# $S^1$-invariant Monopoles on Hyperbolic Three Space

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# Declaration

The work in this thesis is my own except where otherwise stated.

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# Abstract


In this thesis we review various properties of monopoles on three dimensional manifolds. In Chapter 2 we prove some basic properties of the moduli space. We follow Braam's treatment in [B89] and define monopoles on three manifolds as instantons on four manifolds invariant under a rotation action. This correspondence allows us to study monopoles with similar methods developed for instantons. By constructing the Kuranishi map and applying the implicit function theorem, we can obtain a local model of the monopole moduli space. We prove that the moduli space is smooth and orientable. The proof uses the same techniques as the instanton case. The monopole moduli space can be compactified by adding limit ideal connections. This is the method of Uhlenbeck compactification. The treatment is again similar to the four-manifold case with minor modifications regarding the invariance condition. To better understand the structure of the compactified space, we introduce a method of constructing monopoles.

In Chapter 3 we review the ADHM construction and use it to describe the charge 1 instanton moduli space.

In Chapter 4 we calculate two examples: monopoles on hyperbolic 3-space and monopoles on $S^2 \times \mathbb{R}$, in which we construct the $S^1$-equivariant $SU(2)$ bundles, define the charge and mass and describe the local model for the moduli space explicitly using the Kuranishi map. Moreover, we consider another $S^1$ action for both cases and describe the $S^1$-invariant monopole moduli space.

Lastly we finish with a brief introduction of Sen's conjecture.

# Contents

# Chapter 1

# Introduction

The concept of a magnetic monopole was introduced by Paul Dirac in [D31]. A monopole is a point particle-like object of magnetic charge described by the Bogomolny equation

$$F(A) = - * d_A\Phi \tag{1.0.1}$$

where $A$ is a connection on a principal $SU(2)$-bundle $Q$ over $M$ with energy conditions which, translated in language of differential equations, is given by the boundary conditions

$$\begin{aligned} &E(A,\Phi) := (8\pi)^{-1}\int_M |F(A)|^2 + |d_A\Phi|^2 d\mathrm{vol}_M < \infty \\ &|\Phi(x)| \to m_j \ \text{ when } \ x \text{ approaches boundary of } M \ (m_j \in R_{\geq 0}) \end{aligned} \tag{1.0.2}$$

Here $M$ is the three-dimensional Riemannian manifold on which the monopole is defined and $\Phi$ a section of the associated adjoint bundle ad$Q$ of $Q$.

In this thesis we take a pure mathematical point of view. Let $M$ be an oriented complete Riemannian 3-manifold, which is the interior of a compact 3-manifold with boundary consisting of a disjoint union of surfaces $S = \sqcup_{j=1}^N S_j$. Topologically, $M$ is a manifold without boundary with ends $\mathbb{R}_{>0} \times S_j$. The metric on an end $\mathbb{R}_{>0} \times S_j$ in $M$ is approximately of the form $dl^2 + e^{2l}ds^2_{S_j}$. A monopole is a pair $(A_3, \Phi)$ satisfying the Bogomolny equations and boundary conditions.

In Chapter 2 we follow Braam's treatment in [B89] to study various properties of a monopole and the monopole moduli space, which we will apply in examples in Chapter 4. We begin by constructing the 4-manifold $X$ as a conformal compactification of $M \times S^1$ and a principal $SU(2)$-bundle $P$ on $X$. Topologically, $X$ is $M$ spun around the boundary surfaces $\partial\overline{M} = \cup_j S_j$. $M \times S^1$ carries an $S^1$-action which fixes the first part and acts canonically on the second. This action carries over to $X$ and is free away from the fixed surfaces. The connection $A$ on $P$ defined as the pull-back of $(A_3, \Phi)$ by the projection map $\pi : M \times S^1 \to M$ satisfies the anti-self-dual Yang-Mills equation if $(A_3, \Phi)$ satisfies the Bogomolny equation. This construction motivates and justifies the central definition of the chapter. We define a monopole on $(Q, M)$ as an ASD connection on $P$ invariant under the $S^1$-action and the monopole moduli space as the space of $S^1$-invariant instantons quotient by $S^1$-equivariant gauge transformations. The benefit of this definition is a two-fold: it gives a clear geometric meaning of a monopole and the monopole moduli space; more importantly, it hints that most techniques used to study monopoles and moduli spaces are essentially the same as those previously developed for instantons with minor modifications dealing with the $S^1$-invariance condition.

Next we give a mathematical definition of the mass and charge of a monopole. The $S^1$-action on $X$ lifts to a double cover action denoted by $\tilde{S}^1$ on $P$. The associated bundle $P \times_{SU(2)} \mathbb{C}^2$ to the fundamental representation admits a splitting of two complex line bundles $L_j$ and $L_J^*$ with respect to the $\tilde{S}^1$-action. Hence the action on each line bundle is fibre preserving. The mass $m_j$ is then defined as the winding number of the $\tilde{S}^1$-action on the line bundle $L_j$. The charge $k_j$ is defined as the first Chern class of the line bundle $L_j^*$. We prove that the isomorphism classes of $S^1$-equivariant $SU(2)$-bundles on $X$ are in one-to-one correspondence with tuples of integers $(m_j, k_j)$.

In section 2.2 we describe the deformation theory of monopole moduli space, that is the study of a local geometric structure of the moduli space. This requires a notion of differential structure for most relevant spaces. We follow section 4.4 of [M98] and define the Sobolev norm on the space of $C^\infty$-sections on the vector bundle $\mathrm{ad}P \to X$ as

$$\|s\|_{p,k} = \int_X \|s(x)\|^p + \|d_A(s(x))\|^p + \cdots + \|d_A^k(s(x))\|^p$$

for $s \in \Gamma(\mathrm{ad}P)$. The completion of the space with respect to this norm is the Sobolev space $\mathscr{W}^0_{k,p} = L^0_{k,p}(\Gamma(\mathrm{ad}P))$ defined as [1]

$$\mathscr{W}^0_{k,p} = \left\{ s \;\middle|\; \exists s_i \in \Gamma(\mathrm{ad}P) \text{ such that } \lim_{i\to\infty} \|s_i - s\|_{p,k} = 0 \right\}.$$

The space of $\tilde{S}^1$-invariant $L_{p,k}$-connections $\mathscr{A}_{p,k}(P)^{\tilde{S}^1}$ is a closed affine Banach manifold. We have the Atiyah-Hitchin-Singer deformation complex associated to

[1] Here $\Gamma(\mathrm{ad}P)$ is the space of sections of the adjoint bundle of $P$.

an ASD connection $A_0$ given by

$$\mathscr{W}^0_{p,k+1} \xrightarrow{d_{A_0}} \mathscr{W}^1_{p,k} \xrightarrow{d^+_{A_0}} \mathscr{W}^{2,+}_{p,k-1}.$$

where $\mathscr{W}^i_{p,k}$ is the space of $L_{p,k}$ $i$-forms on the adjoint bundle ad$P$ of $P$. It follows from the slice theorem that a slice at $A_0$ is given by $\mathscr{W}^1_{p,k} \cap \ker(d^*_{A_0})$. Finally we use the implicit function theorem to construct the Kuranishi map whose zero set is a local neighbourhood of $[A_0]$.

In section 2.3, 2.4 and 2.5 we prove that the moduli space is smooth and orientable and has a compactification. The smoothness and orientability proof are given by Braam in [B89], the technique is essentially the same as the proof in [FU84] except we utilise a result by Cho in [C91] to incorporate the $S^1$-invariance condition into the proof of smoothness. The compactification for the monopole moduli space uses the same technique (known as Uhlenbeck compactification) as described in [DK90] and [FU84]. We define the ideal connection $([A], (x_1, \ldots, x_l))$ with curvature density given by

$$|F(A)|^2 + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r}.$$

The curvature (or energy) tends to infinity at the removed points $(x_1, \ldots, x_l) \in X^l$. Such points must lie in the fixed surfaces $S_j$ so that the total mass of the curvature is finite. A sequence of points $[A_\alpha]$ in the moduli space converges towards a limit point $([A_\infty], (x_1, \ldots, x_l))$ if the action densities of $\{A_\alpha\}$ converge as measures and a gauge equivalent sequence converge in $C^\infty$ norm on the punctured manifold. The compactification is then defined to be the closure of the moduli space $M_k$ in the space of ideal connections.

This procedure gives limit points $[A_\infty]$ that lose $l$ lumps of energy as instantons, the energy difference corresponds to the charge difference in the monopole case.

To better understand the structure of the moduli space near the limit point, we introduce a gluing theory to construct monopoles near limit points of the compactified moduli space. The core machinery is the connected sum of two manifolds.

In Chapter 3 we introduce ADHM construction to describe instantons and define the ADHM data corresponding to $S^1$-invariant instantons on $\mathbb{R}^4$. We use ADHM data to give explicit description of the monopole moduli space with mass 1 and charge 1.

In chapter 4 we introduce another $\Gamma$-action on $M$ and apply theories developed in chapter 2 to study $\Gamma$-invariant monopoles. The first example is calculating $\Gamma \cong S^1$-invariant monopoles on 3-dimensional hyperbolic space $\mathbb{H}^3$. In particular, we will show that the moduli space $\mathcal{M}(m, 1)$ of mass $m$ and charge 1 monopoles on $\mathbb{H}^3$ has dimension 3 and is parametrized by translations (along $\mathbb{R} \times \{0\}$) and dilations of $\mathbb{R}^4$. That is, $\mathcal{M}(m, 1)$ is diffeomorphic to $S^2 \times \mathbb{R}_{>0}$. From this results we can deduce the result involving the $\Gamma$-action. That is, $\mathcal{M}^\Gamma(k, 1) \cong \mathbb{R}_{>0}$. In the second example we take $M = S^2 \times \mathbb{R}$. The compactification of $M$ can be identified as $S^2 \times S^2$. The $\tilde{S}^1$-equivariant $SU(2)$ bundle $E$ over the compactified space is then constructed from the complex line bundle $\mathcal{O}(1) \to S^2$ (i.e. the associate bundle of the standard Hopf bundle) as follows. The bundle $\mathcal{O}(n)$ defined as $\mathcal{O}^{\otimes n}$ admits an $\tilde{S}^1$-action when the weights $\omega_0$, $\omega_\infty$ satisfy the condition $\omega_0 - \omega_\infty = -2n$, $n \in \mathbb{Z}$. Then we define a bundle $\mathcal{O}(n, m)$ also with $\tilde{S}^1$-action as $\mathcal{O}(n, m) = \pi_1^*\mathcal{O}(n) \otimes \pi_2^*\mathcal{O}(m)$ where $\pi_i$ is the projection $S^2 \times S^2 \to S^2$ onto the $i$-th factor. The $\tilde{S}^1$-equivariant $SU(2)$ bundle $E$ is then defined by

$$E = \mathcal{O}(k, -k) \oplus \mathcal{O}(-k, k)$$

Following the definition in chapter 2, the mass and charge of the monopole can be expressed in terms of the weights $\omega_0$, $\omega_\infty$ and the order $k$ and check the index formula given in [B89]. Moreover, we study the local model of a monopole by calculating the (leading term of the) Kuranishi map. Without the $\tilde{S}^1$-action, we have

$$H^1(X, \mathcal{O}(2k, -2k)) \cong \mathbb{C}^{4k^2-1}$$

and the (leading term of the) Kuranishi map is given by

$$\begin{aligned} q :& H^1(X, \mathcal{O}(2k, -2k)) \oplus H^1(X, \mathcal{O}(-2k, 2k)) \longrightarrow \mathbb{R} \\ (\alpha, \beta) &\longmapsto \frac{1}{2}(||\alpha||_{L^2}^2 - ||\beta||_{L^2}^2) \end{aligned} .$$

Therefore a neighbourhood of $[A]$ in the ASD instanton moduli space is given by

$$q^{-1}(0)/\Gamma_A = \left\{ (\alpha, \beta) \in \mathbb{C}^{4k^2-1} \oplus \mathbb{C}^{4k^2-1}; \|\alpha\|_{L^2}^2 = \|\beta\|_{L^2}^2 \right\} /U(1) \tag{1.0.3}$$

where $\Gamma_A$ is the stabiliser of $A$ and $U(1)$ acts by $\lambda \cdot (\alpha, \beta) = (\lambda\alpha, \lambda^{-1}\beta)$. That is, the neighbourhood is a cone over the manifold

$$S^{8k^2-3} \times S^{8k^2-3}/U(1).$$

This calculation is given in Braam's work [B89] in less detail. In this thesis we move on to calculate the Kuranishi map in the presence of the $\tilde{S}^1$-invariance condition. The addition of the $\tilde{S}^1$-invariance condition does not change the computation of the Kuranishi map. However, the $\tilde{S}^1$-invariant part of the cohomology groups $H_0^i$ depends on the sign of $k$, $\omega_0$ and $\omega_\infty$. In summary, there are five cases to consider:

(1) $\omega_0 > 0$,

(2) $\omega_0 < 0$ and $\omega_\infty = 0$,

(3) $\omega_0 < 0$ and $\omega_\infty > 0$,

(4) $\omega_0 < 0$ and $\omega_\infty < 0$,

(5) $\omega_0 = 0$.

We calculate the (leading term of the) Kuranishi map in (1) and (4) is given by

$$q : \{\text{point}\} \longrightarrow \mathbb{R};$$

in (2) and (5) the map is given by

$$\begin{aligned} q : 0 \oplus \mathbb{C}^{2k-1} &\longrightarrow \mathbb{R} \\ \beta &\longmapsto -\frac{1}{2}\|\beta\|^2_{L^2}. \end{aligned}$$

All four cases are trivial as $q^{-1}(0) = \{\text{point}\}$.The non-trivial case occurs when $\omega_0 < 0$ and $\omega_\infty > 0$, we have

$$\begin{aligned} q : \mathbb{C}^{2k+1} \oplus \mathbb{C}^{2k-1} &\longrightarrow \mathbb{R}, \\ (\alpha, \beta) &\longmapsto \frac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right). \end{aligned}$$

and $q^{-1}(0)/U(1)$ is a cone over the manifold

$$S^{4k+1} \times S^{4k+1}/U(1).$$

Finally for $\Gamma = S^1$ we consider another $\Gamma$-action and describe the $\Gamma$-invariant monopoles on $S^2 \times \mathbb{R}$. We apply the same method as before to calculate the local model of a monopole in the moduli space $\mathcal{M}^\Gamma(E)$. Four of the five cases turn out to be trivial again and in the non-trivial case, we have

$$\begin{aligned} q : \mathbb{C} \oplus \mathbb{C} &\longrightarrow \mathbb{R} \\ (\alpha, \beta) &\longmapsto \frac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right) \end{aligned}$$

and hence

$$q^{-1}(0) = \{(\alpha, \beta) \mid |\alpha|^2 = |\beta|^2\}.$$

and thus $(q^{-1}(0)\backslash\{0\})/\Gamma_A = (q^{-1}(0)\backslash\{0\})/S^1$ is diffeomorphic to $S^1 \times \mathbb{R}^+$. Therefore a neighbourhood of monopole in $\mathcal{M}^\Gamma(E)$ is a cone over $S^1$. These two examples are discussed in [B89] without the addition of $\Gamma$-action.

In the last chapter we introduce briefly Sen's conjecture. It is a topic of recent interest on studying the $L^2$ harmonic forms on the moduli space. The conjecture is originally given in Sen's physics paper [S94] to test the conjectured S-duality in $N = 2$ supersymmetric Yang-Mills theory. The conjecture has been formulated in mathematical language and explained in more detail in [H98], [SS95], [S96] and some recent progress by Singer and . The topic may also be the author's further direction of research.

# Chapter 2

# Monopoles on 3-manifolds

## 2.1 Monopole-instanton correspondence

Following the setup of [B89], we start with a compact, oriented 3-manifold $\bar{M}$ with a disjoint union of boundary surfaces $\partial M = \sqcup_j S_j,\ \ j = 1, \ldots, N$. Each $S_j$ is a compact surface without boundary. Denote by $M = \bar{M} \backslash \partial M$ the interior of $\bar{M}$. We assume $M$ carries a complete Riemannian metric, which means the boundary surfaces $S_j$ lie at infinity.

Let $Q \to M$ be a principal $G$-bundle. In this thesis we take $G = SU(2)$. Let $A_3$ be a connection on $Q$, and $\Phi$ a section of ad$Q$, called the Higgs field.

Consider the bundle $Q \times S^1 \to M \times S^1$ where the $S^1$-action fixes $M$ and act on the $S^1$ part by rotation. Let $Q \times S^1 \to M \times S^1$ be the pullback bundle of $Q$ under the projection map $\pi : M \times S^1 \to M$. Then $Q \times S^1$ inherits the $S^1$-action

$$(q, e^{i\theta})g = (q, e^{i\theta}g)$$

for $(q, e^{i\theta}) \in Q \times S^1$, $g \in S^1$. Denote by $\partial/\partial\theta$ the vector field on $Q \times S^1$ generated by this $S^1$-action, and $d\theta$ the dual one-form. Then every $S^1$-invariant monopole on $Q \times S^1$ can be uniquely written in terms of $A_3$ as

$$A = \pi^* A_3 + (\pi^* \Phi) d\theta \tag{2.1.1}$$

To study non-trivial monopoles we introduce an $S^1$-equivariant conformal compactification $X$ of $M \times S^1$, constructed $X$ as follows: glue $M \times S^1$ and $\sqcup_j S_j \times D^2$ by identifying $S_j \times S^1 \times (1, \infty) \subset M \times S^1$ with $S_j \times S^1 \times (0, 1) \subset \sqcup_j S_j \times D^2$ in an $S^1$-equivariant way. The open set $(1, \infty)$ is identified with $(0, 1)$ via the map

$p \mapsto 1/p$. Here $S^1$ acts on $S_j \times D^2$ by the standard rotation on $D^2$ hence $S_j$ is the fixed point set. That is

$$X = (M \times S^1) \bigcup (\sqcup_j S_j \times D^2)/\sim .$$

Observe that $X \backslash (M \times S^1) = \cup_j S_j$. We can view $X$ as $M$ spun around its boundary surfaces as indicate in Figure (2.1).

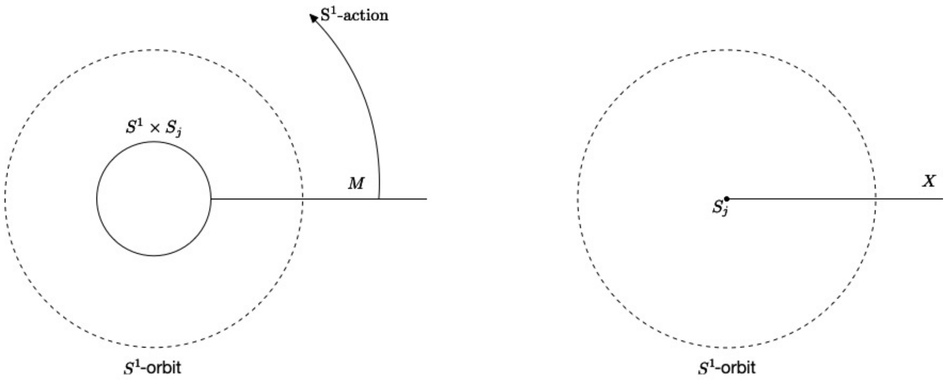



Figure 2.1: compactification of $M \times S^1$

The quotient map $\bar{M} \times S^1 \to X$ identifies $S_j \times S^1$ in $\bar{M} \times S^1$ with the boundary of $S_j \times D^2$. Therefore the construction of the bundle over $X$ requires an appropriate gluing of the fibres over $S_j \times S^1$. The $SU(2)$-bundle $P$ over $X$ is formed by gluing all the fibres on the boundary surfaces to the fibre over $S_j \times \{1\}$. Equivalently, we choose a homomorphism $f : S^1 \to SU(2)$ and each fibre over some point $x$ in $S_j \times S^1$ is identified to the fibre at $S_j \times \{1\}$, $1 = \mathrm{id} \in S^1$, as indicated in Figure (2.2). The bundle $P$ depends on the choice of $f$.

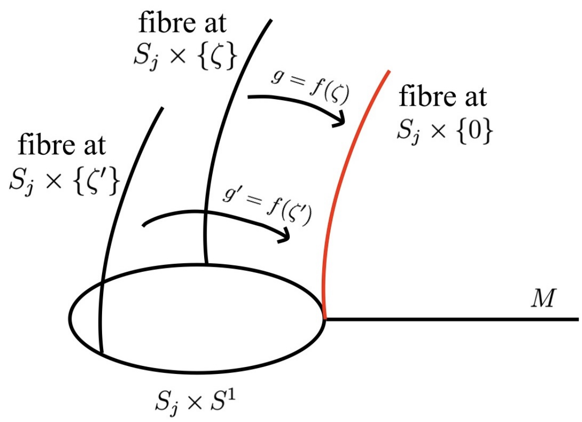



Figure 2.2: Construction of the bundle $P$

Next we want to lift the $S^1$-action on $X$ to an action on $P$. Let $\mathscr{A}(P)$ be the space of connections on $P$ and $\mathscr{G}(P)$ be the group of gauge transformations. Define $\mathscr{H}$ to be the group of all bundle automorphisms of $P$ which cover the action of some element $\theta$ of $S^1$ on $X$, that is

$$\mathscr{H} = \{f : P \to P,\ \pi \circ f = \theta \circ \pi \text{ for some } \theta \in S^1\}.$$

To be more specific, we denote by $f_\theta$ an automorphism covering the element $\theta$. $f_\theta$ is called a lift for the action $\theta$. Since $S^1$ is connected, it is always possible to find such a lift.

**Lemma 2.1.** *$f_\theta$ is unique up to gauge transformations.*

*Proof.* It is clear that two different lifts of the same $\theta \in S^1$ differ by some element in $\varphi \in SU(2)$, that is $f'_\theta = \varphi \cdot f_\theta$. Since $f_\theta$ and $f'_\theta$ are $SU(2)$-invariant,

$$f'_\theta(p \cdot g) = f'_\theta(p) \cdot g$$

and

$$f_\theta(p \cdot g) = f_\theta(p) \cdot g$$

for $p \in P$ and $g \in SU(2)$, then

$$\begin{aligned} & f'_\theta(p \cdot g) = \varphi(p \cdot g) \cdot f_\theta(p \cdot g) \\ \Rightarrow\quad & f'_\theta(p) \cdot g = \varphi(p \cdot g) \cdot f_\theta(p \cdot g). \end{aligned}$$

This implies $\varphi(p) = \varphi(p \cdot g)$ and hence $\varphi$ is a gauge transformation. □

By definition, the gauge group $\mathscr{G}(P)$ can be viewed as a subgroup of $\mathscr{H}$ and we have the following short exact sequence

$$\mathscr{G}(P) \longrightarrow \mathscr{H} \longrightarrow S^1 \tag{2.1.2}$$

where the first map is inclusion and the second is defined by $f_\theta \mapsto \theta$.

The above lemma implies $S^1$ acts on the quotient space $\mathscr{B}^* = \mathscr{A}^*(P)/\mathscr{G}(P)$ by pull-back, that is $\theta \cdot [A] \mapsto [f_\theta^* A]$.

**Proposition 2.2.** *An equivalence class of connections $[A]$ is a fixed point if and only if there is an exact sequence*

$$\mathscr{G}_A \longrightarrow \mathscr{H}_A \longrightarrow S^1 \tag{2.1.3}$$

*where $\mathscr{G}_A$ is the stabiliser of $A$ and $\mathscr{H}_A \subset \mathscr{H}$ is the set of bundle automorphisms fixing $A$.*

*Proof.* If $[A]$ is a fixed point, that is $[f_\theta^* A] = [A]$, then for some $\varphi \in \mathscr{G}(P)$,

$$\begin{aligned} & \varphi^* f_\theta^* A = A \\ \Rightarrow \quad & (f_\theta \circ \varphi)^* A = A. \end{aligned}$$

This means $f_\theta \circ \varphi$ is a lift of $\theta$ and hence the map $\mathscr{H}_A \to S^1$ is surjective. Conversely, for any $\theta \in S^1$, there exists $f_\theta \in \mathscr{H}_A$ which, by definition, fixes $A$. □

If we restrict to irreducible monopoles, then the stabiliser $\mathscr{G}_A = \{\pm 1\}$ (specific to the case where the structure group is $SU(2)$). If the sequence

$$\{\pm 1\} \longrightarrow \mathscr{H}_A \longrightarrow S^1$$

splits, we have $\mathscr{H}_A \cong \mathbb{Z}_2 \times S^1$. Otherwise it follows from the short exact sequence that $\mathscr{H}_A \cong S^1$ is a non-trivial double cover of $S^1$. We denote this double cover by $\tilde{S}^1$. The former case means the $S^1$-action lifts to an $S^1$-action, which is given by the $\tilde{S}^1$-action. Thus we consider only the latter case. Combining with the calculation above, we arrive at the result that if $[A]$ is an $S^1$-fixed point in $\mathscr{B}^*$, then there is a unique $\tilde{S}^1$-action stabilising $A$ (which is represented by $f_\theta \circ \varphi$ above). Also the $S^1$-action lifts to the double cover action on the bundle $P$.

**Lemma 2.3.** *Given a connection $A_3$ and a bundle endomorphism $\Phi$, $(A_3, \Phi)$ satisfies the Bogomolny equation (1.0.1) with boundary condition (1.0.2) if and only if there exists an $S^1$-equivariant bundle $P$ on which $A := A_3 + \Phi d\theta$ is $S^1$-invariant and satisfies the anti-self-dual equation*

$$F_A = - *_4 F_A \tag{2.1.4}$$

*Proof.*

$$\begin{aligned} F_A &= dA + A \wedge A \\ &= d(\pi^* A_3) + d(\pi^*\Phi) \wedge d\theta + \pi^*(A_3 \wedge A_3) + \pi^*(A_3 \wedge \Phi d\theta) + \pi^*(\Phi d\theta \wedge A_3) \\ &= (\pi^* dA_3 + \pi^*(A_3 \wedge A_3)) + \pi^*(d\Phi) \wedge d\theta + \pi^*(A_3 \wedge \Phi d\theta) + \pi^*(\Phi d\theta \wedge A_3) \\ &= \pi^*(F_{A_3}) + \pi^*(d\Phi) \wedge d\theta + \pi^*(A_3 \wedge \Phi d\theta) + \pi^*(\Phi d\theta \wedge A_3) \end{aligned}$$

Acting $*_4$ on both sides gives

$$*_4(F_A) = *_4\big(\pi^*(F_{A_3}) + \pi^*(d\Phi) \wedge d\theta + \pi^*(A_3 \wedge \Phi d\theta) + \pi^*(\Phi d\theta \wedge A_3)\big).$$

Since the first term $\pi^*(F_{A_3})$ is independent of $d\theta$,

$$*_4 \pi^*(F_{A_3}) = *_3(\pi^*(F_{A_3})) \wedge d\theta.$$

Then we have

$$*_4(F_A) = *_3(\pi^*(F_{A_3})) \wedge d\theta + *_3\big(\pi^*(d\Phi) + \pi^*(A_3 \wedge \Phi) + \pi^*(\Phi \wedge A_3)\big).$$

This implies

$$\begin{aligned} F_A + *_4 F_A &= \pi^*(F_{A_3}) + \pi^*(d\Phi) \wedge d\theta + \pi^*(A_3 \wedge \Phi d\theta) + \pi^*(\Phi d\theta \wedge A_3) \\ &\quad + *_3\,(\pi^*(F_{A_3})) \wedge d\theta + *_3\big(\pi^*(d\Phi) + \pi^*(A_3 \wedge \Phi) + \pi^*(\Phi \wedge A_3)\big) \\ &= \pi^*(d_{A_3}\Phi + *_3 F_A) + (d_{A_3}\Phi + *_3 F_A). \end{aligned}$$

It remains to check that an anti-self-dual connection $A$ satisfies the boundary condition (1.0.2).

$$\begin{aligned} YM(A) &= \frac{1}{16\pi^2}\int_X |F_A|^2 dV_X \\ &= \frac{1}{16\pi^2}\int_{U_\epsilon(S_j)} |F_A|^2 dV_{U_\epsilon(S_j)} + \frac{1}{16\pi^2}\int_{X\setminus U_\epsilon(S_j)} |F_A|^2 dV_{X\setminus U_\epsilon(S_j)} \end{aligned}$$

where $U_\epsilon(S_j)$ is a $\epsilon$-neighbourhood of $S_j$ in $X$. Since $A := \pi^*A_3 + (\pi^*\Phi)d\theta$, we have $F_A = \pi^*F_{A_3} + d_{A_3}\pi^*\Phi d\theta$ and $\pi^*F_{A_3}$ does not contain the $d\theta$ component, so

$$\begin{aligned} |F_A|^2 &= |\pi^*F_{A_3} + \pi^* d_{A_3}\Phi d\theta|^2 \\ &= |\pi^*F_{A_3}|^2 + |\pi^* d_{A_3}\Phi d\theta|^2 \\ &= |F_{A_3}|^2 + |d_{A_3}\Phi|^2. \end{aligned}$$

As $\epsilon \to 0$, the first integral tend to zero becasue $|F_A|^2$ is finite and the volume of $U_\epsilon(S_j)$ in $X$ tends to zero and the second integral becomes

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{1}{16\pi^2}\int_{X\setminus U_\epsilon(S_j)} |F_A|^2 dV_{X\setminus U_\epsilon(S_j)} &= \frac{1}{16\pi^2}\int_{M\times S^1} |F_A|^2 dV_{X\setminus U_\epsilon(S_j)} \\ &= \frac{1}{16\pi^2}\int_{S^1}\int_M |F_A|^2 dV_{X\setminus U_\epsilon(S_j)} d\theta \\ &= \frac{1}{8\pi}\int_M |F_A|^2 dV_{X\setminus U_\epsilon(S_j)} \\ &= \frac{1}{8\pi}\int_M |F_{A_3}|^2 + |d_{A_3}\Phi|^2 dvol_M \\ &= E(A_3, \Phi), \end{aligned}$$

Hence $E(A_3, \Phi)$ is finite.

□

**Definition 2.4.** An $SU(2)$*-monopole on* $P$ is an $S^1$-invariant instanton on $P$. The *monopole moduli space* $\mathcal{M}(P)$ is the space

$$\mathcal{M}(P) = \{S^1\text{-invariant instantons}\}/\mathscr{G}(P)^{S^1}.$$

where $\mathscr{G}(P)^{S^1} = \Gamma(X, P \times_{Ad} G)^{S^1}$ is the group of gauge transformation on $P$ satisfying $\pi \circ f = \theta \circ \pi$ where $\pi : P \to M$ is the projection and $\theta \in \mathbf{S}^1$.

To better understand the monopole moduli space, we now aim to study the space $\{S^1\text{-invariant instantons}\}/\mathscr{G}(P)^{S^1}$. The first step is to classify $\tilde{S}^1$-equivariant $SU(2)$ bundles on $X$. We start by introducing two important topological invariants of the equivariant bundle $P$, namely the mass and charge of a monopole.

By definition, the mass $m_j$ is the value of the Higgs field $\Phi$ when $x \to S_j$. That is,

$$m_j = \lim_{x \to S_j} |\Phi(x)| = \lim_{x \to S_j} \left| A(\frac{\delta}{\delta\theta})(x) \right| \tag{2.1.5}$$

where $A := A_3 + \Phi d\theta$ and $j = 1, \dots, N$.

This means $m_j$ corresponds to the value of $|A(\partial/\partial\theta)|$ as $x \to S_j$ where $\partial/\partial\theta$ is the vector field generated by the $S^1$-action.

**Lemma 2.5.** *$m_j$ is integer valued.*

*Proof.* Identify the fibre $P|_x$ at $x$ with $G$ by $p \mapsto id \in G$. The family of $S^1$-actions on the orbit of $p$, which is isomorphic to $S^1$, is parametrized by an integer-valued winding number. That is for $q \in p \cdot S^1$ and $\theta \in S^1$,

$$q \cdot \theta = q e^{wi\theta}$$

for some winding number $w \in \mathbb{Z}_{>0}$.

Since $p$ is identified to $id \in G$, $\frac{\partial}{\partial\theta}|_p \in T_pP|_x \cong T_{id}G = \mathfrak{g}$. Therefore

$$A\left(\frac{\partial}{\partial\theta}|_p\right) = \frac{\partial}{\partial\theta}|_p$$

and by definition

$$\frac{\partial}{\partial\theta}|_p = \frac{\partial}{\partial\theta}|_{\mathrm{id}} := \frac{\partial}{\partial\theta}|_{\theta=0}\mathrm{id}e^{\omega i\theta} = wi.$$

Together we have

$$\left|A\left(\frac{\partial}{\partial\theta}|_p\right)\right| = |wi| = w.$$

Then (2.1.5) implies the desired result. ∎

Let $E = P \times_{SU(2)} \mathbb{C}^2$ where $\tilde{S}^1$ acts on $\mathbb{C}^2$ by fundamental representation of $SU(2)$. The fibres of $E|_{S_j}$ can be identified as $\mathbb{R}^4$ which can be split into two subspaces isomorphic to $\mathbb{R}^2$ with one being rotated by the $\tilde{S}^1$-action and its orthogonal complement. This means $\tilde{S}^1$-action splits $E|_{S_j}$ into two complex line bundles $L_j$ and $L_j^*$, i.e.

$$E|_{S_j} = L_j \oplus L_j^* \tag{2.1.6}$$

The splitting (2.1.6) is well-defined only for non-zero monopole mass. If the mass $m = 0$, then $\tilde{S}^1$-action is trivial on the $L_j$. This means $E|_{S_j}$ is trivial as an $\tilde{S}^1$-equivariant bundle.

The proof above tell us that an alternative definition of $m_j$ is the winding number of the action on the line bundles and the action can be explicitly written as

$$\begin{aligned} \tilde{S}^1 &\longrightarrow Aut(L_j \oplus L_j^*) \\ \theta &\longmapsto \begin{pmatrix} e^{\mathbf{i}m_j\theta} & 0 \\ 0 & e^{-\mathbf{i}m_j\theta} \end{pmatrix}. \end{aligned} \tag{2.1.7}$$

This description amounts to a geometric definition for the mass of a monopole:

**Proposition 2.6.** *The mass $m$ of a monopole is the winding number of the $\tilde{S}^1$-action on the complex line bundle $L_j$.*

Next we define the charge of a monopole:

**Definition 2.7.** The charge $k_j$ of a monopole is defined by the relation

$$k_j \cdot x_j = c_1(L_j^*) \in H^2(S_j; \mathbb{Z}) \cong \mathbb{Z},\ j = 1, \ldots, N.$$

where $x_j$ is a positive generator of $H^2(S_j; \mathbb{Z})$.

If monopole mass $m = 0$ then naturally the charge is undefined.

**Proposition 2.8.** (Proposition 2.1 of [B89])

*Isomorphism classes of $S^1$-equivariant $SU(2)$-bundles on $X$ are in one-to-one correspondence with tuples of integers $((m_1, \ldots, m_N), (k_1, \ldots, k_n)) = (m_j, k_j)_{j=1,\cdots,N}$ where $k_j \in \mathbb{Z}$ is undefined for $m_j = 0$, the $m_j \in \mathbb{Z}_{\geq 0}$ are all either even or odd.*

For the proof of this proposition, we need the following result:

**Lemma 2.9.** *All $SU(2)$-bundles over 3-manifolds are trivial.*

The idea is that any transition function is homotopic to identity.

*Proof.* The idea is to consider the open cover $(M \times S^1) \cup NS_j$, where $NS_j$ is a tubular neighbourhood of $S_j$ (which also is a normal bundle over $S_j$), of $X$ and examine isomorphism classes over each bundle and and consider the gluing map.

Define $P'$ as the pull-back of $P$ by the double cover map $p : M \times \tilde{S}^1 \to M \times S^1$, i.e.

$$P' = p^* P|_{M\times S^1} = \left\{((x,\theta),q) \in (M \times \tilde{S}^1) \times P|_{M\times S^1}; p(\pi,\theta) = (\pi, s\theta), \pi(q)\right\}.$$

There is an $\tilde{S}^1$-action on $P'$ covering the $\tilde{S}^1$-action on $M \times S^1$ and this action is free. Then we can define a bundle $P''$ to be $P'/\tilde{S}^1$, which means $P'$ is the pull-back bundle of $P''$ by the quotient map. According to the following lemma, $P''$ is a trivial bundle, so we can write $P'' \cong M \times SU(2)$. It follows that $P'$ is also trivial (pull-back of a trivial bundle is trivial) and we have $P' = M \times \tilde{S}^1 \times SU(2)$.

We claim that all $\tilde{S}^1$-action on $P'$ is in the same isomorphism class.

Let $\tilde{\phi}_1 : \tilde{S}^1 \to Aut(P')$ be the action that fixes $SU(2)$ and $\tilde{\phi}_2 : \tilde{S}^1 \to Aut(P')$ be any $\tilde{S}^1$-action on $P'$. The claim is true if there exists a gauge transformation $\tilde{h} : M \times \tilde{S}^1 \to SU(2)$ and $\theta \in S^1$ such that $\tilde{\phi}_2(\theta) \circ \tilde{h} = \tilde{h} \circ \tilde{\phi}_1(\theta)$ . In other words, the claim is true if the following diagram commutes.

$$\begin{tikzcd}
M \times \tilde{S}^1 \times SU(2) \arrow[rrr, "\tilde{h}"] \arrow[dddd, "\tilde{\phi}_1(\theta)"'] & & & M \times \tilde{S}^1 \times SU(2) \arrow[dddd, "\tilde{\phi}_2(\theta)"] \\
& (x,\xi,g) \arrow[r, mapsto] \arrow[dd, mapsto] & (x,\xi,\tilde{h}(x,\xi)g) \arrow[d, mapsto] & \\
& & (x,\xi+\theta,\tilde{\phi}_2(\theta)\tilde{h}(x,\xi)g) & \\
& (x,\xi+\theta,g) \arrow[r, mapsto] & (x,\xi+\theta,\tilde{h}(x,\xi+\theta)g) & \\
M \times \tilde{S}^1 \times SU(2) \arrow[rrr, "\tilde{h}"'] & & & M \times \tilde{S}^1 \times SU(2)
\end{tikzcd}$$

For $x \in M$, $\xi,\ \theta \in \tilde{S}^1$ and $g \in SU(2)$

$$\begin{aligned}
\tilde{\phi}_2(\theta) \circ \tilde{h}(x,\xi,g) &= \tilde{\phi}_2(\theta)(x,\xi,\tilde{h}(x,\xi)g) \\
&= (x,\xi+\theta,\tilde{\phi}_2(\theta)\tilde{h}(x,\xi)g)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{h} \circ \tilde{\phi}_1(\theta)(x,\xi,g) &= \tilde{h}(x,\xi+\theta,g) \\
&= (x,\xi+\theta,\tilde{h}(x,\xi+\theta)g).
\end{aligned}$$

We now choose the gauge transformation to be $\tilde{h}(x,\xi) := \tilde{\phi}_2(\xi)$, then

$$\begin{aligned}(x,\xi+\theta,\tilde{\phi}_2(\theta)\tilde{h}(x,\xi)g) &= (x,\xi+\theta,\tilde{\phi}_2(\theta)\tilde{\phi}_2(\xi)g)\\ &= (x,\xi+\theta,\tilde{\phi}_2(\theta+\xi)g)\\ &= (x,\xi+\theta,\tilde{h}(x,\xi+\theta)g).\end{aligned}$$

Hence the diagram commutes.

Now we come back to consider the action on $P$. Let $\phi_1$ and $\phi_2$ be the $\tilde{S}^1$-action on $P|_{M\times S^1}$ induced by $\tilde{\phi}_1$ and $\tilde{\phi}_2$ respectively. Similarly, $\phi_1$ and $\phi_2$ are equivalent if and only if there exists a $\tilde{S}^1$-equivariant gauge transformation $h$ on $P|_{M\times S^1}$ such that $\phi_2(\theta)\circ h = h\circ\phi_1(\theta)$. We have a canonical choice of $h$ as the gauge transformation induced by $\tilde{h}$. Since $P|_{M\times S^1} = P'/\mathbb{Z}_2$, this can happen only when $\tilde{h}$ is $\mathbb{Z}_2$-equivariant, that is $\tilde{h}(-x,-\xi) = \tilde{h}(x,\xi)$.

Note that the $\tilde{S}^1$-action on $P|_{M\times S^1}$ may not be free because $-1\in\tilde{S}^1$ can act either as $+\mathrm{id}$ or $-\mathrm{id}$ on $P|_{M\times S^1}$. We consider two cases:

1. $\phi_1(-1)$ and $\phi_2(-1)$ has the same sign, i.e. $\phi_1(-1) = \phi_2(-1) = \pm\mathrm{id}$,

2. $\phi_1(-1)$ and $\phi_2(-1)$ has different sign, i.e. $\phi_1(-1) = \pm\mathrm{id}$, $\phi_2(-1) = \mp\mathrm{id}$.

Since $\phi_i$ are induced from $\tilde{\phi}_i$, we also have $\tilde{\phi}_1(-1) = \tilde{\phi}_2(-1) = \pm\mathrm{id}$ for the first case, which gives

$$\begin{aligned}\tilde{\phi}_2(-1)\circ\tilde{h}(x,\xi,g) &= \tilde{\phi}_2(-1)(x,\xi,\tilde{h}(x,\xi)g)\\ &= (x,\xi+\pi,\pm\tilde{h}(x,\xi)g)\end{aligned}$$

and

$$\begin{aligned}\tilde{h}\circ\tilde{\phi}_1(-1)(x,\xi,g) &= \tilde{h}(x,\xi+\pi,\pm g)\\ &= (x,\xi+\pi,\pm\tilde{h}(x,\xi+\pi)g).\end{aligned}$$

This means $\tilde{h}$ commutes with $-1$-action, hence $\tilde{h}$ is $\mathbb{Z}_2$-equivariant.

For the second case, assume $\phi_1(-1) = -\mathrm{id}$, $\phi_2(-1) = \mathrm{id}$, and for any gauge transformation $h : P|_{M\times S^1}\to P|_{M\times S^1}$ (we no longer require $h$ to be induced from $\tilde{h}$ in this case) we have

$$h\circ\phi_1(-1)(q) = h(-q) = -h(q)$$

and

$$\phi_2(-1)\circ h(q) = h(q).$$

This mean $h \circ \phi_1(-1)(q) = \phi_2(-1) \circ h(q)$ if and only if $h(q) = -h(q)$ which is never true because the gauge group is isomorphic to $SU(2)$ and $0 \notin SU(2)$.

We conclude that the number of isomorphism classes of $\tilde{S}^1$-action on $P|_{M \times S^1}$ depends on the sign of the $-1$-action.

Next we consider the part $P|_{NS_j}$. Denote by $E|_{NS_j}$ the associate bundle of $P|_{NS_j}$.

Since $S_j$ is the boundary surface, it can be view as an "end" of $M$. This means a tubular neighbourhood of $S_j$ in $M$ can be written as $(N, \infty) \times S_j$ for some $N \in \mathbb{R}_{\geq 0}$. In the manifold $X$, this neighbourhood is

$$(N \times \infty] \times S_j \times S^1/(\infty, x, \theta_1) \sim (\infty, \xi, \theta_2) \cong S_j \times D^2 \cong S_j \times \mathbb{C}.$$

So the normal bundle over $S_j$ is a trivial bundle. Let $p_1 : S_j \times \mathbb{C} \to S_j$ be the projection map. We claim that there exists an $\tilde{S}^1$-equivariant isomorphism between $E|_{NS_j}$ and $p_1^*(E|_{S_j})$.

Denote by $\iota$ the inclusion map $S_j \hookrightarrow S_j \times \mathbb{C}$.

$$\begin{array}{ccccc} p_1^*(E|_{S_j}) & & E|_{S_j} & & E|_{S_j \times \mathbb{C}} \\ \downarrow & & \downarrow & & \downarrow \\ S_j \times \mathbb{C} & \xrightarrow{p_1} & S_j & \xhookrightarrow{\iota} & S_j \times \mathbb{C} \end{array}$$

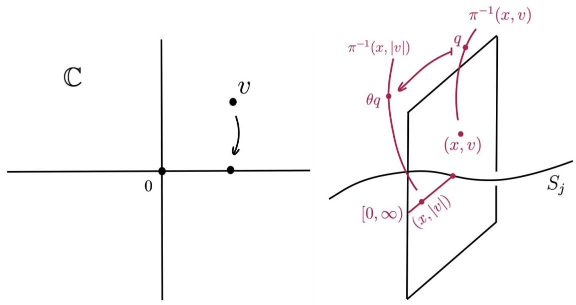
Since $(\iota \circ p_1)^*(E|_{S_j \times \mathbb{C}}) \cong p_1^*(E|_{S_j})$ and $\iota \circ p_1 \simeq id_{S_j \times \mathbb{C}}$, we have $p_1^*(E|_{S_j}) \cong E|_{S_j \times \mathbb{C}}$. The same argument also gives $E|_{S_j \times [0,\infty)} \cong p_1^*(E|_{S_j})$. Denote the isomorphism by $f : E|_{S_j \times [0,\infty)} \to p_1^*(E|_{S_j})$.

We now construct the $\tilde{S}^1$-equivariant isomorphism $\tilde{f} : E|_{S_j \times [0,\infty)} \to p_1^*(E|_{S_j})$ from $f$. For any $q \in E_{S_j \times \mathbb{C}}$ with $\pi(q) = (x, v) \in S_j \times \mathbb{C}$, there exists $\theta \in \tilde{S}^1$ such that $\pi(\theta, q) = (x, |v|)$. In other words, $\theta$ rotates the plane $\mathbb{C}$ such that $v$ is mapped to $[0, \infty)$.

Define $\tilde{f}(q) := \theta^{-1} f(\theta, q)$. We check that

1. $\tilde{f}$ is well-defined and
2. $\tilde{f}$ is $\tilde{S}^1$-equivariant.

For any $q \in E|_{S_j \times \mathbb{C}}$, there exists two elements $\theta, \theta' \in \tilde{S}^1$ that rotates the plane $\mathbb{C}$ such that $v$ is mapped to $[0, \infty)$.




$\tilde{f}$ is well-defined if

$$\theta^{-1} f(\theta q) = \theta'^{-1} f(\theta' q). \tag{2.1.8}$$

It is easy to see that $\theta' = -\theta$, so (2.1.8) amounts to

$$-\theta^{-1} f(-\theta q) = \theta^{-1} f(\theta q). \tag{2.1.9}$$

Since $f$ is fibre-wise linear, it acts by multiplication by $SU(2)$-matrix, and $-id \in \mathbb{Z}(SU(2))$. Therefore $f(-\theta q) = -f(\theta q)$ and (2.1.9) is satisfied.

Note that $\tilde{f}$ is $\tilde{S}^1$-equivariant means $\tilde{f}(\theta_1 q) = \theta_1 \tilde{f}(q)$ for all $\theta_1 \in \tilde{S}^1$.

By construction, we have

$$\theta_1 q \xmapsto{\theta\theta_1^{-1}} \theta q \longmapsto f(\theta q) \longmapsto \theta^{-1} f(\theta q) = \tilde{f}(q) \tag{2.1.10}$$

This proves the $\tilde{S}^1$-equivariance of $\tilde{f}$ and proves the claim.

We show now that the $\tilde{S}^1$-equivariant isomorphism classes of $\tilde{S}^1$-equivariant $SU(2)$-bundles on $S_j$ are in one-to-one correspondence with the set $\{(m_j, k_j) | m_j \in \mathbb{Z}_{\geq 0}\}$. That is the map $\phi \mapsto (m_j, k_j)$ is bijective.

Remark that there is a bijection between the isomorphism classes of line bundles over $X$ and element of $H^2(X, \mathbb{Z})$, which associate a line bundle to its first Chern class. Therefore given any pair $(m_j, k_j)$, there is a unique line bundle $L_j^*$ with $c_1(L_j^*) = k_j \cdot x_j$. This mean the map $\phi \mapsto (m_j, k_j)$ surjective [①].

For injectivity, suppose $\phi_1 \mapsto (m_j, k_j)$ and $\phi_2 \mapsto (m_j, k_j)$ (here $\phi_1$ and $\phi_2$ are not related to the construction above and simply means two different $\tilde{S}^1$-actions). Then same $k_j$ means $\phi_1$ and $\phi_2$ splits $E|_{S_j}$ identically, say into $L_j \oplus L_j^*$ and same $m_j$ means $\phi_1$, $\phi_2$ act on $L_j$ and $L_j^*$ identically. This means $\phi_1 = \phi_2$.

Finally, we need an $\tilde{S}^1$-equivariant transition function to glue the two parts together. Note that $(M \times S^1) \cap NS_j = S_j \times (\mathbb{C} \backslash \{0\})$ and the bundle over $S_j$

[①] Here $j$ is a fixed value, same goes for next paragraph when we discuss further about this map.

is trivial ($SU(2)$ bundles over 2-manifolds are trivial) and so is the bundle over $S_j \times (\mathbb{C}\backslash\{0\})$ because it can be viewed as the pull-back bundle under the projection map. Then we can write the transition function as a map

$$\gamma : S_j \times (\mathbb{C}\backslash\{0\}) \longrightarrow SU(2).$$

Similarly to the argument used before in showing $\tilde{f}$ is $\tilde{S}^1$-equivariant, $\gamma$ being $\tilde{S}^1$-equivariant amounts to the induced map

$$\tilde{\gamma} : S_j \times (\mathbb{C}\backslash\{0\}) \longrightarrow SU(2)$$

being $\mathbb{Z}_2$-equivariant. We claim that such map exists if an only if either of the following condition is true

- $m_j$ is odd and $-1 \in \tilde{S}^1$ acts on $M \times S^1$ as $-id$, or
- $m_j$ is even and $-1 \in \tilde{S}^1$ acts on $M \times S^1$ as $id$.

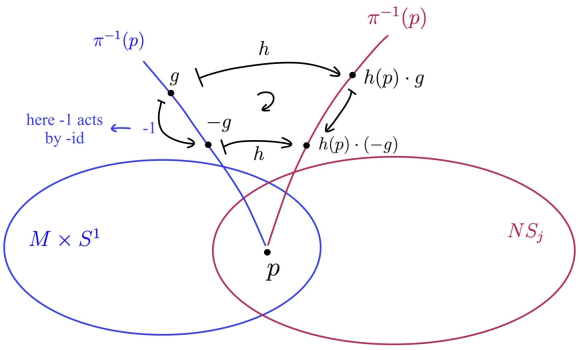



Figure 2.3: gluing of the the bundles over $M \times \tilde{S}^1$ and $NS_j$ ($\theta$ chosen to be $-1$), the commutativity of the diagram determines the action.

It follows from the explicit form (2.1.7) that if $-1 \in \tilde{S}^1$ act as $id$, then $m_j$ must be odd for otherwise the map $h(p)g$ is not mapped to $h(p)(-g)$. Similarly, if $-1 \in \tilde{S}^1$ act as $-id$, then $m_j$ must be even for the diagram to commute.

Suppose now that $m_j$ is odd and $-1 \in \tilde{S}^1$ acts on $M \times S^1$ as $-id$

Since $P|_{S_j \times (\mathbb{C}\backslash\{0\})}$ is trivial, there exists an isomorphism

$$S_j \times [0, \infty) \longrightarrow SU(2).$$

Now if one of two conditions above is satisfied, we can prove this map is $\mathbb{Z}_2$-equivariant in the same way we proved the $\tilde{h}$ before is $\mathbb{Z}_2$-equivariant. This concludes the proof of Proposition 2.8.

$\square$

Denote the $\tilde{S}^1$-actions on $P$ by $\phi_j : \tilde{S}^1 \to \mathcal{H}$. Then the monopole moduli space is

$$\bigcup_{j\in J} (\mathscr{A}_j^*)^{\mathrm{ASD}}/\mathscr{G}_j.$$

Here $J$ denotes the index set from proposition 2.8. Next lemma tells this space is in fact the $S^1$-invariant instantons.

**Lemma 2.10.** *The map* $\bigcup_{j\in J} \mathscr{A}_j/\mathscr{G}_j \longrightarrow (\mathcal{B}^*)^{S^1} = (\mathscr{A}^*/\mathscr{G}(P))^{S^1}$ *is a bijection.*

*Proof.* $[A] \in (\mathcal{B}^*)^{S^1}$ is, by definition, fixed by $S^1$-action and so is fixed by some $\phi_j$, hence the map is surjective.

Suppose $A_0 \in A_{j_0}^*$, $A_1 \in A_{j_1}^*$ are different elements in $\bigcup_{j\in J} \mathscr{A}_j/\mathscr{G}_j$ $(j_0 \neq j_1)$ and they are mapped to the same point in $(\mathcal{B}^*)^{S^1}$, i.e. $[A_0] = [A_1]$ or equivalently $A_0 = g^* A_1$ for some $g \in \mathscr{G}$ .

Since $[A_1]$ is a fixed point in $(\mathscr{B}^*)^{S^1}$, there is a unique action $\phi_{A_1}$ stabilising $A_1$. $A_1 \in A_{j_1}^*$ implies $\phi_{A_1} = \phi_{j_1}$ and similarly $\phi_{A_0}$ is stabilised by $\phi_{j_0}$. It follows that $A_0 = g^* A_1$ is stabilised by $g^{-1}\phi_{j_1} g$. The uniqueness of the stabiliser gives $\phi_{j_0} = g^{-1}\phi_{j_1} g$. Therefore the map is injective.

□

The constraints $m_j$, $k_j$ for monopole and $c_2(P)$ for the instanton are also related. The relation is given explicitly by

**Proposition 2.11.**

$$c_2(P) = \sum_j m_j k_j. \tag{2.1.11}$$

*Proof.* The equivariant second Chern class of $E = P \times_{SU(2)} \mathbb{C}^2$ restricted to $S_j$ is given by

$$\begin{aligned} c_2^{\mathrm{eq}}(E|_{S_j}) &= \sum_j c_2^{\mathrm{eq}}(L_j \oplus L_j^*) \\ &= \sum_j c_1^{\mathrm{eq}}(L_j) c_1^{\mathrm{eq}}(L_j^{-1}) \\ &= (-1)^N \sum_j c_1^{\mathrm{eq}}(L_j^{-1})^2. \end{aligned}$$

$L_j^{-1}$, as a complex line bundle over $S_j$, is trivial, hence

$$c_1^{\mathrm{eq}}(L_j^{-1}) \in H^2_{\tilde{S}^1}(S_j)$$

and

$$c_1^{\mathrm{eq}}(L_j^{-1}) = c_1 + fu$$

where $f$ is a function and $u \in H^2_{S^1}(\text{point}) \subset H^2_{S^1}(S_j) \cong H^2(S_j)[u]$

Let $V$ be the vector field generated by $\tilde{S}^1$-action and $\iota_V$ contraction by $V$, then we have the following deduction

$$\begin{aligned} & (d - \iota_V)(c_1 + fu) = 0 \\ \Rightarrow\ & \iota_V c_1 - df = 0 \\ \Rightarrow\ & df = 0 \\ \Rightarrow\ & f = -m_j \end{aligned}$$

Therefore $c_1^{\mathrm{eq}}(L_j^{-1}) = c_1 - m_j u$.

Since $NS_j \cong S_j \times \mathbb{C}$ is a pull-back of the complex line bundle over a point, $c_1(NS_j) = u$ and thus

$$c_1^{\mathrm{eq}}(L_j^{-1}) = k_j x_j - m_j u.$$

Denote by $\mathfrak{e}_{S_j}$ the Euler class of $NS_j$, the localization formula (cf. [AB84]) gives

$$\begin{aligned} c_2(P) &= c_2(E) = \int_X c_2^{\mathrm{eq}}[(c_2(E)]_{(0)} \\ &= \sum_j \int_{S_j} \left[ \frac{c_2^{\mathrm{eq}}(\mathfrak{e}|_{S_j})}{\mathfrak{e}|_{S_j}} \right]_{(0)} \\ &= \sum_j \int_{S_j} \left[ \frac{-(k_j x_j - m_j u)^2}{\mathfrak{e}|_{S_j}} \right]_{(0)} \\ &= \sum_j \int_{S_j} \left[ \frac{-(k_j x_j - m_j u)^2}{2u} \right]_{(0)} = \sum_j m_j k_j \end{aligned}$$

In the last line of the calculation we have $\mathfrak{e}_{S_j} = 2u$ because the $\tilde{S}^1$-action on the normal bundle $NS_j$ is rotation by 2-cycles.

□

## 2.2 Deformation theory of monopoles

The aim of this section is to study the local geometric structure of the monopole moduli space. We will construct a local slice of the gauge group action.

The space of connections $\mathscr{A}(P)$ is an affine space modelled on the vector space $\Omega^1(\mathrm{ad}P) = \Gamma(T^*X \otimes \mathrm{ad}P)$, that is by fixing some smooth connection $A$ we can write $\mathscr{A}(P) = A + \Omega^1(\mathrm{ad}P)$. The completion of the space $\mathscr{A}(P)$ is defined by the completion of the space $\Omega^1(\mathrm{ad}P)$, that is

$$\mathscr{A}_{p,k} = A + L^p_k(\Omega^1(\mathrm{ad}P)).$$

Again we are interested in the $\tilde{S}^1$-fixed point set. Here we assume $A$ is $S^1$-invariant. This is called the configuration space and is denoted by

$$C_{k,p}(P) = \mathscr{A}_{p,k}(P)^{\tilde{S}^1}.$$

**Proposition 2.12.** *$C_{k,p}(P)$ is a closed affine Banach manifold.*

*Proof.* Let $\theta \in \tilde{S}^1$, from the map

$$\begin{array}{rcl} f_\theta : \mathscr{A}_{p,k}(P) & \longrightarrow & L^p_k(\Omega^1(\mathrm{ad}P)) \\ A & \longmapsto & \theta^* A - A, \end{array}$$

we see that $C_{k,p}(P) = \bigcap_{\theta \in \tilde{S}^1} f_\theta^{-1}(0)$. Since $f_\theta$ is continuous, $f_\theta^{-1}(0)$ is closed and so is the intersection.

To show the affinity part, it suffices to prove that for $A+a$ and $A+b \in C_{k,p}(P)$, $A+\mathrm{const}.a \in C_{p,k}(P)$ and $A + (a+b) \in C_{p,k}(P)$.

$$\begin{array}{rcl} \theta^*(A + \mathrm{const}.a) & = & \theta^*(A) + \theta^*(\mathrm{const}.a)^{①} \\ & = & A + \mathrm{const}.a\ . \end{array}$$

The second equality follows from the linearity of the map $\theta^*$. Since $\theta^*(A + a) = A + a$, then $A + \theta^* a = A + a$ which, in turn, implies $\theta^* a = a$. Therefore $A + \mathrm{const}.\theta^* a = A + \mathrm{const}.a$, which means $A + \mathrm{const}.a$ is fixed by the $\tilde{S}^1$-action. Similar calculation also shows

$$\theta^*(A + a + b) = A + \theta^* a + \theta^* b = A + a + b.$$

□

①Here $\tilde{S}^1$ acts on $a$ not as a 1-form on $X$ but as a Lie algebra-valued 1-form on $P$. The correspondence is defined as follows:

Let $f \in \Omega^1(X, \mathrm{ad}P)$, $f(v) = [p, \xi]$, $v \in T_x X$, $\xi \in \mathfrak{g}$, fix a $p \in P_x$. The correspondence is given by the map

$$\begin{array}{c} \Omega^1(X, \mathrm{ad}P) \to \Omega^1(P, \mathfrak{g}) \\ f \mapsto \left( \omega \xmapsto{d\pi|_p} d\pi|_p(\omega) \xmapsto{f} [p, \xi] \xmapsto{\text{fixing } p} \xi,\ \omega \in T_p P \right). \end{array}$$

Denote the space of $\tilde{S}^1$-invariant $L_{p,k}$ $i$-forms of the bundle ad$P$ as

$$\mathscr{W}^i_{p,k} = L^p_k(\Omega^i(\mathrm{ad}P))^{\tilde{S}^1}.$$

$\mathscr{W}^1_{p,k}$ is the underlying vector space of $C_{k,p}(P)$.

Here the AHS complex associated to an $L_{p,k}$-connection $A$ is given by

$$\mathscr{W}^0_{p,k+1} \xrightarrow{d_A} \mathscr{W}^1_{p,k} \xrightarrow{d^+_A} \mathscr{W}^{2,+}_{p,k-1}, \tag{2.2.1}$$

where $d^+_A$ is the covariant derivative composed with a projection $P_+$ onto the self-dual part.

Denote by $L_{p,k+1}(\Gamma(\mathrm{ad}P))$ the completion of the space of $L^p_{k+1}$-sections of the vector bundle ad$P$. Since $\mathrm{Lie}(\mathscr{G}(P)) \cong \Gamma(\mathrm{ad}P)$, we define the group of $L^p_{k+1}$-gauge transformations $\mathscr{G}_{p,k+1}(P)$ as the image of $\Lambda_{p,k+1}(\Gamma(\mathrm{ad}P))$ under the exponential map. Denote the $\tilde{S}^1$ invariant part $\mathscr{G}_{p,k+1}(P)^{\tilde{S}^1}$ as $GA_{p,k+1}(P)$. By lemma 4.4.3 of [M98], for $k+1-4/p \geq 0$, $\mathscr{G}_{p,k+1}(P)$ acts smoothly on the space $\mathscr{A}_{p,k}(P)$ of $L^p_k$-connections on $P$. The result holds when restricting all spaces to their $\tilde{S}^1$ invariant part.

**Proposition 2.13.** (Theorem 3.1 [B89]) *The slice at a point $[A] \in C_{k,p}(P)$ for the action $GA_{p,k+1}(P)$ exists and is given by the affine space $A+W$ where $W$ is a small neighbourhood of $0 \in H_A \equiv \mathscr{W}^1_{p,k} \cap ker(d^*_A)$.*

*The quotient space $B_{p,k}(P) = C_{p,k}(P)/GA_{p,k+1}(P)$ is a smooth manifold away from the reducible monopoles.*

The proof of the first part is an application of the slice theorem.

To study the anti-self-dual connections, fix some monopole $A_0 \in C_{p,k}(P)$ and consider the map

$$\begin{aligned} K : H_{A_0} &\longrightarrow \mathscr{W}^{2,+}_{p,k-1} \\ a &\longmapsto P_+F(A_0+p) = P_+(F(A_0) + d_{A_0}a + a\wedge a). \end{aligned}$$

By construction the zeros of $K$ are the anti-self-dual connections. We employ the standard Lyaunov-Schmidt reduction technique. The idea is to decompose the domain in a way such that $K$ is isomorphic on one summand.

Denote by $H^i_0$ the $S^1$-invariant part of the $i$-th cohomology of the complex (2.2.1), that is

$$\begin{aligned} H^1_0 &= \ker(d^+_{A_0}) \cap \mathscr{W}^1_{p,k}/\mathrm{im}\ (d_{A_0}) \cap \mathscr{W}^1_{p,k}, \\ H^2_0 &= \mathscr{W}^{2,+}_{p,k-1}/\mathrm{im}(d^+_{A_0}) \cap \mathscr{W}^{2,+}_{p,k-1}. \end{aligned}$$

Consider the decomposition,

$$H_{A_0} = H_0^1 \oplus \left(\mathscr{W}^1_{p,k}/\ker(d_A^+)\right) = H_0^1 \oplus \left(\operatorname{im}(d_A^+)^* \cap \mathscr{W}^1_{k,p}\right)$$
$$\mathscr{W}^{2,+}_{p,k+1} = V_{A_0} \oplus \left(\operatorname{im}(d_A^+) \cap \mathscr{W}^{2,+}_{p,k-1}\right) \tag{2.2.2}$$

where $V_{A_0}$ is a finite dimensional subspace of $\mathscr{W}^{2,+}_{p,k-1}$ such that the $L^2$ projection to the $\tilde{S}^1$-invariant part $H_0^2$ is an isomorphism. Denote

$$C = \operatorname{im}(d_{A_0}^+)^* \cap \mathscr{W}^1_{p,k}, \qquad I = \operatorname{im}(d_{A_0}^+) \cap \mathscr{W}^{2,+}_{p,k-1}.$$

By definition, $d_{A_0}^+ = \frac{dK}{da}|_0$ is the linearisation of $K$, so we have

$$dK_{A_0} = d_{A_0}^+|_C : C \longrightarrow I. \tag{2.2.3}$$

Since $\mathscr{W}^1_{p,k} = \operatorname{im}(d_{A_0}) \oplus H_{A_0} = \operatorname{im}(d_{A_0}) \oplus H_0^1 \oplus C = \ker(d_{A_0}^+) \oplus C$, then $\ker(d_{A_0}^+|_C) = 0$. Also $d_{A_0}^+(C) = d_{A_0}^+(\mathscr{W}^1_{p,k})$ implies $\operatorname{coker}(d_{A_0}^+|_C) = 0$. Therefore $dK_{A_0}$ is an isomorphism. Then we can apply the implicit function theorem to obtain a diffeomorphism

$$\eta : H_0^1 \longrightarrow C$$

which induces the map

$$\begin{aligned} f : H_0^1 &\longrightarrow H_0^1 \oplus C \\ a &\longmapsto \tilde{a} = (a, \eta(a)) \end{aligned}$$

such that $K(\tilde{a}) = K(a, \eta(a)) = 0$.

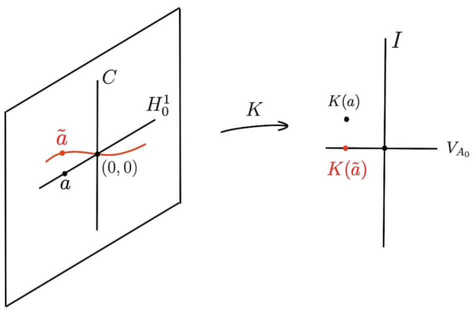



Figure 2.4: the implicit function theorem guarantees that there exists $\tilde{a}$ in a neighbourhood of $a$ such that $K(\tilde{a}) = (\cdot, 0)$ in the decomposition $V_{A_0} \oplus I$.

We also have two important estimates given by the implicit function theorem

$$\begin{aligned} ||a - \tilde{a}|| &= O(a^2) \\ \left|\left|\frac{d\tilde{a}}{da} - \mathrm{id}\right|\right| &= O(a). \end{aligned} \tag{2.2.4}$$

Using this map, we define the Kuranishi map:

$$\begin{array}{rcccl} \phi : & H^1_0 & (\longrightarrow H^1_0 \oplus C) & \longrightarrow & V_{A_0} \\ & a & (\stackrel{\eta}{\longmapsto} \quad \tilde{a} \quad ) & \stackrel{K}{\longmapsto} & K(\tilde{a}) \end{array} \tag{2.2.5}$$

Since $\eta$ is a diffeomorphism, im$\phi$ is diffeomorphic to some local neighbourhood of $0 \in H^1_0$. It follows from estimate (2.2.4) that locally $K^{-1}(0) \cong \phi^{-1}(0)$ and in turn, $\phi^{-1}(0)$ quotient by the stabiliser $\Gamma_A$ of $A$ is a neighbourhood of $[A]$ in the moduli space $\mathcal{M}$.

Finally, the leading term of $\phi$ is described explicitly in chapter IV(i), [D86]. Let $\omega \in H^2_{A_0}$ be a solution of the adjoint equation $d^*_{A_0}\omega = 0$ (i.e. $\omega \in \mathrm{Ker}(d^+_{A_0})^*$), the equation $\phi(a) = 0$ is equivalent to $\langle \omega, \phi(a) \rangle_{L^2} = 0$ and

$$\begin{aligned} \langle \omega, \phi(a) \rangle_{L^2} &= \langle \omega, P_+(d_A \tilde{a} + \frac{1}{2}[\tilde{a}, \tilde{a}]) \rangle_{L^2} \\ &= \langle \omega, \frac{1}{2}[\tilde{a}, \tilde{a}] \rangle_{L^2} \\ &= \langle \omega, \frac{1}{2}[a, a] \rangle_{L^2} + O(a^3). \end{aligned}$$

Hence $\phi$ can be approximated (in $C^1$) by the quadratic term $q_\omega(a) = \langle \omega, \frac{1}{2}[a, a] \rangle_{L^2}$. By the stability theorem for submersions, if 0 is regular value of the map

$$q : H^1_0 \backslash \{0\} \longrightarrow (H^2_0)^*,$$

then the zero set $\phi^{-1}(0)$ of $\phi$ is homeomorphic to the zero set $q^{-1}(0)$ of $q$. This means the local neighbourhood of $[A]$ can be described by studying the map $q$.

## 2.3 Smoothness of the monopole moduli space

In this chapter we prove that for a generic $S^1$ invariant metric on $X$, the module space of irreducible instantons are smooth manifolds. We will follow the proof in [FU84], which gives the result without the $S^1$ invariance condition. The extra treatment needed for the $S^1$ invariance condition is studied in detail in [C91], we will use Cho's result to complete the proof.

The reason we need to consider the metric on $X$ is because of the fact that not all metric gives the moduli space as smooth manifolds. Define the parameter space $\mathcal{D} = C^k(GL(TX))$. Then an element $\varphi \in \mathcal{D}$ is an $C^k$ automorphism of the tangent bundle and therefore acts on a metric $g$ on $X$ by pullback. That is $\varphi(g) := \varphi^* g$ is a metric on $X$. We assume $\varphi$ is symmetric so that the action is unique. To verify this is the metric space we need to check two defining properties of a metric and the fact that all metrics are realised in this fashion.

Let $\varphi \in \mathcal{D}$ and $g$ be a metric on $X$. For any $x_1, x_2 \in X$, $\varphi(g)$ is

- positive-definite because $x_1 = x_2$ implies $\varphi^* g(x_1, x_2) := g(\varphi(x_1), \varphi(x_2)) = 0$;
- linear because $g(\varphi(ax_1), \varphi(x_2)) = ag(\varphi(x_1), \varphi(x_2))$

Let $g, h$ be two metrics on $X$. For all $p \in X$,

$$g_p : T_pX \times T_pX \longrightarrow \mathbb{R}, \qquad h_p : T_pX \times T_pX \longrightarrow \mathbb{R}.$$

By fixing a basis $\{x_1, x_2, \ldots, x_n\}$ on $T_pX$, $g_p$ and $h_p$ can be viewed as $n \times n$ symmetric positive-definite matrices, and so there exists a unique symmetric matrix $A \in M_{n\times n}$ such that $h_p = A^T g_p A$.

Define

$$\begin{aligned} \mathcal{P} : \mathscr{A}^*_{2,l-1} \times \mathcal{D} &\longrightarrow L^2_{l-2}\left(\wedge^2_-(\mathfrak{g}_P)\right) \\ (A, \varphi) &\longmapsto P_-\left((\varphi^{-1})^* F_A\right) \end{aligned}$$

Here $\mathscr{A}^*_{2,l-1}$ is the irreducible part of $\mathscr{A}_{2,l-1}$. Observe that $\mathcal{P}(A, \varphi) = 0$ if and only if $F_A$ is self-dual with respect to $\varphi^* g$. The following proposition, together with the preimage theorem (see section 1.4 of [GP74]), implies that $\mathcal{P}^{-1}(0)$ is a smooth manifold.

**Proposition 2.14.** *$\mathcal{P}$ is smooth and has zero as a regular value for large enough $l$.*

*Proof.* $\mathcal{P}$ is quadratic in $A$ as the map $A \mapsto F_A$ is quadratic in $A$. Since $(\varphi^{-1})^*(dx_i \wedge dx_j) = dx_i \wedge dx_j(\varphi(v_1), \varphi(v_2))$, $\mathcal{P}$ is also quadratic in $\varphi$. Therefore $\mathcal{P}$ is smooth.

We prove 0 is a regular value by showing that $d_{(A,\varphi)}\mathcal{P}$ is surjective for any $(A, \varphi)$ with $\mathcal{P}(A, \varphi) = 0$ or equivalently, $\mathrm{coker}(d\mathcal{P}) = 0$.

By definition, $d\mathcal{P}$ is the map

$$T_A(A^*_{2,l-1}) \oplus T_\varphi \mathcal{D} = \Omega^1(\mathrm{ad}P) \times C^q(\mathrm{End}(TX)) \longrightarrow \Omega^{2,-}(\mathrm{ad}P).$$

Let $\Phi \in \mathrm{coker}(d\mathcal{P}_{(A,\varphi)})$. Our first step is to identify $\mathrm{coker}(d\mathcal{P})$ as a subspace of $\Omega^{2,-,\varphi g}_{l-2}(\mathrm{ad}P)$ in order for some of the calculations in this proof to make sense. That is we want to see $\Phi$ as a function in $\Omega^{2,-,\varphi g}_{l-2}(\mathrm{ad}P)$ as opposed to a functional in the $L^2$-dual of $\Omega^{2,-,\varphi g}_{l-2}(\mathrm{ad}P)$[1].

[1]Recall that cokernel of an operator (with closed image) can be defined as the kernel of the adjoint operator so $\Phi$, by this defintion, is a functional in $L^2$-dual of $\Omega^{2,-,\varphi g}_{l-2}(\mathrm{ad}P)$

We split the map along connections and frames:

$$d_{(A,\varphi)}\mathcal{P} = \partial_1\mathcal{P}_{(A,\varphi)} \oplus \partial_2\mathcal{P}_{(A,\varphi)}$$

where $\partial_1\mathcal{P}$ is the partial derivative map

$$\begin{aligned} \partial_1\mathcal{P}_{(A,\varphi)} : \Omega^1(\mathrm{ad}P) &\longrightarrow \Omega^{2,-}(\mathrm{ad}P) \\ f &\longmapsto P_-\left((\varphi^{-1})^* df\right) \end{aligned}$$

and $\partial_2\mathcal{P}$ is the partial derivative map

$$\begin{aligned} \partial_2\mathcal{P}_{(A,\varphi)} : C^q(\mathrm{End}(TX)) &\longrightarrow \Omega^{2,-}(\mathrm{ad}P) \\ h &\longmapsto P_-\left((\varphi^{-1})^*(h^* F_A)\right). \end{aligned}$$

We claim $\mathrm{im}\partial_1\mathcal{P}_{(A,\varphi)}$ is closed and has finite codimension in $\Omega^{2,-}(\mathrm{ad}P)$ so that $\mathrm{im}d\mathcal{P} \subset \mathrm{im}\partial_1\mathcal{P}$ is also closed of finite codimension. This in turn implies $\mathrm{coker}(d\mathcal{P}) \subset \mathrm{coker}\partial_1\mathcal{P}$. As $\mathrm{coker}(\partial_1\mathcal{P})$ is a subspace of $\Omega^{2,-,\varphi g}_{l-2}(\mathrm{ad}P)$, so is $\mathrm{coker}(d\mathcal{P})$.
*Proof of claim*: Since

$$\begin{aligned} \partial_1\mathcal{P}_{(A,\varphi)}(a) &= \frac{\partial}{\partial A}\left(P_-(\varphi^{-1})^* F_A\right) \\ &= P_-(\varphi^{-1})^* \frac{\partial}{\partial A}(dA + A\wedge A) \\ &= P_-(\varphi^{-1})^* \lim_{t\to 0}\frac{1}{t}\left(d(A+ta) + (A+ta)\wedge(A+ta) - dA - A\wedge A\right) \\ &= P_-(\varphi^{-1})^*\left(da + A\wedge a + a\wedge A\right) \\ &= P_-(\varphi^{-1})^* d_A a \equiv d_A^{-,\varphi g}, \end{aligned}$$

$\partial_1\mathcal{P}$ fits into the elliptic complex

$$\Omega^0(\mathrm{ad}P) \xrightarrow{d_A} \Omega^1(\mathrm{ad}P) \xrightarrow{d_A^{-,\varphi g}} \Omega^{2,-,\varphi g}(\mathrm{ad}P). \tag{2.3.1}$$

Since

$$\mathrm{ker}d_A^* = \Omega^1(\mathrm{ad}P)/\mathrm{im}d_A$$

and $\mathrm{im}d_A$ is mapped to zero by $d_A^{-,\varphi g}$, we have

$$\mathrm{im}(d_A^{-,\varphi g}|_{\mathrm{ker}d_A^*}) = \mathrm{im}(d_A^{-,\varphi g}).$$

Therefore $\mathrm{coker}(d_A^{-,\varphi g})$ has finite codimension.

For $a \in \Omega^1_{l-1}(\mathrm{ad}P)$, we can now have the following calculation

$$\begin{aligned} 0 &= \int_X \langle a, (\partial_1 \mathcal{P}_{(A,\varphi)})^* \Phi \rangle_g \\ &= \int_X \langle (\partial_1 \mathcal{P}_{(A,\varphi)}) a, \Phi \rangle_g \\ &= \int_X \langle (P_-(\varphi^{-1})^* d_A a, \Phi \rangle_g \\ &= \int_X \langle (P_-(\varphi^{-1})^* d_A a, \Phi \rangle_{\varphi g} \end{aligned}$$

Since $\Phi$ is anti-self-dual (and so is $\varphi^*\Phi$), then $\langle \varphi^* P_+(\varphi^{-1})^* d_A a, \varphi^* \Phi \rangle_{\varphi^* g} = 0$. Thus

$$\begin{aligned} & \int_X \langle (P_-(\varphi^{-1})^* d_A a, \Phi \rangle_{\varphi g} \\ =& \int_X \langle (P_-(\varphi^{-1})^* d_A a, \Phi \rangle_{\varphi g} + \langle (P_+(\varphi^{-1})^* d_A a, \Phi \rangle_{\varphi g} \\ =& \int_X \langle d_A^{\varphi g} a, \varphi^* \Phi \rangle_{\varphi g} \\ =& \int_X \langle a, \varphi^* \Phi \rangle_{\varphi g} \end{aligned}$$

By the elliptic regularity theorem, $\Phi$ is continuous, then so is $d_A^{*,\varphi g} \varphi^* \Phi$. Since $a$ is arbitrary, $\int_X \langle a, \varphi^* \Phi \rangle_{\varphi g} = 0$ implies $d_A^{*,\varphi g} \varphi^* \Phi = 0$ for otherwise one can choose $a$ to be a bump function at a point $p$ where $d_A^{*,\varphi g} \varphi^* \Phi$ is non-zero contradicts the premise.

Also $\Phi \in \mathrm{coker}(\partial_2 \mathcal{P}_{(A,\varphi)})$, then fore any $r \in C^q(\mathrm{End}(TX))$

$$\begin{aligned} 0 &= \int_X \langle r, (\partial_2 \mathcal{P}_{(A,\varphi)})^* \Phi \rangle_g \\ &= \int_X \langle \partial_2 \mathcal{P}_{(A,\varphi)}(r), \Phi \rangle_g \\ &= \int_X \langle P_-(\varphi^{-1})^* r^* F_A, \Phi \rangle_g \\ &= \int_X \langle r^* F_A, \Phi \rangle_{\varphi g}. \end{aligned}$$

It follows that $\int_X \langle r^* F_A, \Phi \rangle_{\varphi g} = 0$ for all $r$ by the same reason as above.

Since $F_A \in \Omega^{2,+,\varphi g}(\mathrm{ad}P)$, $\varphi^* \Phi \in \Omega^{2,-,\varphi g}$, lemma 3.7 of [FU84] implies that $\mathrm{im}F_A$ and $\mathrm{im}\varphi^*\Phi$ are perpendicular with respect to inner product in $g$. Let $U \subset X$ be a open subset, we can regard $F_A$ and $\varphi^*\Phi$ as Lie-algebra valued functions

$$F_A : \bigwedge^{2,+,\varphi g} TU \to \mathfrak{g} \quad , \varphi^* \Phi : \bigwedge^{2,-,\varphi g} TU \to \mathfrak{g}.$$

If there exists $p \in U$ at which $\mathrm{im}F_A$ is non-zero, then either $\mathrm{im}F_A$ or $\mathrm{im}\varphi^*\Phi$ at $p$ has dimension one. We also have the following four identities

$$\begin{aligned} d_A F_A &= 0, \\ d_A^* F_A &= 0, \\ d_A \varphi^*\Phi &= 0, \\ d_A^* \varphi^*\Phi &= 0, \end{aligned}$$

which implies that $F_A$ and $\varphi^*\Phi$ are solutions of the operator $d_A d_A^* + d_A^* d_A$. By unique continuation property of elliptic differential operators, neither $F_A$ nor $\varphi^*\Phi$ vanishes on an open set unless they vanish identically. Together, $\mathrm{rank}F_A$ and $\mathrm{rank}\varphi^*\Phi$ must add up to 3. We now show the result in each possible case of rank combination.

Assume $\Phi \neq 0$, this means there exists $p \in U$ such that $\mathrm{rank}\varphi^*\Phi > 0$ at $p$. Suppose first that $\mathrm{rank}F_A = 2$ and $\mathrm{rank}\varphi^* g = 1$ at $p$. This means there is neighbourhood $V$ of $p$ such that $\mathrm{rank}F_A \geq 2$ and so $\mathrm{rank}\varphi^*\Phi = 1$ on $V$ (which in turn implies $\mathrm{rank}F_A = 2$). Thus we can write $\varphi^*\Phi = \alpha \otimes u$ for some $\alpha \in \Omega^{2,-,\varphi^* g}(V)$, $u \in \Gamma(V, \mathrm{ad}(P|_V))$. Set $|u(x)| = 1$ for all $x \in V$[①]. We have

$$0 = d_A(\varphi^*\Phi) = d_A(\alpha \otimes u) = d\alpha \otimes u + \alpha \wedge d_A u$$

and

$$0 = d\langle u, u\rangle_g = \langle du, u\rangle_g + (-1)^{\deg(u)}\langle u, du\rangle_g = 2\langle du, u\rangle_g$$

where $\langle u, u\rangle_g$ refers to the function

$$\begin{aligned} V &\longrightarrow \mathfrak{g} \\ x &\longmapsto \langle u(x), u(x)\rangle_g. \end{aligned}$$

Together, we have

$$\begin{aligned} 0 &= \langle (d\alpha \otimes u \oplus \alpha \otimes d_A u), u\rangle_g \\ &= d\alpha + \alpha \wedge \langle d_A u, u\rangle_g \\ &= d\alpha. \end{aligned}$$

Therefore

$$\alpha \wedge d_A u = 0$$

[①] $\mathrm{rank}\varphi^*\Phi = 1$ means we can choose any non-zero $u$.

for all $\alpha \in \Omega^{2,-,\varphi^* g}(V)$. By lemma 3.5 of [FU84], we can choose a coframe $\theta^i$ at $p$ so that $\alpha = \frac{|\alpha|}{2}(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)$ under which $d_A u = \sum \theta^i \otimes \omega_i$ for $\omega_i \in \mathrm{ad}P$. Then

$$\begin{aligned} 0 &= \alpha \wedge d_A u \\ &= \frac{|\alpha|}{2}\left(\theta^1 \wedge \theta^2 \wedge \theta^3 \otimes \omega_3 + \theta^1 \wedge \theta^2 \wedge \theta^4 \otimes \omega_4 - \theta^3 \wedge \theta^4 \wedge \theta^1 \otimes \omega_1 - \theta^3 \wedge \theta^4 \wedge \theta^2 \otimes \omega_2\right). \end{aligned}$$

This means $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0$, hence $d_A u = 0$. Finally, we employ lemma 3.12 of [FU84] to arrive at the contradiction $d_A^2 \equiv [F_A, u] \neq 0$.

Next suppose that $\mathrm{rank} F_A = 1$. The deduction is mostly similar to the previous case. Denote by $S$ the complement of $F_A^{-1}(0)$ in $X$. Hence $S$ is open and dense and we have $F_A = \alpha \otimes u$ on $S$ with $u \in \Gamma(S, \mathrm{ad}P)$. Again we can arrive at the $d_A = 0$. If $S$ is connected, $u$ can be extended to $F_A^{-1}(0)$ in a covariantly constant fashion. In other words, for any $x \in F_A^{-1}(0)$, there exists a sequence $\{x_1, x_2, \dots\} \subset S$ converging to $x$ such that $u(x) = \lim_{i\to\infty} u(x_i)$. This means $d_A \neq 0$ and we have the contradiction that $A$ is a reducible connection.

If $S$ is disconnected, then we can find a neighbourhood $\mathcal{O}$ in $X$ such that the set of zeros of $F_A$, $\{x \in X | F_A(x) = 0\}$ disconnects $\mathcal{O}$ as shown in Figure 2.5.

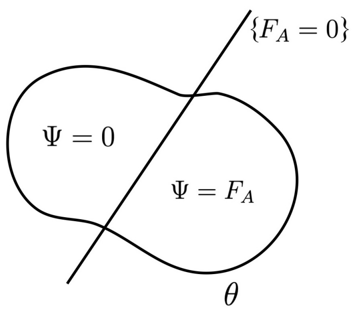



Figure 2.5

Define a 2-form $\Psi$ on $\mathcal{O}$ to be $F_A$ on one component and 0 on the other. Then $\Psi$ is in the kernel of $d_A$ and $d_A^*$, we employ continuation of elliptic differential operator again to deduct that, since $\Psi = 0$ on an open set, $\Psi = 0$ everywhere, contradicting the assumption.

□

So far, we have the space $\mathcal{P}^{-1}(0)$ of irreducible self-dual connections under the metric $\varphi^* g$, then $\mathcal{P}^{-1}(0)/\mathscr{G}_l$ is the parametrized moduli space. Our aim is to show that for generic metric, the connection moduli space is a smooth manifold.

**Theorem 2.15.** *Let $\pi : \mathcal{P}^{-1}(0)/\mathscr{G}_l \longrightarrow \mathcal{D}$ be the projection onto the second factor. There exists a Baire set of $\varphi \in \mathcal{D}$ for which $\pi^{-1}(\varphi)$ is a smooth manifold.*

*Proof.* We prove this by employing the Sard-Smale theorem (c.f. [FU84]). In order to apply the theorem, we need to show that $\pi$ is a Fredholm operator. Let $([A], \varphi) \in \mathcal{P}^{-1}(0)/\mathscr{G}_l$, we have

$$T_{(A,\varphi)}\mathcal{P}^{-1}(0) = \mathrm{ker} d_{(A,\varphi)}\mathcal{P} = \lambda(a,r)|\partial_1\mathcal{P}_{(A,\varphi)}(a) + \partial_2\mathcal{P}_{(A,\varphi)}(r) = 0\rho$$

and

$$T_{(A,\varphi)}\mathcal{P}^{-1}(0)/\mathscr{G} = \{(a,r)|\partial_1\mathcal{P}_{(A,\varphi)}(a) + \partial_2\mathcal{P}_{(A,\varphi)}(r) = 0, d_A^* a = 0\}.$$

Then the kernel and image of

$$\begin{array}{rcl} d\pi_{([A],\varphi)} : T_{([A],\varphi)}\mathcal{P}(0)/\mathscr{G}_l & \longrightarrow & T_\varphi\mathcal{D} \\ (a,r) & \longmapsto & r \end{array}$$

are the set

$$\mathrm{ker} d\pi_{([A],\varphi)} = \left\{(a,0)|d_A^* a = 0, \partial_1\mathcal{P}_{(A,\varphi)}(a) = 0\right\}$$

and

$$\begin{aligned} \mathrm{im}\ d\pi_{([A],\varphi)} &= \left\{r \in T_\varphi\mathcal{D}|\partial_1\mathcal{P}_{(A,\varphi)}(a) + \partial_2\mathcal{P}_{(A,\varphi)}(r) = 0 \text{ and } d_A^* a = 0\right\} \\ &= \partial_2\mathcal{P}^{-1}_{(A,\varphi)}\left(\mathrm{im}\left(\partial_1\mathcal{P}_{(A,\varphi)}|_{\mathrm{ker} d_A^*}\right)\right) \\ &= \partial_2\mathcal{P}^{-1}_{(A,\varphi)}\left(\mathrm{im}\left(\partial_1\mathcal{P}_{(A,\varphi)}\right)\right). \end{aligned} \tag{2.3.2}$$

The last equality follows from the fact that $\partial_1\mathcal{P} \circ d_A = 0$.

Since $d\mathcal{P}$ is surjective, it follows from (2.3.2) and the fact $\Omega^{2,-,\varphi g}(\mathrm{ad}P) = H_A^2 \oplus \mathrm{im}(d_A^{-,\varphi g})$ that $\mathrm{im} d\pi_{([A],\varphi)}$ has codimension equal to dimension of $H_A^2$. Thus $\dim(\mathrm{coker} d\pi_{([A],\varphi)}) = \dim H_A^2$. This completes the set up for Sard-Smale theorem and completes the proof.

□

The result we need from [C91] is the following:

Let $B : V \to W$ be a linear $S^1$-equivariant surjection. Then $B$ restricted to the zero weight space in $V$ maps surjectively to the zero weight space in $W$.

Applying this result on the map $d\mathcal{P}_{(A,\varphi)} : \Omega^1(\mathrm{ad}P) \times C^q(\mathrm{End}(TX)) \to \Omega^{2,-}(\mathrm{ad}P)$. We get that $d\mathcal{P}_{(A,\varphi)}$ restricted to the fixed point set of $\Omega^1(\mathrm{ad}P) \times C^q(\mathrm{End}(TX))$ is a surjection onto the fixed point set of $\Omega^{2,-}(\mathrm{ad}P)$. That is the map

$$d\mathcal{P}^{S^1}_{(A,\varphi)} : \mathscr{W}^1(\mathrm{ad}P) \times C^q(\mathrm{End}(TX))^{S^1} \to \mathscr{W}^{2,-}(\mathrm{ad}P)$$

is a surjection and the rest of the proof follows through to yield the result.

So far we have being focusing on the irreducible connections, the next result gives the condition for which there is no reducible connections present so that results so far apply general moduli space.

Let $\mathcal{M}(m_j, k_j)$ be the monopole moduli space determined by the tuple of monopole mass and charge $\{(m_j, k_j)\}$ (recall Proposition (2.8)) and we denote the moduli space with irreducible monopoles by $\mathcal{M}^*$ and correspondingly $\mathcal{M}^*(m_j, k_j)$ is the moduli space of irreducible monopoles determined by $\{(m_j, k_j)\}$. We will also denote the space of self-dual monopoles by $\mathcal{M}^{\mathrm{SD}}$.

**Theorem 2.16.** *Consider the complex (2.2.1). Let $h^i(M) = dimH^i(M)$, $i = 0, 1, 2$ . If $b^2(M) > b^1(M)$ then for an open dense set of invariant metrics on $X$ there are no reducible monopoles in $\mathcal{M}(m_j, k_j)$, the moduli space determined by the set of tuples $\{(m_j, k_j)\}$, if $\sum m_j k_j \neq 0$, i.e. $\mathcal{M}^*(m_j, k_j) = \mathcal{M}(m_j, k_j)$.*

The proof goes word by word as the proof of Corollary 3.21 of [FU84], which is stated below, except adding $S^1$-invariance in appropriate places.

**Proposition 2.17.** *If the intersection form $\omega$ is indefinite, then for an open dense set of metrics, there are no line bundle solutions to the self-dual or anti-self-dual equations.*

If there exist any reducible connection $A$ over the bundle $E \to X$. The connection can be decomposed into anti-self-dual connections on the line bundle, i.e. $A = A_1 \oplus A_2$ where $A_1$, $A_2$ are anti-self-dual connections on $L$ and $L^{-1}$ respectively, contradicting the corollary. Therefore the theorem follows.

*Proof.* The intersection form $\omega$ being indefinite is equivalent to $b_+ > 0$ and $b_- > 0$. By the Atiyah-Singer index theorem, we have

$$h^1 - h^2 - h^0 = -\frac{1}{2}(\chi - \tau) = -1 - b_-$$

where $\chi$ is the Euler characteristic, $\tau = b_+ - b_-$ is the signature. Index calculation gives that for an open dense set of $C^q$ metrics, $\mathcal{M}(L)^{\mathrm{ASD}}$ is a manifold of dimension $h^1 - h^2$. Therefore $b_- > 0$ means $h^1 - h^2 < 0$. That is $\mathcal{M}(L)$ is empty, which means no reducible self-dual connection present.

By reverting the self-dualness, we have $\dim\mathcal{M}^{\mathrm{SD}} = h^1 - h^2 = -b_+$. Hence $b_+ > 0$ means there is no reducible anti-self-dual connection on $L$. $\square$

## 2.4 Orientability

**Theorem 2.18.** *Assume that $\mathcal{M}^*$ is cut-out transversely. An orientation of $H^1(M,\mathbb{R}) \oplus H^2(M,\mathbb{R})$ gives a canonical orientation of $\mathcal{M}^*(m_j,k_j)$.*

Over the trivial bundle $\mathscr{A} \times_{\mathscr{G}} \Omega^{2,+} \to \mathcal{B}$ there is a section $s$ given by

$$\begin{aligned} s : \mathcal{B} &\longrightarrow \mathscr{A} \times_{\mathscr{G}} \Omega^{2,+} \\ [A] &\longmapsto F_A^+. \end{aligned}$$

The moduli space $\mathcal{M}$ is the preimage of 0 under this map. Then $\mathcal{M}$ being cut out transversely means $s \pitchfork 0$. By definition, $ds = d_A^+$, then $s \pitchfork 0$ means $d_A^+$ is surjective or equivalently $H_A^2 = 0$.

*Proof.* From the universal bundle $(\mathscr{A}^* \times P)/\mathscr{G} \to (\mathscr{A} \times X)/\mathscr{G} = \mathcal{B}^* \times X$ we get a family of Dirac operators

$$D_A := (d_A^+)^* \oplus d_A : \Omega^{2,+}(\mathrm{ad}P) \oplus \Omega^0(\mathrm{ad}P) \longrightarrow \Omega$$

parametrized by $[A] \in \mathcal{B}^*$. Consider the real determinant bundle associated to this operator $\lambda \to \mathcal{B}^*$ whose fibre at $[A]$ is given by

$$\lambda_A = \wedge^{\max}(\ker D_A)^* \otimes \wedge^{\max}(\mathrm{coker} D_A).$$

This is a line bundle because space of top forms has dimension 1, so $\dim \lambda_A = 1$. We aim to show that there exists a nonvanishing section of $\lambda$ (up to pointwise multiplication by a positive function). Therefore the bundle $\lambda \to \mathcal{B}^*$ is trivial.

For any $[A] \in \mathcal{M}^*(X)$,

$$\mathrm{coker} D_A = \ker(d_A^+ \oplus d_A) = \ker(d_A^+)/\mathrm{im}(d_A) = H_A^1 \cong T_{[A]}\mathcal{M}(X)$$

where the last isomorphism is shown in previous chapter, and

$$\begin{aligned} \ker D_A &= \ker(d_A^+)^* \oplus \ker(d_A) \\ &= H_A^2 \oplus H_A^0. \end{aligned}$$

Now $A$ being irreducible means $H_A^0 = 0$, hence $\ker D_A = 0$. By decomposing the space $T_{[A]}\mathcal{M}^*(X)$ along the fixed point part, we have

$$\begin{aligned} T_{[A]}\mathcal{M}^*(X) &= T_{[A]}\mathcal{M}^*(X)^{\tilde{S}^1} \oplus N_A \\ &= T_{[A]}\mathcal{M}^*(m_j,k_j) \oplus N_A. \end{aligned}$$

Fibre of the determinant line bundle (restricted to the anti-self-dual part) can be written as $\lambda_A = T_{[A]}\mathcal{M}^*(m_j, k_j) \oplus N_A$ and so

$$\begin{aligned} \lambda|_{\mathcal{M}^*(X)} &\cong \wedge^{\text{top}}(TM^*(m_j, k_j) \oplus N) \\ &\cong \wedge^{\text{top}}(TM^*(m_j, k_j)) \oplus \wedge^{\text{top}}(N) \end{aligned}$$

By construction, there is an $\tilde{S}^1$-action on $N$ which is fibre-preserving and fixes only one point. Hence the structure group of $N$ is U(1) and so $N$ is a complex bundle. Since complex bundles are orientable as real bundle, then $\wedge^{\text{top}}(N)$ is trivial. Both $\lambda$ and $\wedge^{\text{top}}(N)$ being trivial means $\wedge^{\text{top}}(T\mathcal{M}^*(m_j, k_j))$ is orientable.

□

**Lemma 2.19.** *$H^1(X;\mathbb{R}) = H^1(M;\mathbb{R})$; $H^2_+(X;\mathbb{R}) = H^2(M)$.*

*Proof.* The result is a consequence of Proposition 2.2 in [B88] stated as follows.

**Proposition 2.20.** *(*Prop 2.2 [B88]*)*

*a). $\pi_1(X, m) \cong \pi_1(\bar{M}, m)$ for $m \in S_j$ ($S_j$ a fixed surface of $X$ or boundary surface in $M$);*

*b). There are natural isomorphisms $H_2(\bar{M}, \partial\bar{M}; \mathbb{Z}) \to H_3(X;\mathbb{Z})$ and*

*$H_2(\bar{M};\mathbb{Z}) \oplus H_1(\bar{M};\mathbb{Z}) \to H_2(\mathbb{Z})$.*

*The two summands of $H_2(X;\mathbb{Z})$ (modulo torsion) are isotropic and dual to each other under the intersection form $Q$ on $H_2$; consequently the signature $\sigma(X) = 0$, and $Q = n$ times $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with $n =$ rank $H_2(\bar{M};\mathbb{Z})$.*

We assume the proposition here and derive the result.

Since both $M$ and $X$ in our setup is path-connected, the first homology group $H_1(X)$ and $H_1(M)$ are the abelianization of the fundamental group $\pi_1(X)$ and $\pi_1(M$. Hence part (a) above implies $H_1(X) \cong H_1(M)$. This means $H^1(X) \cong H^1(M)$ and we obtain our result by applying the universal coefficient theorem for cohomology.

Part (b) of the proposition tells us that the intersection form $Q$ is given by the $2n \times 2n$ matrix

$$\begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{pmatrix}.$$

This implies rank $H_{2,+}(X) = $ rank $H_{2,-}(X) = n$. Hence $H_{2,+}(X) = H_2(M)$ and we obtain our result by applying the universal coefficient theorem for cohomology.

□

## 2.5 Uhlenbeck compactification

The method of Uhlenbeck compactification is described in detail by Donaldson and Kronheimer in [DK90] for the instanton moduli space case. We outline the main results here:

**Definition 2.21.** (Definition 4.4.1 [DK90]) An ideal ASD connection over $X$, of Chern class $k$, is a pair

$$([A],(x_1,\dots,x_l))$$

where $[A]$ is an ASD connection with $c_2 = k - l$ and $(x_1,\dots,x_l)$ is an unordered $l$-tuple of points in $X$. The curvature density of $([A],(x_1,\dots,x_l))$ is the measure

$$|F(A)|^2 + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r}.$$

Let $A_\alpha, \alpha \in N$ be a sequence of connections on $SU(2)$ bundle $P_k$ of Chern class $k$. Convergence of the gauge equivalence classes $[A_\alpha]$ is defined as follows:

**Definition 2.22.** The gauge equivalence classes $[A_\alpha]$ converge weakly to a limiting ideal ASD connection $([A],(x_1,\dots,x_l))$ if

1. The action densities converge as measures, i.e. for any continuous function $f$ on $X$,

$$\int_X f|F(A_\alpha)|^2 d\mu \longrightarrow \int_X f|F(A)|^2 d\mu + 8\pi^2 \sum_{r=1}^{l} f(x_r).$$

2. There are bundle maps $\rho_\alpha : P_{k-l}|_{X\setminus\{x_1,\dots,x_l\}} \to P_k|_{X\setminus\{x_1,\dots,x_l\}}$ such that $\rho_\alpha^*(A_\alpha)$ converges to $A$ in $C^\infty$ norm on compact subsets of the punctured manifold.

The bundles over punctured manifolds $P_{k-l}|_{X\setminus\{x_1,\dots,x_l\}}$ and $P_l|_{X\setminus\{x_1,\dots,x_l\}}$ are isomorphic but not over the entire manifold. Thus the map in Condition 2 above is between the bundles over punctured manifolds only.

**Lemma 2.23.** *The total mass of curvature density is finite.*

*Proof.* The second Chern class is given by

$$c_2(P) = \frac{\mathrm{Tr}(F_A^2) - \mathrm{Tr}(F_A)^2}{8\pi^2} \in H^4(X;\mathbb{R}).$$

In the case of $SU(2)$-principal bundle, curvature form is $su(2)$-valued, which means it is trace-free and so the formula simplifies to the Chern-Weil formula

$$c_2(E) = \frac{1}{8\pi^2}\mathrm{Tr}(F_A^2).$$

By definition, the total mass of curvature $F(A)$ is the number

$$\int_X \|F_A\|^2 d\mathrm{vol}. \tag{2.5.1}$$

Since we focus on the ASD connection,

$$\begin{aligned} \mathrm{Tr}(F_A^2) &= -\left(|F_A^+|^2 - |F_A^-|^2\right) d\mu \\ &= |F_A^-|^2 = |F_A|^2 \end{aligned}$$

Thus the Chern-Weil formula applies to give

$$|F_A|^2 = 8\pi^2 k.$$

□

Since the limiting ideal connection is parametrised by a union of products $M_{k-l} \times s^l(X)$, we take the closure of $M_k$ in the space

$$IM_k \equiv M_k \cup (M_{k-1} \times X) \cup (M_{k-2} \times X) \ldots .$$

The topology on $IM_k$ is defined by the convergence of definition 2.22.

**Theorem 2.24.** (Theorem 4.4.4 [D90]) *Any infinite sequence in $M_k$ has a weakly convergent subsequence, with a limit point in $\bar{M}_k$, the closure of $M_k$ in $IM_k$.*

We sketch the proof here. We obtain a control for the connections with sufficiently small curvatures from the next proposition.

**Proposition 2.25.** (Proposition 4.4.9 [D90]) *Let $\Omega$ be an oriented Riemannian four-manifold. Suppose $A_\alpha$ is a sequence of ASD unitary connections on a bundle $E$ over $\Omega$ with the following property. For each point $x \in \Omega$ there is a geodesic ball $D_x$, such that for all large enough $\alpha$,*

$$\int_{D_x} |F_{A_\alpha}|^2 d\mu \le \varepsilon^2 \tag{2.5.2}$$

*for some constant $\varepsilon > 0$. Then there is a subsequence $\{\alpha'\}$ and gauge transformations $u_{\alpha'} \in Aut\ E$ such that $u_{\alpha'}(A_{\alpha'})$ converges over $\Omega$.*

Condition (2.5.2) ensure the limit connection also has finite energy, that is

$$\int_{X\setminus\{x_1,\ldots,x_l\}} |F_A|^2 d\mu \leq 8\pi^2 k.$$

Since the limit connection is defined on bundle over punctured manifold, we need the following Ulenbeck removable singularity theorem to ensure that a gauge equivalent connection exist on the original manifold.

**Theorem 2.26.** (Theorem 4.4.12 [D90]) *Let $A$ be a unitary connection over the punctured ball $B^4\setminus\{0\}$, which is ASD with respect to a smooth metric on $B^4$. If*

$$\int_{B^4\setminus\{0\}} |F(A)|^2 < \infty,$$

*then there is a smooth ASD connection over $B^4$ gauge equivalent to $A$ over the punctured ball.*

The theorem implies the limit connection $A$ extends to a connection $A'$ on $P'$ over $X$. The limit measure is given by $\nu = |F_A|^2 + \sum_{r=1}^{p} n_r \delta_{x_r}$ as

$$\begin{aligned}\int_U d\nu &= \lim_{x_r\to 0}\left(\int_{U\setminus(\sqcup_{r=1}^{p} D_{x_r})} d\nu + \int_{U\setminus(\sqcap_{r=1}^{p} D_{x_r})} d\nu\right)\\ &= \int_{U\setminus\{x_1,\ldots,x_l\}} |F_A|^2 d\mu + \sum_{r=1}^{p} n_r \delta_{x_r}.\end{aligned}$$

where $\delta_{x_r}$ is the delta function

$$\delta_{x_r} = \begin{cases} 1 & \text{if } x_r \in U \\ 0 & \text{if } x_r \notin U \end{cases}.$$

The proof is completed by showing $n_r \in 8\pi^2\mathbb{Z}$. Let $Z$ be a compact oriented four manifold with boundary, $B$ a connection over the boundary manifold $\partial Z$ and $A$ the extension of $B$ over $Z$. Define $\tau_{\partial Z}(B) = \int_Z \mathrm{Tr}(F_A^2) \bmod 8\pi^2\mathbb{Z}$.

Choose small disjoint balls $Z_r$ in $X$ centred on the points $x_r$. From the convergence of measures

$$\int_{Z_r} |F_{A_\alpha}|^2 \mu \longrightarrow \int_{Z_r} |F_A|^2 d\mu + n_r$$

we have

$$\begin{aligned} n_r &= \lim_{r\to 0}\int_{Z_r} |F_{A_\alpha}|^2 d\mu - \int_{Z_r} |F_A|^2 d\mu \\ &= \lim_{r\to 0}\int_{Z_r} \mathrm{Tr}(F_{A_\alpha}^2) - \int_{Z_r} \mathrm{Tr}(F_A^2) \\ &= \lim_{r\to 0} \tau_{\partial Z_r}(A_\alpha) - \tau_{\partial Z_r}(A).\end{aligned}$$

The compactification of the monopole moduli space uses a similar approach. Let $[A_i]$ be a sequence of points in $\mathcal{M}(m_j, k_j)$. The removed points of singularity $(x_1, \ldots, x_l)$ must lie in the fixed surfaces $S_j$ because otherwise the curvature on the orbit of the the connection at $x_i$ blows up and the total mass of the curvature density cannot equal $8\pi^2 k$. Hence we can denote by $x_{j_i}$ the $i$-th point in $S_j$ and $\mu_{j_i}$ its multiplicity.

**Proposition 2.27.** (Proposition 4.8 [B89])

1. *$\mu_{j_i} = \lambda_{j_i} \cdot m_j$ for some $\lambda_{j_i} \in \mathbb{Z}_{>0}$*
2. *$[A_\infty] \in \mathcal{M}(m_j, l_j)$ with $l_j = k_j - \sum_i \lambda_{j_i}$ and $\sum_j m_j \cdot l_j \geq 0$.*

To express the meaning of this proposition precisely: When viewed as monopole moduli space, the limit point has lost $m_j k_j - m_j l_j$ lumps of energy, this change in energy in the instanton case corresponds to the change in charge in monopole case. In other words, the limit point added in this compactification process has the same mass.

The proof of this proposition will require application of Proposition 2.25, so we check first that the premise is satisfied. Given a sequence of ASD unitary connections $A_\alpha$ on a bundle $P$ over $X$, we claim there exist a finite set of points $\{x_1, \ldots, x_p\} \subset X$ such that for each $x \in X \setminus \{x_1, \ldots, x_p\}$ there is a geodesic ball $D_x$, such that for all large enough $\alpha$, $\int_{D_x} |F(A_\alpha)|^2 \mu \leq \varepsilon^2$.

Denote by $\mathcal{V}_\alpha$ the curvature density measure, i.e.

$$\mathcal{V}_\alpha(U) = \int_X 1_U |F_{A_\alpha}|^2 d\mu.$$

Then

$$\int_X d\mathcal{V}_\alpha = \int_X |F_{A_\alpha}|^2 d\mu = 8\pi^2 k < \infty. \tag{2.5.3}$$

By choosing a sequence of functions $f_i$ whose linear span is dense in $C^0(X)$[①], we have for each $i$ a bounded sequence $\{\int_X f_i d\mathcal{V}_1, \int_X f_i d\mathcal{V}_2, \int_X f_i d\mathcal{V}_3, \ldots\}$ in $\mathbb{R}$. Thus it is convergent (upon possible restriction to subsequence). Thus after taking the diagonal subsequence we obtain a convergent sequence $\{\int_X f_i d(V)_{\alpha'}\}_{\alpha'}$ and we have

$$\int_X f d\mathcal{V}_{\alpha'} \longrightarrow \int_X f d\mathcal{V} = 8\pi^2 k \quad \text{for all} \quad f \in C^0(X).$$

[①] such a construction is possible, for example: smoothings of the characteristic functions of balls.

It follows that there are at most $8\pi^2 k/\varepsilon^2$ points in $X$ which do not lie in a geodesic ball of $\mathcal{V}$-measure less than $\varepsilon^2$ for otherwise $\int_{D_{x_i}} d\mathcal{V} \geq \varepsilon^2$ which then implies

$$\int_X d\mathcal{V} \geq \sum_{i=1}^{r} \int_{D_{x_i}} d\mathcal{V} \geq r\varepsilon^2 > 8\pi^2 k.$$

*Proof.* By Proposition 2.25, there exists a sequence of gauge transformations $g_i \in \mathscr{G}(P)$ (not necessarily commuting with the $S^1$-action) such that $g_i \cdot A_i \to A_\infty$ in the $C^\infty$-topology on $X \backslash \{x_1, \cdots, x_p\}$. Let

By Lemma 2.28, convergence (2.5.4), (2.5.5) and identity (2.5.6) implies that there exists $h(u) \in \mathscr{G}(P)$ such that $h_i(u) \to h(u)$ uniformly in $u$ and $h(u)A_\infty = \theta(u) \cdot A_\infty$.

$$g_i \cdot A_i \to A_\infty \tag{2.5.4}$$

$$h_i \circ g_i \cdot A_i \to \theta(u) \cdot A_\infty \tag{2.5.5}$$

$$h_i(u) = \theta(u) \circ g_i \circ \theta(-u) \circ g_i^{-1} \tag{2.5.6}$$

**Lemma 2.28.** (Lemma 4.9 [B89]) *Suppose $A_i$, $A_\infty$ and $B(u)$ are connections on some principal $SU(2)$-bundle over a smooth manifold with $B(u)$ depending smoothly on $u$ ($u$ and element of a compact space). Let $h_i(u)$ be gauge transformations also depending smoothly on $u$. If*

$$A_i \to A_\infty$$

*and*

$$h_i(u) \cdot A_i \to B(u)$$

*both converge uniformly on $u$ in the $C^\infty$-topology (convergence of all derivatives on precompact subsets of $X$), then a subsequence $h_j(u)$ converges uniformly in $u$ in the $C^\infty$-topology to a family of gauge transformations $h(u)$, where $h(u)$ has the property that*

$$h(u) \cdot A_\infty = B(u).$$

It follows from (2.5.5) that $h_i(u)^{-1} h_i(u) g_i A_i$ and $h_i(u)^{-1}\theta(u)A_\infty$ converge to the same limit.

Since

$$g_i \cdot A_i = h_i^{-1}(u) h_i(u) g_i \cdot A_i \to A_\infty$$

and

$$h_i(u)^{-1}\theta(u) A_i nfty \to h(u)^{-1}\theta(u) A_\infty,$$

we deduce that $A_\infty = h(u)^{-1}\theta(u)A_\infty$ and hence the map

$$\begin{array}{rcl} h_i(u) & \to & h(u) \\ u & \mapsto & h(u)^{-1}\circ\theta(u) \end{array}$$

defines an action on $P|_{X\setminus\{x_i\}}$ stabilising $A_\infty$. This action is the limit of the actions $u \to g_i \circ \theta(u) \circ g_i^{-1}$ because

$$\begin{array}{rl} & h_i(u) = \theta(u)\circ g_i \circ \theta(-u)\circ g_i^{-1} \\ \Rightarrow & \theta(-u)\circ h_i(u) = g_i\circ\theta(-u)\circ g_i^{-1} \\ \Rightarrow & h_i(u)^{-1}\circ\theta(u) = g_i\circ\theta(u)\circ g_i^{-1}. \end{array}$$

Therefore the weights are equal to those of $\theta$, which means $m'_j = m_j$ where $m_j$ are weights of the bundle $P'$.

By theorem (2.26), $A_\infty$ can be extended to the bundle $P' \to X$ where $P'$ satisfies $P'|_{X\setminus\{x_j\}} \cong P|_{X\setminus\{x_j\}}$. Thus we have the same $\tilde{S}^1$-action on $P'|_{X\setminus\{x_j\}}$. This action can be extended to $P'$. The next step is to show that $A_\infty$ (extended to $P'$) is invariant with respect to the $\tilde{S}^1$-action on $P'$. This result then applies, together with formula (2.1.11), on a small neighbourhood of only one removed point to give

$$c_2(P'|_{X\setminus\{x_j\}}) = c_2(P|_{S_j}) - \mu_{j_i} = m_j k_j - \mu_{j_i}.$$

This means

$$\begin{array}{rrcl} & m'_j k'_j & = & m_j k_j - \mu_{j_i} \\ \Leftrightarrow & \mu_{j_i} & = & m_j(k_j - k'_j), \end{array} \tag{2.5.7}$$

i.e. multiplicities $\mu_{j_i}$ is integer multiple of $m_j$. It also follows from (2.5.7) that

$$\begin{array}{rrcl} & m'_j k'_j & = & m_j k_j - \lambda_{j_i} m_j \\ \Leftrightarrow & k'_j & = & k_j - \sum_i \lambda_{j_i} \end{array}$$

Let $B$ be a small $S^1$-invariant ball centered at one of the singular points $x_{j_i}$ and $s : B \to P$ be a section such that the action of $\tilde{S}^1$ on $P$ is described by

$$u \cdot s(b) = s(u\cdot b)\cdot\lambda(u)$$

for a homomorphism $\lambda : \tilde{S}^1 \to SU(2)$. Since the $\tilde{S}^1$-invariance of $s^*A_\infty$ is well-defined, the idea here is to identify $s^*A_\infty$ with $A_\infty$ so that we can understand the

$\tilde{S}^1$-invariance of $A_\infty$. $s^*A_\infty$ is a connection on the base $B$ and $A_\infty$ is a connection on $P$, the precise meaning of identifying the two is to identify the pull-back of $A_\infty$ on $B$ by the $\tilde{S}^1$-action with the connection $\lambda(u)A_\infty$, the calculation below shows that the identification amount to the equation

$$u^*A_\infty = \lambda(u)\cdot A_\infty. \tag{2.5.8}$$

Let $\omega$ be a vector at point $u(b)$, $b \in B$, then

$$\begin{aligned}
\lambda(u)\cdot A_\infty(\omega) &= \lambda(u)d\lambda(u)^{-1} + \lambda(u)^{-1}A_\infty(\omega)\lambda(u)\\
&= \lambda(u)^{-1}A_\infty(\omega)\lambda(u)\\
&= \lambda(u)^{-1}A_\infty(s_*\omega)\lambda(u)\\
&= A_\infty(\lambda(u)_* s_*\omega)\\
&= (u^*A_\infty)(\lambda(u)_* s_*\omega)\\
&= (u^*A_\infty)(\omega).
\end{aligned}$$

For the final part of the proof we want to show that $g\cdot A_\infty$ is invariant and $g$ is an $\tilde{S}^1$-equivariant gauge transformation. From (2.5.8) and the set of equations (2.5.9) we see that $g\cdot A_\infty$ is invariant if and only if $h(u)$ is a constant function. Hence the goal is to show the following set of equations in $h(u)$ has a unique solution and the solution is the constant function.

$$\begin{cases} d^*(u^*g\cdot A_\infty) = 0\\ u^*g\cdot A_\infty = h(u)g\cdot A_\infty \end{cases} \tag{2.5.9}$$

First we derive the two identities: Using the definition of adjoints, we have

$$\begin{aligned}
\langle d^*(u^*g\cdot A_\infty),\zeta\rangle &= \langle u^*g\cdot A_\infty, d\zeta\rangle\\
&= \langle g\cdot A_\infty, (u^{-1})^*d\zeta\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle u^*d^*g\cdot A_\infty,\zeta\rangle &= \langle d^*g\cdot A_\infty, (u^{-1})^*\zeta\rangle\\
&= \langle g\cdot A_\infty, (u^{-1})^*d\zeta\rangle.
\end{aligned}$$

Therefore $d^*(u^*g\cdot A_\infty) = u^*g\cdot A_\infty = 0$. The second identity follows from the following deduction:

$$\begin{aligned}
h(u)g\cdot A_\infty &= (u^*g)\circ\lambda(u)\circ g^{-1}\circ g\cdot A_\infty\\
&= (u^*g)\circ\lambda(u)\cdot A_\infty\\
&= u\circ g\circ u^{-1}\circ u\cdot A_\infty\\
&= u^*g\cdot A_\infty
\end{aligned}$$

The identity

$$u^* g \cdot A_\infty = h(u) g \cdot A_\infty = h(u)^{-1} dh(u) + h(u)^{-1} g \cdot A_\infty h(u)$$

implies $h(u)$ satisfies the ordinary differential equation (ODE)

$$h(u)(u^* g \cdot A_\infty) = dh(u) + g \cdot A_\infty h(u)$$

which admits a unique solution. Hence so does (2.5.9). Again we apply due to the Ulenbeck removable singularity theorem to obtain a gauge transformation $g$ such that $g \cdot A_\infty$ is the connection on the entire ball. The energy condition $\|F_{g \cdot A_\infty}\| < \varepsilon$ is met because we choose the $S^1$-invariant ball $B$ to be small enough.

Ulenbeck removable singularity theorem also asserts that if there exists a constant $\varepsilon > 0$ such that $\|F_{g \cdot A_\infty}\| < \varepsilon$, then the connection $g \cdot A_\infty$ also satisfies

1. $d^*(g \cdot A_\infty) = 0$,
2. $g \cdot A_\infty \left(\frac{\partial}{\partial r}\right) = 0$ on $\partial B$,
3. $\|g \cdot A_\infty\|_{L_1^2} \leq M \|F_{g \cdot A_\infty}\|_{L^2}$ and
4. $g$ is smooth on $B \backslash \{0\}$.

Moreover, for suitable constants $\varepsilon_1$, $M$, the connection $g \cdot A_\infty$ is uniquely determined by these properties, up to the transformation $g \cdot A_\infty \to u_0 g \cdot A_\infty u_0^{-1}$ for a constant $u_0$ in $U(n)$ (refer to Theorem 2.3.7 [DK90] for detail).

Since $u^* g \cdot A_\infty$ also satisfies Conditions (1)-(4), then by this uniqueness the second identity of (2.5.9) implies $h(u)$ is constant. This means $g \cdot A_\infty$ is invariant on the entire $P'$. To see $g \cdot A_\infty$ is invariant on the extended points $x_{j_i}$, consider a sequence of vector $v_i$ approaching the limit vector $v$ at $x_{j_i}$, then

$$u^* A_\infty(v) = A_\infty(u_*(v))$$

and

$$\begin{aligned} \lim u^* A_\infty(v_i) &= \lim A_\infty(u_*(v_i)) \\ &= A_\infty(u_*(v)). \end{aligned}$$

This completes the proof.

$\square$

We can now define the compactification of the monopole moduli sapce $\mathcal{M}(m_j, k_j)$ as its closure in the compact space $\mathscr{J}$ given by

$$\bigcup_{\left\{ (l_j) \middle| \begin{array}{l} j \text{ is s.t. } m_j \neq 0 \\ l_j \leq k_j, \sum m_j l_j \geq 0 \end{array} \right\}} \left( \mathcal{M}(m_j, l_j) \times \prod_{j \text{ s.t. } m_j \neq 0} \mathscr{J}^{k_j - l_j}(S_j) \right)$$

where $\mathscr{J}^l(S_j)$ is the $l$-th symmetric product of $S_j$.

## 2.6 Construction of Monopoles

Ulenbeck compactification introduces limit connections that exists in the lower stratum of the space

$$M_k \cup (M_{k-1} \times X) \cup (M_{k-2} \times X) \ldots$$

but the structure of the compactified space near the limit points is not revealed by the construct. We introduce a method to construct monopoles in some neighbourhood of the limit points.

Let $X_0$, $X_1$ be two Riemannian 4-manifolds which are constructed from two 3-manifold $M_0$ and $M_1$ as in chapter 1. Choose points $x_i \in X_i$ and suppose that the Riemannian metrics on $X_i$ are conformally flat near $x_i$.[①] This allows us to identify an annulus near $x_i$ to a half-cylinder via the conformal equivalence

$$\begin{array}{rcl} e : \mathbb{R} \times S^3 & \longrightarrow & \mathbb{R}^4 \backslash \{0\} \\ (t, \omega) & \longmapsto & e^t \omega. \end{array}$$

More explicitly, an annulus $\Omega$ with radii $N\lambda^{1/2}$, $N^{-1}\lambda^{1/2}$ with $\lambda \in \mathbb{R}_{>0}$ is mapped to to the tube

$$e^{-1}(\Omega) = \left( \frac{1}{2} \log(\lambda) - T, \frac{1}{2} \log(\lambda) + T \right) \times S^3.$$

The two tubes of $X_0$ and $X_1$ are identified by reflection as indicated in the diagram below.

①This condition is not necessary for the construct but does simplify the treatment. Since any 4-manifold arising from construction in Chapter 1 satisfies this condition, having this condition has no drawback in this case.

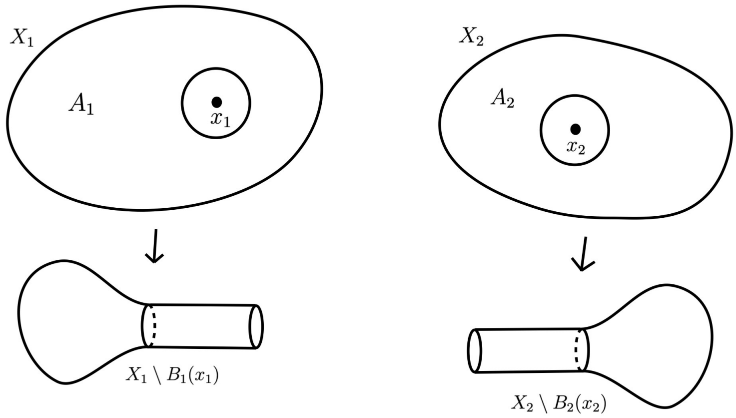



Figure 2.6

The *connected sum* of $X_0$ and $X_1$, denoted by $X = X(\lambda) = X_0 \# X_1$, is defined as

$$\left(X_0 \backslash B(x_0, N^{-1}\lambda^{1/2})\right) \sqcup \left(X_1 \backslash B(x_1, N^{-1}\lambda^{1/2})\right)$$

where $B(x_i, N^{-1}\lambda^{1/2})$ denotes the ball centred at $x_i$ with radius $N^{-1}\lambda^{1/2}$.

A special case is gluing with the manifold $S^4$, it is clear that $X\#S^4$ is equivalent to $X$ as conformal manifolds, that is for metric $g$ on $X$ and $g'$ on $X\#S^4$, there exists a positive function $f$ such that $g' = fg$. In other words, $S^4$ is the identity with respect to the operation of connection sum. In our case, we will also construct a monopole near the limit points by gluing the 4-manifold $X$ with $S^4$.

Now we consider the bundles $P_i$ over $X_i$ and connections. The general idea is to choose a trivialisation on the neighbourhood of $x_i$ and glue the local trivial bundles over the neighbourhood by choosing a $\tilde{S}^1$-equivariant bundle automorphism $\rho : P_0 \to P_1$. Therefore on the connected sum there will be a family of connections parametrised by $\rho$.

We have explained in the previous chapter that $x_i$ must lie in fixes surfaces $S_i$ in $X_i$. Also an identification $\rho$ exists only if the masses of $S_i$ are equal, because different mass means $\tilde{S}^1$ acts on fibres over $S_i$ with different winding number and so the gauge transformation cannot be $\tilde{S}^1$-equivariant.

Following the construction in [B89] to specify the neighbourhoods, define spheres in $X_0$ of radii $N\lambda^{1/2}$ and $N\lambda^{-1/2}$ and shells

$$\begin{aligned} R_{-1} &= \left\{x \in X_0; |x - x_0| \in [kN^{-1}\lambda^{1/2}, N^{-1}\lambda^{1/2}]\right\}, \\ R_1 &= \left\{x \in X_0; |x - x_0| \in [N\lambda^{1/2}, k^{-1}N\lambda^{1/2}]\right\}, \end{aligned}$$

Let $U_0 \subset X_0$ be the complement of the ball $B(x_0, |\xi|)$ of radius $|\xi| \leq kN^{-1}\lambda^{1/2}$ and $\hat{U}_0 \subset X_0$ be the complement of the ball $B(x_0, |\xi|)$ of radius $|\xi| \leq N^{-1}\lambda^{1/2}$. $U_1$ and $\hat{U}_1$ is defined symmetrically, i.e.

$$\begin{aligned} U_1 &= X_1 \backslash B(x_1, |\xi|), \ \xi \leq N^{-1}\lambda^{1/2} \\ \hat{U}_1 &= X_1 \backslash B(x_1, |\xi|), \ \xi \leq kN^{-1}\lambda^{1/2}. \end{aligned}$$

The construction requires $N$ to be large enough. More precisely, $k$ and $N$ must satisfy

$$(k^{-4} - 1) \cdot (1 - k) \cdot N \cdot (N - N^1)^{-3} < 8.$$

Additionally, $\lambda$ also needs to be sufficiently small for the spheres and shells to be well defined.

**Proposition 2.29.** (Proposition 4.11 of [D86])

*The $\tilde{S}^1$-invariant connections on the connected sum can be represented in the form*

$$A = (A_0 + a, A_1 + a', \rho), \tag{2.6.1}$$

*where*

$$\begin{aligned} &a \text{ is defined on } \hat{U}_0, \quad d^*_{A_0} a = 0, \quad \|a\|_{L^{2p}(\hat{U}_0)} < \eta_0 \\ &a' \text{ is defined on } \hat{U}_1, \quad d^*_{A_1} a' = 0, \quad \|a'\|_{L^{2p}(\hat{U}_1)} < \eta_0 \end{aligned} \tag{2.6.2}$$

*and*

$$\rho \in C^\infty\left(\left(\hat{U}_0 \cap \hat{U}_1\right), Hom^{\tilde{S}^1}(P_0|_{x_0}, P_1|_{x_1})\right) \tag{2.6.3}$$

*satisfying*

$$A_1 + a' = \rho \cdot (A_0 + a) \text{ on } \hat{U}_0 \cap \hat{U}_1. \tag{2.6.4}$$

According to Donaldson, any $p > 2$ is sufficient for the construction arguments but taking larger value, say $p > 6$ is more convenient as all maps will be smooth on the relevant Banach spaces.

Now we are in position to construct the monopole. Compare Figure 2.7 for the following arguments.

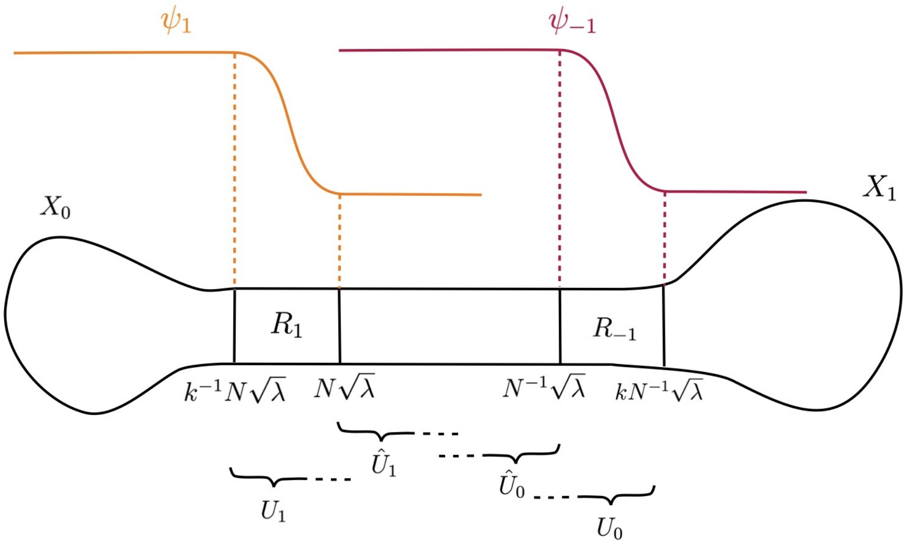



Figure 2.7

Let $A_i$ be monopoles on $X_i$. We identify $P_0|_{U_0\cap U_1}$ and $P_1|_{U_0\cap U_1}$ by choosing a $\tilde{S}^1$-equivariant gauge transformation

$$\rho \in C^\infty\left(\left(\hat{U}_0 \cap \hat{U}_1\right), \mathrm{Hom}^{\tilde{S}^1}(P_0|_{x_0}, P_1|_{x_1})\right).$$

We aim to construct a connection $A^0(\rho)$ form $A_0 + a_0$ and $A_1 + a_0'$ such that

$$A^0(\rho) = \begin{cases} A_0 & \text{on } U_1 \setminus (U_0 \cap U_1) \\ A_1 & \text{on } \hat{U}_1 \end{cases}.$$

The first step is to construct $A_0 + a_0$ and $A_1 + a_0'$ such that $A_0 + a_0$ equals $A_0$ on $U_0 \backslash (U_0 \cap U_1)$ and is gauge equivalent to $A_1 + a_0'$ on $U_0 \cap \hat{U}_1$; $A_1 + a_0'$ equals to $A_1$ on $\hat{U}_1$ and is gauge equivalent to $A_0 + a_0'$ on $R_1$. With the requirement satisfied, the connection given by $(A_0 + a_0, A_1 + a_0', \rho)$ is denoted by $A^0(\rho)$. To achieve this, we introduce cut-off functions $\psi_1$ and $\psi_{-1}$ such that

$$\begin{aligned} &\psi_1|_{X\backslash U_1} = 1,\ \text{supp}\ \psi_1 \subset X\backslash\hat{U}_1,\ \text{supp}\ d\psi_1 \subset R_1, \\ &\psi_{-1}|_{X\backslash U_1} = 1,\ \text{supp}\ \psi_{-1} \subset U_0,\ \text{supp}\ d\psi_{-1} \subset R_{-1} \quad . \end{aligned}$$

Define $a_0$ and $a_0'$ as follows

$$\begin{aligned} a_0 = &\quad (\psi_1 - 1)\cdot A_0 + (1-\psi_1)\cdot \rho^{-1}A_1\rho \quad \text{on} \quad U_0 \cap U_1, \\ a_0' = &\quad -\psi_1 \cdot A_1 + \psi_1 \cdot \rho A_0 \rho^{-1} \qquad\qquad \text{on} \quad U_0 \cap U_1. \end{aligned}$$

$a_0$ and $a_0'$ are extended to $X_0$ and $X_1$ by 0 so that $A_0 + a_0$ and $A_1 + a_0'$ are defined on appropriate subsets of the connected sum $X$. It is easy to check that

$$A_0 + a_0 = \psi_1 \cdot A_0 + (1-\psi_1)\cdot \rho^{-1}A_1\rho$$

and

$$A_1 + a_0' = (1-\psi_1)\cdot A_1 + \psi_1 \cdot \rho A_0 \rho^{-1}$$

satisfy the requirements as intended. Such modification is not yet satisfactory as $A^0(\rho)$ is not necessarily ASD as it is not known that wether $\mathrm{supp} P_+ F_{A^0(P)} \subset R_1 \subset X$ empty. The rest of the construction is about improving the modification factor to ensure that the construction yields an ASD connection on the connected sum.

Let $b_0 \in \mathscr{W}^1_{p,1}(X_0)$ be another modification. If the equation

$$\begin{aligned} P_+F(A_0 + a_0 + b_0) &= 0 && (2.6.5)\\ d^*_{A_0} b_0 &= 0 && (2.6.6)\end{aligned}$$

admits a solution, then $A_0 + a_0$ can be modified to be an ASD connection. Equation (2.6.5) is equivalent to

$$\begin{aligned} & P_+\left(d_{A_0}(a_0+b_0) + \frac{1}{2}[a_0+b_0, a_0+b_0] + F(A_0)\right) = 0 \\ \Leftrightarrow\ & P_+\left(d_{A_0} b_0 + [a_0, b_0] + \frac{1}{2}[b_0, b_0]\right) = 0 \qquad (2.6.7)\end{aligned}$$

Regarding the left side of (2.6.7) as an operator given by

$$\begin{aligned} Q : \mathscr{W}^1_{p,1}(X_0) \times \mathscr{W}^1_{2p,0}(X_0) &\longrightarrow \mathscr{W}^2_{+,p,0}(X_0) \times \mathscr{W}^0_{p,0}(X_0), \\ (b_0, a_0) &\longmapsto \left(P_+\left(d_{A_0} b_0 + [a,b] + \frac{1}{2}[b_0,b_0]\right), d^*_{A_0} b_0\right). \end{aligned}$$

Observe that the derivative with respect to the first variable is the same as the operator 2.2.3 studied in chapter 2 and is invertible. Therefore the implicit function theorem implies that the set of equations (2.6.7) and (2.6.6) admits a solution for small $a_0$ and $\sigma$.

Next we define inductively a sequence $a_j$ and $a_j'$ as

$$\begin{aligned} a_{j+1} &= a_j + \psi_{-1}\cdot b_j, \\ a'_{j+1} &= a'_j + \psi_{-1}\cdot \rho b_j \rho^{-1} \end{aligned}$$

where $b_j$ is the solution of the set of equation (2.6.7) and (2.6.6) with $a_0$ replaced with $a_j$. It is proved by Donaldson (section IV(iv) [D86]) that the sequence $a_j$ coverges to a limit $a_\infty$ in $L^{2p}$ on $X_0$ and $a_j'$ coverges to a limit $a_\infty'$ in $L^{2p}$ on $X_1$.

Also the bound of $P_+(F(A^j(\rho)))$ vanishes as $j \to \infty$, therefore we arrive at an ASD connection

$$A^\infty(\rho) = (A_0 + a_\infty, A_1 + a'_\infty, \rho).$$

Suppose $A_i$ satisfy the following acyclicity condition on the $\tilde{S}^1$-invariant part of the cohomology groups of the AHS complex:

$$H^0_{0,A_i} = H^1_{0,A_i} = H^2_{0,A_i} = 0, \quad i = 0, 1, \tag{2.6.8}$$

then we have the following theorem.

**Theorem 2.30.** (Theorem 5.1 [B89])

*If $A_i$ satisfy condition (2.6.8), and $\lambda$ and $\eta_0$ are sufficiently small, the gauge equivalent classes of ASD connections over $X_0 \# X_1$ which can be represented in the standard form (2.6.1), are smoothly parametrised by Hom$(P_0|_{x_0}, P_1|_{x_1}/)\,\{\pm 1\} \cong$ $SO(3)$.*

The following lemma shows that the construction is exhaustive, that is any other ASD connection in the form (2.6.1) is gauge equivalent to one of this family.

**Lemma 2.31.** (Lemma (4.31) [D86])

*Given the acyclicity condition (2.6.8), $A^\infty(\rho)$ and $A^\infty(\bar{\rho})$ are gauge equivalent if and only if $\bar{\rho} = \pm\rho$.*

Proposition 2.29, Theorem 2.30 and Lemma 2.31 together implies that the map

$$\mathrm{Hom}^{\tilde{S}^1}\left(P_0|_{x_0}, P_1|_{x_1}\right)/\{\pm 1\} \longrightarrow \mathcal{M}(m_j, k_j + k'_j)$$

is injective onto the neighbourhood $W$ of gauge equivalent classes of ASD connections over $X$ at the limit points added at $x_0$ and $x_1$ with charge $k_j$ and $k'_j$ respectively.

There are two conditions assumed for the construction. One is the acyclicity condition, the other is $X_i$ being conformally flat near $x_i$. According to Donaldson (in [D86]), both condition can be relaxed. The acyclicity condition is a very strict condition. As we studied in section 2.2, $H^1$ is the tangent space of the moduli space, thus $H^1 = 0$ means the moduli space is discrete. We omit the detail deduction here and state the result by Donaldson without acyclicity condition:

**Theorem 2.32.** (Theorem 5.2 of [B89])

*Let $g_0$, $g_1$ be $S^1$-invariant metrics on $X_0$, $X_1$, which are conformally flat near $x_0$, $x_1$ and $A_0$, $A_1$ monopoles with respect to these metrics. If $\lambda$ and $\eta_0$ are sufficiently small, and $g$ is the $S^1$-invariant metric on $X = X_0 \# X_1$, then:*

*(1) there is a $(\Gamma_{A_0} \times \Gamma_{A_1})$-invariant open neighbourhood $N$ of $I \times \{0\} \times \{0\}$ in $I \times H^1_{0,A_0} \times H^1_{0,A_0}$.*

*(2) there is a $(\Gamma_{A_0} \times \Gamma_{A_1})$-equivariant map*

$$\Phi = (\varphi_0, \varphi_1) : N \longrightarrow H^2_{0,A_0} \times H^2_{0,A_1} \tag{2.6.9}$$

*such that the monopoles with respect to the metric $g$ on $X$ representable in the standard form (2.6.1) are parametrised (up to gauge equivalence) by $\Phi^{-1}/(\Gamma_{A_0} \times \Gamma_{A_1})$.*

Theorem (2.32) allow us to construct a $4k-1$ parameter family of monopoles on $H^3$ (whose corresponding compactified space $X = S^4$) of any mass $m \in \mathbb{Z}_{>0}$ and charge $k \in \mathbb{Z}_{>0}$ by gluing an $(m, k-1)$-monopole $A_0$ on $H^3$ and a $(m,1)$-monopole on $H^3$. Recall that the moduli space of dimension 1 instanton on $S^4$ is $B^5$. $S^1$-invariant instantons (which correspond to $(m,1)$-monopoles) can only vary on the fixed surface $S^2$. Counting the energy concentration of the instanton which is not affected by the $S^1$-invariance condition, $S^1$-invariant instantons have 3 degrees of freedom. Applying Theorem (2.32): we have $I = S^1$ and $\Gamma_{A_i} = \{\pm 1\}$. Also $H^2_{0,A_i}$ vanish by theorem (2.16), hence

$$\begin{aligned}
\dim \Phi^{-1}/(\Gamma_{A_0} \times \Gamma_{A_1}) &= \dim I + \dim H^1_{A_0} + \dim H^1_{A_0} \\
&= 1 + 4(k-1) - (1 - 0 + 0) + 4 \times 1 - (1 - 0 + 0) \\
&= 4k - 1,
\end{aligned}$$

which means we are able to construct a $4k-1$ parameter family of monopoles on $H^3$.

# Chapter 3

# $S^1$-invariant ADHM construction

Due to the correspondence of monopoles and $S^1$-invariant instantons, we are able to obtain detailed informations about monopole by studying ADHM construction of instantons on $\mathbb{R}^4$. We outline first the construction of an instanton following [MS04], [A79] and [B14].

For the construction, an element $x = (x_1, x_2, x_3, x_4)$ of the Euclidean 4-space $\mathbb{R}^4$ is identified with a quaternion number $x = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$ for $\{1, i, j, k\}$ the standard basis of $\mathbb{H}$; the group $SU(2)$ is identified with the group $Sp(1)$ of unit quaternions via the map

$$\begin{aligned} SU(2) &\longrightarrow Sp(1) \\ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} &\longmapsto \alpha + \beta j \end{aligned}$$

and the Lie algebra $su(2)$ with the imaginary quaternions $sp(1)$ by identifying the basis

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

of $su(2)$ with the basis $\{i, j, k\}$ of imaginary quaternions. This identification is the link between geometric data and algebraic data. For example, locally, a gauge transformation is interpreted as a unit quaternionic-valued function and a connection $A$ corresponds to an imaginary quaternionic 1-form.

The ADHM data for an $SU(2)$-instanton of charge $N$ consists of a $(N+1) \times N$ matrix

$$\Delta(x) = \begin{pmatrix} L \\ M - x \,\mathrm{Id}_N \end{pmatrix}$$

where $L$ is a $1 \times N$ vector of quaternions, $M$ is a symmetric $N \times N$ matrix of quaternions and $\mathrm{Id}_N$ is the $N \times N$ identity matrix, $x \in \mathbb{H}$. $\Delta$ is required to satisfy the invertibility condition:

$$\Delta(x)^\dagger \Delta(x) = R(x), \tag{3.0.1}$$

where $\dagger$ denotes the quaternionic conjugate transpose[①] and $R(x)$ is a real invertible $N \times N$ matrix for all $x \in \mathbb{H}$. To construct the connection form $A$, we pick an $(N+1)$-component vector $\Psi(x)$ of unit length, $\Psi(x)^\dagger \Psi(x) = 1$, which satisfies the equation

$$\Psi(x)^\dagger \Delta(x) = 0. \tag{3.0.2}$$

Condition (3.0.1) ensures $\mathrm{rank}\Delta(x) = N$ and thus the equation always admits a solution. The connection form $A$ is then given by

$$A(x) = \Psi(x)^\dagger d\Psi(x) \tag{3.0.3}$$

The unit length condition of $\Psi(x)$ implies

$$d\left(\Psi(x)^\dagger \Psi(x)\right) = d\Psi(x)^\dagger \Psi(x) + \Psi(x)^\dagger d\Psi(x) = 0 \tag{3.0.4}$$

and thus

$$\begin{aligned} & \mathrm{Re}\left(d\left(\Psi(x)^\dagger \Psi(x)\right)\right) \\ = \; & \mathrm{Re}\left(d\Psi(x)^\dagger \Psi(x) + \Psi(x)^\dagger d\Psi(x)\right) \\ = \; & 2\mathrm{Re}\left(\Psi(x)^\dagger d\Psi(x)\right) = 0. \end{aligned}$$

Hence $\Psi(x)^\dagger d\Psi(x)$ is an imaginary quaternion valued 1-form which can be regarded as an $su(2)$-valued 1-form.

The construction contains some redundency as condition (3.0.2) is preserved under the map $\Psi(x) \mapsto \Psi(x)q(x)$ with $q$ a unit-quaternion valued function. This map corresponds to the transformation

$$\begin{aligned} A(x) = \Psi(x)^\dagger d\Psi(x) \longmapsto \; & q(x)^\dagger \Psi(x)^\dagger d\Psi(x) q(x) + q(x)^\dagger \Psi(x)^\dagger \Psi(x) dq(x) \\ & = q(x)^\dagger A(x) q(x) + q(x)^\dagger dq(x) \end{aligned} \tag{3.0.5}$$

[①] Quaternionic conjugate transpose refers to the operation that transposes the matrix and map each entry $q = q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4$ to $q^\dagger = q_1e_1 - q_2e_2 - q_3e_3 - q_4e_4$ where $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathbb{H}$.

on $A$, which is a gauge transformation. Next we verify that the connection $d+A$ constructed as such is self-dual. Computing the curvature in terms of $\Psi$:

$$\begin{aligned} F_A &= dA + A \wedge A \\ &= d(\Psi^\dagger d\Psi) + (\Psi^\dagger d\Psi) \wedge (\Psi^\dagger d\Psi) \\ \text{(by condition (3.0.4))} \quad &= d\Psi^\dagger d\Psi - (d\Psi^\dagger)(\Psi\Psi^\dagger) \wedge d\Psi \\ &= d\Psi^\dagger \wedge (1 - \Psi\Psi^\dagger) d\Psi \end{aligned} \tag{3.0.6}$$

Since $\Psi\Psi^\dagger$ is orthogonal projection onto span of $\Psi$, $(1 - \Psi\Psi^\dagger)$ is the orthogonal projection onto orthogonal complement of span of $\Psi$, i.e. the column space of $\Psi$. Consider $P = \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger$. Observe that

$$P\Delta = \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger\Delta = \Delta \tag{3.0.7}$$

and for any element $v$ of orthogonal complement of the column space of $\Delta$,

$$Pv = \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger v = 0. \tag{3.0.8}$$

This mean $P$ is the orthogonal projection onto the column space of $\Delta$, i.e. $P = 1 - \Psi^\dagger\Psi$. Therefore we can rewrite (3.0.6) as

$$\begin{aligned} F_A &= (d\Psi^\dagger)\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger(d\Psi) \\ &= \Psi^\dagger d\Delta(\Delta^\dagger\Delta)^{-1}(d\Delta^\dagger)\Psi \end{aligned}$$

Observe that

$$d\Psi^\dagger\Delta = -\Psi^\dagger d\Delta$$

and

$$d\Delta = -\begin{pmatrix} 0 & I_N \end{pmatrix} dx,$$

therefore

$$F_A = \Psi \begin{pmatrix} 0 \\ \mathrm{Id}_N \end{pmatrix} (\Delta^\dagger\Delta)^{-1} dx \wedge d\bar{x} \begin{pmatrix} 0 \\ \mathrm{Id}_N \end{pmatrix}^\dagger \Psi \tag{3.0.9}$$

The result then follows from the fact that

$$\begin{aligned} dx \wedge d\bar{x} &= -2\big[(dx_1 \wedge dx_2 + dx_3 + dx_4)i + (dx_1 \wedge dx_3 - dx^2 \wedge dx_4)j \\ &\quad +(dx_1 \wedge dx_4 + dx_2 \wedge dx_3)k\big] \end{aligned}$$

is a self-dual 2-form.

There are more redundancies than the one corresponding to the gauge transformation, here we introduce one needed for the construction of $S^1$-invariant data. We claim that the following transformation does not change the connection constructed:

$$\Delta(x) \longmapsto \tilde{\Delta}(x) \equiv \begin{pmatrix} q & 0 \\ 0 & \mathcal{O} \end{pmatrix} \Delta(x)\mathcal{O}^{-1} \tag{3.0.10}$$

where $q$ is a constant unit quaternion and $\mathcal{O}$ is a constant real orthogonal $N \times N$ matrix. To check this, let $\Psi(x) = (\Psi_1(x), \dots, \Psi_{N+1}(x))^T$ be given as in (3.0.3) and choose

$$\tilde{\Psi}(x) = \begin{pmatrix} q\Psi_1(x) \\ \mathcal{O}\begin{pmatrix} \Psi_2(x) \\ \vdots \\ \Psi_{N+1}(x) \end{pmatrix} \end{pmatrix}.$$

so that $\tilde{\Psi}(x)$ and $\tilde{\Delta}(x)$ satisfies condition (3.0.2):

$$\begin{pmatrix} q\Psi_1(x) \\ \mathcal{O}\begin{pmatrix} \Psi_2(x) \\ \vdots \\ \Psi_{N+1}(x) \end{pmatrix} \end{pmatrix}^\dagger \begin{pmatrix} q & 0 \\ 0 & \mathcal{O} \end{pmatrix} \Delta(x)\mathcal{O}^{-1} = \Psi^\dagger \Delta(x)\mathcal{O}^{-1} = 0.$$

The connection constructed from $\tilde{\Psi}$ is the same from that construct from $\Psi$ because

$$\begin{aligned} \tilde{\Psi}(x)^\dagger d\tilde{\Psi}(x) &= \begin{pmatrix} \Psi_1^\dagger(x)q^\dagger & \begin{pmatrix} \Psi_2^2(x) & \dots & \Psi_{N+1}^2(x) \end{pmatrix}^\dagger \mathcal{O}^\dagger \end{pmatrix} \begin{pmatrix} qd\Psi_1(x) \\ \mathcal{O}\begin{pmatrix} d\Psi_2(x) \\ \vdots \\ d\Psi_{N+1}(x) \end{pmatrix} \end{pmatrix} \\ &= \Psi(x)^\dagger d\Psi(x). \end{aligned}$$

Now we consider the connection form $A$ that is invariant under some rotation group $S^1 \subset SO(4)$. Via the identification

$$SO(4) \cong Spin(3) \times_{\mathbb{Z}_2} Spin(3) \cong SU(2) \times_{\mathbb{Z}_2} SU(2) \cong Sp(1) \times_{\mathbb{Z}_2} Sp(1),$$

where $(g_1, g_2) \sim (-g_1, -g_2)$, any element of the action group can be represented by a tuple of unit quaternions $[g_1, g_2]$ and the action is given by

$$x \longmapsto g_1 x g_2^{-1}.$$

**Proposition 3.1.** *A connection $A$ is $S^1$-invariant if and only if the $S^1$-action on the ADHM data $\Delta(x)$ of $A$ leads to a gauge equivalent connection*[①]*. That is for each $[g_1, g_2]$ there exist unit quaternion $q$ and real orthogonal matrix $\mathcal{O}$, both depending on $g_1$ and $g_2$, such that*

$$(3.0.11)\quad \Delta(g_1 x g_2^{-1}) = \begin{pmatrix} L \\ M - g_1 x g_2^{-1} Id_N \end{pmatrix} = g_1 \begin{pmatrix} q & 0 \\ 0 & \mathcal{O} \end{pmatrix} \begin{pmatrix} L \\ M - x Id_N \end{pmatrix} g_2^{-1} \mathcal{O}^{-1}.$$

The proposition provide a criteria for the ADHM data to be $S^1$-invariant: we can define the matrix $\Delta(x)$ to be $S^1$-invariant if $\Delta(g_1 x g_2^{-1})$ and $g_1 \Delta(x) g_2^{-1}$ give rise to the same (up to gauge equivalence) connection.

*Proof.* Given (3.0.11), the transformation (3.0.10) implies that

$$g_1 \tilde{\Delta}(x) g_2^{-1} = g_1 \begin{pmatrix} q & 0 \\ 0 & \mathcal{O} \end{pmatrix} \begin{pmatrix} L \\ M - x\mathrm{Id}_N \end{pmatrix} g_2^{-1} \mathcal{O}^{-1}$$

and

$$g_1 \Delta(x) g_2^{-1} = g_1 \begin{pmatrix} L \\ M - x\mathrm{Id}_N \end{pmatrix} g_2^{-1}$$

give rise to gauge equivalent connections. Since $g_1$ and $g_2$ are constant unit quaternions, $g_1 \Delta(x) g_2^{-1}$ and $\Delta(x)$ give the same connection. Therefore $g_1 \tilde{\Delta}(x) g_2^{-1} = \Delta(g_1 x g_2^{-1})$ and $\Delta(x)$ gives gauge equivalent connections. As the former gives the connection $[g_1, g_2] \cdot A$ and latter gives $A$, this means $A$ is $S^1$-invariant.

Conversely, suppose $[g_1, g_2] \cdot A$ is gauge equivalent to $A$. This means $\Delta(x)$ and $\Delta(g_1 x g_2^{-1})$ are equivalent data. Then

$$g_1^{-1} \Delta(g_1 x g_2^{-1}) g_2 = \begin{pmatrix} g_1^{-1} L g_2 \\ g_1^{-1} M g_2 - x\mathrm{Id}_N \end{pmatrix}$$

and $\Delta(x)$ are equivalent data. Denote this matrix by $\hat{\Delta}(x)$. We apply the following theorem of Atiyah, Drinfeld, Hitchin and Manin (ADHM) to conclude that there exists $q$ and $\mathcal{O}$ such that

$$\hat{\Delta}(x) = \begin{pmatrix} q & 0 \\ 0 & \mathcal{O} \end{pmatrix} \Delta(x) \mathcal{O}^{-1}.$$

**Theorem 3.2.** ([ADHM78])

*Every $SU(2)$-instanton of charge $k$ on $\mathbb{R}^4$ arises from parameters $(L, M)$ satisfying condition (3.0.1), the connection form being given by (3.0.3) and with $\Psi$ defined by condition (3.0.2).*

---

[①] We cannot require the action to yield exactly the same connection due to the redundancy (3.0.5).

Therefore

$$\Delta(g_1 x g_2^{-1}) = g_1 \begin{pmatrix} q & 0 \\ 0 & \mathcal{O} \end{pmatrix} \Delta(x) \mathcal{O}^{-1} g_2^{-1}.$$

□

The ADHM construction allows the connection space to be identified with the matrix space

$$\mathfrak{M} = \left\{ \Delta(x) = \begin{pmatrix} L \\ M - x\mathrm{Id}_N \end{pmatrix} \in M_{(N+1)\times N} \middle| M \text{ is symmetric}, \Delta(x)^\dagger \Delta(x) \text{ is invertible.} \right\}.$$

## 3.1 Energy 1 instanton

To describe the instanton moduli space, consider first energy $N = 1$ instantons, then

$$\Delta(x) = \begin{pmatrix} L \\ M - x \end{pmatrix} = \begin{pmatrix} \frac{L}{|L|}|L| \\ M - x \end{pmatrix}$$

Since $\frac{L}{|L|}$ is a unit quaternion, $\Delta(x)$ is equivalent to

$$\begin{pmatrix} |L| \\ M - x \end{pmatrix}.$$

Hence we can identify $\mathfrak{M}$ with $\mathbb{R}^4 \times \mathbb{R}^+$ via the isomorphism

$$\begin{aligned} \mathfrak{M} &\longrightarrow \mathbb{R}^4 \times \mathbb{R}^+ \\ \Delta &\longmapsto (M, |L|). \end{aligned} \tag{3.1.1}$$

The isomorphism is well-defined for $L \neq 0$ as

$$\begin{pmatrix} |L| & (M - x)^\dagger \end{pmatrix} \begin{pmatrix} |L| \\ M - x \end{pmatrix} = |L|^2 + |M - x|^2 \neq 0.$$

Observe that the space $\mathbb{R}^4 \times \mathbb{R}^+$ indicates the mass center and energy concentration of an instanton, which is the geometric meaning of the data $L$ and $M$.

If in addition the instanton is invariant under certain rotation action, then

$$\begin{pmatrix} L \\ M - g_1 x g_2^{-1} \end{pmatrix} = g_1 \begin{pmatrix} qL\mathcal{O}^{-1} \\ M - x \end{pmatrix} g_2^{-1} = \begin{pmatrix} g_1 q L g_2^{-1} \mathcal{O}^{-1} \\ g_1 M g_2^{-1} - g_1 x g_2^{-1} \end{pmatrix}$$

which implies $M = g_1 M g_2^{-1}$ is fixed by $[g_1, g_2]$-action. This implies if the ADHM data leads to the same conclusion as before, then the mass center must be a the fixed point of $[g_1, g_2]$-action.

**Remark 3.3.** 1. The connection $A_0$ corresponds to $(0,1) \in \mathbb{R}^4 \times \mathbb{R}^+$ is calculated explicitly in (3.4.2) [DK90] to be

$$A = \frac{1}{1+|x|^2}(\theta_1 \mathbf{i} + \theta_2 \mathbf{j} + \theta_3 \mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a standard basis for $su(2)$ and

$$\begin{aligned}
\theta_1 &= x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3, \\
\theta_2 &= x_1 dx_3 - x_3 dx_1 - x_4 dx_2 + x_2 dx_4, \\
\theta_3 &= x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2.
\end{aligned}$$

2. The moduli space of 1-instanton can be identified with the space of gauge equivalent connections $\{f^* A_0\}$ where $f$ is generated by translations and dilations of $\mathbb{R}^4$.

# Chapter 4

# Γ-invariant monopoles and examples

In this chapter, we first consider a certain kind of group action $\Gamma$ on $M$, where $\Gamma$ is a Lie group, and the moduli space of $\Gamma$-invariant monopoles and then calculate two examples.

In Section 4.1, we give the definition of $\Gamma$-invariant monopoles, state the $\Gamma$-invariant version of deformation theory and describe the local neighbourhood of a $\Gamma$-invariant monopole through the Kuranishi map.

In Section 4.2 and 4.3, we calculate two examples: $\mathbb{H}^3$ and $S^2 \times \mathbb{R}$. In both cases we take $\Gamma = S^1$. The $\Gamma$-action on $\mathbb{H}^3$ is defined to be rotations of the upper half space around the $z$-axis. We focus on the moduli space of monopoles on $\mathbb{H}^3$ with charge 1. The $\Gamma$-action on $S^2 \times \mathbb{R}$ is the rotation of $S^2$ fixing $\{0, \infty\}$. Braam calculated these two examples without $\Gamma$-actions in [B89]. We first follow Braam's calculation and describe the original moduli spaces, then describe the $\Gamma$-invariant moduli spaces.

## 4.1 Γ-invariant monopoles

In addition to the setup in chapter 1, we consider an action on the 3-manifold $M$. Denote the action group by $\Gamma$. In order to obtain meaningful results, we assume that the $\Gamma$-action

1. extends to $\partial\overline{M}$ and thus induces an action on the compactified space $X$,

2. has fixed surfaces $T_i$ in $X$ and the action is free away from the fixed surfaces.

**Proposition 4.1.** *1. The $\Gamma$-action and $S^1$-action on $X$ commutes.*

2. *The* $\Gamma$*-action induces a well-defined action on* $\mathcal{B}$ *and the moduli space of monopoles.*

*Proof.* Two actions commute on $M \times S^1 \subset X$ because they act on independent parts. On the fixed surface $S_j \subset X$, $S^1$-action is trivial so the result follows.

The group $\mathscr{H}$ is defined as

$$\mathscr{H} = \{f : P \longrightarrow P, \pi \circ f = \gamma \circ \pi \text{ for some } \gamma \in \Gamma\}$$

Similar as in chapter 1, $f_\gamma$ is unique up to gauge transformation and thus the action of $\Gamma$ on the space $\mathscr{B} = \mathscr{A}^*(P)/\mathscr{G}(P)$ of gauge equivalent classes of irreducible monopoles given by $\gamma \cdot [A] \mapsto [f_\gamma^* A]$ is well defined.

□

If there exists a connection $[A]$ that is fixed by the $\Gamma$-action, we obtain the same short exact sequence

$$\mathscr{G}(P)_A \longrightarrow \mathscr{H}_A \longrightarrow S^1.$$

If $A$ is irreducible, then there exists an action on $P$ which is a double cover of the $\Gamma$-action on $X$, which we denote by $\tilde{\Gamma}$.

**Definition 4.2.** $\Gamma$-invariant $SU(2)$-monopoles on $P$ correspond to an $(\tilde{\Gamma} \times \tilde{S}^1)$-instanton on $P$. The $\Gamma$-invariant monopole moduli space $\mathcal{M}(P)$ is given by

$$\mathcal{M}^\Gamma(P) = \left\{\tilde{\Gamma} \times \tilde{S}^1\text{-invariant instantons}\right\}/\mathcal{G}(P)^{\tilde{\Gamma}\times\tilde{S}^1}.$$

**Remark 4.3.** 1. Note that $\tilde{\Gamma} \times \tilde{S}^1$ is a cover of $\Gamma \times S^1$ of degree 4 rather than 2. However one can still apply the theory developed in Chapter 2 because there is an equivalent definition of $\Gamma$-invariant monopole encoding a double cover lift: By lifting the $(\Gamma \times S^1)$-action on $X$ (via some fixed point $[A] \in \mathcal{B}$) to a $\widetilde{\Gamma \times S^1}$-action on $P$, we can define $\Gamma$-invariant monopole as $\Gamma \times S^1$-invariant instantons on $X$ and

$$\mathcal{M}^\Gamma(P) = \left\{\widetilde{\Gamma \times S^1}\text{-invariant instantons}\right\}/\mathcal{G}(P)^{\widetilde{\Gamma\times S^1}}.$$

Recall that a reducible connection $A$ on a complex rank 2 vector bundle $E$ admits a splitting

$$E = L \oplus L^*$$

and $A = A_1 \oplus A_2$. In the presence of the $\Gamma \times S^1$-action, a $\Gamma \times S^1$-invariant reducible connection is a reducible connection such that in the splitting, $L$ and $L^*$ are $\widetilde{\Gamma \times S^1}$-equivariant.

2. In our examples, $\Gamma = S^1$, and the deformation theory of section 2.2 and gluing constructions of section 2.6 can be applied with $S^1$ replaced by $\Gamma \times S^1$. We state here the corresponding results which will be used in later calculations.

**Theorem 4.4.** (Γ-invariant version of Theorem 3.2 on [B89])

*Let $GA^{\Gamma}_{p,k+1}(P)$ be the Γ-invariant part of $GA_{p,k+1}(P)$, $A$ be a Γ-invariant monopole on $P$, $\Gamma_A \subset GA^{\Gamma}_{p,k+1}(P)$ the stabiliser of $A$ and $(H^i_A)^{\widetilde{\Gamma\times S^1}}$ be $\widetilde{\Gamma \times S^1}$-invariant part of the cohomology groups $H^i_A$ associated to the complex*

$$\Omega^0(adP) \xrightarrow{d_A} \Omega^1(adP) \xrightarrow{d^+_A} \Omega^{2,+}(adP).$$

*There exist a $\Gamma_A$-invariant neighourhood $W$ of $0$ in $(H^i_A)^{\widetilde{\Gamma\times S^1}}$ and a smooth map*

$$\phi : W \longrightarrow (H^2_A)^{\widetilde{\Gamma\times S^1}}$$

*such that a neighbourhood of $[A]$ in $\mathcal{M}^{\Gamma}(P)$ is modelled on $\phi^{-1}(0)/\Gamma_A$.*

Let $X_0$, $X_1$ be two Riemannian 4-manifolds with $\Gamma \times S^1$-actions which are constructed from two 3-manifolds $M_0$ and $M_1$ as in Chapter 2 and $P_i$ be the equivariant $SU(2)$-bundles over $X_i$.

**Theorem 4.5.** (Γ-invariant version of theorem 5.2 of [B89])

*Let*

*(1) $g_0$, $g_1$ be $\Gamma \times S^1$-invariant metrics on $X_0$, $X_1$ which is conformally flat near $x_0 \in X_0$, $x_1 \in X_1$;*

*(2) $A_0$, $A_1$ be $\Gamma \times S^1$-invariant monopoles with respect to $g_0$, $g_1$, $\Gamma_{A_i} \subset GA^{\Gamma}_{p,k+1}(P_i)$ the stabiliser of $A_i$;*

*(3) $I$ be the space of $\Gamma \times S^1$-equivariant gluing parameters, that is $I$ is the set of $\Gamma \times S^1$-equivariant homomorphisms from $P_0$ to $P_1$;*

*(4) $g$ be the $\Gamma \times S^1$-invariant metric on $X = X_0 \# X_1$.*

*There exists a $\Gamma_{A_0} \times \Gamma_{A_1}$-invariant neighbourhood $N$ of $I \times \{0\} \times \{0\}$ in $I \times (H^1_{A_0})^{\widetilde{\Gamma\times S^1}} \times (H^1_{A_1})^{\widetilde{\Gamma\times S^1}}$ and a $\Gamma_{A_0} \times \Gamma_{A_1}$-equivariant map*

$$\Phi : N \longrightarrow (H^2_{A_0})^{\widetilde{\Gamma\times S^1}} \times (H^2_{A_1})^{\widetilde{\Gamma\times S^1}}$$

*such that Γ-invariant monopoles with standard form (2.6.1) is parametrized by $\Phi^{-1}(0)/\Gamma_{A_0} \times \Gamma_{A_1}$.*

## 4.2 Example : $M = \mathbb{H}^3$

### 4.2.1 Monopoles on $\mathbb{H}^3$

In this chapter we study the monopoles on $\mathbb{H}^3$. This is an important case as the monopole construction usually involves gluing with such monopoles. In addition to the original set up, we consider $\mathbb{H}^3$ with a group action given by a subgroup $\Gamma$ of isometries of $\mathbb{H}^3$.

Let $M = \mathbb{H}^3$ be the 3-dimensional hyperbolic space. Throughout the section we will use two different coordinate system listed below for the hyperbolic space depending on which one is simpler to work with in different cases.

(1) Identify the hyperbolic 3-space as the Euclidean half-space, that is

$$\mathbb{H}^3 = \left\{(x, y, t) \in \mathbb{R}^3 \mid t > 0\right\},$$

endowed with the metric$ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. We will also sometimes identify the $xy$-plane as the complex numbers and thus use $(z, t)$ to represent an element of $\mathbb{H}^3$.

(2) Identify the hyperbolic 3-space as the Euclidean 3-space with spherical coordinates, that is

$$\mathbb{H}^3 = \{(l, \psi, \phi) \mid l \geq 0, \psi \in [0, \pi], \phi \in [0, 2\pi]\},$$

endowed with the metric $ds^2 = dl^2 + \sinh^2 l d\psi^2 + \sinh^2 l \sin^2 \psi d\phi^2$.

Identify the hyperbolic 3-space as subspace of Euclidean 4-space with the Minkowski metric, that is

$$\mathbb{H}^3 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 - x_4^2 = -1, x_4 > 0\right\},$$

endowed with the metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$. The relation between two coordinates can be calculated using their relation with this one, that is

$$\begin{aligned} x_1 &= \frac{x}{t}, \quad x_2 = \frac{y}{t} \\ x_3 &= \frac{t - t^{-1}}{2} + \frac{x^2 + y^2}{2t} \\ x_4 &= \frac{t + t^{-1}}{2} + \frac{x^2 + y^2}{2t} \end{aligned}$$

and

$$\begin{aligned} x_1 &= \sinh l \sin\psi \cos\phi \\ x_2 &= \sinh l \sin\psi \sin\phi \\ x_3 &= \sinh l \cos\psi \\ x_4 &= \cosh l. \end{aligned}$$

Consider the coordinate system (1), $\partial\overline{M} \cong S^2$. The compactified space can be identified as $S^4$ via the embedding

$$\begin{aligned} \iota : \mathbb{H}^3 \times S^1 &\longrightarrow \left(\mathbb{R}^2 \oplus \mathbb{R}^2\right) \setminus \left(\mathbb{R}^2 \oplus \{0\}\right) S^4 \setminus S^2 \\ (x, y, t, \tau) &\longmapsto (x, y, t\cos\tau, t\sin\tau). \end{aligned} \tag{4.2.1}$$

This embedding gives a geometric picture of the conformal compactification. We identify $X$ as

$$X \cong S^2 \times [0, \infty] \times S^1 / \sim$$

where the equivalence relation is given by $(x, 0, \tau) \sim (x', 0, \tau)$ and $(x, \infty, \tau) \sim (x, \infty, \tau')$ for $x$, $x' \in S^2$ and $\tau$, $\tau' \in S^1$. Then

$$\begin{aligned} & S^2 \times [0, \infty] \times S^1 / \sim \\ \cong\ & \left(S^2 \times [0,1] \times S^1\right) \cup \left(S^2 \times [1, \infty] \times S^1\right) / \sim \\ \cong\ & \frac{S^2 \times [0,1] \times S^1}{(x, 0, \tau) \sim (x', 0, \tau)} \cup \frac{S^2 \times [1, \infty] \times S^1}{(x, \infty, \tau) \sim (x, \infty, \tau')} \\ \cong\ & \left(D^3 \times S^1\right) \cup \left(S^2 \times D^2\right) \end{aligned} \tag{4.2.2}$$

where last identification is given by gluing $\partial D^3$ with $S^2$ and $S^1$ with $\partial D^2$. With this picture in mind, it is clear that the $S^1$-action extended to $X$ has one fixed surface being $S^2$. Regarding the embedding (4.2.1), the fixed surface is

$$S^2 = (\mathbb{R}^2 \oplus \{0\}) \cup \{\infty\} \subset S^4$$

because the action of an element $\theta \in S^1$ is given by

$$\begin{aligned} \theta \cdot (x, y, t\cos\tau, t\sin\tau) &= \iota\left(\theta \cdot ((x, y, t), \tau)\right) \\ &= \iota((x, y, t), \tau + \theta) \\ &= ((x, y, t), t\cos(\tau + \theta), t\sin(\tau + \theta)). \end{aligned}$$

According to Chakrabarti [C84], the basic monopole $(A_3, \Phi)$ of mass $m = \alpha - 1$ with center at $l = 0$ are given by

$$A_3 = \left(\tfrac{\alpha \cdot \sinh l}{\sinh(\alpha l)}\right) \tau_2 \cdot d\psi + \left(\tfrac{\alpha \cdot \sinh l}{\sinh(\alpha l)} \sin(\psi) \cdot \tau_1 + \cos(\psi) \cdot \tau_3\right) \cdot d\phi,$$
$$\Phi = [\coth l - \alpha \cdot \coth(\alpha l)] \cdot \tau_3$$

where $\{\tau_1, \tau_2, \tau_3\}$ is a basis of the lie algebra $su(2)$ satisfying $[\tau_i, \tau_j] = -\varepsilon_{ijk} \cdot \tau_k$.

**Proposition 4.6.** $(A_3, \Phi) \in \mathcal{M}(\alpha - 1, 1)$.

*Proof.* The following calculation follows from example 1.5(2) and example 2.5(1) of [B89].

The curvature is given by computing

$$\begin{aligned} dA_3 &= \frac{\alpha \cosh l \sinh \alpha l - \alpha^2 \sinh l \cosh \alpha l}{\sinh^2 \alpha l} \tau_2 \cdot d\psi \wedge dl \\ &\quad + \frac{\alpha \cosh l \sinh \alpha l - \alpha^2 \sinh l \cosh \alpha l}{\sinh^2 \alpha l} \sin(\psi) \tau_1 \cdot d\phi \wedge \psi \\ A_3 \wedge A_3 &= \left( - \left( \frac{\alpha \sinh l}{\sinh \alpha l} \right)^2 \sin \psi \tau_3 + \frac{\alpha \sinh l}{\sinh \alpha l} \cos \psi \tau_1 \right) d\psi \wedge d\phi. \end{aligned}$$

By taking the limit $l \to \infty$, we obtain the curvature at north pole $\infty$ of the sphere $S^2$ given by

$$\begin{aligned} F(A_3)|_{S^2_\infty} &= \lim_{l \to \infty} \left( dA_3 + \frac{1}{2} A_3 \wedge A_3 \right) \\ &= \sin \psi \cdot \tau_3 d\phi \wedge \psi \end{aligned}$$

The element $\tau_3 \in su(2)$, viewed as an endomorphism of $L_1 \oplus L_1^*$, is the matrix

$$\begin{pmatrix} \frac{1}{2}\mathbf{i} & 0 \\ 0 & \frac{1}{2}\mathbf{i} \end{pmatrix}.$$

By definition, the charge of $A_3$ is given by

$$k_1 = c_1(L_1^*) = -\frac{1}{2\pi \mathbf{i}} \int_{S^2} (-\frac{1}{2}\mathbf{i} \sin \psi \cdot d\phi \wedge \psi) = 1$$

and the mass is

$$m = \lim_{l \to \infty} |\Phi| = \alpha - 1.$$

□

Braam also states in example 1.5(2) of [B89] that all other elements in $\mathcal{M}(\alpha - 1, 1)$ is obtained by applying orientation preserving isometries of $\mathbb{H}^3$, that is

$$\mathcal{M}(m, 1) = \left\{ [f^*(A_3, \Phi)] \mid f \text{ is an orientation preserving isometry of } \mathbb{H}^3 \right\}.$$

Let $\pi : \mathbb{H}^3 \times S^1 \to \mathbb{H}^3$ be the projection map. As described in chapter 1, $A_0 \equiv \pi^* A_3 + (\pi^* \Phi) d\theta$ is an $S^1$-invariant connection on the principal bundle $P$ over $\mathbb{R}^4$.

**Proposition 4.7.** *The moduli space $\mathcal{M}(m,1)$ of charge 1 monopoles correspond to the space of connections of the form $f^*(\pi^* A_3 + \pi^*\Phi d\theta)$ where $f$ is an automorphism of $\mathbb{R}^4$ generated by translations along $\mathbb{R} \times \{0\} \subset \mathbb{R}^4$ and dilations of $\mathbb{R}^4$.*

*Proof.* It suffices to show that the mass center of $A_0$ is at $0 \in \mathbb{R}^4$ and isometries of $\mathbb{H}^3$ corresponds to translation and dilations of $\mathbb{R}^4$ via the the embedding (4.2.1).

The antipotal map $\nu$ on $\mathbb{R}^4$ restricting to $\mathbb{H}^3 \times S^1$ is given by

$$\begin{aligned} \nu(x,y,t,\tau) &= i^{-1}(-i(x,y,t,\tau)) \\ &= i^{-1}(-x,-y,-t\cos(\tau),-t\sin(\tau)) \\ &= (-x,-y,t,\tau+\pi). \end{aligned}$$

Since the coordinate $\{x,y,t\}$ and $\{l,\psi,\phi\}$ is related by

$$\begin{aligned} &\frac{x}{t} = \sinh l \sin\psi\cos\phi \\ &\frac{y}{t} = \sinh l \sin\psi\cos\phi \\ &\frac{t-t^{-2}}{-2} + \frac{x^2+y^2}{2t} = \sinh l \cos\psi, \\ &\frac{t+t^{-1}}{2} + \frac{x^2+y^2}{2t} = \cosh l, \end{aligned}$$

therefore $\nu(l,\psi,\phi) = (l,\psi,\phi+\pi)$. We have

$$\begin{aligned} A_3(\nu(l,\psi,\phi)) &= A_3(l,\psi,\phi+\pi) = A_3(l,\psi,\phi) \\ \Phi(\nu(l,\psi,\phi)) &= \Phi(l,\psi,\phi+\pi) = \Phi(l,\psi,\phi) \end{aligned}$$

which means $(A_3, \Phi)$ is even and hence so is the connection $A_0$. Therefore the mass center of $A_0$ is at 0.

Recall that every linear (or anti-linear) fractional map

$$\varphi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}},$$

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ extends continuously and uniquely to a map of $\hat{\varphi} : \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ whose restriction to $\mathbb{H}^3$ is an isometry of $\mathbb{H}^3$ with the metric given as in coordinate choice (2). Moreover, every isometry of $\mathbb{H}^3$ is obtained this way.[1] In other words, every orientation preserving isometry of $\mathbb{H}^3$ is a composition of translations, dilations and rotations.

[1] See section 9.3, [B09] for detailed discussion on isometries of the hyperbolic space.

The translation maps

$$\begin{aligned} \mathbb{H}^3 &\longrightarrow \mathbb{H}^3 \\ (x, y, t) &\longmapsto (x + a, y + b, t) \end{aligned}$$

of $\mathbb{H}^3$ induces the map

$$\begin{aligned} \mathbb{R}^4 &\longrightarrow \mathbb{R}^4 \\ (x, y, t) &\longmapsto (x + a, y + b, t)\,, \end{aligned}$$

on $\mathbb{R}^4$. Similarly, the dilation map

$$\begin{aligned} \mathbb{H}^3 &\longrightarrow \mathbb{H}^3 \\ (x, y, t) &\longmapsto (\lambda x, \lambda y, \lambda t) \end{aligned}$$

of $\mathbb{H}^3$ induces the map

$$\begin{aligned} \mathbb{R}^4 &\longrightarrow \mathbb{R}^4 \\ (x, y, t) &\longmapsto (\lambda x, \lambda y, \lambda t\cos(\tau), \lambda t\sin(\tau))\,, \end{aligned}$$

on $\mathbb{R}^4$. Using spherical coordinates, it is clear that the rotation map

$$\begin{aligned} \mathbb{H}^3 &\longrightarrow \mathbb{H}^3 \\ (l, \psi, \phi) &\longmapsto (l, \psi, \phi + \theta) \end{aligned}$$

fixes $(A_3, \Phi)$. Hence

(4.2.3)

$$\mathcal{M}(m, 1) = \left\{ [f^*(\pi^* A_3 + \pi^*\Phi d\theta)] \;\middle|\; \begin{array}{l} f : \mathbb{R}^4 \to \mathbb{R}^4 \text{ is a composition of translations} \\ \text{along } \mathbb{R} \times \{0\} \text{ and dilations.} \end{array} \right\}.$$

The proof is concluded.

□

**Remark 4.8.**

1. The proposition shows that $\mathcal{M}(m, 1) \cong S^2 \times \mathbb{R}_{>0}$ and $\dim\mathcal{M}(m, 1) = 3$.
2. Let $P \to S^4$ be the $SU(2)$-bundle with $c_2(P) = m$, then the moduli space of monopoles on $\mathbb{H}^3$ is

$$\mathcal{M}(P) = \bigsqcup_{m'k'=m} \mathcal{M}(m', k')$$

where $m'$ and $k'$ are mass and charge (in integer values). In this section we have only described the connected component $\mathcal{M}(m, 1)$.

3. In the case $m = 1$, the proposition matches the result we obtained using ADHM construction.

### 4.2.2 $\Gamma$-invariant monopoles on $\mathbb{H}^3$

Consider the isometry group generated by

$$\begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{\mathbf{i}\theta} \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{C}^*$$

acting on $\mathbb{H}^3$. This group consists of functions of the form

$$e^{\mathbf{i}\theta} : (z,t) \longmapsto (e^{2\mathbf{i}\theta} z, t).$$

Hence the group $S^1$ acts on $H^3$ by earth rotation, that is rotation around the $t$ axis, and has fixed point set $S_\Gamma = \{(0,0,t) \mid t > 0\}$. To avoid confusion with the previous $S^1$ action, we denote this action group by $\Gamma$. Extending the $\Gamma$-action to $\mathbb{H}^3 \times S^1$, the corresponding fixed point set becomes the cylinder $S_\Gamma \times S^1$. It is clear that the induced action on $S^4$ is fixing the first two entries and has fixed surface $S^2$. This is however not the same $S^2$ given by the fixed surface under the $S^1$ action in Chapter 2. Using the geometric picture (4.2.2), the fixed surface of $S^1$ action is given by the boundary of $D^3$ while the fixed surface of $\Gamma$ is given by $T \times S^1$ where $T \subset D^3$ is the diameter connecting north and south pole. Hence two fixed surface intersect transversally at two points.

**Proposition 4.9.** $\mathcal{M}^\Gamma(m,1) \cong \mathbb{R}_{>0}$.

*Proof.* Observe that the $\Gamma \times S^1$-action on $\mathbb{R}^4$ only fixes the origin $(0,0)$, hence the moduli space is no longer generated by the translations along $\mathbb{R} \times \{0\}$. It follows from proposition (4.7) that $\mathcal{M}^\Gamma(m,1)$ is generated only by dilations of $\mathbb{R}^4$. □

## 4.3 Example : $M = S^2 \times \mathbb{R}$

Consider the three manifold $M = S^2 \times \mathbb{R}$ with the metric

$$ds = \cosh^2 l \cdot ds^2_{S^2} + dl^2$$

where $\cosh x = \frac{e^x + e^{-x}}{2}$ and $ds^2_{S^2}$ is the $SO(3)$-invariant metric of curvature 1 on $S^2$.

The compactified space $M \times S^1$ is then $X = S^2 \times \mathbb{R} \times S^1$ which is identified with $S^2 \times S^2$. Topologically the second $S^2$ is obtained by collapsing two ends of the cylinder $S^1 \times \mathbb{R}$ to two points. The $S^1$ action is the on the second summand by earth rotation, then the fixed surfaces are $S_0 = S^2 \times \{0\}$ and $S_\infty = S^2 \times \{\infty\}$.

### 4.3.1 Construction of $\tilde{S}^1$-equivariant $SU(2)$ bundle $E$ on $X$

Let $\mathcal{O}(1) \to S^2$ be the complex line bundle with $c_1 = 1$ with an $S^1$ action on the base by earth rotation. Define $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. Let $D^2_+$ and $D^2_-$ be the top and bottom closed hemisphere of the base manifold $S^2$, both including the equator.

By choosing the open cover $\{D^2_+, D^2_-\}$ and the transition function

$$\begin{aligned} g : D^2_+ \cap D^2_- &\longrightarrow \mathbb{C}^* \\ x &\longmapsto x^{-n}. \end{aligned}$$

The bundle $\mathcal{O}(n)$ can be represented as

$$\mathcal{O}(n) = \left(D^2_+ \times S^1 \sqcup D^2_- \times S^1\right) / \sim$$

where the equivalence relation is defined by the transition function.

**Lemma 4.10.** *The bundle $\mathcal{O}(n)$ admits an $\tilde{S}^1$-action with weights $\omega_0, \omega_\infty \in \mathbb{Z}$ satisfying $\omega_0 - \omega_\infty = 2n$.*

Here the weights $\omega_0, \omega_\infty$ refers to the winding number of the action on the fibres at north and south pole of $S^2$ respectively.

*Proof.* $\delta \in \tilde{S}^1$ acts on $D^2_+ \times S^1$ by

$$\delta \cdot (x, \xi) = (\delta^2 x, \delta^{\omega_0} \xi)$$

and on $D^2_- \times S^1$ by

$$\delta \cdot (x, \xi) = (\delta^2 x, \delta^{\omega_\infty} \xi).$$

For $(x, \xi) \in \partial D^2_\pm \times S^1$,

$$\begin{aligned} \delta^2 x, \delta^{\omega_0} \xi &\sim (\delta^2 x, \delta^{\omega_0} \xi (\delta^2 x)^{-n}) \\ &= (\delta^2 x, \delta^{\omega_0 - 2n} \xi x^{-n}) \\ &= (\delta^2 x, \delta^{\omega_\infty} \xi x^{-n}). \end{aligned}$$

Hence the action is well-defined on the intersection $D^2_+ \cap D^2_- \cong S^1$. $\square$

Let $\pi_j : X \to S^2$ be the projection onto the $j$-th factor, denote by $O(n, m)$ the bundle $\pi_1^* \mathcal{O}(n) \otimes \pi_2^* \mathcal{O}(m)$ over $X$.

Fix the weights $\omega_0$, $\omega_\infty$ with $\omega_0 - \omega_\infty = -2k$. The $\tilde{S}^1$-action on the bundle $\mathcal{O}(-k) \to S^2$ with weights $\omega_0$ and $\omega_\infty$ extends to the bundle $\mathcal{O}(k, -k) \to X$ and the $\tilde{S}^1$-action on the bundle $\mathcal{O}(k) \to S^2$ with weights $-\omega_0$ and $-\omega_\infty$ extends to the bundle $\mathcal{O}(-k, k) \to X$. Therefore we have a $\tilde{S}^1$-action on the vector bundle

$$E = O(k, -k) \oplus O(-k, k).$$

The line bundles $\mathcal{O}(k, -k)$ carry an anti-self-dual connection, which induces an $\tilde{S}^1$-invariant reducible ASD connection on the bundle $E \to X$. According to our definition in Chapter 2, this corresponds to a reducible monopole on $M$.

Next we will describe the mass and charge of a monopole on $M = S^2 \times \mathbb{R}$.

### 4.3.2 Calculating mass and charge

**Proposition 4.11.** *Given the $\tilde{S}^1$-action on $E$ as described above. The fixed surfaces are $S_0 = \{north\ pole\} \times S^2$ and $S_\infty = \{south\ pole\} \times S^2$. The mass and charge corresponding to $S_0$ and $S_\infty$ are given by*

$$\begin{aligned}(m_0, k_0) &:= (|\omega_0|, -sign(\omega_0)k)\\ (m_1, k_1) &:= (|\omega_\infty|, sign(\omega_\infty)k)\end{aligned}$$

*Proof.* Following the procedure of Chapter 1, we wish to find a splitting of the bundle $E|_{S^2\times\{0\}}$ defined by (2.1.6). Observe that

$$\begin{aligned}E|_{S_0} &= \left(\pi_1^*\mathcal{O}(k) \otimes \pi_2^*\mathcal{O}(-k)|_{\{0\}}\right) \oplus \left(\pi_1^*\mathcal{O}(-k) \otimes \pi_2^*\mathcal{O}(k)|_{\{0\}}\right)\\ &\cong (\mathcal{O}(k) \otimes \mathbb{C}) \oplus (\mathcal{O}(-k) \otimes \mathbb{C})\,.\end{aligned} \tag{4.3.1}$$

and similarly

$$\begin{aligned}E|_{S_\infty} &= \left(\pi_1^*\mathcal{O}(k) \otimes \pi_2^*\mathcal{O}(-k)|_{\{\infty\}}\right) \oplus \left(\pi_1^*\mathcal{O}(-k) \otimes \pi_2^*\mathcal{O}(k)|_{\{\infty\}}\right)\\ &\cong (\mathcal{O}(k) \otimes \mathbb{C}) \oplus (\mathcal{O}(-k) \otimes \mathbb{C})\,.\end{aligned} \tag{4.3.2}$$

Since $E|_{S_j}$ by construction is already a direct sum of two line bundles(one also being the dual of the other), it is natural to expect that $L_j = \mathcal{O}(k)$ or $L_j = \mathcal{O}(-k)$. This is indeed the case. To see this, we trace back to the $\tilde{S}^1$-action on $\mathcal{O}(k, -k)$. The action is trivial on the $\mathcal{O}(k)$ part and has weight equal to $\omega_0$ on the $\mathcal{O}(-k)$ part. Over the fixed surface $S_0$ the $\mathcal{O}(-k)$ is restricted to a point thus the action is simply on the fibre $\mathbb{C}$ is by multiplication by unit complex number or equivalently on the first summand of (4.3.1) with weight $\omega_0$. The same reasoning applies to the action on the second summand and on the restricted bundle over $S_\infty$. Thus the summands of the splitting (4.3.1) and (4.3.2) satisfies the definition of the line bundles $L_j$ and $L_j^*$ of (2.1.6). By definition the weight of the $\tilde{S}^1$-action on $L_j$'s is the mass of monopole, thus

$$m_0 = |\omega_0|$$

and

$$m_\infty = |\omega_\infty|.$$

Note that this also puts a constraint that $m_0$ and $m_\infty$ cannot both be zero for otherwise the condition $\omega_0 - \omega_\infty = -2k$ means $k = 0$ and the bundle $E$ becomes trivial.

There is a freedom in the choice $L_j = \mathcal{O}(k)$ or $L_j = \mathcal{O}(-k)$ depending on the sign of $\omega_0$ and $\omega_\infty$. Here we choose $\omega_0 > 0$ and $\omega_\infty > 0$ and thus $L_1 = \mathcal{O}(k)$, $L_2 = \mathcal{O}(-k)$. The monopole charge is given by

$$\begin{aligned} k_0 \cdot x_1 = c_1(\mathcal{O}(-k)) \in H^2(S^2 \times \{0\}); \\ k_\infty \cdot x_1 = c_1(\mathcal{O}(k)) \in H^2(S^2 \times \{\infty\}). \end{aligned}$$

Observe that the sign of $k_j$ reverts for different sign of $\omega_j$. Moreover, the sign of $\text{sign}(\omega_j)k_j$ depends on the choice of orientation of each fixed surface, the two choice must also differ by a sign. Therefore

$$\begin{aligned} k_0 &= -\text{sign}(\omega_0)k \\ k_\infty &= \text{sign}(\omega_\infty)k. \end{aligned}$$

□

We check formula (2.1.11) to ensure that the correspondence between monopole and instanton are as intended:

$$\begin{aligned} c_2(E) &= c_2(\mathcal{O}(k,-k)) + c_2(\mathcal{O}(-k,k)) + c_1(\mathcal{O}(k,-k))c_1(\mathcal{O}(-k,k)) \\ &= 0 + 0 + (k,-k)\cdot(-k,k) \\ &= 2k^2, \end{aligned}$$

the calculation of monopole mass and charge involves the choice of sign of $\omega_0$ and $\omega_\infty$, the calculation here incorporates all possibilities,

$$\begin{aligned} \sum_j m_j k_j &= m_0 k_0 + m_\infty k_\infty \\ &= \text{sign}(\omega_0)\omega_0(-1)\text{sign}(\omega_0)k + \text{sign}(\omega_\infty)\omega_\infty\text{sign}(\omega_\infty)k \\ &= -k(\omega_0 - \omega_\infty) \\ &= 2k^2, \end{aligned}$$

hence (2.1.11) holds.

### 4.3.3 Index formula associated to $E$

Knowing the charge and mass, we can use the invariant version of the index formula to calculate the dimension of the monopole moduli space, which is given by theorem 2.3 in [B89]:

$$I(m_j, k_j) \ = \textstyle\sum_{m_j \neq 0} 4 \cdot k_j + (g_j - 1) + \sum_{m_j = 0} 3 \cdot (g_j - 1)$$

where $g_j$ is the genus of $S_j$. Hence in this example the formula simplifies to

$$I(m_j, k_j) \ = \textstyle\sum_{m_j \neq 0} 4 \cdot k_j - 1 - \sum_{m_j = 0} 3 \tag{4.3.3}$$

### 4.3.4 Local model of $[A]$ in $\mathcal{M}(E)$

Let $[A]$ be the reducible connection defined by the unique ASD connection on $\mathcal{O}(k, -k)$. Recall in Chapter 2 we studied a model of a neighbourhood of an instanton $[A]$ given by $\phi^{-1}(0)/\Gamma_A$ where $\phi$ is Kuranishi map relation to the complex (2.2.1). Therefore we can get a local picture by calculating the leading term of Kuranishi map.

**Proposition 4.12.** *The cohomology of the Atiyah-Hitchin-Singer complex associated to A*

$$\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{d_A^+} \Omega^{2,+}$$

*is given by*

$$\begin{aligned} H_A^0 &= \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cong \mathbb{R}, \\ H_A^1 &\cong \mathbb{C}^{4k^2-1} \oplus \mathbb{C}^{4k^2-1}, \\ H_A^2 &= \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cong \mathbb{R}. \end{aligned}$$

*Proof.* Let $\omega$ be the self-dual Kahler form of $X = S^2 \times S^2$. We follow section 6.4.2 of [D90] to compute cohomology groups using holomorphic data:

$$\begin{aligned} H_A^1 &= H^1(\mathrm{End}_0 \mathscr{E}) \\ H_A^2 &= H^2(\mathrm{End}_0 \mathscr{E}) \oplus H_A^0 \cdot \omega \end{aligned}$$

where $\mathscr{E}$ is the holomorphic structure on $E$ corresponds to $A$ and $\mathrm{End}_0\mathscr{E}$ is a (trace-free) subbundle of End $\mathscr{E}$. Since $E$ is a sum of two line bundles, and

$$\begin{aligned}
\mathrm{End}(L\oplus L^*) &= (L\oplus L^*)\otimes(L\oplus L^*)^* \\
&= (L\oplus L^*)\otimes(L^*\oplus L) \\
&= (L\otimes L^*)\oplus L^{\otimes 2}\oplus(L^*)^{\otimes 2}\oplus(L\otimes L^*) \\
(L^*=L^{-1}\Rightarrow) &= \mathbb{C}\oplus L^{\otimes 2}\oplus(L^*)^{\otimes 2}\oplus\mathbb{C}.
\end{aligned} \tag{4.3.4}$$

Due to the trace-free condition, the second trivial bundle $\mathbb{C}$ becomes redundant data. We have

$$H^1(\mathrm{End}_0\mathscr{E})\cong H^1_{\text{sheaf}}\left(X, C\oplus\mathcal{O}(2k,-2k)\oplus\mathcal{O}(-2k,2k)\right). \tag{4.3.5}$$

Since cohomology commutes with direct sums (cf. Remark 2.9.1 [H77]), we need only to calculate each summand. Recall that $X=S^2\times S^2$, by Kunneth formula,

$$\begin{aligned}
&H^1(X,\mathcal{O}(2k,-2k)) \\
&= H^0(S^2,\mathcal{O}(2k))\otimes H^1(S^2,\mathcal{O}(-2k))\oplus H^1(S^2,\mathcal{O}(2k))\otimes H^0(S^2,\mathcal{O}(-2k)).
\end{aligned} \tag{4.3.6}$$

According to the result of sheaf cohomology (cf. Theorem 5.1 [H77])

$$H^0(S^2,\mathcal{O}(l)) = \begin{cases} 0, & l<0, \\ \mathbb{C}^{l+1}, & l\geq 0 \end{cases} \tag{4.3.7}$$

$$H^1(S^2,\mathcal{O}(l)) = H^0(S^2,O(-l-2)), \tag{4.3.8}$$

we compute

$$\begin{aligned}
H^0(S^2,\mathcal{O}(2k)) &= \begin{cases} 0, & k<0, \\ \mathbb{C}^{2k+1}, & k\geq 0 \end{cases}, \\
H^0(S^2,\mathcal{O}(-2k)) &= \begin{cases} 0, & k>0, \\ \mathbb{C}^{-2k+1}, & k\leq 0 \end{cases}, \\
H^1(S^2,\mathcal{O}(-2k)) &= H^0(S^2,O(2k-2)) \\
&= \begin{cases} 0, & k<1, \\ \mathbb{C}^{2k-1}, & k\geq 1 \end{cases}, \\
H^1(S^2,\mathcal{O}(2k)) &= H^0(S^2,O(-2k-2)) \\
&= \begin{cases} 0, & k>-1, \\ \mathbb{C}^{-2k-1}, & k\leq -1 \end{cases}.
\end{aligned}$$

Plugging the results into (4.3.6) gives

$$H^1(X, O(2k, -2k)) \cong \mathbb{C}^{4k^2-1} \tag{4.3.9}$$

and hence

$$\begin{aligned} H^1_A = H^1(\mathrm{End}_0\mathscr{E}) &\cong 0 \oplus \mathbb{C}^{4k^2-1} \oplus \mathbb{C}^{4k^2-1} \\ &= \mathbb{C}^{4k^2-1} \oplus \mathbb{C}^{4k^2-1} \end{aligned}$$

Following the computation in section 6.4.3 example (ii) [D90],

$$H^2(\mathrm{End}_0\mathscr{E}) = H^2(\mathbb{C} \oplus O(2k, -2k) \oplus O(-2k, 2k)).$$

Again, we calculate $H^2(\mathrm{End}_0\mathscr{E})$ by computing each summand:

$$\begin{aligned} H^2(\mathcal{O}(2k, -2k)) &= H^1(S^2, \mathcal{O}(2k)) \otimes H^1(S^2, \mathcal{O}(-2k)) \\ &= \begin{cases} 0 & \text{if } k > -1 \\ \mathbb{C}^{-2k-1} \otimes 0 & \text{if } k \leq -1 \end{cases} \\ &= 0 \end{aligned}$$

Then the Kunneth formula for sheaf cohomology implies

$$H^2(\mathrm{End}_0\mathscr{E}) = 0$$

Regarding Theorem 3.1 of [FU84], since $A$ is reducible,we have $\Gamma_A \cong U(1) \subset SU(2)$, then $H^0_A = \ker d_A = \mathrm{Lie}(\Gamma_A)$ can be identified as the real line generated by

$$\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix},$$

that is,

$$H^0_A = \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}.$$

Summarising, we have

$$H^2_A = \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cdot \omega.$$

$\square$

The calculation (4.3.4) gives rise to an explicit isomorphism for the identification (4.3.5) in the following way:

Regard $\alpha \in L \otimes L$ as a map

$$\begin{aligned} L^* &\longrightarrow L \\ v &\longmapsto \alpha \cdot v, \end{aligned}$$

then $\alpha$, as an element of $\mathrm{End}_0\mathscr{E}$, is given by

$$\begin{pmatrix} 0 & \alpha \\ -\alpha^* & 0 \end{pmatrix}$$

and similarly $\beta \in L^* \otimes L^*$ is given by

$$\begin{pmatrix} 0 & \beta^* \\ -\beta & 0 \end{pmatrix}$$

therefore the isomorphism is given by

$$\begin{aligned} H^1(X, \mathcal{O}(2k, -2k)) \oplus H^1(X, \mathcal{O}(-2k, 2k)) &\longrightarrow H^1_A \\ (\alpha, \beta) &\longmapsto \begin{pmatrix} 0 & \alpha - \beta^* \\ \beta - \alpha^* & 0 \end{pmatrix}. \end{aligned}$$

Since $H^1 = \mathrm{ker} d_A^+ / \mathrm{im} d_A$, the matrix

$$\begin{pmatrix} 0 & \alpha - \beta^* \\ \beta - \alpha^* & 0 \end{pmatrix}$$

can be regarded as a $su(2)$-valued one form, that is by choosing a complex local coordinate $\{z_1, z_2\}$ of $S^2 \times S^2$, the matrix can be represented as

$$\begin{aligned} &\begin{pmatrix} 0 & \sum_{i=1}^2 \alpha_i dz_i - \left(\sum_{i=1}^2 \beta_i dz_i\right)^* \\ \sum_{i=1}^2 \beta_i dz_i - \left(\sum_{i=1}^2 \alpha_i dz_i\right)^* & 0 \end{pmatrix} \\ = &\begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix} dz_1 + \begin{pmatrix} 0 & -\beta_1 \\ -\alpha_1 & 0 \end{pmatrix} d\bar{z}_1 + \begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix} dz_z + \begin{pmatrix} 0 & -\beta_2 \\ -\alpha_2 & 0 \end{pmatrix} d\bar{z}_2. \end{aligned} \tag{4.3.10}$$

This identification allows us to calculate the leading term of the Kuranishi map

$$\begin{aligned} q : H^1 &\longrightarrow \mathbb{R} \\ p &\longmapsto \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \omega, \frac{1}{2} P_+[p, p] \right\rangle_{L^2} \end{aligned}$$

where $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ is the Kahler form of $S^2 \times S^2$, as a map from the sheaf cohomologies, that is with $p$ given (locally) by (4.3.10).

We calculate the second term first:

$$\tag{4.3.11} \frac{1}{2}[p,p] = p \wedge p$$

$$= (-\alpha_1\bar{\alpha}_1 + \beta_1\bar{\beta}_1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} dz_1 \wedge d\bar{z}_1 + (-\alpha_2\bar{\alpha}_2 + \beta_2\bar{\beta}_2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} dz_2 \wedge d\bar{z}_2 + \dots$$

$$= \frac{1}{2}\left(-\alpha_1\bar{\alpha}_1 + \beta_1\bar{\beta}_1 - \alpha_2\bar{\alpha}_2 + \beta_2\bar{\beta}_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$$

$$+\frac{1}{2}\left(-\alpha_1\bar{\alpha}_1 - \beta_1\bar{\beta}_1 + \alpha_2\bar{\alpha}_2 - \beta_2\bar{\beta}_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2) + \dots$$

the omitted terms are independent of $dz_1 \wedge d\bar{z}_1$ and $dz_2 \wedge d\bar{z}_2$ and thus gives zero after inner product with $\omega$. The first term of (4.3.11) is self-dual, hence

$$\begin{aligned}\frac{1}{2}P_+[p,p] &= \frac{1}{2}\left(-\alpha_1\bar{\alpha}_1 + \beta_1\bar{\beta}_1 - \alpha_2\bar{\alpha}_2 + \beta_2\bar{\beta}_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).\\ &= -\frac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).\end{aligned}$$

Then

$$\begin{aligned}&\left\langle \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\omega, \frac{1}{2}P_+[p,p]\right\rangle_{L^2}\\ &= \begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right) & 0\\ 0 & \frac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right)\end{pmatrix}\\ &= \frac{1}{4}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right) + \frac{1}{4}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right)\\ &= \frac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}.\right) \qquad (4.3.12)\end{aligned}$$

Therefore the map $q$ is given by

$$\begin{aligned} q : H^1(X, \mathcal{O}(2k,-2k)) \oplus H^1(X, \mathcal{O}(-2k,2k)) &\longrightarrow \mathbb{R}\\ (\alpha,\beta) &\longmapsto \tfrac{1}{2}\left(\|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}\right)\end{aligned}$$

Regarding (4.3.9), by choosing a basis $\{e_i, \dots, e_{2k^2-1}\}$ $\alpha$ and $\beta$ can be viewed as a $1 \times (4k^2 - 1)$ complex vector:

$$\begin{aligned} \alpha &= \sum_{i=1}^{4k^2-1} \alpha_i e_i, \quad \alpha_i \in \mathbb{C} \\ \beta &= \sum_{i=1}^{4k^2-1} \beta_i e_i. \quad \beta_i \in \mathbb{C} \end{aligned}$$

Fix $\alpha$, $\beta \in q^{-1}(0)$ and let $\alpha(t)$ and $\beta(t)$ be two curves with $\alpha(0) = \alpha$ and $\beta(0) = \beta$, then by definition

$$\begin{aligned} dq_{(\alpha(0),\beta(0))}(\alpha(t), \beta(t)) &= \lim_{t\to 0} \frac{q(\alpha(t), \beta(t)) - q(\alpha(0), \beta(0))}{t} \\ &= \lim_{t\to 0} \frac{q(\alpha(t), \beta(t))}{t}. \end{aligned}$$

It is now clear that $dq$ is surjective away from 0 and thus 0 is indeed a regular value of $q$. Therefore $q^{-1}(0) \setminus \{0\}$ is a manifold. We can identify $q^{-1}(0) \setminus \{0\}$ as

$$\left\{(\alpha, \beta) \mid |\alpha|^2 = |\beta|^2 = 1\right\} \times \mathbb{R}$$

via the map

$$(\alpha, \beta) \longmapsto \left(\left(\frac{\alpha}{|\alpha|}, \frac{\beta}{|\beta|}\right), |\alpha|\right).$$

The neighbourhood of $[A]$ in the ASD instanton moduli space is given by

$$q^{-1}(0)/\Gamma_A = \left\{(\alpha, \beta) \in \mathbb{C}^{4k^2-1} \oplus \mathbb{C}^{4k^2-1}; \|\alpha\|^2_{L^2} = \|\beta\|^2_{L^2}\right\}/U(1) \tag{4.3.13}$$

where $U(1)$ acts by $\lambda \cdot (\alpha, \beta) = (\lambda\alpha, \lambda^{-1}\beta)$. Observe that for any real number $\epsilon$, $\epsilon(\alpha, \beta) = (\epsilon\alpha, \epsilon\beta) \in q^{-1}(0)/U(1)$, therefore the neighbourhood is a cone over the manifold

$$S^{8k^2-3} \times S^{8k^2-3}/U(1).$$

Next we take the $\tilde{S}^1$-action invariance into account. There are several cases to be considered concerning the sign of the parameters $k$, $\omega_0$ and $\omega_\infty$. We can assume $k > 0$ without loss of generality because the $k < 0$ case yields $k_0$ and $k_\infty$ of opposite signs, which can be recovered by reversing the roles of 0 and $\infty$. Moreover, $k = 0$ corresponds to the case of a trivial bundle $E$ and hence a flat monopole, which is not interested. So for $k > 0$, the possible choices of signs of weights are:

(1) $\omega_0 > 0$, in this case $k_0 < 0$ and $\omega_\infty = \omega_0 + 2k > 0$, thus $k_\infty > 0$;

(2) $\omega_0 < 0$, in this case $k_0 > 0$ but the sign of $\omega_\infty = \omega_0 + 2k$ is not determined so we have

(2.1) $\omega_\infty = 0$,

(2.2) $\omega_\infty > 0$ and so $k_\infty > 0$,

(2.3) $\omega_\infty < 0$ and so $k_\infty < 0$,

(3) $\omega_0 = 0$, in this case $\omega_\infty = 2k > 0$ and $k_\infty > 0$.

In conclusion, there are five cases to consider:

(1) $\omega_0 > 0$,

(2) $\omega_0 < 0$ and $\omega_\infty = 0$,

(3) $\omega_0 < 0$ and $\omega_\infty > 0$,

(4) $\omega_0 < 0$ and $\omega_\infty < 0$,

(5) $\omega_0 = 0$.

The dimension of the moduli space in each case is given by (using formula (4.3.3)):

$$I = \begin{cases} -2 & \text{in case (1) and (4)} \\ 4k-4 & \text{in case (2) and (5)} \\ 8k-2 & \text{in case (3)} \end{cases}$$

**Proposition 4.13.** *The $\tilde{S}^1$-part of the cohomology is given by*

$$\begin{aligned} H_0^0 &= \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cong \mathbb{R} \\ H_0^1 &\cong \begin{cases} 0 & \text{in case (1) and (4)} \\ \mathbb{C}^{2k-1} \otimes \mathbb{C} \cong \mathbb{C}^{2k-1} & \text{in case (2) and (5)} \\ \mathbb{C}^{2k+1} \oplus \mathbb{C}^{2k-1} & \text{in case (3)} \end{cases} . \\ H_0^2 &= \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cdot \omega \cong \mathbb{R}. \end{aligned}$$

*Proof.* The treatment is similar as before. To evaluate the $\tilde{S}^1$-action on $H^1_{\text{sheaf}}$, rewrite result (4.3.7) in terms of homogenous polynomials of degree $l$:

$$
\begin{aligned}
H^0_{\text{sheaf}}(S^2, O(l)) &\cong \bigoplus_{\substack{a_0, a_1 \geq 0 \\ a_0 + a_1 = l}} \mathbb{C} \cdot x_0^{a_0} x_1^{a_1} \\
H^1_{\text{sheaf}}(S^2, O(l)) &\cong \bigoplus_{\substack{a_0, a_1 < 0 \\ a_0 + a_1 = l}} \mathbb{C} \cdot x_0^{a_0} x_1^{a_1}
\end{aligned}
$$

with generators $x_0$ and $x_1$. The $\tilde{S}^1$-action on the generator is given by

$$
\begin{aligned}
\theta \cdot x_0^{a_0} x_1^{a_1} &= \left(e^{2\pi \mathbf{i} p_\infty} x_0\right)^{a_0} \left(e^{2\pi \mathbf{i} p_0} x_1\right)^{a_1} \\
&= e^{2\pi \mathbf{i}(a_0 p_\infty + a_1 p_0)} \left(x_0^{a_0} x_1^{a_1}\right).
\end{aligned}
$$

where $p_0$ and $p_\infty$ denotes the weights of the action as defined before[①]. Observe that the action does not change the degree $a_0$ and $a_1$, therefore $x_0^{a_0} x_1^{a_1}$ is $\tilde{S}^1$-invariant if and only if the generators $x_0^{a_0} x_1^{a_1}$ are $\tilde{S}^1$-invariant if and only if

$$\theta \cdot x_0^{a_0} x_1^{a_1} = x_0^{a_0} x_1^{a_1}$$

for any $\theta \in \tilde{S}^1$. This means

$$p_\infty a_0 + p_0 a_1 = 0.$$

together with the conditions on the parameters $a_0$, $a_1$, $p_0$ and $p_\infty$

(1) $p_0 - p_\infty = 2l$,

(2) $a_0 + a_1 = l$,

(3) $a_0$, $a_1 > 0$ and

(4) $l > 0$,

we deduce a sign condition on the weights $p_0$ and $p_\infty$

$$p_0 = 2a_0 \geq 0, \quad p_\infty = -2a_1 \leq 0$$

and the polynomial degree that survives the $\tilde{S}^1$-action

$$a_0 = \frac{p_0}{2}, \quad a_1 = -\frac{p_\infty}{2}.$$

[①]For the deduction of this general case we avoid using the same symbol $\omega_0$, $\omega_\infty$ for the weights as when we compute the actual case the action will need to be evaluated for the specific bundle in question.

Therefore the sheaf cohomology is given by

$$\begin{aligned}
H^0_{\text{sheaf}}(S^2, O(l))^{\tilde{S}^1} &\cong \mathbb{C} \cdot x_0^{p_0/2} x_1^{p_\infty/2} \\
\text{(4.3.14)} \qquad &\cong \begin{cases} \mathbb{C}, & \text{if } p_0, p_\infty \text{ are even, } p_0 \geq 0,\ p_\infty \leq 0 \text{ and } l > 0 \\ 0 & \text{otheriwse} \end{cases} \\
H^1_{\text{sheaf}}(S^2, O(l))^{\tilde{S}^1} &\cong \mathbb{C} \cdot x_0^{p_0/2} x_1^{p_\infty/2}. \\
\text{(4.3.15)} \qquad &\cong \begin{cases} \mathbb{C}, & \text{if } p_0, p_\infty \text{ are even, } p_0 + 2 \leq 0,\ -p_\infty + 2 \leq 0 \text{ and } l < -2 \\ 0 & \text{otheriwse} \end{cases}
\end{aligned}$$

With all the data above, we can use the Kunneth formula to compute

$$\begin{aligned}
H^1_0 &\cong H^1\left(X, \mathbb{C} \oplus O(2k, -2k) \oplus \mathcal{O}(-2k, 2k)\right)^{\tilde{S}^1} \\
&\cong \left\{ H^0(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \otimes H^1(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \right\} \\
&\quad \oplus \left\{ H^1(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \otimes H^0(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \right\} \\
&\quad \oplus \left\{ H^0(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \otimes H^1(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \right\} \\
&\quad \oplus \left\{ H^1(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \otimes H^0(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \right\}.
\end{aligned}$$

The $\tilde{S}^1$-action always fixes the first part of the product $S^2 \times S^2$, hence the first term of all four summands are computed same as before. That is, we need to calculate

$$\begin{aligned}
H^1_0 &\cong \left\{ H^0(S^2, \mathcal{O}(2k)) \otimes H^1(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \right\} \\
&\quad \oplus \left\{ H^1(S^2, \mathcal{O}(2k)) \otimes H^0(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \right\} \\
&\quad \oplus \left\{ H^0(S^2, \mathcal{O}(-2k)) \otimes H^1(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \right\} \\
&\quad \oplus \left\{ H^1(S^2, \mathcal{O}(-2k)) \otimes H^0(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \right\}.
\end{aligned}$$

The $\tilde{S}^1$-action on $\mathcal{O}(2k, -2k)$ is the one induced by the action on the bundle $\mathcal{O}(-2k) \to S^2$, which is the action with weight $p_0 = 2\omega_0$ and $p_\infty = 2\omega_\infty$. Then the five cases can be written in terms of $p_0$ and $p_\infty$ as

(1) $p_0 > 0$ (which determines $p_\infty > 0$),

(2) $p_0 < 0$ and $p_\infty = 0$,

(3) $p_0 < 0$ and $p_\infty > 0$,

(4) $p_0 < 0$ and $p_\infty < 0$,

(5) $p_0 = 0$ (which determines $p_\infty > 0$).

Then (4.3.14) implies

$$H^0(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \cong 0 \text{ in all cases} \tag{4.3.16}$$

$$H^0(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \cong \begin{cases} 0 & \text{in case (1) and (4)} \\ \mathbb{C} & \text{in case (2), (3) and (5)} \end{cases} \tag{4.3.17}$$

Similarly, the action on $\mathcal{O}(-2k, 2k)$ is the one induced by the action on the bundle $\mathcal{O}(2k) \to S^2$, which has weight $p_0 = -2\omega_0$ and $p_\infty = -2\omega_\infty$.

(1) $p_0 \leq -2$ (which determines $p_\infty \geq -2$),

(2) $p_0 \geq 2$ and $p_\infty = 0$,

(3) $p_0 \geq 0$ and $p_\infty \leq -2$,

(4) $p_0 \geq 0$ and $p_\infty \geq 2$,

(5) $p_0 = 0$ (which determines $p_\infty \leq -2$).

Then (4.3.15) implies

$$H^1(S^2, \mathcal{O}(-2k))^{\tilde{S}^1} \cong \begin{cases} 0 & \text{in case (1), (2), (4) and (5)} \\ \mathbb{C} & \text{in case (3)} \end{cases} \tag{4.3.18}$$

$$H^1(S^2, \mathcal{O}(2k))^{\tilde{S}^1} \cong 0 \text{ in all cases} \tag{4.3.19}$$

Therefore

$$H_0^1 \cong \begin{cases} 0 & \text{in case (1) and (4)} \\ \mathbb{C}^{2k-1} \otimes \mathbb{C} \cong \mathbb{C}^{2k-1} & \text{in case (2) and (5)} \\ \mathbb{C}^{2k+1} \oplus \mathbb{C}^{2k-1} & \text{in case (3)} \end{cases} .$$

By construction, the splitting of $E$ is $S^1$-equivariant, this means the stabiliser $\Gamma_A \cong U(1)$ is $S^1$-equivariant due to the fact that $U(1)$ is abelian. Therefore the $\tilde{S}^1$-invariant part of $H_A^0 = \mathrm{Lie}(\Gamma_A)$ is the entire group, thus

$$H_0^0 = \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cong \mathbb{R}$$

and hence

$$H_0^2 = \mathbb{R} \cdot \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \cdot \omega \cong \mathbb{R}.$$

□

Now we have all the information to compute the leading term of the Kuranishi map using the formula developed before:

$$q : H_0^1 \longrightarrow \mathbb{R}$$

In case (1) and (4), $H_0^1 = \{\text{point}\}$,

$$q : \{\text{point}\} \longrightarrow \mathbb{R};$$

in case (2) and (5), $H_0^1 = 0 \oplus \mathbb{C}^{2k-1}$,

$$\begin{aligned} q : \mathbb{C}^{2k-1} &\longrightarrow \mathbb{R}, \\ \beta &\longmapsto -\frac{1}{2}\|\beta\|_{L^2}^2; \end{aligned}$$

in case (3),$H_0^1 = \mathbb{C}^{2k+1} \oplus \mathbb{C}^{2k-1}$,

$$\begin{aligned} q : \mathbb{C}^{2k+1} \oplus \mathbb{C}^{2k-1} &\longrightarrow \mathbb{R}, \\ (\alpha, \beta) &\longmapsto \frac{1}{2}\left(\|\alpha\|_{L^2}^2 - \|\beta\|_{L^2}^2\right). \end{aligned}$$

The proof that 0 is a regular value of the map $q$ is the same as before, therefore we can still use zero set of $q$ to study the neighbourhood of a monopole in the moduli space. It is clear that in case (1), (2), (4) and (5), $q^{-1}(0) = \{\text{point}\}$ and in case (3) $q^{-1}(0)/U(1)$ a cone over the manifold

$$S^{4k+1} \times S^{4k-3}/U(1),$$

by the same reasoning as before.

**Remark 4.14.** It is worth mentioning that although $q^{-1}(0) = \{\text{point}\}$ in cases (1), (2), (4) and (5), case (1) and (4) differ from case (2) and (5). Because in case (1) and (4) there exists no irreducible monopole as evident by negative index; in case (2) and (5) the index is positive and thus $q^{-1}(0) = \{\text{point}\}$ means the reducible monopole is isolated but the existence of irreducible monopole is not known. However, it is shown in [B87] that $\mathcal{M}((m_0, k_0), (m_\infty, k_\infty)) = \mathcal{M}((2k, k), (0, 0))$ is a point.

### 4.3.5 $\Gamma$-invariant monopoles on $S^2 \times \mathbb{R}$

Let $\Gamma = S^1$ be the rotation group that acts on $S^2 \times \mathbb{R}$ by earth rotation on $S^2$ and fixes $\mathbb{R}$. The induced $S^1$-action also acts only on the first component of $X = S^2 \times S^2$. Observe that the action is essentially the same as the previous

$S^1$-action except they act on different $S^2$ of $S^2 \times S^2$. This also means the induced action on $E \to X$ is described in the same way as the previous $S^1$-action. To avoid confusion with the previous $S^1$-action, we will still denote the new action by $\Gamma$ (and $\tilde{\Gamma}$ its double cover) knowing it is a $S^1$ just acting on different parts to the previous $S^1$.

**Lemma 4.15.** *The bundle $\mathcal{O}(n)$ admits an $\tilde{\Gamma}$-action with weights $\gamma_0$, $\gamma_\infty \in \mathbb{Z}$ satisfying $\gamma_0 - \gamma_\infty = 2n$*

The proof copies word by word from that of (4.10). To simplify the calculation, we define the action so that $\gamma_0 = \omega_0$ and $\gamma_\infty = \omega_\infty$. The action then extends to $\mathcal{O}(k,-k)$, $\mathcal{O}(-k,k)$ and hence to the bundle $E \to X$.

### 4.3.6 Local model of $[A]$ in $\mathcal{M}^\Gamma(E)$

**Proposition 4.16.** *The $\tilde{\Gamma} \times \tilde{S}^1$-invariant part of the cohomology is given by*

$$\begin{aligned}
(H^0)^{\tilde{\Gamma}\times\tilde{S}^1} &= \mathbb{R}\cdot\begin{pmatrix}\mathbf{i} & 0\\ 0 & -\mathbf{i}\end{pmatrix} \cong \mathbb{R}\\
(H^1)^{\tilde{\Gamma}\times\tilde{S}^1} &\cong \begin{cases}\mathbb{C}\oplus\mathbb{C} & \text{in case (3)}\\ 0 & \text{in case (1), (2), (4) and (5)}\end{cases}.\\
(H^2)^{\tilde{\Gamma}\times\tilde{S}^1} &= \mathbb{R}\cdot\begin{pmatrix}\mathbf{i} & 0\\ 0 & -\mathbf{i}\end{pmatrix}\cdot\omega \cong \mathbb{R}.
\end{aligned}$$

*Proof.* Again we use formula (4.3.14) and (4.3.15) for sheaf cohomology to calculate $H_0^i$. The Kunneth formula gives

$$\begin{aligned}
(H^1)^{\tilde{\Gamma}\times\tilde{S}^1} \cong\ & \left\{H^0(S^2,\mathcal{O}(2k))^{\tilde{\Gamma}\times\tilde{S}^1}\otimes H^1(S^2,\mathcal{O}(-2k))^{\tilde{\Gamma}\times\tilde{S}^1}\right\}\\
&\oplus\left\{H^1(S^2,\mathcal{O}(2k))^{\tilde{\Gamma}\times\tilde{S}^1}\otimes H^0(S^2,\mathcal{O}(-2k))^{\tilde{\Gamma}\times\tilde{S}^1}\right\}\\
&\oplus\left\{H^0(S^2,\mathcal{O}(-2k))^{\tilde{\Gamma}\times\tilde{S}^1}\otimes H^1(S^2,\mathcal{O}(2k))^{\tilde{\Gamma}\times\tilde{S}^1}\right\}\\
&\oplus\left\{H^1(S^2,\mathcal{O}(-2k))^{\tilde{\Gamma}\times\tilde{S}^1}\otimes H^0(S^2,\mathcal{O}(2k))^{\tilde{\Gamma}\times\tilde{S}^1}\right\}.
\end{aligned}$$

Since $\Gamma$ fixes the second part and $S^1$ fixes the first part, then we have

$$\begin{aligned}
(H^1)^{\tilde{\Gamma}\times\tilde{S}^1} \cong\ & \left\{H^0(S^2,\mathcal{O}(2k))^{\tilde{\Gamma}}\otimes H^1(S^2,\mathcal{O}(-2k))^{\tilde{S}^1}\right\}\\
&\oplus\left\{H^1(S^2,\mathcal{O}(2k))^{\tilde{\Gamma}}\otimes H^0(S^2,\mathcal{O}(-2k))^{\tilde{S}^1}\right\}\\
&\oplus\left\{H^0(S^2,\mathcal{O}(-2k))^{\tilde{\Gamma}}\otimes H^1(S^2,\mathcal{O}(2k))^{\tilde{S}^1}\right\}\\
&\oplus\left\{H^1(S^2,\mathcal{O}(-2k))^{\tilde{\Gamma}}\otimes H^0(S^2,\mathcal{O}(2k))^{\tilde{S}^1}\right\}.
\end{aligned}$$

Recall that $\Gamma$ is also an $S^1$-action, therefore the first terms of all four summands can still be computed using formula (4.3.16), (4.3.18), (4.3.17) and (4.3.19). Hence we find that the only non-trivial case is case (3) where

$$(H^1)^{\tilde{\Gamma}\times\tilde{S}^1} \cong (\mathbb{C}\otimes\mathbb{C}) \oplus (\mathbb{C}\otimes\mathbb{C}) \cong \mathbb{C}\oplus\mathbb{C}$$

Observe that comparing with the previous situation where only one ($\tilde{S}^1$) action is present, the result here is completely independent of $k$. This is because in the current case the first terms of all four summands needs to be calculated using the result of $\tilde{S}^1$-invariant sheaf cohomology (see (4.3.14), (4.3.15) ).

For the same reason as before, the stabiliser $\Gamma_A \cong U(1)$ is $\tilde{\Gamma}$-equivariant and therefore the $\tilde{\Gamma}\times\tilde{S}^1$ invariant part of $H^0_A$ is the $H^0_A$. Hence the the new action does not effect $H^0_A$ and $H^2_A$. $\square$

In case (1), (2), (4) and (5), $(H^1)^{\tilde{\Gamma}\times\tilde{S}^1} = 0$,

$$q : \{\text{point}\} \longrightarrow \mathbb{R}$$

and $q^{-1}(0)$ consists of a point. In case (3),

$$q : \mathbb{C}\oplus\mathbb{C} \longrightarrow \mathbb{R}$$

and

$$q^{-1}(0) = \left\{(\alpha,\beta) \mid |\alpha|^2 = |\beta|^2\right\},$$

and thus $q^{-1}(0)\backslash\{0\}$ is diffeomorphic to $S^1\times S^1\times\mathbb{R}^+$ via the map

$$(\alpha,\beta) \longmapsto (\frac{\alpha}{|\alpha|}, \frac{\beta}{|\beta|}, |\alpha|).$$

Recall that that the stabiliser $\Gamma_A$ is the group $U(1)$ acting by $\lambda(\alpha,\beta) = (\lambda\alpha, \lambda^{-1}\beta)$, then $(q^{-1}(0)\backslash\{0\})/\Gamma_A$ is diffeomorphic to $S^1\times\mathbb{R}^+$. Therefore a neighbourhood of monopole in $\mathcal{M}^\Gamma(E)$ is a cone over $S^1$. These two examples are discussed in [B89] without the addition of $\Gamma$-action.

# Chapter 5

# Sen conjecture

In this chapter we consider monopoles on $\mathbb{R}^3$. The subject of Sen's conjecture is the space of $L^2$-harmonic forms on the relevant moduli space. To state the conjecture, we define first the reduced moduli space $\tilde{\mathcal{M}}_k^0$. The group of translations of $\mathbb{R}^3$ acts on the moduli space, hence $\mathcal{M}_k/\mathbb{R}^3$ is the manifold of centred monopoles. It is shown by Donaldson in [D84] that $\mathcal{M}_k^0$, as a smooth manifold, can be identified with the complex manifold $\mathrm{Rat}_q^0/\mathbb{C}^*$, where

$$\mathrm{Rat}_k^0 = \left\{ \frac{f(z)}{g(z)} = \frac{a_1 z^{k-1} + \cdots + a_k}{z^q + b_2 z^{k-2} + \cdots + b_k} \mid f \text{ and } g \text{ have no common roots} \right\}$$

where the action is given by multiplication

$$\begin{array}{rcl} \mathbb{C}^* \times \mathrm{Rat}_k^0 & \to & \mathrm{Rat}_k^0 \\ (\lambda, \frac{f}{g}) & \mapsto & \lambda \frac{f}{g}. \end{array}$$

By this description one can calculate the fundamental group of $\mathrm{Rat}_k^0$ as $\pi_1(\mathrm{Rat}_k^0) = \mathbb{Z}$

Denote by $\mathcal{M}_k^0$ the universal cover of the quotient $\mathcal{M}_k/\mathbb{R}^3 \times S^1$, where $\mathbb{R}^3$ acts by translation and $S^1$ by rotations of the framing. Let $\tilde{\mathcal{M}}_k^0$ be the universal cover of the quotient $\mathcal{M}/\mathbb{R}^3 \times S^1$. In Donaldson's algebraic description,

$$\tilde{\mathcal{M}}_k^0 = \left\{ \frac{f}{g} \in \mathrm{Rat}_k^0 \mid \mathrm{Res(f,g)} = 1 \right\}$$

where $\mathrm{Res}(f, g)$ is defined as follows: For $g(z) = \Pi_{i=1}^q (z - \beta_i)$, $\mathrm{Res}(f, g) = \Pi_{i=1}^q (\beta_i)$. Therefore $\tilde{\mathcal{M}}_k^0$ admits a free action by the group $\mu_q$ of $q^{\text{th}}$ roots of unity.

Sen's conjecture concerns the space $\mathcal{H}_k^i$ of $L^2$-harmonic $i$-forms on $\tilde{\mathcal{M}}_k^0$. This space admits a decomposition with respect to the induced action of the cyclic group action $\mu_q$ of $q^{\text{th}}$ roots of unity:

$$\mathcal{H}_k^i = \oplus \mathcal{H}_{k,p}^i,$$

where $\mathcal{H}_{k,p}^i$ is the part where $z \in \sigma_k$ (the cyclic group of order $k$) act by multiplication by $z^p$.

**Sen's Conjecture**

(i). If $k$ and $p$ are relatively prime, then $\mathcal{H}_{k,p}^i = 0$ except when $i = 2k-2$ in which $\mathcal{H}_{k,p}^{2k-2} \cong \mathbb{C}$, and

(ii). if $k$ and $p$ have a common factor, then $\mathcal{H}_{k,p}^i = 0$ for all $i$.

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