Convergence to Pure Steady States of Linear Quantum Systems

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Abstract— **We study the convergence of density operators of single-mode linear quantum systems towards their steady states in trace norm. We give compact and intuitive necessary and sufficient conditions for such linear quantum systems to have pure Gaussian steady states in terms of the Hamiltonian and coupling operators.**

I. Introduction

Over the last two decades, quantum information processing has received considerable attention. While most of the initial concepts were first developed for discrete quantum variables (e.g., qubits), subsequent developments have also investigated the use of continuous variables; e.g., quadratures of optical fields [1], [2]. Pure Gaussian states are a central resource for continuous-variable quantum information processing [3]. Recent work has shown that reservoir engineering (i.e., the control of dissipative dynamics) is an efficient approach to construct pure Gaussian states [4], [5]. In addition, a physical realizability condition for linear quantum stochastic differential equations (QSDEs) with pure Gaussian steady states has been studied recently in [6]–[8]. These constructions may open up the possibility to generate massive entangled Gaussian states, which are useful in continuous-variable quantum computation.

In [9], the dissipative dynamics of a linear QSDE describing the evolution of position and momentum operators has been analyzed for the case that the density operator of the system is always Gaussian. Using the covariance formalism given in [10], [11] and the fact that Gaussianity is preserved in linear QSDEs dynamics [12], an algebraic condition was formulated to verify whether the steady state of the linear QSDE is a pure state. However, it is not clear whether this result also applies when the initial system density operator is a non-Gaussian state. Non-Gaussian states can arise from a nonlinearity as required in universal quantum computing [3], [13].

For most applications of reservoir engineering, the uniqueness of the steady state is desirable [14]. Moreover, sometimes it is also useful to consider the distance between a steady state and the current system state as well as the rate at which this state converges to the steady state. The evolution of the density operators related to linear QSDEs can be analyzed by analogy to the evolution of probability distributions corresponding to Ornstein-Uhlenbeck processes [12]. A convergence analysis for linear QSDEs has also been considered in the mathematical physics community via the analysis of quantum Fokker-Planck models [15], [16]. In [16], the asymptotic stability properties of quantum Fokker-Planck models have been carefully studied using results from quantum Markovian semigroups. The authors of [16] gave an algebraic condition on when the density operator of a linear QSDE converges to a particular steady state.

In this work, we study the Hurwitz property of the system matrix of linear QSDEs, which can be used to infer the convergence of a quantum system's state towards its steady state. We further give alternative necessary and sufficient conditions so that the steady state is a pure Gaussian state. While the single-mode case is included in the purity criterion given in [9], our criterion is algebraically simpler to validate, and has an intuitive interpretation. This condition is that for the steady state of a single-mode linear quantum system to be pure Gaussian, the Hamiltonian has to commute with $\mathbb{L}^\dagger\mathbb{L}$ (here $\mathbb{L}$ is the system coupling operator). Moreover, we do not assume the Gaussianity of the steady state in our derivations. Using the stability notion in [17], we then show that this steady state is globally exponentially stable in the trace norm, which is a distance measure between two density operators that is frequently used in quantum information processing [18].

The paper is organized as follows. In Section II we give a necessary and sufficient condition for linear QSDEs describing single-mode quantum systems to have a pure Gaussian steady state. After that, we give a convergence analysis for the steady state in trace norm. Section III gives some examples of the purity criterion and the convergence analysis. The last section gives some conclusions.

A. Notation

The imaginary unit $\sqrt{-1}$ is denoted by $\imath$. Hilbert space adjoints, are indicated by $*$. We also use $*$ for complex conjugate. For a matrix whose elements are operators on a Hilbert space, the adjoint transpose will be denoted by $\dagger$; i.e., $(\mathbf{X}^*)^\top = \mathbf{X}^\dagger$. Throughout the article, $\mathscr{H}$ is the system's Hilbert space, and $\mathfrak{B}(\mathscr{H}), \mathfrak{S}(\mathscr{H})$ are the classes of bounded linear operators and density operators on the Hilbert space $\mathscr{H}$ respectively. The quantum master equation for the density operator ρ is determined by $\mathscr{L}_*(\rho)$, while the quantum Markovian generator for an operator X is given by $\mathscr{L}(X)$; see [19] for more detailed definitions.

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II. Stability of Single Mode Linear Quantum Systems and Purity of Their Steady States

In this section, we analyse the stability of linear QSDEs describing the dynamics of a single mode quantum system. Consider a quantum system with Hamiltonian and coupling operator given by

$$\mathbb{H} = \frac{1}{2}\mathbf{x}^\top \mathbf{M}\mathbf{x}, \qquad \mathbb{L} = \mathbf{C}^\top \mathbf{x}, \tag{1}$$

where $\mathbf{x} = [q\ p]^\top$ corresponds to the position and momentum operators, $\mathbf{M} = \mathbf{M}^\top \in \mathbb{R}^{2\times 2}, \mathbf{C} \in \mathbb{C}^{2\times 1}$. The dynamics of this quantum system can be described by the following linear QSDEs,

$$d\mathbf{x} = \mathbf{A}\mathbf{x}dt + \mathbf{B}^* dA + \mathbf{B}dA^*, \tag{2a}$$

where

$$\mathbf{A} = \mathbf{\Sigma}\mathbf{M} - \frac{\gamma}{2}\mathbf{I}, \tag{2b}$$

$$\mathbf{B} = \imath \mathbf{\Sigma}\mathbf{C}, \qquad \gamma = \imath \mathbf{B}^\dagger \mathbf{\Sigma}^\top \mathbf{B}, \tag{2c}$$

and $\mathbf{\Sigma}$ is the skew-symmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The processes dA^*, dA are the annihilation and creation processes of the quantum field [20]. In this section, we will give a condition for the convergence of the density operator corresponding to QSDE (2a) towards a steady state, without assuming that system's density operator is Gaussian. Using a transformation given in [12, Exercise A.25], one can transform the Wigner-Fokker-Planck (WFP) dynamics in [16] into a standard master equation. The corresponding QSDEs (2a) of the transformed master equations have the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\tilde{\gamma}/2 \end{bmatrix}, \tag{3}$$

where ω is the frequency of the harmonic confinement potential, and $\tilde{\gamma}$ is a constant related to the dissipation in the system. Here, we consider a form of QSDEs which commonly appears in quantum optics, and does not necessarily fit into the form (3). In the following proposition, we will give a relation between $\mathbf{A}$ being Hurwitz, and the dissipation characteristics of the matrix $\mathbf{B}$.

Proposition 1: For linear QSDEs of the form (2a), $\mathbf{A}$ is Hurwitz only if $\gamma > 0$. Moreover, if the Hamiltonian $\mathbb{H}$ is either a positive or a negative operator, then $\mathbf{A}$ is Hurwitz if and only if $\gamma > 0$.

Proof: By definition, $\mathbf{A}$ is Hurwitz if $\mathrm{Re}(\lambda(\mathbf{A})) < 0$, for all eigenvalues $\lambda(\mathbf{A})$ of $\mathbf{A}$. Evaluating the characteristic polynomial of $\mathbf{A}$ using (2b), we have

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 + \lambda\gamma + \left(m_{11}m_{22} - m_{12}^2 + \gamma^2/4\right), \tag{4}$$

where m_{ij} denotes the elements of $\mathbf{M}$. When $\mathbf{A}$ is Hurwitz, then we must have $\gamma > 0$. Moreover, when either of $\pm\mathbb{H}$ is positive, then $\pm\mathbf{M}$ is positive semi-definite. Using the assumption that $\pm\mathbf{M}$ is positive semi-definite, it follows that $m_{11}m_{22} - m_{12}^2 \geq 0$ and one can clearly see from (4) that $\mathbf{A}$ is Hurwitz if and only if $\gamma > 0$ which completes the proof. ■

Proposition 1 implies that when the Hamiltonian is either a positive or a negative operator, then it becomes irrelevant to the overall system stability. That is, the linear QSDE stability property is preserved even if the Hamiltonian is changed to another positive or negative operator. Let us define the covariance matrix $\mathbf{P}_t = \frac{1}{2}\mathrm{tr}((\mathbf{x}\mathbf{x}^\top + (\mathbf{x}\mathbf{x}^\top)^\top)\rho_t)$, and define the mapping $\mathbf{V}$ as follow:

$$\mathbf{V}(\mathbf{P}_*) = \mathbf{A}\mathbf{P}_* + \mathbf{P}_*\mathbf{A}^\top + \frac{1}{2}(\mathbf{B}^*\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\dagger). \tag{5}$$

Notice that for linear QSDEs with $\mathbf{A}$ Hurwitz, both $\mathrm{tr}(\mathbf{x}\rho_t)$ and $\mathbf{P}_t$ converge [9]. The steady state covariance matrix is given by the solution of $\mathbf{V}(\mathbf{P}_*) = 0$ in (5). Since the steady state of the linear QSDE (2a) will be a Gaussian state regardless of the initial state [21, Eq. 5.80], [12, §5.6.1] and a Gaussian state is uniquely determined by $\mathrm{tr}(\mathbf{x}\rho_t)$ and $\mathbf{P}_t$, then a linear QSDE with $\mathbf{A}$ is Hurwitz has a unique steady state which is Gaussian. Due to the one to one correspondence between Wigner functions and density operators and the uniqueness of the steady state, by [22, Remark 18], as expected, this steady state is weakly globally asymptotically stable (WGA-stable). That is, for any ρ as an initial state and any bounded operator A, $\lim_{t\to\infty} \mathrm{tr}(A(\rho_t - \rho_*)) = 0$, where ρ_* is the Gaussian steady state. This fact can be restated in the following Proposition.

Proposition 2: The linear QSDEs of the form (2a) with a Hurwitz $\mathbf{A}$ matrix have unique WGA-stable steady states.

In what follows, we will prove that the only case in which a linear QSDE of the form (2a) with $\mathbf{A}$ is Hurwitz has a pure steady state is if the Hamiltonian and the coupling operator satisfy $\left[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}\right] = 0$. Let us write the coupling operator $\mathbb{L}$ in the following form

$$\mathbb{L} = \mu a + \nu a^\dagger, \tag{6}$$

where $\mu, \nu \in \mathbb{C}$ and $\gamma = |\mu|^2 - |\nu|^2$. Now introduce a constant $\hat{\gamma} = \gamma/L$, with $L = |\mu|^2 + |\nu|^2$ and $r = \tanh^{-1}\left(\sqrt{(1-\hat{\gamma})/(1+\hat{\gamma})}\right)$. When $\hat{\gamma} > 0$, it is possible to find the solution $\mathbb{L}|\xi\rangle = 0|\xi\rangle$ which is given by the following squeezed state [23, p 158-161]

$$|\xi\rangle = \frac{1}{\sqrt{\cosh(r)}}\sum_{n=0}^{\infty}(-1)^n \frac{\sqrt{(2n)!}}{2^n n!} e^{\imath n\theta}\tanh(r)^n |2n\rangle. \tag{7}$$

It was argued in [16, Lemma 9.1, Corollary 9.1], that the only possible occasion where the steady state of a certain class of QSDEs (2a) is *pure* is when $\left[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}\right] = 0$. In the following proposition, we also show that this is the case for a linear QSDE in the form (2a). Recall also that the set of *dark states* of the master equation is a subset of system Hilbert space which are not affected by the dissipative dynamics. This subset is defined as $\mathscr{D}_{\mathbb{L},\mathbb{H}} = \{|\psi\rangle \in \mathscr{H} : \mathbb{L}|\psi\rangle = 0, \mathbb{H}|\psi\rangle = \lambda_H|\psi\rangle, \lambda_H \in \mathbb{R}\}$ [4]. Roughly speaking, we show that there exists no subspace $\mathscr{S}$ of $\mathscr{H}$ that is orthogonal to $\mathscr{D}_{\mathbb{L},\mathbb{H}}$, such that $\mathbb{L}|\psi\rangle \in \mathscr{S}, \forall|\psi\rangle \in \mathscr{S}$; implying that the only steady state is the dark state; see also [4, Theorem 2].

Proposition 3: A linear QSDE of the form (2a) with $\mathbf{A}$ Hurwitz has a *pure* steady state if and only if $\left[\mathbb{H}, \mathbb{L}^\dagger\mathbb{L}\right] = 0$.

Proof: Suppose $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$, and let $|\xi\rangle$ be an eigenstate of $\mathbb{L}$ such that $0|\xi\rangle=\mathbb{L}|\xi\rangle$, which exists since $\hat{\gamma}>0$ when $\mathbf{A}$ is Hurwitz by Proposition 1. Since $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$, then $0|\xi\rangle=\mathbb{H}\mathbb{L}^{\dagger}\mathbb{L}|\xi\rangle=\mathbb{L}^{\dagger}\mathbb{L}\mathbb{H}|\xi\rangle$. Therefore $|\xi\rangle$ is also an eigenstate of $\mathbb{H}$. For this $|\xi\rangle$, $\mathscr{L}_*(|\xi\rangle\langle\xi|)=0$. Hence $\rho_*=|\xi\rangle\langle\xi|$ is a steady state of (2a).

Now suppose the steady state is pure. Then there exists a $|\psi\rangle\in\mathscr{H}$ satisfying $\mathscr{L}_*(|\psi\rangle\langle\psi|)=0$. Therefore, we can write

$$Q|\psi\rangle\langle\psi|+|\psi\rangle\langle\psi|Q^{\dagger}=\mathbb{L}|\psi\rangle\langle\psi|\mathbb{L}^{\dagger}, \tag{8}$$

where $Q=\imath\mathbb{H}+\frac{1}{2}\mathbb{L}^{\dagger}\mathbb{L}$. To satisfy (8), since the RHS of (8) has rank one, then $|\psi\rangle$ must be an eigen state of Q. In this case, it is also an eigen state of $\mathbb{L}$, [4, Theorem 1]. Suppose $\mathbb{L}|\psi\rangle=\lambda_l|\psi\rangle$, and $Q|\psi\rangle=\lambda|\psi\rangle$ for $\lambda,\lambda_l\in\mathbb{C}$. Then (8) is equivalent to

$$(\lambda+\lambda^*)|\psi\rangle\langle\psi|=|\lambda_l|^2|\psi\rangle\langle\psi|. \tag{9}$$

Therefore, $\mathrm{Re}\{(\lambda)\}=\frac{|\lambda_l|^2}{2}$. Now, the only possible case that Q and $\mathbb{L}$ have the same eigenvector is when $\lambda_l=0$. Otherwise, if $\lambda_l\neq 0$ then $\lambda_l=\langle\mathbb{L}\rangle=\mathbf{C}^{\top}\langle\mathbf{x}\rangle=0$. Therefore, we observe that

$$0|\psi\rangle\langle\psi|=Q|\psi\rangle\langle\psi|+|\psi\rangle\langle\psi|Q^{\dagger}=\imath\mathbb{H}|\psi\rangle\langle\psi|-|\psi\rangle\langle\psi|\imath\mathbb{H}.$$

Hence $|\psi\rangle$ must also be an eigen-state of $\mathbb{H}$. Now let us rewrite $\mathbb{H}$ in terms of annihilation and creation operators as below:

$$\mathbb{H}=\frac{1}{2}\left[a^{\dagger}a\right]\begin{bmatrix}\hat{m}_{11} & \hat{m}_{12}\\ \hat{m}_{12}^* & \hat{m}_{22}\end{bmatrix}\begin{bmatrix}a\\ a^{\dagger}\end{bmatrix},$$

with $\hat{m}_{11}=\hat{m}_{22}=(m_{11}+m_{22})/2$, $\hat{m}_{12}=\imath m_{12}+(m_{11}-m_{22})/2$. Since $\mathbf{A}$ is Hurwitz, $\mu\neq 0$. Moreover, without loss of generality, assume that $\nu\neq 0$. Otherwise, we will have $\hat{m}_{12}=\hat{m}_{22}=0$, and hence $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$. When $\lambda_l=0$, $|\psi\rangle=|\xi\rangle$ as defined in (7). Expanding $|\xi\rangle$, and using the relation $s_{n+1}=-(\nu/\mu)(\sqrt{n}/\sqrt{n+1})s_{n-1}$, we arrive at the following equation

$$\begin{aligned}2\mathbb{H}|\xi\rangle=\sum_{n=0}^{\infty}\Bigg[&(2n+1)\hat{m}_{11}+\hat{m}_{12}H(n-2)\frac{-\mu}{\nu}n\\ &+\hat{m}_{12}^*\frac{-\nu}{\mu}(n+1)\Bigg]s_n|n\rangle,\end{aligned}$$

where $H(n)$ is the Heavyside function. Equating $\mathbb{H}|\xi\rangle=\lambda_H|\xi\rangle$, for $n=0$ we obtain $\hat{m}_{11}+\hat{m}_{12}^*(-\nu/\mu)=\lambda_H$, and for $n\geq 2$ we obtain $\hat{m}_{12}=2\hat{m}_{11}(\mu^*\nu)/(|\mu|^2+|\nu|^2)$. Hence, we obtain

$$\mathbb{H}=\frac{\hat{m}_{11}}{|\mu|^2+|\nu|^2}\left[a^{\dagger}a\right]\begin{bmatrix}|\mu|^2 & \mu^*\nu\\ \nu^*\mu & |\nu|^2\end{bmatrix}\begin{bmatrix}a\\ a^{\dagger}\end{bmatrix}-\frac{\hat{m}_{11}\left(|\mu|^2-|\nu|^2\right)}{2}.$$

Therefore, $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$. ■

The criterion in Proposition 3 is equivalent the condition that the $\mathbf{M}$ and $\mathbf{C}$ matrices satisfy

$$\mathbf{M}\boldsymbol{\Sigma}\left(\mathbf{C}^*\mathbf{C}^{\top}\right)=\left(\mathbf{C}^*\mathbf{C}^{\top}\right)\boldsymbol{\Sigma}\mathbf{M}. \tag{10}$$

The following corollary is then immediate from the proof of the previous proposition.

Corollary 1: A linear QSDE of the form (2a) has a unique pure Gaussian steady state if and only if the Hamiltonian is either a positive or a negative operator, $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$ and $\mathbf{A}$ is Hurwitz.

Proof: If the Hamiltonian is either positive or a negative operator where $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$ and $\mathbf{A}$ is Hurwitz, then the system has a unique pure steady state by Proposition 3. Now suppose that the steady state is unique and Gaussian. From the existence of the pure Gaussian steady state, the solution $\mathbf{P}_*>0$ of $\mathbf{V}(\mathbf{P}_*)=0$ in (5) exists. Therefore, the matrix $\mathbf{A}$ corresponding to this Hamiltonian and coupling operator is Hurwitz. Moreover, since $\mathbf{A}$ is Hurwitz then $\left[\mathbb{H},\mathbb{L}^{\dagger}\mathbb{L}\right]=0$. Therefore, the Hamiltonian $\mathbb{H}$ is of the form $\mathbb{H}=\mathbf{x}^{\top}\left(\alpha\mathbf{C}^*\mathbf{C}^{\top}+\imath\beta\boldsymbol{\Sigma}\right)\mathbf{x}=\alpha\mathbb{L}^{\dagger}\mathbb{L}+\beta,\alpha,\beta\in\mathbb{R}$. Therefore, the Hamiltonian is either a positive or a negative operator up to a constant β. ■

In Proposition 2, all steady states of the linear QSDEs are asymptotically stable in the weak sense. If the steady state is pure, a stronger stability condition can be obtained if we assume that the quantum dynamical semigroup arising from (2a) is uniformly continuous [24], [25]. Under this assumption, we can use quantum Lyapunov stability to conclude a global exponential stability condition for the steady state ρ_* [17]. That is, for any initial density operator ρ_0 and steady state ρ_*, there exists $\beta,\kappa>0$ such that $\|\rho_t-\rho_*\|_1\leq\beta\|\rho_0-\rho_*\|_1\exp(-\kappa t)$ for all $t\geq 0$, where $\|\cdot\|_1$ is the trace operator norm; see [17, Definition 9]. Let us first recall a result about the quantum Lyapunov stability of [17].

Lemma 1: [17] Let $V\in\mathfrak{B}(\mathscr{H})$ be a self-adjoint operator with strictly increasing spectrum values such that

$$\mathrm{tr}\left(V(\rho-\rho_*)\right)>0,\forall\rho\in\mathfrak{S}(\mathscr{H})\backslash\rho_*. \tag{11}$$

If there exists $k_1>0$ and $k_2\in\mathbb{R}$ such that

$$\mathrm{tr}\left(\mathscr{L}(V)\rho\right)\leq -k_1\,\mathrm{tr}\left(V\rho\right)+k_2<0,\ \forall\rho\in\mathfrak{S}(\mathscr{H})\backslash\rho_*, \tag{12}$$

then ρ_* is globally exponentially stable (GE-stable).

Proposition 4: The pure Gaussian steady states of single mode linear QSDEs of the form (2a) are GE-stable.

Proof: Choose a candidate Lyapunov operator $V=\mathbb{L}^{\dagger}\mathbb{L}$. By Corollary 1, $\mathbf{A}$ is Hurwitz, and hence $\hat{\gamma}>0$. According to Proposition 3, the steady state density operator is given by $\rho_*=|\xi\rangle\langle\xi|$, where $\mathrm{tr}(V\rho_*)=0$. Calculating the generator of V, using $\mathbb{L}$ of the form (6), we obtain

$$\mathscr{L}(V)=\mathscr{L}(\mathbb{L}^{\dagger}\mathbb{L})=\frac{1}{2}\left(\mathbb{L}^{\dagger}\left[\mathbb{L}^{\dagger}\mathbb{L},\mathbb{L}\right]+\left[\mathbb{L}^{\dagger},\mathbb{L}^{\dagger}\mathbb{L}\right]\mathbb{L}\right) \tag{13}$$

$$=-L^2\hat{\gamma}\mathbb{L}^{\dagger}\mathbb{L}=-L^2\hat{\gamma}V. \tag{14}$$

Since V is a positive operator for which $\mathrm{tr}(V\rho)$ can only be zero if $\rho=\rho_*$, it follows that

$$\mathrm{tr}\left(\mathscr{L}(V)\rho\right)=-L^2\hat{\gamma}\mathrm{tr}\left(V\rho\right)<0,\forall\rho\in\mathfrak{S}(\mathscr{H})\backslash\rho_*. \tag{15}$$

Hence ρ_* is GE-stable. ■

Remark 1: In [26, Theorem 2], exponential convergence to the steady state for linear QSDEs with perturbed Hamiltonians has also been established. The main difference between their result and what we obtain here is that the convergence in Proposition 4 is given in trace norm. Also, we only consider Hamiltonians which are quadratic in q and p. In [26] convergence is given in the Hilbert-Schmidt operator norm and also this paper allows for quadratic Hamiltonians with some smooth bounded perturbations.

Remark 2: It is also worth noticing that for the case of single mode passive linear QSDEs, as considered in [27]–[29], corresponds to the case where $\hat{\gamma} = 1$, and $m_{11} = m_{22}, m_{12} = 0$. Using (10), we can easily verify that a passive linear QSDE has a pure Gaussian steady state. In fact, since $\hat{\gamma} = 1$, this Gaussian state is the vacuum state. Notice however that a linear QSDE that has a pure Gaussian steady state need *not* be passive. As an example, using (6), if we choose $0 < \hat{\gamma} < 1$, $\theta = 0$, $L > 0$ and the Hamiltonian $\mathbb{H} = \mathbb{L}^\dagger \mathbb{L}$, by Proposition 1 and Proposition 3, the matrix $\mathbf{A}$ is Hurwitz and the steady state of this linear QSDE is a pure Gaussian state. However, this linear QSDE is *not* passive; see Example 2 for a simulation of the case with $\hat{\gamma} = \frac{1}{2}$.

III. Examples

Example 1: Consider a single mode optical parametric oscillator (OPO) with Hamiltonian $\mathbb{H} = \frac{\imath}{4}\left(\varepsilon a^{\dagger 2} - \varepsilon^* a^2\right), \varepsilon \neq 0, \varepsilon \in \mathbb{C}$. The OPO is coupled to a vacuum field, where the coupling operator is given by $\mathbb{L} = \sqrt{\gamma} a$. The corresponding $\mathbf{M}$ and $\mathbf{C}$ matrices are given by

$$\mathbf{M} = \frac{1}{2}\begin{bmatrix} -\mathrm{Im}\{\varepsilon\} & \mathrm{Re}\{\varepsilon\} \\ \mathrm{Re}\{\varepsilon\} & \mathrm{Im}\{\varepsilon\} \end{bmatrix}, \qquad \mathbf{C}^\top = \sqrt{\frac{\gamma}{2}}\begin{bmatrix} 1 & \imath \end{bmatrix}.$$

Based on Corollary 1, since $\mathbf{M}$ is sign indefinite, we can conclude that the OPO system is not purifiable. In fact, the eigenvalues of $\mathbf{A}$ (2b) are given by $-\gamma \pm \frac{|\varepsilon|}{2}$. Proposition 2 implies that this OPO will only be WGA-stable if and only if $\frac{|\varepsilon|}{2} < \gamma$. Evaluating the algebraic condition in (10), one can verify that the OPO will only have a pure state if $\gamma = 0$. However, if $\gamma = 0$, $\mathbf{A}$ (2b) is not Hurwitz, hence no positive γ can make the OPO system have a pure Gaussian steady state; see also [9]. A straightforward verification also shows that the OPO will never have a pure steady state even if the coupling operator is given by an $\mathbb{L}$ in the form (6).

Example 2: Consider a linear QSDE whose the coupling operator $\mathbb{L}$ is given by (6) with $L = 10$, and $\hat{\gamma} = 0.5$, and $\mathbb{H} = \mathbb{L}^\dagger \mathbb{L}$. The corresponding $\mathbf{A}$ matrix is Hurwitz. To demonstrate the global exponential convergence of the system's density operator towards the steady state, we simulate this linear QSDE using a truncated Hilbert space quantum master equation with several pure random initial density operators [30], [31]. The results of this simulation are given in Figure 1. It can be seen in Figures 1(a) and 1(b) that irrespective of the initial density operators, the expected value of Lyapunov candidate operator $V = \mathbb{H}$ and the distance to the steady state are decreasing exponentially. Moreover, Figure 1(c) also shows that the density operators from different initial condition converge to a pure steady state; see also Figure 1(d) for the Wigner function of the density operator at different times taken from the first sample of the master equation simulation.

IV. Conclusions

In this article, we have given necessary and sufficient conditions for single-mode linear quantum systems to have pure Gaussian steady states. We have also considered whether these states are globally exponentially stable in the trace norm. Extensions of these results into multiple-mode linear quantum systems are likely to be possible using the *nullifier* concept of [32], and will be an interesting future research direction.

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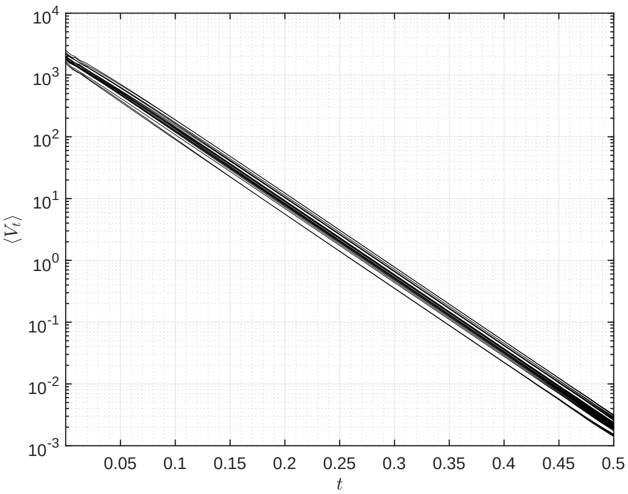
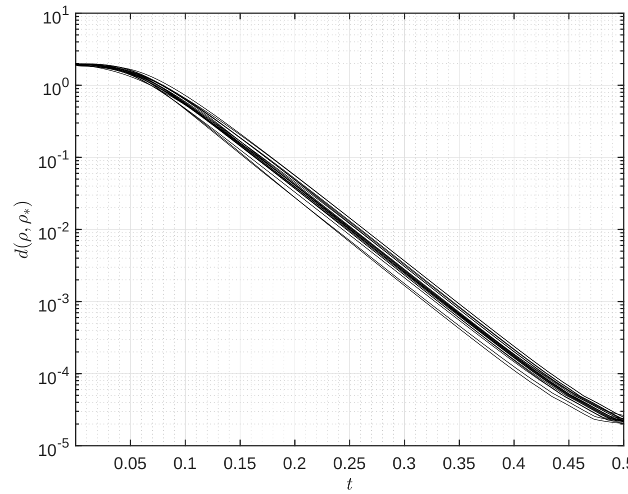
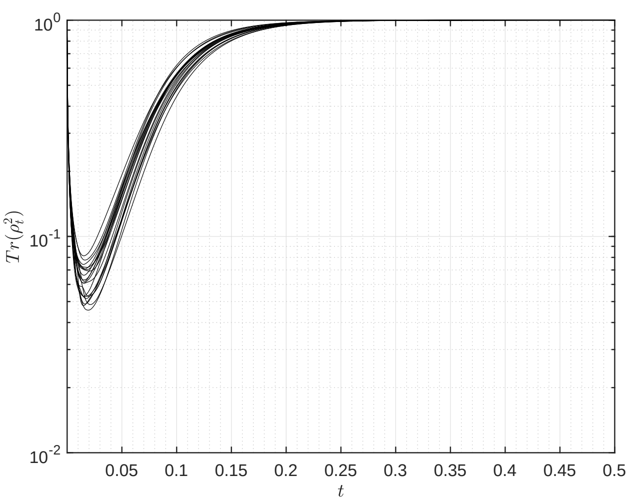


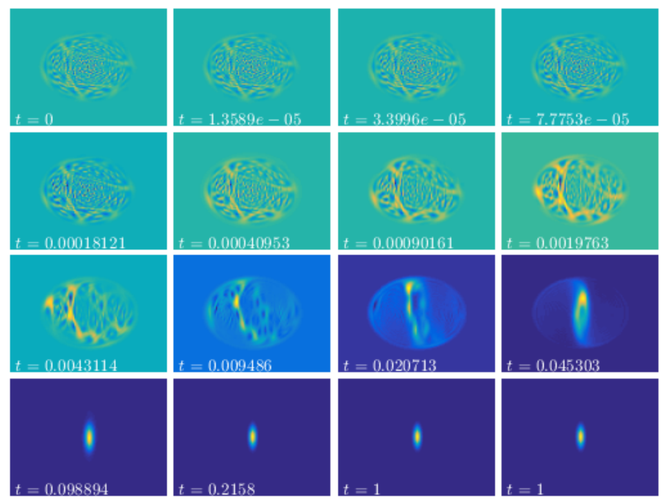
(a) $\langle V_t \rangle$



(b) Trace norm distance between current density operator and the steady state, $d(\rho_t, \rho_*) \equiv \|\rho_t - \rho_*\|_1$



(c) Purity of ρ_t



(d) Wigner function of one sample quantum master equation

Fig. 1: (Color Online) Quantum master equation simulations of the linear QSDE in Example 2 with 20 different initial pure density operators.

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