

Acknowledgements

Many people have influenced my development during my involvement with this thesis.

Professor Leon Simon introduced me to the area of Geometric Measure Theory and provided

technical advice during the analysis, including times when I doubted the
for solutions of geometric variational problems.

I have benefited from numerous discussions with many visitors to the Australian National University during my stay. I have learned much from their interesting seminars. The Department of Mathematics, Research School of Physical Sciences, ANU provided support services during my study. The Centre for Mathematical Analysis and Department of Mathematics, The Faculty, ANU also provided assistance, especially during the preparation of this thesis.

Contact with other students at David Adams National University has been important. I particularly appreciate the help and friendship of Dr David Guller, Dr John Duggan and Eamonn O'Brien. They have helped pass the time.

I deeply value my contact with John and Rhonda Lipson. To them I owe my sanity. I have learned much from them, and from Dr Val Brown, while struggling with this thesis.

My family has provided support during this trying time. Sometimes I

A thesis submitted for the degree of Doctor of Philosophy of using
an Apple Macintosh and The Australian National University.

The thesis was finally finished with the assistance of Dr Cowling.

August 1986.

All work in this thesis is my own, except where indicated.

David Adams

13/8/86



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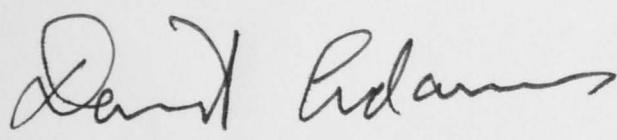
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13/8/86

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Abstract.

Do solutions of differential equations describing the evolution of functions have unique limits? At what rates do they converge?

Existence of unique limits for a class of equations arising from geometric variational problems (for example, extrema of area or energy functionals) is shown in [S]. Solutions converge exponentially if, for example, the linearisation of the equation has no zero eigenvalues, or more generally, if all the stationary solutions of the linearised equation are integrable. Here we find non-exponentially converging solutions in the complementary case, when not all stationary solutions are integrable. We describe some open problems which are related to the description of singularities of analytic functions arising from this work.

In many cases, non-exponentially decaying solutions represent non analytic solutions of the geometrical problems. For example, we show that the non-exponentially decaying solutions for particular examples of minimal surfaces and harmonic maps give *examples of minimal surfaces and harmonic maps with isolated singularities which do not have analytic support*. We list some questions about the existence of such minimal surfaces, for example, are there any minimising or codimension one non analytic minimal surfaces?

- [S] L.Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math., **118** (1983), 525-571

Introduction.

Must minimal surfaces have analytic support?

Milani [M] gave examples which do not. However, these examples still have subanalytic support. (Subanalytic sets are formed from analytic sets by taking closures, complements, unions, intersections and images under proper real analytic maps - see for example [HA], [HI].) Here we give *examples which do not even have subanalytic support*.

We represent a minimal surface with an isolated singularity as a graph over its tangent cone, following [S]. The surface is minimal if and only if its graph is stationary for the area functional. As in [S], we put $t = -\log |x|$ to give a nonlinear evolution equation. We actually study a class of nonlinear evolution equations arising from geometric variational problems and give necessary and sufficient conditions for existence of solutions with non exponential decay. We discuss some problems arising from the study of such equations.

In Chapter 1, we describe the notation, which is similar to that used in [SL]. We use contraction mappings to prove existence of solutions of the equations. In Chapter 2, we prove the necessary estimates for solutions of linear equations. The unusual feature is that we have to use estimates for second order ordinary differential equations with singular points of the second kind (equation 2.15). In Chapter 3, we solve the nonlinear equations. We find exponential solutions as in [CHS]. We repeat the proof that the stationary solutions correspond to an analytic subset of the kernel of linearised operator, which is the space of Jacobi fields. If not all of the Jacobi fields are integrable (that is,

correspond to stationary solutions), then we can use the linear estimates of Chapter 2 to find solutions that converge at a rate which is only some negative power of t . We need more delicate estimates in the kernel of the linearised equation, based on solutions of a finite dimensional ordinary differential equation (equation 3.14). There are many open questions about the solutions of this equation (see Chapter 5). If all Jacobi fields are integrable, then all solutions converge at exponential rates (see Appendix). Hence non exponential solutions exist if and only if there are non integrable Jacobi fields.

In Chapter 4, we analyse particular examples of minimal surfaces and harmonic maps. By direct computation of the third variation of the functional, we show that minimal products of 3 or more spheres, and harmonic maps from the sphere into $(\mathbb{R}^n - \{0\}, g)$ for many metrics g , have non integrable Jacobi fields. Hence there are minimal surfaces and harmonic maps which converge to their tangent cones and tangent maps at a rate given by some power of $1/t = 1/|\log|x||$. However, any subanalytic set must converge to its tangent cone at a rate given by some *rational power of $|x|$* - see e.g. [BR]. Hence these minimal surfaces and harmonic maps do not even have subanalytic support.

In Chapter 5, we list some open problems arising from this work.

In the Appendix, we give a proof due to Professor Simon that if all the Jacobi fields are integrable, then all solutions must converge at exponential rates. This is included for the sake of completeness, since it is the complementary case to the results of Chapter 3.

CHAPTER 1

Notation.

We use the standard notation of [GT] and [SL].

We denote by Σ a smooth compact $(n-1)$ -dimensional submanifold embedded in $\mathbb{R}^P$ and write $T\Sigma = \{T_\omega\Sigma\}_{\omega \in \Sigma}$ for the tangent bundle of Σ . If $e_1, \dots, e_P$ is the standard orthonormal basis for $\mathbb{R}^P$, then we can write e_i^T for the orthogonal projection of e_i onto the subspace $T_\omega\Sigma$ of $\mathbb{R}^P$. By a C^1 function f on Σ we mean the restriction to Σ of a function which is C^1 in a neighbourhood of Σ in $\mathbb{R}^P$. We define the gradient of f on Σ by

$$1.1 \quad \nabla f(\omega) = \sum_{i=1}^P (D_{e_i^T} f)(\omega) e_i^T,$$

where $D_v f$ denotes directional differentiation of f in direction v . We can write

$$1.2 \quad \nabla f(\omega) = (\nabla_1 f(\omega), \dots, \nabla_P f(\omega)),$$

where (since $v = \sum (v \cdot e_j^T) e_j^T$ for any $v \in T_\omega\Sigma$)

$$\nabla_i f(\omega) = \sum_{j=1}^P (D_{e_j^T} f) e_j^T \cdot e_i^T = D_{e_i^T} f.$$

We denote by $\mathbf{V}$ a vector bundle over Σ ; that is, $\mathbf{V} = \{V_\omega\}_{\omega \in \Sigma}$ is a smoothly varying family of subspaces of $\mathbb{R}^M$ with fixed dimension $N \leq M$. We denote by Π_ω the orthogonal projection of $\mathbb{R}^M$ onto V_ω , $\omega \in \Sigma$; which varies smoothly in ω by assumption. By a C^1 section f of $\mathbf{V}$ we mean a C^1 function

$f : \Sigma \rightarrow \mathbb{R}^M$ such that $f(\omega) \in V_\omega$, $\forall \omega \in V_\omega$. We define

$$1.3 \quad \nabla^V f(\omega) = \sum_{i=1}^P e_i^T \otimes \Pi_\omega (D_{e_i} f) ,$$

$$\text{where } D_{e_i} f = (D_{e_i} f^1, \dots, D_{e_i} f^M) .$$

We can write

$$1.4 \quad \nabla^V f(\omega) = \sum_{i=1}^P e_i^T \otimes \nabla_i^V f$$

$$\text{where } \nabla_i^V f(\omega) = \sum_{j=1}^P (e_i^T \cdot e_j^T) \Pi_\omega (D_{e_j} f) = \Pi_\omega (D_{e_i} f) .$$

Thus we can represent $\nabla^V f(\omega)$ by the $P \times M$ matrix

$$[e_\alpha \cdot \nabla_i^V f(\omega)] ; i=1, \dots, P \text{ and } \alpha=1, \dots, M$$

We denote by Q the cylinder with cross section Σ , $Q = \Sigma \times \mathbb{R} \subseteq \mathbb{R}^{P+1}$, with the usual product metric induced from $\mathbb{R}^{P+1}$. We write

$$Q_+ = \Sigma \times (0, \infty) ,$$

$$Q_{a,b} = \Sigma \times (a, b) , \text{ for } -\infty \leq a < b \leq \infty .$$

Let $u = u(\omega, t)$ be a C^1 function from $Q_{a,b}$ to $\mathbb{R}^M$ with $u(\omega, t) \in V_\omega$ for $(\omega, t) \in Q_{a,b}$. For fixed $t \in (a, b)$, we define $u(t)$, $u'(t)$ on Σ by

$$u(t)(\omega) = u(\omega, t) , \quad u'(t)(\omega) = \frac{\partial u(\omega, t)}{\partial t} .$$

Note that $u'(t)(\omega) \in V_\omega$ if $u(\omega, t) \in V_\omega$, for $\omega \in \Sigma$.

We will write $D^V u(\omega, t) = (\nabla^V u(t)(\omega) , u'(t)(\omega))$. Where there is no ambiguity, we will write

$$u'(\omega, t) = u'(t)(\omega) , \quad \nabla^V u(\omega, t) = \nabla^V u(t)(\omega) , \text{ and } \nabla u = \nabla^V u .$$

We write $C^1(\Sigma ; V)$ for the space of C^1 sections of the vector bundle V , and $L^2(\Sigma ; V)$ for the space of functions

$$\{ f : f \in L^2(\Sigma ; \mathbb{R}^M) \text{ and } f(\omega) \in V_\omega \text{ for } H^{n-1} \text{ a.e. } \omega \in \Sigma \} .$$

We also write $C^k(Q_{a,b} ; V)$ for the functions

$$\{ f ; f \in C^k(Q_{a,b} ; \mathbb{R}^M) \text{ and } f(\omega, t) \in V_\omega \text{ for } (\omega, t) \in Q_{a,b} \} .$$

We also define weighted norms for functions on $Q_{a,b}$. Suppose $\psi(t)$ is convex and $\psi(t) > 0$ for $t \in (0, \infty)$. We define

$$\|u\|_{L^2_\psi(Q_{a,b}; V)} = \sup_{a < t < b} \psi(t) \|u(t)\|_{L^2(\Sigma; V)}$$

for $u \in L^2(Q_{a,b} ; V)$. If $\psi(t) = (T+t)^q$ for $T, q > 0$, then we write

$$\|u\|_q \text{ for } \|u\|_{L^2_\psi} .$$

We define

$$|u|_{C_q^{2,\alpha}(Q_{a,b}; \mathbf{V})} = \sup_{a < t < b} (T+t)^q \left(|u(t)|_{C^{2,\alpha}(\Sigma)} + |u'(t)|_{C^{1,\alpha}(\Sigma)} + |u''(t)|_{C^{0,\alpha}(\Sigma)} + [u'']_{\alpha; \Sigma \times (a,b) \cap (t,t+1)} \right).$$

Our notation is that of, for example, [GT], with

$$[u]_{\alpha; \Omega} = \sup_{\substack{x, y \in \Omega \\ x \neq y}} |x - y|^{-\alpha} |u(x) - u(y)|.$$

We write

$$|u|_{C_q^{2,\alpha}(Q; \mathbf{V})} = \sup_{t > 0} |u|_{C_q^{2,\alpha}(Q_{t,t+1}; \mathbf{V})}.$$

If $v : (0, \infty) \rightarrow \mathbb{R}^k$, then we write

$$|v|_{2,q} = \sup_{t > 0} \left[(T+t)^q |v(t)| + (T+t)^{1+q} (|v'(t)| + |v''(t)|) \right].$$

For estimates in Chapter 2, we require that the first derivative decays more quickly than the function, but *not* that the second derivative decays more quickly than the first.

We also write

$$|v|_q = \sup_{t > 0} (T+t)^q |v(t)|.$$

We consider functionals of the form

$$F_{a,b}(u) = \int_{Q_{a,b}} F(\omega, u, \nabla u, u') e^{-mt} d\omega dt$$

where $u \in C^1(Q_{a,b}; \mathbf{V})$, $\nabla u = \nabla^{\mathbf{V}} u$ and m is a constant. We consider F to be defined on $\Sigma \times \mathbb{R}^M \times \mathbb{R}^{PM} \times \mathbb{R}^M$, and write $F = F(\omega, z, p, q)$

for $\omega \in \Sigma$, $z \in \mathbb{R}^M$, $q \in \mathbb{R}^M$ and $p = [p_i^\alpha]$; $i = 1, \dots, P$ and $\alpha = 1, \dots, M$.

We assume further that F is analytic in z and p for $|z| + |p|$ small. This hypothesis is not essential, but the appropriate weaker assumptions are difficult to verify in practice - see Remark 3.43.

We let M be e^{mt} times the Euler-Lagrange operator of $F_{a,b}$, that is, $M(u) \in C^0(Q_{a,b}; \mathbf{V})$ for $u \in C^2(Q_{a,b}; \mathbf{V})$ is defined by

$$1.5 \quad \langle e^{-mt} M(u), \zeta \rangle = - \frac{d}{ds} F_{a,b}(u + s\zeta) \big|_{s=0}$$

for $\zeta \in C_c^1(Q_{a,b}; \mathbf{V})$. Here, $\langle, \rangle$ is the usual inner product for $L^2(Q_{a,b}; \mathbf{V})$ given by

$$\langle f, g \rangle = \int_{Q_{a,b}} f \cdot g \, dH^n.$$

We define the functional F_Σ on $C^2(\Sigma; \mathbf{V})$ by

$$F_\Sigma(u) = \int_\Sigma F(\omega, u, \nabla u, 0) d\omega$$

for $u \in C^1(\Sigma; \mathbf{V})$.

The Euler-Lagrange operator M_Σ is characterised by

$M_\Sigma(u) \in C^0(\Sigma; \mathbf{V})$ for $u \in C^2(\Sigma; \mathbf{V})$ and

$$1.6 \quad \langle M_\Sigma(u), \zeta \rangle = - \frac{d}{ds} F_\Sigma(u+s\zeta) \big|_{s=0}$$

for $\zeta \in C^1(\Sigma; \mathbf{V})$.

As in [SL] we can compute explicit expressions for Mu and $M_\Sigma u$ to get

$$1.7 \quad \langle M(u), \zeta \rangle = \int_{Q_{a,b}} e^{-mt} (F_{p_i^\alpha} \zeta_i^\alpha + F_{q^\alpha} (\zeta^\alpha)' + F_{z^\alpha} \zeta^\alpha) d\omega dt$$

where $\zeta_i^\alpha = e^\alpha \cdot \nabla_i^V \zeta$ and $\zeta(\omega, t) \in V_\omega$.

The repeated Greek indices are summed from 1 to M , and repeated Latin indices are summed from 1 to P . If we integrate by parts, then 1.7 implies

$$1.8 \quad M(u) = (F_{q^\alpha q^\beta} ((u^\beta)'' - m(u^\beta)')) + (F_{p_i^\alpha q^\beta} + F_{q^\alpha p_i^\beta}) (u_i^\beta)' \hat{e}^\alpha + F_{p_i^\alpha p_j^\beta} u_{ij}^\beta + R$$

where R contains derivatives of order at most 1. Here, $\hat{e}^\alpha$ denotes the orthogonal projection of e^α onto V_ω ,

$$u_{ij}^\beta = e^\beta \cdot (\nabla_{i,j}^V u)$$

$$\text{and } \nabla_{i,j}^V u = \nabla_i^V \nabla_j^V u - (e_k^T \cdot D_{e_i} e_j^T) \nabla_k^V u = \nabla_i^V \nabla_j^V u - \Pi_\omega (D_{e_i} (D_{e_j}^T e_i^T)) u$$

is the Hessian operator. Similarly,

$$1.9 \quad M_\Sigma(u) = F_{p_i^\alpha p_j^\beta}(\omega, u, \nabla^V u, 0) u_{ij}^\beta \hat{e}^\alpha + R^1$$

where R^1 contains derivatives of order at most 1.

Thus, M, M_Σ are second order quasilinear operators (see [GT] , Chapter 10) with coefficients depending on $\omega, u, \nabla u$, and u' in the case of M .

We will assume the following ellipticity conditions, 1.10 and 1.11, for M and M_Σ , with $u \in C^2(\Sigma; V)$. We assume the Legendre-Hadamard condition for strong ellipticity of $M_\Sigma u$, (see, for example, [MC]) :

$$1.10 \quad F_{p_i^\alpha p_j^\beta}(\omega, z, p, 0) \eta^\alpha \eta^\beta \xi_i \xi_j > 0$$

for $\eta \in V_\omega - \{0\}$, $\xi \in T_\omega - \{0\}$, $\omega \in \Sigma$, $z \in V_\omega$, $p \in T_\omega \Sigma \otimes V_\omega$.

We also assume :

$$1.11 \quad q \cdot F_q(\omega, z, p, q) > c|q|^2,$$

for some $c > 0$, $q \in V_\omega$, $\omega \in \Sigma$, $z \in V_\omega$, $p \in T_\omega \Sigma \otimes V_\omega$, with $(|z| + |p| + |q|) \leq 1$. This condition implies that

$$F_{q^\alpha q^\beta}(\omega, z, p, 0) \eta^\alpha \eta^\beta > c|\eta|^2,$$

$$F_{q^\alpha p_j^\beta}(\omega, z, p, 0) = 0, \text{ and}$$

$$F_{q^\alpha z^\beta}(\omega, z, p, 0) = 0$$

so that the linearisation L of M has the form 1.14. Conditions 1.9 and 1.10 imply that Mu is strongly elliptic if $u \in C^2(Q_{a,b}; V)$ and $|u'|$ is sufficiently small.

Next we compute the linearisations L, L_Σ of M, M_Σ at 0, which are strongly elliptic by 1.9 and 1.10. We have

$$1.12 \quad Lv = \frac{d}{ds} M(sv) |_{s=0}$$

where $v \in C^2(Q_{a,b}; V)$ and $Lv \in C^0(Q_{a,b}; V)$, and

$$1.13 \quad L_{\Sigma} v = \frac{d}{ds} M_{\Sigma}(sv) \big|_{s=0}$$

where $v \in C^2(\Sigma; V)$ and $L_{\Sigma} v \in C^0(\Sigma; V)$.

Since M and M_{Σ} are Euler-Lagrange operators, L and L_{Σ} are self adjoint with respect to the inner product for $L^2(\Sigma; V)$.

From 1.8, 1.9 we can compute the forms of L, L_{Σ} to give

$$1.14 \quad Lv = a_{\alpha\beta} \left((v^{\beta})'' - m(v^{\beta})' \right) \hat{e}^{\alpha} + L_{\Sigma} v,$$

$$1.15 \quad L_{\Sigma} v = \nabla_i^V (a_{\alpha\beta}^{ij} u_j^{\beta} \hat{e}^{\alpha}) + b_{\alpha\beta}^i u_i^{\beta} \hat{e}^{\alpha} + c_{\alpha\beta} u^{\beta} \hat{e}^{\alpha},$$

$$\text{where } a_{\alpha\beta}(\omega) = F_{q^{\alpha} q^{\beta}}(\omega, 0, 0, 0) \quad \text{and} \quad a_{\alpha\beta}^{ij}(\omega) = F_{p_i^{\alpha} p_j^{\beta}}(\omega, 0, 0, 0).$$

The ellipticity assumptions 1.10, 1.11 imply

$$1.16 \quad a_{\alpha\beta}(\omega) \eta^{\alpha} \eta^{\beta} > 0,$$

$$1.17 \quad a_{\alpha\beta}^{ij}(\omega) \xi_i \xi_j \eta^{\alpha} \eta^{\beta} > 0$$

for $\xi \in T_{\omega} \Sigma - \{0\}$, $\eta \in V_{\omega} - \{0\}$.

For technical convenience in Chapters 2 and 3, we will want to have $a_{\alpha\beta}(\omega) = \delta_{\alpha\beta}$. This can be achieved by defining a new functional F^* to be

$$1.18 \quad F^*(u) = F(Au),$$

where A is self adjoint and $A^2 = a^{-1}$ and $a(\omega) : V_\omega \rightarrow V_\omega$ acts as a smooth positive definite linear transformation given by $a(\omega)(\eta) = a_{\alpha\beta} \eta^\beta \hat{e}^\alpha$. Note that $A(\omega)$ is smooth and positive definite, since $a(\omega)$ is positive definite by 1.16 and smooth because F and V are smooth. If u is an extremum of F^* , then Au is an extremum of F .

We compute the first variation formula for F^* to give

$$\begin{aligned} \langle M^*(u), \zeta \rangle &= d/ds F^*(u+s\zeta)|_{s=0} \\ &= d/ds F(Au+sA\zeta)|_{s=0} \\ &= \langle M(Au), A\zeta \rangle \\ &= \langle A M(Au), \zeta \rangle, \text{ as } A \text{ is self adjoint. Hence} \end{aligned}$$

$$1.19 \quad M^*u = A M(Au).$$

The linearisation of M^* at 0 is

$$1.20 \quad \mathbb{L}^*v = v'' - mv' + \mathbb{L}_\Sigma^*v,$$

with

$$1.21 \quad \mathbb{L}_\Sigma^*v = A \nabla_i^V \left(a_{\alpha\beta}^{ij} (Av)_j^\beta \hat{e}^\alpha \right) + A b_{\alpha\beta} (Av)_i^\beta \hat{e}^\alpha + A^2 c_{\alpha\beta} v^\beta \hat{e}^\alpha$$

where we have used $a.A^2 = \text{identity on } V$. This operator again satisfies ellipticity assumptions of the form 1.17.

Also we can write

$$1.22 \quad M^* u = u'' - mu' + M_{\Sigma}^* u + a_1 \cdot Du'(t) + a_2 \cdot u'(t) ,$$

$$\text{where } |a_i(\omega, u, \nabla u, u')|_{C^{0,\alpha}(Q_{t,t+1})} \leq C |u|_{C^{2,\alpha}(Q_{t,t+1})} ,$$

$$\text{and } |a_i(\omega, u_1, \nabla u_1, u_1') - a_i(\omega, u_2, \nabla u_2, u_2')|_{C^{0,\alpha}(Q_{t,t+1})} \leq C |u_1 - u_2|_{C^{2,\alpha}(Q_{t,t+1})} ,$$

$$\text{for } i = 1, 2, \text{ provided that } |u|_{C^{2,\alpha}(Q_{t,t+1})} \leq 1, |u_j|_{C^{2,\alpha}(Q_{t,t+1})} \leq 1, \text{ for } j = 1, 2.$$

This particular form of M^* will be used in Chapter 3.

From now on, we will assume that F satisfies the condition

$$a_{\alpha\beta} = \delta_{\alpha\beta} , \text{ after replacing } F \text{ by } F^* .$$

CHAPTER 2

Estimates for linear equations.

In this chapter we prove decay estimates for solutions of linear equations. These estimates are used in Chapter 3 for proving the existence of solutions of the nonlinear equations discussed in Chapter 1. We consider linear operators of the forms

$$2.1 \quad \mathbb{L}_1 v = v'' - mv' + Lv \quad \text{for } v \in C^2(Q_{a,b}; V),$$

$$2.2 \quad \mathbb{L}_2 w = w'' - mw' - (T+t)^{-1}M.w \quad \text{for } w \in C^2((a,b); \mathbb{R}^k).$$

In the applications, $\mathbb{L}_1$ will be the operator $\mathbb{L}^*$ of 1.20, which is

$$2.3 \quad \mathbb{L}^* v = v'' - mv' + \mathbb{L}_\Sigma^* v$$

with

$$2.4 \quad \mathbb{L}_\Sigma^* v = A \nabla_i^V \left(a_{\alpha\beta}^{ij} (Av)_j^\beta \hat{e}^\alpha \right) + A b_{\alpha\beta} (Av)_i^\beta \hat{e}^\alpha + A^2 c_{\alpha\beta} v^\beta \hat{e}^\alpha.$$

For our main result, we will choose $\mathbb{L}_1$ to be the projection of $\mathbb{L}^*$ into $(\ker \mathbb{L}_\Sigma^*)^\perp$, which is an elliptic operator with zero kernel, and we will choose $\mathbb{L}_2$ to be an ordinary differential operator on a finite dimensional space.

For operators of the form 2.1, we assume that $u \in C^2(Q_+; V)$, V is as in Chapter 1; and L is an elliptic operator defined on Σ (operating on

$u \in C^2(\Sigma; \mathbf{V})$, with values in $C^0(\Sigma; \mathbf{V})$ such that L is self-adjoint with respect to the usual inner product $\langle \cdot, \cdot \rangle$ for $L^2(\Sigma; \mathbf{V})$; that is,
 $\langle Lu, v \rangle = \langle u, Lv \rangle$ for $u, v \in C^2(\Sigma; \mathbf{V})$.

For operators of the form

We may apply usual elliptic regularity theory (see, for example, [GT]) for solutions of $\mathbb{L}_1 u = f$ and $Lu = f$, to deduce the following estimates

$$2.5 \quad \|u\|_{W^{2,p}(Q_{a,b})} \leq C(\sigma, p) \left(\|\mathbb{L}_1 u\|_{L^p(Q_{a-\sigma, b+\sigma})} + \|u\|_{L^2(Q_{a-\sigma, b+\sigma})} \right),$$

$$|u|_{C^{2,\mu}(Q_{a,b})} \leq C(\sigma, \mu) \left(|\mathbb{L}_1 u|_{C^{0,\mu}(Q_{a-\sigma, b+\sigma})} + \|u\|_{L^2(Q_{a-\sigma, b+\sigma})} \right),$$

$$2.6 \quad \|u\|_{W^{2,p}(\Sigma)} \leq C(p) \left(\|Lu\|_{L^p(\Sigma)} + \|u\|_{L^2(\Sigma)} \right),$$

$$|u|_{C^{2,\mu}(\Sigma)} \leq C(\mu) \left(|Lu|_{C^{0,\mu}(\Sigma)} + \|u\|_{L^2(\Sigma)} \right),$$

for $\sigma > 0$, $p \geq 2$, $\mu \in (0,1)$, and where $C^{2,\mu}(\Sigma)$ stands for $C^{2,\mu}(\Sigma; \mathbf{V})$ etc.

There is a discrete set of eigenvalues λ_j for L on Σ with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$; $\lambda_j \uparrow \infty$ as $j \uparrow \infty$, and corresponding orthonormal eigenfunctions $\phi_1, \phi_2, \phi_3 \dots$; $\phi_j \in C^2(\Sigma; \mathbf{V})$, with $\{\phi_j\}$ a complete set in $L^2(\Sigma; \mathbf{V})$.

We define characteristic values $\gamma_j^\pm$ for $\mathbb{L}_1$ by

$$\gamma_j^\pm = -m/2 \pm \sqrt{m^2/4 + \lambda_j}$$

so that the separated solutions $v(\omega, t) = \exp(\gamma_j^\pm t) \phi_j(\omega)$ of $\mathbb{L}_1 v = 0$ are a complete set in $L^2(Q_{a,b}; \mathbf{V})$ for $-\infty < a < b < \infty$.

For operators of the form

$$2.2 \quad \mathbb{L}_2 v = v'' - mv' - (T+t)^{-1} M.v,$$

we assume $T \geq 0$ and that $M: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is linear and self adjoint, so it has eigenvalues $\mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \mu_k$, and corresponding orthonormal eigenvectors $e_1, e_2, e_3, \dots e_k$.

We will now state existence results and decay estimates as $t \rightarrow \infty$ for solutions of $\mathbb{L}_1 u = f$ on Q_+ . The decay estimates for solutions of these problems depend on the growth of $f(t)$ as $t \rightarrow \infty$. Also the required decay estimate affects the initial conditions (at $t=0$) which can be imposed.

Let $\Pi_J: L^2(\Sigma; \mathbf{V}) \rightarrow L^2(\Sigma; \mathbf{V})$ be the orthogonal projection

$$\sum_{j=1}^{\infty} a_j \phi_j \rightarrow \sum_{j \in J} a_j \phi_j \text{ where } J \text{ is a set of non negative integers. Put } \Pi_J^\perp = \text{identity} - \Pi_J.$$

Let $J_\pm = \{j; \pm \operatorname{Re} \gamma_j^\pm > 0 \text{ and } \lambda_j \neq 0\}$ if $m > 0$ and let $J_+ = \{j; \lambda_j \neq 0\}$, $J_- = \emptyset$ if $m < 0$.

Write $\Pi_\pm = \Pi_{J_\pm}$. Write $\gamma_j = \gamma_j^+$. Write $\Pi^T: L^2(\Sigma; \mathbf{V}) \rightarrow L^2(\Sigma; \mathbf{V})$ for

the orthogonal projection into $\ker L$, $\sum_{j=1}^{\infty} a_j \phi_j \rightarrow \sum_{\{j; \lambda_j=0\}} a_j \phi_j$. Write $\Pi^\perp = \text{identity} - \Pi^T$.

Write $\|f\|_\psi = \sup_{t>0} \psi(t) \|f(t)\|_{L^2(\Sigma; \mathbf{V})}$, where $\psi(t) > 0$ and ψ is convex.

The following result for linear systems was proved in [CHS, Thm 1.1] (with the substitution $t = -\log r$). We do not use this result here, and quote it only for comparison with later results.

LEMMA 2.7.

Suppose $q > \gamma_1^-$, $q \notin \{ \operatorname{Re} \gamma_j^+ ; j \geq 1 \}$, $\psi(t) = e^{qt}$, $\|f\|_\psi < \infty$, and put

$$2.8 \quad \mathbb{J} = \{ j ; \operatorname{Re} \gamma_j > q \}.$$

Then for any $g \in L^2(\Sigma ; \mathbf{V})$ there is a unique solution u of $\mathbb{L}_1 u = f$ with $\lim_{t \downarrow 0} \Pi_{\mathbb{J}} u(t) = \Pi_{\mathbb{J}} g$ (weakly in $L^2(\Sigma)$).

Also, u satisfies the estimates

$$\|u\|_\psi \leq C \left(\|g\|_{L^2(\Sigma)} + \|f\|_\psi \right).$$

PROOF: This is Theorem 1.1 of [CHS].

However, we need to have decay estimates when $f(t)$ does not decay exponentially as $t \rightarrow \infty$. In this case, estimates cannot easily be obtained for the projection of $u(t)$ into $\ker L = \operatorname{span}\{ \phi_j ; \lambda_j = 0 \}$ using the separated solutions $\exp(\gamma_j t) \phi_j(\omega)$. Rather, the projection into $\ker L$ has to be treated specially, which is why equation 2.2 is required.

We now prove decay estimates for the case when $\psi(t) = (T+t)^q$, $T > 0$, that is, when $\sup_{t>0} (T+t)^q \|f(t)\| < \infty$. Write $\|f\|_q$ for $\|f\|_\psi$ in this case.

LEMMA 2.9.

Suppose $f(t) \in \operatorname{image}(\Pi^\perp)$ for $t > 0$, and $\|f\|_q < \infty$.

For any $g \in L^2(\Sigma ; \mathbf{V})$, there is a unique solution u of $\mathbb{L}_1 u = f$ on $(0, \infty)$ with $\|u\|_q < \infty$ and $\lim_{t \downarrow 0} \Pi_+ u(t) = \Pi_+ g$ (weakly in $L^2(\Sigma)$).

Also, u satisfies the estimate

$$\|u\|_q \leq C (T^q \|g\|_{L^2(\Sigma)} + \|f\|_q).$$

PROOF : As in [CHS], we seek a solution as an $L^2(\Sigma)$ convergent series $\sum a_j(t)\phi_j(\omega)$. (Elliptic regularity theory implies this series, and all formal series obtained by termwise differentiation converge uniformly and absolutely on compact subdomains of $(0, \infty)$).

This gives a solution provided $a_j(t)$ satisfies the ordinary differential equation

$$2.10 \quad w'' - mw' - \lambda_j w = f_j, \quad f_j = \langle f, \phi_j \rangle.$$

Note that $f_j=0$ if $\lambda_j=0$ as $\Pi^T f(t) = 0$.

As in [CHS], there is a representation for solutions of 2.10 by

$$2.11 \quad a_j(t) = \operatorname{Re} \left(\alpha_j e^{-\gamma_j t} - e^{-\gamma_j t} \int_{\beta_j}^t e^{(2\gamma_j+m)s} \int_s^\infty e^{-(\gamma_j+m)\tau} f_j(\tau) d\tau ds \right) \text{ if } \operatorname{Re}(\gamma_j+m) > 0, \text{ and}$$

$$2.11' \quad a_j(t) = \operatorname{Re} \left(\alpha_j e^{-\gamma_j t} - e^{-\gamma_j t} \int_{\beta_j}^t e^{(2\gamma_j+m)s} \int_0^s e^{-(\gamma_j+m)\tau} f_j(\tau) d\tau ds \right) \text{ if } \operatorname{Re}(\gamma_j+m) < 0.$$

Here α_j, β_j are arbitrary constants, which are chosen to satisfy the initial conditions ($t=0$) and decay conditions ($t \rightarrow \infty$). For $j \in \mathbb{J}_-$, choose $\alpha_j = 0$ and $\beta_j = \infty$, and for $j \in \mathbb{J}_+$, choose $\beta_j = 0$ and $\alpha_j = \langle g, \phi_j \rangle$.

Estimates for linear equations

To prove the estimates, we write $\sum_{j=1}^{\infty} a_j \phi_j = \sum_{j \in \mathbb{J}_-} a_j \phi_j + \sum_{j \in \mathbb{J}_+} a_j \phi_j$.

If $m > 0$, then $\mathbb{J}_-$ is finite and $\mathbb{J}_+ = \{J, J+1, \dots\}$; while if $m < 0$, then $\mathbb{J}_+ = \{j; \lambda_j \neq 0\}$.

We will give the proof for the case $m > 0$; for $m < 0$, one uses both representations 2.11 and 2.11'. For $m < 0$ however, Π_+ is the identity on $(\ker L)^\perp$.

Let $\langle b_j \rangle \in l^2$ with $\|\langle b_j \rangle\|_{l^2} = 1$. We have

$$\sum_{j \in \mathbb{J}_+} a_j(t) b_j \leq \left| \sum_{j \in \mathbb{J}_+} \alpha_j b_j e^{-\gamma_j t} \right| + \left| \sum_{j \in \mathbb{J}_+} \left(e^{-\gamma_j t} \int_0^t e^{(2\gamma_j+m)s} \int_s^\infty e^{-(\gamma_j+m)\tau} f_j(\tau) b_j d\tau ds \right) \right|$$

$$\begin{aligned} &\leq e^{-\gamma_{J+1}t} \|\Pi_J^\perp g\|_{L^2} + \int_0^t \int_s^\infty \left(\sum_j \left(e^{-\gamma_j t + (2\gamma_j+m)s - (\gamma_j+m)\tau} f_j \right)^2 \right)^{1/2} d\tau ds \\ &\leq e^{-\gamma_{J+1}t} \|\Pi_J^\perp g\|_{L^2} + \int_0^t \int_s^\infty e^{-\gamma_{J+1}t + (2\gamma_{J+1}+m)s - (\gamma_{J+1}+m)\tau} \|f(\tau)\|_{L^2} d\tau ds \end{aligned}$$

since $\gamma_{J+1} > 0$, and $s < \min\{\tau, t\}$,

$$\leq e^{-\gamma_{J+1}t} \|\Pi_J^\perp g\|_{L^2} + \int_0^t \int_s^\infty e^{-\gamma_{J+1}t + (2\gamma_{J+1}+m)s - (\gamma_{J+1}+m)\tau} \|f(\tau)\|_q (T+\tau)^{-q} d\tau ds$$

$$\leq e^{-\gamma_{J+1}t} \|\Pi_J^\perp g\|_{L^2} + C(T+t)^{-q} \|f\|_q,$$

$$\text{since } \int_0^t \frac{e^{c(s-t)}}{(T+s)^q} ds \leq \frac{c'}{(T+t)^q}, \text{ for } c > 0.$$

$$\text{Thus } \sum_{j \in \mathbb{J}_+} a_j(t) b_j \leq e^{-\gamma_{J+1}t} \|\Pi_J^\perp g\|_{L^2} + C(T+t)^{-q} \|f\|_q.$$

Taking the supremum over $\langle b \rangle \in l^2$ with $\|\langle b \rangle\| = 1$, we conclude that

$$\|\langle a_j(t) \rangle\|_2 \leq e^{-\gamma_{j+1}t} \|g\|_2 + C(T+t)^{-q} \|f\|_q.$$

For $\mathbb{J}_-$ we have

$$\begin{aligned} \sum_{j \in \mathbb{J}_-} |a_j(t)|^2 &\leq \sum_{j \in \mathbb{J}_-} \left(\int_t^\infty \int_s^\infty \exp(\operatorname{Re}(-\gamma_j t + (2\gamma_j + m)s - (\gamma_j + m)\tau)) |f(\tau)| d\tau ds \right)^2 \\ &\leq C(\text{cardinality of } \mathbb{J}_-) \sum_{j \in \mathbb{J}_-} \left(\int_t^\infty \int_s^\infty \exp(\operatorname{Re}(-\gamma_j t + (2\gamma_j + m)s - (\gamma_j + m)\tau)) (T+\tau)^{-q} \|f\|_q d\tau ds \right)^2 \end{aligned}$$

since $\mathbb{J}_-$ is finite,

$$\begin{aligned} &\leq C \sum_{j \in \mathbb{J}_-} \left(\int_t^\infty \int_s^\infty \exp(\operatorname{Re}(-\gamma_j t + (2\gamma_j + m)s - (\gamma_j + m)\tau)) (T+\tau)^{-q} \|f\|_q d\tau ds \right)^2 \\ &\leq C (T+t)^{-q} \|f\|_q, \end{aligned}$$

$$\text{since } \int_t^\infty \frac{e^{c(s-t)}}{(T+s)^q} ds \leq \frac{c'}{(T+t)^q}, \text{ for } c < 0.$$

Hence $\sum a_j(t) \phi_j(\omega)$ converges in $L^2(\Sigma)$ for all t , so is a solution. Uniqueness follows from the representation 2.11 for the difference of two solutions.

Thus $u = \sum_{\lambda_j \neq 0} a_j(t) \phi_j(\omega)$ is the required solution.

Q.E.D.

REMARK 2.12: Higher regularity of u follows from higher regularity assumptions for f and g . For $p \geq 2$, write

$$\|f\|_{L_q^p} = \sup_{t>0} (T+t)^q \|f\|_{L^p(Q_{t,t+1})}.$$

If $f \in L_q^p(Q; \mathbf{V})$ and $g \in W^{2,p}(\Sigma; \mathbf{V})$, then

$$\|u\|_{W_q^{2,p}(Q; \mathbf{V})} \leq C(\|f\|_{L_q^p} + T^q \|g\|_{W^{2,p}}).$$

Also, for $\alpha \in (0,1)$,

$$|u|_{C_q^{2,\alpha}(Q)} \leq C(|f|_{C_q^{0,\alpha}(Q)} + T^q |g|_{C^{2,\alpha}(\Sigma)}).$$

Here we have used the following norm from Chapter 1 :

$$|u|_{C_q^{2,\alpha}(Q_{a,b}; \mathbf{V})} = \sup_{a < t < b} (T+t)^q \left(|u(t)|_{C^{2,\alpha}(\Sigma)} + |u'(t)|_{C^{1,\alpha}(\Sigma)} + |u''(t)|_{C^{0,\alpha}(\Sigma)} + [u]_{\alpha; \Sigma \times (a,b) \cap (t,t+1)} \right).$$

Solutions of $\mathcal{L}_1 u = f$ satisfy estimates with the second derivatives decaying at a rate $(T+t)^{-1}$ faster than first derivatives. However, we cannot prove such estimates for solutions of $\mathcal{L}_2 u = f$.

Choose a basis for $\mathbb{R}^k$ of eigenvectors $e_1, \dots, e_k$ of M , and write

$$v = \sum_{j=1}^k v_j e_j.$$

Then M can be represented by the diagonal matrix $[\mu_1, \dots, \mu_k]$, and

$\mathbb{L}_3 v = f$ separates to give

$$2.13 \quad v''_i - m v'_i - \mu_i (T+t)^{-1} v_i = f_i, \quad 1 \leq i \leq k.$$

Writing $v = v_i$, $\mu = \mu_i$, we obtain the homogeneous equation

$$2.14 \quad v'' - m v' - \mu (T+t)^{-1} v = 0.$$

REMARK 2.15 : If we make the substitution $s = (T+t)^{-1}$, then we get

$$s^4 v'' + (2s^3 - s^2) v' - \mu s v = 0.$$

This second order linear O.D.E. has an isolated singularity at $s = 0$ (see, for example, [CL] Chapter 4) which, in the terminology of [CL, p.111], is a singular point of the second kind. Hence the solution to 2.14 may not be representable as a convergent power series in $(T+t)^{-1}$. However, there is a divergent power series which is an "asymptotic series" for the solution (see [CL] 5.3, 5.4). Here, we use the special form of 2.14 and the fact that $m \neq 0$ to get a simpler proof than [CL, Theorem 5.4.1] of the following result.

LEMMA 2.16.

For any δ , $0 < \delta < 1$, there is constant $T_0 > 0$ such that for $T \geq T_0$, there is a solution $b(t)$, $t \in (0, \infty)$ of 2.14 of the form

$$2.17 \quad b(t) = (T+t)^{-\mu/m} (1 + w(t)), \quad \text{with } |w(t)|_{2,\delta} < 1.$$

PROOF : Substituting $b(t) = (T+t)^{-\mu/m} (1+w(t))$ into 2.14, we obtain

$$w'' - \left(2\mu/m (T+t)^{-1} + m \right) w' + \mu/m (\mu/m + 1)(T+t)^{-2} (1+w) = 0.$$

Recall, from Chapter 1, that

$$|w|_{2,\delta} = \sup_{t>0} \left[(T+t)^\delta |w(t)| + (T+t)^{1+\delta} \left(|w'(t)| + |w''(t)| \right) \right]$$

and set $B_\delta = \{ w ; w \in C^2(0,\infty) \text{ and } |w|_{2,\delta} < \infty \}$.

Define a mapping $S : B_\delta \rightarrow B_\delta$ by setting $S.v$ to be the solution of the equation

$$v'' - mv' = 2\mu/m(T+t)^{-1}w' - \mu/m(\mu/m+1)(T+t)^{-2}(1+w),$$

$$v(0) = 0.$$

O.E.D.

Solutions of this equation can be represented as

$$v(t) = \int_0^t e^{ms} \int_s^\infty e^{-m\tau} \left(\frac{2\mu}{m}(T+\tau)^{-1}w'(\tau) - \frac{\mu}{m}\left(\frac{\mu}{m}+1\right)(T+\tau)^{-2}(1+w(\tau)) \right) d\tau ds$$

for $m > 0$ and a similar representation for $m < 0$. By direct computation using these representations, we deduce that, for $m > 0$,

$$|v(t)| \leq C \int_0^\infty e^{ms} \int_s^\infty e^{-m\tau} \left((T+\tau)^{-(2+\delta)} |w|_{2,\delta} + (T+\tau)^{-2} \right) d\tau ds,$$

$$|v'(t)| \leq C e^{mt} \int_t^\infty e^{-m\tau} \left((T+\tau)^{-(2+\delta)} |w|_{2,\delta} + (T+\tau)^{-2} \right) d\tau ds.$$

$$|v''(t)| \leq C \left(|v'(t)| + (T+t)^{-(2+\delta)} |w|_{2,\delta} + (T+t)^{-2} \right).$$

Note that the first derivative of v decays more quickly than v , but that the second derivative does not decay more quickly than the first. This is why we need the special norm $|w|_{2,\delta}$ defined in Chapter 1.

From these estimates, and similar estimates for $m < 0$, we deduce that

$$|Sw|_{2,\delta} \leq C T^{-(1-\delta)} (1 + T^{-\delta} |w|_{2,\delta}),$$

$$|Sw_1 - Sw_2|_{2,\delta} \leq C T^{-1} |w_1 - w_2|_{2,\delta}.$$

Choose $T_0 > 0$ so that $T \geq T_0$ implies S is a contraction on $\{w ; |w|_{2,\delta} < 1\}$. Then, by the contraction mapping principle, S has a unique fixed point, say $w(t)$, with $w(0) = 0$ and $|w(t)|_{2,\delta} < \infty$. Hence, $b(t) = (T+t)^{-\mu/m} (1+w(t))$ is the solution of 2.14 which satisfies the required estimate 2.17, provided T_0 is sufficiently large.

Q.E.D.

The solution $b(t)$ of 2.16 will be used to construct solutions of the inhomogeneous equation $\mathbb{L}_2 u = f$. As with solutions of $\mathbb{L}_1 u = f$, decay estimates for solutions of $\mathbb{L}_2 u = f$ depend on the growth of f , and affect the initial conditions which can be imposed on u .

For $\delta > 0$, $\delta \notin \{\mu_j / m\}$, write $\mathbb{J}_\delta = \{j : \mu_j / m > \delta\}$, and let $P_\delta : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be the projection $a \rightarrow \sum_{j \in \mathbb{J}_\delta} a_j e_j$,

that is, the projection into the span of eigenvectors of M whose eigenvalues are greater than $m\delta$ if $m > 0$, and less than $m\delta$ if $m < 0$.

THEOREM 2.18.

There is a constant $T_0 > 0$ such that if $T > T_0$, $|f|_{1+\delta} < \infty$, and $0 < \delta$ with $\delta \notin \{ \mu_j / m \}$, then for any $g \in \mathbb{R}^k$, there is a unique solution of $\mathbb{L}_2 u = f$ on $(0, \infty)$ with $|u|_{2,\delta} < \infty$ and $P_\delta u(0) = P_\delta g$.

Also, $|u|_{2,\delta} \leq C (T^\delta |g| + |f|_{1+\delta})$.

PROOF : We will use the solutions $b_i(t)$ from 2.16, corresponding to the basis $\{e_i\}$ of $\mathbb{R}^k$. Write $B(t)$ for the diagonal matrix $[b_i(t)]$.

Put $u(t) = B(t).w(t) = \sum b_j(t) w_j(t) e_j$, so w satisfies

$$B^{-1} e^{mt} \frac{d}{dt} (e^{-mt} B^2 w'(t)) = f$$

whence, for $m > 0$, we have the representation

$$w(t) = \sum_j \left(W_j - \left(\int_t^{c_j} e^{ms} B(s)^{-2} \int_s^\infty B(\tau) e^{-m\tau} f(\tau) d\tau ds \right)_j \right) e_j$$

while for $m < 0$, we have

$$w(t) = \sum_j \left(W_j + \left(\int_t^{c_j} e^{ms} B(s)^{-2} \int_0^s B(\tau) e^{-m\tau} f(\tau) d\tau ds \right)_j \right) e_j,$$

with W_j , c_j constants to be chosen. Henceforth, we assume that $m > 0$; the argument for $m < 0$ is similar but uses the appropriate representation for $w(t)$.

We have the following representation for $u(t)$:

$$u(t) = \sum_j b_j(t) \left(W_j - \left(\int_t^{c_j} e^{ms} B(s)^{-2} \int_s^\infty B(\tau) e^{-m\tau} f(\tau) d\tau ds \right)_j \right) e_j.$$

If $\mu_j / m < \delta$, then the requirement that solutions decay at rate $(T+t)^{-\delta}$ implies that the initial conditions cannot be prescribed, so set $W_j = 0$ and $c_j = \infty$. However, if $\mu_j / m > \delta$, then put $W_j = P_j g / b_j(0) = \langle g, e_j \rangle / b_j(0)$ and $c_j = 0$.

From 2.17, $b(t) = (T+t)^{-\mu_j/m} (1 + \mathbf{w}(t))$, with $\|\mathbf{w}(t)\|_{2,\delta} < 1$, so there is a $a > 0$ such that $a^{-1} \leq |b_j(t)| (T+t)^{\mu_j/m} \leq a$, $|b_j'(t)| (T+t)^{(1+\mu_j/m)} \leq a$, and $|b_j''(t)| (T+t)^{(1+\mu_j/m)} \leq a$, for T sufficiently large.

Suppose $\mu_j / m < \delta$. Then, for T sufficiently large,

$$\begin{aligned} |u_j(t)| &\leq a(T+t)^{-\frac{\mu_j}{m}} \int_t^\infty e^{ms} a^{-2}(T+s)^{\frac{2\mu_j}{m}} \int_s^\infty a(T+\tau)^{-\frac{\mu_j}{m}} e^{-m\tau} (T+\tau)^{-(1+\delta)} |f_j|_{1+\delta} d\tau ds \\ &\leq C(T+t)^{-\frac{\mu_j}{m}} |f_j|_{1+\delta} \int_t^\infty (T+s)^{\frac{\mu_j}{m} - (1+\delta)} ds \\ &\leq C |f_j|_{1+\delta} (T+t)^{-\delta}. \end{aligned}$$

Estimates for derivatives of $u_j(t)$ are obtained by differentiating the integral representation and using the estimate 2.17 to give

$$\begin{aligned} |u_j'(t)| &\leq |b_j'(t)| \int_t^\infty e^{ms} a^{-2}(T+s)^{\frac{2\mu_j}{m}} \int_s^\infty a(T+\tau)^{-\frac{\mu_j}{m}} e^{-m\tau} (T+\tau)^{-(1+\delta)} |f_j|_{1+\delta} d\tau ds \\ &\quad + |b_j(t)| e^{mt} a^{-2}(T+t)^{\frac{2\mu_j}{m}} \int_t^\infty a(T+\tau)^{-\frac{\mu_j}{m}} e^{-m\tau} (T+\tau)^{-(1+\delta)} |f_j|_{1+\delta} d\tau \\ &\leq C |f_j|_{1+\delta} (T+t)^{-(1+\delta)}. \end{aligned}$$

By differentiating again, and using 2.17, we obtain

$$|u_j''(t)| \leq C |f_j|_{1+\delta} (T+t)^{-(1+\delta)}.$$

Similar estimates hold if $\mu_j / m > \delta$, except that initial conditions can be imposed. For example,

$$\begin{aligned} |u_j(t)| &\leq \frac{|b_j(t)|}{|b_j(0)|} |g_j| + a(T+t)^{-\frac{\mu_j}{m}} \int_0^t e^{ms} a^{-2}(T+s)^{\frac{2\mu_j}{m}} \int_s^\infty a(T+\tau)^{-\frac{\mu_j}{m}} e^{-m\tau} (T+\tau)^{-(1+\delta)} |f_j|_{1+\delta} d\tau ds \\ &\leq C (T+t)^{-\frac{\mu_j}{m}} \left(\frac{|g_j|}{|b_j(0)|} + |f_j|_{1+\delta} \int_0^t (T+s)^{\frac{\mu_j}{m} - (1+\delta)} ds \right) \\ &\leq C \left((T+t)^{-\frac{\mu_j}{m}} T^{\frac{\mu_j}{m}} |g_j| + (T+t)^{-\delta} |f_j|_{1+\delta} \right). \end{aligned}$$

By combining these inequalities, we obtain the required estimate

$$2.19 \quad |u|_{2,\delta} \leq C (T^\delta |g| + |f|_{1+\delta}).$$

Q.E.D.

CHAPTER 3

Solutions of nonlinear equations.

In this chapter we prove the existence of solutions of nonlinear equations

$$3.1 \quad Mu = 0,$$

satisfying the conditions of Chapter 1.

The technique used is to linearise the equation at some point, and to use the estimates for linear equations described in the previous chapter to define contraction mappings in suitable spaces. The particular linearisation and spaces which are used depend on the type of solution sought.

We first state a result due to [CHS, Theorem 2.1], which shows the existence of solutions which decay exponentially as $t \rightarrow \infty$. Here equation 3.1 is linearised at 0, giving the operator L_1 of 2.1 .

$$3.2 \quad L_1 u = u'' - mu' + Lu = R(u) ,$$

with

$$\|R(u)\|_{C^{0,\alpha}(Q_{t,t+1})} \leq C \|u\|_{C^{2,\alpha}(Q_{t,t+1})}^2 .$$

Suppose q is as in 2.7, that is, $q > \gamma_1^-$, $q \notin \{ \operatorname{Re} \gamma_j^+ ; j \geq 1 \}$, and either $J = 0$ and $q < \operatorname{Re} \gamma_1$, or $J \geq 1$ and $\operatorname{Re} \gamma_J < q < \operatorname{Re} \gamma_{J+1}$, where $\{\gamma_j\}$ are the characteristic exponents of L_1 .

THEOREM 3.3 .

There is a constant $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, and $g \in L^2(\Sigma; V)$ with

$$\|g\|_{C^{2,\alpha}(\Sigma)} \leq \varepsilon,$$

there is a unique solution u of 3.2 on Q_+ with

$$\|u\|_{C_{\psi}^{2,\alpha}(Q)} \leq C \|g\|_{C^{0,\alpha}(\Sigma)},$$

$$\Pi_J^\perp u = \Pi_J^\perp g \text{ on } \Sigma,$$

$$\left\| u - \sum_Z \langle g, \phi_j \rangle \phi_j e^{-\gamma_j^+ t} \right\|_{C_{\psi'}^{2,\alpha}(Q)} < \infty, \text{ where } Z = \{j; \operatorname{Re} \gamma_j^+ = \operatorname{Re} \gamma_{J+1}^+\},$$

and $\psi'(t) = e^{q't}$ for $q' \leq \min(2\operatorname{Re} \gamma_{J+1}, \min\{\operatorname{Re} \gamma_j; \operatorname{Re} \gamma_j > \operatorname{Re} \gamma_{J+1}\})$.

REMARK 3.4 : This result shows that if a solution decays exponentially as $t \rightarrow \infty$, then it must decay at a rate prescribed by the eigenvalues of L . We shall not need this result here, but contrast it with the case when $\psi(t) = (T+t)^q$, for $q > 0$, $T \geq 0$ fixed constants.

PROOF OF 3.3 : This is essentially Theorem 2.1 of [CHS], but a slightly different proof is needed to get the precise decay estimate. We use a contraction mapping to obtain a function $\hat{u}$ with $\|\hat{u}\|_{\psi'} < \infty$ and such that the solution of equation 3.2 has the form :

$$\sum_j \langle g, \phi_j \rangle \phi_j e^{-\gamma_j^+ t} + \hat{u}.$$

Q.E.D.

The existence of solutions to 3.1 which are independent of t is also well known. Again equation 3.1 is linearised at 0, but for solutions independent of t this gives

$$3.5 \quad L_{\Sigma} u = R_{\Sigma}(u),$$

where $R_{\Sigma}(u)$ is the "non-linear" part of $M_{\Sigma} (= L_{\Sigma} - M_{\Sigma})$.

The inverse function theorem can be used to give information about solutions of this equation, as the following theorem shows.

THEOREM 3.6 .

In a $C^{2,\alpha}$ neighbourhood of 0 in $L^2(\Sigma; \mathbf{V})$, the set of solutions of $M_{\Sigma}u = 0$ is an analytic subvariety of a finite dimensional real analytic manifold.

PROOF : Write $K = \ker L_{\Sigma}$ in $L^2(\Sigma; \mathbf{V})$. Since L_{Σ} is elliptic, K has finite dimension, say k , and so it can be identified with $\mathbb{R}^k$ after choice of basis. Recall that $\Pi^T: L^2(\Sigma; \mathbf{V}) \rightarrow L^2(\Sigma; \mathbf{V})$ is the orthogonal projection onto K .

Q.E.D. Define $F: C^{2,\alpha}(\Sigma; \mathbf{V}) \rightarrow L^2(\Sigma; \mathbf{V})$ by $F(u) = \Pi^T u + \Pi^{\perp} M_{\Sigma} u$.

Then F is differentiable at 0 and $DF(0).h = \Pi^T h + \Pi^{\perp} L_{\Sigma} h$. Thus

$\ker DF(0) = \{0\}$, so by the inverse function theorem in Banach spaces (see, for example, [N, 2.7]), there is a unique function G ,

$G: (\text{neighbourhood } U \text{ of } 0 \text{ in } C^{0,\alpha}) \rightarrow (\text{neighbourhood } V \text{ of } 0 \text{ in } C^{2,\alpha})$,

such that

$$FG(u) = u \quad \text{for } u \in U$$

$$GF(u) = u \quad \text{for } u \in V.$$

For $v \in \ker L_{\Sigma}$, write $H(v) = \Pi^{\perp} G(v)$.

Since $v = FG(v) = \Pi^T G(v) + \Pi^\perp M_\Sigma(G(v))$, $\Pi^T G(v) = v$, so $G(v) = v + H(v)$, and

$$3.7 \quad \Pi^\perp M_\Sigma(v + H(v)) = 0.$$

Thus, if $v \in \mathbb{K}$, then $v + H(v)$ is a solution of $M_\Sigma(v + H(v)) = 0$ if and only if

$$3.8 \quad \Pi^T M_\Sigma(v + H(v)) = 0.$$

This consists of a finite number of equations in $\mathbb{K}$, and any solution of 3.8 generates a solution of $M_\Sigma u = 0$.

Conversely, if u satisfies $M_\Sigma u = 0$, then, since $\Pi^T G(v) \equiv v$ for $v \in \mathbb{K}$,
 $u = GF(u) = G(\Pi^T u) = \Pi^T u + H(\Pi^T u)$.

Thus all solutions of 3.5 in a $C^{2,\alpha}$ neighbourhood of 0 in $L^2(\Sigma; \mathbf{V})$ are given by $\{v + H(v) ; v \text{ satisfies 3.8}\}$. Recall that F_Σ is analytic, since $F(\omega, z, p, 0)$ is analytic in z and p . Hence H is analytic, so $\{v ; \Pi^T M_\Sigma(v + H(v)) = 0 \text{ and } |v| < \rho\}$ is analytic, and the set of solutions near 0 is a finite dimensional analytic subvariety of the finite dimensional real analytic manifold $\{v + H(v) ; v \in \mathbb{K} \text{ and } |v| < \rho\}$.
 Q.E.D.

LEMMA 3.9.

In a neighbourhood of 0 in $\mathbb{K}$, the analytic subvariety defined by 3.8 is actually the set of stationary points of the analytic function f , where
 $f(v) = F_\Sigma(v + H(v))$, $v \in \text{neighbourhood of 0 in } \mathbb{K}$; in fact,

$$3.10 \quad \langle \nabla f(v), \zeta \rangle = \langle \Pi^T M_\Sigma(v + H(v)), \zeta \rangle$$

which vanishes precisely when $M_\Sigma(v + H(v)) = 0$ by 3.7.

PROOF : Recall that $M_{\Sigma}(u) = -\nabla F_{\Sigma}(u)$, in the sense of Frechét derivatives, so

$$\langle M_{\Sigma}(u), \zeta \rangle = -\frac{d}{ds} F_{\Sigma}(u+s\zeta) \Big|_{s=0} \quad \text{for } \zeta \in C^2(\Sigma; V).$$

If $\zeta \in K$, then

$$\begin{aligned} \frac{d}{ds} F_{\Sigma}(v+s\zeta+H(v+s\zeta)) \Big|_{s=0} &= \langle M_{\Sigma}(v+H(v)), \zeta+DH(v)(\zeta) \rangle \\ &= \langle M_{\Sigma}(v+H(v)), \zeta \rangle + \langle \Pi^{\perp} M_{\Sigma}(v+H(v)), DH(v)(\zeta) \rangle \end{aligned}$$

since $H : K \rightarrow K^{\perp}$,

$$= \langle \Pi^T M_{\Sigma}(v+H(v)), \zeta \rangle$$

since $\Pi^{\perp} M_{\Sigma}(v+H(v))=0$ from the definition of H .

Thus the set of solutions of 3.8 is the set of stationary points of the analytic function $F_{\Sigma}(v+H(v))$ defined on K .

Q.E.D.

Recall from Chapter 1 that elements of $\ker L_{\Sigma}$ are called Jacobi fields on Σ . A Jacobi field v is said to be integrable if there exists a one parameter family u_s , $u_s \in C^2(\Sigma; V)$ for $s \in (0,1)$, satisfying $M_{\Sigma}(u_s) = 0$ and

$$\lim_{s \downarrow 0} \frac{u_s}{s} = v, \text{ in the } C^1(\Sigma; V) \text{ norm.}$$

LEMMA 3.11 .

All Jacobi fields on Σ are integrable if and only if $f \equiv \text{constant}$, where f is as in 3.10.

PROOF : If $f \equiv \text{constant}$, then for any Jacobi field v , $u = v + H(v)$ is a solution of $M_\Sigma(u) = 0$, so $u_s = sv + H(sv)$ is a one parameter family of solutions. By direct calculation, we can readily show that

$$\frac{d}{ds} H(sv) \big|_{s=0} = 0, \text{ so that } \lim_{s \downarrow 0} \frac{u_s}{s} = v.$$

Conversely, suppose all Jacobi fields on Σ are integrable. If f is not identically constant, then f can be expanded in a power series near 0, say

$$f = f(0) + \sum_{j \geq p} f_j,$$

where f_j is homogeneous of degree j , f_p is not identically zero in $\mathbb{K}$, and $p > 2$. Hence there is some $\hat{u} \in \mathbb{K}$ with

$$\nabla f_p(\hat{u}) \neq 0$$

and a one parameter family u_s with

$$\lim_{s \downarrow 0} \frac{u_s}{s} = \hat{u}, \text{ and } M_\Sigma(u_s) = 0.$$

In particular, by 3.10, $\nabla f(\Pi^T u_s) = 0$, so dividing by s^{p-1} and letting $s \downarrow 0$ gives $\nabla f_p(\hat{u}) = 0$, which is a contradiction.

Q.E.D.

The main result of this chapter is that if there exist Jacobi fields on Σ which are not integrable, (that is, f not identically constant), then there are solutions of 3.1 which decay at a rate which is only a power of t .

THEOREM 3.12.

Let f be as in 3.11, with $p \geq 3$ and f_p not identically constant. Suppose $\hat{u} \in S^{k-1}$ satisfies

$$3.13 \quad \nabla^{S^{k-1}} f_p(\hat{u}) = 0, \\ \frac{f_p(\hat{u})}{m} > 0,$$

$$\text{and } u = \left(\frac{1}{p(p-2)} \frac{m}{f_p(\hat{u})} \right)^{\frac{1}{(p-2)}} \hat{u}.$$

Choose δ with $1/(p-2) < \delta < 2/(p-2)$ and $\delta \notin \{ \mu_j/m ; \mu_j \text{ eigenvalue of Hess } f_p(u) \text{ for } 1 \leq j \leq k \}$.

There is $\varepsilon > 0$ and $T_0 \geq 1$ such that if $T \geq T_0$ and $|g|_{C^{2,\alpha}(\Sigma; V)} \leq \varepsilon T^{-(1+\delta)}$,

then there is a solution $\phi \in C_{1/(p-2)}^2((0, \infty); \mathbb{R}^k)$ of

$$3.14 \quad \phi'' - m\phi' - \nabla f(\phi) = 0,$$

with $|\phi - (T+t)^{-1/(p-2)} u|_{2,\delta} < C\varepsilon$, and $P_\delta(\phi(0)) = P_\delta(g)$.

Also, there is a solution $u \in C_{1/(p-2)}^{2,\alpha}(Q_+; V)$ of $Mu = 0$, with $|u - \phi|_{2,\delta}^* \leq C(\varepsilon)$.

Here $|w|_{2,\delta}^* = |w^T|_{2,\delta} + |w^\perp|_{C_{1+\delta}^{2,\alpha}}.$

Also, u satisfies the boundary condition $\mathbb{P}_\delta u(0) = \mathbb{P}_\delta g$ on Q_+ , where $\mathbb{P}_\delta$ is the orthogonal projection onto the complement of the finite dimensional subspace of $L^2(\Sigma; \mathbf{V})$ spanned by $\{\phi_j; \lambda_j > 0\} \cup \{e_j; \mu_j/m < \delta\}$.

PROOF : The method of proof is to linearise 3.1 at suitable functions of t , and to use the estimates for linear equations from Chapter 2 to define contraction mappings. From now on,

$|u|_{C^{0,\alpha}}, |u|_{C^2}, |u|_{C^{2,\alpha}}$ will denote $|u|_{C^{0,\alpha}(Q_{t,t+1})}, |u|_{C^2(Q_{t,t+1})}, |u|_{C^{2,\alpha}(Q_{t,t+1})}$ respectively. Also, $|\phi|_{C^2} = |\phi|_{C^2((t,t+1))}$.

First we compute the linearisation of 3.1. As in 1.21, write $\mathbb{M}u=0$ as

$$3.15 \quad u''(t) - u'(t) + \mathbb{M}_\Sigma(u(t)) = a_1 Du'(t) + a_2 u'(t),$$

so that the right hand side of 3.15 contains all the nonlinear terms in which a time derivative appears.

We will linearise equation 3.15 about $\Phi(t) = \phi(t) + H(\phi(t))$, where $\phi(t) \in C^{2,\alpha}((0, \infty); \mathbb{K})$ will be chosen later to satisfy 3.14. Write $\Phi = \phi + H(\phi)$, and $u = \phi + w^T + H(\phi + w^T) + w^\perp$, where $w \in C^{2,\alpha}(Q_{t,t+1})$ and $w^T = \Pi^T w$, $w^\perp = \Pi^\perp w$. Write $H(\phi + w^T) = H(\phi) + H^1(\phi, w^T)$, where

$$|H^1(\phi, w^T)|_{C^{2,\alpha}} \leq C(|\phi|_{C^2}) |w^T|_{C^{2,\alpha}},$$

and $\Pi^T(H^1(\phi, w^T)) = 0$. Thus $u = \Phi + w + H^1(\phi, w^T)$. Substitute these into 3.12 to get

$$3.16 \quad (\phi + w^T + H(\phi + w^T) + w^\perp)'' - m(\phi + w^T + H(\phi + w^T) + w^\perp)' + \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T) + w^\perp) \\ = (a_1^1(\phi) + a_1^2(\phi, w))\Phi'' + (a_2^1(\phi) + a_2^2(\phi, w))\Phi' + (a_1 D + a_2) \frac{d}{dt}(w + H^1(\phi, w^T))$$

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where $|a_j^1|_{C^{0,\alpha}} \leq C |\phi|_{C^{2,\alpha}}$ and $|a_j^2|_{C^{0,\alpha}} \leq C (|\phi|_{C^{2,\alpha}} + |w|_{C^{2,\alpha}})$

$$|a_j^2(\phi, w_1) - a_j^2(\phi, w_2)|_{C^{0,\alpha}} \leq C |w_1 - w_2|_{C^{2,\alpha}}$$

for $j = 1, 2$, and provided $|\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1$.

Rewrite the right hand side of 3.14 as $b_1(\phi)(\Phi') + b_2(\phi, w)(w')$,

where $|b_1(\phi)|_{C^{0,\alpha}} \leq C |\phi|_{C^{2,\alpha}}$ and $|b_2(\phi, w)|_{C^{0,\alpha}} \leq C (|\phi|_{C^{2,\alpha}} + |w|_{C^{2,\alpha}})$

$$|b_2(\phi, w_1) - b_2(\phi, w_2)|_{C^{0,\alpha}} \leq C |w_1 - w_2|_{C^{2,\alpha}}$$

provided $|\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1$.

Different estimates are required for the normal and tangential parts of 3.15.

The projection into $\mathbb{K}^\perp$ is

$$\begin{aligned} 3.17 \quad & (H(\phi + w^T) + w^\perp)'' - m(H(\phi + w^T) + w^\perp)' + \Pi^\perp \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T) + w^\perp) \\ & = \Pi^\perp (b_1(\phi)(\Phi') + b_2(\phi, w)(w')) . \end{aligned}$$

Now, linearising at $w^\perp = 0$ gives

$$\begin{aligned} \Pi^\perp \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T) + w^\perp) &= \Pi^\perp \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T)) + \mathbb{L}_\Sigma w^\perp + R_0(\phi, w)(w^\perp) \\ &= \mathbb{L}_\Sigma w^\perp + R_0(\phi, w)(w^\perp) , \\ &\text{since } \Pi^\perp \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T)) = 0 \text{ by 3.7 .} \end{aligned}$$

Here $|R_0(\phi, w)(w^\perp)|_{C^{0,\alpha}} \leq C (|\phi|_{C^{2,\alpha}} + |w|_{C^{2,\alpha}}) |w^\perp|_{C^{2,\alpha}}$

$$|R_0(\phi, w_1)(w_1^\perp) - R_0(\phi, w_2)(w_2^\perp)|_{C^{0,\alpha}} \leq C \left((|w_1^\perp|_{C^{2,\alpha}} + |w_2^\perp|_{C^{2,\alpha}}) |w_1 - w_2|_{C^{2,\alpha}} + (|\phi|_{C^{2,\alpha}} + |w_1|_{C^{2,\alpha}} + |w_2|_{C^{2,\alpha}}) |w_1^\perp - w_2^\perp|_{C^{2,\alpha}} \right)$$

provided $|\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1$.

We thus get an equation of the following form for $w^\perp$:

3.18 $w^{\perp''} - mw^{\perp'} + \mathbb{L}_\Sigma w^\perp = f^\perp(\phi, w)$, where

3.18' $|f^\perp(\phi, w)|_{C^{0,\alpha}} \leq C (|\phi|_{C^{2,\alpha}} + |w|_{C^{2,\alpha}}) (|\phi'|_{C^{1,\alpha}} + |w'|_{C^{1,\alpha}} + |w^\perp|_{C^{2,\alpha}})$

$$|f^\perp(\phi, w_1) - f^\perp(\phi, w_2)|_{C^{0,\alpha}} \leq C \left((|\phi|_{C^{2,\alpha}} + |w_1|_{C^{2,\alpha}} + |w_2|_{C^{2,\alpha}}) (|w_1' - w_2'|_{C^{1,\alpha}} + |w_1^\perp - w_2^\perp|_{C^{2,\alpha}}) + (|\phi'|_{C^{1,\alpha}} + |w_1'|_{C^{1,\alpha}} + |w_2'|_{C^{1,\alpha}} + |w_1^\perp|_{C^{2,\alpha}} + |w_2^\perp|_{C^{2,\alpha}}) |w_1 - w_2|_{C^{2,\alpha}} \right)$$

provided $|\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1$.

The projection into $\mathbb{K}$ is

3.19 $(\phi + w^T)'' - m(\phi + w^T)' + \Pi_\Sigma^T (\phi + w^T + H(\phi + w^T) + w^\perp) = \Pi_\Sigma^T (b_1(\phi)(\Phi') + b_2(\phi, w)(w'))$.

However,

$$\Pi^T \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T) + w^\perp) = \Pi^T \mathbb{M}_\Sigma(\phi + w^T + H(\phi + w^T)) + L_2(\phi, w)(w^\perp),$$

$$\text{where } |L_2(\phi, w)(w^\perp)|_{C^{0,\alpha}} \leq C(|\phi|_{C^{2,\alpha}} + |w|_{C^{2,\alpha}})|w^\perp|_{C^{2,\alpha}}$$

$$\begin{aligned} |L_2(\phi, w_1)(w_1^\perp) - L_2(\phi, w_2)(w_2^\perp)|_{C^{0,\alpha}} &\leq C(|\phi|_{C^{2,\alpha}} + |w_1|_{C^{2,\alpha}} + |w_2|_{C^{2,\alpha}})|w_1^\perp - w_2^\perp|_{C^{2,\alpha}} \\ &\quad + (|w_1^\perp|_{C^{2,\alpha}} + |w_2^\perp|_{C^{2,\alpha}})|w_1 - w_2|_{C^{2,\alpha}} \end{aligned}$$

$$\text{provided } |\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1.$$

Write $\Pi^T \mathbb{M}_\Sigma(z + H(z)) = -\nabla f(z)$ as in 3.10. The estimates in $\mathbb{K}$ are more delicate, and we have to linearise ∇f at ϕ rather than at 0, giving

$$\nabla f(\phi + w^T) = \nabla f(\phi) + \text{Hess } f(\phi) w^T + R^T(\phi, w^T),$$

$$\text{where } |R^T(\phi, w^T)|_{C^{0,\alpha}} \leq C(|\phi|_{C^{2,\alpha}}^{p-3} |w^T|_{C^{2,\alpha}}^2 + |w^T|_{C^{2,\alpha}}^{p-1}),$$

$$|R^T(\phi, w_1^T) - R^T(\phi, w_2^T)|_{C^{0,\alpha}} \leq C(|\phi|_{C^{2,\alpha}}^{p-3} (|w_1^T|_{C^{2,\alpha}} + |w_2^T|_{C^{2,\alpha}}) + (|w_1^T|_{C^{2,\alpha}}^{p-2} + |w_2^T|_{C^{2,\alpha}}^{p-2})) |w_1^T - w_2^T|_{C^{2,\alpha}})$$

$$\text{provided } |\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1.$$

We require that $\phi : (0, \infty) \rightarrow \mathbb{K}$ is a solution of

$$3.14 \quad \phi'' - m\phi' - \nabla f(\phi) = 0,$$

$$\text{with } |\phi(t) - (T+t)^{-1/(p-2)} \mathbf{u}|_{2,\delta} \leq C\varepsilon.$$

The existence of such solutions is discussed later (see Lemma 3.26).

We then have the following equation for w^T :

$$3.20 \quad w^{T''} - mw^{T'} + \text{Hess } f(\phi) w^T = f^T,$$

where $f^T(\phi, w) = \Pi^T(b_1(\phi)(\phi') + b_2(\phi, w)(w')) - L_2(\phi, w)(w^\perp) + R^T(\phi, w)$, so

$$3.20' \quad |f^T(\phi, w)|_{C^{2,\alpha}} \leq C \left((|\phi|_{C^{2,\alpha}} + |w|_{C^{2,\alpha}}) (|\phi'|_{C^{1,\alpha}} + |w'|_{C^{1,\alpha}} + |w^\perp|_{C^{2,\alpha}}) + |w^\perp|_{C^{2,\alpha}}^{p-1} + |\phi|_{C^{2,\alpha}}^{p-3} |w^T|_{C^{2,\alpha}}^2 \right)$$

$$\begin{aligned} & |f^T(\phi, w_1) - f^T(\phi, w_2)|_{C^{0,\alpha}} \\ & \leq C (|\phi|_{C^{2,\alpha}} + |w_1|_{C^{2,\alpha}} + |w_2|_{C^{2,\alpha}}) (|w_1' - w_2'|_{C^{1,\alpha}} + |w_1^\perp - w_2^\perp|_{C^{2,\alpha}}) \\ & + C (|\phi'|_{C^{1,\alpha}} + \sum_{j=1}^2 (|w_j'|_{C^{1,\alpha}} + |w_j^\perp|_{C^{2,\alpha}} + |w_j^T|_{C^{2,\alpha}}^{p-2} + |\phi|_{C^{2,\alpha}}^{p-3} |w_j^T|_{C^{2,\alpha}})) |w_1 - w_2|_{C^{2,\alpha}} \\ & \text{provided } |\phi|_{C^{2,\alpha}}, |w|_{C^{2,\alpha}}, |w_1|_{C^{2,\alpha}}, |w_2|_{C^{2,\alpha}} \leq 1. \end{aligned}$$

Define Sv , for v with $|v|_{2,\delta}^* < \infty$, to be $w^\perp + w^T$, where these are solutions of 3.18, 3.20 with inhomogeneous terms $f^\perp(\phi, v)$, $f^T(\phi, v)$ satisfying 3.18', 3.20' respectively, and with boundary conditions $Pu(0) = Pg$.

REMARK 3.21 : We need $|v^\perp|_{2,1+\delta} < \infty$ since this term appears in f^T along with derivatives of ϕ , v^T and we need $|f^T|_{1+\delta} < \infty$. By the estimates of 2.12, $w^\perp$ has the same decay as $f^\perp$, that is, $(T+t)^{-(1+2/(p-2))}$, so this estimate for $v^\perp$ is preserved under the mapping S .

Since $|\phi|_{2,1/(p-2)} < \infty$ and $|v|_{2,\delta}^* < \infty$,

$$3.22 \quad |f^\perp(\phi, v)|_{C^{0,\alpha}} \leq C \left((T+t)^{-(1+2/(p-2))} |\phi|_{2,1/(p-2)} + (T+t)^{-(1+\delta+1/(p-2))} |v|_{2,\delta}^* \right),$$

$$|f^T(\phi, v)|_{C^{0,\alpha}} \leq C \left((T+t)^{-(1+2/(p-2))} |\phi|_{2,1/(p-2)} + (T+t)^{-(1+2\delta-1/(p-2))} |v|_{2,\delta}^* \right),$$

$$|f^\perp(\phi, v_1) - f^\perp(\phi, v_2)|_{C^{0,\alpha}} \leq C (T+t)^{-(1+\delta+1/(p-2))} |v_1 - v_2|_{2,\delta}^*,$$

$$|f^T(\phi, v_1) - f^T(\phi, v_2)|_{C^{0,\alpha}} \leq C (T+t)^{-(1+2\delta-1/(p-2))} |v_1 - v_2|_{2,\delta}^*,$$

provided that $|\phi|_{2,1/(p-2)}, |v|_{2,\delta}^* \leq 1$, so that, for example, $|\phi|^{p-3} |w^T|^2$ can be bounded by $C(T+t)^{-(1+2\delta-1/(p-2))} |v|_{2,\delta}^*$.

We can use the estimates of 2.12 and 2.18 since $\delta \notin \{ \mu_j / m ; \mu_j \text{ an eigenvalue of Hess } f_p(u) \}$. Hence

$$3.23 \quad |w^\perp|_{C_{1+\delta}^{2,\alpha}} \leq C \left(T^{1+\delta} |\Pi_+ g|_{C^{2,\alpha}(\Sigma; V)} + T^{(\delta-2/(p-2))} |\phi|_{2,1/(p-2)} T^{-1/(p-2)} |v|_{2,\delta}^* \right),$$

$$|w^T|_{2,\delta} \leq C \left(T^\delta |P_\delta g| + T^{(\delta-2/(p-2))} |\phi|_{2,1/(p-2)} T^{-(\delta-1/(p-2))} |v|_{2,\delta}^* \right),$$

$$|w_1^\perp - w_2^\perp|_{C_{1+\delta}^{2,\alpha}} \leq C T^{-1/(p-2)} |v_1 - v_2|_{2,\delta}^*,$$

$$|w_1^T - w_2^T|_{2,\delta} \leq C T^{-(\delta-1/(p-2))} |v_1 - v_2|_{2,\delta}^*,$$

provided that $|\phi|_{2,1/(p-2)}, |v|_{2,\delta}^* \leq 1$.

Choose $\varepsilon > 0$ and $T_0 \geq 1$ so that, for $T \geq T_0$ and $|g|_{2,\alpha} \leq \varepsilon T^{-(1+\delta)}$, S maps the ball of radius 1 in $\{ w ; |w|_{2,\delta}^* < \infty \}$ into itself and S is a contraction.

The unique fixed point of S is the solution of $Mu = 0$ satisfying the conditions of Theorem 3.12.

Q.E.D.

REMARK 3.24 : We have finally to prove that there are solutions $\phi(t)$ of

$$3.14 \quad \phi'' - m\phi' - \nabla f(\phi) = 0.$$

Solutions of this equation seem to generate solutions of

$$3.1 \quad Mu = 0,$$

although we can only prove this for solutions of the form

$$u(t) = (T+t)^{-1/(p-2)} u + w(t), \text{ with } |w|_{2,\delta}^* < \infty.$$

Also, this is the only case in which we can prove existence of solutions to 3.14. However, it seems likely that there are other solutions of 3.1 and 3.14 (see Problem 5.4).

REMARK 3.25 : Equation 3.14 can be viewed as a special case of 3.1 in which Σ is a single point. It is important to understand this special case. Also, the conditions 3.13 for existence of a solution have a geometric interpretation in this case (see Lemma 5.6). For these reasons, and on the basis of Remark 3.24, we consider the existence of solutions to 3.14 separately, in Lemma 3.26. Additional properties are discussed in Problems 5.1 - 5.4 .

LEMMA 3.26.

Suppose $\hat{u} \in S^{k-1}$ satisfies 3.13,

$$3.13 \quad \nabla^{S^{k-1}} f_p(\hat{u}) = 0,$$

$$\frac{f_p(\hat{u})}{m} > 0,$$

$$\text{and } \mathbf{u} = \left(\frac{1}{p(p-2)} \frac{m}{f_p(\hat{u})} \right)^{\frac{1}{(p-2)}} \hat{u} .$$

Choose δ with $1/(p-2) < \delta < 2/(p-2)$ and $\delta \notin \{ \mu_j/m ; \mu_j \text{ eigenvalue of Hess } f_p(\mathbf{u}) \text{ for } 1 \leq j \leq k \}$.

There is $\varepsilon > 0$ and $T_0 \geq 1$ such that if $T \geq T_0$ and $\|g\|_{C^{2,\alpha}(\Sigma; \mathbf{V})} \leq \varepsilon T^{-\delta}$,

then there is a solution $\phi \in C_{1/(p-2)}^2((0, \infty); \mathbb{R}^k)$ of

$$3.14 \quad \phi'' - m\phi' - \nabla f(\phi) = 0, \text{ with}$$

$$3.27 \quad \|\phi - (T+t)^{-1/(p-2)} \mathbf{u}\|_{2,\delta} < C\varepsilon, \text{ and } P_\delta(\phi(0)) = P_\delta(g) .$$

REMARK 3.28 : This condition for the Jacobi field $\hat{u}$ involves the *sign* of the analytic function f_p/m at a stationary point. If the sign condition is not satisfied, then we can replace t by $-t$ to get

$$3.29 \quad \phi'' + m\phi' - \nabla f(\phi) = 0 ,$$

which is 3.14 with m replaced by $-m$, so that the sign condition is satisfied. We can thus in this case solve the original problem on $Q_- = (-\infty, 0) \times \Sigma$ rather than $Q_+ = (0, \infty) \times \Sigma$. However, in some cases (in particular if p is odd), we can find solutions to 3.24 on *both* Q_- and Q_+ .

Corollary 3.30.

If p is even, then f_p has either a positive maximum or negative minimum, and so either $f_p(\hat{u})/m > 0$ or $f_p(\hat{u})/(-m) > 0$, giving a solution on either Q_+ or Q_- .

If p is odd, then f_p has both a positive maximum and a negative minimum on S^{k-1} and so both $f_p(\hat{u})/m > 0$ and $f_p(-\hat{u})/(-m) > 0$, giving solutions on both Q_+ and Q_- .

Example 3.31 : Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = -\frac{1}{8}x^4 + \frac{1}{8}x^6.$$

For $T > 0$, $(T-t)^{-1/2}$ is a solution of $x'' - x' \frac{df}{dx}(x) = x'' - x' + \frac{1}{2}x^3 - \frac{3}{4}x^5$ on $(-\infty, 0)$.

However, our method does not give solutions on $(0, \infty)$.

PROOF OF 3.26 : Suppose $\hat{u}$ is a stationary point of f_p restricted to S^{k-1} with $f_p(\hat{u})/m > 0$, and put

$$u = \left(\frac{1}{p(p-2)} \frac{m}{f_p(\hat{u})} \right)^{1/(p-2)} \hat{u}.$$

Write $q = p-2$, so $q \geq 1$ by 3.11.

We will find a function w with $|w|_{2,\delta} < \infty$ so that the solution of 3.14 has the form

$$3.32 \quad \phi(t) = (T+t)^{-1/q} u + w(t),$$

$$\text{whence} \quad \phi'(t) = -1/q (T+t)^{-(1+1/q)} u + w',$$

$$\text{and} \quad \phi''(t) = 1/q (1 + 1/q) (T+t)^{-(2+1/q)} u + w''.$$

Substitute these into 3.14 to get

$$3.33 \quad 1/q (1+1/q) (T+t)^{-(2+1/q)} \mathbf{u} + w'' + m/q (T+t)^{-(1+1/q)} \mathbf{u} - mw' - \nabla f((T+t)^{-1/q} \mathbf{u} + w).$$

Linearise about $(T+t)^{-1/q} \mathbf{u}$ to get

$$3.34 \quad w'' - mw' - \text{Hess } f((T+t)^{-1/q} \mathbf{u}) w = \\ 1/q (1+1/q) (T+t)^{-(2+1/q)} \mathbf{u} + m/q (T+t)^{-(1+1/q)} \mathbf{u} - \nabla f((T+t)^{-1/q} \mathbf{u}) + R_1(t, w).$$

where, provided $|w|, |w_1|, |w_2| < 1$,

$$|R_1(t, w)| \leq C \left((T+t)^{-(1-1/q)} |w|^2 + |w|^{p-1} \right), \text{ and}$$

$$|R_1(t, w_1) - R_1(t, w_2)| \leq C \left((T+t)^{-(1-1/q)} (|w_1| + |w_2|) + (|w_1|^{p-2} + |w_2|^{p-2}) \right) |w_1 - w_2|$$

Note that

$$\begin{aligned} \text{Hess } f((T+t)^{-1/q} \mathbf{u}) &= \sum_{j \geq p} (T+t)^^{-(j-2)/q} \text{Hess } f_j(\mathbf{u}) \\ &= (T+t)^{-1} \text{Hess } f_p(\mathbf{u}) + O((T+t)^{-(1+1/q)}). \end{aligned}$$

Also,

$$\begin{aligned} \nabla f((T+t)^{-1/q} \mathbf{u}) &= \sum_{j \geq p} (T+t)^^{-(j-1)/q} \nabla f_j(\mathbf{u}) \\ &= (T+t)^{-(1+1/q)} \nabla f_p(\mathbf{u}) + O((T+t)^{-(1+2/q)}). \end{aligned}$$

Hence we can rewrite 3.34 as

$$3.35 \quad w'' - mw' - (T+t)^{-1} \text{Hess } f_p(\mathbf{u}) w = (T+t)^{-(1+1/q)} (m/q \mathbf{u} - \nabla f_p(\mathbf{u})) + R_1(t, w) + R_2(t, w),$$

where, provided $|w|, |w_1|, |w_2| < 1$,

$$|R_2(t, w)| \leq C \left((T+t)^{-(1+2/q)} + (T+t)^{-(1+1/q)} |w| \right), \text{ and}$$

$$|R_2(t, w_1) - R_2(t, w_2)| \leq C (T+t)^{-(1+1/q)} |w_1 - w_2|.$$

We are seeking a solution of the form

$$3.32 \quad \phi(t) = (T+t)^{-1/q} \mathbf{u} + w(t),$$

where $|w(t)|_{2,\delta} < \infty$. We would like to apply the linear estimates for systems of ordinary differential equations from Theorem 2.18 to obtain a contraction mapping in a neighbourhood of 0 in $\{w; |w|_{2,\delta} < \infty\}$. For this, we need that

$$3.36 \quad \text{the right hand side of 3.35 has finite } (1+\delta) \text{ norm.}$$

This is satisfied if and only if

$$3.37 \quad -m\mathbf{u} + q \nabla f_p(\mathbf{u}) = 0.$$

However, by using the homogeneity of f_p and taking the inner product with $\hat{u}$, and with tangents to S^{k-1} , we see that 3.37 is equivalent to 3.38 and 3.39 :

$$3.38 \quad \frac{m}{q} |\mathbf{u}|^{-q} = p f_p(\hat{u}),$$

$$3.39 \quad \nabla^{S^{k-1}} f_p(\hat{u}) = 0.$$

Notice that 3.38 means that $\mathbf{u} = (1/p(p-2) m / f_p(\hat{u}))^{1/(p-2)} \hat{u}$ and 3.39 means that $\hat{u}$ is a stationary point of f_p restricted to the sphere. There is thus a point $\mathbf{u}$ as in 3.37 if and only if there exists a stationary point $\hat{u}$ of f_p on the sphere

where $f_p(\hat{u})/m > 0$. If p is odd, then we automatically get suitable $\hat{u}$ by taking the point $\hat{u} \in S^{k-1}$ where f_p restricted to S^{k-1} is maximum.

Define Sw for w with $|w|_{2,\delta} < \infty$ to be the solution of

$$3.40 \quad v'' - mv' - (T+t)^{-1} \text{Hess } f_p(u)v = R_1(t,w) + R_2(t,w),$$

with initial values $P_\delta v(0) = P_\delta g$, as in Theorem 2.18.

From the estimates of 2.18 we have

$$|Sw|_{2,\delta} \leq C (T^\delta |g| + |R_1(t,w)|_{1+\delta} + |R_2(t,w)|_{1+\delta})$$

$$|Sw_1 - Sw_2|_{2,\delta} \leq C (|R_1(t,w_1) - R_1(t,w_2)|_{1+\delta} + |R_2(t,w_1) - R_2(t,w_2)|_{1+\delta}),$$

provided $|w|_{2,\delta}, |w_1|_{2,\delta}, |w_2|_{2,\delta} \leq 1$.

Hence

$$|Sw|_{2,\delta} \leq C (T^\delta |g| + T^{-\varepsilon} |w|_{2,\delta}^2 + T^{-(p-1)\varepsilon} |w|_{2,\delta}^{p-1} + T^{\varepsilon-1/q} + T^{-1/q} |w|_{2,\delta}) \text{ and}$$

$$|Sw_1 - Sw_2|_{2,\delta} \leq C (T^{-\varepsilon} (|w_1|_{2,\delta} + |w_2|_{2,\delta}) + T^{-(p-1)\varepsilon} (|w_1|_{2,\delta}^{(p-2)} + |w_2|_{2,\delta}^{(p-2)}) + T^{-1/q}) |w_1 - w_2|_{2,\delta},$$

provided $|w|_{2,\delta}, |w_1|_{2,\delta}, |w_2|_{2,\delta} \leq 1$.

Choose $\varepsilon > 0$ and $T_0 \geq 1$ so that, for $T \geq T_0$ and $|g| < \varepsilon T^{-\delta}$, S is a contraction in the set $\{w; |w|_{2,\delta} \leq 1\}$. Hence S has a unique fixed point, say $w(t)$, and $\phi(t) = (T+t)^{-1/(p-2)} u + w(t)$ is the required solution of 3.24.

Q.E.D.

Remark 3.41: If f_p restricted to S^{k-1} has a maximum or minimum at $\hat{u}$, then at least one of the $\mu_j/m > 1/(p-2)$. Hence there is a one parameter family ϕ_s of solutions of 3.14 satisfying 3.27 and $P_\delta \phi_s(0) = sP_\delta g$ for $g \in \mathbb{R}^k$, where the notation is as in 2.18.

To see this, note that at any point ζ in S^{k-1} , we can choose coordinates $\xi_1, \dots, \xi_{k-1}$ for S^{k-1} and radial coordinate $\xi_k = r$. If f_p has an extremum at ζ , then

$$\frac{\partial f_p}{\partial \xi_j} = 0 \quad \text{for } j = 1, \dots, k-1.$$

Since f_p is homogeneous,

$$\frac{\partial}{\partial r} \frac{\partial f_p}{\partial \xi_j} = 0.$$

Hence $[\text{Hess } f_p(\zeta)]_{ik} = 0$ unless $i = k$, and

$$[\text{Hess } f_p]_{kk} = \frac{\partial^2 f_p}{\partial r^2} = p(p-1) f_p(\zeta).$$

Then

$$\begin{aligned} \mu_k &= [\text{Hess } f_p(u)]_{kk} \\ &= |u|^{p-2} [\text{Hess } f_p(\hat{u})]_{kk} \\ &= \frac{1}{p(p-2)} \frac{m}{f_p(\hat{u})} p(p-1) f_p(\hat{u}) \\ &= m(1 + 1/(p-2)). \end{aligned}$$

$$3.42 \quad \mu_k = m(1 + 1/(p-2)).$$

Hence, $\mu_k / m = 1 + 1/(p-2)$. However, we cannot say anything about the other eigenvalues.

Non Integrable Jacobi fields

REMARK 3.43 : The existence proofs in this chapter do not really require that $F(\omega, z, p, 0)$ be real analytic in (z, p) . It would suffice that

In Chapter 3, we proved that if not all of the Jacobi fields on Σ are

$$3.44 \quad f(u) = f_p(u) + R_{p+1}(u) ,$$

this chapter, we show that there are examples of minimal surfaces and harmonic

where f_p is homogeneous of degree p , f_p not identically 0, $p > 2$, and

minimal surfaces and harmonic maps which converge at a rate given by some

$$\frac{R_{p+1}(su)}{s^p} \rightarrow 0 \text{ as } s \rightarrow 0 ,$$

minimal surfaces and harmonic maps in [54]. Culler and White [CW] also have

an example of such an harmonic map with slow decay, but not a general method

that is, that $F_\Sigma(sv + H(sv))$ does not vanish to infinite order in s for some $v \in \ker L_\Sigma$.

Remark 3.45 : I cannot show that if f is analytic, then all non exponential solutions must converge at some power rate (see Problem 4, Chapter 5).

the third variation of F_Σ

$$\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} F_\Sigma(s, v + H(s, v)) \Big|_{s=0} ,$$

where $s = (s_1, s_2, s_3)$ and $sv = \sum_{i=1}^3 s_i v_i$.

All our examples below will have the property that this third variation is

non zero. In this case, by 3.30, there are solutions on both $(-\infty, 0)$ and $(0, \infty)$. Also,

we do not have to calculate $H : \ker L_\Sigma \rightarrow (\ker L_\Sigma)^\perp$ alone

$$\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} F_\Sigma(s, v) = \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} F_\Sigma(s, v + H(s, v)) .$$

CHAPTER 4

Non integrable Jacobi fields.

In Chapter 3, we proved that if not all of the Jacobi fields on Σ are integrable, then there are solutions of 3.1 which decay at only power rates. In this chapter, we show that there are examples of minimal surfaces and harmonic maps where not all the Jacobi fields are integrable. These provide examples of minimal surfaces and harmonic maps which converge at a rate given by some specific power of $|\log |x||^{-1}$ as $|x| \rightarrow 0$, since $t = \log |x|^{-1}$ for the applications to minimal surfaces and harmonic maps in [SL]. Gulliver and White [GW] also have an example of such an harmonic map with slow decay, but not a general method for constructing examples.

From Lemma 3.11, all Jacobi fields on Σ are integrable if and only if $f \equiv \text{constant}$, where $f(v) = F_{\Sigma}(v + H(v))$, for $v \in \mathbb{K}$, as in 3.10. We will compute the third variation of F_{Σ} ,

$$\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} F_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v}))|_{\mathbf{s}=0},$$

where $\mathbf{s} = (s_1, s_2, s_3)$ and $\mathbf{s} \cdot \mathbf{v} = \sum_{i=1}^3 s_i v_i$.

All our examples below will have the property that this third variation is non zero. In this case, by 3.30, there are solutions on both $(-\infty, 0)$ and $(0, \infty)$. Also, we do not have to calculate $H : \ker L_{\Sigma} \rightarrow (\ker L_{\Sigma})^{\perp}$, since

$$\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} F_{\Sigma}(\mathbf{s} \cdot \mathbf{v}) = \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} F_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v})).$$

We prove this by direct computation, using 1.5:

$$\begin{aligned} \frac{\partial}{\partial s_1} \mathbb{F}_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v}))|_{s_1=0} &= - \langle \mathbb{M}_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v})), \mathbf{v}_1 + DH(\mathbf{s} \cdot \mathbf{v}) \cdot \mathbf{v}_1 \rangle|_{s_1=0} \\ &= - \langle \mathbb{M}_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v})), \mathbf{v}_1 \rangle|_{s_1=0}, \end{aligned}$$

since $H : \ker \mathbb{L}_{\Sigma} \rightarrow (\ker \mathbb{L}_{\Sigma})^{\perp}$, and $\Pi^{\perp} \mathbb{M}_{\Sigma}(z + H(z)) = 0$.

$$\frac{\partial^2}{\partial s_1 \partial s_2} \mathbb{F}_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v}))|_{s_1=s_2=0} = - \langle D\mathbb{M}_{\Sigma}(s_3 \mathbf{v}_3 + H(s_3 \mathbf{v}_3)) \cdot (\mathbf{v}_2 + DH(s_3 \mathbf{v}_3) \cdot \mathbf{v}_2), \mathbf{v}_1 \rangle|_{s_1=s_2=0}$$

where $D\mathbb{M}_{\Sigma}(z)$ is the linearisation at z of $\mathbb{M}_{\Sigma}$.

$$\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} \mathbb{F}_{\Sigma}(\mathbf{s} \cdot \mathbf{v} + H(\mathbf{s} \cdot \mathbf{v}))|_{\mathbf{s}=0} = - \langle D^2 \mathbb{M}_{\Sigma}(0)(\mathbf{v}_2, \mathbf{v}_3), \mathbf{v}_1 \rangle - \langle \mathbb{L}_{\Sigma}(D^2 H(\mathbf{v}_2, \mathbf{v}_3)), \mathbf{v}_1 \rangle$$

since $H(0) = 0$, $DH(0) = 0$,

$$= - \langle D^2 \mathbb{M}_{\Sigma}(0)(\mathbf{v}_2, \mathbf{v}_3), \mathbf{v}_1 \rangle$$

since $\langle \mathbb{L}_{\Sigma}(z), \mathbf{v}_1 \rangle = \langle z, \mathbb{L}_{\Sigma} \mathbf{v}_1 \rangle = 0$,

$$= \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} \mathbb{F}_{\Sigma}(\mathbf{s} \cdot \mathbf{v})|_{\mathbf{s}=0}, \text{ as required.}$$

Minimal Surfaces.

First we compute the area functional for a submanifold of the sphere $S^{N-1} \subseteq \mathbb{R}^N$, following the computation in [SL, Appendix to Part 1]. Let Σ be a smooth compact $(n-1)$ -dimensional minimal submanifold of the sphere S^{N-1} , that is, Σ has zero mean curvature in S^{N-1} . Let $\mathbf{V} = \{V_{\omega}\}_{\omega \in \Sigma}$ be the normal bundle over Σ relative to S^{N-1} , where for $\omega \in \Sigma$, V_{ω} is the $N-n$ dimensional

subspace normal to Σ and tangent to S^{N-1} at ω . Since we are studying surfaces which can be written as graphs over Σ , we consider the map G ,

$G : \omega \rightarrow (1+|g|^2)^{-1/2} (\omega + g(\omega))$, where $g \in C^2(\Sigma; \mathbf{V})$. The image of G is a subset of S^{N-1} , but we can view G as a map from Σ to $\mathbb{R}^N$.

We compute the linear map $dG_\omega : T_\omega \Sigma \rightarrow \mathbb{R}^N$ as

$$dG_\omega(\tau) = (1+|g|^2)^{-1/2} (\tau + D_\tau g A(\omega, g)) ,$$

where $p A(\omega, g) = p \cdot (1+|g|^2)^{-1} \langle p, g \rangle (\omega + g)$ for $p \in V_\omega$. Here $D_\tau g$ denotes the directional derivative in $\mathbb{R}^N$, so that

$$D_\tau g = \nabla_\tau^V g + (D_\tau g)^T ,$$

where ∇^V is the connection in $\mathbf{V}$ given by $\nabla_\tau^V Y = (D_\tau Y)^\perp$ for any $Y \in (T_\omega \Sigma)^\perp \equiv V_\omega$, and X^T is the orthogonal projection of X onto $T_\omega \Sigma$.

From the definition of the second fundamental form B_ω for Σ at ω ,

$$(D_\tau g)^T = - \sum_{j=1}^{n-1} g \cdot B_\omega(\tau_j, \tau_j) \tau_j ,$$

where $\tau_1, \dots, \tau_{n-1}$ is an orthonormal basis for $T_\omega \Sigma$. We can write

$(D_\tau g)^T = -g \cdot B_\omega(\tau)$, with B_ω viewed as a linear transformation from $T_\omega \Sigma \rightarrow \text{Hom}(T_\omega \Sigma, (T_\omega \Sigma)^\perp)$. Hence

$$dG_\omega(\tau) = (1+|g|^2)^{-1/2} (\tau + (\nabla_\tau^V g) A(\omega, g) - g \cdot B_\omega(\tau)) ,$$

so the area functional for $G(\Sigma)$ is

$$H^{n-1}(G(\Sigma)) = \int_{\Sigma} (1+|g|^2)^{-(n-1)/2} \left| \left(1 + (\nabla^V g) A(\omega, g) - g \cdot B_{\omega} \right)_{\#} v(\omega) \right| dH^{n-1}$$

where $v(\omega) = \tau_1 \wedge \dots \wedge \tau_{n-1}$.

Hence

$$\begin{aligned} 4.1 \quad H^{n-1}(G(\Sigma)) &= \int_{\Sigma} (1+|g|^2)^{-(n-1)/2} \left| \bigwedge_{i=1}^{n-1} \left(\tau_i + (\nabla_{\tau_i}^V g) A(\omega, g) - g \cdot B_{\omega}(\tau_i) \right) \right| dH^{n-1} \\ &= \int_{\Sigma} (1+|g|^2)^{-(n-1)/2} \det [b_{ij}]^{1/2} dH^{n-1}, \end{aligned}$$

where $b_{ij} = \sum_{l=1}^N a_{il} a_{jl}$, and

$$\begin{aligned} a_{il} &= (1+|g|^2)^{1/2} dG_{\omega}(\tau_i) \cdot e_l \\ &= \tau_i^l + (\nabla_{\tau_i}^V g) \cdot e_l - (g \cdot B_{\omega}(\tau_i)) \cdot e_l - (1+|g|^2)^{-1} (g \cdot (\nabla_{\tau_i}^V g)) (g \cdot e_l + \omega_l) . \end{aligned}$$

Thus

$$b_{ij} = \delta_{ij} - [\tau_j \cdot (g \cdot B_{\omega}(\tau_i)) + \tau_i \cdot (g \cdot B_{\omega}(\tau_j))] + (\nabla_{\tau_i}^V g) \cdot (\nabla_{\tau_j}^V g) + (g \cdot B_{\omega}(\tau_i)) \cdot (g \cdot B_{\omega}(\tau_j)) + O(|g|^4)$$

We have to compute the expansion for $\det [b_{ij}]$ near $|g| = 0$. Suppose $a_{ij}(t)$ is a symmetric $m \times m$ matrix which is a function of t . Write A_{ij} for the (i,j) th cofactor of A , and $a = \det [a_{ij}]$. From the relations

$$\frac{da}{dt} = \sum_{i,j=1}^n a_{ij}' A_{ij} ,$$

$$a_{ik} A_{kj} = a \delta_{ij} ,$$

$$a A_{ij}' = a' A_{ij} - A_{ik} a_{kl}' A_{lj},$$

we conclude, for $a \neq 0$, that

$$a'' = a_{ij}'' A_{ij} + a^{-1}(a')^2 - a^{-1}(a_{ij}' A_{jk})(a_{il}' A_{lk}), \text{ and}$$

$$a''' = a_{ij}''' A_{ij} + a^{-1}a_{ij}''(a' A_{ij} - A_{ik} a_{kl}' A_{lj}) + 2a^{-1}a'a'' - a^{-2}(a')^3 + a^{-2}a'(a_{ij}' A_{jk})(a_{il}' A_{lk}) + \\ - 2a^{-1}(a_{ij}'' A_{jk})(a_{il}' A_{lk}) - 2a^{-2}a_{ij}'(a_{iq}' A_{qk})(a' A_{jk} - A_{jl} a_{lp}' A_{pk}).$$

In particular, if $[a_{ij}(t)] = I + tP + t^2Q + O(t^4)$, then $A_{ij}(0) = \delta_{ij}$, so that

$$\det[a_{ij}(t)] \\ = 1 + \text{tr}P t + (\text{tr}Q + (\text{tr}P)^2 - \text{tr}P^2)t^2/2 + (6\text{tr}P\text{tr}Q - 6\text{tr}(PQ) - 3\text{tr}P\text{tr}P^2 + (\text{tr}P)^3 + 2\text{tr}P^3)t^3/6 + O(t^4).$$

Hence,

$$4.2 \quad \det [b_{ij}] = 1 + |\nabla^V g|^2 - |g \cdot B_\omega|^2 - 2 \langle g, B_\omega(\tau_i, \tau_j) \rangle \nabla_{\tau_i}^V g \cdot \nabla_{\tau_j}^V g + \\ + \frac{2}{3} \langle g, B_\omega(\tau_i, \tau_j) \rangle \langle g, B_\omega(\tau_j, \tau_k) \rangle \langle g, B_\omega(\tau_i, \tau_k) \rangle + O(|g|^4)$$

because $H = \text{tr} B_\omega = 0$, since Σ is minimal.

Substitute this expansion for $\det [b_{ij}]$ into 4.1 to deduce

$$4.3 \quad H^{n-1}(G(\Sigma)) = \int_{\Sigma} \left(1 + \frac{1}{2}(|\nabla^V g|^2 - |g \cdot B_\omega|^2 - (n-1)|g|^2) - \langle g, B_\omega(\tau_i, \tau_j) \rangle \nabla_{\tau_i}^V g \cdot \nabla_{\tau_j}^V g + \right. \\ \left. \frac{1}{3} \langle g, B_\omega(\tau_i, \tau_j) \rangle \langle g, B_\omega(\tau_j, \tau_k) \rangle \langle g, B_\omega(\tau_i, \tau_k) \rangle \right) dH^{n-1}$$

Remark 4.4 : It follows from this expression that the linearisation $\mathbb{L}_\Sigma$ of the operator $\mathbb{M}_\Sigma$ discussed in Chapter 1 has the form 1.15 (see [SL, Part 1]).

The terms of $O(|g|^4)$ are not important, since we are only computing the third variation of $H^{n-1}(G_s(\Sigma))$, where G_s represents the graph of $\mathbf{s.v} = s_1 v_1 + s_2 v_2 + s_3 v_3$, with $v_i \in \ker \mathbb{L}_\Sigma$, $i = 1, 2, 3$.

The third variation is

$$4.5 \quad \int_{\Sigma} \left(2 \langle v_1, B_{ij} \rangle \langle v_2, B_{jk} \rangle \langle v_3, B_{ki} \rangle - 2 \langle v_1, B_{ij} \rangle \nabla_{\tau_i}^v v_2 \nabla_{\tau_j}^v v_3 + \right. \\ \left. - 2 \langle v_2, B_{ij} \rangle \nabla_{\tau_i}^v v_1 \nabla_{\tau_j}^v v_3 - 2 \langle v_3, B_{ij} \rangle \nabla_{\tau_i}^v v_1 \nabla_{\tau_j}^v v_2 \right) dH^{n-1}$$

where $B_{ij} = B(\tau_i, \tau_j)$.

We now give an example which shows that this expression may be non zero. The following example of products of spheres was investigated in [AA]. The eigenvalues and eigenspaces of $\mathbb{L}_\Sigma$ in $C^2(\Sigma; \mathbf{V})$ were analysed, and the Jacobi fields (in $\ker \mathbb{L}_\Sigma$) were identified. It was shown that all Jacobi fields arise from rotations of Σ for the product of 2 spheres ; but there are other Jacobi fields for the product of 3 or more spheres. Using the properties of Jacobi fields given in [AA], we show that there are some Jacobi fields which are not integrable. A similar computation was made by [PL].

Example 4.6 :

Let $\Sigma = S_{r_1}^{m_1} \times \dots \times S_{r_n}^{m_n}$ be a product of n spheres.

We require that $\sum_{i=1}^n r_i^2 = 1$, and put $m = \sum_{i=1}^n m_i$,

so that $\Sigma^m \subseteq S^{m+n-1} \subseteq \mathbb{R}^{m+n}$. We also require that Σ be minimal, so that

$$m_i/r_i^2 = m.$$

Let $\{\tau_{11}, \dots, \tau_{1m_1}; \tau_{21}, \dots; \tau_{n1}, \dots, \tau_{nm_n}\}$ be an orthonormal basis for

$$T_x \Sigma \cong \bigoplus_{i=1}^n TS^{m_i}$$

so that $\{\tau_{ia}; 1 \leq a \leq m_i\}$ restricts to an orthonormal basis for TS_i .

The normal space to Σ is spanned by vectors of the form

$$4.7 \quad \bigoplus_{i=1}^n c_i \frac{x_i}{r_i}, \quad \text{where } \frac{x_i}{r_i} \text{ is normal to } S_{r_i}^{m_i} \subseteq \mathbb{R}^{m_i+1}, \quad 1 \leq i \leq n; c_i \in \mathbb{R},$$

and $\bigoplus c_i x_i / r_i$ stands for the vector in $\mathbb{R}^{m+n}$ with components :

$$\left(\frac{c_1 x_{11}}{r_1}, \dots, \frac{c_1 x_{1m_1+1}}{r_1}; \frac{c_2 x_{21}}{r_2}, \dots, \frac{c_n x_{nm_n+1}}{r_n} \right).$$

The normal space V_x to $T_x \Sigma$ in $T_x S^{n-1}$ is spanned by vectors of the form 4.7 with the additional restriction that $\sum c_i r_i = 0$.

The second fundamental form B of Σ is

$$B(\tau_{ia}, \tau_{jb}) = - \sum_{k=1}^n (\tau_{ia} \cdot D_{\tau_{jb}} v_k) v_k,$$

where $v_k = \bigoplus \delta_{lk} x_l / r_l$. Thus

$$4.8 \quad B(\tau_{ia}, \tau_{jb}) = \bigoplus_{k=1}^n \frac{\delta_{ik} \delta_{jk}}{r_k} B_{ka, kb} v_k ,$$

where $B_{ka, kb} v_k$ is the second fundamental form of S_k .

Jacobi fields on Σ are of the form

$$u(x) [\bigoplus (c_i x_i) / r_i] ,$$

where $u : \Sigma \rightarrow \mathbb{R}$ satisfies the equation $\Delta_\Sigma u + 2mu = 0$, and $\sum c_i r_i = 0$.

A basis for the Jacobi fields is $x_{ia} x_{jb} v$; where x_{ia} is a linear function on the i -th sphere, $x_{ia} \neq x_{jb}$, and v is a normal of the form 4.7 with $\sum c_i r_i = 0$.

We also have

$$4.9 \quad \nabla_{\tau_{kc}}^v (x_{ia} x_{jb} v) = \tau_{kc} (x_{ia} x_{jb}) v \quad \text{since} \quad \nabla_{\tau_{kc}} \left(\bigoplus_{l=1}^n c_l \frac{x_l}{r_l} \right) \text{ is tangent to } \Sigma .$$

Write

$$v_1 = u \cdot \left(\bigoplus U_i x_i / r_i \right) ,$$

$$v_2 = v \cdot \left(\bigoplus V_i x_i / r_i \right) ,$$

$$v_3 = w \cdot \left(\bigoplus W_i x_i / r_i \right) .$$

Then

$$\langle v_1, B_{ia, jb} \rangle = - \bigoplus_k \frac{U_k}{r_k} B_{ka, kb}$$

and so

$$4.10 \quad \langle v_1, B_{ia,jb} \rangle \langle v_2, B_{jb,kc} \rangle \langle v_3, B_{kc,ia} \rangle = -m \sum_k \frac{U_k V_k W_k}{r_k} uvw$$

as $m_k / r_k^2 = m$. Also,

$$\begin{aligned} \langle v_1, B_{ia,jb} \rangle \nabla_{\tau_{ia}}^v v_2 \cdot \nabla_{\tau_{jb}}^v v_3 &= - \sum_k u \frac{U_k}{r_k} \sum_a \tau_{ka}(v) \tau_{ka}(w) \left(\sum_l V_l W_l \right) \\ &= - \sum_k u \frac{U_k}{r_k} V \cdot W \nabla^{S^{m_k}} v \cdot \nabla^{S^{m_k}} w \end{aligned}$$

Thus the third variation is

$$\begin{aligned} 4.11 \quad \int_{\Sigma} &-2m uvw \sum_k \frac{U_k V_k W_k}{r_k} + 2u V \cdot W \sum_k \frac{U_k}{r_k} (\nabla^{S^{m_k}} v) \cdot (\nabla^{S^{m_k}} w) \\ &+ 2v U \cdot W \sum_k \frac{V_k}{r_k} (\nabla^{S^{m_k}} u) \cdot (\nabla^{S^{m_k}} w) + 2w U \cdot V \sum_k \frac{W_k}{r_k} (\nabla^{S^{m_k}} u) \cdot (\nabla^{S^{m_k}} v) \end{aligned}$$

Write x_{ia} for a linear function in $\mathbb{R}^{m_i+1}$ and extend it to $\mathbb{R}^{m_1+1} \times \dots \times \mathbb{R}^{m_n+1}$.

Write $\nabla^k = \nabla^{S^{m_k}}$.

Then $\nabla^k x_{ia} = 0$ for $i \neq k$, and

$$4.12 \quad \nabla^i x_{ia} = \left(\delta_{ab} - \frac{x_{ia} x_{ib}}{r_i^2} \right).$$

Hence $\nabla^k x_{ia} - \nabla^k x_{jb} = (\delta_{ab} - \frac{x_{ia} x_{ib}}{r_i^2})$ provided $i = j = k$, and is zero otherwise.

Now we can choose $u = x_{ia} x_{jb}$, $v = x_{ia} x_{kc}$, $w = x_{jb} x_{kc}$, where i, j, k are all different. Then $uvw = x_{ia}^2 x_{jb}^2 x_{kc}^2$, and it can be shown that

$$\int_{\Sigma} uvw = \frac{r_i^2}{(m_i+1)} \frac{r_j^2}{(m_j+1)} \frac{r_k^2}{(m_k+1)},$$

(see, for example, [FE, 3.2.13, page 251]).

For each l , $u (\nabla^l v \cdot \nabla^l w) = x_{ia}^2 x_{jb}^2 (1 - x_{kc}^2 / r_k^2) \delta_{lk}$, where there is no sum over l .

Also,

$$\begin{aligned} 4.13 \quad \int u (\nabla^l v) \cdot (\nabla^l w) &= \delta_{lk} \frac{r_i^2}{(m_i+1)} \frac{r_j^2}{(m_j+1)} \left(1 - \frac{1}{(m_k+1)}\right), \\ &= \delta_{lk} m \frac{r_i^2}{(m_i+1)} \frac{r_j^2}{(m_j+1)} \frac{r_k^2}{(m_k+1)}, \end{aligned}$$

where we have used $m_k / r_k^2 = m$ again.

Substituting 4.12, 4.13 into 4.11, we get, for each i, j, k :

$$4.14 \quad -2 \frac{r_i^2}{(m_i+1)} \frac{r_j^2}{(m_j+1)} \frac{r_k^2}{(m_k+1)} \left(\sum_l \frac{U_l V_l W_l}{r_l} - \frac{U_k}{r_k} V \cdot W - \frac{V_j}{r_j} U \cdot W - \frac{W_i}{r_i} U \cdot V \right)$$

(where there is no sum over i, j, k).

Hence, there are Jacobi fields on $S_{r_1}^{m_1} \times \dots \times S_{r_n}^{m_n}$

for which the third variation 4.5 is not zero, provided we can find U, V, W with

$$\sum U_i r_i = \sum V_i r_i = \sum W_i r_i = 0, \text{ but } 4.14 \text{ not zero.}$$

For example, choose $n = 3$, $r_1 = r_2 = r_3$ (so $m_1 = m_2 = m_3$),

$U = (1, -1, 0)$, $V = (1, -1, 0)$, $W = (1, 1, -2)$ and $k=1, j=2, i=3$.

Remark 4.15 : For these examples, we have $p = 3$ (in the notation of 3.30), so that we get examples of minimal surfaces which converge at rate $1 / (\log |x|)$ as $|x| \rightarrow \infty$ and examples which converge at rate $1 / |(\log |x|)|$ as $|x| \rightarrow 0$.

Example 4.16 : Nagura [NT] showed that if $\Sigma \subseteq S^{N-1}$ is a minimal submanifold arising from the full equivariant minimal isometric immersion induced from the harmonic polynomials homogeneous of degree at least 2 on S^2 , then there is a non integrable Jacobi field on Σ (Remark 12.4.2 of [NT]).

Remark 4.17 : In both examples 4.6 and 4.16, the minimal submanifold Σ has codimension at least 2 in S^{N-1} . I do not know of any examples of nonintegrable Jacobi fields on minimal submanifolds of the sphere with codimension 1 (see Problem 5, Chapter 5).

Harmonic maps

We consider the energy functional E^γ for maps $u = (u_1, \dots, u_m)$ from the unit ball $B_1(0)$ in Euclidean space $\mathbb{R}^n$ into an open subset U of $\mathbb{R}^m$ with analytic metric γ . It is shown in [SL] that the functional $F(v) = E^\gamma(u_0 + v)$, with u_0 a fixed homogeneous degree zero harmonic map, is of the form described in Chapter 1, so the results of Chapter 3 apply, provided there are non integrable Jacobi fields.

We think of u_0 as a homogeneous degree zero (that is, $u_0(x) = u_0(x/|x|)$) harmonic map $\mathbb{R}^n \rightarrow (U, \gamma)$ and also as a harmonic map $S^{n-1} \rightarrow (U, \gamma)$. We will show that many harmonic maps have corresponding non integrable Jacobi fields, by perturbing standard examples.

The energy of a map $u : S^{n-1} \rightarrow \mathbb{R}^n$ is

$$4.18 \quad E^\gamma(u) = \frac{1}{2} \int_{\Sigma} \gamma(du_x(\tau_i), du_x(\tau_i)) dH^{n-1}$$

where $\tau_1, \dots, \tau_n$ is a basis for $\mathbb{R}^n$. We choose a basis $\tau_1, \dots, \tau_{n-1}$ tangent to $\Sigma = S^{n-1}$, and put τ_n = the unit vector in the radial direction. We suppose that $u_0 : S^{n-1} \rightarrow \mathbb{R}^n$ is harmonic, and let $F_\Sigma(u) = E^\gamma_\Sigma(u_0 + u)$.

We perturb the metric γ , by putting $g_{ij} = \gamma_{ij} + h_{ij}$, where h_{ij} is an analytic function on U , so that g is an analytic metric. Then

$$\begin{aligned} E^g(u) &= \frac{1}{2} \int_{B_1} g(du_x(\tau_i), du_x(\tau_i)) \\ &= E^\gamma(u) + \frac{1}{2} \int_{B_1} h(du_x(\tau_i), du_x(\tau_i)) \end{aligned}$$

and we have the new functional

$$F^g(u) = E^g(u_0 + u) = F(u) + \frac{1}{2} \int_{B_1} h(d(u_0 + u)_x \tau_i, d(u_0 + u)_x \tau_i) .$$

We assume further that

$$4.19 \quad h_{ij}(u_0(x)) = 0, \nabla h_{ij}(u_0(x)) = 0, \nabla^2 h_{ij}(u_0(x)) = 0 ,$$

where these are derivatives for maps from U to $\mathbb{R}$.

If $|h|_{C^2}$ is sufficiently small, then E^g also satisfies the conditions of Chapter 1, since E^γ does.

The first variation of E^g at u_0 , (that is, the first variation of F^g at 0), is

$$4.20 \quad \frac{\partial}{\partial s} E^g(u_s) |_{s=0} = \frac{\partial}{\partial s} E^\gamma(u_s) |_{s=0} + \frac{1}{2} \int_{B_1} \frac{\partial}{\partial s} (h(u_s)) |_{s=0} (du_x(\tau_i), du_x(\tau_i)) + \\ \int_{B_1} (h(u_s)) \left(\frac{\partial}{\partial s} d(u_s)_x(\tau_i) |_{s=0}, du_x(\tau_i) \right)$$

where u_s is a 1-parameter family with $u_s = u_0$ at $s=0$.

Evidently, the first variation of E^g is equal to the first variation of E^γ , since $h(u_0(x))=0$, and $\nabla h(u_0(x))=0$. Hence u_0 is also harmonic with respect to the metric g . Similarly, the second variation of E^g is equal to the second variation of E^γ , so the Jacobi field operators with respect to the two metrics are equal at u_0 .

However, the third variation of E^g is

$$4.21 \quad \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} E^g(u_s) |_{s=0} = \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} E^\gamma(u_s) |_{s=0} + \int_{\Sigma} \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} (h(u_s)) (du_x(\tau_i), \cdot)$$

where u_s is a 3-parameter family with $u_s = u_0$ at $s=0$.

Hence there are non integrable Jacobi fields on Σ for the metric g provided

$$4.22 \quad \int_{\Sigma} \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} (h(u_s))|_{s=0} (d(u_0)_x(\tau_i), d(u_0)_x(\tau_j)) \neq 0,$$

for some family u_s of the form $u_s = u_0 + s.v$.

We will give examples where 4.22 holds.

Example 4.23 : This is an example of harmonic maps from the ball into the cylinder $\mathbb{R} \times S^{n-1}$. Here $U = \mathbb{R}^n \sim \{0\}$ and the coordinate chart for the cylinder is $(t, \omega) \in \mathbb{R} \times S^{n-1} \rightarrow e^{t\omega} \in U$; and $\gamma(r, \omega) = r^2 dr^2 + \sigma(\omega)$, where σ is the metric on the unit sphere S^{n-1} , is the metric for U which makes this coordinate chart an isometry.

The energy E^γ for a map $u : B_1 \rightarrow U$ is then

$$E^\gamma = \frac{1}{2} \int_{B_1} \frac{|\nabla u|^2}{|u|^2} dx.$$

It is well known that $id : S^{n-1} \rightarrow S^{n-1} \subseteq \mathbb{R}^n$ with Euclidean metric is harmonic. Define $u_0(\omega) = \omega$, so that $u_0 : S^{n-1} \rightarrow \{0\} \times S^{n-1}$ is harmonic.

The metric γ is invariant under scaling $r \rightarrow \lambda r$ ($\lambda > 0$) and so the energy is also invariant under scaling of u . Hence $u_s(\omega) = (1 + s)\omega$ is a one parameter family of harmonic maps, with initial velocity ω , which is a Jacobi field at u_0 . The metric is also invariant under rotations of S^{n-1} , so $u_s(\omega) = \exp(sA).\omega$ with A a skew symmetric linear transformation is also a one parameter family of harmonic maps having Jacobi field $A.\omega$. We observe that it was shown in [SR] that all Jacobi fields on the sphere are integrable, and correspond to rotations, although we do not use this here.

We need to check that there is some h and $u_s = u_0 + \mathbf{s} \cdot \mathbf{v}$ for which

$$4.22 \quad \int_{\Sigma} \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} (h(u_s))|_{s=0} (d(u_0)_x(\tau_i), d(u_0)_x(\tau_j)) \neq 0.$$

For example, choose $h_{ij}(x) = (1 + (|x| - 1)^3) \delta_{ij}$. Choose $v(\omega) = \omega$ (which is a Jacobi field) and put $u_s = u_0 + s v$. Then

$$\frac{\partial^3}{\partial s_3} h_{ij}(u_s(x))|_{s=0} = 6 \delta_{ij}$$

so 4.22 holds.

Example 4.24 : This is an example of harmonic maps from the n dimensional ball into the sphere S^n . Here $U = \mathbb{R}^n$ is the coordinate chart for $S^n - \{\text{north pole}\}$, with respect to stereographic projection

$$P : x \rightarrow \frac{x'}{(1 - x_{n+1})}, \text{ where } x = (x', x_{n+1}) \in S^n.$$

Then the metric γ on $\mathbb{R}^n$, for which the inverse stereographic projection is an isometry, is

$$\gamma(r, \omega) = \frac{4 (dr^2 + r^2 \sigma(\omega))}{(1 + r^2)^2},$$

and the energy for a map $u : B_1 \rightarrow \mathbb{R}^n$ is then

$$E^\gamma(u) = \frac{1}{2} \int_{B_1} \frac{4}{(1+|u|^2)^2} |\nabla u|^2 dx .$$

Clearly, $u_0(\omega) = \omega$ as a map from S^{n-1} to $S^{n-1} \subset S^n$ is harmonic, so that $Pu_0 : S^{n-1} \rightarrow (\mathbb{R}^n, \gamma)$ is also harmonic.

Let A be a skew symmetric linear transformation of $\mathbb{R}^{n+1}$.

Then $u_s(\omega) = \exp(sA) \cdot (\omega, 0)$ is a one - parameter family of harmonic maps into S^n , and $P(u_s(\omega))$ is a one - parameter family of harmonic maps into $(\mathbb{R}^n, \gamma)$.

Choose A so that $A \cdot x = (0, \dots, 0, x_{n+1}, -x_n)$.

$$\text{Then } Pu_s(\omega) = \frac{\omega + s (A \cdot (\omega, 0))' + O(s^2)}{1 + s\omega_n + O(s^2)} = \omega (1 - s\omega_n) + O(s^2) .$$

Choose $h_{ij}(x) = x_n (|x| - 1)^3 \delta_{ij}$ for $x \in \mathbb{R}^n$, which is analytic in a neighbourhood of the image of u_0 .

$$\text{Then } \frac{\partial^3}{\partial s^3} h_{ij} (Pu_s(\omega))|_{s=0} = -6 \omega_n^4 \delta_{ij} ,$$

which implies

$$4.22 \quad \int_{\Sigma} \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} (h(u_s))|_{s=0} (du_x(\tau_i), du_x(\tau_i)) \neq 0 .$$

Remark 4.25 : In examples 4.23 and 4.24, we demonstrate analytic metrics on coordinate charts in $\mathbb{R}^n$ such that not all Jacobi fields are integrable. As in Remark 4.15, since $p = 3$ (in the notation of 3.30), the corresponding examples of harmonic maps from $B_1(0)$ into (U, γ) approach their tangent maps at the rate $1/(\log |x|^{-1})$.

The following open problems arise from the preceding work. The first four are related to the description of singularities of analytic functions, and the rest relate to examples of minimal surfaces and harmonic maps with non integrable Jacobi fields.

Problem 1

The solutions up to found in Chapter 3 all have a unique "blow-up" as $t \rightarrow \infty$, that is, the limit of $u(t) / |u(t)|$ exists as $t \rightarrow \infty$. Is this always so?

Conjecture: yes.

Problem 2

A simpler version of Problem 1 is the finite dimensional equation

$$(5.1) \quad x' = -\nabla g(x), \text{ with } g \text{ analytic.}$$

This was considered by R. Thom [TR]. He proposed the following argument to show that the limit of $x(t) / |x(t)|$ exists. Unfortunately, the argument is not complete, although Thom conjectures that the limit exists (personal communication).

Argument :

Łojasiewicz [L] showed that any solution $x(t)$ of (5.1) that is bounded as $t \rightarrow \infty$ has a unique limit. Without loss of generality, suppose that $x(t) \rightarrow 0$. Put $r(t) = |x(t)|$ and $\phi(t) = x(t) / r(t)$, and calculate their equations to get

CHAPTER 5

Some open problems.

The following open problems arise from the preceding work. The first four are related to the description of singularities of analytic functions, and the rest relate to examples of minimal surfaces and harmonic maps with non integrable Jacobi fields.

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The solutions $u(t)$ found in Chapter 3 all have a unique "blow-up" as $t \rightarrow \infty$, that is, the limit of $u(t) / |u(t)|$ exists as $t \rightarrow \infty$. Is this always so ?

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A simpler version of Problem 1 is the finite dimensional equation

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Lojasiewicz [L] showed that any solution $x(t)$ of 5.1 that is bounded as $t \rightarrow \infty$ has a unique limit. Without loss of generality, suppose that $x(t) \rightarrow 0$.

Put $r(t) = |x(t)|$ and $\omega(t) = x(t) / r(t)$, and calculate their equations to get

Some open problems

$$5.2 \quad \frac{d}{dt} r(t) = \frac{x \cdot x'}{|x|} = -\omega \cdot \nabla g(r\omega) \\ = - \sum_{p \geq p} p r^{p-1} g_p(\omega)$$

where g_p is homogeneous of degree p . Also,

$$5.3 \quad \frac{d}{dt} \omega(t) = \frac{1}{|x|} \left(x' - \frac{(x \cdot x')x}{|x|^2} \right) \\ = -\frac{1}{r} \nabla^{S^{n-1}} g(r\omega) \\ = - \sum_{p \geq p} r^{p-2} \nabla^{S^{n-1}} g_p(\omega) ,$$

where $\nabla^{S^{n-1}}$ is the gradient taken in S^{n-1} .

Next, change time scale by putting

$$\tau(t) = \int_0^t r^{p-2}(s) ds$$

If $\tau(t)$ is bounded as t tends to ∞ , then

$$\int |\omega'(t)| dt \text{ is finite, so } \frac{x}{|x|} \text{ converges.}$$

If not, then the equations become

$$5.4 \quad \frac{d}{d\tau} r(\tau) = -p r(\tau) g_p(\omega) - \sum_{q>0} (p+q) r^{1+q}(\tau) g_{p+q}(\omega)$$

$$5.5 \quad \frac{d}{d\tau} \omega(\tau) = -\nabla^{S^{n-1}} g_p(\omega) - \sum_{q>0} r^q \nabla^{S^{n-1}} g_{p+q}(\omega)$$

LEMMA 5.6.

As $\tau \rightarrow \infty$, $\omega(\tau) \rightarrow$ a component of the set $\{\omega; \nabla^S g_p(\omega) = 0\}$.

PROOF : If not, then g_p would not be bounded below on $\{\omega(\tau)\}$, but

$$\begin{aligned} \frac{d}{d\tau} g_p(\omega(\tau)) &= -\nabla^{S^{n-1}} g_p \cdot \left(\nabla^{S^{n-1}} g_p + \sum_{q>0} r^q \nabla^{S^{n-1}} g_{p+q} \right) \\ &= -|\nabla^{S^{n-1}} g_p|^2 - r(\tau) \left[\sum_{q>0} r^{q-1} \nabla^{S^{n-1}} g_p \cdot \nabla^{S^{n-1}} g_{p+q} \right]. \end{aligned}$$

Suppose $\nabla^S g_p(\omega)$ does not approach 0. Then there exists $\tau_k \rightarrow \infty$ and $W \in S^{n-1}$ such that $\omega(\tau_k) \rightarrow W$ and $|\nabla^S g_p(W)| = \delta > 0$. By hypothesis, $r(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$, so for $\tau > \tau_k$,

$$\left| r(\tau) \sum_{q>0} r^{q-1} \nabla^{S^{n-1}} g_p \cdot \nabla^{S^{n-1}} g_{p+q} \right| < \frac{\delta}{4}.$$

Hence, there is $\varepsilon > 0$ and a neighbourhood U of $\{\omega; \nabla^S g_p(\omega) = 0\}$ such that $W \notin U$, and for $\tau > \tau_k$, $g_p(\omega(\tau))$ decreases by at least ε while $\omega(\tau) \notin U$, and can increase by at most $\varepsilon/2$ while $\omega(\tau) \in U$. Since $W \notin U$, and $\omega(\tau_k) \rightarrow W$, $g_p(\omega)$ is not bounded below on $\{\omega(\tau)\} \subseteq S^{n-1}$, which is a contradiction.

QED.

From Lemma 5.6, we see that $g_p(\omega(\tau)) \rightarrow$ constant, say γ , as $\tau \rightarrow \infty$.

From 5.4, $\gamma \geq 0$ since $r(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$.

If $\gamma > 0$, then $r(\tau) \leq C e^{-\delta t}$, for $\delta < \gamma$ and $\tau > \tau_k$. Hence, equation 5.5 has the form

$$5.7 \quad \frac{d}{d\tau} \omega(\tau) = -\nabla^{S^{n-1}} g_p(\omega) - R(\tau),$$

with $|R(\tau)| \leq C e^{-\delta\tau}$ as $\tau \rightarrow \infty$. Then the results of [S] (particularly Theorem 2) imply that $\omega(\tau)$ has a unique limit as $\tau \rightarrow \infty$.

REMARK 5.8 : This is the case in Chapter 3, i.e. $\omega(\tau) \rightarrow \hat{u}$ and $f_p(\hat{u}) \neq 0$ (see Lemma 3.26).

If $\gamma = 0$ however, then there are difficulties. The remainder terms

$$\sum_{q>0} r^q \nabla^{S^{n-1}} g_{p+q}(\omega)$$

may not be exponentially small compared with $\nabla^{S^{n-1}} g_p(\omega)$, and the results of [S] do not directly apply. Thom [TR] suggested reducing dimensions and arguing inductively, but it is not clear how to do this.

Problem 3

A similar question has been asked by T.C.Kuo (personal communication). If $x(t)$ satisfies 5.1, then does

$$\lim_{t \rightarrow \infty} \frac{x'(t)}{|x'(t)|} \text{ exist ?}$$

Problem 4

In Remark 5.8, we noted that the solutions found in Chapter 3 all have

$$\frac{u(t)}{|u(t)|} \rightarrow \hat{u}$$

where $\hat{u}$ is a stationary point for f_p , and all converge at rates given by $t^{-1/(p-2)}$.

APPENDIX A

Must all non exponential solutions be of this form ? It seems likely that there are examples where $f_p(\hat{u}) = 0$.

Must all non exponential solutions converge at some power rate ?

Problem 5

In Chapter 4, the minimal surfaces with non integrable Jacobi fields have codimension 2 or more in the unit sphere. Are there any codimension 1 minimal submanifolds of the unit sphere with non integrable Jacobi fields?

Problem 6

Are there any area minimising surfaces which do not have subanalytic support ?

Problem 7

Do small perturbations of area minimising cones with isolated singularities in any codimension give area minimising surfaces ?

REMARK 5.9 : Small perturbations of strictly minimising hypercones (codimension 1) with isolated singularities give area minimising surfaces by a result of [HS]. Hence a codimension 1 strictly minimising example for Problem 5 would be an example for Problem 6.

APPENDIX A

Exponential decay.

Here we prove exponential decay in the case when Jacobi fields are integrable. This was proved in [AA] for the case of minimal surfaces, and in [SL] for functionals of the type considered in Chapter 1. The method here is a simplification of [SL, Part II, §6], due to Professor Simon; it is based on De Giorgi's "blow-up" method.

We let F_Σ be a functional of the form

$$F_\Sigma(u) = \int_\Sigma F(\omega, u, \nabla u)$$

where Σ is a compact submanifold of $\mathbb{R}^N$, and $F(\omega, z, p)$ is a smooth function of $(\omega, z, p) \in \Sigma \times \mathbb{R}^N \times \mathbb{R}^{NM}$, real analytic at 0 in (z, p) . We also assume the Legendre-Hadamard condition 1.10, so that the Euler-Lagrange operator M_Σ is quasilinear elliptic with a self-adjoint linearisation L_Σ at 0 which is elliptic.

We consider general equations on $\Sigma \times (0, \infty)$ of the form

$$A.1 \quad u'' - u' - M_\Sigma u = R(u),$$

where $R(u) = a_1 \cdot Du' + a_2 \cdot u'$, with $D = (\nabla, \partial/\partial t)$ as in Chapter 1, and

$$A.2 \quad |a_1| + |a_2| \leq C(|u| + |\nabla u| + |u'|).$$

(Notice of course that this includes the equations considered in Chapter 3.)

As in Chapter 3, we note that $\ker \mathbb{L}_\Sigma$ is a finite dimensional subspace $\mathbb{K}$ of $L^2(\Sigma; \mathbf{V})$. Here we assume that

A.3 every element of $\mathbb{K}$ is integrable, in the same sense as Chapter 3, 3.11 :

that is, for all $v \in \mathbb{K}$, there is a one parameter family $\{u_s\}_{s \in (0,1)}$ with $u_s/s \rightarrow v$ and $\mathbb{M}_\Sigma(u_s) = 0$ for all $s \in (0,1)$. As noted in Theorem 3.6, this implies there is a real analytic function H , defined in a neighbourhood $\{v; |v| < \rho\}$ of 0 in $\mathbb{K}$ with values in $\mathbb{K}^\perp$ such that $|H(v)| \leq C|v|^2$ and such that the function $\mathbb{F}_\Sigma(v+H(v))$ is constant for $|v| < \rho$ (and, in particular, $\mathbb{M}_\Sigma(v+H(v)) \equiv 0$ for $|v| < \rho$, $v \in \mathbb{K}$). We let $S \subset L^2(\Sigma; \mathbf{V})$ be the real analytic manifold $\{v+H(v); |v| < \rho\}$ and observe that this manifold has tangent space $\mathbb{K}$ at $v = 0$.

Subject to the assumptions A.1 - A.3, we want to show that

if $\|u(t)\|_{C^1} + \|u'(t)\|_{C^0} \rightarrow 0$ as $t \rightarrow \infty$, then the convergence is actually exponential :

that is, there is a $q > 0$ such that

$$\|u(t)\|_{C^1(\Sigma; \mathbf{V})} + \|u'(t)\|_{C^0(\Sigma; \mathbf{V})} \leq C e^{-qt} \text{ as } t \rightarrow \infty.$$

Of course, by elliptic estimates, it is enough to prove $\|u(t)\|_{C^0} \leq C e^{-qt}$ because we already know that $\|u(t)\|_{C^1} \rightarrow 0$.

To prove the result, take any $T \geq 1$, and a sequence $v_j \rightarrow 0$ in $\mathbb{K}$ with

$$A.4 \quad \sup_{t \in [jT, (j+1)T]} \|V_j - u(t)\|_{C^0(\Sigma; \mathbf{V})} = \inf_{v \in \mathbb{K}} \sup_{t \in [jT, (j+1)T]} \|V - u(t)\|,$$

where, for $v \in \mathbb{K}$, $V = v+H(v)$, where H is as in 3.6.

We claim that if T is sufficiently large, then

$$A.5 \quad \sup_{t \in [(j+1)T, (j+2)T]} |V_{j+1} - u(t)|_{C^0(\Sigma; V)} \leq \frac{1}{2} \sup_{t \in [jT, (j+1)T]} |V_j - u(t)|_{C^0(\Sigma; V)}.$$

We will iterate this inequality to obtain the required exponential convergence.

Write I_j for the interval $[jT, (j+1)T]$. We first note that by iterating A.5, for any j we obtain

$$\sup_{I_j} |V_j - u(t)| \leq C \left(\frac{1}{2}\right)^j = C e^{-qjT}, \text{ for some } q > 0.$$

Hence, $|V_j - V_{j+1}| \leq 2 C e^{-qjT}$, since this is a sup estimate and $(j+1)T \in I_j \cap I_{j+1}$. However $V_j \rightarrow 0$ so we deduce $|V_j| \leq C' e^{-qjT}$ by summing the series $\sum (V_j - V_{j+1})$. Substituting back into the previous inequality, we conclude that $|u(t)| \leq C e^{-qt}$.

For the proof of A.5, notice that, since $\mathbb{M}_\Sigma(V_j) = 0$, we can rewrite A.1 to give

$$w_j'' - w_j' - \mathbb{L}_\Sigma w_j = a_1^{(j)} \cdot D^2 w_j + a_2^{(j)} \cdot Dw_j + a_3^{(j)} \cdot w_j,$$

where $|a^{(j)}| \leq C(|u| + |\nabla u| + |u'|)$ and $w_j = u - V_j$ so $w_j' = u'$. Since $w_j \rightarrow 0$, we know by A.2 and the general growth estimates ([S] or [SL, Part 2, §3]) that for any $\varepsilon > 0$, T sufficiently large ($T \geq T_0(\varepsilon, \mathbb{L})$), there are j_0, τ_0 such that for all $\tau \geq \tau_0$ and all $j \geq j_0$,

$$A.6 \quad \sup_{[\tau, \tau+T]} |w_j(t)| \leq e^{\varepsilon T} \sup_{[\tau-T, \tau]} |w_j(t)|.$$

(From now on, $|w(t)|$ denotes $|w(t)|_{C^0(\Sigma; V)}$.)

Exponential decay

Now let $\underline{w}_j(t) = \varepsilon_j^{-1} w_j(t+jT)$, where $\varepsilon_j = \sup_{[jT, (j+1)T]} |V_j - u(t)|$. We assume that $\varepsilon_j > 0$; otherwise A.6 trivially implies A.5.

To get a contradiction, we suppose that A.5 is false for infinitely many j . Then, by A.6 and standard ellipticity estimates (for example, L^p and continuity estimates [GT]), we can find a subsequence j' such that

A.5 fails with $j = j'$,

$\underline{w}_{j'} \rightarrow w$ locally on $\Sigma \times (0, \infty)$ with $w'' - w' - \mathbb{L}_\Sigma w = 0$,

$\sup_{(0,T]} |w(t)| \leq 1$,

$\sup_{[T, 2T]} |w(t)| \geq 1/2 > 0$,

and $\sup_{[\tau, \tau+T]} |w| \leq e^{\varepsilon T} \sup_{[\tau-T, \tau]} |w|$, for each $\tau > 0$.

Then, taking $\varepsilon < 1/2 \min \{ \operatorname{Re} \gamma_j ; \operatorname{Re} \gamma_j > 0 \}$ (notation as in Chapter 2), and using L^2 estimates [GT] and standard eigenfunction expansions, we have

$$w = v + \operatorname{Re} \sum_J a_j e^{-\gamma_j t} \phi_j,$$

where $v \in \mathbb{K}$, ϕ_j are eigenfunctions, $\sum a_j^2 < C < \infty$, and $J = \{ j ; \operatorname{Re} \gamma_j \geq \alpha > 0 \}$. Thus by taking T sufficiently large, it evidently follows that

$$\sup_{t \in [3T/4, 3T]} \frac{\|w - v\|}{L^2(\Sigma; \mathbf{V})} \leq C e^{-\alpha T/4} \sup_{t \in [T/4, T/2]} \frac{\|w\|}{L^2(\Sigma; \mathbf{V})}.$$

Then by elliptic estimates

$$\sup_{[T, 2T]} |w - v| \leq C' e^{-\alpha T/4} \sup_{[T/4, T/2]} |w|.$$

Exponential decay

Choose T so that $C' e^{-\alpha T/4} \leq 1/4$. Hence, for sufficiently large j' , since w is not identically 0 on $[T/4, T/2]$ by virtue of the eigenfunction expansion and the fact that w is not identically 0 on $[T, 2T]$,

$$\sup_{t \in [(j'+1)T, (j'+2)T]} |u(t) - V_{j'} - \varepsilon_{j'} v| \leq \frac{1}{2} \sup_{t \in [j'T, (j'+1)T]} |u - V_{j'}|.$$

Since S is a C^1 manifold with $\mathbb{K} = T_0 S$, and since $V_{j'} \rightarrow 0$, we have a sequence $\xi_{j'}$,

$$\xi_{j'} \in T_{V_{j'}} S$$

with $\eta_{j'} := |\xi_{j'} - v| \rightarrow 0$. Then, since $|\xi_{j'}| \leq |v| + \eta_{j'}$, and since S is a C^2 manifold, we have $\zeta_{j'} \in S$ such that

$$|\zeta_{j'} - (V_{j'} + \varepsilon_{j'} v)| \leq |\zeta_{j'} - (V_{j'} + \varepsilon_{j'} \xi_{j'})| + \eta_{j'} \varepsilon_{j'}$$

$$\leq C (|v| + \eta_{j'})^2 + \eta_{j'} \varepsilon_{j'},$$

where C depends on the curvature of S near 0.

Thus A.7 gives

$$\sup_{t \in [(j'+1)T, (j'+2)T]} |u - \zeta_{j'}| \leq 1/3 \varepsilon_{j'} + \varepsilon_{j'} ((|v| + \eta_{j'}) \varepsilon_{j'} + \eta_{j'})$$

$$\leq 1/2 \varepsilon_{j'},$$

for sufficiently large j' , which is a contradiction.

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