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Recursive generation of simple planar 5-regular graphs and pentangulations

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Abstract

We describe how the 5-regular simple planar graphs can all be obtained from an elementary family of starting graphs by repeatedly applying a few local expansion operations. The proof uses an amalgam of theory and computation. By incorporating the recursion into the canonical construction path method of isomorph rejection, a generator of non-isomorphic embedded 5-regular planar graphs is obtained with time complexity $O(n^2)$ per isomorphism class. A similar result is obtained for simple planar pentangulations with minimum degree 2.

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1 Introduction

By a *planar graph* we mean a graph embedded in the sphere without edge crossings. We do not distinguish an outer face.

On account of Euler's formula, a simple planar graph has average degree less than 6, and therefore a regular simple planar graph can have degree at most 5. However, although much is known about the structure of 3-regular and 4-regular simple planar graphs, little is known about the 5-regular case.

Similarly, there is a large literature on planar graphs with every face of size 3 (that is, each face bounded by 3 edges), or every face of size 4, but very little for every face of size 5. The latter are *planar pentangulations*.

Our aim in this paper is to present recursive constructions of all connected 5-regular simple planar graphs, and all connected simple planar pentangulations without vertices of degree 1.

By a CSPG5 we mean a connected 5-regular simple planar graph. The dual of a CSPG5 is a connected planar graph of minimum degree at least 3, with each face of size 5, having the additional property that no two faces share more than one edge of their boundaries.

Since the dual of a CSPG5 is not necessarily simple (and vice-versa), we also consider the construction of simple planar pentangulations. Technically, these can have vertices of degree 1, but we will not consider that case. By an *SP2* we mean a simple planar pentangulation with no vertices of degree 1. Such graphs are necessarily 2-connected.

Two planar graphs are regarded as the same if there is an embedding-preserving isomorphism (possibly reflectional) between them. That is, we are not concerned with abstract graph isomorphisms.

Let $\mathcal{C}$ be a class of planar graphs, $\mathcal{S}$ a subset of $\mathcal{C}$, and $\mathcal{F}$ a set of mappings from subsets of $\mathcal{C}$ to the power set $2^{\mathcal{C}}$. We say that $(\mathcal{S},\mathcal{F})$ *recursively generates* $\mathcal{C}$ if for every $G \in \mathcal{C}$ there is a sequence $G_1, G_2, \ldots, G_k = G$ in $\mathcal{C}$ where $G_1 \in \mathcal{S}$ and, for each i, $G_{i+1} \in F(G_i)$ for some $F \in \mathcal{F}$. In many practical examples including that in this paper, there is some nonnegative integral graph parameter (such as the number of vertices) which is always increased by mappings in $\mathcal{F}$, and $\mathcal{S}$ consists of those graphs which are not in the range of any $F \in \mathcal{F}$. In this case, we refer to the elements of $\mathcal{F}$ as *expansions*, their inverses as *reductions*, and the graphs in $\mathcal{S}$ as *irreducible*. In this circumstance, $(\mathcal{S},\mathcal{F})$ recursively generates $\mathcal{C}$ if every graph in $\mathcal{C} - \mathcal{S}$ is reducible.

Recursive generation algorithms for many classes of planar graphs have appeared in the literature. Expansions usually take the form of replacing some small subgraph by a larger subgraph. We mention the examples of 3-connected [13], 3-regular [5], minimum degree 4 [1], 4-regular [4, 11], minimum degree 5 [2], and fullerenes (whose duals have minimum degree 5 and maximum degree 6) [7]. Such construction theorems can be used to prove properties of graph classes by induction as well as to produce actual generators for practical use. Conspicuously missing from this list are the classes of 5-regular planar graphs or planar pentangulations, which are somewhat harder than the others. In this paper we will fill these gaps for simple graphs.

We are not aware of any previous results on generation of simple planar pentangulations. For CSPG5s, the best previous work is that of Kanno and Kriesell [10]. They define two families of reduction and show that they suffice to reduce any CSPG5 with connectivity less than 3, and any CSPG5 with an edge lying on three 3-cycles.

2 Planar 5-regular graphs

The numbers of isomorphism types of CSPG5s of order up to 36 appear in Table 1.

		vertex connectivity					
vertices	faces	1	2	3	4	5	total
12	20					1	1
14	23						0
16	26					1	1
18	29					1	1
20	32				1	5	6
22	35				1	13	14
24	38		2	3	15	78	98
26	41		11	24	76	418	529
28	44	5	113	252	711	2954	4035
30	47	53	1135	2562	5717	21542	31009
32	50	573	11383	24965	49935	165530	252386
34	53	5780	110607	236101	429835	1291446	2073769
36	56	55921	1054596	2187742	3726718	10252136	17277113

Table 1: Counts of connected 5-regular simple planar graphs of small order

Our starting set $\mathcal{S}$ consists of the 5 graphs M, C, J, T, B and the infinite family $\{D_i \mid i \geq 1\}$ described in Figure 1.

Our main result employs 6 expansions, each of which involve replacing a small subgraph by a larger subgraph. We define them via their corresponding reductions, with the help of Figure 2. In order for the operations shown in the figure to be reductions, they must obey the following requirements.

1. The 5-regular graph resulting from the operation must be connected and simple. (Example: for reduction B, vertices D, I cannot be equal or adjacent before the reduction, since that would imply a loop or double edge after the reduction.)
2. The vertices shown in Figure 2 with uppercase names $A, B, \ldots$ need not be distinct except as necessary for requirement 1. However their cyclic order around the vertices shown with lowercase names $u, v, \ldots$ must be as illustrated.
3. Before reduction B, vertices w, x must not be adjacent. Before reductions C_1, C_2, or C_3, vertices u, w must not be adjacent.

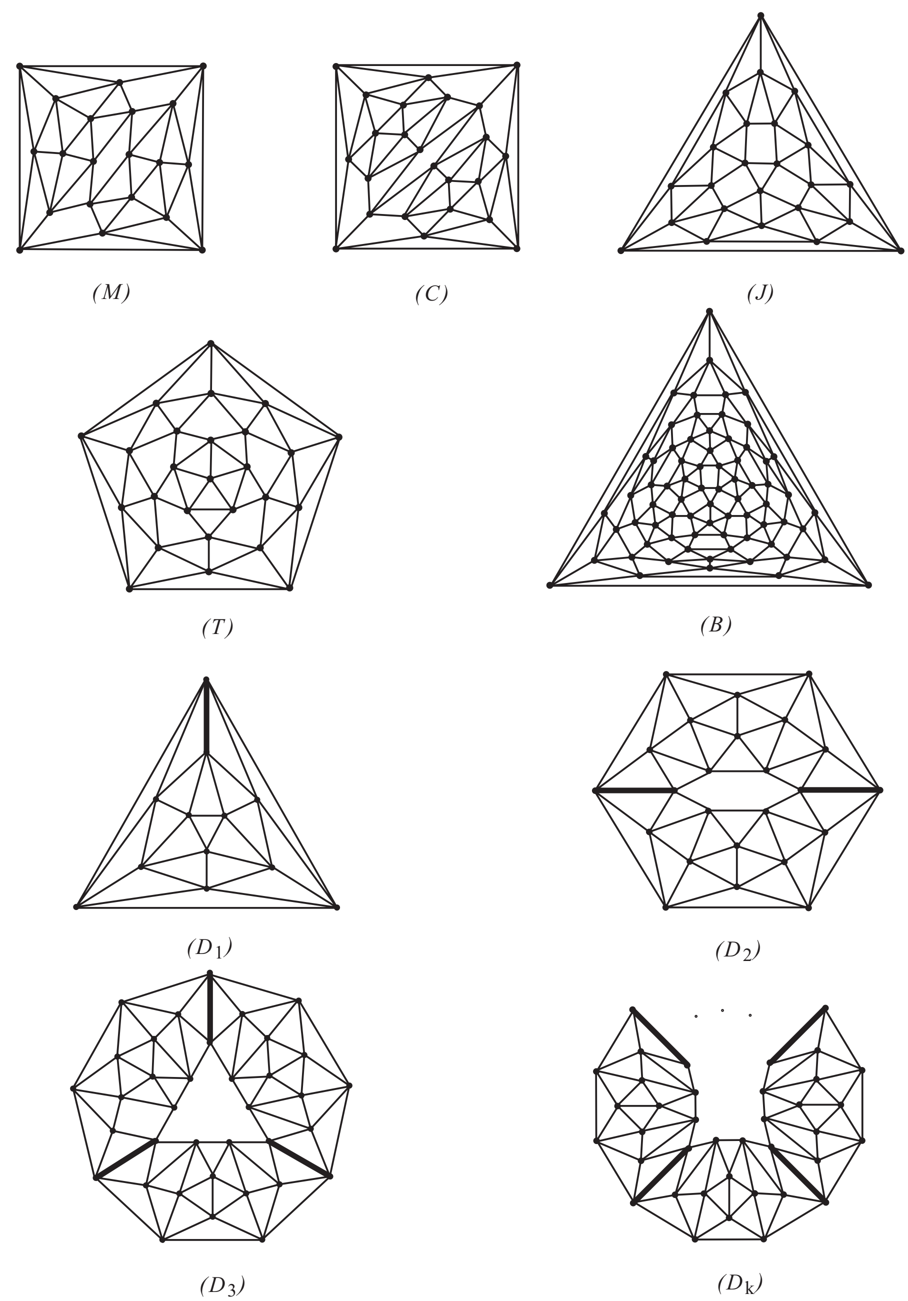

Figure 1: Irreducible graphs, including the infinite sequence $D_1, D_2, \ldots$

Since we do not distinguish between a graph and its mirror image, the mirror image of a reduction is also a reduction. (For example, the mirror image of A_2 inserts edges CD, BE, AF, HG.) Note that our A_1 and A_2 reductions are special cases of what Kanno and Kriesell called a D-reduction. Let $\mathcal{F}$ be the set of expansions inverse to the reductions $\{A_1, A_2, B, C_1, C_2, C_3\}$ shown in Figure 2. We can now state our main result.

Theorem 1 *The class of all CSPG5s is generated by* $(\mathcal{S}, \mathcal{F})$.

To prove the theorem, we need to show that every CSPG5 not in $\mathcal{S}$ is reducible by one of the reductions $\mathcal{R} = \{A_1, A_2, B, C_1, C_2, C_3\}$. We abbreviate this to "$\mathcal{R}$-reducible". The structure of the remainder of this section is as follows. In Section 2.1 we show that CSPG5s with a cut-vertex are $\mathcal{R}$-reducible, and in the following section we do the same for graphs of connectivity 2, partly with computer assistance. In Section 2.3, we first show that 3-connected CSPG5s with a separating 3-cycle are $\mathcal{R}$-reducible. Then we show that 3-connected CSPG5s not in $\{M, C, J, T, B, D_1, D_2\}$ are $\mathcal{R}'$-reducible, where $\mathcal{R}'$ is $\mathcal{R}$ with an additional reduction D added. Finally, we show that the extra reduction D is unnecessary if D_i $(i \geq 3)$ are added to the starting set. This will complete the proof of Theorem 1.

2.1 Cut-vertices

In this section we consider the case that the graph has a cut-vertex. Reductions will be specified according to the labelling in Figure 2. In the case of A and B reductions, we can also specify the mirror image with a notation like, for example, $A_2^R(v, w)$.

Lemma 1 *Every CSPG5 with a cut-vertex is* $\mathcal{R}$*-reducible.*

Proof: Take such a cut-vertex v incident with an end-block. Three cases can occur, as illustrated in Figure 3(a–c), where the end-block is drawn on the left. In case (a), reduction $A_1(v, w)$ applies. (Multiple edges are impossible, and the reduced graph is connected because end-blocks are connected. We will generally omit such detail in our description.)

In case (c), either reduction $C_1(u, v, w)$ or $C_1(x, v, y)$ applies unless the situation in Figure 3(c$_1$) occurs. However, in this case $C_1(y, v, x)$ applies. In case (b), either reduction $C_1(u, v, w)$ or $C_1(x, v, y)$ applies. A situation mirror to that in Figure 3(c$_1$) cannot occur since the left side of v is an end-block. $\square$

2.2 2-vertex Cuts

In this section we show that 2-connected CSPG5s with a 2-cut are $\mathcal{R}$-reducible. We divide 2-cuts according to whether the two vertices are adjacent or not, but first we take care of the special cases of when there is a cut consisting of two edges or an edge and a vertex.

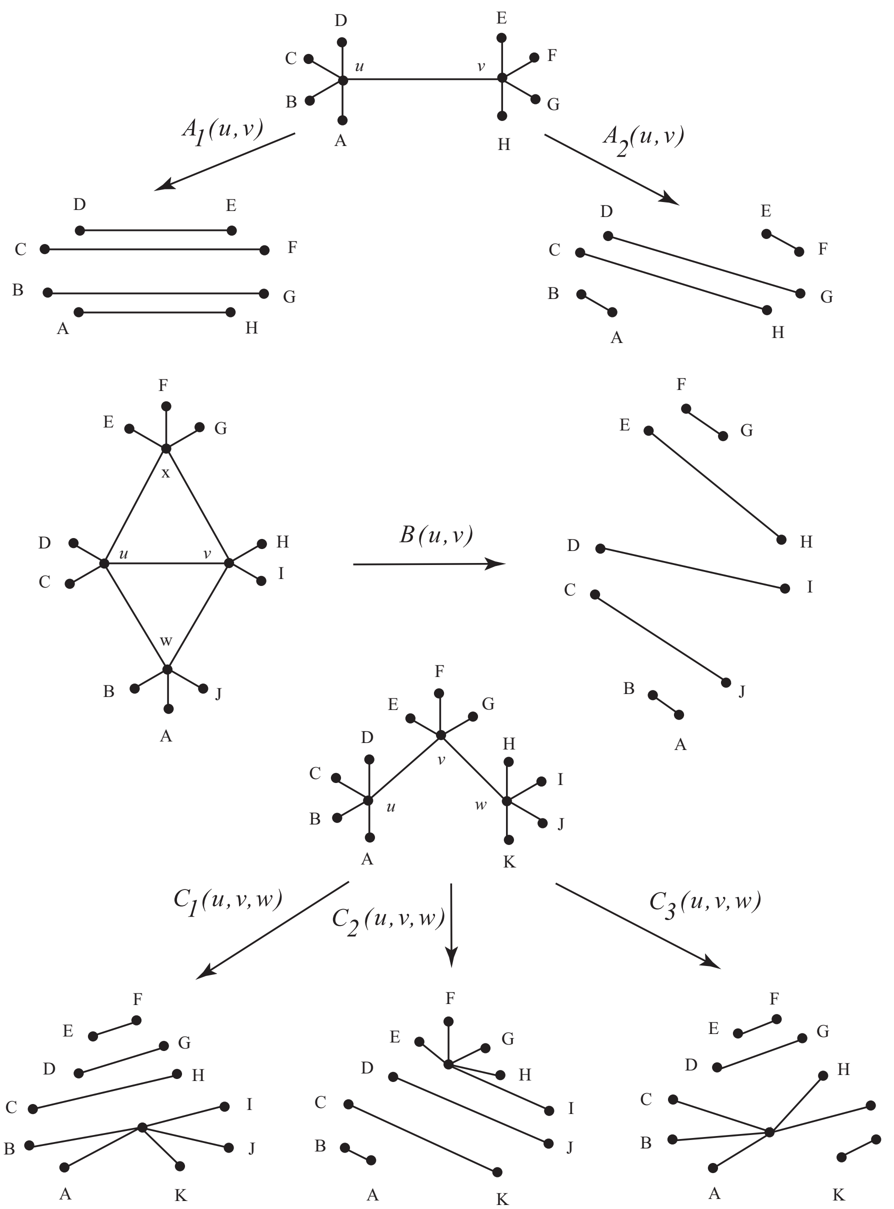

Figure 2: Reductions A_1, A_2, B, C_1, C_2 and C_3 (subject to rules 1–4)

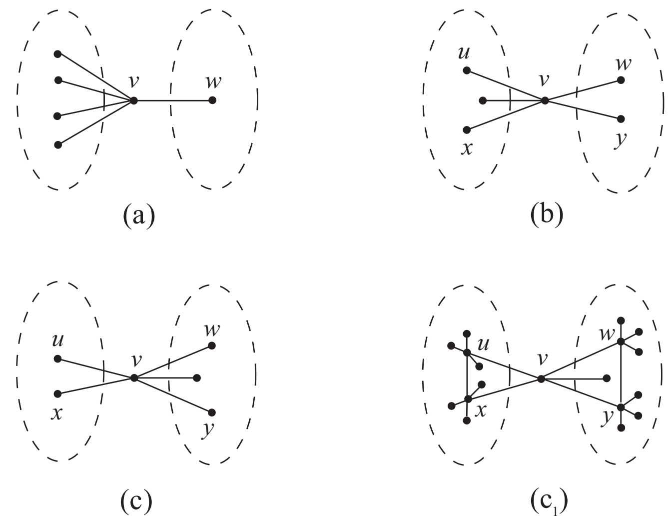

Figure 3: Possible cases for a 1-cut

Lemma 2 *Every 2-connected CSPG5 with a cut consisting of two edges or an edge and a vertex is $\mathcal{R}$-reducible.*

Proof: In the case of cuts of two edges (Figure 4(a)), we can apply $A_1(x, y)$ unless $x_i = w$ and $y_i = z$ for some i. If $x_1 = w$ and $y_1 = z$, $C_2(w, x, y)$ can be applied instead. If $x_2 = w$ and $y_2 = z$, either $A_2^R(w, x)$ or $C_3(x_1, x, w)$ can be applied. The other two cases are equivalent to these.

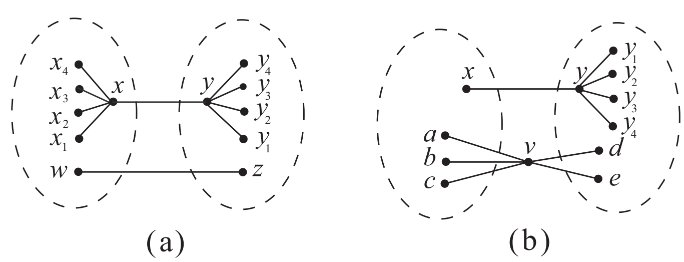

Figure 4: Cuts of two edges or one edge and a vertex

Now consider the case of a cut consisting of an edge and a vertex (Figure 4(b)). If $y \neq d$ and $y \neq e$, we consider whether x is the same as a, b or c. The case $x = b$ implies a two-edge cut, which is treated above, and $x = c$ is equivalent to $x = a$. If $x = a$, apply $C_1(e, v, c)$, while if $x \neq a, b, c$ apply $A_1(x, y)$. If $y = d$, 2-connectivity implies that $y_4 = v$, then if also $y_1 = e$ and $x = c$, the reduction $C_2(e, y, x)$ applies; otherwise $A_2(v, y)$ applies. The case of

$y = e$ is equivalent to $y = d$. □

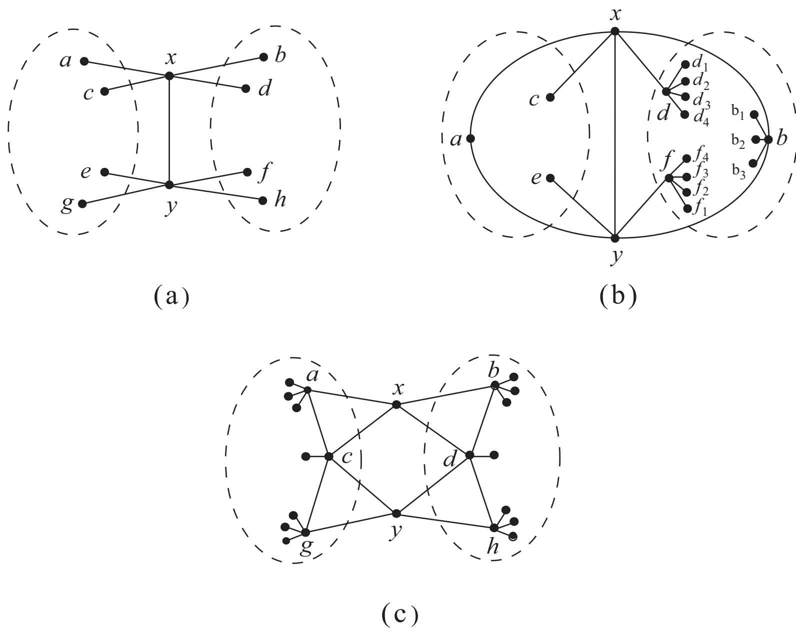

Figure 5: Remaining case of cuts of 2 adjacent vertices

Lemma 3 *Every 2-connected CSPG5 with a cut of two adjacent vertices is $\mathcal{R}$-reducible.*

Proof: The only possibility not covered by Lemma 2 is shown in Figure 5(a). We start by noting that $b = f$, $d = h$, $a = e$ or $c = g$ imply a 2-cut covered by Lemma 2. If we do not have $a = g$ and $h = b$, or $d = f$ and $c = e$, then either $C_1(a, x, b)$ or $C_1(b, x, a)$ apply. We now divide the argument into the case when $a = g$ and $h = b$ and the case when $d = f$ and $c = e$, in which cases it is possible that $C_1(a, x, b)$ and $C_1(b, x, a)$ create multiple edges.

If $a = g$, $h = b$, $d = f$ and $c = e$, either $A_2(c, x)$ or $A_2(c, y)$ applies. If $a = g$, $h = b$, $c \neq e$ and $d = f$, $C_3(a, y, b)$ or $C_3(a, x, b)$ applies.

Suppose $a = g$, $h = b$, $c \neq e$ and $d \neq f$, as shown in Figure 5(b). If $f_4 \neq d$, $C_1(f, y, x)$ applies. If $f_4 = d$ and $d_1 \neq b$, $C_3(f, y, x)$ applies, whereas if $f_4 = d$ and $f_1 \neq b$, $C_3(d, y, x)$ applies. Therefore, from now on we assume that $f_4 = d$ and $d_1 = f_1 = b$. If $d_2 \neq b_2$, $A_2(f, b)$ applies, whereas if $f_2 \neq b_2$, $A_2(d, b)$ applies. In the case that $d_2 = f_2 = b_2$, if f_3 is not d_3 then $A_2(y, f)$ applies and otherwise $B(f, b)$ applies.

If $a \neq g$, $h = b$, $c = e$ and $d = f$, one of $A_2^R(x, d)$ and $A_2^R(y, d)$ applies.

The remaining case is that $a \neq g$, $h \neq b$, $c = e$ and $d = f$. One of $A_2(y, d)$, $A_2(x, d)$, $A_2(y, c)$ and $A_2(x, c)$ applies unless we have the situation

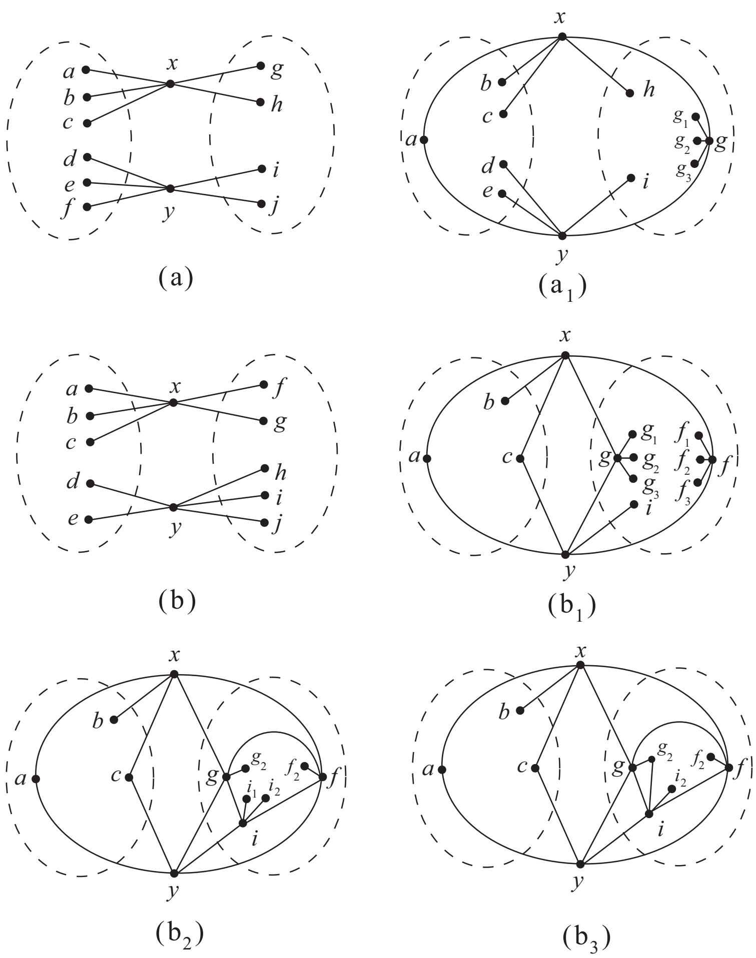

Figure 6: Remaining cases of cuts of 2 non-adjacent vertices

that appears in Figure 5(c). Removal of the cut $\{x, y\}$ clearly results in exactly 2 components, so assume that (i) no other types of adjacent 2-cuts are present (since we already considered them above), and (ii) the component to the right of $\{x, y\}$ is a smallest of the components resulting from a cut of 2 adjacent vertices.

Although this case can be completed by hand, it is time-consuming and complicated so a computer program was employed. The initial configuration shown in the figure was expanded one vertex at a time. At each step, the program had an induced subgraph, some of whose vertices had additional edges whose other endpoint was not yet constructed. Such "incomplete edges" are distinct (since the subgraph is induced) but their endpoints might coincide. Expanding the subgraph consisted of choosing an incomplete edge (choosing the oldest on the right side of the cut proved a good heuristic), adding a new vertex to its incomplete end, then deciding all the additional adjacencies of the new vertex to the previous induced subgraph. Those sets of adjacencies that implied a 1-cut, a 2-cut of two adjacent vertices to the right of $\{x, y\}$, or a reduction in $\mathcal{R}$, were rejected. This expansion process finished after less than one second, never making an induced subgraph larger than 18 vertices. This completes the proof of the lemma. □

Lemma 4 *Every 2-connected CSPG5 with a cut of two non-adjacent vertices is $\mathcal{R}$-reducible.*

Proof: The two cases not covered by Lemma 2 are shown in Figure 6(a,b).

Consider case (a) first. If $g \neq j$ or $a \neq f$, then $C_1(g, x, a)$ applies. If $g = j$ and $a = f$, we have the situation of Figure 6(a$_1$). If $g_3 \neq i$, $A_2(x, g)$ applies, whereas if $g_1 \neq h$, $A_2(y, g)$ applies. If $g_3 = i$ and $g_1 = h$, then $C_1(h, x, c)$ applies.

Now consider case (b). If $a \neq e$ or $f \neq j$, $C_1(f, x, a)$ applies, while if $a = e$, $f = j$ and either $c \neq d$ or $g \neq h$, $C_1(g, x, c)$ applies. This leaves the case that $a = e$, $f = j$, $c = d$ and $g = h$, as in Figure 6(b$_1$). In that case we find that $C_3(c, y, g)$ applies if $g_1 \neq f$, $A_2(x, f)$ applies if $g_1 = f$ and $i \neq f_3$, and $A_2^R(x, g)$ applies if $g_1 = f$, $i = f_3$ and $g_3 \neq i$. The remaining situation is as shown in Figure 6(b$_2$). We find that $A_2(y, i)$ applies if $g_2 \neq i_1$. If $g_2 = i_1$ then $i_2 \neq f_2$, since otherwise $\{g_2, f_2\}$ would be a type of 2-cut that was already considered. Therefore we can apply $C_1(g, f, f_2)$ if g_2 and f_2 are not adjacent, and $A_2(i, g_2)$ if they are adjacent. □

2.3 Completion of the Proof

A *separating k-cycle* is a k-cycle which is not the boundary of a face.

Lemma 5 *Every 3-connected CSPG5 with a separating 3-cycle is $\mathcal{R}$-reducible.*

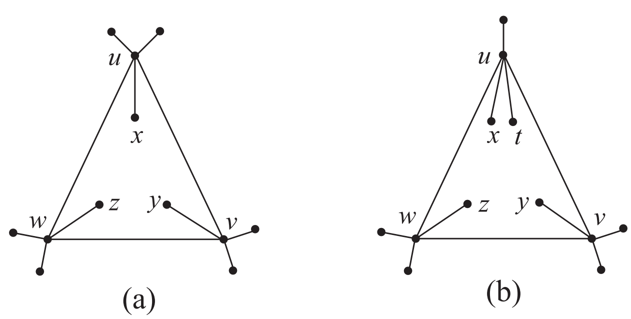

Figure 7: Cases for a separating 3-cycle

Proof: By the symmetry between the inside and outside of the 3-cycle, two cases can occur as shown in Figure 7. In case (a), 3-connectivity requires x, y, z to be distinct, so $C_2(v, u, x)$ applies (if the reduced graph is disconnected then

x is a cut-vertex). In case (b), 3-connectivity requires $y \neq z$. If $x \neq z$ and $t \neq y$, $C_2(w, v, y)$ applies, while if $x = z$ we must have $t \neq y$ by connectivity, so $A_2^R(u, x)$ applies. □

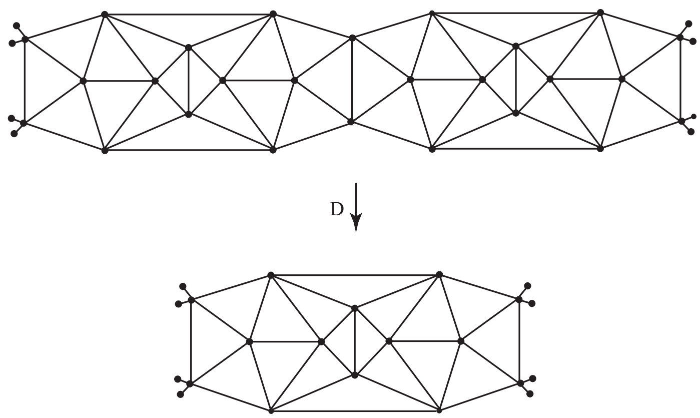

Figure 8: The reduction D

To complete the proof of Theorem 1 we will first show that 3-connected CSPG5s without separating 3-cycles, other than a finite set of CSPG5s, are $\mathcal{R}'$-reducible. Here $\mathcal{R}' = \mathcal{R} \cup \{D\}$, where D is the additional reduction shown in Figure 8.

Together with the results of Sections 2.1–2.3, this shows that all CSPG5s other than elements of $\mathcal{S}$ are $\mathcal{R}'$-reducible. We will then argue that reduction D is not actually required, thereby proving Theorem 1.

Lemma 6 *Every 3-connected CSPG5 without a separating 3-cycle is $\mathcal{R}'$-reducible, except for M, C, J, T, B, D_1 and D_2.*

Proof: The proof of this lemma is tedious and was carried out by a computer program similar to that described in Lemma 3.

Since the average face size (that is, the length in edges of the boundary of the face) of a CSPG5 is greater than 3 (except for the dodecahedron), there is a face of size at least 4. Therefore, we grew induced subgraphs starting with a 4-face, and then starting with a path of 4 vertices on the boundary of a larger face. In the latter case, we forbade 4-faces since they were already covered by the former case. As the induced subgraph was grown, we rejected those that implied cuts of size less than 3, separating 3-cycles (on account of Lemma 5), or $\mathcal{R}'$-reductions.

It is easy to see that the result of applying a reduction in $\mathcal{R}'$ to a 3-connected CSPG5 results in a connected graph. So in all cases the program does not need to verify connectivity.

The program completed execution in 21 seconds. In total, 39621 induced subgraphs were found which did not evidently have connectivity problems or

$\mathcal{R}'$-reductions. These had at most 72 vertices. Of these subgraphs, 23 were regular but all of these were isomorphic to one of M, C, J, T, B, D_1, D_2. This completes the proof. □

Proof of Theorem 1: According to Lemmas 1–6, every CSPG5 is reducible by a reduction in $\mathcal{R}'$, except for the graphs M, C, J, T, B, D_1, D_2. Now consider the smallest CSPG5 G, not in the above list, that is not $\mathcal{R}$-reducible. Let G' be the result of reducing G by a D reduction. Since D reductions preserve regularity, simplicity and connectivity, G' is a CSPG5. Moreover, any reduction in $\mathcal{R}$ that applies to G' must also apply to G. Therefore, G' contradicts the minimality of G unless G' is one of M, C, J, T, B, D_1, D_2. Of these possibilities, only D_2 has the configuration that results from a D reduction and the only graph which reduces to it is D_3. Arguing in the same manner produces the sequence $D_4, D_5, \ldots$. This completes the proof of the theorem. □

As a partial check of the theorem, we found an $\mathcal{R}$-reduction for each of the 19.6 million graphs listed in Table 1, apart from the known irreducible graphs. These were made very slowly using a modified version of the program `plantri` [3]. The present, very much faster, algorithm will be incorporated into `plantri` in due course.

3 Planar pentangulations

In this section, we study the generation of SP2s. The starting set $\mathcal{S}'$ consists of the dodecahedron and the graphs C_5, A and F (Figure 9) and $\mathcal{F}'$ is the set of expansions which are the inverses of the reductions shown in Figure 10.

In Figure 10 and later figures in this section, each vertex shown is distinct. A small triangle attached to a vertex indicates the possibility of zero or more incident edges in that position. The absence of a small triangle in some position indicates that no extra edges are incident there.

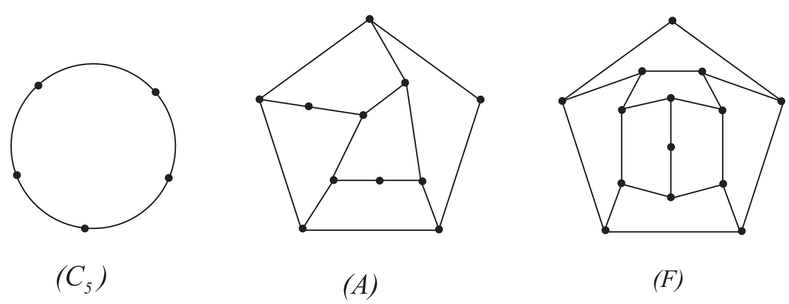

Figure 9: Irreducible SP2s

SP2s exist for all orders $3k-1$ for $k \geq 2$. For 5, 8, 11 and 14 vertices, the number of isomorphism types of SP2s is 1, 3, 30 and 855, respectively.

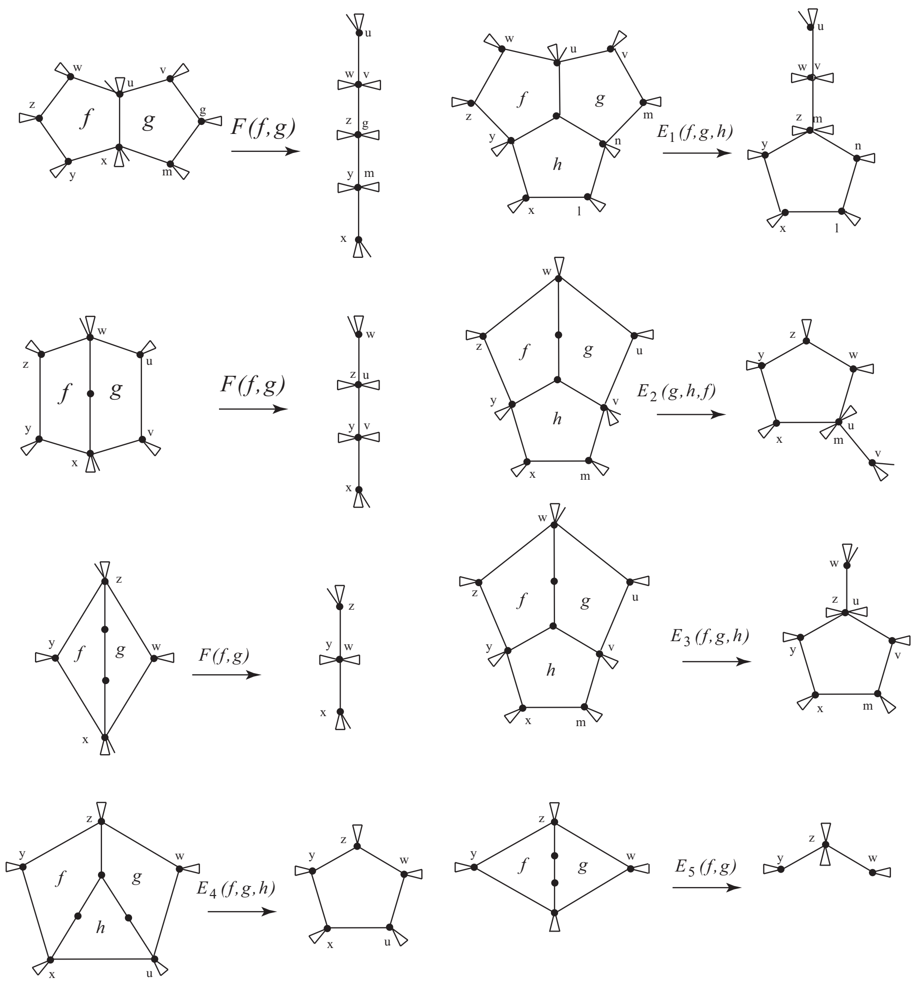

Figure 10: Reductions for SP2s

Theorem 2 *The class of all SP2s is generated by* $(\mathcal{S}', \mathcal{F}')$.

To prove the theorem, we need to show that every SP2 which is not in the set $\mathcal{S}'$ is reducible. In the following lemmas, we prove that if an SP2 has a separating cycle with length less than 7 then it is reducible. Then we complete the proof for general case by using the lemmas.

A separating k-cycle C is called *minimal* if there is either no other separating k-cycle whose interior lies inside C or no other separating k-cycle whose exterior lies outside C. It is not hard to see that if k is the shortest length of a separating cycle, then there is some minimal separating k-cycle.

Lemma 7 *Every SP2 with a separating 3-cycle is reducible.*

Proof: Let C be a minimal separating 3-cycle of G. The possible cases are shown in Figure 11(a,b). In Case (a), because of the planarity of G and the minimality of C, one of $F(f,g)$ and $F(f',g')$ applies (see Figure 11(a1)). In Case (b), by the symmetry between the outside and inside of C we can suppose that there are no separating 3-cycles in the interior of C. If the degree of x is greater than 3, then $F(f,g)$ applies (Figure 11(b1)) and otherwise $E_1(f,g,h)$ applies (Figure 11(b2)). $\square$

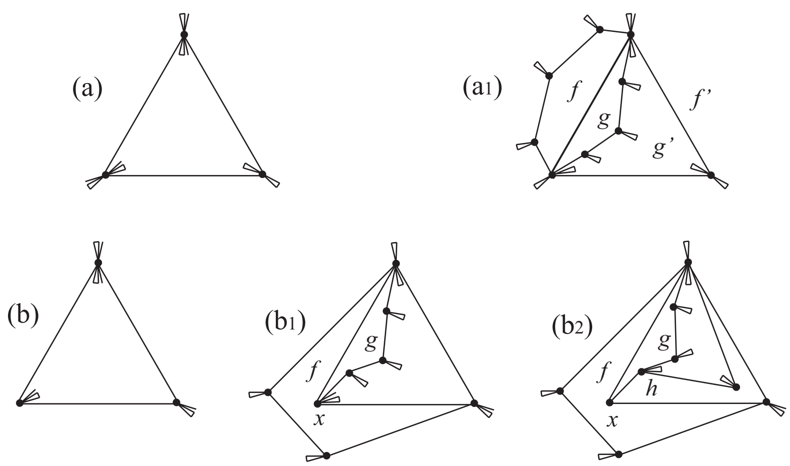

Figure 11: Cases for a separating 3-cycle

Lemma 8 *Every SP2 with a separating 4-cycle is reducible.*

Proof: By Lemma 7, we can assume G has no 3-cycles but has a minimal separating 4-cycle C. By the absence of separating 3-cycles and the symmetry between the outside and inside of C, the possible cases are classified as shown in Figures 12(a–e) and 13(f, g).

In Cases (a) and (b), $F(f,g)$ applies (Figure 12(a1,b1)) and in Case (c) $E_5(f,h)$.

In Case (d), if the degrees of x and w are 3 or the degrees of y and z are 3, then it is the same as Case (c) and so G is reducible. So, we suppose the degree of one of x and w and the degree of one of y and z (say y) is greater than 3. Because of the minimality of C and the symmetry between the outside and inside of C we can suppose that there is no separating 4-cycle inside C and so the degree of both of x and w is greater than 3 and $F(f, g)$ applies (Figure 12(d1)).

In Case (e), because of the symmetry of inside and outside of C we can suppose that there is no separating 4-cycles outside C. If degree of x is greater than 3, then $F(f, g)$ applies and otherwise one of $E_1(g, f, h)$ and $E_2(g, f, h)$ applies (Figure 12(e1)).

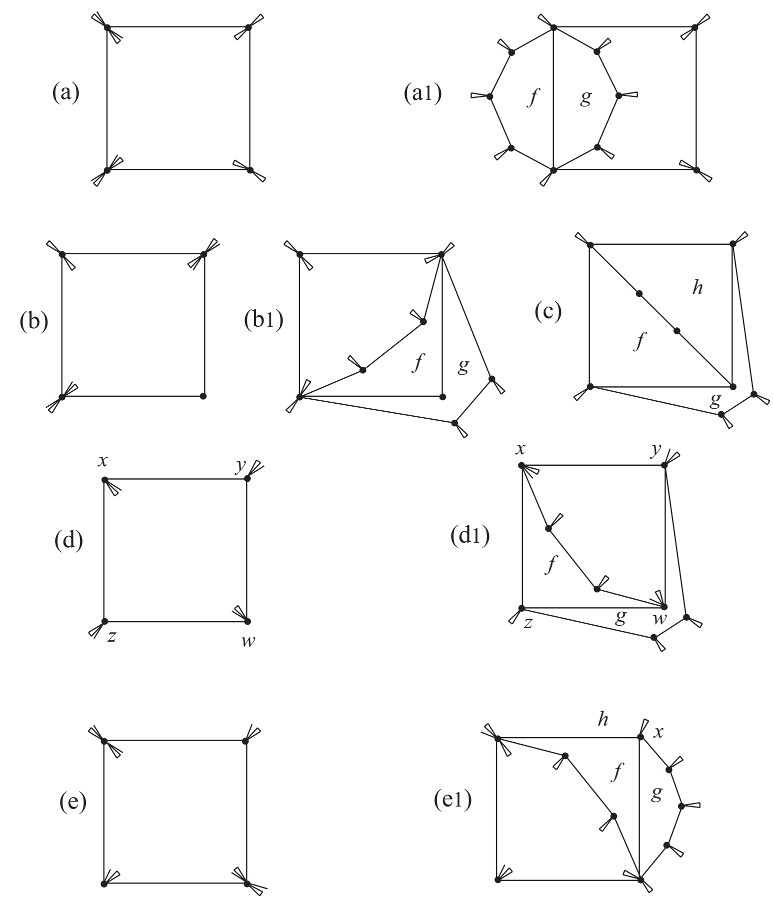

Figure 12: Cases for a separating 4-cycle (part 1)

In Case (f), if the degree of w is 3, then one of $E_1(f, g, h)$ and $E_2(g, f, h)$ applies (Figure 13(f1)). Suppose that the degree of w is greater than 3. If x and v are not adjacent, then $F(f, g)$ applies (Figure 13(f2)). Suppose x and v are

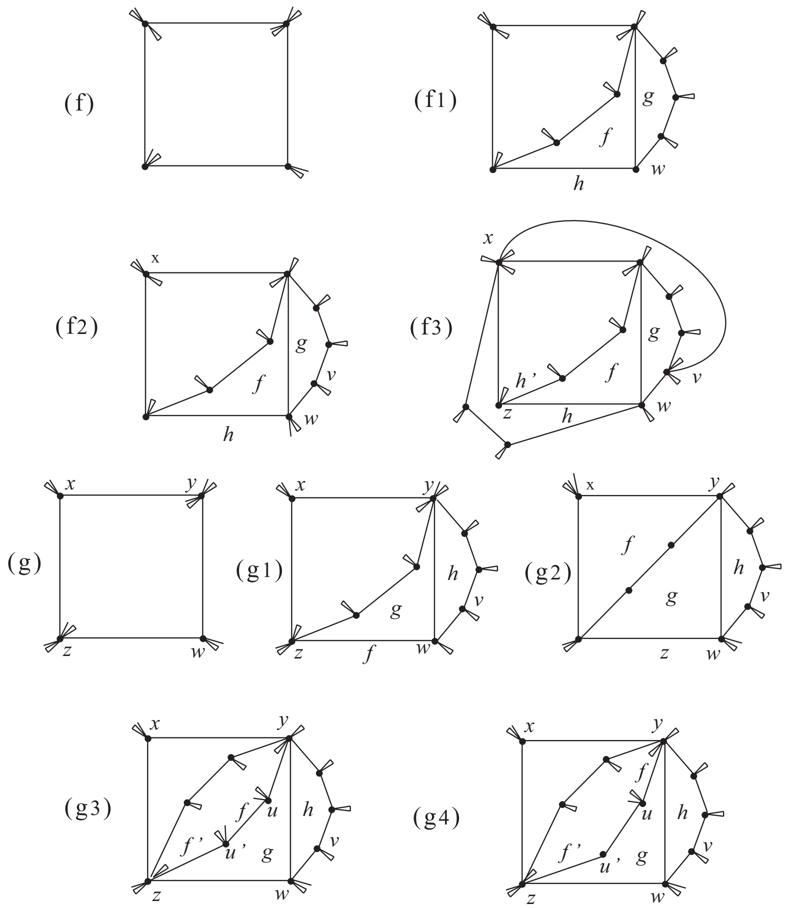

Figure 13: Cases for a separating 4-cycle (part 2)

adjacent. If the degree of z is greater than 3, then $F(f,h)$ applies (Figure 13(f3)) and otherwise one of $E_1(f,h,h')$ and $E_5(h',f)$ applies.

In Case (g), if the degree of w is 3, then one of $E_1(g,h,f)$ and $E_2(g,h,f)$ applies and otherwise if v and z do not have any common neighbours apart from w, then $F(g,h)$ applies (Figure 13(g1)). Suppose that v and z have some common neighbours. By minimality of C, there is no separating 4-cycle inside C. The possible subcases are shown in Figure 13(g2,g3,g4). In Case (g2), by the absence of separating 3-cycles w and x do not have any common neighbours and so $F(f,g)$ applies. In Case (g3), if the degrees of u and u' are greater than 3, then $F(f,g)$ applies. Suppose that the degree of one of them (say u') is 3. In this case, one of $E_1(g,f',f)$ and $E_2(g,f',f)$ applies. In Case (g4), if the degree of u is greater than 3, then $F(f',g)$ applies and otherwise one of $E_3(g,f',f)$ and $E_4(f,g,f')$ applies. □

Lemma 9 *Every SP2 with a separating 5-cycle is reducible.*

Proof: By Lemmas 7 and 8, we can assume G has no 3-cycles or 4-cycles but has a minimal separating 5-cycle C. By the symmetry between the outside and inside of C, in every possible case, G has two faces f and g as one of pictures (a–c) in Figure 14. In Case (a), if z is not the common neighbour of x and y then $F(f,g)$ applies and otherwise by minimality of C we can suppose there is no separating 5-cycle inside C and so $E_4(h,h',g)$ applies (Figure 14(a1)). In Case (b), $F(f,g)$ applies and in Case (c), if the degree of w is greater than 3 then $F(f,g)$ applies and otherwise one of $E_1(g,f,h)$, $E_2(g,f,h)$ and $E_5(f,h)$ applies. □

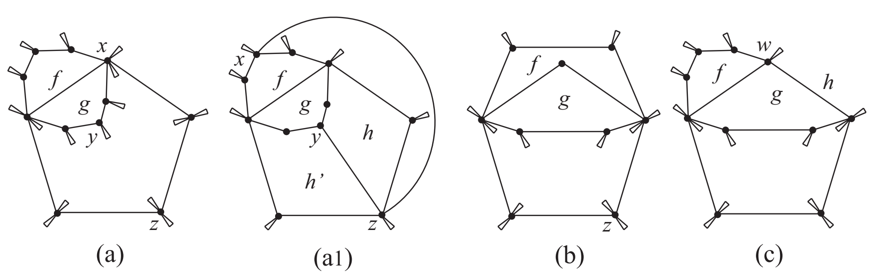

Figure 14: Cases for a separating 5-cycle

Lemma 10 *Every SP2 with a separating 6-cycle is reducible.*

Proof: By Lemmas 7, 8 and 9, we can assume G has no 3-cycles, 4-cycles or separating 5-cycles, but has a minimal separating 6-cycle C. By the symmetry between the outside and inside of C, in every possible case G has two faces f and g as in one of the pictures (a–e) in Figure 14. In Cases (a) and (b), $F(f,g)$ applies. In Case (c), if the degree of w is greater than 3 then $F(f,g)$ applies and otherwise one of $E_1(g,f,h)$ and $E_2(g,f,h)$ applies.

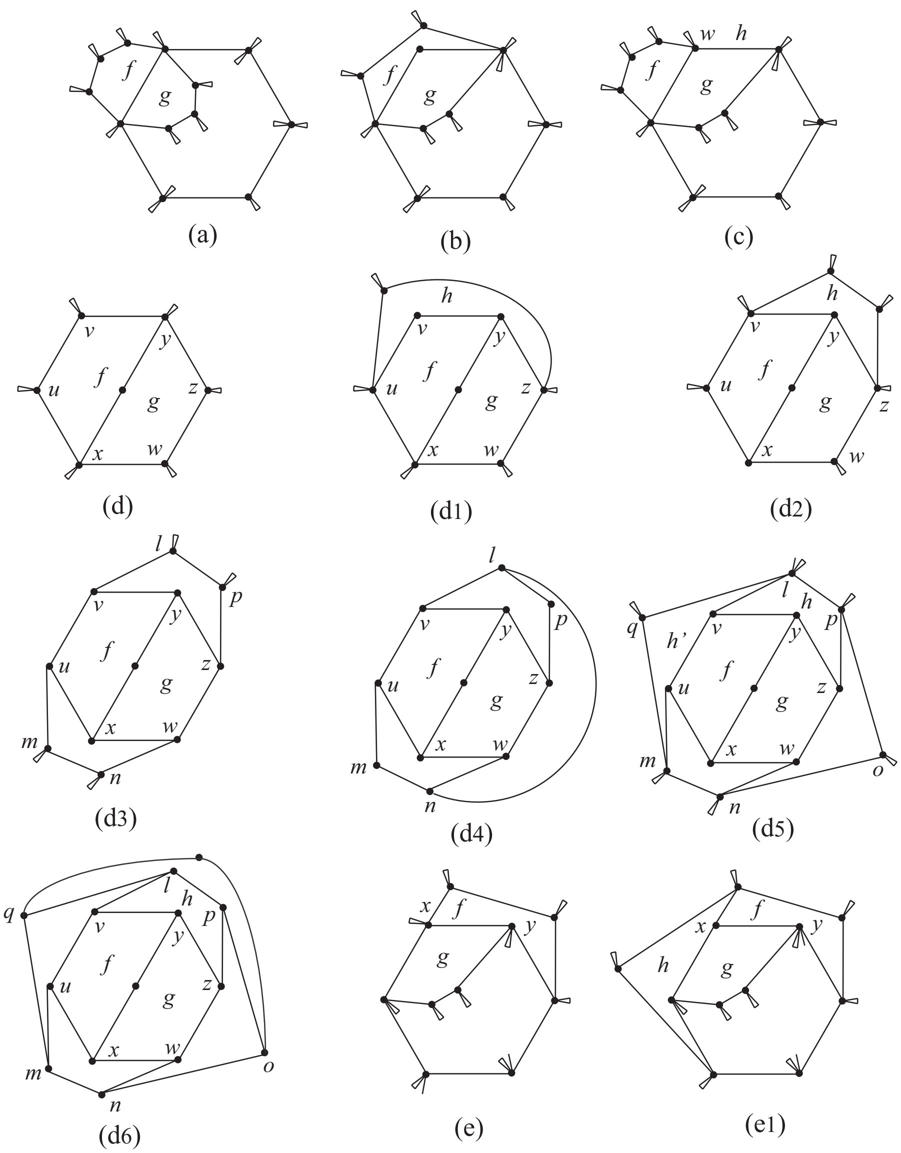

Figure 15: Cases for a separating 6-cycle

In Case (d), if the degrees of x and y are greater than 3 then $F(f, g)$ applies. Suppose the degree of one of them (say y) is 3. If the degree of v is 2, then $E_4(h, g, f)$ applies (Figure 15(d1)) and similarly G is reducible if the degree of z is 2. So, suppose the degrees of v and z are at least 3. If the degree of one of v, z and x is greater than 3, then one of $E_2(f, h, g)$, $E_2(h, g, f)$ and $E_3(g, f, h)$ applies (Figure 15(d2)). Suppose that the degrees of v, z and x are 3. Since the degree of x is 3, by a similar discussion if the degree of one of w and u is not 3 then G is reducible and otherwise G has a subgraph as Figure 15(d3). If the degree of one of l, p, m, n, say p, is 2, since the graph does not have any separating 5-cycle G is graph A (Figure 15(d4)). Otherwise, if the degree of l is greater than 3, then $E_1(h', h, f)$ applies (Figure 15(d5)). By symmetry G is reducible if the degree of one of p, m and n is greater than 3, then G is reducible and otherwise, G is graph F (Figure 15(d6)).

In Case (e), if the degree of x and y is greater than 4 then $F(f, g)$ applies. Otherwise, because of minimality of C, the degree of one of x and y (say y) is greater than 3. So $E_1(f, g, h)$ applies (Figure 15(e1)). □

Proof of Theorem 2: Let G be an SP2 which is not C_5, A or F. If G has a separating cycle with length less than 7 then by Lemmas 7, 8, 9 and 10 G is reducible. So, suppose that G does not have any separating 3, 4, 5 and 6-cycles. This proves that G does not have any vertices with degree 2. A simple calculation shows that the average degree of G is greater than 3 and less than 4. So, there is a vertex x with degree 3 which is adjacent to a vertex y with degree greater than 3. Let f be the face on the left side of xy, g be the face on the right side of xy and h be the face other than f and g whose boundary includes x. By the absence of short separating cycles in G, $E_1(f, g, h)$ applies. This completes the proof. □

4 Concluding Remarks

Theorems 1 and 2 can be used in conjunction with the method of [12] to produce a generator of non-isomorphic CSPG5s or SP2s. Briefly the method works as follows. For each graph G, one expansion is attempted from each equivalence class of expansions under the automorphism group of G. If the new larger graph is H, then H is accepted if the reduction inverse to the expansion by which H was constructed is equivalent under the automorphism group of H to a “canonical” reduction of H; otherwise it is rejected. The essential algorithmic requirements are computation of automorphism groups and canonical labelling, which can be done in linear time [6, 9]. The number of reductions can be applied to one graph is clearly $O(n)$, so by [12, Theorem 3], the amortised time per output graph is at most $O(n^2)$. This does not reflect the likely practical performance; as with all the graph classes mentioned in [3], a careful use of heuristics is likely to make the amortised time per graph approximately constant within the range of sizes for which examination of all the graphs is plausible.

Theorem 1 appeared first in [8]. We wish to thank Mohammadreza Jooyandeh, Gordon Royle, and the anonymous referees for useful advice.

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