Schrödinger Operators and the Kato Square Root Problem

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Declaration

I hereby declare that the material in this thesis is my own original work except where stated otherwise.

Part I of this thesis has been published as “The Hardy-Littlewood Operator Adapted to the Harmonic Oscillator” in Revista de la Unión Matemática Argentina [12].

Chapters 3 and 5 are based on the material in the preprint [13] and will be submitted for publication at a later date.

Julian Bailey

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Abstract

The general theme of this thesis is the harmonic analysis of Schrödinger operators and its applications. We will focus on two distinct but related open problems in this field.

The first problem is the construction of potential dependent averaging operators and will be primarily considered in the first part of this thesis. Here, a Hardy-Littlewood type maximal operator adapted to the Schrödinger operator $\mathcal{L} := -\Delta + |x|^2$ and acting on $L^2(\mathbb{R}^n)$ is constructed. This is achieved through the use of the Gaussian grid Δ_0^γ, constructed in [42] with the Ornstein-Uhlenbeck operator in mind. At the scale of this grid, the maximal operator will resemble the classical Hardy-Littlewood operator. At a larger scale, the constituent averaging operators of the maximal function are decomposed over the cubes from Δ_0^γ and weighted appropriately. Through this maximal function, a new class of weights is defined, A_p^+, with the property that for any $w \in A_p^+$ the heat maximal operator associated with $\mathcal{L}$ is bounded from $L^p(w)$ to itself. This class contains any other known class that possesses this property and contains weights of exponential growth. In particular, it is strictly larger than A_p.

The second problem that we consider is the Kato square root problem for divergence form elliptic operators with potential $V : \mathbb{R}^n \to \mathbb{C}$. This is the equivalence statement $\left\|(L+V)^{\frac{1}{2}} u\right\|_{L^2(\mathbb{R}^n)} \simeq \|\nabla u\|_{L^2(\mathbb{R}^n)} + \left\|V^{\frac{1}{2}} u\right\|_{L^2(\mathbb{R}^n)}$, where $L + V := -\mathrm{div}\,(A\nabla) + V$ and the perturbation A is an L^∞ complex matrix-valued function satisfying an ellipticity condition. One possible path to a solution for this problem is by proving square function estimates for perturbations of associated non-homogeneous Dirac-type operators. At present, there is no general method to obtain such square function estimates other than for potentials bounded both from above and below (cf. [10]). We develop such a method by adapting the homogeneous framework introduced by A. Axelsson, S. Keith and A. McIntosh in their seminal paper [11]. Two distinct approaches will be considered when adapting this framework. The second such approach will yield a satisfying solution to the potential dependent Kato problem for a large class of potentials. This class will include any potential V with range contained in some sector of angle $\omega_V \in [0, \frac{\pi}{2})$ and for which $|V|$ belongs either to the reverse Hölder class RH_2 in any dimension or $L^{\frac{n}{2}}(\mathbb{R}^n)$ for $n > 4$.

Contents

Introduction

Classical harmonic analysis is inextricably linked to the Laplacian operator Δ. Many of its fundamental objects are intimately related to the functional calculus of this all important operator. A current area of active research is the study of the harmonic analysis associated with differential operators other than the Laplacian. A first step in this line of inquiry is to attempt to define direct counterparts to the fundamental objects from classical harmonic analysis in this new setting. Following this, one might then go on to ask whether these adapted objects share similar properties to their classical counterparts. In this thesis, the differential operators of primary interest are Schrödinger operators.

Throughout this thesis, n will be used to denote the dimension of the underlying Euclidean space $\mathbb{R}^n$. For a locally integrable function $V : \mathbb{R}^n \to \mathbb{C}$ with range contained in the right-half plane, one can define the sum

$$\mathcal{L}_V := V - \Delta$$

viewed as an unbounded operator on $L^2(\mathbb{R}^n)$. This operator can be defined rigorously through its corresponding sesquilinear form. $\mathcal{L}_V$ is called the Schrödinger operator for the potential V, named in deference to its indelible role in quantum mechanics and its appearance in the Schrödinger equation.

Part I: Potential Dependent Averaging Operators

Consider some of the standard operators affiliated with the Laplacian such as the heat operator

$$P_t := e^{t\Delta} \qquad t > 0,$$

and the Riesz transform

$$R := \nabla (-\Delta)^{-\frac{1}{2}}.$$

These operators can be defined rigorously through the functional calculus of the Laplacian (see for example §3.1). For such operators, it seems almost obvious how one could define potential dependent counterparts. Simply swap the negative Laplacian with the

Schrödinger operator to obtain the potential dependent heat operator

$$P_t^V := e^{-t\mathcal{L}_V} \qquad t > 0,$$

and Riesz transform

$$R_V := \nabla \left(\mathcal{L}_V\right)^{-\frac{1}{2}}.$$

The dyadic cubes of $\mathbb{R}^n$, labelled Δ, is a collection of cubes defined as follows,

$$\Delta_m := \{2^m \left(k + (0,1]^n\right) : k \in \mathbb{Z}^n\} \;\; for \; m \in \mathbb{Z}, \qquad \Delta_t := \Delta_m \; for \; t \in \left(2^{m-1}, 2^m\right],$$

$$\Delta := \bigcup_{m=-\infty}^{\infty} \Delta_m.$$

The notation Δ is ambiguously used for both the Laplacian and the system of dyadic cubes but it will be clear from context which object is being referred to. For $f \in L^1_{loc}\left(\mathbb{R}^n\right)$ and $t > 0$, define the dyadic averaging operator

$$A_t f(x) := \frac{1}{Q(t,x)} \int_{Q(t,x)} f(y)\, dy, \tag{1}$$

where $Q(t,x)$ is the unique dyadic cube in Δ_t that contains the point $x \in \mathbb{R}^n$. It is evident from their definition that the averaging operators are not easily expressible in terms of the functional calculus of the Laplacian. Thus a potential dependent counterpart cannot be defined through the simple direct swap process that was used to define the potential dependent heat operator and Riesz transform. A recurring theme throughout this thesis is the development of averaging operators more suited to the Schrödinger operator setting.

Averaging operators are ubiquitous in classical harmonic analysis. From the Lebesgue differentiation theorem to Calderón Zygmund theory, the importance of these ever-present operators can hardly be overstated. It then stands to reason that the discovery of potential dependent averaging operators would be of pivotal importance to the advancement of the theory of Schrödinger operators through the transfer of numerous ideas and proofs from the classical setting. In addition, these operators could also illuminate the underlying intrinsic geometry of the Schrödinger operator. Averaging operators should ultimately be thought of as being connected to the underlying intrinsic geometry of the differential operator under consideration. Our lack of knowledge of the averaging operator for $\mathcal{L}_V$ thus reflects our lack of knowledge of the geometry of Schrödinger operators.

The dyadic Hardy-Littlewood maximal operator is the sublinear operator defined through

$$Mf(x) := \sup_{t>0} A_t \left|f\right|(x)$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Similar to its constituent averaging operators, the Hardy-Littlewood operator plays a very influential role in classical harmonic analysis. Since no appropriate analogue for the averaging operators has yet been discovered, at present there is no suitable analogue for the Hardy-Littlewood operator in the Schrödinger operator setting. The development of an adapted Hardy-Littlewood operator can be seen as almost equivalent to the development of averaging operators

The first part of this thesis is dedicated to the construction of a Hardy-Littlewood operator, and thus averaging operators, for the Schrödinger operator with harmonic oscillator potential,

$$\mathcal{L} := \mathcal{L}_{|x|^2} := |x|^2 - \Delta.$$

It is hoped that such a construction for the specific potential $V(x) = |x|^2$ can be used as a prototype to model the development of averaging operators adapted to a more general potential. The harmonic oscillator potential has been chosen on the grounds that $\mathcal{L}$ is perhaps the simplest example of a Schrödinger operator with unbounded potential and in particular the heat kernel of $\mathcal{L}$ is known precisely. This exact information on the heat kernel will form a solid foundation on which we can base our construction.

Part II: The Kato Square Root Problem with Potential

For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ denote the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and set $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ and suppose that the ellipticity condition

$$\int_{\mathbb{R}^n} \langle A\nabla u, \nabla u\rangle \, dx \geq \kappa_A \|\nabla u\|^2 \tag{2}$$

is satisfied for some $\kappa_A > 0$, for all $u \in H^1(\mathbb{R}^n) := W^{1,2}(\mathbb{R}^n)$. Consider the divergence form operator

$$L := -\mathrm{div} A\nabla$$

viewed as an unbounded operator on $L^2(\mathbb{R}^n)$. Such an operator can be defined rigorously through its corresponding sesquilinear form. For operators of this form, it is possible to define a square root operator $\sqrt{L}$, with domain $D(L)$, that satisfies $\sqrt{L} \cdot \sqrt{L} = L$. The domain $D(L)$ is not a natural domain of definition for the square root operator since it is a second-order domain and the square root operator is first-order and since the operator $\sqrt{L}$ is not closed on $D(L)$. The Kato square root problem asks what is the natural domain of definition of the square root operator or, equivalently, is the square root operator closable and what is the domain of the closure? This problem, first posed by Tosio Kato over 50 years ago, was conjectured to have the following solution.

Theorem (Kato Square Root)**.** *The natural domain of* $\sqrt{L}$ *is*

$$D(\sqrt{L}) = H^1\left(\mathbb{R}^n\right).$$

In particular, for any $u \in H^1\left(\mathbb{R}^n\right)$

$$\left\|\sqrt{L}u\right\| \simeq \|\nabla u\|. \tag{3}$$

This long-standing problem withstood solution until 2002 where it was proved using local $T(b)$ methods by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [7]. Four years following this, building upon the technology developed in [7], Andreas Axelsson, Stephen Keith and Alan McIntosh developed a general framework for proving quadratic estimates for perturbations of Dirac-type operators in [11]. The Kato problem was shown to be an easy consequence of this framework.

Many classical problems from harmonic analysis have a direct counterpart in the Schrödinger operator setting. Adhering with this theme, in the second part of this thesis we consider the Kato square root problem with potential. In a potential dependent context the Kato square root problem takes the form of the following conjecture.

Conjecture (Kato Square Root with Potential)**.** *Let* $V : \mathbb{R}^n \to \mathbb{C}$ *be a locally integrable function with range contained in the right-half plane. Then there must exist some constant* $C_V > 0$ *such that for all* $u \in D\left(L+V\right)$*,*

$$C_V^{-1}\left(\left\|V^{\frac{1}{2}}u\right\| + \|\nabla u\|\right) \leq \left\|\sqrt{L+V}u\right\| \leq C_V\left(\left\|V^{\frac{1}{2}}u\right\| + \|\nabla u\|\right). \tag{KP}$$

This problem is actually a statement concerning the domain of the square root operator $\sqrt{L+V}$. Indeed, (KP) implies the equality

$$D\left(\sqrt{L+V}\right) = H^{1,V}\left(\mathbb{R}^n\right) := H^1\left(\mathbb{R}^n\right) \cap D\left(|V|^{\frac{1}{2}}\right).$$

In this thesis, the Kato problem with potential will be solved for a large class of potentials that includes any potential V with range contained in some sector of angle $\omega_V \in [0, \frac{\pi}{2})$ and for which $|V|$ belongs either to the reverse Hölder class RH_2 in any dimension or $L^{\frac{n}{2}}\left(\mathbb{R}^n\right)$ for $n > 4$. This will be achieved by adapting the proof by Axelsson, Keith and McIntosh in [11]. The potential free proofs of both [7] and [11] involve habitual use of the averaging operators A_t. Adapting the argument from [11] will thus once again reintroduce us to the question of how to adapt the averaging operators in an appropriate manner or, alternatively, how to circumvent their use in order to transfer the classical proof to the potential dependent setting.

I

Potential Dependent Averaging Operators

Abstract

This part of the thesis sets out to construct a Hardy-Littlewood maximal operator, and thus averaging operators, adapted to the Schrödinger operator with harmonic oscillator potential,

$$\mathcal{L} := \mathcal{L}_{|x|^2} := |x|^2 - \Delta.$$

This is achieved through the use of the Gaussian grid Δ_0^γ, constructed in [42] with the Ornstein-Uhlenbeck operator in mind. At the scale of this grid, the maximal operator will resemble the classical Hardy-Littlewood operator. At a larger scale, the constituent averaging operators of the maximal function are decomposed over the cubes from Δ_0^γ and weighted appropriately.

There is a natural correspondence between Hardy-Littlewood type maximal operators and generalised Muckenhoupt weight classes. As such, through the construction of an adapted Hardy-Littlewood type maximal operator associated with $\mathcal{L}$, we will obtain an adapted Muckenhoupt class, A_p^+, affilitated with the harmonic oscillator potential. This weight class will have the property that for any $w \in A_p^+$ the heat maximal operator associated with $\mathcal{L}$ is bounded from $L^p(w)$ to itself. Moreover, it will be proved that this class contains any other known class that possesses this property and contains weights of exponential growth. In particular, it will be shown that our weight class is strictly larger, and thus closer to optimal, than the classical Muckenhoupt class A_p and another adapted weight class, $A_p^{V,\infty}$, developed in [15].

Chapter 1

Prelude

In this chapter we will provide background material required to understand and motivate our construction of an adapted Hardy-Littlewood operator. This includes a brief survey of classical weight theory, Schrödinger operators, the reverse Hölder class of potentials, the adapted weight class $A_p^{V,\infty}$ and the adapted space of functions $BMO_{V,\infty}$ as defined in [16]. The final section of this chapter will provide an original proof of the fact that the class $A_p^{V,\infty}$ does not satisfy an exponential-log link with $BMO_{V,\infty}$, a well known property that holds for the potential free counterparts. This gives further credence to the argument that the adapted weight class $A_p^{V,\infty}$, and by the transitive property the Hardy-Littlewood operator attached to $A_p^{V,\infty}$, is not optimal in a sense that will be discussed in §1.4. The details of our adapted operator and the weight class affilitated with it will be provided in the chapter to follow.

Notation. Throughout this part of the thesis, the notation $A \lesssim B$ and $A \simeq B$ will be used to denote that there exists $C > 0$ for which $A \leq C \cdot B$ and $C^{-1} \cdot B \leq A \leq C \cdot B$ respectively.

Acknowledgements

This part of the thesis is inspired by discussions of my supervisor Pierre Portal with Paco Villarroya, aiming to better understand how to adapt harmonic analysis to the hidden geometry of a differential operator. In most cases, this involves situations beyond the reach of Calderón-Zygmund theory (see e.g. [43, 35]). However, this can also be done within Calderón-Zygmund theory by proving stronger properties of smaller classes of singular integral operators than the Calderón-Zygmund class. Paco Villaroya has particularly focused on describing compact (as opposed to merely bounded) singular integral operators (see e.g. [31]). To do so, he had to refine classical dyadic approaches, in order to understand which cubes particularly affect compactness. I follow a similar path here, modifying standard dyadic arguments in a way that aims to reveal the "hidden geome-

try" of the harmonic oscillator. This is done by attempting to find the largest class of weights for which the corresponding heat maximal operator is bounded. Perhaps surprisingly, such a question seems to be rarely formulated in the context of standard weighted Calderón-Zygmund theory (generic questions involving all singular integrals and all related maximal functions are considered instead), but quite common in the context of two weights inequalities (where studying just the Hilbert transform is hard enough, and natural).

I would like to thank the anonymous referee of the article version of this part of the thesis for their suggestion that the improvement of the inclusion $A_p^{V,\infty} \subset A_p^+$ to a strict inclusion would be of benefit to the article if it could be proved.

1.1. Classical Weight Theory

A weight on $\mathbb{R}^n$ is any non-negative measurable function $w : \mathbb{R}^n \to \mathbb{R}$ that is non-zero almost everywhere. For a weight w and $p \in [1, \infty)$, let $L^p(w)$ denote the weighted Lebesgue space

$$L^p(w) := L^p(\mathbb{R}^n, w) := \left\{ f : \mathbb{R}^n \to \mathbb{C} : \|f\|_{L^p(w)} := \int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx < \infty \right\}.$$

The weighted weak Lebesgue space is defined through

$$L^{p,\infty}(w) := L^{p,\infty}(\mathbb{R}^n, w) := \left\{ f : \mathbb{R}^n \to \mathbb{C} : \|f\|_{L^{p,\infty}(w)} := \sup_{\lambda > 0} \lambda \cdot w\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right)^{\frac{1}{p}} < \infty \right\},$$

where $w(E) := \int_E w(x) \, dx$ for a measurable subset $E \subset \mathbb{R}^n$. Weight theory is a branch of harmonic analysis that considers inequalities of the form

$$\|Tf\|_{L^p(w)} \leq C \cdot \|f\|_{L^q(\nu)} \quad \textit{and} \quad \|Tf\|_{L^{p,\infty}(w)} \leq C \cdot \|f\|_{L^q(\nu)},$$

for weights w and ν on $\mathbb{R}^n$, $1 \leq p,\, q \leq \infty$, constant $C > 0$ and some operator T. By far the most important object in classical weight theory is the Muckenhoupt class of weights. We recall the definition of this fundamental weight class and some of the more pertinent results connected to it. For a more thorough account of weight theory the reader is referred to [30], [27], or [52]. For a contemporary survey of weight theory refer to [48].

Definition 1.1.1. *A weight w on $\mathbb{R}^n$ is said to belong to the Muckenhoupt class A_p for $p \in (1, \infty)$ if there exists $C > 0$ that satisfies*

$$w(B)^{\frac{1}{p}} \cdot w^{-\frac{1}{p-1}}(B)^{\frac{p-1}{p}} \leq C \, |B|$$

for all balls $B \subset \mathbb{R}^n$. w is said to belong to A_1 if there exists $C > 0$ for which

$$\frac{w(B)}{|B|} \leq C \cdot w(x)$$

for all balls $B \subset \mathbb{R}^n$ and $x \in B$. Finally, w is said to belong to A_∞ if there exists $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\exp\left(\frac{1}{|B|}\int_B \log w(x)^{-1}\,dx\right) \leq C$$

for all balls $B \subset \mathbb{R}^n$. The smallest constant C for which the A_p inequality is satisfied for $p \in [1,\infty]$ is denoted by $[w]_{A_p}$.

Remark 1.1.1. In the above definition, balls in $\mathbb{R}^n$ can be swapped for cubes and this will not alter the resulting weight class. See [52, §V.1.6] for further details.

The Muckenhoupt classes $\{A_p\}_{p\in[1,\infty]}$ form a nested increasing collection of weight classes that satisfy a self-improvement property in the following sense.

Theorem 1.1.1 ([30, Cor. 9.2.6])**.** *For $1 \leq p \leq q \leq \infty$ we have*

$$A_1 \subset A_p \subset A_q \subset A_\infty.$$

Let $w \in A_p$. There exists some $\varepsilon > 0$ for which $w \in A_{p-\varepsilon}$. In other words,

$$A_p = \bigcup_{p'\in[1,p)} A_{p'}.$$

Example 1.1.1. We list some typical examples of Muckenhoupt weights for the reader to gain some insight into the type of weights that are contained within this class.

1. Any weight bounded both from above and below must be contained in A_p for any $p \in [1,\infty]$.

2. The weight $|x|^\alpha$ is contained in A_p for $\alpha \in (-n, n\,(p-1))$ (cf. [30, Exm. 9.1.7]).

3. For any polynomial P, the function $|P|$ is contained in A_∞ (cf. [30, pg. 308]).

4. The function
$$w_0(x) := \begin{cases} \log\frac{1}{|x|} & \textit{when } |x| < \frac{1}{e}, \\ 1 & \textit{otherwise}, \end{cases}$$
is contained in A_p for all $p \in [1,\infty]$. This example is taken from [30, Exm. 9.1.8]

One does not have to search for too long to encounter examples of functions that are not contained in A_p. Simply consider functions that either grow or decay at a fast rate.

1. The weight $w(x) = |x|^{\alpha}$ is not contained in A_p for $\alpha \leq -n$ or $\alpha \geq n\,(p-1)$.

2. The exponential function $w(x) = e^{|x|}$ is not contained in A_∞.

From looking at the previous examples, it becomes apparent that the spirit of the Muckenhoupt inequalities is to characterise weights that neither grow nor decay too rapidly on average. The constant $[w]_{A_p}$ thus quantifies the rate of growth and decay. If the growth or decay of a weight is too great then $[w]_{A_p} = \infty$ for all $p \in [1,\infty]$ and the weight will not be Muckenhoupt. Another way to characterise this attribute is through the bounded mean oscillation property.

Definition 1.1.2. *A function $u \in L^1_{loc}(\mathbb{R}^n)$ is said to be of bounded mean oscillation if there exists a constant $C > 0$ such that*

$$\frac{1}{|B|}\int_B |u(y) - \langle u\rangle_B|\, dy \leq C$$

for all balls $B \subset \mathbb{R}^n$, where the notation $\langle u\rangle_E$ is used to denote the average of u over a subset $E \subset \mathbb{R}^n$. The space of all functions of bounded mean oscillation on $\mathbb{R}^n$ is denoted by $BMO(\mathbb{R}^n)$, or more compactly BMO.

A weight will not decay or grow too rapidly if and only if its logarithm does not oscillate too wildly. This connection between the Muckenhoupt weights and the function space BMO is called the exponential-log link.

Theorem 1.1.2 (Exponential-Log Link, [52])**.** *A weight w is in the class A_∞ if and only if* $\log(w)$ *is contained in BMO.*

Undoubtedly one of the most important theorem from classical weight theory is Muckenhoupt's theorem. This result characterises the A_p class in terms of the boundedness of the Hardy-Littlewood maximal operator.

Theorem 1.1.3 (Muckenhoupt's Theorem, [52])**.** *Let w be a weight on $\mathbb{R}^n$. For $p \in (1,\infty)$,*

$$w \in A_p \quad \Leftrightarrow \quad \|M\|_{L^p(w)\to L^p(w)} < \infty.$$

Moreover,

$$w \in A_1 \quad \Leftrightarrow \quad \|M\|_{L^1(w)\to L^{1,\infty}(w)} < \infty.$$

Let $e^{t\Delta}$ be the heat operator for the Laplacian for $t > 0$. For a rigorous definition of this operator using holomorphic functional calculus refer to §3.1. For now, it suffices to know that this is an integral operator with representation

$$e^{t\Delta}f(x) = \int_{\mathbb{R}^n} h_t(x,y)\cdot f(y)\,dy,$$

where $h_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the heat kernel for the Laplacian given by

$$h_t(x,y) := \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

See for example [24, pg. 46] for a derivation of this kernel. Another fundamental operator from classical harmonic analysis is the heat maximal operator. For $f \in L^1_{loc}(\mathbb{R}^n)$, this is the sublinear operator defined through

$$T^* f(x) := \sup_{t>0} e^{t\Delta} |f|(x)$$

for $x \in \mathbb{R}^n$. As it turns out, Muckenhoupt's theorem is not simply restricted to the Hardy-Littlewood operator. The weight class A_p can also be characterised in terms of this heat maximal operator or, in fact, any Calderón-Zygmund operator.

Theorem 1.1.4 ([52]). *Let w be a weight on $\mathbb{R}^n$, $p \in (1,\infty)$ and S be a Calderón-Zygmund operator on $\mathbb{R}^n$ in the sense of [30, Def. 8.1.8]. Then*

$$w \in A_p \quad \Leftrightarrow \quad \|T^*\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftrightarrow \quad \|M\|_{L^p(w)\to L^p(w)} < \infty$$

$$\Leftrightarrow \quad \|S\|_{L^p(w)\to L^p(w)} < \infty.$$

Moreover,

$$w \in A_1 \quad \Leftrightarrow \quad \|T^*\|_{L^1(w)\to L^{1,\infty}(w)} < \infty \quad \Leftrightarrow \quad \|M\|_{L^1(w)\to L^{1,\infty}(w)} < \infty$$

$$\Leftrightarrow \quad \|S\|_{L^1(w)\to L^{1,\infty}(w)} < \infty.$$

It is this fundamental characterisation of A_p weights that our construction of the Hardy-Littlewood operator for $\mathcal{L}$ will be based on.

1.2. Schrödinger Operators

The Schrödinger operator $\mathcal{L}_V$ has previously been defined to be the formal sum of the negative Laplacian $-\Delta$ and the potential V. However, as it stands this object is not well-defined and the domain of definition is not clear. As this is a thesis concerning Schrödinger operators it is important to provide a precise and rigorous definition of these operators.

Let $V : \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function with range contained in the right half-plane. V can be viewed as a densely defined closed multiplication operator on $L^2(\mathbb{R}^n)$ with domain

$$D(V) = \left\{u \in L^2(\mathbb{R}^n) : V \cdot u \in L^2(\mathbb{R}^n)\right\}.$$

The density of $D(V)$ follows from the measurability of V. Define the subspace

$$H^{1,V}(\mathbb{R}^n) := H^1(\mathbb{R}^n) \cap D\left(V^{\frac{1}{2}}\right) := \left\{u \in H^1(\mathbb{R}^n) : V^{\frac{1}{2}} \cdot u \in L^2(\mathbb{R}^n)\right\}.$$

Here the complex square root $V^{\frac{1}{2}}$ is defined via the principal branch cut along the negative real axis. To see that $H^{1,V}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, simply note that since $V \in L^1_{loc}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |V|\,|\varphi|^2 \le \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \int_{\operatorname{supp}\varphi} |V| < \infty$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. This shows that $C_0^\infty(\mathbb{R}^n) \subset D\left(|V|^{\frac{1}{2}}\right)$. Since $C_0^\infty(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it then follows that $H^{1,V}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

Consider the sesquilinear form $\mathfrak{l}_V : H^{1,V}(\mathbb{R}^n) \times H^{1,V}(\mathbb{R}^n) \to \mathbb{C}$ defined through

$$\mathfrak{l}_V[u,v] := \int_{\mathbb{R}^n} \langle \nabla u(x), \nabla v(x)\rangle\, dx + \int_{\mathbb{R}^n} \langle V(x)u(x), v(x)\rangle\, dx$$

for u, $v \in H^{1,V}(\mathbb{R}^n)$. A well-known representation theorem from classical form theory (cf. [39, Thm. VI.2.1]) asserts the existence of an associated linear operator $\mathcal{L}_V : D(\mathcal{L}_V) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for which

$$\mathfrak{l}_V[u,v] = \langle \mathcal{L}_V u, v\rangle$$

for all $v \in H^{1,V}(\mathbb{R}^n)$ and u in the domain of $\mathcal{L}_V$,

$$D(\mathcal{L}_V) = \left\{u \in H^{1,V}(\mathbb{R}^n) : \exists w \in L^2(\mathbb{R}^n)\ s.t.\ \mathfrak{l}_V[u,v] = \langle w, v\rangle\ \forall\ v \in H^{1,V}(\mathbb{R}^n)\right\}.$$

The operator $\mathcal{L}_V$ is denoted

$$\mathcal{L}_V = V - \Delta,$$

since the two sides of the above relation will naturally coincide whenever the right-hand side makes senses. $\mathcal{L}_V$ is a densely defined maximal accretive operator called the Schrödinger operator with potential V.

1.3. The Reverse Hölder Class

The study of the Schrödinger operator $\mathcal{L}_V$ has presented an important and difficult challenge for the progress of harmonic analysis. One factor that contributes to the intractability of this problem is the growth and decay of the potential V. When V either grows or decays too fast then the situation can become quite complicated when this zero-order term is mixed with the second-order derivative Δ. However, if suitable growth conditions are imposed upon the potential, then matters become simpler. This tactic, pioneered by Z. Shen in [50], led to the introduction of the Reverse Hölder class of potentials.

Definition 1.3.1. *A non-negative and locally integrable function $V : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the reverse Hölder inequality with index $q \in (1, \infty)$ if there exists $C > 0$ such that*

$$\left(\frac{1}{|B|} \int_B V^q(x)\, dx \right)^{\frac{1}{q}} \leq C \left(\frac{1}{|B|} \int_B V(x)\, dx \right) \tag{1.1}$$

for every ball $B \subset \mathbb{R}^n$. It is said to satisfy the reverse Hölder inequality with index $q = \infty$ if there exists $C > 0$ for which

$$\sup_{x \in B} V(x) \leq C \left(\frac{1}{|B|} \int_B V(x)\, dx \right)$$

for every ball $B \subset \mathbb{R}^n$. The reverse Hölder class of index q, denoted by RH_q, is then defined to be the class of all potentials that satisfy the reverse Hölder inequality with index q. Define the reverse Hölder class RH_1 as

$$RH_1 := \bigcup_{q \in (1, \infty]} RH_q.$$

Remark 1.3.1. Similar to the definition of Muckenhoupt weights, balls can be replaced by cubes in the above definition and identical potential classes will result.

Remark 1.3.2. Note that the reverse Hölder inequalities are appropriately named since the reverse inequality to (1.1) can be obtained with constant $C = 1$ directly from Hölder's inequality.

The reverse Hölder inequalities present another quantifiable way of measuring the growth or decay of a function. Given this strong connection between Muckenhoupt weights and reverse Hölder potentials, it should then come as no surprise that the reverse Hölder class RH_1 and A_∞ are simply two ways of viewing the same class of functions.

Theorem 1.3.1 ([27, Cor. 2.13])**.** *The classes RH_1 and A_∞ are identical,*

$$RH_1 = A_\infty.$$

Another result that highlights the relationship between these two sets of classes is the following useful theorem.

Theorem 1.3.2 ([53, Cor. 6.2])**.** *For any $p \in (1, \infty)$ we have the following equivalence*

$$w \in RH_p \quad \Leftrightarrow \quad w^p \in A_\infty$$

Extending the analogy between the Muckenhoupt weights and reverse Hölder potentials even further, the reverse Hölder classes $\{RH_q\}_{q \in [1,\infty]}$ form a nested decreasing collection that satisfies a self-improvement property in the following sense.

Theorem 1.3.3 ([30])**.** *Let $1 \le p \le q \le \infty$. Then*

$$RH_1 \supset RH_p \supset RH_q \supset RH_\infty.$$

Let $V \in RH_p$. There exists some $\varepsilon > 0$ for which $V \in RH_{p+\varepsilon}$. In other words,

$$RH_p = \bigcup_{p' \in (p,\infty]} RH_{p'}.$$

Theorem 1.3.1, together with Example 1.1.1, automatically gives us the following examples.

Example 1.3.1.

1. Any potential that is bounded from both above and below is contained in RH_q for any $q \in [1, \infty]$.

2. For any polynomial P, $|P|$ will be contained in RH_q for any $q \in (1, \infty)$. To see this, note that by Theorem 1.3.2 we will have $|P| \in RH_q$ if and only if $|P|^q \in A_\infty$. From Example 1.1.1, we know that $|P| \in A_\infty$. The exponential-log connection then implies that $\log |P| \in BMO$ and therefore

$$\log |P|^q = q \log |P| \in BMO.$$

   The exponential-log connection can then be applied once more to obtain $|P|^q \in A_\infty$.

3. For any $q \in [1, \infty]$, the function w_0 from Example 1.1.1 is contained in RH_q.

4. The potential $e^{|x|}$ is not contained in RH_1.

5. Any compactly supported potential is not contained in RH_1.

For over the past two decades, reverse Hölder potentials have played a very influential role in the development of the harmonic analysis of Schrödinger operators. Indeed, the fixation on this class has not been without warrant since these potentials form a natural class for the construction of numerous harmonic analytic objects associated with the Schrödinger operator. The key development that initiated this boom of interest and slew of results was the introduction of the critical radius function by Z. Shen in [49].

Definition 1.3.2. *For $V \in RH_{\frac{n}{2}}$, define the critical radius function $\rho_V : \mathbb{R}^n \to \mathbb{R}$ through*

$$\rho_V(x) := \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V \leq 1 \right\}$$

for $x \in \mathbb{R}^n$; where the notation $B(x,r)$ is used to denote the open ball in $\mathbb{R}^n$, centered at x and of radius r.

Example 1.3.2. For the harmonic oscillator potential $V(x) = |x|^2$, the critical radius function is given by

$$\rho(x) := \rho_{|x|^2} = \min \left\{ 1, \frac{1}{|x|} \right\}$$

for $x \in \mathbb{R}^n$. More generally, for a polynomial potential $V(x) = P(x)$ of degree k, the critical radius function is given by

$$\rho_V(x)^{-1} \simeq M(x,V) := \sum_{|\alpha| \leq k} |\partial^\alpha P(x)|^{\frac{1}{|\alpha|+2}},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ are multi-indices. See [50] for further details.

This radius function, to a large extent, determines the natural geometry associated with a reverse Hölder potential. At a scale smaller than the critical radius, the operators affiliated with $\mathcal{L}_V$ behave locally like their classical counterparts for the Laplacian. This intuitive idea will come to life in the statement of Theorem B in the next chapter.

An important inequality satisfied by the critical radius function and used extensively in the harmonic analysis of Schrödinger operators is the Fefferman-Phong Lemma. Physically speaking, it is a form of uncertainty principle which gives a lower bound on the sum of the potential and kinetic energy of a state.

Lemma 1.3.1 (Fefferman-Phong Lemma, [50])**.** *Let $V \in RH_{\frac{n}{2}}$. Then there exists $C > 0$ such that*

$$\int_{\mathbb{R}^n} |u(x)|^2 \frac{1}{\rho_V(x)^2}\, dx \leq C \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}^n} |u(x)|^2 V(x)\, dx \right)$$

for all $u \in C_c^1(\mathbb{R}^n)$.

In classical harmonic analysis, the $L^2(\mathbb{R}^n)$-boundedness of the Riesz transform $R := \nabla(-\Delta)^{-\frac{1}{2}}$ is frequently used as a tool to swap the derivative ∇ for $(-\Delta)^{\frac{1}{2}}$ when evaluating

L^2 estimates from above. Similarly, for the Schrödinger operator, we could consider first-order Riesz transforms such as

$$V^{\frac{1}{2}} (V - \Delta)^{-\frac{1}{2}}, \quad (-\Delta)^{\frac{1}{2}} (V - \Delta)^{-\frac{1}{2}} \quad \textit{and} \quad \nabla (V - \Delta)^{-\frac{1}{2}},$$

or their second-order counterparts such as

$$V (V - \Delta)^{-1}, \quad V^{\frac{1}{2}} \nabla (V - \Delta)^{-1}, \quad \nabla V^{\frac{1}{2}} (V - \Delta)^{-1} \quad \textit{and} \quad \Delta (V - \Delta)^{-1}.$$

Proving the L^2-boundedness of such operators would enable us to transition between different derivatives. In this way, we consider the multiplication operator $V^{\frac{1}{2}}$ to act as a derivative.

Out of these potential dependent Riesz transforms, the three first-order Riesz transforms are trivially bounded on $L^2(\mathbb{R}^n)$. This is given in the below proposition.

Proposition 1.3.1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a non-negative locally integrable potential. Then*

$$\left\| V^{\frac{1}{2}} u \right\|, \ \left\| (-\Delta)^{\frac{1}{2}} u \right\|, \ \|\nabla u\| \leq \left\| (V - \Delta)^{\frac{1}{2}} u \right\|$$

for any $u \in D\left((V - \Delta)^{\frac{1}{2}} \right)$.

Proof. For the third inequality, it follows from the non-negativity of V that

$$\begin{aligned} \|\nabla u\|^2 &= \langle \nabla u, \nabla u \rangle \\ &= \langle u, -\mathrm{div} \nabla u \rangle \\ &\leq \langle u, V - \mathrm{div} \nabla u \rangle \\ &= \langle (V - \Delta)^{\frac{1}{2}} u, (V - \Delta)^{\frac{1}{2}} u \rangle \\ &= \left\| (V - \Delta)^{\frac{1}{2}} u \right\|^2. \end{aligned}$$

The second inequality follows in an identical manner. The first inequality also follows from a similar argument but the positivity of the Laplacian $(-\Delta)$ must be used instead of the non-negativity of the potential. ◻

The second-order Riesz transforms, however, are not trivially bounded on L^2. Indeed, there exist examples of non-negative potentials for which such Riesz transforms are unbounded.

Proposition 1.3.2 ([50, §7])**.** *Let $n = 3$ and consider the potential $V(x) := |x|^{-\frac{3}{2}} \in RH_p$ for any $p < 2$. The estimate*

$$\|Vu\| \lesssim \|(V - \Delta) u\|$$

for $u \in D(V-\Delta)$ does not hold for this potential.

In [50], Z. Shen used the critical radius function that he developed to prove boundedness of second-order Riesz transforms on L^2 for reverse Hölder potentials of limited growth and decay. The most important Riesz transform for our purposes is $V(V-\Delta)^{-1}$, it will be used extensively in the second part of this thesis. Z. Shen proved the boundedness of this Riesz transform in dimension $n \geq 3$ for $V \in RH_q$ with $q \geq \max\left(\frac{n}{2}, 2\right)$. This result was later improved to RH_q for $q \geq 2$ and extended to arbitrary dimension by P. Auscher and B. Ben Ali in [6].

Theorem 1.3.4 ([50], [6]). *For any $V \in RH_q$ with $q \geq 2$, there exists a $c_V > 0$ for which*

$$\|Vu\|_2 \leq c_V \cdot \|(V-\Delta)\,u\|_2$$

for all $u \in D(V-\Delta)$.

1.4. The Weight Class $A_p^{V,\infty}$

Consider the heat operators for $\mathcal{L}_V$ denoted by $e^{-t\mathcal{L}_V}$. Once again, the precise definition of these operators is made clear in §3.1. These are integral operators with representation

$$e^{-t\mathcal{L}_V} f(x) = \int_{\mathbb{R}^n} k_t(x,y) \cdot f(y)\, dy,$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where $k_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the heat kernel for $\mathcal{L}_V$. The exact form of the heat kernel is rarely known except for a few special cases. For reverse Hölder potentials, however, we at the very least have the following upper bound.

Lemma 1.4.1 ([40]). *Let $V \in RH_{\frac{n}{2}}$. For any $N > 0$, there exists a constant $C_N > 0$ such that*

$$k_t(x,y) \leq C_N \cdot t^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right)\left(1 + \frac{\sqrt{t}}{\rho_V(x)} + \frac{\sqrt{t}}{\rho_V(y)}\right)^{-N} \tag{1.2}$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$.

The above estimate shows, in particular, that the heat operators $e^{-t\mathcal{L}_V}$ are pointwise bounded from above by the classical operators $e^{t\Delta}$. Consider the heat maximal operator for the Schrödinger operator $\mathcal{L}_V$ defined through

$$\mathcal{T}_V^* f(x) := \sup_{t>0} e^{-t\mathcal{L}_V} |f|(x),$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. This operator is pointwise bounded from above by T^* and, moreover, if the potential is suitably large then it will be noticeably smaller. We should then expect for the first equivalence in Theorem 1.1.4 to break if the operator T^* is replaced by its potential dependent counterpart $\mathcal{T}_V^*$. Specifically, it is reasonable to expect the implication

$$w \in A_p \quad \Rightarrow \quad \|\mathcal{T}_V^* f\|_{L^p(w)\to L^p(w)} < \infty$$

to hold but the opposite implication should no longer be valid. Indeed, the heat maximal operator for $\mathcal{L}_V$ should be bounded on $L^p(w)$ for a larger class of weights than the classical A_p class. This begs the question, is there a potential dependent weight class that will produce this equivalence?

A significant source of inspiration for our investigation stemmed from the article [15]. In this paper, B. Bongioanni, E. Harboure and O. Salinas defined a new class of weights more suited to the Schrödinger operator setting than classical A_p.

Definition 1.4.1. *Let $V \in RH_{\frac{n}{2}}$ for $n \geq 3$. Define, for $\theta \geq 0$, the map*

$$\psi_\theta^V(B) := \left(1 + \frac{r_B}{\rho_V(c_B)}\right)^\theta$$

for balls $B \subset \mathbb{R}^n$, where the notation r_B and c_B is used to denote, respectively, the radius and center of a ball B. A weight w on $\mathbb{R}^n$ is said to belong to the class $A_p^{V,\theta}$ if there exists $C > 0$ such that

$$w(B)^{\frac{1}{p}} \cdot w^{-\frac{1}{p-1}}(B)^{\frac{p-1}{p}} \leq C\,|B|\,\psi_\theta^V(B)$$

for all balls $B \subset \mathbb{R}^n$. Define

$$A_p^{V,\infty} := \bigcup_{\theta\geq 0} A_p^{V,\theta}.$$

It is apparent from the above definition that the classical A_p weight class is contained in $A_p^{V,\infty}$ for any potential $V \in RH_{\frac{n}{2}}$ since $A_p^{V,0} = A_p$. Moreover, if the potential is suitably large then this inclusion will become strict. To see this, notice that for a large potential the critical radius function will be smaller than unity and therefore more growth or decay will be allowed in the weight.

What makes this weight class so interesting is that not only is it larger than the classical A_p weight class but the heat maximal operator along with other standard operators associated with $\mathcal{L}_V$ are bounded on $L^p(w)$ for any weight $w \in A_p^{V,\infty}$.

Theorem 1.4.1 ([15])**.** *Fix $p \in (1,\infty)$ and $n \geq 3$. Let $V \in RH_{\frac{n}{2}}$ and $w \in A_p^{V,\infty}$. The* *heat maximal operator $\mathcal{T}_V^*$, the Riesz transforms*

$$\mathcal{R}_V := \nabla \left(\mathcal{L}_V\right)^{-\frac{1}{2}},$$

their adjoints

$$\mathcal{R}_V^* := \left(\mathcal{L}_V\right)^{-\frac{1}{2}} \nabla$$

and the $\mathcal{L}_V$-square function

$$\left(\int_0^\infty \left|\frac{d}{dt} e^{-t\mathcal{L}_V}(f)(x)\right|^2 t\, dt\right)^{\frac{1}{2}}$$

are all bounded from $L^p(w)$ to itself. Moreover, the $\mathcal{L}_V$-Riesz potential

$$\mathcal{I}_\alpha f(x) := \left(\mathcal{L}_V\right)^{-\frac{\alpha}{2}} = \int_0^\infty e^{-t\mathcal{L}_V} f(x) t^{\frac{\alpha}{2}} \frac{dt}{t}$$

is bounded from $L^p(w)$ into $L^\nu\left(w^{\frac{\nu}{p}}\right)$ when $w^{\frac{\nu}{p}} \in A^{V,\infty}_{1+\frac{\nu}{p'}}$ and ν, p and α satisfies $1 < p \leq \frac{n}{\alpha}$ and $\frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{n}$. The notation p' has been introduced to denote the Hölder conjugate to the index p, $p' := \frac{p}{p-1}$.

The above theorem indicates that $A_p^{V,\infty}$ is a more suitable class of weights than the classical Muckenhoupt class in the potential dependent setting. Although this is clearly a step in the right direction, there is reason to suspect that this weight class is still not optimal. That is, membership in this class does not completely characterise the boundedness of the standard operators attached to $\mathcal{L}_V$ on $L^p(w)$. A larger class is still required to fully capture the harmonic analytic aspects of $\mathcal{L}_V$.

In the chapter to follow, for the harmonic oscillator potential, a larger class of weights will be constructed for which the heat maximal operator is bounded on $L^p(w)$ for any weight in this class. It will be found that even for a potential of limited growth such as $V(x) = |x|^2$, the correct weight class should increase in size so dramatically as to include exponential type weights of the form $e^{|x|}$. Such weights are not included in the class $A_p^\infty := A_p^{|x|^2,\infty}$, let alone A_p.

Theorem 1.1.4 indicates that if one is to construct an adapted Hardy-Littlewood maximal operator M_V then it should satisfy the equivalence

$$\|M_V\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftrightarrow \quad \|\mathcal{T}_V^*\|_{L^p(w)\to L^p(w)} < \infty$$

for any weight w and $p \in (1,\infty)$. It can be gleaned from this that the construction of an adapted Muckenhoupt class is intrinsically related to the construction of an adapted Hardy-Littlewood maximal operator. In [54], the author developed a maximal function

M_V^θ corresponding to the class $A_p^{V,\theta}$ in the sense that $M_V^\theta : L^p(w) \to L^p(w)$ is bounded if and only if $w \in A_p^{V,\theta}$. This operator is defined through

$$M_V^\theta f(x) := \sup \frac{1}{\psi_\theta^V(B)\,|B|} \int_B |f(y)|\,dy, \tag{1.3}$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where the supremum is taken over all balls $B = B(c_B, r_B) \subset \mathbb{R}^n$ that contain the point x. Unfortunately, owing to the fact that the classes $A_p^{V,\theta}$ are not optimal, this Hardy-Littlewood maximal operator will still not be optimally adapted to the potential. Our aim through this research is to improve upon the weight class $A_p^{V,\infty}$ and the operator M_V^θ for the specific case of the harmonic oscillator potential. This will provide insight into what the correct objects should be for a more general potential.

1.5. The Gaussian Grid

As is quite frequent in mathematics, when studying a particular object it can be fruitful to change perspective by studying an isomorphic object in a different setting. Let $d\gamma(x) := \pi^{-\frac{n}{2}} e^{-|x|^2} dx$ denote the Gaussian measure on $\mathbb{R}^n$. Gaussian harmonic analysis is the study of the Ornstein-Uhlenbeck operator, $\mathcal{O} := -\Delta + 2x \cdot \nabla$, on the space $L^2(\gamma)$ and its associated harmonic analysis. Its relevance to the study of $\mathcal{L}$ is that through the isometry $U : L^2(dx) \to L^2(\gamma)$, defined by

$$Uf(x) := \pi^{-\frac{n}{4}} e^{-\frac{|x|^2}{2}} f(x)$$

for $f \in L^2(dx)$ and $x \in \mathbb{R}^n$, the operators $\mathcal{L}$ and $\mathcal{O}$ become, more-or-less, similar. See [1] for further details. This similarity allows for the transfer of geometric ideas between the Gaussian and the harmonic oscillator setting.

A vital attribute of the Lebesgue measure is that when the radius of a ball is doubled the volume of the enlarged ball will be a fixed dimensional multiple of the volume of the original ball. This attribute for a measure is called the doubling property.

Definition 1.5.1. *A measure μ on $\mathbb{R}^n$ is said to be doubling if there exists some $C > 0$ such that*

$$\mu\left(B(x, 2r)\right) \leq C \cdot \mu\left(B(x, r)\right) \tag{1.4}$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

Example 1.5.1. It is well known that the measure $w(x)\,dx$ is doubling for any $w \in A_\infty$ (see [30]). This also implies, through Theorem 1.3.1, that for any $V \in RH_1$ the measure $V(x)\,dx$ is doubling. The class A_p^+, to be defined in the next chapter, will contain some non-doubling weights. For example, it will be proved that $e^{|x|} \in A_p^+$, which is easily seen to be non-doubling.

Many of the constructions from classical harmonic analysis directly rely on the fact that the Lebesgue measure is doubling. Indeed, a substantial proportion of classical harmonic analysis can be translated over to a more general metric measure space when the measure is doubling. A fundamental obstruction in the development of Gaussian harmonic analysis is that, due to the non-doubling nature of the Gaussian measure, many of these constructions will not directly translate to the Gaussian setting. In their seminal paper [43], G. Mauceri and S. Meda made a crucial step in this development by transposing the critical radius over to Gaussian harmonic analysis. They introduced their concept of admissibility. The intuition behind this move was that the isomorphism between the Gaussian and harmonic oscillator settings should allow for the free flow of geometric ideas between these two separate but parallel worlds.

As stated in Example 1.3.2, $\rho(x) = \min\{1, 1/|x|\}$. A ball $B(x, r)$ is then said to be admissible if $r \le \rho(x)$. The collection of all admissible balls in $\mathbb{R}^n$, $\mathcal{B}$, possesses the desirable property that there exists some $C > 0$ such that the Gaussian measure satisfies the doubling condition (1.4) for all balls in $\mathcal{B}$. As such, by restricting their attention to the collection $\mathcal{B}$, Mauceri and Meda were able to construct Gaussian analogues of the spaces BMO and H^1. A similar construction for the harmonic oscillator, also based on the distinction between local and non-local scales, was developed by J. Dziubanski and J. Zienkiewicz in [21] and [20].

The collection of dyadic cubes Δ is an immensely important mathematical object that forms a cornerstone of classical harmonic analysis. Due to their non-overlappling nature, the dyadic cubes are commonly and efficaciously used to decompose quantities in the spatial variable and analyse the quantity restricted to each cube separately. Then, the separate components are summed in a meaningful way to achieve a desired estimate. This form of argument is repeated in classical harmonic analysis countlessly many times. It is therefore clear that the construction of a Gaussian analogue for Δ would be of tremendous use.

In [42], J. Maas, J. van Neerven and P. Portal extended the idea of admissibility by constructing an admissible dyadic grid Δ^γ. Just as the the critical radius function was transposed from the harmonic oscillator to the Gaussian setting, the Gaussian grid will be fed back through to the harmonic oscillator setting. It is this grid that will form the foundation for our construction. We recall some pertinent details. Define the layers

$$L_0 := [-1, 1)^n, \quad L_l := [-2^l, 2^l)^n \setminus [-2^{l-1}, 2^{l-1})^n,$$

for $l \in \mathbb{N}^*$. Then define, for $k \in \mathbb{Z}$ and $l \in \mathbb{N}$,

$$\Delta^\gamma_{k,l} := \{Q \in \Delta_{-l-k} : Q \subseteq L_l\}, \quad \Delta^\gamma_k := \bigcup_{l \geq 0} \Delta^\gamma_{k,l}, \quad \Delta^\gamma := \bigcup_{k \geq 0} \Delta^\gamma_k.$$

The collection Δ^γ is called the Gaussian grid and will be used extensively throughout this part of the thesis. As the definition of the grid Δ^γ_0 is based on the critical radius function, this grid will define a notion of locality for which operators affilitated with $\mathcal{L}_V$ will behave similarly to their classical counterparts at this local scale. For reference, a depiction of the grid Δ^γ_0 is provided below.

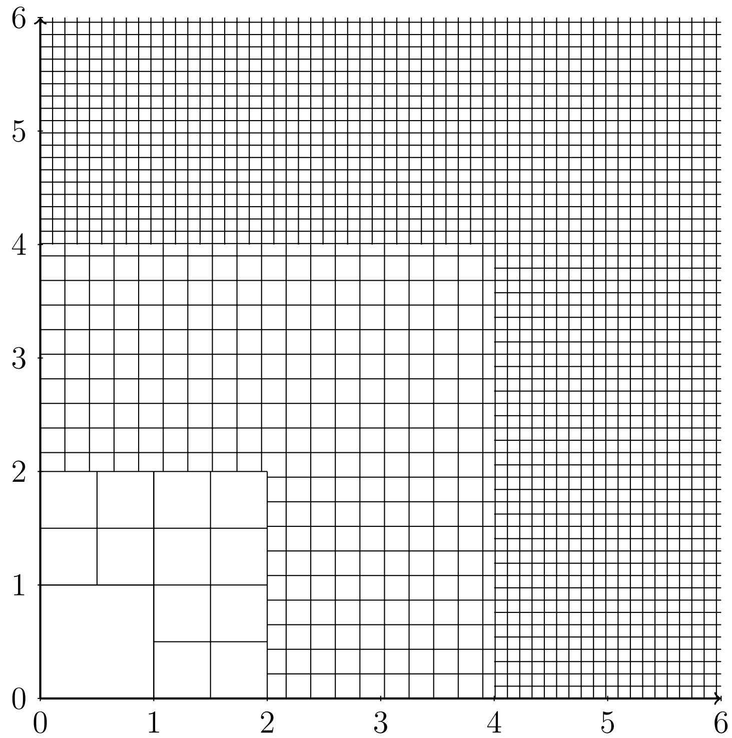

Figure 1.1: The Gaussian grid Δ^γ_0. The layers L_l are depicted as the regions of differently sized cubes.

Observe that the further you are from the origin, the smaller the cubes become. This is in scale with the critical radius function ρ.

Notation. Let's introduce some notation that can be used in conjunction with this grid. For any $x \in \mathbb{R}^n$, R_x will be used to denote the unique cube in Δ^γ_0 that contains the point x. For any $R \in \Delta^\gamma_0$, $j(R)$ is defined to be the unique integer such that $R \subset L_{j(R)}$. The more commonly used notation, c_Q and $l(Q)$, representing the center and side-length of a cube Q respectively, will also be used.

1.6. The $BMO_{V,\infty}$ Class

Continuing in the spirit of defining potential dependent analogues for classical harmonic analytic objects, one can define a BMO class adapted to a Schrödinger operator with potential in the reverse Hölder class. The following adapted BMO class was introduced in [16].

Definition 1.6.1. *Let $V \in RH_{\frac{n}{2}}$. Define $BMO_{V,\infty}$ to be the union*

$$BMO_{V,\infty} := \bigcup_{\theta \geq 0} BMO_{V,\theta},$$

where $BMO_{V,\theta}$ consists of all functions f such that

$$\|f\|_{BMO_{V,\theta}} := \sup \frac{1}{\psi_\theta^V(B)\,|B|} \int_B |f(y) - \langle f \rangle_B|\, dy < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Recall the exponential-log link from classical weight theory as given in Theorem 1.6.1. An obvious question to ask with the construction of this adapted BMO class is whether such a connection holds between $BMO_{V,\infty}$ and the weight class $A_p^{V,\infty}$. It will in fact be shown that such a link does not hold. This gives further merit to the argument that the classes $A_p^{V,\infty}$ and $BMO_{V,\infty}$ are not necessarily the correct classes associated with the Schrödinger operator. Indeed, the class $A_p^{V,\infty}$ is too small to be optimal and the class $BMO_{V,\infty}$ is too large to suit $A_p^{V,\infty}$. To the best of my knowledge, this result was originally proved by myself and it runs counter to what some researchers in the field might expect.

Theorem 1.6.1. *Consider the harmonic oscillator potential $V(x) = |x|^2$. Introduce the shorthand notation $BMO_\infty := BMO_{|x|^2,\infty}$. The exponential-log link between A_p^∞ and BMO_∞ does not hold. In particular, there exists a weight w such that $\log w \in BMO_\infty$ but $w^\eta \notin A_p^\infty$ for any $\eta > 0$ and $p > 1$.*

To formulate a counterexample we work in dimension $n = 3$ for simplicity and since this is the lowest dimension for which the class $A_p^{V,\infty}$ was considered in [15]. We also work with the equivalent definitions of $BMO_{V,\infty}$ and $A_p^{V,\infty}$ that substitutes balls in $\mathbb{R}^n$ for cubes. Consider the weight $w(x) = e^{|x_1|}$. It will be proved that $\log w = |x_1| \in BMO_\infty$ but $w^\eta = e^{\eta|x_1|} \notin A_p^\infty$ for any $\eta > 0$ and $p > 1$. The proof will be split into these two parts.

Lemma 1.6.1. *The function $f(x) := |x_1|$ is contained in BMO_∞.*

Proof. It suffices to prove that $f \in BMO_\theta$ for some $\theta \geq 0$. That is, it will be shown that there exists some constant $C > 0$ such that

$$\frac{1}{\psi_\theta(Q)\,|Q|}\int_Q |f(y) - \langle f\rangle_Q|\,dy \le C$$

for all cubes $Q \subset \mathbb{R}^3$, where

$$\psi_\theta(Q) := \psi_\theta^{|x|^2}(Q) := \left(1 + \frac{l(Q)}{\rho(c_Q)}\right)^\theta.$$

For dimension $n = 3$, any cube will be of the form $Q = (a_1, a_1+l] \times (a_2, a_2+l] \times (a_3, a_3+l]$ for some a_1, a_2, $a_3 \in \mathbb{R}$ and $l > 0$. Suppose first that $a_1 \ge 0$. Note that

$$\begin{aligned}\langle f\rangle_Q &= \frac{1}{l^3}\int_{a_3}^{a_3+l}\int_{a_2}^{a_2+l}\int_{a_1}^{a_1+l} x_1\,dx_1\,dx_2\,dx_3\\ &= a_1 + \frac{l}{2}.\end{aligned}$$

Then

$$\begin{aligned}\int_Q |f - \langle f\rangle_Q| &= l^2\int_{a_1}^{a_1+l}\left|x_1 - \left(a_1 + \frac{l}{2}\right)\right|\,dx_1\\ &= l^2\int_{a_1}^{a_1+\frac{l}{2}}\left(a_1+\frac{l}{2}\right) - x_1\,dx_1 + l^2\int_{a_1+\frac{l}{2}}^{a_1+l} x_1 - \left(a_1+\frac{l}{2}\right)\,dx_1\\ &= l^2\left[\left(a_1+\frac{l}{2}\right)x_1 - \frac{x_1^2}{2}\right]_{a_1}^{a_1+\frac{l}{2}} + l^2\left[\frac{x_1^2}{2} - \left(a_1+\frac{l}{2}\right)x_1\right]_{a_1+\frac{l}{2}}^{a_1+l}\\ &= \frac{l^4}{4}.\end{aligned}$$

This gives us

$$\begin{aligned}\int_Q |f - \langle f\rangle_Q| &\lesssim l^4\\ &\le (1+l)\,|Q|\\ &\le \left(1 + \frac{l}{\rho(Q)}\right)|Q|.\end{aligned}$$

The case for when $a_1 + l \le 0$ will obviously be identical to $0 \le a_1$. Lastly, let's consider cubes for which $a_1 < 0$ and $a_1 + l > 0$. For this type of cube, the average will be given by

$$\begin{aligned}
\langle f\rangle_Q &= \frac{1}{l}\int_{a_1}^{a_1+l} |x_1|\ dx_1 \\
&= \frac{1}{l}\left(\int_{a_1}^{0} -x_1\, dx_1 + \int_{0}^{a_1+l} x_1\, dx_1\right) \\
&= \frac{1}{l}\left(\left[-\frac{x_1^2}{2}\right]_{a_1}^{0} + \left[\frac{x_1^2}{2}\right]_{0}^{a_1+l}\right) \\
&= \frac{a_1^2 + (a_1+l)^2}{2l}.
\end{aligned}$$

Define

$$c := \langle f\rangle_Q = \frac{a_1^2 + (a_1+l)^2}{2l}.$$

Then

$$\begin{aligned}
\int_Q |f - \langle f\rangle_Q| &= l^2 \int_{a_1}^{a_1+l} ||x_1| - c|\ dx_1 \\
&= l^2\left(\int_{a_1}^{-c} (-x_1) - c\, dx_1 + \int_{-c}^{0} c - (-x_1)\ dx_1 + \int_{0}^{c} c - x_1\, dx_1 + \int_{c}^{a_1+l} x_1 - c\, dx_1\right) \\
&= l^2\left(\left[-\frac{x_1^2}{2} - cx_1\right]_{a_1}^{-c} + \left[cx_1 + \frac{x_1^2}{2}\right]_{-c}^{0} + \left[cx_1 - \frac{x_1^2}{2}\right]_{0}^{c} + \left[\frac{x_1^2}{2} - cx_1\right]_{c}^{a_1+l}\right) \\
&= \frac{1}{2}\left(a_1^2 + (a_1+l)^2\right)^2
\end{aligned}$$

We have

$$\begin{aligned}
\int_Q |f - \langle f\rangle_Q| &\leq \left(a_1^2 + (a_1+l)^2\right)^2 \\
&\lesssim l^4 \\
&\leq (1+l)\,|Q| \\
&\leq \left(1 + \frac{l}{\rho(Q)}\right)|Q|\,.
\end{aligned}$$

Therefore we can conclude that $f(x) = |x_1| \in BMO_1 \subset BMO_\infty$. $\square$

To complete the proof of our theorem, it then remains to prove the following lemma.

Lemma 1.6.2. *For any $p > 1$ and $\eta > 0$, $w^\eta = e^{\eta|x_1|} \notin A_p^\infty$.*

PROOF. Suppose that $w^\eta \in A_p^\infty := \cup_{\theta\geq 0} A_p^\theta$ for some $p > 1$, $\eta > 0$. Then $w^\eta \in A_p^\theta$ for some $\theta \geq 0$. There must then exist some constant $C > 0$ such that

$$\left(\frac{1}{\psi_\theta(Q)\,|Q|}\int_Q w^\eta(x)dx\right)\left(\frac{1}{\psi_\theta(Q)\,|Q|}\int_Q w^{-\frac{\eta}{p-1}}(x)dx\right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^3$. In particular, this inequality must be true for any cube of the form $Q = (-N, N]^3$ for $N \in \mathbb{N}^*$. That is,

$$\left(\int_{-N}^{N} e^{\eta|x_1|}dx_1\right)\cdot\left(\int_{-N}^{N} e^{-\frac{\eta}{p-1}|x_1|}dx_1\right)^{p-1} \leq C\left(\psi_\theta(Q)2N\right)^p.$$

Since

$$\psi_\theta(Q) = \left(1 + \frac{l(Q)}{\rho(c_Q)}\right)^\theta = (1+2N)^\theta.$$

This is equivalent to the bound

$$\left(\int_{-N}^{N} e^{\eta|x_1|}dx_1\right)\cdot\left(\int_{-N}^{N} e^{-\frac{\eta}{p-1}|x_1|}dx_1\right)^{p-1} \leq C\,(1+2N)^{p\theta}\cdot(2N)^p \tag{1.5}$$

for all $N \in \mathbb{N}$. Now,

$$\begin{aligned}
\left(\int_{-N}^{N} e^{\eta|x_1|}dx_1\right)\cdot\left(\int_{-N}^{N} e^{-\frac{\eta}{p-1}|x_1|}dx_1\right)^{p-1} &\geq \left(\int_{0}^{N} e^{\eta x_1}dx_1\right)\cdot\left(\int_{0}^{N} e^{-\frac{\eta}{p-1}x_1}dx_1\right)^{p-1}\\
&= \left[\frac{1}{\eta}e^{\eta x_1}\right]_0^N\cdot\left(\left[-\frac{p-1}{\eta}e^{-\frac{\eta}{p-1}x_1}\right]_0^N\right)^{p-1}\\
&\simeq \left[e^{\eta N}-1\right]\cdot\left[1-e^{-\frac{\eta}{p-1}N}\right]^{p-1}\\
&\geq \left[e^{\eta N}-1\right]\cdot\left[1-e^{-\frac{\eta}{p-1}}\right]^{p-1}.
\end{aligned}$$

The above taken together with (1.5) then implies that

$$\left(e^{\eta N}-1\right) \lesssim C\,(1+2N)^{p\theta}\cdot(2N)^p$$

for all $N \in \mathbb{N}$. This is clearly a contradiction since an exponential can't be bounded by a constant multiple of a polynomial for all $N \in \mathbb{N}$. □

Chapter 2

Hardy-Littlewood Adapted to the Harmonic Oscillator

As alluded to earlier, the critical radius function ρ_V for a reverse Hölder potential V determines a scale below which the operators associated with $\mathcal{L}_V$ will behave classically. This indicates that if we are to construct a Hardy-Littlewood type maximal operator for $\mathcal{L}$, then our construction should resemble the classical Hardy-Littlewood operator at a local scale of the size of the Gaussian grid Δ_0^γ. The following definition establishes what will be considered to be our local region in the Gausian grid.

Definition 2.0.1. *For a cube $R \in \Delta_0^\gamma$, fix a subcollection $\mathcal{N}(R) \subset \Delta_0^\gamma$ that satisfies the following two properties.*

- *$\mathcal{N}(R)$ contains all cubes $R' \in \Delta_0^\gamma$ satisfying*

$$d(R, R') < 2^{-j(R)},$$

  where $d(R, R') := \mathrm{dist}(R, R') := \inf \{|x - y| : x \in R \text{ and } y \in R'\}$.

- *The region*

$$N(R) := \bigsqcup_{R' \in \mathcal{N}(R)} R',$$

  is a cube of sidelength $2^2 l(R)$.

The notation $\mathcal{F}(R) := \Delta_0^\gamma \setminus \mathcal{N}(R)$ and $F(R) := \mathbb{R}^n \setminus N(R)$ will also be employed.

It is obvious that such a subcollection must exist for each cube. There might even be more than one such example. This, however, is unimportant. What is important, is that we fix $\mathcal{N}(R)$ from the outset. Examples of subcollections that satisfy these properties are illustrated below.

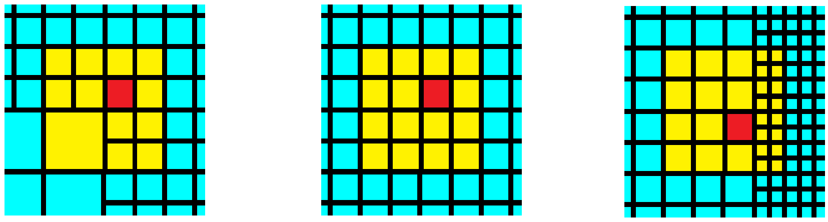

Figure 2.1: Each of the above illustrations depicts a cube R, coloured in red, contained in the grid Δ_0^γ in dimension two. The near region, $N(R)$, consists of all cubes highlighted in yellow together with the cube R. The far region, $F(R)$, is coloured blue and extends out to infinity.

As our operator is expected to behave differently at large scales than at local scales, it is desirable to split it up into local and non-local components. For any sub-linear operator B, define

$$B_{loc}f(x) := B\left(f \cdot \chi_{N(R_x)}\right)(x) \quad \textit{and}$$

$$B_{far}f(x) := B\left(f \cdot \chi_{F(R_x)}\right)(x)$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Such decompositions into local and global components for Schrödinger operators have their origins in the work of Shen, [50]. Notice that if B satisfies the property that $|f| \leq |g|$ implies $|B(f)| \leq |B(g)|$ then due to sub-linearity, for any weight w on $\mathbb{R}^n$, to bound the quantity $\|Bf\|_{L^p(w)}$ it is both sufficient and necessary to bound $\|B_{loc}f\|_{L^p(w)}$ and $\|B_{far}f\|_{L^p(w)}$.

As noted previously, Δ_0^γ acts as a mediator between the local and non-local worlds. It is then appropriate to consider maximal functions of the below general form as candidates for an adapted maximal function for $\mathcal{L}$.

Definition 2.0.2. *For $Q \in \Delta$, let $\mathcal{G}(Q)$ be the collection of cubes*

$$\mathcal{G}(Q) := \begin{cases} \{Q\} & \text{if } Q \in \Delta^\gamma \\ \{R' \in \Delta_0^\gamma : R' \subset Q\} & \text{otherwise.} \end{cases}$$

Then for $c : \Delta \times \Delta_0^\gamma \times \Delta_0^\gamma \to \mathbb{R}_{\geq 0}$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in R \in \Delta_0^\gamma$, define the operator $\mathcal{M}_c$ by

$$\mathcal{M}_c f(x) := \sup_{Q \in \Delta,\ Q \ni x} \frac{1}{|Q|} \sum_{R' \in \mathcal{G}(Q)} c(Q, R, R') \int_{R'} |f(y)|\, dy. \tag{2.1}$$

This looks promising but how do we determine what the right c-coefficients are? Any candidate for an adapted Hardy-Littlewood should share similar properties to the classical Hardy-Littlewood operator. We will determine appropriate coefficients from one of these properties. In particular, we will make use of Theorem 1.1.4. This theorem indicates that if we are to construct a Hardy-Littlewood type maximal operator for $\mathcal{L}$, then the correct c-coefficients should satisfy the below equivalence for each $1 < p < \infty$,

$$\|\mathcal{T}^*\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftrightarrow \quad \|\mathcal{M}_c\|_{L^p(w)\to L^p(w)} < \infty,$$

where $\mathcal{T}^*$ is the semigroup maximal operator associated with $\mathcal{L}$,

$$\mathcal{T}^* f(x) := \mathcal{T}^*_{|x|^2} f(x) := \sup_{t>0} e^{-t\mathcal{L}} |f| (x).$$

The coefficients for our generalised maximal function will be optimised in an attempt to produce the above equivalence.

It can be inferred from the fact that the operator $\mathcal{T}^*$ is smaller than the classical heat operator T^* that the coefficients for $\mathcal{M}_c$ should be smaller than unity. This aligns with the intuition from §1.4 since the weight class A_p^∞ is larger than the classical Muckenhoupt class and therefore the adapted maximal operator should be smaller.

Remark 2.0.1. Notice that if $c(Q, R, R') = 1$ for all $R' \in \mathcal{G}(Q)$ and $Q \in \Delta$, then the operator $\mathcal{M}_c$ is identical to the classical dyadic Hardy-Littlewood operator. The dyadic version of the maximal function $M_{|x|^2,\theta}$ from §1.4 is also an example of the general class from Definition 2.0.2 with $c(Q, R, R') = \psi_\theta(Q)^{-1} < 1$. This allows for more weights in the class A_p^θ. However, it does not take into account the fact that if the cubes R and R' are far apart, then the potential should have a larger effect and therefore the coefficient $c(Q, R, R')$ should be smaller. The coefficients that we define for our maximal function will account for this.

The main theorem of this chapter is stated below.

Theorem A. *There exists maximal functions, $\mathcal{M}^-_{far}$ and $\mathcal{M}^+_{far}$, of similar form to Definition 2.0.2, that satisfy the chain of implications*

$$\left\|\mathcal{M}^+_{far}\right\|_{L^p(w)} < \infty \quad \Rightarrow \quad \left\|\mathcal{T}^*_{far}\right\|_{L^p(w)} < \infty \quad \Rightarrow \quad \left\|\mathcal{M}^-_{far}\right\|_{L^p(w)} < \infty,$$

for any weight w on $\mathbb{R}^n$ and $1 < p < \infty$.

A precise definition of the above maximal functions, $\mathcal{M}^-_{far}$ and $\mathcal{M}^+_{far}$, and a proof of this statement, will be deferred to §2.2. A secondary result of this chapter that characterises the local behaviour of an adapted maximal function is the following theorem.

Theorem B. *For any weight w on $\mathbb{R}^n$ and $1 < p < \infty$,*

$$\|M_{loc}\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftrightarrow \quad \|\mathcal{T}^*_{loc}\|_{L^p(w)\to L^p(w)} < \infty.$$

This theorem will be proved in §2.1. This is a manifestation of the general guiding principle that the operators for $\mathcal{L}_V$ should behave classically at a scale smaller than the critical radius. Theorems A and B together demonstrate that for any weight in the class

$$A_p^+ := \left\{ w \text{ weight on } \mathbb{R}^n : \left\|\mathcal{M}^+_{far}\right\|_{L^p(w)\to L^p(w)} < \infty \ \textit{and} \ \|M_{loc}\|_{L^p(w)\to L^p(w)} < \infty \right\},$$

we have $\|\mathcal{T}^*\|_{L^p(w)\to L^p(w)} < \infty$.

It is then natural to ask how our weight class compares with the class A_p^∞. Section 2.3 provides an answer to this question in the form of the following proposition.

Proposition C. *The following chain of strict inclusions holds for any $1 < p < \infty$,*

$$A_p \subsetneq A_p^\infty \subsetneq A_p^+.$$

The above inclusion indicates that our coefficients serve as an improvement upon the constant coefficients of (1.3).

Finally, in §2.4, the techniques developed throughout this paper will be used to show that the heat maximal operator for $\mathcal{L}$ can be safely truncated when considering weighted questions.

2.1. The Local Class

In this section, a local version of the A_p class is introduced, A_p^{loc}. This class is a dyadic variation of a similar class introduced in [15]. Through this class, and a few preliminary lemmas, Theorem B will be proved.

Consider a cube in $\mathbb{R}^n$, $Q_0 := [a_1, a_1+l(Q_0))\times\cdots\times[a_n, a_n+l(Q_0))$, where $\{a_1, \cdots, a_n\} \subset \mathbb{R}$. In the usual manner, this cube can be divided into 2^n congruent disjoint cubes with half the side-length of the original cube. These cubes can themselves be divided into 2^n disjoint cubes each and so on ad infinitum. If a cube $Q \subset \mathbb{R}^n$ can be obtained in this manner from Q_0, then it is called a dyadic subcube of the cube Q_0. Note that we did not require our initial cube Q_0 to be a member of the standard dyadic grid and that Q_0 is a dyadic subcube of itself.

Definition 2.1.1. *Fix a weight w on $\mathbb{R}^n$ and $1 < p < \infty$. For a cube $Q_0 \subset \mathbb{R}^n$, the* *weight w is said to belong to the class $A_p(Q_0)$ if there exists a constant $C > 0$ such that*

$$w^{-\frac{1}{p-1}}(Q)^{\frac{p-1}{p}} w(Q)^{\frac{1}{p}} \leq C\,|Q| \tag{2.2}$$

for all dyadic subcubes $Q \subseteq Q_0$. The smallest such C is denoted $[w]_{A_p(Q_0)}$.

A variation of the next statement was originally proved in [33]. It is an extension lemma for weights that satisfy the A_p property when restricted to a cube.

Lemma 2.1.1. *Fix a cube $Q_0 \subset \mathbb{R}^n$, $1 < p < \infty$ and a weight $w \in A_p(Q_0)$. Then there exists a weight $w_{Q_0} \in A_p(\mathbb{R}^n)$ that coincides with w on Q_0 such that $[w_{Q_0}]_{A_p} = [w]_{A_p(Q_0)}$.*

Proof. Our proof proceeds by construction. Let $\mathcal{D}^{Q_0}$ denote a dyadic system of cubes on $\mathbb{R}^n$ for which Q_0 is a member. This can be explicitly constructed as follows. First, scale the standard dyadic grid by a factor of $l(Q_0)$ to form the collection $l(Q_0)\cdot\Delta$ that consists of all cubes of the form

$$\left[m_1 2^k l(Q_0), (m_1+1)\,2^k l(Q_0)\right) \times \cdots \times \left[m_n 2^k l(Q_0), (m_n+1)\,2^k l(Q_0)\right)$$

where $k, m_1, \cdots, m_n \in \mathbb{Z}$. Then, if we let b_{Q_0} denote the corner of the cube Q_0 closest to the origin, we can translate this scaled grid to Q_0,

$$\mathcal{D}^{Q_0} := l(Q_0)\cdot\Delta + b_{Q_0} := \left\{Q + b_{Q_0} : Q \in l(Q_0)\cdot\Delta\right\}.$$

Let $\mathcal{D}_0^{Q_0}$ denote the subcollection that consists of all cubes in $\mathcal{D}^{Q_0}$ of the same size as Q_0. A weight, w_{Q_0} on $\mathbb{R}^n$, will be constructed for which there exists $B > 0$ such that

$$w_{Q_0}^{-\frac{1}{p-1}}(Q)^{\frac{p-1}{p}} w_{Q_0}(Q)^{\frac{1}{p}} \leq B\,|Q| \tag{2.3}$$

for all $Q \in \mathcal{D}^{Q_0}$. As the dyadic description of $A_p(\mathbb{R}^n)$ is scale and translation invariant, this criteria will be sufficient to determine that $w_{Q_0} \in A_p(\mathbb{R}^n)$.

Fix $Q \in \mathcal{D}_0^{Q_0}$. Let $\varphi_Q : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation that takes the cube Q to the cube Q_0. Then, for $x \in Q$, define

$$w_{Q_0}(x) := w\left(\varphi_Q(x)\right).$$

As the cubes in $\mathcal{D}_0^{Q_0}$ partition $\mathbb{R}^n$, this description defines a unique function w_{Q_0} on $\mathbb{R}^n$. Moreover, it is clear that this function will be a weight that coincides with w on Q_0.

By definition, as $w \in A_p(Q_0)$, it follows that there must exist a $C > 0$ such that (2.3) is satisfied for all dyadic subcubes $Q \subset Q_0$. Fix a cube $Q \in \mathcal{D}^{Q_0}$. Suppose that Q is a dyadic subcube of a cube from $\mathcal{D}_0^{Q_0}$. Then (2.3) must be satisfied automatically with constant C. So suppose that Q is not a dyadic subcube of any cube in $\mathcal{D}_0^{Q_0}$. Then, since a parent cube is always decomposable into its children, there must exist finitely many cubes $\{Q_i\}_{i=1}^N \subset \mathcal{D}_0^{Q_0}$ such that $Q = \sqcup_{i=1}^N Q_i$. We then have

$$\begin{aligned} w_{Q_0}^{-\frac{1}{p-1}}(Q)^{\frac{p-1}{p}} w_{Q_0}(Q)^{\frac{1}{p}} &= \left(\int_Q w_{Q_0}(y)^{-\frac{1}{p-1}}\, dy\right)^{\frac{p-1}{p}} \left(\int_Q w_{Q_0}(y)\, dy\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^N \int_{Q_i} w_{Q_0}(y)^{-\frac{1}{p-1}}\, dy\right)^{\frac{p-1}{p}} \left(\sum_{i=1}^N \int_{Q_i} w_{Q_0}(y)\, dy\right)^{\frac{1}{p}} \\ &= \left(N \int_{Q_0} w^{-\frac{1}{p-1}}(y)\, dy\right)^{\frac{p-1}{p}} \left(N \int_{Q_0} w(y)\, dy\right)^{\frac{1}{p}} \\ &\leq CN\, |Q_0| \\ &= C\, |Q|\, . \end{aligned}$$

□

Definition 2.1.2. *Fix $1 < p < \infty$. A weight w on $\mathbb{R}^n$ is said to be in the class A_p^{loc} if there exists a constant $C > 0$ such that*

$$[w]_{A_p(N(R))} \leq C$$

for all $R \in \Delta_0^\gamma$. The smallest such constant will be denoted by $[w]_{A_p^{loc}}$.

The subsequent lemma will be used numerous times throughout this investigation. It states the exact form of the heat kernel corresponding to $\mathcal{L}$. Its proof can be found in [51] in dimension 1. Higher dimensions follow from this case by taking tensor products of Hermite functions.

Lemma 2.1.2. *For $t > 0$, define the map $k_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ through*

$$k_t(x, y) = h_t(x, y) \cdot \exp\left(-\alpha(t)\left(|x|^2 + |y|^2\right)\right), \tag{2.4}$$

where h_t is the classical heat kernel

$$h_t(x, y) := \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right)$$

and α is defined by

$$\alpha(t) := \frac{\sqrt{1+t^2} - 1}{2t}$$

for all x and y in $\mathbb{R}^n$. The operator $\mathcal{T}^*$ is then given by

$$\mathcal{T}^* f(x) := \sup_{t>0} \int_{\mathbb{R}^n} k_t(x,y)\,|f(y)|\,dy$$

for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Note that the heat kernel for $\mathcal{L}$ is actually $k_{\sinh 2t}$. We have chosen to rescale the kernel for simplicity. An expanded version of Theorem B is presented and proved below.

Theorem B. *Let T^* and M denote the classical heat maximal operator and Hardy-Littlewood operator respectively. Let w be a weight on $\mathbb{R}^n$. For any $1 < p < \infty$, the following statements are equivalent.*

1. $\|M_{loc}\|_{L^p(w)\to L^p(w)} < \infty$.

2. $w \in A_p^{loc}$.

3. $\|T^*_{loc}\|_{L^p(w)\to L^p(w)} < \infty$.

4. $\|\mathcal{T}^*_{loc}\|_{L^p(w)\to L^p(w)} < \infty$.

Proof. We will prove the following chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Fix a cube $R \in \Delta_0^\gamma$, Q a dyadic subcube of $N(R)$ and $f \in L^1_{loc}(\mathbb{R}^n)$. Define $C := \|M_{loc}\|_{L^p(w)\to L^p(w)}$. Then, using standard techniques from weighted theory,

$$\begin{aligned}
\left(\int_Q w\right)\left(\frac{1}{|Q|}\int_Q |f|\right)^p &= \int_Q \left(\frac{1}{|Q|}\int_Q |f|\right)^p w(y)\,dy \\
&\le \int_Q M_{loc}(f\cdot\chi_Q)(y)^p w(y)\,dy \\
&\le \|M_{loc}(f\cdot\chi_Q)\|^p_{L^p(w)} \\
&\le C^p\,\|f\cdot\chi_Q\|^p_{L^p(w)} \\
&= C^p\left(\int_Q |f|^p\,w\right).
\end{aligned}$$

Take $f := (w+\varepsilon)^{-\frac{1}{p-1}}$ for some $\varepsilon > 0$. Then

$$w(Q)\left(\frac{1}{|Q|}\int_Q (w(y)+\varepsilon)^{-\frac{1}{p-1}}\,dy\right)^p \le C^p \int_Q \frac{w(y)}{(w(y)+\varepsilon)^{\frac{p}{p-1}}}\,dy,$$

which implies that

$$w(Q)\left(\int_Q (w(y)+\varepsilon)^{-\frac{1}{p-1}}\,dy\right)^p \le C^p\,|Q|^p\int_Q \frac{(w(y)+\varepsilon)}{(w(y)+\varepsilon)^{\frac{p}{p-1}}}\,dy,$$

$$\Rightarrow\; w(Q)\left(\int_Q (w(y)+\varepsilon)^{-\frac{1}{p-1}}\,dy\right)^{p-1} \le C^p\,|Q|^p$$

for each $\varepsilon > 0$. An application of the Lebesgue monotone convergence theorem then produces the desired result.

(2) $\Rightarrow$ (3). Lemma 2.1.1 states that for any cube $R \in \Delta_0^\gamma$ the restriction $w|_{N(R)}$ can be extended to an A_p weight $w_{N(R)}$. As $w_{N(R)} \in A_p$, we know from Theorem 1.1.4 that $\|T^*\|_{L^p(w_{N(R)})\to L^p(w_{N(R)})} \lesssim \left[w_{N(R)}\right]_{A_p} < \infty$. Then, for $f \in L^p(w)$,

$$\begin{aligned}
\|T^*_{loc}f\|^p_{L^p(w)} &= \int_{\mathbb{R}^n} T^*_{loc}f(x)^p w(x)\,dx\\
&= \sum_{R\in\Delta_0^\gamma}\int_R T^*(f\cdot\chi_{N(R)})(x)^p w(x)\,dx\\
&\le \sum_{R\in\Delta_0^\gamma}\int_{\mathbb{R}^n} T^*(f\cdot\chi_{N(R)})(x)^p w_{N(R)}(x)\,dx\\
&\lesssim \sum_{R\in\Delta_0^\gamma}\left[w_{N(R)}\right]^p_{A_p}\int_{N(R)} |f(x)|^p\,w_{N(R)}(x)\,dx\\
&\le [w]^p_{A_p^{loc}}\sum_{R\in\Delta_0^\gamma}\int_{N(R)} |f(x)|^p\,w(x)dx\\
&\lesssim [w]^p_{A_p^{loc}}\int_{\mathbb{R}^n} |f(x)|^p\,w(x)\,dx,
\end{aligned}$$

where the final inequality was obtained from the bounded overlap property of the cubes $\{N(R)\}_{R\in\Delta_0^\gamma}$.

(3) $\Rightarrow$ (4). This follows trivially from the inequality $k_t(x,y) \le h_t(x,y)$ for all x, $y \in \mathbb{R}^n$ and $t > 0$.

(4) $\Rightarrow$ (1). Fix $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in R \in \Delta_0^\gamma$. Let Q be any cube containing x that satisfies $Q \subseteq N(R)$. We first observe that for any $y \in Q$,

$$\exp\left(-\frac{|x-y|^2}{2l(Q)^2}\right) \simeq 1.$$

To see this, note that

$$|x-y| \le \sqrt{n}\cdot l(Q).$$

This implies that

$$-\frac{|x-y|^2}{2l(Q)^2} \geq -\frac{n}{2},$$

and therefore

$$\exp(-\frac{|x-y|^2}{2l(Q)^2}) \gtrsim 1.$$

Moreover, we trivially have

$$\exp(-\frac{|x-y|^2}{2l(Q)^2}) \leq 1.$$

Note that for any x, $y \in Q$, since $l(Q) \leq 4l(R)$, we have the bound

$$|x|, |y| \leq 2\sqrt{n}2^{j(R)} = \frac{2\sqrt{n}}{l(R)} \leq \frac{8\sqrt{n}}{l(Q)}.$$

This then implies that

$$\exp\left(-\frac{\left(\sqrt{1+l(Q)^4}-1\right)}{2l(Q)^2}\left(|x|^2+|y|^2\right)\right) \geq \exp\left(-\frac{8^2n\left(\sqrt{1+l(Q)^4}-1\right)}{l(Q)^4}\right).$$

It is easy to show that the bound

$$\frac{\sqrt{1+t^4}-1}{t^4} \leq \frac{1}{2}$$

is satisfied for all $t > 0$. This then gives us

$$\exp\left(-\frac{\left(\sqrt{1+l(Q)^4}-1\right)}{2l(Q)^2}\left(|x|^2+|y|^2\right)\right) \geq e^{-\frac{8^2n}{2}}.$$

For $t := l(Q)^2$, we then have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |f(y)|\,dy \\
&\quad\lesssim \frac{1}{l(Q)^n}\int_Q \exp\left(-\frac{\left(\sqrt{1+l(Q)^4}-1\right)}{2l(Q)^2}\left(|x|^2+|y|^2\right)\right)\exp\left(-\frac{|x-y|^2}{2l(Q)^2}\right)|f(y)|\,dy \\
&\quad= \int_Q \frac{1}{t^{n/2}}\exp\left(-\frac{(\sqrt{1+t^2}-1)}{2t}\left(|x|^2+|y|^2\right)\right)\exp\left(-\frac{|x-y|^2}{2t}\right)|f(y)|\,dy \\
&\quad\lesssim \int_Q k_t(x,y)\,|f(y)|\,dy \\
&\quad\lesssim \mathcal{T}^*_{loc}f(x).
\end{aligned}$$

On taking the supremum over all such Q, we obtain $M_{loc}f(x) \lesssim \mathcal{T}^*_{loc}f(x)$. □

2.2. The Far Class

In this section, the adapted operators $\mathcal{M}^-_{far}$ and $\mathcal{M}^+_{far}$ are defined and Theorem 4.1.2 is proved. With this, a sufficient condition for the boundedness of $\|\mathcal{T}^*\|_{L^p(w)\to L^p(w)}$ is obtained. Prior to presenting these definitions, it is necessary to introduce a collection of cubes that represent the regions over which our averaging operators will act.

Definition 2.2.1. *For each $R \in \Delta_0^\gamma$, define the following subsets of $\mathbb{R}^n$.*

- *$Q_0(R)$ is the smallest cube containing the region*

$$\left\{y \in \mathbb{R}^n : |y| \leq 2^{16} n^4 2^{j(R)}\right\},$$

that can be decomposed into cubes from the grid Δ_0^γ.

- *For $t \leq 2^4 n^2$, $Q_t(R) := Q_0(R)$.*

- *For $t \geq 2^4 n^2$, $Q_t(R)$ is the smallest cube containing the region*

$$\left\{y \in \mathbb{R}^n : |y| \leq 2^8 t^2 2^{j(R)}\right\},$$

that can be decomposed into cubes from the grid Δ_0^γ.

Notation. For sets A and B contained in $\mathbb{R}^n$, introduce the notation $k_t^+(A, B)$ and $k_t^-(A, B)$ to denote respectively the supremum and infimum of $k_t(x, y)$ over all $x \in A$ and $y \in B$.

Definition 2.2.2. *For $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in R \in \Delta_0^\gamma$, define the operators $\mathcal{M}^+_{far}$ and $\mathcal{M}^-_{far}$ through*

$$\begin{aligned}
\mathcal{M}^+_{far} f(x) &:= \sup_{t>0} \sum_{R' \in \mathcal{F}(R),\ R' \subset Q_t(R)} k_t^+(R, R') \int_{R'} |f(y)|\, dy \qquad \textit{and} \\
\mathcal{M}^-_{far} f(x) &:= \sup_{t>0} \sum_{R' \in \mathcal{F}(R),\ R' \subset Q_t(R)} k_t^-(R, R') \int_{R'} |f(y)|\, dy.
\end{aligned} \tag{2.5}$$

With the introduction of our maximal functions, it is a straightforward matter to define their corresponding weight classes.

Definition 2.2.3. *For $1 < p < \infty$, the classes of weights on $\mathbb{R}^n$, A_p^{far+} and A_p^{far-}, are* *defined through*

$$A_p^{far+} := \left\{ w \text{ weight on } \mathbb{R}^n : \left\| \mathcal{M}_{far}^{+} \right\|_{L^p(w) \to L^p(w)} < \infty \right\} \quad \text{and}$$
$$A_p^{far-} := \left\{ w \text{ weight on } \mathbb{R}^n : \left\| \mathcal{M}_{far}^{-} \right\|_{L^p(w) \to L^p(w)} < \infty \right\}. \tag{2.6}$$

We then define $A_p^+ := A_p^{far+} \cap A_p^{loc}$ *and* $A_p^- := A_p^{far-} \cap A_p^{loc}$.

In order to verify our main result, a string of technical lemmas must first be proved. The first two of these provide some valuable estimates concerning the maximum of the function $t \mapsto k_t(x,y)$ for fixed x and y in $\mathbb{R}^n$.

Lemma 2.2.1. *Fix points* $x \in R \in \Delta_0^\gamma$ *and* $y \notin Q_0(R)$. *There is precisely one maximum for the function* $t \mapsto k_t(x,y)$. *Denote this point by* $t_m(x,y)$. *Then for* R *not contained in the first layer,* $t_m(x,y)$ *must satisfy*

$$\frac{|y|}{9 \cdot n\,|x|} \leq t_m(x,y) \leq \frac{|x-y|^2}{n}. \tag{2.7}$$

For R *contained in the first layer,* $t_m(x,y)$ *will satisfy*

$$\frac{|y|}{9 \cdot n} \leq t_m(x,y) \leq \frac{|x-y|^2}{n}. \tag{2.8}$$

Proof. On differentiating expression (2.4) with respect to t we obtain

$$\frac{\partial}{\partial t} k_t(x,y) = \frac{1}{2t^2} g(t) k_t(x,y),$$

where the function g is defined to be

$$g(t) := -n \cdot t + \frac{\left(|x|^2 + |y|^2\right)}{\sqrt{1+t^2}} - 2\langle x, y \rangle.$$

As the kernel $k_t(x,y)$ is always positive, it follows that the sign of the derivative will be identical to the sign of the function $g(t)$. Suppose that g is negative. Then we must have

$$(n \cdot t + 2\langle x, y \rangle)\sqrt{1+t^2} > \left(|x|^2 + |y|^2\right)$$

That is, the derivative of the kernel will be negative if and only if the above inequality holds. Likewise, the derivative of the kernel will be positive if and only if

$$(n \cdot t + 2\langle x, y\rangle)\sqrt{1+t^2} < \left(|x|^2 + |y|^2\right) \tag{2.9}$$

and the derivative will vanish if and only if equality holds.

It is simple to show that $|x-y|^2/n$ serves as the only maximum of the function $t \mapsto h_t(x,y)$. This implies that $h_t(x,y)$ is decreasing for $t > |x-y|^2/n$. As the function $\alpha(t)$ is strictly increasing, we have that

$$\exp\left(-\alpha(t)\left(|x|^2+|y|^2\right)\right)$$

is strictly decreasing for all t. This shows that $k_t(x,y)$ is strictly decreasing for $t > |x-y|^2/n$. It then follows that any maximum for $t \mapsto k_t(x,y)$ must be less than $|x-y|^2/n$. As this function must approach 0 as t approaches 0, continuity of the derivative then implies that there must exist at least one maximum in the interval $\left[0, |x-y|^2/n\right]$.

Let $t_m(x,y)$ denote the largest maximum in the above interval. It will be shown that $t_m(x,y)$ is the only maximum. From our previous argument, equality will hold in (2.9) for the value $t_m(x,y)$. Suppose that $t_0 < t_m(x,y)$. Then $t_0 = t_m(x,y) - a$ for some $a > 0$. We then have

$$\begin{aligned}(n \cdot t_0 + 2\langle x, y\rangle)\sqrt{1+t_0^2} &= (n \cdot t_m(x,y) - n \cdot a + 2\langle x, y\rangle)\sqrt{1+t_0^2}\\ &= (n \cdot t_m(x,y) + 2\langle x, y\rangle)\sqrt{1+t_0^2} - n \cdot a\sqrt{1+t_0^2}.\end{aligned}$$

As equality holds in expression (2.9) for $t_m(x,y)$, it follows that the factor $n \cdot t_m(x,y) + 2\langle x, y\rangle$ must be positive. Therefore

$$\begin{aligned}(n \cdot t_0 + 2\langle x, y\rangle)\sqrt{1+t_0^2} &\leq (n \cdot t_m(x,y) + 2\langle x, y\rangle)\sqrt{1+t_m(x,y)^2} - n \cdot a\sqrt{1+t_0^2}\\ &= \left(|x|^2+|y|^2\right) - n \cdot a\sqrt{1+t_0^2}\\ &< \left(|x|^2+|y|^2\right).\end{aligned}$$

This demonstrates that the derivative must be positive for any $t_0 < t_m(x,y)$.

Let's now show the lower bound for $t_m(x,y)$. First suppose that R is not contained in the first layer. It will be shown that for any $t_1 < |y|/(9 \cdot n|x|)$, inequality (2.9) holds. From our previous argument, this will then imply that the function is increasing on the interval $[0, |y|/(9 \cdot n|x|))$. As $y \notin Q_0(R)$, it follows that y satisfies the bound $|y| > 3|x|$.

We know that

$$\begin{aligned}
1+t_1^2 &< 1+\frac{1}{9}\left(\frac{|y|}{3\,|x|}\right)^2 \\
&= 1+\left(\frac{|y|}{3\,|x|}\right)^2-\frac{8}{9}\left(\frac{|y|}{3\,|x|}\right)^2 \\
&\leq 1+\left(\frac{|y|}{3\,|x|}\right)^2-\frac{8}{9} \\
&= \frac{1}{9}\left(1+\frac{|y|^2}{|x|^2}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
(n\cdot t_1+2\langle x,y\rangle) &\leq (n\cdot t_1+2\,|\langle x,y\rangle|) \\
&\leq \left(\frac{|y|}{9\,|x|}+2\,|x|\,|y|\right) \\
&\leq \left(\frac{|y|}{|x|}\right)\left(\frac{1}{9}+2\,|x|^2\right) \\
&\leq \left(\frac{|y|}{|x|}\right)3\,|x|^2.
\end{aligned}$$

This demonstrates that

$$\begin{aligned}
(n\cdot t_1+2\langle x,y\rangle)\sqrt{1+t_1^2} &< (3\,|x|\,|y|)\cdot\frac{1}{3}\sqrt{1+\frac{|y|^2}{|x|^2}} \\
&= |y|\sqrt{|x|^2+|y|^2} \\
&\leq \left(|x|^2+|y|^2\right).
\end{aligned}$$

Now suppose that R is in the first layer and $y\notin Q_0(R)$. Then $|y|\geq 2^{16}n^4$. Let $t_2 < |y|\,/\,(9n)$. Then

$$\begin{aligned}
\left(1+t_2^2\right) &< \left(1+\left(\frac{|y|}{9n}\right)^2\right) \\
&\leq \left(\frac{|y|^2}{2^{32}n^8}+\frac{|y|^2}{9^2n^2}\right) \\
&\leq \frac{2\,|y|^2}{9^2n^2}.
\end{aligned}$$

On noting that $|x| \leq \sqrt{n}$,

$$\begin{aligned}(n \cdot t_2 + 2\langle x, y\rangle) \leq (n \cdot t_2 + 2\,|\langle x, y\rangle|) &\leq \left(\frac{|y|}{9} + 2\,|x|\,|y|\right) \\ &\leq \left(\frac{1}{9} + 2\,|x|\right)|y| \leq \left(\frac{1}{9} + 2\sqrt{n}\right)|y| \\ &\leq 3\sqrt{n}\,|y|\,.\end{aligned}$$

This finally leads to

$$\begin{aligned}(n \cdot t_2 + 2\langle x, y\rangle)\sqrt{1 + t_2^2} &< \left(3\sqrt{n}\,|y|\right)\left(\frac{\sqrt{2}\,|y|}{9n}\right) \\ &\leq |y|^2 \\ &\leq \left(|x|^2 + |y|^2\right),\end{aligned}$$

which validates our lower bound. □

Lemma 2.2.2. *Fix cubes R and R' in Δ_0^γ with $R' \subset Q_0(R)^c$. Fix points $x \in R$ and $y \in R'$. The maximum $t_m(x, y)$ satisfies the inequality,*

$$2 \leq 8 \cdot t_m(x, y)\sqrt{\frac{2^{j(R)+j(R')}}{|x|^2 + |y|^2}} \leq \frac{t_m(x, y)}{2^4 n^2}. \tag{2.10}$$

Proof. As $y \notin Q_0(R)$, we have $|y| \geq 2^{16}n^4 2^{j(R)}$ and also $|y| \geq 2^{j(R')-1}$. The upper inequality then follows from

$$\begin{aligned}|x|^2 + |y|^2 &\geq |y|^2 \\ &\geq 2^{j(R')-1}2^{16}n^4 2^{j(R)} \\ &= 2^{15}n^4 2^{j(R)+j(R')}.\end{aligned}$$

As for the lower bound, first consider when R is not in the first layer. On applying Lemma 2.2.1 and recalling that $|y| \geq |x|$,

$$\begin{aligned}t_m(x, y)\sqrt{\frac{2^{j(R)+j(R')}}{|x|^2 + |y|^2}} &\geq \frac{|y|}{9n\,|x|}\sqrt{\frac{2^{j(R)+j(R')}}{|x|^2 + |y|^2}} \\ &\geq \frac{1}{9n}\sqrt{\frac{|y|^2\,2^{j(R)+j(R')}}{2\,|x|^2\,|y|^2}}.\end{aligned}$$

Then, on applying the bounds $|x| \leq \sqrt{n}2^{j(R)}$, $|y| \leq \sqrt{n}2^{j(R')}$ and $|y| \geq 2^{16}n^4 2^{j(R)}$ successively we obtain

$$\begin{aligned} t_m(x,y)\sqrt{\frac{2^{j(R)+j(R')}}{|x|^2+|y|^2}} &\geq \frac{1}{9n}\sqrt{\frac{2^{j(R)+j(R')}}{2n2^{2j(R)}}} \\ &\geq \frac{1}{9n}\sqrt{\frac{|y|}{2n^{3/2}2^{j(R)}}} \\ &\geq \frac{1}{9n}\sqrt{\frac{2^{16}n^4 2^{j(R)}}{2n^{3/2}2^{j(R)}}} \\ &\geq 2. \end{aligned}$$

Next, consider when R is in the first layer. Once again apply Lemma 2.2.1 and $|y| \geq |x|$ to obtain

$$\begin{aligned} t_m(x,y)\sqrt{\frac{2^{j(R)+j(R')}}{|x|^2+|y|^2}} &\geq \frac{|y|}{9n}\sqrt{\frac{2^{j(R')}}{2|y|^2}} \\ &= \frac{1}{9n}\sqrt{\frac{2^{j(R')}}{2}}. \end{aligned}$$

Then, on successively applying the bounds $|y| \leq \sqrt{n}2^{j(R')}$ and $|y| \geq 2^{16}n^4$,

$$\begin{aligned} t_m(x,y)\sqrt{\frac{2^{j(R)}+2^{j(R')}}{|x|^2+|y|^2}} &\geq \frac{1}{9n}\sqrt{\frac{|y|}{2\sqrt{n}}} \\ &\geq \frac{1}{9n}\sqrt{\frac{2^{16}n^4}{2\sqrt{n}}} \\ &\geq 2. \end{aligned}$$

This concludes the proof. $\square$

The next lemma obtains an estimate on ratios of the form $k_t(x,y)\cdot k_{t_m(x,y)}(x,y)^{-1}$ for fixed x and y. It will play a key role in the proof of Theorem 4.1.2.

Lemma 2.2.3. *Fix cubes R and R' in Δ_0^γ with $R' \subset Q_0(R)^c$. Fix the points $x \in R$ and $y \in R'$. Introduce the shorthand notation $t_m := t_m(x,y)$. Define*

$$M := 8 \cdot t_m\sqrt{\frac{2^{j(R)+j(R')}}{|x|^2+|y|^2}}.$$

Then for all $t \leq t_m/M = \frac{1}{8}\sqrt{\frac{|x|^2+|y|^2}{2^{j(R)+j(R')}}}$ we must have the bound

$$k_t(x,y)\cdot k_{t_m}(x,y)^{-1} \lesssim \frac{1}{2^{(j(R)+j(R'))(n+1)}} \tag{2.11}$$

Proof. According to Lemma 2.2.2, $t_m/M \leq t_m$. As $t \mapsto k_t(x,y)$ is increasing for $t \leq t_m(x,y)$, it follows that it is sufficient to show (2.11) for the value t_m/M. We then have

$$k_{t_m/M}(x,y)\cdot k_{t_m}(x,y)^{-1} = M^{d/2}\exp\left((\alpha(t_m)-\alpha(t_m/M))\left(|x|^2+|y|^2\right)\right)\cdot\exp\left(-\frac{|x-y|^2}{2t_m}(M-1)\right).$$

Let's find a bound on the function $\alpha(t_m)-\alpha(t_m/M)$ in terms of t_m and M. Define the function $\beta : (0,\infty)\to\mathbb{R}$ through

$$\beta(u) := \alpha\left(\frac{1}{u}\right) = \frac{\sqrt{1+\frac{1}{u^2}}-1}{2/u} = \frac{\sqrt{1+u^2}-u}{2}.$$

For any $u \leq 1$, perform a Taylor expansion about the origin for β to obtain

$$\beta(u) = \frac{1}{2}\left(1-u+\frac{u^2}{2}-\frac{u^4}{8}+\frac{u^6}{16}-\cdots\right).$$

According to Lemma 2.2.2, both t_m and t_m/M are greater than 1. The above formula will therefore apply to these values.

$$\alpha(t_m) = \beta(1/t_m) = \frac{1}{2}\left(1-\frac{1}{t_m}+\frac{1}{2t_m^2}-\frac{1}{8t_m^4}+\frac{1}{16t_m^6}-\cdots\right),$$

$$\alpha(t_m/M) = \frac{1}{2}\left(1-\frac{M}{t_m}+\frac{M^2}{2t_m^2}-\frac{M^4}{8t_m^4}+\frac{M^6}{16t_m^6}-\cdots\right).$$

Which gives

$$\begin{aligned}\alpha(t_m)-\alpha(t_m/M) &= \frac{(M-1)}{2t_m}-\frac{(M^2-1)}{4t_m^2}+\frac{(M^4-1)}{16t_m^4}-\frac{(M^6-1)}{32t_m^6}+\cdots\\ &\leq \frac{(M-1)}{2t_m}-\frac{(M^2-1)}{4t_m^2}+\frac{(M^4-1)}{16t_m^4}.\end{aligned}$$

As $M^2-1 \geq \frac{M^2}{2}$ and $\frac{(M^4-1)}{16t_m^4} \leq \frac{M^2}{16t_m^2}$, we obtain

$$\alpha(t_m)-\alpha(t_m/M) \leq \frac{(M-1)}{2t_m}-\frac{M^2}{16t_m^2}. \tag{2.12}$$

Once more from Lemma 2.2.2, we have that

$$\begin{aligned}M^{n/2}2^{(j(R)+j(R'))(n+1)} &\leq t_m^{n/2}2^{(j(R)+j(R'))(n+1)}\\ &\lesssim |y-x|^{n/2}\,2^{(j(R)+j(R'))(n+1)}\\ &\leq (|y|+|x|)^{n/2}2^{(j(R)+j(R'))(n+1)}\\ &\lesssim (2^{j(R)}+2^{j(R')})^{n/2}2^{(j(R)+j(R'))(n+1)}.\end{aligned}$$

It is easy to see that there must exist some $A \geq 0$, independent of both R and R', such that

$$\left(2^{j(R)} + 2^{j(R')}\right)^{n/2} 2^{(j(R)+j(R'))(n+1)} \leq Ae^{2^{j(R)+j(R')}}.$$

This would then give

$$M^{n/2}2^{(j(R)+j(R'))(n+1)} \lesssim e^{2^{j(R)+j(R')}}.$$

On applying (2.12) and the above,

$$\begin{aligned}
&k_{t_m/M}(x,y) \cdot k_{t_m}(x,y)^{-1}2^{(j(R)+j(R'))(n+1)} \\
&\quad\lesssim M^{n/2}2^{(j(R)+j(R'))(n+1)} \exp\left((\alpha(t_m) - \alpha(t_m/M))\left(|x|^2 + |y|^2\right)\right) \cdot \exp\left(-\frac{|x-y|^2}{2t_m}(M-1)\right) \\
&\quad\lesssim \exp\left(2^{j(R)+j(R')}\right) \cdot \exp\left(\left(\frac{(M-1)}{2t_m} - \frac{M^2}{16t_m^2}\right)\left(|x|^2 + |y|^2\right)\right) \cdot \exp\left(-\frac{|x-y|^2}{2t_m}(M-1)\right) \\
&\quad= \exp\left(2^{j(R)+j(R')} + \frac{(M-1)}{t_m}\langle x, y\rangle - \frac{M^2}{16t_m^2}\left(|x|^2 + |y|^2\right)\right) \\
&\quad\leq \exp\left(2^{j(R)+j(R')} + \frac{(M-1)}{t_m}|x|\,|y| - \frac{M^2}{16t_m^2}\left(|x|^2 + |y|^2\right)\right).
\end{aligned}$$

On applying $M/t_m \leq 1/\left(2^4n^2\right)$,

$$\begin{aligned}
k_{t_m/M}(x,y) \cdot k_{t_m}(x,y)^{-1}2^{(j(R)+j(R'))(n+1)}& \\
&\lesssim \exp\left(2^{j(R)+j(R')} + \frac{|x|\,|y|}{2^4n^2} - \frac{M^2}{16t_m^2}\left(|x|^2 + |y|^2\right)\right) \\
&\lesssim \exp\left(2^{j(R)+j(R')} + \frac{2^{j(R)+j(R')}}{2^4n} - \frac{M^2}{16t_m^2}\left(|x|^2 + |y|^2\right)\right).
\end{aligned}$$

From which the definition of M then provides

$$k_{t_m/M}(x,y) \cdot k_{t_m}(x,y)^{-1}2^{(j(R)+j(R'))(n+1)} \lesssim 1.$$

□

The next result is a direct analogue for A_p^+ of the defining condition for the classic A_p class. It is unlikely that this condition is enough to completely characterise A_p^+.

Lemma 2.2.4. *Let w be a weight on $\mathbb{R}^n$ and suppose that $\mathcal{M}_{far}^+ : L^p(w) \to L^p(w)$ is bounded for some $1 < p < \infty$. Fix cubes R and R' in Δ_0^γ with $R' \not\subset Q_0(R)$. Then there must exist some constant $C > 0$, independent of both R and R', such that for all $\tilde{x} \in R$ and $\tilde{y} \in R'$ we have*

$$w(R)^{\frac{1}{p}} \cdot w^{-\frac{1}{p-1}}(R')^{\frac{p-1}{p}} \leq C \cdot k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^{-1}. \tag{2.13}$$

Proof. It shall first be shown that

$$R' \subset Q_{t_m(\tilde{x},\tilde{y})}(R).$$

Fix any point $y \in R'$. From the definition of $Q_t(R)$, it will be sufficient to show that

$$|y| \le 2^2 t_m(\tilde{x},\tilde{y})^2 2^{j(R)}.$$

First suppose that R is not in the first layer. Then

$$2^{j(R)} \cdot t_m(\tilde{x},\tilde{y})^2 \ge 2^{j(R)} \frac{|\tilde{y}|^2}{9^2 n^2 |\tilde{x}|^2}.$$

As $|\tilde{x}| \le \sqrt{n} \cdot 2^{j(R)}$, $|y| \le \sqrt{n} \cdot 2\,|\tilde{y}|$ and $|\tilde{y}| \ge n^4 2^{16} 2^{j(R)}$, we have that

$$\begin{aligned} 2^{j(R)} \cdot t_m(\tilde{x},\tilde{y})^2 &\ge \frac{2^{j(R)}}{9^2 n^2} \cdot \frac{|y|}{2\sqrt{n}} \cdot \frac{n^4 2^{16} 2^{j(R)}}{n 2^{2j(R)}} \\ &\ge |y|\,. \end{aligned}$$

Next suppose that R is contained in the first layer. Then

$$\begin{aligned} 2^{j(R)} \cdot t_m(\tilde{x},\tilde{y})^2 &\ge \frac{|\tilde{y}|^2}{9^2 n^2} \\ &\ge \frac{|y|}{2\sqrt{n}} \cdot \frac{2^{16} n^4}{9^2 n^2} \\ &\ge |y|\,. \end{aligned}$$

This demonstrates that $R' \subset Q_{t_m(\tilde{x},\tilde{y})}(R)$. Then, for any $\tilde{x} \in R$ and $\tilde{y} \in R'$,

$$\begin{aligned} w(R)\left(\int_{R'} |f(y)|\,dy\right)^p &= \int_R w(x)\,dx \cdot \frac{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p} \left(\int_{R'} |f(y)|\,dy\right)^p \\ &= \frac{1}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p} \int_R \left(k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y}) \int_{R'} |f(y)|\,dy\right)^p w(x)\,dx \\ &\le \frac{1}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p} \int_R \mathcal{M}^+_{far}(f \cdot \chi_{R'})(x)^p w(x)\,dx. \end{aligned}$$

From the boundedness of $\mathcal{M}^+_{far}$, we then obtain

$$w(R)\left(\int_{R'} |f(y)|\,dy\right)^p \lesssim \frac{1}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p} \int_{R'} |f(y)|^p\, w(y)\,dy.$$

Take $f := (w+\varepsilon)^{-\frac{1}{p-1}}$ for some $\varepsilon > 0$. Then

$$w(R)\left(\int_{R'} (w(y)+\varepsilon)^{-\frac{1}{p-1}}\, dy\right)^p \lesssim \frac{1}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p} \int_{R'} \frac{w(y)}{(w(y)+\varepsilon)^{\frac{p}{p-1}}}\, dy$$

for all $\varepsilon > 0$. Which implies that

$$w(R)\left(\int_{R'} (w(y)+\varepsilon)^{-\frac{1}{p-1}}\, dy\right)^p \lesssim \frac{1}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p} \int_{R'} \frac{(w(y)+\varepsilon)}{(w(y)+\varepsilon)^{\frac{p}{p-1}}}\, dy$$

$$\Rightarrow \quad w(R)\left(\int_{R'} (w(y)+\varepsilon)^{-\frac{1}{p-1}}\, dy\right)^{p-1} \lesssim \frac{1}{k_{t_m(\tilde{x},\tilde{y})}(\tilde{x},\tilde{y})^p}$$

for each $\varepsilon > 0$. An application of the Lebesgue monotone convergence theorem then produces the desired result. □

Finally, enough machinery is in place to prove our main result.

Theorem A. *Let w be a weight on $\mathbb{R}^n$ and $1 < p < \infty$. Then we have*

$$\left\|\mathcal{M}^+_{far}\right\|_{L^p(w)} < \infty \quad \Rightarrow \quad \left\|\mathcal{T}^*_{far}\right\|_{L^p(w)} < \infty \quad \Rightarrow \quad \left\|\mathcal{M}^-_{far}\right\|_{L^p(w)} < \infty. \tag{2.14}$$

Proof. The second implication follows quickly from the pointwise bound

$$\begin{aligned}
\mathcal{T}^*_{far} f(x) &= \sup_{t>0} \int_{F(R)} k_t(x,y)\, |f(y)|\, dy \\
&= \sup_{t>0} \sum_{R' \in \mathcal{F}(R)} \int_{R'} k_t(x,y)\, |f(y)|\, dy \\
&\geq \sup_{t>0} \sum_{R' \in \mathcal{F}(R),\ R' \subset Q_t(R)} k^-_t(R,R') \int_{R'} |f(y)|\, dy \\
&= \mathcal{M}^-_{far} f(x)
\end{aligned}$$

for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in R \in \Delta^\gamma_0$.

As for the first implication, suppose that $\left\|\mathcal{M}^+_{far}\right\|_{L^p(w)\to L^p(w)} < \infty$. Then

$$\begin{aligned}
\left\|\mathcal{T}^*_{far} f\right\|_{L^p(w)} &= \left[\int_{\mathbb{R}^n} \left|\mathcal{T}^*_{far} f(x)\right|^p w(x)\, dx\right]^{1/p} \\
&= \left[\int_{\mathbb{R}^n} \left(\sup_{t>0} e^{-t\mathcal{L}} \left(f \cdot \chi_{F(R_x)}\right)(x)\right)^p w(x)\, dx\right]^{1/p} \\
&= \left[\int_{\mathbb{R}^n} \left(\sup_{t>0} \int_{F(R_x)} k_t(x,y)\, |f(y)|\, dy\right)^p w(x)\, dx\right]^{1/p}.
\end{aligned}$$

The heat operators can be expanded dyadically to obtain

$$\begin{aligned}
\left\|\mathcal{T}_{far}^{*}f\right\|_{L^p(w)} &= \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x)}\int_{R'}k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{1/p} \\
&\lesssim \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx\right]^{1/p} \\
&\lesssim \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\subset Q_t(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right.\right. \\
&\qquad\left.\left.+\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\not\subset Q_t(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx\right]^{1/p}.
\end{aligned}$$

On applying Minkowski's inequality,

$$\begin{aligned}
\left\|\mathcal{T}_{far}^{*}f\right\|_{L^p(w)} &\lesssim \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\subset Q_t(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx\right]^{1/p} \\
&\quad+\left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\not\subset Q_t(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx\right]^{1/p} \\
&= \left\|\mathcal{M}_{far}^{+}f\right\|_{L^p(w)}+\left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\not\subset Q_t(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx\right]^{1/p}.
\end{aligned}$$

It remains to bound the tail end term on the right-hand side of the above expression. On expanding dyadically once more,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\not\subset Q_t(R_x)}k_t^+(R_x,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx \\
&\qquad= \sum_{R\in\Delta_0^\gamma}\int_R\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)}k_t^+(R,R')\,\|f\|_{L^1(R')}\right)^p w(x)\,dx \\
&\qquad= \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)}k_t^+(R,R')\,\|f\|_{L^1(R')}\,w(R)^{1/p}\right)^p.
\end{aligned}$$

Let x_R^t and $y_{R'}^t$ denote points contained in R and R' respectively that satisfy

$$k_t^+(R,R')\leq 2\cdot k_t(x_R^t,y_{R'}^t).$$

On applying Hölder's inequality and Lemma 2.2.4 we obtain

$$\begin{aligned}
&\sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)} k_t^+(R,R')\,\|f\|_{L^1(R')}\,w(R)^{1/p}\right)^p\\
&\qquad\lesssim \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)} k_t^+(R,R')w^{-\frac{1}{p-1}}(R')^{\frac{p-1}{p}}w(R)^{\frac{1}{p}}\,\|f\|_{L^p(R',w)}\right)^p\\
&\qquad\lesssim \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)} k_t(x_R^t,y_{R'}^t)\cdot k_{t_m(x_R^t,y_R^t)}(x_R^t,y_{R'}^t)^{-1}\,\|f\|_{L^p(R',w)}\right)^p.
\end{aligned}$$

Note that since $|y_{R'}^t|\geq 2^8t^2 2^{j(R)}$, it follows that

$$\begin{aligned}
\frac{1}{8}\sqrt{\frac{|x_R^t|^2+|y_{R'}^t|^2}{2^{j(R)+j(R')}}} &\geq \frac{1}{8}\sqrt{\frac{|y_{R'}^t|^2}{2^{j(R)+j(R')}}}\\
&\geq \frac{1}{8}\sqrt{\frac{2^{j(R')-1}\cdot 2^8t^22^{j(R)}}{2^{j(R)+j(R')}}}\\
&\geq t.
\end{aligned}$$

This implies that Lemma 2.2.3 can be applied to obtain

$$\begin{aligned}
&\sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)} k_t(x_R^t,y_{R'}^t)\cdot k_{t_m(x_R^t,y_{R'}^t)}(x_R^t,y_{R'}^t)^{-1}\,\|f\|_{L^p(R',w)}\right)^p\\
&\qquad\lesssim \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)} 2^{-(j(R)+j(R'))(n+1)}\,\|f\|_{L^p(R',w)}\right)^p\\
&\qquad\lesssim \|f\|_{L^p(w)}^p\sum_{k=0}^{\infty}\sum_{R\in L_k}\left(\sum_{l=0}^{\infty}\sum_{R'\in L_l}2^{-(k+l)(n+1)}\right)^p\\
&\qquad\lesssim \|f\|_{L^p(w)}^p\sum_{k=0}^{\infty}2^{kn}\left(\sum_{l=0}^{\infty}2^{ln}\cdot 2^{-(k+l)(n+1)}\right)^p\\
&\qquad\lesssim \|f\|_{L^p(w)}^p,
\end{aligned}$$

since the number of cubes in a layer L_k is bounded by a constant multiple of 2^{kn}. $\square$

Theorems A and B, together with the fact that $\|\mathcal{T}^*\|_{L^p(w)}<\infty$ if and only if both $\|\mathcal{T}^*_{loc}\|_{L^p(w)\to L^p(w)}<\infty$ and $\|\mathcal{T}^*_{far}\|_{L^p(w)\to L^p(w)}<\infty$ for any weight w on $\mathbb{R}^n$, lead to the below corollary.

Corollary 2.2.1. *The following chain of inclusions holds for any* $1 < p < \infty$,

$$A_p^+ \subseteq \left\{ w \text{ weight on } \mathbb{R}^n : \|\mathcal{T}^*\|_{L^p(w)\to L^p(w)<\infty} \right\} \subseteq A_p^-. \tag{2.15}$$

The class of weights in the middle of the above chain of inclusions is a natural candidate for the A_p class associated with the harmonic oscillator. The above corollary indicates that our A_p classes are honing in on what should be the correct class.

2.3. Relation to the A_p^∞ Class

Recall the definitions of the classes A_p^∞ and A_p^θ from §1.4. This section is devoted to the proof of the strict inclusion $A_p^\infty \subsetneq A_p^+$. This will be accomplished by first showing, for any $\theta \geq 0$, that the pointwise bound $\mathcal{M}_{far}^+ f(x) \lesssim M^\theta f(x)$ holds for all $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, thereby demonstrating the inclusion $A_p^\theta \subseteq A_p^{far+}$. Recall the heat kernel estimate for $\mathcal{L}_V$ given in Lemma 1.4.1. Due to the rescaling of $\sinh 2t$ introduced following Lemma 2.1.2, our kernels will satisfy

$$k_{\sinh 2t}(x,y) \leq C_N t^{-n/2} \exp\left(-\frac{|x-y|^2}{2t}\right)\left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \tag{2.16}$$

for any $N \in \mathbb{N}$, for some $C_N > 0$, for all x, $y \in \mathbb{R}^n$ and $t > 0$.

Proposition 2.3.1. *For any* $\theta \geq 0$, *there exists some* $C_\theta > 0$ *so that*

$$\mathcal{M}_{far}^+ f(x) \leq C_\theta M^\theta f(x)$$

for every locally integrable function f *on* $\mathbb{R}^n$ *and* $x \in \mathbb{R}^n$.

Proof. For $R \in \Delta_0^\gamma$ and $k \geq 0$, define $\mathcal{C}_k(R)$ to be the collection of cubes $R' \in \Delta_0^\gamma$ that satisfy $d(R,R') < 2^k l(R)$. As $\mathcal{F}(R) \subset \Delta_0^\gamma \setminus \mathcal{C}_0(R)$, the operator $\mathcal{M}_{far}^+$ can be decomposed as

$$\begin{aligned} \mathcal{M}_{far}^+ f(x) &\leq \sup_{t>0} \sum_{R' \in \Delta_0^\gamma \setminus \mathcal{C}_0(R)} k_t^+(R,R') \int_{R'} |f(y)|\,dy \\ &= \sup_{t>0} \sum_{R' \in \Delta_0^\gamma \setminus \mathcal{C}_0(R)} k_{\sinh 2t}^+(R,R') \int_{R'} |f(y)|\,dy \\ &\leq \sup_{t>0} \sum_{k=1}^\infty \sum_{R' \in \mathcal{C}_k(R)\setminus\mathcal{C}_{k-1}(R)} k_{\sinh 2t}^+(R,R') \int_{R'} |f(y)|\,dy \end{aligned}$$

for $x \in R$. Let's find a bound on the values $k_{\sinh 2t}^+(R,R')$ for $R' \subset \mathcal{C}_k(R) \setminus \mathcal{C}_{k-1}(R)$. Suppose that $x \in R$ and $y \in R' \in \mathcal{C}_k(R) \setminus \mathcal{C}_{k-1}(R)$ where $k \geq 1$. Then, $|x-y| \geq 2^{k-1}2^{-j(R)}$. From this bound, (2.16) and the inequality $\rho(x) \leq 2^{1-j(R)}$,

$$\begin{aligned} k_{\sinh 2t}(x,y) &\lesssim t^{-n/2} \exp\left(-\frac{|x-y|^2}{2t}\right)\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-N} \\ &\lesssim t^{-n/2}\frac{t^{M/2}}{|x-y|^M}\left(1+2^{j(R)-1}\sqrt{t}\right)^{-N} \\ &\lesssim t^{-n/2}\left(2^{j(R)}\sqrt{t}\right)^M 2^{-kM}\left(1+2^{j(R)-1}\sqrt{t}\right)^{-N} \\ &\lesssim 2^{j(R)n}2^{-kM}\left(2^{j(R)-1}\sqrt{t}\right)^{M-n}\left(1+2^{j(R)-1}\sqrt{t}\right)^{-N} \end{aligned}$$

for any $M > 0$. Therefore

$$k^{+}_{\sinh 2t}(R,R') \lesssim 2^{j(R)n}2^{-kM}\left(2^{j(R)-1}\sqrt{t}\right)^{M-n}\left(1+2^{j(R)-1}\sqrt{t}\right)^{-N} \tag{2.17}$$

for any $R' \subset \mathcal{C}_k(R) \setminus \mathcal{C}_{k-1}(R)$. On applying this bound to our previous decomposition we find that $\mathcal{M}^{+}_{far}f(x)$ can be estimated above by

$$\sup_{t>0}\sum_{k=1}^{\infty} 2^{j(R)n}2^{-kM}\left(2^{j(R)-1}\sqrt{t}\right)^{M-n}\left(1+2^{j(R)-1}\sqrt{t}\right)^{-N}\sum_{R'\in\mathcal{C}_k(R)\setminus\mathcal{C}_{k-1}(R)}\int_{R'}|f(y)|\,dy.$$

Define R_k to be the smallest cube that contains every cube in the collection $\mathcal{C}_k(R)$. Then

$$\mathcal{M}^{+}_{far}f(x) \lesssim 2^{j(R)n}\sup_{s>0} s^{M-n}(1+s)^{-N}\sum_{k=1}^{\infty}2^{-kM}\int_{R_k}|f(y)|\,dy,$$

where we set $s := 2^{j(R)-1}\sqrt{t}$. It is obvious that if we set $N \geq M-n$, then the supremum term must be bounded by 1. We then obtain

$$\mathcal{M}^{+}_{far}f(x) \lesssim 2^{j(R)n}\sum_{k=1}^{\infty}2^{-kM}\int_{R_k}|f(y)|\,dy.$$

On noting that $l(R_k) \approx 2^k 2^{-j(R)}$ and $\psi_\theta(R_k) \lesssim 2^{k\theta}$, we have

$$\begin{aligned} \mathcal{M}^{+}_{far}f(x) &\lesssim \sum_{k=1}^{\infty}2^{-k(M-n-\theta)}\frac{1}{2^{k\theta}}\left(\frac{2^{j(R)n}}{2^{kn}}\right)\int_{R_k}|f(y)|\,dy \\ &\lesssim \sum_{k=1}^{\infty}2^{-k(M-n-\theta)}\frac{1}{\psi_\theta(R_k)\,|R_k|}\int_{R_k}|f(y)|\,dy \\ &\leq M_\theta f(x) \end{aligned}$$

for $M \geq n+\theta$. □

Proposition C. *The following chain of strict inclusions holds for any* $1 < p < \infty$*,*

$$A_p \subsetneq A_p^\infty \subsetneq A_p^+.$$

Proof. The strict inclusion $A_p \subsetneq A_p^\infty$ has already been proved in [15]. As for the upper inclusion, the previous proposition demonstrates that $A_p^\infty \subseteq A_p^{far+}$. It will now be proved that $A_p^\infty \subseteq A_p^{loc}$. Fix $w \in A_p^\infty$. Then there must exist some $\theta \geq 0$ such that $w \in A_p^\theta$. It must be shown that there exists some $B > 0$ that satisfies

$$[w]_{A_p(N(R))} \leq B \tag{2.18}$$

for every $R \in \Delta_0^\gamma$. Fix any cube $R \in \Delta_0^\gamma$ and Q a dyadic subcube of $N(R)$. As $w \in A_p^\theta$, there must exist some $C > 0$ such that

$$w(Q)^{\frac{1}{p}} w^{-\frac{1}{p-1}}(Q)^{\frac{p-1}{p}} \leq C\,|Q| \left(1 + \frac{l(Q)}{\rho(c_Q)}\right)^\theta.$$

As Q is a dyadic subcube of $N(R)$, we have that $l(Q) \leq 4\rho(c_R)$ and $\rho(c_Q) \geq \rho(c_R)/2$. Therefore

$$\begin{aligned} w(Q)^{\frac{1}{p}} w^{-\frac{1}{p-1}}(Q)^{\frac{p-1}{p}} &\leq C\,|Q|\,(1+8)^\theta \\ &\leq 9^\theta C\,|Q|\,. \end{aligned}$$

This demonstrates that (2.18) holds with constant $B := 9^\theta C$.

It will now be proved that the inclusion of A_p^∞ in A_p^+ is in fact strict. In particular, the weight defined by

$$w(x) = w(x_1, \cdots, x_n) = e^{|x_1|}$$

for $x \in \mathbb{R}^n$ will be shown to belong to the class A_p^+ but not A_p^∞.

Let's first show that $w \in A_p^{loc}$. That is, it will be proved that there exists $C > 0$ such that for any $R \in \Delta_0^\gamma$ and dyadic subcube Q of $N(R)$

$$w(Q) w^{-\frac{1}{p-1}}(Q)^{p-1} \leq C\,|Q|^p\,. \tag{2.19}$$

Note that for any $x = (x_1, \cdots, x_n) \in Q$ we must have the bound

$$\left|c_R^{(1)}\right| - 4 \cdot 2^{-j(R)} \leq |x_1| \leq \left|c_R^{(1)}\right| + 4 \cdot 2^{-j(R)},$$

where $c_R = \left(c_R^{(1)}, \cdots, c_R^{(d)}\right)$. This gives

$$\begin{aligned} w(Q) &= \int_Q e^{|x_1|}\,dx \\ &\lesssim e^{\left|c_R^{(1)}\right|}\,|Q|\,. \end{aligned}$$

Similarly,

$$\begin{aligned} w^{-\frac{1}{p-1}}(Q)^{p-1} &= \left(\int_Q e^{-\frac{|x_1|}{p-1}}\,dx\right)^{p-1} \\ &\lesssim e^{-\left|c_R^{(1)}\right|}\,|Q|^{p-1}\,. \end{aligned}$$

This gives estimate (2.19) and proves that $w \in A_p^{loc}$.

Next let's prove that $w \in A_p^{far+}$. That is, it must be shown that

$$\left\|\mathcal{M}_{far}^{+} f\right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$$

for any $f \in L^p(w)$.

$$\begin{aligned} \left\|\mathcal{M}_{far}^{+} f\right\|_{L^p(w)}^p &= \int_{\mathbb{R}^n} \mathcal{M}_{far}^{+} f(x)^p w(x)\,dx \\ &= \int_{\mathbb{R}^n} \left(\sup_{t>0} \sum_{\substack{R'\in\mathcal{F}(R_x),\\ R'\subset Q_t(R_x)}} k_t^{+}(R_x, R') \int_{R'} |f(y)|\,dy\right)^p w(x)\,dx \\ &= \int_{\mathbb{R}^n} \sup_{t>0} \left(\int_{Q_t(R_x)\cap F(R_x)} k_t^{+}(R_x, R_y)\,|f(y)|\,w(y)^{\frac{1}{p}} w(y)^{-\frac{1}{p}}\,dy\right)^p w(x)\,dx. \end{aligned}$$

On applying Hölder's inequality we obtain

$$\begin{aligned} \left\|\mathcal{M}_{far}^{+} f\right\|_{L^p(w)}^p &\lesssim \left(\int_{\mathbb{R}^n} \sup_{t>0} \left(\int_{Q_t(R_x)\cap F(R_x)} k_t^{+}(R_x, R_y)^{p'} w(y)^{-\frac{p'}{p}}\,dy\right)^{\frac{p}{p'}} w(x)\,dx\right) \|f\|_{L^p(w)}^p \\ &\le \left(\int_{\mathbb{R}^n} \sup_{t>0} \left(\int_{Q_t(R_x)\cap F(R_x)} k_t^{+}(R_x, R_y)^{p'}\,dy\right)^{\frac{p}{p'}} w(x)\,dx\right) \|f\|_{L^p(w)}^p\,. \end{aligned} \tag{2.20}$$

Let $M \ge 1$, the exact value to be determined at a later time. It will now be proved that the function

$$(t, x) \ \mapsto\ \left(\int_{Q_t(R_x)\cap F(R_x)} k_t^{+}(R_x, R_y)^{p'}\,dy\right)^{\frac{p}{p'}} \tag{2.21}$$

is uniformly bounded for $t > 0$ and $x \in [-M, M]^n$. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, let $\tilde{x}$ and $\tilde{y}$ denote points in R_x and R_y respectively that satisfy $k_t(R_x, R_y) \le 2k_t(\tilde{x}, \tilde{y})$. As

$\tilde{y} \in F(R_x) = F(R_{\tilde{x}})$ we must have $|\tilde{x} - \tilde{y}| \geq 2^{-j(R_x)}$. This implies that

$$\begin{aligned} k_t(R_x, R_y) &\lesssim \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2t}\right) \cdot \exp\left(-\alpha(t)\left(|\tilde{x}|^2 + |\tilde{y}|^2\right)\right) \\ &\lesssim \frac{1}{t^{\frac{n}{2}}} \exp\left(-\frac{2^{-2j(R_x)}}{2t}\right) \\ &\lesssim \frac{1}{t^{\frac{n}{2}}} \cdot \frac{1}{(2^{-2j(R_x)}/2t)^{\frac{n}{2}}} \\ &\simeq 2^{nj(R_x)}. \end{aligned}$$

As x is restricted to $[-M, M]^n$, the layer number $j(R_x)$ is bounded implying that $(t, x, y) \mapsto k_t(R_x, R_y)$ is bounded. For $t \leq 1$ the size of $Q_t(R_x)$ is bounded proving that (2.21) is bounded for $t \leq 1$ and $x \in [-M, M]^n$. For $t > 1$ note that

$$\begin{aligned} k_t(R_x, R_y) &\lesssim \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2t}\right) \cdot \exp\left(-\alpha(t)\left(|\tilde{x}|^2 + |\tilde{y}|^2\right)\right) \\ &\lesssim \exp\left(-\alpha(t)\,|\tilde{y}|^2\right). \end{aligned}$$

Since $|y| \leq 2\left(|\tilde{y}| + \sqrt{n}\right)$ and α is an increasing function,

$$\int_{Q_t(R_x)\cap F(R_x)} k_t(R_x, R_y)^{p'}\,dy \lesssim \int_{\mathbb{R}^n} \exp\left(-\alpha(1)\left(\frac{|y|}{2} - \sqrt{n}\right)^2 p'\right) dy$$

which is clearly integrable. This shows that (2.21) is uniformly bounded for $x \in [-M, M]^n$ and $t > 0$. Therefore, to complete the proof of $w \in A_p^{far+}$ it is sufficient to show that

$$\int_{\mathbb{R}^n\setminus[-M,M]^n} \sup_{t>0} \left(\int_{Q_t(R_x)\cap F(R_x)} k_t^+(R_x, R_y)^{p'}\,dy\right)^{\frac{p}{p'}} w(x)\,dx$$

is finite. In fact, due to the form of the kernel, this can be further reduced to proving that

$$\int_{\mathbb{R}_+^n\setminus[0,M]^n} \sup_{t>0} \left(\int_{\mathbb{R}_+^n\cap F(R_x)} k_t^+(R_x, R_y)^{p'}\,dy\right)^{\frac{p}{p'}} w(x)\,dx \tag{2.22}$$

is finite. Note that for any $x \in \mathbb{R}_+^n \setminus [0, M]^n$, $y \in \mathbb{R}_+^n \cap F(R_x)$ we will have the bounds $|x| \leq 4\sqrt{n}\,|\tilde{x}|$, $|y| \leq 4\sqrt{n}\,|\tilde{y}|$ and $|x - y| \leq 4\sqrt{n}\,|\tilde{x} - \tilde{y}|$. This then leads to

$$\begin{aligned} k_t^+(R_x, R_y) &\lesssim k_t(\tilde{x}, \tilde{y}) \\ &\lesssim \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp\left(-\frac{\alpha(t)}{4^2 n}\left(|x|^2 + |y|^2\right)\right) \cdot \exp\left(-\frac{|x-y|^2}{4^2 n \cdot 2t}\right), \end{aligned}$$

implying that (2.22) is bounded from above by a constant multiple of

$$\int_{\mathbb{R}^n_+\setminus[0,M]^n}\sup_{t>0}\left(\int_{\mathbb{R}^n_+}\frac{1}{(2\pi t)^{\frac{np'}{2}}}\exp\left(-\frac{p'\alpha(t)}{4^2n}\left(|x|^2+|y|^2\right)\right)\cdot\exp\left(-\frac{p'|x-y|^2}{4^2n\cdot 2t}\right)dy\right)^{p-1}w(x)\,dx. \tag{2.23}$$

For $t>0$ and $x\in\mathbb{R}^n_+$, define the function

$$\begin{aligned}
f_t(x) &:= \int_{\mathbb{R}^n_+}\frac{1}{(2\pi t)^{\frac{np'}{2}}}\exp\left(-\frac{p'\alpha(t)}{4^2n}\left(|x|^2+|y|^2\right)\right)\cdot\exp\left(-\frac{p'|x-y|^2}{4^2n\cdot 2t}\right)dy\\
&\simeq \frac{1}{t^{\frac{np'}{2}}}\exp\left(-\frac{p'|x|^2}{4^2n}\left(\alpha(t)+\frac{1}{2t}\right)\right)\cdot\int_0^\infty\exp\left(-\frac{p'y_1^2}{4^2n}\left(\alpha(t)+\frac{1}{2t}\right)+\frac{p'x_1y_1}{4^2nt}\right)dy_1\\
&\qquad\qquad\cdots\int_0^\infty\exp\left(-\frac{p'y_n^2}{4^2n}\left(\alpha(t)+\frac{1}{2t}\right)+\frac{p'x_ny_n}{4^2nt}\right)dy_n\\
&\simeq \frac{t^{n/2}}{t^{\frac{np'}{2}}\left(1+t^2\right)^{n/4}}\exp\left(-\frac{p't}{32n\sqrt{1+t^2}}|x|^2\right)\operatorname{erfc}\left(\sqrt{\frac{p'}{32nt\sqrt{1+t^2}}}x_1\right)\cdots\operatorname{erfc}\left(\sqrt{\frac{p'}{32nt\sqrt{1+t^2}}}x_n\right),
\end{aligned}$$

where $\operatorname{erfc}(a):=\frac{2}{\sqrt{\pi}}\int_a^\infty e^{-s^2}ds$ is the complementary error function. To prove that the integral (2.23) is finite it is sufficient to prove that there exists $c>0$ such that

$$f_t(x)\le e^{-c|x|^2} \tag{2.24}$$

for all $t>0$ and $x\in\mathbb{R}^n_+\setminus[0,M]^n$. For $t\ge 1$ this bound follows easily from

$$f_t(x)\lesssim\exp\left(-\frac{p't}{8\sqrt{1+t^2}}|x|^2\right),$$

for all $x\in\mathbb{R}^n_+\setminus[0,M]^n$. For $t\le 1$ and $x\in\mathbb{R}^n_+\setminus[0,M]^n$ we have

$$\begin{aligned}
f_t(x) &\lesssim \sqrt{\frac{p'}{32nt\sqrt{1+t^2}}}^{\,n(p'-1)}\operatorname{erfc}\left(\sqrt{\frac{p'}{32nt\sqrt{1+t^2}}}x_1\right)\cdots\operatorname{erfc}\left(\sqrt{\frac{p'}{32nt\sqrt{1+t^2}}}x_n\right)\\
&= \frac{1}{u^{(p'-1)}}\operatorname{erfc}\left(\frac{x_1}{u}\right)\cdots\frac{1}{u^{(p'-1)}}\operatorname{erfc}\left(\frac{x_n}{u}\right)\\
&\lesssim \frac{1}{(u/x_1)^{(p'-1)}}\operatorname{erfc}\left(\frac{1}{(u/x_1)}\right)\cdots\frac{1}{(u/x_n)^{(p'-1)}}\operatorname{erfc}\left(\frac{1}{(u/x_n)}\right)
\end{aligned}$$

where we have set $u:=\sqrt{\frac{32nt\sqrt{1+t^2}}{p'}}$. This gives

$$\sup_{t\le 1}f_t(x)\lesssim\sup_{u\le 8n}\frac{1}{(u/x_1)^{(p'-1)}}\operatorname{erfc}\left(\frac{1}{u/x_1}\right)\cdots\sup_{u\le 8n}\frac{1}{(u/x_n)^{p'-1}}\operatorname{erfc}\left(\frac{1}{u/x_n}\right).$$

Applying a simple integration by parts argument to the complementary error function yields the estimate $\operatorname{erfc}(x)\le e^{-x^2}$ for $x>1$. From this it is not difficult to see that there must exist $0 < \varepsilon < 1$ small enough so that the derivative of the function

$$\frac{1}{s^{p'-1}}\operatorname{erfc}\left(\frac{1}{s}\right)$$

is positive on $[0, \varepsilon]$. Therefore if we set $M \geq \frac{8n}{\varepsilon}$ the function

$$u \mapsto \frac{1}{(u/z)^{(p'-1)}}\operatorname{erfc}\left(\frac{1}{u/z}\right)$$

will be increasing on $[0, 8n]$ for any $z \geq M$. This then gives

$$\sup_{t\leq 1} f_t(x) \lesssim x_1^{(p'-1)}\operatorname{erfc}\left(\frac{x_1}{8n}\right)\cdots x_n^{(p'-1)}\operatorname{erfc}\left(\frac{x_n}{8n}\right).$$

Bounding the above complementary error functions by Gaussian functions completes the proof of (2.24) and we can therefore conclude that $w \in A_p^{far+}$.

Lastly, it must be proved that w is not contained in the class A_p^∞. Consider the cube $Q := [l, 2l) \times \cdots \times [l, 2l)$ where $l > 1$. We have

$$\begin{aligned} w(Q) &= \int_l^{2l} \cdots \int_l^{2l} e^{x_1}\,dx_1 \cdots dx_n \\ &\gtrsim \int_l^{2l} e^{x_1} dx_1 \\ &= e^{2l} - e^l. \end{aligned}$$

Similarly,

$$\begin{aligned} w^{-\frac{1}{p-1}}(Q)^{p-1} &= \left(\int_l^{2l} \cdots \int_l^{2l} e^{-\frac{x_1}{p-1}}\,dx_1 \cdots dx_n\right)^{p-1} \\ &\gtrsim \left(\int_l^{2l} e^{-\frac{x_1}{p-1}}\,dx_1\right)^{p-1} \\ &\approx \left(e^{-\frac{l}{p-1}} - e^{-\frac{2l}{p-1}}\right)^{p-1} \\ &\gtrsim e^{-l}. \end{aligned}$$

This implies that

$$\begin{aligned} w(Q)w^{-\frac{1}{p-1}}(Q)^{p-1} &\gtrsim \left(e^{2l} - e^l\right) \cdot e^{-l} \\ &= e^l - 1. \end{aligned}$$

It is impossible to bound this exponential of l in terms of a polynomial of l. Therefore a bound of the type required for $w \in A_p^\theta$ is impossible for any $\theta \geq 0$. This proves that $w \notin A_p^\infty$. □

2.4. Truncating the Heat Operators.

As a by-product of the techniques developed in this paper we now show that, in searching for the appropriate weight class for the maximal function associated with the harmonic oscillator, one can safely truncate the maximal function.

Definition 2.4.1. *The truncated heat maximal operator $\mathcal{T}^{\#}$ is defined through*

$$\mathcal{T}^{\#} f(x) := \sup_{t>0} e^{-t\mathcal{L}} \left| f \cdot \chi_{Q_t(R_x)} \right| (x)$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Lemma 2.4.1. *Fix $x \in R \in \Delta_0^\gamma$ and $y \in R' \in \Delta_0^\gamma$ where $R' \subset Q_0(R)^c$. Then for any $\tilde{x} \in R$ and $\tilde{y} \in R'$,*

$$k_{t_m(x,y)}(x, y) \le C \cdot k_{t_m(x,y)} \left(\tilde{x}, \tilde{y}\right),$$

for some constant $C > 0$ independent of both R and R'.

Proof. Introduce the shorthand notation $t_m := t_m(x, y)$. Evidently

$$|x - y| \ge |\tilde{x} - \tilde{y}| - \sqrt{n} \left(l(R) + l(R')\right).$$

This implies that

$$|x - y|^2 \ge |\tilde{x} - \tilde{y}|^2 - 2\sqrt{n} \left|\tilde{x} - \tilde{y}\right| \left(l(R) + l(R')\right) + n \left(l(R) + l(R')\right)^2$$

and therefore

$$\begin{aligned} \exp\left(-\frac{|x-y|^2}{2t_m}\right) &\le \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2t_m}\right) \cdot \exp\left(\frac{\sqrt{n}\,|\tilde{x}-\tilde{y}|\left(l(R)+l(R')\right)}{t_m}\right) \cdot \exp\left(-\frac{n\left(l(R)+l(R')\right)^2}{2t_m}\right) \\ &\le \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2t_m}\right) \cdot \exp\left(\frac{\sqrt{n}\,|\tilde{x}-\tilde{y}|\left(l(R)+l(R')\right)}{t_m}\right). \end{aligned} \tag{2.25}$$

Suppose first that R is not contained in the first layer. On recalling that $|\tilde{x}| \le |\tilde{y}|$ and applying the bound $t_m \ge |y| / (9n\,|x|)$,

$$\begin{aligned} \frac{|\tilde{x}-\tilde{y}|\left(l(R)+l(R')\right)}{t_m} &\le \frac{\left(|\tilde{x}|+|\tilde{y}|\right)\left(l(R)+l(R')\right)}{t_m} \\ &\le \frac{2\,|\tilde{y}|\left(l(R)+l(R')\right)}{|y| / (9n\,|x|)}. \end{aligned}$$

Then, from applying $|\tilde{y}| \leq 2|y|$ and $l(R') \leq l(R)$ in succession,

$$\begin{aligned}\frac{|\tilde{x}-\tilde{y}|\,(l(R)+l(R'))}{t_m} &\leq \frac{4\cdot 9n\,|x|\,|y|\,(l(R)+l(R'))}{|y|}\\ &\leq 8\cdot 9n\,|x|\,l(R)\\ &\leq 8\cdot 9n^{3/2}2^{j(R)}2^{-j(R)}\\ &= 8\cdot 9n^{3/2}.\end{aligned}$$

Next consider the case when R is contained in the first layer. On applying the bound $t_m \geq |y|/(9n)$,

$$\begin{aligned}\frac{|\tilde{x}-\tilde{y}|\,(l(R)+l(R'))}{t_m} &\leq \frac{2\,|\tilde{y}|\,(l(R)+l(R'))}{|y|/(9n)}\\ &\leq \frac{4\cdot 9n\,|y|\,(l(R)+l(R'))}{|y|}\\ &\leq 8\cdot 9n^{3/2}.\end{aligned}$$

This demonstrates that the above bound is independent of layer number. On applying this estimate to (2.25) we obtain

$$\exp\left(-\frac{|x-y|^2}{2t_m}\right) \lesssim \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{2t_m}\right). \tag{2.26}$$

Let's switch our attention to bounding the second exponential term in the kernel. First consider the case when R is not in the first layer. Note that

$$|x| \geq |\tilde{x}| - \sqrt{n}l(R) \quad \textit{and} \quad |y| \geq |\tilde{y}| - \sqrt{n}l(R'). \tag{2.27}$$

From this we obtain

$$\begin{aligned}-|x|^2 &\leq -|\tilde{x}|^2 + 2\sqrt{n}\cdot l(R)\,|\tilde{x}| - d\cdot l(R)^2\\ &\leq -|\tilde{x}|^2 + 2n\cdot 2^{-j(R)}2^{j(R)} - n\cdot l(R)^2\\ &\leq -|\tilde{x}|^2 + 2n,\end{aligned}$$

and similarly $-|y|^2 \leq -|\tilde{y}|^2 + 2n$. We then obtain

$$\exp\left(-\alpha(t_m)\left(|x|^2+|y|^2\right)\right) \leq \exp\left(-\alpha(t_m)\left(|\tilde{x}|^2+|\tilde{y}|^2\right)\right)\cdot\exp\left(4n\cdot\alpha(t_m)\right).$$

As the function α is uniformly bounded by 1, we then have

$$\exp\left(-\alpha(t_m)\left(|x|^2+|y|^2\right)\right) \lesssim \exp\left(-\alpha(t_m)\left(|\tilde{x}|^2+|\tilde{y}|^2\right)\right).$$

Combining this with (2.26) leads to our result.

Next consider the case when R is in the first layer. As $R' \not\subset Q_0(R)$, it follows that R' can't also be contained in the first layer. For this scenario, the bound (2.27) might not be true for x and $\tilde{x}$, but it must hold for y and $\tilde{y}$. We do, however, have the bounds $|x|, |\tilde{x}| \leq \sqrt{n}$. Then

$$\begin{aligned}\exp\left(-\alpha(t_m)\left(|x|^2+|y|^2\right)\right) &\leq \exp\left(-\alpha(t_m)|y|^2\right)\\ &\leq \exp\left(-\alpha(t_m)|\tilde{y}|^2\right)\cdot\exp\left(2n\cdot\alpha(t_m)\right).\end{aligned}$$

Once again, on applying the uniform bound for α we obtain

$$\exp\left(-\alpha(t_m)\left(|x|^2+|y|^2\right)\right) \lesssim \exp\left(-\alpha(t_m)|\tilde{y}|^2\right).$$

Note that since $|\tilde{x}| \leq \sqrt{n}$ we must have $-\alpha(t_m)|\tilde{x}|^2 \geq -n$. Then

$$\begin{aligned}\exp\left(-\alpha(t_m)|\tilde{y}|^2\right) &= e^n e^{-n}\exp\left(-\alpha(t_m)|\tilde{y}|^2\right)\\ &\leq e^n\exp\left(-\alpha(t_m)\left(|\tilde{x}|^2+|\tilde{y}|^2\right)\right).\end{aligned}$$

This leads to the desired bound and concludes our proof. ☐

In direct analogy to Lemma 2.2.4, the following Lemma provides an estimate for weights in the A_p^- class.

Lemma 2.4.2. *Let w be a weight on $\mathbb{R}^n$ and suppose that $\mathcal{M}_{far}^- : L^p(w) \to L^p(w)$ is bounded for some $1 < p < \infty$. Fix cubes R and R' in Δ_0^γ with $R' \not\subset Q_0(R)$. Then there must exist some constant $C > 0$, independent of both R and R', such that*

$$w(R)^{\frac{1}{p}}\cdot w^{-\frac{1}{p-1}}(R')^{\frac{p-1}{p}} \leq C\cdot k^-_{t_m(x_0,y_0)}(R,R')^{-1}$$

for any $x_0 \in R$ and $y_0 \in R'$.

Proof. Recall that $R' \subset Q_{t_m(x_0,y_0)}(R)$. Refer to the proof of Lemma 2.2.4 for why this statement is true. Then

$$\begin{aligned}w(R)\left(\int_{R'}|f(y)|\,dy\right)^p &= \int_R\left(\int_{R'}|f(y)|\,dy\right)^p w(x)\,dx\\ &= k^-_{t_m(x_0,y_0)}(R,R')^{-p}\int_R\left(k^-_{t_m(x_0,y_0)}(R,R')\int_{R'}|f(y)|\,dy\right)^p w(x)\,dx\\ &\leq k^-_{t_m(x_0,y_0)}(R,R')^{-p}\int_R \mathcal{M}_{far}^-\left(f\cdot\chi_{R'}\right)(x)^p w(x)dx\\ &\lesssim k^-_{t_m(x_0,y_0)}(R,R')^{-p}\int_{R'}|f(y)|^p\,w(y)\,dy.\end{aligned}$$

Then from arguments identical to that of Lemma 2.2.4, our result is obtained. $\square$

With Lemmas 2.4.1 and 2.4.2 in hand, the following result can be proved in a similar manner to Theorem A.

Theorem D. *Fix $1 < p < \infty$. For any weight w on $\mathbb{R}^n$, the following equivalence holds*

$$\|\mathcal{T}^*\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftrightarrow \quad \|\mathcal{T}^\#\|_{L^p(w)\to L^p(w)} < \infty.$$

Proof. It is trivially true that the equivalence holds for the local components of these operators. That is, for any weight w on $\mathbb{R}^n$,

$$\|\mathcal{T}^*_{loc}\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftrightarrow \quad \left\|\mathcal{T}^\#_{loc}\right\|_{L^p(w)\to L^p(w)} < \infty.$$

This leaves the far equivalence. The forward implication of the far equivalence follows from the bound $\mathcal{T}^\# f(x) \le \mathcal{T}^* f(x)$ for all $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

It remains to show that for any weight w on $\mathbb{R}^n$,

$$\|\mathcal{T}^*_{far}\|_{L^p(w)\to L^p(w)} < \infty \quad \Leftarrow \quad \left\|\mathcal{T}^\#_{far}\right\|_{L^p(w)\to L^p(w)} < \infty.$$

Fix a weight w and suppose that $\mathcal{T}^\#_{far} : L^p(w) \to L^p(w)$ is bounded. Fix $f \in L^1_{loc}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|\mathcal{T}^*_{far} f\|_{L^p(w)} &= \left[\int_{\mathbb{R}^n} \mathcal{T}^*_{far} f(x)^p w(x)\,dx\right]^{\frac{1}{p}} \\
&= \left[\int_{\mathbb{R}^n} \left(\sup_{t>0} e^{-t\mathcal{L}} \left|f \cdot \chi_{N(R_x)^c}\right|\right)^p w(x)\,dx\right]^{\frac{1}{p}} \\
&= \left[\int_{\mathbb{R}^n} \left(\sup_{t>0} \int_{\mathbb{R}^n\setminus N(R_x)} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{\frac{1}{p}} \\
&= \left[\int_{\mathbb{R}^n} \left(\sup_{t>0} \int_{Q_t(R_x)\setminus N(R_x)} k_t(x,y)\,|f(y)|\,dy\right.\right. \\
&\qquad\qquad \left.\left. + \int_{\mathbb{R}^n\setminus Q_t(R_x)} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{\frac{1}{p}} \\
&\le \left[\int_{\mathbb{R}^n} \left(\sup_{t>0} \int_{Q_t(R_x)\setminus N(R_x)} k_t(x,y)\,|f(y)|\,dy\right.\right. \\
&\qquad\qquad \left.\left. + \sup_{t>0} \int_{\mathbb{R}^n\setminus Q_t(R_x)} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{\frac{1}{p}}.
\end{aligned}$$

On applying Minkowsi's inequality and expanding dyadically,

$$\begin{aligned}
\left\|\mathcal{T}^*_{far} f\right\|_{L^p(w)} &\lesssim \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\int_{Q_t(R_x)\setminus N(R_x)} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{\frac{1}{p}} \\
&\quad + \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\int_{\mathbb{R}^n\setminus Q_t(R_x)} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{\frac{1}{p}} \\
&= \left\|\mathcal{T}^{\#}_{far} f\right\|_{L^p(w)} + \left[\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\not\subset Q_t(R_x)}\int_{R'} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx\right]^{\frac{1}{p}}.
\end{aligned}$$

It remains to bound the tail end term on the right-hand side of the above expression. On expanding dyadically once more,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R_x),\ R'\not\subset Q_t(R_x)}\int_{R'} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx \\
&\qquad = \sum_{R\in\Delta_0^\gamma}\int_R\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),\ R'\not\subset Q_t(R)}\int_{R'} k_t(x,y)\,|f(y)|\,dy\right)^p w(x)\,dx \\
&\qquad \lesssim \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),R'\not\subset Q_t(R)} k_t^+(R,R')\,\|f\|_{L^1(R')}\right)^p w(R).
\end{aligned}$$

For each $t>0$, let x_R^t and $y_{R'}^t$ denote points contained in R and R' respectively that satisfy

$$k_t^+(R,R') \le 2\cdot k_t(x_R^t, y_{R'}^t).$$

Note that since $\mathcal{T}^{\#}: L^p(w)\to L^p(w)$ is bounded, it is obvious that $\mathcal{M}^-_{far}: L^p(w)\to L^p(w)$ is bounded as well. On applying Hölder's inequality and Lemma 2.4.2,

$$\begin{aligned}
&\sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),R'\not\subset Q_t(R)} k_t^+(R,R')\,\|f\|_{L^1(R')}\right)^p w(R) \\
&\quad \lesssim \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),R'\not\subset Q_t(R)} k_t(x_R^t,y_{R'}^t)\,w^{-\frac{1}{p-1}}(R')^{\frac{p-1}{p}}\,w(R)^{\frac{1}{p}}\,\|f\|_{L^p(R',w)}\right)^p \\
&\quad \lesssim \sum_{R\in\Delta_0^\gamma}\left(\sup_{t>0}\sum_{R'\in\mathcal{F}(R),R'\not\subset Q_t(R)} k_t(x_R^t,y_{R'}^t)\cdot k^-_{t_m(x_R^t,y_{R'}^t)}(R,R')^{-1}\,\|f\|_{L^p(R',w)}\right)^p.
\end{aligned} \tag{2.28}$$

We know from Lemma 2.2.3 that

$$k_t(x_R^t,y_{R'}^t) \lesssim k_{t_m(x_R^t,y_{R'}^t)}(x_R^t,y_{R'}^t)\cdot 2^{-(j(R)+j(R'))(n+1)}.$$

Lemma 2.4.1 can then be applied to acquire

$$k_t(x_R^t, y_{R'}^t) \lesssim k_{t_m(x_R^t, y_{R'}^t)}(\tilde{x}, \tilde{y}) \cdot 2^{-(j(R)+j(R'))(n+1)}$$

for all $\tilde{x} \in R$ and $\tilde{y} \in R'$. Therefore

$$k_t(x_R^t, y_{R'}^t) \lesssim k^-_{t_m(x_R^t, y_{R'}^t)}(R, R') 2^{-(j(R)+j(R'))(n+1)}$$

This can be applied to (2.28) to obtain

$$\begin{aligned}
\sum_{R \in \Delta_0^\gamma} &\left(\sup_{t>0} \sum_{R' \not\subset Q_t(R)} k_t^+(R, R') \|f\|_{L^1(R')} \right)^p w(R) \\
&\lesssim \sum_{R \in \Delta_0^\gamma} \left(\sup_{t>0} \sum_{R' \not\subset Q_t(R)} 2^{-(j(R)+j(R'))(n+1)} \|f\|_{L^p(R', w)} \right)^p \\
&\lesssim \|f\|_{L^p(w)},
\end{aligned}$$

which concludes our proof. □

II

The Kato Square Root Problem with Potential

Abstract

The Kato square root problem for divergence form elliptic operators with potential $V : \mathbb{R}^n \to \mathbb{C}$ is the equivalence statement $\left\|(L+V)^{\frac{1}{2}} u\right\|_{L^2(\mathbb{R}^n)} \simeq \|\nabla u\|_{L^2(\mathbb{R}^n)} + \left\|V^{\frac{1}{2}} u\right\|_{L^2(\mathbb{R}^n)}$, where $L + V := -\mathrm{div}\,(A\nabla) + V$ and the perturbation A is an L^∞ complex matrix-valued function satisfying an ellipticity condition. One possible path to a solution for this problem is by proving square function estimates for perturbations of associated non-homogeneous Dirac-type operators. At present, there is no general method to obtain such square function estimates other than for potentials bounded both from above and below (cf. [10]). We develop such a method by adapting the homogeneous framework introduced by A. Axelsson, S. Keith and A. McIntosh in their seminal paper [11]. Two distinct approaches will be considered when adapting this framework.

The first approach alters the initial hypotheses of [11] to allow for Dirac-type operators that depend on a scalar zero-order potential $V \in L^1_{loc}(\mathbb{R}^n)$. This approach will not impose any additional additional assumptions on the perturbations or the algebraic structure of the operators and will therefore remain quite general. The end result, however, will be a somewhat unwieldy square function estimate that will require rather restrictive conditions on the potential in order to obtain boundedness.

The second approach assumes a concrete three-by-three matrix form for the Dirac-type operators and imposes additional restrictions on the perturbations. Generality is lost through these additional restrictions but then subsequently gained by allowing the scalar zero-order potential V to be replaced by any suitable operator J that is not necessarily first-order homogeneous. The algebraic structure of these Dirac-type operators will be exploited to obtain square function estimates for a large class of potentials that includes any potential V with range contained in some sector

$$S_{\omega_V +} := \{z \in \mathbb{C} \cup \{\infty\} : |\mathrm{arg} z| \leq \mu \text{ or } z = 0,\, \infty\}$$

with $\omega_V \in \left[0, \frac{\pi}{2}\right)$ and such that $|V|$ belongs to either the reverse Hölder class RH_2 in any dimension or $L^{\frac{n}{2}}(\mathbb{R}^n)$ for $n > 4$. Thus a satisfying solution to the Kato problem with potential will be obtained for any such potential.

Chapter 3

Prelude

Introduction

Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ and consider the sesquilinear form $\mathfrak{l}^A : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$\mathfrak{l}^A[u, v] := \int_{\mathbb{R}^n} \langle A(x)\nabla u(x), \nabla v(x)\rangle \, dx$$

for u, $v \in H^1(\mathbb{R}^n)$. Suppose that $\mathfrak{l}^A$ satisfies the Gårding inequality

$$\operatorname{Re}\left(\mathfrak{l}^A[u, u]\right) \geq \kappa_A \|\nabla u\|^2 \tag{3.1}$$

for all $u \in H^1(\mathbb{R}^n)$, for some $\kappa_A > 0$. A well-known representation theorem from classical form theory (cf. [39, Thm. VI.2.1]) asserts the existence of an associated operator $L : D(L) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for which

$$\mathfrak{l}^A[u, v] = \langle Lu, v\rangle$$

for all $v \in H^1(\mathbb{R}^n)$ and u in the domain of L,

$$D(L) = \left\{u \in H^1(\mathbb{R}^n) : \exists w \in L^2(\mathbb{R}^n) \ \ s.t. \ \mathfrak{l}^A[u, v] = \langle w, v\rangle \ \forall \ v \in H^1(\mathbb{R}^n)\right\}.$$

The operator L is denoted

$$L = -\mathrm{div} A\nabla,$$

since the two sides of the above relation will naturally coincide whenever the right-hand side makes sense. The operator L will be a densely defined maximal accretive operator. As such, it is possible to define a square root operator $\sqrt{L}$, with domain $D(L)$, that satisfies $\sqrt{L} \cdot \sqrt{L} = L$. The domain $D(L)$ is not a natural domain of definition for the square root operator since it is a second-order domain and the square root operator is first-order and since the operator $\sqrt{L}$ is not closed on $D(L)$. The Kato square root problem asks what is the natural domain of definition of the square root operator or, equivalently, is the square root operator closable and what is the domain of the closure? This problem, first posed by Tosio Kato over 50 years ago, was conjectured to have the following solution.

Theorem (Kato Square Root). *The natural domain of $\sqrt{L}$ is*

$$D(\sqrt{L}) = H^1(\mathbb{R}^n).$$

In particular, for any $u \in H^1(\mathbb{R}^n)$

$$\left\| \sqrt{L} u \right\| \simeq \|\nabla u\|. \tag{3.2}$$

This long-standing problem withstood solution until 2002 where it was proved using local $T(b)$ methods by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [7]. We will be interested in an alternate method of proof that was built from similar principles and appeared a few years later.

Let $\Pi := \Gamma + \Gamma^*$ be a Dirac-type operator on a Hilbert space $\mathcal{H}$ and $\Pi_B := \Gamma + B_1 \Gamma^* B_2$ be a perturbation of Π by bounded operators B_1 and B_2. Typically, Π is considered to be a first-order system acting on $\mathcal{H} := L^2(\mathbb{R}^n; \mathbb{C}^N)$ for some n, $N \in \mathbb{N}^*$ and the perturbations B_1 and B_2 are multiplication by matrix-valued functions B_1, $B_2 \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$. In their seminal paper [11], A. Axelsson, S. Keith and A. McIntosh developed a general framework for proving that the perturbed operator Π_B possessed a bounded holomorphic functional calculus. This ultimately amounted to obtaining square function estimates of the form

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \simeq \|u\|^2, \tag{3.3}$$

where $Q_t^B := t\Pi_B (I + t^2 \Pi_B^2)^{-1}$ and u is contained in the range $\overline{R(\Pi_B)}$. They proved that this estimate would follow entirely from a set of simple conditions imposed upon the operators Γ, B_1 and B_2, labelled (H1) - (H8). Then, by checking this list of simple conditions, the Axelsson-Keith-McIntosh framework, or AKM framework by way of abbreviation, could be used to conclude that the particular selection of operators

$$\Gamma := \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad B_1 = I, \quad B_2 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \tag{3.4}$$

defined on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n)$, would satisfy (3.3) and therefore possess a bounded holomorphic functional calculus. The Kato square root estimate then followed almost trivially from this.

Many classical problems from harmonic analysis will have a direct counterpart in the

Schrödinger operator setting. Adhering with this theme, one can consider the Kato square root problem with potential. Let $V : \mathbb{R}^n \to \mathbb{C}$ be a measurable function that is contained in $L^1_{loc}(\mathbb{R}^n)$. V can be viewed as a densely defined closed multiplication operator on $L^2(\mathbb{R}^n)$ with domain

$$D(V) = \left\{u \in L^2(\mathbb{R}^n) : V \cdot u \in L^2(\mathbb{R}^n)\right\}.$$

The density of $D(V)$ in $L^2(\mathbb{R}^n)$ follows from the measurability of V. Define the subspace

$$H^{1,V}(\mathbb{R}^n) := H^1(\mathbb{R}^n) \cap D\left(V^{\frac{1}{2}}\right) := \left\{u \in H^1(\mathbb{R}^n) : V^{\frac{1}{2}} \cdot u \in L^2(\mathbb{R}^n)\right\}. \tag{3.5}$$

Here the complex square root $V^{\frac{1}{2}}$ is defined via the principal branch cut along the negative real axis. As explained in §1.2, $H^{1,V}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ since $C_0^\infty(\mathbb{R}^n) \subset H^{1,V}(\mathbb{R}^n)$. Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ be as before with (3.1) satisfied for some $\kappa_A > 0$. Consider the sesquilinear form $\mathfrak{l}^A_V : H^{1,V}(\mathbb{R}^n) \times H^{1,V}(\mathbb{R}^n) \to \mathbb{C}$ defined through

$$\mathfrak{l}^A_V[u, v] := \mathfrak{l}^A[u, v] + \int_{\mathbb{R}^n} \langle V(x)u(x), v(x)\rangle \, dx$$

for u, $v \in H^{1,V}(\mathbb{R}^n)$. Suppose that there exists some $\kappa^V_A > 0$ for which $\mathfrak{l}^A_V$ satisfies the associated Gårding inequality

$$\mathrm{Re}\left(\mathfrak{l}^A_V[u, u]\right) \geq \kappa^V_A \left(\left\|V^{\frac{1}{2}}u\right\|^2 + \|\nabla u\|^2\right), \tag{3.6}$$

for all $u \in H^{1,V}(\mathbb{R}^n)$.

Remark 3.0.1. If the range of V is contained in some sector

$$S_{\mu+} := \{z \in \mathbb{C} \cup \{\infty\} : |\arg(z)| \leq \mu \ or \ z = 0, \infty\}$$

for some $\mu \in [0, \frac{\pi}{2})$, then (3.6) will follow automatically from (3.1).

Once again, the accretivity of $\mathfrak{l}^A_V$ implies the existence of a maximal accretive operator associated with this form denoted by

$$L + V = -\mathrm{div} A\nabla + V,$$

defined on

$$D(L + V) = \left\{u \in H^{1,V}(\mathbb{R}^n) : \exists w \in L^2(\mathbb{R}^n) \ s.t. \ \mathfrak{l}^A_V[u, v] = \langle w, v\rangle \ \forall \ v \in H^{1,V}(\mathbb{R}^n)\right\}.$$

In this potential dependent context the Kato square root problem takes the form of the following conjecture.

Conjecture (Kato Square Root with Potential)**.** *Let $V : \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function with range contained in $S_{\omega_V +}$ for some $\omega_V \in \left[0, \frac{\pi}{2}\right)$. There must exist some constant $C_V > 0$ for which*

$$C_V^{-1}\left(\left\|V^{\frac{1}{2}}u\right\| + \|\nabla u\|\right) \leq \left\|\sqrt{L+V}u\right\| \leq C_V\left(\left\|V^{\frac{1}{2}}u\right\| + \|\nabla u\|\right). \tag{KP}$$

for all $u \in D\left(L+V\right)$.

This problem is actually a statement concerning the domain of the square root operator $\sqrt{L+V}$. Indeed, (KP) implies the equality

$$D\left(\sqrt{L+V}\right) = H^{1,V}\left(\mathbb{R}^n\right).$$

In direct analogy to the potential free case, the Kato problem with potential will be solved by constructing appropriate potential dependent Dirac-type operators and demonstrating that they retain a bounded holomorphic functional calculus under perturbation. In particular, this strategy will be applied to the Dirac-type operator

$$\Pi_{|V|^{\frac{1}{2}}} := \Gamma_{|V|^{\frac{1}{2}}} + \Gamma^*_{|V|^{\frac{1}{2}}} := \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & |V|^{\frac{1}{2}} & -\mathrm{div} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

defined on $L^2(\mathbb{R}^n) \oplus L^2\left(\mathbb{R}^n\right) \oplus L^2\left(\mathbb{R}^n; \mathbb{C}^n\right)$, under the perturbation

$$B_1 = I, \quad B_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i \cdot \arg V} & 0 \\ 0 & 0 & A \end{pmatrix}. \tag{3.8}$$

It should be observed that the operator $\Gamma_{|V|^{\frac{1}{2}}}$ is not first-order homogeneous due to the presence of the zero-order potential term. It will therefore not necessarily satisfy the two conditions of the original framework that are intended to capture the first-order homogeneity property, the cancellation and coercivity conditions, (H7) and (H8). As such, the original framework developed by Axelsson, Keith and McIntosh cannot be directly applied. The key difficulty in proving our result is then to alter the original framework in order to allow for such operators. This part of the thesis is dedicated to the construction of a non-homogeneous AKM framework and its applications to the Kato square root problem with potential. We consider two distinct approaches to such a construction.

The first approach, considered in Chapter 4, alters the initial hypotheses of [11] to allow for Dirac-type operators that depend on a scalar zero-order potential. This approach will not impose any additional algebraic structure or any restrictions on the perturbations and will therefore remain quite general. The end result, however, is a somewhat unwieldy square function estimate that will require rather restrictive conditions on the potential in order to obtain boundedness.

The second approach, that will be introduced in Chapter 5, suffers from a reduction in generality by imposing a concrete three-by-three matrix form for the Dirac-type operator. Additional restrictions will also be placed upon the perturbations B_1 and B_2. This reduction in generality is compensated for by allowing the scalar zero-order potential $|V|^{\frac{1}{2}}$ to be replaced by any suitable operator J that is not necessarily first-order homogeneous. The algebraic structure of these Dirac-type operators will be exploited to obtain square function estimates for a large class of potentials that includes any potential V with range contained in some sector $S_{\omega_V +}$ with $\omega_V \in [0, \frac{\pi}{2})$ and such that $|V|$ belongs to either RH_2 in any dimension or $L^{\frac{n}{2}}(\mathbb{R}^n)$ for $n > 4$. Thus a satisfying solution to the Kato problem with potential will be obtained for this class. Along the way, the explicit dependence of the constant in (KP) on the potential will be determined.

The proof of the original AKM framework heavily relies on a square function equivalence between the A_t and $P_t := (I + t^2\Pi^2)^{-1}$ operators,

$$\int_0^\infty \|(A_t - P_t)\, u\|^2 \, \frac{dt}{t} \lesssim \|u\|^2 .$$

This estimate allows one to interchange the A_t and P_t operators freely when evaluating upper square function estimates, allowing one to take advantage of some of the more enviable properties of the averaging operators. Unfortunately the removal of the cancellation and coercivity conditions will force such an equivalence to break. Indeed, given that the P_t operators become potential dependent in the Schrödinger setting, it seems intuitively clear that the potential free averaging operator A_t will not satisfy the previous equivalence and will distinctly be the wrong operator to use in this context. This brings us back to the central problem of the first part of this thesis on the construction of potential dependent averaging operators. Although we do not construct general adapted averaging operators in order to solve this problem, we will demonstrate how the breakdown of such a square function equivalence can be bypassed entirely.

Chapter Outline

In this chapter we will outline any preliminaries that will be needed to understand the original AKM framework and the construction of its non-homogeneous counterparts. The opening two sections are quite classical in nature. The first provides a brief survey of the natural functional calculus for bisectorial operators. It also describes how to prove the boundedness of holomorphic functional calculus from square function estimates for a bisectorial operator. These classical proofs are repeated here in order to precisely keep track of how the dependence on various constants traces through these classical arguments. This attention to detail will pay off later when we want to determine how the constant in the Kato estimate with potential (KP) depends on the potential. The second section will provide a couple of important results from the classical theory of Carleson measures that will be needed for our proof.

In §3.3 we will provide a short review of the potential free Kato problem. Then, in §3.4, we will go on to give a brief description of the original AKM framework. It is here that a precise definition of the hypotheses (H1) - (H8) will be provided. A proof of the classical potential free Kato square root problem from this framework will also be outlined. Then, by adapting this classical argument, the path to the solution of the Kato problem with potential through a non-homogeneous AKM framework will be delineated in §3.5.

As the operator $\Gamma_{|V|^{\frac{1}{2}}}$ does not satisfy (H7) or (H8), it will be fruitful to see what happens to the original AKM framework when these two conditions are removed. This will be considered in the final section of this chapter. In particular, we will list which results from [11] remain valid without the hypotheses (H7) and (H8) and which ones will no longer hold.

Acknowledgements

The research for this part of the thesis was initiated when, years ago, El Maati Ouhabaz suggested to my supervisor Pierre Portal that Kato's estimates for Schrödinger type operators on $\mathbb{R}^n$ with a control on the constant would have interesting applications. While this research was in progress, it was found that Andrew Morris and Andrew Turner from the University of Birmingham were also working on the Kato problem with potential on $\mathbb{R}^n$. After meeting them and discussing their research, it appears that their approach differs from both of my approaches in their assumptions, results and, more substantially, their proofs.

I would like to thank Moritz Egert for his comments on Proposition 5.3.2. I would also like to thank the anonymous referee from Mathematisch Nachrichten for providing such a detailed and thoughtful critique of the article version of Chapters 3 and 5. Reflecting on their comments led me to several significant improvements.

3.1. Functional Calculus for Bisectorial Operators

Let's outline the construction of the natural functional calculus associated with a bisectorial operator. This functional calculus constitutes the bedrock upon which the solution to the Kato problem through the AKM framework is built. The treatment of functional calculi found here follows closely to [32] with significant detail omitted. Appropriate changes are made to account for the fact that we consider bisectorial operators instead of sectorial operators. Other thorough treatments of functional calculus for sectorial operators can be found in [37], [3], [18] and [45].

For $\mu \in [0, \pi)$ define the open and closed sectors

$$S^o_{\mu+} := \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu\} & \mu \in (0, \pi) \\ (0, \infty) & \mu = 0 \end{cases}$$

and

$$S_{\mu+} := \begin{cases} \{z \in \mathbb{C} \cup \{\infty\} : |\arg(z)| \leq \mu \text{ or } z = 0,\ \infty\} & \mu \in (0, \pi) \\ [0, \infty] & \mu = 0. \end{cases}$$

Then, for $\mu \in \left[0, \frac{\pi}{2}\right)$, define the open and closed bisectors

$$S^o_\mu := \left(S^o_{\mu+}\right) \cup \left(-S^o_{\mu+}\right)$$

and

$$S_\mu := (S_{\mu+}) \cup (-S_{\mu+})$$

respectively. Throughout this section we consider bisectorial operators defined on a Hilbert space $\mathcal{H}$.

Definition 3.1.1 (Bisectorial Operator). *A linear operator $T : D(T) \subseteq \mathcal{H} \to \mathcal{H}$ is said to be ω-bisectorial for $\omega \in \left[0, \frac{\pi}{2}\right)$ if the spectrum $\sigma(T)$ is contained in the bisector S_ω and if for any $\mu \in \left(\omega, \frac{\pi}{2}\right)$, there exists $C_\mu > 0$ such that the resolvent bound*

$$|\zeta| \left\|(\zeta I - T)^{-1}\right\| \leq C_\mu \tag{3.9}$$

holds for all $\zeta \in \mathbb{C} \setminus S_\mu$. T is said to be bisectorial if it is ω-bisectorial for some $\omega \in \left[0, \frac{\pi}{2}\right)$.

Sectorial operators are defined identically except with the sector $S_{\mu+}$ performing the role of the bisector S_μ. An important fact concerning bisectorial operators is the following decomposition result.

Proposition 3.1.1 ([18, Thm. 3.8]). *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a bisectorial operator. Then T is necessarily densely defined and the Hilbert space $\mathcal{H}$ admits the following decomposition*

$$\mathcal{H} = N(T) \oplus \overline{R(T)}.$$

Let T be an ω-bisectorial operator for $\omega \in \left[0, \frac{\pi}{2}\right)$ and $\mu \in \left(\omega, \frac{\pi}{2}\right)$. Our aim is to construct a functional calculus associated with the operator T. That is, for appropriate functions f we would like to assign an operator $f(T)$. In order for a functional calculus to be of use, the following basic principles should hold.

1. The operators in the range of the functional calculus can be manipulated like they are functions. That is, something akin to the relations $f(T) + g(T) = (f + g)(T)$ and $(f \cdot g)(T) = f(T) \cdot g(T)$ holds.
2. The algebra of functions for which $f(T)$ is defined is not too small. The usual operators such as T, $\sqrt{T^2}$ and $(T - i)^{-1}$ are contained within the range of the functional calculus.
3. It is consistent. That is, it maps the functions z, 1 and $(z - i)^{-1}$ respectively to the operators T, I and $(T - i)^{-1}$ for example.

Let $\mathcal{M}\left(S_\mu^o\right)$ denote the algebra of all meromorphic functions on the open bisector S_μ^o and define the following subalgebras,

$$H\left(S_\mu^o\right) := \left\{f \in \mathcal{M}\left(S_\mu^o\right) : f \textit{ holomorphic on } S_\mu^o\right\},$$

$$H^\infty\left(S_\mu^o\right) := \left\{f \in H\left(S_\mu^o\right) : \|f\|_\infty := \sup_{z \in S_\mu^o} |f(z)| < \infty\right\}$$

and

$$H_0^\infty\left(S_\mu^o\right) := \left\{f \in H^\infty\left(S_\mu^o\right) : \exists\, C, \alpha > 0 \textit{ s.t. } |f(z)| \leq C \cdot \frac{|z|^\alpha}{1 + |z|^{2\alpha}} \;\forall\, z \in S_\mu^o\right\}.$$

For any $f \in H_0^\infty\left(S_\mu^o\right)$, one can associate an operator $f(T)$ as follows. For $u \in \mathcal{H}$, define

$$f(T)\,u := \frac{1}{2\pi i}\oint_\gamma f(z)\,(zI - T)^{-1}\,u\,dz,$$

where the curve

$$\gamma := \left\{\pm r e^{\pm i\nu} : 0 \leq r < \infty\right\}$$

for some $\nu \in (\omega, \mu)$ is traversed anticlockwise.

Theorem 3.1.1. *The map* $\Phi_0^T : H_0^\infty\left(S_\mu^o\right) \to \mathcal{L}(\mathcal{H})$ *defined through*

$$\Phi_0^T(f) := f(T)$$

is a well-defined algebra homomorphism. Moreover, it is independent of of the value of ν.

PROOF. The resolvent bounds of our operator and the size estimates on f imply that the above integral will converge absolutely ensuring that $f(T)$ is a well-defined bounded operator. An application of the Cauchy integral formula will give us the independence of the definition of $f(T)$ from the value of ν. For a proof of the homomorphism property refer to [32, Lem. 2.3.1]. $\square$

Since the functions in $H_0^\infty\left(S_\mu^o\right)$ approach zero at the origin we should expect that the null space of the newly formed operator will be larger than the null space of the original operator. This is indeed the case as stated in the below proposition.

Proposition 3.1.2 ([32, Thm. 2.3.3]). *For a bisectorial operator $T : D(T) \subseteq \mathcal{H} \to \mathcal{H}$, the null-space inclusion*

$$N(T) \subseteq N(f(T))$$

holds for all $f \in H_0^\infty\left(S_\mu^o\right)$.

One serious deficiency with this functional calculus, as it is currently defined, is that many important operators affiliated with T are not contained in the range of Φ_0^T. For example, the resolvent operator $(T+i)^{-1}$, constant multiples of the identity and T itself are all out of reach of this functional calculus. The astounding benefit of working with a functional calculus is that operators in its range can be treated symbollically as if they were functions. This makes for easy manipulation of what should be quite complex objects. However, if even the most basic operator T itself is not contained in the range then we will quickly run into problems with our symbolic manipulation and the functional calculus will cease to be of any use. The bottom line is that the algebra of functions $H_0^\infty\left(S_\mu^o\right)$ is too small for Φ_0^T to be useful. A step in the right direction is to extend Φ_0^T to the subalgebra

$$\mathcal{E}\left(S_\mu^o\right) := H_0^\infty\left(S_\mu^o\right) \oplus \langle (z+i)^{-1} \rangle \oplus \langle 1 \rangle.$$

Specifically, define the extension

$$\Phi_p^T : \mathcal{E}\left(S_\mu^o\right) \to \mathcal{L}(H)$$

through

$$g(T) := \Phi_p^T(g) := f(T) + c \cdot (T+i)^{-1} + d \cdot I$$

for $g = f + c \cdot (z+i)^{-1} + d \in \mathcal{E}\left(S_\mu^o\right)$, where $f \in H_0^\infty\left(S_\mu^o\right)$ and $c,\ d \in \mathbb{C}$.

Theorem 3.1.2 ([32, Thm. 2.3.3]). *The map Φ_p^T is an algebra homomorphism called the primary functional calculus associated with T.*

Since the range of Φ_p^T is restricted to the bounded operators, for unbounded T the range of the functional calculus will still lack even the basic operator T. Fortunately, one further extension through the process of regularization will solve this problem and increase the range beyond the bounded operators. A function $f \in \mathcal{M}\left(S_\mu^o\right)$ is said to be regularizable with respect to the primary functional calculus $\Phi_p^T : \mathcal{E}\left(S_\mu^o\right) \to \mathcal{L}(\mathcal{H})$ if there exists $e \in \mathcal{E}\left(S_\mu^o\right)$ such that $e(T)$ is injective and $e \cdot f \in \mathcal{E}\left(S_\mu^o\right)$. The notation $\mathcal{E}\left(S_\mu^o\right)_r$ will be used to denote the algebra of regularizable functions. Let $\mathcal{C}(\mathcal{H})$ denote the set of closed operators from $\mathcal{H}$ to itself. Then define the extension

$$\Phi^T : \mathcal{E}\left(S_\mu^o\right)_r \to \mathcal{C}(H)$$

through

$$f(T) := \Phi^T(f) := \Phi_p^T(e)^{-1} \cdot \Phi_p^T(e \cdot f)$$

for $f \in \mathcal{E}\left(S_\mu^o\right)_r$ and $e \in \mathcal{E}\left(S_\mu^o\right)$ a regularizing function for f. This definition is independent of the chosen regularizer e for f and therefore Φ^T is well-defined. We have the following important theorem that establishes the desired properties of a functional calculus for this extension. Thus the map Φ^T will be known as the natural functional calculus for the operator T.

Theorem 3.1.3 ([32, Thm. 1.3.2])**.** *Let T be an ω-bisectorial operator on a Hilbert space $\mathcal{H}$ for some $\omega \in \left[0, \frac{\pi}{2}\right)$. Let $\mu \in \left(\omega, \frac{\pi}{2}\right)$. The following assertions hold.*

1. $\mathbf{1}(T) = I$ *and* $(z)(T) = T$, *where* $\mathbf{1} : S_\mu^o \to \mathbb{C}$ *is the constant function defined by* $\mathbf{1}(z) := 1$ *for* $z \in S_\mu^o$.
2. *Let* $f, g \in \mathcal{E}\left(S_\mu^o\right)_r$. *Then*

$$f(T) + g(T) \subset (f+g)(T), \qquad f(T)g(T) \subset (f \cdot g)(T)$$

and $D(f(T)g(T)) = D((f \cdot g)(T)) \cap D(g(T))$. *One will have equality in these relations if* $g(T) \in \mathcal{L}(H)$.

The ensuing definition plays a vital role in the solution method to the Kato square root problem using the AKM framework.

Definition 3.1.2. *Let $0 \leq \omega < \mu < \frac{\pi}{2}$. An ω-bisectorial operator $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is said to have a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus if there exists $c > 0$ such that*

$$\|f(T)\| \leq c \cdot \|f\|_\infty \tag{3.10}$$

for all $f \in H_0^\infty\left(S_\mu^o\right)$. T is said to have a bounded holomorphic functional calculus if it has a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus for some μ.

Remark 3.1.1. Note that a more intuitive definition for a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus would be to require that (3.10) hold for all $f \in H^\infty\left(S_\mu^o\right)$. Unfortunately at this stage it is impossible to ascertain whether $H^\infty\left(S^o_\mu\right) \subset \mathcal{E}\left(S^o_\mu\right)_r$. When this inclusion does not hold, the operator $f(T)$ will not be well-defined for all $f \in H^\infty\left(S^o_\mu\right)$. If T so happens to be injective, then each $f \in H^\infty\left(S^o_\mu\right)$ is in fact regularizable by $z\left(1+z^2\right)^{-1}$ and the estimate (3.10) makes sense for all $f \in H^\infty\left(S^o_\mu\right)$. Fortunately, in this situation the two definitions coincide. That is, (3.10) will be true for all $f \in H^\infty_0\left(S^o_\mu\right)$ if and only if it is true for all $f \in H^\infty\left(S^o_\mu\right)$ when T is injective.

Definition 3.1.3 (Square Function Norms)**.** *Let $\psi \in H^\infty_0\left(S^o_\mu\right)$. For $t > 0$, define $\psi_t : S^o_\mu \to \mathbb{C}$ to be the function $\psi_t(z) := \psi(tz)$ for $z \in S^o_\mu$. The square function norm associated with ψ and T is defined through*

$$\|u\|_{\psi,T} := \left(\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

for $u \in \mathcal{H}$. Let $q : S^o_\mu \to \mathbb{C}$ be the function given by

$$q(z) := \frac{z}{1+z^2}, \quad z \in S^o_\mu.$$

$\|\cdot\|_{q,T}$ is called the canonical square function norm for the operator T.

For injective T, true to its name, the square function norms $\|\cdot\|_{\psi,T}$, for $\psi \in H^\infty_0\left(S^o_\mu\right)$ not identically equal to zero on either $S^o_{\mu+}$ or $\left(-S^o_{\mu+}\right)$, are indeed norms on $\mathcal{H}$. For non-injective T however, they are at most seminorms on $\mathcal{H}$ and will only be norms following a restriction to the subspace $\overline{R(T)}$.

Definition 3.1.4 (Square Function Estimates)**.** *A bisectorial operator T on a Hilbert space $\mathcal{H}$ is said to satisfy square function estimates if there exists a constant $C_{SF} > 0$ such that*

$$C_{SF}^{-\frac{1}{2}} \cdot \|u\| \leq \|u\|_{q,T} \leq C_{SF}^{\frac{1}{2}} \cdot \|u\| \tag{3.11}$$

for all $u \in \overline{R(T)}$.

The above definition is the same as saying that the canonical square function norm $\|\cdot\|_{q,T}$ is norm equivalent to $\|\cdot\|_{\mathcal{H}}$ when restricted to the Hilbert subspace $\overline{R(T)}$.

Remark 3.1.2. The use of the canonical norm $\|\cdot\|_{q,T}$ in the above definition of square function estimates is somewhat arbitrary. Indeed, it can be swapped with $\|\cdot\|_{\psi,T}$ for any $\psi \in H^\infty_0\left(S^o_\mu\right)$ not identically equal to zero on either $S^o_{\mu+}$ or $\left(-S^o_{\mu+}\right)$. This follows from the equivalence of these two norms as stated in [32, Thm. 7.3.1]. The implicit constant in the norm equivalence will depend on the functions under consideration.

Proposition 3.1.3 (Resolution of the Identity). *For $\psi \in H_0^\infty\left(S_\mu^o\right)$ and any $u \in \mathcal{H}$,*

$$c_\psi\left(I - \mathbb{P}_{N(T)}\right) u = \int_0^\infty \psi_t(T)^2 u \, \frac{dt}{t}, \tag{3.12}$$

where $\mathbb{P}_{N(T)}$ denotes the projection operator onto the subspace $N(T)$ and

$$c_\psi := \int_0^\infty \psi(t)^2 \, \frac{dt}{t}.$$

PROOF. Equality follows from Proposition 3.1.2 for $u \in N(T)$. For $u \in \overline{R(T)}$ this is given by Theorem 5.2.6 of [32] in the sectorial case. The bisectorial case can be proved similarly. □

Corollary 3.1.1. *Suppose that T is self-adjoint and $\psi \in H_0^\infty\left(S_\mu^o\right)$. Then for any $u \in \mathcal{H}$,*

$$\int_0^\infty \left\|\psi_t(T)u\right\|^2 \frac{dt}{t} \leq c_\psi \left\|u\right\|^2$$

where c_ψ is as defined in the previous proposition. Equality will hold if $u \in \overline{R(T)}$.

PROOF. As T is self-adjoint, it follows from the definition of $\psi_t(T)$ that it must also be self-adjoint. On expanding the square function norm,

$$\begin{aligned}\int_0^\infty \left\|\psi_t(T)u\right\|^2 \frac{dt}{t} &= \int_0^\infty \langle \psi_t(T)u, \psi_t(T)u\rangle \frac{dt}{t} \\ &= \left\langle u, \int_0^\infty \psi_t(T)^2 u \frac{dt}{t}\right\rangle.\end{aligned}$$

The previous proposition then gives

$$\begin{aligned}\int_0^\infty \left\|\psi_t(T)u\right\|^2 \frac{dt}{t} &= \left\langle u, c_\psi\left(I - \mathbb{P}_{N(T)}\right) u\right\rangle \\ &\leq c_\psi \left\|u\right\|^2.\end{aligned}$$

Equality will clearly hold in the above if $u \in \overline{R(T)}$. □

Theorem 3.1.4. *Let T be an ω-bisectorial operator on $\mathcal{H}$ for $\omega \in \left[0, \frac{\pi}{2}\right)$. Suppose that T satisfies square function estimates with constant $C_{SF} > 0$. Then T must have a bounded $H^\infty\left(S_\mu^o\right)$-functional calculus for any $\mu \in \left(\omega, \frac{\pi}{2}\right)$. In particular, there exists a constant $c > 0$, independent of T, such that*

$$\left\|f(T)\right\| \lesssim \left(c \cdot C_{SF} \cdot C_\mu\right) \cdot \left\|f\right\|_\infty$$

for all $f \in H_0^\infty\left(S_\mu^o\right)$, where $C_\mu > 0$ is the constant from the resolvent estimate (3.9).

Proof. Let $f \in H_0^\infty\left(S_\mu^o\right)$. For $u \in N(T)$, the bound

$$\|f(T)\,u\| \leq (c \cdot C_{SF} \cdot C_\mu) \cdot \|f\|_\infty \cdot \|u\| \tag{3.13}$$

follows trivially from Proposition 3.1.2 for any $c > 0$. Fix $u \in \overline{R(T)}$. On applying the lower square function estimate to $f(T)u \in \overline{R(T)}$,

$$\begin{aligned}\|f(T)u\|^2 &\leq C_{SF}\int_0^\infty \|q_s(T)f(T)\,u\|^2\,\frac{ds}{s}\\ &= 2C_{SF}\int_0^\infty \left\|q_s(T)f(T)\int_0^\infty (q_t(T))^2\,u\frac{dt}{t}\right\|^2\frac{ds}{s}\\ &\leq 2C_{SF}\int_0^\infty \left(\int_0^\infty \|q_s(T)f(T)\,q_t(T)\|\,\|q_t(T)u\|\,\frac{dt}{t}\right)^2\frac{ds}{s},\end{aligned}$$

where in the second line we used the resolution of the identity Proposition 3.1.3. The Cauchy-Schwarz inequality then gives

$$\|f(T)\,u\|^2 \leq 2C_{SF}\int_0^\infty \left(\int_0^\infty \|q_s(T)f(T)\,q_t(T)\|\,\frac{dt}{t}\right)\left(\int_0^\infty \|q_s(T)f(T)\,q_t(T)\|\,\|q_t(T)u\|^2\,\frac{dt}{t}\right)\frac{ds}{s}. \tag{3.14}$$

From the homomorphism property for the $H_0^\infty\left(S_\mu^o\right)$-functional calculus,

$$q_s(T)f(T)\,q_t(T) = (q_s \cdot f \cdot q_t)(T)\,.$$

Since our operator T satisfies resolvent bounds with constant $C_\mu > 0$,

$$\begin{aligned}\|q_s(T)f(T)q_t(T)\| &= \|(q_s \cdot f \cdot q_t)(T)\|\\ &= \frac{1}{2\pi}\left\|\int_\gamma (q_s \cdot f \cdot q_t)(z)\,(T - zI)^{-1}\,dz\right\|\\ &\leq \frac{C_\mu}{2\pi}\cdot\|f\|_\infty\cdot\int_\gamma |q_s(z)|\,|q_t(z)|\,\frac{|dz|}{|z|}.\end{aligned}$$

On noting that $q \in H_0^\infty\left(S_\mu^o\right)$,

$$\|q_s(T)f(T)\,q_t(T)\| \leq c \cdot C_\mu \cdot \|f\|_\infty \cdot \int_\gamma \frac{|sz|^\alpha}{1 + |sz|^{2\alpha}}\frac{|tz|^\alpha}{1 + |tz|^{2\alpha}}\frac{|dz|}{|z|}$$

for some $\alpha > 0$ and constant $c > 0$ independent of T. Thus we obtain the estimate

$$\|q_s(T)f(T)\,q_t(T)\| \leq c \cdot C_\mu \cdot \|f\|_\infty \cdot \begin{cases} \left(\frac{t}{s}\right)^\alpha \left(1 + \log\left(\frac{s}{t}\right)\right) & \text{for } 0 < t \leq s < \infty \\ \left(\frac{s}{t}\right)^\alpha \left(1 + \log\left(\frac{t}{s}\right)\right) & \text{for } 0 < s < t < \infty, \end{cases}$$

where the value of the T independent constant c is allowed to change. This then implies that

$$\sup_{s>0} \int_0^\infty \|q_s(T)f(T)q_t(T)\| \frac{dt}{t}, \quad \sup_{t>0} \int_0^\infty \|q_s(T)f(T)q_t(T)\| \frac{ds}{s} \leq c \cdot C_\mu \cdot \|f\|_\infty .$$

On applying this estimate to (3.14),

$$\begin{aligned} \|f(T)\,u\|^2 &\leq c \cdot C_\mu \cdot C_{SF} \cdot \|f\|_\infty \int_0^\infty \int_0^\infty \|q_s(T)f(T)\,q_t(T)\| \, \|q_t(T)u\|^2 \frac{dt}{t}\frac{ds}{s} \\ &= c \cdot C_\mu \cdot C_{SF} \cdot \|f\|_\infty \int_0^\infty \|q_t(T)u\|^2 \int_0^\infty \|q_s(T)f(T)\,q_t(T)\| \frac{ds}{s}\frac{dt}{t} \\ &\leq c^2 \cdot C_\mu^2 \cdot C_{SF} \cdot \|f\|_\infty^2 \int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} \\ &\lesssim c^2 \cdot C_\mu^2 \cdot C_{SF}^2 \cdot \|f\|_\infty^2 \|u\|^2 . \end{aligned}$$

$\square$

Finally, the following Kato type estimate follows from a well-known classical argument.

Corollary 3.1.2. *Suppose that the bisectorial operator T satisfies square function estimates with constant $C_{SF} > 0$ and the constant in the resolvent estimate (3.9) is $C_\mu > 0$. Then there exists a constant $c > 0$, independent of T, such that*

$$(c \cdot C_{SF} \cdot C_\mu)^{-1} \cdot \|Tu\| \leq \left\|\sqrt{T^2}u\right\| \leq (c \cdot C_{SF} \cdot C_\mu) \cdot \|Tu\| \tag{3.15}$$

for any $u \in D(T)$.

Proof. Consider the restriction $S := T|_{\overline{R(T)}}$. S is an injective bisectorial operator that satisfies square function estimates with constant $C_{SF} > 0$. Since S is injective it follows that $f(S)$ is well-defined for any $f \in H^\infty\left(S^o_\mu\right)$ by Remark 3.1.1. This allows us to define the operators $f_1(S)$ and $f_2(S)$, where the functions f_1 and f_2 are defined by

$$f_1(z) := \frac{\sqrt{z^2}}{z} \quad \text{and} \quad f_2(z) := \frac{z}{\sqrt{z^2}} \quad \text{for} \quad z \in S^o_\mu.$$

The previous theorem allows us to deduce that both of these operators are norm bounded by $c \cdot C_{SF} \cdot C_\mu$ for some T independent constant $c > 0$. Applying the multiplicative part of Theorem 3.1.3 to the functions $f = f_1$ and $g(z) = z$ implies that

$$\frac{\sqrt{S^2}}{S} \cdot S = \sqrt{S^2} \tag{3.16}$$

on

$$D(S) = D\left(\frac{\sqrt{S^2}}{S} \cdot S\right) = D\left(\sqrt{S^2}\right) \cap D(S). \tag{3.17}$$

Similarly, applying the multiplicative part of Theorem 3.1.3 to $f = f_2$ and $g(z) = \sqrt{z^2}$ gives

$$\frac{S}{\sqrt{S^2}} \cdot \sqrt{S^2} = S \tag{3.18}$$

on

$$D\left(\sqrt{S^2}\right) = D\left(\frac{S}{\sqrt{S^2}} \cdot \sqrt{S^2}\right) = D(S) \cap D\left(\sqrt{S^2}\right). \tag{3.19}$$

Equations (3.17) and (3.19) together imply that the domains $D\left(\sqrt{S^2}\right)$ and $D(S)$ coincide and therefore both (3.16) and (3.18) will remain valid on all of $D(S)$.

Let $u \in D(T)$. Proposition 3.1.1 states that u has the decomposition $u = u_1 \oplus u_2 \in N(T) \oplus \overline{R(T)}$. Then

$$\begin{aligned}
\|Tu\| &= \|Su_2\| \\
&= \left\| \frac{S}{\sqrt{S^2}} \cdot \sqrt{S^2} u_2 \right\| \\
&\leq c \cdot C_{SF} \cdot C_\mu \cdot \left\| \sqrt{S^2} u_2 \right\| \\
&= c \cdot C_{SF} \cdot C_\mu \cdot \left\| \sqrt{T^2} u \right\|,
\end{aligned}$$

where in the last line we used the fact that the functional calculus commutes with the restriction map as given in Proposition 2.6.5 of [32]. Also,

$$\begin{aligned}
\left\| \sqrt{T^2} u \right\| &= \left\| \sqrt{S^2} u_2 \right\| \\
&= \left\| \frac{\sqrt{S^2}}{S} \cdot S u_2 \right\| \\
&\leq c \cdot C_{SF} \cdot C_\mu \cdot \|Su_2\| \\
&= c \cdot C_{SF} \cdot C_\mu \cdot \|Tu\|.
\end{aligned}$$

□

3.2. Carleson Measures

The AKM framework relies on a few fundamental results from classical harmonic analysis concerning Carleson measures. These results are listed here for the sake of completeness and since they will also be required for the potential dependent case. First recall the definition of a Carleson measure and the non-tangential maximal operator.

Definition 3.2.1 (Carleson Measure)**.** *Let μ be a measure on $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0,\infty)$. The Carleson norm of μ is defined to be the quantity*

$$\|\mu\|_{\mathcal{C}} := \sup_{Q \in \Delta} \frac{\mu\left(R_Q\right)}{|Q|},$$

where $R_Q := Q \times [0, l(Q))$ is the Carleson box over Q. μ is said to be a Carleson measure if $\|\mu\|_{\mathcal{C}} < \infty$.

Definition 3.2.2 (Non-Tangential Maximal Operator)**.** *The non-tangential maximal operator $\mathcal{N}$ is the operator defined through*

$$\mathcal{N}\left(F\right)(x) := \sup_{(y,t):|y-x|<t} |F(y,t)|$$

for $F : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ and any $x \in \mathbb{R}^n$.

Theorem 3.2.1 (Carleson's Theorem, [30, Cor. 7.3.6])**.** *There exists a constant $c > 0$ for which*

$$\int_{\mathbb{R}^{n+1}_+} |F(x,t)|^2 \, d\mu(x,t) \leq c \left\|\mathcal{N}\left(F\right)\right\|_2^2 \cdot \|\mu\|_{\mathcal{C}}$$

for all Carleson measures μ on $\mathbb{R}^{n+1}_+$ and μ-measurable F.

The following lemma was used in the original solution to the Kato problem in [7]. It states that in order to prove a Carleson measure estimate it is sufficient to replace the Carleson boxes with subsets $E^*_Q \subset R_Q$ that make up a sufficiently large proportion of R_Q.

Lemma 3.2.1 (The John-Nirenberg Lemma for Carleson Measures)**.** *Let μ be a measure on $\mathbb{R}^{n+1}_+$ and $\beta > 0$. Suppose that for every $Q \in \Delta$ there exists a collection $\{Q_k\}_k \subset \Delta$ of disjoint subcubes of Q such that $E_Q := Q \setminus \cup_k Q_k$ satisfies $|E_Q| > \beta\,|Q|$ and such that*

$$\sup_{Q \in \Delta} \frac{\mu(E^*_Q)}{|Q|} \leq C \tag{3.20}$$

for some $C > 0$*, where* $E_Q^* := R_Q \setminus \cup_k R_{Q_k}$*. Then*

$$\sup_{Q\in\Delta} \frac{\mu(R_Q)}{|Q|} \le \frac{C}{\beta}. \tag{3.21}$$

PROOF. Fix $Q \in \Delta$ and let $\{Q_{k_1}\}_{k_1}$ be a collection of subcubes as in the hypotheses of the lemma. Apply the bound (3.20) to the decomposition

$$\mu(R_Q) = \mu\left(E_Q^*\right) + \sum_{k_1} \mu\left(R_{Q_{k_1}}\right)$$

to obtain

$$\mu(R_Q) \le C\,|Q| + \sum_{k_1} \mu\left(R_{Q_{k_1}}\right).$$

For each k_1, let $\{Q_{k_1,k_2}\}_{k_2}$ be a collection of subcubes of Q_{k_1} that satisfy the hypotheses of the lemma. Decompose $\mu(R_{Q_{k_1}})$ and once again apply (3.20) to obtain

$$\begin{aligned}
\mu(R_Q) &\le C\,|Q| + \sum_{k_1} \left(\mu(E^*_{Q_{k_1}}) + \sum_{k_2} \mu(Q_{k_1,k_2})\right) \\
&\le C\,|Q| + \sum_{k_1} C\,|Q_{k_1}| + \sum_{k_1,k_2} \mu(Q_{k_1,k_2}) \\
&\le C\,|Q| + C\,|Q|\,(1-\beta) + \sum_{k_1,k_2} \mu(Q_{k_1,k_2}),
\end{aligned}$$

where the last line follows from $|E_Q| > \beta\,|Q|$. Iterating this process and summing the resulting geometric series gives (3.21). □

3.3. The Kato Square Root Problem

We will now present a brief survey on the Kato square root problem. For a more thorough account of the problem, its general history and motivation, the reader is referred to [47]. The Kato square root problem has a long and varied history extending back to 1961 when T. Kato first posed this problem in a more abstract form for general m-accretive operators in [38]. A year following this, in [41], a counterexample to this general question was found by J. L. Lions. This led to a reformulation of the problem for operators A associated with sesquilinear nonsymmetric forms $\mathfrak{a}$. In this form, the problem amounted to proving that the domain of the square root operator $D\left(A^{\frac{1}{2}}\right)$ was equal to the domain of the form $D\left(\mathfrak{a}\right)$. Once again, the general solution to this problem was answered in the negative by A. McIntosh in [44] almost ten years later forcing the problem to be reposed in the form that it it is famous for today, the Kato square root problem for divergence form elliptic operators $(-\mathrm{div} A\nabla)$. This problem was solved in one-dimension by R. Coifman,

A. McIntosh and Y. Meyer in 1982 in [17]. Then, it wasn't until a full 20 years following this that the problem was solved by S. Hofmann, M. Lacey and A. McIntosh, [34], in arbitrary dimension with the additional assumption that the heat kernel of L satisfied Gaussian upper bounds. Consecutively, the full solution was found for divergence form elliptic operators on $\mathbb{R}^n$ by P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and P. Tchamitchian in [7].

This, however, was far from the end of the story. Indeed, the problem remained open in a number of different directions. For instance, in [8] the solution was generalised to elliptic systems. The solution was also extended to domains in [10] and [22] but all such results in this vein required some degree of regularity of the boundary. The Kato problem was considered in a more geometric setting on submanifolds in [46] and on vector bundles in [14]. In yet another direction, the Kato problem and quadratic estimates were considered in a Banach space setting in [36] and more recently [26]. The Kato problem and quadratic estimates were also considered in the weighted setting for degenerate elliptic operators in [19] and [9]. Finally, non-homogeneous Kato type estimates have been considered before in [10, 22, 28, 29]. An expanded account of these references will be provided in §6.1.

The Kato square root problem presents as an important critical case of a more general problem. For the operator L on $L^2(\mathbb{R}^n)$ and $\alpha \in (0,1)$, one could consider the domain of the fractional operator L^α and ask whether it is true that

$$D(L^\alpha) = W^{2\alpha,2}(\mathbb{R}^n).$$

It was proved by Kato in [38] that for $\alpha \in \left(0, \frac{1}{2}\right)$ this statement will be true but for $\alpha \in \left(\frac{1}{2}, 1\right)$ it will be false.

It is interesting to note that the solution to the Kato square root problem for the operator L is actually trivial in the case that A is real symmetric. Observe that in this situation L will be symmetric and therefore so will the square root operator $\sqrt{L}$. This gives

$$\begin{aligned}\left\|\sqrt{L}u\right\|^2 &= \langle\sqrt{L}u, \sqrt{L}u\rangle \\ &= \langle\sqrt{L}\sqrt{L}u, u\rangle \\ &= \langle -\mathrm{div}A\nabla u, \nabla u\rangle \\ &= \langle A\nabla u, \nabla u\rangle.\end{aligned}$$

Applying the ellipticity and boundedness properties of A then gives the Kato estimate.

Remark 3.3.1. For a proof of the Kato estimate for L on $\mathbb{R}^n$, it is enough to prove the upper estimate

$$\left\|\sqrt{L}u\right\| \leq C \cdot \|\nabla u\|$$

for all $u \in D(L)$. The lower estimate can be obtained from this using a duality argument. On applying the Gårding inequality for $\mathfrak{l}^A$, (3.1),

$$\begin{aligned}\kappa_A \|\nabla u\|^2 &\leq \mathrm{Re}\left(\mathfrak{l}^A[u,u]\right)\\ &\leq \langle Lu, u\rangle\\ &= \langle \sqrt{L}u, \left(\sqrt{L}\right)^* u\rangle\\ &\leq \left\|\sqrt{L}u\right\| \cdot \left\|\sqrt{L^*}u\right\|.\end{aligned}$$

Since the adjoint operator L^* is a divergence form elliptic operator, the upper estimate can be applied to obtain

$$\kappa_A \|\nabla u\|^2 \leq \left\|\sqrt{L}u\right\| \cdot \|\nabla u\|.$$

This gives the lower estimate.

There are numerous motivations and interactions of the Kato problem with other areas in Harmonic Analysis including, but not limited to, the boundedness of the Cauchy singular integral operator [17], solvability of boundary value problems on Lipschitz domains [5, 9], quadratic estimates and T(b) type theorems. We present a brief motivating example taken from [47].

Example 3.3.1. Consider the Neumann problem in $\mathbb{R}^n \times (0,\infty)$,

$$\frac{\partial^2 u}{\partial t^2} = Lu, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = f$$

for some $f \in L^2(\mathbb{R}^n)$. The boundedness of the holomorphic functional calculus of L would allow us to form the solution

$$u := -L^{-\frac{1}{2}} e^{-tL^{\frac{1}{2}}} f$$

to this problem. This solution would clearly satisfy the bound

$$\|\partial_t u\| = \left\|e^{-tL^{\frac{1}{2}}} f\right\| \leq \|f\|$$

uniformly in t. The Kato square root estimate would then also give us the bound

$$\|\nabla u\| \lesssim \left\|L^{\frac{1}{2}} u\right\| = \left\|e^{-tL^{\frac{1}{2}}} f\right\| \lesssim \|f\|.$$

3.4. Potential Free AKM

We now recall the original AKM framework, as introduced in [11], and its applications to the Kato square root problem. Let $\Pi := \Gamma + \Gamma^*$ be a Dirac-type operator on a Hilbert space

$\mathcal{H}$ and $\Pi_B := \Gamma + B_1\Gamma^* B_2$ be a perturbation of Π by bounded operators B_1 and B_2. As Π is a self-adjoint operator, it follows from Corollary 3.1.1 that it must satisfy square function estimates and therefore possess a bounded holomorphic functional calculus. The aim of of the AKM framework is to demonstrate that, under suitable conditions imposed on the operators Γ, B_1 and B_2, the perturbed operator Π_B will retain a bounded holomorphic functional calculus. The hypotheses used form a set of eight conditions. These are listed below for the convenience of the reader.

(H1) $\Gamma : D(\Gamma) \to \mathcal{H}$ is a closed, densely defined, nilpotent operator.

(H2) B_1 and B_2 satisfy the accretivity conditions

$$\mathrm{Re}\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2 \qquad \text{and} \qquad \mathrm{Re}\langle B_2 v, v\rangle \geq \kappa_2 \|v\|^2$$

for all $u \in R(\Gamma^*)$ and $v \in R(\Gamma)$ for some κ_1, $\kappa_2 > 0$.

(H3) The operators Γ and Γ^* satisfy

$$\Gamma^* B_2 B_1 \Gamma^* = 0 \qquad \text{and} \qquad \Gamma B_1 B_2 \Gamma = 0.$$

(H4) The Hilbert space is $\mathcal{H} = L^2\left(\mathbb{R}^n; \mathbb{C}^N\right)$ for some $n, N \in \mathbb{N}$.

(H5) The operators B_1 and B_2 represent multiplication by matrix-valued functions. That is,

$$B_1(f)(x) = B_1(x) \cdot f(x) \qquad \text{and} \qquad B_2(f)(x) = B_2(x) \cdot f(x)$$

for all $f \in \mathcal{H}$ and $x \in \mathbb{R}^n$ where B_1, $B_2 \in L^\infty\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^N\right)\right)$.

(H6) For every bounded Lipschitz function $\eta : \mathbb{R}^n \to \mathbb{C}$, we have that $\eta D(\Gamma) \subset D(\Gamma)$ and $\eta D(\Gamma^*) \subset D(\Gamma^*)$. Moreover, the commutators $[\Gamma, \eta I]$ and $[\Gamma^*, \eta I]$ are multiplication operators that satisfy the bound

$$|[\Gamma, \eta I](x)|, \ |[\Gamma^*, \eta I](x)| \leq c\, |\nabla \eta(x)|$$

for all $x \in \mathbb{R}^n$ and some constant $c > 0$.

(H7) For any $u \in D(\Gamma_0)$ and $v \in D(\Gamma_0^*)$, both compactly supported,

$$\int_{\mathbb{R}^n} \Gamma u = 0 \quad \textit{and} \quad \int_{\mathbb{R}^n} \Gamma_0^* v = 0.$$

(H8) There exists $c > 0$ such that

$$\|\nabla u\| \leq c \|\Pi_0 u\|$$

for all $u \in R(\Pi_0) \cap D(\Pi_0)$.

The main theorem of [11] then states that under these hypotheses the operator Π_B will possess a bounded holomorphic functional calculus.

Theorem 3.4.1 ([11, Thm. 2.7]). *Let Γ, B_1 and B_2 be operators that satisfy the conditions (H1) - (H8). The perturbed Dirac-type operator $\Pi_B := \Gamma + B_1 \Gamma^* B_2$ will satisfy square function estimates. That is, there must exist $C > 0$ for which*

$$C^{-1} \cdot \|u\|^2 \leq \int_0^\infty \left\|Q_t^B u\right\|^2 \frac{dt}{t} \leq C \cdot \|u\|^2$$

for all $u \in \overline{R(\Pi_B)}$, where $Q_t^B := q_t(\Pi_B) = t\Pi_B \left(I + t^2 \Pi_B^2\right)^{-1}$.

Let us discuss how Theorem 3.4.1 is used to prove the Kato square root estimate (3.2). Set the Hilbert space $\mathcal{H}$ to be

$$\mathcal{H} := L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n).$$

Brand our operators as

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad B_1 = I \quad \textit{and} \quad B_2 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},$$

where $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ is a perturbation matrix that satisfies the Gårding inequality (3.1). The perturbed Dirac-type operator will then be

$$\Pi_B := \begin{pmatrix} 0 & -\mathrm{div} A \\ \nabla & 0 \end{pmatrix},$$

with square

$$\Pi_B^2 = \begin{pmatrix} -\mathrm{div} A \nabla & 0 \\ 0 & -\nabla \mathrm{div} A \end{pmatrix}.$$

The square root of this operator will then be given by

$$\sqrt{\Pi_B^2} = \begin{pmatrix} \sqrt{-\mathrm{div} A \nabla} & 0 \\ 0 & \sqrt{-\nabla \mathrm{div} A} \end{pmatrix}.$$

It is straightforward to check that the operators $\{\Gamma, B_1, B_2\}$ satisfy all eight hypotheses (H1) - (H8). Indeed, (H1), (H3), (H4) and (H5) follow trivially from the definitions of our operators, (H2) follows from the Gårding inequality (3.1), (H6) follows from the product rule, (H7) is given by the homogeneity of the operator ∇ and (H8) follows from the boundedness of the Riesz transforms $\partial_j \partial_k (-\Delta)^{-1}$ on $L^2(\mathbb{R}^n)$ for $j, k = 1, \cdots, n$. Theorem 3.4.1 then implies that Π_B must satisfy square function estimates and therefore possess a bounded holomorphic functional calculus. Corollary 3.1.2 then gives the estimate

$$C^{-1} \|\Pi_B u\| \leq \left\| \sqrt{\Pi_B^2} u \right\| \leq C \|\Pi_B u\|.$$

for all $u \in D(\Pi_B)$, for some $C > 0$. If we apply this estimate to an element of the form $u = (u_1, 0) \in D(\Pi_B)$ where $u_1 \in D(L)$ we obtain the Kato estimate

$$C^{-1} \|\nabla u_1\| \leq \left\| \sqrt{L} u_1 \right\| \leq C \|\nabla u_1\|.$$

3.5. A Path to the Kato Problem with Potential

We will now delineate a path to the Kato estimate with potential from the boundedness of holomorphic functional calculus for certain Dirac-type operators. The process used to obtain the potential free Kato estimate in the previous section will be adapted. Define our Hilbert space to be

$$\mathcal{H} := L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n).$$

Let our operator Γ be given by

$$\Gamma = \Gamma_{|V|^{\frac{1}{2}}} = \begin{pmatrix} 0 & 0 \\ \nabla_{|V|^{\frac{1}{2}}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix},$$

defined on the dense domain $H^{1,V}(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n)$, where $H^{1,V}(\mathbb{R}^n)$ is as defined in (3.5) and $\nabla_{|V|^{\frac{1}{2}}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n)$ is the operator

$$\nabla_{|V|^{\frac{1}{2}}} = \begin{pmatrix} |V|^{\frac{1}{2}} \\ \nabla \end{pmatrix}.$$

It is tempting to write the adjoint of the operator $\Gamma_{|V|^{\frac{1}{2}}}$ as

$$\Gamma^*_{|V|^{\frac{1}{2}}} = \begin{pmatrix} 0 & |V|^{\frac{1}{2}} & -\mathrm{div} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, this is not strictly speaking true and this notation should be used with caution since the domain of $\Gamma^*_{|V|^{\frac{1}{2}}}$ is not necessarily $L^2 \oplus D\left(|V|^{\frac{1}{2}}\right) \oplus D\left(\nabla^*\right)$. Instead, it is more accurate to write

$$\Gamma^*_{|V|^{\frac{1}{2}}} = \begin{pmatrix} 0 & \nabla^*_{|V|^{\frac{1}{2}}} \\ 0 & 0 \end{pmatrix}.$$

Let $A \in L^\infty\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^n\right)\right)$ be a matrix-valued multiplication operator and suppose that the Gårding inequalities (3.1) and (3.6) are satisfied with constants $\kappa_A > 0$ and $\kappa^V_A > 0$ respectively. Define our perturbations B_1 and B_2 through

$$B_1 = I \qquad \textit{and} \qquad B_2 := \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} := \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i \cdot \arg V} & 0 \\ 0 & 0 & A \end{pmatrix},$$

where

$$\hat{A} = \begin{pmatrix} e^{i \cdot \arg V} & 0 \\ 0 & A \end{pmatrix}.$$

Our perturbed Dirac-type operator then becomes

$$\Pi_{|V|^{\frac{1}{2}},B} := \Gamma_{|V|^{\frac{1}{2}}} + B_1 \Gamma^*_{|V|^{\frac{1}{2}}} B_2 = \begin{pmatrix} 0 & \nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \\ \nabla_{|V|^{\frac{1}{2}}} & 0 \end{pmatrix}.$$

It is straightforward to check that

$$\nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \nabla_{|V|^{\frac{1}{2}}} = V - \mathrm{div} A \nabla = L + V$$

with correct domains, where $V - \mathrm{div} A \nabla$ is as defined in Section 3 using sesquilinear forms. Thus the square of our perturbed Dirac-type operator is

$$\Pi^2_{|V|^{\frac{1}{2}},B} = \begin{pmatrix} V - \mathrm{div} A \nabla & 0 \\ 0 & \nabla_{|V|^{\frac{1}{2}}} \nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \end{pmatrix}.$$

Suppose that it can be proved that $\Pi_{|V|^{\frac{1}{2}},B}$ is a bisectorial operator that satisfies square function estimates with constant $C_V > 0$. Corollary 3.1.2 then states that there must exist some $c > 0$, independent of the potential, such that

$$(c \cdot C_\mu \cdot C_V)^{-1} \left\| \Pi_{|V|^{\frac{1}{2}},B} u \right\| \leq \left\| \sqrt{\Pi^2_{|V|^{\frac{1}{2}},B}} u \right\| \leq c \cdot C_\mu \cdot C_V \cdot \left\| \Pi_{|V|^{\frac{1}{2}},B} u \right\|$$

for all $u \in D\left(\Pi_{B,|V|^{\frac{1}{2}}}\right)$, where C_μ is the constant from the resolvent estimates for $\Pi_{|V|^{\frac{1}{2}}}$ (cf. Definition 3.1.1). For $u_1 \in D(L+V)$ we evidently have $u := (u_1, 0, 0) \in D\left(\Pi_{|V|^{\frac{1}{2}},B}\right)$. Applying the previous inequality to the vector u will then give the Kato estimate with potential (KP).

This shifts the burden of proof to showing that the perturbed Dirac-type operator $\Pi_{|V|^{\frac{1}{2}},B}$ is a bisectorial operator that satisfies square function estimates. As the operators under consideration are no longer first-order homogeneous, the conditions (H7) and (H8) will clearly not be satisfied. Therefore the AKM framework cannot be applied directly. This part of the thesis is dedicated to the construction of a non-homogeneous AKM framework that can be applied to non-homogeneous Dirac-type operators.

3.6. AKM without Cancellation and Coercivity

The operators that we wish to consider will satisfy the first six conditions of [11]. However, they will not necessarily satisfy the cancellation condition (H7) and the coercivity condition (H8). It will therefore be fruitful to see what happens to the original AKM framework when the cancellation and coercivity conditions are removed.

Similar to the original result, we begin by assuming that we have operators that satisfy the hypotheses (H1) - (H3) from [11]. In Section 4 of [11], the authors assume that they have operators that satisfy the hypotheses (H1) - (H3) and they derive several important operator theoretic consequences from only these hypotheses. As our operators Γ, B_1 and B_2 also satisfy (H1) - (H3), it follows that any result proved in Section 4 of [11] must also be true for our operators and can be used with impunity. In the interest of making this thesis as self-contained as possible, we will now restate any such result that is to be used in this thesis.

Proposition 3.6.1 (The Hodge Decomposition, [11]). *Define the perturbation dependent operators*

$$\Gamma_B^* := B_1 \Gamma^* B_2, \quad \Gamma_B := B_2^* \Gamma B_1^* \quad \textit{and} \quad \Pi_B := \Gamma + \Gamma_B^*.$$

Suppose that the operators $\{\Gamma, B_1, B_2\}$ satisfy (H1) - (H3). The Hilbert space $\mathcal{H}$ has the following Hodge decomposition into closed subspaces:

$$\mathcal{H} = N(\Pi_B) \oplus \overline{R(\Gamma_B^*)} \oplus \overline{R(\Gamma)}. \tag{3.22}$$

Moreover, we have $N(\Pi_B) = N(\Gamma_B^*) \cap N(\Gamma)$ *and* $\overline{R(\Pi_B)} = \overline{R(\Gamma_B^*)} \oplus \overline{R(\Gamma)}$*. When* $B_1 = B_2 = I$ *these decompositions are orthogonal, and in general the decompositions are topological. Similarly, there is also a decomposition*

$$\mathcal{H} = N(\Pi_B^*) \oplus \overline{R(\Gamma_B)} \oplus \overline{R(\Gamma^*)}.$$

Proposition 3.6.2 ([11])**.** *Suppose that the operators* $\{\Gamma, B_1, B_2\}$ *satisfy (H1) - (H3). The perturbed Dirac-type operator* Π_B *is an* ω*-bisectorial operator with* $\omega := \frac{1}{2}(\omega_1 + \omega_2)$ *where*

$$\omega_1 := \sup_{u \in R(\Gamma^*) \setminus \{0\}} |\arg\langle B_1 u, u\rangle| < \frac{\pi}{2}$$

and

$$\omega_2 := \sup_{u \in R(\Gamma) \setminus \{0\}} |\arg\langle B_2 u, u\rangle| < \frac{\pi}{2}.$$

The bisectoriality of Π_B ensures that the following operators will be well-defined.

Definition 3.6.1. *Suppose that the operators* $\{\Gamma, B_1, B_2\}$ *satisfy (H1) - (H3). For* $t \in \mathbb{R} \setminus \{0\}$*, define the perturbation dependent operators*

$$R_t^B := (I + it\Pi_B)^{-1}, \quad P_t^B := \left(I + t^2 (\Pi_B)^2\right)^{-1},$$

$$Q_t^B := t\Pi_B P_t^B \quad \textit{and} \quad \Theta_t^B := t\Gamma_B^* P_t^B.$$

When there is no perturbation, i.e. when $B_1 = B_2 = I$*, the* B *will dropped from the superscript or subscript. For example, instead of* Θ_t^I *or* Π_I *the notation* Θ_t *and* Π *will be employed.*

Remark 3.6.1. An easy consequence of Proposition 3.6.2 is that the operators R_t^B, P_t^B and Q_t^B are all uniformly $\mathcal{H}$-bounded in t. Furthermore, on taking the Hodge decomposition Proposition 3.6.1 into account, it is clear that the operators Θ_t^B will also be uniformly $\mathcal{H}$-bounded in t.

The next result tells us how the operators Π_B and P_t^B interact with Γ and Γ_B^*.

Lemma 3.6.1 ([11])**.** *Suppose that the operators* $\{\Gamma, B_1, B_2\}$ *satisfy (H1) - (H3). The following relations are true.*

$$\Pi_B \Gamma u = \Gamma_B^* \Pi_B u \quad \textit{for all } u \in D(\Gamma_B^* \Pi_B),$$

$$\Pi_B \Gamma_B^* u = \Gamma \Pi_B u \quad \textit{for all } u \in D(\Gamma \Pi_B),$$

$$\Gamma P_t^B u = P_t^B \Gamma u \quad \textit{for all } u \in D(\Gamma), \quad \textit{and}$$

$$\Gamma_B^* P_t^B u = P_t^B \Gamma_B^* u \quad \textit{for all } u \in D(\Gamma_B^*) .$$

The subsequent lemma provides a square function estimate for the unperturbed Dirac-type operator Π. When considering square function estimates for the perturbed operator, there will be several instances where the perturbed case can be reduced with the assistance of this unperturbed estimate. Its proof follows directly from the self-adjointness of the operator Π and Corollary 3.1.1.

Lemma 3.6.2 ([11])**.** *Suppose that the operators $\{\Gamma, B_1, B_2\}$ satisfy (H1) - (H3). The quadratic estimate*

$$\int_0^\infty \|Q_t u\|^2 \frac{dt}{t} \leq \frac{1}{2} \|u\|^2 \tag{3.23}$$

holds for all $u \in \mathcal{H}$. Equality holds on $\overline{R(\Pi)}$.

The following result will play a crucial role in the reduction of the square function estimate (3.3).

Proposition 3.6.3 ([11])**.** *Suppose that the operators $\{\Gamma, B_1, B_2\}$ satisfy (H1) - (H3). Assume that the estimate*

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \leq c \cdot \|u\|^2 \tag{3.24}$$

holds for all $u \in R(\Gamma)$ and some constant $c > 0$, together with three similar estimates obtained on replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$ and $\{\Gamma, B_1^*, B_2^*\}$. Then Π_B satisfies the quadratic estimate

$$(c \cdot C)^{-1} \cdot \|u\|^2 \leq \int_0^\infty \left\|Q_t^B u\right\|^2 \frac{dt}{t} \leq c \cdot C \cdot \|u\|^2 \tag{3.25}$$

for all $u \in \overline{R(\Pi_B)}$, for some $C > 0$ entirely dependent on (H1) - (H3).

The constant dependence of (3.25) is not explicitly mentioned in Proposition 4.8 of [11], but it is relatively easy to trace through their argument and record where (3.24) is used. The following corollary is proved during the course of the proof of Proposition 4.8 of [11].

Corollary 3.6.1 (High Frequency Estimate)**.** *Suppose that the operators $\{\Gamma, B_1, B_2\}$ satisfy (H1) - (H3). For any $u \in R(\Gamma)$, there exists a constant $c > 0$ for which*

$$\int_0^\infty \left\|\Theta_t^B (I - P_t) u\right\|^2 \frac{dt}{t} \leq c \cdot \|u\|^2 .$$

From this point onwards, it will also be assumed that our operators satisfy the additional hypotheses (H4) - (H6). In contrast to the original result, our operators will not be assumed to satisfy the cancellation condition (H7) and the coercivity condition (H8). Without these two conditions, many of the results from Section 5 of [11] will fail. One notable exception to this is that the bounded operators associated with our perturbed Dirac-type operator Π_B will satisfy off-diagonal estimates.

Definition 3.6.2 (Off-Diagonal Bounds). *Define $\langle x \rangle := 1 + |x|$ for $x \in \mathbb{C}$ and $\operatorname{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$ for E, $F \subset \mathbb{R}^n$.*

Let $\{U_t\}_{t>0}$ be a family of operators on $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^N)$. This collection is said to have off-diagonal bounds of order $M > 0$ if there exists $C_M > 0$ such that

$$\|U_t u\|_{L^2(E)} \leq C_M \langle \operatorname{dist}(E, F)/t \rangle^{-M} \|u\| \tag{3.26}$$

whenever E, $F \subset \mathbb{R}^n$ are Borel sets and $u \in \mathcal{H}$ satisfies $\operatorname{supp} u \subset F$.

Proposition 3.6.4 ([11]). *Suppose that the operators $\{\Gamma, B_1, B_2\}$ satisfy (H1) - (H6). Let U_t be given by either R_t^B, R_{-t}^B, P_t^B, Q_t^B or Θ_t^B for every $t > 0$. The collection of operators $\{U_t\}_{t>0}$ has off-diagonal bounds of every order $M > 0$. Moreover, the constant C_M in the estimate (3.26) depends only on M and the hypotheses (H1) - (H6).*

Recall the dyadic decomposition of $\mathbb{R}^n$. Let $\Delta = \cup_{j=-\infty}^{\infty} \Delta_{2^j}$ where $\Delta_t := \{2^j(k + (0,1]^n) : k \in \mathbb{Z}^n\}$ if $2^{j-1} < t \leq 2^j$. Define the averaging operator $A_t : \mathcal{H} \to \mathcal{H}$ through

$$A_t u(x) := \frac{1}{|Q(x,t)|} \int_{Q(x,t)} u(y)\, dy$$

for $x \in \mathbb{R}^n$, $t > 0$ and $u \in \mathcal{H}$, where $Q(x,t)$ is the unique dyadic cube in Δ_t that contains the point x.

For an operator family $\{U_t\}_{t>0}$ that satisfies off-diagonal bounds of every order, there exists an extension $U_t : L^\infty(\mathbb{R}^n; \mathbb{C}^N) \to L^2_{loc}(\mathbb{R}^n; \mathbb{C}^N)$ for each $t > 0$. This is constructed by defining

$$U_t u(x) := \lim_{r \to \infty} \sum_{\substack{R \in \Delta_t \\ \operatorname{dist}(Q,R) < r}} U_t(\mathbb{1}_R u)(x),$$

for $x \in Q \in \Delta_t$ and $u \in L^\infty(\mathbb{R}^n; \mathbb{C}^N)$. The convergence of the above limit is guaranteed by the off-diagonal bounds of $\{U_t\}_{t>0}$. Further detail on this construction can be found in [11], [23], [46] or [26]. The above extension then allows us to introduce the principal part of the operator U_t.

Definition 3.6.3. *Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every* *order. For $t > 0$, the principal part of U_t is the operator $\zeta_t : \mathbb{R}^n \to \mathcal{L}\left(\mathbb{C}^N\right)$ defined through*

$$[\zeta_t(x)]\,(w) := (U_t w)\,(x)$$

for each $x \in \mathbb{R}^n$ and $w \in \mathbb{C}^N$.

The following generalisation of Corollary 5.3 of [11] will also be true with an identical proof.

Proposition 3.6.5. *Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. Let $\zeta_t : \mathbb{R}^n \to \mathcal{L}\left(\mathbb{C}^N\right)$ denote the principal part of the operator U_t. Then there exists $c > 0$ such that*

$$\fint_Q |\zeta_t(y)|^2 \, dy \leq c$$

for all $Q \in \Delta_t$ and $t > 0$. Moreover, the operators $\zeta_t A_t$ are uniformly L^2-bounded in t.

The ensuing partial result will also be valid. Its proof follows in an identical manner to the first part of the proof of Proposition 5.5 of [11].

Proposition 3.6.6. *Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. Let $\zeta_t : \mathbb{R}^n \to \mathcal{L}\left(\mathbb{C}^N\right)$ denote the principal part of U_t. Then there exists $c > 0$ such that*

$$\|(U_t - \zeta_t A_t)\, v\| \leq c \cdot \|t\nabla v\| \,. \tag{3.27}$$

for any $v \in H^1\left(\mathbb{R}^n; \mathbb{C}^N\right) \subset \mathcal{H}$ and $t > 0$.

The following lemma is entirely independent of the operators $\{\Gamma, B_1, B_2\}$ and will be used in our potential dependent setting.

Lemma 3.6.3 (The Weighted Poincaré Inequality, [11, Lem. 5.4])**.** *There exists a constant $c > 0$ such that*

$$\int_{\mathbb{R}^n} |u(x) - \langle u\rangle_Q|^2 \langle \mathrm{dist}\,(x, Q)\,/t\rangle^{\beta} \, dx \leq c \cdot \int_{\mathbb{R}^n} |t\nabla u(x)|^2 \langle \mathrm{dist}\,(x, Q)\,/t\rangle^{2n+\beta} \, dx$$

for $Q \in \Delta_t$, $\beta < -2n$ and for every u in the Sobolev space $H^1\left(\mathbb{R}^n; \mathbb{C}^N\right)$.

The cancellation and coercivity conditions are both an indelible part of the original AKM framework. We cannot naively expect their removal to have no serious repercussions. The most dire of these consequences is the failure of the estimate

$$\int_0^{\infty} \|(A_t - P_t)\, u\|^2 \, \frac{dt}{t} \leq C \cdot \|u\|^2 \tag{3.28}$$

for $u \in R(\Pi)$. This square function equivalence between the A_t and P_t operators is used in [11] to interchange the P_t operators with the A_t operators, granting use of some of the more enviable properties of the A_t operators. The failure of this estimate in the potential dependent setting is one of two significant obstacles that must be overcome in order for a non-homogeneous AKM framework to be developed. The second significant obstacle is the construction of the test functions in the local T(b) argument.

Chapter 4

Abstract AKM with Potential

In this chapter, additional structure will be imposed upon our Dirac-type operator which assumes that it is dependent on a scalar non-negative potential $V \in L^1_{loc}(\mathbb{R}^n)$ and that it satisfies three additional potential dependent hypotheses intended to replace (H7) and (H8). From this newly imposed structure, we will investigate how close we can get to the usual square function estimate,

$$\int_0^\infty \left\| Q_t^B u \right\|^2 \frac{dt}{t} \simeq \|u\|^2 \qquad u \in \overline{R(\Pi_B)}, \tag{4.1}$$

without imposing additional constraints on our operators and potential. It will be proved that the above perturbation dependent square function norm can be estimated by quantities involving only perturbation free operators. Then, with further restrictions on the potential and operators, the full square function estimate (4.1) will result. This theory will then be applied to the Kato square root problem with potential. In particular, it will be proved that the Kato estimate holds for a restricted class of potentials.

4.1. Potential Dependent Axelsson-Keith-McIntosh

In this section we introduce our potential dependent hypotheses. Our main results for this framework will also be stated. Similar to the original result, it is assumed that we have operators Γ, B_1 and B_2 that satisfy the hypotheses (H1) - (H6) from [11].

For this chapter, our operators will be assumed to satisfy a potential dependent version of (H7). It is labelled (H7V) below to distinguish it from the potential free case.

(H7V) There exists $c > 0$, independent of the potential, for which

$$\left| \int_{\mathbb{R}^n} \Gamma u \right| \leq c \, \|V\|^{\frac{1}{2}}_{L^1(Q)} \cdot \|u\|_{L^2(Q)} \qquad \textit{and} \qquad \left| \int_{\mathbb{R}^n} \Gamma^* v \right| \leq c \, \|V\|^{\frac{1}{2}}_{L^1(Q)} \cdot \|v\|_{L^2(Q)}$$

for every cube $Q \subset \mathbb{R}^n$, $u \in D(\Gamma)$ and $v \in D(\Gamma^*)$ both with compact support in Q.

We introduce two distinct notions of coercivity to replace (H8). These conditions will not be automatically imposed upon our operators.

(H8V1) There exists $A_V > 0$ such that

$$\|\nabla u\| \leq A_V \|\Pi u\|$$

for any $u \in R(\Pi) \cap D(\Pi)$.

(H8V2) There exists $B_V > 0$ such that

$$\left\| V^{\frac{1}{2}} v \right\| \leq B_V \|\Pi v\|$$

for any $v \in R(\Pi) \cap D(\Pi)$.

A weaker form of these coercivity conditions is given below.

(wH8V1) There exists $C_V > 0$ such that

$$\int_0^\infty \|t\nabla P_t u\|^2 \frac{dt}{t} \leq C_V \|u\|^2$$

for any $u \in R(\Pi) \cap D(\Pi)$.

(wH8V2) There exists $D_V > 0$ such that

$$\int_1^\infty \left\| V^{\frac{1}{2}} P_t u \right\|^2 \frac{dt}{t} \leq D_V \|u\|^2$$

for any $u \in R(\Pi) \cap D(\Pi)$.

Remark 4.1.1. It is not too difficult to prove that the strong coercivity conditions (H8V1) and (H8V2) will respectively imply the weak coercivity conditions (wH8V1) and (wH8V2). Indeed, on applying (H8V1) to the left-hand side of (wH8V1),

$$\begin{aligned} \int_0^\infty \|t\nabla P_t u\|^2 \frac{dt}{t} &\leq A_V^2 \int_0^\infty \|t\Pi P_t u\|^2 \frac{dt}{t} \\ &= A_V^2 \int_0^\infty \|Q_t u\|^2 \frac{dt}{t} \\ &= \frac{1}{2} A_V^2 \|u\|^2, \end{aligned}$$

where the last line follows from Lemma 3.6.2. On applying (H8V2) to the left-hand side of (wH8V2),

$$\begin{aligned}\int_1^\infty \left\| V^{\frac{1}{2}} P_t u \right\|^2 \frac{dt}{t} &\leq B_V^2 \int_1^\infty \|t\Pi P_t u\|^2 \frac{dt}{t} \\ &= \frac{1}{2} B_V^2 \|u\|^2 .\end{aligned}$$

In the hopes of proving a result that is as general as possible, we will not automatically assume that our potential dependent operators satisfy any coercivity conditions. In fact, when we come to apply our results to the potential dependent Kato problem, it will be found that the particular operators given in (3.7) satisfy the strong coercivity condition if and only if the higher-order Riesz transforms associated with the potential are bounded on $L^2(\mathbb{R}^n)$ (cf. Lemma 4.3.1). It is known that there exist potentials for which these Riesz transforms are unbounded (recall Proposition 1.3.2 for example). It then follows that the coercivity conditions will not necessarily be satisfied for these operators unless additional restrictions are imposed upon the potential.

Notation. Throughout this chapter, the notation $A \lesssim B$ and $A \simeq B$ will be used to denote that there exists some constant $C > 0$, independent of the potential, for which $A \leq C \cdot B$ and $C^{-1} B \leq A \leq C \cdot B$ respectively.

4.1.1. General Potential

Unlike the original AKM framework, we will not presume that our operators satisfy coercivity conditions. As a result of this and the change in the cancellation condition (H7), we cannot expect for the square function equivalence (3.28) to hold between the operators A_t and P_t. However, a weaker form of this equivalence can be proved and it is this estimate that will ripple through the proof and lead to a weaker form of the square function estimate (3.24).

Theorem 4.1.1. *Define, for all $t > 0$, the functions*

$$w_1(t) := 1 + t^\alpha \left(\sup_{Q \in \Delta_t} \fint_Q |V| \right)^{\frac{1}{2}}, \quad w_2(t) := 1 + t^{1+\delta} \left(\sup_{Q \in \Delta_t} \fint_Q |V| \right)^{\frac{1}{2}},$$

$$g_1(t) := w_1(t)^{-1} \cdot \mathbb{1}_{[0,1]}(t), \qquad \textit{and} \qquad g_2(t) := w_2(t)^{-1} \cdot \mathbb{1}_{(1,\infty)}(t),$$

where α and δ are small numbers such that $0 \leq \alpha < 1$ and $0 < \delta < 1/4$. There exists a *constant $C > 0$, independent of the potential, such that*

$$\int_0^\infty \|(A_t - P_t)\, u\|^2\, g_i(t)^2 \frac{dt}{t} \leq C \cdot \left[\int_0^\infty \|t \nabla P_t u\|^2 \frac{dt}{t} + \|u\|^2 \right] \tag{4.2}$$

for any $u \in \mathcal{H}$, for both $i = 1$ and 2.

The above aligns with our intuition. The classical heat operators $e^{t\Delta}$ are well-known to satisfy a square function equivalence with the A_t operators. The potential dependent P_t operators are related to the heat operators $e^{t(\Delta - V)}$. Thus, for a sufficiently large potential, the P_t operators should become much smaller than the classical heat operators. This could force the difference $\|(P_t - A_t)\, u\|$ to increase, potentially breaking the estimate (3.28) and preventing the A_t and P_t operators from being freely interchanged without paying a price. This price is quantified in the weights of the above theorem and different weights are required for small and large times.

Notice that the above estimate does not tell us anything interesting when V is a potential that grows to infinity as x increases to infinity. Indeed, in such a situation the weights w_1 and w_2 will both be infinite and, as a result, the left-hand side of (4.2) will be zero. It is also useful to note that the boundedness of the term on the right-hand side of (4.2) would normally follow from the coercivity condition (wH8V1). However, since this condition has not been automatically imposed, it must remain in this form.

Remark 4.1.2. At this stage, one might point out that the problem with (3.28) holding for general potentials is that P_t is potential dependent while A_t is not. Perhaps the estimate (3.28) could be saved by replacing the A_t operator with a potential dependent counterpart A_t^V. This is further motivation for our work in Part I of this thesis where potential dependent averaging operators were constructed for the harmonic oscillator potential.

In order to provide a concise statement of our main result, we introduce the following non-tangential maximal functions.

Definition 4.1.1. *Define the weighted local non-tangential maximal function $\mathcal{N}_1$ through*

$$\mathcal{N}_1(u)(x) := \mathcal{N}\left((t, y) \mapsto \mathbb{1}_{[0,1]}(t) w_1(t) A_t P_t u(y)\right)(x) \tag{4.3}$$

and the weighted global non-tangential maximal function $\mathcal{N}_2$ through

$$\mathcal{N}_2(u)(x) := \mathcal{N}\left((t, y) \mapsto \mathbb{1}_{(1,\infty)}(t) w_2(t) A_t P_t u(y)\right)(x), \tag{4.4}$$

for $u \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

The effects of the weakened equivalence (4.2) permeate through the proof of Axelsson-Keith-McIntosh and lead to the below main result that states what price must be paid in the square function estimate (3.24). This result provides a perturbation free quadratic estimate for our operators.

Theorem 4.1.2. *Suppose that the operators $\{\Gamma, B_1, B_2\}$ satisfy (H1) - (H6) and (H7V). There must exist a constant $C > 0$, independent of the potential, and functions $(u_Q)_{Q\in\Delta}$ that satisfy $\|u_Q\|^2 \lesssim |Q|$ for all $Q \in \Delta$ such that*

$$\begin{aligned}\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \leq C \cdot \Bigg[&\left(1 + \sup_{Q\in\Delta} \frac{1}{|Q|} \int_0^{l(Q)} \|t\nabla P_t u_Q(x)\|^2 w_1(t)^{-2} \mathbb{1}_{[0,1]}(t) \frac{dt}{t}\right) \cdot \|\mathcal{N}_1(u)\|^2 \\ &+ \left(1 + \sup_{Q\in\Delta} \frac{1}{|Q|} \int_0^{l(Q)} \|t\nabla P_t u_Q(x)\|^2 w_2(t)^{-2} \mathbb{1}_{(1,\infty)}(t) \frac{dt}{t}\right) \cdot \|\mathcal{N}_2(u)\|^2 \\ &+ \|u\|^2 + \int_0^\infty \|t\nabla P_t u\|^2 \frac{dt}{t}\Bigg]\end{aligned} \tag{4.5}$$

for any $u \in R(\Gamma)$.

The proof of both Theorem 4.1.1 and Theorem 4.1.2 will be presented in detail in §4.2. At first glance, the above estimate might seem complicated and unhelpful. However, there are a few important things to note that will lead to a better understanding of the estimate and its significance.

First of all, observe that the right-hand side of the estimate is independent of the perturbation B. This estimate squeezes out any dependence on the perturbation while remaining as small as possible for this method of proof. Another thing to note is the presence of the localized functions $(u_Q)_{Q\in\Delta}$. The precise definition of these functions is quite technical and will be given in §4.2. The proof of [11] is heavily reliant on a local T(b) type argument and these functions are a remnant of such an argument.

A final observation is that the right-hand side of (4.5) can be greatly simplified if the weak coercivity condition (wH8V1) is satisfied. Indeed if (wH8V1) is satisfied, then each term that contains the operator $t\nabla P_t$ will vanish. The estimate will then reduce to

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim C_V \left(\|u\|^2 + \|\mathcal{N}_1(u)\|^2 + \|\mathcal{N}_2(u)\|^2\right),$$

for some constant $C_V > 0$ that is allowed to depend on the potential. Boundedness will then follow from boundedness of the local and global weighted non-tangential maximal functions. Before stating this in the form of a corollary, some additional notation will be introduced to help us keep track of the dependence of the constants on the potential. In the below definition it is important to note that each of these quantities is reliant not only on the choice of the potential V but also on the operator Γ. $d(V)$ is also dependent on both B_1 and B_2. This dependence on the operators is not explicitly represented in the notation in order to promote readability.

Definition 4.1.2. *Define the quantities*

$$a_1(V) := \sup_{u \in R(\Pi) \cap D(\Pi)} \frac{\|\nabla u\|}{\|\Pi u\|}, \qquad a_2(V) := \sup_{u \in R(\Pi) \cap D(\Pi)} \frac{\left\|V^{\frac{1}{2}} u\right\|}{\|\Pi u\|},$$

$$a_1^w(V) := \sup_{u \in R(\Gamma)} \frac{1}{\|u\|} \left(\int_0^\infty \|t \nabla P_t u\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \qquad a_2^w(V) := \sup_{u \in R(\Gamma)} \frac{1}{\|u\|} \left(\int_1^\infty \left\|V^{\frac{1}{2}} P_t u\right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

$$b_1(V) := \sup_{u \in R(\Gamma)} \frac{\|\mathcal{N}_1(u)\|}{\|u\|}, \quad b_2(V) := \sup_{u \in R(\Gamma)} \frac{\|\mathcal{N}_2(u)\|}{\|u\|}$$

Also define

$$d(V) := \sup_{u \in R(\Gamma)} \left(\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \|u\|^{-1}.$$

Let us now consisely state what conditions will lead to boundedness in estimate (4.5). The below result is an immediate corollary of Theorem 4.1.2.

Corollary 4.1.1. *Suppose that the following condition is satisfied by the operator Γ and potential V.*

(V1) The quantities $a_1^w(V)$, $b_1(V)$ and $b_2(V)$ are all finite.

Then there must exist a constant $C_V > 0$ such that

$$\int_0^\infty \left\|\Theta^B P_t u\right\|^2 \frac{dt}{t} \leq C_V \|u\|^2$$

for all $u \in R(\Gamma)$. Moreover,

$$d(V) \lesssim \left(1 + a_1^w(V)^2\right) \left(1 + b_1(V)^2 + b_2(V)^2\right).$$

The above corollary provides conditions on the potential and the perturbation free Dirac operator that are sufficient to imply the square function estimates for the perturbation dependent operators. Although this achieves one of our goals for this chapter, it must be emphasised that the problem of determining whether these conditions are satisfied for a particular potential and operator Γ is not immediately tractible. In particular, it is non-trivial to determine whether the maximal functions $\mathcal{N}_1$ and $\mathcal{N}_2$ are bounded on $L^2(\mathbb{R}^n)$. This point will be discussed further in §4.1.4.

For the above reason, in the remainder of this paper, we will attack the problem of bounding the square function for $\Theta_t^B P_t$ from two different directions in order to obtain more tangible but less general results. Assumptions will be placed on our potential and operator at the beginning of the proof that will enable us to circumvent the technical hurdles and obtain boundedness.

4.1.2. Potentials Close to Zero

The first such simplifying assumption that can be imposed on the potential V and operator Γ is that the equivalence between the A_t and P_t operators does in fact hold. That is, the unweighted square function estimate (3.28) holds. The below corollary provides a sufficient condition for this to occur. Its proof follows trivially from Theorem 4.1.1.

Corollary 4.1.2. *Define the quantities*

$$\|V\|_\alpha := \sup_{t\le 1}\sup_{Q\in\Delta_t} t^{2\alpha}\fint_Q |V|$$

and

$$\|V\|_\delta := \sup_{t>1}\sup_{Q\in\Delta_t} t^{2(1+\delta)}\fint_Q |V|.$$

Suppose that (wH8V1) is satisfied. If $\|V\|_\alpha < \infty$ then the local estimate

$$\int_0^1 \|(A_t - P_t)\, u\|^2 \frac{dt}{t} \lesssim \left(1 + \|V\|_\alpha + a_1^w(V)^2\right)\|u\|^2$$

holds for any $u \in R(\Pi)$. If $\|V\|_\delta < \infty$ then the global estimate

$$\int_1^\infty \|(A_t - P_t)\, u\|^2 \frac{dt}{t} \lesssim \left(1 + \|V\|_\delta + a_1^w(V)^2\right)\|u\|^2$$

will hold for all $u \in R(\Pi)$.

As the equivalence between the A_t and P_t operators was a key component of the classical proof of the AKM framework, it stands to reason that if this equivalence holds in the potential dependent setting then the entire argument from [11] has a good chance of flowing through unchanged and unimpeded. It turns out that the conditions of the previous corollary are indeed sufficient for this to occur. This is stated in the below theorem and will be proved in §4.2.2.

Theorem 4.1.3. *Suppose that (wH8V1) is satisfied. If* $\|V\|_\alpha < \infty$ *then the local estimate*

$$\int_0^1 \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim \left(1 + a_1^w(V)^2 + \|V\|_\alpha\right)\left(1 + a_1^w(V)^2\right)\|u\|^2 \tag{4.6}$$

is satisfied for all $u \in R(\Gamma)$. *If* $\|V\|_\delta < \infty$ *then the global estimate*

$$\int_1^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim \left(1 + a_1^w(V)^2 + \|V\|_\delta\right)\left(1 + a_1^w(V)^2\right)\|u\|^2 \tag{4.7}$$

must be true for any $u \in R(\Gamma)$. *As a result, if* V *satisfies the condition*

(V2) The quantities $a_1^w(V)$, $\|V\|_\alpha$ *and* $\|V\|_\delta$ *are all finite,*

then

$$d(V) \lesssim \left(1 + a_1^w(V)^2 + \|V\|_\alpha + \|V\|_\delta\right)\left(1 + a_1^w(V)^2\right) < \infty.$$

Unfortunately, the class of potentials that satisfy (V2) is rather restrictive. Moreover, it is quite likely that this class is near optimal if we are to demand that the square function equivalence (3.28) hold. Intuitively this makes sense since, as stated previously, the P_t operators are related to the heat operators $e^{t(\Delta - V)}$. If a unit of heat energy were to be placed at a point, then the spread of the heat distribution should be unaffected for small times if the potential is bounded from above. If the potential satisfies some form of decay condition then for large time scales the potential will be barely visible and the heat distribution will again be unaffected. Conversely, if a decay condition is not satisfied then $e^{t(\Delta - V)}$ will be comparatively much smaller than A_t for large times and the global square function estimate will resemble

$$\int_1^\infty \left\|\left(A_t - e^{t(\Delta - V)}\right)u\right\|^2 \frac{dt}{t} \simeq \int_1^\infty \|A_t u\|^2 \frac{dt}{t}$$

and will then be unbounded.

4.1.3. Potentials Far From Zero

The second assumption that can be placed on our potential in order to simplify the proof is that the potential is essentially bounded from below. Indeed, if the potential is essentially bounded from below and (wH8V2) is satisfied then it is not too difficult to show that the global square function estimate must hold.

Proposition 4.1.1. *Suppose that (wH8V2) is satisfied and* $\|V^{-1}\|_\infty < \infty$. *Then*

$$\int_1^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim \left\|V^{-1}\right\|_\infty a_2^w(V)^2 \|u\|^2 \qquad \forall\ u \in R(\Pi).$$

Proof. As V is essentially bounded from below,

$$\begin{aligned}\int_1^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} &\lesssim \int_1^\infty \|P_t u\|^2 \frac{dt}{t}\\ &\lesssim \left\|V^{-1}\right\|_\infty \int_1^\infty \left\|V^{\frac{1}{2}} P_t u\right\|^2 \frac{dt}{t},\end{aligned}$$

where the uniform L^2 boundedness of the operators Θ_t^B was used in the first line. On applying the coercivity condition (wH8V2) and Lemma 3.6.2 we obtain the desired estimate.

□

Recall from Theorem 4.1.3 that if (wH8V1) is satisfied and $\|V\|_\alpha < \infty$ then the local square function estimate (4.6) will hold. The following theorem follows trivially from this and the preceding proposition.

Theorem 4.1.4. *Suppose that the following condition is satisfied.*

(V3) $a_1^w(V)$, $a_2^w(V)$, $\|V\|_\alpha$ and $\left\|V^{-1}\right\|_\infty < \infty$.

Then there must exist $C_V > 0$ such that

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \leq C_V \|u\|^2 \tag{4.8}$$

for all $u \in R(\Gamma)$. Moreover,

$$d(V) \lesssim \left(1 + a_1^w(V)^2 + \|V\|_\alpha\right)\left(1 + a_1^w(V)^2\right) + \left\|V^{-1}\right\|_\infty a_2^w(V)^2 < \infty.$$

Combining Theorems 4.1.3, 4.1.4 and Corollary 4.1.1 together with Proposition 3.6.3 and Theorem 3.1.4 then form a complete proof of the following theorem.

Theorem 4.1.5. *Suppose that V and Γ satisfy one of the conditions (V1) - (V3). Then the operator Π_B has a bounded holomorphic functional calculus with*

$$\|f(\Pi_B)\| \lesssim d(V) \sup_{\zeta \in S_\mu^o} |f(\zeta)|.$$

As a result, there exists a constant $c > 0$, independent of the potential, for which

$$(c \cdot d(V))^{-1} \cdot \|\Pi_B u\| \leq \left\|\sqrt{\Pi_B^2} u\right\| \leq c \cdot d(V) \cdot \|\Pi_B u\|$$

for all $u \in D(\Pi_B)$.

4.1.4. Further Remarks

Let us discuss the problem of bounding the perturbation independent operators $\mathcal{N}_1$ and $\mathcal{N}_2$. As the operators P_t can be thought of as being related to the operators $e^{t(\Delta - V)}$, we can gain some insight into the conditions $b_1(V) < \infty$ and $b_2(V) < \infty$ by studying the operators

$$\mathcal{N}\left((t, y) \mapsto w_i(t) A_t e^{t(\Delta - V)} u(y)\right)$$

for $i = 1$ and 2. For either $i = 1$ or 2, it is straightforward to obtain the bound

$$\left\|\mathcal{N}\left((t, y) \mapsto w_i(t) A_t e^{t(\Delta - V)} u(y)\right)\right\| \lesssim \left\|\sup_{t>0} w_i(t) e^{t(\Delta - V)} |u|\right\|.$$

If $V = V_0 + N$ for some $N > 0$ and non-negative potential V_0, then the boundedness of the above operator is obvious. Indeed, we would obtain a bound

$$\left\|\mathcal{N}\left((t, y) \mapsto w_i(t) A_t e^{t(\Delta - V)} u(y)\right)\right\| \lesssim \left\|\sup_{t>0} w_i(t) e^{-Nt} e^{t(\Delta - V_0)} |u|\right\|.$$

The exponential decay would then more than compensate for the at most polynomial growth of the weight w_i and boundedness would be obtained as a result. This leads to the following open question: can we compensate for this polynomial growth with a less demanding condition on V other than boundedness from below?

4.2. Square Function Estimates

In this section, the square function estimate of Theorem 4.1.2 will be proved for a general non-negative potential. Following this, in §4.2.2, we will consider operators and potential that satisfy the square function equivalence between the A_t and P_t operators. This will simplify the problem, allowing us to obtain boundedness for the square function of $\Theta_t^B P_t$.

4.2.1. General Potential

Let's begin by first generalising a result from the original Axelsson-Keith-McIntosh over to the potential dependent setting.

Lemma 4.2.1. *Let Υ be either Π, Γ or Γ^*. Then, for any $Q \in \Delta$ and $u \in D(\Upsilon)$,*

$$\left|\fint_Q \Upsilon u\right|^2 \lesssim \frac{1}{l(Q)} \left(\fint_Q |u|^2\right)^{\frac{1}{2}} \left(\fint_Q |\Upsilon u|^2\right)^{\frac{1}{2}} + \frac{1}{|Q|^2} \|V\|_{L^1(Q)} \|u\|_{L^2(Q)}^2. \tag{4.9}$$

PROOF. Let $u \in D(\Upsilon)$ and $Q \in \Delta$. Define the quantity

$$\tau := \left(\int_Q |u|^2\right)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2\right)^{-\frac{1}{2}}.$$

If $\tau \geq l(Q)/4$, then (4.9) will follow directly from the Cauchy-Schwarz inequality. So suppose that $\tau < l(Q)/4$. Let $\eta \in C_0^\infty(Q)$ be a real-valued bump function with $\eta(x) = 1$ when $\operatorname{dist}(x, \mathbb{R}^n \setminus Q) > \tau$ and $|\nabla \eta| \lesssim 1/\tau$. We have that

$$\begin{aligned}
\left|\int_Q \Upsilon u\right| &= \left|\int_{\mathbb{R}^n} \eta \Upsilon u + \int_Q (1-\eta)\, \Upsilon u\right| \\
&= \left|\int_Q [\eta, \Upsilon]\, u + \int_Q (1-\eta)\, \Upsilon u + \int_{\mathbb{R}^n} \Upsilon(\eta u)\right| \\
&\lesssim \|\nabla \eta\|_\infty \left(\tau l(Q)^{n-1}\right)^{\frac{1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{2}} + \left(\tau l(Q)^{n-1}\right)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2\right)^{\frac{1}{2}} + \left|\int_{\mathbb{R}^n} \Upsilon(\eta u)\right|.
\end{aligned}$$

On substituting in the definition of τ,

$$\left|\int_Q \Upsilon u\right| \lesssim l(Q)^{\frac{n-1}{2}} \left(\int_Q |u|^2\right)^{\frac{1}{4}} \left(\int_Q |\Upsilon u|^2\right)^{\frac{1}{4}} + \left|\int_{\mathbb{R}^n} \Upsilon(\eta u)\right|. \tag{4.10}$$

(H7V) states that

$$\left|\int_{\mathbb{R}^n} \Upsilon(\eta u)\right| \lesssim \|V\|_{L^1(Q)}^{\frac{1}{2}} \cdot \|u\|_{L^2(Q)}. \tag{4.11}$$

is satisfied for $\Upsilon = \Gamma$ and Γ^*. This is also true for $\Upsilon = \Pi$ by the triangle inequality. Applying (4.11) to (4.10) then gives us our result.

□

The Relationship between A_t and P_t

As stated previously, the equivalence of the A_t and P_t operators in the sense of the square function estimate (3.28) forms a key component of the proof of Axelsson-Keith-McIntosh. In the potential dependent setting, there is no reason to expect for this equivalence to hold. In this section, Theorem 4.1.1 will be proved. This theorem quantifies how the square function equivalence between the A_t and P_t operators becomes weaker in this new setting.

Proposition 4.2.1. *For any $u \in \mathcal{H}$,*

$$\int_0^\infty \|A_t(P_t - I)u\|^2 g_i(t)^2 \frac{dt}{t} \lesssim \|u\|^2 \tag{4.12}$$

for $i = 1$ and 2.

Proof. The estimate is trivially satisfied for any $u \in N(\Pi)$ since

$$\begin{aligned}(P_t - I)\, u &= \left(\left(I + t^2\Pi^2\right)^{-1} - I\right) u \\ &= \left(I + t^2\Pi^2\right)^{-1}\left(I - \left(I + t^2\Pi^2\right)\right) u \\ &= 0\end{aligned}$$

for any $t > 0$. So suppose that $u \in \overline{R(\Pi)}$. On applying the resolution of the identity, Proposition 3.1.3,

$$\begin{aligned}\int_0^\infty \left\| A_t\,(P_t - I)\, u\right\|^2 g_i(t)^2 \frac{dt}{t} &= \int_0^\infty \left\| A_t\,(P_t - I)\, 2\int_0^\infty Q_s^2 u \frac{ds}{s}\right\|^2 g_i(t)^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left(\int_0^\infty \left\| A_t\,(P_t - I)\, Q_s^2 u\right\| \frac{ds}{s}\right)^2 g_i(t)^2 \frac{dt}{t}.\end{aligned}$$

The Cauchy-Schwarz inequality leads to

$$\begin{aligned}&\int_0^\infty \left\| A_t\,(P_t - I)\, u\right\|^2 g_i(t)^2 \frac{dt}{t} \lesssim \\ &\qquad\int_0^\infty \left(\int_0^\infty \left\| A_t\,(P_t - I)\, Q_s\right\| g_i(t) \frac{ds}{s}\right) \cdot \left(\int_0^\infty \left\| A_t\,(P_t - I)\, Q_s\right\| \left\| Q_s u\right\|^2 g_i(t) \frac{ds}{s}\right) \frac{dt}{t}.\end{aligned} \tag{4.13}$$

Let's estimate the term $\left\| A_t\,(P_t - I)\, Q_s\right\| g_i(t)$. First assume that $t \leq s$. On noting that $(I - P_t)\, Q_s = \frac{t}{s} Q_t\,(I - P_s)$ we obtain

$$\left\| A_t\,(P_t - I)\, Q_s\right\| \lesssim \left\| (P_t - I)\, Q_s\right\| \lesssim \frac{t}{s}\left\| Q_t\,(I - P_s)\right\| \lesssim \frac{t}{s}. \tag{4.14}$$

Since $g_i(t) \leq 1$ for all $t > 0$, for $i = 1$ and 2,

$$\left\| A_t\,(P_t - I)\, Q_s\right\| g_i(t) \lesssim \frac{t}{s}. \tag{4.15}$$

Next, suppose that $t > s$. Then the equality $P_t Q_s = \frac{s}{t} Q_t P_s$ gives

$$\left\| A_t\,(P_t - I)\, Q_s\right\| \lesssim \left\| P_t Q_s\right\| + \left\| A_t Q_s\right\| \lesssim \frac{s}{t} + \left\| A_t Q_s\right\|.$$

On applying Lemma 4.2.1,

$$
\begin{aligned}
\|A_t Q_s u\|^2 &= \sum_{Q\in\Delta_t} |Q| \left| \fint_Q s\Pi \left(I + s^2\Pi^2\right)^{-1} u \right|^2 \\
&\lesssim \sum_{Q\in\Delta_t} |Q|\, s^2 \left[\frac{1}{l(Q)} \left(\fint_Q |P_s u|^2 \right)^{\frac{1}{2}} \left(\fint_Q |\Pi P_s u|^2 \right)^{\frac{1}{2}} + \frac{1}{|Q|^2} \|V\|_{L^1(Q)} \|P_s u\|^2_{L^2(Q)} \right] \\
&\lesssim \sum_{Q\in\Delta_t} \frac{s}{t} \left(\int_Q |P_s u|^2 \right)^{\frac{1}{2}} \left(\int_Q |Q_s u|^2 \right)^{\frac{1}{2}} + \frac{s^2}{|Q|} \|V\|_{L^1(Q)} \|P_s u\|^2_{L^2(Q)} \\
&\lesssim \left[\sum_{Q\in\Delta_t} \frac{s}{t} \left(\int_Q |P_s u|^2 \right)^{\frac{1}{2}} \left(\int_Q |Q_s u|^2 \right)^{\frac{1}{2}} \right] + s^2 \left(\frac{1}{|Q|} \sup_{Q\in\Delta_t} \|V\|_{L^1(Q)} \right) \|u\|^2_{L^2(\mathbb{R}^n)} .
\end{aligned}
$$

As in Proposition 5.7 of [11], the first term can be bounded by $(s/t)\|u\|^2$ leading to

$$
\|A_t Q_s u\|^2 \lesssim \frac{s}{t} \|u\|^2 + s^2 \left(\frac{1}{|Q|} \sup_{Q\in\Delta_t} \|V\|_{L^1(Q)} \right) \|u\|^2_{L^2(\mathbb{R}^n)} . \tag{4.16}
$$

This then gives

$$
\|A_t (P_t - I) Q_s\| \, g_i(t) \lesssim \left(\left(\frac{s}{t}\right)^{\frac{1}{2}} + s \left(\sup_{Q\in\Delta_t} \fint_Q |V| \right)^{\frac{1}{2}} \right) g_i(t). \tag{4.17}
$$

For $i = 1$ we have

$$
\begin{aligned}
\|A_t (P_t - I) Q_s\| \, g_1(t) &\lesssim \left(\left(\frac{s}{t}\right)^{\frac{1}{2}} + s \left(\sup_{Q\in\Delta_t} \fint_Q |V| \right)^{\frac{1}{2}} \right) \left(1 + t^{\alpha} \left(\sup_{Q\in\Delta_t} \fint_Q |V| \right)^{\frac{1}{2}} \right)^{-1} \mathbb{1}_{[0,1]}(t) \\
&\lesssim \left(\left(\frac{s}{t}\right)^{\frac{1}{2}} + \frac{s}{t^{\alpha}} \right) \mathbb{1}_{[0,1]}(t) \\
&\lesssim \left(\left(\frac{s}{t}\right)^{\frac{1}{2}} + \frac{s}{t} \right) \mathbb{1}_{[0,1]}(t) \\
&\lesssim \left(\frac{s}{t}\right)^{\frac{1}{2}} .
\end{aligned}
$$

For $i = 2$,

$$
\begin{aligned}
\|A_t (P_t - I) Q_s\| \, g_2(t) &\lesssim \left(\left(\frac{s}{t}\right)^{\frac{1}{2}} + s \left(\sup_{Q\in\Delta_t} \fint_Q |V| \right)^{\frac{1}{2}} \right) \left(1 + t^{1+\delta} \left(\sup_{Q\in\Delta_t} \fint_Q |V| \right)^{\frac{1}{2}} \right)^{-1} \mathbb{1}_{[1,\infty)}(t) \\
&\lesssim \left(\left(\frac{s}{t}\right)^{\frac{1}{2}} + \frac{s}{t} \right) \mathbb{1}_{[1,\infty)}(t) \\
&\lesssim \left(\frac{s}{t}\right)^{\frac{1}{2}} .
\end{aligned}
$$

Putting everything together gives

$$\|A_t (P_t - I) Q_s\| \, g_i(t) \lesssim \min\left\{\frac{t}{s}, \frac{s}{t}\right\}^{\frac{1}{2}}. \tag{4.18}$$

This bound can then be applied to (4.13) to give (4.12). ◻

Remark 4.2.1. The above square function estimate for $A_t (P_t - I)$ actually holds trivially for the weight $g_3(t) := t^{-(1+\delta)} \mathbb{1}_{(1,\infty)}(t)$. Indeed, this follows from the fact that the operators $A_t (P_t - I)$ are uniformly L^2-bounded in t. Depending on the decay conditions of the potential, this weight could be more or less demanding than g_2. However, even for potentials where g_3 provides a stronger square function estimate, the weight g_2 will always be used. The precise form of g_2 and, specifically, its potential dependence will be required later to circumvent a second obstruction that appears when proving the Carleson measure estimate.

Let γ_t^B denote the principal part of the operator Θ_t^B as defined in Definition 3.6.3.

Corollary 4.2.1. *For any $u \in \mathcal{H}$,*

$$\int_0^\infty \left\|\gamma_t^B A_t (P_t - I) u\right\|^2 g_i(t)^2 \frac{dt}{t} \lesssim \|u\|^2 \tag{4.19}$$

for $i = 1$ and 2.

Proof. Combining Proposition 3.6.5 with the fact that $A_t^2 = A_t$ leads to

$$\begin{aligned}\left\|\gamma_t^B A_t (P_t - I) u\right\| &= \left\|\gamma_t^B A_t A_t (P_t - I) u\right\| \\ &\lesssim \|A_t (P_t - I) u\| .\end{aligned}$$

The result then follows immediately from the previous proposition. ◻

Proposition 4.2.2. *For any $u \in \mathcal{H}$,*

$$\int_0^\infty \|(I - A_t) P_t u\|^2 \frac{dt}{t} \lesssim \int_0^\infty \|t \nabla P_t u\|^2 \frac{dt}{t}. \tag{4.20}$$

Proof. On decomposing dyadically,

$$\begin{aligned}\|(I-A_t)\,P_tu\|^2 &= \sum_{Q\in\Delta_t}\int_Q |P_tu(x)-\langle P_tu\rangle_Q|^2\,dx\\ &= \sum_{Q\in\Delta_t}\int_Q |P_tu(x)-\langle P_tu\rangle_Q|^2\,(1+\mathrm{dist}(x,Q)/t)^{-(3n+1)}\,dx,\end{aligned}$$

where $\mathrm{dist}\,(x,Q) := \inf_{y\in Q}|x-y|$ and $\langle f\rangle_Q := \fint_Q f$ for $f\in L^1_{loc}(\mathbb{R}^n)$. On applying the weighted Poincaré inequality, Lemma 3.6.3,

$$\begin{aligned}\|(I-A_t)\,P_tu\|^2 &\lesssim \sum_{Q\in\Delta_t}\int_{\mathbb{R}^n} |t\nabla P_tu(x)|^2\,(1+\mathrm{dist}(x,Q)/t)^{-(n+1)}\,dx\\ &\lesssim \|t\nabla P_tu\|^2,\end{aligned}$$

which gives (4.20). □

The above proposition, together with Proposition 4.2.1, then forms a complete proof of Theorem 4.1.1.

Carleson Measure Estimate

This section is dedicated to the proof of the following proposition.

Proposition 4.2.3. *Let $g:(0,\infty)\to[0,1]$ satisfy*

$$\int_0^{l(Q)} tg(t)^2dt \lesssim C_V\left(\fint_Q |V|\right)^{-1} \tag{4.21}$$

for some $C_V>0$, for all $Q\in\Delta$. Suppose that the estimate

$$\int_0^\infty \|(A_t-P_t)\,u\|^2\,g(t)^2\frac{dt}{t} \lesssim D_V\left(\|u\|^2+\int_0^\infty \|t\nabla P_tu\|^2\,g(t)^2\frac{dt}{t}\right)$$

holds for all $u\in R\,(\Pi)$. Then there exist functions $(u_Q)_{Q\in\Delta}$ with $\|u_Q\|^2\lesssim|Q|$ such that

$$\left\|(t,x)\mapsto \gamma_t^B(x)^2g(t)^2\,\frac{dx\,dt}{t}\right\|_{\mathcal{C}} \lesssim B_V\left(1+\sup_{Q\in\Delta}\frac{1}{|Q|}\int_0^{l(Q)} \|t\nabla P_tu_Q\|^2\,g(t)^2\,\frac{dt}{t}\right), \tag{4.22}$$

where $B_V=1+C_V+D_V$. Moreover, the functions $(u_Q)_{Q\in\Delta}$ are independent of g.

The precise definition of the collection of functions $(u_Q)_{Q\in\Delta}$ will be determined throughout the course of the proof. Let $\sigma>0$ be a constant to be determined at a later time. For

$\nu \in \mathcal{L}(\mathbb{C}^N)$, define the set

$$K_\nu := \left\{ \nu' \in \mathcal{L}\left(\mathbb{C}^N\right) : \left| \frac{\nu'}{|\nu'|} - \nu \right| \le \sigma \right\}.$$

Let $\mathcal{V}$ be a finite set such that

$$\bigcup_{\nu \in \mathcal{V}} K_\nu = \mathcal{L}\left(\mathbb{C}^N\right) \setminus \{0\},$$

where each $\nu \in \mathcal{V}$ is of unit length, $|\nu| = 1$. Then, in order to prove (4.22), it suffices to prove for any $\nu \in \mathcal{V}$ that

$$\sup_{Q\in\Delta} \frac{1}{|Q|} \int\int_{R_Q} \left|\gamma_t^B(x)\right|^2 g(t)^2 \mathbb{1}_{\gamma_t^B(x)\in K_\nu} \frac{dx\,dt}{t} \lesssim B_V \left(1 + \sup_{Q\in\Delta} \frac{1}{|Q|} \int_0^{l(Q)} \|t\nabla P_t u_Q(x)\|^2 g(t)^2 \frac{dt}{t} \right). \tag{4.23}$$

The John-Nirenberg Lemma for Carleson measures, Lemma 3.2.1, can now be used to reduce the proof of (4.23) to the following proposition.

Proposition 4.2.4. *Let $g : (0,\infty) \to [0,1]$ satisfy the hypotheses of Proposition 4.2.3. There exists $\beta > 0$, a choice of $\sigma > 0$ and a collection of functions $(u_Q)_{Q\in\Delta}$ with $\|u_Q\|^2 \lesssim |Q|$ that satisfy the following conditions. For every $\nu \in \mathcal{V}$ and $Q \in \Delta$, there is a collection $\{Q_k\}_k \subset \Delta$ of disjoint subcubes of Q such that $E_{Q,\nu} := Q \setminus \cup_k Q_k$ satisfies $|E_{Q,\nu}| > \beta |Q|$ and such that*

$$\sup_{Q\in\Delta} \frac{1}{|Q|} \int\int_{\substack{(x,t)\in E^*_{Q,\nu} \\ \gamma_t(x)\in K_\nu}} \left|\gamma_t^B(x)\right|^2 g(t)^2 \frac{dx\,dt}{t} \lesssim B_V \left(1 + \sup_{Q\in\Delta} \frac{1}{|Q|} \int_0^{l(Q)} \|t\nabla P_t u_Q(x)\|^2 g(t)^2 \frac{dt}{t} \right), \tag{4.24}$$

where $E^*_{Q,\nu} := R_Q \setminus \cup_k R_{Q_k}$ and B_V is as given in Proposition 4.2.3. Moreover β and σ are entirely dependent on (H1) - (H6) and independent of (H7V) and the functions $(u_Q)_{Q\in\Delta}$ are independent of g.

For now, fix $\nu \in \mathcal{V}$ and $Q \in \Delta$. Let $w^\nu, \hat{w}^\nu \in \mathbb{C}^N$ with $|\hat{w}^\nu| = |w^\nu| = 1$ and $\nu^*(\hat{w}^\nu) = w^\nu$. To simplify notation, when superfluous, this dependence will be kept implicit by defining $w := w^\nu$ and $\hat{w} := \hat{w}^\nu$. For $\epsilon > 0$ the function $f^w_{Q,\epsilon}$ can be defined in an identical manner to [11]. Specifically, let $\eta_Q : \mathbb{R}^d \to [0,1]$ be a smooth cutoff function equal to 1 on $2Q$, with support in $4Q$ and with $\|\nabla \eta_Q\|_\infty \le \frac{1}{l}$, where $l := l(Q)$. Then define $w_Q := \eta_Q \cdot w$ and

$$f^w_{Q,\epsilon} := w_Q - \epsilon l i \Gamma \left(I + \epsilon l i \Pi_B\right)^{-1} w_Q = \left(I + \epsilon l i \Gamma_B^*\right)\left(I + \epsilon l i \Pi_B\right)^{-1} w_Q.$$

Lemma 4.2.2. *There exists a constant $C > 0$, that depends only on (H1) - (H6), that* *satisfies* $\left\|f^w_{Q,\epsilon}\right\| \leq C\,|Q|^{\frac{1}{2}}$ *and*

$$\left|\fint_Q f^w_{Q,\epsilon} - w\right| \leq C\left(\epsilon^{\frac{1}{2}} + \epsilon l\left(\fint_Q |V|\right)^{\frac{1}{2}}\right), \tag{4.25}$$

for any $\epsilon > 0$.

Proof. The proof of the first claim is unchanged from Lemma 5.10 of [11] and relies only on potential independent results. For the second estimate, on applying Lemma 4.2.1,

$$\begin{aligned}
\left|\fint_Q f^w_{Q,\epsilon} - w\right| &= \left|\fint_Q \epsilon l\Gamma\left(I + \epsilon li\Pi_B\right)^{-1} w_Q\right| \\
&\lesssim \epsilon^{\frac{1}{2}}\left(\fint_Q \left|\left(I + \epsilon li\Pi_B\right)^{-1} w_Q\right|^2\right)^{\frac{1}{4}}\left(\fint_Q \left|\epsilon l\Gamma\left(I + \epsilon li\Pi_B\right)^{-1} w_Q\right|^2\right)^{\frac{1}{4}} \\
&\qquad + \frac{\epsilon l}{|Q|}\left\|V\right\|^{\frac{1}{2}}_{L^1(Q)}\left(\int_Q \left|\left(I + \epsilon li\Pi_B\right)^{-1} w_Q\right|^2\right)^{\frac{1}{2}} \\
&\lesssim \epsilon^{\frac{1}{2}} + \epsilon l\left(\fint_Q |V|\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used the Hodge decomposition and the uniform L^2-boundedness of our operators to obtain the last estimate. That is, there must exist some $C > 0$ such that (4.25) is satisfied. ◻

From this point forward, with C as in the previous lemma, set $\epsilon = \frac{1}{16(C+1)^2}\left(1 + l\left(\fint_Q |V|\right)^{\frac{1}{2}}\right)^{-1}$ and introduce the notation $f^w_Q := f^w_{Q,\epsilon}$.

Lemma 4.2.3. *There exists a constant* $D > 0$*, dependent only on (H1) - (H6), such that*

$$\int\int_{R_Q} \left|\Theta^B_t f^w_Q\right|^2 g(t)^2\,\frac{dx\,dt}{t} \leq D(1 + C_V)\,|Q|\,. \tag{4.26}$$

Proof. Notice that

$$\Theta^B_t f^w_Q = \frac{t}{\epsilon l}\left(I + t^2\Pi_B^2\right)^{-1}\epsilon l\Gamma^*_B\left(I + \epsilon li\Pi_B\right)^{-1} w_Q.$$

This leads to

$$\begin{aligned}
\int\int_{R_Q} \left|\Theta_t^B f_Q^w\right|^2 g(t)^2 \frac{dxdt}{t} &= \int_0^l \left(\frac{t}{\epsilon l}\right)^2 g(t)^2 \left(\int_Q \left|\left(I + t^2\Pi_B^2\right)^{-1} \epsilon l \Gamma_B^* \left(I + \epsilon l i \Pi_B\right)^{-1} w_Q\right|^2 dx\right) \frac{dt}{t} \\
&\lesssim |Q| \int_0^l \left(\frac{t}{\epsilon l}\right)^2 g(t)^2 \frac{dt}{t} \\
&\lesssim |Q| \int_0^l \frac{t}{l^2} \left(1 + l\left(\fint_Q |V|\right)^{\frac{1}{2}}\right)^2 g(t)^2 dt \\
&\lesssim |Q| \int_0^l \frac{t}{l^2} dt + |Q| \left(\fint_Q |V|\right) \int_0^l t g(t)^2 dt,
\end{aligned}$$

where the second line follows from the uniform L^2-boundedness of our operators and the Hodge decomposition. At this stage, we can apply the weight condition (4.21) to obtain the estimate (4.26). ◻

Given our previous choice of ϵ, it must be true that

$$\begin{aligned}
\left|\fint_Q f_Q^w - w\right| &\leq C\left(\epsilon^{\frac{1}{2}} + \epsilon l \left(\fint_Q |V|\right)^{\frac{1}{2}}\right) \\
&\leq C\left(\frac{1}{4(C+1)} + \frac{1}{16(C+1)^2}\right) \\
&\leq \frac{1}{2}.
\end{aligned}$$

That is,

$$\begin{aligned}
1 - 2\mathrm{Re}\left\langle \fint_Q f_Q^w, w\right\rangle &= |w|^2 - 2\mathrm{Re}\left\langle \fint_Q f_Q^w, w\right\rangle \\
&\leq \left|\fint_Q f_Q^w - w\right|^2 \\
&\leq \frac{1}{4}.
\end{aligned}$$

On rearranging we find that

$$\mathrm{Re}\left\langle \fint_Q f_Q^w, w\right\rangle \geq \frac{1}{4}.$$

From Lemma 4.2.2 and the above estimate, Lemmas 5.11 and 5.12 of [11] will follow in our case by applying an identical argument. Moreover, since their proofs do not depend on any other results, the constants obtained will be independent of the potential and only dependent on (H1) - (H6). Combining these two lemmas into a single statement gives the following result.

Lemma 4.2.4. *There exists a choice of $\sigma > 0$ and $\beta > 0$, entirely dependent on (H1) - (H6) and independent of (H7V), that will satisfy the following conditions. For $\nu \in \mathcal{V}$ and $Q \in \Delta$, there exists a collection $\{Q_k\}_k \subset \Delta$ such that $|E_{Q,\nu}| > \beta |Q|$, where $E_{Q,\nu} = Q \setminus \cup_k Q_k$. Also, for $(x,t) \in E^*_{Q,\nu}$ and $\gamma_t^B(x) \in K_\nu$ we must have*

$$\left|\gamma_t^B(x)\right| \lesssim \left|\gamma_t^B(x)\left(A_t f_Q^w(x)\right)\right|,$$

where $E^*_{Q,\nu} = R_Q \setminus \cup_k R_{Q_k}$.

We are now well equipped to tackle Proposition 4.2.4 thereby completing our chain of results.

Proof of proposition 4.2.4. Let β, σ, $(E_{Q,\nu})_{Q\in\Delta}$ and $\left(E^*_{Q,\nu}\right)_{Q\in\Delta}$ be as given in Lemma 4.2.4. Fix $Q \in \Delta$ and $\nu \in \mathcal{V}$. Then we must have

$$\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t^B(x)\in K_\nu}} \left|\gamma_t^B(x)\right|^2 g(t)^2 \frac{dx\,dt}{t} \lesssim \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t^B(x)\in K_\nu}} \left|\Theta_t^B f_Q^w - \gamma_t^B A_t f_Q^w\right|^2 g(t)^2 \frac{dx\,dt}{t} + \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t^B(x)\in K_\nu}} \left|\Theta_t^B f_Q^w\right|^2 g(t)^2 \frac{dx\,dt}{t}.$$

Lemma 4.2.3 can be used to bound the last term by $D(1+C_V)\,|Q|$, where D is a constant independent of V. For the first term, note that

$$\Theta_t^B f_Q^w - \gamma_t^B A_t f_Q^w = -\left(\Theta_t^B - \gamma_t^B A_t\right)\epsilon li\Gamma\left(1 + \epsilon li\Pi_B\right)^{-1} w_Q + \left(\Theta_t^B - \gamma_t^B A_t\right) w_Q.$$

Then

$$\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t^B(x)\in K_\nu}} \left|\Theta_t^B f_Q^w - \gamma_t^B A_t f_Q^w\right|^2 g(t)^2 \frac{dx\,dt}{t} \lesssim \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t^B(x)\in K_\nu}} \left|\left(\Theta_t^B - \gamma_t^B A_t\right) u_Q^\nu\right|^2 g(t)^2 \frac{dx\,dt}{t} + \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t^B(x)\in K_\nu}} \left|\left(\Theta_t^B - \gamma_t^B A_t\right) w_Q\right|^2 g(t)^2 \frac{dx\,dt}{t},$$

where $u_Q^\nu := \epsilon li\Gamma\left(I + \epsilon li\Pi_B\right)^{-1} w_Q^\nu$. The second term can be bounded by a V independent multiple of $|Q|$ by applying the argument from the proof of [11, Prop. 5.9] verbatim. We can then restrict our attention to the square integral for the term $\left(\Theta_t^B - \gamma_t^B A_t\right) u_Q^\nu$. We have that

$$\begin{aligned}\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\left(\Theta^B_t - \gamma^B_t A_t\right) u^\nu_Q\right| g(t)^2 \,\frac{dx\,dt}{t} \lesssim \\ \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\Theta^B_t \left(I - P_t\right) u^\nu_Q\right|^2 \frac{dx\,dt}{t} + \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\left(\Theta^B_t P_t - \gamma^B_t A_t P_t\right) u^\nu_Q\right|^2 \frac{dx\,dt}{t} \\ + \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\gamma^B_t A_t \left(P_t - I\right) u^\nu_Q\right|^2 g(t)^2 \,\frac{dx\,dt}{t}.\end{aligned} \tag{4.27}$$

As stated in Corollary 3.6.1, the high frequency estimate

$$\int_0^\infty \left\|\Theta^B_t \left(I - P_t\right) u^\nu_Q\right\|^2 \frac{dt}{t} \lesssim \left\|u^\nu_Q\right\|^2$$

must be true with constant independent of the potential. For the second term, apply Proposition 3.6.6 to obtain

$$\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\left(\Theta^B_t P_t - \gamma^B_t A_t P_t\right) u^\nu_Q\right|^2 g(t)^2 \,\frac{dx\,dt}{t} \lesssim \int_0^{l(Q)} \left\|t\nabla P_t u^\nu_Q\right\|^2 g(t)^2 \frac{dt}{t}. \tag{4.28}$$

Consider the collection of functions $\mathcal{U} := \left\{u^\nu_Q : \nu \in \mathcal{V}\right\}$. As $\mathcal{V}$ is by definition finite it follows that $\mathcal{U}$ must be finite. There must then exist a unique $u_Q \in \mathcal{U}$ for which the right-hand side of (4.28) attains a maximum.

For the third term of (4.27), the uniform L^2-boundedness of the operators $\gamma^B_t A_t$ together with the hypothesis on g produces

$$\begin{aligned}&\int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\gamma^B_t A_t (P_t - I) u^\nu_Q\right|^2 g(t)^2 \,\frac{dx\,dt}{t} \\ &= \int\int_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma^B_t(x)\in K_\nu}} \left|\gamma^B_t A_t A_t (P_t - I) u^\nu_Q\right|^2 g(t)^2 \,\frac{dx\,dt}{t} \lesssim \int_0^\infty \left\|A_t (P_t - I) u^\nu_Q\right\|^2 g(t)^2 \,\frac{dx\,dt}{t} \\ &\lesssim D_V \left\|u^\nu_Q\right\|^2 + \int_0^\infty \left\|t\nabla P_t u^\nu_Q\right\|^2 g(t)^2 \frac{dt}{t} \lesssim D_V \left\|u^\nu_Q\right\|^2 + \int_0^\infty \left\|t\nabla P_t u_Q\right\|^2 g(t)^2 \frac{dt}{t}\end{aligned}$$

We have

$$\begin{aligned}\left\|u^\nu_Q\right\| &= \left\|\epsilon li \Gamma \left(I + \epsilon li \Pi_B\right)^{-1} w^\nu_Q\right\| \lesssim \left\|\epsilon li \Pi_B \left(I + \epsilon li \Pi_B\right)^{-1} w^\nu_Q\right\| \\ &= \left\|\left(\left(I + \epsilon li \Pi_B\right) - I\right)\left(I + \epsilon li \Pi_B\right)^{-1} w^\nu_Q\right\| \lesssim \left\|w^\nu_Q\right\| + \left\|\left(I + \epsilon li \Pi_B\right)^{-1} w^\nu_Q\right\| \\ &\lesssim \left\|w^\nu_Q\right\| \lesssim |Q|.\end{aligned}$$

Putting everything together gives (4.24). ◻

A Perturbation Free Estimate

With the proof of the Carleson measure estimate complete, we can now move on to the proof of our main result, Theorem 4.1.2. To this end, it must first be proved that g_1 and g_2 both satisfy (4.21). This claim will follow from the below lemma.

Lemma 4.2.5. *There exists $C > 0$, independent of V, such that for any $Q \in \Delta$ and $t > 0$ with $t < l(Q)$,*

$$\fint_Q |V| \leq C \cdot \sup_{\substack{R\in\Delta_t \\ R\cap Q\neq\emptyset}} \fint_R |V| . \tag{4.29}$$

Proof. Let $Q \in \Delta$ and $t > 0$ with $t < l(Q)$. Then

$$\begin{aligned}
\fint_Q |V| &\leq \frac{1}{|Q|} \sum_{\substack{R\in\Delta_t \\ R\cap Q\neq\emptyset}} \int_R |V| \\
&= \frac{t^n}{|Q|} \sum_{\substack{R\in\Delta_t \\ R\cap Q\neq\emptyset}} \fint_R |V| \\
&\leq \left(\sup_{\substack{R\in\Delta_t \\ R\cap Q\neq\emptyset}} \fint_R |V| \right) \frac{t^n}{|Q|} \left|\{R \in \Delta_t : R \cap Q \neq \emptyset\}\right| .
\end{aligned}$$

It remains to show that

$$\left|\{R \in \Delta_t : R \cap Q \neq \emptyset\}\right| \lesssim \frac{|Q|}{t^n} \tag{4.30}$$

with constant independent of Q and t. Let $3Q$ be the cube with the same center as Q but three times the length. Then evidently it must be true that

$$\bigcup_{\substack{R\in\Delta_t \\ R\cap Q\neq\emptyset}} R \subset 3Q.$$

Therefore

$$\begin{aligned}
\left|\{R \in \Delta_t : R \cap Q \neq \emptyset\}\right| &\leq \left|\{R \in \Delta_t : R \subseteq 3Q\}\right| \\
&\leq \frac{|3Q|}{t^n} \\
&= 3^n \frac{|Q|}{t^n}.
\end{aligned}$$

□

On applying the above lemma for $i=1$,

$$\begin{aligned}\left(\fint_Q |V|\right)\int_0^{l(Q)} t g_1(t)^2 dt &= \left(\fint_Q |V|\right)\int_0^{l(Q)\wedge 1} t\left(1+t^{\alpha}\left(\sup_{R\in\Delta_t}\fint_R |V|\right)^{\frac{1}{2}}\right)^{-2} dt \\ &\le \int_0^{l(Q)\wedge 1}\frac{t}{t^{2\alpha}}\left(\fint_Q |V|\right)\left(\sup_{R\in\Delta_t}\fint_R |V|\right)^{-1} dt \\ &\lesssim \int_0^{l(Q)\wedge 1} t^{1-2\alpha}\, dt \\ &\lesssim 1,\end{aligned}$$

which shows that g_1 satisfies (4.21). For $i=2$,

$$\begin{aligned}\left(\fint_Q |V|\right)\int_0^{l(Q)} t g_2(t)^2 dt &= \left(\fint_Q |V|\right)\int_1^{l(Q)} t\left(1+t^{1+\delta}\left(\sup_{R\in\Delta_t}\fint_R |V|\right)^{\frac{1}{2}}\right)^{-2} dt \\ &\le \int_1^{l(Q)}\frac{t}{t^{2(1+\delta)}}\left(\fint_Q |V|\right)\left(\sup_{R\in\Delta_t}\fint_R |V|\right)^{-1} dt \\ &\lesssim \int_1^{l(Q)}\frac{1}{t^{1+2\delta}} dt \\ &\lesssim 1,\end{aligned}$$

which gives (4.21) for g_2.

Given that g_1 and g_2 both satisfy (4.21) with constants independent of the potential, Proposition 4.2.3 taken with Theorem 4.1.1 then gives us the Carleson estimates

$$\left\|(t,x)\mapsto \gamma_t^B(x)^2 g_i(t)^2\,\frac{dx\,dt}{t}\right\|_{\mathcal{C}} \lesssim 1+\sup_{Q\in\Delta}\frac{1}{|Q|}\int_0^{l(Q)}\|t\nabla P_t u_Q\|^2 g_i(t)^2\frac{dt}{t}, \qquad (4.31)$$

for functions $(u_Q)_{Q\in\Delta}$ with $\|u_Q\|^2\lesssim|Q|$ for all $Q\in\Delta$, for both $i=1$ and 2.

Split our main square function norm from above by

$$\int_0^{\infty}\|\Theta_t^B P_t u\|^2\frac{dt}{t}\lesssim\int_0^{\infty}\left\|\left(\Theta_t^B P_t-\gamma_t^B A_t P_t\right)u\right\|^2\frac{dt}{t}+\int_0^{1}\|\gamma_t^B A_t P_t u\|^2\frac{dt}{t}+\int_1^{\infty}\|\gamma_t^B A_t P_t u\|^2\frac{dt}{t}.$$

On applying Proposition 3.6.6 we obtain

$$\int_0^{\infty}\|\Theta_t^B P_t u\|^2\frac{dt}{t}\lesssim\int_0^{\infty}\|t\nabla P_t u\|^2\frac{dt}{t}+\int_0^{1}\|\gamma_t^B A_t P_t u\|^2\frac{dt}{t}+\int_1^{\infty}\|\gamma_t^B A_t P_t u\|^2\frac{dt}{t}.$$

Carleson's theorem can then be applied to give

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \|t\nabla P_t u\|^2 \frac{dt}{t} + \left\|(t,x) \mapsto \gamma_t^B(x)^2 g_1(t)^2 \frac{dx\,dt}{t}\right\|_{\mathcal{C}} \left\|\mathcal{N}\left(\mathbb{1}_{[0,1]} w_1 A_t P_t u\right)\right\|^2 + \left\|(t,x) \mapsto \gamma_t^B(x)^2 g_2(t)^2 \frac{dx\,dt}{t}\right\|_{\mathcal{C}} \left\|\mathcal{N}\left(\mathbb{1}_{(1,\infty)} w_2 A_t P_t u\right)\right\|^2 .$$

Applying (4.31) to the above estimate then proves Theorem 4.1.2.

4.2.2. Potentials Close to Zero

In this section, Theorem 4.1.3 will be proved. Namely, it will be shown that the square function for $\Theta_t^B P_t$ is bounded for potentials that satisfy (V2).

Proposition 4.2.5. *Suppose that the coercivity condition (wH8V1) is satisfied. Let $g : (0,\infty) \to [0,1]$ be such that*

$$\int_0^{l(Q)} t g(t)^2 dt \lesssim C_V \left(\fint_Q |V| \right)^{-1} \tag{4.32}$$

for some $C_V > 0$, for all $Q \in \Delta$. Suppose that the estimate

$$\int_0^\infty \|(A_t - P_t)\, u\|^2 g(t)^2 \frac{dt}{t} \lesssim D_V \|u\|^2$$

holds for all $u \in R(\Pi)$. Then

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 g(t)^2 \frac{dt}{t} \lesssim (1 + C_V + D_V)\,(1 + a_1^w(V)^2)\, \|u\|^2$$

for all $u \in R(\Gamma)$.

Proof. Split the square function up according to

$$\int_0^\infty \left\|\Theta_t^B P_t u\right\|^2 g(t)^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\left(\Theta_t^B P_t - \gamma_t^B A_t P_t\right) u\right\|^2 g(t)^2 \frac{dt}{t} + \int_0^\infty \left\|\gamma_t^B A_t\,(P_t - I)\, u\right\|^2 g(t)^2 \frac{dt}{t} + \int_0^\infty \left\|\gamma_t^B A_t u\right\|^2 g(t)^2 \frac{dt}{t}. \tag{4.33}$$

For the first term, we have by Proposition 3.6.6 and (wH8V1),

$$\int_0^\infty \left\|\left(\Theta_t^B P_t - \gamma_t^B A_t P_t\right) u\right\|^2 g(t)^2 \frac{dt}{t} \lesssim \int_0^\infty \|t\nabla P_t u\|^2 \frac{dt}{t} \lesssim a_1^w(V)^2 \|u\|^2 .$$

The third term of (4.33) is bounded by Carleson's theorem, an application of Proposition 4.2.3 and (wH8V1),

$$\begin{aligned}\int_0^\infty \left\|\gamma_t^B A_t u\right\|^2 g(t)^2 \frac{dt}{t} &\lesssim \left\|(t,x)\mapsto \gamma_t^B(x)^2 g(t)^2\,\frac{dx\,dt}{t}\right\|_{\mathcal{C}} \|u\|^2 \\ &\lesssim (1+C_V+D_V)\left(1+\sup_{Q\in\Delta}\frac{1}{|Q|}\int_0^{l(Q)} \|t\nabla P_t u_Q\|^2 g(t)^2\,\frac{dt}{t}\right)\|u\|^2 \\ &\lesssim (1+C_V+D_V)(1+a_1^w(V)^2)\,\|u\|^2 .\end{aligned}$$

This reduces our result to bounding the second term of (4.33). Applying the L^2 uniform boundedness of the $\gamma_t^B A_t$ operators and splitting from above,

$$\begin{aligned}\int_0^\infty \left\|\gamma_t^B A_t\,(P_t-I)\,u\right\|^2 g(t)^2\frac{dt}{t} &= \int_0^\infty \left\|\gamma_t^B A_t A_t\,(P_t-I)\,u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \|A_t\,(P_t-I)\,u\|^2\, g(t)^2\frac{dt}{t} \\ &\lesssim \int_0^\infty \|(I-A_t)\,P_t u\|^2\,\frac{dt}{t} + \int_0^\infty \|(P_t-A_t)\,u\|^2\, g(t)^2\frac{dt}{t}.\end{aligned}$$

The first term on the right-hand side of the above estimate is bounded by $a_1^w(V)^2\,\|u\|^2$ by Proposition 4.2.2 and (wH8V1). The second term is bound by $D_V\,\|u\|^2$ by hypothesis. □

Our proof of Theorem 4.1.3 can now be concluded. Suppose first that (wH8V1) and $\|V\|_\alpha < \infty$. Then the weight $g(t) = \mathbb{1}_{(0,1)}(t)$ will satisfy (4.32) since

$$\begin{aligned}\int_0^{l(Q)} t g(t)^2\,dt &= \int_0^{l(Q)\wedge 1} t^{1-2\alpha}t^{2\alpha}\left(\sup_{\substack{R\in\Delta_t\\ R\cap Q\neq\emptyset}} \fint_R |V|\right)\left(\sup_{\substack{R\in\Delta_t\\ R\cap Q\neq\emptyset}} \fint_R |V|\right)^{-1} dt \\ &\leq \|V\|_\alpha \int_0^{l(Q)\wedge 1} t^{1-2\alpha}\left(\sup_{\substack{R\in\Delta_t\\ R\cap Q\neq\emptyset}} \fint_R |V|\right)^{-1} dt \\ &\lesssim \|V\|_\alpha\left(\fint_Q |V|\right)^{-1}\int_0^{l(Q)\wedge 1} t^{1-2\alpha} dt \\ &\lesssim \|V\|_\alpha\left(\fint_Q |V|\right)^{-1},\end{aligned}$$

where the second line follows from the definition of $\|V\|_\alpha$ and the third line follows from Lemma 4.2.5. It then follows from the previous proposition and Corollary 4.1.2 that the local estimate (4.6) will hold.

Similarly, if instead (wH8V1) and $\|V\|_\delta < \infty$ then the weight $g(t) = \mathbb{1}_{(1,\infty)}(t)$ will satisfy

(4.32). This follows from

$$
\begin{aligned}
\int_0^{l(Q)} tg(t)^2 dt &= \int_0^{l(Q)} t^{1-2(1+\delta)} t^{2(1+\delta)} \left(\sup_{\substack{R \in \Delta_t \\ R \cap Q \neq \emptyset}} \fint_R |V| \right) \left(\sup_{\substack{R \in \Delta_t \\ R \cap Q \neq \emptyset}} \fint_R |V| \right)^{-1} \mathbb{1}_{[1,\infty)}(t)\, dt \\
&\leq \|V\|_\delta \int_0^{l(Q)} t^{1-2(1+\delta)} \left(\sup_{\substack{R \in \Delta_t \\ R \cap Q \neq \emptyset}} \fint_R |V| \right)^{-1} \mathbb{1}_{[1,\infty)}(t)\, dt \\
&\lesssim \|V\|_\delta \left(\fint_Q |V| \right)^{-1} \int_0^{l(Q)} t^{1-2(1+\delta)} \mathbb{1}_{[1,\infty)}(t)\, dt \\
&\lesssim \|V\|_\delta \left(\fint_Q |V| \right)^{-1},
\end{aligned}
$$

where the second line follows from the definition of $\|V\|_\delta$ and the third line follows from Lemma 4.2.5. It then follows from the previous proposition and Corollary 4.1.2 that (4.7) will hold.

4.3. Applications to Kato with Potential

Let $\Gamma_{|V|^{\frac{1}{2}}}$, B_1 and B_2 be as defined in §3.5. As proved in that section, $\left\{\Gamma_{|V|^{\frac{1}{2}}}, B_1, B_2\right\}$ satisfies the properties (H1) - (H6). They also satisfy (H7V) since by Cauchy-Schwarz

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} \Gamma u \right| &= \left| \int_{\mathbb{R}^n} \begin{bmatrix} 0 \\ V^{\frac{1}{2}} u_1 \\ \nabla u_1 \end{bmatrix} \right| \\
&= \left| \int_{\mathbb{R}^n} V^{\frac{1}{2}} u_1 \right| \\
&\leq \|V\|_{L^1(Q)}^{\frac{1}{2}} \cdot \|u\|_{L^2(Q)}
\end{aligned}
\tag{4.34}
$$

for $u = (u_1, u_2, u_3) \in D(\Gamma)$ compactly supported. The bound for Γ^* follows similarly,

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} \Gamma^* u \right| &= \left| \int_{\mathbb{R}^n} \begin{bmatrix} V^{\frac{1}{2}} u_2 - \operatorname{div} u_3 \\ 0 \\ 0 \end{bmatrix} \right| \\
&= \left| \int_{\mathbb{R}^n} V^{\frac{1}{2}} u_2 \right| \\
&\leq \|V\|_{L^1(Q)}^{\frac{1}{2}} \cdot \|u\|_{L^2(Q)} .
\end{aligned}
$$

With this selection of operators, define the classes

$$
\mathcal{A}_i := \{ V : \mathbb{R}^n \to [0, \infty) : (Vi) \text{ is satisfied} \}
$$

for $i = 1$, 2 and 3. Set

$$\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3.$$

The Kato estimate with potential for $V \in \mathcal{A}$ then follows by applying the estimate from Theorem 4.1.5 to an element $u = (u_1, 0, 0) \in D\left(\Pi_{|V|^{\frac{1}{2}},B}\right)$ with $u_1 \in D(L + V)$.

Theorem 4.3.1 (Kato with Potential)**.** *For any $V \in \mathcal{A}$, the Kato estimate*

$$d(V)\left(\left\|V^{\frac{1}{2}}u\right\| + \|\nabla u\|\right) \lesssim \left\|\sqrt{L+V}u\right\| \lesssim d(V)\left(\left\|V^{\frac{1}{2}}u\right\| + \|\nabla u\|\right) \tag{4.35}$$

is true for any $u \in D(L + V)$.

The following lemma sheds some light on what type of potentials will satisfy the coercivity conditions for this set of operators.

Lemma 4.3.1. *Let $V : \mathbb{R}^n \to [0, \infty)$ be a potential and let $\Gamma_{|V|^{\frac{1}{2}}}$, B_1 and B_2 be as defined in §3.5.*

1. *The coercivity condition (H8V1) will be satisfied if and only if the higher-order Riesz transforms $\nabla^2 (V - \Delta)^{-1}$ and $\nabla V^{\frac{1}{2}} (V - \Delta)^{-1}$ are bounded from $L^2(\mathbb{R}^n)$ to itself. In which case,*

$$a_1(V) \leq \left\|\nabla^2 (V - \Delta)^{-1}\right\| + \left\|\nabla V^{\frac{1}{2}} (V - \Delta)^{-1}\right\|.$$

2. *The coercivity condition (H8V2) will be satisfied if and only if the higher-order Riesz transforms $V (V - \Delta)^{-1}$ and $V^{\frac{1}{2}}\nabla (V - \Delta)^{-1}$ are bounded from $L^2(\mathbb{R}^n)$ to itself. In which case,*

$$a_2(V) \leq \left\|V (V - \Delta)^{-1}\right\| + \left\|V^{\frac{1}{2}}\nabla (V - \Delta)^{-1}\right\|.$$

Proof. Let $w \in R(\Pi)$. Then w is of the form

$$w = \begin{pmatrix} V^{\frac{1}{2}}v_2 - \mathrm{div}v_3 \\ V^{\frac{1}{2}}v_1 \\ \nabla v_1 \end{pmatrix}$$

for some $v = (v_1, v_2, v_3) \in D(\Pi)$. Note that

$$\|\nabla w\|^2 = \left\|\nabla\left(V^{\frac{1}{2}}v_2 - \mathrm{div}v_3\right)\right\|^2 + \left\|\nabla V^{\frac{1}{2}}v_1\right\|^2 + \left\|\nabla^2 v_1\right\|^2.$$

Also,

$$\|\Pi w\|^2 = \left\|\nabla\left(V^{\frac{1}{2}}v_2 - \mathrm{div}v_3\right)\right\|^2 + \|(V - \Delta)v_1\|^2 + \left\|V^{\frac{1}{2}}\left(V^{\frac{1}{2}}v_2 - \mathrm{div}v_3\right)\right\|^2.$$

It is obvious by inspecting the previous two equalities that (H8V1) will hold if and only if the higher-order Riesz transforms $\nabla^2 (V-\Delta)^{-1}$ and $\nabla V^{\frac{1}{2}} (V-\Delta)^{-1}$ are both bounded on $L^2(\mathbb{R}^n)$.

The second statement follows similarly. Simply compare

$$\left\|V^{\frac{1}{2}} w\right\|^2 = \left\|V^{\frac{1}{2}}\left(V^{\frac{1}{2}} v_2 - \mathrm{div} v_3\right)\right\|^2 + \|V v_1\|^2 + \left\|V^{\frac{1}{2}} \nabla v_1\right\|^2$$

and $\|\Pi w\|$ for $w \in R(\Pi) \cap D(\Pi)$. □

Corollary 4.3.1. *For any $V \in RH_q$ with $q \geq \frac{n}{2}$ the condition (H8V2) is satisfied.*

Proof. In [50], it was proved that the higher-order Riesz transforms $V(V-\Delta)^{-1}$ and $V^{\frac{1}{2}} \nabla (V-\Delta)^{-1}$ are bounded on $L^2(\mathbb{R}^n)$ when V is in the reverse Hölder class RH_q for $q \geq \frac{n}{2}$. The claim then follows from the preceding Lemma. □

Remark 4.3.1. The reverse Hölder condition may or may not be sufficient to imply (H8V1). In [50] and [6], the Riesz transform $\nabla^2 (V-\Delta)^{-1}$ is proved to be bounded but there is no mention of the $\nabla V^{\frac{1}{2}} (V-\Delta)^{-1}$.

Chapter 5

Non-Homogeneous AKM with 3×3 Structure

In this chapter we consider our second approach to the non-homogeneous Axelsson-Keith-McIntosh framework that, although imposes more on the algebraic structure of the operators, will yield a surprisingly effective solution to the Kato problem with potential. For $\alpha \in [1,2]$, define $\mathcal{W}_\alpha$ to be the class of all $V \in L^1_{loc}(\mathbb{R}^n)$ for which

$$[V]_\alpha := \sup_{u \in C_0^\infty(\mathbb{R}^n)} \frac{\left\| |V|^{\frac{\alpha}{2}} u \right\| + \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|}{\left\| (|V| - \Delta)^{\frac{\alpha}{2}} u \right\|} < \infty.$$

As will be proved in §5.3.2, the collection of potential classes $\{\mathcal{W}_\alpha\}_{\alpha \in [1,2]}$ is decreasing. The largest class $\mathcal{W}_1$ consists of all locally integrable potentials with no additional restrictions and the smallest class $\mathcal{W}_2$ contains RH_2 in any dimension and $L^{\frac{n}{2}}(\mathbb{R}^n)$ in dimension $n > 4$. Our approach to non-homogeneous AKM in this chapter will provide a proof of the following theorem.

Theorem 5.0.1 (Kato with Potential)**.** *Let $V \in \mathcal{W}_\alpha$ for some $\alpha \in (1,2]$ and $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$. Suppose that the Gårding inequalities (3.1) and (3.6) are both satisfied with constants $\kappa_A > 0$ and $\kappa_A^V > 0$ respectively. There exists a constant $C_V > 0$ such that*

$$C_V^{-1} \left(\left\| V^{\frac{1}{2}} u \right\| + \|\nabla u\| \right) \leq \left\| \sqrt{L+V} u \right\| \leq C_V \left(\left\| V^{\frac{1}{2}} u \right\| + \|\nabla u\| \right) \tag{5.1}$$

for all $u \in D(L+V)$. Moreover, the constant C_V depends on V and α through

$$C_V = \tilde{C}_V (\alpha - 1)^{-1} \left(1 + [V]_\alpha^2 \right),$$

where $\tilde{C}_V$ only depends on V through κ_A^V and is independent of α.

The above theorem tells us, in particular, that the Kato estimate with potential, (5.1), is valid for any potential V with range contained in some sector $S_{\mu+}$ for $\mu \in [0, \frac{\pi}{2})$ and with $|V|$ either contained in RH_2 for any dimension or in $L^{\frac{n}{2}}(\mathbb{R}^n)$ in dimension $n > 4$. This theorem will be proved by imposing additional constraints on our non-homogeneous operator Γ in order to mimic the three-by-three nature of the operator $\Gamma_{|V|^{\frac{1}{2}}}$. In particular, a non-homogeneous Axelsson-Keith-McIntosh framework will be developed to handle operators of the form

$$\Gamma_J := \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ D & 0 & 0 \end{pmatrix}, \tag{5.2}$$

where D is a homogeneous first-order differential operator and J is a possibly non-homogeneous differential operator of order less than or equal to one. With this additional structure, it will be shown that the perturbed Dirac-type operator will satisfy square function estimates under modest coercivity conditions.

The technical challenge presented by the inclusion of the non-homogeneous part J will be overcome by separating our square function norm into components and demonstrating that the non-homogeneous term will allow for the first two components to be bounded while the third component can be bound using an argument similar to the classical argument of [11].

Since the operator Γ_J is of a more general form than $\Gamma_{|V|^{\frac{1}{2}}}$, the non-homogeneous AKM framework that we develop will have applications not confined to zero-order scalar potentials. Indeed, the non-homogeneous framework will also be used to prove Kato estimates for systems of equations with zero-order potential and for scalar equations with first-order potentials. It takes no great leap of imagination to see that it is also possible to apply our framework to systems of equations with first-order potentials. This, however, will be left to the readers discretion.

The structure of this chapter is as follows. Section 5.1 describes the non-homogeneous AKM framework and states the main results associated with it. Section 5.2 contains most of the technical machinery and is dedicated to a proof of our main result. Section 5.3 will apply the non-homogeneous AKM framework to the scalar Kato problem with potential, the Kato problem for systems with zero-order potential and the scalar Kato problem with first-order potential. It is here that a proof of Theorem 5.0.1 will be completed.

5.1. Three-by-Three Structure

At this point, further structure will be imposed upon our operators in order to generalise the non-homogeneous operator $\Gamma_{|V|^{\frac{1}{2}}}$ defined in (3.7). This additional structure will later be exploited in order to obtain square function estimates of the form (3.24).

Let $\mathbb{C}^N = V_1 \oplus V_2 \oplus V_3$ where V_1, V_2 and V_3 are finite-dimensional complex Hilbert spaces. Let $\mathbb{P}_i : \mathbb{C}^N \to \mathbb{C}^N$ be the projection operator onto the space V_i for $i = 1$, 2 and 3. Our Hilbert space will have the following orthogonal decomposition

$$\mathcal{H} := L^2\left(\mathbb{R}^n; \mathbb{C}^N\right) = L^2\left(\mathbb{R}^n; V_1\right) \oplus L^2\left(\mathbb{R}^n; V_2\right) \oplus L^2(\mathbb{R}^n; V_3).$$

The notation $\mathbb{P}_i$ will also be used to denote the natural projection operator from $\mathcal{H}$ onto $L^2\left(\mathbb{R}^n; V_i\right)$. For a vector $v \in \mathcal{H}$, $v_i \in L^2\left(\mathbb{R}^n; V_i\right)$ will denote the ith component for $i = 1$, 2 or 3.

Let Γ_J be an operator on $\mathcal{H}$ of the form

$$\Gamma_J := \begin{pmatrix} 0 & 0 \\ D_J & 0 \end{pmatrix} := \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ D & 0 & 0 \end{pmatrix},$$

where J and D are closed densely defined operators,

$$J : L^2\left(\mathbb{R}^n; V_1\right) \to L^2\left(\mathbb{R}^n; V_2\right) \ \textit{and}$$

$$D : L^2\left(\mathbb{R}^n; V_1\right) \to L^2\left(\mathbb{R}^n; V_3\right),$$

and $D_J : L^2(\mathbb{R}^n; V_1) \to L^2(\mathbb{R}^n; V_2) \oplus L^2(\mathbb{R}^n; V_3)$ is the operator $D_J = \begin{pmatrix} J \\ D \end{pmatrix}$. Define the operators

$$\Gamma_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D & 0 & 0 \end{pmatrix}, \qquad M_J := \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_0 := \Gamma_0 + \Gamma_0^*, \quad S_J := M_J + M_J^* \quad \textit{and} \quad \Pi_J := \Gamma_J + \Gamma_J^*.$$

Let $B_1, B_2 \in L^\infty\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^N\right)\right)$ be matrix-valued multiplication operators. The following key assumption will be imposed on our operators throughout the entirety of this chapter.

Key Assumption. *The family of operators $\{\Gamma_0, B_1, B_2\}$ satisfies the conditions (H1) - (H8) of [11] while $\{\Gamma_J, B_1, B_2\}$ satisfies only (H1) - (H6).*

Implicit in the assumption that $\{\Gamma_J, B_1, B_2\}$ satisfies (H1) is the condition that $D(J) \cap D(D)$ is dense in $\mathcal{H}$.

Example 5.1.1. Typical examples of operators that satisfy the previous key assumption are when both D and J are partial differential operators of order less than or equal to one. If the perturbations B_1 and B_2 satisfy suitable accretivity conditions then the families of operators $\{\Gamma_0, B_1, B_2\}$ and $\{\Gamma_J, B_1, B_2\}$ will both satisfy (H1) - (H6). If, in addition, D is first-order homogeneous and there exists $c > 0$ for which

$$\|\nabla u\| \leq c \cdot \|Du\|$$

for all $u \in R(D^*) \cap D(D)$ and

$$\|\nabla u\| \leq c \cdot \|D^* u\|$$

for all $u \in R(D) \cap D(D^*)$ then $\{\Gamma_0, B_1, B_2\}$ will also satisfy (H7) and (H8). A particular example of such a situation is given by the operator $\Gamma_{|V|^{\frac{1}{2}}}$ together with perturbations B_1 and B_2 as defined in (3.7) and (3.8) with (3.1) and (3.6) satisfied.

Remark 5.1.1. Since the operator Γ_0, together with the perturbations B_1 and B_2, satisfy all eight conditions (H1) - (H8) of [11], it follows that any result from that paper must be valid for these operators.

Definition 5.1.1. *For $t \in \mathbb{R} \setminus \{0\}$, define the perturbation dependent operators*

$$\Gamma_{J,B} := B_2^* \Gamma_J B_1^*, \quad \Gamma_{J,B}^* := B_1 \Gamma_J^* B_2, \quad \Pi_{J,B} := \Gamma_J + \Gamma_{J,B}^*,$$

$$R_t^{J,B} := (I + it\Pi_{J,B})^{-1}, \quad P_t^{J,B} := \left(I + t^2 (\Pi_{J,B})^2\right)^{-1},$$

$$Q_t^{J,B} := t\Pi_{J,B} P_t^{J,B} \quad \text{and} \quad \Theta_t^{J,B} := t\Gamma_{J,B}^* P_t^{J,B}.$$

When there is no perturbation, i.e. when $B_1 = B_2 = I$, the B will be dropped from the superscript or subscript. For example, instead of $\Theta_t^{J,I}$ the notation Θ_t^J will be employed.

We now introduce coercivity conditions to serve as a replacement for (H8) for the operators $\{\Gamma_J, B_1, B_2\}$. These conditions will not be automatically imposed upon our operators but, rather, will be taken as hypotheses for our main results.

(H8Dα) Let $\alpha \in (1, 2]$. The domain inclusion

$$D\left((D_J^* D_J)^{\frac{\alpha}{2}}\right) \subset D\left((D^* D)^{\frac{\alpha}{2}}\right)$$

holds and there exists a constant $C > 0$ such that

$$\left\| (D^* D)^{\frac{\alpha}{2}} u \right\| \leq C \cdot \left\| (D_J^* D_J)^{\frac{\alpha}{2}} u \right\|$$

for all $u \in D\left((D_J^* D_J)^{\frac{\alpha}{2}}\right)$.

(H8J) B_2 is of the form

$$B_2 = \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} := \begin{pmatrix} I & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}, \tag{5.3}$$

where $A_{ij} \in L^\infty(\mathbb{R}^n; \mathcal{L}(V_j, V_i))$ for i, $j = 2$ or 3. The inclusion

$$D(D_J^* D_J) \subset D\left(D_J^* \hat{A} \begin{pmatrix} J \\ 0 \end{pmatrix}\right)$$

is satisfied. Furthermore, there exists a constant $C > 0$ such that for all $u \in D(D_J^* D_J)$,

$$\left\| D_J^* \hat{A} \begin{pmatrix} J \\ 0 \end{pmatrix} u \right\| \leq C \cdot \| D_J^* D_J u \|.$$

Remark 5.1.2. The situation of most interest to us is when $A_{32} = 0$ and

$$\| J^* A_{22} J u \| = \| J^* J u \|$$

for all $u \in D(J^* A_{22} J) = D(J^* J)$. In this case, the Riesz transform condition of (H8J) becomes the perturbation free condition

$$\| J^* J u \| \leq C \cdot \| D_J^* D_J u \|$$

for all $u \in D(D_J^* D_J)$ and the domain inclusion of (H8J) becomes $D(D_J^* D_J) \subset D(J^* J)$. Furthermore, when this occurs, the estimate from (H8J) will be equivalent to the condition

$$\| S_J u \| \leq C \cdot \| \Pi_J u \|$$

or equivalently

$$\| \Pi_0 u \| \leq C \cdot \| \Pi_J u \|$$

for all $u \in D(\Pi_J) \cap R(\Pi_J)$.

The Kato square root estimate is a first-order Riesz transform condition. To some extent, it then seems intuitively unnatural to use a second-order Riesz transform condition as in (H8J) as our hypotheses. Indeed, when the conditions of the above remark are satisfied and J is a positive operator, it will be sufficient to consider a lower-order version of (H8J) as given below.

(H8Jα) Let $\alpha \in (1, 2]$. The perturbation B_2 is of the form (5.3) with $A_{32} = 0$. J is a positive operator and

$$\|JA_{22}Ju\| = \left\|J^2u\right\|$$

for all $u \in D(JA_{22}J) = D(J^2)$. The domain inclusion

$$D\left((D_J^* D_J)^{\frac{\alpha}{2}}\right) \subset D(J^\alpha)$$

holds. Furthermore, there exists a constant $C > 0$ such that

$$\|J^\alpha u\| \leq C \cdot \left\|(D_J^* D_J)^{\frac{\alpha}{2}} u\right\|$$

for all $u \in D\left((D_J^* D_J)^{\frac{\alpha}{2}}\right)$.

Notation. For $\alpha \in (1, 2]$, let b_α^D, b^J and b_α^J denote the smallest constant for which (H8Dα), (H8J) or (H8Jα) are satisfied respectively. If the criteria for one of these conditions is not met then the corresponding constant will be set to infinity. For example, if (H8J) is not satisfied then $b^J = \infty$. Also define

$$c_\alpha^J := \left(1 + \left(b_\alpha^D\right)^2 + \left(\min\left\{b^J, b_\alpha^J\right\}\right)^2\right) \cdot (\alpha - 1)^{-1}.$$

For the remainder of this chapter we introduce the notation $A \lesssim B$ and $A \simeq B$ to denote that there exists a constant $C > 0$, independent of (H8Dα), (H8J) and (H8Jα), for which $A \leq C \cdot B$ and $C^{-1} \cdot B \leq A \leq C \cdot B$ respectively. C is still allowed to depend on (H1) - (H8) for $\{\Gamma_0, B_1, B_2\}$ and (H1) - (H6) for $\{\Gamma_J, B_1, B_2\}$. Note that this implies that C will only depend on J through the constants in (H2) for $\{\Gamma_J, B_1, B_2\}$ and it will be independent of α.

We are now in a position where the main result of the non-homogeneous AKM framework can finally be stated.

Theorem 5.1.1. *Let $\{\Gamma_J, B_1, B_2\}$ be as defined above. Consider the following square function estimate:*

$$\int_0^\infty \left\|\Theta_t^{J,B} \mathbb{P}_i P_t^J u\right\|^2 \frac{dt}{t} \lesssim C \cdot \|u\|^2 \tag{5.4}$$

for all $u \in R(\Gamma_J)$, for some $C > 0$, for $i = 1$, 2 or 3.

(i) The estimate is trivially satisfied for $i = 1$ for any $C \geq 0$.

(ii) If (H8J) is satisfied then (5.4) is satisfied for $i = 2$ with $C = \left(b^J\right)^2$.

(iii) If (H8Jα) is satisfied for some $\alpha \in (1, 2]$ then (5.4) is satisfied for $i = 2$ with $C = \left(1 + \left(b^J_\alpha\right)^2\right)(\alpha - 1)^{-1}$.

(iv) If (H8Dα) is satisfied for some $\alpha \in (1, 2]$ and either (H8J) or (H8Jα) is also satisfied then (5.4) holds for $i = 3$ with constant $C = c^J_\alpha$.

Proof. The proof of (iii) and (iv) will be postponed until §5.2. For (i), simply note that since Γ_J commutes with the operator P^J_t by Lemma 3.6.1 we must have $\mathbb{P}_1 P^J_t u = 0$ for any $u \in R(\Gamma_J)$.

It remains to consider (ii). Suppose that (H8J) holds. First it will be proved that for $u \in R(\Gamma_J)$ we have $\mathbb{P}_2 P^J_t u \in D\left(\Gamma^*_{J,B}\right)$. Since $u \in R(\Gamma_J)$, $u = \Gamma_J v$ for some $v \in D(\Gamma_J)$. As

$$P^J_t u = P^J_t \Gamma_J v = \Gamma_J P^J_t v$$

by Lemma 3.6.1 and $P^J_t u \in D(\Pi_J)$, it follows that $\left(P^J_t v\right)_1 \in D(D^*_J D_J)$, which by (H8J) is contained in $D\left(D^*_J \hat{A}\begin{pmatrix} J \\ 0 \end{pmatrix}\right)$. This implies that

$$\begin{pmatrix} J(P^J_t v)_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (P^J_t u)_2 \\ 0 \end{pmatrix} \in D(D^*_J \hat{A})$$

and therefore $\mathbb{P}_2 P^J_t u \in D(\Gamma^*_{J,B})$

Since $\mathbb{P}_2 P^J_t u \in D\left(\Gamma^*_{J,B}\right)$, it follows from Lemma 3.6.1 that

$$\Theta^{J,B}_t \mathbb{P}_2 P^J_t u = P^{J,B}_t t \Gamma^*_{J,B} \mathbb{P}_2 P^J_t.$$

The estimate in (H8J) gives

$$\left\|\Gamma^*_{J,B} \mathbb{P}_2 v\right\| \leq b^J \cdot \|\Gamma^*_J v\|$$

for any $v \in R(\Gamma_J) \cap D(\Pi_J)$. Since P^J_t and Γ_J commute by Lemma 3.6.1, it follows that

$$\left\|\Gamma^*_{J,B} \mathbb{P}_2 P^J_t u\right\| \leq b^J \cdot \left\|\Gamma^*_J P^J_t u\right\|$$

for $u \in R(\Gamma_J)$. On applying the uniform L^2-boundedness of the $P_t^{J,B}$ operators,

$$\begin{aligned}\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_2 P_t^J u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|P_t^{J,B} t\Gamma_{J,B}^*\mathbb{P}_2 P_t^J u\right\|^2 \frac{dt}{t}\\ &\lesssim \int_0^\infty \left\|t\Gamma_{J,B}^*\mathbb{P}_2 P_t^J u\right\|^2 \frac{dt}{t}\\ &\leq \left(b^J\right)^2 \int_0^\infty \left\|t\Gamma_J^* P_t^J u\right\|^2 \frac{dt}{t}.\end{aligned}$$

On successively applying the Hodge decomposition Proposition 3.6.1 and Lemma 3.6.2 we obtain

$$\begin{aligned}\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_2 P_t^J u\right\|^2 \frac{dt}{t} &\lesssim \left(b^J\right)^2 \int_0^\infty \left\|t\Pi_J P_t^J u\right\|^2 \frac{dt}{t}\\ &= \left(b^J\right)^2 \int_0^\infty \left\|Q_t^J u\right\|^2 \frac{dt}{t}\\ &= \frac{1}{2}\left(b^J\right)^2 \|u\|^2.\end{aligned}$$

This shows that (5.4) is valid for $i = 2$ with constant $C = \left(b^J\right)^2$. ◻

Let's consider an estimate that serves as a dual to (5.4).

Proposition 5.1.1. *For $t > 0$, define the operator*

$$\underline{P}_t^{J,B} := \left(I + t^2\left(\Gamma_J^* + B_2\Gamma_J B_1\right)^2\right)^{-1}.$$

The square function estimate

$$\int_0^\infty \left\|tB_2\Gamma_J B_1 \underline{P}_t^{J,B} P_t^J u\right\|^2 \frac{dt}{t} \lesssim \|u\|^2 \tag{5.5}$$

will hold for all $u \in \mathcal{H}$ when $B_1 = I$.

Proof. Since $\{\Gamma_J, B_1, B_2\}$ satisfies (H1) - (H6) it follows that $\{\Gamma_J^*, B_2, B_1\}$ must also satisfy (H1) - (H6). Proposition 3.6.2 then implies that the operators $\underline{P}_t^{J,B}$ are well-defined and uniformly L^2-bounded. Since $B_1 = I$, it follows that $P_t^J u \in D(B_2\Gamma_J B_1 u)$ for any $u \in \mathcal{H}$ and therefore, by Lemma 3.6.1,

$$B_2\Gamma_J B_1 \underline{P}_t^{J,B} P_t^J u = \underline{P}_t^{J,B} B_2\Gamma_J B_1 P_t^J u = \underline{P}_t^J B_2\Gamma_J P_t^J u.$$

This together with the uniform L^2-boundedness of the $\underline{P}_t^{J,B}$ operators implies that

$$\begin{aligned}\int_0^\infty \left\| tB_2\Gamma_J B_1 \underline{P}_t^{J,B} P_t^J u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\| \underline{P}_t^{J,B} tB_2\Gamma_J P_t^J u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\| t\Gamma_J P_t^J u\right\|^2 \frac{dt}{t} \\ &\leq \int_0^\infty \left\| t\Pi_J P_t^J u\right\|^2 \frac{dt}{t} \\ &= \int_0^\infty \left\| Q_t^J u\right\|^2 \frac{dt}{t} \\ &= \frac{1}{2}\left\| u\right\|^2,\end{aligned}$$

where the inequality $\|\Gamma_J v\| \leq \|\Pi_J v\|$ for $v \in D(\Pi_J)$ follows immediately from the three-by-three matrix form of the operators and Lemma 3.6.2 was applied to obtain the last line. □

From our main result, Theorem 5.1.1, and the previous proposition, the upper and lower square function estimates for $Q_t^{J,B}$ can be proved.

Theorem 5.1.2. *Suppose that $B_1 = I$. Suppose further that (H8Dα) is satisfied for some $\alpha \in (1, 2]$ and either (H8J) or (H8Jα) is satisfied. Then*

$$\left(c_\alpha^J\right)^{-1} \cdot \|u\|^2 \lesssim \int_0^\infty \left\| Q_t^{J,B} u\right\|^2 \frac{dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2 \tag{5.6}$$

for all $u \in \overline{R(\Pi_J)}$.

Proof. Proposition 3.6.3 states that in order to prove the square function estimate (5.6), it is sufficient for the estimate (5.4) to be valid for all $i = 1$, 2 and 3 for the permutations of operators $\{\Gamma_J, B_1, B_2\}$, $\{\Gamma_J, B_1^*, B_2^*\}$, $\{\Gamma_J^*, B_2, B_1\}$ and $\{\Gamma_J^*, B_2^*, B_1^*\}$. The permutations $\{\Gamma_J, B_1, B_2\}$ and $\{\Gamma_J, B_1^*, B_2^*\}$ both come under the umbrella of Theorem 5.1.1 and the permutations $\{\Gamma_J^*, B_2, B_1\}$ and $\{\Gamma_J^*, B_2^*, B_1^*\}$ are handled by Proposition 5.1.1. □

From the upper and lower estimate of the previous theorem, Theorem 3.1.4 then implies that $\Pi_{J,B}$ has a bounded holomorphic functional calculus.

Theorem 5.1.3. *Suppose that $B_1 = I$. Suppose further that (H8Dα) is satisfied for some $\alpha \in (1, 2]$ and either (H8J) or (H8Jα) is satisfied. Then $\Pi_{J,B}$ has a bounded $H^\infty\left(S_\mu^o\right)$-holomorphic functional calculus for any $\mu \in \left(\omega_J, \frac{\pi}{2}\right)$, where*

$$\omega_J := \frac{1}{2}\left(\sup_{u\in R\left(\Gamma_J^*\right)\setminus\{0\}} |\arg\langle B_1 u, u\rangle| + \sup_{u\in R(\Gamma_J)\setminus\{0\}} |\arg\langle B_2 u, u\rangle|\right).$$

In particular,

$$\|f(\Pi_{J,B})\| \lesssim c_\alpha^J \cdot \sup_{\zeta \in S_\mu^o} |f(\zeta)|$$

for any $f \in H_0^\infty\left(S_\mu^o\right)$.

Corollary 5.1.1. *Suppose that $B_1 = I$. Suppose further that (H8Dα) is satisfied for some $\alpha \in (1, 2]$ and either (H8J) or (H8Jα) is satisfied. The operator*

$$L_B^J := D_J^* \hat{A} D_J$$

is a $2\omega_J$-sectorial operator with a bounded $H^\infty\left(S_{2\mu+}^o\right)$-functional calculus for any $\mu \in \left(\omega_J, \frac{\pi}{2}\right)$. Moreover

$$\left\| \sqrt{L_B^J} u \right\| \simeq c_\alpha^J \cdot (\|Ju\| + \|Du\|) \tag{5.7}$$

for all $u \in D\left(L_B^J\right)$.

Proof. The bounded $H^\infty\left(S_{2\mu+}^o\right)$-functional calculus of L_B^J follows from the bounded $H^\infty\left(S_\mu^o\right)$-functional calculus of $\Pi_{J,B}$ and that $\Pi_{J,B}^2$ is of the form

$$\Pi_{J,B}^2 = \begin{pmatrix} L_B^J & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

The estimate (5.7) follows from Corollary 3.1.2 applied to the operator $\Pi_{J,B}$ and an element $(u, 0, 0) \in \mathcal{H}$ with $u \in D\left(L_B^J\right)$. □

5.2. Square Function Estimates

In this section, a proof of our main result, Theorem 5.1.1, will be provided. The first part of the proof consists in showing that the operators P_t^J can effectively be diagonalised when estimating square function norms from above. This diagonalisation will be applied to bound the second component of our square function norm when (H8Jα) is satisfied thus proving the third part of Theorem 5.1.1. To prove the fourth and most challenging part of Theorem 5.1.1 we will use this diagonalisation and an argument similar to the original result [11]. That is, a $T(1)$-type reduction will be applied to reduce the third component of the square function norm to a Carleson measure norm which will subsequently be proved to be bounded.

5.2.1. Diagonalisation of the P_t^J Operators

Define the bounded operator $\mathcal{P}_t^J : \mathcal{H} \to \mathcal{H}$ through

$$\mathcal{P}_t^J := \begin{pmatrix} (I + t^2 D_J^* D_J)^{-1} & 0 & 0 \\ 0 & (I + t^2 J J^*)^{-1} & 0 \\ 0 & 0 & (I + t^2 D D^*)^{-1} \end{pmatrix},$$

for $t > 0$. Observe that since the operators $D_J^* D_J$, JJ^* and DD^* are all self-adjoint, it follows from Corollary 3.1.1 that square function estimates hold for each of these operators with constant independent of J and D. Therefore each of these operators possesses a bounded holomorphic functional calculus with constant independent of J and D. It can be deduced from this that the operators $\mathcal{P}_t^J$ are uniformly L^2-bounded with constant independent of J and D.

Let us prove that the operator P_t^J can be effectively diagonalised when evaluating square function estimates. Specifically, the following theorem will be proved.

Theorem 5.2.1. *Suppose that (H8Dα) is satisfied for some $\alpha \in (1, 2]$ then*

$$\int_0^\infty \left\| \mathbb{P}_3 \left(\mathcal{P}_t^J - P_t^J \right) u \right\|^2 \frac{dt}{t} \lesssim \left(1 + (b_\alpha^D)^2 \right) \cdot \|u\|^2 \tag{5.8}$$

for all $u \in R(\Gamma_J)$. Suppose, in addition, that (H8Jα) is also satisfied. Then

$$\int_0^\infty \left\| \left(\mathcal{P}_t^J - P_t^J \right) u \right\|^2 \frac{dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2 . \tag{5.9}$$

Such a diagonalisation will aid us tremendously in the bounding of our main square function estimate (5.4) for the second and third component. This theorem will be proved by inspecting each component separately.

Remark 5.2.1. It is easy to see that the diagonalisation estimate (5.9) is trivially satisfied on the first component for any $u \in \mathcal{H}$ since $\mathbb{P}_1 \mathcal{P}_t^J = \mathbb{P}_1 P_t^J$.

Proposition 5.2.1. *For any $u \in \mathcal{H}$,*

$$\int_0^\infty \left\| \mathcal{P}_t^J \left(P_t^J - I \right) u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 . \tag{5.10}$$

PROOF. The estimate is trivially satisfied for any $u \in N(\Pi_J)$ since

$$\begin{aligned}\left(P_t^J - I\right) u &= \left(\left(I + t^2\Pi_J^2\right)^{-1} - I\right) u \\ &= \left(I + t^2\Pi_J^2\right)^{-1} \left(I - \left(I + t^2\Pi_J^2\right)\right) u \\ &= 0\end{aligned}$$

for any $t > 0$. Suppose that $u \in \overline{R\left(\Pi_J\right)}$. On applying the resolution of the identity, Proposition 3.1.3,

$$\begin{aligned}\int_0^\infty \left\|\mathcal{P}_t^J \left(P_t^J - I\right) u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\mathcal{P}_t^J \left(P_t^J - I\right) 2 \int_0^\infty \left(Q_s^J\right)^2 u \frac{ds}{s}\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left(\int_0^\infty \left\|\mathcal{P}_t^J \left(P_t^J - I\right) \left(Q_s^J\right)^2 u\right\| \frac{ds}{s}\right)^2 \frac{dt}{t}.\end{aligned}$$

The Cauchy-Schwarz inequality leads to

$$\begin{aligned}&\int_0^\infty \left\|\mathcal{P}_t^J \left(P_t^J - I\right) u\right\|^2 \frac{dt}{t} \lesssim \\ &\quad \int_0^\infty \left(\int_0^\infty \left\|\mathcal{P}_t^J \left(P_t^J - I\right) Q_s^J\right\| \frac{ds}{s}\right) \cdot \left(\int_0^\infty \left\|\mathcal{P}_t^J \left(P_t^J - I\right) Q_s^J\right\| \left\|Q_s^J u\right\|^2 \frac{ds}{s}\right) \frac{dt}{t}.\end{aligned} \tag{5.11}$$

Let's estimate the term $\left\|\mathcal{P}_t^J \left(P_t^J - I\right) Q_s^J\right\|$. First assume that $t \leq s$. On noting that $\left(P_t^J - I\right) Q_s^J = \frac{t}{s} Q_t^J \left(P_s^J - I\right)$ we obtain

$$\left\|\mathcal{P}_t^J \left(P_t^J - I\right) Q_s^J\right\| \lesssim \left\|\left(P_t^J - I\right) Q_s^J\right\| \lesssim \frac{t}{s} \left\|Q_t^J \left(P_s^J - I\right)\right\| \lesssim \frac{t}{s}. \tag{5.12}$$

Next, suppose that $t > s$. Then the equality $P_t^J Q_s^J = \frac{s}{t} Q_t^J P_s^J$ gives

$$\left\|\mathcal{P}_t^J \left(P_t^J - I\right) Q_s^J\right\| \lesssim \left\|P_t^J Q_s^J\right\| + \left\|\mathcal{P}_t^J Q_s^J\right\| \lesssim \frac{s}{t} + \left\|\mathcal{P}_t^J Q_s^J\right\|.$$

The term $\mathcal{P}_t^J Q_s^J$ will be considered component-wise. For the first component, recall that $\mathbb{P}_1 \mathcal{P}_t^J = \mathbb{P}_1 P_t^J$ and observe that

$$\begin{aligned}\left\|\mathbb{P}_1 \mathcal{P}_t^J Q_s^J\right\| &= \left\|\mathbb{P}_1 P_t^J s \Pi_J P_s^J\right\| \\ &= \frac{s}{t} \left\|\mathbb{P}_1 P_t^J t \Pi_J P_s^J\right\| \\ &= \frac{s}{t} \left\|\mathbb{P}_1 Q_t^J P_s^J\right\| \\ &\lesssim \frac{s}{t}.\end{aligned}$$

For the second component, note that

$$\mathbb{P}_2 \mathcal{P}_t^J = \left(I + t^2 S_J^2\right)^{-1} \mathbb{P}_2.$$

Also observe

$$\mathbb{P}_2 \Pi_J u = \mathbb{P}_2 \Pi_J \mathbb{P}_1 u = \mathbb{P}_2 S_J \mathbb{P}_1 u$$

for $u \in D(\Pi_J)$. This gives

$$\begin{aligned} \left\|\mathbb{P}_2 \mathcal{P}_t^J Q_s^J\right\| &= \left\|\left(I + t^2 S_J^2\right)^{-1} \mathbb{P}_2 s \Pi_J P_s^J\right\| \\ &= \left\|\left(I + t^2 S_J^2\right)^{-1} \mathbb{P}_2 s S_J \mathbb{P}_1 P_s^J\right\| \\ &= \frac{s}{t} \left\|\mathbb{P}_2 t S_J \left(I + t^2 S_J^2\right)^{-1} \mathbb{P}_1 P_s^J\right\| \\ &\lesssim \frac{s}{t}, \end{aligned}$$

where the last line follows from the fact that S_J is self-adjoint and therefore possesses a bounded holomorphic functional calculus with constant independent of J. Lastly, for the third component, we have

$$\mathbb{P}_3 \mathcal{P}_t^J = \mathbb{P}_3 P_t^0 = P_t^0 \mathbb{P}_3$$

and

$$\mathbb{P}_3 \Pi_J u = \mathbb{P}_3 \Pi_J \mathbb{P}_1 u = \mathbb{P}_3 \Pi_0 \mathbb{P}_1 u$$

for $u \in D(\Pi_J)$. This leads to

$$\begin{aligned} \left\|\mathbb{P}_3 \mathcal{P}_t^J Q_s^J\right\| &= \left\|P_t^0 \mathbb{P}_3 s \Pi_J P_s^J\right\| \\ &= \left\|P_t^0 \mathbb{P}_3 s \Pi_0 \mathbb{P}_1 P_s^J\right\| \\ &= \frac{s}{t} \left\|\mathbb{P}_3 t \Pi_0 P_t^0 \mathbb{P}_1 P_s^J\right\| \\ &\lesssim \frac{s}{t}. \end{aligned}$$

Putting everything together gives

$$\left\|\mathcal{P}_t^J \left(P_t^J - I\right) Q_s^J\right\| \lesssim \min\left\{\frac{t}{s}, \frac{s}{t}\right\}^{\frac{1}{2}}. \tag{5.13}$$

This bound can then be applied to (5.11) to give (5.10). ◻

Proposition 5.2.2. *Suppose that the condition (H8Jα) is satisfied for some $\alpha \in (1, 2]$. Then*

$$\int_0^\infty \left\|\mathbb{P}_2 \left(I - \mathcal{P}_t^J\right) P_t^J u\right\|^2 \frac{dt}{t} \lesssim (\alpha - 1)^{-1} \left(b_\alpha^J\right)^2 \cdot \|u\|^2$$

for any $u \in R(\Gamma_J)$.

PROOF. It will first be proved that $\left(P_t^J u\right)_2 \in D\left(J^{\alpha-1}\right)$. Since Γ_J commutes with P_t^J we must have

$$P_t^J u = \Gamma_J \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$$

for some $(v, 0, 0) \in D\left(\Gamma_J\right)$. This then gives $\left(P_t^J u\right)_2 = Jv$. Therefore $\left(P_t^J u\right)_2 \in D(J^{\alpha-1})$ if and only if $v \in D(J^{\alpha})$. We know that $P_t^J u \in D(\Pi_J)$ which implies that $v \in D(D_J^* D_J)$ and therefore $v \in D\left((D_J^* D_J)^{\frac{\alpha}{2}}\right)$. Our hypothesis (H8J$\alpha$) then tells us that $v \in D(J^{\alpha})$ which allows us to conclude, using the previous reasoning, that $\left(P_t^J u\right)_2 \in D(J^{\alpha-1})$.

Since $\left(P_t^J u\right)_2 \in D(J^{\alpha-1})$, it follows that

$$\begin{aligned} \mathbb{P}_2\left(I - \mathcal{P}_t^J\right) P_t^J u &= \left(0, t^2 J^2 \left(I + t^2 J^2\right)^{-1} \left(P_t^J u\right)_2, 0\right) \\ &= \left(0, g_\alpha^t (J)\, t^{\alpha-1} J^{\alpha-1} \left(P_t^J u\right)_2, 0\right), \end{aligned} \tag{5.14}$$

where $g_\alpha^t : S_\mu^o \to \mathbb{C}$ is the bounded holomorphic function defined through

$$g_\alpha^t(z) := \frac{t^2 z^2}{\left(I + t^2 z^2\right) t^{\alpha-1} \left(\sqrt{z^2}\right)^{\alpha-1}}. \tag{5.15}$$

As J is self-adjoint, it follows from Corollary 3.1.1 that J possesses a bounded holomorphic functional calculus with constant independent of J. This, together with (5.14), gives

$$\left\|\mathbb{P}_2\left(I - \mathcal{P}_t^J\right) P_t^J u\right\| \lesssim \left\|t^{\alpha-1} J^{\alpha-1}\left(P_t^J u\right)_2\right\|.$$

On applying (H8Jα),

$$\begin{aligned} \left\|J^{\alpha-1}\left(P_t^J u\right)_2\right\| &= \left\|J^{\alpha} v\right\| \\ &\lesssim b_\alpha^J \cdot \left\|\left(D_J^* D_J\right)^{\frac{\alpha}{2}} v\right\| \\ &= b_\alpha^J \cdot \left\| |\Pi_J|^{\alpha} \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \right\| \\ &\simeq b_\alpha^J \cdot \left\| |\Pi_J|^{\alpha-1} \Pi_J \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \right\| \\ &= b_\alpha^J \cdot \left\| |\Pi_J|^{\alpha-1} P_t^J u\right\|, \end{aligned}$$

where $|\Pi_J| := \sqrt{\Pi_J^2}$ and in the fourth line we applied the bounded holomorphic functional calculus of the operator Π_J. Therefore

$$\begin{aligned}\int_0^\infty \left\|\mathbb{P}_2\left(I-\mathcal{P}_t^J\right)P_t^J u\right\|^2 \frac{dt}{t} &\lesssim \left(b_\alpha^J\right)^2 \cdot \int_0^\infty \left\|t^{\alpha-1}\left|\Pi_J\right|^{\alpha-1} P_t^J u\right\|^2 \frac{dt}{t}\\ &\lesssim \left(b_\alpha^J\right)^2 (\alpha-1)^{-1}\cdot \|u\|^2,\end{aligned}$$

where we used the fact that Π_J is self-adjoint and Corollary 3.1.1 in the last line. □

Proposition 5.2.3. *Suppose that (H8Dα) is satisfied for some $\alpha \in (1,2]$. Then*

$$\int_0^\infty \left\|\mathbb{P}_3\left(I-\mathcal{P}_t^J\right)P_t^J u\right\|^2 \frac{dt}{t} \lesssim (\alpha-1)^{-1}\left(b_\alpha^D\right)^2 \cdot \|u\|^2 \tag{5.16}$$

for all $u \in R(\Gamma_J)$.

Proof. First note that the left-hand side of (5.16) can be re-written as

$$\begin{aligned}\int_0^\infty \left\|\mathbb{P}_3\left(I-\mathcal{P}_t^J\right)P_t^J u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\mathbb{P}_3\left(I-P_t^0\right)P_t^J u\right\|^2 \frac{dt}{t}\\ &= \int_0^\infty \left\|\left(I-P_t^0\right)\mathbb{P}_3 P_t^J u\right\|^2 \frac{dt}{t}.\end{aligned}$$

It will be shown that $\mathbb{P}_3 P_t^J u \in D\left(|\Pi_0|^{\alpha-1}\right)$. Since Γ_J commutes with the operator P_t^J and $u \in R(\Gamma_J)$, we must have $P_t^J u = \Gamma_J P_t^J(v,0,0)$ for some $(v,0,0) \in D(\Gamma_J)$. This implies that

$$\mathbb{P}_3 P_t^J u = \mathbb{P}_3 \Gamma_J P_t^J(v,0,0) = \mathbb{P}_3 \Pi_0 P_t^J(v,0,0)$$

and therefore $\mathbb{P}_3 P_t^J u \in D\left(|\Pi_0|^{\alpha-1}\right)$ will follow from $P_t^J(v,0,0) \in D(|\Pi_0|^{\alpha-1}\mathbb{P}_3\Pi_0) = D(|\Pi_0|^{\alpha-1}\Pi_0)$. The bounded holomorphic functional calculus of the operator Π_0 tells us that $D(|\Pi_0|^{\alpha-1}\Pi_0) = D(|\Pi_0|^{\alpha})$ and it is therefore sufficient to prove that $P_t^J(v,0,0) \in D(|\Pi_0|^{\alpha})$. Since $P_t^J(v,0,0)$ is non-zero only in the first component, this in turn is equivalent to proving that

$$\left(P_t^J(v,0,0)\right)_1 \in D\left((D^*D)^{\frac{\alpha}{2}}\right).$$

This however follows directly from our hypothesis (H8Dα) and the fact that $\left(P_t^J(v,0,0)\right)_1 \in D\left(D_J^* D_J\right) \subset D\left((D_J^* D_J)^{\frac{\alpha}{2}}\right)$. This completes the proof of our claim that $\mathbb{P}_3 P_t^J u \in D\left(|\Pi_0|^{\alpha-1}\right)$.

Since $\mathbb{P}_3 P_t^J u \in D\left(|\Pi_0|^{\alpha-1}\right)$, we must have

$$\left(I-P_t^0\right)\mathbb{P}_3 P_t^J u = g_\alpha^t\left(\Pi_0\right) t^{\alpha-1}\left|\Pi_0\right|^{\alpha-1}\mathbb{P}_3 P_t^J u,$$

where g_α^t is as defined in (5.15). From the bounded holomorphic functional calculus of Π_0

we then obtain

$$\int_0^\infty \left\|\mathbb{P}_3 \left(I - \mathcal{P}_t^J\right) P_t^J u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\mathbb{P}_3 t^{\alpha-1} \left|\Pi_0\right|^{\alpha-1} P_t^J u\right\|^2 \frac{dt}{t}.$$

On recalling that $P_t^J u = \Gamma_J P_t^J (v,0,0)$ for some $(v,0,0) \in D(\Gamma_J)$,

$$\begin{aligned}\int_0^\infty \left\|\mathbb{P}_3 t^{\alpha-1} \left|\Pi_0\right|^{\alpha-1} P_t^J u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\mathbb{P}_3 t^{\alpha-1} \left|\Pi_0\right|^{\alpha-1} \Gamma_J P_t^J (v,0,0)\right\|^2 \frac{dt}{t} \\ &= \int_0^\infty \left\|\mathbb{P}_3 t^{\alpha-1} \left|\Pi_0\right|^{\alpha-1} \Pi_0 P_t^J (v,0,0)\right\|^2 \frac{dt}{t}.\end{aligned}$$

On exploiting the bounded holomorphic functional calculus of the operator Π_0 once more,

$$\int_0^\infty \left\|\mathbb{P}_3 t^{\alpha-1} \left|\Pi_0\right|^{\alpha-1} \Pi_0 P_t^J (v,0,0)\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|t^{\alpha-1} \left|\Pi_0\right|^{\alpha} P_t^J (v,0,0)\right\|^2 \frac{dt}{t}.$$

Observe that since $P_t^J(v,0,0)$ is non-zero only in the first entry,

$$\begin{aligned}\left\|\left|\Pi_0\right|^\alpha P_t^J (v,0,0)\right\| &= \left\|(D^*D)^{\frac{\alpha}{2}} \left(P_t^J (v,0,0)\right)_1\right\| \\ &\leq b_\alpha^D \cdot \left\|(D^*D + J^*J)^{\frac{\alpha}{2}} \left(P_t^J (v,0,0)\right)_1\right\| \\ &= b_\alpha^D \cdot \left\|\left|\Pi_J\right|^\alpha P_t^J (v,0,0)\right\|.\end{aligned}$$

On then applying the bounded holomorphic functional calculus of the operator Π_J,

$$\begin{aligned}\left(b_\alpha^D\right)^2 \cdot \int_0^\infty \left\|t^{\alpha-1} \left|\Pi_J\right|^\alpha P_t^J (v,0,0)\right\|^2 \frac{dt}{t} &\lesssim \left(b_\alpha^D\right)^2 \int_0^\infty \left\|t^{\alpha-1} \left|\Pi_J\right|^{\alpha-1} \Pi_J P_t^J (v,0,0)\right\|^2 \frac{dt}{t} \\ &= \left(b_\alpha^D\right)^2 \cdot \int_0^\infty \left\|t^{\alpha-1} \left|\Pi_J\right|^{\alpha-1} P_t^J u\right\|^2 \frac{dt}{t} \\ &\lesssim \left(b_\alpha^D\right)^2 (\alpha-1)^{-1} \cdot \left\|u\right\|^2,\end{aligned}$$

where we used the fact that Π_J is self-adjoint and Corollary 3.1.1 in the final line. □

Combining Propositions 5.2.1, 5.2.2 and 5.2.3 together then gives Theorem 5.2.1. With this diagonalisation in hand we can now return to our proof of Theorem 5.1.1. In particular the second component of our square function norm will now be bounded.

Proof of Theorem 5.1.1.(iii). On splitting the second component of our square function norm from above,

$$\int_0^\infty \left\|\Theta_t^{J,B} \mathbb{P}_2 P_t^J u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\Theta_t^{J,B} \mathbb{P}_2 \left(\mathcal{P}_t^J - P_t^J\right) u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\Theta_t^{J,B} \mathbb{P}_2 \mathcal{P}_t^J u\right\|^2 \frac{dt}{t}.$$

The uniform L^2-boundedness of the operators $\Theta_t^{J,B}$ together with Propositions 5.2.1 and

5.2.2 give

$$\int_0^\infty \left\| \Theta_t^{J,B} \mathbb{P}_2 \left(\mathcal{P}_t^J - P_t^J \right) u \right\|^2 \frac{dt}{t} \lesssim \left(1 + \left(b_\alpha^J \right)^2 \right) (\alpha - 1)^{-1} \cdot \|u\|^2 .$$

It remains to bound the second term in the splitting. In order to do do so, it will first be shown that $\mathbb{P}_2 \mathcal{P}_t^J u \in D(\Gamma_{J,B}^*)$. From the definition of the adjoint, $\mathbb{P}_2 \mathcal{P}_t^J u \in D(\Gamma_{J,B}^*)$ if and only if there exists some $u' \in \mathcal{H}$ such that

$$\langle \Gamma_{J,B} w, \mathbb{P}_2 \mathcal{P}_t^J u \rangle = \langle w, u' \rangle$$

for all $w \in D(\Gamma_{J,B})$, where $\Gamma_{J,B} = B_2^* \Gamma_J B_1^*$. For $w \in D(\Gamma_{J,B})$,

$$\begin{aligned} \langle \Gamma_{J,B} w, \mathbb{P}_2 \mathcal{P}_t^J u \rangle &= \langle B_2^* \Gamma_J B_1^* w, \mathbb{P}_2 \mathcal{P}_t^J u \rangle \\ &= \langle \Gamma_J B_1^* w, B_2 \mathbb{P}_2 \mathcal{P}_t^J u \rangle \\ &= \langle M_J B_1^* w, B_2 \mathbb{P}_2 \mathcal{P}_t^J u \rangle, \end{aligned} \tag{5.17}$$

where in the last line we used the fact that $A_{32} = 0$ by (H8Jα) and therefore $B_2 \mathbb{P}_2 = \mathbb{P}_2 B_2 \mathbb{P}_2$. This proves that $\mathbb{P}_2 \mathcal{P}_t^J u \in D(\Gamma_{J,B}^*)$ will follow from $B_2 \mathbb{P}_2 \mathcal{P}_t^J u \in D\left(M_J^*\right)$ which, in turn, will follow from $(\mathcal{P}_t^J u)_2 \in D(J^* A_{22}) = D(J A_{22})$.

Note that $u \in R\left(\Gamma_J\right)$ implies that $u = \Gamma_J(v, 0, 0)$ for some $(v, 0, 0) \in D\left(\Gamma_J\right)$. Then

$$\begin{aligned} \left(\mathcal{P}_t^J u \right)_2 &= \left(I + t^2 J^2 \right)^{-1} J v \\ &= J \left(I + t^2 J^2 \right)^{-1} v. \end{aligned}$$

(H8Jα) states that $D(J^2) = D(J A_{22} J)$ with $\|J^2 \tilde{u}\| = \|J A_{22} J \tilde{u}\|$ for $\tilde{u} \in D\left(J^2\right)$. Since $\left(I + t^2 J^2\right)^{-1} v \in D\left(J^2\right)$ we must have $\left(I + t^2 J^2\right)^{-1} v \in D\left(J A_{22} J\right)$ and therefore $\left(\mathcal{P}_t^J u\right)_2 \in D\left(J A_{22}\right)$. This allows us to conclude that $\mathbb{P}_2 \mathcal{P}_t^J u \in D\left(\Gamma_{J,B}^*\right)$. Moreover, from (5.17) we know that $\Gamma_{J,B}^* \mathbb{P}_2 \mathcal{P}_t^J u = B_1 M_J^* B_2 \mathbb{P}_2 \mathcal{P}_t^J u$ and therefore

$$\begin{aligned} \left\| \Gamma_{J,B}^* \mathbb{P}_2 \mathcal{P}_t^J u \right\| &= \left\| B_1 M_J^* B_2 \mathbb{P}_2 \mathcal{P}_t^J u \right\| \\ &\lesssim \left\| M_J^* B_2 \mathbb{P}_2 \mathcal{P}_t^J u \right\| \\ &= \left\| J A_{22} J \left(I + t^2 J^2 \right)^{-1} v \right\| \\ &= \left\| J^2 \left(I + t^2 J^2 \right)^{-1} v \right\| \\ &= \left\| J \left(I + t^2 J^2 \right)^{-1} u_2 \right\| . \end{aligned}$$

This together with Lemma 3.6.1 gives

$$\begin{aligned}\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_2\mathcal{P}_t^J u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|P_t^{J,B} t\Gamma_{J,B}^*\mathbb{P}_2\mathcal{P}_t^J u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\|t\Gamma_{J,B}^*\mathbb{P}_2\mathcal{P}_t^J u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\|tJ\left(I+t^2J^2\right)^{-1} u_2\right\| \frac{dt}{t}.\end{aligned}$$

The theorem then follows from the fact that J is self-adjoint and therefore satisfies square function estimates with constant independent of J by Corollary 3.1.1. ◻

5.2.2. The Third Component

This section is dedicated to bounding the third component of our square function norm and thus proving the fourth and final part of Theorem 5.1.1. Specifically, it will be proved that when (H8Dα) is satisfied for some $\alpha \in (1,2]$ and either (H8J) or (H8Jα) is satisfied the estimate

$$\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_3 P_t^J u\right\|^2 \frac{dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2 \tag{5.18}$$

will hold for any $u \in R(\Gamma_J)$. A similar argument to that of [11] will be used, but one will need to keep track of the effect of the projection $\mathbb{P}_3$.

$T(1)$-Reduction

Our first step towards a $T(1)$-reduction is to use the splitting

$$\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_3 P_t^J u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_3\left(P_t^J - \mathcal{P}_t^J\right)u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_3\mathcal{P}_t^J u\right\|^2 \frac{dt}{t}.$$

The uniform L^2-boundedness of the operators $\Theta_t^{J,B}$ and Theorem 5.2.1 can be applied to the first term to obtain

$$\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_3\left(\mathcal{P}_t^J - P_t^J\right)u\right\|^2 \frac{dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2.$$

On recalling that $\mathbb{P}_3\mathcal{P}_t^J = \mathbb{P}_3 P_t^0$, this reduces the task of proving our square function estimate to obtaining the bound

$$\int_0^\infty \left\|\Theta_t^{J,B}\mathbb{P}_3 P_t^0 u\right\|^2 \frac{dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2.$$

Introduce the notation $\tilde{\Theta}_t^{J,B}$ to denote the operators $\tilde{\Theta}_t^{J,B} := \Theta_t^{J,B}\mathbb{P}_3$. Let $\gamma_t^{J,B}$ and $\tilde{\gamma}_t^{J,B}$ denote the principal parts of the operators $\Theta_t^{J,B}$ and $\tilde{\Theta}_t^{J,B}$ respectively. That is, they are the multiplication operators defined through

$$\gamma_t^{J,B}(x)w := \Theta_t^{J,B}(w)(x) \quad \textit{and} \quad \tilde{\gamma}_t^{J,B}(x)(w) := \left(\Theta_t^{J,B}\mathbb{P}_3\right)(w)(x),$$

for $w \in \mathbb{C}^N$ and $x \in \mathbb{R}^n$. Evidently we must have $\tilde{\gamma}_t^{J,B}(x)w = \gamma_t^{J,B}(x)\mathbb{P}_3 w$.

Our square function norm can be reduced to this principal part by applying the splitting

$$\int_0^\infty \left\|\tilde{\Theta}_t^{J,B} P_t^0 u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\left(\tilde{\Theta}_t^{J,B} - \tilde{\gamma}_t^{J,B} A_t\right) P_t^0 u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t P_t^0 u\right\|^2 \frac{dt}{t}. \quad (5.19)$$

Since the operator $\Theta_t^{J,B}$ satisfies the conditions of Proposition 3.6.6, it follows that

$$\begin{aligned}
\int_0^\infty \left\|\left(\tilde{\Theta}_t^{J,B} - \tilde{\gamma}_t^{J,B} A_t\right) P_t^0 u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\left(\Theta_t^{J,B} - \gamma_t^{J,B} A_t\right) \mathbb{P}_3 P_t^0 u\right\|^2 \frac{dt}{t} \\
&\lesssim \int_0^\infty \left\|t\nabla \mathbb{P}_3 P_t^0 u\right\|^2 \frac{dt}{t} \\
&\lesssim \int_0^\infty \left\|t\Pi_0 P_t^0 u\right\|^2 \frac{dt}{t} \\
&= \int_0^\infty \left\|Q_t^0 u\right\|^2 \frac{dt}{t} \\
&= \frac{1}{2}\left\|u\right\|^2,
\end{aligned}$$

where the estimate $\|\nabla \mathbb{P}_3 P_t^0 u\| \lesssim \|\Pi_0 P_t^0 u\|$ follows from (H8) for the operator Γ_0. It should be noted that in order to use (H8) we had to use the fact that $u = \Gamma_J v$ for some $v \in D(\Gamma_J)$ and therefore

$$\mathbb{P}_3 P_t^0 u = P_t^0 \mathbb{P}_3 \Gamma_J v = P_t^0 \mathbb{P}_3 \Gamma_0 v = \Gamma_0 P_t^0 v \in R(\Gamma_0).$$

Our theorem has thus been reduced to a proof of the following square function estimate

$$\int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t P_t^0 u\right\|^2 \frac{dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2.$$

On splitting from above using the triangle inequality,

$$\int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t P_t^0 u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} + \int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t u\right\|^2 \frac{dt}{t}. \quad (5.20)$$

Proposition 3.6.4 states that the uniform estimate $\left\|\tilde{\gamma}_t^{J,B} A_t\right\| \lesssim 1$ is true for all $t > 0$. Furthermore, notice that $A_t^2 = A_t$ and $\mathbb{P}_3 A_t = A_t \mathbb{P}_3$ for all $t > 0$. These facts combine together to produce

$$\begin{aligned}\int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} &= \int_0^\infty \left\|\gamma_t^{J,B} A_t \mathbb{P}_3 A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty \left\|\mathbb{P}_3 A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t}.\end{aligned}$$

According to the argument from Proposition 5.7 of [11], this final term can be bound by

$$\int_0^\infty \left\|A_t \left(P_t^0 - I\right) u\right\|^2 \frac{dt}{t} \lesssim \|u\|^2,$$

since $\{\Gamma_0, B_1, B_2\}$ by hypothesis satisfies (H1) - (H8). For the second term in (5.20), apply Carleson's theorem, Theorem 3.2.1, to obtain

$$\int_0^\infty \left\|\tilde{\gamma}_t^{J,B} A_t u\right\|^2 \frac{dt}{t} \lesssim \|\mu\|_{\mathcal{C}} \cdot \|\mathcal{N}\left(A_t u\right)\|^2 \lesssim \|\mu\|_{\mathcal{C}} \cdot \|u\|^2,$$

where μ is the measure on $\mathbb{R}^{n+1}$ defined through

$$d\mu(x,t) := \left|\tilde{\gamma}_t^{J,B}(x)\right|^2 \frac{dx\, dt}{t}$$

for $x \in \mathbb{R}^n$ and $t > 0$. The proof of our theorem has thus been reduced to showing that the measure μ is a Carleson measure with constant smaller than a multiple of c_α^J.

Carleson Measure Estimate

Our goal now is to prove the following Carleson measure estimate,

$$\sup_{Q \in \Delta} \frac{1}{|Q|} \int_0^{l(Q)} \int_Q \left|\tilde{\gamma}_t^{J,B}(x)\right|^2 \frac{dx\, dt}{t} \lesssim c_\alpha^J < \infty. \tag{5.21}$$

Let $\mathcal{L}_3$ denote the subspace

$$\mathcal{L}_3 := \left\{\nu \in \mathcal{L}\left(\mathbb{C}^N\right) \setminus \{0\} : \nu \mathbb{P}_3 = \nu\right\}. \tag{5.22}$$

By construction, we have $\tilde{\gamma}_t^{J,B}(x) \in \mathcal{L}_3$ for any $t > 0$ and $x \in \mathbb{R}^n$ since

$$\begin{aligned}\tilde{\gamma}_t^{J,B}(x) \mathbb{P}_3 w &= \left(\Theta_t^{J,B} \mathbb{P}_3\right) \left(\mathbb{P}_3 w\right)(x) \\ &= \left(\Theta_t^{J,B} \mathbb{P}_3\right)(w)(x) \\ &= \tilde{\gamma}_t^{J,B}(x)(w).\end{aligned}$$

Let $\sigma > 0$ be a constant to be determined at a later time. Let $\mathcal{V}$ be a finite set consisting of $\nu \in \mathcal{L}_3$ with $|\nu| = 1$ such that $\cup_{\nu\in\mathcal{V}} K_\nu = \mathcal{L}_3 \setminus \{0\}$, where

$$K_\nu := \left\{\nu' \in \mathcal{L}_3 \setminus \{0\} : \left|\frac{\nu'}{|\nu'|} - \nu\right| \le \sigma\right\}.$$

Then, in order to prove our Carleson measure estimate (5.21), it is sufficient to fix $\nu \in \mathcal{V}$ and prove that

$$\sup_{Q\in\Delta} \frac{1}{|Q|} \int\int_{\substack{(x,t)\in R_Q \\ \tilde{\gamma}_t^{J,B}(x)\in K_\nu}} \left|\tilde{\gamma}_t^{J,B}(x)\right|^2 \frac{dx\,dt}{t} \lesssim c_\alpha^J < \infty, \tag{5.23}$$

where $R_Q := Q \times [0, l(Q))$. Recall the John-Nirenberg lemma for Carleson measures as given in 3.2.1. With this tool at our disposal, the proof of our theorem is reduced to the following proposition.

Proposition 5.2.4. *There exists $\beta > 0$ and $\sigma > 0$ that will satisfy the following conditions. For every $\nu \in \mathcal{V}$ and $Q \in \Delta$, there is a collection $\{Q_k\}_k \subset \Delta$ of disjoint subcubes of Q such that $E_{Q,\nu} = Q \setminus \cup_k Q_k$ satisfies $|E_{Q,\nu}| > \beta\,|Q|$ and such that*

$$\sup_{Q\in\Delta} \frac{1}{|Q|} \int\int_{\substack{(x,t)\in E^*_{Q,\nu} \\ \tilde{\gamma}_t^{J,B}(x)\in K_\nu}} \left|\tilde{\gamma}_t^{J,B}(x)\right|^2 \frac{dx\,dt}{t} \lesssim c_\alpha^J < \infty, \tag{5.24}$$

where $E^*_{Q,\nu} := R_Q \setminus \cup_k R_{Q_k}$. Moreover, β and σ are entirely independent of the conditions (H8Dα), (H8J) and (H8Jα).

For now, fix $\nu \in \mathcal{V}$ and $Q \in \Delta$. Let w^ν, $\hat{w}^\nu \in \mathbb{C}^N$ with $|\hat{w}^\nu| = |w^\nu| = 1$ and $\nu^*(\hat{w}^\nu) = w^\nu$. To simplify notation, when superfluous, this dependence will be kept implicit by defining $w := w^\nu$ and $\hat{w} := \hat{w}^\nu$. Notice that since ν satisfies $\nu = \nu\mathbb{P}_3$, w must satisfy $\mathbb{P}_3 w = w$.

For $\epsilon > 0$ the function $f^w_{Q,\epsilon}$ can be defined in an identical manner to [11]. Specifically, let $\eta_Q : \mathbb{R}^N \to [0,1]$ be a smooth cutoff function equal to 1 on $2Q$, with support in $4Q$ and with $\|\nabla\eta_Q\|_\infty \le \frac{1}{l}$, where $l := l(Q)$. Then define $w_Q := \eta_Q \cdot w$ and

$$f^w_{Q,\epsilon} := w_Q - \epsilon l i \Gamma_J (I + \epsilon l i \Pi_{J,B})^{-1} w_Q = \left(I + \epsilon l i \Gamma^*_{J,B}\right)(I + \epsilon l i \Pi_{J,B})^{-1} w_Q.$$

Lemma 5.2.1. *There exists a constant $C > 0$, independent of (H8Dα), (H8J) and (H8Jα), that satisfies $\left\|f^w_{Q,\epsilon}\right\| \le C\,|Q|^{\frac{1}{2}}$ and*

$$\left|\fint_Q \mathbb{P}_3 f^w_{Q,\epsilon} - w\right| \le C \cdot \epsilon^{\frac{1}{2}}, \tag{5.25}$$

for any $\epsilon > 0$.

PROOF. The first claim follows from

$$\begin{aligned}\left\|f_{Q,\epsilon}^{w}\right\| &\lesssim \|w_Q\| + \left\|\epsilon li\Gamma_J\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right\| \\ &\lesssim |Q|^{\frac{1}{2}} + \left\|\epsilon li\Pi_{J,B}\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right\| \\ &\lesssim |Q|^{\frac{1}{2}}.\end{aligned}$$

On recalling that w is zero in the first two components,

$$\left|\fint_Q \mathbb{P}_3 f_{Q,\epsilon}^{w} - w\right|^2 = \left|\fint_Q \mathbb{P}_3\epsilon li\Gamma_J\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2 = \left|\fint_Q \epsilon li\Gamma_0\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2.$$

At this point, apply Lemma 5.6 of [11] to the operator $\Upsilon = \Gamma_0$ to obtain

$$\begin{aligned}\left|\fint_Q \epsilon li\Gamma_0\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2 &\lesssim \frac{(\epsilon l)^2}{l}\left(\fint_Q\left|\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2\right)^{\frac{1}{2}}\cdot\left(\fint_Q\left|\Gamma_0\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2\right)^{\frac{1}{2}} \\ &\lesssim \epsilon\left(\fint_Q\left|\epsilon li\Gamma_0\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2\right)^{\frac{1}{2}} \\ &\leq \epsilon\left(\fint_Q\left|\epsilon li\Gamma_J\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2\right)^{\frac{1}{2}} \lesssim \epsilon,\end{aligned}$$

where the inequality $\|\Gamma_0 v\| \leq \|\Gamma_J v\|$ for $v \in D(\Gamma_J)$ follows trivially from the matrix form of Γ_0 and Γ_J. □

Lemma 5.2.2. *There exists a constant $D > 0$, independent of (H8Dα), (H8J) and (H8Jα), such that*

$$\int\int_{R_Q}\left|\Theta_t^{J,B} f_{Q,\epsilon}^{w}(x)\right|^2\frac{dx\,dt}{t} \leq D\frac{|Q|}{\epsilon^2}. \tag{5.26}$$

PROOF. First observe that

$$\begin{aligned}\Theta_t^{J,B} f_{Q,\epsilon}^{w} &= P_t^{J,B} t\Gamma_{J,B}^{*}\left(I+\epsilon li\Gamma_{J,B}^{*}\right)\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q \\ &= \frac{t}{\epsilon l}P_t^{J,B}\epsilon l\Gamma_{J,B}^{*}\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^l\int_Q\left|\Theta_t^{J,B} f_{Q,\epsilon}^{w}(x)\right|^2\frac{dx\,dt}{t} &= \int_0^l\left(\frac{t}{\epsilon l}\right)^2\int_Q\left|P_t^{J,B}\epsilon l\Gamma_{J,B}^{*}\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right|^2 dx\,\frac{dt}{t} \\ &\lesssim \int_0^l\left(\frac{t}{\epsilon l}\right)^2\left\|\epsilon li\Gamma_{J,B}^{*}\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\right\|^2\frac{dt}{t} \\ &\lesssim \frac{|Q|}{(\epsilon l)^2}\int_0^l t\,dt \simeq \frac{|Q|}{\epsilon^2}.\end{aligned}$$

□

From this point forward, with C as in Lemma 5.2.1, set $\epsilon := \frac{1}{4C^2}$ and introduce the notation $f_Q^w := f_{Q,\epsilon}^w$. With this choice of ϵ it must be true that

$$\left| \fint_Q \mathbb{P}_3 f_Q^w - w \right| \leq \frac{1}{2}.$$

That is,

$$\begin{aligned} 1 - 2\mathrm{Re} \left\langle \fint_Q \mathbb{P}_3 f_Q^w, w \right\rangle &= |w|^2 - 2\mathrm{Re} \left\langle \fint_Q \mathbb{P}_3 f_Q^w, w \right\rangle \\ &\leq \left| \fint_Q \mathbb{P}_3 f_Q^w - w \right|^2 \\ &\leq \frac{1}{4}. \end{aligned}$$

On rearranging we find that

$$\mathrm{Re} \left\langle \fint_Q \mathbb{P}_3 f_Q^w, w \right\rangle \geq \frac{1}{4}. \tag{5.27}$$

In this context, Lemma 5.11 of [11] will take on the below form.

Lemma 5.2.3. *There exists β, c_1, $c_2 > 0$ and a collection $\{Q_k\}$ of dyadic cubes of Q such that $|E_{Q,\nu}| > \beta |Q|$ and such that*

$$\mathrm{Re} \left\langle w, \fint_{Q'} \mathbb{P}_3 f_Q^w \right\rangle \geq c_1 \quad \text{and} \quad \fint_{Q'} \left| \mathbb{P}_3 f_Q^w \right| \leq c_2$$

for all dyadic subcubes $Q' \in \Delta$ of Q which satisfy $R_{Q'} \cap E_{Q,\nu}^* \neq \emptyset$. Moreover, β, c_1 and c_2 are independent of (H8Dα), (H8J), (H8Jα), Q, σ and ν.

The proof of this statement follows in an identical manner to the argument in [11]. If we set $\sigma = \frac{c_1}{2c_2}$, then the following pointwise estimate can be deduced.

Lemma 5.2.4. *If $(x,t) \in E_{Q,\nu}^*$ and $\tilde{\gamma}_t^{J,B}(x) \in K_\nu$ then*

$$\left| \tilde{\gamma}_t^{J,B}(x) \left(A_t f_Q^w(x) \right) \right| \geq \frac{1}{2} c_1 \left| \tilde{\gamma}_t^{J,B}(x) \right|. \tag{5.28}$$

Proof. First observe that

$$\begin{aligned} \left| \nu \left(A_t f_Q^w(x) \right) \right| &\geq \mathrm{Re} \left\langle \hat{w}, \nu \left(A_t f_Q^w(x) \right) \right\rangle \\ &= \mathrm{Re} \left\langle w, A_t f_Q^w(x) \right\rangle \\ &= \mathrm{Re} \left\langle w, A_t \mathbb{P}_3 f_Q^w(x) \right\rangle \\ &\geq c_1. \end{aligned}$$

Then

$$\begin{aligned}
\left|\frac{\tilde{\gamma}_t^{J,B}(x)}{\left|\tilde{\gamma}_t^{J,B}(x)\right|}\left(A_t f_Q^w(x)\right)\right| &= \left|\frac{\tilde{\gamma}_t^{J,B}(x)}{\left|\tilde{\gamma}_t^{J,B}(x)\right|}\left(A_t \mathbb{P}_3 f_Q^w(x)\right)\right| \\
&\geq \left|\nu\left(A_t f_Q^w(x)\right)\right| - \left|\frac{\tilde{\gamma}_t^{J,B}(x)}{\left|\tilde{\gamma}_t^{J,B}(x)\right|} - \nu\right| \left|A_t \mathbb{P}_3 f_Q^w(x)\right| \\
&\geq c_1 - \sigma c_2 \\
&= \frac{1}{2} c_1.
\end{aligned}$$

□

Proof of Proposition 5.2.4. From the pointwise bound of the previous lemma,

$$\int\int_{\substack{(x,t)\in E_{Q,\nu}^* \\ \tilde{\gamma}_t^{J,B}(x)\in K_\nu}} \left|\tilde{\gamma}_t^{J,B}(x)\right|^2 \frac{dx\,dt}{t} \lesssim \int\int_{R_Q} \left|\tilde{\gamma}_t^{J,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t}.$$

At this stage we can begin to unravel our square function norm,

$$\begin{aligned}
\int\int_{R_Q} \left|\tilde{\gamma}_t^{J,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t} \lesssim \int\int_{R_Q} &\left|\Theta_t^{J,B} f_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\
&+ \int\int_{R_Q} \left|\Theta_t^{J,B} f_Q^w(x)\right|^2 \frac{dx\,dt}{t}.
\end{aligned} \tag{5.29}$$

Lemma 5.2.2 states that the final term in the above estimate will be bounded from above by a multiple of $|Q|$. This reduces the task of proving the proposition to bounding the first term of the above splitting. Recall that f_Q^w can be expressed in the form

$$f_Q^w := w_Q - u_Q^w,$$

where $u_Q^w \in R(\Gamma_J)$ is given by

$$u_Q^w := \epsilon li \Gamma_J \left(I + \epsilon li \Pi_{J,B}\right)^{-1} w_Q.$$

An application of the triangle inequality then leads to

$$\begin{aligned}
\int\int_{R_Q} &\left|\Theta_t^{J,B} f_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t f_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\
&\lesssim \int\int_{R_Q} \left|\Theta_t^{J,B} w_Q(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x)\right|^2 \frac{dx\,dt}{t} \\
&\quad + \int\int_{R_Q} \left|\Theta_t^{J,B} u_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t u_Q^w(x)\right|^2 \frac{dx\,dt}{t}.
\end{aligned} \tag{5.30}$$

On noticing that for every $x \in Q$ and $0 < t < l(Q)$

$$\begin{aligned}\Theta_t^{J,B} w_Q(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x) &= \Theta_t^{J,B} w_Q(x) - \Theta_t^{J,B}\left(A_t w_Q(x)\right)(x) \\ &= \Theta_t^{J,B}\left(\left(\eta_Q - 1\right) w\right)(x),\end{aligned}$$

it is clear that the first term in (5.30) can be handled in an identical manner as in the proof of Proposition 5.9 from [11]. Specifically, since $(\operatorname{supp}\left(\eta_Q - 1\right) w) \cap 2Q = \emptyset$, the off-diagonal estimates of the operator $\Theta_t^{J,B}$ lead to

$$\int_Q \left|\Theta_t^{J,B}\left(\left(\eta_Q - 1\right) w\right)(x)\right|^2 dx \lesssim \frac{t\,|Q|}{l},$$

which implies that

$$\int\!\!\int_{R_Q} \left|\Theta_t^{J,B} w_Q(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x)\right|^2 \frac{dx\,dt}{t} \lesssim |Q|\,.$$

As for the second term in (5.30),

$$\begin{aligned}&\int\!\!\int_{R_Q} \left|\Theta_t^{J,B} u_Q^w - \tilde{\gamma}_t^{J,B}(x) A_t u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\ &\qquad\lesssim \int\!\!\int_{R_Q} \left|\Theta_t^{J,B}\left(I - P_t^J\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\ &\qquad\quad + \int\!\!\int_{R_Q} \left|\Theta_t^{J,B} P_t^J u_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t u_Q^w(x)\right|^2 \frac{dx\,dt}{t}.\end{aligned} \tag{5.31}$$

Since $u_Q^w \in R\left(\Gamma_J\right)$, Corollary 3.6.1 gives

$$\begin{aligned}\int\!\!\int_{R_Q} \left|\Theta_t^{J,B}\left(I - P_t^J\right) u_Q^w\right|^2 \frac{dx\,dt}{t} &\lesssim \left\|u_Q^w\right\|^2 \\ &\lesssim |Q|\,.\end{aligned}$$

For the remaining term in (5.31),

$$\begin{aligned}&\int\!\!\int_{R_Q} \left|\Theta_t^{J,B} P_t^J u_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\ &\qquad\lesssim \int\!\!\int_{R_Q} \left|\Theta_t^{J,B}\left(I - \mathbb{P}_3\right) P_t^J u_Q^w\right|^2 \frac{dx\,dt}{t} \\ &\qquad\quad + \int\!\!\int_{R_Q} \left|\tilde{\Theta}_t^{J,B} P_t^J u_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t u_Q^w(x)\right|^2 \frac{dx\,dt}{t}.\end{aligned} \tag{5.32}$$

Since we have already proved the boundedness of the first and second components,

$$\begin{aligned}\int\int_{R_Q}\left|\Theta_t^{J,B}\left(I-\mathbb{P}_3\right)P_t^J u_Q^w\right|^2\frac{dx\,dt}{t}&\lesssim c_\alpha^J\cdot\left\|u_Q^w\right\|^2\\&\lesssim c_\alpha^J\cdot|Q|.\end{aligned}$$

For the second term in (5.32),

$$\begin{aligned}&\int\int_{R_Q}\left|\tilde{\Theta}_t^{J,B}P_t^J u_Q^w(x)-\tilde{\gamma}_t^{J,B}(x)A_t u_Q^w(x)\right|^2\frac{dx\,dt}{t}\\&\quad\lesssim\int\int_{R_Q}\left|\tilde{\Theta}_t^{J,B}P_t^J u_Q^w(x)-\tilde{\gamma}_t^{J,B}(x)A_tP_t^J u_Q^w(x)\right|^2\frac{dx\,dt}{t}\\&\qquad+\int\int_{R_Q}\left|\gamma_t^{J,B}(x)\mathbb{P}_3\left(A_tP_t^J-A_t\right)u_Q^w(x)\right|^2\frac{dx\,dt}{t}.\end{aligned}\tag{5.33}$$

To bound the first term on the right-hand side of the above estimate notice that

$$\tilde{\Theta}_t^{J,B}P_t^J u_Q^w(x)-\tilde{\gamma}_t^{J,B}(x)A_tP_t^J u_Q^w(x)=\left(\Theta_t^{J,B}-\gamma_t^{J,B}A_t\right)\mathbb{P}_3P_t^J u_Q^w(x).$$

Theorem 5.2.1 then allows us to diagonalise our P_t^J operators in the first term of (5.33) to get

$$\int_0^{l(Q)}\left\|\left(\Theta_t^{J,B}-\gamma_t^{J,B}A_t\right)\mathbb{P}_3P_t^J u_Q^w\right\|_{L^2(Q)}^2\frac{dt}{t}\lesssim c_\alpha^J|Q|+\int_0^\infty\left\|\left(\Theta_t^{J,B}-\gamma_t^{J,B}A_t\right)\mathbb{P}_3\mathcal{P}_t^J u_Q^w\right\|^2\frac{dt}{t}.$$

From Proposition 3.6.6 we know that

$$\begin{aligned}\int_0^\infty\left\|\left(\Theta_t^{J,B}-\gamma_t^{J,B}A_t\right)\mathbb{P}_3P_t^J u_Q^w\right\|^2\frac{dt}{t}&\lesssim\int_0^\infty\left\|t\nabla\mathbb{P}_3\mathcal{P}_t^J u_Q^w\right\|^2\frac{dt}{t}=\int_0^\infty\left\|t\nabla\mathbb{P}_3P_t^0 u_Q^w\right\|^2\frac{dt}{t}\\&\lesssim\int_0^\infty\left\|t\Pi_0P_t^0 u_Q^w\right\|^2\frac{dt}{t}=\int_0^\infty\left\|Q_t^0 u_Q^w\right\|^2\frac{dt}{t}\\&\lesssim|Q|,\end{aligned}$$

where in the second line we applied (H8) for the operators $\{\Gamma_0,B_1,B_2\}$. It should be noted that in order to use (H8), we had to use the fact that

$$\begin{aligned}\mathbb{P}_3P_t^0u_Q^w&=P_t^0\mathbb{P}_3u_Q^w=P_t^0\mathbb{P}_3\epsilon li\Gamma_J\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\\&=P_t^0\mathbb{P}_3\epsilon li\Gamma_0\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q=\epsilon li\Gamma_0P_t^0\left(I+\epsilon li\Pi_{J,B}\right)^{-1}w_Q\in R\left(\Gamma_0\right).\end{aligned}$$

It remains to bound the second term in (5.33),

$$\int\int_{R_Q}\left|\gamma_t^{J,B}(x)\mathbb{P}_3A_t\left(P_t^J-I\right)u_Q^w(x)\right|^2\frac{dx\,dt}{t}=\int\int_{R_Q}\left|\gamma_t^{J,B}A_t\mathbb{P}_3A_t\left(P_t^J-I\right)u_Q^w(x)\right|^2\frac{dx\,dt}{t}$$

On noting the uniform L^2-boundedness of the $\gamma_t^{J,B} A_t$ operators and applying the triangle inequality,

$$\begin{aligned}\int\int_{R_Q} \left|\gamma_t^{J,B} A_t \mathbb{P}_3 A_t \left(P_t^J - I\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t} &\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left|\mathbb{P}_3 A_t \left(P_t^J - I\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\ &\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left|\mathbb{P}_3 A_t \left(P_t^J - \mathcal{P}_t^J\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \\ &\quad + \int_0^\infty \int_{\mathbb{R}^n} \left|\mathbb{P}_3 A_t \left(\mathcal{P}_t^J - I\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t}.\end{aligned}$$

Applying Theorem 5.2.1 and recalling that $\mathbb{P}_3 \mathcal{P}_t^J = \mathbb{P}_3 P_t^0$,

$$\int\int_{R_Q} \left|\gamma_t^{J,B} A_t \mathbb{P}_3 A_t \left(P_t^J - I\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \lesssim c_\alpha^J \cdot \|u\|^2 + \int_0^\infty \int_{\mathbb{R}^n} \left|\mathbb{P}_3 A_t \left(P_t^0 - I\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t}.$$

From the proof of Proposition 5.7 of [11] we know that

$$\int_0^\infty \int_{\mathbb{R}^n} \left|A_t \left(P_t^0 - I\right) u_Q^w(x)\right|^2 \frac{dx\,dt}{t} \lesssim |Q|,$$

allowing us to finally conclude our proof. □

5.3. Applications

Our non-homogeneous framework will now be applied to three different contexts. We begin with the case that serves as the primary motivation for this part of the thesis, the scalar Kato square root problem with zero-order potential.

5.3.1. Scalar Kato with Zero-Order Potential

Theorem 5.0.1, the promised result of the introductory section, will be proved. Fix $V \in \mathcal{W}_\alpha$ for some $\alpha \in (1,2]$. Brand the definition of the operators Γ_J, B_1 and B_2 to be as given in §3.5. That is, define our Hilbert space to be

$$\mathcal{H} := L^2\left(\mathbb{R}^n\right) \oplus L^2\left(\mathbb{R}^n\right) \oplus L^2(\mathbb{R}^n;\mathbb{C}^n)$$

for some $n \in \mathbb{N}^*$. Set $J = |V|^{\frac{1}{2}}$ and $D = \nabla$. Our operator Γ_J is then given by

$$\Gamma_J = \Gamma_{|V|^{\frac{1}{2}}} = \begin{pmatrix} 0 & 0 \\ \nabla_{|V|^{\frac{1}{2}}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix},$$

defined on the dense domain $H^{1,V}\left(\mathbb{R}^n\right) \oplus L^2\left(\mathbb{R}^n\right) \oplus L^2\left(\mathbb{R}^n;\mathbb{C}^n\right)$, where $H^{1,V}\left(\mathbb{R}^n\right)$ is as defined in (3.5).

Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ be a matrix-valued multiplication operator and suppose that the Gårding inequalities (3.1) and (3.6) are satisfied with constants $\kappa_A > 0$ and $\kappa_A^V > 0$ respectively. Define our perturbations B_1 and B_2 through

$$B_1 = I \qquad \textit{and} \qquad B_2 := \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} := \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i \cdot \arg V} & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Our perturbed Dirac-type operator then becomes

$$\Pi_{B,|V|^{\frac{1}{2}}} := \Gamma_{|V|^{\frac{1}{2}}} + \Gamma^*_{|V|^{\frac{1}{2}}} \begin{pmatrix} I & 0 \\ 0 & \hat{A} \end{pmatrix} = \begin{pmatrix} 0 & \nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \\ \nabla_{|V|^{\frac{1}{2}}} & 0 \end{pmatrix}.$$

The square of our perturbed Dirac-type operator is then given by

$$\Pi^2_{B,|V|^{\frac{1}{2}}} = \begin{pmatrix} \nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \nabla_{|V|^{\frac{1}{2}}} & 0 \\ 0 & \nabla_{|V|^{\frac{1}{2}}} \nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \end{pmatrix} = \begin{pmatrix} V - \operatorname{div} A \nabla & 0 \\ 0 & \nabla_{|V|^{\frac{1}{2}}} \nabla^*_{|V|^{\frac{1}{2}}} \hat{A} \end{pmatrix}.$$

It is clear from the form of our operator Γ_0 and the fact that A satisfies (3.1) that the operators $\{\Gamma_0, B_1, B_2\}$ satisfy (H1) - (H8). Similarly, since A and V satisfy (3.6), it follows that $\{\Gamma_J, B_1, B_2\}$ satisfy the properties (H1) - (H6).

Lemma 5.3.1. *For $V \in \mathcal{W}_\alpha$, the conditions (H8Dα) and (H8Jα) are both satisfied and $c_\alpha^J \lesssim \left(1 + [V]_\alpha^2\right)(\alpha - 1)^{-1}$.*

Proof. The condition $V \in \mathcal{W}_\alpha$ tells us that

$$\left\| |V|^{\frac{\alpha}{2}} u \right\| + \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\| \leq [V]_\alpha \left\| (|V| - \Delta)^{\frac{\alpha}{2}} u \right\| \tag{5.34}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Let's first prove that (H8Dα) is satisfied. In order to prove this condition, it is sufficient to show that $D\left((|V| - \Delta)^{\frac{\alpha}{2}}\right) \subset D\left((-\Delta)^{\frac{\alpha}{2}}\right)$ and

$$\left\| (-\Delta)^{\frac{\alpha}{2}} u \right\| \leq [V]_\alpha \left\| (|V| - \Delta)^{\frac{\alpha}{2}} u \right\| \tag{5.35}$$

for all $u \in D\left((|V| - \Delta)^{\frac{\alpha}{2}}\right)$. Fix $u \in D\left((|V| - \Delta)^{\frac{\alpha}{2}}\right)$. Since $C_0^\infty(\mathbb{R}^n)$ is a core for $(|V| - \Delta)^{\frac{\alpha}{2}}$, there must exist some $\{u_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ with

$$\|u_n - u\| + \left\| (|V| - \Delta)^{\frac{\alpha}{2}} (u_n - u) \right\| \xrightarrow{n \to \infty} 0. \tag{5.36}$$

We then have for n, $m \in \mathbb{N}$,

$$\|u_n - u_m\| + \left\| (-\Delta)^{\frac{\alpha}{2}} (u_n - u_m) \right\| \leq \|u_n - u_m\| + [V]_\alpha \left\| (|V| - \Delta)^{\frac{\alpha}{2}} (u_n - u_m) \right\|.$$

This proves that $\{u_n\}_{n=1}^{\infty}$ is Cauchy in the graph norm of $(-\Delta)^{\frac{\alpha}{2}}$. The sequence $\{u_n\}_{n=1}^{\infty}$ must therefore must converge to some $\tilde{u} \in D\left((-\Delta)^{\frac{\alpha}{2}}\right)$ in the graph norm of $(-\Delta)^{\frac{\alpha}{2}}$,

$$\|u_n - \tilde{u}\| + \left\|(-\Delta)^{\frac{\alpha}{2}}(u_n - \tilde{u})\right\| \xrightarrow{n\to\infty} 0.$$

This combined with (5.36) shows that $u = \tilde{u}$ and therefore $u \in D\left((-\Delta)^{\frac{\alpha}{2}}\right)$. Moreover, we have that

$$\begin{aligned}\left\|(-\Delta)^{\frac{\alpha}{2}}u\right\| &= \lim_{n\to\infty}\left\|(-\Delta)^{\frac{\alpha}{2}}u_n\right\| \\ &\leq [V]_\alpha \lim_{n\to\infty}\left\|(|V|-\Delta)^{\frac{\alpha}{2}}u_n\right\| \\ &= [V]_\alpha \left\|(|V|-\Delta)^{\frac{\alpha}{2}}u\right\|,\end{aligned}$$

completing the proof of (H8Dα) with $b_\alpha^D \lesssim [V]_\alpha$. An identical proof can be used to obtain the condition (H8Jα) with constant $b_\alpha^J \lesssim [V]_\alpha$. □

Combining the above Lemma with Corollary 5.1.1 completes the proof of Theorem 5.0.1.

5.3.2. The Class $\mathcal{W}$

Define the potential class

$$\mathcal{W} := \bigcup_{\alpha\in(1,2]} \mathcal{W}_\alpha.$$

It has so far been proved that the Kato estimate holds for any potential in the class $\mathcal{W}$ with range contained in some sector $S_{\mu+}$ with $\mu \in [0, \frac{\pi}{2})$. At this stage, however, the unperturbed condition $V \in \mathcal{W}$ remains in quite an abstract form. It will therefore be instructive to unpack this condition and compare $\mathcal{W}$ with other commonly used classes of potentials. It is first interesting to note that $\mathcal{W}_1$ is the collection of all potentials with no additional restrictions.

Proposition 5.3.1. *For any locally integrable $V : \mathbb{R}^n \to \mathbb{C}$ we have $[V]_1 \leq 2$. That is, $\mathcal{W}_1 = L^1_{loc}(\mathbb{R}^n)$.*

Proof. Recall from Proposition 1.3.1 that

$$\left\||V|^{\frac{1}{2}}u\right\|, \left\|(-\Delta)^{\frac{1}{2}}u\right\| \leq \left\|(|V|-\Delta)^{\frac{1}{2}}u\right\|$$

for all $u \in D\left((|V|-\Delta)^{\frac{1}{2}}\right)$. This immediately proves our claim. □

Using this as an endpoint, it can then be proved using an interpolation style argument that the potential classes $\{\mathcal{W}_\alpha\}_{\alpha\in[1,2]}$ form a decreasing collection.

Proposition 5.3.2. *Suppose that the potential $V:\mathbb{R}^n\to\mathbb{C}$ is in $\mathcal{W}_\alpha$ for some $\alpha\in(1,2]$. Then $V\in\mathcal{W}_\beta$ for any $\beta\in[1,\alpha]$ with*

$$[V]_\beta \le 2\,[V]_\alpha^{\frac{\beta-1}{\alpha-1}}. \tag{5.37}$$

Therefore for $1\le\beta\le\alpha$

$$\mathcal{W}_\beta \supset \mathcal{W}_\alpha.$$

Proof. Assume that $V\in\mathcal{W}_\alpha$ for some $\alpha\in(1,2]$. It will be shown through the Hadamard three-lines theorem that $V\in\mathcal{W}_\beta$ for $\beta\in[1,\alpha]$ with the constant given in (5.37).

Let $St:=\{z\in\mathbb{C}:0<\operatorname{Re}z<1\}$ and set $\theta:=\frac{\beta-1}{\alpha-1}$. Fix $u\in C_0^\infty(\mathbb{R}^n)$ and define $f:\overline{St}\to\mathbb{R}$ to be the function given by

$$f(z):=\left\||V|^{\frac12+\left(\frac{\alpha-1}{2}\right)z}\left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-z)}u\right\|.$$

f is holomorphic on St and continuous on the closed strip $\overline{St}$. In order to apply the three-lines theorem, it must first be proved that f is bounded on $\overline{St}$. For $z=s+it$ where $0\le s\le1$ and $t\in\mathbb{R}$ we have

$$\begin{aligned} f(z)&=\left\||V|^{\frac12+\left(\frac{\alpha-1}{2}\right)(s+it)}\left(|V|-\Delta\right)^{\frac{\beta-1}{2}-\left(\frac{\alpha-1}{2}\right)(s+it)}u\right\|\\ &=\left\||V|^{\frac12+\left(\frac{\alpha-1}{2}\right)s}\left(|V|-\Delta\right)^{-\frac12-\left(\frac{\alpha-1}{2}\right)s}\left(|V|-\Delta\right)^{-\left(\frac{\alpha-1}{2}\right)it}\left(|V|-\Delta\right)^{\frac{\beta}{2}}u\right\|. \end{aligned} \tag{5.38}$$

The function

$$s\mapsto\left\||V|^{\frac12+\left(\frac{\alpha-1}{2}\right)s}\left(|V|-\Delta\right)^{-\frac12-\left(\frac{\alpha-1}{2}\right)s}v\right\|$$

is continuous on $[0,1]$ for $v\in L^2(\mathbb{R}^n)$. This, together with (5.38) then gives

$$\begin{aligned} f(z)&\lesssim\left\|\left(|V|-\Delta\right)^{-\left(\frac{\alpha-1}{2}\right)it}\left(|V|-\Delta\right)^{\frac{\beta}{2}}u\right\|\\ &\le\left\|\left(|V|-\Delta\right)^{\frac{\beta}{2}}u\right\|, \end{aligned}$$

where we used the fact that $\left(|V|-\Delta\right)^{ia}$ is a contraction operator on $L^2(\mathbb{R}^n)$ for $a\in\mathbb{R}$. This demonstrates that f is bounded on $\overline{St}$.

For $t\in\mathbb{R}$, Proposition 1.3.1 implies that

$$\begin{aligned} f(it)&=\left\||V|^{\frac12+\left(\frac{\alpha-1}{2}\right)it}\left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-it)}u\right\|\\ &=\left\||V|^{\frac12}\left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-it)}u\right\|\\ &\le\left\|\left(|V|-\Delta\right)^{\frac12}\left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-it)}u\right\|\\ &\le\left\|\left(|V|-\Delta\right)^{\frac{\beta}{2}}u\right\|. \end{aligned}$$

We also have

$$\begin{aligned}
f(1+it) &= \left\| |V|^{\frac{\alpha}{2}+\left(\frac{\alpha-1}{2}\right)it} \left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-1-it)} u \right\| \\
&= \left\| |V|^{\frac{\alpha}{2}} \left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-1-it)} u \right\| \\
&\leq [V]_\alpha \left\| \left(|V|-\Delta\right)^{\frac{\alpha}{2}} \left(|V|-\Delta\right)^{\left(\frac{\alpha-1}{2}\right)(\theta-1-it)} u \right\| \\
&\leq [V]_\alpha \left\| \left(|V|-\Delta\right)^{\frac{\alpha}{2}+\left(\frac{\alpha-1}{2}\right)(\theta-1)} u \right\| \\
&= [V]_\alpha \left\| \left(|V|-\Delta\right)^{\frac{\beta}{2}} u \right\|.
\end{aligned}$$

The Hadamard three-lines theorem then gives the bound

$$f(\theta) = \left\| |V|^{\frac{\beta}{2}} u \right\| \leq [V]_\alpha^{\left(\frac{\beta-1}{\alpha-1}\right)} \left\| \left(|V|-\Delta\right)^{\frac{\beta}{2}} u \right\|. \tag{5.39}$$

A similar argument can be applied to obtain the bound

$$\left\| (-\Delta)^{\frac{\beta}{2}} u \right\| \leq [V]_\alpha^{\left(\frac{\beta-1}{\alpha-1}\right)} \left\| \left(|V|-\Delta\right)^{\frac{\beta}{2}} u \right\| \tag{5.40}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, one must simply remember that the imaginary powers of the positive self-adjoint operator $(-\Delta)$ are contraction operators on $L^2(\mathbb{R}^n)$. Combining (5.39) and (5.40) then gives $[V]_\beta \leq 2\,[V]_\alpha^{\frac{\beta-1}{\alpha-1}}$. □

Recall the definition of the reverse Hölder class of potentials from §1.3. The following theorem amounts to a simple restatement of Theorem 1.3.4.

Theorem 5.3.1 ([50], [6]). *Let $V \in L^1_{loc}(\mathbb{R}^n)$ and suppose that $|V| \in RH_q$ for some $q \geq 2$. Then $V \in \mathcal{W}_2 \subset \mathcal{W}$.*

On recalling Example 1.3.1 we then immediately obtain the following corollary.

Corollary 5.3.1. *For any polynomial P, we have $P \in \mathcal{W}_2$. In particular, the Kato estimate holds for any polynomial with range contained in $S_{\mu+}$ for some $\mu \in [0, \frac{\pi}{2})$.*

The ensuing proposition demonstrates that the inclusion of the reverse Hölder potentials in $\mathcal{W}_2$ is strict, at least in dimension $n > 4$.

Proposition 5.3.3. *For $n > 4$,*

$$L^{\frac{n}{2}}(\mathbb{R}^n) \subset \mathcal{W}_2.$$

PROOF. Fix $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Hölder's inequality gives us

$$\left\| |V| \left(|V| - \Delta\right)^{-1} u \right\|_2 \leq \|V\|_{\frac{n}{2}} \cdot \left\| \left(|V| - \Delta\right)^{-1} u \right\|_{\frac{2n}{n-4}}.$$

It is well-known that the Riesz potential $(-\Delta)^{-1}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^{\frac{2n}{n-4}}(\mathbb{R}^n)$. Since the kernel of $(|V| - \Delta)^{-1}$ is pointwise bounded from above by the kernel of $(-\Delta)^{-1}$, it follows that $(|V| - \Delta)^{-1}$ is also bounded from $L^2(\mathbb{R}^n)$ to $L^{\frac{2n}{n-4}}(\mathbb{R}^n)$. There must then exist some $C_V > 0$ for which

$$\left\| |V| \left(|V| - \Delta\right)^{-1} u \right\|_2 \leq \|V\|_{\frac{n}{2}} \cdot C_V \cdot \|u\|_2$$

The bound $\left\| \Delta \left(|V| - \Delta\right)^{-1} \right\| < \infty$ then follows simply from the triangle inequality. $\square$

The above statements demonstrate clearly that the class of potentials $\mathcal{W}_2$ is quite large. In light of Proposition 5.3.2, however, it is also evident that $\mathcal{W}_2$ is the smallest class out of the collection $\{\mathcal{W}_\alpha\}_{\alpha \in (1,2]}$. One can then neatly surmise that $\mathcal{W}$, the class of all potentials for which the Kato estimate holds, is certain to be quite large itself.

This chapter has so far proved that the Kato estimate is valid for any scalar zero-order potential in the class $\mathcal{W}$ that is also μ-sectorial with $\mu \in [0, \frac{\pi}{2})$. In effect we have reduced a perturbed Kato type estimate with potential to the unperturbed condition

$$\left\| |V|^{\frac{\alpha}{2}} u \right\| + \left\| (-\Delta)^{\frac{\alpha}{2}} \right\| \lesssim c_V^\alpha \left\| \left(|V| - \Delta\right)^{\frac{\alpha}{2}} u \right\| \tag{5.41}$$

for some $c_V^\alpha > 0$, for all $u \in C_0^\infty(\mathbb{R}^n)$ and for some $\alpha \in (1, 2]$. Worded differently, we have reduced a perturbed Riesz transform type estimate to an unperturbed higher-order Riesz transform estimate for the Schrödinger operator. In one sense, such a reduction seems quite natural since, intuitively, it is unlikely that a perturbed Riesz transform estimate will hold without an associated unperturbed Riesz transform estimate holding. However, in another sense, it could still be possible to improve upon this condition so that the Kato estimate is given by (5.41) when $\alpha = 1$ so that the order matches the order of the Kato estimate. As shown in Proposition 1.3.1, (5.41) is in fact valid for any potential $V : \mathbb{R}^n \to \mathbb{C}$ for $\alpha = 1$. Heuristically speaking, we might then expect the Kato estimate to be satisfied for any μ-sectorial potential with $\mu \in [0, \frac{\pi}{2})$.

One possible route to such a result through our current work would be if our potential class $\mathcal{W}$ is already the class of all potentials. That is, $\mathcal{W} = \mathcal{W}_1$. A sufficient condition for this to occur is if the Riesz transform condition (5.41) is open-ended. This tautological implication is stated in the following proposition.

Proposition 5.3.4. *Suppose that for any potential V there exists some $\varepsilon_V \in (0, 1]$ for* *which* (5.41) *is satisfied for* $\alpha = 1 + \varepsilon_V$. *Then* $\mathcal{W} = \mathcal{W}_1$ *will consist of all potentials and the Kato estimate will be satisfied by any* μ*-sectorial potential with* $\mu \in \left[0, \frac{\pi}{2}\right)$.

The proof of such an openness condition is unfortunately beyond the scope of this thesis.

5.3.3. Systems with Zero-Order Potential

Fix $m \in \mathbb{N}^*$ and $A \in L^\infty\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^n \otimes \mathbb{C}^m\right)\right)$. Let $V : \mathbb{R}^n \to \mathcal{L}\left(\mathbb{C}^m\right)$ be a measurable matrix-valued function contained in $L^1_{loc}\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^m\right)\right)$. V can be viewed as a densely defined closed multiplication operator on $L^2\left(\mathbb{R}^n; \mathbb{C}^m\right)$ with domain

$$D(V) = \left\{u \in L^2\left(\mathbb{R}^n; \mathbb{C}^m\right) : V \cdot u \in L^2\left(\mathbb{R}^n; \mathbb{C}^m\right)\right\}.$$

Similar to the scalar case, one can define forms $\mathfrak{l}^A$ and $\mathfrak{l}^A_V$ defined respectively through

$$\mathfrak{l}^A[u, v] := \int_{\mathbb{R}^n} \langle A(x) \nabla u(x), \nabla v(x) \rangle \, dx$$

for u, $v \in H^1(\mathbb{R}^n)$ and

$$\mathfrak{l}^A_V[u', v'] := \mathfrak{l}^A[u', v'] + \int_{\mathbb{R}^n} \langle V(x) u'(x), v'(x) \rangle \, dx$$

for u' and v' contained in

$$H^{1,V}\left(\mathbb{R}^n; \mathbb{C}^m\right) := H^1\left(\mathbb{R}^n; \mathbb{C}^m\right) \cap D(|V|^{\frac{1}{2}}),$$

where $|V(x)|^{\frac{1}{2}} := \sqrt{V(x)^* V(x)}$ for $x \in \mathbb{R}^n$. The density of $H^{1,V}\left(\mathbb{R}^n; \mathbb{C}^m\right)$ in $L^2\left(\mathbb{R}^n; \mathbb{C}^m\right)$ follows from the fact that $C^\infty_0\left(\mathbb{R}^n; \mathbb{C}^m\right) \subset H^{1,V}\left(\mathbb{R}^n; \mathbb{C}^m\right)$. Assume that the forms $\mathfrak{l}^A$ and $\mathfrak{l}^A_V$ satisfy the Gårding inequalities (3.1) and (3.6) with constants $\kappa_A > 0$ and $\kappa^V_A > 0$ respectively. Then $\mathfrak{l}^A$ and $\mathfrak{l}^A_V$ will both have a unique associated maximal accretive operator, L and $L + V$. Define $[V]_\alpha$ and $\mathcal{W}_\alpha$ for $\alpha \in [1, 2]$ to be the system analogues of the corresponding scalar objects,

$$[V]_\alpha := \sup_{u \in C^\infty_0(\mathbb{R}^n; \mathbb{C}^m)} \frac{\left\| |V|^{\frac{\alpha}{2}} u \right\| + \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|}{\left\| (|V| - \Delta)^{\frac{\alpha}{2}} u \right\|} \quad \textit{and}$$

$$\mathcal{W}_\alpha := \left\{V \in L^1_{loc}\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^m\right)\right) : [V]_\alpha < \infty\right\}.$$

In the below theorem, our non-homogeneous framework will be applied to determine the domain of $\sqrt{L + V}$ for normal potentials.

Theorem 5.3.2. *Suppose that* $V(x)$ *is a normal matrix for almost every* $x \in \mathbb{R}^n$ *and*

$V \in \mathcal{W}_\alpha$ for some $\alpha \in (1,2]$. Suppose that the Gårding inequalities (3.1) and (3.6) are both satisfied with constants $\kappa_A > 0$ and $\kappa_A^V > 0$ respectively. Then there must exist some $C_V > 0$ such that

$$C_V^{-1}\left(\left\||V|^{\frac{1}{2}}u\right\| + \|\nabla u\|\right) \leq \left\|\sqrt{L+V}u\right\| \leq C_V\left(\left\||V|^{\frac{1}{2}}u\right\| + \|\nabla u\|\right)$$

for all $u \in D(L+V)$. Moreover, the constant C_V depends on V and α through

$$C_V = \tilde{C}_V(\alpha-1)^{-1}\left(1+[V]_\alpha^2\right),$$

where $\tilde{C}_V$ only depends on V through κ_A^V and is independent of α.

Proof. The polar decomposition theorem asserts the existence of some $U : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^m)$, with $U(x)$ unitary for all $x \in \mathbb{R}^n$, such that

$$V(x) = U(x)\,|V(x)|$$

for all $x \in \mathbb{R}^n$. As $V(x)$ is normal, the matrices $U(x)$ and $|V(x)|$ are well-known to commute. We therefore have the decomposition

$$V(x) = |V(x)|^{\frac{1}{2}}\,U(x)\,|V(x)|^{\frac{1}{2}} \tag{5.42}$$

for almost every $x \in \mathbb{R}^n$. Set

$$D := \nabla : H^1(\mathbb{R}^n;\mathbb{C}^m) \subset L^2(\mathbb{R}^n;\mathbb{C}^m) \to L^2(\mathbb{R}^n;\mathbb{C}^n \otimes \mathbb{C}^m)$$

and

$$J := |V|^{\frac{1}{2}} : D(|V|^{\frac{1}{2}}) \subset L^2(\mathbb{R}^n;\mathbb{C}^m) \to L^2(\mathbb{R}^n;\mathbb{C}^m),$$

both defined as operators on $L^2(\mathbb{R}^n;\mathbb{C}^m)$. Define the perturbation matrices

$$B_1 := I \qquad \textit{and} \qquad B_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & A \end{pmatrix}.$$

It is not too difficult to see that the operators $\{\Gamma_0, B_1, B_2\}$ will satisfy conditions (H1) - (H8) and $\{\Gamma_J, B_1, B_2\}$ will satisfy (H1) - (H6). Indeed, the only non-trivial condition for both sets of operators is (H2) and this follows from the respective Gårding inequalities (3.1) and (3.6). It is also clear from the fact that $V \in \mathcal{W}_\alpha$ that (H8Dα) and (H8Jα) will both be satisfied with $c_\alpha^J \lesssim (1+[V]_\alpha^2)(\alpha-1)^{-1}$. This follows in an identical manner to Lemma 5.3.1. The Kato estimate then follows from Corollary 5.1.1 with constant $\tilde{C}_V(\alpha-1)^{-1}\left(1+[V]_\alpha^2\right)$. It should be noted that (5.42) was needed so that we would have $L_B^J = L + V$. □

In analogy to the scalar case, it is quite likely that a similar reverse Hölder type condition will be sufficient to imply the boundedness of the operator $|V|\left(|V| - \Delta\right)^{-1}$ for $m > 1$. However, as far as the author is aware, this is still an open problem for $m > 1$. What is apparent is that the condition that the potential belongs to $L^{\frac{n}{2}}$ will once again be sufficient to imply that it belongs to $\mathcal{W}_2$. The following proposition has an identical proof to that of Proposition 5.3.3.

Proposition 5.3.5. *For $n > 4$,*

$$L^{\frac{n}{2}}\left(\mathbb{R}^n; \mathcal{L}\left(\mathbb{C}^m\right)\right) \subset \mathcal{W}_2.$$

5.3.4. First-Order Potentials

Let $b : \mathbb{R}^n \to \mathbb{C}^n$ be contained in $L^1_{loc}\left(\mathbb{R}^n; \mathbb{C}^n\right)$ and $A \in L^\infty\left(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n)\right)$. Define the Hilbert space to be

$$\mathcal{H} := L^2\left(\mathbb{R}^n\right) \oplus L^2\left(\mathbb{R}^n; \mathbb{C}^n\right) \oplus L^2\left(\mathbb{R}^n; \mathbb{C}^n\right).$$

Then set

$$J := \nabla + b : L^2\left(\mathbb{R}^n; \mathbb{C}\right) \to L^2\left(\mathbb{R}^n; \mathbb{C}^n\right) \quad \textit{and} \quad D := \nabla : L^2\left(\mathbb{R}^n; \mathbb{C}\right) \to L^2\left(\mathbb{R}^n; \mathbb{C}^n\right).$$

Also let $B_1 = I$ as usual and

$$B_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A \end{pmatrix}.$$

The operator L_B^J as in Corollary 5.1.1 is then given by

$$\mathcal{L}_A^b := L_{J,B} = (b + \nabla)^* (b + \nabla) - \mathrm{div} A \nabla.$$

Suppose that A satisfies the standard Gårding inequality

$$\mathrm{Re} \int_{\mathbb{R}^n} \langle A(x) \nabla u(x), \nabla u(x) \rangle \, dx \geq \kappa \cdot \|\nabla u\|^2$$

for all $u \in H^1\left(\mathbb{R}^n; \mathbb{C}\right)$, for some $\kappa > 0$. Then $\{\Gamma_0, B_1, B_2\}$ and $\{\Gamma_J, B_1, B_2\}$ will both satisfy (H2). This in turn implies that $\{\Gamma_0, B_1, B_2\}$ satisfies (H1) - (H8) and $\{\Gamma_J, B_1, B_2\}$ satisfies (H1) - (H6). The non-homogeneous framework, in the form of Corollary 5.1.1, applied to these operators then produces the following theorem.

Theorem 5.3.3. *Suppose that* $D\left((\nabla+b)^*(\nabla+b)-\Delta\right)\subset D(-\Delta)$ *and there exists some* $c_b>0$ *such that*

$$\|\Delta u\|\leq c_b\left\|\left[(\nabla+b)^*(\nabla+b)-\Delta\right]u\right\| \tag{5.43}$$

for all $u\in D\left((\nabla+b)^*(\nabla+b)-\Delta\right)$. *Then there exists some constant* $c>0$, *independent of* b, *for which*

$$(c\cdot(1+c_b))^{-2}\left(\|(\nabla+b)\,u\|+\|\nabla u\|\right)\leq\left\|\sqrt{\mathcal{L}_A^b}u\right\|\leq(c\cdot(1+c_b))^2\left(\|(\nabla+b)\,u\|+\|\nabla u\|\right)$$

for all $u\in D\left(\mathcal{L}_A^b\right)$.

To see that the above theorem is true, simply note that (5.43) implies both (H8Dα) and (H8Jα) for $\alpha=1$ in this context. The independence of the constant c from b follows from the fact that (H2) is satisfied by $\{\Gamma_J,B_1,B_2\}$ with constant independent of b.

Chapter 6

Final Remarks

6.1. Literature Comparison

It is important to note that this is not the first time that Kato type estimates have been studied for non-homogeneous operators. We will now take some time to outline how our work differs in technique and results from each of these previous forays.

Recently, in [28] and [29], F. Gesztesy, S. Hofmann and R. Nichols studied the domain of the square root operator $\sqrt{L+V}$ using techniques distinct from those developed in [11]. The article [28] considers potentials in the class $L^p + L^\infty$ but is not directly relevant since it considers bounded domains. On the other hand, [29] does not impose a boundedness assumption on the domain and considers the potential class $L^{\frac{n}{2}} + L^\infty$. There is already an immediate comparison with our potential class $\mathcal{W}$ from Chapter 5 since it was shown in Proposition 5.3.3 that $L^{\frac{n}{2}} \subset \mathcal{W}$ in dimension $n > 4$.

Axelsson, Keith and McIntosh themselves considered non-homogeneous operators on Lipschitz domains with mixed boundary conditions in [10]. The potentials that they considered were, however, bounded both from above and below and thus contained in $RH_2 \subset \mathcal{W}$. In [22] and [23], M. Egert, R. Haller-Dintelmann and P. Tolksdorf generalised this to certain non-smooth domains.

The articles [10], [23] and [22] are all built upon the original AKM framework, similar to this part of the thesis. A key step in the original proof of the AKM framework is the proof of the estimate

$$\int_0^\infty \|(A_t - P_t)\, u\|^2 \, \frac{dt}{t} \lesssim \|u\|^2 . \tag{6.1}$$

This estimate allows for the A_t and P_t operators to be freely interchanged at several stages in the proof granting use of some of the more enviable properties of the A_t operator. As has been demonstrated in this thesis through the diagonalisation theorem, Theorem 5.2.1, (6.1) will not hold for a general potential. The articles [10], [23] and [22] circumvent this problem by imposing boundedness of the potential from above and below. In the language of our setup, the boundedness of the potential from below allows one to reduce the main square function estimate to the local square function estimate

$$\int_0^1 \left\| \Theta^{B,|V|^{\frac{1}{2}}} P_t^{|V|^{\frac{1}{2}}} u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for all $u \in R\left(\Gamma_{|V|^{\frac{1}{2}}}\right)$. Then one only requires a local version of (6.1) to hold, namely

$$\int_0^1 \left\| \left(A_t - P_t^{|V|^{\frac{1}{2}}} \right) u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for all $u \in R\left(\Gamma_{|V|^{\frac{1}{2}}}\right)$. Such an estimate will be true for any potential bounded from above. This will be proved in the section to follow.

This is a crude explanation as to why the techniques developed in [10] cannot be directly applied for a general potential that is not bounded both from above and below. There are similar obstructions, for example, in the selection of test functions in the Carleson measure proof. However, these also disappear when the potential is bounded both from above and below.

In this thesis, we have applied two different methods to solve this problem. Each method has its own advantages and disadvantages. The first method is closer to [10] in spirit. In [10] the usual cancellation condition from potential free AKM is swapped with the following cancellation condition.

(H7*) There exists $c_V > 0$ such that

$$\left| \int_{\mathbb{R}^n} \Gamma_{|V|^{\frac{1}{2}}} u \right| \leq c_V \, |Q|^{\frac{1}{2}} \cdot \|u\|_{L^2(Q)} \quad \textit{and} \quad \left| \int_{\mathbb{R}^n} \Gamma^*_{|V|^{\frac{1}{2}}} v \right| \leq c_V \, |Q|^{\frac{1}{2}} \cdot \|v\|_{L^2(Q)}$$

for every cube Q in $\mathbb{R}^n$, $u \in D\left(\Gamma_{|V|^{\frac{1}{2}}}\right)$ and $v \in D\left(\Gamma^*_{|V|^{\frac{1}{2}}}\right)$ both with compact support in Q.

Our condition (H7V) from Chapter 4 is quite similar in nature to the above condition. Indeed, if the potential is bounded from above then (H7V) will imply (H7*). Our method in this chapter has been to hopefully let the non-tangential maximal functions $\mathcal{N}_1$ and $\mathcal{N}_2$ compensate for this weakening of the cancellation condition and the failure of the square function equivalence between A_t and $P_t^{|V|^{\frac{1}{2}}}$.

In Chapter 5 on the other hand, our method has been quite distinct to [10]. Instead of introducing a potential dependent cancellation and coercivity condition we instead exploit the algebraic structure of the operators $\Gamma_{|V|^{\frac{1}{2}}}$, B_1 and B_2. This exploitation allowed us to conclude that the estimate (6.1) will at least hold on the third component. Similar obstructions in the proof of the main square function estimate also vanish when considered component-wise. As a consequence of this three-by-three mindset we have been able to obtain square function estimates for potentials that aren't necessarily bounded from above or below and, moreover, are not contained in $L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$ (cf. Corollary 5.3.1).

6.2. Miscellaneous Square Function Estimates

In this section, we include some square function estimates that were not directly required for the proof of the main square function estimates in the previous chapters, but are still interesting in their own right.

6.2.1. Local SFE for Bounded Potential

It will now be shown that for bounded potentials the $P_t^{|V|^{\frac{1}{2}}}$ operators can be approximated by the A_t operators locally on the interval $[0, 1]$.

Proposition 6.2.1. *Suppose that (H8Dα) is satisfied for some $\alpha \in (1, 2]$. Then*

$$\int_0^\infty \left\| \left(P_t^0 - I\right) P_t^J u \right\|^2 \frac{dt}{t} \lesssim (\alpha - 1)^{-1} \left(b_\alpha^D\right)^2 \cdot \|u\|^2 \tag{6.2}$$

for all $u \in R(\Gamma_J)$.

Proof. This is essentially just Proposition 5.2.3 in disguise. Indeed the third component,

$$\int_0^\infty \left\| \mathbb{P}_3 \left(P_t^0 - I\right) P_t^J u \right\|^2 \frac{dt}{t} \lesssim (\alpha - 1)^{-1} \left(b_\alpha^D\right)^2 \cdot \|u\|^2,$$

follows from Proposition 5.2.3 since $\mathbb{P}_3 \left(P_t^0 - I\right) = \mathbb{P}_3 \left(\mathcal{P}_t^J - I\right)$. The first and second components are trivial for $u \in R(\Gamma_J)$. $\square$

Proposition 6.2.2. *Suppose that J is a bounded operator. Then*

$$\int_0^1 \left\| P_t^0 \left(I - P_t^J\right) u \right\|^2 \frac{dt}{t} \lesssim \left(1 + \|J\|^2\right) \cdot \|u\|^2 \tag{6.3}$$

for all $u \in R(\Gamma_J)$.

PROOF. We have

$$\begin{aligned}\left\|P_t^0\left(I-P_t^J\right)u\right\| &\simeq \left\|P_t^0\int_0^t \left(Q_s^J\right)^2 u\frac{ds}{s}\right\| \\ &\lesssim \int_0^t \left\|P_t^0\left(Q_s^J\right)^2 u\right\| \frac{ds}{s}.\end{aligned}$$

Also note that since $u\in R\left(\Gamma_J\right)$,

$$P_t^0\left(Q_s^J\right)^2 u = P_t^0 s\Pi_J \mathbb{P}_1 P_s^J Q_s^J u.$$

Therefore

$$\begin{aligned}\int_0^1 \left\|P_t^0\left(I-P_t^J\right)u\right\|^2\frac{dt}{t} &\lesssim \int_0^1\left(\int_0^t \left\|P_t^0\left(Q_s^J\right)^2 u\right\|\frac{ds}{s}\right)^2\frac{dt}{t} \\ &\lesssim \int_0^1\left(\int_0^t \left\|P_t^0 s\Pi_0\mathbb{P}_1 P_s^J Q_s^J u\right\|\frac{ds}{s}\right)^2\frac{dt}{t} \\ &\quad+\int_0^1\left(\int_0^t \left\|P_t^0 sS_J\mathbb{P}_1 P_s^J Q_s^J u\right\|\frac{ds}{s}\right)^2\frac{dt}{t}.\end{aligned}\tag{6.4}$$

For the first term,

$$\begin{aligned}\int_0^1\left(\int_0^t \left\|P_t^0 s\Pi_0\mathbb{P}_1 P_s^J Q_s^J u\right\|\frac{ds}{s}\right)^2\frac{dt}{t} &\lesssim \int_0^1\int_0^t \left\|P_t^0 \Pi_0\mathbb{P}_1 P_s^J Q_s^J u\right\|^2 ds\,dt \\ &= \int_0^1\int_0^t st\left\|P_t^0 \Pi_0\mathbb{P}_1 P_s^J Q_s^J u\right\|^2 \frac{ds}{s}\frac{dt}{t} \\ &\lesssim \int_0^1\int_0^1 \left\|Q_t^0 \mathbb{P}_1 P_s^J Q_s^J u\right\|^2 \frac{dt}{t}\frac{ds}{s}.\end{aligned}$$

On successively applying the square function estimates for Q_t^0, the uniform L^2-boundedness of P_s^J and the square function estimates for Q_s^J,

$$\begin{aligned}\int_0^1\left(\int_0^t \left\|P_t^0 s\Pi_0\mathbb{P}_1 P_s^J Q_s^J\right\|\frac{ds}{s}\right)^2\frac{dt}{t} &\lesssim \int_0^1 \left\|P_s^J Q_s^J u\right\|^2\frac{ds}{s} \\ &\lesssim \int_0^1 \left\|Q_s^J u\right\|^2\frac{ds}{s} \\ &\lesssim \|u\|^2.\end{aligned}$$

For the second term in (6.4), on applying the uniform boundedness of P_t^0 and the boundedness of S_J,

$$\begin{aligned}\int_0^1 \left(\int_0^t \left\| P_s^0 s S_J \mathbb{P}_1 P_s^J Q_s^J u \right\| \frac{ds}{s} \right)^2 \frac{dt}{t} &\lesssim \int_0^1 \left(\int_0^t \left\| s S_J \mathbb{P}_1 P_s^J Q_s^J u \right\| \frac{ds}{s} \right)^2 \frac{dt}{t} \\ &\lesssim \|J\|^2 \int_0^1 \left(\int_0^t \left\| P_s^J Q_s^J u \right\| ds \right)^2 \frac{dt}{t} \\ &\lesssim \|J\|^2 \int_0^1 \int_0^t \left\| P_s^J Q_s^J u \right\|^2 ds\, dt \\ &\lesssim \|J\|^2 \int_0^\infty \left\| Q_s^J u \right\|^2 \frac{ds}{s} \\ &\lesssim \|J\|^2 \cdot \|u\|^2 .\end{aligned}$$

□

The below corollary then follows as a direct result of the previous two propositions and the triangle inequality.

Corollary 6.2.1. *Suppose that $V \in \mathcal{W}_\alpha \cap L^\infty(\mathbb{R}^n)$ for some $\alpha \in (1, 2]$. Then*

$$\int_0^1 \left\| \left(A_t - P_t^{|V|^{\frac{1}{2}}} \right) u \right\|^2 \frac{dt}{t} \lesssim \left(1 + \|V\|_\infty + \left(b_\alpha^D \right)^2 \cdot (\alpha - 1)^{-1} \right) \cdot \|u\|^2$$

for all $u \in R\left(\Gamma_{|V|^{\frac{1}{2}}} \right)$.

6.2.2. A Candidate for the A_t^V Operators

Recall from §1.3 that for reverse Hölder potentials a form of uncertainty principle holds through the Fefferman-Phong lemma. Specifically, the critical radius function provides a lower bound on the sum of the kinetic and potential energy of a state. This concept is generalised in the below definition.

Definition 6.2.1. *Let $V \in L^1_{loc}(\mathbb{R}^n)$ be a non-negative potential. A non-negative function $E : \mathbb{R}^n \to \mathbb{R}$ is said to be an effective potential for V if the inequality*

$$\int_{\mathbb{R}^n} E\, |u|^2 \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^2 + V\, |u|^2 \, dx \tag{6.5}$$

holds for all $u \in H^{1,V}(\mathbb{R}^n)$.

Example 6.2.1. The functions $E \equiv 0$ and $E \equiv V$ both provide trivial examples of effective potentials. The critical radius function for $V \in RH_{\frac{n}{2}}$ provides a non-trivial example of an effective potential through the Fefferman-Phong lemma.

Ideally, the effective potential should take into account not only the energy coming from the potential but also the kinetic term. It then follows that the larger the effective potential E, the closer to optimal it will become.

Effective potentials are featured in the theory of S. Agmon (cf. [2]), used to determine the localization of eigenfunctions of Schrödinger operators. For an effective potential E, Agmon's method can be used to show that an eigenfunction of $\mathcal{L}_V$ with value λ, $\mathcal{L}_V \psi = \lambda \psi$, will have most of their mass confined to a region

$$\{x \in \mathbb{R}^n : E(x) \leq \lambda + \delta\}$$

for some small $\delta > 0$ and decay exponentially outside of this region.

For an ordered potential such as the harmonic oscillator, the previous phenomenon for $E \equiv V$ essentially captures the physical process of trapping a state in a quantum well. For disordered potentials, it has been known since the 50's that localization will still occur but the mathematical reasoning behind this process was shrouded in mystery. Such localization for disordered potentials is known in the Physics community as Anderson localization. Only very recently, in the papers [25] and [4], was a sound mathematical explanation discovered. In these articles an effective potential is introduced for a disordered potential that takes into account the kinetic term and accurately describes Anderson localization. Specifically, the following theorem is proved.

Theorem 6.2.1 (The Landscape Function, [4])**.** *Let Ω be the bi-Lipschitz image of a C^∞ bounded domain in $\mathbb{R}^n$ and $M = \overline{\Omega}$. Let $V \in L^\infty(M)$. The solution to $\mathcal{L}_V u = 1$ on M exists and is unique. This function, called the landscape function, satisfies the uncertainty principle*

$$\int_M \left(|\nabla f|^2 + V|f|^2\right) dx = \int_M \left(u^2 \left|\nabla\left(\frac{f}{u}\right)\right|^2 + \frac{1}{u}|f|^2\right) dx$$

for all $f \in W^{1,2}(M)$.

Recall from the previous chapter that for $V \in \mathcal{W}$ the operators $P_t^{|V|^{\frac{1}{2}}}$ can effectively be diagonalised when considering square function estimates. Moreover, the second and third components of the diagonalisation of this operator consist of only one form of derivative, namely

$$\left(\mathcal{P}_t^{|V|^{\frac{1}{2}}} u\right)_2 = \left(I + t^2 V\right)^{-1} u_2 \quad \textit{and} \quad \left(\mathcal{P}_t^{|V|^{\frac{1}{2}}} u\right)_3 = \left(I - t^2 \nabla \mathrm{div}\right)^{-1} u_3.$$

The first component

$$\left(\mathcal{P}_t^{|V|^{\frac{1}{2}}} u\right)_1 = \left(I + t^2 (V - \Delta)\right)^{-1} u_1$$

is more problematic since the two forms of derivatives become mixed. We now introduce a potential dependent averaging type operator that can be swapped for this first component for certain potentials.

Fix non-negative potential $V : \mathbb{R}^n \to \mathbb{R}$ and let E be an effective potential for V. For each $t > 0$, define the operator

$$A_t^E f(x) := \fint_{Q(t,x)} \frac{f(y)}{1 + tE^{\frac{1}{2}}(y)}\, dy$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. It will be shown that under certain restrictions on the potential, A_t^E can be used as a square function approximation for the first component of $P_t^{|V|^{\frac{1}{2}}}$.

Proposition 6.2.3. *For all $u \in \mathcal{H}$,*

$$\int_0^\infty \left\| \mathbb{P}_1 \left(I - A_t^E\right) P_t^{|V|^{\frac{1}{2}}} u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 .$$

Proof. Since $\left(P_t^{|V|^{\frac{1}{2}}} u\right)_1 = \left(I + t^2 (V - \Delta)\right)^{-1} u_1$, this is the same as showing

$$\int_0^\infty \left\| \left(I - A_t^E\right) \left(I + t^2 (V - \Delta)\right)^{-1} u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for $u \in L^2(\mathbb{R}^n)$. It should first be noted that the estimate

$$\left\| (I - A_t) \left(I + t^2 (V - \Delta)\right)^{-1} u \right\| \lesssim \left\| t\nabla \left(I + t^2 (V - \Delta)\right)^{-1} u \right\|$$

holds through an application of Poincaré's inequality as in the proof of Proposition 5.5 of [11]. On applying the estimate $\|\nabla u\| \leq \left\| (V - \Delta)^{\frac{1}{2}} u \right\|$, one then obtains

$$\int_0^\infty \left\| (I - A_t) \left(I + t^2 (V - \Delta)\right)^{-1} u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| t (V - \Delta)^{\frac{1}{2}} \left(I + t^2 (V - \Delta)\right)^{-1} u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 .$$

This reduces the proof of our proposition to proving the square function estimate

$$\int_0^\infty \left\| \left(A_t - A_t^E\right) \left(I + t^2 (V - \Delta)\right)^{-1} u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2 .$$

We have for any $v \in H^{1,V}(\mathbb{R}^n)$,

$$\begin{aligned}\left\|\left(A_t - A_t^E\right)v\right\|^2 &= \sum_{Q\in\Delta_t} |Q| \left| \fint_Q v(y)\,dy - \fint_Q \frac{v(y)}{1+tE^{\frac{1}{2}}(y)}\,dy \right|^2 \\ &= \sum_{Q\in\Delta_t} \frac{1}{|Q|} \left| \int_Q \frac{tE^{\frac{1}{2}}(y)v(y)}{1+tE^{\frac{1}{2}}(y)}\,dy \right|^2 \\ &\lesssim \sum_{Q\in\Delta_t} \int_Q \left|tE^{\frac{1}{2}}(y)v(y)\right|^2 dy \\ &\lesssim \left\|tE^{\frac{1}{2}}v\right\| \\ &\lesssim \left\|t\left(V-\Delta\right)^{\frac{1}{2}}v\right\|,\end{aligned}$$

where the last line was obtained from the uncertainty principle for our effective potential (6.5). Applying this to our square function norm gives

$$\begin{aligned}\int_0^\infty \left\|\left(A_t - A_t^E\right)\left(I+t^2\left(V-\Delta\right)\right)^{-1}u\right\|^2 \frac{dt}{t} &\lesssim \int_0^\infty \left\|t\left(V-\Delta\right)^{\frac{1}{2}}\left(I+t^2\left(V-\Delta\right)\right)^{-1}u\right\|^2 \frac{dt}{t} \\ &\lesssim \|u\|^2.\end{aligned}$$

□

Proposition 6.2.4. *Let $V \in \mathcal{W}_2$. Suppose that the effective potential satisfies $V \lesssim E$ and*

$$\left|\nabla\left(\frac{1}{1+tE^{\frac{1}{2}}}\right)(x)\right| \lesssim \frac{1}{t} \tag{6.6}$$

for all $t > 0$ and $x \in \mathbb{R}^n$. Then there exists some $C_V > 0$ for which

$$\int_0^\infty \left\|A_t^E\left(I - P_t^{|V|^{\frac{1}{2}}}\right)u\right\|^2 \frac{dt}{t} \le C_V \cdot \|u\|^2 \tag{6.7}$$

for all $u \in \overline{R(\Gamma_{|V|^{\frac{1}{2}}})} \oplus N(\Pi_{|V|^{\frac{1}{2}}})$.

Proof. The proof will begin similarly to the proof of Proposition 4.2.1. In particular, note that the estimate will be trivial on $N(\Pi_{|V|^{\frac{1}{2}}})$ since $(I-P_t^{|V|^{\frac{1}{2}}})u = 0$ for $u \in N(\Pi_{|V|^{\frac{1}{2}}})$. Fix $u \in R(\Gamma_{|V|^{\frac{1}{2}}})$. On applying the resolution of the identity, Proposition 3.1.3,

$$\int_0^\infty \left\|A_t^E\left(I - P_t^{|V|^{\frac{1}{2}}}\right)u\right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left(\int_0^\infty \left\|A_t^E(P_t^{|V|^{\frac{1}{2}}} - I)\left(Q_s^{|V|^{\frac{1}{2}}}\right)^2 u\right\| \frac{ds}{s}\right)^2 \frac{dt}{t}.$$

Since $u \in R\left(\Gamma_{|V|^{\frac{1}{2}}}\right)$, we have that $(Q_s^{|V|^{\frac{1}{2}}})^2 u = Q_s^{|V|^{\frac{1}{2}}}\tilde{\mathbb{P}}Q_s^{|V|^{\frac{1}{2}}}u$, where $\tilde{\mathbb{P}} := \mathbb{P}_1 \cdot \mathbb{P}'$ and $\mathbb{P}'$

is the projection onto the subspace $\overline{R(\Pi_{|V|^{\frac{1}{2}}})}$. The Cauchy-Schwarz inequality then gives

$$\int_0^\infty \left\| A_t^E \left(I - P_t^{|V|^{\frac{1}{2}}} \right) u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left(\int_0^\infty \left\| A_t^E \left(P_t^{|V|^{\frac{1}{2}}} - I \right) Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| \frac{ds}{s} \right) \cdot \left(\int_0^\infty \left\| A_t^E \left(P_t^{|V|^{\frac{1}{2}}} - I \right) Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| \left\| Q_s^{|V|^{\frac{1}{2}}} u \right\|^2 \frac{ds}{s} \right) \frac{dt}{t}.$$

Then, to prove our desired estimate, it is sufficient to show that

$$\left\| A_t^E \left(I - P_t^{|V|^{\frac{1}{2}}} \right) Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| \lesssim \min \left\{ \frac{t}{s}, \frac{s}{t} \right\}^{\frac{1}{2}}. \tag{6.8}$$

For $t \leq s$, since A_t^E is bounded on $L^2(\mathbb{R}^n)$ and $(I - P_t^{|V|^{\frac{1}{2}}}) Q_s^{|V|^{\frac{1}{2}}} = \frac{t}{s} Q_t^{|V|^{\frac{1}{2}}} \left(I - P_s^{|V|^{\frac{1}{2}}} \right)$, we have

$$\begin{aligned} \left\| A_t^E \left(I - P_t^{|V|^{\frac{1}{2}}} \right) Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| &\lesssim \left\| \left(I - P_t^{|V|^{\frac{1}{2}}} \right) Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| \\ &\lesssim \frac{t}{s} \left\| Q_t^{|V|^{\frac{1}{2}}} \left(I - P_s^{|V|^{\frac{1}{2}}} \right) \tilde{\mathbb{P}} \right\| \\ &\lesssim \frac{t}{s}. \end{aligned}$$

Assume that $t > s$. Then the equality $P_t^{|V|^{\frac{1}{2}}} Q_s^{|V|^{\frac{1}{2}}} = \frac{s}{t} Q_t^{|V|^{\frac{1}{2}}} P_s^{|V|^{\frac{1}{2}}}$ gives

$$\begin{aligned} \left\| A_t^E \left(I - P_t^{|V|^{\frac{1}{2}}} \right) Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| &\lesssim \left\| P_t^{|V|^{\frac{1}{2}}} Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| + \left\| A_t^E Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| \\ &\lesssim \frac{s}{t} + \left\| A_t^E Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\|. \end{aligned}$$

Our proof has now been reduced to the task of showing the estimate

$$\left\| A_t^E Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| \lesssim \left(\frac{s}{t} \right)^{\frac{1}{2}}.$$

A first step in this direction is to apply the splitting

$$\begin{aligned} \left\| A_t^E Q_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} \right\| &= \left\| A_t^E s \Pi_{|V|^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' \right\| \\ &\lesssim \left\| A_t^E s \Pi_0 \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' \right\| + \left\| A_t^E s S_{|V|^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' \right\|. \end{aligned} \tag{6.9}$$

For the second term, for $v \in \mathcal{H}$,

$$\begin{aligned}
\left\| A_t^E s S_{|V|^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right\|^2 &\lesssim \sum_{Q \in \Delta_t} s^2 |Q| \left| \fint_Q \frac{V^{\frac{1}{2}}(y)(P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v)_1(y)}{1 + tE^{\frac{1}{2}}(y)} dy \right|^2 \\
&\lesssim \sum_{Q \in \Delta_t} \left(\frac{s}{t}\right)^2 \int_Q \left| \frac{tV^{\frac{1}{2}}(y)(P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v)_1(y)}{1 + tE^{\frac{1}{2}}(y)} \right|^2 dy \\
&\lesssim \left(\frac{s}{t}\right)^2 \left\| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right\|^2 \\
&\lesssim \left(\frac{s}{t}\right)^2 \|v\|^2 ,
\end{aligned}$$

where the condition $V \lesssim E$ was used to obtain the second to last line. For the first term in (6.9), note that condition (6.6) implies that the commutator $\left[\Pi_0, \left(1 + tE^{\frac{1}{2}}\right)^{-1}\right]$ is a multiplication operator with bound

$$\left| \left[\Pi_0, \frac{1}{1 + tE^{\frac{1}{2}}}\right](y) \right| \lesssim \frac{1}{t}. \tag{6.10}$$

We then have

$$\begin{aligned}
\left\| A_t^E s \Pi_0 \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right\|^2 &= \sum_{Q \in \Delta_t} s^2 |Q| \left| \fint_Q \frac{1}{1 + tE^{\frac{1}{2}}(y)} \Pi_0 \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v(y) \right|^2 \\
&\lesssim \sum_{Q \in \Delta_t} s^2 |Q| \left(\left| \fint_Q \Pi_0 \left(\frac{1}{1 + tE^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right)(y) \right|^2 + \left| \fint_Q \left[\Pi_0, \frac{1}{1 + tE^{\frac{1}{2}}}\right] \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v(y) \right|^2 \right).
\end{aligned}$$

From the bound (6.10),

$$\begin{aligned}
\sum_{Q \in \Delta_t} s^2 |Q| \left| \fint_Q \left[\Pi_0, \frac{1}{1 + tE^{\frac{1}{2}}}\right] \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v(y) \right|^2 &\lesssim \left(\frac{s}{t}\right)^2 \sum_{Q \in \Delta_t} \int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v(y) \right|^2 dy \\
&\lesssim \left(\frac{s}{t}\right)^2 \|v\|^2 .
\end{aligned}$$

For the other term, apply Lemma 5.6 of [11] to obtain

$$
\begin{aligned}
&\sum_{Q\in\Delta_t} s^2 |Q| \left| ⨍_Q \Pi_0 \left(\frac{1}{1+tE^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right)(y) \right|^2 \\
&\quad \lesssim \sum_{Q\in\Delta_t} |Q| \frac{s^2}{t} \left(⨍_Q \left| \frac{1}{1+tE^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} \cdot \left(⨍_Q \left| \Pi_0 \left(\frac{1}{1+tE^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right) \right|^2 \right)^{\frac{1}{2}} \\
&\quad \lesssim \sum_{Q\in\Delta_t} \frac{s^2}{t} \left(\int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} \cdot \left(\int_Q \left| \Pi_0 \left(\frac{1}{1+tE^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right) \right|^2 \right)^{\frac{1}{2}} \\
&\quad \lesssim \sum_{Q\in\Delta_t} \frac{s^2}{t} \left(\int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} \\
&\qquad \cdot \left[\left(\int_Q \left| \left[\Pi_0, \frac{1}{1+tE^{\frac{1}{2}}} \right] \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} + \left(\int_Q \left| \frac{1}{1+tE^{\frac{1}{2}}} \Pi_0 \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} \right] \\
&\quad \lesssim \sum_{Q\in\Delta_t} \frac{s^2}{t^2} \left(\int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right) + \frac{s^2}{t} \left(\int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} \left(\int_Q \left| \Pi_0 P_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} v \right|^2 \right)^{\frac{1}{2}} \\
&\quad = \frac{s^2}{t^2} \left\| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right\|^2 + \frac{s}{t} \sum_{Q\in\Delta_t} \left(\int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 \right)^{\frac{1}{2}} \left(\int_Q \left| s\Pi_0 P_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} v \right|^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

It is not difficult to see that $V \in \mathcal{W}_2$ implies that $\|\Pi_0 \tilde{v}\| \lesssim \left\| \Pi_{|V|^{\frac{1}{2}}} \tilde{v} \right\|$ for $\tilde{v} \in R\left(\Pi_{|V|^{\frac{1}{2}}}\right) \cap D\left(\Pi_{|V|^{\frac{1}{2}}}\right)$. Apply this coercivity condition to obtain

$$
\begin{aligned}
&\sum_{Q\in\Delta_t} s^2 |Q| \left| ⨍_Q \Pi_0 \left(\frac{1}{1+tE^{\frac{1}{2}}} \mathbb{P}_1 P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right)(y) \right|^2 \\
&\quad \lesssim \frac{s^2}{t^2} \|v\|^2 + \frac{s}{t} \sum_{Q\in\Delta_t} \int_Q \left| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right|^2 + \int_Q \left| s\Pi_0 P_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} v \right|^2 \\
&\quad = \frac{s^2}{t^2} \|v\|^2 + \frac{s}{t} \left(\left\| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right\|^2 + \left\| s\Pi_0 P_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} v \right\|^2 \right) \\
&\quad \lesssim \frac{s^2}{t^2} \|v\|^2 + \frac{s}{t} \left(\left\| P_s^{|V|^{\frac{1}{2}}} \mathbb{P}' v \right\|^2 + \left\| s\Pi_{|V|^{\frac{1}{2}}} P_s^{|V|^{\frac{1}{2}}} \tilde{\mathbb{P}} v \right\|^2 \right) \\
&\quad \lesssim \left(\frac{s^2}{t^2} + \frac{s}{t} \right) \|v\|^2 \\
&\quad \lesssim \frac{s}{t} \|v\|^2.
\end{aligned}
$$

This completes the proof of the estimate (6.8) and therefore concludes the proof of our proposition. ∎

Corollary 6.2.2. *Let $V \in \mathcal{W}_2$. Suppose that the effective potential E satisfies $V \lesssim E$ and*

$$\left| \nabla \left(\frac{1}{1 + tE^{\frac{1}{2}}} \right)(x) \right| \lesssim \frac{1}{t}$$

for all $t > 0$ and $x \in \mathbb{R}^n$. Then for some $C_V > 0$ we have

$$\int_0^\infty \left\| \mathbb{P}_1 \left(A_t^E - P_t^{|V|^{\frac{1}{2}}} \right) u \right\|^2 \frac{dt}{t} \leq C_V \cdot \|u\|^2 .$$

for all $u \in \overline{R(\Gamma_{|V|^{\frac{1}{2}}})} \oplus N(\Pi_{|V|^{\frac{1}{2}}})$.

For $V \in RH_{\frac{n}{2}}$, fix the effective potential to be the critical radius function $E := \rho_V^{-2}$. Then our averaging operators would be given by

$$A_t^E f(x) = \fint_{Q(t,x)} \frac{f(y)}{1 + \frac{t}{\rho_V(y)}} \, dy.$$

For the harmonic oscillator potential, the critical radius function ρ satisfies all of the conditions of the previous corollary. It then follows that A_t^E acts as a good approximation for the potential dependent averaging operator in this context.

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Notation

General

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$	Real numbers, complex numbers, natural numbers and integers
$\mathbb{N}^*$	$\mathbb{N}^* := \mathbb{N} \setminus \{0\}$
$L^p(\mathbb{R}^n)$	Lebesgue space of p-integrable functions on $\mathbb{R}^n$ with respect to the Lebesgue measure
$H^k(\mathbb{R}^n)$, $W^{k,2}(\mathbb{R}^n)$	Sobolev space of index $(k, 2)$
$C^k_c(\Omega)$	Space of k-times continuously differentiable and compactly supported functions on the domain $\Omega \subset \mathbb{R}^n$
$\mathcal{L}(\mathcal{H}; K)$	Bounded linear operators between the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ . . . 3
$\mathcal{L}(\mathcal{H})$	Bounded linear operators from $\mathcal{H}$ to itself . . . 3
$\mathcal{L}_V$	Schrödinger operator $\mathcal{L}_V := V - \Delta$ . . . 14
L	Divergence form operator $L := -\mathrm{div} A \nabla$ . . . 67
$L + V$	$L + V := -\mathrm{div} A \nabla + V$ . . . 69
$\mathfrak{l}_V$, $\mathfrak{l}^A$, $\mathfrak{l}^A_V$	Sesquilinear forms corresponding to the operators $\mathcal{L}_V$, L, $L + V$ . . . 14, 67, 69

Classical Harmonic Analysis

Cubes and Balls

Schrödinger Operators

The Harmonic Oscillator

AKM Framework

D_J	The operator $D_J := \begin{pmatrix} J \\ D \end{pmatrix}$	125
M_J	Non-homogeneous part of Γ_J	125
S_J	$S_J := M_J + M_J^*$	125
Γ_0	Homogeneous first-order part of Γ_J	125
Π_0	$\Pi_0 := \Gamma_0 + \Gamma_0^*$	125
R_t^J, P_t^J, Q_t^J, Θ_t^J	Operators affiliated with the operator Π_J	126
$\mathcal{P}_t^J$	Diagonalisation of P_t^J	133
$\Gamma_{J,B}^*$, $\Pi_{J,B}$	Perturbed operators $\Gamma_{J,B}^* := B_1\Gamma_J^* B_2$, $\Pi_{J,B} := \Gamma_J + \Gamma_{J,B}^*$	126
$R_t^{J,B}$, $P_t^{J,B}$, $Q_t^{J,B}$, $\Theta_t^{J,B}$	Operators affiliated with $\Pi_{J,B}$	126
$\gamma_t^{J,B}$	Principal part of $\Theta_t^{J,B}$	140
$\tilde{\Theta}_t^{J,B}$	$\tilde{\Theta}_t^{J,B} := \Theta_t^{J,B}\mathbb{P}_3$	140
$\tilde{\gamma}_t^{J,B}$	Principal part of $\tilde{\Theta}_t^{J,B}$	140
L_B^J	$L_B^J u$ is the first component of $\Pi_{J,B}^2(u,0,0)$	132
b_α^D, b^J, b_α^J	Smallest constant for which (H8Dα), (H8J) and (H8Jα) are satisfied respectively	128
c_α^J	$c_\alpha^J := 1 + \left(b_\alpha^D\right)^2 + \left(\min\left\{b^J, b_\alpha^J\right\}\right)^2 (\alpha - 1)^{-1}$	128
$\mathcal{N}_1$, $\mathcal{N}_2$	Weighted local and global non-tangential maximal operators	99
$\mathbb{P}_i$	Projection onto the ith subspace of $\mathcal{H} = \bigoplus_{i=1,2,3} L^2\left(\mathbb{R}^n; V_i\right)$	125