KAON-NUCLEON FORWARD DISPERSION RELATIONS

by

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CONTENTS

Abstract

Forward dispersion relations, indexed by continuous parameters β and ω_0, are written for the class of functions

$$B_\pm(\omega) = F_\pm(\omega) \Big/ \left[(\omega \pm \omega_1)^\beta (\omega \pm \omega_0)^{1-\beta}\right] \quad ,$$
$$0 \leqslant \beta \leqslant 1 \quad , \quad 0 \leqslant \omega_0 < \omega_1 \quad ,$$

defined in terms of the $K^\pm N$ forward scattering amplitude $F_\pm = A_\pm + i\, D_\pm$. The magnitude of the dispersion integral involving F_- over the 'unphysical' region of energies below the K^-N threshold is sensitively controlled by β and ω_0. Thus, this class of dispersion relations is well suited to test the extrapolation of a model amplitude into the unphysical region. It is found that none of the models tested - a constant scattering length analysis (6 parameters), a zero-range K-matrix analysis (9 parameters), and an effective-range K-matrix analysis (44 parameters) - adequately represents F_- when extrapolated to unphysical energies, but that the first two of these models are superior to the last. The data currently available on D_- is seriously in doubt; with substantially improved D_- data, the methods described in this work would almost certainly provide a valuable test of the extrapolations to unphysical energies of the models for F_-.

STATEMENT

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and that, to the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text.

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1.1 Introduction

Recent years have seen a steady improvement in the experimental data relating to kaon-nucleon scattering. This has prompted the application of kaon-nucleon forward dispersion relations to the determination of the kaon-nucleon-hyperon coupling constants, $g^2_{\bar{K}N\Lambda}$ and $g^2_{\bar{K}N\Sigma}$, in a manner analogous to that in which $g^2_{\pi NN}$ was found from pion-nucleon dispersion relations. However, the KN dispersion relations do not enjoy the simplicity of those for πN scattering. There is a region below the elastic scattering threshold in which the imaginary part of the $\bar{K}N$ scattering amplitude is different from zero. As this region is inaccessible to experiment, models must be used to predict the amplitude here, and the uncertainties inherent in the models lead to large uncertainties in the values computed from the dispersion relations for the coupling constants $g^2_{\bar{K}N\Lambda}$ and $g^2_{\bar{K}N\Sigma}$.

The reason why the imaginary part of the $\bar{K}N$ amplitude fails to vanish in this 'unphysical region' is readily seen from the general structure of dispersion relations and the unitarity relation when applied to $\bar{K}N$ scattering. Dispersion relations relate the real part of the scattering amplitude to integrals involving the imaginary part over all energies. The imaginary part of the amplitude is in turn related by the unitarity equations to the amplitudes for transitions to the initial and the final states from all those states coupled to both. In the particular case of $\bar{K}N$ scattering, the imaginary part of the amplitude for

$$\bar{K}N \longrightarrow \bar{K}N$$

is related by unitarity to the amplitudes for

$$\pi\Lambda \longrightarrow \bar{K}N$$

and

$$\pi\Sigma \longrightarrow \bar{K}N$$

,

since the internal quantum numbers of the $\bar{K}N$, $\pi\Lambda$ and $\pi\Sigma$ states coincide. As the thresholds of the $\pi\Lambda$ and $\pi\Sigma$ states, 1250 MeV/c and 1330 MeV/c respectively, lie well below that of the $\bar{K}N$ state, 1432 MeV/c, the imaginary part of the $\bar{K}N$ forward scattering amplitude is required at 'unphysical energies' in the kaon-nucleon forward dispersion relations.

In addition to the above contribution to the $\bar{K}N$ forward scattering amplitude below threshold, there is another associated with the exchange of Λ and Σ particles between the initial and final $\bar{K}N$ pairs. Simple poles occur in the amplitude at positions related to the masses of the Λ and Σ particles, and the residues at these poles are directly related to the $\bar{K}N\Lambda$ and $\bar{K}N\Sigma$ coupling constants. Thus, if all the other terms appearing in the KN dispersion relations can be calculated, the values of $g^2_{KN\Lambda}$ and $g^2_{KN\Sigma}$, or at least a linear combination of these values, can be estimated.

Now the imaginary part, $A_{\pm}(\omega)$, of the $K^{\pm}N$ forward scattering amplitude in the laboratory frame is related at energies above the elastic threshold to the total cross section $\sigma_{\pm}(\omega)$ by the optical theorem,

$$q(\omega)\,\sigma_{\pm}(\omega)/4\pi = A_{\pm}(\omega) ,$$

where

$$q(\omega) = (\omega^2 - m_K^2)^{1/2} ,$$

and ω is the kaon total energy, again in the laboratory frame. Thus, dispersion integrals involving $A_{\pm}(\omega)$ over the range of 'physical energies' can be replaced by integrals involving the cross sections $\sigma_{\pm}(\omega)$. Reliable data on $\sigma_{\pm}$ are available in the range from roughly $q = 400$ MeV/c to $q = 20$ GeV/c ; q is the momentum of the kaon beam in the laboratory frame. At higher momenta the cross sections can be estimated from Regge-pole models, and at lower momenta from theoretically based parametrisations of the rather poor data in this region. It is one of these low energy parametrisations which must be continued to obtain the $\bar{K}N$ scattering amplitude in the unphysical region. Beside this data on the imaginary part of the forward scattering amplitude, the value of the real part at one or more energies is required. It can be obtained in magnitude from total cross section and angular distribution data and in sign from coulomb interference, or, alternatively, at low energies, from the existing partial wave analyses of the scattering data.

The material presented herein is organised into three chapters. The first reviews the more recent partial wave analyses of $\bar{K}N$ and KN scattering data, presents the derivation of a forward dispersion relation of which many appearing in the literature are but special cases, and finally, summarises the efforts made to determine the kaon-nucleon-hyperon coupling constants from KN dispersion relations. Chapter two expands on the work completed for this thesis, giving details of the dispersion relations evaluated and the data employed in them. The final chapter contains a discussion of the results of the calculations.

1.2 Partial Wave Analyses of the Low Energy $\bar{K}N$ Data

The analysis of the low energy $\bar{K}N$ scattering data is impeded because the $\bar{K}N$ channel, even at its threshold, is coupled to several others. In the state of zero isospin, the channels open are the $\bar{K}N$ and $\pi\Sigma$ channels. A further channel, $\pi\Lambda$, is open for the I = 1 amplitude. The only three particle channel coupled to the $\bar{K}N$ system at low energies is that for $\Lambda\pi\pi$ production, but, since the phase space factor associated with this channel is small, $\Lambda\pi\pi$ production may be ignored.

The theoretically most acceptable parametrisation of the $\bar{K}N$ scattering data at low energies is in terms of the K-matrix, the formalism for which has been developed by Ross and Shaw (1) for the general situation with many coupled two-body channels. For a single partial wave, labelled by the eigenvalues of the constants of motion, the K and T-matrices are only distinguished by the boundary conditions which characterise the particular scattering states used to define the elements of the K and T-matrices. When the 'outgoing wave' boundary condition is imposed in channel j, with the incident beam in channel i, we have the scattering state

$$\psi_i^{(+)}(r) \xrightarrow[r\to\infty]{} (p_j/p_i)^{1/2}\left[(\delta_{ij} + i\,T_{ij})\,j_{\ell_j}(k_j r) + T_{ij}\,n_{\ell_j}(k_j r)\right]\mathcal{M}_j ,$$

whereas, for the 'principal value' boundary condition, we have

$$\psi_i^{(P)}(r) \xrightarrow[r\to\infty]{} (p_j/p_i)^{1/2}\left[\delta_{ij}\,j_{\ell_i}(k_i r) + K_{ij}\,n_{\ell_j}(k_j r)\right]\mathcal{M}_j \quad ;$$

here, ρ_i is the product of the momentum k_i and the reduced energy E_i in the centre of mass system for channel i, and η_i are normalised simultaneous eigenfunctions of the constants of motion. The relation between the K and T-matrices, therefore, does not depend on any knowledge of the dynamics, and is

$$T = K(1 - iK)^{-1} \quad .$$

Although explicit reference need not be made to the interaction, the normalisation of the K and T-matrices is conveniently stated in terms of the interaction Hamiltonian H_I:

$$T_{ij} = -2(\rho_i \rho_j)^{1/2} \langle \varphi_i , H_I \psi_j^{(+)}(r) \rangle$$

$$K_{ij} = -2(\rho_i \rho_j)^{1/2} \langle \varphi_i , H_I \psi_j^{(P)}(r) \rangle \quad ,$$

where φ_i is the appropriate partial wave projection of a plane wave incident in channel i. The partial cross section from channel i to channel j is related to T_{ij} by

$$\sigma_{ij} = \frac{4\pi}{k_i^2} \left(\frac{2J+1}{2S_i+1} \right) |T_{ij}|^2 \quad ,$$

where S_i is the spin of the initial (unpolarised) channel. At energies for which all the coupled channels are open, the unitarity and the invariance of the S-matrix under time reversal respectively secure the hermiticity and symmetry of the K-matrix. Thus, K is both real and symmetric, properties to which it owes its importance.

The kinematical branch points occurring at the $\bar{K}N$, $\pi\Sigma$ and $\pi\Lambda$ thresholds are eliminated in the quantity

$$M(W) = Q^{\ell+1/2} K^{-1} Q^{\ell+1/2}$$
,

where

$$(Q^{\ell+1/2})_{ij} = \delta_{ij} k_i^{\ell_i+1/2}$$
,

and W is the total energy in the centre of mass frame. M(W) is an analytic function of the energy, and, as explained by Ross and Shaw, can be expanded in a power series about the $\bar{K}N$ threshold at $W_o = m_{\bar{K}} + m_N$:

$$M(W) = M(W_o) + \tfrac{1}{2}\left[Q^2(W) - Q^2(W_o)\right]^{1/2} R(W_o) \left[Q^2(W) - Q^2(W_o)\right]^{1/2}$$
$$+ \cdots\cdots$$

In this expansion, the square root of an entry in a diagonal matrix is to be interpreted as $+i\kappa$ $(\kappa > 0)$ if the entry is negative and as the positive square root otherwise. M(Wo) and R(Wo) are energy independent, real, symmetric matrices in channel space, whose elements can be taken as free parameters in fitting the $\bar{K}N$ scattering data. Retention of only M(Wo) is the zero-range approximation; the effective-range approximation also included R(Wo).

(a) Constant Scattering Length Analyses

At momenta blow 300 MeV/c , K^-p elastic scattering angular distributions are found to be isotropic, so that an s-wave analysis of the data is sufficient. The simplest s-wave K-matrix fit is the zero-range fit and involves three parameters for the I=0 channel and six for I=1. However, until recently, the data on K^-p and K_2^o p processes did not suffice to determine these nine parameters uniquely. Instead a modified parametrisation, the constant scattering length fit, in terms of just six parameters, had to be used.

The constant scattering length fit (CSL) derives from the conventional effective-range expansion of the phase shift δ_ℓ:

$$k_K^{2\ell+1} \cot \delta_{\ell I} = 1/A_I + 1/2\, R_I k_K^2 ,$$

where k_K is the centre of mass momentum in the $\bar{K}N$ channel and I labels the isospin. When the scattering lengths, A_I, and the effective-ranges, R_I , are allowed to assume complex values, this expansion gives a simple phenomenological description of $\bar{K}N$ scattering. The CSL fit sets the effective-ranges to zero and estimates the two complex scattering lengths, A_0 and A_1 , the ratio of Λ production to total hyperon production in the I = 1 state at the K^-p threshold,

$$\varepsilon = \frac{\sigma(\pi\Lambda)}{\left[\sigma(\pi\Sigma) + \sigma(\pi\Lambda)\right]_{I=1}} ,$$

and, ϕ , the phase angle between the I = 0 and I= 1 amplitudes for $\pi\Sigma$ production at the $\bar{K}^0 n$ threshold. The penalty for estimating just six parameters instead of the minimum of nine required by a K-matrix model is that the correct analytic behaviour at the $\pi\Sigma$ and $\pi\Lambda$ thresholds is lost in the CSL fit.

The CSL parameters predict an I=0, s-wave resonance just below the $\bar{K}N$ threshold, with characteristics (position and width) which match well with that of the Yo*(1405) known to occur in the $\pi\Sigma \rightarrow \pi\Sigma$ channel. The Yo*(1405) is believed to dominate the K^-p amplitude below threshold, so the CSL fit at least reproduces the main structure of the amplitude.

Recently it has been found that the differential cross section for $K_2^0 p \rightarrow \pi^+ \Lambda$ scattering is not isotropic and that proper analysis of the data requires the inclusion of a p-wave. Since the $\bar{K}N$ interaction in $K_2^0 p$ collisions occurs solely in the I=1 state, the p-wave term is readily explained as the high energy tail of the $Y_1^*(1385)$ resonance, which is coupled to the $\bar{K}N$ channel and occurs in the $p_{3/2}$ state. In the calculations with forward dispersion relations, the effect of the Y_1*(1385) can be estimated by treating it in the zero-width approximation (see section 3.3).

(b) Zero-Range K-Matrix Analyses

Using recent data on the reactions

$$K^- p \longrightarrow K^- p\,,\ \bar{K}^0 n$$

$$\longrightarrow \Sigma^+ \pi^-\,,\ \Sigma^0 \pi^0\,,\ \Sigma^- \pi^+$$

$$\longrightarrow \Lambda \pi^0$$

and

$$K_2^0 p \longrightarrow K_1^0 p\,,\ \Sigma^0 \pi^+\,,\ \Lambda \pi^+ \quad ,$$

A.D. Martin and G.G. Ross (2) have estimated the nine s-wave, zero-range K-matrix parameters. They were able to find a good fit to the data; χ^2 was 168 for 185 degrees of freedom. Three of their parameters, however, were only poorly determined, since large, correlated variations of these parameters neither significantly changed the χ^2 value from its minimum nor appreciably altered the values of the remaining parameters. Furthermore, the errors on the K-matrix parameters were so correlated that the six CSL parameters were well determined.

As a test of the quality of their K-matrix fit, Martin and Ross first ensured that an equally good fit could not be obtained if finite effective-ranges were permitted in the CSL fit. Secondly, they tested the K-matrix fit against an increase in the number of parameters by including multi-channel effective-range terms. They failed to find any solution which improved the fit, and, in those fits they did find, the effective-ranges were always so small that the character of the solution was unchanged. Thus, they concluded, the zero-range K-matrix parametrisation adequately described the data on $\bar{K}N$ processes below 280 MeV/c.

B.R. Martin and M. Sakitt (3) have also estimated the nine s-wave, zero-range K-matrix parameters. However, the data on $K_2^0 p$ reactions were not used. A unique solution was found, which exhibited the same correlations noticed by A.D. Martin and G.G. Ross.

(c) Effective-Range K-Matrix Analyses

Kim (4) attempted a far more ambitious K-matrix analysis of all the data available at the time (1967) on $K^- p$ and $K_2^0 p$ induced reactions up to a kaon momentum of 550 MeV/c. Since an important $d_{3/2}$ resonance, Yo*(1520), falls within this range, s, p and d-waves had to be incorporated in the analysis. Kim represented these amplitudes in the following way.

(1) The s-waves and resonant I=1, $p_{3/2}$ - wave he fitted using multi-channel effective-range theory, in which the effective-range matrices were constrained to be diagonal.

(2) The remaining p-waves, $p_{1/2}$ and $p_{3/2}$ for I=0 and $p_{1/2}$ for I=1, were included in the zero-range approximation.

(3) The $d_{3/2}$ -wave in the I=1 state was represented by a CSL approximation, while the Yo*(1520) in the I=0 $d_{3/2}$ -wave was parametrised as a resonance of Breit-Wigner form with energy dependent width. Other d-waves were omitted.

This fit, which involves 44 parameters, predicts that the Yo*(1405) is considerably broader than to be expected from CSL and zero-range K-matrix fits to the data. Also it suggests that the couplings of the Y_1*(1385) in the $p_{3/2}$-wave to the $\pi\Lambda$, $\pi\Sigma$ and $\bar{K}N$ channels are respectively strong, weak and negligible.

Kim's s-wave parameters are uniquely determined, but not so those pertaining to higher partial waves. This alone suggests that perhaps the data has been over-parametrised or that the choice of parameters has not been the best. Other doubts can be cast on Kim's work.

(1) The width predicted for the Yo*(1405) resonance is known (5) to be very sensitive to variations in the effective-range terms, so that the omission of all off-diagonal elements from the effective-range matrices is probably quite serious. Furthermore, Kim included d-waves in his analysis, but not the third term of the s-wave effective-range expansion, even though this term has the same momentum dependence. How this term would have affected the s-wave effective ranges is difficult to tell; it is certainly not clear that the effect would have been negligible.

(2) Kim's fits to the total cross sections are not good at higher energies; his predictions for $\sigma(K^-p)$ are systematically low by a few mb , while those for $\sigma(K^-n)$ are systematically high. In fact, when $\sigma(K^-n)$ is extrapolated 40 MeV/c past the limit of validity quoted for the parametrisation by Kim to 590 MeV/c, the lowest momentum point measured by Bugg et al (6), Kim's prediction is found to be about 12 mb too large. The K^-n cross section is thought to be slowly varying in this region, so this discrepancy is difficult to explain. Furthermore, the simple nine parameter fit obtained by Martin and Ross predicts cross sections for $\pi\Lambda$ production that are in better agreement with experiment than those from Kim's effective-range K-matrix fit.

(3) Neglect of the $\Lambda\pi\pi$ channel for momenta above 300 MeV/c is dubious, since the observed cross section for $\Lambda\pi\pi$ production beyond this energy is comparable with the other production cross sections. Thus, to analyse the data up to 550 MeV/c as Kim has attempted, the $\Lambda\pi\pi$ channel should either be explicitly included or certain of the K-matrix elements should be allowed complex values to represent the leakage into the $\Lambda\pi\pi$ channel.

1.3 Partial Wave Analyses of the Low Energy KN Data

Analysis of the data on low energy KN scattering is relatively straight-forward because there is but one open channel and the angular distributions show only slight departures from isotropy.

Goldhaber et al (7) found that their data over the momentum range 140 to 642 MeV/c on K^+p scattering, that is, KN scattering in the I=1 state, could be adequately described by an s-wave scattering length and an s-wave effective-range. This simple parametrisation fails at momenta above about 800 MeV/c , where the angular distributions are anisotropic, but phase shift analyses for the higher partial waves present at these energies have not yet yielded unique solutions.

Data on K^+n processes are required if the KN interaction in the I=0 state is to be isolated. Such data are obtained from K^+d scattering data by Glauber's technique, but, because uncertainties are introduced in so doing, the I=0 component of the KN interaction is known less reliably than the I=1 component. Stenger et al (8) analysed K^+d data up to a momentum of 812 MeV/c , and concluded that s, p and d-waves were necessary to described the KN interaction in the I=0 state.

These few remarks conclude this sketch of the more recent partial wave analyses of data on $\overline{K}N$ and KN processes. We now turn to the dispersion relations in which these analyses will later be employed.

1.4 Kaon-Nucleon Forward Dispersion Relations

(a) Forward Dispersion Relations for $K^{\pm}p$ Scattering

Let $$F_{\pm}(\omega) = D_{\pm}(\omega) + iA_{\pm}(\omega)$$

denote the $K^{\pm}p$ forward scattering amplitude in the laboratory system; the variable ω is the total energy of the kaon, also in the laboratory frame.

From the optical theorem,

$$q(\omega)\,\sigma_{\pm}(\omega)/4\pi = A_{\pm}(\omega)$$,

where $$q(\omega) = (\omega^2 - m_K^2)^{1/2}$$,

we see that integrals of the Cauchy type containing $A_{\pm}(\omega)$,

$$\int^{\infty} \frac{A_{\pm}(\omega')\,d\omega'}{(\omega' - \omega)}$$,

cannot converge, if, as we believe, $\sigma_{\pm}(\omega)$ approach non-zero constants as $\omega \to \infty$. Convergence is assured if the dispersion relation is written for a function which behaves asymptotically as $F_{\pm}(\omega)/\omega$. Even this is not obvious; the convergence of the dispersion relation we shall ultimately write down depends crucially on crossing symmetry and the constant limits of $\sigma_{\pm}(\omega)$ as $\omega \to \infty$.

Consider the function

$$B_{-}(\zeta) = F_{-}(\zeta)/[(\zeta-\omega_1)^{\beta}(\zeta-\omega_0)^{1-\beta}]$$,

where

$$0 < \beta < 1$$,

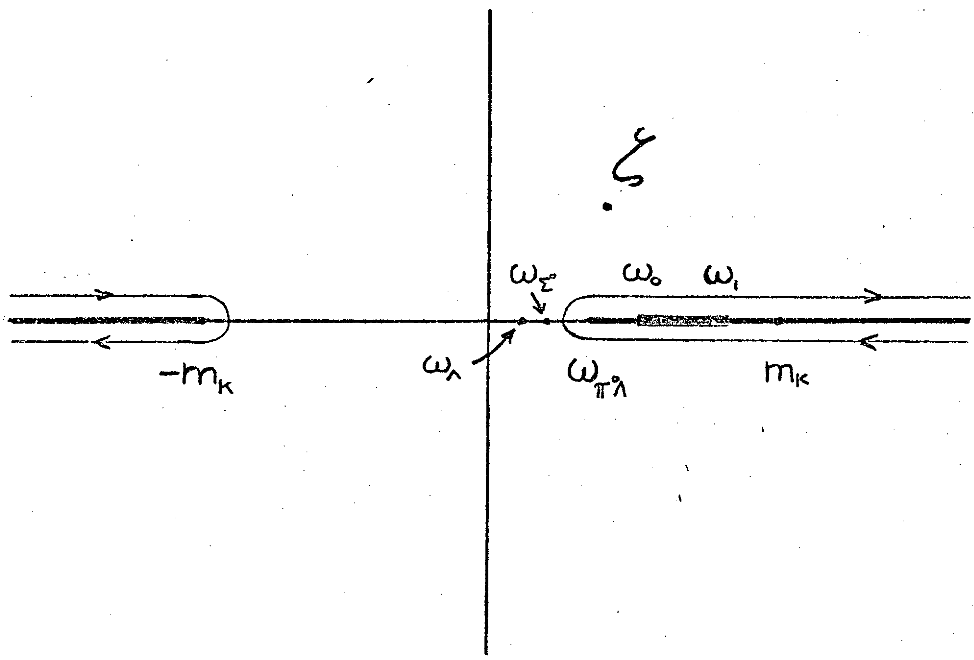
$$0 \leqslant \omega_0 < \omega_1$$,

and the variable ζ is the continuation of the kaon lab energy into the complex plane. Field theory 'proofs' show that $F_{-}(\omega)$, for real $\omega \geqslant m_K$, is the boundary value (as ζ approaches the real axis from above) of an analytic function $F_-(\zeta)$, whose singularity structure will now be given. $F_-(\zeta)$ has poles arising from Λ and Σ° exchange plus a series, beginning at the threshold of the $\pi^\circ\Lambda$ channel, of branch points along the positive real axis at the thresholds of all channels to which the K^-p system is coupled. Crossing symmetry,

$$F_-^*(\omega_+) = F_+(-\omega_+)$$

,

requires that there should be a similar series of threshold branch points along the negative real axis at the thresholds of channels coupled to the K^+p system. These begin with the K^+p threshold itself. Therefore, $F_-(\zeta)$ is single-valued in the ζ plane cut from $-\infty$ to $-m_K$ and from $\omega_{\pi^\circ\Lambda}$ to $+\infty$. The denominator $(\zeta-\omega_1)^\beta(\zeta-\omega_0)^{1-\beta}$ has branch points at ω_0 and ω_1 , but an additional cut joining these points will secure its single-valuedness.



To integrate $B_-(\zeta)$ around the contour shown above, we need the discontinuity of $B_-(\zeta)$ across the cuts in the ζ plane.

$$\mathrm{Disc}\,[B_-(\omega)] = B_-(\omega_+) - B_-(\omega_-)$$

$$= \frac{F_-(\omega_+)}{(\omega_+-\omega_1)^{\beta}(\omega_+-\omega_0)^{1-\beta}} - \frac{F_-(\omega_-)}{(\omega_--\omega_1)^{\beta}(\omega_--\omega_0)^{1-\beta}} ,$$

where ω_+ and ω_- denote limits taken as the point ω on the real axis is approached from above and below respectively. $F_-(\zeta)$ satisfies the Schwarz reflection property,

$$F_-(\zeta^*) = F_-^*(\zeta) ,$$

so

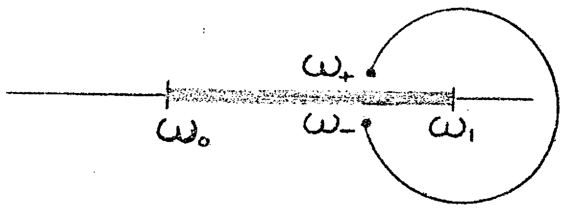
$$\mathrm{Disc}\,[B_-(\omega)] = \frac{F_-(\omega_+)}{(\omega_+-\omega_1)^{\beta}(\omega_+-\omega_0)^{1-\beta}} - \frac{F_-^*(\omega_+)}{(\omega_--\omega_1)^{\beta}(\omega_--\omega_0)^{1-\beta}}$$

(1) Suppose $\omega > \omega_1$. Then

$$\mathrm{Disc}\,[B_-(\omega)] = \frac{F_-(\omega_+) - F_-^*(\omega_+)}{(\omega-\omega_1)^{\beta}(\omega-\omega_0)^{1-\beta}}$$

$$= 2i \frac{A_-(\omega)}{(\omega-\omega_1)^{\beta}(\omega-\omega_0)^{1-\beta}} .$$

(2) Suppose $\omega_0 < \omega < \omega_1$. We have



$$(\omega_- - \omega_1)^{\beta}(\omega_- - \omega_0)^{1-\beta} = (\omega_1 - \omega)^{\beta}(\omega - \omega_0)^{1-\beta}\, e^{-i\pi\beta}$$

$$(\omega_+ - \omega_1)^{\beta}(\omega_+ - \omega_0)^{1-\beta} = (\omega_1 - \omega)^{\beta}(\omega - \omega_0)^{1-\beta}\, e^{+i\pi\beta}$$

so that

$$\mathrm{Disc}\,[B_-(\omega)] = \frac{F_-(\omega_+)\, e^{-i\pi\beta} - F_-^{*}(\omega_+)\, e^{+i\pi\beta}}{(\omega_1 - \omega)^{\beta}\,(\omega - \omega_0)^{1-\beta}}$$

$$= 2i\, \frac{\mathrm{Im}\,[\,F_-(\omega)\, e^{-i\pi\beta}\,]}{(\omega_1 - \omega)^{\beta}(\omega - \omega_0)^{1-\beta}}$$

$$= 2i\, \frac{[\cos(\pi\beta)\, A_-(\omega) - \sin(\pi\beta)\, D_-(\omega)]}{(\omega_1 - \omega)^{\beta}\,(\omega - \omega_0)^{1-\beta}}$$

(3) Suppose $\omega < \omega_0$. Then

$$(\omega_{\pm} - \omega_1)^{\beta}(\omega_{\pm} - \omega_0)^{1-\beta} = -(\omega_1 - \omega)^{\beta}(\omega_0 - \omega)^{1-\beta}$$

and

$$\mathrm{Disc}\,[B_-(\omega)] = -2i\, \frac{A_-(\omega)}{(\omega_1 - \omega)^{\beta}(\omega_0 - \omega)^{1-\beta}} .$$

The function $B_{-}(\zeta)$ satisfies conditions sufficient for the following integral representation to be written (for a detailed justification, see Woolcock (44)):

$$B_{-}(\zeta) = \sum_{Y} \frac{X(Y)}{(\omega_1-\omega_Y)^{\beta}(\omega_0-\omega_Y)^{1-\beta}(\omega_Y-\zeta)}$$

$$+\frac{1}{\pi}\int_{\omega_1}^{\infty} \frac{d\omega' \quad A_{-}(\omega')}{(\omega'-\omega_1)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'-\zeta)}$$

$$+\frac{1}{\pi}\int_{\omega_0}^{\omega_1} \frac{d\omega' \left[\cos(\pi\beta)A_{-}(\omega')-\sin(\pi\beta)D_{-}(\omega')\right]}{(\omega_1-\omega')^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'-\zeta)}$$

$$-\frac{1}{\pi}\left\{\int_{-\infty}^{-m_K}+\int_{\omega_{\pi^0\Lambda}}^{\omega_0}\right\}\frac{d\omega' \quad A_{-}(\omega')}{(\omega_1-\omega')^{\beta}(\omega_0-\omega')^{1-\beta}(\omega'-\zeta)}$$

Here, ω_0 and ω_1 have been chosen to satisfy

$$\omega_{\pi^0\Lambda} \leqslant \omega_0 < \omega_1 ,$$

and $X(\Lambda)$ and $X(\Sigma)$ are the residues of $E(\zeta)$ at the locations, ω_Λ and ω_Σ, of the Λ and Σ poles. The integral of $A_{-}(\omega')$ over negative values of ω' can be converted to an integral of $A_{+}(\omega')$ over positive (physical) energies, since crossing symmetry implies that

$$A_{+}(\omega') = -A_{-}(-\omega') , \quad \omega' \geqslant m_K .$$

Thus,

$$B_-(\zeta) = -\sum_Y \frac{X(Y)}{(\omega_1-\omega_Y)^\beta(\omega_0-\omega_Y)^{1-\beta}(\zeta-\omega_Y)}$$

$$+\frac{1}{\pi}\int_{\omega_1}^{\infty}\frac{d\omega'\ A_-(\omega')}{(\omega'-\omega_1)^\beta(\omega'-\omega_0)^{1-\beta}(\omega'-\zeta)}$$

$$-\frac{1}{\pi}\int_{m_K}^{\infty}\frac{d\omega'\ A_+(\omega')}{(\omega'+\omega_1)^\beta(\omega'+\omega_0)^{1-\beta}(\omega'+\zeta)}$$

$$+\frac{1}{\pi}\int_{\omega_0}^{\omega_1}\frac{d\omega'\left[\cos(\pi\beta)A_-(\omega')-\sin(\pi\beta)D_-(\omega')\right]}{(\omega_1-\omega')^\beta(\omega'-\omega_0)^{1-\beta}(\omega'-\zeta)}$$

$$-\frac{1}{\pi}\int_{\omega_{\pi\Lambda}}^{\omega_0}\frac{d\omega'\ A_-(\omega')}{(\omega_1-\omega')^\beta(\omega_0-\omega')^{1-\beta}(\omega'-\zeta)}\ .$$

Finally, we allow ζ to approach from above a value ω, which is real and at least equal to m_K, and again exploit crossing symmetry to derive the K^+p forward dispersion relation from that for K^-p scattering.

$$\frac{D_\pm(\omega)}{(\omega\pm\omega_1)^\beta(\omega\pm\omega_0)^{1-\beta}} = -\sum_Y \frac{X(Y)}{(\omega_1-\omega_Y)^\beta(\omega_0-\omega_Y)^{1-\beta}(\omega\pm\omega_Y)}$$

$$\mp\frac{1}{\pi}\int_{\omega_1}^{\infty}\frac{d\omega'\ A_-(\omega')}{(\omega'-\omega_1)^\beta(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)}$$

$$\pm\frac{1}{\pi}\int_{m_K}^{\infty}\frac{d\omega'\ A_+(\omega')}{(\omega'+\omega_1)^\beta(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega)}$$

$$\mp \frac{1}{\pi}\int_{\omega_0}^{\omega_1} \frac{d\omega' \left[\cos(\pi\beta) A_-(\omega') - \sin(\pi\beta) D_-(\omega')\right]}{(\omega_1-\omega')^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)}$$

$$\pm \frac{1}{\pi}\int_{\omega_{\pi^0\Lambda}}^{\omega_0} \frac{d\omega' \quad A_-(\omega')}{(\omega_1-\omega')^{\beta}(\omega_0-\omega')^{1-\beta}(\omega'\pm\omega)} \quad . \qquad \mathrm{I}$$

Note that the integrals involving a singularity in the range of integration are to be interpreted as Cauchy principal value integrals.

The relation between the residues $X(Y)$, $Y=\Lambda,\Sigma^0$, and the corresponding $\bar{K}NY$ coupling constants is found to be

$$4\pi X(Y) = g_Y^2 \left[m_K^2 - (m_Y - m_p)^2\right]/4m_p^2 \quad .$$

This is obtained by computing the Born term from the conventional interaction Lagrangian

$$g_\Lambda \bar{\psi}_p \gamma_5 \psi_\Lambda \phi_{K^-} + g_{\Sigma^0} \bar{\psi}_p \gamma_5 \psi_{\Sigma^0} \phi_{K^-} + h.c. \quad ,$$

in which g_Y is the renormalised, rationalised, pseudoscalar K^-pY coupling constant.

A more familiar dispersion relation is obtained as a limiting case of the above relation.

When $\beta \rightarrow 0$,

$$\mp \frac{1}{\pi} \int_{\omega_0}^{\omega_1} \frac{d\omega' \left[\cos(\pi\beta)\, A_-(\omega') - \sin(\pi\beta)\, D_-(\omega') \right]}{(\omega_1 - \omega')^{\beta} (\omega' - \omega_0)^{1-\beta} (\omega' \pm \omega)}$$

$$\pm \frac{1}{\pi} \int_{\omega_{\pi^0\Lambda}}^{\omega_0} \frac{d\omega' \quad A_-(\omega')}{(\omega_1 - \omega')^{\beta} (\omega_0 - \omega')^{1-\beta} (\omega' \pm \omega)} \longrightarrow$$

$$\frac{D_-(\omega_0)}{(\omega \pm \omega_0)} \quad \mp \frac{1}{\pi} \int_{\omega_{\pi^0\Lambda}}^{\omega_1} \frac{d\omega' \, A_-(\omega')}{(\omega' - \omega_0)(\omega' \pm \omega)} ,$$

so that we have

$$\frac{D_\pm(\omega)}{(\omega \pm \omega_0)} = \frac{D_-(\omega_0)}{(\omega \pm \omega_0)} - \sum_{Y} \frac{X(Y)}{(\omega_0 - \omega_Y)(\omega \pm \omega_Y)}$$

$$\mp \frac{1}{\pi} \int_{\omega_1}^{\infty} \frac{d\omega' \, A_-(\omega')}{(\omega' - \omega_0)(\omega' \pm \omega)}$$

$$\pm \frac{1}{\pi} \int_{m_K}^{\infty} \frac{d\omega' \, A_+(\omega')}{(\omega' + \omega_0)(\omega' \mp \omega)}$$

$$\mp \frac{1}{\pi} \int_{\omega_{\pi^0\Lambda}}^{\omega_1} \frac{d\omega' \, A_-(\omega')}{(\omega' - \omega_0)(\omega' \pm \omega)} . \qquad \boxed{\text{II}}$$

This is the usual singly subtracted forward dispersion relation for $K^{\pm}p$ scattering, with the subtraction point at ω_0. Note that when $\omega_0 > \omega_{\pi^0\Lambda}$, the last integral is also principal-valued.

A dispersion relation frequently appearing in the literature (9,12,13,14,16-20,22) follows from the last relation upon setting $\omega_0 = 0$ and computing $D_-(\bar{\omega}) - D_+(\omega)$, where ω and $\bar{\omega}$ are arbitrary physical energies. Explicitly, this dispersion relation for the difference of the amplitudes is

$$D_-(\bar{\omega}) - D_+(\omega) = -\sum_Y \frac{(\omega+\bar{\omega})\, X(Y)}{(\omega_Y-\bar{\omega})(\omega_Y+\omega)}$$

$$+\frac{1}{\pi}\int_{\omega_{\pi^0\Lambda}}^{\infty} \frac{(\omega+\bar{\omega})\, A_-(\omega')\, d\omega'}{(\omega'-\bar{\omega})(\omega'+\omega)}$$

$$-\frac{1}{\pi}\int_{m_K}^{\infty} \frac{(\omega+\bar{\omega})\, A_+(\omega')\, d\omega'}{(\omega'+\bar{\omega})(\omega'-\omega)} . \qquad \text{III}$$

(b) Forward Dispersion Relations for $K^{\pm}n$ Scattering

Dispersion relations for $K^{\pm}n$ scattering can be directly written down, since the only single particle with the same internal quantum numbers as the $K^{-}n$ system is Σ^{-}, and, under the assumption of exact charge independence, $g^2_{K^-n\Sigma^-} = 2\, g^2_{K^-p\Sigma^0}$. Thus, the Λ pole must be omitted from the $K^{-}p$ relation and the Σ^{0} pole doubled in strength. All $K^{\pm}p$ amplitudes and cross sections must be re-interpreted as those for $K^{\pm}n$ scattering.

The Σ and Λ pole terms are isolated if dispersion relations are written for each of the amplitudes of definite isospin. Only $X(\Lambda)$ appears when a dispersion relation is evaluated for the combination $(K^{\pm}p) - 1/2\,(K^{\pm}n)$ of amplitudes; only $X(\Sigma)$ appears in the $K^{\pm}n$ relations.

1.5 Applications of KN and $\bar{K}N$ Forward Dispersion Relations

The principal application of kaon-nucleon forward dispersion relations at the present time is to the determination of the coupling constants g^2_{Λ} and g^2_{Σ} which appear in these relations. Various symmetry groups, such as SU(3) and SU(6), relate these coupling constants to $g^2_{\pi NN}$, so their computation provides a measure of the symmetry breaking. Also, they are of interest in the field of weak interactions.

To obtain the $\bar{K}N$ amplitude in the unphysical region between the $\pi\Lambda$ and $\bar{K}N$ thresholds, one of the parametrisations of the low energy $\bar{K}N$ scattering data must be continued below the $\bar{K}N$ threshold. Consequently, any determination of the coupling constants will be dependent upon the parametrisation used. A second use, related to the first, of the dispersion relations is, therefore, to test the models used for the $\bar{K}N$ amplitude in the unphysical region.

Consistency of the predictions for the high energy scattering amplitude from forward dispersion relations and from Regge models leads to the formulation of "sum rules". These have wide application in the estimation of Regge parameters, but, as this field overlaps only slightly with the work presented here, it will not be considered further.

(a) Determination of g_Λ^2 and g_Σ^2.

If the residues, $X(\Lambda)$ and $X(\Sigma)$, of the $\bar{K}N$ amplitude at the Λ and Σ poles are to be estimated from forward dispersion relations, the following data must be available:

(1) the real part, $D_\pm(\omega)$, of the $K^\pm N$ lab forward scattering amplitude at one or more energies:

(2) the $K^\pm N$ total cross sections for all energies above the respective thresholds;

(3) the imaginary part, and perhaps also the real part, of the $\bar{K}N$ forward scattering amplitude in the unphysical region between the $\pi\Lambda$ and $\bar{K}N$ thresholds.

$D_\pm(\omega)$ is usually computed from low energy partial wave analyses or directly from the experimental data. The method used in the latter case is the following. At those energies where both the total cross section $\sigma_\pm$ and the angular distribution $\frac{d\sigma_\pm}{d\Omega}(\omega, \theta_{CM})$ have been measured, $|D_\pm(\omega)|$ can be found, since the optical theorem,

$$q(\omega)\, \sigma_\pm(\omega)/4\pi = A_\pm(\omega) ,$$

provides the imaginary part, and the differential cross section extrapolated to the forward direction gives the magnitude of the total amplitude,

$$\left(\frac{m_p^2 + m_K^2 + 2m_p\omega}{m_p^2}\right) \frac{d\sigma_\pm}{d\Omega}(\omega, \theta_{cm}=0) = |F_\pm(\omega)|^2 ,$$

and

$$|D_\pm(\omega)|^2 = |F_\pm(\omega)|^2 - |A_\pm(\omega)|^2 .$$

In most instances, the sign of $D_\pm$ may be determined from forward dispersion relations (see the discussion in Perrin and Woolcock (23)). Since $|D_\pm(\omega)|$ is found as the difference of two quantities of comparable magnitude, values computed in this matter are subject to large uncertainties.

The $K^\pm N$ cross section data between momenta 400 Me V/c and 20 Ge V/c are reliable, so that over this range $A_\pm(\omega)$ is well determined by $\sigma_\pm(\omega)$ via the optical theorem. However, at lower momenta, the measurements of $\sigma_\pm$ are few in number and are statistically poor, so $A_\pm$ is best obtained from theoretically based fits to the data. Above machine energies, Regge pole models, whose parameters have been estimated from cross section data at the highest energies accessible, are used to continue the cross sections out to infinity.

Finally, the $\bar{K}N$ amplitude in the unphysical region is obtained as the continuation in energy of some parametrisation of the low energy $\bar{K}N$ scattering data. As the uncertainty introduced by this extrapolation outweighs those arising from poor data, the need for tests which discriminate between the low energy parametrisations is clear.

Dispersion Relations for the Difference of Amplitudes

Many attempts to determine the kaon-nucleon-hyperon coupling constants have employed the dispersion relation obtained as a special case of $\overline{\underline{\text{III}}}$ with $\omega = \bar{\omega}$, namely,

$$D_-(\omega) - D_+(\omega) = -\sum_Y \frac{2\omega X(Y)}{(\omega_Y^2 - \omega^2)} + \frac{2\omega}{\pi}\int_{\omega_{\pi\Lambda}}^{m_K} \frac{A_-(\omega')\,d\omega'}{(\omega'^2 - \omega^2)} + \frac{2\omega}{\pi}\int_{m_K}^{\infty} \frac{(A_-(\omega') - A_+(\omega'))\,d\omega'}{(\omega'^2 - \omega^2)},$$

and with the variable ω set to the K^-p threshold $(\omega = m_K)$. The results of these analyses are summarised in Table 1, and the differences seen there arise from the use of different models for the $\bar{K}N$ amplitude in the unphysical region and from the gradual improvement of the data on $K^{\pm}N$ total cross-sections.

The error quoted on each determination is based only on known errors in the experimental data, and takes no account of the limitations of the unphysical region model. In fact, comparison of the predictions obtained using different models shows that the systematic errors in the models exceed the statistical errors arising from the data. Kim's effective-range K-matrix leads to results that are incompatible with those from other models; this discrepancy emphasises the importance of the contribution from the Yo*(1405) resonance. A common failing of these determinations is that they all require the values of $D_{\pm}(\omega)$ at the $K^{\pm}N$ thresholds. Perrin and Woolcock (23) have stressed that these quantities are only poorly known, and this work reinforces their warning.

Analysis (Year)	$\bar{K}N$ Parametrisation	g_Λ^2	g_Σ^2
Zovko (9) (1966)	Kim CSL (10)	$g_\Lambda^2 + g_\Sigma^2 \approx 121$	
	Sakitt Et Al CSL (11)	$g_\Lambda^2 + g_\Sigma^2 \approx 103$	
Lusignoli Et Al (12) (1966)	Kim CSL (10)	49 ± 10	$\lesssim$ 31
	Sakitt Et Al CSL (11)	53 ± 11	$\lesssim$ 5
Carter (13) (1967)	Kim CSL (10)	62 ± 11	$\lesssim$ 38
Davies Et Al (14) (1967)	Kim CSL (15)	63 ± 21	$\lesssim$ 38
Kim (16) (1967)	Kim CSL (10)	50 ± 11	$\lesssim$ 30
	Kim K-Matrix (4)	170 ± 26	3 ± 5
Martin & Poole (17) (1967)	Kim CSL (15)	$\approx$ 52	-
Rood (18) (1967)	Rood K-Matrix (18)	78 ± 13	-
Carter (19) (1968)	Kim CSL (10)	65 ± 11	$\lesssim$ 38
Queen Et Al (20) (1968)	Kim CSL (15)	49 ± 8	$\lesssim$ 39
	Kittel-Otter Effective Range (21)	65 ± 11	$\lesssim$ 13
	Kim K-Matrix (4)	150 ± 34	5 ± 8
Granovskii & Starikov (22) (1968)	Kim CSL (10)	74 ± 16	$\lesssim$ 16

TABLE 1

Davies et al (14) attempted to reduce the error from this source by evaluating the dispersion relation for several values of ω above threshold. At these energies, $D_{\pm}(\omega)$ was computed from the low energy parametrisation used in the unphysical region. They found that the various constant scattering length analyses (10,15,24) gave comparable results, that the inclusion of an I=0 effective-range term increased the estimate for g_{Λ}^{2}, and that the effective-range K-matrix analysis predicted a value for g_{Λ}^{2} incompatible with the rest. The statistical errors on g_{Λ}^{2} were substantially larger than the variation of g_{Λ}^{2} with energy, for the case of the CSL analyses. This is not true for the Kim effective-range K-matrix analysis (see the section on consistency tests, page 32).

Subtracted Dispersion Relations

The once subtracted dispersion relation with the subtraction point at the $\bar{K}N$ threshold,

$$\frac{D_+(\omega)-D_+(m_K)}{(\omega-m_K)} = \sum_{Y} \frac{X(Y)}{(m_K+\omega_Y)(\omega+\omega_Y)}$$

$$+\frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \left\{ \frac{A_+(\omega')}{(\omega'-m_K)(\omega'-\omega)} - \frac{A_-(\omega')}{(\omega'+m_K)(\omega'+\omega)} \right\}$$

$$-\frac{1}{\pi}\int_{\omega_{\pi\Lambda}}^{m_K} d\omega' \frac{A_-(\omega')}{(\omega'+m_K)(\omega'+\omega)}$$

was used by Davies et al (14) in the hope that for physical energies the denominator $(\omega'+m_K)(\omega'+\omega)$ would suppress the contribution from the unphysical region. Although such is the case, the numerical results are particularly sensitive to the low energy K^+N interaction when $\omega \approx m_K$. Thus, they were able to conclude that a K^+p effective-range term was required to achieve compatibility with the predictions for g_Λ^2 and g_Σ^2 from unsubtracted dispersion relations.

These authors attempted to use the same dispersion relation at higher energies where the cancellations which cause this sensitivity are less severe. Data on $D_+(\omega)$ were drawn from phase-shift and optical-model analyses of K^+p and K^+n scattering data at several energies up to 2 Ge V/c, and were employed in the relations for the K^+p and K^+n amplitudes to find a certain linear combination of g_Λ^2 and g_Σ^2 from one and a bound on g_Σ^2 from the other. Only a bound was computed because the contribution of the $Y_1^*(1385)$ was neglected in the calculations. Although the errors were rather large, the results were in accord with those from unsubtracted dispersion relations.

Perrin and Woolcock (23,25) utilised all the data on $D_\pm(\omega)$ that they would glean from experimental reports in a dispersion relation subtracted at $\omega_0 = 0$:

$$\frac{D_\pm(\omega)}{\omega} = \frac{C}{\omega} \mp \sum_Y \frac{X(Y)}{\omega(\omega \pm \omega_Y)} \mp \frac{1}{\pi}\int_{\omega_{\pi\Lambda}}^{m_K} \frac{d\omega' A_-(\omega')}{\omega'(\omega' \pm \omega)}$$

$$+ \frac{1}{\pi}\int_{m_K}^{\infty} \frac{d\omega'}{\omega'} \left\{ \frac{A_\pm(\omega')}{(\omega' - \omega)} - \frac{A_\mp(\omega')}{(\omega' + \omega)} \right\},$$

where

$$C = D_-(0) + \sum_Y \frac{X(Y)}{\omega_Y} .$$

The poles at ω_Λ and ω_Σ were placed by an effective pole at $\bar{\omega}$ with residue $R = X(\Lambda) + X(\Sigma)$. After rewriting the relation as

$$D_\pm(\omega) - I_\pm(\omega) = C \mp R/(\omega \pm \bar{\omega})$$

,

where $I_\pm(\omega)$ contains the integral terms, they computed the left hand side for each of the 66 energies at which data on $D_\pm(\omega)$ had been found, and performed a least squares fit for the two constants C and R. For the CSL and effective-range K-matrix models, they found estimates of 100±21 and 137±21 respectively for the combination $g_\Lambda^2 + 0{\cdot}847\, g_\Sigma^2$.

Once the best values of C and R had been found, Perrin and Woolcock attempted to check the reliability of the unphysical region models used by inserting these values into the dispersion relation to compute $D_\pm(m_K)$. In their first paper (23), the input value of $D_-(m_K)$ from the unphysical region model was in poor agreement with the computed output value, but, when they repeated their work with substantially better data for $D_\pm(\omega)$ and slightly improved data on $\sigma_\pm(\omega)$, they found that the input-output discrepancy disappeared. This clearly indicates the importance of the data on $D_\pm(\omega)$.

Dispersion Relations Weakly Dependent upon Unphysical Region Model

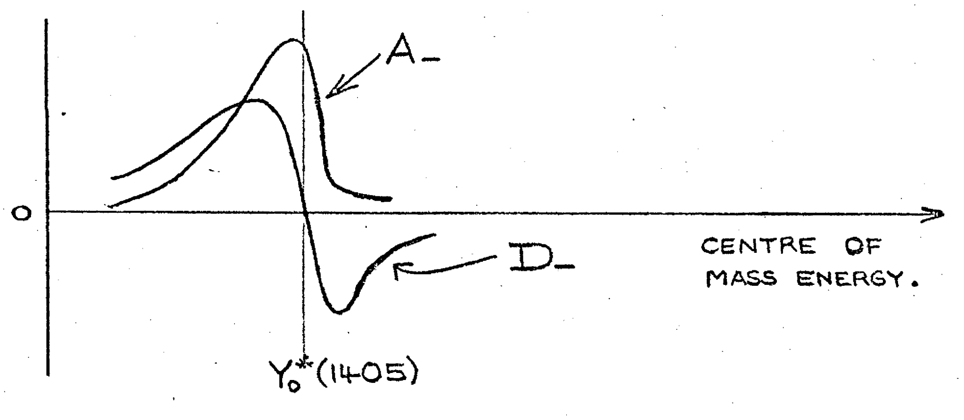
Martin and Poole (17) considered the relation

$$\frac{-D_+(m_K)}{(2m_K)^{1/2}(m_K+\omega_0)^{1/2}} = \sum_Y \frac{X(Y)}{(m_K-\omega_Y)^{1/2}(\omega_0-\omega_Y)^{1/2}(m_K+\omega_Y)}$$
$$+\frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \left\{ \frac{A_-(\omega')}{(\omega'-m_K)^{1/2}(\omega'-\omega_0)^{1/2}(\omega'+m_K)} - \frac{A_+(\omega')}{(\omega'+m_K)^{1/2}(\omega'+\omega_0)^{1/2}(\omega'-m_K)} \right\}$$

$$-\frac{1}{\pi}\int_{\omega_0}^{m_K} d\omega' \frac{D_-(\omega')}{(m_K-\omega')^{1/2}(\omega'-\omega_0)^{1/2}(\omega'+m_K)}$$

$$-\frac{1}{\pi}\int_{\omega_{\pi\Lambda}}^{\omega_0} d\omega' \frac{A_-(\omega')}{(m_K-\omega')^{1/2}(\omega_0-\omega')^{1/2}(\omega'+m_K)}$$

obtained from $\bar{I}$ when $\beta = 1/2$ and $\omega = -m_K$. Both the $K^{\pm}p$ and $K^{\pm}n$ relations were considered, and from the usual combination of these they estimated g_Λ^2. Their method reduces the contribution from the unphysical region integral, which now involves $D_-(\omega')$ whose behaviour near the Yo*(1405) resonance is compared with that of $A_-(\omega')$ in the figure below. It is not clear whether the uncertainty in the unphysical region integral is reduced, as they claim it is.



They found using this method and the CSL parameters (15) a value of 69 ± 19 for g_Λ^2 and noted that this prediction was insensitive to variation of ω_0 within the range corresponding to total centre of mass energies from 1340 MeV to 1395 MeV.

More recently, Restignoli and Violini (26) have written a dispersion relation for the function

$$\tilde{F}_\pm(\omega) = \left(\frac{\omega-\omega_0}{\omega+\bar{\omega}}\right) F_\pm(\omega)$$

,

defined in terms of the I=0 $\bar{K}N$ amplitude,

$$F_-(\omega) = F_{K^-p} - 1/2\, F_{K^-n}$$

,

and the amplitude $F_+(\omega)$ obtained from it by crossing. In detail, their relation is

$$\frac{-2m_K(\omega_o-\omega_\Lambda)}{(\omega_\Lambda^2-m_K^2)(\omega_\Lambda+\bar{\omega})}\, X(\Lambda) = \left(\frac{\omega_o-m_K}{\bar{\omega}+m_K}\right)\mathrm{Re}\, F_-(m_K)$$

$$+\left(\frac{m_K+\omega_o}{m_K-\bar{\omega}}\right)\mathrm{Re}\, F_+(m_K) + \frac{2m_K(\bar{\omega}+\omega_o)}{(\bar{\omega}^2-m_K^2)}\mathrm{Re}\, F_+(\bar{\omega})$$

$$+\frac{2m_K}{\pi}\int_{\omega_{\pi\Sigma}}^{\infty} d\omega'\left(\frac{\omega'-\omega_o}{\omega'+\bar{\omega}}\right)\frac{\mathrm{Im}\, F_-(\omega')}{(\omega'^2-m_K^2)}$$

$$-\frac{2m_K}{\pi}\int_{m_K}^{\infty} d\omega'\left(\frac{\omega'+\omega_o}{\omega'-\bar{\omega}}\right)\frac{\mathrm{Im}\, F_+(\omega')}{(\omega'^2-m_K^2)} \quad .$$

ω_o was chosen to coincide with the location of Yo*(1405) resonance, so that the zero of $\tilde{F}_-(\omega)$ at ω_o would reduce the unphysical region integral. The possible values of $\bar{\omega}$ had to be restricted to the range from 450 MeV to 533 MeV in order that $\mathrm{Re}\, F_\pm(m_K)$ and $\mathrm{Re}\, F_+(\bar{\omega})$

could be obtained with (hopefully) some degree of confidence from parametrisations of the low energy $\bar{K}N$ and KN data. Both the K^-p and the K^-n amplitudes in the unphysical and low energy regions were taken from the same K-matrix model (Martin and Sakitt (3)), and, as only K^-p data is used in estimating the parameters of this model, Restignoli and Violini argued that the K^-n amplitude obtained in this way should be reliable. To determine

$$F_+(\omega) = F_{K^+p}(\omega) - \tfrac{1}{2} F_{K^+n}(\omega) ,$$

they used Martin and Perrin's (27) effective-range analysis of the K^+p data and the K^+n scattering length quoted by Stenger et al (8). Although the latter is of limited reliability, their hope was that the I=0 KN amplitude was small.

Restignoli and Violini obtained $g_\Lambda^2 = 90 \pm 50$, and found that had they used Kim's K-matrix solution this figure would only have been increased to about 100. That dispersion relations which reduce the importance of the unphysical region should lead to estimates for g_Λ^2 differing substantially from those from other relations only when Kim's effective-range K-matrix model is used suggests that this model is inadequate.

This inference has also been drawn when the effective-range K-matrix analysis has been used in sum rules which reduce the unphysical region contribution. Martin and Ross (28), for example, have considered the sum rules obtained by requiring consistency between the real part of the high energy amplitude as predicted by a dispersion relation and by a Regge-pole model. The effective-range K-matrix led to values for the coupling constants which agreed well with those from the other models simply because the sum rules reduced the importance of the Yo*(1405) resonance.

Lest Kim's effective-range K-matrix analysis be discarded on these grounds alone, it is worth noting that Martin and Michael (29) have analysed in the framework of Regge theory the very recent cross section data on $K^- n \to \pi^- \Lambda$ at $3{\cdot}9$ GeV/c, and, quite independently of any unphysical region model, have computed $g_\Lambda^2 = 182 \pm 63$.

(b) <u>Consistency Tests</u>

The process $\bar{K}N \to \bar{K}N$ cannot, of course, be observed experimentally at energies below the $\bar{K}N$ threshold, so the models for the amplitude in this region can only be tested indirectly by comparing with experiment their predictions concerning production processes such as $\pi\Lambda \to \bar{K}N$ and $\pi\Sigma \to \bar{K}N$. But it is from this very data that the parameters of the models are estimated! Dispersion relations do, however, provide a way of testing the models if a class of relations can be found in which the relative importance of the unphysical region integral can be varied by means of a free parameter. The stability of the predictions for the coupling constants under variations of this parameter then indicates the reliability of the model.

The most obvious free parameter is just the energy variable ω. Queen et al (20) computed the gradient of the coupling constants with respect to energy by evaluating the derivative sum rule, obtained by differentiating with respect to ω the relations for the $\bar{K}N$ amplitudes of definite isospin, namely,

$$-X(Y) = (\omega_Y^2 - \omega^2)\left(D_-(\omega) - D_+(\omega)\right)/4\omega$$

$$+ \frac{(\omega_Y^2 - \omega^2)}{2\pi} \left\{ \int_{m_K}^{\infty} \frac{d\omega' A_+(\omega')}{(\omega'^2 - \omega^2)} - \int_{\omega_{\pi\Lambda}}^{\infty} \frac{d\omega' A_-(\omega')}{(\omega'^2 - \omega^2)} \right\} .$$

The subscripts (±) refer to the combination $(K^\pm p) - 1/2\,(K^\pm n)$ of amplitudes for $Y = \Lambda$, and to the $K^\pm n$ amplitudes for $Y = \Sigma^0$. To differentiate the principal valued integrals, they had recourse to the identity

$$g'(\omega) = \frac{h(\omega_0)}{(\omega_0 - \omega)} - \frac{h(\omega_1)}{(\omega_1 - \omega)} + P\int_{\omega_0}^{\omega_1} \frac{h'(\omega')\, d\omega'}{(\omega' - \omega)}$$

where

$$g(\omega) = P\int_{\omega_0}^{\omega_1} \frac{h(\omega')}{(\omega' - \omega)}\, d\omega'$$

and

$$\omega_0 < \omega < \omega_1 .$$

This identity holds if $h(\omega)$ is differentiable on $[\omega_0, \omega_1]$ and its derivative satisfies a Lipschitz condition at each point of (ω_0, ω_1). They cautioned that because there are threshold cusps in the K^-p amplitude, the energy derivatives of terms which depend on $D_-(\omega)$ and $A_-(\omega)$ have singularities at $\omega = m_K$ and at the charge - exchange threshold. Consequently, care had to be exercised in the choice of energy ω at which the derivative sum rule was evaluated, and also in the range $[\omega_0, \omega_1]$ over which the identity given above was applied.

Large cancellations occur among the terms of the derivative sum rules, but yet the statistical errors on the derivatives

$\partial(g_Y^2)/\partial\omega$ they calculated are surprisingly small. This indicates that the errors on determinations of the coupling constants at neighbouring energies are correlated. They found that for the I=0 amplitude, the derivative values computed with all the unphysical region models are incompatible with a value of zero. The effective-range K-matrix model led to the largest slope, which was of opposite sign to that of the other parametrisations. However, only this model gave an I=1 amplitude which satisfied the test.

A.D. Martin, Queen and Violini (5) considered the predictions for g_Λ^2 obtained from the combination $(K^\pm p) - \frac{1}{2}(K^\pm n)$ of dispersion relations for $F_-(\omega) - F_+(\omega_0)$,

$$\frac{-X(\Lambda)}{(\omega_\Lambda-\omega)(\omega_\Lambda+\omega_0)} = \frac{D_-(\omega)-D_+(\omega_0)}{(\omega+\omega_0)} - \frac{1}{\pi}\int_{\omega_{\Sigma\pi}}^{\infty} d\omega' \frac{A_-(\omega')}{(\omega'-\omega)(\omega'+\omega_0)} + \frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \frac{A_+(\omega')}{(\omega'+\omega)(\omega'-\omega_0)}$$

for various values of ω just below the $\bar{K}N$ threshold and for a fixed value of ω_0 . Variation of ω near the Yo*(1405) resonance substantially alters the contribution from this resonance, and so the consistency of the predictions for g_Λ^2 should provide a test of the Yo*(1405) parameters. Once again the effective-range K-matrix model proved inferior to the simpler parametrisations.

Quite the reverse of this result was found by Chan and Meiere (30), who evaluated at $\omega = -m_K$ the dispersion relations obtained from $\bar{I}$ for amplitudes of definite isospin. ω_0 was located at the threshold of the lowest open channel, that is, $\omega_{\pi\Lambda}$ for I=1 and $\omega_{\pi\Sigma}$ for I=0. The dependence of the unphysical region integral,

$$\mp \frac{1}{\pi} \int_{\omega_0}^{m_K} d\omega' \frac{\left[\cos(\pi\beta) A_-(\omega') - \sin(\pi\beta) D_-(\omega') \right]}{(m_K - \omega')^{\beta} (\omega' - \omega_0)^{1-\beta} (\omega' \pm m_K)}$$

upon β is quite strong, since β controls the ratio in which $A_-(\omega)$ and $D_-(\omega)$ are mixed and also the behaviour of the integrand near ω_0 and near m_K . Chan and Meiere found that, to within the statistical errors on any one determination, the effective-range K-matrix led to stable predictions for g_Λ^2 and g_Σ^2 ,whereas the CSL parameters did not.

There is, however, a flaw in their argument. The CSL parametrisation does not have the correct analytic behaviour at the $\pi\Lambda$ and $\pi\Sigma$ thresholds, so the magnification of the contribution from the region just above these thresholds, which occurs for low values of β , should not be expected to lead to stable results.

The dispersion relation $\bar{\text{I}}$, which Chan and Meiere evaluated at $\omega = -m_K$, certainly offers a high degree of flexibility in the unphysical region term. It is a more thorough study of this relation which constitutes the following two chapters of this thesis.

2.1 Dispersion Relations Evaluated

The objective of this work has been the investigation of the merits of the $K^{\pm}p$ dispersion relation

$$
\begin{aligned}
\frac{D_{\pm}(\omega)}{(\omega \pm m_K)^{\beta}(\omega \pm \omega_0)^{1-\beta}} = & - \frac{(m_K^2-(m_\Lambda-m_p)^2)\left(g_\Lambda^2 + G(\beta,\omega_0)\, g_\Sigma^2\right)}{16\pi m_p^2 (\omega \pm \omega_\Lambda)(m_K-\omega_\Lambda)^{\beta}(\omega_0-\omega_\Lambda)^{1-\beta}} \\
& \pm \frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \, \frac{A_+(\omega')}{(\omega'+m_K)^{\beta}(\omega'+\omega_0)^{1-\beta}(\omega' \mp \omega)} \\
& \mp \frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \, \frac{A_-(\omega')}{(\omega'-m_K)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega' \pm \omega)} \\
& \mp \frac{1}{\pi}\int_{\omega_0}^{m_K} d\omega' \, \frac{\left[\cos(\pi\beta)\, A_-(\omega') - \sin(\pi\beta)\, D_-(\omega')\right]}{(m_K-\omega')^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega' \pm \omega)} \\
& \pm \frac{1}{\pi}\int_{\omega_{\pi^0\Lambda}}^{\omega_0} d\omega' \, \frac{A_-(\omega')}{(m_K-\omega')^{\beta}(\omega_0-\omega')^{1-\beta}(\omega' \pm \omega)} ,
\end{aligned}
$$

where

$$
G(\beta,\omega_0) = \left(\frac{m_K^2-(m_{\Sigma^0}-m_p)^2}{m_K^2-(m_\Lambda-m_p)^2}\right)\left(\frac{m_K-\omega_\Lambda}{m_K-\omega_{\Sigma^0}}\right)^{\beta}\left(\frac{\omega_0-\omega_\Lambda}{\omega_0-\omega_{\Sigma^0}}\right)^{1-\beta} ,
$$

derived earlier, when used as a test of the models for the $\bar{K}N$ amplitude between the $\pi\Lambda$ and $\bar{K}N$ thresholds. The contribution to the relation from the unphysical region is exceedingly sensitive to β , since the integrand in this region contains not just $A_-(\omega')$ but $[\cos(\pi\beta)\, A_-(\omega') - \sin(\pi\beta)\, D_-(\omega')]$ and is magnified near ω_0 and m_K when β

is close to 0 and 1 respectively. Thus, it would be expected that the values obtained for $C = g_\Lambda^2 + G(\beta,\omega_0)\, g_\Sigma^2$ from the dispersion relation would vary rapidly with β if the unphysical region model were inadequate. In addition to the above dispersion relation for which $0<\beta<1$, the subtracted relation obtained as the limiting case $\beta \to 0$ was studied. Explicitly this relation is

$$\frac{D_\pm(\omega)}{(\omega\pm\omega_0)} = -\frac{\left(m_K^2-(m_\Lambda-m_p)^2\right)\left(g_\Lambda^2+G(0,\omega_0)g_\Sigma^2\right)}{16\pi\, m_p^2\,(\omega\pm\omega_\Lambda)\,(\omega_0-\omega_\Lambda)}$$

$$+\frac{D_-(\omega_0)}{(\omega\pm\omega_0)} \mp \frac{1}{\pi}\int_{\omega_{\pi^0\Lambda}}^{\infty} d\omega' \frac{A_-(\omega')}{(\omega'-\omega_0)(\omega'\pm\omega)}$$

$$\pm\frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \frac{A_+(\omega')}{(\omega'+\omega_0)(\omega'\mp\omega)},$$

in which

$$G(0,\omega_0) = \left(\frac{m_K^2-(m_{\Sigma^0}-m_p)^2}{m_K^2-(m_\Lambda-m_p)^2}\right)\left(\frac{\omega_0-\omega_\Lambda}{\omega_0-\omega_{\Sigma^0}}\right).$$

Note that a slight approximation has been introduced in the above equations by transferring the location of the Σ^0 pole to that of the Λ pole. Since the energies ω at which the dispersion relations are evaluated lie above threshold, and since g_Σ^2 is almost certainly much smaller than g_Λ^2, the error introduced in this way is probably very slight. At worst, it is still negligible in comparison with the errors arising from poor data and the uncertain unphysical region contribution.

The quantity sought from both the subtracted and unsubtracted dispersion relations is $C = g_\Lambda^2 + G(\beta, \omega_0)\, g_\Sigma^2$. Fortunately, $G(\beta, \omega_0)$ is only weakly dependent upon β and ω_0 , provided $\omega_0 \neq 0$.

ω_0 \ β	0.0	0.2	0.4	0.5	0.6	0.8
0.0	.3411					
.725	1.244	1.210	1.177	1.160	1.144	1.113
.842	1.154	1.139	1.125	1.117	1.110	1.096
.902	1.122	1.114	1.106	1.102	1.098	1.090
1.0	1.082					

TABLE OF VALUES OF $G(\beta, \omega_0)$.

Since g_Σ^2 is thought to be small, this slow variation of $G(\beta, \omega_0)$, $\omega_0 \neq 0$, may be neglected, and $G(\beta, \omega_0)$ may be regarded as a constant.

The combinations of values of β and ω_0 for which the dispersion relations were evaluated are those for which the table above has non-blank entries; there are twenty in all.

The data used for $D_\pm(\omega)$ consisted of five groups. These, and the notations used for them, were:

(1) $D^{(1)}$, a set, tabulated by A.D. Martin and R. Perrin (31), containing values of $D_+(\omega)$ at 29 momenta between 140 MeV/c and 3.0 GeV/c;

(2) $D^{(2)}$, a set from the same source of 35 values of $D_-(\omega)$ at momenta between 617 MeV/c and 1183 MeV/c;

(3) $D^{(3)}$ obtained by combining $D^{(1)}$ and $D^{(2)}$;

(4) $D^{(4)}$ a set of 45 uncorrected values of $D_-(\omega)$ taken directly from the experimentalists' reports (32) at momenta between 426 MeV/c and 1183 MeV/c; and

(5) $D^{(5)}$, the union of sets $D^{(1)}$ and $D^{(4)}$.

With an unsubtracted dispersion relation, characterised by a particular pair (β,ω_0) (with $\beta\neq 0$), and a definite model for the K^-p amplitude in the unphysical region, as many estimates for $C = g_\Lambda^2 + G(\beta,\omega_0)\, g_\Sigma^2$ were obtained as there were values of $D_\pm(\omega)$ in the set $D^{(i)}$, $i=1,\cdots,5$, of data employed in the dispersion relation. A statistical weight for each of these estimates was approximated by taking as the total error in each determination of C only the error arising from the experimental value of $D_\pm(\omega)$. The weighted mean, denoted $C^{(i)}$, of the several estimates for C was then taken.

To state more precisely the steps mentioned in the above paragraph, we cast the dispersion relation for $0<\beta<1$ into the form

$$v_\ell = u_\ell\, C$$

Here

$$v_\ell = \frac{D_\pm(\omega_\ell)}{(\omega_\ell \pm m_K)^\beta(\omega_\ell \pm \omega_0)^{1-\beta}} \mp \frac{1}{\pi}\int_{m_K}^{\infty} \frac{d\omega' \quad A_+(\omega')}{(\omega'+m_K)^\beta(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega_\ell)}$$
$$\pm \frac{1}{\pi}\int_{m_K}^{\infty} \frac{d\omega' \quad A_-(\omega')}{(\omega'-m_K)^\beta(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega_\ell)}$$

$$\pm \frac{1}{\pi} \int_{\omega_0}^{m_K} d\omega' \frac{\left[\cos(\pi\beta) A_-(\omega') - \sin(\pi\beta) D_-(\omega')\right]}{(m_K-\omega')^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega' \pm \omega_\ell)}$$

$$\mp \frac{1}{\pi} \int_{\omega_{\pi\Lambda}}^{\omega_0} d\omega' \frac{A_-(\omega')}{(m_K-\omega')^{\beta}(\omega_0-\omega')^{1-\beta}(\omega' \pm \omega_\ell)}$$

and

$$\mathcal{U}_\ell = -\frac{(m_K^2-(m_\Lambda-m_P)^2)}{16\pi m_P^2} \Big/ \left[(m_K-\omega_\Lambda)^{\beta}(\omega_0-\omega_\Lambda)^{1-\beta}(\omega_\ell \pm \omega_\Lambda)\right],$$

where ω_ℓ is the ℓ^{th} of the energies at which $D_\pm(\omega)$ has been measured. Thus, $N^{(i)}$ values, denoted C_ℓ, $\ell = 1, \cdots, N^{(i)}$, are obtained for C if the set $D^{(i)}$ is used. If $\Delta D_\pm(\omega_\ell)$ is the quoted error on $D_\pm(\omega_\ell)$, the statistical weight ascribed to C_ℓ is

$$p_\ell = \left[\frac{\Delta D_\pm(\omega_\ell)}{(\omega_\ell \pm m_K)^{\beta}(\omega_\ell \pm \omega_0)^{1-\beta}\mathcal{U}_\ell}\right]^{-2},$$

and the weighted mean of the $N^{(i)}$ observations is

$$C^{(i)} = \sum_{\ell=1}^{N^{(i)}} p_\ell C_\ell \Big/ \sum_{\ell=1}^{N^{(i)}} p_\ell .$$

The procedure followed in the case of the subtracted relations is similar, except that 'weighted mean' is to be re-interpreted as the weighted least squares estimate for two parameters, the subtraction constant $D_-(\omega_0)$ and the combination $C = g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$. We write the subtracted dispersion relation in the form

$$b_\ell = \mathcal{U}_{\ell 1} D_-(\omega_0) + \mathcal{U}_{\ell 2} C ,$$

where

$$\mathcal{U}_{\ell 1} = (\omega_\ell \pm \omega_0)^{-1} \quad ,$$

$$\mathcal{U}_{\ell 2} = -\frac{(m_K^2-(m_\Lambda-m_P)^2)}{16\pi\, m_P^2} \Big/ \left[(\omega_0-\omega_\Lambda)(\omega_\ell \pm \omega_\Lambda)\right]$$

and

$$b_\ell = \frac{D_\pm(\omega_\ell)}{(\omega_\ell \pm \omega_0)} \pm \frac{1}{\pi}\int_{\omega_{\pi^0\Lambda}}^{\infty} d\omega' \frac{A_-(\omega')}{(\omega'-\omega_0)(\omega' \pm \omega_\ell)}$$

$$\mp \frac{1}{\pi}\int_{m_K}^{\infty} d\omega' \frac{A_+(\omega')}{(\omega'+\omega_0)(\omega' \mp \omega_\ell)} \quad .$$

With each such equation there is a weight p_ℓ whose value is approximated by

$$\left(\frac{\Delta D_\pm(\omega_\ell)}{(\omega_\ell \pm \omega_0)}\right)^{-2} \quad .$$

If b_ℓ', $\mathcal{U}_{\ell 1}'$, $\mathcal{U}_{\ell 2}'$ respectively denote $p_\ell^{1/2} b_\ell$, $p_\ell^{1/2}\mathcal{U}_{\ell 1}$, $p_\ell^{1/2}\mathcal{U}_{\ell 2}$, the weighted set of $N^{(i)}$ equations (corresponding to the $N^{(i)}$ pieces of data in the set $D^{(i)}$) can be collected into a vector form as

$$\underline{b}' = U' \begin{pmatrix} D_-(\omega_0) \\ C \end{pmatrix},$$

in which the $N^{(i)}\times 2$ matrix U' has rows $(\mathcal{U}_{\ell 1}', \mathcal{U}_{\ell 2}')$ and the ℓ^{th} element of $\underline{b}'$ is b_ℓ'. The least squares estimates, $D_-^{(i)}(\omega_0)$ and $C^{(i)}$, of the subtraction constant and the coupling constant combination are then found by standard techniques as those values which minimise $\underline{r}'^T \underline{r}'$ in the equation

$$\underline{b}' = U' \begin{pmatrix} D_-^{(i)}(\omega_0) \\ C^{(i)} \end{pmatrix} + \underline{r}' \quad ;$$

$\underline{r}'$ is the weighted vector of residuals.

There is a further method of analysing the $D_\pm$ results by means of the dispersion relations with $\beta=0$ and $\omega_0 > \omega_{\pi^0\Lambda}$. $D_-(\omega_0)$ can be obtained from the model used to extrapolate the K^-p amplitude below threshold, so that there remains but one parameter to be estimated. The averaging procedure used for these cases was as for the unsubtracted dispersion relations.

The remainder of this chapter will be devoted to a detailed description of the data used in the dispersion relations and the techniques applied to it.

2.2 Data on the Real Part of the Forward Scattering Amplitude

As mentioned earlier, the data on $D_\pm(\omega)$ were grouped into five sets, labelled $D^{(i)}$, $i=1,\cdots,5$, the first two of which were taken directly from the tabulations performed by Martin and Perrin (31), and the fourth from the experimentalists' reports. Sets three and five simply compound the data on $D_+(\omega)$ with the two collections of data on $D_-(\omega)$.

The values of $D_-(\omega)$ in set four were computed by means of the prescription

$$|D_-(\omega)|^2 = \left(\frac{W(\omega)}{m_p}\right)^2 \frac{d\sigma_-}{d\Omega}(\omega, \theta_{cm}=0) - \left(\frac{q_L(\omega)\, \sigma_-^{LAB}(\omega)}{4\pi}\right)^2 ,$$

derived from the relations

$$\left(\frac{W(\omega)}{m_p}\right)^2 \frac{d\sigma_-}{d\Omega}(\omega, \theta_{CM}=0) = |F_-(\omega)|^2 \quad ,$$

$$q(\omega)\, \sigma_-^{LAB}(\omega)/4\pi = A_-(\omega) \quad ,$$

and

$$D_-(\omega) + i\, A_-(\omega) = F_-(\omega) \quad .$$

Note that

$$q(\omega) = (\omega^2 - m_K^2)^{1/2}$$

and

$$W(\omega) = (m_p^2 + m_K^2 + 2 m_p \omega)^{1/2} \quad .$$

The extrapolation of the differential cross section to the forward direction was performed using a Legendre series fitted to the angular distribution:

$$\frac{d\sigma_-}{d\Omega}(\omega, \theta_{CM}) = \sum_{\ell=0}^{L} a_\ell\, P_\ell(\cos\theta_{CM}) \quad ,$$

so that

$$\frac{d\sigma_-}{d\Omega}(\omega, \theta_{CM}=0) = \sum_{\ell=0}^{L} a_\ell \quad .$$

Thus, the calculation of $|D_-(\omega)|$ is quite straightforward. However, its sign is more elusive. The signs given to $D_-(\omega)$ by Martin and Perrin were copied at those energies in common to the two sets of data on $D_-(\omega)$. At the remaining energies the signs selected were those for which the best least squares fit was obtained (smallest value of χ^2).

Martin and Perrin refined the data from the experimental papers in two ways.

(1) Instead of a selecting a definite degree, L , for a Legendre fit, they used several values of L at each energy, and selected as the most appropriate that which minimised the weighted sum of squares of the residuals of the Legendre fit.

(2) They made some effort to allow for Coulomb corrections. Firstly, the values of $D_+(\omega)$ between 140 MeV/c and 642 MeV/c were obtained not by extrapolating the differential cross section to the forward direction and comparing this with the optical limit, but rather by using the s-wave analysis of the low energy K^+p data performed by Goldhaber et al (7). They also considered the Coulomb enhancement of the K^-p cross sections to be so large below 600 MeV/c that K^-p data below this momentum could not be used to determine $D_-(\omega)$. They discarded this data. At momenta just above 600 MeV/c they omitted from the Legendre fits to the angular distributions these experimental points in the near forward direction which could possibly have non-negligible Coulomb correction.

The errors they estimated after considering:

(1) the value of $\Delta D_\pm$ calculated using the error matrix from the least squares fit with the optimum value of L , denoted L_0 :

(2) the spread in the values of $|D_\pm|$ obtained by fitting the angular distributions with Legendre polynomials of degree $L_0 - 1$, L_0, $L_0 + 1$; and

(3) $|D_\pm|$ and $\Delta D_\pm$ obtained from fits to angular distributions measured in the same experiment at neighbouring energies.

There seems to be little doubt from the phase shift analysis of Lea at et (33) that the sign of $D_+(\omega)$ is negative over the whole range considered by Martin and Perrin.

The D_- amplitude however possesses zeros , in the neighbourhood of which the magnitude of $D_-(\omega)$ is so small that predictions for the sign of $D_-(\omega)$ from forward dispersion relations are not unambiguous. Martin and Perrin resolved the ambiguity of sign in the data they quoted for $D_-(\omega)$ by an iterative method. Where only $|D_-(\omega)|$ was known, a sign was chosen arbitrarily. A fit to all the D_- data was performed with a finite energy sum rule and the fitted curve was used to predict the values of $D_-(\omega)$. The process was repeated until agreement in sign was obtained between input and output.

In table 2 are listed the three sets, $D^{(1)}$, $D^{(2)}$ and $D^{(4)}$, of data on $D_\pm(\omega)$; the further sets $D^{(3)}$ and $D^{(5)}$ are just the unions $D^{(1)} \cup D^{(2)}$ and $D^{(1)} \cup D^{(4)}$. Note especially the large differences between the values in $D^{(2)}$ and $D^{(4)}$, resulting from the corrections applied by Martin and Perrin.

TABLE 2.

$D_\pm$ DATA SETS.

$D_\pm$ UNITS : GeV^{-1}.

q : KAON LAB. MOMENTUM (MeV/c).

q	$D_+^{(1)}$	$\pm\Delta D_+^{(1)}$
140	-2.11	0.48
175	-2.43	0.43
205	-2.34	0.35
235	-2.31	0.33
265	-2.19	0.35
355	-2.33	0.24
520	-2.31	0.26
642	-2.23	0.23
778	-2.40	0.20
810	-2.16	0.40
864	-2.23	0.20
900	-1.90	0.20
910	-2.04	0.40
969	-2.11	0.20
978	-2.21	0.25
1060	-1.93	0.25
1124	-1.87	0.35
1207	-2.49	0.30
1216	-2.55	0.70
1253	-3.21	0.60
1317	-2.50	0.40
1376	-2.60	0.50
1450	-3.81	0.30
1477	-3.30	0.60
2110	-6.31	1.50
2310	-6.08	1.50
2530	-4.32	1.50
2720	-6.37	1.50
3000	-5.58	2.50

q	$D_-^{(4)}$	$+\Delta D_-^{(4)}$	$-\Delta D_-^{(4)}$
436	3.11	1.00	1.55
455	2.57	0.95	1.62
475	3.30	0.71	0.88
495	2.05	0.98	2.05
514	3.00	0.82	1.07
534	3.45	0.71	0.88
554	3.47	0.67	0.80
573	3.04	0.71	0.92
597	3.51	0.76	0.98
617	3.90	0.73	0.92
637	2.18	0.90	2.18
658	3.92	0.61	0.76
677	3.98	0.66	0.80
699	3.93	0.66	0.85
705	1.70	1.19	1.19
719	4.68	0.68	0.78
725	1.88	1.34	1.34
740	4.51	0.78	0.84
741	2.03	1.43	1.43
761	4.38	0.77	0.95
768	2.27	1.61	1.61
773	5.14	0.71	0.77
777	5.05	0.88	1.14
793	5.22	0.66	0.79
802	2.56	1.80	1.80
806	5.84	1.02	1.22
838	6.20	0.82	1.16
853	5.41	0.91	1.12
874	4.79	0.97	1.27
894	5.19	1.02	1.26
904	5.09	1.05	1.29
916	3.41	1.58	3.41
935	4.36	1.13	1.48
954	3.46	1.46	3.46
970	2.56	1.80	2.56
991	4.38	1.19	1.69
1022	3.22	1.61	3.22
1044	3.44	1.44	3.44
1061	2.43	2.10	2.43
1102	-1.03	2.58	1.03
1117	-1.41	2.02	1.41
1134	-2.75	1.67	2.75
1153	-3.26	1.44	3.26
1174	-2.23	1.86	2.23
1183	-2.76	1.66	2.76

q	$D_-^{(2)}$	$\pm\Delta D_-^{(2)}$
617	2.6	1.5
637	3.0	1.5
658	1.3	1.5
699	2.1	1.5
705	3.4	2.0
719	2.2	1.5
725	1.8	2.0
740	2.2	1.5
741	1.7	2.0
761	3.6	1.5
768	2.7	2.0
773	3.3	1.5
777	2.8	1.5
793	3.4	1.5
802	4.7	2.0
806	1.7	1.5
853	4.2	1.5
874	3.6	1.5
894	4.0	1.5
904	3.5	1.5
916	3.3	1.5
954	3.5	1.5
970	3.2	1.5
985	4.1	1.5
991	1.7	1.5
1022	2.5	1.5
1044	0.6	1.5
1061	1.7	1.5
1080	-1.3	1.5
1102	-2.3	1.5
1117	-2.4	1.5
1134	-3.4	1.5
1153	-4.7	1.5
1174	-2.6	1.5
1183	-3.9	1.5

2.3 Cross Section Integrals

For the purposes of calculation, the physical region integrals, namely,

$$\mp \frac{1}{\pi} \int_{m_K}^{\infty} d\omega' \frac{A_-(\omega')}{(\omega'-m_K)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)}$$

$$\pm \frac{1}{\pi} \int_{m_K}^{\infty} d\omega' \frac{A_+(\omega')}{(\omega'+m_K)^{\beta}(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega)},$$

were partitioned by points $\omega_1^{(\pm)}$ and ω_2 into integrals over regions of low, high and very high energies:

$$\mp \frac{1}{\pi} \left\{ \int_{m_K}^{\omega_1^{(-)}} + \int_{\omega_2}^{\infty} \right\} d\omega' \frac{A_-(\omega')}{(\omega'-m_K)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)}$$

$$\pm \frac{1}{\pi} \left\{ \int_{m_K}^{\omega_1^{(+)}} + \int_{\omega_2}^{\infty} \right\} d\omega' \frac{A_+(\omega')}{(\omega'+m_K)^{\beta}(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega)}$$

$$\mp \frac{1}{4\pi^2} \int_{\omega_1^{(-)}}^{\omega_2} d\omega' \frac{q(\omega')\,\sigma_-(\omega')}{(\omega'-m_K)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)}$$

$$\pm \frac{1}{4\pi^2} \int_{\omega_1^{(+)}}^{\omega_2} d\omega' \frac{q(\omega')\,\sigma_+(\omega')}{(\omega'+m_K)^{\beta}(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega)}.$$

Note that, over the intermediate range from $\omega_1^{(\pm)}$ to ω_2 , $A_\pm(\omega)$ have been expressed in terms of the total cross sections $\sigma_\pm(\omega)$ via the optical theorem:

$$q(\omega)\,\sigma_\pm(\omega)\,/4\pi \;=\; A_\pm(\omega)\;.$$

(1) Very High Energy Region

The present limit of reliable data on $\sigma_\pm$ is 20 GeV/c, and ω_2 was chosen to correspond to this momentum. Above ω_2, $\sigma_\pm(\omega)$ was predicted from the Regge model recently proposed by Barger and Phillips (34). In addition to the Regge poles P, P', A_2, ρ, ω usually incorporated in models of the $K^\pm p$ scattering amplitudes at high energies, this model included a leading vacuum cut, whose purpose was, in essence, to displace the asymptotic region (approximate equality of σ_+ and σ_-) to higher energies than previously anticipated, and hence to secure compatibility of the recent Serpukhov total cross section data with the Pomeranchuk theorem.

Barger and Phillips parametrised the contribution to the forward scattering amplitude from the vacuum Regge cut as

$$F^c(\omega) \;=\; \frac{-\gamma_c}{4\pi}\int_{-\infty}^{\alpha_c} d\alpha\,\rho(\alpha)\,\exp(-\tfrac{1}{2}i\pi\alpha)\,\omega^\alpha\;.$$

For the spectral function, $\rho(\alpha)$, they chose

$$\rho(\alpha) \;=\; (\alpha_c-\alpha)^\lambda \exp\left[b(\alpha_c-\alpha)\right],$$

with λ and b as parameters to be varied. Upon integration this gave for F^c the following simple form

$$F^c(\omega) = -\frac{\gamma_c}{4\pi}\,\Gamma(\lambda+1)\,\exp(-\tfrac{1}{2}i\pi\alpha_c)\,\omega^{\alpha_c}\left[\ln\omega - b - \tfrac{1}{2}i\pi\right]^{-\lambda-1}.$$

The pole amplitudes were represented as

$$F^{j}(\omega) = -\frac{\gamma_j}{4\pi} \exp\left(-\tfrac{1}{2} i \pi \alpha_j\right) \omega^{\alpha_j} , \quad \text{FOR } j = P, P', A_2 ,$$

and as

$$F^{j}(\omega) = +\frac{i\gamma_j}{4\pi} \exp\left(-\tfrac{1}{2} i \pi \alpha_j\right) \omega^{\alpha_j} , \quad \text{FOR } j = \rho, \omega .$$

The quantities α_j were not taken as free parameters; instead their values were fixed at $\alpha_P = 1$ and $\alpha_j = 0{\cdot}4 \,,\, j \neq P$.

The parameters were estimated by best fit to the $\pi^{\pm}p$ and $K^{\pm}p$ scattering data, with continuous moment sum rules serving as constraints. Their values, in units of $\hbar = c = \text{GeV} = 1$ and with the signs of γ_ρ and γ_ω appropriate to $K^{-}p$ scattering, were

$$\begin{aligned}
\gamma_P &= 65.4 \\
\gamma_{P'} &= 74.3 \\
\gamma_{A_2} &= 6.5 \\
\gamma_\rho &= 6.8 \\
\gamma_\omega &= 26.8 \\
\gamma_c &= -110 \\
b &= 0 .
\end{aligned}$$

The solution was insensitive to the value of λ , provided $\lambda \approx 0$, so Barger and Phillips fixed $\lambda = 0{\cdot}1$ to fulfil the theoretical requirement $\rho(\alpha_c) = 0$. However, because of the insensitivity to λ , the formulae can be simplified with scarcely any sacrifice of accuracy by setting $1 + \lambda \simeq 1$. With this approximation, the total forward scattering amplitude is

$$F_{\pm}(\omega) = -(\gamma_c/4\pi)\exp(-\tfrac{1}{2}i\pi\alpha_c)\omega^{\alpha_c}(\ln\omega + \tfrac{1}{2}i\pi)/(\ln^2\omega + \pi^2/4)$$
$$-\sum_{j=P,P',A_2}(\gamma_j/4\pi)\exp(-\tfrac{1}{2}i\pi\alpha_j)\,\omega^{\alpha_j}$$
$$\mp i\sum_{j=\omega,\rho}(\gamma_j/4\pi)\exp(-\tfrac{1}{2}i\pi\alpha_j)\,\omega^{\alpha_j} \quad .$$

For leading cuts, $\alpha_c = 1$, and so

$$A_{\pm}(\omega) = (\gamma_c/4\pi)\;\omega\ln\omega/(\ln^2\omega + \pi^2/4)$$
$$+\sum_{j=P,P',A_2}(\gamma_j/4\pi)\sin(\tfrac{1}{2}\pi\alpha_j)\,\omega^{\alpha_j}$$
$$\mp\sum_{j=\omega,\rho}(\gamma_j/4\pi)\cos(\tfrac{1}{2}\pi\alpha_j)\,\omega^{\alpha_j} \quad .$$

The quantity required in the dispersion integrals is

$$H^{(\pm)}(\omega) = \mp\frac{1}{\pi}\int_{\omega_2}^{\infty}d\omega'\left\{\frac{A_-(\omega')}{(\omega'-m_K)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)} - \frac{A_+(\omega')}{(\omega'+m_K)^{\beta}(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega)}\right\}$$

If ω_3 is an energy above which m_K , ω_0 and ω are negligible compared with ω', $H^{(\pm)}(\omega)$ may be rewritten as $H_1^{(\pm)}(\omega) + H_2^{(\pm)}$ where

$$H_1^{(\pm)}(\omega) = \mp\frac{1}{\pi}\int_{\omega_2}^{\omega_3}d\omega'\left\{\frac{A_-(\omega')}{(\omega'-m_K)^{\beta}(\omega'-\omega_0)^{1-\beta}(\omega'\pm\omega)} - \frac{A_+(\omega')}{(\omega'+m_K)^{\beta}(\omega'+\omega_0)^{1-\beta}(\omega'\mp\omega)}\right\}$$

and

$$H_2^{(\pm)} = \mp \frac{1}{\pi} \int_{\omega_3}^{\infty} d\omega' \left(A_-(\omega') - A_+(\omega') \right) / \omega'^2$$

$$= \mp \frac{1}{2\pi^2} \sum_{j=\rho,\omega} \gamma_j \cos(\tfrac{1}{2}\pi\alpha_j) \int_{\omega_3}^{\infty} d\omega' \, \omega'^{\alpha_j - 2}$$

$$= \mp \frac{1}{2\pi^2} \sum_{j=\rho,\omega} \gamma_j \cos(\tfrac{1}{2}\pi\alpha_j) \quad \omega_3^{(\alpha_j - 1)} / (1-\alpha_j) \; .$$

In the calculations, ω_2 and ω_3 were chosen to be 41 m_K and 4100 m_K respectively. $H_1^{(\pm)}(\omega)$ was obtained by numerical integration for each value of ω, with the Barger and Phillips model used for $A_{\pm}(\omega)$.

(2) Experimental Region

The technique used to integrate

$$\mp \frac{1}{4\pi^2} \int_{\omega_1^{(-)}}^{\omega_2} d\omega' \; \frac{q(\omega')\,\sigma_-(\omega')}{(\omega' - m_K)^{\beta} (\omega' - \omega_0)^{1-\beta} (\omega' \pm \omega)}$$

and

$$\pm \frac{1}{4\pi^2} \int_{\omega_1^{(+)}}^{\omega_2} d\omega' \; \frac{q(\omega')\,\sigma_+(\omega')}{(\omega' + m_K)^{\beta} (\omega' + \omega_0)^{1-\beta} (\omega' \mp \omega)}$$

was to fit a curve to 'measurements' of the function

$$\frac{1}{4\pi^2} \qquad q(\omega')\,\sigma_\pm(\omega')\,/(\omega'\pm\omega_0)$$

,

computed from the data on $\sigma_\pm(\omega')$ in the range $[\omega_1^{(\pm)}, \omega_2]$, and using this curve to perform the integrations analytically. For this method to be successful, the functions used to fit the data must be 'flexible' in order to reproduce the rapidly varying σ_- cross section, and must have a simple form so that the integrals are amenable to standard formulae. While polynomials fulfil the second requirement, they, at least in low orders, vary too slowly to satisfy the first. The usual parametrisations in terms of Breit-Wigner resonances satisfy the first but not the second. Examples of functions with both requisites are the polynomial spline functions, and in particular, the cubic splines.

Polynomial splines are piece-wise polynomials in which the constituent polymonials are smoothly joined. More specifically, the function $\zeta(x)$ is a cubic spline with knots $\xi_1, \xi_2, \cdots\cdots \xi_{N+1}$ if:
(1) on the interval $[\xi_i , \xi_{i+1}]$, $i = 1, 2, \cdots\cdots N$, $\zeta(x)$ reduces to a cubic polymonial; and
(2) the polynomials in the intervals $[\xi_{i-1} , \xi_i]$ and $[\xi_i, \xi_{i+1}]$, $i = 2, \cdots\cdots, N$, are joined so that $\zeta(x)$ possesses left and right third derivatives at ξ_i , even though they may not be equal.

Splines have the simple analytic form required, since between knots they reduce to polymonials. Furthermore, they are capable of following extremely rapid variations, since any rapid, but smooth, variation can be approximated by a polynomial over a sufficiently small region. Thus, where the function being fitted has bad behaviour, the knots of the spline will be close together.

M.J.D. Powell (35) has experimented with splines with the 'free knots', that is, splines whose knot locations are not preassigned, but instead are regarded as quantities to be determined by the data being fitted. Determination of the knot locations requires a means of detecting genuine trends in the data and a mechanism for suppressing spurious oscillations. As a problem in numerical analysis this is quite involved, but, fortunately, he has made available programs which automatically select the knot locations and find the least squares estimates of the remaining spline parameters. These programs are the ones that have been used throughout this work.

Both the principal value integral and the integral over the crossed channel cross-section can be obtained from

$$U^{(\pm)}(\lambda\omega) = \frac{1}{4\pi^2}\int_{\omega_1^{(\pm)}}^{\omega_2} d\omega' \frac{q(\omega')\ \sigma_\pm(\omega')}{(\omega'\pm m_K)^\beta(\omega'\pm\omega_0)^{1-\beta}(\omega'+\lambda\omega)},$$

where $\lambda=\pm1$, so the technique of integration using splines will be illustrated with $U^{(\pm)}(\lambda\omega)$.

Set

$$\zeta^{(\pm)}(\omega) = (1/4\pi^2)\, q(\omega)\sigma_\pm(\omega)/(\omega\pm\omega_0) ,$$

so that

$$U^{(\pm)}(\lambda\omega) = \int_{\omega_1^{(\pm)}}^{\omega_2} d\omega' \frac{\zeta^{(\pm)}(\omega')}{(\omega'+\lambda\omega)}\left(\frac{\omega'\pm\omega_0}{\omega'\pm m_K}\right)^\beta .$$

The interval $[\omega_1^{(\pm)},\omega_2]$ is sectioned by knots $\omega_1^{(\pm)}=\xi_1^{(\pm)}<\xi_2^{(\pm)}<\cdots<\xi_{N^{(\pm)}+1}^{(\pm)}=\omega_2$ and between each pair of knots $\zeta^{(\pm)}$ is represented as a cubic:

$$\zeta^{(\pm)}(\omega') = \sum_{j=1}^{4} a_{ji}^{(\pm)}\omega'^{\,j-1} \quad \text{ON} \quad [\xi_i^{(\pm)}, \xi_{i+1}^{(\pm)}],\ i=1,\cdots,N^{(\pm)} .$$

Thus ,

$$U^{(\pm)}(\lambda\omega) = \sum_{i=1}^{N^{(\pm)}} \sum_{j=1}^{4} \int_{\xi_i^{(\pm)}}^{\xi_{i+1}^{(\pm)}} d\omega' \, \frac{a_{ji}^{(\pm)} \omega'^{\,j-1}}{(\omega' + \lambda\omega)} \left(\frac{\omega' \pm \omega_0}{\omega' \pm m_K} \right)^{\beta} .$$

A change of variable to $\quad u = (\omega' \pm \omega_0)/(\omega' \pm m_K)$

leads, upon rearrangement, to

$$U^{(\pm)}(\lambda\omega) = \sum_{i=1}^{N^{(\pm)}} \sum_{j=1}^{4} \frac{(\mp)^j a_{ji}^{(\pm)}}{(\lambda\omega \mp m_K)} \sum_{\ell=1}^{j} (\omega_0 - m_K)^{\ell} \, m_K^{\,j-\ell} \binom{j-1}{\ell-1}$$

$$\times \int_{(\xi_i^{(\pm)} \pm \omega_0)/(\xi_i^{(\pm)} \pm m_K)}^{(\xi_{i+1}^{(\pm)} \pm \omega_0)/(\xi_{i+1}^{(\pm)} \pm m_K)} du \; \frac{u^{\beta}}{(1-u)^{\ell} \left(\frac{\lambda\omega \mp \omega_0}{\lambda\omega \mp m_K} - u \right)} .$$

If now β is permitted only rational values ($\beta = m/n$, where m and n are positive integers), a closed form can be obtained for $U^{(\pm)}(\lambda\omega)$ by means of the following identities:

(a)

$$\int du \, \frac{u^{\beta}}{(1-u)^{\ell}(c-u)} = \left[\int du \, \frac{u^{\beta}}{(1-u)^{\ell-1}(c-u)} - \int du \, \frac{u^{\beta}}{(1-u)^{\ell}} \right] \Big/ (1-c) , \quad \begin{matrix} \ell \geqslant 1 \\ c \neq 1 \end{matrix} ;$$

(b)

$$\int_a^b du \, \frac{u^{\beta}}{(c-u)} = c^{\beta} \int_{a/c}^{b/c} du \, \frac{u^{\beta}}{(1-u)}$$

(c) if n is even and $\quad 1 \leqslant m < n$,

$$\int du \, \frac{u^{m/n}}{(1-u)} = -\frac{u^{m/n}}{m/n} + (-)^{m+1} \ln|1+u^{1/n}|$$

$$- \ln|1-u^{1/n}|$$

$$- \sum_{k=1}^{\frac{n}{2}-1} \cos\left(\frac{km\pi}{n/2}\right) \ln\left|1-2u^{1/n}\cos\left(\frac{k\pi}{n/2}\right)+u^{2/n}\right|$$

$$+2\sum_{k=1}^{\frac{n}{2}-1} \sin\left(\frac{km\pi}{n/2}\right) \operatorname{arctg}\left(\frac{u^{1/n} - \cos\frac{k\pi}{n/2}}{\sin\frac{k\pi}{n/2}}\right) \quad ;$$

(d) if n is odd and $1 \leqslant m < n$,

$$\int du \, \frac{u^{m/n}}{(1-u)} = -\frac{u^{m/n}}{m/n} - \ln|1-u^{1/n}|$$

$$+(-)^{m+1} \sum_{k=1}^{(n-1)/2} \cos\left(\frac{m\pi(2k-1)}{n}\right) \ln\left|1+2u^{1/n}\cos\left(\frac{(2k-1)\pi}{n}\right)+u^{2/n}\right|$$

$$+(-)^{m+1} 2 \sum_{k=1}^{(n-1)/2} \sin\left(\frac{m\pi(2k-1)}{n}\right) \operatorname{arctg}\left(\frac{u^{1/n}+\cos\left(\frac{(2k-1)\pi}{n}\right)}{\sin\left(\frac{(2k-1)\pi}{n}\right)}\right) ;$$

(e)

$$\int_a^b du \, \frac{u^\beta}{(1-u)^\ell} = \left[\frac{b^{\beta+1}}{(1-b)^{\ell-1}} - \frac{a^{\beta+1}}{(1-a)^{\ell-1}} \right.$$

$$\left. - \left(\beta+1-(\ell-1)\right) \int_a^b du \, \frac{u^\beta}{(1-u)^{\ell-1}} \right] \Big/ (\ell-1),$$

$$\ell > 1 .$$

An objection which might be raised is that the spline functions were fitted to

$$(1/4\pi^2) \quad q(\omega)\,\sigma_\pm(\omega)\,/(\,\omega \pm \omega_0)$$

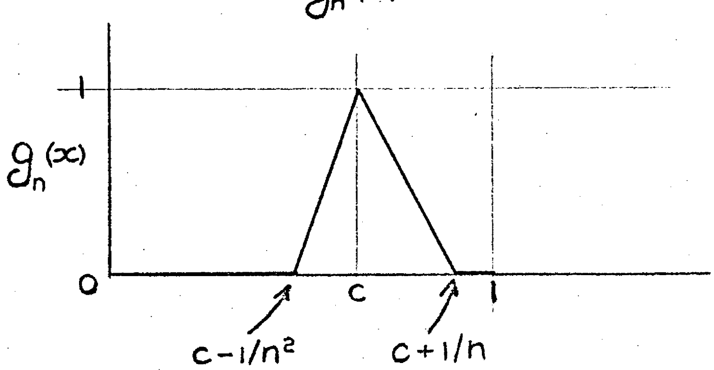
and not to the cross section data itself, so, for every value of ω_0 , the fitting process had to be repeated, and the fits to $\sigma_\pm(\omega)$ computed from those to $(1/4\pi^2)q(\omega)\sigma_\pm(\omega)/(\omega\pm\omega_0)$ might not be the same. In practice, however, it was found that the variation with ω_0 of the fit to $\sigma_\pm(\omega)$ is so small as to be negligible.

Figures 1 and 2 are given as an indication of the quality of the spline fits. The former shows the experimental data (36) on σ_- and the curve obtained by fitting to $\zeta^{(-)}(\omega) = (1/4\pi^2)q(\omega)\sigma_-(\omega)/(\omega-\omega_0)$. Graphed in the latter are the 'measurements' (37) of the function $\zeta^{(+)}(\omega) = (1/4\pi^2)\,q(\omega)\,\sigma_+(\omega)\,/(\omega+\omega_0)$ and the spline fitted to them. For later comparison, the fit obtained using the analysis of the K^+p data by Martin and Perrin is also shown.

One cautionary remark (45) that should be made concerning the numerical estimation of principal value integrals is that there exist sequences $\{f_n(x)\}$ of functions which converge in norm, but yet the sequence $\{\varphi_n(c)\}$, defined by

$$\varphi_n(c) = P\int_a^b \frac{f_n(x)}{(x-c)}\,dx\ , \quad a<c<b,$$

diverges. For example if $g_n(x)$ is the function



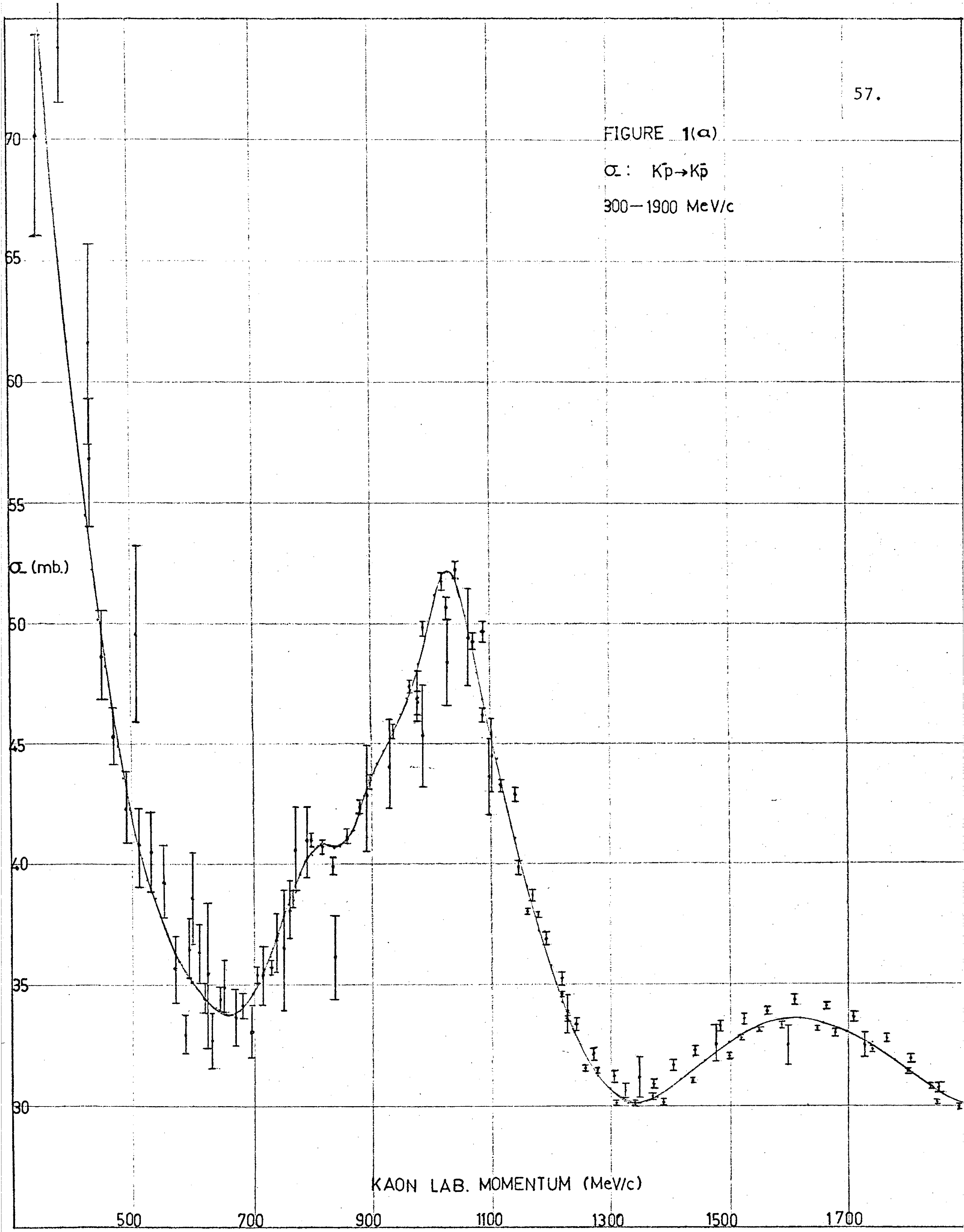


FIGURE 1(a)
σ_-: $K^-p \to K^-p$
300—1900 MeV/c
σ_- (mb.)
70
65
60
55
50
45
40
35
30
KAON LAB. MOMENTUM (MeV/c)
500
700
900
1100
1300
1500
1700

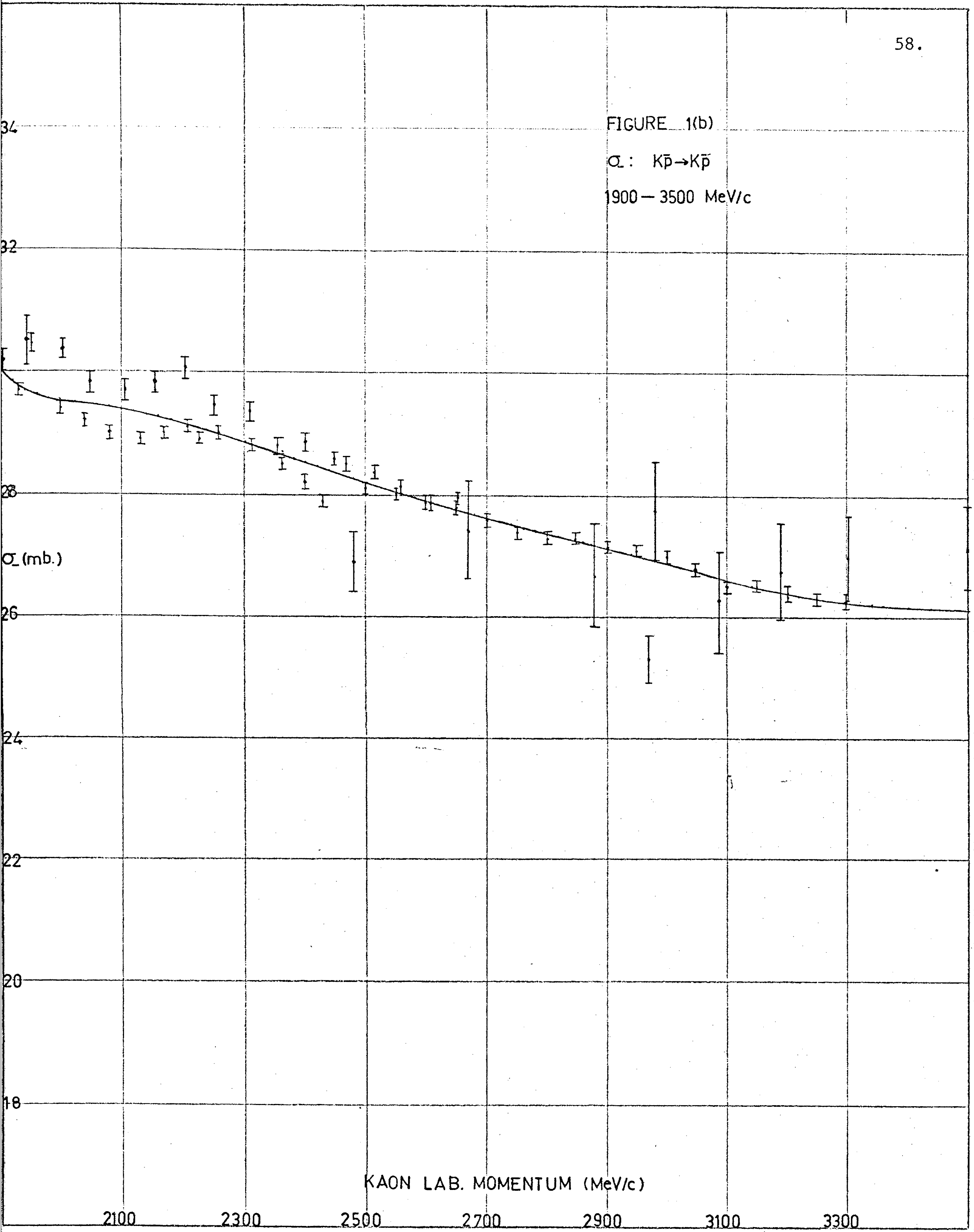


FIGURE 1(b)
σ_: K p̄→K p̄
1900 — 3500 MeV/c
34
32
28
σ_(mb.)
26
24
22
20
18
KAON LAB. MOMENTUM (MeV/c)
2100
2300
2500
2700
2900
3100
3300

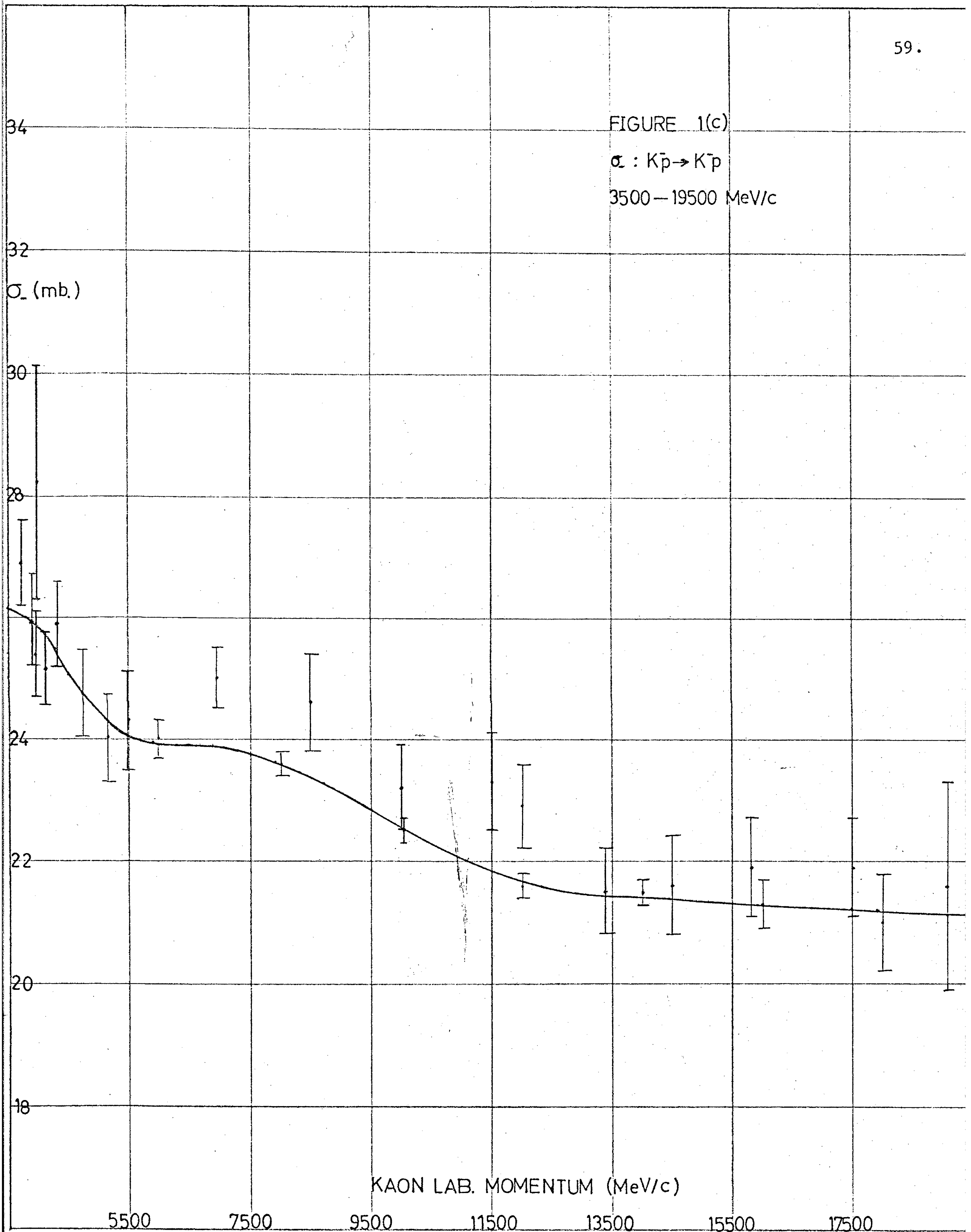


FIGURE 1(c)
σ_- : $K^-p \to K^-p$
3500 — 19500 MeV/c
σ_- (mb.)
34
32
30
28
24
22
20
18
KAON LAB. MOMENTUM (MeV/c)
5500
7500
9500
11500
13500
15500
17500

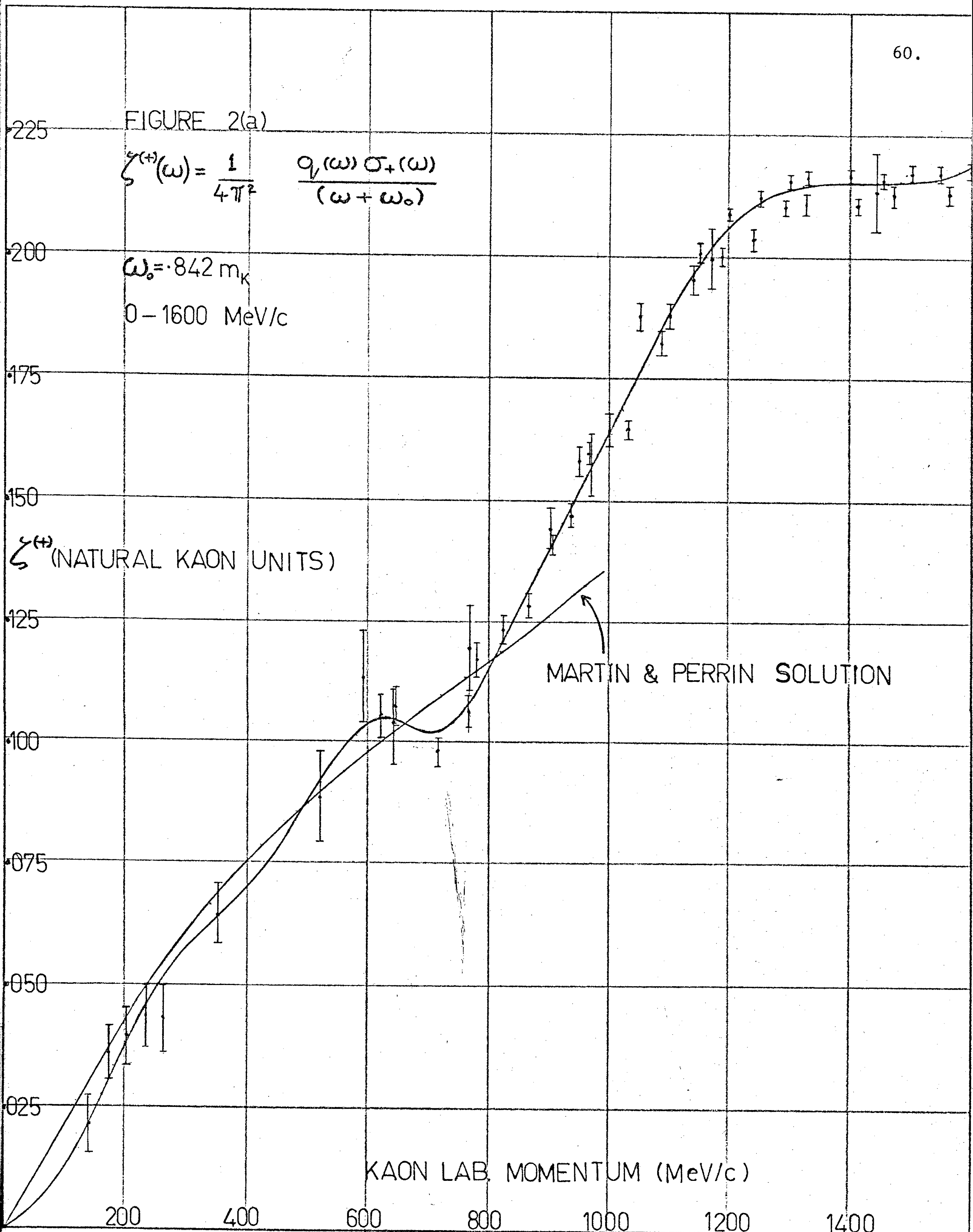


FIGURE 2(a)
$\zeta^{(+)}(\omega) = \frac{1}{4\pi^2} \frac{q(\omega)\sigma_+(\omega)}{(\omega+\omega_0)}$
$\omega_0 = \cdot 842\, m_K$
0 – 1600 MeV/c
·225
·200
·175
·150
·125
·100
·075
·050
·025
$\zeta^{(+)}$ (NATURAL KAON UNITS)
MARTIN & PERRIN SOLUTION
KAON LAB. MOMENTUM (MeV/c)
200
400
600
800
1000
1200
1400

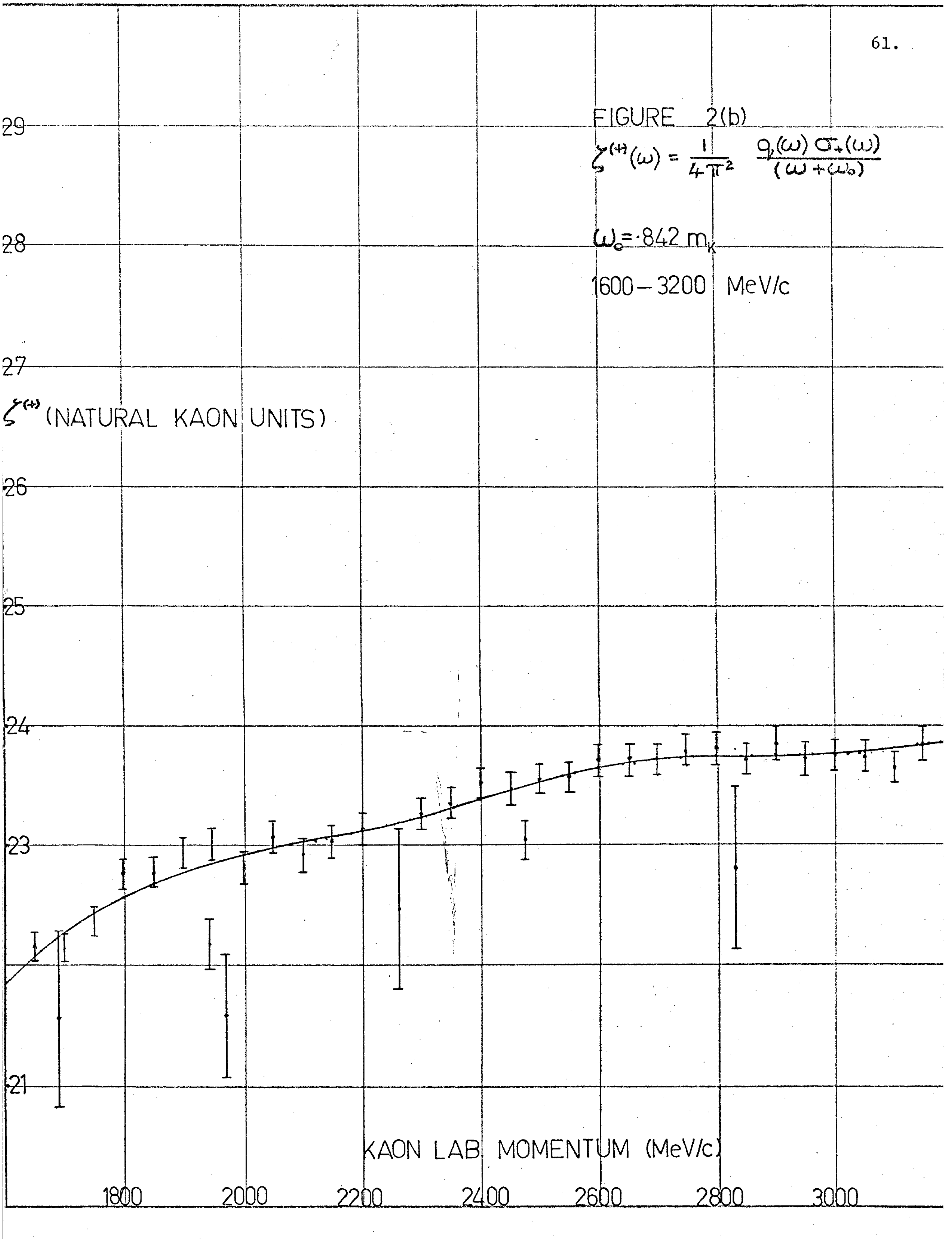
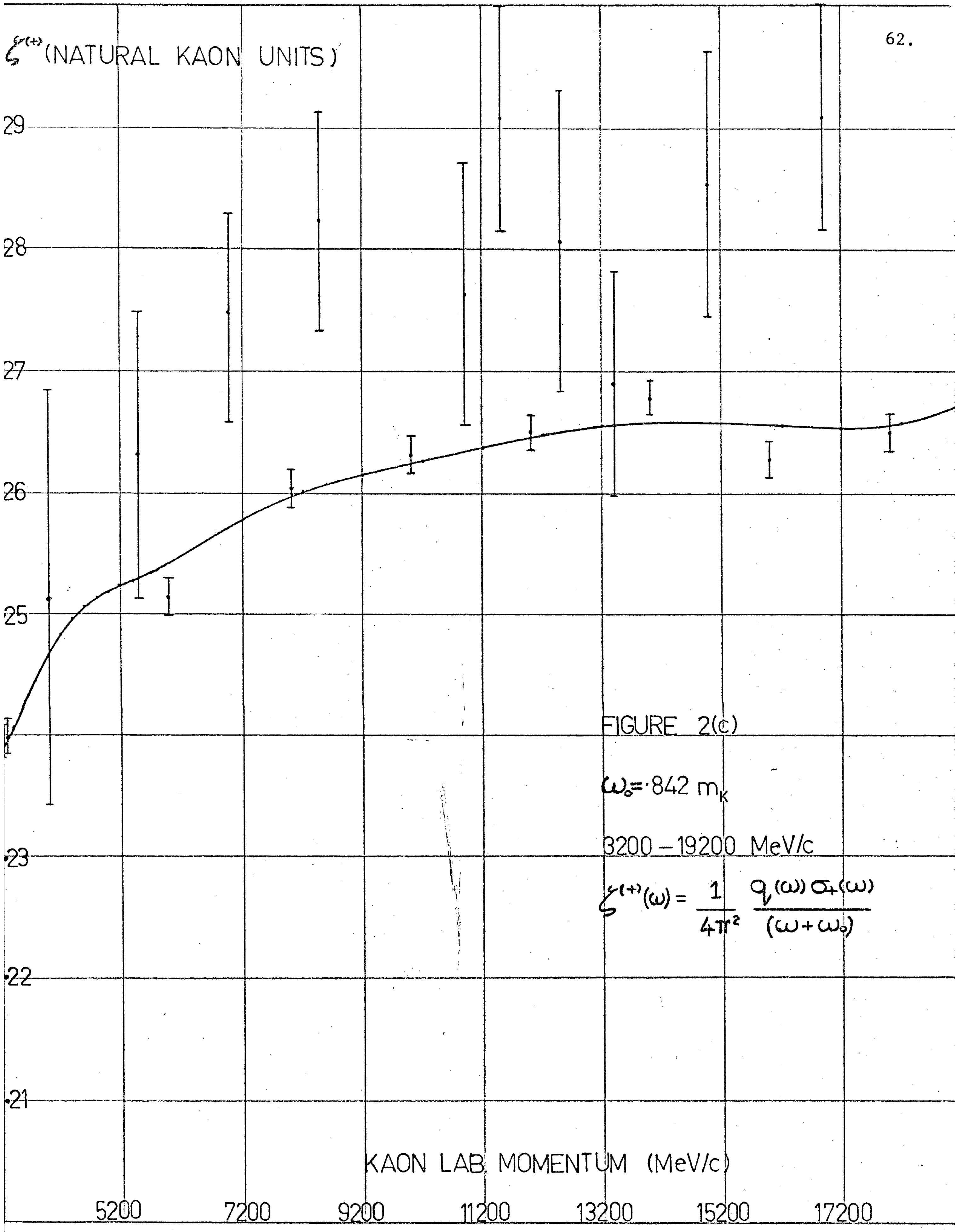


FIGURE 2(b)
$\zeta^{(+)}(\omega) = \frac{1}{4\pi^2} \frac{q_L(\omega)\,\sigma_+(\omega)}{(\omega+\omega_0)}$
$\omega_0 = \cdot 842\, m_K$
1600 – 3200 MeV/c
$\zeta^{(+)}$ (NATURAL KAON UNITS)
29
28
27
26
25
24
23
21
KAON LAB. MOMENTUM (MeV/c)
1800
2000
2200
2400
2600
2800
3000


$\zeta^{(+)}$ (NATURAL KAON UNITS)
29
28
27
26
25
23
22
21
FIGURE 2(c)
$\omega_0 = \cdot 842\ m_K$
3200 – 19200 MeV/c
$\zeta^{(+)}(\omega) = \frac{1}{4\pi^2} \frac{q(\omega)\,\sigma_+(\omega)}{(\omega+\omega_0)}$
KAON LAB. MOMENTUM (MeV/c)
5200
7200
9200
11200
13200
15200
17200

then

$$P\int_0^1 \frac{g_n(x)}{(x-c)}\,dx = \ln n \quad ,$$

so for the sequence $\quad f_n(x) = g_n(x)/(\ln n)^{1/2} \quad ,$

we have

$$\|f_n(x)\|_\infty \to 0 \quad \text{AND} \quad P\int_0^1 \frac{f_n(x)}{(x-c)}\,dx \to \infty \quad \text{AS } n \to \infty .$$

The numerical importance of this type of behaviour was tested by generating a smooth, in fact thrice differentiable, approximation $g(x)$ to the function $f(x) = 1$ on $[1,2]$, such that $|f(x)-g(x)| < 0.004$ for all $x \in [1,2]$. With c located at 1·51, the value of the integral

$$\varphi(c) = -\int_1^2 dx\, \frac{f(x)}{(x-c)}$$

was .0400, but, when $f(x)$ was replaced by $g(x)$, the integral value changed markedly to .0244.

(3) Low Energy Region

The integral over $A_-(\omega')$ in the low energy region from threshold to $\omega_1^{(-)}$, chosen to correspond to a kaon lab momentum of 280 MeV/c, was evaluated using for $A_-(\omega')$ the low energy parametrisation also used to extrapolate below the K^-p threshold.

Reference to figure 2 shows that the spline fit to

$$\zeta^{(+)}(\omega) = (1/4\pi^2)\; q(\omega)\,\sigma_+(\omega)/(\omega+\omega_0)$$

follows more closely the trends in the data at low energies than the fit obtained using Martin and Perrin's analysis (27) of the K^+p data.

Although the latter is probably more acceptable on theoretical grounds, the spline fit has been used right down to the K^+p threshold, since the dispersion relations are relatively insensitive to the low energy K^+p interaction. Thus, $\omega_1^{(+)}$ was set to m_K .

2.4 Unphysical Regional Models Tested

The models tested in dispersion relations $\underline{\mathrm{I}}$ and $\underline{\mathrm{II}}$ were Kim's effective-range K-matrix (ERK), the zero-range K-matrix analysis (ZRK) performed by A.D. Martin and G.G. Ross, and a constant scattering length analysis (CSL) using parameters also estimated by Kim. The last two of these analyses are restricted to the s-wave amplitude, so, to maintain consistency, only the s-wave parameters were incorporated in the ERK model. This is acceptable because the s-wave resonance Yo*(1405) dominates the unphysical region, and because the Y_1*(1385) which occurs in the p-wave is very weakly coupled to the $\bar{K}N$ channel in this model.

The s-wave T-matrix for transitions between states of total isospin I is expressed as a function of the total centre of mass energy, W, by the effective-range expansion

$$T^{(I)^{-1}}(W) = \left[Q^{(I)}(W)\right]^{-1/2}\left[M^{(I)}(W_0) + \tfrac{1}{2}\left(Q^{(I)^2}(W) - Q^{(I)^2}(W_0)\right)^{1/2}\right.$$
$$\left.\times R^{(I)}(W_0)\left(Q^{(I)^2}(W) - Q^{(I)^2}(W_0)\right)^{1/2}\right]\left[Q^{(I)}(W)\right]^{-1/2} - i .$$

$M^{(I)}(W_0)$ and $R^{(I)}(W_0)$ are real, symmetric, energy independent matrices in channel space, and, with Wo = $(m_K + m_p)$, are taken as the free parameters in fitting the data. $Q^{(I)}(W)$ is the diagonal matrix of channel centre of mass momenta. In the I=0 state, the $\pi\Sigma$ channel is the only one coupled to the $\bar{K}N$ channel at low energies. Thus, matrices $M^{(0)}$, $R^{(0)}$ and $Q^{(0)}$ are two dimensional:

$$M^{(0)} = \begin{pmatrix} m_{KK}^{(0)} & m_{K\Sigma}^{(0)} \\ m_{K\Sigma}^{(0)} & m_{\Sigma\Sigma}^{(0)} \end{pmatrix}, \quad R^{(0)} = \begin{pmatrix} r_{KK}^{(0)} & r_{K\Sigma}^{(0)} \\ r_{K\Sigma}^{(0)} & r_{\Sigma\Sigma}^{(0)} \end{pmatrix}, \quad Q^{(0)} = \begin{pmatrix} k_K & 0 \\ 0 & k_\Sigma \end{pmatrix}.$$

The $\pi\Lambda$ channel is also coupled to the $\bar{K}N$ system in the I=1 state, so $M^{(1)}$, $R^{(1)}$ and $Q^{(1)}$ are three dimensional:

$$M^{(1)} = \begin{pmatrix} m_{KK}^{(1)} & m_{K\Sigma}^{(1)} & m_{K\Lambda}^{(1)} \\ m_{K\Sigma}^{(1)} & m_{\Sigma\Sigma}^{(1)} & m_{\Sigma\Lambda}^{(1)} \\ m_{K\Lambda}^{(1)} & m_{\Sigma\Lambda}^{(1)} & m_{\Lambda\Lambda}^{(1)} \end{pmatrix}, \quad R^{(1)} = \begin{pmatrix} \gamma_{KK}^{(1)} & \gamma_{K\Sigma}^{(1)} & \gamma_{K\Lambda}^{(1)} \\ \gamma_{K\Sigma}^{(1)} & \gamma_{\Sigma\Sigma}^{(1)} & \gamma_{\Sigma\Lambda}^{(1)} \\ \gamma_{K\Lambda}^{(1)} & \gamma_{\Sigma\Lambda}^{(1)} & \gamma_{\Lambda\Lambda}^{(1)} \end{pmatrix}, \quad Q^{(1)} = \begin{pmatrix} k_K & & \\ & k_\Sigma & \\ & & k_\Lambda \end{pmatrix}.$$

Note that the subscripts K, Σ and Λ respectively label the $\bar{K}N$, $\pi\Sigma$ and $\pi\Lambda$ channels.

In terms of the s-wave $\bar{K}N$ scattering lengths, A_I, defined such that

$$k_K \cot \delta_I = A_I^{-1},$$

where δ_I are the s-wave $\bar{K}N$ phase shifts of isospin I, the $\bar{K}N$ T-matrix elements at very low energies can be written

$$T_{KK}^{(I)} = k_K A_I / (1 - i k_K A_I).$$

This form for $T_{KK}^{(I)}$ is the most convenient in which to apply corrections for the K^-, $\bar{K}^0$ and p, n mass differences. The prescription for doing so is discussed by Dalitz and Tuan (38). The net effect of these corrections is the replacement

$$(1 - i k_K A_{0,1})^{-1} \rightarrow (1 - i k_K' A_{1,0})/D,$$

where k_K' is the centre of mass momentum in the $\bar{K}^0 n$ channel and

$$D = 1 - \tfrac{1}{2} i (A_0 + A_1)(k_K + k_K') - k_K k_K' A_0 A_1 .$$

Thus, the $K^- p$ s-wave amplitude in the centre of mass system is finally given by

$$F_-^{cm} = \tfrac{1}{2} \left(A_0 + A_1 - 2 i k_K' A_0 A_1 \right) / D .$$

There is some uncertainty in the literature as to whether the K^-, $\bar{K}^0$ and p, n mass differences should be taken into account, since they are electromagnetic effects and are neglected in the formulation of dispersion relations for the strong interactions. The common belief is that the amplitudes obtained from fits to the data with the formulae which include the empirical mass differences should be used in the dispersion relations, but sound reasons why this is correct have not been advanced.

The numerical values of the parameters obtained by Kim in his effective-range analysis are listed below. Effective-ranges are quoted in fm and the M-matrix elements in fm^{-1} .

$$M^{(0)} = \begin{pmatrix} -0.00 \pm 0.02 & -1.11 \pm 0.04 \\ & +2.04 \pm 0.10 \end{pmatrix}, \quad R^{(0)} = \begin{pmatrix} 0.54 \pm 0.08 & \\ & -0.89 \pm 0.31 \end{pmatrix},$$

$$M^{(1)} = \begin{pmatrix} -3.60 \pm 0.02 & -2.86 \pm 0.03 & +2.08 \pm 0.07 \\ & -1.40 \pm 0.06 & +1.81 \pm 0.04 \\ & & -2.31 \pm 0.11 \end{pmatrix},$$

$$R^{(1)} = \begin{pmatrix} -0.13 \pm 0.07 & & \\ & -0.78 \pm 0.23 & \\ & & -1.22 \pm 0.45 \end{pmatrix}.$$

In units of fm , the inverses of the M-matrices of the zero-range K-matrix analysis by A.D. Martin and G.G. Ross are

$$M^{(0)-1} = \begin{pmatrix} -2.40 & -1.21 \\ & -1.05 \end{pmatrix}$$

and

$$M^{(1)-1} = \begin{pmatrix} -0.01 & -0.71 & -0.38 \\ & +0.34 & -0.21 \\ & & +0.17 \end{pmatrix} .$$

Finally, the values used for the scattering lengths in the CSL analysis were taken from an early paper (10) published by Kim. He quoted for the six CSL parameters

$$A_0 = (-1.674 \pm 0.038) + i\,(0.722 \pm 0.040)$$

$$A_1 = (-0.003 \pm 0.058) + i\,(0.688 \pm 0.033)$$

$$\varepsilon = +0.318 \pm 0.021$$

$$\phi = -53.8^\circ .$$

It was pointed out earlier that the amplitude predicted by the CSL model does not have the correct analytic behaviour at the $\pi\Sigma$ and $\pi\Lambda$ thresholds. To compensate for this defect, the imaginary parts, $A_-^{(0)}(\omega)$ and $A_-^{(1)}(\omega)$, of the amplitudes of isopin 0 and 1 were replaced, respectively, by

$$\tilde{A}_-^{(0)}(\omega) = \left(\frac{\omega - \omega_{\pi\Sigma}}{\bar{\omega}^{(0)} - \omega_{\pi\Sigma}}\right)^{1/2} A_-^{(0)}(\omega) , \text{ FOR } \omega_{\pi\Sigma} \leqslant \omega \leqslant \bar{\omega}^{(0)}$$

FIGURE 3

GRAPHS OF $D_{-}^{CM}(\omega) = \frac{m_p}{W(\omega)} D_{-}^{LAB}(\omega)$

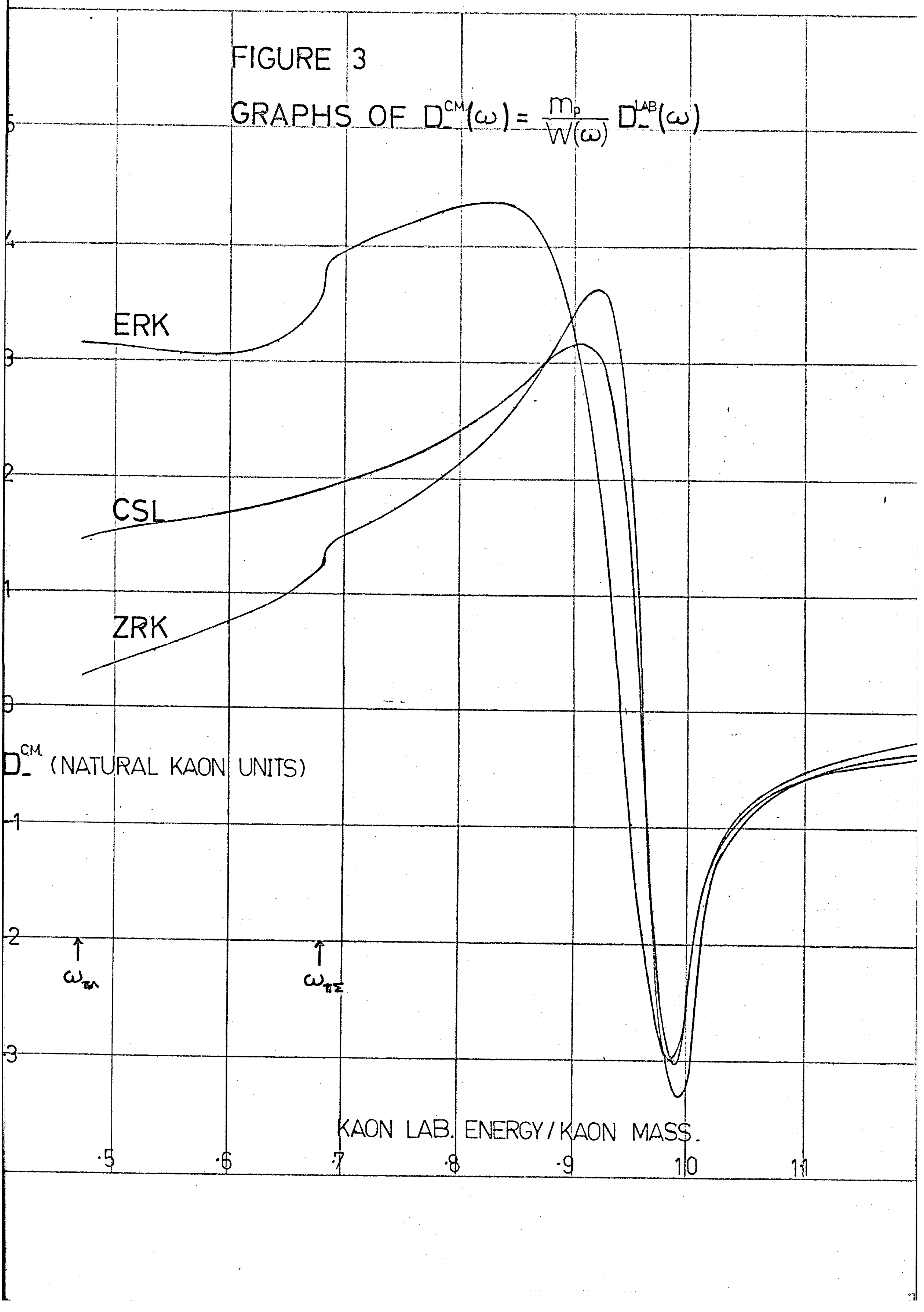
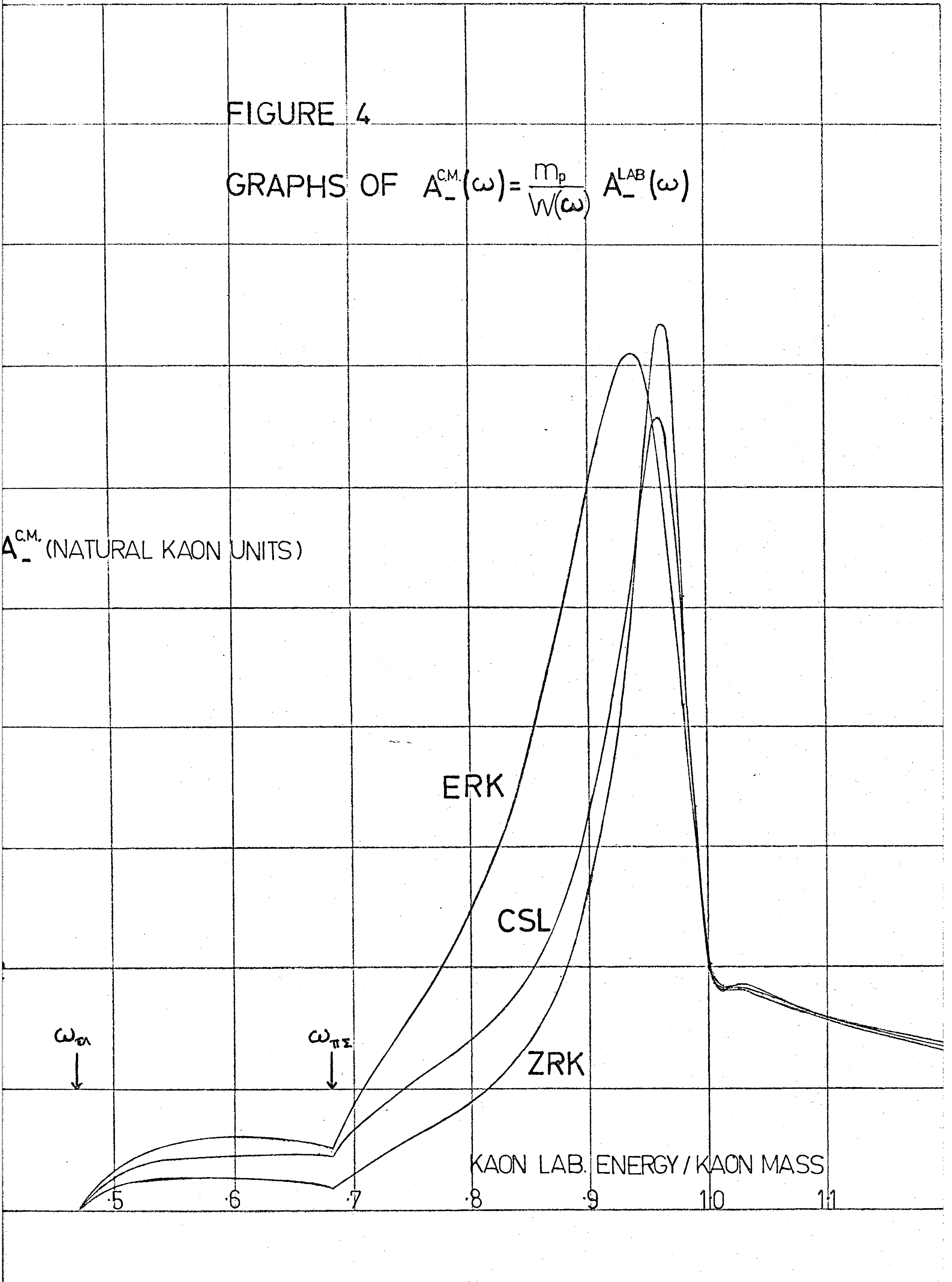


FIGURE 4

GRAPHS OF $A_-^{C.M.}(\omega) = \frac{m_p}{W(\omega)} A_-^{LAB}(\omega)$



and

$$\tilde{A}_{-}^{(1)}(\omega) = \left(\frac{\omega - \omega_{\pi\Lambda}}{\bar{\omega}^{(1)} - \omega_{\pi\Lambda}}\right)^{1/2} A_{-}^{(1)}(\omega) \quad \text{FOR} \quad \omega_{\pi\Lambda} \leqslant \omega \leqslant \bar{\omega}^{(1)}.$$

$\bar{\omega}^{(0)}$ and $\bar{\omega}^{(1)}$ were chosen at $(3\omega_{\Sigma\Lambda} + m_K)/4$ and $(3\omega_{\pi\Lambda} + \omega_{\pi\Sigma})/4$ respectively. This modification ensures that these imaginary parts vanish at the appropriate thresholds in the manner required by unitarity.

Graphs of both the real and imaginary parts of the $K^{-}p$ forward scattering amplitude from $\omega_{\pi^0\Lambda}$ to $1.2m_K$, for each of the models, are given in figures 3 and 4. Note especially the larger width predicted for the Yo*(1405) resonance by the ERK model.

One final remark concerns the accuracy with which all numerical integrations were performed. The programs used were adaptive versions of Newton-Coates quadrature schemes, and were able to detect rapidly varying behaviour of the integrand and adjust the mesh size so as to obtain a prescribed error bound. Throughout, all numerical integrations were performed to an absolute accuracy of 1×10^{-5}. The magnitudes of the terms were about unity, so that this accuracy was more than adequate.

3.1 Results and Discussion

Before the results obtained for the coupling constant combination $C = g_\Lambda^2 + G(\beta,\omega_0)\, g_\Sigma^2$ are presented, it is probably advisable to recall the quantities with which each determination of C must be labelled. They are the following:

(1) the model employed for the $K^{-}p$ amplitude in the low energy and unphysical regions;

(2) the pair (β,ω_0) which characterises the function

$$B_{\pm}(\omega) = F_{\pm}(\omega) \Big/ \left[(\omega \pm m_K)^{\beta}(\omega \pm \omega_0)^{1-\beta}\right]$$

for which the dispersion relation is written; and

(3) the collection of data for $D_{\pm}(\omega)$ used in the dispersion relation. The dependence upon the first of these quantities will be exhibited by collecting all the estimates for C obtained with a given model into a single table labelled by that model. For every combination of β and ω_o appearing in these tables, there will be inserted five values, denoted $C^{(i)}, i=1,\cdots,5$, resulting from the analyses of the dispersion relation with the five sets, $D^{(i)}$, of data on $D_{\pm}(\omega)$ described in the previous chapter.

The figures presented in tables 3, 4, and 5 are the values estimated for $C^{(i)}, i=1,\cdots,5$, from those dispersion relations in which only the combination of coupling constants was unknown. In the cases $\beta=0$ and $\beta=1$, the dispersion relation is subtracted, but the option of providing the subtraction constant, $D_{-}(\omega_o)$ for $\beta=0$ and $D_{-}(m_K)$ for $\beta=1$, from the unphysical region model has been selected, so that only one parameter remained to be estimated. Because the χ^2 value and the standard deviation vary slowly across any row of the tables, only averages have been quoted.

Contrary to all expectation, the estimates for $C^{(i)}, i=1,\cdots,5$, are reasonably stable against variations of β, even though such variations mix real and imaginary parts of the K^-p amplitude in the unphysical region, and emphasise for β near 0 and 1 the peaked behaviour of $B_{-}(\omega)$ around ω_o and m_K respectively. In fact the least variation occurs when ω_o is positioned directly underneath the Yo*(1405) resonance!

MODEL : J.K.KIM CONSTANT SCATTERING LENGTH ANALYSIS (MODIFIED).
****

	$C^{(i)}$	BETA=0.	BETA=.2	BETA=.4	BETA=.5	BETA=.6	BETA=.8	BETA=1.	χ^2/D.F.	STD.DEV.
ω_0 = .725	1	76	84	92	95	98	101	94	19/28	3
	2	50	65	80	87	92	99	81	25/34	32
	3	76	84	92	95	98	101	94	45/63	3
	4	-177	-191	-212	-225	-240	-280	-345	80/44	21
	5	67	76	84	88	90	94	87	337/73	3
ω_0 = .842	1	104	105	105	105	104	103	94	20/28	3
	2	103	105	106	106	106	104	81	25/34	36
	3	104	105	105	105	104	103	94	45/63	3
	4	-193	-214	-238	-251	-265	-297	-345	81/44	24
	5	96	97	98	98	97	96	87	317/73	3
ω_0 = .902	1	107	106	105	105	104	103	94	20/28	4
	2	111	111	109	108	107	104	81	25/34	39
	3	107	106	106	105	104	103	94	45/63	4
	4	-230	-248	-266	-276	-286	-298	-345	83/44	26
	5	100	99	98	98	98	96	87	315/73	3

TABLE 3.

MODEL : A.D.MARTIN & G.G.ROSS ZERO RANGE K-MATRIX ANALYSIS.
****

	$C^{(i)}$	BETA=0.	BETA=.2	BETA=.4	BETA=.5	BETA=.6	BETA=.8	BETA=1.	χ^2/D.F.	STD.DEV.
ω_0 = .725	1	49	53	57	58	59	61	54	17/28	3
	2	12	17	21	22	23	22	0	26/34	32
	3	49	53	56	58	59	60	54	45/63	3
	4	-216	-242	-275	-294	-316	-355	-425	83/44	21
	5	40	45	49	50	51	53	47	378/73	3
ω_0 = .842	1	62	62	63	63	63	62	54	17/28	3
	2	29	29	28	27	26	22	0	26/34	36
	3	62	62	62	62	62	61	54	44/63	3
	4	-273	-297	-324	-338	-344	-377	-425	85/44	24
	5	53	54	55	55	55	54	47	368/73	3
ω_0 = .902	1	64	64	63	63	63	62	54	18/28	4
	2	32	30	28	26	25	21	0	26/34	39
	3	64	64	63	63	62	61	54	44/63	4
	4	-317	-336	-346	-356	-367	-390	-425	88/44	26
	5	56	56	56	56	55	54	47	363/73	3

TABLE 4.

MODEL : J.K.KIM EFFECTIVE RANGE K-MATRIX ANALYSIS.
****

	$C^{(l)}$	BETA=0.	BETA=.2	BETA=.4	BETA=.5	BETA=.6	BETA=.8	BETA=1.	χ^2/D.F.	STD.DEV.
ω_0 = .725	1	227	228	229	228	228	225	213	38/28	3
	2	298	312	324	330	335	342	330	23/34	32
	3	228	229	230	229	229	226	214	71/63	3
	4	86	71	50	37	22	-13	-80	68/44	21
	5	222	223	224	224	223	221	209	193/73	3
ω_0 = .842	1	233	232	230	229	227	224	213	38/28	3
	2	337	341	344	346	347	347	330	23/34	36
	3	234	233	231	230	228	225	214	71/63	3
	4	57	40	20	10	-2	-29	-80	70/44	24
	5	229	227	226	224	223	220	209	191/73	3
ω_0 = .902	1	232	230	228	227	225	223	213	38/28	4
	2	349	350	350	350	350	349	330	23/34	39
	3	233	231	229	228	226	224	214	71/63	4
	4	27	13	-3	-11	-20	-38	-80	71/44	26
	5	227	226	223	222	221	218	209	194/73	3

TABLE 5.

Of the slight variations with β that are apparent in the table, probably only two are of significance. The first trend is one of rising magnitude with β, and occurs in the tables for the CSL and zero-range K-matrix models when ω_0, at $.725\, m_K$, lies just above the threshold of the $\pi\Sigma$ channel. For small values of β, the contribution to the unphysical region integral from around ω_0 is enhanced, so that, if this gradation with β can be taken seriously, these models do not accurately reproduce the K^-p amplitude as far below threshold as the entrance to the $\pi\Sigma$ channel. Although the ERK model appears superior to the other two models in this respect, it is to be noted that it is inferior in that the χ^2 value associated with the determination of $C^{(1)}$ is substantially larger for the ERK model than for the others. The second peculiarity is that in all the tables the cases with $\beta=1$ are singular. A possible cause for this is that when $\beta=1$ the subtraction point is at threshold, above the Yo*(1405) in energy, whereas ω_0, the point of enhancement if $0<\beta<1$ and of subtraction if $\beta=0$, has its values spread over the lower side of the Yo*(1405) resonance.

It is difficult to known how much importance to attach to these variations with β once the sizeable discrepancies between the estimates $C^{(i)}$, $i=1,\cdots,5$, have been noted. This failing is worst in the effective-range K-matrix table where $C^{(2)}$ exceeds $C^{(1)}$ by about 110; for the zero-range K-matrix these quantities differ by about 35 in the other direction. Such a discrepancy is not found with the CSL model. Tempting though it is to conclude on this basis that the CSL model provides a superior extrapolation of the K^-p amplitude into the unphysical region, it is probably not correct to do so. Comparison of $C^{(4)}$ and $C^{(2)}$, respectively obtained when the 'raw' and corrected D_- data were employed in the dispersion relation, shows the startling effect of Martin and Perrin's rather haphazard corrections to the D_- data, and suggests that the discrepancy between $C^{(1)}$ and $C^{(2)}$ might only be a reflection of the poor quality of this data. It appears, however, that were substantially improved D_- data available, the discrepancy between the values of C obtained from the D_+ and the D_- data separately would provide a critical test of the reliability of the unphysical region model used.

Martin and Perrin have done well to attempt to include Coulomb corrections in the data for $D_-(\omega)$. The values $C^{(4)}$ obtained when the uncorrected D_- data is analysed in the dispersion relations are meaningless, and so no further consideration will be given $C^{(4)}$ and $C^{(5)}$. The set of corrected D_- data leads to more reasonable values, $C^{(2)}$, for the coupling constant combination, but, when compared with $C^{(1)}$, they are still statistically useless.

Typical values for $C^{(3)}$ in tables 3, 4 and 5 for the three models are

CSL	100
ZRK	62
ERK	230

The first two of these are in general agreement with those obtained by other authors, but the third is substantially higher. No remark on this will be passed here, for later it will be seen that the same data do reproduce the results of Perrin and Woolcock (25) when used in the dispersion relation corresponding to theirs, namely, that characterised by $\beta = 0$ and $\omega_0 = 0$.

As explained earlier, the subtracted dispersion relations represented in table 3, 4 and 5, namely, those with either $\beta = 0$ or $\beta = 1$, had the subtraction constant supplied from the unphysical region model. In tables 6, 7 and 8, these relations and the additional one with $\omega_0 = 0$ are used to fit the data, with both the subtraction constant $D_-(\omega_0)$ and the coupling constant combination

$C = g_\Lambda^2 + G(0,\omega_0) g_\Sigma^2$ as free parameters. For easy reference, the values of $D_-(\omega_0)$ obtained from the model used for the unphysical region amplitude are listed across the tops of the tables; $D_-(0)$ is not included because it is not available. The error matrix quoted represents an average value for the cases with $\omega_0 > 0$; the matrix for the case $\omega_0 = 0$ has entries of roughly the same magnitude, but the sign of its off-diagonal elements is negative.

To illustrate the main feature of the six tables combined, we collate a subset of the values of $C^{(1)}$ - these because we believe the data on $D_+(\omega)$ to be reliable - computed with the CSL model amplitude in the unphysical region.

ω_0	$\longrightarrow$	.725	.842	.902	1.0
$D_-(\omega_0)$ FROM CSL MODEL	$\longrightarrow$	2.89	3.96	4.72	-4.16
$C^{(1)}$ FITTED.	$C^{(1)}$	76	104	107	94
	χ^2	17.9	20.0	20.7	19.4
	d.f.	28	28	28	28
$C^{(1)}$ & $D_-^{(1)}(\omega_0)$ FITTED.	$C^{(1)}$	22	22	22	22
	$D_-^{(1)}(\omega_0)$	2.32	3.20	3.98	-4.74
	χ^2	16.1	16.1	16.1	16.1
	d.f.	27	27	27	27

If the CSL model truly represented the K^-p amplitude, and if, as we have assumed, the data on $D_+(\omega)$ is accurate, the input value to the dispersion relation of $D_-(\omega_0)$ from the CSL model would agree with the output estimate for $D_-(\omega_0)$, regarded as a parameter, and hence $C^{(1)}$ and the ratio of χ^2 to the number of degrees of freedom would not change when the fitting process is allowed an extra degree of freedom. This, however, is clearly not the case. When two parameters are

MODEL : J.K.KIM CONSTANT SCATTERING LENGTH ANALYSIS (MODIFIED).
****

	ω_0 →	0.0	0.725	0.842	0.902	1.0			
$D_-(\omega_0)$ FROM THE MODEL →			2.89	3.96	4.72	-4.16			
$D_\pm$ DATA SET $D^{(i)}$	PARAMETERS ESTIMATED.						χ^2/D.F.	ERROR MATRIX ($\omega_0 \neq 0.0$)	
$D^{(1)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-1.73 27	2.32 22	3.20 22	3.98 22	-4.74 22	16/27	0.13	14 1570
$D^{(2)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-36.95 930	7.81 930	7.50 933	7.78 929	-1.56 936	19/33	2.04	468 113000
$D^{(3)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-3.85 88	3.01 87	3.81 87	4.55 87	-4.22 87	45/62	0.04	4 488
$D^{(4)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-30.30 780	7.71 779	7.57 775	7.96 776	-1.29 775	35/43	0.42	106 29100
$D^{(5)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-10.21 273	4.98 273	5.53 272	6.17 272	-2.72 273	105/72	0.02	2 206

TABLE 6.

MODEL : A.D.MARTIN & G.G.ROSS ZERO RANGE K-MATRIX ANALYSIS.
****

	$\omega_0 \rightarrow$	0.0	0.725	0.842	0.902	1.0			
$D_-(\omega_0)$ FROM THE MODEL $\rightarrow$			2.34	3.76	5.12	-4.38			
$D_\pm$ DATA SET $D^{(i)}$	PARAMETERS ESTIMATED.						χ^2/D.F.	ERROR MATRIX ($\omega_0 \neq 0.0$)	
$D^{(1)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0) g_\Sigma^2$	-1.54 19	1.98 15	3.32 15	4.68 15	-4.70 15	16/27	0.13	14 1570
$D^{(2)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0) g_\Sigma^2$	-35.90 900	7.30 900	7.48 903	8.36 900	-1.62 906	19/33	2.04	468 113000
$D^{(3)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0) g_\Sigma^2$	-3.48 76	2.61 74	3.88 75	5.21 74	-4.22 75	44/62	0.04	4 488
$D^{(4)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0) g_\Sigma^2$	-28.99 743	7.17 742	7.53 738	8.52 740	-1.38 738	35/43	0.42	106 29100
$D^{(5)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0) g_\Sigma^2$	-9.73 257	4.55 257	5.57 257	6.80 257	-2.75 257	101/72	0.02	2 206

TABLE 7.

MODEL : J.K.KIM EFFECTIVE RANGE K-MATRIX ANALYSIS.
****

ω_0 →		0.0	0.725	0.842	0.902	1.0			
$D_-(\omega_0)$ FROM THE MODEL →			5.78	6.41	4.74	-4.16			
$D_\pm$ DATA SET $D^{(i)}$	PARAMETERS ESTIMATED.						χ^2/D.F.	ERROR MATRIX ($\omega_0 \neq 0.0$)	
$D^{(1)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-2.20 44	3.80 40	4.62 40	3.07 40	-5.56 40	16/27	0.13	14 1570
$D^{(2)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-40.22 1020	9.82 1020	9.34 1030	7.26 1020	-2.04 1030	19/33	2.04	468 113000
$D^{(3)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-4.89 122	4.67 121	5.37 121	3.78 121	-4.90 121	48/62	0.04	4 488
$D^{(4)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-34.68 900	9.87 898	9.52 894	7.53 896	-1.70 895	35/43	0.42	106 29100
$D^{(5)}$	$D_-(\omega_0)$ $g_\Lambda^2 + G(\beta,\omega_0)g_\Sigma^2$	-11.57 317	6.74 317	7.18 316	5.49 316	-3.33 316	117/72	0.02	2 206

TABLE 8.

estimated instead of one, the quality of the fit, evidenced by the ratio of χ^2 to the number of degrees of freedom, is improved, the estimate for $C^{(1)}$ shifts from about 100 to near 20 and $D_-^{(1)}(\omega)$ adjusts accordingly. It is therefore quite clear that the CSL model is inadequate.

Similar remarks apply to the ZRK and ERK models, especially the latter for which the ratio of χ^2 to the number of degrees of freedom improves from 38/28 to 16/27 with the extra degree of freedom. To conclude, none of the present partial wave analyses of low energy $\bar{K}N$ data is reliable when extrapolated below threshold, but certainly the CSL and ZRK models are superior to the ERK model.

In the light of these remarks, any determination of the couplings g_Λ^2 and g_Σ^2 must be regarded as suspect if data on $D_\pm(\omega)$ for the dispersion relation were extracted from the model for the $\bar{K}N$ amplitude below or near threshold or if the dispersion relation were strongly dependent upon the unphysical region amplitude. The relation used by Restignoli and Violini (26) certainly minimises the importance of the unphysical region integral, but fails the first condition rather badly. Perrin and Woolcock (23,25) and Martin and Perrin (27) seem to be the only authors whose work passes the first of these tests, since, in both cases, all the data available from the experimental reports on $D_\pm(\omega)$ were used. However, the D_+ and D_- data seem always to have been combined, and in so doing an important fact was missed.

A typical estimate for $C^{(1)}$ in tables 6, 7 and 8 is 30 ± 40, whereas that for $C^{(2)}$ is 900 ± 300, so the two are quite incompatible. The first figure is far lower and the second far higher than the generally accepted value for the couplings, and the question is whether the discrepancy between them arises from the model or from the $D_\pm$ data. That the discrepancy should be qualitatively the same for all the models suggests that the cause lies in the $\mathcal{D}_{\pm}$ data, but, since the models are known to be inadequate from the arguments above, a definite conclusion cannot be drawn.

The fits to the combination of couplings and the subtraction constant are statistically excellent when data sets $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ are used separately, but the two estimates $C^{(1)}$ and $C^{(2)}$ thereby obtained are incompatible. Thus, it might be expected that $C^{(3)}$, the estimate obtained with the union of these sets of data, would be statistically insignificant. In practice this is not so, and the fit to $C^{(3)}$ is almost as good as those to $C^{(1)}$ and $C^{(2)}$. The reason why this is the case, as a glance at the error matrix will show, is that the two parameters estimated are highly correlated - the error matrix is almost singular. This reinforces the earlier conclusion that the extrapolations of the $\bar{K}N$ amplitude below threshold are inadequate; the earlier argument hinged upon the improvement of the fit with an extra parameter, whereas here the estimate for this parameter is found to be strongly correlated to that for the first parameter.

It is amazing that the fits to $C^{(1)}$ and $C^{(2)}$ and the associated error matrices should be such that the estimate $C^{(3)}$, based upon all the data on $\mathcal{D}_{\pm}$ tabulated by Martin and Perrin, reproduces the results of Perrin and Woolcock. For the several models, the values of $C^{(3)}$ are

CSL	87 ± 22
ZRK	75 ± 22
ERK	121 ± 22 .

The $\mathcal{D}_{-}$ data in set $\mathcal{D}^{(2)}$, which seemed statistically useless in the analysis of the unsubtracted dispersion relations, is here seen to be crucial to a sane estimate for the coupling constant combination. Since the $\mathcal{D}_{-}$ data could certainly still be in error - the striking effect of Martin and Perrin's corrections to the raw $\mathcal{D}_{-}$ data suggests this - the method of Perrin and Woolcock for determining the usual combination of the couplings cannot be applied with confidence until the reliability of the D_- data has been tested. Finally we note that the input - output discrepancy observed by Perrin and Woolcock was but the residual of a far larger discrepancy which they would have noticed had they analysed the D_+ and D_- data separately.

To summarise the observations of this chapter, we list the points for and against the various models. Since the CSL and ZRK models lead to almost indistinguishable results, we need only contrast the ERK model with either of these.

(1) In favour of the ERK model (as against the CSL and ZRK models), there is the least variation in $C^{(3)}$ with variations in β and ω_0 .

(2) Against the ERK model, we note that:

(i) when only one parameter is estimated, a much better χ^2 value is found for the determination of $C^{(3)}$ with each of the CSL and the ZRK models;

(ii) only for the ERK model is there a substantial improvement in the value of χ^2 when a second parameter is fitted; and

(iii) when data set $D^{(3)}$ is used, the input-output discrepancy for the ERK model is much larger than for the CSL and the ZRK models, and the value of $C^{(3)}$ is very different from that for the one-parameter fits.

Any inferences based upon these observations must, of course, be very weak. However, with better data on $D(\omega)$, they would almost certainly be strengthened.

3.2 Effect of the Y_1*(1385) Resonance

All the models used in this work have included only the s-wave amplitude, and hence have neglected the contribution due to the Y_1*(1385) resonance which occurs in the I=1, $p_{3/2}$ channel.

This resonance is most simply treated in the zero-width approximation. The Rarita-Schwinger (42) formalism with the Lagrangian

$$(g_{Y^*}/2m_p)(\partial_\mu \bar{\Psi}_p \phi_K - \bar{\Psi}_p \partial_\mu \phi_K)\gamma_5 \underline{\tau} \cdot \underline{\Psi}^\mu_{Y^*} + \text{h.c.},$$

in which g_{Y^*} is the rationalised $K^-p\ Y_1^*$ coupling constant, leads in Born approximation to the contribution

$$A_-(\omega) = -g^2_{Y^*} \frac{(\omega^2_{Y^*} - m_K^2)}{(24\, m_p^4)} (m_K^2 - (m_{Y^*} + m_p)^2)\, \delta(\omega - \omega_{Y^*})$$

to the imaginary part of the K^-p amplitude in the laboratory frame. $g^2_{Y^*}$ may be computed from the observed width of the N*(1236), to which it is related by SU(3) symmetry. According to Bunnell et al (43), this symmetry is only slightly, if at all, broken.

Because of the sign of the residue at the $Y_1^*(1385)$ pole, the estimates for $C = g_\Lambda^2 + G(\beta,\omega_0)\, g_\Sigma^2$ computed with only the s-wave amplitude in the unphysical region are actually upper bounds for C. To approximate the effect of the $Y_1^*(1385)$, Perrin and Woolcock used the zero-width approximation discussed above, and found that their estimates of $g_\Lambda^2 + 0{\cdot}847\, g_\Sigma^2$ were reduced by about 20. Had the $Y_1^*(1385)$ been included in this work, there would have been a reduction of comparable magnitude in the estimates from the subtracted dispersion relations evaluated with all the data on $D_\pm(\omega)$ compiled by Martin and Perrin.

3.3 Comparison with Symmetry Predictions

SU(3) symmetry predicts (39,40) that the $K^-p\Lambda$ and $K^-p\Sigma^0$ coupling constants are related by

$$g^2_{K^-p\Lambda} = (1+2\alpha)^2 \, g^2_{\pi^-pn} / 6$$

$$g^2_{K^-p\Sigma^0} = (1-2\alpha)^2 \, g^2_{\pi^-pn} / 2 \qquad ,$$

where $g^2_{\pi^-pn} = 370 \pm 8$ is the usual rationalised, pseudoscalar coupling constant and the parameter $\alpha = f/(f+d)$ is the fraction of F-type coupling. The combination $C = g_\Lambda^2 + G(\beta, \omega_0)\, g_\Sigma^2$ is therefore a quadratic function of α and, if 1.15 is taken as a typical value of $G(\beta, \omega_0)$, has a minimum of 192 when $\alpha = \cdot 276$. ~~.276~~ Thus, the estimates for C obtained from the CSL and zero-range K-matrix models are definitely incompatible with the SU(3) prediction. With the effective-range model and a one-parameter fit, agreement with SU(3) is secured, but, since consistency tests discriminate against this model and since χ^2 is poor for this fit, this is of little consequence.

The more restrictive symmetry of the SU(6) group fixes the parameter α at .4 (41) so that $g_\Lambda^2 = 200$ and $g_\Sigma^2 = 8$. Once again the only model consistent with estimates so large is the effective-range K-matrix model.

The evidence from the dispersion relations is that the SU(3) and SU(6) symmetries are strongly broken when applied to the meson-baryon coupling constants. This conclusion is re-inforced by the observation of section 3.2 that what we have calculated is an upper bound on C. If the $Y_1^*(1385)$ contribution reduces the calculated values of C by around 20, then SU(3) symmetry for the coupling constants is even more obviously broken.

3.4 Concluding Remarks

The unsubtracted dispersion relations for the function

$$B_{\pm}(\omega) = F_{\pm}(\omega)/\left[(\omega \pm m_K)^{\beta}(\omega \pm \omega_0)^{1-\beta}\right], \quad 0<\beta<1$$

fail to discriminate between the several models for the unphysical region amplitude on the basis of the stability of the predictions for the coupling constants against variations of β and ω_0. This failure is thought to be partly due to the poor quality of the available data on $D_-(\omega)$; with greatly improved D_- data (comparable in accuracy with the D_+ data), these dispersion relations for $B_{\pm}(\omega)$ would almost certainly provide a searching test. However, even with the present data on $D_-(\omega)$, the subtracted relations, that is, those with $\beta = 0$ do provide a sensitive test. With the assumption that the data on $D_+(\omega)$ are accurate, this test indicates that the constant scattering length and the zero and effective-range K-matrix analyses are all inadequate, but that the first two are superior to the last.

The reliability of the D_- data is in serious doubt, and, until better data on D_- is available, methods such as those used by Martin and Perrin and by Perrin and Woolcock (and improved and extended in this present work) cannot be applied with confidence.

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