact support requirement and choose $\phi = e^h$ in the above inequality we obtain

$$\int_{\Omega} |u|^2 (\lambda - |\nabla h|^2) e^{2h} \leq \int_{\Omega} f u e^{2h}.$$

Multiplying the expression on the right hand side by 1 gives

$$\begin{aligned} \int_{\Omega} f u e^{2h} &= \int_{\Omega} u e^h (\lambda - |\nabla h|^2)^{\frac{1}{2}} f e^h (\lambda - |\nabla h|^2)^{-\frac{1}{2}} \\ &\leq \left(\int_{\Omega} |u|^2 e^{2h} (\lambda - |\nabla h|^2) \right)^{\frac{1}{2}} \left(\int_{\Omega} |f|^2 e^{2h} (\lambda - |\nabla h|^2)^{-1} \right)^{\frac{1}{2}} \end{aligned}$$

where the last line follows by Cauchy Schwarz.

If we divide both sides by $\left(\int_{\Omega} |u|^2 e^{2h} (\lambda - |\nabla h|^2) \right)^{\frac{1}{2}}$ and then square we obtain

$$\begin{aligned} \frac{(\int_{\Omega} f u e^{2h})^2}{\int_{\Omega} |u|^2 e^{2h} (\lambda - |\nabla h|^2)} &\leq \int_{\Omega} |f|^2 e^{2h} (\lambda - |\nabla h|^2)^{-1} \\ \therefore \frac{(\int_{\Omega} |u|^2 e^{2h} (\lambda - |\nabla h|^2))^2}{\int_{\Omega} |u|^2 e^{2h} (\lambda - |\nabla h|^2)} &\leq \int_{\Omega} |f|^2 e^{2h} (\lambda - |\nabla h|^2)^{-1} \\ \therefore \int_{\Omega} |u|^2 e^{2h} (\lambda - |\nabla h|^2) &\leq \int_{\Omega} |f|^2 e^{2h} (\lambda - |\nabla h|^2)^{-1} \end{aligned}$$

which is the desired result.

Notice that we did not use (3.6.2) (the growth assumption of u) when we set $\phi = e^h$. This assumption becomes important in the following proof, when we approximate e^h with Lipschitz and compactly supported ϕ .

PROOF: As above, suppose that u is a weak solution to $Lu = f$ on Ω . We have

$$\int_{\Omega} \nabla u \cdot \nabla \varphi + V u \varphi = \int_{\Omega} f \varphi \quad (3.6.5)$$

for $\varphi \in W_0^{1,2}(\Omega)$.

If ϕ is Lipschitz and compactly supported, then $u\phi^2 \in W_0^{1,2}(\Omega)$. Thus, we set φ from (3.6.5) to be $\varphi = u\phi^2$ and get

$$\int_{\Omega} \nabla u \cdot \nabla(u\phi^2) + Vu^2\phi^2 = \int_{\Omega} fu\phi^2.$$

If we rewrite the above using the important identity (1) we get

$$\int_{\Omega} |\nabla(u\phi)|^2 - |u|^2 |\nabla\phi|^2 + Vu^2\phi^2 = \int_{\Omega} fu\phi^2. \quad (3.6.6)$$

As $u\phi^2 \in W_0^{1,2}(\Omega)$ we may invoke (3.6.1), giving

$$\int_{\Omega} |\nabla(u\phi)|^2 + V|u|^2\phi^2 \geq \int_{\Omega} \lambda|u\phi|^2$$

applying this to (3.6.5) we have

$$\int_{\Omega} \lambda|u\phi|^2 - |u|^2 |\nabla\phi|^2 \leq \int_{\Omega} fu\phi^2. \quad (3.6.7)$$

Recall that we are free to choose ϕ Lipschitz and compactly supported. We would like to choose $\phi = e^h$, however this choice of ϕ is not compactly supported. Hence we will approximate e^h with $\phi = e^g\chi$ where g and χ are Lipschitz and χ has compact support with $0 \leq \chi \leq 1$ and $|\nabla g|^2 < \lambda$ a.e.. This choice of g and χ give

$$|\nabla\phi|^2 = e^{2g}\chi^2|\nabla g|^2 + 2e^{2g}\chi|\nabla\chi \cdot \nabla g| + e^{2g}|\nabla\chi|^2$$

and so (3.6.7) implies

$$\int_{\Omega} \lambda|ue^g\chi|^2 - |u|^2(e^{2g}\chi^2|\nabla g|^2 + 2e^{2g}\chi|\nabla\chi \cdot \nabla g| + e^{2g}|\nabla\chi|^2) \leq \int_{\Omega} fu(e^g\chi)^2$$

rearranging gives

$$\begin{aligned} \int_{\Omega} |u\chi|^2(\lambda - |\nabla g|^2)e^{2g} &\leq \int_{\Omega} fu(e^g\chi)^2 \\ &\quad + \int_{\Omega} |u|^2(e^{2g}(2\chi|\nabla\chi \cdot \nabla g| + |\nabla\chi|^2)). \end{aligned}$$

Estimating the terms on the right hand side,

$$\int_{\Omega} fu(e^g\chi)^2 = \int_{\Omega} u\chi(\lambda - |\nabla g|^2)^{\frac{1}{2}}e^g f\chi(\lambda - |\nabla g|^2)^{-\frac{1}{2}}e^g$$

an application of Cauchy Schwarz gives

$$\begin{aligned} \int_{\Omega} u\chi(\lambda - |\nabla g|^2)^{\frac{1}{2}}e^g f\chi(\lambda - |\nabla g|^2)^{-\frac{1}{2}}e^g \\ \leq \left(\int_{\Omega} |u\chi|^2(\lambda - |\nabla g|^2)e^{2g} \right)^{\frac{1}{2}} \left(\int_{\Omega} |f\chi|^2(\lambda - |\nabla g|^2)^{-1}e^{2g} \right)^{\frac{1}{2}} \end{aligned}$$

and thus

$$\begin{aligned} \int_{\Omega} |u\chi|^2(\lambda - |\nabla g|^2)e^{2g} \leq \left(\int_{\Omega} |u\chi|^2(\lambda - |\nabla g|^2)e^{2g} \right)^{\frac{1}{2}} \left(\int_{\Omega} |f\chi|^2(\lambda - |\nabla g|^2)^{-1}e^{2g} \right)^{\frac{1}{2}} \\ + \int_{\Omega} |u|^2(e^{2g}(2\chi|\nabla\chi \cdot \nabla g| + |\nabla\chi|^2)). \end{aligned}$$

This inequality is of the form

$$a \leq a^{\frac{1}{2}}b^{\frac{1}{2}} + c$$

where a , b and c are positive. It can be shown that this implies

$$a \leq b + 2c$$

thus we have

$$\begin{aligned} \int_{\Omega} |u\chi|^2(\lambda - |\nabla g|^2)e^{2g} \leq \int_{\Omega} |f\chi|^2(\lambda - |\nabla g|^2)^{-1}e^{2g} \\ + 2 \int_{\Omega} |u|^2(e^{2g}(2\chi|\nabla\chi \cdot \nabla g| + |\nabla\chi|^2)). \end{aligned} \quad (3.6.8)$$

We note that using this identity does not exclude the case where $b = \infty$.

We've obtained an inequality that appears close to (3.6.3) barring the last term, we construct sequences $\chi_{d,j}$ and h_i which satisfy the hypotheses for χ and g as well as having the property that $\chi_{d,j}e^{h_i}$ approximates e^h . Once we've constructed these functions we will show that the last term can be controlled.

Let $\{K_i\}$ be a sequence of compact sets in Ω such that $K_i \subset K_j$ if $i < j$ and

$$\bigcup_{i=1}^{\infty} \text{int } K_i = \Omega$$

given $d > 0$ we define $\eta_d(t)$ on $[0, \infty)$ by

$$\eta_d(t) = \begin{cases} t/d & \text{if } t \in [0, d] \\ 1 & \text{if } t \in (d, \infty). \end{cases}$$

Let

$$\chi_{d,j}(x) = \eta_d(\rho_\lambda(x, \Omega \setminus K_j)).$$

By how we've defined $\chi_{d,j}$, for x outside of K_j , $\chi_{d,j}(x) = 0$. Hence $\chi_{d,j}$ has compact support.

Also, by definition, for $x \in K_j$ we either have $\chi_{d,j}(x) = 1$ when x is further than d in the metric ρ_λ to exterior K_j , or $\chi_{d,j}(x) = \frac{\rho_\lambda(x, \Omega \setminus K_j)}{d}$ when x is closer than d in the metric ρ_λ to exterior K_j .

To see that $\chi_{d,j}$ is Lipschitz, for $t, s \in [0, \infty)$ we have

$$|\eta_d(t) - \eta_d(s)| \leq \frac{|t - s|}{d}$$

which implies

$$\begin{aligned} |\chi_{d,j}(x) - \chi_{d,j}(y)| &\leq \frac{|\rho_\lambda(x, \Omega \setminus K_j) - \rho_\lambda(y, \Omega \setminus K_j)|}{d} \\ &\leq \frac{\rho_\lambda(x, y)}{d} && \text{(triangle inequality)} \\ &\leq \frac{C}{d} |x - y|. && \text{(recall } \rho_\lambda \text{ is Lipschitz by 3.1.1)} \end{aligned}$$

Hence $\chi_{d,j}$ can be used in (3.6.8). Furthermore, by the above application of the triangle inequality, we also have that $d\chi_{d,j}$ satisfies Lemma 3.1. Applying Lemma 3.1 we have

$$\begin{aligned} d^2 |\nabla_A \chi_{d,j}|^2 &\leq \lambda \\ \therefore |\nabla_A \chi_{d,j}|^2 &\leq d^{-2} \lambda. \end{aligned} \tag{3.6.9}$$

We have a suitable candidate for χ , we move to constructing h_i . Recall the growth assumption on u , that is, there exists $\delta > 0$ such that

$$\int_{\Omega} |u|^2 \lambda e^{-2(1-\delta)\rho_\lambda} < \infty.$$

Without loss of generality we can assume $0 < \delta < 1$ (if $\delta \geq 1$, any $0 < \delta' < 1$ satisfies the above). We use this δ to define the following sequence of functions. For $i \in \mathbb{N}$ set

$$f_i(x) = -1(1 - \delta)\rho_\lambda(x) + i.$$

We define

$$h_i(x) = \min\{h(x), f_i\}$$

for $i \in \mathbb{N}$. Notice that h_i is Lipschitz as the minimum of Lipschitz functions.

To estimate derivatives of h_i observe that for any i , the functions h , h_i and $-1(1 - \delta)\rho_\lambda(x) + i$ are all Lipschitz. Hence the differentials of all three functions exist a.e.. We claim that when it exists, ∇h_i is equal to either ∇h or ∇f_i .

Let x^0 be a point where ∇h , ∇f_i and ∇h_i exist. When $f_i(x^0) \neq h(x^0)$, we either have $\nabla h_i = \nabla h$ or $\nabla h_i = \nabla f_i$, verifying the claim. When $f_i(x^0) = h(x^0)$, for $\omega \in \mathbb{R}^n$ and $t > 0$ small enough we have

$$\frac{h_i(x^0 + t\omega) - h_i(x^0)}{t} \leq \frac{h(x^0 + t\omega) - h(x^0)}{t}$$

and as $t \rightarrow 0$ we have

$$\langle \omega, [\nabla h_i(x^0) - \nabla h(x^0)] \rangle \leq 0$$

for all $\omega \in \mathbb{R}^n$. Hence we must have $\nabla h_i(x^0) = \nabla h(x^0)$.

So for any x^0 where ∇h , ∇f_i and ∇h_i all exist either

$$|\nabla h_i|^2 = |\nabla h|^2$$

or

$$\begin{aligned} |\nabla h_i|^2 &= |\nabla f_i|^2 \\ &= (1 - \delta)^2 |\nabla \rho_\lambda|^2. \end{aligned}$$

Recall that $|\nabla h|^2 < \lambda$ a.e. and by Lemma 3.1 we have

$$\begin{aligned} (1 - \delta)^2 |\nabla \rho_\lambda|^2 &\leq (1 - \delta)^2 \lambda \\ &\leq \lambda \end{aligned} \quad (\text{since } \delta < 1)$$

almost everywhere. Thus we conclude

$$|\nabla h_i|^2 < \lambda \tag{3.6.10}$$

almost everywhere, and hence h_i is a suitable choice for g in (3.6.8). If we combine the estimates (3.6.10) and (3.6.9) we obtain

$$\begin{aligned} |\nabla \chi_{d,j} \cdot \nabla h_i| &\leq |\nabla \chi_{d,j}| |\nabla h_i| && \text{(Cauchy Schwarz)} \\ &\leq \frac{\lambda}{d}. && (3.6.11) \end{aligned}$$

Replacing χ with $\chi_{d,j}$ and g by h_i in (3.6.8) we obtain

$$\begin{aligned} \int_{\Omega} |u \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2) e^{2h_i} &\leq \int_{\Omega} |f \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} \\ &\quad + 2 \int_{\Omega} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)). \end{aligned} \quad (3.6.12)$$

Now, $\nabla \chi_{d,j} = 0$ except on $K_j \setminus K_{j,d}$ where $K_{j,d} = \{x \in K_j : \rho_{\lambda}(x, \Omega \setminus K_j) > d\}$. Let $\mathcal{K}_{j,d}$ be the characteristic function of $K_j \setminus K_{j,d}$. Since any compact set K is contained in the interior of K_j for large enough j we have

$$\lim_{j \rightarrow \infty} \mathcal{K}_{j,d}(x) = 0$$

for all $x \in \Omega_d$ since Ω_d is complete in ρ_{λ} . We can rewrite the final term of (3.6.12)

$$\begin{aligned} &2 \int_{\Omega} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) \\ &= 2 \int_{\Omega_d} \mathcal{K}_{j,d} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) \\ &\quad + 2 \int_{\Omega_d} (1 - \mathcal{K}_{j,d}) |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) \quad (\star) \\ &\quad + 2 \int_{\Omega \setminus \Omega_d} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) \end{aligned}$$

since $\mathcal{K}_{j,d}$ as $j \rightarrow \infty$ we have

$$2 \int_{\Omega_d} \mathcal{K}_{j,d} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) = 0.$$

Recall that h_i satisfies

$$\int_{\Omega} |u|^2 \lambda e^{2h_i} < \infty.$$

and $|\nabla \chi_{d,j}|^2$, $|\nabla \chi_{d,j} \cdot \nabla h_i|$ are finite by (3.6.9) and (3.6.11) we can apply the dominated convergence theorem to the second integral of the right hand side of $(\star)$. Letting $j \rightarrow \infty$ we find that

$$2 \int_{\Omega_d} (1 - \mathcal{K}_{j,d}) |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) = 0.$$

This implies that letting $j \rightarrow \infty$ in (3.6.12) gives

$$\begin{aligned} \limsup_{j \rightarrow \infty} \int_{\Omega} |u \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2) e^{2h_i} &\leq \limsup_{j \rightarrow \infty} \int_{\Omega} |f \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} \\ &\quad + 2 \limsup_{j \rightarrow \infty} \int_{\Omega \setminus \Omega_d} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)). \end{aligned}$$

Using estimates (3.6.9), (3.6.11) and $h_i \leq h$ for all i we deduce that

$$2 \limsup_{j \rightarrow \infty} \int_{\Omega \setminus \Omega_d} |u|^2 (e^{2h_i} (2\chi_{d,j} |\nabla \chi_{d,j} \cdot \nabla h_i| + |\nabla \chi_{d,j}|^2)) \leq \frac{2(1+2d)}{d^2} \int_{\Omega \setminus \Omega_d} e^{2h} |u|^2 \lambda$$

and because $0 \leq \chi_{d,j} \leq 1$ we also have

$$\limsup_{j \rightarrow \infty} \int_{\Omega} |f \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} \leq \int_{\Omega} |f|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i}.$$

Putting the above estimates together we conclude

$$\begin{aligned} \limsup_{j \rightarrow \infty} \int_{\Omega} |u \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2) e^{2h_i} \\ \leq \int_{\Omega} |f|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} + \frac{2(1+2d)}{d^2} \int_{\Omega \setminus \Omega_d} e^{2h} |u|^2 \lambda. \end{aligned} \quad (3.6.13)$$

This is starting to appear close to (3.6.4). We note that $|u \chi_{d,j}| \rightarrow |u|$ because $\chi_{d,j} \rightarrow 1$ on Ω as $j \rightarrow \infty$. Hence we apply Fatou's lemma to obtain

$$\int_{\Omega} |u|^2 (\lambda - |\nabla h_i|^2) e^{2h_i} \leq \limsup_{j \rightarrow \infty} \int_{\Omega} |u \chi_{d,j}|^2 (\lambda - |\nabla h_i|^2) e^{2h_i}.$$

Combining this with (3.6.13) yields

$$\int_{\Omega} |u|^2 (\lambda - |\nabla h_i|^2) e^{2h_i} \leq \int_{\Omega} |f|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} + \frac{2(1+2d)}{d^2} \int_{\Omega \setminus \Omega_d} e^{2h} |u|^2 \lambda. \quad (3.6.14)$$

We claim that

$$\lim_{i \rightarrow \infty} \int_{\Omega} |f|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} = \int_{\Omega} |f|^2 (\lambda - |\nabla h|^2)^{-1} e^{2h} \quad (3.6.15)$$

This would follow by the dominated convergence theorem if

$$(\lambda - |\nabla h_i|^2)^{-1} \leq \delta^{-1} (\lambda - |\nabla h|^2)^{-1} \quad (*)$$

and

$$\delta^{-1} \int_{\Omega} |f|^2 (\lambda - |\nabla h|^2)^{-1} e^{2h} < \infty. \quad (**)$$

We may assume $(**)$ since otherwise, $\int_{\Omega} |f|^2 (\lambda - |\nabla h|^2)^{-1} e^{2h} = \infty$ and so (3.6.3) and (3.6.4) are trivially true.

We notice that $(*)$ is true when $|\nabla h_i|^2 = |\nabla h|^2$. If not, we must have $|\nabla h_i|^2 = (1 - \delta)^2 |\nabla \rho_{\lambda}|^2$, implying

$$\begin{aligned}
 \lambda - |\nabla h_i|^2 &= \lambda - (1 - \delta)^2 |\nabla \rho_{\lambda}|^2 \\
 &\geq \lambda - (1 - \delta)^2 \lambda. && (\text{since } \lambda \geq |\nabla \rho_{\lambda}|^2 \text{ by Lemma 3.1}) \\
 &= \delta \lambda (2 - \delta) \\
 &\geq \delta \lambda && (\text{since } \delta < 1) \\
 &\geq \delta (\lambda - |\nabla h|^2).
 \end{aligned}$$

In both cases $(\star)$ holds, so (3.6.15) follows by the dominated convergence theorem. Recall that

$$\int_{\Omega} |u|^2 (\lambda - |\nabla h|^2) e^{2h} \leq \int_{\Omega} |u|^2 (\lambda - |\nabla h_i|^2) e^{2h_i}$$

for every i . Applying Fatou's lemma to the right hand side, we deduce

$$\begin{aligned}
 \int_{\Omega} |u|^2 (\lambda - |\nabla h|^2) e^{2h} &\leq \limsup_{i \rightarrow \infty} \int_{\Omega} |u|^2 (\lambda - |\nabla h_i|^2) e^{2h_i} && (\text{by 3.6.14}) \\
 &\leq \limsup_{i \rightarrow \infty} \int_{\Omega} |f|^2 (\lambda - |\nabla h_i|^2)^{-1} e^{2h_i} \\
 &\quad + \frac{2(1+2d)}{d^2} \int_{\Omega \setminus \Omega_d} |u(x)|^2 \lambda(x) e^{h(x)} \\
 &= \int_{\Omega} |f|^2 (\lambda - |\nabla h|^2)^{-1} e^{2h} \\
 &\quad + \frac{2(1+2d)}{d^2} \int_{\Omega \setminus \Omega_d} |u(x)|^2 \lambda(x) e^{h(x)} && (\text{by (3.6.15)})
 \end{aligned}$$

This completes the proof for (3.6.4). Notice that (3.6.3) follows since if Ω is complete in ρ_{λ} then $\Omega = \Omega_d$.

□

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