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Local Utility Functions and Local Probability Transformations

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Abstract
The local utility function and the local probability transformation are derived from the benefit function and the distance function.

Machina (1982) presented a systematic basis for generalizations of expected utility theory by introducing the idea of local utility functions such that preferences in a neighborhood of a given distribution may be approximated by an expected utility functional. An important class of generalised expected utility functional is the rank-dependent class (Quiggin 1982, 1993). Yaari's (1987) dual model, in which preferences are determined by a linear utility function and a nonlinear probability transformation is an important and influential member of the rank-dependent class. It is natural to ask whether Machina's local utility function representation of general preferences has an analog in terms of local probability transformations. One way of viewing the latter question is in terms of 'shadow' or 'risk-neutral' probabilities (Peleg and Yaari 1975, Nau 2001; Chambers and Quiggin, 2001). Under appropriate conditions, preferences may be locally approximated by the linear utility or risk-neutral preference functional associated with a local probability transformation.

In this note, the local utility function and the local probability transformation are derived from alternative representations of ordinal preferences, the benefit function and the distance function, which have been applied to problems of choice under uncertainty by Quiggin and Chambers (1998) and Chambers and Quiggin (2000, 2001).

1 Setup

We are concerned with preferences over state-contingent income or consumption vectors represented as mappings from a state space Ω to an outcome space $Y \subseteq \Re$. We focus on the case where Ω is a finite set $\{1...S\}$, the space of random variables is $Y^S \subseteq \Re^S$. We will make particular use of the unit vector $\mathbf{1} = (1, 1, ...1)$. We assume that preferences over Y^S are given by a total ordering $\succeq$, with an associated indifference ordering $\approx$ and that $\succeq$ may be represented by a strictly increasing, continuous function W, of the form $W : Y^S \rightarrow \Re$, where $Y \subseteq \Re_+$.

Machina's analysis is developed for preferences over spaces of cumulative distribution functions. In order to extend a local utility function analysis to preferences expressed in terms of random variables, it is necessary to restrict attention to functions W satisfying symmetry conditions to ensure that any two random variables with the same cumulative distribution function are judged equivalent. We will pay particular attention to the case when W is generalized Schur-concave with respect to some probability distribution $\boldsymbol{\pi}$ (Quiggin and Chambers, 1998; Chambers and Quiggin, 2000). Given a vector of probabilities $\boldsymbol{\pi}$, a function $W : \Re^S \to \Re$ is said to be *generalized Schur concave* if $\mathbf{y} \preceq_{\pi} \mathbf{y}'$ implies $W(\mathbf{y}) \geq W(\mathbf{y}')$, where the notation $\mathbf{y} \preceq_{\pi} \mathbf{y}'$ means that $\mathbf{y}$ and $\mathbf{y}'$ have the same mean and $\mathbf{y}$ is less risky than $\mathbf{y}'$ in the sense of Rothschild and Stiglitz. In particular, generalized Schur-concavity implies that if $\mathbf{y}$ and $\mathbf{y}'$ have the same probability distribution, $\mathbf{y} \preceq_{\pi} \mathbf{y}'$ and $\mathbf{y}' \preceq_{\pi} \mathbf{y}$ so that $W(\mathbf{y}) = W(\mathbf{y}')$.

For $\mathbf{g} \in Y^S$, the benefit function, $B : Y^S \times \Re \to \Re$, is defined by:

$$B(\mathbf{y}, w) = \max\{\beta \in \Re : W(\mathbf{y} - \beta\mathbf{g}) \geq w\}$$

if $W(\mathbf{y} - \beta\mathbf{g}) \geq w$ for some β and $-\infty$ otherwise. Following Blackorby and Donaldson (1980), for the special case $\mathbf{g} = \mathbf{1}$, the benefit function is referred to as the *translation function.* Henceforth, $\mathbf{g}$ is always taken to equal $\mathbf{1}$.[1]

The distance function, $D : Y^S \to \Re_+$ is defined

$$D(\mathbf{y}, w) = \sup\left\{\lambda > 0 : W\left(\frac{\mathbf{y}}{\lambda}\right) \geq w\right\}.$$

The properties of both of these functions are well known (Blackorby and Donaldson, 1980; Luenberger, 1992; Chambers, Chung, and Färe, 1996; Färe, 1988):

Lemma 1: $B(\mathbf{y}, w)$ satisfies:

a) $B(\mathbf{y}, w)$ is nonincreasing in w and nondecreasing in $\mathbf{y}$;

b) $B(\mathbf{y} + \alpha\mathbf{1}, w) = B(\mathbf{y}, w) + \alpha$, $\alpha \in \Re$ (the translation property);

c) $B(\mathbf{y}, w) \geq 0 \Leftrightarrow W(\mathbf{y}) \geq w$, and $B(w, \mathbf{y}) = 0 \Leftrightarrow W(\mathbf{y}) = w$; and

d) $B(\mathbf{y}, w)$ is jointly continuous in $\mathbf{y}$ and w in the interior of the region $Y^S \times \Re$ where $B(\mathbf{y}, w)$ is finite.

Lemma 2: $D(\mathbf{y}, w)$ satisfies:

a) $D(\mathbf{y}, w)$ is nonincreasing in w and nondecreasing in $\mathbf{y}$;

[1] Results generalize in an obvious manner to the more general case where $\mathbf{g} \neq \mathbf{1}$.

b) $D(\mu\mathbf{y}, w) = \mu D(\mathbf{y}, w)$, $\mu > 0$;

c) $D(\mathbf{y}, w) \geq 1 \Leftrightarrow W(\mathbf{y}) \geq w$, and $D(\mathbf{y}, w) = 1 \Leftrightarrow W(\mathbf{y}) = w$;

d) $D(\mathbf{y}, w)$ is continuous in w and $\mathbf{y}$ jointly in the interior of the region $Y^S \times \Re$ where $D(\mathbf{y}, w)$ is finite.

These properties are self-explanatory, but two prove particularly important in what follows: The first is the fact that, by 1.c and 2.c, both the benefit function and the distance function are alternative representations of the preference structure. The second is the translation property of the benefit function (1.b) and the corresponding homogeneity property (2.b) of the distance function. These properties are key to establishing the dual relationship between the local utility functions and the local probability transformations.

2 Local utility functions

The local utility function in a neighborhood of $\mathbf{y}$ is determined (up to an affine transformation) in a neighborhood of the support of $\mathbf{y}$ by the condition $\pi_i u'(y_i)/\pi_j u'(y_j) = W_i/W_j$. Since u is determined only up to an affine transformation, we can select for any W, a local utility function representation such that

$$u'(y_i) = W_i/\pi_i$$

and

$$\sum \pi_i u(y_i) = W(\mathbf{y})$$

with the latter equality holding approximately in a neighborhood of $\mathbf{y}$.

A particularly convenient, canonical representation of the local utility function can be derived from the translation function. Observe that $W(\mathbf{y}') \geq W(\mathbf{y})$ if and only if

$$
\begin{aligned}
0 &\leq B\left(\mathbf{y}', W\left(\mathbf{y}\right); \mathbf{1}\right) \\
&= B\left(\mathbf{y}', W\left(\mathbf{y}\right); \mathbf{1}\right) - B\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right) \\
&= \sum B_i\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right)\left(y_i' - y_i\right) + o||\mathbf{y}' - \mathbf{y}|| \\
&= \sum \pi_i \frac{B_i\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right)}{\pi_i}\left(y_i' - y_i\right) + o||\mathbf{y}' - \mathbf{y}||
\end{aligned}
$$

Hence the local utility function may be represented by the derivative condition

$$u'(y_i; \mathbf{y}) = B_i\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right)/\pi_i.$$

By Lemma 1.b:

$$\sum_s B_s\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right) = 1 = \sum_i \pi_s u'(y_s; \mathbf{y}).$$

The local utility function is defined only up to an affine transformation, as is the global utility function if preferences are consistent with expected utility. If preferences are consistent with expected utility maximization for some given u,

$$B_i\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right) = \frac{\pi_i u'(y_i)}{\sum \pi_j u'(y_j)}$$

for all $\mathbf{y}$.

By the properties of the distance function $W(\mathbf{y}') \geq W\left(\mathbf{y}\right)$ if and only if

$$
\begin{aligned}
0 &\leq \ln\left(D\left(\mathbf{y}', W\left(\mathbf{y}\right)\right)\right) - \ln D\left(\mathbf{y}, W\left(\mathbf{y}\right)\right) \\
&= \sum_s D_s\left(\mathbf{y}, W\left(\mathbf{y}\right)\right)\left(y_s' - y_s\right) + o||\mathbf{y}' - \mathbf{y}|| \\
&= \sum_s \pi_s \frac{D_s\left(\mathbf{y}, W\left(\mathbf{y}\right)\right)}{\pi_s}\left(y_s' - y_s\right) + o||\mathbf{y}' - \mathbf{y}||,
\end{aligned}
$$

where the first equality follows because

$$\frac{\partial \ln D\left(\mathbf{y}, w\right)}{\partial y_s} = \frac{D_s\left(\mathbf{y}, w\right)}{D\left(\mathbf{y}, w\right)},$$

and by Lemma 2.c. Hence, the local utility function may be represented by the derivative condition

$$u'(y_s; \mathbf{y}) = \frac{D_s\left(\mathbf{y}, W\left(\mathbf{y}\right)\right)}{\pi_s}.$$

By Lemma 2 (b and c), this local utility representation satisfies

$$\sum_s D_s\left(\mathbf{y}, W\left(\mathbf{y}\right)\right) y_s = 1 = \sum_i \pi_s u'(y_s; \mathbf{y}) y_s.$$

If preferences are consistent with expected utility then

$$D_s\left(\mathbf{y}, W\left(\mathbf{y}\right)\right) = \frac{\pi_i u'(y_i)}{\sum \pi_j u'(y_j) y_j}$$

3 The Local Probability Transformation

Now consider rank-dependent expected-utility preferences with linear utility; that is, the dual model examined by Yaari (1987). For such preferences, and $\mathbf{y}$, $\mathbf{y}'$ such that $y_1 \leq y_2 \leq ... \leq y_S$, and $y_1' \leq y_2' \leq ... \leq y_S'$, $W(\mathbf{y}^{'}) \geq W(\mathbf{y})$ if and only if/

$$\sum h_i y_i' \geq \sum h_i y_i$$

where the h_i are probability weights derived from a probability transformation function $q : [0,1] \to [0,1]$ with $q(0) = 0$ and $q(1) = 1$,

$$h_{[s]}(\boldsymbol{\pi}) = q\left(\sum_{t=1}^{s} \pi_{[t]}\right) - q\left(\sum_{t=1}^{s-1} \pi_{[t]}\right)$$

where the subscript [] denotes the ordering of the states derived from the increasing re-arrangement of $\mathbf{y}$.

Noting the requirement that $\sum h_i = 1$, the analysis above shows that any generalized Schur-concave preference structure may be locally approximated in a neighborhood of $\mathbf{y}$ by linear utility rank-dependent preferences $q(\bullet; \mathbf{y})$ such that

$$h_s = B_s\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right),$$

because generalized Schur concavity of the preference structure implies (Marshall and Olkin, 1979)

$$\left[\frac{B_s\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right)}{\pi_s} - \frac{B_k\left(\mathbf{y}, W\left(\mathbf{y}\right); \mathbf{1}\right)}{\pi_k}\right] (y_s - y_k) \leq 0.$$

In this setting, the h_i are equivalent to the 'shadow probabilities' or risk-neutral probabilities (Nau, 2001; Chambers and Quiggin, 2001).

Having established the duality between the local utility function and the local probability transformation, we turn attention to the extent to which global preferences inherit the properties of the local probability transformation. Machina's central insight was that when all the local utility functions shared certain properties, such as preservation of first-order or second-order stochastic dominance,these properties were inherited by the global preference functional W. The equivalent result on first-order stochastic dominance for fixed probability transformations are proved by Quiggin (1982) and Chew, Karni and Safra (1987). It is easy to see that the local utility function is monotonic if and only if the local probability transformation function $q(;\mathbf{y})$ is motononic. Machina's analysis of second-order stochastic dominance also may be extended to take account of the dual representation in terms of local probability transformations derived from the distance function.

Proposition 1 *For given* $\mathbf{y}$ *and probabilities* π *the following are equivalent:*

(i) the local utility function $U(;\mathbf{y})$ is concave

(ii) the local probability transformation function $q(;\mathbf{y})$ is concave

(iii) preferences are Generalized-Schur concave in a neighborhood of $\mathbf{y}$

Proof: The equivalence of (i) and (iii) is proved by Machina (1982). For the equivalence of (i) and (ii), observe that

$$\begin{aligned} U(;\mathbf{y}) \text{ is concave } &\Leftrightarrow (y_i - y_j)(B_i/\pi_i - B_j/\pi_j) \leq 0 \quad \forall i,j \\ &\Leftrightarrow (y_i - y_j)(h_i/\pi_i - h_j/\pi_j) \leq 0 \quad \forall i,j \\ &\Leftrightarrow q(;\mathbf{y}) \text{ is concave} \end{aligned}$$

4 Concluding comments

The central object of this paper has been the exploration of the links between local characterizations of preferences based on concepts familiar from simple parametric representations such as the expected-utility and Yaari dual models and those derived from translation and distance functions. Because the translation function is a special case of the more general benefit function, our results can immediately be generalized by considering directions in Y^S other than $\mathbf{1}$. This observations implies the existence of a wide range of local representations of risk attitudes consistent, for example, with state-dependent preferences or uninsurable background risk.

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