Bayesian Analysis of Claim Run-off Triangles

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This thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, contains no material published or written by another person, except where due reference is made in the thesis.

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Kar Wai Lim

4 November 2011

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Abstract

This dissertation studies Markov chain Monte Carlo (MCMC) methods, and applies them to actuarial data, with a focus on claim run-off triangles. After reviewing a classical model for run-off triangles proposed by Hertig (1985) and improved by de Jong (2004), who incorporated a correlation structure, a Bayesian analogue is developed to model an actuarial dataset, with a view to estimating the total outstanding claim liabilities (also known as the required reserve). MCMC methods are used to solve the Bayesian model, estimate its parameters, make predictions, and assess the model itself. The resulting estimate of reserve is compared to estimates obtained using other methods, such as the chain-ladder method, a Bayesian over-dispersed Poisson model, and the classical development correlation model of de Jong.

The thesis demonstrates that the proposed Bayesian correlation model performs well for claim reserving purposes. This model yields similar results to its classical counterparts, with relatively conservative point estimates. It also gives a better idea of the uncertainties involved in the estimation procedure.

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CHAPTER 1

Introduction

Markov chain Monte Carlo (MCMC) methods play an important role in Bayesian statistics, especially when inference cannot be made directly due to the complexity of the Bayesian model, for example, when there is no closed form solution to the *posterior distribution* of a target parameter. MCMC methods allow one to sample random values from the posterior distribution; these values are subsequently used to estimate quantities of interest, such as the posterior means of model parameters. MCMC methods are often easy and quick to implement, and provide an alternative approach to the analysis of Bayesian models even when an analytic solution is possible.

This thesis demonstrates the usefulness of MCMC methods when applied to the Bayesian analysis of actuarial data, with a focus on claim run-off triangles. A run-off triangle shows the claim liabilities for accidents occurring in certain years and the delays in claim reporting. Traditionally, deterministic algorithms such as the chain-ladder (CL) method (Harnek, 1966) and the Bornhuetter-Ferguson method (Bornhuetter & Ferguson, 1972) were used to forecast the future claim liabilities to determine the outstanding claims reserve. With improvement in technology and the need to identify the variability underlying the future claim liabilities, stochastic models were developed and analysed to justify these deterministic algorithms; the most notable of these stochastic models is the stochastic CL method developed by Mack (1993). A summary of the stochastic CL method can be found in Mack (2006). An extensive literature on claims reserving is available in the book by Taylor (2000).

Several Bayesian models for claim run-off triangles were considered by Verrall (1990), de Alba (2002, 2006), England & Verrall (2002), Lamps (2002a, 2002b, 2002c), Scollnik (2004), de Alba & Nieto-Barajas (2008), England, Verrall, & Wuthrich (2010) and other researchers. Verrall (1990) analysed the traditional CL method using the theory of Bayesian linear models, by transforming the multiplicative CL model into linear model by taking logarithms. Verrall utilises a Kalman filter (state space) approach in the Bayesian analysis. de Alba (2002, 2006) presented Bayesian approach for several models using direct Monte Carlo (MC) method. England and Verrall (2002) proposed a Bayesian analysis using an over-dispersed Poisson CL model, they compared and contrasted this approach with other reserving methods. Details on the Bayesian over-dispersed Poisson model are available in England *et al.* (2010). Lamps (2002a, 2002b, 2002c) discussed various MCMC models to deal with claim run-off triangles, and Scollnik (2004) performed MCMC methods on the CL model using statistical package *WinBUGS* (to be discussed further in Chapter 2). As mentioned in Scollnik (2004), the Bayesian approach is useful because Bayesian models allow prior information to be included in the analysis, if available. Bayesian models allow parameter uncertainty and model uncertainty to be incorporated in the analysis and predictive inference. They also yield complete posterior distributions for quantities of interest rather than just point estimates and confidence intervals.

This thesis considers the Bayesian analogue of a frequentist (classical) model proposed by de Jong (2004), which in turn is an extension of a model proposed by Hertig (1985). Hertig's model is different from all the models mentioned in the previous paragraph because it models the log-link ratios of the cumulative liabilities instead of the claim liabilities directly. de Jong extended Hertig's model by introducing a correlation

structure. These models of Hertig and de Jong will be discussed further in Chapter 4. This thesis contributes to the existing literature by developing a Bayesian model for de Jong's classical model and comparing the two. MCMC methods will be used to perform the Bayesian analysis.

The next chapter provides an overview of the Bayesian approach and MCMC methods generally. Chapter 3 describes claim run-off triangles, in particular the data to be analysed. In Chapter 4, the classical models proposed by Hertig (1985) and de Jong (2004) are studied and Bayesian analogues thereof are developed. Chapter 5 discusses how MCMC methods can be used to perform the Bayesian analysis; the results of the analysis are then presented. Chapter 6 assesses the Bayesian models in terms of goodness-of-fit and the appropriateness of the estimated reserve. Comparison of the Bayesian results with those of previous studies is made in Chapter 7. Finally, Chapter 8 provides a summary of the thesis, discusses limitations of the approach taken and suggests several avenues for further research.

CHAPTER 2

Bayesian Inference and MCMC Methods

2.1 Bayesian modelling

A classical model treats its unknown parameters as constants that need to be estimated, whereas a Bayesian model regards the same parameters as random variables, each of them having a *prior distribution*. It is assumed that the readers of the thesis understand the basics of Bayesian methods and hence discussion will focus on Bayesian inference and results. Readers may find the introductory text "Bayesian Data Analysis" by Gelman, Carlin, Stern, and Rubin (2003) useful. Important formulae, results and examples will now be presented as a brief overview of Bayesian methods.

Consider the following Bayesian model:

$$(y|a,b) \sim f(y|a,b)$$

$$(a|b) \sim f(a|b)$$

$$b \sim f(b)$$

where $f(\cdot)$ denotes the *probability density function* (pdf).

In this Bayesian model, the joint posterior density of a and b can be written as $f(a,b|y) = f(y|a,b)f(a,b)/f(y)$, where $f(y) = \iint f(y|a,b)f(a,b)\,da\,db$, and $f(a,b) = f(a|b)f(a)$.

It is often convenient to write the *joint* posterior density up to a proportionality constant, namely $f(a,b|y) \propto f(y|a,b)f(a,b)$, since $f(y)$ can be hard to determine and does not

depend on a and b (the arguments in $f(a,b|y)$). The *marginal* posterior densities can be derived by integrating out the relevant nuisance parameter, in this case:

$$f(a|y) = \int f(a,b|y)\, db\,, \qquad f(b|y) = \int f(a,b|y)\, da$$

These integrals are usually difficult to express in closed form when the model is complicated or there are many parameters (note that a and b are possibly vectors). Hence inference on a and b may be impractical, if not impossible; one may then consider approximating the solutions of the equations involved using special techniques such as numerical integration. However, these can be tedious and time consuming.

2.2 Markov chain Monte Carlo methods

MCMC methods are useful because they can provide simple but typically very good approximations to integrals and other equations that are very difficult or impossible to obtain directly. With these methods, knowing only the joint posterior density, $f(a,b|y)$ (up to a proportionality constant) is sufficient for inference to be made on the marginal posterior densities, $f(a|y)$ and $f(b|y)$. Briefly, this is achieved by alternately simulating random observations from the *conditional* marginal posterior densities, $f(a|y,b)$ and $f(b|y,a)$ (each of which is proportional to the joint posterior density, $f(a,b|y)$), so as to ultimately produce random samples (as detailed below) from the (unconditional) marginal posterior densities, $f(a|y)$ and $f(b|y)$. Estimates of a and b can then be obtained from these latter samples. Moreover, these samples can also be used to estimate *any*, possibly very complicated, functions of a and b, *e.g.* $c = ab/(a+b)$.

MCMC methods were first proposed by Metropolis *et al.* (1953), and subsequently generalised by Hastings (1970), leading to the Metropolis-Hastings (MH) algorithm. The MH algorithm can be summarised as follows:

i. Specify an initial value for (a, b), call it (a_0, b_0), which will be the starting point for the algorithm's simulation process.

ii. Define *driver distributions* for the parameters a and b, from which the next simulated values will be sampled. Let the pdfs of the driver distributions for a and b be $g(t|y, a, b)$ and $h(t|y, a\ , b)$, respectively.

iii. Sample a candidate value of a from $g(t|y, a_i, b_i)$, call it a', and accept it with probability

$$p = \frac{f(a', b_i|y)}{f(a_i, b_i|y)} \times \frac{g(a_i|y, a', b_i)}{g(a'|y, a_i, b_i)}$$

Then the value of a_{i+1} is updated to be a' if a' is accepted. Otherwise, a_{i+1} retains the previous value, *i.e.* $a_{i+1} = a_i$.

To decide whether a' is accepted, generate $u \sim \text{Uniform}(0,1)$. Then accept a' if $u \leq p$. (Note that a' is automatically accepted if $p \geq 1$.)

iv. Sample a candidate value of b from $h(t|y, a_{i+1}\ , b_i)$, call it b', and accept it with probability

$$q = \frac{f(a_{i+1}, b'|y)}{f(a_{i+1}, b_i|y)} \times \frac{h(b_i|y, a_{i+1}, b')}{h(b'|y, a_{i+1}, b_i)}$$

Then the value of b_{i+1} is updated to be b' if b' is accepted. Otherwise, b_{i+1} retains the previous value, *i.e.* $b_{i+1} = b_i$. To decide whether b' is accepted, generate $v \sim \text{Uniform}(0,1)$. Then accept b' if $v \leq q$. This concludes the first iteration.

v. Repeat steps *iii* and *iv* again and again until a desired total large sample of size K is created, *i.e.* $\{(a_i, b_i): i = 0, 1, \dots, K\}$. This concludes the MH algorithm.

vi. Decide on a suitable *burn-in* length B (see below). Next, relabel $a_{B+1}, \dots, a_K$ as $a_1, \dots, a_J$ and $b_{B+1}, \dots, b_K$ as $b_1, \dots, b_J$. Then take $(a_1, b_1), \dots, (a_J, b_J)$ as an approximately iid sample from $f(a, b|y)$.

A drawback of the MH algorithm is that the simulated values are not truly independent (hence the word 'approximate' in step *vi* above); their correlation comes from the Markov chain method where the next simulated value is obtained from its predecessor. Also, a bad choice of initial values distorts the sampling distribution. This means that a truly random sample of the posterior density is not available. Fortunately, the simulated values converge to their marginal posterior density as the number of iterations, K gets larger. Hence, these problems with MCMC methods can be addressed by setting K to be very large, say 11000, and discarding the initial portion of the simulated data, for example, a burn-in of $B = 1000$, for a final sample of size $J = 10000$.

The theory of convergence and how to choose B and K will not be discussed in the thesis; refer to Raftery & Lewis (1995) for details regarding these issues. To ensure a *good mixing* of the simulated values, where a_i and b_i as a whole represent the true marginal posterior distributions and each a_i and b_i is a random realisation from the distributions, a conservative approach will be used in choosing the number of simulations K and burn-in B. Convergence can then be assessed from the traces of the algorithm, which show the time series values of simulated a and b; a cut-off point can then be chosen conservatively for a suitable burn-in.

The driver distributions are usually chosen so that the candidate values will be easy to sample from; examples include the uniform and normal distributions. A suitable choice of driver distribution allows the candidate values to be accepted more frequently with higher probability of acceptance, which is desirable as the algorithm produces a better mixing of simulated values with lower *wastage* (rejections of a' and b').

Note that when a driver distribution is chosen to be a *symmetric distribution*, in the sense that $g(t|y,a,b) = g(a|y,t,b)$ and $h(t|y,a,b) = h(b|y,a,t)$, the acceptance probabilities simplify to

$$p = \frac{f(a', b_i|y)}{f(a_i, b_i|y)} \quad \text{and} \quad q = \frac{f(a_{i+1}, b'|y)}{f(a_{i+1}, b_i|y)}$$

The MH algorithm reduces to the Gibbs sampler when the driver distributions are chosen to be the conditional posterior distributions, *i.e.* when $g(t|y,a,b)$ is set to be $f(a = t|y,b)$ and $h(t|y,a,b)$ is set equal to $f(b = t|y,a)$. In this case, the acceptance probability for a reduces to

$$p = \frac{f(a'|y,b_i)f(b_i|y)}{f(a_i|y,b_i)f(b_i|y)} \times \frac{f(a_i|y,b_i)}{f(a'|y,b_i)} = 1$$

and likewise q reduces to 1.

The Gibbs sampler is preferred to the general MH algorithm because it produces no wastage. However, sometimes a considerable amount of effort is needed to derive the distribution of the conditional density and/or to sample from it. Thus there may be a trade-off between efficiency and simplicity.

2.3 MCMC inference

From a large sample with appropriate burn-in, the simulated values of a and b can be used to make inferences on a and b, as well as on any function of a and b. For example, one may wish to estimate a by its posterior mean, $\hat{a} = E(a|y)$, but this involves solving a difficult or impossible integral to determine the posterior mean. Therefore, in turn, $\hat{a}$ is estimated by the Monte Carlo sample mean, $\bar{a} = (1/J)\sum_{j=1}^{J} a_j$.

This estimate is unbiased because

$$E(\bar{a}|y) = \frac{1}{J}\sum_{j=1}^{J} E(a_j|y) = \frac{1}{J}\sum_{j=1}^{J} E(a|y) = \frac{1}{J}J\hat{a} = \hat{a} \qquad \text{since } a_j \sim f(a|y)$$

With the aid of a statistical package such as *S-Plus* or *R*, the entire marginal posterior distributions can also be estimated from the simulated values. The estimated marginal posterior distributions can then be displayed on a graph as a representation of the true marginal posterior distributions. The approximation improves with the number of simulations. This provides a simple alternative to deriving the exact distributions analytically, typically by integration. Inference on functions of a and b, such as $c = ab/(a+b)$ mentioned above, can be performed in a similar manner; this is achieved simply by calculating values $c_j = a_j b_j/(a_j + b_j)$ and applying the method of Monte Carlo, as before. (*i.e.* to estimate $\hat{c} = E(c|y)$ by $\bar{c} = (1/J)\sum_{j=1}^{J} c_j$).

MCMC methods also allow *predictive* inference to be made in a convenient manner, when one is interested in some quantity x for which $f(x|y,a,b)$ is known. For instance, x could be a future independent replicate of y, in which case $f(x|y,a,b)$ is the same as $f(y|a,b)$ with y changed to x.

Then the predictive density is

$$f(x|y) = \iint f(x, a, b|y)\, da\, db = \iint f(x|y, a, b) f(a, b|y)\, da\, db$$

Usually it is difficult to derive the predictive density analytically. However, observations of the predictive quantity x can often be sampled easily from the predictive distribution using the *method of composition*. This is done by sampling x_j from $f(x|y, a_j, b_j)$, where a_j and b_j are taken from the MCMC sample described above. The triplet (a_j, b_j, x_j) is then a sample value from $f(x, a, b|y)$; also, x_j is a random observation from $f(x|y)$. Thus, inference on x, or any function of a, b and x, can be performed much more conveniently without deriving the predictive density directly.

There are two general approaches to obtaining a Bayesian estimate of a general quantity of interest z using a MCMC sample. First, there is the 'normal method' (as mentioned above) whereby the posterior mean of the quantity, $\hat{z} = E(z|y)$ is estimated by the respective MCMC sample mean, $\bar{z}$; *e.g.* if $z = \omega(a, b, x)$, then $\hat{z}$ is estimated by $\bar{z} = (1/J)\sum_{j=1}^{J} z_j = (1/J)\sum_{j=1}^{J} \omega(a_j, b_j, x_j)$. Alternatively, one may apply the 'Rao-Blackwell method' (see McKeague & Wefelmeyer, 2000) and estimate $\hat{z}$ by the sample mean of a *conditional* posterior expected value of z. An example of the Rao-Blackwell method is shown below in Subsection 2.5.5. This method is often more precise than the normal method; however, it is typically more complicated and requires further derivations.

In addition to using the mean for predictive inference, one could also consider the median and mode. Comparing these statistics may give an idea of the underlying distribution of the quantity of interest, z. In Bayesian decision theory, the mean

minimises the quadratic error loss function, the median minimises the absolute error loss function and the mode minimises the zero-one error loss function. An actuary may need to consider, in his or her application, the costs associated with making errors of various magnitudes. In practice, the quadratic error loss function is usually chosen.

2.4 *WinBUGS*

WinBUGS (Bayesian inference Using Gibbs Sampling for Windows) is a software package which is useful for analysing Bayesian models *via* MCMC methods (Lunn, Thomas, Best, & Spiegelhalter, 2000). This software utilises the Gibbs sampler to produce the simulated values. Using *WinBUGS* to estimate the quantities of interest is much quicker and simpler than writing the algorithm codes manually (*e.g.* in *S-plus* or *R*). This is because Gibbs sampling is done internally by *WinBUGS* without the need to derive the posterior distribution of the parameters. Refer to Sheu & O'Curry (1998) and Merkle & Zandt (2005) for an introduction to *WinBUGS*.

2.5 Application of MCMC methods to time series

In the following example, MCMC methods are used to analyse a randomly generated time series data with statistical package *R*. This is done by coding the MH algorithm in *R*. MCMC simulations are also performed using *WinBUGS* at the end of the section for comparison purpose.

2.5.1 Underlying model and data

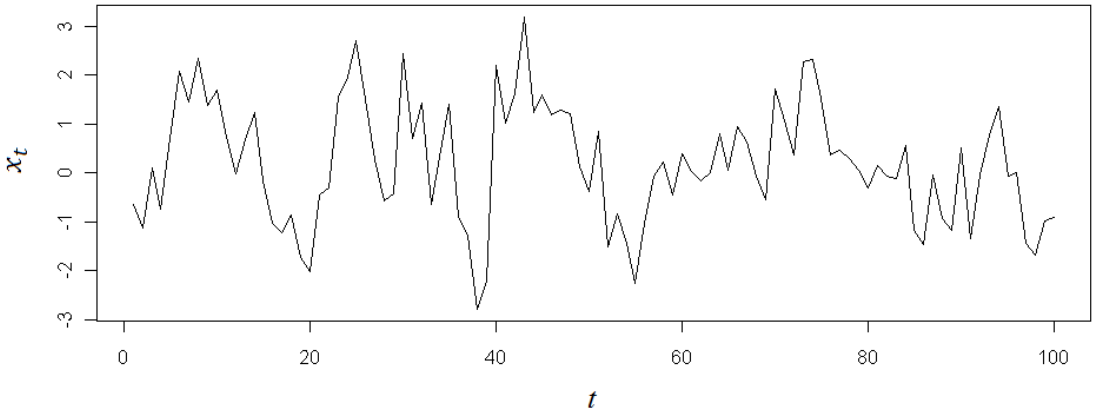
Consider the following stationary AR(1) model (autoregressive model of order 1):

$$x_t = \alpha + \beta x_{t-1} + e_t, \qquad t = 1, \dots, n, \qquad e_t \sim iid \text{ Normal}(0, \sigma^2)$$

A time series $x = (x_1, x_2, \ldots, x_n)$ was generated according to this model with $n = 100$, $\alpha = 0.2$, $\beta = 0.6$ and $\sigma^2 = 1$. x_1 was simulated directly from the normal density with mean $\alpha/(1-\beta) = 0.5$ and variance $\sigma^2/(1-\beta) = 2.5$; see Appendix A-1 for details on the derivation of the mean and the variance. The other x values were generated according to $(x_t|x_{t-1}) \sim \text{Normal}(\alpha + \beta x_{t-1}, \sigma^2)$. The *R* code for generating these x values is presented in Appendix B-1.

Note that for the model to be a stationary AR(1) model, the condition $|\beta| < 1$ has to be satisfied. The values of the time series x are shown in Figure 2.1.

Figure 2.1: Simulated time series values of x



2.5.2 Priors and the joint posterior distribution

This example assumes *a priori* ignorance and independence regarding α, β and σ^2, with prior densities defined as:

$$f(\alpha) \propto 1, \qquad \alpha \in \Re$$

$$f(\beta) = \frac{1}{2}, \qquad -1 < \beta < 1$$

$$f(\sigma^2) \propto \frac{1}{\sigma^2}, \qquad \sigma^2 > 0$$

Note that the prior distributions of α and σ^2 are *improper*, for which their densities do not integrate to 1. These priors (including β) are *uninformative* (or *non-informative*) in the sense that they do not provide any real information of their underlying values. Care should be taken when using improper priors, as they might produce improper posterior distributions, which are nonsensical for making inferences, see Hobert & Casella (1996) for a detailed analysis on the dangers of using improper priors. According to Hobert & Casella, "the fact that it is possible to implement the Gibbs sampler without checking that the posterior is proper is dangerous".

Denoting $\theta = (\alpha, \beta, \sigma^2)$, the posterior density of θ (also the joint density of α, β and σ^2) is:

$$
\begin{aligned}
f(\theta|x) &\propto f(\theta) f(x|\theta) \\
&= f(\alpha) f(\beta) f(\sigma^2) f(x_1, x_2, \dots, x_n|\theta) \\
&= 1 \times \frac{1}{2} \times \frac{1}{\sigma^2} \times f(x_1|\theta) f(x_2|x_1, \theta) \dots f(x_n|x_{n-1}, \dots, x_1, \theta) \\
&\propto \frac{1}{\sigma^2} f(x_1|\theta) \prod_{t=2}^{n} f(x_t|x_{t-1}, \theta) \\
&= \frac{1}{\sigma^2} \emptyset\left(x_1, \frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta}\right) \prod_{t=2}^{n} \emptyset(x_t, \alpha + \beta x_{t-1}, \sigma^2)
\end{aligned}
$$

where $\emptyset(x, a, b^2)$ denotes the pdf of the normal distribution, with mean a and variance b^2, evaluated at x, namely

$$
\emptyset(x, a, b^2) = \frac{1}{b\sqrt{2\pi}} exp\left[-\frac{1}{2b^2}(x-a)^2\right], \qquad x \in \Re, \qquad a \in \Re, \qquad b > 0
$$

Deriving the marginal posterior densities from the joint posterior density is impractical given the complex nature of the joint posterior density. Nevertheless, the marginal posterior density can be estimated *via* MCMC methods, as detailed in the following subsections.

2.5.3 A Metropolis-Hastings algorithm

A MH algorithm was designed and implemented (see the *R* code in Appendix B-2) so as to generate a sample of 1100 values of θ with symmetric uniform drivers. The mean of the driver distributions were chosen to be the last simulated values of θ, with starting points $\alpha_0 = 0$, $\beta_0 = 0$ and $\sigma_0^2 = 0.5$. Specifically, the driver distributions are:

$$g(\alpha_{i+1}|y, \alpha_i, \beta_i, \sigma_i^2) \sim \text{Uniform}(\alpha_i - \delta_\alpha, \alpha_i + \delta_\alpha)$$

$$g(\beta_{i+1}|y, \alpha_{i+1}, \beta_i, \sigma_i^2) \sim \text{Uniform}(\beta_i - \delta_\beta, \beta_i + \delta_\beta)$$

$$g(\sigma_{i+1}^2|y, \alpha_{i+1}, \beta_{i+1}, \sigma_i^2) \sim \text{Uniform}(\sigma_i^2 - \delta_{\sigma^2}, \sigma_i^2 + \delta_{\sigma^2})$$

where $\delta_\alpha = 0.2$, $\delta_\beta = 0.2$ and $\delta_{\sigma^2} = 0.5$ are called the *tuning parameters*.

The *acceptance rates* were found to be 62.5% for α, 55.3% for β and 47.5% for σ^2. (For example, 62.5% of the 1100 proposed values of α were accepted.) In the algorithm, the values of β were restricted between -1 and 1, while the values of σ^2 were restricted to be positive. Figure 2.2, 2.3 and 2.4 show the traces of the sampling process. These are plots of the simulated values; the dotted lines show the cut-off point for burn-in, $B = 100$. These figures show that the simulated values of α, β and σ^2 converge very quickly and exhibit a good mixing.

Figure 2.2: Trace of α

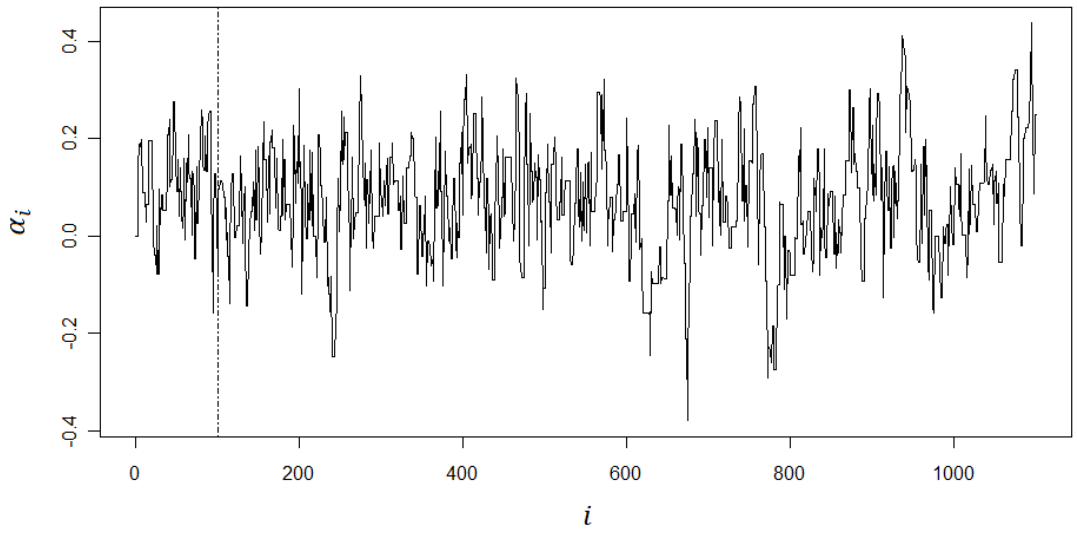


Figure 2.3: Trace of β

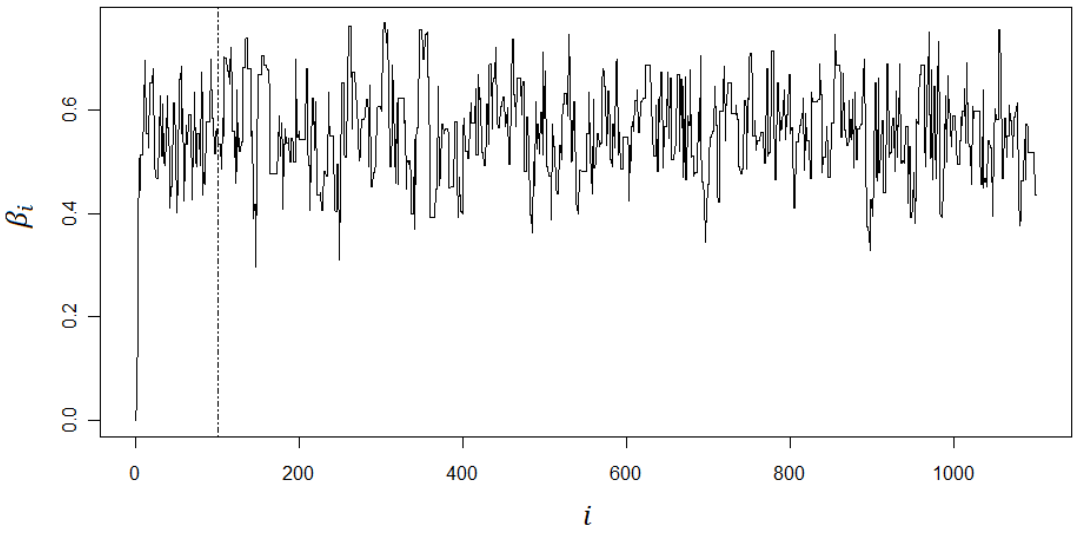
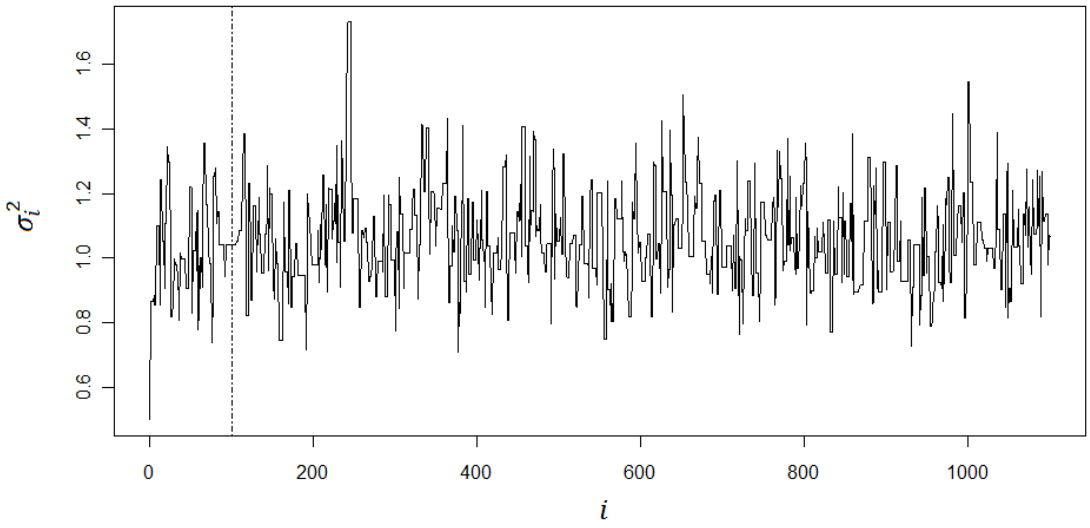


Figure 2.4: Trace of σ^2



2.5.4 Inference on marginal posterior densities

After burn-in, the simulated values of α, β and σ^2 can be thought of being sampled directly from the marginal posterior densities. This allows inference on the marginal posterior densities to be made from these simulated values. Table 2.1 shows the estimates of the posterior mean of α (alpha), β (beta), σ^2 (sigma^2), $\eta = E(x_i) = \alpha/(1-\beta)$ (eta) and $\tau^2 = Var(x_i) = \sigma^2/(1-\beta)$ (tau^2), together with their 95% confidence intervals (CI) and 95% central posterior density regions (CPDR). The first CI is obtained via the 'ordinary' method which assumes no autocorrelations, while the second CI is the batch means confidence interval (see Pawlikowski, 1990). The batch means CI is better since the simulated values generated via MH algorithm are correlated with varying degree, depending on the driver distributions used. The true

value of α, β and σ^2 are included in the table for comparison. The *R* code used to produce the posterior inference is in Appendix B-3.

Note that η and τ^2 are functions of α, β and σ^2. The posterior inference of η and τ^2 are made from $\eta_j = \alpha_j/(1-\beta_j)$ and $\tau_j^2 = \sigma_j^2/(1-\beta_j)$, this is as discussed in Section 2.3.

Table 2.1: Estimated posterior quantities (*R* output)

			95% CI1		95% CI2		95% CPDR	
	true	mean	lower	upper	lower	upper	2.5%	97.5%
alpha	0.2	0.07094	0.0637	0.07818	0.04619	0.09569	-0.1593	0.3020
beta	0.6	0.56014	0.5547	0.56554	0.54870	0.57157	0.3934	0.7397
sigma^2	1.0	1.06224	1.0530	1.07146	1.04029	1.08419	0.8020	1.3926
eta	0.5	0.15717	0.1400	0.17434	0.09968	0.21466	-0.4694	0.6996
tau^2	2.5	2.52140	2.4798	2.56302	2.43336	2.60943	1.6181	4.2473

Note that, with the exception of τ^2, both the ordinary and the batch means confidence intervals fail to contain the true values; this is because the true posterior mean of these parameters is not the same as the true value. The posterior mean is clearly dependent on the values of x generated. In this case the generated x values cause the posterior mean to deviate from their unconditional mean. By increasing n, the number of x values, the posterior means will be closer to the true values.

Figure 2.5, 2.6 and 2.7 show the frequency histograms of the simulated values of α, β and σ^2, overlayed with the non-parametric estimate of the marginal posterior densities. The frequency histograms of the simulated values of η and τ^2 are similar, they are shown in Appendix D-1. Note that in these graphs, the 'dots' point toward the true values and the dotted lines represent the estimated means, 95% CI's and 95% CPDR's of the posterior distributions (from Table 2.1).

Figure 2.5: Frequency histogram of simulated α_j

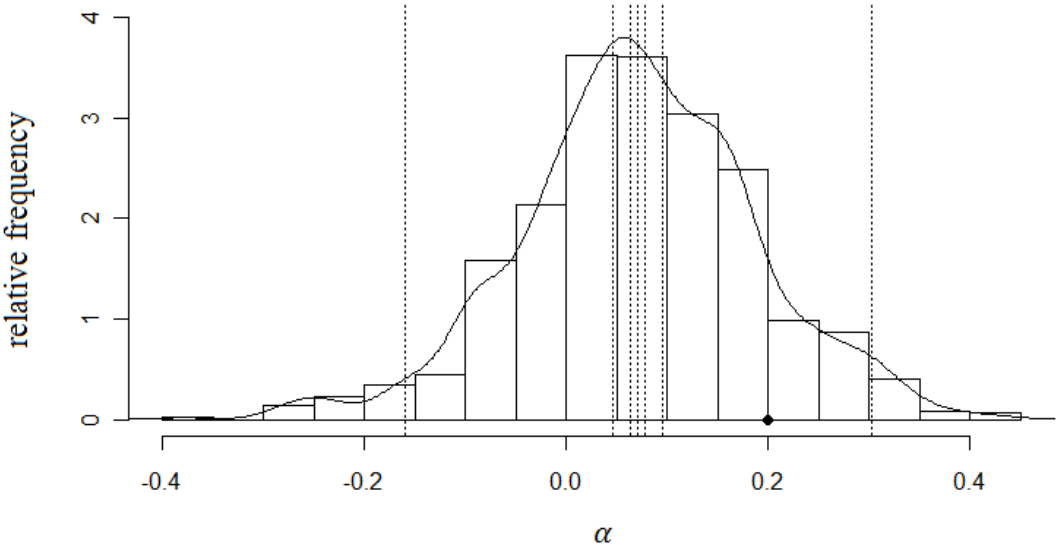


Figure 2.6: Frequency histogram of simulated β_j

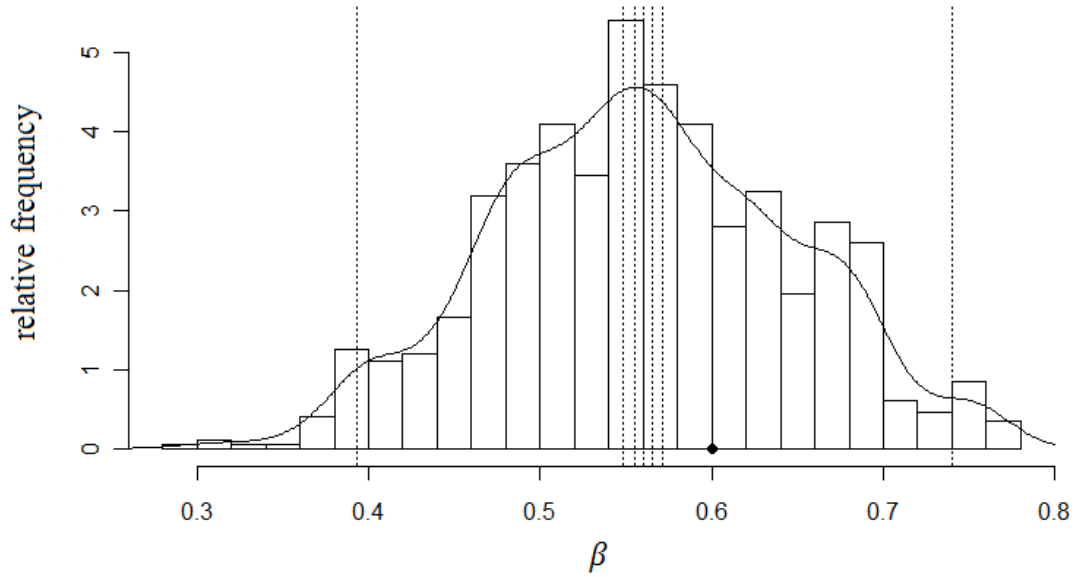
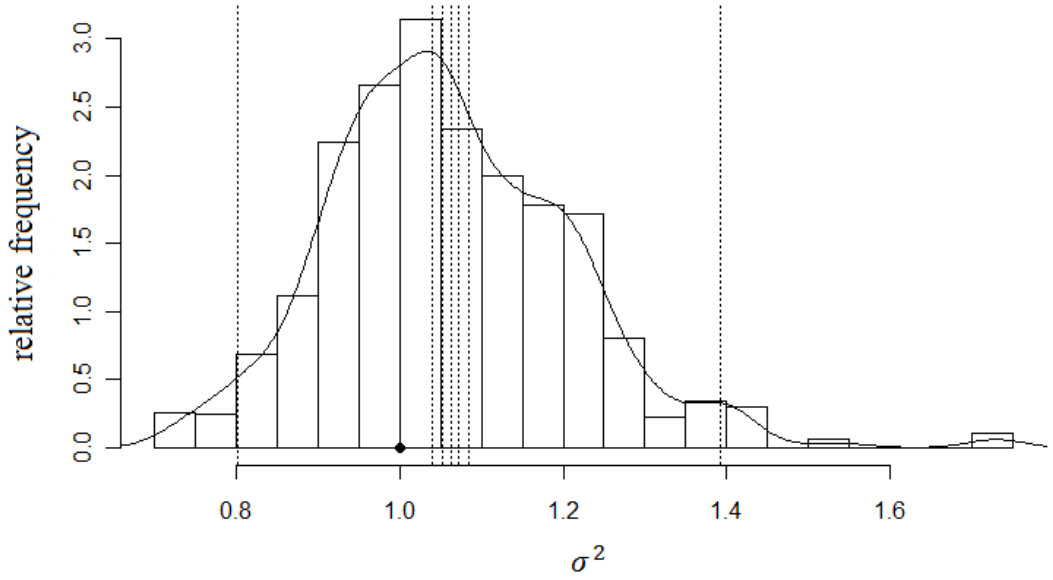


Figure 2.7: Frequency histogram of simulated σ_j^2



2.5.5 Predictive inference

From the simulated values of $\theta = (\alpha, \beta, \sigma^2)$, future x values can be forecasted from each set of $\theta\ (\theta_1, \ldots, \theta_J)$. In this example, the predicted values of $x_{n+1}, \ldots, x_{n+10}$ are generated, according to the following distribution:

$$\left(x_{n+t}^{(j)} \middle| x_{n+t-1}^{(j)}\right) \sim \text{Normal}\left(\alpha_j + \beta_j x_{n+t-1}^{(j)},\ {\sigma_j}^2\right), \quad t = 1, \ldots, 10, \quad j = 1, \ldots, J$$

$$x_n^{(j)} = x_n\,, \qquad j = 1, \ldots, J$$

These generated values of $x_{n+1}, \ldots, x_{n+10}$ are analysed in the same way as α, β and σ^2, giving the predictive inference shown in Table 2.2. The associated sample variances (s^2) and batch means variances (labelled s2b) are also included to illustrate the increasing uncertainties in the prediction as t increases. Note that the 95% CPDR

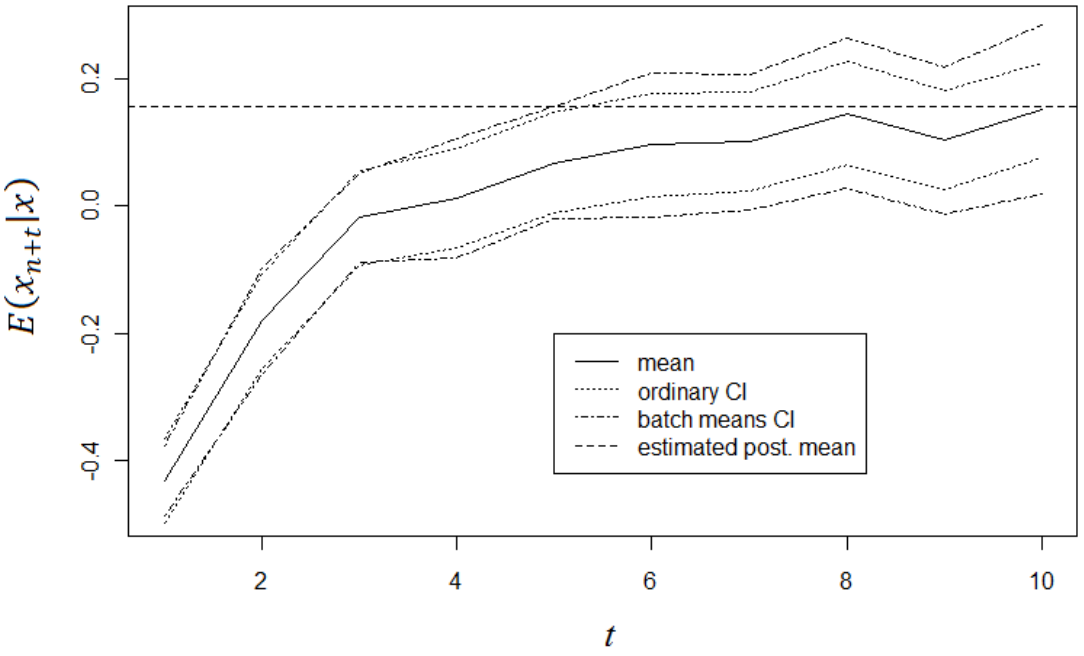
here corresponds to the 95% prediction interval. See the *R* codes in Appendix B-4 on how these estimates are obtained.

Table 2.2: Predictive inference on future x values (*R* output)

		95% CI1		95% CI2		95% CPDR			
	mean	lower	upper	lower	upper	2.5%	97.5%	s2	s2b
x(n+1)	-0.4314	-0.4978	-0.3650	-0.48673	-0.3761	-2.46	1.77	1.15	0.796
x(n+2)	-0.1810	-0.2556	-0.1064	-0.26380	-0.0983	-2.51	2.20	1.45	1.783
x(n+3)	-0.0188	-0.0942	0.0566	-0.08894	0.0514	-2.53	2.29	1.48	1.281
x(n+4)	0.0121	-0.0668	0.0910	-0.08257	0.1067	-2.66	2.47	1.62	2.331
x(n+5)	0.0675	-0.0110	0.1460	-0.02037	0.1553	-2.46	2.62	1.60	2.009
x(n+6)	0.0958	0.0154	0.1763	-0.01799	0.2097	-2.60	2.57	1.68	3.373
x(n+7)	0.1009	0.0229	0.1790	-0.00558	0.2075	-2.40	2.46	1.59	2.954
x(n+8)	0.1457	0.0649	0.2265	0.02825	0.2632	-2.45	2.68	1.70	3.592
x(n+9)	0.1030	0.0252	0.1807	-0.01233	0.2182	-2.39	2.47	1.57	3.460
x(n+10)	0.1512	0.0765	0.2258	0.01836	0.2839	-2.06	2.52	1.45	4.590

The mean of the forecasted future x values converges to the estimated posterior mean of η, ($\hat{\eta} = 0.1572$), as illustrated in Figure 2.8. Note that the 95% CPDR's are too wide to be included, and hence they are omitted.

Figure 2.8: Estimated posterior mean of future x values



An alternative method for predicting the future x values is through the Rao-Blackwell approach, as mentioned in Section 2.3. Using this method, the unnecessary variability which arose from generating x values randomly through its distribution is eliminated. The Rao-Blackwell estimates of the mean of the forecasted x values can be calculated from the following formulae (refer to Appendix A-2 for proof):

$$E(x_{n+t}|x) = E_\theta\big(E(x_{n+t}|x,\theta)\big|x\big) \cong \frac{1}{J}\sum_{j=1}^{J} E\big(x_{n+t}\big|x,\theta_j\big), \qquad t = 1,\dots,10$$

where $E\big(x_{n+t}\big|x,\theta_j\big) = \alpha_j\big(1+\beta_j+\cdots+\beta_j^{t-1}\big)+\beta_j^t x_n\,, \qquad t = 2,\dots,10$

and $E\big(x_{n+1}\big|x,\theta_j\big) = \alpha_j + \beta_j x_n$

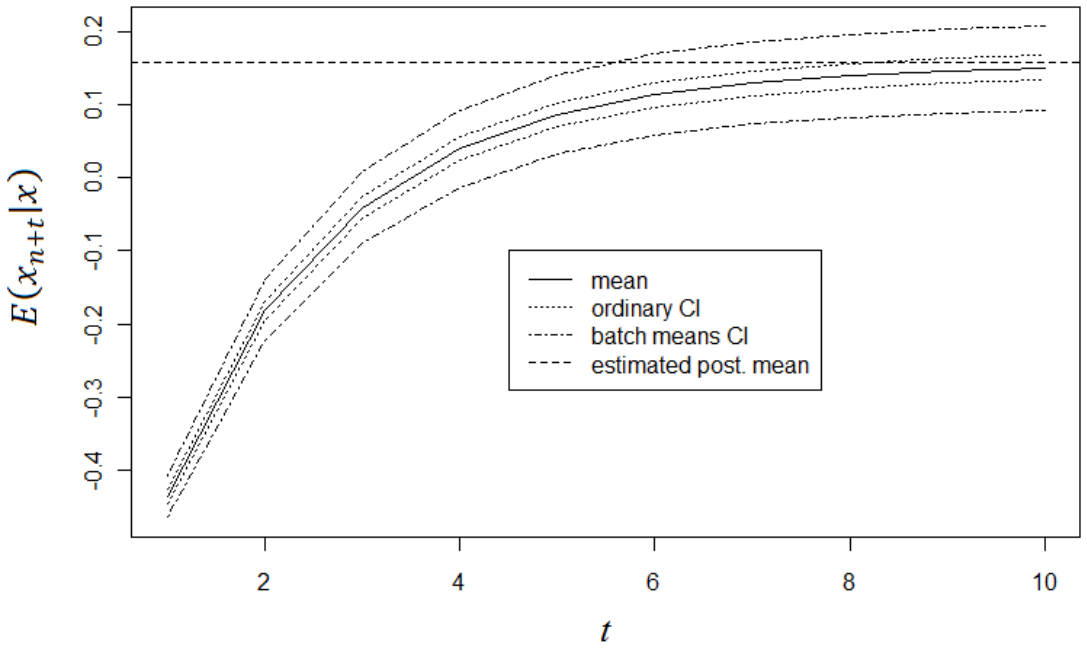
The Rao-Blackwell estimates, together with associated 95% CI's are shown in Table 2.3. Note that the 95% CPDR is irrelevant here since the Rao-Blackwell approach aims to estimate the mean of the future x values directly, *i.e.* the 95% CPDR is not the 95% prediction interval when Rao-Blackwell method is used.

Table 2.3: Rao-Blackwell estimates of posterior mean of future x values

		95% CI1		95% CI2	
	mean	**lower**	**upper**	**lower**	**upper**
x(n+1)	-0.4351	-0.4445	-0.4257	-0.4634	-0.40685
x(n+2)	-0.1814	-0.1946	-0.1681	-0.2232	-0.13953
x(n+3)	-0.0406	-0.0556	-0.0256	-0.0894	0.00814
x(n+4)	0.0392	0.0234	0.0551	-0.0132	0.09166
x(n+5)	0.0855	0.0691	0.1018	0.0310	0.13995
x(n+6)	0.1128	0.0962	0.1295	0.0572	0.16844
x(n+7)	0.1293	0.1124	0.1461	0.0730	0.18558
x(n+8)	0.1393	0.1224	0.1563	0.0826	0.19607
x(n+9)	0.1456	0.1286	0.1626	0.0886	0.20260
x(n+10)	0.1496	0.1325	0.1667	0.0925	0.20673

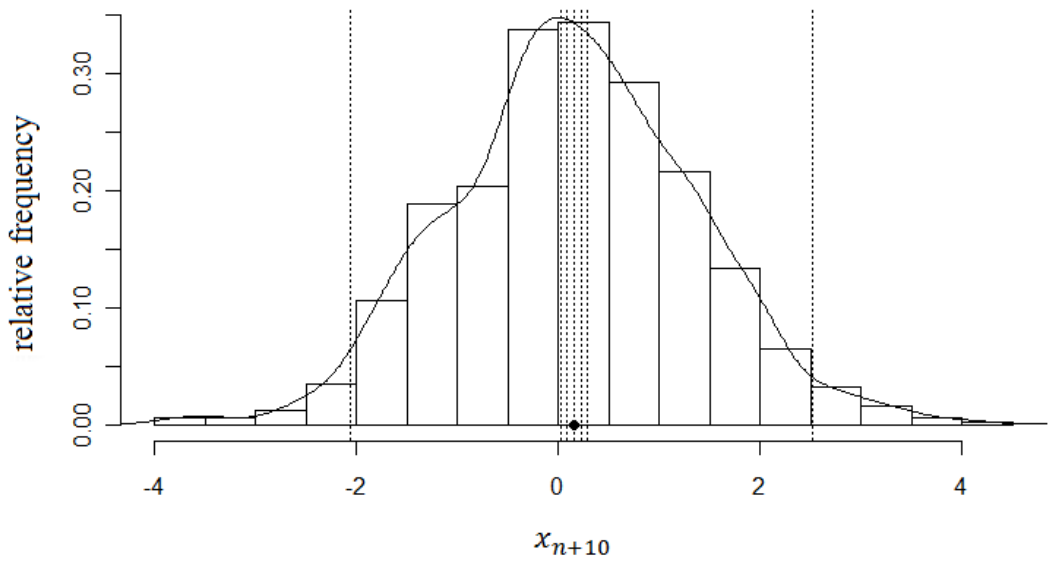
Again, the estimated mean converges to the estimated posterior mean of η, $(\hat{\eta} = 0.1572)$, but in a steadier manner, as opposed to the predictive inference produced through the 'normal approach' (see point $t = 9$ in Figure 2.8). Rao-Blackwell estimation is more precise due to the narrower CI's produced, as can be seen in Figure 2.9.

Figure 2.9: Rao-Blackwell posterior mean of future x values



Note that the predictive distribution of x_{n+t} can be estimated in a similar way as in Subsection 2.5.4, by plotting frequency histograms of the generated values $x_{n+t}^{(j)}$. As an example, the frequency histogram for $x_{n+10}^{(j)}$ is shown in Figure 2.10.

Figure 2.10: Frequency histogram of $x_{n+10}^{(j)}$



2.5.6 Hypothesis testing

From Figure 2.1, one may suggest that x values appear to be random rather than follow a time series model, implying $\beta = 0$. With the Bayesian approach, hypothesis testing can be performed by inspecting the *posterior predictive p-value* (ppp-value), which is analogous to the classical p-value. This subsection carries out hypothesis testing to test the null hypothesis that $\beta = 0$, against the alternative hypothesis $\beta \neq 0$.

Under the null hypothesis, the time series model can be written as $x_i = \alpha + e_i$, or $x_i \sim iid$ Normal(α, σ^2). The ppp-value for the null hypothesis is

$$p = P\big(T(x', \theta) \leq T(x, \theta) \big| x, \beta = 0\ \big)$$

where x' is an independent replicate of x and $T(x, \theta)$ is a suitable test statistic.

The ppp-value can then be estimated by

$$\hat{p} = \frac{1}{J}\sum_{j=1}^{J} I\left(T(x', \theta_j) \leq T(x, \theta_j) | x, \beta = 0\right)$$

where $I(\cdot)$ is the standard indicator function.

Two different test statistics $T(x, \theta)$ are here considered; the first one is the number of runs above or below α; the other is the number of runs above or below $\overline{x}$, the mean of $x = (x_1, \ldots, x_n)$. The ppp-value is estimated to be $\hat{p}_1 = 0.000$ using the first test statistic, and $\hat{p}_2 = 0.000$ using the second; hence the null hypothesis is rejected at 5% significant level. The number of runs for the original dataset x is never higher than the replicated dataset x' generated under the null hypothesis, suggesting the autocorrelation between the x values is highly significant.

Note that to facilitate the estimation of the ppp-values, a simpler MH algorithm is written and run (which assumes $\beta = 0$). The *R* code for the hypothesis testing and also the simpler MH algorithm is presented in Appendix B-5.

2.5.7 Assessment of the MCMC estimates

In order to assess the MCMC estimates of the posterior means, the exact posterior means need to be calculated. Since it is almost impossible to derive the marginal posterior densities, approximation techniques will be employed to determine the exact quantities. First, the joint posterior density, up to a proportional constant, is written in term of its kernel, $k(\alpha, \beta, \sigma^2)$:

$$f(\theta|x) \propto \frac{1}{\sigma^2}\, \emptyset\left(x_1, \frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta}\right) \prod_{i=2}^{n} \emptyset(x_i, \alpha + \beta x_{i-1}, \sigma^2) = k(\alpha, \beta, \sigma^2)$$

Then, the kernel of the marginal posterior densities can be obtained by integrating out the nuisance parameters. For instance, the kernel of the posterior density of α is

$$f(\alpha|x) \propto \int_0^{\infty} \int_{-1}^{1} k(\alpha, \beta, \sigma^2)\, d\beta\, d\sigma^2$$

However, since this integral is intractable, numerical integration is needed to approximate the posterior density at each value of α. The value of $f(\alpha|x)$ is estimated numerically using R for α between -1 and 1, with an increment of 0.02 (*i.e.* for $\alpha = -1, -0.98, \ldots, 0.98, 1$). Note that this method is computational intensive; about 46 minutes were spent just to approximate the exact marginal posterior density of α (numerical integration is performed with processor "Intel® Core™ 2 Duo CPU T5450 @1.66GHz 1.67GHz"). *R* code for the implementation of the numerical integration is presented in Appendix B-6.

Figure 2.11, 2.12 and 2.13 show the approximated exact posterior densities, compared to the non-parametric densities estimated from the simulated values. In these diagrams, the vertical dotted lines represent the posterior means, which were found to be $E(\alpha|x) = 0.081$, $E(\beta|x) = 0.56$ and $E(\sigma^2|x) = 1.05$, while the MCMC estimates are $\hat{\alpha} = 0.07094$, $\hat{\beta} = 0.56014$ and $\widehat{\sigma^2} = 1.06224$. Hence it is apparent that the MCMC methods provide reasonably good estimates. Note that the ordinary CI's of the parameter α and σ^2 (see Table 2.1) do not contain the approximated true posterior means, giving credit to the batch means CI since all the posterior means are contained in the batch means CI's.

Figure 2.11: Approximated exact marginal posterior density of α

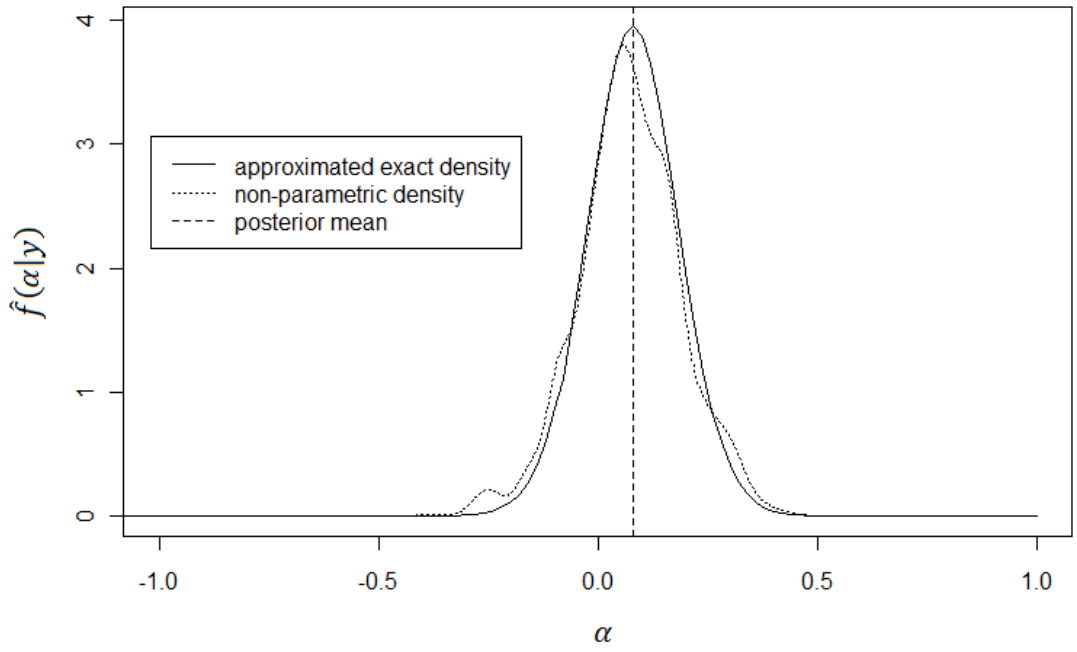


Figure 2.12: Approximated exact marginal posterior density of β

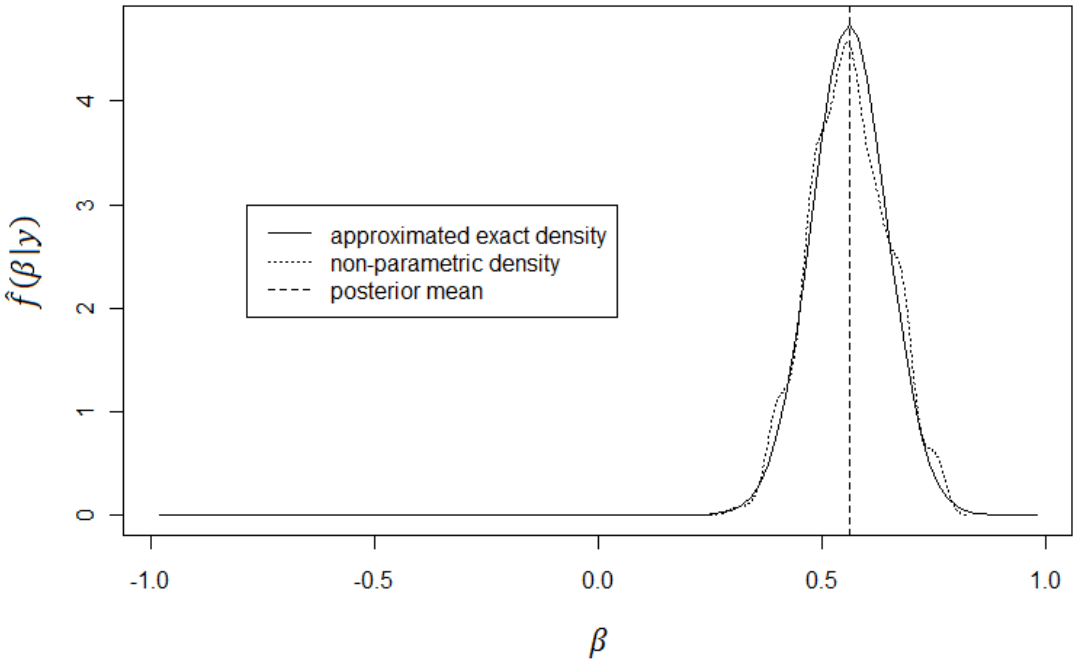
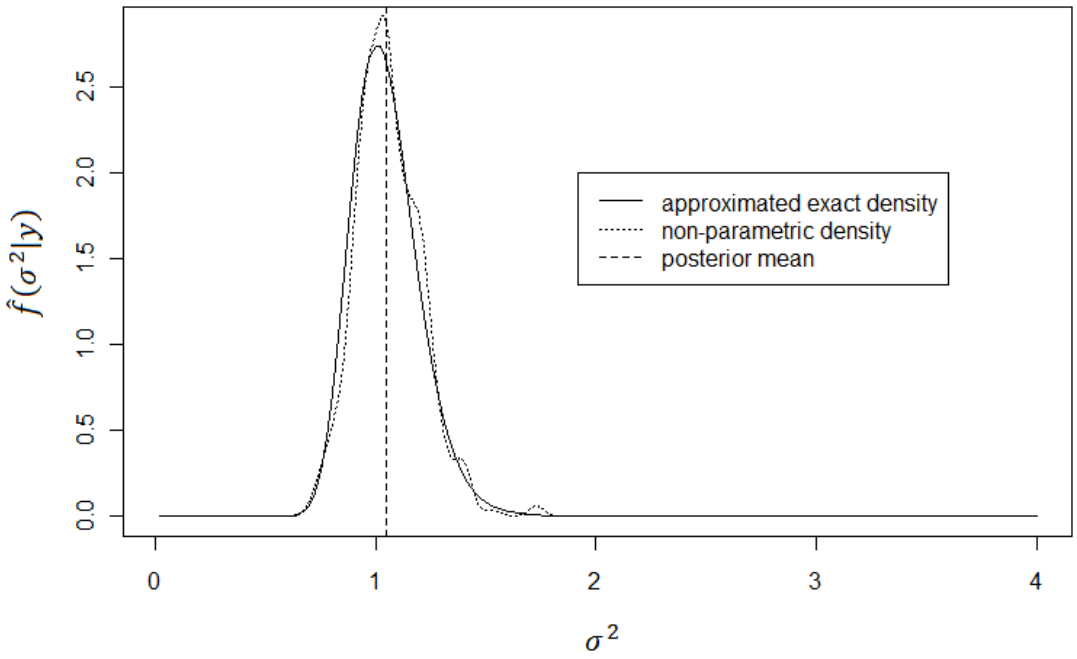


Figure 2.13: Approximated exact marginal posterior density of σ^2



Next, to assess the coverage of the 95% CPDR of α, the MH algorithm was performed again and again to obtain thousand 95% CPDR's of α, and the number of the 95% CPDR which contains the true value of α was determined. Such assessment is computationally intensive (the process took 1 hour and 48 minutes, using the same processor as mentioned above). The *R* code of the assessment is shown in Appendix B-7. The proportion of the CPDR's containing the true value is found to be 0.947, with 95% CI of $(0.933, 0.961)$. It appears that the 95% CPDR of α is reasonable. Similar can be performed for β and σ^2, the respective proportions for β and σ^2 were found to be 0.942 and 0.938, with 95% CI of $(0.928, 0.956)$ and $(0.923, 0.953)$.

The 95% prediction interval (PI) is also assessed in the similar way, for example, to evaluate the coverage of the 95% PI of the mean of the next ten future values of x,

$\tilde{x} = (1/10)\sum_{t=1}^{10} x_{n+t}$, a thousand of these PI's were obtained to estimate the proportion containing the true value of $\tilde{x}$. Note that the true value of $\tilde{x}$ is known in advance through the same generating process as in Subsection 2.5.1, but never used in the MH algorithm. It is found that the proportion of the 95% PI's containing $\tilde{x}$ is 0.933, with 95% CI $(0.9175,\ 0.9485)$. This interval does not contain 0.95, suggesting that the prediction interval is not appropriate. However, it is important to understand that the proportion is just an estimate based on a thousand runs of the MH algorithm, and that the PI itself is an estimate from the MCMC methods; due to the random nature of the simulation process, such estimates are rarely exactly the same as the true values.

2.5.8 *WinBUGS* implementation of the MCMC methods

This subsection repeats the above analysis with *WinBUGS*, for which Gibbs sampler will be used. Note that the minimum burn-in allowed in *WinBUGS* is 500, hence, to obtain an effective sample size of $J = 1000$, $K = 1500$ simulations were performed. The priors are also modified slightly as *WinBUGS* does not allow the priors to be improper; they are now

$$f(\alpha) \sim \text{Normal}(0, 0.001)$$

$$f(\beta) \sim \text{Uniform}(-1, 1)$$

$$f(\sigma^2) \sim \text{Gamma}(0.001, 0.001)$$

WinBUGS code of the time series model is presented in Appendix C-1. The estimated posterior means of α (alpha), β (beta), σ^2 (sigma2), $\eta = E(x_i) = \alpha/(1-\beta)$ (eta) and $\tau^2 = Var(x_i) = \sigma^2/(1-\beta)$ (tau2) are shown in Table 2.4, the 95% CPDR's are

shown under the labels '2.5%' and '97.5%'. The 'MC error' in Table 2.4 corresponds to the batch means approach:

$$\text{MC error} = \frac{\text{batch means standard deviation}}{\sqrt{J}}$$

Table 2.4: Estimated posterior quantities (*WinBUGS* output)

node	mean	sd	MC error	2.5%	median	97.5%
alpha	0.07605	0.1035	0.003158	-0.1255	0.07713	0.2849
beta	0.5626	0.08371	0.003017	0.3966	0.5616	0.7299
eta	0.1724	0.2491	0.007584	-0.3379	0.1784	0.646
sigma2	1.055	0.1581	0.005542	0.8016	1.037	1.408
tau2	2.517	0.7089	0.0239	1.595	2.382	4.286

Note that with the exception of parameter α, the estimated posterior means are very close to the estimates obtained via MH algorithm. With reference to the true posterior mean of α from Subsection 2.5.7, *WinBUGS* performs better in estimating the posterior mean of α, owing to the fact that Gibbs sampler is more efficient.

Performing predictive inference with *WinBUGS* is much simpler than writing the code in *R*, in which one only need to specify the formulae and/or the distributions of the desired quantities. The following shows the next ten forecasted values of x, together with their 95% PI's, shown under the labels '2.5%' and '97.5%'.

Table 2.5: Predictive inference of future x values (*WinBUGS* output)

node	mean	sd	MC error	2.5%	median	97.5%
xnext[1]	-0.3766	1.038	0.03256	-2.35	-0.379	1.766
xnext[2]	-0.1549	1.254	0.03523	-2.691	-0.1786	2.245
xnext[3]	-0.05483	1.255	0.04311	-2.569	-0.03766	2.399
xnext[4]	-0.04479	1.285	0.0486	-2.701	-0.07862	2.49
xnext[5]	0.07476	1.271	0.03215	-2.346	0.04934	2.539
xnext[6]	0.1162	1.296	0.03326	-2.324	0.1212	2.567
xnext[7]	0.1167	1.291	0.03609	-2.569	0.1745	2.726
xnext[8]	0.1504	1.237	0.03873	-2.245	0.1324	2.626
xnext[9]	0.1583	1.238	0.03996	-2.286	0.1846	2.535
xnext[10]	0.159	1.265	0.03908	-2.293	0.1755	2.685

This result is similar to the Rao-Blackwell forecast of the future x values.

Hypothesis testing can be slightly harder to implement in *WinBUGS*, as the posterior predictive p-value is determined using the 'step' function; refer to the *WinBUGS* user manual by Spiegelhalter *et al.* (2003) for more details. Carrying out the same hypothesis test as Subsection 2.5.6, where the null hypothesis is $\beta = 0$, the ppp-value is again found to be zero. The *WinBUGS* code for the alternative time series model for hypothesis testing is shown in Appendix C-2.

Finally, the assessment of the 95% CPDR and the 95% PI are performed. This is done in *R*, as there is no function in *WinBUGS* that facilitates such assessment (see Appendix B-8 for the implementation details). From the thousand 95% CPDR's and 95% PI's obtained via *WinBUGS*, it is found that 93.7% of the CPDR's for α contain the true value of α, with 95% CI $(0.924, 0.954)$. This estimated proportion is very close to the one from the MH algorithm. As for the PI, the proportion containing $\tilde{x}$ is 0.926, with 95% CI $(0.9097,\ 0.9423)$; again, the 95% CI does not contain 0.95.

CHAPTER 3

Claim Run-off Triangles and Data

3.1 Claim run-off triangles

A claim run-off triangle is a presentation used by actuaries, showing the claim liabilities which are long-tailed in nature, meaning that the claims usually take months or years to be fully realised. In a claim run-off triangle, the rows correspond to the years in which accidents or claim events occurred (accident years), while the columns show the years elapsed in which the claims are paid out (development years). An example of a claim run-off triangle is presented in Table 3.1 in the next section.

Note that a claim run-off triangle contains empty cells, which are associated to claim liabilities that are not yet realised. The area in which the cells are empty is generally referred as the *lower triangle*. The total (sum) of the values in the lower triangle equals the required reserve.

3.2 The AFG data

The Automatic Facultative General Liability data from the Historical Loss Development study, known as the AFG data, will be studied. The data were considered by Mack (1994), England & Verrall (2002), and de Jong (2004); hence choosing these data allows comparison to be made across different reserving methods. The AFG data are displayed in Table 3.1 as a run-off triangle. The entries in the run-off triangle represent the *cumulative* claim amounts c_{ij} for accident years i and development years j.

Since it is unclear on the exact timing of these claims, the calendar year for which the first claim (corresponding to entry $(i, j) = (1, 0)$, which has a value of 5012) occurred is defined to be year 1. Consequently, the calendar year associated with each entry (i, j) is $i + j$, for example, the calendar year associates with entry $(i, j) = (4, 2)$ is year 6. Also note that the value of the c_{ij} in the run-off triangle is in unit of \$'000, for instance, the cumulative incurred claim amounts for entry $(i, j) = (4, 2)$ is $\$15766000$.

Table 3.1: AFG data - cumulative incurred claim amounts (c_{ij})

Accident year *i*	Development Year *j*									
	0	1	2	3	4	5	6	7	8	9
1	**5,012**	8,269	10,907	11,805	13,539	16,181	18,009	18,608	18,662	18,834
2	106	4,285	5,396	10,666	13,782	15,599	**15,496**	16,169	16,704	
3	3,410	8,992	13,873	16,141	18,735	22,214	22,863	23,466		
4	5,655	11,555	**15,766**	21,266	23,425	26,083	27,067			
5	1,092	9,565	15,836	22,169	25,955	26,180				
6	1,513	6,445	11,702	12,935	15,852					
7	557	4,020	10,946	12,314						
8	1,351	6,947	13,112							
9	3,133	5,395								
10	2,063									

The exact claim amounts can be obtained from the difference between successive cumulative claim amounts c_{ij}. These are presented in Table 3.2. To illustrate, the exact claim amount for $(i, j) = (4, 2)$ is $15766 - 11555 = 4211$, that means, the claim amount incurred at year 6 due to events originated from year 4 is $\$4211000$. Note that the exact claim amounts generally decrease with development year j for $j \geq 1$, which makes sense intuitively as most claims are usually settled within a couple of years after accidents or claim events occurred. Also note that the claim amount corresponding to

entry $(i, j) = (2, 6)$ is negative; such a negative claim amount could be the result of salvage recoveries, rejection of claims or cancellation due to initial overestimation of the claim liabilities.

Table 3.2: AFG data - exact incurred claim amounts

Accident year *i*	Development Year *j*									
	0	1	2	3	4	5	6	7	8	9
1	5,012	3,257	2,638	898	1,734	2,642	1,828	599	54	172
2	106	4,179	1,111	5,270	3,116	1,817	**-103**	673	535	
3	3,410	5,582	4,881	2,268	2,594	3,479	649	603		
4	5,655	5,900	**4,211**	5,500	2,159	2,658	984			
5	1,092	8,473	6,271	6,333	3,786	225				
6	1,513	4,932	5,257	1,233	2,917					
7	557	3,463	6,926	1,368						
8	1,351	5,596	6,165							
9	3,133	2,262								
10	2,063									

CHAPTER 4

Bayesian Modelling of the AFG Data

4.1 Reserving methods

As discussed in Chapter 1, deterministic approaches to reserving include the traditional chain-ladder (CL) method and the Bornhuetter-Ferguson method. However, these approaches only provide point estimates of the required reserve. Stochastic reserving methods were considered by Scollnik (2004), England & Verrall (2002) and several others.

The required reserve is equal to the value of outstanding claims liabilities, in which the liabilities may or may not discounted to present value. In the thesis, the outstanding claims liabilities are not discounted to present value, this is to facilitate comparison between different reserving methods. Note that the actual reserve held in practice might not be the same as the required reserve; it is conservative to ensure that the actual reserve is at least as great as the required reserve. For convenience, the term 'reserve' will be used to mean the required reserve instead of actual reserve in this thesis.

More precisely, in a claim run-off triangle showing exact claim amounts (such as the one in Table 3.2), the reserve is equal to the sum of the claim liabilities associated with the lower triangle, which need to be estimated. Equivalently, for a cumulative claim run-off triangle (such as in Table 3.1), the reserve (r) has the following formula:

$$r = \sum_{i=1}^{n} \left(c_{i,n-1} - c_{i,n-i} \right)$$

where n is the number of rows or columns (assuming a square run-off triangle).

It is important to note that the accident year i starts from 1 while the development year j starts from 0.

This thesis develops Bayesian models based on classical reserving models first proposed by Hertig (1985) and later extended by de Jong (2004). These Bayesian models, together with their classical counterparts, are discussed in the following section.

4.2 Bayesian models

Consider a run-off triangle with entries c_{ij} which represents the cumulative claim liabilities for accident years i and development years j. Define the development factors δ_{ij} for $j \geq 1$ as the continuous growth rates in cumulative claim liabilities of accident years i in development year j, and formulate this quantity as the log-link ratio. Also define δ_{i0} as the natural logarithm of the initial claim liabilities of accident year i. Thus:

$$\delta_{ij} = \ln\left(\frac{c_{ij}}{c_{i,j-1}}\right), \qquad i = 1, \dots, n, \qquad j = 1, \dots, n-1$$

$$\delta_{i0} = \ln(c_{i0}), \qquad i = 1, \dots, n$$

The variable n denotes number of years for which data are available; note that the claim run-off triangle considered in the thesis (AFG data) has equal numbers of rows and columns.

The development factors δ_{ij} are calculated and displayed in Table 4.1. Note that one of the development factors (δ_{26}) is negative, which corresponds to the negative incremental claim. Besides, the δ_{ij} for which $j = 0$ appears to be exceptionally large compared to the rest; this is due to the difference in the definition of the development

factors, for which δ_{i0} are not the growth rates of the cumulative claims. Also note that the development factors tend to decrease across development years.

Table 4.1: AFG data - development factors (δ_{ij})

Accident year *i*	Development Year *j*									
	0	1	2	3	4	5	6	7	8	9
1	8.520	0.501	0.277	0.079	0.137	0.178	0.107	0.033	0.003	0.009
2	4.663	3.699	0.231	0.681	0.256	0.124	**-0.007**	0.043	0.033	
3	8.134	0.970	0.434	0.151	0.149	0.170	0.029	0.026		
4	8.640	0.715	0.311	0.299	0.097	0.107	0.037			
5	6.996	2.170	0.504	0.336	0.158	0.009				
6	7.322	1.449	0.596	0.100	0.203					
7	6.323	1.976	1.002	0.118						
8	7.209	1.637	0.635							
9	8.050	0.543								
10	7.632									

Hertig (1985) made an assumption that the development factors δ_{ij} are uncorrelated and can be written in the following form:

$$\delta_{ij} = \mu_j + h_j \varepsilon_{ij} \tag{1}$$

$$\varepsilon_{ij} \sim iid \text{ Normal}(0, \sigma^2), \qquad i = 1, \dots, n, \qquad j = 0, \dots, n-1$$

Hence, each δ_{ij} has a normal distribution with mean μ_j and variance $h_j^2\sigma^2$; that is,

$$\delta_{ij} \sim \text{Normal}\left(\mu_j,\ h_j^2\sigma^2\right), \qquad i = 1, \dots, n, \qquad j = 0, \dots, n-1$$

In this model (which will hereafter be called Hertig's model), h_0 is set to 1 to avoid over-parameterisation. Note that this model allows the incremental claim liabilities to take negative values, since it is possible for the development factors to be negative.

Under the classical approach, the parameters μ_j, h_j and σ^2 are assumed to be unknown constants that need to be estimated, either by maximum likelihood estimation or the

method of moments, as, for example, in de Jong (2004). In contrast, the Bayesian approach treats the parameters μ_j, h_j and σ^2 as random variables, with a joint prior distribution. Often these parameters are taken as *a priori* independent, uninformative and improper, as follows:

$$f(\mu_j) \propto 1, \qquad \mu_j \in \Re, \qquad j = 0, \dots, n-1$$

$$f(h_j) \propto \frac{1}{h_j}, \qquad h_j > 0, \qquad j = 1, \dots, n-1$$

$$f(\sigma^2) \propto \frac{1}{\sigma^2}, \qquad \sigma^2 > 0$$

As mentioned before, care must be taken when using improper priors since this might lead to posterior distributions being improper, giving nonsensical inference. Note that actuaries typically have some prior knowledge regarding the parameters, which may be available through past experience, or simply based on their actuarial judgement; the prior distributions can then be modified accordingly.

Hertig's model assumes that the growth rates across accident years i have the same distribution on each development year j. However, de Jong found that the development factors are correlated and suggested adding correlation terms to Hertig's model. The following are three different modifications to Hertig's model (1) suggested by de Jong.

i. $$\delta_{ij} = \mu_j + h_j(\varepsilon_{ij} + \theta_j \varepsilon_{i,j-1}) \qquad (2)$$

This defines the *development correlation model*. Here, correlation of the development factors δ_{ij} across development years j is captured by the parameter θ_j. This model asserts that the magnitude of each δ_{ij} directly influences the development factors for the next year, $\delta_{i,j+1}$.

ii. $\delta_{ij} = \mu_{ij} + h_j \varepsilon_{ij}$, $\mu_{i+1,j} = \mu_{ij} + \lambda_j \eta_{ij}$

Under this *accident correlation model*, correlation of the development factors δ_{ij} across accident years i is modelled by the parameters λ_j and η_{ij}.

iii. $\delta_{ij} = \mu_j + h_j(\tau_{i+j} + \varepsilon_{ij})$, $\tau_{i+j+1} = \tau_{i+j} + \kappa \eta_{i+j}$

This *calendar correlation model* introduces the parameters τ_{i+j}, κ and η_{i+j} to model the correlation of δ_{ij} across calendar years.

Bayesian implementation of Hertig's model and the development correlation model (2) will be discussed in detail in Chapter 5. Note that, for simplicity, the term "Hertig's model" will refer to the Bayesian implementation of Hertig's model rather than its classical counterpart.

CHAPTER 5

Analysis and Results

5.1 Hertig's model

Hertig's model, introduced in Section 4.2, will be used in this chapter to model the AFG data. Ideally, the prior distributions of this model's parameters are uninformative and improper, as follows:

$$\mu_j \sim \text{Normal}(0, \infty), \qquad j = 0, \dots, n-1$$

$$h_j \sim \text{Gamma}(0, 0), \qquad j = 1, \dots, n-1$$

$$\sigma^2 \sim \text{Gamma}(0, 0)$$

However, as the modelling will be done in *WinBUGS*, which does not allow priors to be improper, the following *vague* prior distributions are chosen in an attempt to give the best representation of the uninformative and improper priors:

$$\mu_j \sim \text{Normal}(0, 100000), \qquad j = 0, \dots, n-1$$

$$h_j \sim \text{Gamma}(0.1, 0.1), \qquad j = 1, \dots, n-1$$

$$\sigma^2 \sim \text{Gamma}(0.1, 0.1)$$

Note that choosing a prior distribution which is more diffuse for h_j and σ^2, such as $\text{Gamma}(0.0001, 0.0001)$, leads to an error message in *WinBUGS*. This is because the software cannot accept priors that are too diffuse. Since there are 20 parameters to be estimated from only 55 data points, the resulting posterior distributions end up being improper. (If there were many more data points to estimate the same number of parameters, the posterior might be proper.)

The *WinBUGS* code for this model is presented in Appendix C-3. With the starting points chosen to be $\mu_j = 0$, $h_j = 1$ and $\sigma^2 = 1$, $K = 11000$ simulated values of each parameter were created (in 6 seconds, with the same processor described above) *via* the Gibbs sampler. By burning-in the first 1000 simulated values, estimates of the posterior means were obtained from the remaining 10000 simulations; the *WinBUGS* output is displayed in Table 5.1. Classical estimates by de Jong are included for comparison. Note that in Table 5.1, each h[j] corresponds to parameter h_{j-1}, while each mu[j] corresponds to parameter μ_{j-1} and sigma2 corresponds to parameter σ^2.

Table 5.1: Posterior estimates of μ_j, h_j and σ^2 (Hertig's model)

node	mean	sd	MC error	2.5%	median	97.5%	de Jong's
h[2]	0.9654	0.3503	0.008836	0.4693	0.9088	1.793	0.8571
h[3]	0.2448	0.09868	0.002577	0.1122	0.2254	0.4892	0.2143
h[4]	0.2142	0.09374	0.002222	0.09399	0.195	0.4497	0.1786
h[5]	0.05783	0.02786	6.26E-4	0.02417	0.05152	0.1309	0.0446
h[6]	0.07323	0.04005	9.275E-4	0.02874	0.06386	0.1738	0.0536
h[7]	0.05735	0.04021	8.36E-4	0.01953	0.04666	0.1623	0.0357
h[8]	0.01322	0.01739	5.151E-4	0.003171	0.008864	0.04989	0.0089
h[9]	**0.09772**	**0.3435**	0.01146	0.006977	0.02854	0.6174	0.0089
h[10]	**0.8656**	**3.246**	0.1542	3.468E-10	0.001739	9.444	**0.00**
mu[1]	**7.346**	0.4009	0.003863	6.549	7.345	8.147	7.35
mu[2]	1.518	0.3938	0.004067	0.7263	1.518	2.311	1.52
mu[3]	0.4981	0.1079	0.001066	0.2839	0.4984	0.7141	0.50
mu[4]	0.2515	0.1006	0.00104	0.04582	0.2527	0.4482	0.25
mu[5]	0.1666	0.0297	3.031E-4	0.1069	0.1667	0.226	0.17
mu[6]	0.1182	0.04235	4.402E-4	0.03564	0.1178	0.2026	0.12
mu[7]	0.0409	0.04056	4.143E-4	-0.03797	0.04104	0.118	0.04
mu[8]	0.03362	0.01372	1.166E-4	0.01118	0.03389	0.05449	0.03
mu[9]	0.01951	**0.2885**	0.001898	-0.1811	0.01824	0.24	0.02
mu[10]	-0.01621	**3.873**	0.03726	-3.992	0.009174	4.147	0.01
sigma2	1.612	0.8962	0.03325	0.626	1.387	3.965	1.2544

There are a few points that should be noted. First, the posterior estimate of μ_0 is exceptionally large ($\hat{\mu}_0 = 7.346$, bolded above) compared to other μ_j. This is purely due to the fact that the development factors (δ_{ij}) for $j = 1$ are fundamentally different in definition to the other development factors. Secondly, μ_j is estimated to be smaller as development year j gets larger; this makes sense intuitively, because total claims tend to

become smaller with time after the accident year. The estimates of h_j follow the same pattern, with the exception of h_5, h_8 and h_9.

Thirdly, note that the estimates of h_8 and h_9 are very large, with extremely large standard deviations and wide 95% CPDR's compared to the other h_j (see the bolded figures in Table 5.1). The reason behind this is that there are only a few data points available to estimate these parameters, as can be seen on the right hand side of the claim run-off triangle. Since uninformative priors are used, lack of data points leads to the posterior distributions being highly disperse. Note that this problem is also present in the estimates of μ_8 and μ_9.

Finally, the estimates of μ_j are found to be very close to the classical estimates obtained by de Jong. On the other hand, the variability of the development factors, captured by the parameters h_j and σ^2, are found to be higher than the classical estimates. However, the classical approach tends to underestimate uncertainty regarding the parameters. This is most apparent for h_9, de Jong took h_9 as 0 since there is only one data point corresponding to $j = 9$ (in which case the variability cannot be estimated).

Traces of the simulated values of μ_1, μ_5 and μ_9 produced *via WinBUGS* are shown in Figure 5.1, 5.2 and 5.3 respectively. These traces show that the simulated values of μ_1 exhibit good mixing; the simulated values of μ_5 exhibit a relatively worse mixing, with some of the simulated values being either too large or too small; the simulated values of μ_9 clearly do not exhibit good mixing, since a significantly large number of the simulated values are exceptionally far from their mean. This shows that the parameters μ_j get progressively imprecise as j increases, mainly due to the decreasing amount of data points available to estimate the parameters. Note that the existence of spikes

(extremely large values) in Figure 5.3 is the primary reason for the high standard deviation of the estimate μ_9 (see Table 5.1). The traces of the simulated h_j are similar and give the same inference for h_j. The traces of simulated μ_j, h_j and σ^2 are shown in Appendix D-2 for completeness.

Figure 5.1: Trace of simulated μ_1

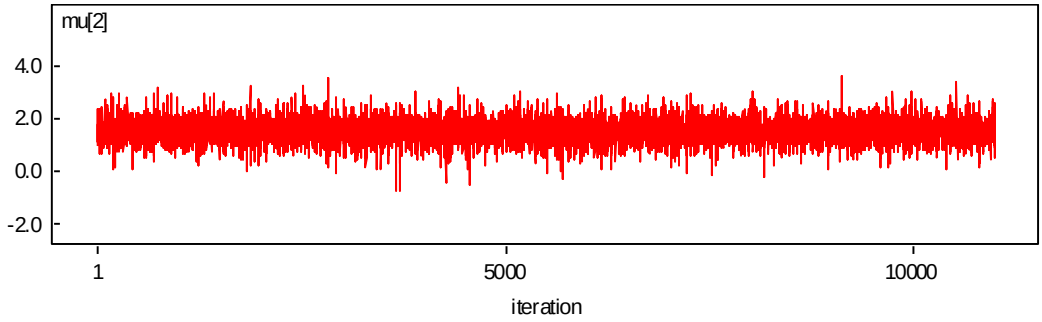


Figure 5.2: Trace of simulated μ_5

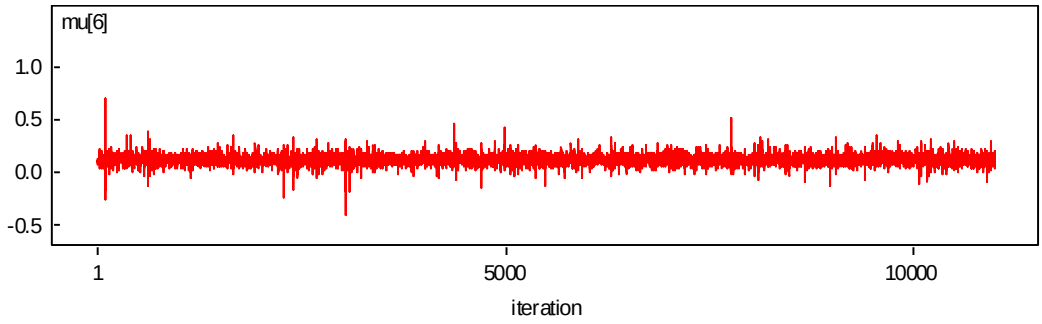
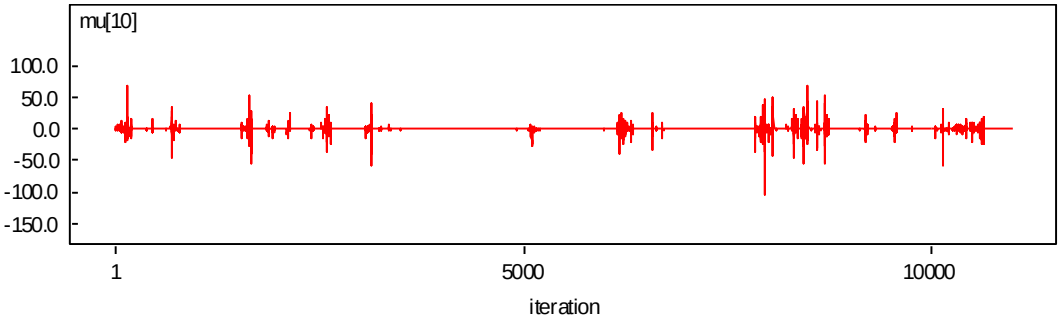


Figure 5.3: Trace of simulated μ_9



Predictive inference on the future development factors can be obtained by generating random observations of the future development factors directly. In *WinBUGS*, this is easily done by specifying the distribution of the future development factors, which is no different from the distribution of the current development factors, as defined in Section 4.2. In addition, the future cumulative claim liabilities and the reserve are readily obtained since they can be expressed as explicit functions of the future development factors. The predictive inference is presented in Table 5.2 in the same format as Table 5.1; again, note that each delta[i,j] corresponds to the development factor $\delta_{i,j-1}$.

Most of the predicted development factors are reasonable predictions of the future, except for development years 8 and 9, where the high variability in predictions is caused by the imprecise posterior estimates of the underlying parameters. For instance, the prediction of $\delta_{10,9}$ (delta[10,10] in Table 5.2) is accompanied by an extremely large standard deviation (5.535) and a very wide 95% prediction interval $(-5.71, 5.75)$. These large standard deviations and prediction intervals are bolded in Table 5.2; they correspond to the estimates where $j = 9$. These predictions do not provide any real information regarding the future claim amounts, because a single data point is insufficient to project future claim payments. For example, the 95% prediction interval for $\delta_{10,9}$ suggests that the cumulative claim liabilities for next year will be between $e^{-5.71} = 0.3\%$ and $e^{5.75} = 31400\%$ of the current cumulative claim.

Table 5.2: Predictive inference on the future δ_{ij} (Hertig's model)

node	mean	sd	MC error	2.5%	median	97.5%
delta[2,10]	-0.06639	**5.792**	0.05582	**-6.386**	0.009174	**4.653**
delta[3,9]	0.02169	0.4779	0.005613	-0.3242	0.01859	0.4005
delta[3,10]	-0.0766	**5.8**	0.05948	**-5.966**	0.009174	**5.193**
delta[4,8]	0.03373	0.02775	2.721E-4	-0.01056	0.03376	0.07774
delta[4,9]	0.0194	0.5035	0.003882	-0.3461	0.01726	0.3832
delta[4,10]	-0.04602	**5.278**	0.05102	**-5.723**	0.009174	**5.258**
delta[5,7]	0.04168	0.08435	7.927E-4	-0.1306	0.04108	0.2113
delta[5,8]	0.03362	0.02701	2.492E-4	-0.01203	0.03359	0.07488
delta[5,9]	0.02299	0.4981	0.004636	-0.3235	0.01821	0.4023
delta[5,10]	0.01581	**5.405**	0.03206	**-5.733**	0.009174	**5.721**
delta[6,6]	0.1181	0.1037	0.00111	-0.08619	0.1187	0.3224
delta[6,7]	0.0414	0.0922	9.204E-4	-0.1322	0.04117	0.225
delta[6,8]	0.03348	0.02793	2.81E-4	-0.01262	0.03348	0.07792
delta[6,9]	0.0274	0.6109	0.004985	-0.3262	0.01809	0.4015
delta[6,10]	-0.02916	**6.729**	0.05375	**-5.362**	0.009174	**5.28**
delta[7,5]	0.1662	0.07821	7.713E-4	0.006882	0.1655	0.3289
delta[7,6]	0.1186	0.1057	0.001188	-0.09621	0.1186	0.3297
delta[7,7]	0.04133	0.0885	8.649E-4	-0.1293	0.04182	0.2084
delta[7,8]	0.03307	0.03147	3.291E-4	-0.01337	0.03376	0.07435
delta[7,9]	0.02044	0.5108	0.005545	-0.3044	0.01844	0.3958
delta[7,10]	-0.01812	**5.94**	0.05174	**-6.015**	0.009174	**6.19**
delta[8,4]	0.2552	0.2903	0.003011	-0.3112	0.2522	0.8456
delta[8,5]	0.1674	0.07901	7.884E-4	0.01359	0.1667	0.3249
delta[8,6]	0.1174	0.105	9.475E-4	-0.08331	0.1173	0.3176
delta[8,7]	0.04268	0.09128	9.016E-4	-0.1322	0.04109	0.224
delta[8,8]	0.03385	0.02704	2.47E-4	-0.009508	0.03392	0.07699
delta[8,9]	0.01833	0.5527	0.004303	-0.3269	0.01807	0.4282
delta[8,10]	-0.01818	**5.352**	0.04204	**-6.319**	0.009174	**5.553**
delta[9,3]	0.5033	0.3174	0.002829	-0.1371	0.5049	1.135
delta[9,4]	0.2473	0.2803	0.003139	-0.3194	0.2479	0.7994
delta[9,5]	0.166	0.08086	8.958E-4	-0.002365	0.1663	0.3278
delta[9,6]	0.1175	0.1075	0.001102	-0.09249	0.1169	0.3274
delta[9,7]	0.0402	0.09023	9.666E-4	-0.1353	0.03959	0.2169
delta[9,8]	0.03385	0.03023	2.84E-4	-0.01223	0.03376	0.0774
delta[9,9]	0.01652	0.4597	0.005298	-0.3345	0.01874	0.3518
delta[9,10]	-0.002851	**5.936**	0.07171	**-5.806**	0.009174	**5.566**
delta[10,2]	1.506	1.225	0.01138	-0.977	1.508	3.946
delta[10,3]	0.4966	0.3173	0.003139	-0.1283	0.4977	1.132
delta[10,4]	0.2549	0.289	0.002871	-0.3159	0.256	0.8419
delta[10,5]	0.1672	0.07973	8.097E-4	0.009221	0.1675	0.3259
delta[10,6]	0.1174	0.1036	0.001107	-0.09371	0.1187	0.3224
delta[10,7]	0.0425	0.08824	8.206E-4	-0.1273	0.04155	0.2132
delta[10,8]	0.0334	0.02738	3.026E-4	-0.01053	0.03362	0.07786
delta[10,9]	0.01795	0.4781	0.00527	-0.3285	0.0181	0.3751
delta[10,10]	0.02923	**5.535**	0.07309	**-5.71**	0.009174	**5.75**

Note that attempting to predict the future cumulative claim liabilities c_{ij} and reserve r in *WinBUGS* results in an error, due to the very imprecise predictions of $\delta_{i,9}$, since by chance the generated development factors can be very large, leading to even larger claim liabilities. In fact, the error is caused by some of the cumulative claim liabilities reaching a very large number that is not supported by *WinBUGS*. However, there are

ways to achieve the predictive inference on the cumulative claim liabilities and the reserve. One method is to reduce the number of effective simulations J. The other is to explicitly set an upper bound for the predicted development factors. The predictive inference obtained via the first method (with $J = 2000$) is shown in Appendix D-3 for completeness. A snapshot of the predictive inference is shown in Table 5.3. This inference is not entirely useful, since the predicted reserve is unreasonably large.

Table 5.3: Predictive inference on reserve (Hertig's model)

node	mean	sd	MC error	2.5%	median	97.5%
reserve	**1.086E+33**	4.406E+34	1.067E+33	-75100.0	86260.0	5.208E+11

5.2 A modified Hertig's model

As can be seen in the previous section, using a single data point for estimation leads to extremely large standard deviations and hence imprecise estimates. This calls for a modification to Hertig's model that rectifies this problem. Since the parameters μ_j and h_j decrease with development years j, one possible approach is to model the structure of μ_j and h_j directly. Assuming μ_j and h_j decrease exponentially for $j > 2$, the following parameterisation was used as a 'fix' to Hertig's model:

$$\delta_{ij} \sim \text{Normal}\left(\mu_j,\ h_j^2\sigma^2\right), \qquad i = 1, \dots, n, \qquad j = 0, \dots, n-1$$

$$\mu_j = M^{j-2} \cdot \mu_2, \qquad 0 < M < 1, \qquad j > 2$$

$$h_j = N^{j-2} \cdot h_2, \qquad 0 < N < 1, \qquad j > 2$$

The vague prior distributions associated with this modification are as follows:

$$\mu_j \sim \text{Normal}(0, 100000)\, I(0,), \qquad j = 0, 1, 2$$

$$h_j \sim \text{Gamma}(0.0001, 0.0001), \qquad j = 1, 2$$

$$\sigma^2 \sim \text{Gamma}(0.0001, 0.0001)$$

$$M \sim \text{Uniform}(0,1)$$

$$N \sim \text{Uniform}(0,1)$$

where $I(0,)$ forces the simulated values of μ_j to be positive (this notation is consistent with *WinBUGS* syntax).

These priors are not really uninformative; for instance, the parameters μ_j, M and N are deliberately restricted to be positive since it is assumed that the mean of the development factors are positive (a development factor can still be negative under this model). This assumption is supported by the MCMC estimates in Table 5.1.

This modification reduces the number of parameters that need to be estimated from 20 to 8 and assumes that μ_j and h_j decrease exponentially with a constant rate. (This assumption could be relaxed by allowing μ_j and h_j to decrease with varying rates, but this would increase the required number of parameters.)

As in Section 5.1, Gibbs sampling was used to simulate $K = 11000$ values of each parameter. The starting points were chosen to be $\mu_j = 0.1$, $h_j = 1$, $\sigma^2 = 1$, $M = 0.5$ and $N = 0.5$. The *WinBUGS* code for this model is presented in Appendix C-4. Again, a burn-in of $B = 1000$ was chosen so that a final sample of $J = 10000$ simulated values were used in producing the posterior mean estimates, as shown by the *WinBUGS*

output in Table 5.4. Note that h[j] represents h_{j-1}, mu[j] represents μ_{j-1} and sigma2 represents σ^2.

Table 5.4: Posterior estimates of μ_j, h_j, σ^2, M and N (modified Hertig's model)

node	mean	sd	MC error	2.5%	median	97.5%
M	0.5715	0.03239	5.412E-4	0.5051	0.5723	0.6342
N	0.5989	0.03991	8.131E-4	0.5254	0.5972	0.6818
h[2]	0.9303	0.3557	0.009583	0.4297	0.8665	1.83
h[3]	0.2341	0.07353	0.00266	0.1165	0.2249	0.4024
h[4]	0.1388	0.03954	0.001446	0.07263	0.1347	0.2253
h[5]	0.08266	0.02243	7.974E-4	0.0441	0.08074	0.1313
h[6]	0.04945	0.01365	4.485E-4	0.0261	0.0484	0.07896
h[7]	0.02972	0.008864	2.588E-4	0.01517	0.02881	0.05011
h[8]	0.01794	0.006021	1.539E-4	0.008543	0.01715	0.03239
h[9]	0.01088	0.004188	9.444E-5	0.004783	0.01017	0.02096
h[10]	0.006628	0.002942	5.969E-5	0.002635	0.00606	0.01388
mu[1]	7.356	0.4189	0.004546	6.536	7.356	8.191
mu[2]	1.521	0.3949	0.003977	0.7353	1.518	2.319
mu[3]	0.4954	0.07502	0.001277	0.355	0.4927	0.6517
mu[4]	0.2814	0.03336	4.899E-4	0.2161	0.2813	0.3493
mu[5]	0.1604	0.01719	1.755E-4	0.1259	0.1605	0.1946
mu[6]	0.09167	0.01119	9.605E-5	0.06856	0.09184	0.1135
mu[7]	0.05256	0.008153	8.652E-5	0.03586	0.05257	0.06861
mu[8]	0.03023	0.005968	7.427E-5	0.01827	0.03014	0.04223
mu[9]	0.01744	0.004263	5.811E-5	0.009349	0.01729	0.02636
mu[10]	0.01009	0.002971	4.277E-5	0.00475	0.009922	0.01647
sigma2	1.802	1.082	0.04033	0.664	1.511	4.661

Although the output shows 22 parameter estimates, there are only 8 free parameters since μ_j and h_j for $j = 3, \dots, 9$ are essentially functions of the other parameters, namely μ_2, h_2, M and N. Note that the problem with extremely large standard deviations was eliminated with the reduction in the number of free parameters; hence overall the estimates are more precise, with smaller standard deviations and narrower 95% CPDR's.

The traces of the simulated values suggest quick convergence and good mixing. Since these traces are similar to the trace in Figure 5.1, they are not displayed here. These traces are available in Appendix D-4 for completeness.

The predictive inference on the development factors δ_{ij} is omitted in this section as they are not of particular interest; rather, attention is given to the predicted cumulative claim

liabilities c_{ij} and the reserve r. Unlike in the preceding section, obtaining the predictive inference with *WinBUGS* did not produce any error messages under modified Hertig's model. The predictive inference is shown in Table 5.5, where c.pred[i,j] represents $c_{i,j-1}$.

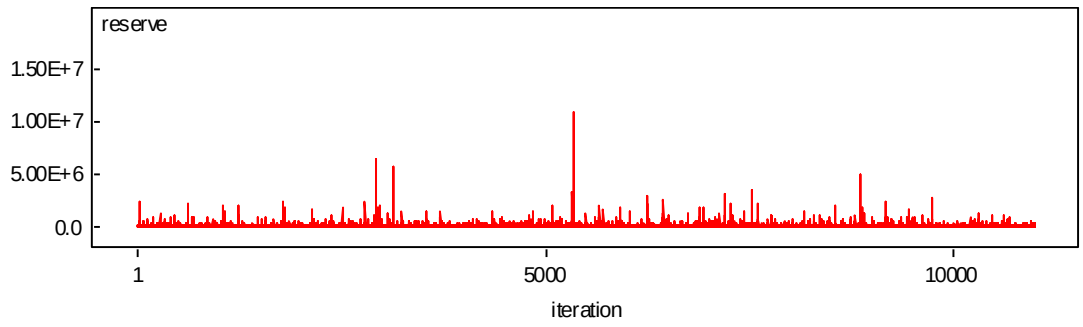
Table 5.5: Predictive inference on future c_{ij} and r (modified Hertig's model)

node	mean	sd	MC error	2.5%	median	97.5%
c.pred[2,10]	16880.0	156.3	1.714	16560.0	16870.0	17190.0
c.pred[3,9]	23880.0	353.2	3.316	23180.0	23880.0	24590.0
c.pred[3,10]	24130.0	439.0	4.899	23260.0	24120.0	25030.0
c.pred[4,8]	27910.0	655.7	6.95	26640.0	27900.0	29270.0
c.pred[4,9]	28410.0	813.4	8.835	26860.0	28380.0	30080.0
c.pred[4,10]	28700.0	894.5	9.416	26990.0	28670.0	30520.0
c.pred[5,7]	27620.0	1043.0	11.59	25630.0	27600.0	29750.0
c.pred[5,8]	28470.0	1292.0	13.94	2.6E+4	28430.0	31170.0
c.pred[5,9]	28970.0	1414.0	15.03	26280.0	28930.0	31950.0
c.pred[5,10]	29270.0	1488.0	15.95	26450.0	29240.0	32410.0
c.pred[6,6]	17400.0	1082.0	12.43	15350.0	17390.0	19600.0
c.pred[6,7]	18360.0	1363.0	14.68	15820.0	18340.0	21140.0
c.pred[6,8]	18940.0	1497.0	16.04	16130.0	18890.0	2.2E+4
c.pred[6,9]	19280.0	1576.0	17.08	16310.0	19230.0	22480.0
c.pred[6,10]	19480.0	1620.0	17.67	16460.0	19430.0	22800.0
c.pred[7,5]	14510.0	1495.0	14.65	11750.0	14440.0	17650.0
c.pred[7,6]	15930.0	1933.0	16.63	12430.0	15800.0	20100.0
c.pred[7,7]	16800.0	2148.0	19.32	12930.0	16680.0	21380.0
c.pred[7,8]	17330.0	2278.0	20.47	13220.0	17200.0	22210.0
c.pred[7,9]	17640.0	2349.0	21.58	13430.0	17500.0	22610.0
c.pred[7,10]	17820.0	2394.0	22.12	13520.0	17670.0	22850.0
c.pred[8,4]	17660.0	3083.0	29.1	12320.0	17440.0	24550.0
c.pred[8,5]	20870.0	4328.0	44.65	13630.0	20410.0	30770.0
c.pred[8,6]	22940.0	5010.0	51.31	14710.0	22360.0	34300.0
c.pred[8,7]	24190.0	5397.0	55.24	15300.0	23560.0	36520.0
c.pred[8,8]	24930.0	5605.0	57.34	15750.0	24330.0	37490.0
c.pred[8,9]	25380.0	5731.0	57.99	15980.0	24760.0	38320.0
c.pred[8,10]	25640.0	5809.0	58.94	16100.0	25010.0	38800.0
c.pred[9,3]	9257.0	2985.0	29.61	4822.0	8820.0	16360.0
c.pred[9,4]	12460.0	4767.0	51.21	5722.0	11650.0	23820.0
c.pred[9,5]	14740.0	5955.0	62.2	6419.0	13640.0	29150.0
c.pred[9,6]	16170.0	6634.0	69.65	6983.0	14940.0	32230.0
c.pred[9,7]	17070.0	7058.0	72.4	7357.0	15750.0	34270.0
c.pred[9,8]	17610.0	7296.0	74.11	7542.0	16230.0	35350.0
c.pred[9,9]	17920.0	7428.0	74.93	7676.0	16500.0	36020.0
c.pred[9,10]	18110.0	7505.0	75.27	7761.0	16680.0	36380.0
c.pred[10,2]	22130.0	**88610.0**	1015.0	782.1	9544.0	109500.0
c.pred[10,3]	38080.0	**160200.0**	1827.0	1212.0	15580.0	190200.0
c.pred[10,4]	51080.0	**205700.0**	2417.0	1554.0	20680.0	2.6E+5
c.pred[10,5]	60160.0	**234100.0**	2763.0	1813.0	24260.0	313600.0
c.pred[10,6]	66330.0	**2.59E+5**	3071.0	2014.0	26690.0	347300.0
c.pred[10,7]	70180.0	**280900.0**	3318.0	2104.0	28120.0	368700.0
c.pred[10,8]	72380.0	**289400.0**	3424.0	2167.0	29050.0	377700.0
c.pred[10,9]	73690.0	**295300.0**	3485.0	2215.0	29620.0	383600.0
c.pred[10,10]	74390.0	**296400.0**	3498.0	2219.0	29850.0	386700.0
reserve	**112300.0**	**296700.0**	3498.0	30740.0	**69700.0**	424500.0

At first glance, the predictions seem reasonable and the standard deviations associated with each predicted value do not appear to be very large; however, this is only true for c_{ij} where $i \leq 9$. For the cumulative claim liabilities at the bottom row of the claim run-off triangle (*i.e.* $i = 10$), the standard deviations are larger than the associated forecasted liabilities, giving imprecise estimates (see the bolded figures in Table 5.5). Consequently, the estimated reserve is also imprecise.

Note that the forecasted means in the predictive inference are larger than the medians, suggesting that the predictive distributions are positively skewed. However, the means corresponding to the last row of the cumulative claim liabilities and reserve appear to be too large compared to their median, suggesting highly dispersed and extremely skewed distributions. This can be seen in Figure 5.4, which depicts the trace of the simulated values of reserve; the extremely large simulated values, which correspond to large spikes in the trace, cause the mean to be much larger than the median. In this case, perhaps the median is a more appropriate choice for predictive purposes.

Figure 5.4: Trace for simulated values of reserve (modified Hertig's model)



The large standard deviations corresponding to the bottom row of the cumulative claim liabilities is most likely to be caused by the imprecise forecast of $c_{10,1}$ (c.pred[10,2] in Table 5.5) initially, and this causes the following cumulative claim liabilities to be highly variable, ultimately the estimated reserve as well. It is possible that the model is lacking something in explaining the development factors when $j = 1$, such as the correlation *between* the development factors. In fact, ignoring the correlation between δ_{i0} and δ_{i1} can lead to sizable forecast errors (de Jong, 2004). Correlation between the development factors will be implemented in the following section.

5.3 Development correlation model

In this section, the development correlation model (see Equation 2 in Section 4.2) is further discussed. Note that this model assumes first order correlation between each consecutive pair of the development factors, in the sense that each δ_{ij} is correlated with $\delta_{i.j-1}$ for $j = 1, \dots, 9$; this is captured by the correlation parameters θ_j.

The same modification from Section 5.2 will be applied to this model (*i.e.* applying structural constraints to μ_j and h_j for $j > 2$), to avoid large standard deviations that accompany the MCMC estimates (for μ_j and h_j when $j = 8, 9$). The introduction of the correlation terms will increase the number of free parameters by 9, which is undesirable given the number of data points. For purposes of *parsimony*, only the correlation between δ_{i0} and δ_{i1} will be considered, meaning that only one extra parameter (θ_1) will be added to the model. This is consistent with de Jong's assumption that only the correlation between development years 0 and 1 is significant.

The resulting development correlation model may be summarised as follows:

$$\delta_{ij} \sim \text{Normal}\left(\mu_j,\ h_j^2\sigma^2\right), \qquad i = 1, \dots, n, \qquad j = 0, 2, \dots, n-1$$

$$\delta_{i1} \sim \text{Normal}\left(\mu_1 + h_1\theta_1(\delta_{i0} - \mu_0),\ h_1^2\sigma^2\right), \qquad i = 1, \dots, n$$

$$\mu_j = M^{j-2} \cdot \mu_2, \qquad 0 < M < 1, \qquad j > 2$$

$$h_j = N^{j-2} \cdot h_2, \qquad 0 < N < 1, \qquad j > 2$$

The prior distributions in this model are exactly the same as those in Section 5.2, with the addition of the prior distribution for θ_1, as follows:

$$\theta_1 \sim \text{Normal}(0, 100000)$$

Note that there are only 9 free parameters in this model. The *WinBUGS* code of this model is in Appendix C-5. Again, Gibbs sampling was used to simulate $K = 11000$ values of each parameter, the starting points were chosen to be $\mu_j = 0.1$, $h_j = 1$, $\sigma^2 = 1$, $M = 0.5$, $N = 0.5$ and $\theta_1 = 1$, and $B = 1000$ values were burnt-in. The posterior estimates are shown below in Table 5.6. Note that the estimate of $\rho = \theta_1/\sqrt{1+\theta_1^2} = Corr(\delta_{i0}, \delta_{i1})$ is also presented in Table 5.6. As above, h[j] represents h_{j-1}, mu[j] represents μ_{j-1}, and sigma2 represents σ^2. Also, theta and rho represent θ_1 and ρ, respectively.

The results in Table 5.6 are similar to those in Table 5.4, but with two extra rows. The estimated θ_1 and ρ show that the development factors between development years 0 and 1 are negatively and significantly correlated, also note that none of the standard deviations in Table 5.6 seem to be unduly large.

Table 5.6: Posterior estimates (development correlation model)

node	mean	sd	MC error	2.5%	median	97.5%
M	0.5711	0.03266	5.617E-4	0.5053	0.5718	0.6351
N	0.5986	0.04001	9.132E-4	0.5266	0.5965	0.6829
h[2]	0.2221	0.08752	0.004816	0.09138	0.21	0.4286
h[3]	0.2238	0.075	0.003573	0.1025	0.2147	0.3959
h[4]	0.1325	0.04036	0.002011	0.06373	0.1292	0.2226
h[5]	0.07883	0.02281	0.001146	0.03887	0.07742	0.128
h[6]	0.04711	0.01375	6.633E-4	0.02317	0.04623	0.07715
h[7]	0.02828	0.008831	3.906E-4	0.01361	0.02742	0.04804
h[8]	0.01706	0.005944	2.344E-4	0.007901	0.01626	0.03096
h[9]	0.01034	0.00411	1.435E-4	0.00446	0.009662	0.02039
h[10]	0.006295	0.002878	8.963E-5	0.002477	0.005728	0.01349
mu[1]	7.38	0.4209	0.01947	6.547	7.382	8.21
mu[2]	1.467	0.3476	0.01601	0.7851	1.463	2.147
mu[3]	0.4967	0.07618	0.001229	0.3518	0.4949	0.6543
mu[4]	0.2819	0.03399	4.692E-4	0.2153	0.2815	0.35
mu[5]	0.1606	0.01754	1.775E-4	0.1259	0.1607	0.1951
mu[6]	0.09172	0.01138	1.087E-4	0.06868	0.09181	0.1135
mu[7]	0.05256	0.008263	9.531E-5	0.03594	0.05265	0.06871
mu[8]	0.03021	0.006035	7.938E-5	0.01847	0.03017	0.0425
mu[9]	0.01742	0.004305	6.115E-5	0.009435	0.01728	0.02645
mu[10]	0.01008	0.002996	4.467E-5	0.004837	0.009887	0.01657
rho	**-0.9576**	0.03492	0.001646	-0.9936	-0.9666	-0.8645
sigma2	2.082	1.597	0.09705	0.7158	1.653	6.188
theta	**-4.141**	1.836	0.1221	-8.785	-3.769	-1.72

The traces of the simulated values illustrate that the simulations converge very quickly and produce reasonably good mixing. Figures 5.5, 5.6 and 5.7 shows the traces of simulated μ_0, μ_1 and θ_1, while the rest of the traces are placed in Appendix D-5. However, most of the simulated values appear to have higher autocorrelation, causing the standard deviations to underestimate the true variability of the parameters; hence, the MC errors are more suitable in measuring the variability. Note that the introduction of θ_1 to the model causes the simulated values of μ_0 and μ_1 to be negatively correlated, which can be seen from the traces in Figure 5.5 and 5.6. In fact, by plotting the simulated values of μ_0 against those of μ_1, the negative relationship is obvious (the plot is shown in Figure 8.6 in Appendix D-6).

Figure 5.5: Trace of μ_0

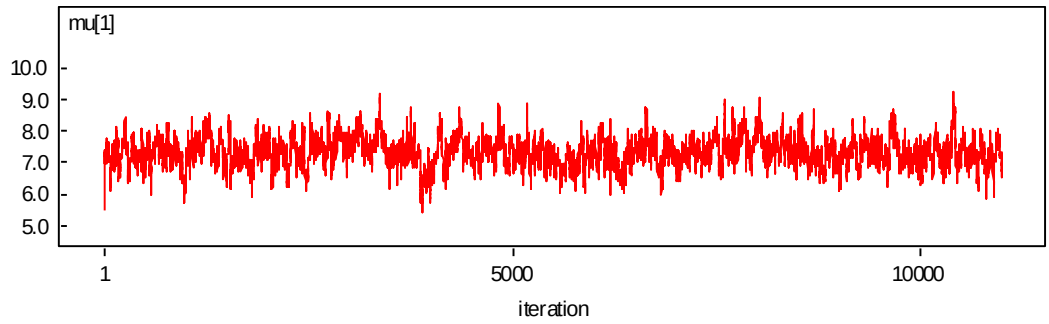


Figure 5.6: Trace of μ_1

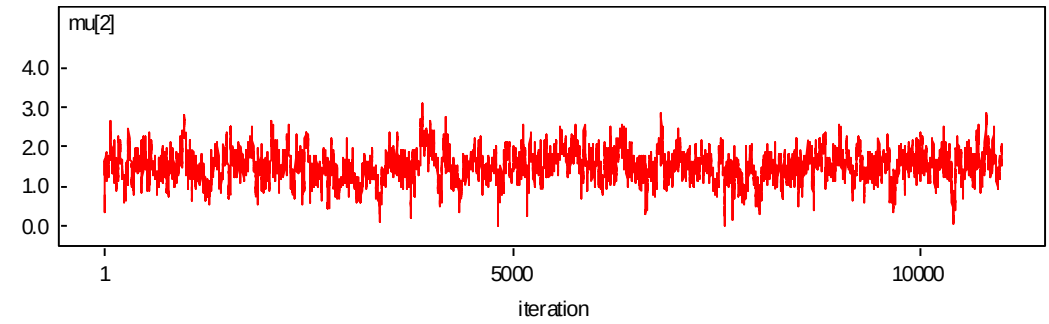
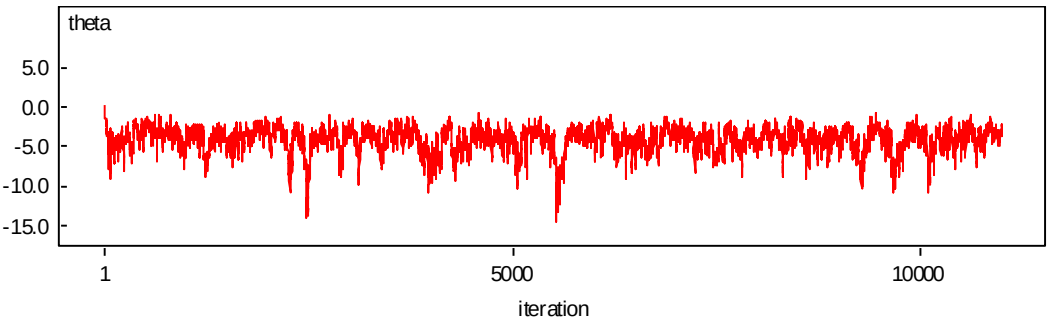


Figure 5.7: Trace of θ_1



Predictive inference on the individual future cumulative claim liabilities and the total reserve are presented in Table 5.7.

Table 5.7: Predictive inference on c_{ij} and r (development correlation model)

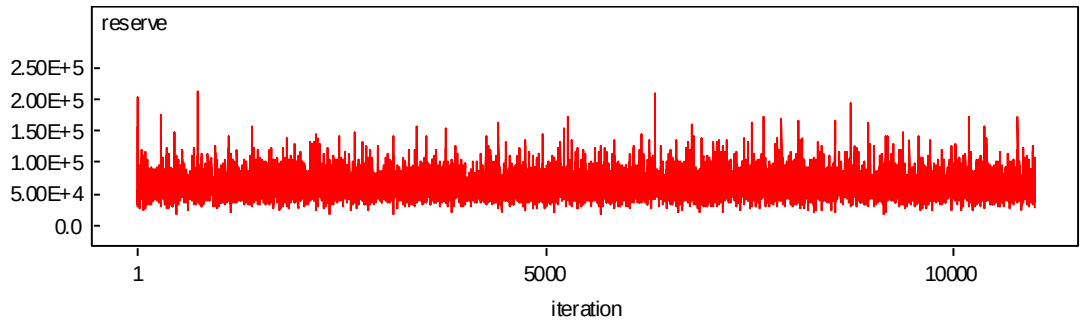
node	mean	sd	MC error	2.5%	median	97.5%
c.pred[2,10]	16880.0	159.1	1.748	16560.0	16870.0	17200.0
c.pred[3,9]	23880.0	355.1	3.685	23180.0	23880.0	24590.0
c.pred[3,10]	24130.0	439.8	4.638	23250.0	24120.0	25020.0
c.pred[4,8]	27920.0	655.2	7.266	26650.0	27900.0	29220.0
c.pred[4,9]	28410.0	815.3	8.964	26840.0	28390.0	30060.0
c.pred[4,10]	28700.0	896.5	10.07	26920.0	28680.0	30530.0
c.pred[5,7]	27610.0	1044.0	9.948	25590.0	27590.0	29740.0
c.pred[5,8]	28470.0	1307.0	11.45	25960.0	28450.0	31190.0
c.pred[5,9]	28980.0	1424.0	12.59	26240.0	28950.0	31900.0
c.pred[5,10]	29270.0	1495.0	13.34	26380.0	29240.0	32360.0
c.pred[6,6]	17390.0	1080.0	10.48	15340.0	17350.0	19590.0
c.pred[6,7]	18330.0	1348.0	12.66	15840.0	18280.0	21100.0
c.pred[6,8]	18910.0	1494.0	13.91	16170.0	18850.0	22060.0
c.pred[6,9]	19240.0	1571.0	14.24	16340.0	19180.0	22590.0
c.pred[6,10]	19440.0	1616.0	14.91	16470.0	19360.0	22870.0
c.pred[7,5]	14530.0	1508.0	13.32	11810.0	14440.0	17700.0
c.pred[7,6]	15940.0	1953.0	18.26	12490.0	15830.0	20140.0
c.pred[7,7]	16820.0	2192.0	19.77	12970.0	16660.0	21490.0
c.pred[7,8]	17350.0	2326.0	20.86	13200.0	17180.0	22350.0
c.pred[7,9]	17660.0	2402.0	21.42	13420.0	17500.0	22840.0
c.pred[7,10]	17840.0	2450.0	21.77	13490.0	17680.0	23080.0
c.pred[8,4]	17600.0	3120.0	32.63	12200.0	17330.0	24560.0
c.pred[8,5]	20790.0	4332.0	44.85	13490.0	20320.0	30540.0
c.pred[8,6]	22820.0	4999.0	49.97	14510.0	22300.0	34030.0
c.pred[8,7]	24050.0	5385.0	53.24	15130.0	23480.0	36200.0
c.pred[8,8]	24810.0	5618.0	55.25	15530.0	24210.0	37260.0
c.pred[8,9]	25260.0	5754.0	55.92	15770.0	24640.0	37970.0
c.pred[8,10]	25520.0	5829.0	56.5	15940.0	24890.0	38490.0
c.pred[9,3]	9248.0	2873.0	27.15	4924.0	8836.0	15990.0
c.pred[9,4]	12460.0	4627.0	46.75	5759.0	11720.0	23380.0
c.pred[9,5]	14740.0	5793.0	62.45	6544.0	13800.0	28850.0
c.pred[9,6]	16190.0	6488.0	71.32	7012.0	15140.0	31890.0
c.pred[9,7]	17080.0	6873.0	74.87	7348.0	15970.0	33770.0
c.pred[9,8]	17610.0	7097.0	76.51	7549.0	16440.0	34820.0
c.pred[9,9]	17910.0	7224.0	77.09	7693.0	16710.0	35510.0
c.pred[9,10]	18100.0	7297.0	77.59	7774.0	16870.0	35860.0
c.pred[10,2]	7670.0	**2460.0**	28.83	3913.0	7324.0	13520.0
c.pred[10,3]	13160.0	**5982.0**	66.39	5094.0	12050.0	27790.0
c.pred[10,4]	17750.0	**8853.0**	98.44	6293.0	15960.0	39680.0
c.pred[10,5]	20980.0	**10800.0**	121.4	7109.0	18760.0	48430.0
c.pred[10,6]	23070.0	**12060.0**	135.9	7753.0	20570.0	53220.0
c.pred[10,7]	24330.0	**12770.0**	143.6	8114.0	21670.0	55910.0
c.pred[10,8]	25080.0	**13180.0**	147.7	8334.0	22350.0	57720.0
c.pred[10,9]	25530.0	**13420.0**	150.1	8488.0	22760.0	58910.0
c.pred[10,10]	25790.0	**13560.0**	151.5	8575.0	22960.0	59540.0
reserve	**63500.0**	**18640.0**	205.0	35050.0	61090.0	107200.0

The predicted c_{ij} for $i \leq 9$ are almost identical to those in the modified Hertig's model (see Table 5.5); this is because the parameter estimates of μ_j in both models are very

close to one another. More importantly, the future cumulative claim liabilities and the reserve have reasonable standard deviations which are not unduly large (see bolded figures). Although the distributions of the predicted claim liabilities and reserve are still positively skewed in this case, the magnitudes of the estimated means are not very large compared to the estimated median.

As can be seen in Figure 5.8, which portrays the trace of the simulated reserve, there appears to be no highly extreme values.

Figure 5.8: Trace of simulated reserve r



Comparing Hertig's model, modified Hertig's model, and the development correlation model, it seems that the development correlation model performs best in modelling the claim run-off triangle (AFG data). In the next chapter, these three models will be assessed formally through hypothesis testing.

CHAPTER 6

Model Diagnostics and MCMC Assessment

6.1 Hypothesis testing

In Chapter 5, some imprecise estimates obtained via MCMC methods were identified, simply by looking at the posterior estimates and associated standard deviations. On that basis, it was concluded that Hertig's model (Section 5.1) and its modification (Section 5.2) are not suitable for reserving purposes. More formally, one could perform hypothesis testing to assess the models. In this section, the theory of posterior predictive p-values (ppp-values) will be used to perform hypothesis tests for assessing *goodness-of-fit*. The strategy will be to generate independent replicated development factors δ'_{ij} from their posterior predictive densities and compare them with the observed development factors δ_{ij} from the data. The behaviour of the replicates relative to the original data will be used to decide whether the model is adequate.

Generally, $\delta_{ij} \sim \text{Normal}(a_{ij}, b_{ij})$, where a_{ij} and b_{ij} are the underlying parameters that generate the development factors. In Hertig's model and its modification, these parameters are

$$a_{ij} = \mu_j$$

$$b_{ij} = h_j^2 \sigma^2$$

Whereas under the development correlation model, they are:

$$a_{ij} = \mu_j, \qquad j \neq 1$$

$$a_{i1} = \mu_1 + h_1 \theta_1 (\delta_{i0} - \mu_0)$$

$$b_{ij} = h_j^2 \sigma^2$$

The replicated development factors δ'_{ij} have the same joint distribution as that of the δ_{ij}. This joint distribution may be represented by writing $\delta'_{ij} \sim \text{Normal}\left(a'_{ij}, b'_{ij}\right)$, where, for Hertig's model and its modification:

$$a'_{ij} = a_{ij} = \mu_j$$

$$b'_{ij} = b_{ij} = h_j^2 \sigma^2$$

Whereas for the development correlation model:

$$a'_{ij} = a_{ij} = \mu_j\,, \qquad j \neq 1$$

$$a'_{i1} = \mu_1 + h_1 \theta_1 (\delta'_{i0} - \mu_0)\,, \qquad (\neq a_{i1})$$

$$b'_{ij} = b_{ij} = h_j^2 \sigma^2$$

Note that δ, the set of all δ_{ij}, and δ', the set of all δ'_{ij}, are independent, conditional on the set of all parameters, ψ (*e.g.* $\psi = (\mu_0, \mu_1, \mu_2, h_1, h_2, \sigma^2, M, N, \theta_1)$ under the development correlation model). That is, $(\delta \perp \delta' | \psi)$.

For general goodness-of-fit testing, the null hypothesis $H_0 = model\ is\ adequate$ is tested against the alternative hypothesis $H_A = model\ is\ inadequate$; by considering two different test statistics as follows:

$$T_1(\delta, \psi) = \sum_{i=1}^{10} \sum_{j=i}^{10-i} \frac{\left(q_{1,ij} - E\left(q_{1,ij}\right)\right)^2}{Var\left(q_{1,ij}\right)}, \qquad q_{1,ij} = \delta_{ij} - \delta_{i,j-1}$$

$$T_2(\delta, \psi) = \sum_{i=1}^{10} \sum_{j=i}^{10-i} \frac{\left(q_{2,ij} - E\left(q_{2,ij}\right)\right)^2}{Var\left(q_{2,ij}\right)}, \qquad q_{2,ij} = \frac{\delta_{ij}}{\delta_{i,j-1}}$$

These test statistics measure the overall deviations of the quantities $q_{ij} = (q_{1,ij}, q_{2,ij})$ from their respective expected values, scaled by an associated variance. Note that $q_{1,ij}$ is the successive *difference* between the development factors, while $q_{2,ij}$ is the corresponding successive *ratio*.

To test whether correlation is present in the model, first define

$$q_{3,ij} = I\left(\frac{\delta_{ij} - a_{ij}}{\delta_{i,j-1} - a_{i,j-1}} < 0\right)$$

Thus, $q_{3,ij}$ takes a value of 1 when $(\delta_{ij} - a_{ij})$ and $(\delta_{i,j-1} - a_{i,j-1})$ are of different sign, and otherwise $q_{3,ij} = 0$.

Under Hertig's model and its modification, the development factors are assumed to be independently distributed, and hence there is no correlation between the development factors. Thus it makes sense to test

$$H_0 = development\ factors\ are\ not\ correlated$$

versus $H_A = development\ factors\ are\ correlated$

Under H_0, it is equally likely for $q_{3,ij}$ to be either 0 or 1; however, a simple calculation of $q_{3,ij}$ from the AFG data suggests otherwise for development year $j = 1$. The $q_{3,ij}$ are calculated from δ_{ij} of the AFG data, using the classical estimates of a_{ij}, *i.e.* $a_{ij} = \hat{\mu}_j$, the mean of the development factors correspond to each column. Table 6.1 shows the values of the $q_{3,ij}$.

Table 6.1: AFG data - $q_{3,ij}$

Accident year *i*	Development Year *j*									
	0	1	2	3	4	5	6	7	8	9
1	-	**1**	0	0	0	1	0	1	0	0
2	-	**1**	1	1	0	0	1	1	0	
3	-	**1**	0	0	0	1	1	0		
4	-	**1**	0	1	1	0	0			
5	-	**1**	0	0	1	0				
6	-	**0**	1	1	1					
7	-	**1**	0	1						
8	-	**1**	0							
9	-	**1**								
10	-									

As can be seen from the first column of Table 6.1, eight of the nine $q_{3,ij}$ values are 1, implying that the development factors are actually correlated, *i.e.* when δ_{i0} is larger than μ_0, δ_{i1} will be lower than μ_1, and *vice versa*. These observations suggest another (third) test statistic, which can be used to formally test the significance of the correlation:

$$T_3(\delta, \psi) = \sum_{i=1}^{10} q_{3,i1}$$

This test statistic counts the number of sign changes corresponding to $(\delta_{ij} - a_{ij})$ and $(\delta_{i,j-1} - a_{i,j-1})$ for development year 1.

For the general assessment of goodness-of-fit, the ppp-value in favour of H_0 against H_A is defined to be:

$$p_k = Pr(T_k' \leq T_k | D), \qquad k = 1, 2$$

where $T_k = T_k(\delta, \psi)$ and $T_k' = T_k(\delta', \psi)$, T_k' denotes the test statistic corresponding to the replicates and D is the set of original data.

The ppp-values can be estimated *via* MCMC methods, by the proportion of simulations for which T_k' is smaller than T_k:

$$\hat{p}_k = \frac{1}{J}\sum_{s=1}^{J} I_k^{(s)}$$

where $I_k^{(s)} = I\left(T_k'^{(s)} \leq T_k^{(s)}\right)$, $T_k^{(s)} = T_k\left(\delta, \psi^{(s)}\right)$, $T_k'^{(s)} = T_k\left(\delta'^{(s)}, \psi^{(s)}\right)$, $k = 1, 2$, $\psi^{(s)}$ is the set of all parameters at iteration s, and $\delta'^{(s)}$ is the set of all δ_{ij}', obtained *via* the method of composition at iteration s.

For the correlation test, the ppp-value in favour of H_0 against H_A is defined as

$$p_3 = Pr(T_3' \geq T_3 | D)$$

where $T_3 = T_3(\delta, \psi)$ and $T_3' = T_3(\delta', \psi)$

In this case, p_3 is estimated by

$$\hat{p}_3 = \frac{1}{J}\sum_{s=1}^{J} I_3^{(s)}$$

where $I_3^{(s)} = I\left(T_3'^{(s)} \geq T_3^{(s)}\right)$, $T_3^{(s)} = T_3\left(\delta, \psi^{(s)}\right)$, and $T_3'^{(s)} = T_3\left(\delta'^{(s)}, \psi^{(s)}\right)$.

All hypothesis tests are conducted with 5% significance level; *i.e.* the null hypothesis H_0 is rejected if the ppp-value is less than 0.05.

The implementation of the hypothesis tests for each model is discussed in the following subsections.

6.1.1 Hertig's model

The expected value and the variance of q_{ij} need to be derived in order to evaluate the test statistic $T_k(\delta, \psi)$. Under the null hypothesis, it can be shown that for $k = 1$ (where $q_{1,ij} = \delta_{ij} - \delta_{i,j-1}$):

$$E(q_{1,ij}) = \mu_j - \mu_{j-1}$$

$$Var(q_{1,ij}) = (h_j^2 + h_{j-1}^2)\sigma^2$$

For $k = 2$, it is not so simple to determine the expected value and the variance of $q_{2,ij} = \delta_{ij}/\delta_{i,j-1}$. However, with the assumption of independence between the development factors, the expected value and the variance can be well approximated using the delta method, as follows:

$$E(q_{2,ij}) \cong \frac{\mu_j}{\mu_{j-1}}$$

$$Var(q_{2,ij}) \cong \frac{\sigma^2}{\mu_{j-1}^2}\left[\left(\frac{h_{j-1}^2}{\mu_{j-1}^2}\right)(\mu_j^2 + h_j^2\sigma^2) + h_j^2\right]$$

Proof of these formulae can be found in Appendix A-3. Note that the *WinBUGS* code used to implement the hypothesis testing is presented with the model specification in Appendix C-3.

For consistency, the same number of simulations ($K = 11000$) and burn-in ($B = 1000$) are chosen. The ppp-values are estimated and presented in Table 6.2.

Table 6.2: Posterior predictive p-values (Hertig's model)

	node	mean	sd	MC error
Goodness-of-fit test:	p1	0.6263	0.4838	0.005816
	p2	0.3833	0.4862	0.007932
Correlation test:	p3	0.0941	0.292	0.003198

These ppp-values imply that the overall goodness-of-fit is acceptable, where the null hypothesis is accepted with $\hat{p}_1 = 0.6263$ and $\hat{p}_2 = 0.3833$. For the correlation test, the null hypothesis that the development factors are independent (not correlated) is rejected at the 10% level but not at the 5% level. Therefore, it is concluded that Hertig's model is adequate in modelling the claim run-off triangle. However, note that with scarce amount of data, these hypothesis tests are not very strong, especially for the third hypothesis testing, where only nine data points are used (for $j = 1$).

6.1.2 Modified Hertig's model

Note that the modified Hertig's model was introduced to overcome the issue of scarce data in estimating the parameters. This model is very similar to Hertig's model, since the only changes to the model is the limiting feature of the parameter estimates for development years $j > 2$. As the fundamental underlying parameters are the same, the expected value and the variance of q_{ij} are the same as in the previous subsection.

The *WinBUGS* code for the hypothesis testing is integrated with the model specification in Appendix C-4. Again, with the same number of simulations and burn-in, the estimated ppp-values are shown in Table 6.3.

Table 6.3: Posterior predictive p-values (modified Hertig's model)

	node	mean	sd	MC error
Goodness-of-fit test:	p1	0.7045	0.4563	0.005104
	p2	0.2943	0.4557	0.003942
Correlation test:	p3	0.0946	0.2927	0.00268

The ppp-values give the same conclusion as Subsection 6.1.1 for goodness-of-fit test and correlation test. Note that the estimate of p_3 is very close to estimate in the preceding subsection; this is because the test statistic associated with this ppp-value

only involves development year $j = 1$, for which the parameter estimates are not affected by the modification.

6.1.3 Development correlation model

Note that the development correlation model is fundamentally different from Hertig's model with an additional correlation term (θ_1); this additional term changes the parameter estimates and ultimately causes the replicated development factors δ'_{ij} to be different as well. Also, the development factors δ_{ij} are no longer independent.

To facilitate the assessment of the goodness-of-fit of this model, the expected value and the variance of $q_{1,ij}$ are derived as followed:

$$E(q_{1,ij}) = a_{ij} - a_{i,j-1}$$

$$Var(q_{1,ij}) = (h_j^2 + h_{j-1}^2)\sigma^2\,, \qquad j \neq 1$$

$$Var(q_{1,ij}) = (h_j^2 + h_{j-1}^2)\sigma^2 - \theta_1 h_0^2 \sigma^2\,, \qquad j = 1$$

Proof of these formulae can be found in Appendix A-3. Note that due to the difficulty in deriving the expected values and variances of $q_{2,ij}$, the associated hypothesis test is not pursued in this subsection.

As in the two previous subsections, the ppp-values are estimated via MCMC methods with *WinBUGS*. Again, the hypothesis testing code is presented with its model specification in Appendix C-5. Table 6.4 displays the estimated ppp-values.

Table 6.4: Posterior predictive p-values (development correlation model)

	node	mean	sd	MC error
Goodness-of-fit test:	p1	0.601	0.4897	0.006086
Correlation test:	p3	0.1989	0.3992	0.005344

For the goodness-of-fit test, the ppp-value suggests that there is no problem with the model and accepts the null hypothesis that the model is adequate.

Since the model incorporates correlation between the development factors, the correlation test in this case is not testing whether correlation is significant. Nevertheless, this is still a useful test. In previous subsections, the hypothesis testing rejected Hertig's model and its modification at the 10% significance level, by looking at the models' inability to capture the correlation between the development factors. While in the development correlation model the relationship between $(\delta_{i1} - a_{i1})$ and $(\delta_{i0} - a_{i0})$ is used to assess the adequacy of the model.

At the 5% significance level, p_3 is consistent with the null hypothesis that the model is adequate. In contrast to the previous models, the adequacy of the development correlation model is not rejected at the 10% significance level. Note that the null hypothesis is rejected at the 20% level. Therefore, in relation to the hypothesis testing of the other models, it is concluded that the development correlation model is the best model in explaining the AFG Data.

6.2 Reserve assessment

In Section 6.1, emphasis was given to the testing of different models to determine their adequacy and ultimately select the best model for reserving purposes. In this section, the focus is shifted to the assessment of the MCMC methods, aiming to answer the question: "How well do MCMC methods predict, provided the model is correct?"

For the purpose of this section, the development correlation model is assumed to be correct and will be used in assessing the MCMC predictions. Note that MCMC methods

were used to predict the future development factors, the future cumulative claims and the reserve in Section 5.3. Since the reserve is the sum of all the outstanding claims for each accident year, it is a function of the current and future cumulative claims; therefore, it is most appropriate to use the predicted reserve for the assessment.

To facilitate the assessment of MCMC predictions, the following procedures will be carried out:

i. Generate a claim run-off triangle similar to the AFG data, by first generating the development factors randomly using the parameter estimates acquired by MCMC methods from Table 5.6, rounded to the nearest 2 significant figures.
ii. Perform MCMC methods to the generated claim run-off triangle to predict the outstanding claims to determine the reserve. (See Section 5.3 for implementation details.)
iii. Fill out the entries for the lower triangle by randomly generating the future development factors from the same parameters as step *i* and determining the future cumulative claims.
iv. Calculate the 'true' required reserve (called true reserve hereafter), which is the overall outstanding claims, from the full 'run-off triangle' (*i.e.* the complete generated data; see example in Table 6.5).
v. Define I_r as 1 if the true reserve is contained within the 95% prediction interval of reserve from the MCMC methods, and 0 otherwise. I_r is the indicator that the true reserve is in the 95% prediction interval.

vi. Define and calculate e_1, e_2 and e_3 as the proportional errors of the predicted mean, median and mode of the reserve respectively:

$$e_1 = \frac{mean - true\ reserve}{true\ reserve}$$

$$e_2 = \frac{median - true\ reserve}{true\ reserve}$$

$$e_3 = \frac{mode - true\ reserve}{true\ reserve}$$

Note that the mode is estimated from the MCMC simulated values of reserve, by fitting a non-parametric density to the simulated reserve.

vii. Repeat steps *i* - *vi* for a total of $L = 1000$ times, recording the values of I_r, e_1, e_2 and e_3 for each iteration.

Table 6.5 shows a randomly generated cumulative claim run-off triangle using the parameter estimates described in step *i* above, with entries for the lower triangle filled out. Note that this is just an example from one of the iteration. The total outstanding claim in this particular example is $\sum_{i=2}^{10}\left(c_{i,11-i} - c_{i,10-i}\right) = 111183$ and hence the true reserve is 111183 as well. Note that the entries in the lower triangle are assumed to be the true realisation of the future cumulative claims.

Table 6.5: Simulated data - cumulative incurred claim amounts (c_{ij})

Accident year *i*	Development Year *j*									
	0	1	2	3	4	5	6	7	8	9
1	4478	8257	17044	27571	40198	50715	55646	56742	56834	57009
2	866	5695	11918	14076	17312	21186	21755	21756	22221	22288
3	3918	13256	20514	25653	33174	36122	41925	42128	42157	43197
4	361	4126	13886	21786	24056	26616	28576	28765	29374	29174
5	1693	8042	28397	46664	52898	58733	65313	65846	66995	66927
6	18974	8361	7214	14748	15347	17340	18926	19661	20072	20388
7	2008	4188	7951	8000	9344	10735	11953	11931	12086	12279
8	478	9149	16792	40459	42203	50845	50089	50721	51628	52355
9	347	12722	25421	25324	29935	32255	36269	39035	39764	40131
10	3726	6580	18089	22381	23450	29470	30500	31387	31911	32689

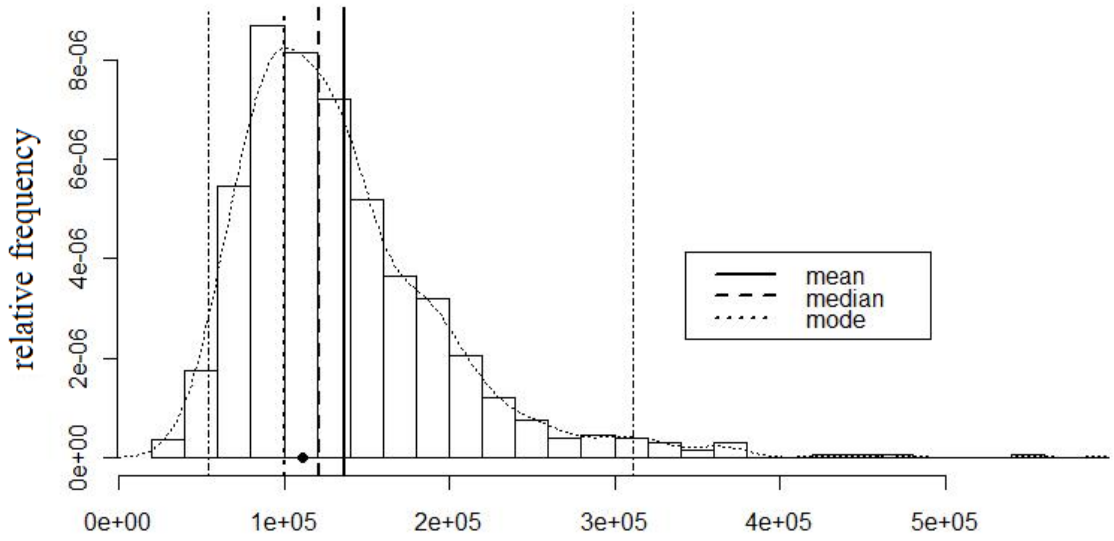
Performing MCMC methods for the above simulated data, the 95% prediction interval for reserve is found to be $(54376, 311404)$, which contains the true reserve; thus I_r is given a value of 1. The mean and median of the reserve estimate are readily available from the *WinBUGS* output as 136497 and 121350 respectively; also the mode is approximated to be 100506. From these values, the proportional errors are calculated to be $e_1 = 0.2277$, $e_2 = 0.09144$ and $e_3 = -0.09603$. Note that the mean and median overestimate the true reserve, while the mode underestimates it.

These results are summarised in Figure 6.1 which shows the histogram of the MCMC simulated values of the required reserve. Note that the histogram does not show some of the simulated values which are unusually large. (Inclusion of these values would make the main part of the histogram looks very thin and hence not presentable.)

In Figure 6.1, the black dot shows the value of the true reserve; the mean, median and mode are displayed as vertical lines. Note that the 95% prediction interval (the vertical dotted lines on the furthest left and right) contains the true reserve. Also note that the

mode corresponds to the highest point of the non-parametric density for the predicted reserve.

Figure 6.1: Frequency histogram of simulated reserve



On average, a 95% prediction interval should be able to capture the true quantity with a probability of 0.95. However, in this complicated model the 95% prediction interval cannot be determined exactly, and has to be estimated via MCMC methods; this raises a question on how well does the 95% prediction interval estimated with MCMC methods represent the exact 95% prediction interval. The MCMC estimate of the 95% prediction interval can be assessed by determining the true coverage of the prediction interval; this is achieved by performing the MCMC methods again and again each time to a different set of simulated run-off triangle.

Denote ϕ as the coverage of the MCMC 95% prediction interval. ϕ is defined to be the proportion of the 95% prediction intervals containing the true reserve:

$$\phi = \frac{1}{L}\sum_{s=1}^{L} I_r^{(s)}$$

Note that the superscript (s) represents the iteration number.

It is important to determine if the point estimates of the required reserve are reasonably close to the true reserve. Due to the random nature of the claims, it is certain that the estimates will be higher or lower than the true reserve. However, hopefully on average, the positive deviations and the negative deviations will even out. The accuracy of the estimates will be assessed by calculating the biases, denoted by b_1, b_2 and b_3 for mean, median and mode respectively.

Define the bias as the average of the proportional errors:

$$b_x = \frac{1}{L}\sum_{s=1}^{L} e_x^{(s)} \ , \qquad x = 1, 2, 3$$

Proportional error is chosen instead of absolute error because the true reserve for each iteration can be very large or very small, for which the absolute errors will be distorted by the size of the true reserve. For example, if the true reserve is very large, the absolute error will be very large hence this particular absolute error is not comparable to that of other iterations.

After performing $L = 1000$ MCMC predictions with *WinBUGS* in *R* (see the implementation code in Appendix B-9), the coverage ϕ is found to be 0.951 with confidence interval $(0.938, 0.964)$ which contains 0.95. This suggests that the MCMC's 95% prediction interval is a good approximation to the exact 95% prediction interval. Note that the coverage ϕ is itself an approximation, meaning that the true coverage might not be 95.1% (but there is 95% confidence that the true coverage is between 93.8% and 96.4%).

Bias of the estimates are found to be $b_1 = 0.4497$, $b_2 = 0.1202$ and $b_3 = -0.03524$ for mean, median and mode respectively. It appears that the usual MCMC prediction (the mean) tends to overestimate the true reserve by 45% on average, while the median only overestimates the true reserve by 12% on average and the mode underestimates the true reserve only by a relatively small percentage (3.5%). Note that the mean is likely to be affected by a couple of unusually large simulated values of reserve in MCMC sampling whereas the median and mode are not; Figure 6.1 shows that the reserve is positively skewed, hence it is very likely that there are some extremely large simulated values for some of the iterations.

From the above results, one should ask whether the median is a more suitable choice for reserving purposes than the mean, since the median is better in predicting the true reserve compared to the mean. The mode tends to underestimate the required reserve; hence it is not so suitable for reserving purposes. In practice, the reserve should never be less than expected claims. As discussed in Chapter 2, the mean minimises the quadratic error loss function, the median minimises the absolute error loss function and the mode minimises the zero-one error loss function. Hence, if one wishes to be conservative, the mean is the most appropriate for reserving purposes; however, the median is also of value.

In conclusion, the reserve assessment in this section suggests that the MCMC approach taken in this thesis works fairly well.

CHAPTER 7

Comparison to Previous Studies

As mentioned above, the AFG Data were considered by Mack (1994), England and Verrall (2002), and de Jong (2004); these data were analysed using the chain-ladder method, a Bayesian over-dispersed Poisson model, and a classical development correlation model, respectively. The forecasted liabilities for reserving purpose derived from these models will be compared with the predicted reserve discussed in the thesis.

Table 7.1 shows the central estimates of the forecasted liabilities and their standard errors associated with the above-mentioned models, together with the result from the Bayesian implementation of the development correlation model discussed in Chapter 5.

Table 7.1: Forecasted liabilities for the AFG data ($′000)

Accident Year	Chain Ladder Method		Bayesian Poisson Model		Classical Development Correlation Model		**Bayesian Development Correlation Model**	
2	154	(206)	243	(486)	154	(146)	**171**	**(155)**
3	617	(623)	885	(984)	642	(375)	**657**	**(431)**
4	1,636	(753)	2,033	(1,589)	1,701	(753)	**1,624**	**(895)**
5	2,747	(1,456)	3,582	(2,216)	2,843	(1,271)	**3,109**	**(1,504)**
6	3,649	(2,007)	3,849	(2,301)	3,948	(1,462)	**3,633**	**(1,627)**
7	5,435	(2,228)	5,393	(2,873)	5,941	(2,290)	**5,569**	**(2,421)**
8	10,907	(5,344)	11,091	(4,686)	12,243	(5,463)	**12,530**	**(5,784)**
9	10,650	(6,284)	10,568	(5,563)	12,475	(6,747)	**12,700**	**(7,288)**
10	16,339	(24,509)	17,654	(12,801)	22,957	(11,551)	**23,940**	**(14,060)**
Total	52,135	(26,909)	55,297	(17,357)	62,982	(16,260)	**63,930**	**(18,900)**

The forecasted liabilities derived from the Bayesian development correlation model are of similar scale compared to the others, especially the classical development correlation model, while the standard errors associated with each accident year are greater than their classical counterparts.

Note that generally the classical approach tends to underestimate the variability of the parameters, especially when the data are scarce. For instance, when only a single data point is present, the classical approach of de Jong (2004) effectively estimates the variance as zero (see Table 5.1), whereas the Bayesian approach estimates the variance as infinite; similar treatment was applied to the development factor associated with the last column of the claim run-off triangle (the highest development year). Besides, in the classical development correlation model, de Jong did not take account of the uncertainties in parameter estimation, the standard errors derived (see Table 7.1) only account for the uncertainties in prediction.

Also note that the total liability is the sum of the forecasted liabilities associated with each accident year. Subject to rounding errors, the total liabilities shown in Table 7.1 are consistent with the sum of the individual liabilities for all models except for the classical development correlation model, for which they differ by $78000; this might be due to a possible fault associated with the prediction method used in the classical development correlation model. For the Bayesian development correlation model, the total liability differs from the sum by $3000; this is because the estimates are rounded to the nearest 4 significant figures in *WinBUGS*.

CHAPTER 8

Summary and Discussion

8.1 Limitations of the Bayesian development correlation model

Note that to facilitate comparison, the required reserve in the model is determined before discounting. In practice, the actual reserve held is the discounted outstanding claims liabilities plus any adjustment. Albeit tedious, it is easy and straightforward to incorporate discounting into the model; this can be achieved by calculating each claim liability (not the cumulative claim liability) associated with the lower claim run-off triangle and discounting it accordingly. The discount factors have to be explicitly assumed; they can be either fixed for each year or vary across the calendar years. Treatment of discounting the liabilities can be seen in Scollnik (2004).

Another limitation is that inflation rates are not explicitly considered in the model. If inflation is assumed to be known, it can be incorporated into the model by first adjusting the AFG Data. Then the same modelling technique discussed in previous chapters can be applied to the adjusted AFG Data, and the claims liabilities can be inflated in the same way as how the discount factors apply.

If the inflation rates are not known and need to be estimated from the data, the whole modelling procedure becomes much more difficult; this is because the inflation rates are partly captured by the development correlation factors and the structure of the inflation rates need to be understood so that the inflation rates can be determined. Inflation is not incorporated in the model due to lack of data; estimation of inflation would reduce the accuracy and precision of other parameter estimates.

8.2 Suggestions for future research

The Bayesian development correlation model from Section 5.3 could be improved by incorporating discount rates and inflation rates as discussed in the previous section. This would give a more realistic model to be used for reserving purposes. If inflation rates are estimated from the model, care must be taken to ensure that the inflation rates are sensible, especially when the data are scarce.

Extensions to the model suggested by de Jong were discussed in Section 4.2. As only one of the extensions is implemented in the thesis, the other two are left for future research. These extensions are the accident correlation model and the calendar correlation model which incorporate correlation across accident years and calendar years, respectively.

8.3 Conclusion

This thesis has explored a Bayesian analysis of three models: Hertig's (1985) model, a modification of that model, and de Jong's (2004) development correlation model. Outstanding claims liabilities associated with a claim run-off triangle, known as the AFG Data, were forecasted for reserving purposes, using MCMC simulations generated with *WinBUGS*.

Of these three models, the development correlation model is selected as the best for reserving purposes. This judgement is based on model diagnostics and the reserve assessments discussed in Chapter 6.

The estimated reserve from this ‘best’ model is then compared to the estimated reserve obtained using the chain-ladder method (Mack, 1993), a Bayesian over-dispersed Poisson model (England & Verrall, 2002), and the classical development correlation model (de Jong, 2004) in Chapter 7. It is concluded that the Bayesian development correlation model provides a better estimation of outstanding claim reserve than the classical development correlation model.

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Appendix

Appendix A: Derivations and proofs

A-1: Mean and variance of AR(1) model

x_i can be written as:

$$x_i = \alpha + \beta x_{i-1} + e_i = \alpha + \beta(\alpha + \beta x_{i-2} + e_{i-1}) + e_i$$

$$= \alpha + \alpha\beta + \beta^2 x_{i-2} + \beta e_{i-1} + e_i$$

$$= \alpha + \alpha\beta + \alpha\beta^2 + \beta^3 x_{i-3} + \beta^2 e_{i-2} + \beta e_{i-1} + e_i$$

$$= \quad \vdots$$

$$= \alpha(1 + \beta + \beta^2 + \cdots + \beta^k) + \beta^{k+1} x_{i-(k+1)} + e_i + \beta e_{i-1} + \cdots + \beta^k e_{i-k}$$

$$= \frac{\alpha}{1-\beta} + 0 + \sum_{j=0}^{\infty} \beta^j e_{i-j} \qquad \text{(as k tends to infinity)}$$

Since $e_i \sim iid$ Normal$(0, \sigma^2)$, x_i has a normal distribution with mean

$$\eta = E(x_i) = \frac{\alpha}{1-\beta}$$

because $E(e_i) = 0$, and variance

$$\tau^2 = V(x_i) = \sum_{j=0}^{\infty} \beta^j = \frac{\sigma^2}{1-\beta}$$

A-2: Derivation of expected values for Rao-Blackwell estimate

Since $(x_{n+t}|x_{n+t-1}) \sim \text{Normal}(\alpha + \beta x_{n+t-1},\ \sigma^2)$, the expected value of x_{n+t} given x_{n+t-1} is $\alpha + \beta x_{n+t-1}$. Conditioning on $x = (x_1, \ldots, x_n)$ and θ_j:

$$E(x_{n+1}|x, \theta_j) = \alpha_j + \beta_j x_n$$

$$\begin{aligned} E(x_{n+2}|x, \theta_j) &= E(E(x_{n+2}|x, \theta_j, x_{n+1})|x, \theta_j) \\ &= E(\alpha_j + \beta_j x_{n+1}|x, \theta_j) \\ &= \alpha_j + \beta_j E(x_{n+1}|x, \theta_j) \\ &= \alpha_j + \alpha_j \beta_j + \beta_j x_n \end{aligned}$$

More generally, for $t \geq 3$:

$$\begin{aligned} E(x_{n+t}|x, \theta_j) &= \alpha_j + \beta_j E(x_{n+t-1}|x, \theta_j) \\ &= \alpha_j + \alpha_j \beta_j + \beta_j E(x_{n+t-2}|x, \theta_j) \\ &= \quad \vdots \\ &= \alpha_j(1 + \beta_j + \cdots + \beta_j^{t-2}) + \beta_j^{t-1} E(x_{n+1}|x, \theta_j) \\ &= \alpha_j(1 + \beta_j + \cdots + \beta_j^{t-1}) + \beta_j^t x_n \end{aligned}$$

A-3: Derivation of $E(q_{k,ij})$ and $Var(q_{k,ij})$

If random variables X and Y are independent, with simple algebra, the variance of XY can be written as followed:

$$Var(XY) = Var(X)Var(Y) + E(X)^2Var(Y) + E(Y)^2Var(X)$$

Delta method:

Utilising the first order Taylor expansion, write $f(x) \cong f(a) + (x-a)f'(a)$. Letting $x = X$ and $a = E(X) = \mu$, the expected value $E[f(X)]$ and variance $Var[f(X)]$ can be derived as:

$$E[f(X)] \cong E[f(\mu) + (X-\mu)f'(\mu)] = f(\mu) + (E(\mathrm{X}) - \mu)f'(\mu) = f(\mu)$$

$$Var[f(X)] \cong Var[(X-\mu)f'(\mu)] = [f'(\mu)]^2Var(X-\mu) = [f'(\mu)]^2Var(X)$$

If $f(x) = 1/x$, then $f'(x) = -1/x^2$ & $[f'(x)]^2 = 1/x^4$, using the above result gives

$$E\left(\frac{1}{\delta_{i,j-1}}\right) \cong \frac{1}{E(\delta_{i,j-1})}$$

$$Var\left(\frac{1}{\delta_{ij}}\right) \cong \frac{1}{E(\delta_{ij})^4} Var(\delta_{ij})$$

For Hertig's basic model and its modification, using the above results and independence between δ_{ij} and $\delta_{i,j-1}$, the expected value and the variance of $q_{1,ij}$ are:

$$q_{1,ij} = \delta_{ij} - \delta_{i,j-1}$$

$$E(q_{1,ij}) = E(\delta_{ij}) - E(\delta_{i,j-1}) = a_{ij} - a_{i,j-1} = \mu_j - \mu_{j-1}$$

$$Var(q_{1,ij}) = Var(\delta_{ij}) + Var(\delta_{i,j-1}) = b_{ij} + b_{i,j-1} = (h_j^2 + h_{j-1}^2)\sigma^2$$

$$q_{2,ij} = \frac{\delta_{ij}}{\delta_{i,j-1}}$$

$$E(q_{2,ij}) = E(\delta_{ij})E\left(\frac{1}{\delta_{i,j-1}}\right) \cong E(\delta_{ij})\frac{1}{E(\delta_{i,j-1})} = a_{ij}\left(\frac{1}{a_{i,j-1}}\right) = \frac{\mu_j}{\mu_{j-1}}$$

$$Var(q_{2,ij}) = E(\delta_{ij})^2 Var\left(\frac{1}{\delta_{i,j-1}}\right) + E\left(\frac{1}{\delta_{i,j-1}}\right)^2 Var(\delta_{ij}) + Var(\delta_{ij})Var\left(\frac{1}{\delta_{i,j-1}}\right)$$

$$= a_{ij}^2\left(\frac{1}{a_{i,j-1}^4}\right)b_{i,j-1} + \frac{1}{a_{i,j-1}^2}b_{ij} + b_{ij}\left(\frac{1}{a_{i,j-1}^4}\right)b_{i,j-1}$$

$$= \frac{\sigma^2}{\mu_{j-1}^2}\left[\left(\frac{h_{j-1}^2}{\mu_{j-1}^2}\right)(\mu_j^2 + h_j^2) + h_j^2\right]$$

In the development correlation model, δ_{ij} and $\delta_{i,j-1}$ are not independent, and the covariance between δ_{i1} and δ_{i0} can be derived as:

$$Cov(\delta_{i0}, \delta_{i1}) = Cov(\delta_{i0}, \mu_1 + h_1\theta_1(\delta_{i0} - \mu_0) + \varepsilon_{i1})$$

$$= Cov(\delta_{i0}, h_1\theta_1(\delta_{i0} - \mu_0)) + Cov(\delta_{i0}, \varepsilon_{i1})$$

$$= h_1\theta_1 Var(\delta_{i0}) + 0$$

$$= h_1\theta_1 h_0^2\sigma^2$$

Note that the covariance for δ_{ij} and $\delta_{i,j-1}$ is zero for $j \neq 1$. Hence:

$$E(q_{1,ij}) = E(\delta_{ij}) - E(\delta_{i,j-1}) = a_{ij} - a_{i,j-1}$$

$$Var(q_{1,ij}) = Var(\delta_{ij}) + Var(\delta_{i,j-1}) - 2Cov(\delta_{ij}, \delta_{i,j-1})$$

$$= (h_j^2 + h_{j-1}^2)\sigma^2 - 2Cov(\delta_{ij}, \delta_{i,j-1})$$

i.e. for $j = 1$, $\quad Var(q_{1,ij}) = (h_j^2 + h_{j-1}^2)\sigma^2 - 2h_1\theta_1 h_0^2\sigma^2$

for $j \neq 1$, $\quad Var(q_{1,ij}) = (h_j^2 + h_{j-1}^2)\sigma^2$

Appendix B: *R* code

B-1: Generation of time series values of $x = (x_1, \ldots, x_n)$

```
n <- 100
alpha.real <- 0.2
beta.real <- 0.6
sigma2.real <- 1
eta.real <- alpha.real/(1-beta.real)
tau2.real <- sigma2.real/(1-beta.real)
x1 <- rnorm(1,eta.real,sqrt(tau2.real))
xvec <- 1:n; xvec[1] <- x1
for(i in 2:n) {
  xvec[i] <- rnorm(1,alpha.real+beta.real*xvec[i-1],sqrt(sigma2.real))
}

# Time Series of x (Figure 2.1)
X11(width=10,height=4); par(mfrow=c(1,1)); par(mai=c(1,1,0.2,0.2))
plot(xvec,type="l",xlab="t",ylab=~x[t], cex.lab=1.7)
```

B-2: The MH algorithm

```
# log posterior density of theta
LOGPOSFUN <- function(alpha,beta,sigma2,x.vector) {
  n <- length(x.vector)
  logf.xi <- x.vector[2:n]
  for(i in 2:n) {
    logf.xi[i] <- -log(sigma2)/2 -((x.vector[i]-alpha-beta*x.vector[i-1])^2)/(2*sigma2)
  }
  eta <- alpha/(1-beta)
  tau2 <- sigma2/(1-beta)
  logf.x1 <- -log(tau2)/2 -((x.vector[1]-eta)^2)/(2*tau2)
  logposfun <- -log(sigma2) + logf.x1 + sum(logf.xi)
  logposfun
}

MH.TS <- function(alpha,beta,sigma2,a.tuning,b.tuning,s.tuning,xvec,burn=100,J=1000) {
  alpha.vec <- alpha
  beta.vec <- beta
  sigma2.vec <- sigma2
  alpha.count <- 0; beta.count <- 0; sigma2.count <- 0
  n <- length(xvec)
  for(i in 1:(burn+J)) {
    alpha1 <- runif(1,alpha-a.tuning,alpha+a.tuning)
    p <- exp(LOGPOSFUN(alpha1,beta,sigma2,xvec) - LOGPOSFUN(alpha,beta,sigma2,xvec))
    if(runif(1) < p) {
      alpha <- alpha1
      alpha.count <- alpha.count + 1
    }
    beta1 <- runif(1,beta-b.tuning,beta+b.tuning)
    if((beta1 < 1)&&(beta1 > -1)) {
      p <- exp(LOGPOSFUN(alpha,beta1,sigma2,xvec) - LOGPOSFUN(alpha,beta,sigma2,xvec))
      if(runif(1) < p) {
        beta <- beta1
        beta.count <- beta.count + 1
        }
    }
    sigma21 <- runif(1,sigma2-s.tuning,sigma2+s.tuning)
    if(sigma21 > 0) {
      p <- exp(LOGPOSFUN(alpha,beta,sigma21,xvec) - LOGPOSFUN(alpha,beta,sigma2,xvec))
      if(runif(1) < p) {
        sigma2 <- sigma21
        sigma2.count <- sigma2.count + 1
      }
    }
```

```
  alpha.vec <- c(alpha.vec,alpha)
  beta.vec <- c(beta.vec,beta)
  sigma2.vec <- c(sigma2.vec,sigma2)
  }
alpha.ar <- alpha.count/(burn+J)
beta.ar <- beta.count/(burn+J)
sigma2.ar <- sigma2.count/(burn+J)
list(alpha.vec=alpha.vec, beta.vec=beta.vec, sigma2.vec=sigma2.vec,
     alpha.ar=alpha.ar, beta.ar=beta.ar, sigma2.ar=sigma2.ar)
}

burn <- 100
J <- 1000
ts1 <- MH.TS(0,0,0.5,0.2,0.2,0.5,xvec,burn,J)
c(ts1$alpha.ar, ts1$beta.ar, ts1$sigma2.ar)

avec <- ts1$alpha.vec; bvec <- ts1$beta.vec; s2vec <- ts1$sigma2.vec

# Simulation Traces (Figure 2.2, 2.3, 2.4)
X11(width=10,height=5); par(mai=c(1,1.2,0.2,0.2))
plot(0:(burn+J),avec,type="l", ann=F)
title(xlab="i",ylab=~alpha[i], cex.lab=1.7, font.lab=6)
abline(v=burn,lty=4)

X11(width=10,height=5); par(mai=c(1,1.2,0.2,0.2))
plot(0:(burn+J),bvec,type="l", ann=F)
title(xlab="i",ylab=~beta[i], cex.lab=1.7, font.lab=6)
abline(v=burn,lty=4)

X11(width=10,height=5); par(mai=c(1,1.2,0.2,0.2))
plot(0:(burn+J),s2vec,type="l", ann=F)
title(xlab="i",ylab=expression(paste(sigma[i]^2)), cex.lab=1.7, font.lab=6)
abline(v=burn,lty=4)
```

B-3: Inference on posterior quantities

```
BURN <- function(vector,n) {
  # return a vector after a burn-in of n
  vector[-(1:(n+1))]
}

INF <- function(vector) {
  # return the mean, 95% CI, 95% CPDR of vector
  m <- mean(vector)
  s2 <- var(vector)
  n <- length(vector)
  ci <- m + c(-1,1)*qnorm(0.975)*sqrt(s2/n)
  cpdr <- quantile(vector,c(0.025,0.975))
  names(m) <- "mean"
  names(ci) <- c("lower","upper")
  c(m,ci,cpdr)
}

INF2 <- function(vector,sub,showS2=F) {
  # return the mean, 95% CI, 95% CPDR and sample variances
  # CI is calculator in 2 ways: CI1: 'ordinary', CI2: 'batch means'
  inf <- INF(vector)
  m <- length(vector)/sub    ## how many in a group
  yv <- rep(NA,sub)
  for(i in 1:sub) {
    yv[i] <- mean(vector[(m*(i-1)+1):(m*i)])
  }
  s2.batch <- m*var(yv)
  ci.batch <- inf[1] + c(-1,1)*qnorm(0.975)*sqrt(s2.batch/length(vector))
  names(ci.batch) <- c("lower","upper")
  result = c(inf[1:3],ci.batch,inf[4:5])
  if(showS2) { result=c(result,"s2"=var(vector),"s2b"=s2.batch) }
  result
}
```

```
avec <- BURN(avec,burn); bvec <- BURN(bvec,burn); s2vec <- BURN(s2vec,burn)
evec <- avec/(1-bvec)
tvec <- s2vec/(1-bvec)

options(digits=4)
rbind(alpha=c(true=alpha.real,INF2(avec,20)),beta=c(true=beta.real,INF2(bvec,20)),
      "sigma^2"=c(true=sigma2.real,INF2(s2vec,20)),
      "eta"=c(true=eta.real,INF2(evec,20)),"tau^2"=c(true=tau2.real,INF2(tvec,20)))

# Frequency Histograms (Figure 2.5, 2.6, 2.7, 8.1, 8.2)
X11(width=8,height=4.5); par(mai=c(1,1.2,0.4,0.2))
hist(avec,prob=T,ylim=c(0,4),breaks=20,xlab=~alpha,ylab="relative frequency",main=NA)
lines(density(avec),lty=1); abline(v=INF2(avec,20),lty=3); points(alpha.real,0,pch=19)

X11(width=8,height=4.5); par(mai=c(1,1.2,0.2,0.2))
hist(bvec,prob=T,breaks=20,xlab=~beta,ylab="relative frequency",main=NA)
lines(density(bvec),lty=1); abline(v=INF2(bvec,20),lty=3); points(beta.real,0,pch=19)

X11(width=8,height=4.5); par(mai=c(1,1.2,0.2,0.2))
hist(s2vec,prob=T,breaks=20,xlab=expression(paste(sigma^2)),
                            ylab="relative frequency",main=NA)
lines(density(s2vec),lty=1); abline(v=INF2(s2vec,20),lty=3); points(sigma2.real,0,pch=19)

X11(width=8,height=4.5); par(mai=c(1,1.2,0.2,0.2))
hist(evec,prob=T,breaks=20,xlab=~eta,ylab="relative frequency",main=NA)
lines(density(evec),lty=1); abline(v=INF2(evec,20),lty=3); points(eta.real,0,pch=19)

X11(width=8,height=4.5); par(mai=c(1,1.2,0.4,0.2))
hist(tvec,prob=T,breaks=20,xlab=expression(paste(tau^2)),
                           ylab="relative frequency",main=NA)
lines(density(tvec),lty=1); abline(v=INF2(tvec,20),lty=3); points(tau2.real,0,pch=19)
```

B-4: Predictive inference

```
M <- 10
lastx <- xvec[length(xvec)]; lastx
xs.matrix <- matrix(NA,nrow=M,ncol=J)
for(i in 1:J) {
  xs.matrix[1,i] <- rnorm(1,avec[i]+bvec[i]*lastx,sqrt(s2vec[i]))
  for(m in 2:M) {
    xs.matrix[m,i] <- rnorm(1,avec[i]+bvec[i]*xs.matrix[m-1,i],sqrt(s2vec[i]))
  }
}

xs.inf <- matrix(NA,nrow=M,ncol=9)
xs.name <- rep(NA,M)
for(m in 1:M) {
  xs.inf[m,] <- INF2(xs.matrix[m,],20,showS2=T)
  xs.name[m] <- paste("x(n+",m,")",sep="")
}
label <- names(INF2(1,1,showS2=T))
dimnames(xs.inf) <- list(xs.name,label)
options(digits=3)
xs.inf

X11(width=8,height=5); par(mai=c(1,1,0.2,0.2))  # (Figure 2.8)
plot(xs.inf[,1],ylim=range(xs.inf[,4:5]),type="l",
     xlab="t",ylab="Forecasted mean of " ~x[n+t])
abline(h=mean(evec),lty=2)
lines(xs.inf[,2],lty=3); lines(xs.inf[,3],lty=3)
lines(xs.inf[,4],lty=4); lines(xs.inf[,5],lty=4)
lines(xs.inf[,6],lty=8); lines(xs.inf[,7],lty=8)
legend(5,-0.2,c("mean","ordinary CI","batch means CI",
                "estimated post. mean"),lty=c(1,3,4,2))

xs.range <- range(xs.matrix)
xs.min <- floor(xs.range[1])
xs.max <- ceiling(xs.range[2])
```

```
X11(width=8,height=4.5); par(mai=c(1,1.2,0.4,0.2))  # (Figure 2.10)
hist(xs.matrix[m,],prob=T,breaks=20,xlab=~x[n+10],ylab="relative frequency",main=NA)
lines(density(xs.matrix[k,]),lty=1); abline(v=xs.inf[k,][-(8:9)],lty=3);
points(mean(evec),0,pch=19)

# Rao Blackwell approach
xs.matrix2 <- matrix(NA,nrow=M,ncol=J)
for(i in 1:J) {
  xs.matrix2[1,i] <- avec[i]+bvec[i]*lastx
  for(m in 2:M) {
    xs.matrix2[m,i] <- avec[i]+bvec[i]*xs.matrix2[m-1,i]
  }
}

xs.inf <- matrix(NA,nrow=M,ncol=5)
xs.name <- rep(NA,M)
for(m in 1:M) {
  xs.inf[m,] <- INF2(xs.matrix2[m,],20)[1:5]
  xs.name[m] <- paste("x(n+",m,")",sep="")
}
label <- names(INF2(1,1)[1:5])
dimnames(xs.inf) <- list(xs.name,label)
xs.inf

X11(width=8,height=5); par(mai=c(1,1,0.2,0.2))  # (Figure 2.9)
plot(xs.inf[,1],ylim=range(xs.inf[,4:5]),type="l",
     xlab="t",ylab="Estimated mean of " ~x[n+t])
abline(h=mean(evec),lty=2)
lines(xs.inf[,2],lty=3); lines(xs.inf[,3],lty=3)
lines(xs.inf[,4],lty=4); lines(xs.inf[,5],lty=4)
legend(4.5,-0.1,c("mean","ordinary CI","batch means CI",
                  "estimated post. mean"),lty=c(1,3,4,2))
```

B-5: Hypothesis testing

```
RUN.COUNT <- function(xvec,mid=mean(xvec)) {
  # this function counts the number of runs above or below 'mid'
  xvec <- xvec - mid
  sign <- sign(xvec[1])
  count <- 1
  for(i in 2:length(xvec)) {
    if(sign(xvec[i]) != sign) {
      sign <- sign(xvec[i])
      count <- count + 1
    }
  }
  count
}

LOGPOSFUN2 <- function(alpha,sigma2,x.vector) {
  n <- length(x.vector)
  logf.xi <- rep(NA,n)
  for(i in 1:n) {
    logf.xi[i] <- -log(sigma2)/2 -((x.vector[i]-alpha)^2)/(2*sigma2)
  }
  logposfun <- -log(sigma2) + sum(logf.xi)
  logposfun
}

MH.TS2 <- function(alpha,sigma2,a.tuning,s.tuning,xvec,burn=100,J=1000) {
  alpha.vec <- alpha
  sigma2.vec <- sigma2
  alpha.count <- 0; sigma2.count <- 0
  n <- length(xvec)
  for(i in 1:(burn+J)) {
    alpha1 <- runif(1,alpha-a.tuning,alpha+a.tuning)
    p <- exp(LOGPOSFUN2(alpha1,sigma2,xvec) - LOGPOSFUN2(alpha,sigma2,xvec))
    if(runif(1) < p) {
```

```
      alpha <- alpha1
      alpha.count <- alpha.count + 1
    }

    sigma21 <- runif(1,sigma2-s.tuning,sigma2+s.tuning)
    if(sigma21 > 0) {
      p <- exp(LOGPOSFUN2(alpha,sigma21,xvec) - LOGPOSFUN2(alpha,sigma2,xvec))
      if(runif(1) < p) {
        sigma2 <- sigma21
        sigma2.count <- sigma2.count + 1
      }
    }
    alpha.vec <- c(alpha.vec,alpha)
    sigma2.vec <- c(sigma2.vec,sigma2)
  }
  alpha.ar <- alpha.count/(burn+J)
  sigma2.ar <- sigma2.count/(burn+J)
  list(alpha.vec=alpha.vec,sigma2.vec=sigma2.vec, alpha.ar=alpha.ar,sigma2.ar=sigma2.ar)
}

burn <- 200
J <- 10000
xlist <- c(100)
ppp1 <- rep(NA,length(xlist)); ppp2 <- ppp1
for(j in 1:length(xlist)) {
  new.xvec <- xvec[1:xlist[j]]
  n <- length(new.xvec)
  ts <- MH.TS2(0,1,0.3,0.5,xvec,burn,J)
  avec2 <- BURN(ts$alpha.vec,burn); s2vec2 <- BURN(ts$sigma2.vec,burn)

  indicator1 <- rep(0,J)
  for(i in 1:J) {
    xrep <- rnorm(n,avec2[i],sqrt(s2vec2[i]))
    T1 <- RUN.COUNT(xrep,avec2[i])
    T0 <- RUN.COUNT(new.xvec,avec2[i])
    if(T1 <= T0) { indicator1[i] <- 1 }
  }
  ppp1[j] <- sum(indicator1)/J

  indicator2 <- rep(0,J)
  T0 <- RUN.COUNT(new.xvec)
  for(i in 1:J) {
    xrep <- rnorm(n,avec2[i],sqrt(s2vec2[i]))
    T1 <- RUN.COUNT(xrep)
    if(T1 <= T0) { indicator2[i] <- 1 }
  }
  ppp2[j] <- sum(indicator2)/J
}
cbind('m'=xlist,ppp1,ppp2)
```

B-6: Approximation of exact posterior densities

```
INTEG <- function(xvec,yvec,a=min(xvec),b=max(xvec)) {
  # Integrates numerically under a spline through the points given by
  # the vectors xvec and yvec, from a to b.
  fit <- smooth.spline(xvec, yvec)
  spline.f <- function(x){ predict(fit, x)$y }
  integrate(spline.f, a, b)$value
}

D.integrate <- function(fxy,xlower,xupper,ylower,yupper,...,precision=0.02) {
  # Perform double integration on the function fxy
  yvec <- seq(ylower,yupper,precision)
  gy <- rep(NA,length(yvec)) # integrand of fxy with respect to x
  for(i in 1:length(yvec)) {
    xvec <- seq(xlower,xupper,precision)
    gx <- rep(NA,length(xvec))
    for(j in 1:length(xvec)) {
      gx[j] <- fxy(xvec[j],yvec[i],...)
```

```
    }
    gy[i] <- INTEG(xvec,gx,xlower,xupper)
  }
  INTEG(yvec,gy,ylower,yupper)
}

alpha.values <- seq(-5,1,0.02)
poskern.alpha <- rep(NA,length(alpha.values))
for(i in 1:length(alpha.values)) {
  fxy <- function(sigma2,beta) { exp(LOGPOSFUN(alpha.values[i],beta,sigma2,xvec)) }
  poskern.alpha[i] <- D.integrate(fxy,0.09,20,-0.99,0.99,precision=0.09)
  plot(alpha.values[i],0,xlim=range(alpha.values)) # for checking the progress
}
pos.alpha <- poskern.alpha/(INTEG(alpha.values,poskern.alpha))
pos.mean.alpha <- INTEG(alpha.values,alpha.values*pos.alpha)
pos.mean.alpha  #  0.0812
pos.var.alpha <- INTEG(alpha.values,((alpha.values-pos.mean.alpha)^2)*pos.alpha)
pos.var.alpha   #  0.0110

X11(width=8,height=5); par(mai=c(1,1,0.2,0.2))  # (Figure 2.11)
plot(alpha.values,pos.alpha,type="l",xlim=c(-1,1),xlab=~alpha,ylab="density",main=NA)
lines(density(avec),lty=3); abline(v=pos.mean.alpha,lty=2); legend(-1,3,c("approximated
exact density","non-parametric density","posterior mean"),lty=c(1,3,2))

beta.values <- seq(-0.98,0.98,0.02)
poskern.beta <- rep(NA,length(beta.values))
for(i in 1:length(beta.values)) {
  fxy <- function(sigma2,alpha) { exp(LOGPOSFUN(alpha,beta.values[i],sigma2,xvec)) }
  poskern.beta[i] <- D.integrate(fxy,0.1,20,-1,1,precision=0.1)
  plot(beta.values[i],0,xlim=range(beta.values)) # for checking the progress
}
pos.beta <- poskern.beta/(INTEG(beta.values,poskern.beta))
pos.mean.beta <- INTEG(beta.values,beta.values*pos.beta)
pos.mean.beta  #  0.56
pos.var.beta <- INTEG(beta.values,((beta.values-pos.mean.beta)^2)*pos.beta)
pos.var.beta   #  0.00721

X11(width=8,height=5); par(mai=c(1,1,0.2,0.2))  # (Figure 2.12)
plot(beta.values,pos.beta,type="l",xlab=~beta,ylab="density",main=NA)
lines(density(bvec),lty=3); abline(v=pos.mean.beta,lty=2); legend(-0.75,3,
c("approximated exact density","non-parametric density","posterior mean"),lty=c(1,3,2))

sigma2.values <- seq(0.02,4,0.02)
poskern.sigma2 <- rep(NA,length(sigma2.values))
for(i in 1:length(sigma2.values)) {
  fxy <- function(alpha,beta) { exp(LOGPOSFUN(alpha,beta,sigma2.values[i],xvec)) }
  poskern.sigma2[i] <- D.integrate(fxy,-1,1,-0.99,0.99,precision=0.09)
  plot(sigma2.values[i],0,xlim=range(sigma2.values)) # for checking the progress
}
pos.sigma2 <- poskern.sigma2/(INTEG(sigma2.values,poskern.sigma2))
pos.mean.sigma2 <- INTEG(sigma2.values,sigma2.values*pos.sigma2)
pos.mean.sigma2  #  1.05
pos.var.sigma2 <- INTEG(sigma2.values,((sigma2.values-pos.mean.sigma2)^2)*pos.sigma2)
pos.var.sigma2   #  0.0235

X11(width=8,height=5); par(mai=c(1,1,0.2,0.2))  # (Figure 2.13)
plot(sigma2.values,pos.sigma2,type="l",ylim=c(0,2.85),xlab=expression(paste(sigma^2)),
ylab="density",main=NA)
lines(density(s2vec),lty=3); abline(v=pos.mean.sigma2,lty=2); legend(2,2,c("approximated
exact density","non-parametric density","posterior mean"),lty=c(1,3,2))
```

B-7: Coverage assessment

```
m <- 10
trial <- 1000
I <- rep(0,trial); Ib <- Is <- rep(0,trial)
data <- matrix(NA,nrow=trial,ncol=3); xrep <- matrix(NA,nrow=m,ncol=J)

for(i in 1:trial) {
```

```
  x1 <- rnorm(1,eta.real,sqrt(tau2.real))
  xvec <- rep(NA,n); xvec[1] <- x1
  for(j in 2:(n+m)) {
    xvec[j] <- rnorm(1,alpha.real+beta.real*xvec[j-1],sqrt(sigma2.real))
  }

  ts1 <- MH.TS(0,0,0.5,0.25,0.25,0.5,xvec[1:n],burn,J)
  avec <- ts1$alpha.vec; avec <- BURN(avec,burn)
  aCI <- INF(avec)[4:5]
  if((alpha.real>aCI[1])&&(alpha.real<aCI[2])) { I[i] <- 1 }

  bvec <- BURN(ts1$beta.vec,burn); bCI <- INF(bvec)[4:5]
  if((beta.real>bCI[1])&&(beta.real<bCI[2])) { Ib[i] <- 1 }

  s2vec <- BURN(ts1$sigma2.vec,burn); sCI <- INF(s2vec)[4:5]
  if((sigma2.real>sCI[1])&&(sigma2.real<sCI[2])) { Is[i] <- 1 }

  xmean <- rep(NA,m)
  for(j in 1:J) {
    xrep[1,j] <- rnorm(1,avec[j]+bvec[j]*xvec[n],sqrt(s2vec[j]))
    for(k in 2:m) {
      xrep[k,j] <- rnorm(1,avec[j]+bvec[j]*xrep[k-1,j],sqrt(s2vec[j]))
    }
    xmean[j] <- mean(xrep[,j])
  }

  PI <- INF(xmean)[4:5]
  data[i,] <- c(mean(xvec[(n+1):(n+m)]),PI)
}
p = sum(I/trial); p  # 0.947
p + c(-1,1)*qnorm(0.975)*sqrt(p*(1-p)/trial)    # 0.9331145 0.9608855
pb = sum(Ib/trial); pb  # 0.942
pb + c(-1,1)*qnorm(0.975)*sqrt(pb*(1-pb)/trial)    # 0.9275127 0.9564873
ps = sum(Is/trial); ps  # 0.938
ps + c(-1,1)*qnorm(0.975)*sqrt(ps*(1-ps)/trial)    # 0.923 0.953

I1 <- I2 <- rep(NA,trial)
for(i in 1:trial) {
  I1[i] <- ( data[i,1] > data[i,2] ) && ( data[i,1] < data[i,3] )
  xvec <- rep(NA,n)
  xvec[1] <- x1
  for(j in 2:(n+m)) {
    xvec[j] <- rnorm(1,alpha.real+beta.real*xvec[j-1],sqrt(sigma2.real))
  }
  xmean <- mean(xvec[(n+1):(n+m)])
  I2[i] <- ( xmean > data[i,2] ) && ( xmean < data[i,3] )
}
p1 <- mean(I1); p2 <- mean(I2); c(p1,p2)  # 0.933 0.918
p1 + c(-1,1)*qnorm(0.975)*sqrt(p1*(1-p1)/trial)  # 0.9175 0.9485
p2 + c(-1,1)*qnorm(0.975)*sqrt(p2*(1-p2)/trial)  # 0.901 0.935
```

B-8: Coverage assessment with *WinBUGS*

```
library("R2WinBUGS")

I <- rep(0,trial)
data <- matrix(NA,nrow=trial,ncol=3)
for(i in 1:trial) {
  x1 <- rnorm(1,eta.real,sqrt(tau2.real))
  x <- rep(NA,n); x[1] <- x1
  for(j in 2:n) {
    x[j] <- rnorm(1,alpha.real+beta.real*x[j-1],sqrt(sigma2.real))
  }
  xrep <- rep(NA,m)
  xrep[1] <- rnorm(1,alpha.real+beta.real*x[n],sqrt(sigma2.real))
  for(j in 2:m) {
    xrep[j] <- rnorm(1,alpha.real+beta.real*xrep[j-1],sqrt(sigma2.real))
  }
```

```
  # MCMC via winbugs
  bugdata <- list("n","x","m")
  buginits <- function() { list( beta=0, sigma2=1) }
  bugparameters <- c("alpha","xmean")
  bugsim <- bugs(bugdata, buginits, bugparameters, model.file=
            "C:/Users/Carlo/Documents/R/Q1Predict.txt",
            n.chains = 1, n.iter = J+burn, n.burnin = burn, DIC = FALSE,
            bugs.directory = "C:/Program Files/WinBUGS14/",
            working.directory = "C:/Users/Carlo/Documents/R/Q1/")

  aCI <- c(bugsim$summary[1,3], bugsim$summary[1,7])
  if((alpha.real>aCI[1])&&(alpha.real<aCI[2])) { I[i] <- 1 }

  PI <- c(bugsim$summary[2,3], bugsim$summary[2,7])
  data[i,] <- c(mean(xrep),PI)
}

p = sum(I/trial); p  # 0.937
p + c(-1,1)*qnorm(0.975)*sqrt(p*(1-p)/trial)  # 0.9219413 0.9520587

I1 <- I2 <- rep(NA,trial)
for(i in 1:trial) {
  I1[i] <- ( data[i,1] > data[i,2] ) && ( data[i,1] < data[i,3] )
  xvec <- rep(NA,n)
  xvec[1] <- x1
  for(j in 2:(n+m)) {
    xvec[j] <- rnorm(1,alpha.real+beta.real*xvec[j-1],sqrt(sigma2.real))
  }
  xmean <- mean(xvec[(n+1):(n+m)])
  I2[i] <- ( xmean > data[i,2] ) && ( xmean < data[i,3] )
}
p1 <- mean(I1); p2 <- mean(I2); c(p1,p2)  # 0.926 0.903
p1 + c(-1,1)*qnorm(0.975)*sqrt(p1*(1-p1)/trial)  # 0.9097756 0.9422244
p2 + c(-1,1)*qnorm(0.975)*sqrt(p2*(1-p2)/trial)  # 0.8846567 0.9213433
```

B-9: Reserve assessment with *WinBUGS*

```
library("R2WinBUGS")

MODE <- function(vector) {
        # estimate the mode from a density of vector by smoothing
        den <- density(vector)
        den$x[den$y==max(den$y)]
}

# parameters
year <- 10; n <- year; M <- 0.57; N <- 0.60
mu <- rep(0,n); mu[1:3] <- c(7.3, 1.5, 0.49)
h <- rep(0,n); h[1:3] <- c(1,0.21,0.22)

for(i in 4:n) {
  mu[i] <- M^(i-3) * mu[3]
  h[i] <- N^(i-3) * h[3]
}
theta <- -4.3; sigma2 <- 2.1

L <- 1000  # running the Gibbs sampler again and again for L times
I <- ppp.list <- error.mean <- error.median <- error.mode <- rep(0,L)
reserve.position <- 78; ppp.location <- 79; options(digits=4)

for(k in 1:L) {
  # generating claim run off data
  c <- matrix(NA, nrow=n, ncol=n)
  delta <- matrix(NA, nrow=n, ncol=n)

  for(i in 1:n) {
      delta[i,1] <- rnorm(1, mu[1], h[1]*sqrt(sigma2))
      delta[i,2] <- rnorm(1, mu[2]+h[2]*theta*(delta[i,1]-mu[1]), h[2]*sqrt(sigma2))
    for(j in 3:n) {
```

```
      delta[i,j] <- rnorm(1, mu[j], h[j]*sqrt(sigma2))
    }
  }
  for(i in 1:n) {
      c[i,1] <- exp(delta[i,1])
    for(j in 2:n) {
      c[i,j] <- exp(delta[i,j]) * c[i,j-1]
    }
  }
  true.reserve = 0
  for(i in 2:n) {
      true.reserve = true.reserve + (c[i,n] - c[i,11-i])
  }

  # MCMC
  bugdata <- list("year","c")
  buginits <- function() {
    list( mu = c(0.1,0.1,0.1,NA,NA,NA,NA,NA,NA,NA), h = c(NA,1,1,NA,NA,NA,NA,NA,NA,NA),
          M = 0.5 , N = 0.5 , sigma2 = 1, theta = 1, delta.rep = structure(.Data = c(
            0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
            0, 0, 0, 0, 0, 0, 0, 0, 0,NA,
            0, 0, 0, 0, 0, 0, 0, 0,NA,NA,
            0, 0, 0, 0, 0, 0, 0,NA,NA,NA,
            0, 0, 0, 0, 0, 0,NA,NA,NA,NA,
            0, 0, 0, 0, 0,NA,NA,NA,NA,NA,
            0, 0, 0, 0,NA,NA,NA,NA,NA,NA,
            0, 0, 0,NA,NA,NA,NA,NA,NA,NA,
            0, 0,NA,NA,NA,NA,NA,NA,NA,NA,
            0,NA,NA,NA,NA,NA,NA,NA,NA,NA), .Dim = c(10,10) ))
  }
  Bugparameters <- c("mu","h","sigma2","M","N","theta",
                     "c.pred","reserve.part","reserve","p1","p3")
  bugsim <- bugs(bugdata, buginits, bugparameters, model.file=
            "C:/Users/xxxxx/Documents/R/Claim.txt",
            n.chains = 1, n.iter = 11000, n.burnin = 1000, DIC = FALSE,
            bugs.directory = "C:/Program Files/WinBUGS14/",
            working.directory = "C:/Users/xxxxx/Documents/R/Claim/")

  mean.reserve <- bugsim$mean$reserve; median.reserve <- bugsim$median$reserve
  mode.reserve <- MODE(bugsim$sims.list$reserve)
  PI.reserve <- c(bugsim$summary[reserve.position,3],bugsim$summary[reserve.position,7])
  if((true.reserve > PI.reserve[1]) && (true.reserve < PI.reserve[2])) { I[l]=1 }
  error.mean[l] <- (mean.reserve - true.reserve)/true.reserve
  error.median[l] <- (median.reserve - true.reserve)/true.reserve
  error.mode[l] <- (mode.reserve - true.reserve)/true.reserve
  ppp.list <- bugsim$summary[ppp.location,1]

  # progress tracking
  if(l %% 10 == 0) { print(cat(l/L*100,"percent completed! ")) }
}
# Results
bugsim$summary[reserve.position,]; bugsim$mean$p1; bugsim$mean$p3
mode.reserve; true.reserve; p=mean(I); p; p + c(-1,1)*1.96*sqrt(p*(1-p)/L)

# Hypothesis testing
ppp.mean = mean(ppp.list); ppp.mean;
sum(ppp.mean[ppp.mean<0.05]); sum(ppp.mean[ppp.mean>0.95])
INF(error.mean)[1:3]; INF(error.median)[1:3]; INF(error.mode)[1:3]
reserve.vector <- bugsim$sims.list$reserve

X11(width=8,height=5); par(mai=c(1,1.2,0.2,0.2))  # (Figure 6.1)
hist(reserve.vector,breaks=40,prob=T,xlab="reserve",main=NA,xlim=c(min(reserve.vector),
     quantile(reserve.vector,0.9975)))
abline(v=bugsim$summary[reserve.position,1],lty=1, lwd=2)
abline(v=bugsim$summary[reserve.position,5],lty=2, lwd=2)
abline(v=MODE(reserve.vector),lty=3, lwd=2); abline(v=PI.reserve,lty=4)
points(true.reserve,0,pch=19); lines(density(reserve.vector),lty=3);
legend(1.1*PI.reserve[2],0.5*max(density(reserve.vector)$y),c("mean","median","mode"),
       lty=c(1,2,3),lwd=c(2,2,2))
```

Appendix C: *WinBUGS* Code

C-1: Time series model

MODEL

```
{
   x[1] ~ dnorm(eta,t)
   for( i in 2 : n ) {
      x[i] ~ dnorm(mu[i],s)
      mu[i] <- alpha+beta*x[i-1]
   }
   t <- 1/tau2
   s <- 1/sigma2
   tau2 <- sigma2/(1-beta)
   eta <- alpha/(1-beta)
   alpha ~ dnorm(0,0.001)
   beta ~ dunif(-1,1)
   sigma2 ~ dgamma(0.001,0.001)

   # Prediction
   for( j in 1 : m) {
      xnext[j] ~ dnorm(munext[j],s)
   }
   munext[1] <- alpha+beta*x[n]
   for( j in 2 : m) {
      munext[j] <- alpha+beta*xnext[j-1]
   }
}
```

DATA

```
list(x =
c( -0.651134500293269,  -1.13297671629464,    0.0902583995422782,  -0.740492555568816,
    0.713285477012912,   2.07994350899636,    1.45421438904982,     2.3381697281389,
    1.37655363797294,    1.70107119781436,    0.721806461126196,   -0.0219793729577730,
    0.675082788161775,   1.23874412208321,   -0.163227938659239,   -1.03468020484809,
   -1.2249661451836,    -0.853068555277069,  -1.73295486859530,    -2.02798270492891,
   -0.44366365835555,   -0.305014271649753,   1.54102341989711,     1.96516193269735,
    2.71260199074518,    1.49709675778887,    0.273603377570765,   -0.577364695257764,
   -0.421633068043591,   2.43096891928915,    0.691620566378797,    1.42058905389156,
   -0.65768699060883,    0.460487372435315,   1.41358849385181,    -0.892847049043457,
   -1.29419561926962,   -2.78297653575327,   -2.21560748836394,     2.19632117391132,
    1.02677193374616,    1.61495879538274,    3.19549635315063,     1.22857740530504,
    1.60048437669280,    1.18059004472060,    1.26998139649821,     1.21220822584111,
    0.12784681367819,   -0.390724777992149,   0.841417275022684,   -1.50778568978362,
   -0.84289599914087,   -1.4333697880734,    -2.26213309168342,    -1.02578547250705,
   -0.0652596684436959,  0.227139857634588,  -0.450604540675496,    0.39469840547178,
    0.0226121794600207, -0.162096211508090,   0.005735236727031 22,  0.78843378855997,
    0.044661381446038,   0.94228199051853,    0.627176242272651,   -0.0676376442225574,
   -0.552602300807993,   1.71968604606597,    1.06298339935968,     0.376287370396052,
    2.28118839342955,    2.32838018294078,    1.51419221969344,     0.359300452079709,
    0.450986128926995,   0.285715539965825,   0.0634575029461997,  -0.314828299900130,
    0.143103691731009,  -0.0699157299604746, -0.116372930898608,    0.561361459142413,
   -1.18346979620212,   -1.46430885733973,   -0.0366906364238819,  -0.928593920178746,
   -1.17409023323741,    0.521128086916033,  -1.34089457936068,    -0.0515133312649343,
    0.779648231613477,   1.35909656451904,   -0.0656115327245324,   0.0084512260436784,
   -1.44338475168051,   -1.67820910177067,   -0.979369032809375,   -0.90346362304396),
xnext = c(NA,NA,NA,NA,NA,NA,NA,NA,NA,NA), n = 100, m = 15)
```

INITS

```
list( beta = 0, sigma2 = 1)
```

C-2: Hypothesis testing on time series model

```
MODEL
{
   for( i in 1 : n ) {
      x[i] ~  dnorm(alpha,s)
   }
   s <- 1/sigma2
   alpha ~ dnorm(0,0.001)
   sigma2 ~ dgamma(0.001,0.001)

   # Prediction
   for( j in 1 : n) {
      xrep[j] ~ dnorm(alpha,s)
   }

   # Hypothesis Test
   mx <- mean(x[])
   for( i in 1:m) {
      y[i] <- (x[i] - mx) / abs(x[i] - mx)
   }
   for( i in 1:(m-1)) {
      z[i] <- abs(y[i+1] - y[i]) / 2
   }
   xcount <- sum(z[]) + 1
   mxrep <- mean(xrep[])
   for( j in 1:m) {
      yrep[j] <- (xrep[j] - mxrep) / abs(xrep[j] - mxrep)
   }
   for( j in 1:(m-1)) {
      zrep[j] <- abs(yrep[j+1] - yrep[j]) / 2
   }
   xrepcount <- sum(zrep[]) + 1
   ppp <- step(xcount – xrepcount)
}

DATA

# Same data as in Appendix C-1

INITS

list( sigma2 = 1)
```

C-3: Hertig's model (Section 5.1)

```
MODEL
{
   for( i in 1 : year) {
      delta[i,1] <- log( c[i,1] )
      for( j in 2 : (year+1-i) ) {
         delta[i,j] <- log( c[i,j] ) - log( c[i,j-1] )
      }
      for( j in 1 : (year+1-i) ) {
         delta[i,j] ~ dnorm( mu.d[i,j], tau.d[i,j] )
         mu.d[i,j] <- mu[j]
         tau.d[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
         c.pred[i,j] <- 0
      }
   }

   # Prior Distributions
   for( j in 1 : year) {
      mu[j] ~ dnorm(0,0.00001)
   }
   sigma2 ~ dgamma(0.1,0.1)
   h[1] <- 1
```

```
for( j in 2 : year) {
   h[j] ~ dgamma(0.1,0.1)
}

# Prediction

for( i in 2 : year) {
   delta[i,year+2-i] ~ dnorm( mu.d[i,year+2-i], tau.d[i,year+2-i])
   mu.d[i,year+2-i] <- mu[year+2-i]
   tau.d[i,year+2-i] <- 1 / ( pow(h[year+2-i],2) * sigma2 )
   c.pred[i,year+2-i] <- c[i,year+1-i] * exp(delta[i,year+2-i])
}
for( i in 3 : year) {
   for( j in (year+3-i) : year) {
      delta[i,j] ~ dnorm( mu.d[i,j], tau.d[i,j] )
      mu.d[i,j] <- mu[j]
      tau.d[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
      c.pred[i,j] <- c.pred[i,j-1] * exp(delta[i,j])
   }
}
reserve.part[1] <- 0
for( i in 2 : year) {
     reserve.part[i] <- c.pred[i,year] - c[i,year+1-i]
}
reserve <- sum(reserve.part[])

# Hypothesis Test
for( i in 1 : year) {
   for( j in 1 : (year+1-i) ) {
      delta.rep[i,j] ~ dnorm( mu.d.rep[i,j], tau.d.rep[i,j] )
      mu.d.rep[i,j] <- mu[j]
      tau.d.rep[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
   }
}
for( i in 1 : (year-1) ) {
   for( j in 2 : (year+1-i) ) {
      Q1[i,j] <- delta[i,j] - delta[i,j-1]
      Q2[i,j] <- delta[i,j] / delta[i,j-1]
      Q1.rep[i,j] <- delta.rep[i,j] - delta.rep[i,j-1]
      Q2.rep[i,j] <- delta.rep[i,j] / delta.rep[i,j-1]
      mean.Q1[i,j] <- mu[j] - mu[j-1]
      mean.Q2[i,j] <- mu[j] / mu[j-1]
      var.Q1[i,j] <- ( pow(h[j],2) + pow(h[j-1],2) ) * sigma2
      var.Q2[i,j] <- sigma2/pow(mu[j-1],2)*((pow(h[j-1],2)/pow(mu[j-1],2))
                     *(pow(mu[j],2) + pow(h[j],2)*sigma2) + pow(h[j],2) )
      t1[i,j] <- pow(( Q1[i,j] - mean.Q1[i,j] ),2) / var.Q1[i,j]
      t1.rep[i,j] <- pow(( Q1.rep[i,j] - mean.Q1[i,j] ),2) / var.Q1[i,j]
      t2[i,j] <- pow(( Q2[i,j] - mean.Q2[i,j] ),2) / var.Q2[i,j]
      t2.rep[i,j] <- pow(( Q2.rep[i,j] - mean.Q2[i,j] ),2) / var.Q2[i,j]
   }
}
for( i in 1 : year ) {
   t1[i,1] <- 0
   t1.rep[i,1] <- 0
   t2[i,1] <- 0
   t2.rep[i,1] <- 0
}
for( i in 2 : year ) {
   for( j in (year+2-i) : year ) {
      t1[i,j] <- 0
      t1.rep[i,j] <- 0
      t2[i,j] <- 0
      t2.rep[i,j] <- 0
   }
}
Ts1 <- sum(t1[,])
Ts1.rep <- sum(t1.rep[,])
Ts2 <- sum(t2[,])
Ts2.rep <- sum(t2.rep[,])
p1 <- step( Ts1 - Ts1.rep )  # 1 if Ts1 >= Ts1.rep
p2 <- step( Ts2 - Ts2.rep )  # 1 if Ts2 >= Ts2.rep
for( i in 1 : (year-1) ) {
```

```
      Q3[i] <- ( delta[i,2] - mu[2] ) / ( delta[i,1] - mu[1] )
      Q3.rep[i] <- ( delta.rep[i,2] - mu[2] ) / ( delta.rep[i,1] - mu[1] )
      t3[i] <- step( - Q3[i] )
      t3.rep[i] <- step( - Q3.rep[i] )
   }
   Ts3 <- sum(t3[])
   Ts3.rep <- sum(t3.rep[])
   p3 <- step( Ts3.rep - Ts3 )  # 1 if Ts3.rep >= Ts3
}
```

DATA

```
list(year = 10, c = structure(.Data = c(
5012, 8269, 10907, 11805, 13539, 16181, 18009, 18608, 18662, 18834,
 106, 4285,  5396, 10666, 13782, 15599, 15496, 16169, 16704,    NA,
3410, 8992, 13873, 16141, 18735, 22214, 22863, 23466,    NA,    NA,
5655, 1555, 15766, 21266, 23425, 26083, 27067,    NA,    NA,    NA,
1092, 9565, 15836, 22169, 25955, 26180,    NA,    NA,    NA,    NA,
1513, 6445, 11702, 12935, 15852,    NA,    NA,    NA,    NA,    NA,
 557, 4020, 10946, 12314,    NA,    NA,    NA,    NA,    NA,    NA,
1351, 6947, 13112,    NA,    NA,    NA,    NA,    NA,    NA,    NA,
3133, 5395,    NA,    NA,    NA,    NA,    NA,    NA,    NA,    NA,
2063,   NA,    NA,    NA,    NA,    NA,    NA,    NA,    NA,    NA), .Dim = c(10,10) ))
```

INITS

```
list( mu = c(0,0,0,0,0,0,0,0,0,0) , h = c(NA,1,1,1,1,1,1,1,1,1) , sigma2 = 1,
      delta.rep = structure(.Data = c(
0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0,NA,
0, 0, 0, 0, 0, 0, 0, 0,NA,NA,
0, 0, 0, 0, 0, 0, 0,NA,NA,NA,
0, 0, 0, 0, 0, 0,NA,NA,NA,NA,
0, 0, 0, 0, 0,NA,NA,NA,NA,NA,
0, 0, 0, 0,NA,NA,NA,NA,NA,NA,
0, 0, 0,NA,NA,NA,NA,NA,NA,NA,
0, 0,NA,NA,NA,NA,NA,NA,NA,NA,
0,NA,NA,NA,NA,NA,NA,NA,NA,NA), .Dim = c(10,10) ))
```

C-4: Modified Hertig's model (Section 5.2)

MODEL

```
{
   for( i in 1 : year) {
      delta[i,1] <- log( c[i,1] )
      for( j in 2 : (year+1-i) ) {
         delta[i,j] <- log( c[i,j] ) - log( c[i,j-1] )
      }
      for( j in 1 : (year+1-i) ) {
         delta[i,j] ~ dnorm( mu.d[i,j], tau.d[i,j] )
         mu.d[i,j] <- mu[j]
         tau.d[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
         c.pred[i,j] <- 0
      }
   }

   # Prior Distributions
   for( j in 1 : 3) {
      mu[j] ~ dnorm(0,0.00001) I(0,)
   }
   for( j in 4 : year) {
      mu[j] <- pow(M,j-3) * mu[3]
   }
   h[1] <- 1
   for( j in 2 : 3) {
      h[j] ~ dgamma(0.0001,0.0001)
   }
   for( j in 4 : year) {
      h[j] <- pow(N,j-3) * h[3]
```

```
}
M ~ dunif(0,1)
N ~ dunif(0,1)
sigma2 ~ dgamma(0.0001,0.0001)

# Prediction
for( i in 2 : year) {
  delta[i,year+2-i] ~ dnorm( mu.d[i,year+2-i], tau.d[i,year+2-i] )
   mu.d[i,year+2-i] <- mu[year+2-i]
   tau.d[i,year+2-i] <- 1 / ( pow(h[year+2-i],2) * sigma2 )
  c.pred[i,year+2-i] <- c[i,year+1-i] * exp(delta[i,year+2-i])
}
for( i in 3 : year) {
   for( j in (year+3-i) : year) {
     delta[i,j] ~ dnorm( mu.d[i,j], tau.d[i,j] )
      mu.d[i,j] <- mu[j]
      tau.d[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
     c.pred[i,j] <- c.pred[i,j-1] * exp(delta[i,j])
   }
}
reserve.part[1] <- 0
for( i in 2 : year) {
   reserve.part[i] <- c.pred[i,year] - c[i,year+1-i]
}
reserve <- sum(reserve.part[])

# Hypothesis Test
for( i in 1 : year) {
   for( j in 1 : (year+1-i) ) {
      delta.rep[i,j] ~ dnorm( mu.d.rep[i,j], tau.d.rep[i,j] )
      mu.d.rep[i,j] <- mu[j]
      tau.d.rep[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
   }
}
for( i in 1 : (year-1) ) {
   for( j in 2 : (year+1-i) ) {
      Q1[i,j] <- delta[i,j] - delta[i,j-1]
      Q2[i,j] <- delta[i,j] / delta[i,j-1]
      Q1.rep[i,j] <- delta.rep[i,j] - delta.rep[i,j-1]
      Q2.rep[i,j] <- delta.rep[i,j] / delta.rep[i,j-1]
      mean.Q1[i,j] <- mu[j] - mu[j-1]
      mean.Q2[i,j] <- mu[j] / mu[j-1]
      var.Q1[i,j] <- ( pow(h[j],2) + pow(h[j-1],2) ) * sigma2
      var.Q2[i,j] <- sigma2 / pow(mu[j-1],2) * ( ( pow(h[j-1],2) / pow(mu[j-1],2) )
                          *( pow(mu[j],2) + pow(h[j],2)*sigma2 ) + pow(h[j],2) )
      t1[i,j] <- pow(( Q1[i,j] - mean.Q1[i,j] ),2) / var.Q1[i,j]
      t1.rep[i,j] <- pow(( Q1.rep[i,j] - mean.Q1[i,j] ),2) / var.Q1[i,j]
      t2[i,j] <- pow(( Q2[i,j] - mean.Q2[i,j] ),2) / var.Q2[i,j]
      t2.rep[i,j] <- pow(( Q2.rep[i,j] - mean.Q2[i,j] ),2) / var.Q2[i,j]
   }
}
for( i in 1 : year ) {
   t1[i,1] <- 0
   t1.rep[i,1] <- 0
   t2[i,1] <- 0
   t2.rep[i,1] <- 0
}
for( i in 2 : year ) {
   for( j in (year+2-i) : year ) {
      t1[i,j] <- 0
      t1.rep[i,j] <- 0
      t2[i,j] <- 0
      t2.rep[i,j] <- 0
   }
}
Ts1 <- sum(t1[,])
Ts1.rep <- sum(t1.rep[,])
Ts2 <- sum(t2[,])
Ts2.rep <- sum(t2.rep[,])
p1 <- step( Ts1 - Ts1.rep )  # 1 if Ts1 >= Ts1.rep
p2 <- step( Ts2 - Ts2.rep )  # 1 if Ts2 >= Ts2.rep
for( i in 1 : (year-1) ) {
```

```
      Q3[i] <- ( delta[i,2] - mu[2] ) / ( delta[i,1] - mu[1] )
      Q3.rep[i] <- ( delta.rep[i,2] - mu[2] ) / ( delta.rep[i,1] - mu[1] )
      t3[i] <- step( - Q3[i] )
      t3.rep[i] <- step( - Q3.rep[i] )
   }
   Ts3 <- sum(t3[])
   Ts3.rep <- sum(t3.rep[])
   p3 <- step( Ts3.rep - Ts3 )  # 1 if Ts3.rep >= Ts3
}
```

DATA

```
# Same data as in Appendix C-3
```

INITS

```
list( mu = c(0.1,0.1,0.1,NA,NA,NA,NA,NA,NA,NA), h = c(NA,1,1,NA,NA,NA,NA,NA,NA,NA),
      M = 0.5 , N = 0.5 , sigma2 = 1, delta.rep = structure(.Data = c(
0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0,NA,
0, 0, 0, 0, 0, 0, 0, 0,NA,NA,
0, 0, 0, 0, 0, 0, 0,NA,NA,NA,
0, 0, 0, 0, 0, 0,NA,NA,NA,NA,
0, 0, 0, 0, 0,NA,NA,NA,NA,NA,
0, 0, 0, 0,NA,NA,NA,NA,NA,NA,
0, 0, 0,NA,NA,NA,NA,NA,NA,NA,
0, 0,NA,NA,NA,NA,NA,NA,NA,NA,
0,NA,NA,NA,NA,NA,NA,NA,NA,NA), .Dim = c(10,10) ))
```

C-5: Development correlation model (Section 5.3)

MODEL

```
{
   for( i in 1 : year) {
      delta[i,1] <- log( c[i,1] )
      for( j in 2 : (year+1-i) ) {
         delta[i,j] <- log( c[i,j] ) - log( c[i,j-1] )
      }
      for( j in 1 : (year+1-i) ) {
         delta[i,j] ~ dnorm( mu.d[i,j], tau.d[i,j] )
         tau.d[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
         c.pred[i,j] <- 0
      }
   }
   for( i in 1 : year) {
      mu.d[i,1] <- mu[1]
   }
   for( i in 1 : year-1) {
      mu.d[i,2] <- mu[2] + h[2] * theta * (delta[i,1] - mu[1])
   }
   for( i in 1 : year-2) {
      for( j in 3 : (year+1-i) ) {
         mu.d[i,j] <- mu[j]
      }
   }

   # Prior Distributions
   for( j in 1 : 3) {
      mu[j] ~ dnorm(0,0.00001) I(0,)
   }
   for( j in 4 : year) {
      mu[j] <- pow(M,j-3) * mu[3]
   }
   h[1] <- 1
   for( j in 2 : 3) {
      h[j] ~ dgamma(0.0001,0.0001)
   }
   for( j in 4 : year) {
      h[j] <- pow(N,j-3) * h[3]
```

```
}
M ~ dunif(0,1)
N ~ dunif(0,1)
sigma2 ~ dgamma(0.0001,0.0001)
theta ~ dnorm(0,0.00001)
rho <- theta/sqrt(1+pow(theta,2))

# Prediction
for( i in 2 : year) {
  delta[i,year+2-i] ~ dnorm( mu.d[i,year+2-i], tau.d[i,year+2-i] )
   tau.d[i,year+2-i] <- 1 / ( pow(h[year+2-i],2) * sigma2 )
  c.pred[i,year+2-i] <- c[i,year+1-i] * exp(delta[i,year+2-i])
}
for( i in 2 : year-1) {
   mu.d[i,year+2-i] <- mu[year+2-i]
}
mu.d[year,2] <- mu[2] + h[2] * theta * (delta[year,1] - mu[1])
for( i in 3 : year) {
   for( j in (year+3-i) : year) {
     delta[i,j] ~ dnorm( mu.d[i,j], tau.d[i,j] )
      mu.d[i,j] <- mu[j]
      tau.d[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
     c.pred[i,j] <- c.pred[i,j-1] * exp(delta[i,j])
   }
}
reserve.part[1] <- 0
for( i in 2 : year) {
   reserve.part[i] <- c.pred[i,year] - c[i,year+1-i]
}
reserve <- sum(reserve.part[])

# Hypothesis Test
for( i in 1 : year) {
   for( j in 1 : (year+1-i) ) {
      delta.rep[i,j] ~ dnorm( mu.d.rep[i,j], tau.d.rep[i,j] )
      tau.d.rep[i,j] <- 1 / ( pow(h[j],2) * sigma2 )
   }
}
for( i in 1 : year) {
   mu.d.rep[i,1] <- mu[1]
}
for( i in 1 : (year-1)) {
   mu.d.rep[i,2] <- mu[2] + h[2] * theta * (delta.rep[i,1] - mu[1])
}
for( i in 1 : (year-2)) {
   for( j in 3 : (year+1-i)) {
      mu.d.rep[i,j] <- mu[j]
   }
}
for( i in 1 : (year-1) ) {
   for( j in 2 : (year+1-i) ) {
      Q1[i,j] <- delta[i,j] - delta[i,j-1]
      Q1.rep[i,j] <- delta.rep[i,j] - delta.rep[i,j-1]
      mean.Q1[i,j] <- mu.d[i,j] - mu.d[i,j-1]
      mean.Q1.rep[i,j] <- mu.d.rep[i,j] - mu.d.rep[i,j-1]
      t1[i,j] <- pow(( Q1[i,j] - mean.Q1[i,j] ),2) / var.Q1[i,j]
      t1.rep[i,j] <- pow(( Q1.rep[i,j] - mean.Q1.rep[i,j] ),2) / var.Q1[i,j]
   }
}
for( i in 1 : (year-1) ) {
   var.Q1[i,2] <- (pow(h[2],2) + pow(h[1],2) - 2*h[2] * theta * pow(h[1],2)) * sigma2
}
for( i in 1 : (year-1) ) {
   for( j in 3 : (year+1-i) ) {
      var.Q1[i,j] <- ( pow(h[j],2) + pow(h[j-1],2) ) * sigma2
   }
}
for( i in 1 : year ) {
   t1[i,1] <- 0
   t1.rep[i,1] <- 0
}
for( i in 2 : year ) {
```

```
      for( j in (year+2-i) : year ) {
         t1[i,j] <- 0
         t1.rep[i,j] <- 0
      }
   }
   Ts1 <- sum(t1[,])
   Ts1.rep <- sum(t1.rep[,])
   p1 <- step( Ts1 - Ts1.rep )  # 1 if Ts1 >= Ts1.rep
   for( i in 1 : (year-1) ) {
      Q3[i] <- ( delta[i,2] - mu.d.rep[i,2] ) / ( delta[i,1] - mu.d.rep[i,1] )
      Q3.rep[i] <- (delta.rep[i,2] - mu.d.rep[i,2]) / ( delta.rep[i,1] - mu.d.rep[i,1] )
      t3[i] <- step( - Q3[i] )
      t3.rep[i] <- step( - Q3.rep[i] )
   }
   Ts3 <- sum(t3[])
   Ts3.rep <- sum(t3.rep[])
   p3 <- step( Ts3.rep - Ts3 )  # 1 if Ts3.rep >= Ts3
}
```

DATA

```
# Same data as in Appendix C-3
```

INITS

```
list( mu = c(0.1,0.1,0.1,NA,NA,NA,NA,NA,NA,NA), h = c(NA,1,1,NA,NA,NA,NA,NA,NA,NA),
      M = 0.5 , N = 0.5 , sigma2 = 1, theta = 1, delta.rep = structure(.Data = c(
0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0,NA,
0, 0, 0, 0, 0, 0, 0, 0,NA,NA,
0, 0, 0, 0, 0, 0, 0,NA,NA,NA,
0, 0, 0, 0, 0, 0,NA,NA,NA,NA,
0, 0, 0, 0, 0,NA,NA,NA,NA,NA,
0, 0, 0, 0,NA,NA,NA,NA,NA,NA,
0, 0, 0,NA,NA,NA,NA,NA,NA,NA,
0, 0,NA,NA,NA,NA,NA,NA,NA,NA,
0,NA,NA,NA,NA,NA,NA,NA,NA,NA), .Dim = c(10,10) ))
```

Appendix D: Additional tables and figures

D-1: Frequency histograms for simulated values of η and τ^2

Figure 8.1: Frequency histogram of simulated η

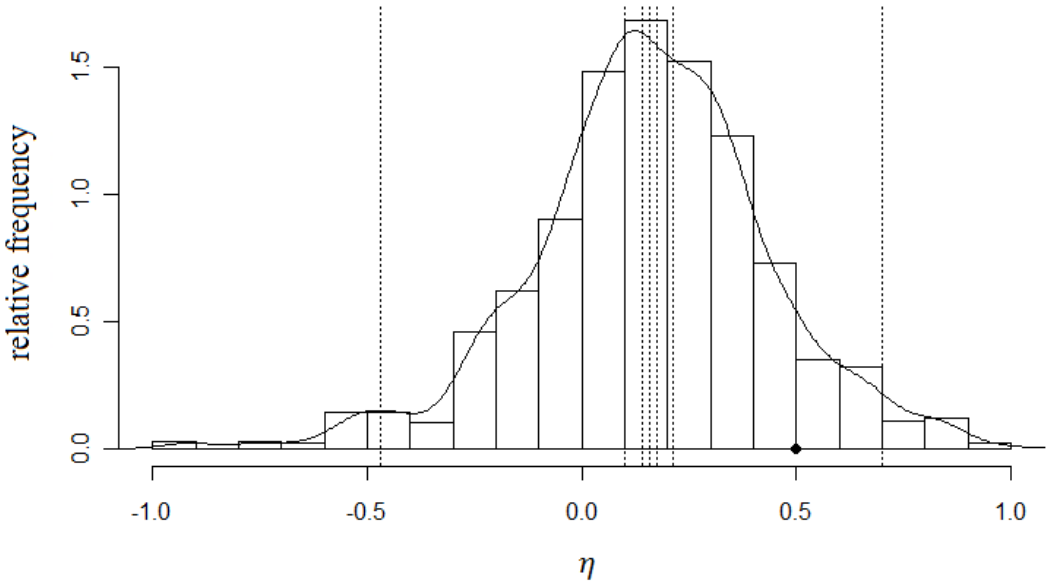
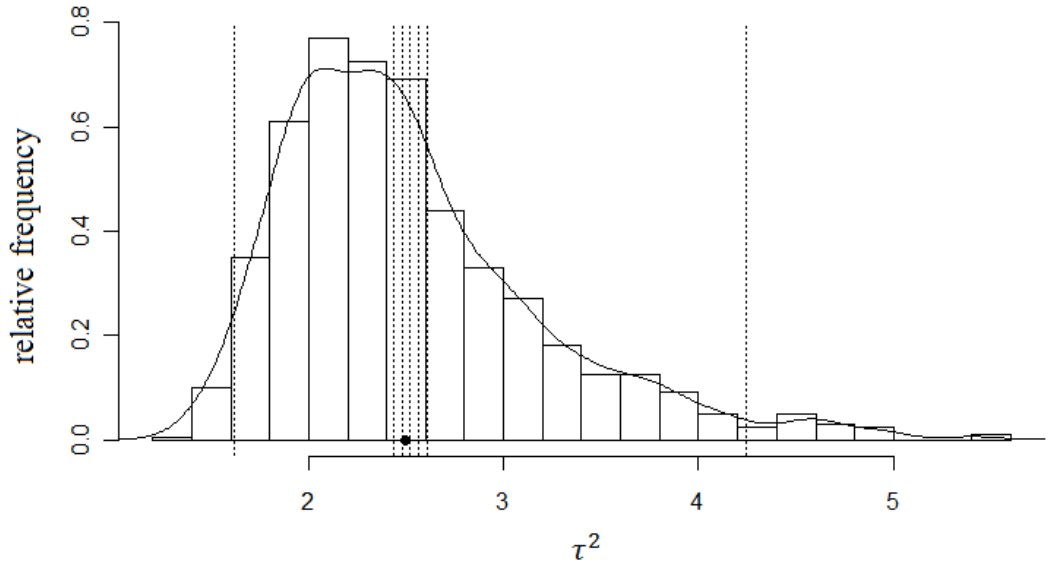
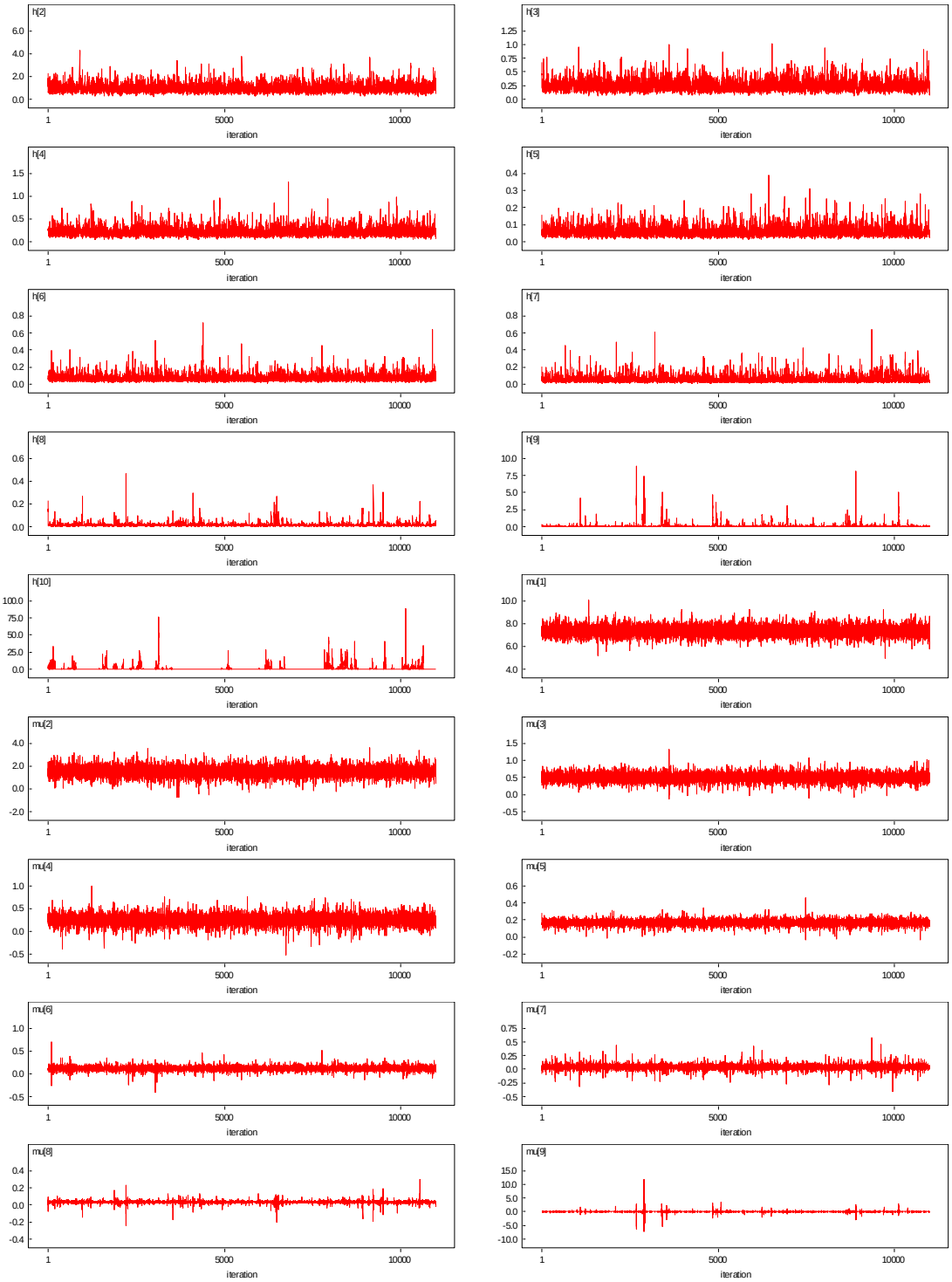


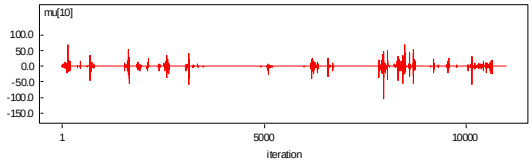
Figure 8.2: Frequency histogram of simulated τ^2

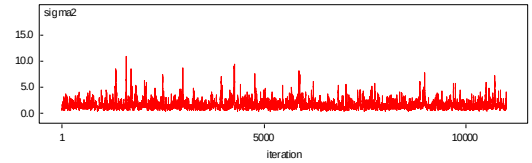


D-2: Traces of simulated values for Hertig's model

Figure 8.3: Traces of simulated μ_j, h_j and σ^2 (Hertig's model)







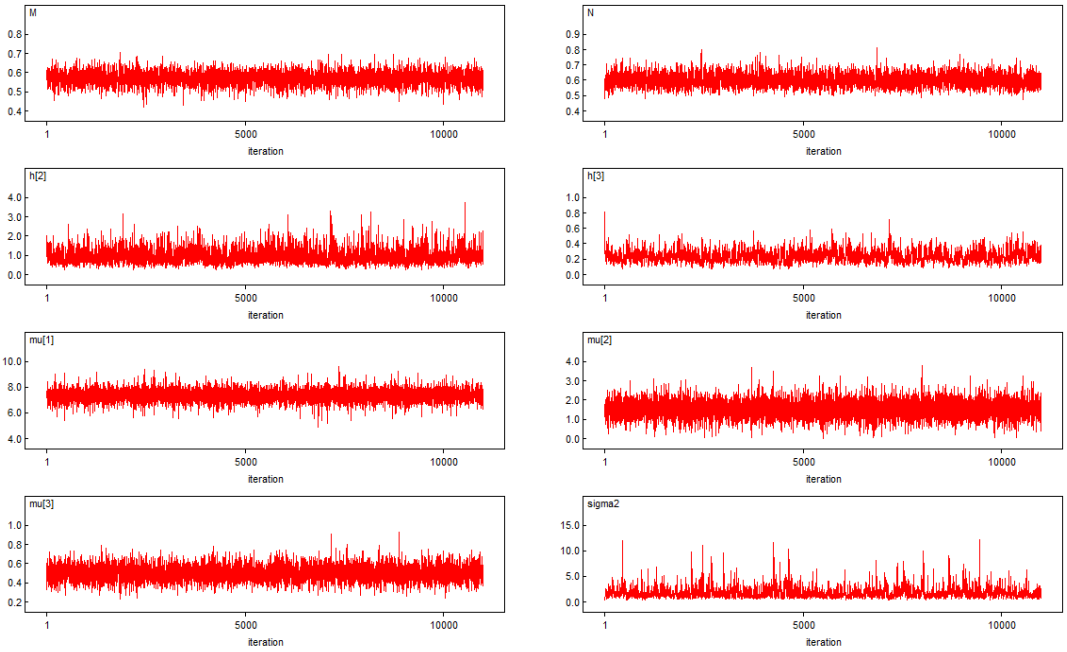
D-3: Predictive inference on future claims (Hertig's model)

Table 8.1: Predictive inference on cumulative claim liabilities and reserve

node	mean	sd	MC error	2.5%	median	97.5%
c.pred[2,10]	**1.02E+20**	4.381E+21	9.839E+19	14.61	16860.0	4.009E+6
c.pred[3,9]	39780.0	641700.0	15180.0	18110.0	23880.0	34470.0
c.pred[3,10]	**1.255E+23**	5.611E+24	1.233E+23	50.6	24060.0	6.399E+6
c.pred[4,8]	2.8E+4	1089.0	18.94	26720.0	27990.0	29290.0
c.pred[4,9]	31110.0	64080.0	1969.0	21250.0	28500.0	39640.0
c.pred[4,10]	**1.602E+19**	7.151E+20	1.572E+19	38.64	28690.0	6.264E+6
c.pred[5,7]	27470.0	2587.0	67.25	22890.0	27330.0	33080.0
c.pred[5,8]	28410.0	2740.0	65.48	23370.0	28230.0	34290.0
c.pred[5,9]	41920.0	503200.0	11450.0	19920.0	28740.0	42920.0
c.pred[5,10]	**1.056E+32**	4.72E+33	1.037E+32	36.9	28970.0	1.31E+7
c.pred[6,6]	17950.0	1880.0	42.52	14550.0	17880.0	22060.0
c.pred[6,7]	18770.0	2661.0	62.45	13880.0	18570.0	24720.0
c.pred[6,8]	19400.0	2841.0	64.16	14310.0	19250.0	25590.0
c.pred[6,9]	63400.0	1.638E+6	42080.0	12770.0	19590.0	29510.0
c.pred[6,10]	**1.577E+18**	5.265E+19	1.159E+18	51.34	19730.0	1.457E+7
c.pred[7,5]	14590.0	1148.0	24.53	12500.0	14550.0	17130.0
c.pred[7,6]	16590.0	2291.0	46.37	12700.0	16380.0	21710.0
c.pred[7,7]	17400.0	3013.0	70.21	12560.0	17130.0	24060.0
c.pred[7,8]	1.8E+4	3275.0	78.47	12900.0	17700.0	24890.0
c.pred[7,9]	32160.0	528600.0	12300.0	11720.0	18090.0	29850.0
c.pred[7,10]	**1.473E+20**	6.583E+21	1.447E+20	9.776	18240.0	4.02E+7
c.pred[8,4]	17760.0	5743.0	117.9	9966.0	16990.0	30960.0
c.pred[8,5]	21050.0	7079.0	152.1	11630.0	19960.0	37440.0
c.pred[8,6]	23820.0	8457.0	178.7	12440.0	22590.0	43560.0
c.pred[8,7]	25020.0	9236.0	171.5	12760.0	23650.0	46690.0
c.pred[8,8]	25900.0	9661.0	183.3	13210.0	24360.0	48400.0
c.pred[8,9]	6.79E+6	3.023E+8	6.646E+6	12460.0	24750.0	52310.0
c.pred[8,10]	**3.22E+21**	1.121E+23	2.673E+21	28.04	25110.0	5.39E+7
c.pred[9,3]	9395.0	3258.0	64.35	4759.0	8958.0	17010.0
c.pred[9,4]	12580.0	5968.0	127.9	4861.0	11400.0	27700.0
c.pred[9,5]	14880.0	7184.0	162.5	5675.0	13410.0	32880.0
c.pred[9,6]	16840.0	8520.0	194.2	6227.0	15150.0	37840.0
c.pred[9,7]	17640.0	9022.0	198.6	6400.0	15890.0	40440.0
c.pred[9,8]	18270.0	9344.0	203.5	6612.0	16420.0	41760.0
c.pred[9,9]	56420.0	1.644E+6	36420.0	6121.0	16710.0	46830.0
c.pred[9,10]	**6.024E+27**	1.936E+29	4.277E+27	30.71	16630.0	8.127E+6
c.pred[10,2]	35310.0	524800.0	11440.0	651.6	9154.0	115800.0
c.pred[10,3]	62900.0	1.115E+6	24540.0	1157.0	14970.0	205700.0
c.pred[10,4]	88200.0	1.732E+6	38260.0	1277.0	19320.0	2.73E+5
c.pred[10,5]	106200.0	2.088E+6	46180.0	1470.0	23290.0	309500.0
c.pred[10,6]	125100.0	2.579E+6	57120.0	1692.0	25800.0	347800.0
c.pred[10,7]	134900.0	2.885E+6	63830.0	1833.0	26700.0	383500.0
c.pred[10,8]	139800.0	2.996E+6	66280.0	1889.0	27560.0	399100.0
c.pred[10,9]	204700.0	4.093E+6	91680.0	1770.0	28330.0	422200.0
c.pred[10,10]	**9.802E+32**	4.381E+34	9.628E+32	86.33	28440.0	1.944E+7
reserve	**1.086E+33**	**4.406E+34**	1.067E+33	**-75100.0**	86260.0	**5.208E+11**

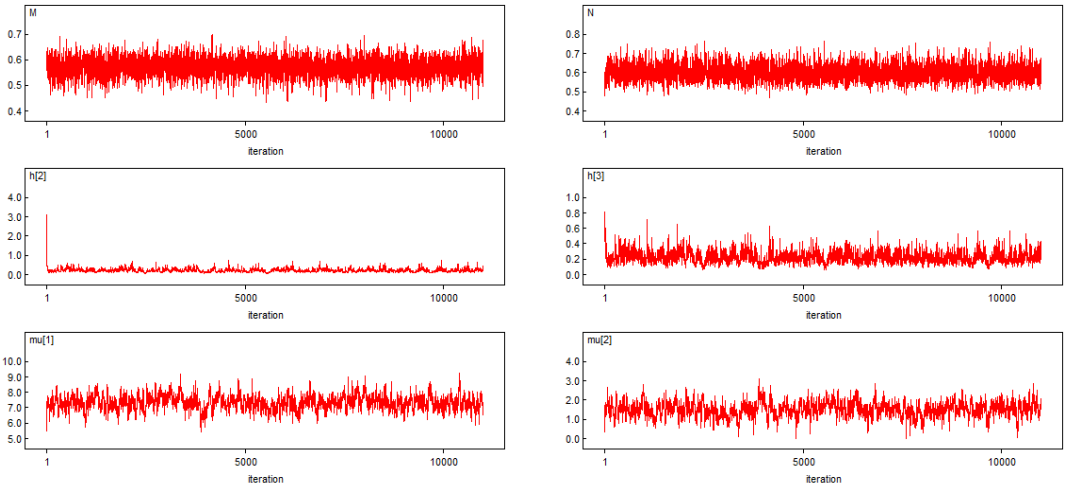
D-4: Traces of simulated values for modified Hertig's model

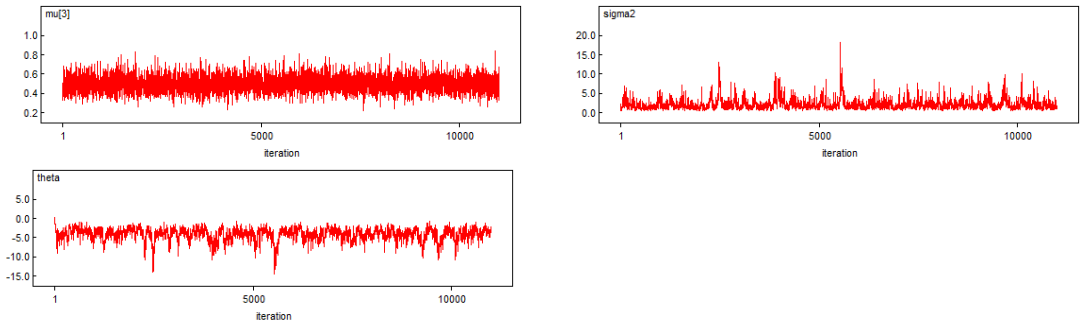
Figure 8.4: Traces of simulated μ_j, h_j, σ^2, M and N (modified Hertig's model)



D-5: Traces of simulated values for development corr. model

Figure 8.5: Traces of simulated μ_j, h_j, σ^2, M, N and θ_1 (dev. corr. model)





D-6: Simulated values of μ_1 *vs* simulated values of μ_0

Figure 8.6: Simulated values of μ_1 *vs* simulated values of μ_0