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Applications of Malliavin Calculus to Limit Theorems

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Preface

The subject of this thesis is limit theorems on the Wiener chaos, using Malliavin calculus and Stein's method.

The work in this thesis is my own except where otherwise stated. I do not claim originality of results presented in this thesis. However, some topics and proofs have been presented differently or I have attempted to fill in some gaps. I hope this makes the exposition clearer to the reader.

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Introduction

Let ν be the standard normal measure and $L^2(\mathbb{R}, \mathcal{B}, \nu)$ be the space of functions f such that $\mathrm{E}(f(N)^2) < \infty$, where $N \sim \mathcal{N}(0, 1)$. It is well known that the Hermite polynomials can be used to form an orthonormal basis for $L^2(\mathbb{R}, \mathcal{B}, \nu)$. We can generalize and extend this idea to the space of square integrable random variables, $L^2(\Omega, \mathcal{F}, \mathrm{P})$, which we will decompose into orthogonal subspaces, known as the Wiener chaoses. In particular, for every $F \in L^2(\Omega, \mathcal{F}, \mathrm{P})$, we have the Wiener-Itô expansion

$$F = \sum_{q=0}^{\infty} I_q(f_q),$$

where $I_q(f_q)$ is in the qth Wiener chaos, but is also a Wiener-Itô integral. In fact, it turns out that all elements of the Wiener chaos are such integrals.

Originally developed by Wiener [Wie38] and Itô [Itô51], the concept of the Wiener chaos allows us to use Gaussian analysis in the study of non-Gaussian phenomena. In particular, the main topic of this text is limit theorems on the Wiener chaos. Unlike the setting of the classical central limit theorem, we deal with the convergence of random variables that are generally neither independent nor identically distributed. The limits are Gaussian in some cases, but not always.

In 2005, Nualart and Peccati [NP05] proved a remarkable result which is now known as the fourth moment theorem. Suppose that $(I_q(f_n))_{n\geq 1}$ is a sequence of random variables on the qth Wiener chaos with $\mathrm{E}(I_q(f_n)^2) \to 1$ as $n \to \infty$, Nualart and Peccati showed that the convergence in distribution of $(I_q(f_n))_{n\geq 1}$ to a standard normal random variable is equivalent to the fourth moment condition $\mathrm{E}(I_q(f_n)^4) \to 3$ as $n \to \infty$.

Later, Nourdin and Peccati [NP09] provided an alternative proof by combining Malliavin calculus with Stein's method, which has the advantage of being able to derive explicit distance bounds and rates of convergence. Their idea for proving limit theorems can be understood heuristically. Stein's method is used to write the total variation distance between a random variable F_n and a standard normal random variable N in terms of differential operators

$$d_{\mathrm{TV}}(F_n, N) \approx \sup_{f \in C^1(\mathbb{R})} \left| \mathrm{E}(f'(F_n) - \mathrm{E}(F f(F_n)) \right|.$$

Then this distance can be computed using formulas for Malliavin calculus and bounded. Finally, $F_n \xrightarrow{d} N$ is proved by showing that the bound for $d_{\mathrm{TV}}(F_n, N)$ converges to 0.

The aim of this text is to develop the theory underlying the fourth moment theorem. We will make the above argument rigorous. We also discuss some applications of this theory to limit theorems for partial sum processes and parameter estimation for Hermite processes.

The first two chapters are used to develop the tools that we will use throughout the text. In Chapter 1, the Wiener chaos is shown to be the subspaces in an orthogonal direct sum decomposition of $L^2(\Omega, \mathcal{F}, \mathrm{P})$. We will show that the elements of the Wiener chaos are Wiener-Itô integrals and derive some of their properties.

In Chapter 2, we introduce Malliavin calculus, an infinite-dimensional differential calculus for functions of Gaussian processes. The three Malliavin operators, including a derivative for random variables, are developed.

Chapter 3 introduces the total variation distance between probability measures. Stein's method and Malliavin calculus is combined to prove the fourth moment theorem in both the univariate and multivariate context.

The final two chapters can be viewed as applications of the techniques and limit theorems derived in the first three chapters. In Chapter 4, we study the limit of the partial sum processes of $(f(X_n))_{n\in\mathbb{Z}}$, where $(X_n)_{n\in\mathbb{Z}}$ is a stationary Gaussian process and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function. It is well known that the limiting process must be self-similar with stationary increments. Furthermore, the limit depends on the covariance structure of the process $(f(X_n))_{n\in\mathbb{Z}}$. In the case where it exhibits short range dependence, we use a corollary of the fourth moment theorem to give a modern proof of the Breuer-Major theorem [BM83], showing that the limiting process is Brownian motion. But in the case of long range dependence, the limit is a Hermite process, which is non-Gaussian.

In Chapter 5, we further explore the Hermite process, which turns out to be a Wiener-Itô integral and a generalization of fractional Brownian motion with Hurst parameter $H > 1/2$ and the Rosenblatt process. In particular, following the work of Chronopoulou, Tudor and Veins [CTV11, TV09], the limit of the leading term in the Wiener Itô expansion for the quadratic variation is used to show that an estimator of the Hurst parameter of a Hermite process converges to a Rosenblatt distribution.

Chapter 1

Wiener Chaos

In this chapter, we introduce the Wiener chaos which are orthogonal subspaces in the decomposition of the space of square integrable random variables. This lays the foundation for the theory developed in later sections. The elements of the Wiener chaos are our primary object of study and we show that they coincide with Wiener-Itô integrals. We also derive some useful properties of Wiener-Itô integrals and their connection with Hermite polynomials.

The main references for this chapter are Nualart [Nua06], Nourdin and Peccati [NP12], and Janson [Jan97].

1.1 Isonormal Gaussian Processes

Let $\mathcal{H}$ be a real separable Hilbert space.

Definition 1.1.1. A stochastic process $W = (W(f))_{h\in\mathcal{H}}$ defined on the probability space $(\Omega, \mathcal{G}, \mathrm{P})$ is called an **isonormal Gaussian process** indexed by $\mathcal{H}$, if $W(f)$ are centered Gaussian random variables and $\mathrm{E}(W(f)W(g)) = \langle f, g\rangle_{\mathcal{H}}$, for all $f, g \in \mathcal{H}$.

Isonormal Gaussian processes originated from the work of R.M. Dudley [Dud67] and they are a generalization of Gaussian measures and allow us to bring Hilbert space techniques into the theory. They will be useful in later sections, for example, in constructing the Wiener-Itô integrals and defining Malliavin operators.

Let $\mathcal{F} = \sigma(X)$ be the sigma field generated by W. We will denote the space of square integrable random variables measurable with respect to $\mathcal{F}$ by $L^2(\Omega) := L^2(\Omega, \mathcal{F}, \mathrm{P})$. An isonormal Gaussian process W is a closed subspace of $L^2(\Omega)$. Since Gaussian processes are determined by their mean and covariance functions, the random variables in W are characterized by

$$\begin{aligned} \mathrm{E}(W(f)) &= 0, \\ \mathrm{Var}(W(f)) &= \|f\|_{\mathcal{H}}^2, \\ \mathrm{Cov}(W(f), W(g)) &= \langle f, g\rangle_{\mathcal{H}}, \end{aligned}$$

for all $f, g \in \mathcal{H}$. Also, the linear map $f \mapsto W(f)$ is an isometry from $\mathcal{H}$ onto W.

The next proposition shows that given a real separable Hilbert space $\mathcal{H}$, it is always possible to construct an isonormal Gaussian process indexed by $\mathcal{H}$.

Proposition 1.1.2. *Let $\mathcal{H}$ be a real separable Hilbert space, then there exists an isonormal Gaussian process indexed by $\mathcal{H}$.*

Proof. Let $(Z_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed standard Gaussian random variables on a probability space $(\Omega, \mathcal{G}, \mathrm{P})$ and let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. We will show that $W = (W(f))_{f\in\mathcal{H}}$ is an isonormal Gaussian process indexed by $\mathcal{H}$, where

$$W(f) := \sum_{i=0}^{\infty} \langle f, e_i\rangle_{\mathcal{H}} Z_i.$$

Note that Parseval's identity says $\sum_{i=0}^{\infty} \langle f_i, e_i\rangle_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 < \infty$. So for $n > m$, using orthogonality, we have

$$\left\| \sum_{i=0}^{n} \langle f, e_i\rangle_{\mathcal{H}} Z_i - \sum_{i=0}^{m} \langle f, e_i\rangle_{\mathcal{H}} Z_i \right\|_{\mathcal{H}}^2 = \sum_{i=m+1}^{n} \langle f, e_i\rangle_{\mathcal{H}}^2 \to 0$$

as $n, m \to \infty$. Therefore, the partial sum sequence $\sum_{i=0}^{n} \langle f, e_i\rangle_{\mathcal{H}} Z_i$ is Cauchy in $L^2(\Omega, \mathcal{G}, \mathrm{P})$ and $W(f)$ is convergent.

By construction, $W(f)$ is a centered Gaussian random variables for all $f \in \mathcal{H}$. Using orthogonality, the covariance is

$$\begin{aligned}
\mathrm{E}(W(f)W(g)) &= \mathrm{E}\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \langle f, e_i\rangle_{\mathcal{H}} \langle g, e_i\rangle_{\mathcal{H}} Z_i Z_j \right) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \langle f, e_i\rangle_{\mathcal{H}} \langle g, e_i\rangle_{\mathcal{H}} \\
&= \langle f, g\rangle_{\mathcal{H}}
\end{aligned}$$

for all $f, g \in \mathcal{H}$, where the last equality is a basic property of orthonormal bases. □

We now provide some examples of isonormal Gaussian processes.

Example 1.1.3. Consider the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, where ν is the standard Gaussian probability measure, defined for all $A \in \mathcal{B}(\mathbb{R})$ by

$$\nu(A) := \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt. \tag{1.1.1}$$

Let $\mathcal{H} = \mathbb{R}$ and set $(W(f))(x) = fx$ for all $f \in \mathcal{H}$. Then $W = (W(f))_{f\in\mathcal{H}}$ is an isonormal Gaussian process with $W(f) \sim \mathcal{N}(0, f^2)$ and $\mathrm{Cov}(W(f), W(g)) = fg$.

Example 1.1.4. Consider the isonormal Gaussian process W indexed by $\mathcal{H} = L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, where μ is the Lebesgue measure. For all $t \geq 0$, let $B_t := W(1_{[0,t]})$. Then we have

$$\mathrm{E}((B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2})) = \left\langle 1_{[s_1,t_1]}, 1_{[s_2,t_2]} \right\rangle_{\mathcal{H}} = \mu([s_1, t_1] \cap [s_2, t_2]) = 0$$

for all $0 \leq s_1 < t_1 < s_2 < t_2$. Thus, $(B_t)_{t\geq 0}$ has independent increments.

We also have

$$\mathrm{Var}(B_t) = \left\| 1_{[0,t]} \right\|_{\mathcal{H}}^2 = t.$$

This implies that $(B_t)_{t\geq 0}$ has stationary increments as $B_t - B_s \sim N(0, t-s)$, for all $0 \leq s < t$. Furthermore, $\mathrm{E}((B_t - B_s)^4) = 3(t-s)^2$, so $(B_t)_{t\geq 0}$ also satisfies the Kolmogorov continuity criterion, implying that there exists an almost surely continuous version of B_t. Finally, $B_0 = 0$ almost surely. So there is a version of $(B_t)_{t\geq 0}$ that is Brownian motion.

1.2 Hermite Polynomials

In this section, we will introduce the Hermite polynomials, which will play a key role throughout the rest of this thesis.

Definition 1.2.1. Let $q \in \mathbb{N}$. The qth **Hermite polynomial** is defined as

$$H_q(x) = (-1)^q \exp\left(\frac{x^2}{2}\right) \frac{\partial^q}{\partial x^q} \exp\left(-\frac{x^2}{2}\right),$$

for $x \in \mathbb{R}$. We set $H_0 = 1$.

The first few Hermite polynomials are

$$\begin{aligned} H_1(x) &= x, \\ H_2(x) &= x^2 - 1, \\ H_3(x) &= x^3 - 3x. \end{aligned}$$

The following lemma represents Hermite polynomials as the coefficient of a generating function, which will be useful to prove further results about these polynomials.

Lemma 1.2.2. *For all $x, t \in \mathbb{R}$, we have*

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{q=0}^{\infty} \frac{t^q}{q!} H_q(x).$$

Proof. Since $\exp\left(-(x-t)^2/2\right)$ is a symmetric function of t and x,

$$\frac{\partial^q}{\partial t^q} \exp\left(-\frac{(x-t)^2}{2}\right) = \frac{\partial^q}{\partial x^q} \exp\left(-\frac{(x-t)^2}{2}\right),$$

for all $x, t \in \mathbb{R}$. Thus, the Taylor series expansion of $\exp((x-t)^2/2)$ as a function of t can be written as

$$\begin{aligned} \exp\left(-\frac{(x-t)^2}{2}\right) &= \sum_{q=0}^{\infty} \frac{t^q}{q!} \frac{\partial^q}{\partial t^q} \exp\left(-\frac{(x-t)^2}{2}\right)\bigg|_{t=0} \\ &= \sum_{q=0}^{\infty} \frac{t^q}{q!} (-1)^q \frac{\partial^q}{\partial x^q} \exp\left(-\frac{x^2}{2}\right). \end{aligned}$$

Now multiplying both sides by $\exp(x^2/2)$ and using the definition of the Hermite polynomial gives the desired result. □

We now give some basic properties of Hermite polynomials.

Proposition 1.2.3. *Let $p, q \in \mathbb{N}$ and H_q be the qth Hermite polynomial.*

(i) The sequence $(H_q/\sqrt{q!})_{q \in \mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, where ν is the standard Gaussian measure defined in (1.1.1).

(ii) All functions $f \in L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ *have an orthogonal expansion given by*

$$f = \sum_{q=0}^{\infty} a_q H_q,$$

where

$$a_q = \int_{-\infty}^{\infty} f(x) H_q(x)\, d\nu(x).$$

(iii) For all $q \geq 1$*, we have* $H_q'(x) = qH_{q-1}(x)$*.*

(iv) For all $q \geq 1$*, we have* $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$*.*

(v) Let $X, Y \sim \mathcal{N}(0,1)$ *with covariance* $\mathrm{E}(XY) = \rho$*, then*

$$\mathrm{E}(H_p(X)H_q(Y)) = \begin{cases} 0 & \text{if } p \neq q \\ p!\rho^p & \text{if } p = q. \end{cases}$$

Proof. (i) Using Lemma 1.2.2, we have

$$\exp\left((s+t)x - \frac{s^2+t^2}{2}\right) = \sum_{p,q=0}^{\infty} \frac{s^p t^q}{p!q!} H_p(x) H_q(x).$$

Now integrating both sides with respect to ν gives

$$e^{st} = \sum_{p,q=0}^{\infty} \frac{s^p t^q}{p!q!} \int_{-\infty}^{\infty} H_p(x) H_q(x)\, d\nu(x).$$

By comparing the coefficients of st on both sides, we have $\langle H_p, H_q \rangle_{L^2(\mathbb{R},\mathcal{B}(\mathbb{R}),\nu)} = 0$ when $p \neq q$, otherwise $\langle H_p, H_q \rangle_{L^2(\mathbb{R},\mathcal{B}(\mathbb{R}),\nu)} = p!^2$. This proves that $(H_q/\sqrt{q!})_{q\in\mathbb{N}}$ is an orthonormal set.

It remains to show that the span $(H_q/\sqrt{q!})_{q\in\mathbb{N}}$ is dense in $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$. Since H_q is a polynomial of degree q, the span $\{H_0, \ldots, H_q\} = \text{span}\,\{1, x, \ldots, x^q\}$, so this reduces to showing that the span of $(x^q)_{q\in\mathbb{N}}$ is dense in $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, which is true (see Proposition 1.1.5 in [NP12]). Therefore, $(H_q/\sqrt{q!})_{q\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$.

(ii) This is an immediate corollary of (i).

(iii) Using Lemma 1.2.2 and differentiating with respect to x, we have

$$\sum_{q=0}^{\infty} \frac{t^q}{q!} H_q'(x) = t \exp\left(tx - \frac{t^2}{2}\right) = \sum_{q=1}^{\infty} \frac{t^q}{(q-1)!} H_{q-1}(x).$$

The result follows by applying the derivative $\partial^q/\partial t^q$ at $t = 0$ to both sides.

(iv) Using Lemma 1.2.2 and differentiating with respect to t, we have

$$\sum_{q=0}^{\infty} \frac{t^q}{q!} H_{q+1}(x) = (x-t)\exp\left(tx - \frac{t^2}{2}\right) = \sum_{q=0}^{\infty} \frac{t^q}{q!} x H_q(x) - \sum_{q=1}^{\infty} \frac{t^q}{(q-1)!} H_{q-1}(x).$$

The result follows by applying the derivative $\partial^q/\partial t^q$ at $t = 0$ to both sides.

(v) The moment generating function for (X, Y) is $\mathrm{E}(e^{sX+tY}) = e^{(s^2+2\rho st+t^2)/2}$. Using this fact we have

$$\begin{aligned} \mathrm{E}\left(\exp\left(sX - \frac{s^2}{2}\right)\exp\left(tY - \frac{t^2}{2}\right)\right) &= \exp\left(-\frac{s^2+t^2}{2}\right)\mathrm{E}(e^{sX+tY}) \\ &= e^{\rho st} \\ &= \sum_{q=0}^{\infty}\frac{s^q t^q}{q!}\rho^q. \end{aligned} \tag{1.2.1}$$

On the other hand, using Lemma 1.2.2

$$\mathrm{E}\left(\exp\left(sX - \frac{s^2}{2}\right)\exp\left(tY - \frac{t^2}{2}\right)\right) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{s^p t^q}{p!q!}\mathrm{E}(H_p(X)H_q(Y)). \tag{1.2.2}$$

Now applying the derivative $\partial^{p+q}/(\partial s^p \partial t^q)$ at $s = t = 0$ to (1.2.1) and (1.2.2) gives the desired result. □

We will also state the following obvious fact because it will be used a few times in the following sections.

Lemma 1.2.4. *For all $q \in \mathbb{N}$, there exists constants $c_r \in \mathbb{R}$, $r = 0, \ldots, q$ such that*

$$x^q = \sum_{r=0}^{q} c_r H_r(x).$$

Proof. Starting with the qth Hermite polynomial, H_q, we can subtract $c_{q-1}H_{q-1}$, for some $c_{q-1} \in \mathbb{R}$ to cancel the x^{q-1} term, and so on. □

1.3 Wiener Chaos Decomposition

We refer the reader to Appendix B for properties of direct sums and tensor products of Hilbert spaces, which will be used throughout this section.

We saw in the previous section that Hermite polynomials can be used to construct an orthonormal basis for $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ and that every $f \in L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ has an orthogonal expansion in Hermite polynomials. Our aim in introducing the Wiener chaos is to generalize and extend this idea to the space of square integrable random variables $L^2(\Omega)$.

Recall that $\mathcal{H}$ is a real separable Hilbert space and $W = (W(f))_{f\in\mathcal{H}}$ is an isonormal Gaussian process indexed by $\mathcal{H}$.

Definition 1.3.1. Let $q \in \mathbb{N}$. The qth **Wiener chaos** of W, denoted by $\mathcal{H}_q$, is the closed subspace of $L^2(\Omega)$ spanned by $\{H_q(W(f)) \mid f \in \mathcal{H}, \|f\|_{\mathcal{H}} = 1\}$. We set $\mathcal{H}_0 = \mathbb{R}$.

The elements of the Wiener chaos and their properties will be the main object of study. The concept of the Wiener chaos was originally introduced using polynomial chaos by Wiener [Wie38] to study statistical mechanics, who also provided the first proof of the Wiener chaos decomposition in the below theorem. A modern proof of this result can be found in Nourdin [NP12]. The proceeding theorem expresses $L^2(\Omega)$ as an orthogonal direct sum of Hilbert spaces, the definition of this concept can be found in Section B.3.

Theorem 1.3.2 (Wiener Chaos Decomposition). *The space $L^2(\Omega)$ has the orthogonal direct sum decomposition*

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q.$$

Proof. By Proposition 1.2.3 (v), $\mathcal{H}_p$ is orthogonal to $\mathcal{H}_q$ for all $p \neq q$. Now recalling Definition B.3.1, it remains to show that the span of $\{H_q(W(f)) \,|\, q \in \mathbb{N}, f \in \mathcal{H}, \|f\|_{\mathcal{H}} = 1\}$ is dense in $L^2(\Omega)$.

Suppose that $F \in L^2(\Omega)$ such that $\mathrm{E}(FH_q(W(f))) = 0$ for all $q \in \mathbb{N}$, $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$, then by Proposition B.2.2, it suffices to prove that $F = 0$ almost surely.

We can write x^q as a linear combination of Hermite polynomials, so $\mathrm{E}(FW(f)^q) = 0$ for all $q \in \mathbb{N}$, $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$, which implies that

$$\mathrm{E}\left(Fe^{iW(f)}\right) = 0 \tag{1.3.1}$$

for all $f \in \mathcal{H}$. Now fix $n \geq 1$, let $e_1, \dots, e_n$ be elements from an orthonormal basis of $\mathcal{H}$, and let $\mathcal{F}_n$ be the σ-field generated by $W(e_j)$ for $j \leq n$. Using the law of iterated expectation, followed by the linearity of $f \mapsto W(f)$ and (1.3.1), we have

$$\mathrm{E}\left(\mathrm{E}\left(F \,|\, \mathcal{F}_n\right) \exp\left(i \sum_{j=1}^{n} t_j W(e_j)\right)\right) = \mathrm{E}\left(F \exp\left(i \sum_{j=1}^{n} t_j W(e_j)\right)\right) = 0 \tag{1.3.2}$$

for all $t_1, \dots, t_n \in \mathbb{R}$. There exists a measurable function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{E}(F \,|\, \mathcal{F}_n) = \phi(W(e_1), \dots, W(e_n))$ because $\mathrm{E}(F \,|\, \mathcal{F}_n)$ is measurable with respect to $\mathcal{F}_n$. Let

$$\psi(x_1, \dots, x_n) := \phi(x_1, \dots, x_n) \exp\left(-\frac{1}{2} \sum_{j=1}^{n} x_j^2\right).$$

The Fourier transform of ψ is

$$\begin{aligned}\hat{\psi}(t_1, \dots, t_n) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \phi(x_1, \dots, x_n) \exp\left(-\frac{1}{2} \sum_{j=1}^{n} x_j^2\right) \exp\left(i \sum_{j=1}^{n} t_j x_j\right) dx_1 \dots dx_n \\ &= 0,\end{aligned}$$

where the last equality follows from (1.3.2) and noting that the integration in (1.3.2) is respect to a standard multivariate Gaussian measure. Since the $\hat{\psi} = 0$, we have that $\phi = 0$ and $\mathrm{E}(F \,|\, \mathcal{F}_n) = 0$ almost surely for all $n \geq 1$, which implies that $\mathrm{E}(F \,|\, \mathcal{F}) = 0$, where $\mathcal{F}$ is generated by the σ-fields $\mathcal{F}_n$, $n \geq 1$. Since F is $\mathcal{F}$-measurable, we conclude that $F = \mathrm{E}(F \,|\, \mathcal{F}) = 0$ almost surely. □

Due to Proposition B.3.2, the Wiener chaos decomposition implies that all $F \in L^2(\Omega)$ has an orthogonal expansion in the form

$$F = \sum_{q=0}^{\infty} F_q,$$

where $F_q \in \mathcal{H}_q$ so that F_p is orthogonal to F_q for all $p \neq a$. Also, $F_0 = \mathrm{E}(F)$.

1.4 The Wiener-Itô Integral

In 1951 Itô [Itô51] introduced what are now known as Wiener-Itô integrals. While the Wiener chaos was introduce earlier and was not defined in terms of these integrals, it turns out that the elements of the Wiener chaos are Wiener-Itô integrals. In this section, we develop the theory of these integrals in a similar fashion to Lebesgue integration, starting on a set of elementary functions and then extending the definition using denseness, we also derive some elementary properties that will be extended in Section 1.5.

Throughout this section we will work exclusively in the Hilbert space $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B}, \mu)$ is a measure space and μ is a σ-finite and nonatomic measure.

Let W be an isonormal Gaussian process indexed by $\mathcal{H}$. For all $A \in \mathcal{B}$, define $W(A) := W(1_A)$, then $W(A) \sim \mathcal{N}(0, \mu(A))$ because $\text{Var}(W(A)) = \|1_A\|^2_{\mathcal{H}} = \mu(A)$. The process $G = \{W(A) \,|\, A \in \mathcal{B}, \mu(A) < \infty\}$ is called **Gaussian white noise** or a Gaussian random measure.

Recall from Proposition B.4.10 that the qth tensor power of $\mathcal{H}$, denoted by $\mathcal{H}^{\otimes q}$, is isomorphic to $L^2(T^q, \mathcal{B}^q, \mu^q)$. We will define the Wiener-Itô integral with respect to Gaussian white noise, G, for functions $f \in \mathcal{H}^{\otimes q}$ as a linear operator

$$I_q : \mathcal{H}^{\otimes q} \to L^2(\Omega)$$

which will sometimes be denoted by

$$I_q(f) = \int_{T^q} f(t_1, \ldots, t_q)\, dW(t_1) \ldots dW(t_q).$$

Note that this notation is not interpreted as the the Itô integral of f. Indeed, such an Itô integral is not defined since the iterated integral forms a nonadapted stochastic process. The key to avoiding this problem is to use off-diagonal simple functions as the elementary functions in the definition of the Wiener-Itô integral.

Definition 1.4.1. A simple function on T^q of the form

$$f(t_1, \ldots, t_q) = \sum_{i_1, \ldots, i_q = 1}^{n} a_{i_1 \ldots i_q} 1_{A_{i_1} \times \cdots \times A_{i_q}}(t_1, \ldots, t_q) \tag{1.4.1}$$

is called an **off-diagonal simple function** if $A_1 \ldots A_n \in \{A \in \mathcal{B} \,|\, \mu(A) < \infty\}$ are disjoint and $a_{i_1 \ldots i_q} = 0$ when $i_r = i_s$ for some $r \neq s$.

Off-diagonal simple functions are just simple functions that vanish on $D := \{(t_1, \ldots, t_q) \in T^q \,|\, t_i = t_j \text{ for some } i = j\}$, set of diagonals of T^q. Note that as with any simple function, it is always possible to choose $A_1, \ldots, A_n$ in a way that partitions T. We will assume that all off-diagonal simple functions are in this form.

For all off-diagonal simple functions, f, given by (1.4.1), define the Wiener-Itô integral of f as

$$I_q(f) := \sum_{i_1, \ldots, i_q = 1}^{n} a_{i_1 \ldots i_q} W(A_{i_1}) \ldots W(A_{i_q}).$$

This integral is well-defined and satisfies some basic properties given below. Here, $\widetilde{f}$ is the symmetrization of f, defined in (B.4.1). The symmetrization transforms a function to a symmetric function, that is a function that is invariant under permutations of its arguments.

Proposition 1.4.2. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. For all $p, q \geq 1$, if $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$ are off-diagonal simple functions, then:*

(i) The operator I_p is linear.

(ii) $I_p(f) = I_q(\widetilde{f})$.

(iii) $\mathrm{E}(I_p(f)) = 0$.

(iv) $\mathrm{E}(I_p(f)I_q(g)) = \begin{cases} 0 & \text{if } p \neq q \\ q!\langle \widetilde{f}, \widetilde{g}\rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q. \end{cases}$

Proof. To show (i), note that for all off-diagonal simple functions, f, g, defined on the partitions of T, $\{A_i\}_{i=1}^n$ and $\{B_i\}_{j=1}^m$, respectively, we can write both f, g as off-diagonal simple functions on the partition fo T formed by the set $A_i \cap B_j$. Then, (i) follows from the linearity of $f \mapsto W(f)$.

Next, we prove (ii). If $f = 1_{A_{i_1}\times\cdots\times A_{i_q}}$, then its symmetrization is given by

$$\begin{aligned}\tilde{f} &= \frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} 1_{A_{i_1}\times\cdots\times A_{i_q}}(t_{i_{\sigma(1)}}, \ldots, t_{i_{\sigma(q)}}) \\ &= \frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} 1_{A_{i_{\sigma(1)}}\times\cdots\times A_{i_{\sigma(q)}}}(t_1, \ldots, t_q).\end{aligned}$$

Then

$$I_q(\widetilde{f}) = \frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} W(A_{i_{\sigma(1)}})\ldots W(A_{i_{\sigma(q)}}),$$

where each of the $q!$ summands are equal to $W(A_{i_1})\ldots W(A_{i_q})$, so

$$I_q(\widetilde{f}) = W(A_{i_1})\ldots W(A_{i_q}) = I_q(f).$$

As every off-diagonal simple function can be written as a linear combination of the indicator functions f, then (ii) follows from (i).

Using the facts that W is an isonormal Gaussian process and $\{A_i\}_{i=1}^n$ is a partition of T, we have $\mathrm{E}(W(A_{i_r})W(A_{i_s})) = \mu(A_{i_r}\cap A_{i_s}) = 0$, for all $r \neq s$. Thus, $W(A_{i_1}), \ldots, W(A_{i_q})$ are mutually independent and

$$\mathrm{E}(W(A_{i_1})\ldots W(A_{i_q})) = \mathrm{E}(W(A_{i_1}))\ldots \mathrm{E}(W(A_{i_q})) = 0,$$

So for any off-diagonal simple function, f, we have $\mathrm{E}(I_q(f)) = 0$, which proves (iii).

Finally, we prove (iv). First, we will deal with the $p = q$ case. Let $f, g \in \mathcal{H}^{\otimes q}$ be off-diagonal simple functions of the form given by (1.4.1). As in the proof of (i), we can assume without loss of generality that f, g are both defined on the partition $\{A_i\}_{i=1}^n$. Furthermore, we can assume that f, g are symmetric functions due to (ii), so that

$$\begin{aligned}f(t_1, \ldots, t_q) &= \sum_{i_1,\ldots,i_q=1}^{n} a_{i_1\ldots i_q} 1_{A_{i_1}\times\cdots\times A_{i_q}}(t_1, \ldots, t_q), \\ g(t_1, \ldots, t_q) &= \sum_{i_1,\ldots,i_q=1}^{n} b_{i_1\ldots i_q} 1_{A_{i_1}\times\cdots\times A_{i_q}}(t_1, \ldots, t_q),\end{aligned}$$

where $a_{i_1\dots i_q} = a_{i_{\sigma(1)}\dots i_{\sigma(q)}}$ and $b_{i_1\dots i_q} = b_{i_{\sigma(1)}\dots i_{\sigma(q)}}$ for all permutations $\sigma \in \mathcal{S}_q$. Then,

$$I_q(f) = I_q(\widetilde{f}) = q! \sum_{1\le i_1<\dots<i_q\le n} a_{i_1\dots i_p} W(A_{i_1})\dots W(A_{i_q}), \tag{1.4.2}$$

and a similar result holds for g. By independence shown in the proof of (iii), we have that

$$\begin{aligned} \mathrm{E}((W(A_{i_1})\dots W(A_{i_q}))^2) &= \mathrm{E}(W(A_{i_1})^2)\dots \mathrm{E}(W(A_{i_q})^2) \\ &= \mu(A_1)\dots\mu(A_n). \end{aligned} \tag{1.4.3}$$

Now using Equation 1.4.2 and 1.4.3, we have

$$\begin{aligned} \mathrm{E}(I_q(f)I_q(g)) &= \mathrm{E}(I_q(\widetilde{f})I_q(\widetilde{g})) \\ &= \mathrm{E}\left(\left(q! \sum_{1\le i_1<\dots<i_q\le n} a_{i_1\dots i_q} W(A_{i_1})\dots W(A_{i_q})\right)\right. \\ &\qquad \left.\left(q! \sum_{1\le i_1<\dots<i_q\le n} b_{i_1\dots i_q} W(A_{i_1})\dots W(A_{i_q})\right)\right) \\ &= p!^2 \sum_{1\le i_1<\dots<i_q\le n} a_{i_1\dots i_q} b_{i_1\dots i_q} \mu(A_{i_1})\dots\mu(A_{i_q}) \\ &= p!\langle \widetilde{f}, \widetilde{g}\rangle_{\mathcal{H}^{\otimes q}}. \end{aligned}$$

In the case where $p \neq q$, for all $1 \le i_1 < \dots < i_p \le n$ and $1 \le i_j < \dots < j_q \le n$, the fact that the sets $\{A_i\}_{i=1}^n$ are disjoint implies that when we get the term $\mathrm{E}(W(A_{i_1})...W(A_{i_p})W(A_{j_1})W(A_{j_q}))$, in the above calculation we can factor out at least one term $\mathrm{E}(W(A_r)) = 0$. Hence, $I_p(f)I_q(g) = 0$. □

As a result of Proposition 1.4.2 (iii), we can always assume without loss of generality that $f \in \mathcal{H}^{\otimes q}$ is a symmetric function, so that $f \in \mathcal{H}^{\odot q}$, the qth symmetric tensor power of $\mathcal{H}$ (see Definition B.4.6). By Proposition B.4.10, $\mathcal{H}^{\odot q}$ is isomorphic to $L^2_s(T^q, B^q, \mu^q)$, the subspace of symmetric functions in $L^2(T^q, B^q, \mu^q)$.

To extend this Wiener-Itô integral to all $f \in \mathcal{H}^{\otimes q}$, we show that the set of off-diagonal simple functions are dense using the fact the μ is a nonatomic measure. The following lemma is from Kuo [Kuo06].

Lemma 1.4.3. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. Let $f \in \mathcal{H}^{\otimes q}$, then there exists a sequence of off-diagonal simple functions $(f_n)_{n\ge1}$ such that $\|f_n - f\|_{\mathcal{H}^{\otimes q}} \to 0$ as $n \to \infty$.*

Proof. Let $f \in \mathcal{H}^{\otimes q}$ and $D_\delta := \{t \in T^q \mid \|t - d\| < \delta \text{ for all } d \in D\}$, where D is the diagonal of T^m. Since μ is a nonatomic measure, for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_{D_\delta} f^2 \, d\mu^q < \frac{\epsilon}{2}.$$

On $D^c_\delta = T^m \backslash D_\delta$, there also exist simple functions f_ϵ, such that

$$\int_{D^c_\delta} (f - f_\epsilon)^2 \, d\mu^q < \frac{\epsilon}{2}.$$

Now δ and f_ϵ can be chosen such that f_ϵ vanished on D_δ, so that it is an off-diagonal simple function, and so that the above two equations imply

$$\int_{T^q} \|f - f_\epsilon\|^2, d\mu < \epsilon.$$

Taking a sequence of ϵ approaching 0 gives the required approximating sequence $f_k = f_\epsilon$. □

Let $g \in \mathcal{H}^{\otimes q}$ be an off-diagonal simple function. Using the orthogonality property in Proposition 1.4.2 (iv) and the triangle inequality

$$\|I_q(g)\|^2_{L^2(\Omega)} = q!\, \|g\|^2_{\mathcal{H}^{\otimes q}} \leq q!\, \|g\|^2_{\mathcal{H}^{\otimes q}}.$$

Now let $f \in \mathcal{H}^{\otimes q}$. By Lemma 1.4.3, there exists a sequence of off-diagonal simple functions $(f_n)_{n\geq 0}$ that converges to f in $\mathcal{H}^{\otimes q}$. Using linearity and the above inequality, $\|I_q(f_n) - I_q(f_m)\| \leq q!\|f_n - f_m\| \to 0$ as $n, m \to \infty$. So $(I_q(f_n))_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$. Since this space is complete, the limit of the sequence exists.

Definition 1.4.4. Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. For all $f \in \mathcal{H}^{\otimes q}$, we define the **Wiener-Itô integral** of order q by

$$I_q(f) := \lim_{n\to\infty} I_q(f_n),$$

where $(f_n)_{n\geq 1}$ is a sequence of off-diagonal simple functions such that $\|f_n - f\|_{\mathcal{H}^{\otimes q}} \to 0$ as $n \to \infty$.

Using the denseness of the off-diagonal simple functions, the properties of Wiener Itô integrals listed in Proposition 1.4.2 can be extended to any $f \in \mathcal{H}^{\otimes q}$.

Proposition 1.4.5. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. The conclusions of Proposition 1.4.2 hold for all $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$.*

Next, we will show a product formula for Wiener-Itô integrals. In the theorems below, $f \otimes_r g$ is the rth contraction of $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$ defined by

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{T^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r)$$
$$g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r)\, d\mu(s_1) \ldots d\mu(s_r).$$

See Appendix B.5 for more details. The following lemma will be used to derive the product formula and the proofs are based on [Nua06].

Lemma 1.4.6. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. If $q \geq 1$ and $f \in \mathcal{H}^{\odot q}$, then*

$$I_p(f)I_1(g) = I_{p+1}(f \otimes g) + pI_{p-1}(f \otimes_1 g).$$

Proof. See [Nua06]. □

Theorem 1.4.7 (Product formula). *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. If $p, q \geq 1$, $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, then*

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g).$$

Proof. See [Nua06]. □

There are several alternative proofs of the product formula, all of which are quite involved. For example, see Proposition 6.4.1 in [PT11] which uses combinatorial properties of Wiener-Itô integrals or Theorem 2.7.10 in [NP12] which uses Malliavin calculus.

The product formula will often be used to compute the square of Wiener-Itô integrals in later chapters.

Corollary 1.4.8. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. If $q \geq 1$ and $f \in \mathcal{H}^{\odot q}$, then*

$$I_q(f)^2 = q!\, \|f\|^2_{\mathcal{H}^{\otimes q}} + \sum_{r=0}^{q-1} r! \binom{p}{r}^2 I_{2q-2r}(f \otimes_r f).$$

Proof. Since I_0 is the identity map for constants, $I_0(f \otimes_r f) = I_0\left(q!\, \|f\|^2_{\mathcal{H}^{\otimes q}}\right) = q!\, \|f\|^2_{\mathcal{H}^{\otimes q}}$. Then the result follows from Theorem 1.4.7. □

We end this section with a fundamental result that relates Wiener-Itô integrals to Hermite polynomials and shows that these integrals are an isomorphism from the qth symmetric tensor power of $\mathcal{H}$ onto the qth Wiener chaos. From Appendix B.4, recall that the inner product on $\mathcal{H}^{\otimes q}$ is $q!\, \langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes q}}$ and the qth tensor power in our setting is given by $f^{\otimes q}(t_1, \ldots, t_q) = f(t_1) \ldots f(t_q)$ (see Proposition B.4.10).

Theorem 1.4.9. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. Let W be an isonormal Gaussian process indexed by $\mathcal{H}$, then for all $q \geq 1$ and $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$, we have*

$$H_q(W(f)) = I_q(f^{\otimes q}), \tag{1.4.4}$$

where $f^{\otimes q}$ is the qth tensor power of f. Moreover, $I_q : \mathcal{H}^{\odot p} \to \mathcal{H}_q$ is an isomorphism.

Proof. Observe that for all $q \geq 1$,

$$(f^{\otimes q} \otimes_1 f)(t_1, \ldots, t_{q-1}) = \int_{T^2} f(t_1) \ldots f(t_{q-1}) f(s)^2 \, d\mu(s) = f^{\otimes q-1}(t_1, \ldots, t_{q-1}), \tag{1.4.5}$$

where $f \in \mathcal{H}$ such that $\|f\|^2_{\mathcal{H}}$.

We will prove (1.4.4) by induction. In the case when $q = 1$, $W(f) = I_1(f)$. Assume that (1.4.4) holds for q. Using Lemma 1.4.6, followed by (1.4.5) and the induction hypothesis, we have

$$\begin{aligned} I_{q+1}\left(f^{\otimes q+1}\right) &= I_q(f^{\otimes q}) I_1(f) - q I_{q-1}(f^{\otimes q} \otimes_1 f) \\ &= H_q(W(f)) W(f) - q H_{q-1}(W(f)) \\ &= H_{q+1}(W(f)), \end{aligned}$$

where we used Proposition 1.2.3 (iv) in the last equality. This completes the proof of (1.4.4).

To prove the second statement, it suffices to show that I_q is an isometry and onto. Proposition 1.4.2 (iv) and Proposition 1.4.5 gives us the isometry property $\mathrm{E}(I_p(f) I_q(g)) = \langle f, g \rangle_{\mathcal{H}^{\odot q}}$, for all $f, g \in \mathcal{H}^{\otimes q}$. It also implies that $I_q(g)$ is orthogonal to $I_p(f^{\otimes p}) = H_p(W(f))$, for all $p \neq q$, $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$ and $g \in \mathcal{H}^{\odot q}$. Since $\{H_p(W(f)) \mid f \in \mathcal{H}, \|f\|_{\mathcal{H}} = 1\}$ generates $\mathcal{H}_p$, $I_q(\mathcal{H}^{\odot p})$ is orthogonal to $\mathcal{H}_p$ for all $p \neq q$. Therefore, by Theorem 1.3.2, $I_q(\mathcal{H}^{\odot p}) \subseteq \mathcal{H}_q$.

Conversely, let $F \in \mathcal{H}_q$. Then F is the L^2 limit of linear combinations of functions of the form $H_q(W(f)) = I_q(f^{\otimes q})$, where $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$. However, by the isometry property, $\mathrm{E}(I_q(f)^2) = \|f\|^2_{\mathcal{H}^{\odot q}}$, so that $I_q(\mathcal{H}^{\odot p})$ is closed under taking linear combination and limits and $F \in I_q(\mathcal{H}^{\odot p})$. Thus, $\mathcal{H}_q \subseteq I_q(\mathcal{H}^{\odot p})$. □

1.5 Relation Between Wiener-Itô Integrals and the Wiener Chaos

Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis for the underlying Hilbert space $\mathcal{H}$. We will now find a orthonormal basis for the qth Wiener chaos.

Definition 1.5.1. An infinite sequence $a = (a_1, a_2, a_3, \dots)$, where $a_i \in \mathbb{N}$ and only a finite number of elements are nonzero, is called a **multi-index**. We use the notation $|a| := \sum_{i=1}^{\infty} a_i$ and $a! := \prod_{i=1}^{\infty} a_i!$. The set of all multi-indexes, a, such that $|a| = q$ will be denoted by $\mathcal{A}_q$, and the set of all multi-indexes will be denoted by $\mathcal{A}$.

Define the random variable

$$E_a := \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)), \tag{1.5.1}$$

where a is a multi-index. Note that this is actually a finite product as only finitely many terms are different from 1. Moreover, this definition does not depend on the choice of orthonormal basis in the sense that $(W(e_i))_{i\geq 1}$ will always be a sequence of independent and identically distributed standard Gaussian random variables.

The following proof that $(E_a)_{a\in\mathcal{A}_q}$ is a orthonormal basis of the qth Wiener chaos is partly inspired by related proofs in [Nua09, Maj81].

Lemma 1.5.2. *Let $\mathcal{P}_q$ be the closure in $L^2(\Omega)$ of the set*

$$\{p(W(e_1), \dots, W(e_n)) \,|\, p \text{ is a polynomial, } \deg p \leq q,\ (e_i)_{i=1}^n \subseteq \mathcal{H} \text{ orthonormal set}\}.$$

Then,

$$\mathcal{P}_q = \bigoplus_{i=0}^{q} \mathcal{H}_i.$$

Consequently, $E_a \in \mathcal{H}_q$ when $|a| = q$.

Proof. Choose $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = 1$, and let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ with $e_1 = f$. Let a, b be multi-indices with $|a| = q$ and $|b| \leq r$, where $r < q$, and define

$$F_b := \prod_{i=1}^{\infty} W(e_i)^{b_i}. \tag{1.5.2}$$

Since E_a and F_b are products of finitely many terms, we have

$$\mathrm{E}(E_a F_b) = \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} \mathrm{E}\left(H_{a_i}(W(e_i)) W(e_i)^{b_i}\right). \tag{1.5.3}$$

Now note that there exists at least one $i \geq 1$ such that $b_i < a_i$, and by Lemma 1.2.4, we can write $W(e_i)^{b_i}$ as a linear combination of Hermite polynomials with degree less than or equal to b_i and strictly less than a_i. Therefore, using Proposition 1.2.3 (v) in (1.5.4) gives

$$\mathrm{E}(E_a F_b) = 0. \tag{1.5.4}$$

We will now prove the first statement. It is obvious that $\bigoplus_{i=1}^{r} \mathcal{H}_i \subseteq \mathcal{P}_r$. In the particular case where a is such that $a_1 = q$ and $a_i = 0$ for all $i \neq 1$, (1.5.4) implies that for all $q > r$, F_b is orthogonal to $H_q(W(h))$, so it is also orthogonal to $\mathcal{H}_q$. By definition, the span of $\{F_b \,|\, |b| = r\}$ is dense in $\mathcal{P}_r$, so if $p \in \mathcal{P}_r$, then p is orthogonal to $H_q(W(h))$ for all $q > r$, thus $\mathcal{P}_r \subseteq \bigoplus_{i=1}^{r} \mathcal{H}_i$.

We now prove the second statement. In the particular case where $r = q-1$ and a is a multi-index with $|a| = q$, (1.5.4) implies that E_a is orthogonal to F_b. Using density again, we have that E_a is orthogonal to $\mathcal{P}_{q-1}$. But clearly, $E_a \in \mathcal{P}_q$. Then using (1.5.2), we have $E_a \in \mathcal{P}_q \backslash \mathcal{P}_{q-1} = \mathcal{H}_q$. □

Theorem 1.5.3. *The sequence $(E_a)_{a\in\mathcal{A}}$ is an orthonormal basis of $L^2(\Omega)$. For all $q \in \mathbb{N}$, $(E_a)_{a\in\mathcal{A}_q}$ is an orthonormal basis of $\mathcal{H}_q$.*

Proof. Let a, b be multi-indices. Using Proposition 1.2.3 (v), we have

$$\mathrm{E}(E_a E_b) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{a!b!}} \mathrm{E}\left(H_{a_i}(W(e_i)) H_{b_i}(W(e_i))\right) = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b. \end{cases}$$

This proves that the elements of $(E_a)_{a\in\mathcal{A}_q}$ are orthonormal. Now we need to show that the span of $(E_a)_{a\in\mathcal{A}}$ is dense in $L^2(\Omega)$. Again, by Proposition B.2.2, it suffices to show that if $\mathrm{E}(FE_a) = 0$ for all $a \in \mathcal{A}$, then $F = 0$ almost surely.

By taking a as the multi-index where $a_i = q$ and 0 elsewhere, the assumption implies that $\mathrm{E}(FH_q(W(e_i))) = 0$ for all $q \in \mathbb{N}$ and $i \geq 1$. From this point, we can argue similarly as the proof of Theorem 1.3.2 to show that $F = 0$. This completes the proof of the first statement.

Next, to show that $(E_a)_{a\in\mathcal{A}_q}$ is an orthonormal basis of $\mathcal{H}_q$, it only remains to show that $(E_a)_{a\in\mathcal{A}_q}$ is total. Fix $q \in \mathbb{N}$. Since $(E_a)_{a\in\mathcal{A}}$ is an orthonormal basis for $L^2(\Omega)$, we have in particular that all $F \in \mathcal{H}_q$ can be written as

$$F = \sum_{a\in\mathcal{A}} \mathrm{E}(FE_a)E_a = \sum_{a\in\mathcal{A}\backslash\mathcal{A}_q} \mathrm{E}(FE_a)E_a + \sum_{a\in\mathcal{A}_q} \mathrm{E}(FE_a)E_a.$$

By Lemma 1.5.2 and the orthogonality of the Wiener chaos, applying the projection onto J_q to the above equation yields

$$F = \sum_{a\in\mathcal{A}_q} \mathrm{E}(FE_a)E_a.$$

Therefore, Proposition B.2.2 says that $(E_a)_{a\in\mathcal{A}}$ is total in $\mathcal{H}_q$. □

We are now in a position to show that the elements of the Wiener chaos are Wiener-Itô integrals. We will use the notation and concepts summarized in Appendix B.4.

Let $\mathcal{H}^{\odot q}$ be the qth symmetric tensor product of $\mathcal{H}$ and a be a multi-index with $|a| = q$. Define the linear operator

$$\widetilde{I}_q : \mathcal{H}^{\odot q} \to L^2(\Omega),$$

by the map

$$\widetilde{I}_q\left(\frac{1}{\sqrt{a!}} \operatorname{sym}\left(e^{\otimes a}\right)\right) = \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

By Proposition B.4.7 and Theorem 1.5.3, $\widetilde{I}_q$ actually maps the orthonormal basis of $\mathcal{H}^{\odot q}$ onto the orthonormal basis of the qth Wiener chaos, $\mathcal{H}_q$. Therefore, I_q has a continuous linear extension that is a bounded operator and which is also a Hilbert space isomorphism from $\mathcal{H}^{\odot q}$ onto $\mathcal{H}_q$. This extension will continue to be denoted as $\widetilde{I}_q$ and is known as a **Wiener-Itô integral** or a multiple stochastic integral.

The Wiener-Itô integral introduced in Section 1.4 is a special case of $\widetilde{I}_q$. Note however, that $\widetilde{I}_q$ is a generalization of I_q as it allows us to define the Wiener-Itô integral for any real separable Hilbert space not necessarily an $L^2(T, \mathcal{B}, \mu)$. So after the proposition below, we will denote them both as I_q and the underlying Hilbert space will be assumed to be any real separable Hilbert space $\mathcal{H}$, unless otherwise stated.

Proposition 1.5.4. *Suppose that $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where (T, B, μ) is a measure space and μ is a σ-finite and nonatomic measure with orthonormal basis $(e_i)_{i\in\mathbb{N}}$. Then for all $q \in \mathbb{N}$, the Wiener Itô integral, I_q from Definition 1.4.4 coincides with $\widetilde{I}_q$.*

Proof. Firstly, we will compute the rth contraction of $e_i^{\otimes a_i}$ and $e_j^{\otimes a_j}$. For all $i \neq j$ and $r = 1, \ldots, a_i \wedge a_j$, using the orthogonality of e_i and e_j, we have

$$e_i^{\otimes a_i} \otimes_r e_j^{\otimes a_j} = e_i^{\otimes a_i - r} e_j^{\otimes a_j - r} \left(\int_T e_i(s) e_j(s)\, d\mu(s) \right)^r = 0.$$

But in the case when $r = 0$, the contraction reduces to the tensor product, $e_i^{\otimes a_i} \otimes e_j^{\otimes a_j}$. Therefore,

$$I_{a_i}\left(e_i^{\otimes a_i}\right) I_{a_j}\left(e_j^{\otimes a_j}\right) = I_{a_i + a_j}\left(\operatorname{sym}\left(e_i^{\otimes a_i} \otimes e_j^{\otimes a_j}\right)\right), \tag{1.5.5}$$

because all the summands in the product formula from Theorem 1.4.7 vanish, except when $r = 0$.

Using Theorem 1.4.9 and repeated applications of (1.5.5) gives

$$\begin{aligned} \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)) &= \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} I_{a_i}(e_i^{\otimes a_i}) \\ &= I_q\left(\frac{1}{\sqrt{a!}} \operatorname{sym}\left(e^{\otimes a}\right) \right), \end{aligned}$$

for all multi-indices a with $|a| = q$. Thus, we see that in the case where $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, $\widetilde{I}_q$ coincides with I_q on an orthonormal basis, and by boundedness, on all of $\mathcal{H}_q$. □

Most of the results about Wiener-Itô integrals in Section 1.4 are true for I_q in the general case where $\mathcal{H}$ is any real separable Hilbert space. Since, $\mathcal{H}$ is isomorphic to $L^2(T, \mathcal{B}, \mu)$, we can use an isometry argument to extend the product formula. These results are summarized the following proposition.

Proposition 1.5.5. *Let $\mathcal{H}$ be any real separable Hilbert space. The conclusions of Proposition 1.4.2 hold for all $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$. The conclusions of Theorem 1.4.7 hold for all $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, however the contraction, $f \otimes_r g$, is given by Definition B.5.1. The conclusions of Theorem 1.4.9 hold for all $f \in \mathcal{H}^{\odot q}$.*

Thus, we can give a simple criterion for the convergence of Wiener-Itô integrals, $I_q(f_n)$, in terms of the convergence of f_n.

Proposition 1.5.6. *Let $q \geq 1$ and $f, f_n \in \mathcal{H}^{\odot q}$ for all $n \geq 1$. If $\|f_n - f\|_{\mathcal{H}^{\otimes q}} \to 0$, then $I_q(f_n) \xrightarrow{L^2(\Omega)} I_q(f)$.*

Proof. This immediately follows from the isometry property $\|I_q(f_n) - I_q(f)\|^2_{L^2(\Omega)} = q! \|f_n - f\|^2_{\mathcal{H}^{\otimes q}}$. □

Now combining Theorem 1.5.4 with the Wiener chaos decomposition given in Theorem 1.3.2, we see that the elements of the Wiener chaos are Wiener-Itô integrals. Thus, all $F \in L^2(\Omega)$ have an expansion in Wiener-Itô integrals.

Theorem 1.5.7 (Wiener chaos expansion). *For any random variable $F \in L^2(\Omega)$, for all $q \in \mathbb{N}$, there exists $f_q \in \mathcal{H}^{\odot q}$, such that*

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{1.5.6}$$

where $I_0(f_0) = \mathrm{E}(X)$. In particular, we can choose $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where (T, B, μ) is a measure space and μ is a σ-finite and nonatomic measure, in which case f_q is a symmetric function in q variables that is square integrable with respect to μ.

In the next chapter, we will give an explicit formula that can be used to computer the Wiener-Itô expansion in terms of Malliavin derivatives.

Chapter 2

Malliavin Calculus

The origins of Malliavin calculus began in a paper by P. Malliavin [Mal76] in which he used probabilistic techniques to prove conditions for the smoothness of solutions to a stochastic differential equation, a result known as Hörmander's theorem. His method used derivatives of random variables and an integration by parts formula. Since then, the theory has been extensively developed, for example, see Malliavin [Mal97], Nualart [Nua06], or Bogachev [Bog10].

In brief, Malliavin calculus is an infinite-dimensional differential calculus with operators acting on functions of Gaussian processes. Since Malliavin calculus originated with its applications to stochastic differential equations, we will see that there are many analogues to the theory of partial differential equations on Sobolev spaces. Our interest in Malliavin calculus is to study limit theorems on the Wiener chaos. We will prove these limit theorems in Chapter 3, where the key insight is that Stein's method allows us to bound the distance between probability measures by use of differential operators, so that Malliavin calculus can then be used to explicitly compute these bounds.

In this chapter, we will introduce the three Malliavin operators, the derivative, divergence and Ornstein-Uhlenbeck operators. This chapter mostly follows [Nua06] and [NP12].

2.1 Malliavin Derivative

Recall that $W = (W(f))_{f \in \mathcal{H}}$ is an isonormal Gaussian process indexed by a real separable Hilbert space $\mathcal{H}$. In this section, we will define the derivative for a class of smooth random variables and outline some of its properties.

Let $C^\infty(\mathbb{R}^n)$ be the space of all infinitely differentiable functions of $\mathbb{R}^n$. Let $C_p^\infty(\mathbb{R}^n)$ denote the space of functions $g \in C^\infty(\mathbb{R}^n)$ where g and all its partial derivatives have at most polynomial growth. Similarly, if g and all its partial derivatives are bounded, it will be denoted as $C_b^\infty(\mathbb{R}^n)$.

The condition that $g \in C_p^\infty(\mathbb{R}^n)$ means that for all multi-indexes a, there exists constants $c, d_a > 0$ such that

$$\left| \frac{\partial^{|a|}}{\partial x^a} g(x) \right| < c(1 + |x|)^{d_a}$$

(see Definition 2.3.15 [Gra08]).

Definition 2.1.1. A random variable $F : \Omega \to \mathbb{R}^n$ is **smooth** if it has the form

$$F = g(W(f_1), \dots, W(f_n)),$$

where $g \in C_p^\infty(\mathbb{R}^n)$, and $f_1, \dots f_n \in \mathcal{H}$. The set of smooth random variables is denoted $\mathcal{S}$. The set of F where $g \in C_b^\infty(\mathbb{R}^n)$ instead is denoted by $\mathcal{S}_b$.

Lemma 2.1.2. *The spaces $\mathcal{S}$ and $\mathcal{S}_b$ are dense in $L^2(\Omega)$.*

Proof. The span of $\{H_q(W(f)) \,|\, q \in \mathbb{N}, f \in \mathcal{H}, \|f\|_{\mathcal{H}} = 1\}$ is a subset of $\mathcal{S}$, and dense in $L^2(\Omega)$ due to the Wiener chaos decomposition in Theorem 1.3.2. Hence, $\mathcal{S}$ is also dense in $L^2(\Omega)$.

The space of functions in $C^\infty(\mathbb{R}^n)$ with compact support, $C_0^\infty(\mathbb{R}^n)$, is a subset of $C_b^\infty(\mathbb{R}^n)$. But since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see Exercise 2.2.5 in [Gra08]), $C_b^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is also dense. Then the second statement follows from the fact that we can use Theorem 1.5.3 to estimate any $F \in L^2(\Omega)$ by polynomial functions of $W(f_1), \dots, W(f_n)$, where $f_1, \dots, f_n \in \mathcal{H}$. □

We will begin by defining the derivative only for $F \in \mathcal{S}$. This requires some basic concepts about Hilbert space valued functions as outlined in Appendix B.1. The space $L^2(\Omega \to \mathcal{H})$ is the set of $\mathcal{H}$-valued random variables, X, that are $\mathcal{F}$-measurable such that $\mathrm{E}(\|X\|_{\mathcal{H}}^2) < \infty$. The inner product on this space is $\langle X, Y \rangle_{L^2(\Omega \to \mathcal{H})} = E(\langle X, Y \rangle_{\mathcal{H}})$. We will see in Section 2.2 that in a particular case, a $\mathcal{H}$-valued random variable can be interpreted as a $\mathbb{R}$-valued stochastic process.

Definition 2.1.3. The operator $D : \mathcal{S} \to L^2(\Omega \to H)$ defined by

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(f_1), \dots, W(f_n)) f_i,$$

is known as the **Malliavin derivative**.

We often write expressions such as $\langle DF, f \rangle_{\mathcal{H}}$. Note that DF is not an element of $\mathcal{H}$, but $DF(\omega) \in \mathcal{H}$ for all $\omega \in \Omega$, so we should interpret $\langle DF, f \rangle_H : \Omega \to \mathbb{R}$ as a random variable such that $\omega \mapsto \langle DF(\omega), f \rangle_H$.

The polynomial growth condition in Definition 2.1.1 ensure that for all $F \in \mathcal{S}$ and $f \in \mathcal{H}$, all moments of F and $\langle DF, f \rangle_{\mathcal{H}}$ are finite.

Example 2.1.4. Continuing on from Example 1.1.4, have Brownian motion written as an isonormal Gaussian process $B_t := W(1_{[0,t]})$. Then $D \sin(B_t) = \cos(B_t) 1_{[0,t]}$.

Clearly, the Malliavin derivative is a linear operator, although it is unbounded. However, it is possible to extend the domain of the derivative by showing that it is a closable operator. It may also be helpful to review the definitions and properties about closed and closable operators as listed in Appendix A.2. First, we need the following lemma.

Lemma 2.1.5 (Integration by parts). *Let $F \in \mathcal{S}$ and $f \in \mathcal{H}$, then*

$$\mathrm{E}(\langle DF, f \rangle_{\mathcal{H}}) = \mathrm{E}(FW(f)).$$

Proof. We have $F = g(X)$, where $g \in \mathcal{S}$ and $X := (W(f_1), \dots, W(f_n))$ is a centered multivariate normal random variable with covariance matrix Σ. Using the spectral decomposition $\Sigma = UDU^T$, we can write $X = UD^{1/2}Z$ where Z is a vector of n independent and identically distributed standard normal random variables. Thus, $F = g(UD^{1/2}Z) = h(Z)$, where $h \in \mathcal{S}$. Since $h(Z) = h(W(e_1), \dots, W(e_n))$, for some orthogonal set $\{e_1, \dots, e_n\} \subseteq \mathcal{H}$, we can assume without loss of generality that $F = f(W(e_1), \dots, W(e_n))$, where $\{e_1, \dots, e_n\} \subseteq \mathcal{H}$ is an orthonormal set and $e_1 = f$.

Now $(W(e_1), \dots, W(e_n))$ is a n dimensional standard normal distribution, so we have

$$\begin{aligned}
\mathrm{E}(\langle DF, h\rangle_{\mathcal{H}}) &= \mathrm{E}\left(\frac{\partial f}{\partial x_1}(W(e_1), \dots, W(e_n))\right) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x_1, \dots, x_n) \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right) dx_1 \dots dx_n \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} x_1 f(x_1, \dots, x_n) \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right) dx_1 \dots dx_n \\
&= \mathrm{E}(FW(e_1)) \\
&= \mathrm{E}(FW(f)),
\end{aligned}$$

where the usual integration by parts formula has been used in the third line. □

Proposition 2.1.6 (Closability). *The Malliavin derivative $D : \mathcal{S} \subseteq L^p(\Omega) \to L^p(\Omega \to \mathcal{H})$ is a closable operator.*

Proof. It is clear from Definition 2.1.3 that the product rule for the partial derivative implies that the Malliavin derivative also satisfies the product rule $D(FG) = FDG + GDF$ for all $F, G \in \mathcal{S}$. Combining this Lemma 2.1.5, we get the formula

$$\mathrm{E}(G\langle DF, f\rangle_H) = -\mathrm{E}(F\langle DG, f\rangle_{\mathcal{H}}) + \mathrm{E}(W(f)FG), \tag{2.1.1}$$

for all $f \in \mathcal{H}$.

We will use Proposition A.2.7 (ii) to show that D is closable. Suppose that $(F_n)_{n\geq 1} \subseteq \mathcal{S}$ is a sequence that converges to 0 in $L^2(\Omega)$ and $(DF_n)_{n\geq 1}$ converges to some $G \in L^2(\Omega \to \mathcal{H})$. Let $H \in \mathcal{S}_b$ and $f \in \mathcal{H}$.

The convergence of $(DF_n)_{n\geq 1}$ in $L^2(\Omega \to \mathcal{H})$ means that $\mathrm{E}\left(\|DF_n - G\|_{\mathcal{H}}^2\right) \to 0$ as $n \to \infty$. Then using the Cauchy-Schwarz inequality, $\mathrm{E}\left(|\langle DF_n - G, f\rangle_{\mathcal{H}}|^2\right) \to 0$ as $n \to \infty$, for all $f \in \mathcal{H}$. So we have that $\langle DF_n, h\rangle_{\mathcal{H}} \to \langle G, f\rangle_{\mathcal{H}}$ and $H\langle DF_n, f\rangle_{\mathcal{H}} \to H\langle G, f\rangle_{\mathcal{H}}$ in $L^2(\Omega)$ as $n \to \infty$, since H is bounded. Then using Proposition A.3.2 yields

$$\begin{aligned}
\mathrm{E}(H\langle G, f\rangle_{\mathcal{H}}) &= \lim_{n\to\infty} \mathrm{E}(H\langle DF_n, f\rangle_{\mathcal{H}}) \\
&= \lim_{n\to\infty} -\mathrm{E}(F_n\langle DH, f\rangle_{\mathcal{H}}) + \mathrm{E}(W(f)F_nH) \\
&\leq \lim_{n\to\infty} \mathrm{E}(|F_n|^2)^{1/2}\,\mathrm{E}(|W(f)H - \langle DH, f\rangle_{\mathcal{H}}|^2)^{1/2}
\end{aligned}$$

using Equation 2.1.1 followed by Hölder's inequality. Now $W(f)H \in L^2(\Omega)$ since H is bounded and $\langle DH, f\rangle_{\mathcal{H}} \in L^2(\Omega)$ by, for example, the Cauchy-Schwarz inequality. Also, $F_n \to 0$ in $L^2(\Omega)$ by Proposition A.3.3, therefore $\mathrm{E}(H\langle G, f\rangle_{\mathcal{H}}) = 0$ for all $H \in \mathcal{S}_b$ and $f \in \mathcal{H}$. Hence, $\langle G, f\rangle_{\mathcal{H}} = 0$ almost surely, in particular for all $f = e_i$, where $(e_i)_{i\geq 1}$ is a orthonormal basis for $\mathcal{H}$. So $G = 0$ almost surely. □

Since D is a closable operator, we can construct its closed extension. Using Proposition A.2.5 (iii), the domain of the closed extension, denoted $\mathbb{D}^{1,2}$ will be the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|^2_{\mathbb{D}^{1,2}} = E(F^2) + E(\|DF\|^2_{\mathcal{H}}), \tag{2.1.2}$$

and for all $F_n \in \mathbb{D}^{1,2}$ converging to $F \in L^2(\Omega)$ with DF_n converging in to some $G \in L^2(\Omega \to \mathcal{H})$, we define $DF = G$. From now on, when we refer to the Malliavin derivative we mean the the closed extension of the Malliavin derivative, which we will continue to denote as D.

The Malliavin derivative satisfies a chain rule.

Proposition 2.1.7 (Chain rule). *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives and $F \in \mathbb{D}^{1,2}$, then $D(g(F)) = g'(F)DF$.*

Proof. We will give a sketch of this proof, as a complete justification requires various technical properties of approximate identities (see Section 1.2.4 in [Gra08]).

Let $F \in \mathcal{S}$, so we can write

$$F = f(W(f_1, \ldots, f_m)),$$

for some $f_1, \ldots, f_m \in \mathcal{H}$, where $f \in C_p^\infty(\mathbb{R}^m)$. Let $(\phi_\epsilon)_{\epsilon>0}$ be an approximate identity, then we can let $g_\epsilon := g * \phi_\epsilon$ where g_ϵ is infinitely differentiable with bounded partial derivatives and $g_\epsilon \xrightarrow{L^2} g$. Now since $g_\epsilon \circ f_n \in C_p^\infty(\mathbb{R}^m)$, using the usual chain rule we have

$$\begin{aligned} D(g_\epsilon(F)) &= \sum_{i=1}^m \frac{\partial(g_\epsilon \circ f)}{\partial x_i}(W(f_1), \ldots, W(f_m))f_i \\ &= g'_\epsilon(F)DF. \end{aligned}$$

Now since g' has bounded derivatives and $F \in \mathcal{S}$, we can apply the dominated convergence theorem see that in the $L^2(\Omega \to \mathcal{H})$ norm, $\|(g'_\epsilon(F) - g'(F))DF\|_{L^2(\Omega\to\mathcal{H})} \to 0$, as $\epsilon \to 0$. So we have that $g_\epsilon(F) \to g(F)$ in $L^2(\Omega)$ and $g'_\epsilon(F)DF \to g'(F)DF$ in $L^2(\Omega \to \mathcal{H})$ as $\epsilon \to 0$. This proves the chain rule for $F \in \mathcal{S}$.

To extend it to all $F \in \mathbb{D}^{1,2}$, we note since D is closed, there exists a sequence $(F_n)_{n\geq1} \subseteq \mathcal{S}$ such that $F_n \to F$ in $L^2(\Omega)$ and $DF_n \to DF$ in $L^2(\Omega \to \mathcal{H})$ as $n \to \infty$. The chain rule is satisfied for all $F_n \in \mathcal{S}$ and it can be shown that $g'(F_n)DF_n \to g'(F)DF$ in $L^2(\Omega \to \mathcal{H})$ as $n \to \infty$. Therefore, due to D being a closed operator, $D(g(F)) = g'(F)DF$. □

The next proposition shows that all Wiener-Itô integrals, $I_q(f)$ for $f \in \mathcal{H}^{\odot q}$, are in the domain of the Malliavin derivative $\mathbb{D}^{1,2}$, and we also provides a characterization for $\mathbb{D}^{1,2}$.

Proposition 2.1.8. *Let $q \in \mathbb{N}$ and $\mathcal{H}$ be a real separable Hilbert space. Suppose that $F \in L^2(\Omega)$ with Wiener-Itô expansion given by (1.5.6) and $G \in L^2(\Omega)$ has a similar expansion. Then:*

(i) For all $f \in \mathcal{H}^{\odot q}$, $I_q(f) \in \mathbb{D}^{1,2}$.

(ii) $F \in \mathbb{D}^{1,2}$ *if and only if*

$$\mathrm{E}(\|DF\|_{\mathcal{H}}^2) = \sum_{q=1}^{\infty} qq!\,\|f_q\|_{\mathcal{H}^{\otimes q}}^2 < \infty.$$

(iii) If $F, G \in \mathbb{D}^{1,2}$*, then*

$$\mathrm{E}\left(\langle DF, DG\rangle_{\mathcal{H}}\right) = \sum_{q=0}^{\infty} qq!\,\langle f, g\rangle_{\mathcal{H}^{\otimes q}}.$$

Proof. Since D is closed, Proposition A.2.5 (iii) says that $F \in \mathbb{D}^{1,2}$ is equivalent to (2.1.2) being finite. As F is already in $L^2(\Omega)$, this reduces to the condition that $\mathrm{E}\left(\|DF\|_{\mathcal{H}}^2\right) < \infty$.

Recall the definition of E_a from (1.5.1), where $(e_i)_{i\geq 1}$ is an orthonormal basis for $\mathcal{H}$. Since $E_a \in \mathcal{S}$, we can use Proposition 1.2.3 (iii) to compute the derivative

$$DE_a = \frac{1}{\sqrt{a!}} \sum_{j=1}^{\infty} \prod_{\substack{i=1\\ i\neq j}}^{\infty} H_{a_i}(W(e_i)) a_j H_{a_j-1}(W(e_j)) e_j.$$

Next, note that sums and products over the elements of a multi-index are finite and that $\mathrm{E}(h_1(W(e_i))h_2(W(e_j))) = \mathrm{E}(h_1(W(e_i)))\,\mathrm{E}(h_2(W(e_j)))$ because $W(e_i)$ and $W(e_j)$ are independent. Thus, taking the $L^2(\Omega \to \mathcal{H})$ norm gives

$$\begin{aligned}
\mathrm{E}\left(\|DE_a\|_{\mathcal{H}}^2\right) &= \frac{1}{a!} \sum_{j=1}^{\infty} a_j \prod_{\substack{i=1\\ i\neq j}}^{\infty} \mathrm{E}\left(H_{a_i}(W(e_i)^2)\right) \mathrm{E}\left(H_{a_j-1}(W(e_j)^2)\right) \\
&= \frac{1}{a!} \sum_{j=1}^{\infty} \prod_{\substack{i=1\\ i\neq j}}^{\infty} a_j a_i! (a_j - 1)! \\
&= |a|, \qquad (2.1.3)
\end{aligned}$$

where we have used Proposition 1.2.3 (v) at the second line.

Now let $\mathcal{A}_q^n := \{a \in \mathcal{A}_q \,|\, a_i = 0 \text{ for all } i > n\}$, $F := F_n$ and

$$F_n := \sum_{a\in\mathcal{A}_q^n} \mathrm{E}(I_q(f)E_a)E_a.$$

For $a \neq b$, the inner product $\mathrm{E}\left(\langle DE_a, DE_b\rangle_{\mathcal{H}}^2\right) = 0$, because we can use similar arguments as above to see that in each product we have at least one term $\mathrm{E}\left(H_{a_i}(W(e_i))H_{b_i}(W(e_i))\right) = 0$ since there is at least one $i \geq 1$ such that $a_i \neq b_i$. Therefore, DE_a is orthogonal to DE_b in $L^2(\Omega \to \mathcal{H})$ if $a \neq b$, so we have

$$\begin{aligned}
\mathrm{E}\left(\|DF_n\|\right)_{\mathcal{H}}^2 &= \sum_{a\in\mathcal{A}_q^n} \mathrm{E}(I_q(f)E_a)^2\,\mathrm{E}(\|DE_a\|_{\mathcal{H}}^2) \\
&= q \sum_{a\in\mathcal{A}_q^n} \mathrm{E}(I_q(f)E_a)^2, \qquad (2.1.4)
\end{aligned}$$

where (2.1.3) is used at the last equality.

Recall that $I_q(f) \in \mathcal{H}_q$ and $(E_a)_{a\in\mathcal{A}_q}$ is a orthonormal basis for $\mathcal{H}_q$ due to Proposition 1.5.3, so

$$\|I_q(f)\|_{L^2(\Omega)} = \sum_{a\in\mathcal{A}_q} \mathrm{E}(I_q(f)E_a)^2 < \infty. \tag{2.1.5}$$

Hence, by taking $n \to \infty$ in (2.1.4), and noting that $F_n \to F$ in $L^2(\Omega)$, the fact that D is closed implies that

$$DI_q(f) = \sum_{a\in\mathcal{A}_q} \mathrm{E}(I_q(f)E_a)DE_a. \tag{2.1.6}$$

and $\mathrm{E}\left(\|DI_q(f)\|^2_{\mathcal{H}}\right) = q\,\|I_q(f)\|_{L^2(\Omega)} < \infty$. Thus, $I_q(f) \in \mathbb{D}^{1,2}$ for all $f \in \mathcal{H}^{\odot q}$. Using the orthogonality property for Wiener-Itô integrals, we also have

$$\mathrm{E}\left(\|DI_q(f)\|^2_{\mathcal{H}}\right) = qq!\,\|f\|_{\mathcal{H}^{\otimes q}}. \tag{2.1.7}$$

Next, we prove (ii). Now let $F \in L^2(\Omega)$ with Wiener chaos expansion given by (1.5.6) and

$$F_n := \sum_{q=0}^{n} I_q(f_q).$$

Now since DE_a and DE_b are orthogonal for $a \neq b$, it is clear from (2.1.6) that $DI_p(f)$ and $DI_q(f)$ are orthogonal in $L^2(\Omega \to \mathcal{H})$ when $p \neq q$, so we have

$$\mathrm{E}\left(\|DF_n\|^2_{\mathcal{H}}\right) = \sum_{q=0}^{n} \mathrm{E}\left(\|DI_q(f_q)\|^2_{\mathcal{H}}\right).$$

Now take $n \to \infty$, by using 2.1.7, noting that $F_n \to F$ in $L^2(\Omega)$ and D is closed, we get

$$\mathrm{E}(\|DF\|^2_{\mathcal{H}}) = \sum_{q=1}^{\infty} qq!\,\|f_q\|^2_{\mathcal{H}^{\otimes q}}.$$

So $F \in \mathbb{D}^{1,2}$ if and only if the above expression is finite, which proves (ii).

Finally, we prove (iii). Define $G_n, G \in L^2(\Omega)$ in a similar way as F. Then using orthogonality again, we have

$$\mathrm{E}(\langle DF_n, GF_m\rangle_{\mathcal{H}}) = \sum_{q=0}^{n\wedge m} \mathrm{E}\left(\langle DI_q(f_q), DI_q(g_q)\rangle_{\mathcal{H}}\right).$$

Now take $n \to \infty$ and then $m \to \infty$. In (ii), we have shown that if $F \in \mathbb{D}^{1,2}$, then $DF_n \to DF$ in $L^2(\Omega \to \mathcal{H})$. Since $F, G \in \mathbb{D}^{1,2}$ by assumption and strong convergence implies weak convergence in $L^2(\Omega \to \mathcal{H})$, the left hand side converges $\mathrm{E}(\langle DF, GF\rangle_{\mathcal{H}})$. On the other hand, it can be shown using the very similar arguments as in (2.1.4) and (2.1.5) that $\mathrm{E}\left(\langle I_q(f), I_q(g)\rangle_{\mathcal{H}}\right) = qq!\,\langle f, g\rangle_{\mathcal{H}^{\otimes q}}$, and so (iii) is established. □

Also, it is clear from the above proof that the derivative of $I_q(f)$ is an element of $\mathcal{H}_{q-1} \otimes \mathcal{H}$. It is possible to write $DI_q(f) = qI_{q-1}(f)$ for all $\mathcal{H}^{\otimes q}$, which is done in, for example, [NP12]. But in order to make sense of this, we would need to properly define

$I_{q-1}(f)$. While we will not need to use this, further details can be found in [NP12]. In the next section, we will see that it is quite simple to understand the $DI_q(f)$ in the case where $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B}, \mu)$ is a measure space and μ is a σ-finite and nonatomic measure.

We finish this section with a brief discussion of higher order Malliavin derivatives, which will appear only in Section 2.2 in a small role. Recall that the Malliavin derivative is a linear operator $D : \mathbb{D}^{1,2} \subseteq L^2(\Omega) \to L^2(\Omega \to \mathcal{H})$. Since the domain and codomain are different, in order to define the iterated derivative, we note that $L^2(\Omega \to \mathcal{H})$ is isomorphic to $L^2(\Omega) \otimes \mathcal{H}$ by Proposition B.4.9 (ii), and define the derivative on $L^2(\Omega) \otimes \mathcal{U}$, where $\mathcal{U}$ is a real separable Hilbert space. Let

$$\mathcal{S}_{\mathcal{U}} := \left\{ F = \sum_{i=1}^{n} F_i \otimes u_i \,\middle|\, F_i \in \mathcal{S}, u_i \in \mathcal{U} \right\}.$$

and note that set of all F of this form is dense in the Hilbert space $L^2(\Omega) \otimes \mathcal{U}$ by Proposition . Then we can define the Malliavin derivative on $\mathcal{S}_{\mathcal{U}}$ by

$$DF := \sum_{i=1}^{n} DF_i \otimes u_i, \tag{2.1.8}$$

and higher order derivatives inductively

$$D^p F = \sum_{i=1}^{n} D^p F_i \otimes u_i$$

for all $p \geq 1$, where the derivative $D^p F_i$ is defined inductively using (2.1.8) by replacing $\mathcal{U}$ with $\mathcal{H}^{\otimes p-1}$ and F with $D^{p-1}F$. Using a similar proof as Proposition 2.1.6, it can be shown the derivative of order p has a closed extension

$$D^p : \mathbb{D}^{p,q}(\mathcal{U}) \subseteq L^q(\Omega) \otimes \mathcal{U} \to L^q(\Omega \to \mathcal{H}^{\otimes p}) \otimes \mathcal{U},$$

for all $p \geq 1, q \in [1, \infty)$, where $\mathbb{D}^{p,q}(\mathcal{U})$ the closure of $\mathcal{S}_{\mathcal{U}}$ with respect to the norm

$$\|F\|^q_{\mathbb{D}^{p,q}(\mathcal{U})} = \mathrm{E}\left(\|F\|^q_{\mathcal{U}}\right) + \sum_{i=1}^{p} \mathrm{E}\left(\|F\|^q_{\mathcal{H}^{\otimes i} \otimes \mathcal{U}}\right).$$

Note that $\mathbb{D}^{1,2}(\mathbb{R}) = \mathbb{D}^{1,2}$ and when the case when $\mathcal{U} = \mathbb{R}$, the derivative reduces to earlier definition, we write $\mathbb{D}^{p,q}$ instead of $\mathbb{D}^{p,q}(\mathcal{U})$.

2.2 Malliavin Derivative for Gaussian White Noise

In this section, we consider the Malliavin derivative in the white noise case, that is when the underlying Hilbert space is $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B}, \mu)$ is a measure space and μ is a σ-finite and nonatomic measure. We give explicit formulas for the derivative of Wiener-Itô integrals in this case and Stroock's formula for computing the Wiener-Itô decomposition.

Let $F \in \mathcal{D}^{1,2}$. By Proposition B.4.9, $DF \in L^2(\Omega \to H)$ is isomorphic to $L^2(\Omega \times T)$. Thus, Malliavin derivative maps the random variable F into the stochastic process

$$\begin{aligned} DF : \Omega \times T &\to \mathbb{R} \\ (\omega, t) &\mapsto D_t F(\omega). \end{aligned}$$

Let us define the notation $f_q^t(t_1, \ldots, t_{q-1}) := f_q(t_1, \ldots, t_{q-1}, t)$.

Proposition 2.2.1. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$. Let $F \in L^2(\Omega)$ with Wiener-Itô expansion given by (1.5.6). If $F \in \mathbb{D}^{1,2}$, then $DF = (D_t F)_{t \in T}$, where*

$$D_t F = \sum_{q=1}^{\infty} q I_{q-1}(f_q^t)$$

Proof. Let $f_q \in \mathcal{H}^{\otimes q}$ be an off-diagonal simple function and a symmetric function. We will compute the derivative of

$$I_q(f_q) = \sum_{i_1=1}^{n} \cdots \sum_{i_q=1}^{n} a_{i_1 \dots i_q} W(A_{i_1}) \dots W(A_{i_q}),$$

where $\{A_i\}_{i=1}^n$ is a partition of T and $a_{i_1 \dots i_q} = 0$ when $i_r = i_s$ for some $r \neq s$.

Note that by $I_q(f_q) \in \mathcal{D}^{1,2}$ by Proposition 2.1.8 (i), so we can apply the Malliavin derivative which gives

$$\begin{aligned} D_t F &= \sum_{j=1}^{q} \sum_{i_1, \dots, i_q=1}^{n} a_{i_1 \dots i_q} W(A_{i_1}) \dots 1_{A_{i_j}}(t) \dots W(A_{i_q}) \\ &= q I_{q-1}(f_q^t). \end{aligned}$$

Now the set of off-diagonal simple functions is dense $L^2(T, \mathcal{B}, \mu)$, so using similar arguments as Proposition 2.1.8, the result can be extended to all $F \in \mathbb{D}^{1,2}$. □

So the Malliavin derivative can be viewed as an inverse of integration in the sense that the derivative of $I_q(f)$ is a Wiener-Itô of order $q-1$. The integrand, f_q^t is treated a function of $q-1$ in the integral, so that we do not integrate with respect to the variables in the superscript.

Suppose that we can apply Malliavin derivative p times to F, that is $F \in \mathbb{D}^{p,2}$, then the pth Malliavin derivative of F would be the stochastic process

$$\begin{aligned} D^p F &: \Omega \times T^p \to \mathbb{R} \\ (\omega, t_1, \dots, t_p) &\mapsto D^p_{t_1, \dots, t_p} F(\omega). \end{aligned}$$

Let $\mathbb{D}^{\infty,2} := \bigcap_{p \geq 1} \mathbb{D}^{p,2}$. The following result was first proved by Stroock in [Str87] and can be used to compute the Wiener-Itô expansion of any random variable $F \in \mathbb{D}^{\infty,2}$. Note that in the formula below, $\mathrm{E}(D^q F)$ is a function of q variables and I_q is integrates with respect to these q variables.

Theorem 2.2.2 (**Stroock's formula**)**.** *Let $F \in \mathbb{D}^{\infty,2}$, then*

$$F = \mathrm{E}(F) + \sum_{q=1}^{\infty} I_q \left(\frac{1}{q!} \mathrm{E}(D^q F) \right).$$

Proof. Since $F \in \mathbb{D}^{\infty,2}$ it is also in $L^2(\Omega)$, so it has Wiener-Itô expansion given by (1.5.6). When only need to compute the functions f_q for $q \geq 1$. Now applying Proposition 3.3 q times we get

$$D^q_{t_1, \dots, t_q} F = \begin{cases} \displaystyle\sum_{p=q}^{\infty} \frac{p!}{(p-q)!} I_{p-q}(f_p^{t_1, \dots, t_q}) & \text{if } p \leq q \\ 0 & \text{if } p > q. \end{cases}$$

Now take expectations of both sides using by Proposition 1.4.2 (iii), we have $\mathrm{E}(I_q(f)) = 0$ for all $q \neq 0$. Thus,

$$\mathrm{E}(D^q_{t_1,\ldots,t_q} F) = q! I_0(f_q^{t_1,\ldots,t_q}) = q! f_q^{t_1,\ldots,t_q}$$

since I_0 is the identity map. □

Example 2.2.3. Continuing on from Example 1.1.4, we have Brownian motion written as an isonormal Gaussian process $B_t := W(1_{[0,t]})$. We will find the Wiener-Itô expansion of $F := B_1^3$. First, compute the derivatives

$$\begin{aligned} D_{t_1}(B_1^3) &= 3B_1^2 1_{[0,1]}(t_1) \\ D^2_{t_1,t_2}(B_1^3) &= 6B_1 1_{[0,1]^2}(t_1, t_2) \\ D^3_{t_1,t_2,t_3}(B_1^3) &= 6 1_{[0,1]^3}(t_1, t_2, t_3), \end{aligned}$$

and all higher derivatives are zero. Applying Theorem 2.2.2 and noting that $B_1 \sim \mathcal{N}(0, 1)$, we have

$$\begin{aligned} f_1(t_1) &= 3 1_{[0,1]}(t_1) \\ f_2(t_1, t_2) &= 0 \\ f_3(t_1, t_2, t_3) &= 1_{[0,1]^3}(t_1, t_2, t_3). \end{aligned}$$

Also, $\mathrm{E}(B_1^3) = 0$. Therefore, the Wiener-Itô expansion of F is

$$B_1^3 = I_1\left(3 1_{[0,1]}\right) + I_3\left(1_{[0,1]^3}\right).$$

2.3 Divergence Operator and Ornstein-Uhlenbeck Operator

Recall that $\mathcal{S}$ is dense in $L^2(\Omega)$ by Lemma 2.1.2. Thus, Malliavin operator $D : \mathbb{D}^{1,2} \subseteq L^2(\Omega) \to L^2(\Omega \to \mathcal{G})$ is densely defined and closed linear operator so that Proposition A.2.9 can be used to define its adjoint.

Definition 2.3.1. Let $\delta : \mathrm{dom}(\delta) \subseteq L^2(\Omega \to \mathcal{H}) \to L^2(\Omega)$ be defined for $u \in \mathrm{dom}(\delta)$ by the duality property

$$E(F\delta(u)) = E(\langle DF, u\rangle_{\mathcal{H}}), \tag{2.3.1}$$

for all $F \in \mathbb{D}^{1,2}$, where $\mathrm{dom}(\delta)$ is the set of $u \in L^2(\Omega \to \mathcal{H})$ such that there exists a constant $c(u)$ satisfying

$$|E(\langle DF, u\rangle_{\mathcal{H}})| \leq c(u) \|F\|_{L^2(\Omega)}, \tag{2.3.2}$$

for all $F \in \mathbb{D}^{1,2}$. We call δ the **divergence operator**.

Note that (2.3.1) is equivalent to $\langle F, \delta(u)\rangle_{L^2(\Omega)} = \langle DF, u\rangle_{L^2(\Omega\to\mathcal{H})}$, which is the standard definition of an adjoint, while (2.3.2) is equivalent to (A.2.1), which is the condition that is required for δ to be uniquely defined closed operator.

Suppose that $F \in L^2(\Omega)$ is written in its Wiener chaos expansion as given by (1.5.6), then we can we introduce the Ornstein-Uhlenbeck operator.

Definition 2.3.2. The linear operator $L : \mathrm{dom}(L) \to L^2(\Omega)$ with

$$\mathrm{dom}(L) = \left\{ F \in L^2(\Omega) \,\middle|\, \sum_{q=1}^{\infty} q^2 \left\| I_q(f_q) \right\|^2_{L^2(\Omega)} < \infty \right\}$$

and

$$LF = \sum_{q=1}^{\infty} -qI_q(f_q)$$

is called the **Ornstein-Uhlenbeck operator**.

Note that the domain of L is simply the set of F where the sum in the definition of L converges. The following result gives an alternative definition.

Proposition 2.3.3. *The Ornstein-Uhlenbeck operator is equivalent to the operator* L *where* $\mathrm{dom}(L) = \{F \in \mathbb{D}^{1,2} \,|\, DF \in \mathrm{dom}(\delta)\}$ *and* $LF = -\delta DF$.

Proof. We start by proving a useful formula. Let $F, G \in \mathbb{D}^{1,2}$, using the orthogonality property of Wiener-Itô integrals, we have

$$\begin{aligned} \mathrm{E}\left(G \sum_{q=1}^{\infty} qI_q(f_q) \right) &= \sum_{q=1}^{\infty} q\, \mathrm{E}(I_q(g_q)I_q(f_q)) \\ &= \sum_{q=1}^{\infty} qq!\, \langle f_q, g_q \rangle_{\mathcal{H}} \\ &= \mathrm{E}\left(\langle DF, DG \rangle_{\mathcal{H}} \right) & (2.3.3) \\ &= \mathrm{E}(G\delta(DF)), & (2.3.4) \end{aligned}$$

where at the second last equality, we used Proposition 2.1.8 (iii), and at the last equality we used the definition of δ, which only holds if $DF \in \mathrm{dom}(\delta)$.

Now, using Definition 2.3.2 and (2.3.3), we have $-\mathrm{E}(GLF) = \mathrm{E}\left(\langle DF, DG \rangle_{\mathcal{H}} \right)$, where $F \in \mathrm{dom}(L)$ and $G \in \mathbb{D}^{1,2}$. By Cauchy-Schwarz, $\left| \mathrm{E}\left(\langle DF, DG \rangle_{\mathcal{H}} \right) \right| \le \|G\|_{L^2(\Omega)} \|LF\|_{L^2(\Omega)}$, where both norms are finite as $F \in \mathrm{dom}(L)$ and $G \in \mathbb{D}^{1,2}$. Thus, (2.3.2) is satisfied and $DF \in \mathrm{dom}(\delta)$. Now we are allowed to use (2.3.4), which gives $\mathrm{E}(GLF) = -\mathrm{E}(G\delta(DF))$ for all $G \in \mathbb{D}^{1,2}$, so that $LF = -\delta DF$.

Conversely, assume that $\mathrm{dom}(L) = \{F \in \mathbb{D}^{1,2} \,|\, DF \in \mathrm{dom}(\delta)\}$ and $LF = -\delta DF$. Then using (2.3.4),

$$\mathrm{E}(GLF) = \mathrm{E}\left(G \sum_{q=0}^{\infty} -qI_q(f) \right),$$

for all $G \in \mathbb{D}^{1,2}$. Since both sides are finite due to $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{dom}(\delta)$, Definition 2.3.2 follows. □

Definition 2.3.4. Let $F \in L^2(\Omega)$. The **pseudo-inverse Ornstein-Uhlenbeck operator**, L^{-1}, is defined by

$$L^{-1}F = -\sum_{q=1}^{\infty} \frac{1}{q} I_q(F).$$

Proposition 2.3.5. *If $F \in L^2(\Omega)$, then $L^{-1}F \in \operatorname{dom}(L)$ and $LL^{-1}F = F - \mathrm{E}(F)$.*

Proof. Firstly, $L^{-1}F \in \operatorname{dom}(L)$ because $\sum_{q=1}^{\infty} \|I_q(f)\|_{L^2(\Omega)} < \infty$. From the definition of L and L^{-1} we have

$$LL^{-1}F = \sum_{q=1}^{\infty} I_q(f_q) = F - \mathrm{E}(F),$$

where the last equality follows from Theorem 1.5.7. □

The next formula is a very important result. In Chapter 3, it will act as the bridge between Malliavin calculus and Stein's method by enabling us to bound distance between probability laws in terms of Malliavin operators.

Lemma 2.3.6. *Let $F, G \in \mathbb{D}^{1,2}$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with bounded derivatives. Then*

$$\mathrm{E}(Ff(G)) = \mathrm{E}(F)\,\mathrm{E}(f(G)) + \mathrm{E}(f'(G)\left\langle DG, -DL^{-1}F\right\rangle_{\mathcal{H}}).$$

Proof. Using in order, Lemma 2.3.4, Proposition 2.3.3, (2.3.1), and the chain rule in Proposition 2.1.7, we have

$$\begin{aligned}
\mathrm{E}\left((F - \mathrm{E}(F))\, f(G)\right) &= \mathrm{E}\left(LL^{-1}Ff(G)\right) \\
&= \mathrm{E}\left(\delta\left(-DL^{-1}F\right) f(G)\right) \\
&= \mathrm{E}\left(\left\langle Df(G), -DL^{-1}F\right\rangle_H\right) \\
&= \mathrm{E}\left(f'(G)\left\langle DG, -DL^{-1}F\right\rangle_H\right).
\end{aligned}$$

□

Finally, we state an important inequality, which will be used prove the main theorem of Chapter 3.

Theorem 2.3.7. *Let $F \in \mathbb{H}_p$, the pth Wiener chaos, where $p \geq 1$. For all $1 < q < r$,*

$$\|F\|_{L^q(\Omega)} \leq \|F\|_{L^r(\Omega)} \leq \left(\frac{r-1}{q-1}\right)^{p/2} \|F\|_{L^q(\Omega)}.$$

A long, but elementary proof of this result can be found in Section 2.8 of [NP12], which follows Nelson's [Nel73] proof of the hypercontractivity of the Ornstein-Uhlenbeck semigroup, which generates the Ornstein-Uhlenbeck operator introduced above. Theorem 2.3.7 is a corollary. Alternatively, another proof uses the logarithmic Sobolev inequality proved by Gross [Gro75] that appears in the theory of partial differential equations.

This bound shows that on a fixed Wiener chaos, all $L^q(\Omega)$ norms, where $q > 1$ are equivalent. It also shows that elements of the Wiener chaos have finite moments of all order $q > 1$.

Chapter 3

Limit Theorems on the Wiener Chaos

Stein's lemma [Ste86] states that $N \sim \mathcal{N}(0,1)$ if and only if $\mathrm{E}(f'(N) - Nf(N)) = 0$ for all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{E}(f'(N)), \mathrm{E}(Nf(N))$ are finite. Perhaps this expectation could act as the distance between a random variable F and the normal distribution, such that if $\mathrm{E}(f'(F) - Nf(F))$ were approximately 0, then F was approximately normal. The computation of such an expectation could be achieved by applying Proposition 2.3.6, thereby bringing Malliavin calculus into the task of measuring the distance between random variables and deriving limit theorems.

The aim of this chapter is to make the above heuristic argument rigorous. By doing so, we will combine Stein's method and Malliavin calculus to prove the celebrated fourth moment theorem, which is arguably the most important result in this text.

The main references for this chapter are Nourdin and Peccati [NP09, NP12], and Nualart and Ortiz-Latorre [NOL08].

Unless otherwise stated, we assume that $\mathcal{H}$ is any real separable Hilbert space.

3.1 Stein's Method with Malliavin Calculus

Let μ_1 and μ_2 be probability measures. For the signed measure $\mu_1 - \mu_2$, on the Borel field $\mathcal{B}$, we can defined the total variation distance as

$$\|\mu_1 - \mu_2\| = \sup_{B \in \mathcal{B}} |\mu_1(B) - \mu_2(B)| .$$

This idea of introducing a distance between probability measures can be extended to random variables associated with the probability measures.

Definition 3.1.1. Let F and G be $\mathbb{R}^d$-valued random variables on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The **total variation distance** between the random variables F and G is defined as

$$d_{\mathrm{TV}}(F, G) = \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mathrm{P}(F \in B) - \mathrm{P}(G \in B)| ,$$

where $\mathcal{B}(\mathbb{R}^d)$ is the Borel field of $\mathbb{R}^d$.

It is immediate from this definition that the total variation distance is a metric which induces a topology that is stronger than convergence in distribution.

Proposition 3.1.2. *Let F_n, F and G be $\mathbb{R}^d$-valued random variables on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then*

(i) $d_{TV}(F, G)$ is a metric.

(ii) If $d_{\mathrm{TV}}(F_n, F) \to 0$ as $n \to \infty$ then $F_n \xrightarrow{d} F$ as $n \to \infty$.

Proof. (i) If F and G have the same law, then clearly $\mathrm{P}(F \in B) - \mathrm{P}(G \in B) = 0$ for all Borel sets B. If $d_{\mathrm{TV}}(F, G) = 0$, then $\mathrm{P}(F \in B) = \mathrm{P}(G \in B)$ for all Borel sets B, so that F and G have the same law. Thus, $d_{\mathrm{TV}}(F, G) = 0$ if and only if F and G have the same law. Clearly, $d_{\mathrm{TV}}(F, G)$ is symmetric. Using the triangle inequality for real numbers we have that

$$\begin{aligned}\sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mathrm{P}(F \in B) - \mathrm{P}(G \in B)| &\leq \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mathrm{P}(F \in B) - \mathrm{P}(H \in B)| + |\mathrm{P}(H \in B) - \mathrm{P}(G \in B)| \\ &\leq \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mathrm{P}(F \in B) - \mathrm{P}(H \in B)| \\ &\quad + \sup_{B \in \mathcal{B}(\mathbb{R}^d)} |\mathrm{P}(H \in B) - \mathrm{P}(G \in B)| .\end{aligned}$$

(ii) If $d_{\mathrm{TV}}(F_n, F) = 0$, then $\mathrm{P}(F_n \in B) \to \mathrm{P}(F \in B)$ for all Borel sets of the form $B = (-\infty, x_1] \times \cdots \times (-\infty, x_d]$, which implies that $F_n \xrightarrow{d} F$. □

Note that the converse of Proposition 3.2 (ii) is not true in general. For example, taking $F_n = 1/n$, we have that F_n converges in distribution to $F = 0$, but $d_{\mathrm{TV}}(F_n, F) = 1$.

Let $N \sim \mathcal{N}(0, 1)$ and let $h : \mathbb{R} \to \mathbb{R}$ with $\mathrm{E}(|h(N)|) < \infty$. The differential equation

$$f'(x) - xf(x) - h(x) + \mathrm{E}(h(N)) = 0 \tag{3.1.1}$$

is known as **Stein's equation**.

Lemma 3.1.3. *Let $h : \mathbb{R} \to [0, 1]$ be a continuous function. Then the solution to Stein's equation is*

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} (h(y) - E(h(N)))e^{-y^2/2}\, dy. \tag{3.1.2}$$

Moreover, $f \in C^1(\mathbb{R})$, the class of continuously differentiable functions,

$$\|f\|_\infty \leq \sqrt{\pi/2} \text{ and } \|f'\|_\infty \leq 2. \tag{3.1.3}$$

Proof. By seeing that (3.1.1) can be written as

$$e^{x^2/2} \frac{\partial}{\partial x} \left(e^{-x^2/2} f(x) \right) - h(x) + \mathrm{E}(h(N)) = 0,$$

it is clear that (3.1.2) is a solution.

Now $\mathrm{E}\left(h(N) - \mathrm{E}(h(N))\right) = 0$ can be written as

$$\int_{-\infty}^{x} (h(y) - \mathrm{E}(h(N)))e^{-y^2/2}\, dy + \int_{x}^{\infty} (h(y) - \mathrm{E}(h(N)))e^{-y^2/2}\, dy = 0,$$

so that

$$f(x) = -e^{x^2/2} \int_x^\infty (h(y) - \mathrm{E}(h(N)))e^{-y^2/2}\, dy. \tag{3.1.4}$$

Since $h(x) \in [0,1]$ for all $x \in \mathbb{R}$, $\mathrm{E}(h(N)) \in [0,1]$, which implies that $|h(x) - \mathrm{E}(h(N))| \leq 1$. Now combining this with (3.1.2) and (3.1.4), we have

$$\begin{aligned} |f(x)| &\leq e^{x^2/2} \min\left(\int_{-\infty}^x e^{-y^2/2}\, dy, \int_x^\infty e^{-y^2/2}\, dy\right) \\ &= e^{x^2/2} \int_{|x|}^\infty e^{-y^2/2}\, dy. \end{aligned}$$

Let us define the last expression as $g(x)$. For $x > 0$,

$$g'(x) = xe^{x^2/2} \int_x^\infty e^{-y^2/2}\, dy - 1 \leq e^{x^2/2} \int_x^\infty ye^{-y^2/2}\, dy - 1 = 0$$

So g is decreasing for $x > 0$ and by symmetry, it is increasing for $x < 0$. Thus, the maximum of g is $g(0) = \sqrt{\pi/2}$ and $\|f\|_\infty \leq \sqrt{\pi/2}$.

Using (3.1.1) and (3.1.2), we have that

$$f'(x) = h(x) - \mathrm{E}(h(N)) + xe^{x^2/2} \int_{-\infty}^x h(y) - \mathrm{E}(h(N))e^{-y^2/2}\, dy. \tag{3.1.5}$$

Then using similar arguments as above,

$$|f'(x)| \leq 1 + |x|\, e^{x^2/2} \int_{|x|}^\infty e^{-y^2/2}\, dy \leq 1 + e^{x^2/2} \int_{|x|}^\infty ye^{-y^2/2}\, dy = 2.$$

Finally, (3.1.5) is continuous because h is continuous, so f is continuously differentiable. □

While we will work exclusively with the total variation distance, it is possible to use other distance functions, such as the Kolmogorov distance. Using different distance functions will require making different assumptions on h, which will change the properties for the solution of Stein's equation.

Using the above lemma we can bound the total variation distance from a normal random variable in terms of Malliavin operators.

Proposition 3.1.4. *If $F \in \mathbb{D}^{1,2}$ with $\mathrm{E}(F) = 0$ and $N \sim \mathcal{N}(0,1)$, then*

$$d_{\mathrm{TV}}(F, N) \leq 2\,\mathrm{E}\left(\left|1 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}}\right|\right). \tag{3.1.6}$$

Proof. Let B be a Borel set in $\mathbb{R}$. Using Lemma A.1.1, taking $\mu = \mathrm{law}(N) + \mathrm{law}(F)$, there exists a sequence of continuous function $(g_n)_{n\geq 1}$ with $g_n(x) \in [0,1]$ which converges to 1_B almost everywhere. So by the dominated convergence theorem, $E(g_n(F)) \to P(F \in B)$ and $E(g_n(N)) \to P(N \in B)$.

Now let f_n be the solution to Stein's equation with h_n. Then using Stein's equation, followed by (3.1.3), we have

$$\begin{aligned} |\mathrm{E}(g_n(F)) - E(g_n(N))| &= |E(f_n'(F)) - E(Ff_n(N))| \\ &= \left|\mathrm{E}\left(f_n'(F)\left(1 - \langle DF, -DL^{-1}F\rangle\right)\right)\right| \\ &\leq \mathrm{E}\left(2\left|1 - \langle DF, -DL^{-1}F\rangle_{\mathcal{H}}\right|\right), \end{aligned}$$

where the last inequality is due to (3.1.3). Now letting $n \to \infty$ and then taking the supremum over all Borel sets B gives the desired result. □

3.2 Fourth Moment Theorem

We know specialize to the case where F in an element of the Wiener chaos and prove the fourth moment theorem. This proof follows Nourdin and Peccati [NP09, NP12].

Lemma 3.2.1. *Let σ be a permutation of $\{1,\dots,2q\}$ and $\mathcal{S}_{2q}$ be the set of all such permutations, and let the cardinality of $\{1,\dots,q\}\cap\{\sigma(1),\dots,\sigma(q)\}$ be denoted by $r\in\{0,\dots,q\}$. Then:*

(i) There are $\binom{q}{r}^2 q!^2$ permutations $\sigma\in\mathcal{S}_{2q}$ such that the cardinality of $\{1,\dots,q\}\cap\{\sigma(1),\dots,\sigma(q)\}$ is r.

(ii) We have that $\{1,\dots,q\}\setminus\{\sigma(1),\dots,\sigma(q)\}=\{\sigma(q+1),\dots,\sigma(2q)\}\setminus\{q+1,\dots,2q\}$ and $\{\sigma(1),\dots,\sigma(q)\}\setminus\{1,\dots,q\}=\{q+1,\dots,2q\}\setminus\{\sigma(q+1),\dots,\sigma(2q)\}$.

(iii) The cardinality of $\{q+1,\dots,2q\}\cap\{\sigma(q+1),\dots,\sigma(2q)\}$ is also r.

Proof. First, we prove (i). Consider a permutation $\sigma\in\mathcal{S}_{2q}$ formed by the following procedure. Firstly, choose r distinct elements $x_1,\dots,x_r$ of $\{1,\dots,q\}$, which can be done in $\binom{q}{r}$ ways. Secondly, choose $q-r$ distinct elements of $x_{r+1},\dots,x_q$ of $\{q+1,\dots,2q\}$, which can be done in $\binom{q}{r}$ ways. Thirdly, choose a bijection from $\{1,\dots,q\}$ to $\{x_1,\dots,x_q\}$, which can be done in $q!$ ways. Lastly, choose a bijection from $\{q+1,\dots,2q\}$ to $\{1,\dots,2q\}\setminus\{x_1,\dots,x_q\}$, which can be done in $q!$ ways. So there are $\binom{q}{r}^2 q!^2$ ways to form a permutation using this procedure and the first three steps ensures that the cardinality of $\{1,\dots,q\}\cap\{\sigma(1),\dots,\sigma(q)\}$ is r.

Next, we prove (ii). If $x\in\{1,\dots,q\}\setminus\{\sigma(1),\dots,\sigma(q)\}$, then $x\notin\{\sigma(1),\dots,\sigma(q)\}$, so $x\in\{\sigma(q+1),\dots,\sigma(2q)\}$. Also, $x\in\{1,\dots,q\}$, so $x\notin\{q+1,\dots,2q\}$. Conversely, if $x\in\{\sigma(q+1),\dots,\sigma(2q)\}\setminus\{q+1,\dots,2q\}$, then $x\notin\{q+1,\dots,2q\}$, so $x\in\{q+1,\dots,2q\}$. Also $x\in\{\sigma(q+1),\dots,\sigma(2q)\}$, so $x\notin\{\sigma(q+1),\dots,\sigma(2q)\}$. This proves the first equality. After swapping $1,\dots,q$ with $q+1\dots,2p$, the same argument can be used to prove the second equality.

Finally, we prove (iii). Since $|\{1,\dots,q\}\cap\{\sigma(1),\dots,\sigma(q)\}|=r$, we have $|\{1,\dots,q\}\setminus\{\sigma(1),\dots,\sigma(q)\}|=q-r$. By (ii), $|\{\sigma(q+1),\dots,\sigma(2q)\}\setminus\{q+1,\dots,2q\}|=q-r$, so $|\{q+1,\dots,2q\}\cap\{\sigma(q+1),\dots,\sigma(2q)\}|=r$. □

Lemma 3.2.2. *Let $f\in\mathcal{H}^{\odot q}$ and $F=I_q(f)$ be a Wiener-Itô integral of order $q\geq 2$. Then we have:*

$$\frac{1}{q}\left\|DF\right\|_{\mathcal{H}}^2 = q!\left\|f\right\|_{\mathcal{H}^{\otimes q}} + q\sum_{r=1}^{q-1}(r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}(f\widetilde{\otimes}_r f) \tag{3.2.1}$$

$$\operatorname{Var}\left(\frac{1}{q}\left\|DF\right\|_{\mathcal{H}}^2\right) = \frac{1}{q^2}\sum_{r=1}^{q-1} r^2 r!^2\binom{q}{r}^4(2q-2r)!\left\|f\widetilde{\otimes}_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}}^2 \tag{3.2.2}$$

$$\mathrm{E}(F^4)-3\,\mathrm{E}(F^2)^2 = \frac{3}{q}\sum_{r=1}^{q-1} r r!^2\binom{q}{r}^4(2q-2r)!\left\|f\widetilde{\otimes}_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}}^2 \tag{3.2.3}$$

$$= \sum_{r=1}^{q-1} q^2\binom{q}{r}^2\left(\left\|f\otimes_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}}^2 + \binom{2q-2r}{q-r}\left\|f\widetilde{\otimes}_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}}^2\right). \tag{3.2.4}$$

Proof. Without loss of generality, we can work on the Hilbert space $L^2(T, \mathcal{B}, \mu)$. Recall that by Proposition , $DF = qI_{q-1}(\bar{f}_t)$, where $\bar{f}_t(t_1, \dots, t_{q-1}) := f(t_1, \dots, t_{q-1}, t)$. Using the product formula in Corollary 1.4.8, the stochastic Fubini theorem [PT10] and the formula of the contraction given in (B.5.1), we have

$$\begin{aligned}
\frac{1}{q}\|DF\|_{\mathcal{H}}^2 &= q\int_T I_{q-1}(\bar{f}_t)^2\, d\mu(t) \\
&= q\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2q-2-2r}\left(\int_T \bar{f}_t \otimes_r \bar{f}_t\, d\mu(t)\right) \\
&= q\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2q-2-2r}\left(f \otimes_{r+1} f\right) \\
&= q\sum_{r=1}^{q} (r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}\left(f\widetilde{\otimes}_r f\right).
\end{aligned}$$

Now since I_0 is the identity map for constants, the summand when $r = q$ is $(q-1)!I_0(f \otimes_q f) = (q-1)!\,\|f\|_{\mathcal{H}^{\otimes q}}$, so (3.2.1) follows.

Now using the orthogonality of Wiener-Itô integrals and (3.2.1), we have

$$\begin{aligned}
\operatorname{Var}\left(\frac{1}{q}\|DF\|_{\mathcal{H}}^2\right) &= q^2\sum_{r=1}^{q} (r-1)!^2\binom{q-1}{r-1}^4 (2q-2r)!\,\left\|f\widetilde{\otimes}_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}} \\
&= \frac{1}{q^2}\sum_{r=1}^{q-1} r^2 r!^2\binom{q}{r}^4 (2q-2r)!\,\left\|f\widetilde{\otimes}_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}}^2
\end{aligned}$$

.

Next, we prove (3.2.4). From Definition 2.3.2 and Proposition 2.3.3, $F = -LF/q = \delta(DF)/q$. Together with (2.3.1), the fourth moment can be expressed as

$$\mathrm{E}(F^4) = \frac{1}{q}\mathrm{E}(F^3\delta(DF)) = \frac{1}{q}\mathrm{E}\left(\langle D(F^3), DF\rangle_{\mathcal{H}}\right) = \frac{3}{q}\mathrm{E}\left(F^2\|DF\|_{\mathcal{H}}^2\right),$$

where the last equality follows from $D(F^3) = 3F^2DF$. This can be obtained by using the product formula followed by Proposition . Alternatively, it follows from an extended chain rule for Lipschitz functions (see Proposition 1.2.4 in [Nua06]), since the version in Proposition 2.1.7 requires the function to have bounded partial derivatives. The random variables F^2 and $\|DF\|_{\mathcal{H}}^2$ can be expressed using the product formula and 3.2.1, then using the orthogonality of Wiener-Itô integrals gives

$$\begin{aligned}
\mathrm{E}(F^4) &= \frac{3}{q}\left(q\left(q!\,\|f\|_{\mathcal{H}^{\otimes n}}\right)^2 + \sum_{r=1}^{q-1} r!\binom{q}{r}^2 q^2(r-1)!\binom{q-1}{r-1}^2 \mathrm{E}\left(I_{2q-2r}(f\widetilde{\otimes}_r f)^2\right)\right) \\
&= 3\,\mathrm{E}(F^2)^2 + \frac{3}{q}\sum_{r=1}^{q-1} rr!\binom{q}{r}^4 (2q-2r)!\,\left\|f\widetilde{\otimes}_r f\right\|_{\mathcal{H}^{\otimes 2q-2r}}.
\end{aligned}$$

Finally, we will prove (3.2.3). Let $\sigma \in \mathcal{S}_{2q}$ and consider the operator

$$L_\sigma(f) := \int_{T^{2q}} f(t_1, \dots, t_q)f(t_{\sigma(1)}, \dots, t_{\sigma(q)})f(t_{q+1}, \dots, t_{2q})f(t_{\sigma(q+1)}, \dots, t_{\sigma(2q)})\, d\mu(t_1)\dots d\mu(t_{2q}).$$

Recalling that f is a symmetric function and using Lemma 3.2.1 (iii), we can assume without loss of generality that the last r arguments of $f(t_1,\dots,t_q)$ and $f(t_{\sigma(1)},\dots,t_{\sigma(q)})$ are equal, and the same holds for $f(t_{q+1},\dots,t_{2q})$ and $f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})$. From Lemma 3.2.1 (ii), we can also assume that the first $q-r$ arguments of $f(t_1,\dots,t_q)$ and $f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})$ are equal, and the same holds for $f(t_{\sigma(1)},\dots,t_{\sigma(q)})$ and $f(t_{q+1},\dots,t_{2q})$. Therefore,

$$\begin{aligned} L_\sigma(f) = &\int_{T^{2q-2r}} \left(\int_{T^r} f(t_1,\dots,t_q) f(t_{\sigma(1)},\dots,t_{\sigma(q)})\, d\mu(t_{q-r+1})\dots d\mu(t_r)\right) \\ &\left(\int_{T^r} f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)}) f(t_{q+1},\dots,t_{2q})\, d\mu(t_{2q-r+1})\dots d\mu(t_{2q})\right) \\ &d\mu(t_1)\dots d\mu(t_{q-r}) d\mu(t_{q+1})\dots d\mu,(t_{2q-1}) \\ = &\int_{T^{2q-2r}} (f\otimes_r f)(u_1,\dots,u_{2q-2r})^2\, d\mu(u_1)\dots d\mu(u_{2q-2r}) \\ = &\|f\otimes_r f\|^2_{\mathcal{H}^{\otimes 2q-2r}}\,. \end{aligned} \tag{3.2.5}$$

The symmetrization of $f\otimes f$ is

$$(f\widetilde{\otimes} f)(t_1,\dots,t_{2q}) = \frac{1}{(2q)!}\sum_{\sigma\in\mathcal{S}_{2q}} f(t_{\sigma(1)},\dots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)}).$$

Then,

$$\begin{aligned} \left\|f\widetilde{\otimes} f\right\|^2_{\mathcal{H}^{\otimes 2q}} = &\frac{1}{(2q)!^2}\sum_{\sigma,\sigma'\in\mathcal{S}_{2q}}\int_{T^{2q}} f(t_{\sigma(1)},\dots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)}) \\ &f(t_{\sigma'(1)},\dots,t_{\sigma'(q)}) f(t_{\sigma'(q+1)},\dots,t_{\sigma'(2q)})\, d\mu(t_1)\dots d\mu(t_{2q}) \\ = &\frac{1}{(2q)!}\sum_{\sigma\in\mathcal{S}_{2q}}\int_{T^{2q}} f(t_1,\dots,t_q) f(t_{q+1},\dots,t_{2q}) \\ &f(t_{\sigma(1)},\dots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})\, d\mu(t_1)\dots d\mu(t_{2q}) \\ = &\frac{1}{(2q)!}\sum_{r=0}^{q}\sum_{\substack{\sigma\in\mathcal{S}_{2q}\\ |\{1,\dots,q\}\cap\{\sigma(1),\dots,\sigma(q)\}|=r}}\int_{T^{2q}} f(t_1,\dots,t_q) f(t_{q+1},\dots,t_{2q}) \\ &f(t_{\sigma(1)},\dots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})\, d\mu(t_1)\dots d\mu(t_{2q}). \end{aligned}$$

The second equality above follows from the fact that there are $(2q)!$ ways to permute the dummy variables $t_1,\dots,t_{2q}$ without changing the value of the integral. Combining this with (3.2.5) and Lemma (3.2.1) (i) gives

$$(2q)!\left\|f\widetilde{\otimes} f\right\|^2_{\mathcal{H}^{\otimes 2q}} = 2q!^2\,\|f\|^4_{\mathcal{H}^{\otimes q}} + q!^2\sum_{r=1}^{q-1}\binom{q}{r}^2\|f\otimes_r f\|^2_{\mathcal{H}^{\otimes 2q-2r}}\,, \tag{3.2.6}$$

since $\|f\otimes_0 f\|^2_{\mathcal{H}^{\otimes 2q}} = \|f\|^4_{\mathcal{H}^{\otimes q}} = \|f\otimes_q f\|^2_{\mathcal{H}^{\otimes 0}}$. We can evaluate the fourth moment using the Corollary 1.4.8 and the orthogonality property of Wiener-Itô integrals, which gives

$$\mathrm{E}(F^4) = (2q)!\left\|f\widetilde{\otimes} f\right\|^2_{\mathcal{H}^{\otimes 2q}} + q!^2\,\|f\|^4_{\mathcal{H}^{\otimes q}} + \sum_{r=1}^{q-1} r!^2\binom{q}{r}^4(2q-2r)!\left\|f\widetilde{\otimes}_r f\right\|^2_{\mathcal{H}^{\otimes 2q-2r}}\,.$$

Substituting (3.2.6) into this equation and noting that $\mathrm{E}(F^2)^2 = q!^2 \|f\|^4_{\mathcal{H}^{\otimes q}}$, we deduce that

$$\begin{aligned}\mathrm{E}(F^4) &= 3\,\mathrm{E}(F^2) + \sum_{r=1}^{q-1}\left(q!^2\binom{q}{r}^2 \|f\otimes_r f\|^2_{\mathcal{H}^{\otimes 2q-2r}} + r!^2\binom{q}{r}^4 (2q-2r)!\,\left\|f\widetilde{\otimes}_r f\right\|^2_{\mathcal{H}^{\otimes 2q-2r}}\right)\\ &= 3\,\mathrm{E}(F^2) + \sum_{r=1}^{q-1} q^2\binom{q}{r}^2\left(\|f\otimes_r f\|^2_{\mathcal{H}^{\otimes 2q-2r}} + \binom{2q-2r}{q-r}\left\|f\widetilde{\otimes}_r f\right\|^2_{\mathcal{H}^{\otimes 2q-2r}}\right).\end{aligned}$$

□

Corollary 3.2.3. *Under the assumptions of Lemma 3.2.2,*

$$\operatorname{Var}\left(\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right) \le \frac{q-1}{3q}\left(\mathrm{E}(F^4) - 3\,\mathrm{E}(F^2)^2\right).$$

Proof. Using (3.2.2),

$$\operatorname{Var}\left(\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right) \le \frac{q-1}{q^2}\sum_{r=1}^{q-1} r r!^2\binom{q}{r}^4 (2q-2r)!\,\left\|f\widetilde{\otimes}_r f\right\|^2_{\mathcal{H}^{\otimes 2q-2r}}.$$

Then the result immediately follows from (3.2.3). □

We can now show a total variation distance bound for elements of the Wiener chaos.

Proposition 3.2.4 (Nourdin and Peccati). *Let $f \in \mathcal{H}^{\odot q}$ and $F = I_q(f)$ be a Wiener-Itô integral of order $q \ge 2$, $E(F^2) = 1$ and $N \sim \mathcal{N}(0,1)$. Then we have*

$$d_{TV}(F,N) \le 2\sqrt{\operatorname{Var}\left(\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right)} \le 2\sqrt{\frac{q-1}{3q}\left(\mathrm{E}(F^4)-3\right)}.$$

Proof. Since $L^{-1}F = -F/q$ by Definition 2.3.4, we have $\langle DF, -DL^{-1}F\rangle_{\mathcal{H}} = \|DF\|^2_{\mathcal{H}}/q$ and Proposition 3.1.4 implies

$$d_{\mathrm{TV}}(F,N) \le 2\,\mathrm{E}\left(\left|1 - \frac{1}{q}\|DF\|^2_{\mathcal{H}}\right|\right). \tag{3.2.7}$$

Now, using Jensen's inequality

$$\begin{aligned}\left(\mathrm{E}\left(\left|1-\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right|\right)\right)^2 &\le \mathrm{E}\left(\left(1-\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right)^2\right)\\ &= \operatorname{Var}\left(1-\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right) + \mathrm{E}\left(1-\frac{1}{q}\|DF\|^2_{\mathcal{H}}\right)^2.\end{aligned}$$

However, $\mathrm{E}\left(1 - \|DF\|^2_{\mathcal{H}}/q\right) = 0$ due to (3.2.1), so combining this with and Corollary 3.2.3 gives the required bound. □

Theorem 3.2.5 (Fourth moment theorem). *Let $(F_n)_{n\ge 1}$ be a sequence of Wiener-Itô integrals, where $F_n = I_q(f_n)$ and $f \in \mathcal{H}^{\odot q}$, such that $\mathrm{E}(F_n^2) \to \sigma^2$ as $n \to \infty$. Then, as $n \to \infty$, the following are equivalent:*

(i) $F_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.

(ii) $\mathrm{E}(F_n^4) \to 3\sigma^2$.

(iii) $\mathrm{Var}(\|DF_n\|_{\mathcal{H}}^2) \to 0$.

(iv) *For all* $1 \leq r \leq q-1$*, we have* $\left\|f_n \widetilde{\otimes}_r f_n\right\|_{\mathcal{H}^{\otimes 2q-2r}} = 0$.

(v) *For all* $1 \leq r \leq q-1$*, we have* $\|f_n \otimes_r f_n\|_{\mathcal{H}^{\otimes 2q-2r}} = 0$.

Proof. We can assume without loss of generality that $\sigma^2 = 1$.

First, we prove (i) $\Rightarrow$ (ii). The assumption $\mathrm{E}(F_n^2) \to 1$ and the equivalence of $L^p(\Omega)$ norm on the Wiener chaos by Theorem 2.3.7, implies $\sup_{n\geq 0} E(|F_n|^r) < \infty$ for all $r > 2$. Then by Proposition A.3.1 and (i), we have that $\mathrm{E}(F_n^r) \to \mathrm{E}(N^r)$ for all $r > 2$, where $N \sim \mathcal{N}(0, \sigma^2)$. In the case of $r = 4$, $\mathrm{E}(N^4) = 3$, which gives (ii).

Next, (ii) $\Rightarrow$ (iii) follows immediately from Corollary 3.2.3. Then (iii) $\Rightarrow$ (iv) follows immediately from Equation 3.2.2. Then (iv) $\Rightarrow$ (v) follows immediately from Equation 3.2.4 and 3.2.3.

Finally, we prove (v) $\Rightarrow$ (i). Since $\left\|f_n \widetilde{\otimes}_r f_n\right\|_{\mathcal{H}^{\otimes 2q-2r}} \leq \|f_n \otimes_r f_n\|_{\mathcal{H}^{\otimes 2q-2r}} \to 0$, it follows from Equation 3.2.2 that and 3.2.4 that $d_{\mathrm{TV}}(F_n, N) \to 0$ as $n \to \infty$, which implies convergence in distribution. □

Condition (iii) in Theorem 3.2.5 is often written as $\|DF_n\|_{\mathcal{H}}^2 \xrightarrow{L^2} q\sigma^2$ as $n \to \infty$, since (3.2.1) implies that $\mathrm{E}\left(\|DF_n\|_{\mathcal{H}}^2\right) \to q\sigma^2$ as $n \to \infty$.

The original proof of the fourth moment theorem was due to Nualart and Peccati [NP05] using the the Dambis-Dubins-Schwarz theorem. A proof using Malliavin calculus was given by Nualart and Ortiz-Latorre [NOL08]. The proof we follow here, using Malliavin calculus and Stein's method, was discovered by Nourdin and Peccati [NP09]. A few other proofs have also been discovered.

The fourth moment theorem has led to a simplification for proving limit theorems on the Wiener chaos. Prior to its discovery, such limits were proved using method of moments, which required showing that all moments of the sequence converged to the moments of the normal distribution. The computation of these moments was usually done using diagram formulas. Janson [Jan97], provides an overview of these methods. Now instead of needing to show convergence in all moments, we only need to consider the second and fourth moments.

Corollary 3.2.6. *Under the assumptions of Theorem 3.2.5,* $d_{\mathrm{TV}}(F_n, N) \to 0$ *as* $n \to \infty$*, where* $N \sim \mathcal{N}(0, 1)$*, is equivalent to* $F_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ *as* $n \to \infty$.

Proof. Convergence in total variation implies convergence in distribution by Proposition (ii). Conversely, assume that $F_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ as $n \to \infty$. Then by Theorem 3.2.5, $\mathrm{E}(F_n^4) \to 3\sigma^2$ as $n \to \infty$. This implies convergence in total variation by Proposition 3.2.4. □

The fourth moment theorem is surprising in the sense that it shows that on the Wiener chaos, convergence in total variation, which is usually a strong mode of convergence, is equivalent to convergence in distribution, which is usually a weak mode of convergence.

3.3 Multivariate Limit Theorems

We will now generalize the fourth moment theorem to the multivariate setting, as well as give sufficient conditions for random vectors, not necessarily on the Wiener chaos, to converge to the multivariate normal distribution. In this section, we follow the proofs from Nualart and Ortiz-Latorre [NOL08].

Lemma 3.3.1. *Fix $d \geq 2$ and $q_1, \dots, q_d \geq 1$ and let $F_n = (I_{q_1}(f_n^1), \dots, I_{q_d}(f_n^d))$ for all $n \geq 1$ such that*

$$\lim_{n\to\infty} \mathrm{E}(F_n^i F_n^j) = \delta_{ij}. \tag{3.3.1}$$

for all $i, j = 1, \dots, d$. If $\|DF_n^i\|_{\mathcal{H}}^2 \xrightarrow{L^2} q_i$ as $n \to \infty$ for all $i = 1, \dots, d$, then $\langle DF_n^i, DF_n^j\rangle_{\mathcal{H}}^2 \xrightarrow{L^2} \sqrt{q_i q_j}\delta_{ij}$ as $n \to \infty$ for all $i, j = 1, \dots, d$.

Proof. When $i = j$, the result follows immediately from the assumption. So it remains to show the result in the case when $i < j$. Without loss of generality, we can work on the Hilbert space $L^2(T, \mathcal{B}, \mu)$. Recall that by Proposition , $DF = qI_{q-1}(\bar{f}_t)$, where $\bar{f}_t(t_1, \dots, t_{q-1}) := f(t_1, \dots, t_{q-1}, t)$. Using the product formula in Corollary 1.4.8, the stochastic Fubini theorem [PT10] and the formula of the contraction given in (B.5.1), we have

$$\begin{aligned}
\langle DF_n^i, DF_n^j\rangle_{\mathcal{H}} &= q_i q_j \int_T I_{q_i-1}(\bar{f}_{n,t}^i) I_{q_j-1}(\bar{f}_{n,t}^j)\, d\mu(t) \\
&= q_i q_j \sum_{r=0}^{q_i\wedge q_j-1} r! \binom{q_i-1}{r}\binom{q_j-1}{r} I_{q_i+q_j-2-2r}\left(\int_T \bar{f}_{n,t}^i \otimes_r \bar{f}_{n,t}^j\, d\mu(t)\right) \\
&= q_i q_j \sum_{r=0}^{q_i\wedge q_j-1} r! \binom{q_i-1}{r}\binom{q_j-1}{r} I_{q_i+q_j-2-2r}\left(f_n^i \otimes_{r+1} f_n^j\right) \\
&= q_i q_j \sum_{r=1}^{q_i\wedge q_j} (r-1)! \binom{q_i-1}{r-1}\binom{q_j-1}{r-1} I_{q_i+q_j-2r}\left(f_n^i \widetilde{\otimes}_r f_n^j\right).
\end{aligned}$$

Using the orthogonality of Wiener-Itô integrals the fact that we can assume without loss of generality that $q_i \leq q_j$, we have that

$$E\left(\langle DF_n^i, DF_n^j\rangle_{\mathcal{H}}^2\right) = q_i^2 q_j^2 \sum_{r=1}^{q_i} (r-1)!^2 \binom{q_i-1}{r-1}^2 \binom{q_j-1}{r-1}^2 (q_i+q_j-2r)! \left\|f_n^i \widetilde{\otimes}_r f_n^j\right\|_{\mathcal{H}^{\otimes q_i+q_j-2r}}^2.$$

Now since $\left\|f_n^i \widetilde{\otimes}_r f_n^j\right\|_{\mathcal{H}^{\otimes q_i+q_j-2r}}^2 \leq \|f_n^i \otimes_r f_n^j\|_{\mathcal{H}^{\otimes q_i+q_j-2r}}^2$, it suffices to prove that for all $r = 1, \dots, q_i$,

$$\lim_{n\to\infty} \left\|f_n^i \otimes_r f_n^j\right\|_{\mathcal{H}^{\otimes q_i+q_j-2r}}^2 = 0.$$

From the definition of the contraction,

$$\left\|f_n^i \otimes_r f_n^j\right\|_{\mathcal{H}^{\otimes q_i+q_j-2r}}^2 = \left\langle f_n^i \otimes_{q_i-r} f_n^i, f_n^j \otimes_{q_j-r} f_n^j\right\rangle_{\mathcal{H}^{\otimes 2r}}.$$

There are 3 cases to consider. When $r = 1, \dots, q_i - 1$, we have using the Cauchy–Schwarz inequality that

$$\left\|f_n^i \otimes_r f_n^j\right\|_{\mathcal{H}^{\otimes q_i+q_j-2r}}^2 \leq \left\|f_n^i \otimes_{q_i-r} f_n^i\right\|_{\mathcal{H}^{\otimes 2r}} \left\|f_n^j \otimes_{q_j-r} f_n^j\right\|_{\mathcal{H}^{\otimes 2r}} \to 0,$$

as $n \to \infty$, due to (3.3.1) and Theorem 3.2.5.

When $r = q_i < q_j$, the Cauchy-Schwarz inequality gives

$$\left\| f_n^i \otimes_r f_n^j \right\|^2_{\mathcal{H}^{\otimes q_i+q_j-2r}} \leq \left\| f_n^i \right\|_{\mathcal{H}^{\otimes 2r}} \left\| f_n^j \otimes_{q_j-r} f_n^j \right\|_{\mathcal{H}^{\otimes 2r}} \to 0,$$

as $n \to \infty$, since (3.3.1) implies that $\sup_{n\geq 1} \|f_n^i\|_{\mathcal{H}^{\otimes 2r}} < \infty$, while (3.3.1) and Theorem 3.2.5 ensures that $\left\| f_n^j \otimes_{q_j-r} f_n^j \right\|_{\mathcal{H}^{\otimes 2r}} \to 0$.

Finally, when $r = q_i = q_j$ using the orthogonality property we have that

$$\left\| f_n^i \otimes_r f_n^j \right\|_{\mathcal{H}^{\otimes q_i+q_j-2r}} = q_i!\, \mathrm{E}(F_n^i F_n^j) \to 0$$

as $n \to \infty$, due to (3.3.1).

A similar argument can be used in the case where $i > j$. This completes the proof. □

A vector of Wiener-Itô integrals converges component-wise to a Gaussian distribution if and only if joint convergence holds. The following result was originally due to Peccati and Tudor [PT05].

Theorem 3.3.2 (Peccati and Tudor). *Fix $d \geq 2$ and $q_1, \ldots, q_d \geq 1$ and let $F_n = (I_{q_1}(f_n^1), \ldots, I_{q_d}(f_n^d))$ for all $n \geq 1$ and I_d be the $d \times d$ identity matrix. Assume that*

$$\lim_{n\to\infty} \mathrm{E}(F_n^i F_n^j) = \delta^{i,j} \tag{3.3.2}$$

for all $i, j = 1, \ldots, d$. Then the follow conditions are equivalent:

(i) $F_n \xrightarrow{d} \mathcal{N}_d(0, I_d)$ as $n \to \infty$

(ii) $F_n^i \xrightarrow{d} \mathcal{N}(0, 1)$ as $n \to \infty$, for all $i = 1, \ldots, d$.

Proof. It is a basic fact of convergence in distribution that (i) ⇒ (ii). So we only need to prove that (ii) ⇒ (i). Since (3.3.2) ensures that $(F_n)_{n\geq 1}$ is bounded in $L^2(\Omega)$, this sequence is tight. So Prokhorov's Theorem implies that there is a subsequence $(F_{n_m})_{m\geq 1}$ and random variable G, such that $F_{n_m} \xrightarrow{d} G$ as $m \to \infty$. Now, let $\phi_n(t) = \mathrm{E}\left(e^{\mathrm{i}\langle t, F_n\rangle}\right)$, be the characteristic function of F_n, and ϕ be the characteristic function of G. Then we have that

$$\lim_{m\to\infty} \phi_{n_m}(t) = \phi(t) \tag{3.3.3}$$

for all $t \in \mathbb{R}^d$. Due the uniqueness of limits, it suffices to $G \sim \mathcal{N}_d(0, I_d)$ and we will do this by showing that ϕ is the characteristic function of $\mathcal{N}_d(0, I_d)$.

Now, for all $i = 1, \ldots, d$, $\partial\phi_n(t)/\partial t_i = \mathrm{i}\,\mathrm{E}(F_n^i e^{\mathrm{i}\langle t, F_n\rangle})$, so we can apply the continuous mapping theorem to (3.3.3) to get

$$\lim_{m\to\infty} \frac{\partial \phi_{n_m}}{\partial t_i}(t) = \frac{\partial \phi}{\partial t_i}(t). \tag{3.3.4}$$

Now from Definition 2.3.2 and Proposition 2.3.4, $F = -LF/q = \delta(DF)/q$. Together with (2.3.1), we get

$$\mathrm{E}\left(F_n^i e^{\mathrm{i}\langle t, F_n\rangle}\right) = \frac{1}{q_i}\, \mathrm{E}\left(\delta(DF_n^i) e^{\mathrm{i}\langle t, F_n\rangle}\right)$$

$$= \frac{1}{q_i} \mathrm{E}\left(\left\langle DF_k^j, D\left(e^{\mathrm{i}\langle t, F_n\rangle}\right)\right\rangle_{\mathcal{H}}\right)$$
$$= \frac{\mathrm{i}}{q_i} \sum_{j=1}^{d} t_j \,\mathrm{E}\left(e^{\mathrm{i}\langle t, F_n\rangle} \left\langle DF_n^i, DF_n^j\right\rangle_{\mathcal{H}}\right),$$

where we used the definition of the Malliavin derivative at the last line. As a result of (ii) and (3.3.2), we can apply Lemma 3.3.1 to get

$$\frac{\partial \phi_{n_m}}{\partial t_i}(t) = -\frac{1}{q_i} t_i \,\mathrm{E}\left(e^{\mathrm{i}\langle t, F_{n_m}\rangle} q_i\right)$$

Finally, taking the $m \to \infty$ and combining this with 3.3.4 give

$$\frac{\partial \phi}{\partial t_i}(t) = -t_i \phi(t), \tag{3.3.5}$$

for all $i = 1, \ldots, d$. This system of partial differential equations has solution $\phi(t) = \exp\left(-\langle t, t\rangle_{\mathbb{R}^d}/2\right)$ which is the characteristic function of $\mathcal{N}_d(0, I_d)$. □

The fourth moment theorem gives conditions for the convergence of random variables on the Wiener chaos. However, we can also examine the conditions for which any square integrable random variable converges to a Gaussian distribution. To do this we make use of the Wiener-Ito expansion. The following theorem gives sufficient conditions.

Theorem 3.3.3 (**Nualart and Ortiz-Latorre**). *Let $(F_n)_{n\geq 1}$ be a sequence of centered d-dimensional random vector in $L^2(\Omega)$. Then for all $n \geq 1$ and $i = 1, \ldots, d$, we have the Wiener-Ito expansion*

$$F_n^i = \sum_{q=1}^{\infty} I_q(f_{n,q}^i).$$

Suppose that in addition, for all $i, j = 1, \ldots, d$:

(i) For all $q \geq 1$, we have $\sigma_q^{i,j} := \lim_{n\to\infty} q! \left\langle f_{n,q}^i, f_{n,q}^j\right\rangle_{\mathcal{H}^{\otimes q}}$ exists.

(ii) $\sum_{q=1}^{\infty} |\sigma_q^{i,j}| < \infty$.

(iii) For all $q \geq 2$, $r = 1, \ldots, q-1$, we have $\left\| f_{n,q}^i \otimes_r f_{n,q}^i\right\|^2_{\mathcal{H}^{\otimes 2q-2r}} \to 0$ as $n \to \infty$.

(iv) $\lim_{N\to\infty} \sup_{n\geq 1} \sum_{q=N+1}^{\infty} q! \left\| f_{n,q}^i\right\|^2_{\mathcal{H}^{\otimes q}} = 0$.

Then $F_n \xrightarrow{d} \mathcal{N}(0, \Sigma)$, where Σ is a $d \times d$ matrix with (i, j)-th entry

$$\sigma^{i,j} = \sum_{q=1}^{\infty} \sigma_q^{i,j}.$$

Note that condition (iii) can be replaced with any of the equivalent conditions listed in Theorem 3.2.5.

Proof. It suffices to prove that for all $a \in \mathbb{R}^d$, $a^T F_n \xrightarrow{D} \mathcal{N}(0, a^T \Sigma a)$ as $n \to \infty$. Now from (i) and (iii) and Theorem 3.2.5, we have $I_q(f^i_{n,q}) \xrightarrow{D} \mathcal{N}(0, \sigma^{i,i}_q)$ as $n \to \infty$, for all $i = 1, \ldots, d$ and $q \geq 1$. Using (i) amd Theorem 3.3.2 gives

$$(I_1(a^T f_{n,1}), \ldots, I_q(a^T f_{n,q})) \xrightarrow{D} (G_1, \ldots, G_q), \tag{3.3.6}$$

as $n \to \infty$, where $f_{n,k} = (f^1_{n,k}, \ldots, f^d_{n,k})$, $G_k \sim \mathcal{N}(0, a^T \Sigma_k a)$ are independent, and Σ_k is a $d \times d$ matrix with (i,j) entry $\sigma^{i,j}_k$, for all $k = 1, \ldots, q$. Now define

$$F^N_n := \sum_{q=1}^{N} I_q(f_{n,q}), \quad G^N := \sum_{q=1}^{N} G_q, \quad G := \sum_{q=1}^{\infty} G_q.$$

Let g be a continuously differentiable function with derivative bounded by C. Then, using the mean value theorem [DC99], there is a H between $a^T F_n$ and $a^T F^N_n$ such that

$$\begin{aligned}
\mathrm{E}(g(a^T F_n)) - \mathrm{E}(g(a^T F^N_n)) &= \mathrm{E}\left(g'(H) a^T \sum_{q=N+1}^{\infty} I_q(f_{n,q}) \right) \\
&\leq \mathrm{E}\left(C|a| \left| \sum_{q=N+1}^{\infty} I_q(f_{n,q}) \right| \right) \\
&\leq C|a| \left(\mathrm{E}\left(\left| \sum_{q=N+1}^{\infty} I_q(f_{n,q}) \right|^2 \right) \right)^{1/2} \\
&= C|a| \left(\sum_{i=1}^{d} \mathrm{E}\left(\left(\sum_{q=N+1}^{\infty} I_q(f^i_{n,q}) \right)^2 \right) \right)^{1/2} \\
&= C|a| \left(\sum_{i=1}^{d} \sum_{q=N+1}^{\infty} \mathrm{E}\left(I_q(f^i_{n,q})^2 \right) \right)^{1/2}
\end{aligned}$$

where we have use the Cauchy-Schwarz inequality for the Euclidean norm, and then the Cauchy-Schwarz inequality for the $L^2(\Omega)$ norm, and the last line follows from the orthogonality property of Wiener-Itô integrals. Now

$$\begin{aligned}
\left| \mathrm{E}(g(a^T F_n)) - \mathrm{E}(g(G)) \right| \leq& \left| \mathrm{E}(g(a^T F_n)) - \mathrm{E}(g(a^T F^N_n)) \right| + \left| \mathrm{E}(g(a^T F^N_n)) - \mathrm{E}(g(G^N)) \right| \\
&+ \left| \mathrm{E}(g(G^N)) - \mathrm{E}(g(G)) \right| \\
\leq& C|a| \left(\sum_{i=1}^{d} \sum_{q=N+1}^{\infty} \mathrm{E}\left(I_q(f^i_{n,q})^2 \right) \right)^{1/2} + \left| \mathrm{E}(g(a^T F^N_n)) - \mathrm{E}(g(G^N)) \right| \\
&+ \left| \mathrm{E}(g(G^N)) - \mathrm{E}(g(G)) \right|.
\end{aligned}$$

Now if we take $n \to \infty$, followed by $N \to \infty$, then the first term approaches 0 due to (iv) and the fact that $\mathrm{E}\left(I_q(f^i_{n,q})^2 \right) = q! \left\| f^i_{n,q} \right\|_{\mathcal{H}^{\otimes q}}$, the second term approaches 0 due to (3.3.6), and the third term approaches 0 due to (ii). This implies that $a^T F_n \xrightarrow{D} G$ as $n \to \infty$. Now since G_q are independent, $G \sim \mathcal{N}(0, a^T \Sigma a)$. This completes the proof. □

Chapter 4

Convergence of Partial Sum Processes

In this chapter we examine the limits of partial sum processes. The general setup is described in Section 4.1. The limit theorems discussed here can be viewed as a generalization of Donsker's theorem to the case where the process is not independent. It turns out that the limiting distribution of these partial sum processes are Brownian motion when underlying process exhibits short range dependence, as proved by Breuer and Major [BM83], or a Hermite process when it exhibits long range dependence, as proved by Taqqu [Taq79], and Dobrushin and Major [DM79].

We will provide a modern proof of the Breuer-Major theorem using Theorem 3.3.3, instead of the method of moments and diagram formula that was originally used to prove it. Then we will discuss self-similar processes with stationary increments, such as fractional Brownian motion and the Hermite processes.

4.1 Partial Sum Processes

Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary Gaussian process and let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\mathrm{Var}(f(X_n)) < \infty$ and denote its autocovariance function by $\rho(n) = E(X_0X_n)$. Without loss of generality, we will further assume that $\mathrm{E}(X_n) = 0$, $\mathrm{Var}(X_n) = 1$ and $\mathrm{E}(f(X_n)) = 0$ for all $n \in \mathbb{Z}$.

Consider the process

$$V_{t,n} = \sum_{k=1}^{[nt]} f(X_k). \tag{4.1.1}$$

Note that $[x] = \lfloor x\rfloor$ if $x \geq 0$, and $[x] = \lfloor x\rfloor + 1$, if $x < 0$. Let $V_n = (V_{t,n})_{t\in\mathbb{R}}$.

In this section, we are interested in the limit as $n \to \infty$ of V_n/a_n, where a_n is a deterministic real sequence, as well as the condition on the autocovariance function under which the process has a nondegenerate limit.

By Proposition 1.2.3 (ii), the assumption that $\mathrm{Var}(f(X_n)) < \infty$ implies that f has a Hermite expansion

$$f(x) = \sum_{q=1}^{\infty} a_q H_q(x), \tag{4.1.2}$$

where H_q is the qth Hermite polynomial.

Definition 4.1.1. Let $f : \mathbb{R} \to \mathbb{R}$ have the Hermite expansion (4.1.2). The smallest integer m such that the coefficient $a_m \neq 0$ is known as the **Hermite rank** of f.

Before we can apply the fourth moment theorem we need to write X as an isonormal Gaussian process. Denote the closed linear subspace of $L^2(\Omega)$ spanned by X as $\mathcal{H}$. Typically, $\mathcal{H}$ is infinite-dimensional, but if this is not the case, then add Gaussian white noise to the spanning set so that $\mathcal{H}$ becomes infinite-dimensional. Since $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, there exists an isometry $\psi : \mathcal{H} \to L^2(\mathbb{R})$. Put $g_k = \psi(X_k)$, then due to the isometry we have that for all $k, l \in \mathbb{N}$,

$$\rho(k-l) = \mathrm{E}(X_k X_l) = \langle g_k, g_l \rangle_{L^2(\mathbb{R})} . \tag{4.1.3}$$

Then the isonormal Gaussian process $(X(g))_{g \in L^2(\mathbb{R})}$ satisfies $(X(g_k))_{k\geq 0} \stackrel{d}{=} (X_k)_{k \geq 0}$. Therefore, without loss of generality we can write

$$X_k = X(g_k) = I_1(g_k). \tag{4.1.4}$$

4.2 Central Limit Theorem for Partial Sum Processes

In this section we use Theorem 3.3.3 to show that the partial sum of a subordinated Gaussian process, $f(X)$, converges to a normal random variable. This result is known as the Breuer-Major theorem.

We follow the proof in [Nou12].

Theorem 4.2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function in (4.1.2) with Hermite rank m, and V_n be defined in (4.1.1). If $\sum_{n\in\mathbb{Z}} |\rho(n)|^m < \infty$, then*

$$\frac{1}{\sqrt{n}} V_n \stackrel{fdd}{\to} \sigma B,$$

where B is standard Brownian motion and

$$\sigma^2 = \sum_{q=d}^{\infty} q! a_q^2 \sum_{n\in\mathbb{Z}} \rho(n)^q < \infty.$$

Proof. Since X_k is a Gaussian process, due to (4.1.4) and (4.1.3), we can assume $X_k = X(e_k)$ and $\langle e_k, e_l \rangle_{\mathcal{H}} = \rho(k-l)$. Using the Hermite expansion of f, we get

$$\frac{1}{\sqrt{n}} V_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \sum_{q \geq m} a_q H_q(X_k) = \sum_{q \geq d} I_q(f_{t,n,q}),$$

where the last equality follows by applying Theorem 1.4.9 so that for all $n \geq 1$ and $q \geq m$,

$$f_{t,n,q} = \frac{a_q}{\sqrt{n}} \sum_{k=1}^{[nt]} g_k^{\otimes q} \in \mathcal{H}^{\odot q}.$$

Let $d \geq 1$ and $F_n^i = \sum_{q\geq m} I_q(f_{t_i,n,q})$, for $i = 1, \dots, d$. Recall that $\mathrm{Cov}(\sigma B(t), \sigma B(s)) = \sigma^2 (t \wedge s)$. So in order to prove that F_n converges in finite dimensional distribution to a scaled Brownian motion, we need to show that as $n \to \infty$, $F_n = (F_n^1, \dots, F_n^d) \xrightarrow{d} \mathcal{N}_d(0, \Sigma)$, where the (i,j) entry of Σ is

$$\sigma^{i,j} = (t_i \wedge t_j) \sum_{q=m}^{\infty} q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q.$$

We do this by checking the sufficient conditions in Theorem 3.3.3.

Condition (i). For all $q \geq m$ and $t > s \geq 0$, using the parallelogram law

$$\begin{aligned} 2 \langle f_{t,n,q}, f_{s,n,q} \rangle_{\mathcal{H}^{\otimes q}} &= \| f_{t,n,q} \|_{\mathcal{H}^{\otimes q}} + \| f_{s,n,q} \|_{\mathcal{H}^{\otimes q}} - \| f_{t,n,q} - f_{s,n,q} \|_{\mathcal{H}^{\otimes q}} \\ &= \frac{a_q^2}{n} \sum_{i,j=1}^{[nt]} \rho(i-j)^q + \frac{a_q^2}{n} \sum_{i,j=1}^{[ns]} \rho(i-j)^q - \frac{a_q^2}{n} \sum_{i,j=1}^{[nt]-[ns]} \rho(i-j)^q \\ &= a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q \frac{[nt] - |v|}{n} 1_{\{|v| < [nt]\}} + a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q \frac{[ns] - |v|}{n} 1_{\{|v| < [nt]\}} \\ &\quad - a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q \frac{[nt] - [ns] - |v|}{n} 1_{\{|v| < [nt]-[ns]\}}. \end{aligned}$$

The assumption $\sum_{v \in \mathbb{Z}} |\rho(v)|^m < \infty$ and the indicator functions ensures that this expression is bounded, and noting that $\frac{[nt]-|v|}{n} 1_{\{|v|<[nt]\}} \to t$ so we can apply the dominated convergence theorem as $n \to \infty$ which implies that

$$q! \langle f_{t,n,q}, f_{s,n,q} \rangle_{\mathcal{H}^{\otimes q}} \to (t \wedge s) q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q.$$

Condition (ii). Since $\mathrm{E}(X_k^2) = 1$ for all $k \geq 1$, the Cauchy-Schwarz inequality gives $\mathrm{E}(X_{k+l} X_l)^2 \leq \mathrm{E}(X_{k+l}^2) \, \mathrm{E}(X_l^2)$, which implies that $|\rho(k)| \leq 1$. Therefore,

$$\sum_{q=m}^{\infty} q! a_q^2 \sum_{v \in \mathbb{Z}} \rho(v)^q \leq \sum_{q=m}^{\infty} q! a_q^2 \sum_{v \in \mathbb{Z}} |\rho(v)|^m = \mathrm{E}(f(X_1)^2) \sum_{v \in \mathbb{Z}} |\rho(v)|^m < \infty,$$

since by assumption, both expressions in the last equality are finite.

Condition (iii). Let $q \geq m$ and $q \neq 1$. Then

$$\begin{aligned} f_{t,n,q} \otimes_r f_{t,n,q} &= \frac{a_q^2}{n} \sum_{i,j=1}^{[nt]} g_i^{\otimes q} \otimes_r g_j^{\otimes q} \\ &= \frac{a_q^2}{n} \sum_{i,j=1}^{[nt]} \langle g_i, g_j \rangle_{\mathcal{H}}^r g_i^{\otimes q-r} g_j^{\otimes q-r} \\ &= \frac{a_q^2}{n} \sum_{i,j=1}^{[nt]} \rho(i-j)^r g_i^{\otimes q-r} g_j^{\otimes q-r}, \end{aligned}$$

for all $n \geq 1$ and $r = 1, \dots, q-1$. Therefore,

$$\|_{t,n,q} \otimes_r f_{t,n,q} \|^2_{\mathcal{H}^{\otimes 2q-2r}} = \frac{a_q^4}{n^2} \sum_{i,j,k,l=1}^{[nt]} \rho(k-l)^r \rho(i-j)^r \left\langle g_i^{\otimes q-r} \otimes g_j^{\otimes q-r}, g_k^{\otimes q-r} \otimes g_l^{\otimes q-r} \right\rangle_{\mathcal{H}^{\otimes 2q-2r}}$$

$$= \frac{a_q^4}{n^2} \sum_{i,j,k,l=1}^{[nt]} \rho(k-l)^r \rho(i-j)^r \rho(k-i)^{q-r} \rho(l-j)^{q-r}.$$

Now $|\rho(k-l)^r \rho(k-i)^{q-r}| \le |\rho(k-l)|^q + |\rho(k-i)|^q$, using this we have

$$\begin{aligned}
\|f_{t,n,q} \otimes_r f_{t,n,q}\|^2_{\mathcal{H}^{\otimes 2q-2r}} &\le \frac{a_q^4}{n^2} \sum_{i,j,k,l=1}^{[nt]} |\rho(k-l)|^q \left(|\rho(i-j)|^r |\rho(l-j)|^{q-r} + |\rho(i-j)|^{q-r} |\rho(l-j)|^r\right) \\
&\le \frac{a_q^4}{n^2} \sum_{k\in\mathbb{Z}} |\rho(k)|^q \sum_{i,j,l=1}^{[nt]} \left(|\rho(i-j)|^r |\rho(l-j)|^{q-r} + |\rho(i-j)|^{q-r} |\rho(l-j)|^r\right) \\
&\le \frac{2a_q^4}{n} \sum_{k\in\mathbb{Z}} |\rho(k)|^m \sum_{|i|\le[nt]} |\rho(i)|^r \sum_{|j|\le[nt]} |\rho(j)|^{q-r} \\
&= 2a_q^4 \sum_{k\in\mathbb{Z}} |\rho(k)|^m n^{-1+r/q} \sum_{|i|\le[nt]} |\rho(i)|^r n^{-1+(q-r)/q} \sum_{|j|\le[nt]} |\rho(j)|^{q-r}.
\end{aligned}$$

Choose $\delta \in (0,t)$. In order to show that $\|f_{t,n,q} \otimes_r f_{t,n,q}\|^2_{\mathcal{H}^{\otimes 2q-2r}} \to 0$ as $n \to \infty$, it suffices to show that

$$n^{-1+r/q} \sum_{|j|\le[nt]} |\rho(j)|^r = n^{-1+r/q} \left(\sum_{|j|\le[n\delta]} |\rho(j)|^r + \sum_{[n\delta]<|j|\le[nt]} |\rho(j)|^r \right) \to 0,$$

since this would also imply that $n^{-1+(q-r)/q} \sum_{|j|\le[nt]} |\rho(j)|^{q-r} \to 0$, while $\sum_{k\in\mathbb{Z}} |\rho(k)|^m < \infty$ by assumption. Using Hölder's inequality gives

$$n^{-1+r/q} \sum_{|j|\le[n\delta]} |\rho(j)|^r \le n^{-1+r/q} (2[n\delta]+1)^{1-r/q} \left(\sum_{j\in\mathbb{Z}} |\rho(j)|^q \right)^{r/q} \le K\delta^{1-r/q}, \tag{4.2.1}$$

where $K > 0$ is a constant and the last inequality follows from $\sum_{j\in\mathbb{Z}} |\rho(j)|^q \le \sum_{j\in\mathbb{Z}} |\rho(j)|^m < \infty$. We also have

$$n^{-1+r/q} \sum_{[n\delta]<|j|\le[nt]} |\rho(j)|^r \le \left(\sum_{[n\delta]<|j|\le[nt]} |\rho(j)|^r \right)^{r/q}. \tag{4.2.2}$$

Since $1 \le r \le q-1$, taking $n \to \infty$, followed by $\delta \to 0$, we get (4.2.1) and (4.2.2) converging to 0, as required.

Condition (iv). Let $N \ge m$. Then,

$$\begin{aligned}
\sum_{q=k+1}^{\infty} q!\, \|f_{t,n,q}\|^2_{\mathcal{H}^{\otimes q}} &= \frac{1}{n} \sum_{q=k+1}^{\infty} a_q^2 q! \sum_{i,j=1}^{[nt]} \rho(i-j)^q \\
&\le \sum_{q=k+1}^{\infty} a_q^2 q! \sum_{v\in\mathbb{Z}} \rho(v)^q \\
&\le \sum_{v\in\mathbb{Z}} \rho(v)^m \sum_{q=k+1}^{\infty} a_q^2 q!.
\end{aligned}$$

Recalling the assumption $\mathrm{Var}(f(N')) = \sum_{q\geq m} a_q^2 q! < \infty$, where $N' \sim \mathcal{N}(0,1)$, we conclude that

$$\lim_{k\to\infty} \sup_{n\geq 1} \sum_{q=k+1}^{\infty} q! \, \|f_{t,n,q}\|^2_{\mathcal{H}^{\otimes q}} \to 0.$$

□

4.3 Self-Similar Processes with Stationary Increments

In this section we introduce the concept of self-similar processes with stationary increments. These will be important for understanding the noncentral limit theorem in following section because these are the only candidates for the limit of partial sum processes. This section is based off [GKS12].

Definition 4.3.1. A stochastic process $(X_t)_{t\in T}$ is **self-similar** if there exists a $H > 0$, such that for all $a > 0$, $(X_{at})_{t\in T} \overset{d}{=} (a^H X_t)_{t\in T}$. We call H the self-similarity parameter.

Definition 4.3.2. A stochastic process $(X_t)_{t\in T}$ has **stationary increments** if

$$(X_{t+s} - X_s)_{t\in T} \overset{d}{=} (X_t - X_0)_{t\in T}$$

for all $s \in T$.

Consider the covariance function

$$R_H(t,s) := \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{4.3.1}$$

Proposition 4.3.3. *For $H \in (0,1)$, let $X = (X_t)_{t\in T}$ be a H-self-similar process with stationary increments and $\mathrm{E}(X_1^2) = 1$. Then*

(i) $X_0 = 0$ *almost surely,*

(ii) $\mathrm{E}(X_t) = 0$ *for all* $t \in T$,

(iii) $\mathrm{E}(X_s X_t) = R_H(s,t)$ *for all* $s, t \in T$.

Proof. (i) If $0 \in T$, the self-similarity property implies that $X_0 \overset{d}{=} c^H X_0$ for all $c > 0$, so $X_0 = 0$ almost surely.

(ii) Fix $s \in T$. Using self-similarity and the stationary increments property, for all nonzero $t \in T$ we have

$$\mathrm{E}(X_s) = \mathrm{E}(X_{t+s} - X_t) = \left(\left(\frac{t+s}{t}\right)^H - 1\right)\mathrm{E}(X_t).$$

For $t = s$, this implies that $\mathrm{E}(X_s) = 0$ because $H > 0$. Therefore, $\mathrm{E}(X_t) = 0$ for all nonzero $t \in T$. When $t = 0$, (i) implies $\mathrm{E}(X_0) = 0$.

(iii) The self-similarity property implies that $X_t \overset{d}{=} t^H X_1$, so $E(X_t^2) = |t|^{2H}$. Using the stationary increments property, we have

$$E(X_s X_t) = \frac{1}{2}\left(\mathrm{E}(X_t^2) + \mathrm{E}(X_s^2) - \mathrm{E}\left((X_t - X_s)^2\right)\right)$$

$$= \frac{1}{2}\left(\mathrm{E}(X_t^2) + \mathrm{E}(X_s^2) - \mathrm{E}\left((X_{t-s})^2\right)\right)$$
$$= \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

for all $s, t \in T$. □

If X is a H-self-similar process with stationary increments and finite variance, then using the triangle inequality we have that

$$2^H\,\mathrm{E}(X_t^2)^{1/2} = \mathrm{E}(X_{2t}^2)^{1/2} \le \mathrm{E}((X_{2t} - X_t)^2)^{1/2} + \mathrm{E}(X_t^2)^{1/2} = 2\,\mathrm{E}(X_t^2)^{1/2}.$$

Thus, we have $2^H \le 2$, which implies that $H \le 1$. Note that when $H = 1$, using Proposition 4.3.3 (iii), we have

$$\mathrm{E}\left((X_t - tX_1)^2\right) = \mathrm{E}(X_t^2) - 2t\,\mathrm{E}(X_tX_1) + t^2\,\mathrm{E}(X_1^2) = t^2 - t(t^2 + 1 - (1-t)^2) + t^2 = 0.$$

Thus, $X_t = tX_1$ almost surely. Since this is an uninteresting stochastic process, we have will always assume that $H \in (0,1)$. Without loss of generality, we will also assume that $\mathrm{E}(X_1^2) = 1$.

Consider the increment process of X defined by $Y = (Y_n)_{n\in\mathbb{Z}}$ where $Y_n := X_{n+1} - X_n$. We now introduce the notion of short range dependence and long range dependence.

Definition 4.3.4. Let $Y = (Y_n)_{n\in\mathbb{Z}}$ be a covariance stationary stochastic process with autocovariance function $\rho(n) = \mathrm{E}(Y_0Y_n)$.

(i) If $0 < \sum_{k\in\mathbb{Z}} |\rho(k)| < \infty$, then Y is said to exhibit **short range dependence**.

(ii) If $\sum_{k\in\mathbb{Z}} |\rho(k)| = \infty$, then Y is said to exhibit **long range dependence**.

Note that the case where $\sum_{k\in\mathbb{Z}} |\rho(k)| = 0$ is excluded since in such a case, Y_n and Y_m are uncorrelated for $n \neq m$. .

Proposition 4.3.5. *Let $Y_n := X_{n+1} - X_n$ for all $n \in \mathbb{Z}$, where X is a H-self-similar process with stationary increments. Then Y is weakly stationary with autocovariance function*

$$\rho(n) = \frac{1}{2}\left(|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H}\right)$$
$$\sim H(2H-1)n^{2H-2}.$$

Moreover, if $H \in (0, 1/2)$ then Y exhibits short range dependence, and if $H \in (1/2, 1)$ then Y exhibits long range dependence.

Proof. We have that

$$\begin{aligned}\mathrm{E}(Y_{k+n}Y_k) &= \mathrm{E}\left((X_{k+n+1} - X_{k+n})(X_{k+1} - X_k)\right)\\ &= R_H(k+n+1, k+1) - R_H(k+n+1, k) - R_H(k+n, k+1) + R_H(k+n, k)\\ &= \frac{1}{2}\left(|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H}\right),\end{aligned}$$

for all $k, n \in \mathbb{Z}$. Therefore $\rho(n) = \mathrm{E}(Y_nY_0) = \mathrm{E}(Y_{k+n}Y_k)$, which implies that Y is weakly stationary.

For $n \geq 0$, we can write $\rho(n) = n^{2H-2}L(n)/2$, where

$$L(n) := n^2 \left(\left(1 + \frac{1}{n}\right)^{2H} - 2 + \left(1 - \frac{1}{n}\right)^{2H} \right).$$

Applying l'Hôpital's rule twice to $L(n)$ shows that $L(n) \to 2H(2H-1)$ as $n \to \infty$. Thus, $\rho(n) \sim H(2H-1)n^{2H-2}$.

Thus, for some positive constant K,

$$\sum_{n \in \mathbb{Z}} |\rho(n)| < K \sum_{n=0}^{\infty} n^{2H-2}.$$

If $H \in (0, 1/2)$, this sum converges so Y exhibits short range dependence. If $H \in (1/2, 1)$, this sum diverges so Y exhibits long range dependence. □

The canonical example of a self-similar process with stationary increments is fractional Brownian motion. This class of processes also includes standard Brownian motion.

Definition 4.3.6. Let $H \in (0, 1)$. A H-self-similar Gaussian process with stationary increments is known as **fractional Brownian motion** and H is called the **Hurst parameter**.

From Proposition 4.3.3, it is clear that fractional Brownian motion is the only Gaussian process that is also a H-self-similar process with stationary increments, in the sense that every other such process is of the form σB^H, for some $\sigma > 0$. We will only work with the case where $\sigma = 1$, which is also known as standard fractional Brownian motion. There are also non-Gaussian examples of self-similar processes with stationary increments, such as Hermite processes which will be introduced in later sections.

When $H = 1/2$, fractional Brownian motion reduces to standard Brownian motion on $\mathbb{R}$. This immediately follows from the fact that centered Gaussian processes are determined by the covariance function which becomes $\mathrm{E}(B_t^{1/2} B_s^{1/2}) = t \wedge s$, the covariance function for standard Brownian motion.

The next result gives an alternative definition for fractional Brownian motion.

Proposition 4.3.7. *Let $H \in (0, 1)$. A stochastic process is fractional Brownian motion if and only if it is a centered Gaussian process $B^H = (B_t^H)_{t \in \mathbb{R}}$ with covariance function*

$$\mathrm{E}(B_t^H B_s^H) = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right). \tag{4.3.2}$$

Proof. One direction follows from Proposition 4.3.3. For the other direction, suppose that B^H is a centered Gaussian process with covariance function R_H. Then since $R_H(at, as) = a^{2H} R_H(t, s)$ and centered Gaussian processes are determined by their covariance function, it follows that B^H is similar. Similarly, it can be shown that $\mathrm{E}\left((B_t^H - B_s^H)^2\right) = |t-s|^{2H}$ so that it has stationary increments. □

It is not immediately obvious that fractional Brownian motion exists. To establish existence, it suffices to show that R_H is a valid covariance function, meaning that R_H is positive semi-definite. A proof of this fact can be found in [Nou12]. Next we show the continuity of the sample paths.

Proposition 4.3.8. *Let B^H be fractional Brownian motion with Hurst parameter H. There exists a version of B^H with locally Hölder continuous paths of order $\alpha < H$.*

Proof. Using the self-similarity and stationary increments property, we have

$$\mathrm{E}\left(\left|B_t^H - B_s^H\right|^q\right) = \mathrm{E}\left(\left|B_{t-s}^H\right|^q\right) = \mathrm{E}\left(\left|B_1^H\right|^q\right)|t-s|^{qH}.$$

Then by the Kolmogorov continuity criteria, there exists a version of B^H with Hölder continuous paths of order $\alpha < (qH-1)/q$. Then letting $q \to \infty$ gives the required result. □

It is well-known that an appropriately normalized sum of independent and identically distributed random variables can only converge in distribution to a stable random variable. In fact an analogous result holds in the case of convergence in finite dimension distribution for stochastic processes. Lamperti [Lam62] proved a theorem which says that the limit of any normalized partial sum must be a self-similarity. Moreover, it motivates the introduction of self-similar processes and their use in various applications.

Theorem 4.3.9. *Suppose that $(X_n)_{n\in\mathbb{Z}}$ is a stationary process and there exists a deterministic sequence $a_n \to \infty$ and a nonzero stochastic process $(Z_t)_{t\geq 0}$ such that*

$$\frac{1}{a_n}\sum_{k=1}^{\lfloor nt\rfloor} X_k \overset{ffd}{\to} Z_t.$$

Then $(Z_t)_{t\geq 0}$ is a continuous and H-self-similar process with stationary increments, for some $H > 0$, and $a_n = n^H L(n)$, where L is a slowly varying function. Furthermore, every self-similar process is the limit of such a partial sum process.

In the next section, we will see that the partial sum processes under the assumptions we have set out in Section 4.1 converges either to Brownian motion or the Hermite process based on the dependence structure of $(X_n)_{n\in\mathbb{Z}}$. This theorem explains why we will assume that the autocovariance is in the form $\rho(n) \sim n^H L(n)$, where L is a slowly varying function.

4.4 Noncentral Limit Theorem for Partial Sum Processes

We have the following fact.

Proposition 4.4.1. *Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stochastic process with autocovariance function $\rho(n) = n^{-\alpha}L(n)$, where L is slowly varying function.*

(i) If $0 < \alpha < 1$, then Y has long range dependence.

(ii) If $\alpha > 1$, then Y has short range dependence.

Proof. See Section 3.1 in [GKS12]. □

The condition from Theorem 4.2.1, $\sum_{n\in\mathbb{Z}} |\rho(n)|^m < \infty$ is actually equivalent to $\sum_{n\in\mathbb{Z}} |\rho_f(n)| < \infty$, where ρ_f is the autocovariance function of $f(X)$. Thus, the Gaussian subordinated process $f(X)$ exhibits short range dependence. It is interesting to ask how applying f affects the short or long range dependence of X. Not surprisingly, the answer depends on the Hermite rank.

If X exhibits short range dependence then, so does $f(X)$. But if $\rho(n) \sim n^{-\alpha}L(n)$ where L is a slowly varying function and $0 < \alpha < 1$ then X exhibits long range dependence. If $1/m < \alpha < 1$, where m is the Hermite rank of f, then $f(X)$ exhibits short range dependence. So Theorem 4.2.1 is applicable when the underlying Gaussian process X has short range dependence or even long range dependence as long as the $f(X)$ exhibits short range dependence. On the other hand if $\alpha \leq 1/m$, then we have that $f(X)$ exhibits long range dependence. The mathematical details of the above discussion can be found in Section 4.6 of [GKS12].

The critical case $\alpha = 1/m$ is interesting, because it turns out that that we can still get a central limit theorem, but instead of dividing V_n by $\sqrt{n}$ as in Theorem 4.2.1, we use $\sqrt{n\log(n)}$. This is proved in [BM83].

In this section, we will discuss the remaining case, $\alpha < 1/m$. Here we assume the underlying Gaussian process X has long range dependence with autocovariance function of the form

$$\rho(n) \sim n^{-\alpha}L(n), \tag{4.4.1}$$

where $0 < \alpha < 1$.

Definition 4.4.2. Let $q \geq 1$ and $H \in (1/2, 1)$. The **Hermite process** of order q and self-similarity parameter H denoted $(Z_t^{q,H})_{t\in[0,1]}$ is defined as $Z_t^{q,H} = I_q(L_t)$, where

$$L_t(y_1, \ldots, y_q) = d_{q,H} \int_0^t \prod_{i=1}^q (u - y_i)_+^{-\frac{1}{2} - \frac{1-H}{q}} \, du,$$

$$d_{q,H} = \left(\frac{H(2H-1)}{q!\beta\left(\frac{1}{2} - \frac{1-H}{q}, \frac{2(1-H)}{q}\right)^q} \right)^{1/2}.$$

The parameter $d_{q,H}$ has been chosen so that $\mathrm{Var}(Z_1^{q,H}) = 1$. In Chapter 5 we prove the existence of the Hermite process and derive its main properties.

Now we can state the noncentral limit theorem.

Theorem 4.4.3 (Taqqu, Dobrushin and Major). *Let $f : \mathbb{R} \to \mathbb{R}$ be the function in (4.1.2) with Hermite rank m, and V_n be defined in (4.1.1). If (4.4.1) holds and $\alpha < 1/m$, then*

$$\frac{c_{m,\alpha}}{n^H L(n)^{m/2}} V_n \xrightarrow{fdd} Z^{m,H},$$

where $Z^{m,H}$ is a Hermite process of order m and self-similarity parameter $H = 1 - (m\alpha)/2$, and

$$c_{m,\alpha} = \frac{1}{a_m} \left(\frac{m!(1-m\alpha)(2-m\alpha)}{2} \right)^{1/2}.$$

Proof. See Taqqu [Taq79] or Dobrushin and Major [DM79]. □

Let us now summarize the results that we have proven in this chapter.

Corollary 4.4.4. *Using the notation in Theorem 4.2.1 and Theorem 4.4.3, we have:*

(i) If X exhibits short range dependence, then $f(X)$ exhibits short range dependence and as $n \to \infty$,

$$\frac{1}{\sqrt{n}} V_n \xrightarrow{fdd} \sigma B.$$

(ii) If X exhibits long range dependence and satisfies (4.4.1) with $\alpha > 1/m$, then $f(X)$ exhibits short range dependence and as $n \to \infty$,

$$\frac{1}{\sqrt{n}} V_n \xrightarrow{fdd} \sigma B.$$

(iii) If X exhibits long range dependence and satisfies (4.4.1) with $\alpha = 1/m$, then $f(X)$ exhibits long range dependence and as $n \to \infty$,

$$\frac{1}{\sqrt{n \log(n)}} V_n \xrightarrow{fdd} \sigma B.$$

(iii) If X exhibits long range dependence and satisfies (4.4.1) with $\alpha < 1/m$, then $f(X)$ exhibits long range dependence and as $n \to \infty$,

$$\frac{c_{m,\alpha}}{n^H L(n)^{m/2}} V_n \xrightarrow{fdd} Z^{m,H}.$$

For more recent results in the continuous time analogue of these limit theorems see Buchmann and Chan [BC09], which makes use of the fourth moment theorem. Also see Hariz [Har02].

4.5 Simulating the Hermite Process

It is possible to use Theorem 4.4.3 to simulate the Hermite process by setting the underlying Gaussian process to fractional Gaussian noise, $X_n^H = B_n^H - B_{n-1}^H$, the increment process of fractional Brownian motion,

From Proposition 4.3.5, we showed that for $H > 1/2$, the increments of fraction Brownian motion exhibits long range dependance. Then we can apply Theorem 4.4.3 to fractional Gaussian noise with $f = H_q$.

Corollary 4.5.1. *Fix $q \geq 1$ and $H \in (1/2, 1)$. Let $X^{H'}$ be fractional Gaussian noise with Hurst parameter $H' = 1 - (1-H)/q$. Then*

$$\frac{b_{q,H}}{n^H} \sum_{k=1}^{[nt]} H_q(X_k^{H'}) \xrightarrow{fdd} Z^{q,H},$$

where

$$b_{q,H} = \frac{1}{a_q} \left(\frac{q! H(2H-1)}{\left(1 - \frac{1-H}{q}\right)^q \left(1 - \frac{2(1-H)}{q}\right)^q} \right)^{1/2}.$$

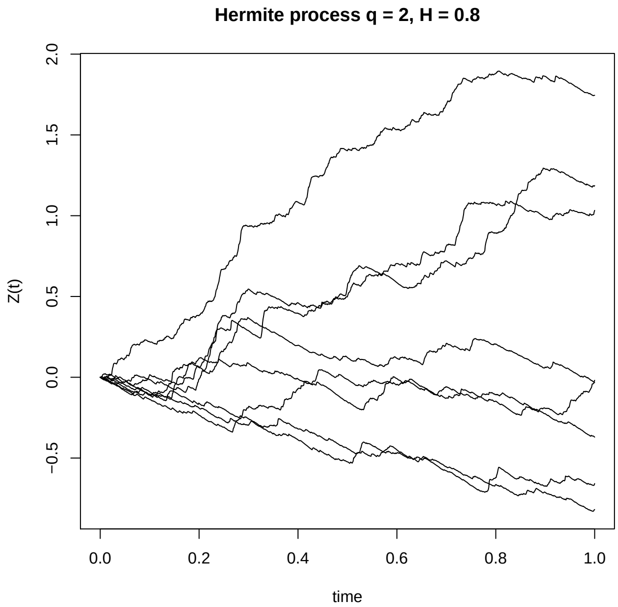

Figure 4.1: Sample paths of the Hermite process of order 2 and self-simiarlity parameter $H = 0.8$.

Example 4.5.2. We will simulate the Hermite process for $q = 2, 3$ and $H = 0.8$. To do this, we first simulate fractional Brownian motion with Hurst parameter H'. This can simply be done using any technique to simulate multivariate normal random variables, such as the Cholesky decomposition. Next we take the difference to form fractional Gaussian noise, $X^{H'}$. Lastly, calculate $\frac{b_{q,H}}{n^H} \sum_{k=1}^{[nt]} H_q(X_k^{H'})$.

The following graphs plot the sample path of the Hermite process $Z_t^{q,H}$ and an estimate of the density of the Hermite distribution $Z_1^{q,H}$. We used $n = 1,000$ grid points. In the case of the distribution, we used 100,000 simulations of the sample path, and applied a kernel smoother to approximate the density function. It appears from these simulations that the Hermite distribution of order 3 is bimodal (see Figure 4.4).

Not much is known about the Hermite process in the literature and its numerical properties. However, recently Veillette and Taqqu [VT13] have release results from a numerical evaluation of the Rosenblatt process, the Hermite process when $q = 2$. Their method involves calculating cumulants and using Fourier inversion.

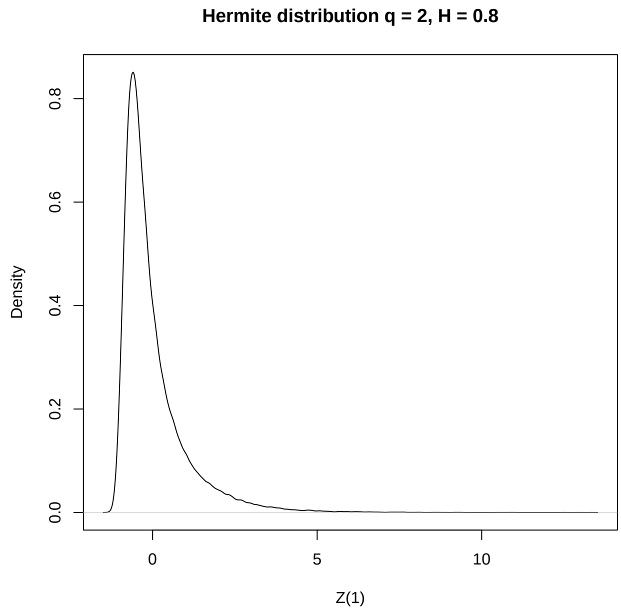

Figure 4.2: Stimulated Hermite distribution of order 2 and self-simiarlity parameter $H = 0.8$.

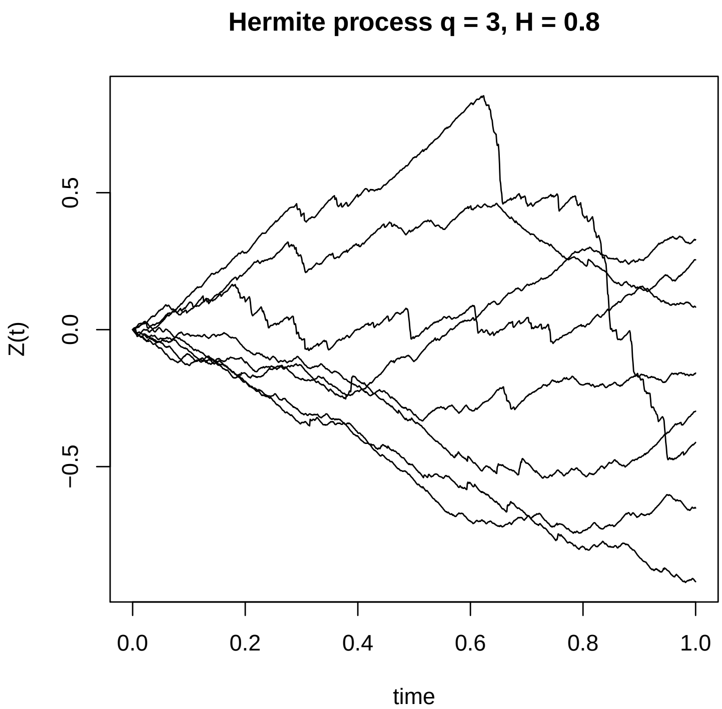

Figure 4.3: Sample paths of the Hermite process of order 3 and self-simiarlity parameter $H = 0.8$.

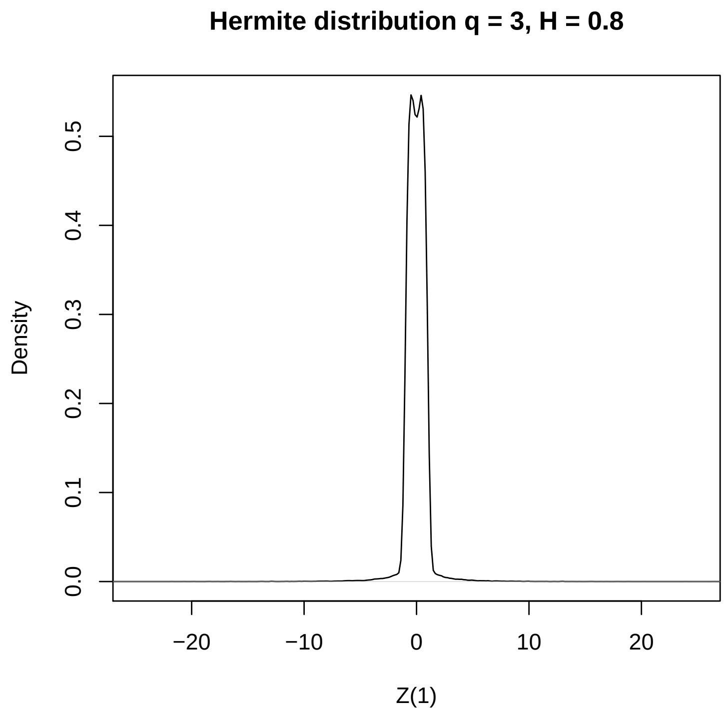

Figure 4.4: Stimulated Hermite distribution of order 3 and self-simiarlity parameter $H = 0.8$.

Chapter 5

Hermite Process

The study of self-similar processes with stationary increments and long range dependence has been of considerable interest in many fields of applications, such as finance, hydrology, and Internet traffic. Such processes may naturally arise. For example, it is conjectured [AN13] that stock prices exhibit self-similarity, that is fractal behavior, due to agents differing interpretations of information and investment horizons. Thus, these processes can play a useful role in modeling a wide range of phenomena.

In this chapter, we examine the properties of the Hermite process, a class of self-similar processes with stationary increments, which includes fractional Brownian motion, and also non-Gaussian processes. We will construct an estimator for the Hurst parameter using the quadratic variation. Our main purpose is to apply the properties of the Wiener chaos and the central and noncentral limit theorems from Chapter 4 to compute the limiting distribution of a estimator for the Hurst parameter.

The main reference for this section are Chronopoulou, Tudor and Veins [CTV11, TV09].

5.1 Hermite Process

Let B^H be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and define the kernel of fractional Brownian motion as

$$K_H(t,s) = c_H 1_{[0,t]}(s) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du \tag{5.1.1}$$

$$c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{1/2}.$$

Throughout this chapter, we will make use of various identities in the case where $H \in (1/2, 1)$ relating to K_H and R_H defined in (4.3.2). These identities are listed below and can be easily verified using elementary calculus techniques.

$$\int_0^{t\wedge s} K_H(t,u) K_H(s,u)\, du = R_H(t,s) \tag{5.1.2}$$

$$H(2H-1) \int_0^t \int_0^s |u-v|^{2H-2}\, dudv = R_H(t,s) \tag{5.1.3}$$

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} \tag{5.1.4}$$

$$\int_0^{u\wedge v} \frac{\partial K_H}{\partial u}(u,y)\frac{\partial K_H}{\partial v}(v,y)\,dy = H(2H-1)|u-v|^{2H-2}. \tag{5.1.5}$$

In this chapter, we will work on the Hilbert space $\mathcal{H} = L^2([0,1])$. The primary object of study is the Hermite process introduced in Definition 4.4.2. Here we will define the Hermite process as a sequence in the qth Wiener chaos, using an integral representation on a finite interval.

Definition 5.1.1. Let $q \geq 1$ and $H \in (1/2, 1)$. The **Hermite process** of order q and self-similarity parameter H denoted $(Z_t^{q,H})_{t\in[0,1]}$ is defined as $Z_t^{q,H} = I_q(L_{t,q})$, where

$$\begin{aligned} L_{t,q}(t_1,\dots,t_q) &= d_H 1_{[0,t]^q}(t_1,\dots,t_q)\int_{t_1\vee\cdots\vee t_q}^{t} \frac{\partial K_{H'}}{\partial u}(u,t_1)\dots\frac{\partial K_{H'}}{\partial u}(u,t_q)\,du,\\ d_H &= \left(\frac{H(2H-1)}{q!(H'(2H'-1))^q}\right)^{1/2},\\ H' &= 1+\frac{H-1}{q}, \end{aligned}$$

and $K_{H'}$ is the kernel of fractional Brownian motion in (5.1.1).

Note that the Hermite process is only defined for $H \in (1/2, 1)$, which is equivalent to $H' \in (1-1/(2q), 1)$. When $H \in (0, 1/2]$, the kernel $L_{t,q}$ is not in $L^2([0,1]^q)$ so that the Hermite process is not defined. The parameter d_H has been chosen so that $\mathrm{E}\left((Z_1^{q,H})^2\right) = 1$. For simplicity, the notation $\partial K(u,t)$ will be used to denote $\partial K_{H'}(u,t)/\partial u$.

Next we will show that the two definitions of the Hermite process given are equivalent.

Proposition 5.1.2. *The Hermite process as defined in Definition 4.4.2 and Definition 5.1.1 are equivalent.*

Proof. See Theorem 1.1 in [PT10]. □

The following result will be useful in a few proofs we give.

Lemma 5.1.3. *Let $q \geq 1$ and $s,t,x,y \in [0,1]$. Suppose that f, g are Riemann-integrable functions. Then*

$$\begin{aligned} &\int_{[0,t\wedge s]^q}\int_{y\vee t_1\vee\cdots\vee t_q}^{t}\int_{x\vee t_1\vee\cdots\vee t_q}^{s} g(u,v,x,y)\prod_{i=1}^{q} f(u,v,t_i)\,du dv dt_1\dots dt_q\\ &= \int_y^t\int_x^s g(u,v,x,y)\left(\int_0^{u\wedge v} f(u,v,t)\,dt\right)^q du dv. \end{aligned}$$

Proof. The result immediately follows from Fubini's theorem, so we only need to discuss how the limits in the integrals change. Since $u \in [x\vee t_1\vee\cdots\vee t_q, s]$ and $v \in [y\vee t_1\vee\cdots\vee t_q, t]$, we have that $t_i \in [0, u\wedge v]$ for all $i = 1,\dots q$, where $u \in [x,s]$ and $v \in [y,t]$. □

The Hermite process is an example of a non-Gaussian self-similar process with stationary increments when $q \geq 2$. We outline of its properties below.

Proposition 5.1.4. *Let $(Z_t^{q,H})_{t\in[0,1]}$ be a Hermite process of order q and self-similarity parameter H. Then* *(i)* *The Hermite process exists if and only if* $H \in (1/2, 1)$.

(ii) $Z^{q,H}$ *is self-similar with parameter* H.

(iii) $Z^{q,H}$ *has stationary increments.*

(iv) $\mathrm{E}\left(Z_t^{q,H}\right) = 0$.

(v) $\mathrm{E}\left(Z_s^{q,H} Z_t^{q,H}\right) = \frac{1}{2}\left(s^{2H} + t^{2H} + |s-t|^{2H}\right)$.

(vi) $Z^{q,H}$ *has locally Hölder-continuous path of all orders* $\alpha < H$.

(vii) *The increments of* $Z^{q,H}$ *have long-range dependence*

Proof. First, we prove (i). Since the Hermite process is defined as a Wiener-Itô integral, we need to show that kernel $L_{t,q} \in L^2([0,1]^q)$ if and only if $H \in (1/2, 1)$. Suppose that $H \in (1/2, 1)$, then

$$\|L_{t,q}\|^2_{\mathcal{H}^{\otimes q}} = q! d_H^2 \int_{[0,t]^q} \int_{t_1 \vee \cdots \vee t_q}^t \int_{t_1 \vee \cdots \vee t_q}^t \partial K(u, t_1) \partial K(v, t_1) \ldots \partial K(u, t_q) \partial K(v, t_q) \, du dv dt_1 \ldots dt_q.$$

Now using Lemma 5.1.3, then (5.1.5) and (5.1.3), we have

$$\begin{aligned}
\|L_{t,q}\|^2_{\mathcal{H}^{\otimes q}} &= q! d_H^2 \int_0^t \int_0^s \left(\int_0^{u \wedge v} \partial K(u, y) \partial K(v, y) \, dy \right)^q du dv \\
&= q! d_H^2 (H'(2H'-1))^q \int_0^t \int_0^s |u-v|^{2H-2} \, du dv \\
&= \frac{q! d_H^2 (H'(2H'-1))^q}{H(2H-1)} R_H(t,t) \\
&= t^{2H} \\
&< \infty.
\end{aligned}$$

So the Hermite process exists for $H \in (1/2, 1)$. Conversely, if $H \geq 1$ then $H' \geq 1$ which implies that the integral in the first equality diverges. If $H \leq 1/2$, then the integral in the second equality diverges.

Now we prove (ii). Let $c > 0$, we have

$$\begin{aligned}
Z_{ct}^{q,H} &= \int_0^{ct} \cdots \int_0^{ct} \int_{y_1 \vee \cdots \vee y_q}^{ct} \partial_1 K_{H'}(u, y_1) \ldots \partial_1 K_{H'}(u, y_q) \, du dW_{y_1} \ldots dW_{y_q} \\
&= c \int_0^{ct} \cdots \int_0^{ct} \int_{\frac{y_1}{c} \vee \cdots \vee \frac{y_q}{c}}^{t} \partial_1 K_{H'}(cu, y_1) \ldots \partial_1 K_{H'}(cu, y_q) \, du dW_{y_1} \ldots dW_{y_q} \\
&= c \int_0^{t} \cdots \int_0^{t} \int_{y_1 \vee \cdots \vee y_q}^{t} \partial_1 K_{H'}(cu, cy_1) \ldots \partial_1 K_{H'}(cu, cy_q) \, du dW_{cy_1} \ldots dW_{cy_q}.
\end{aligned}$$

Now using the self-similarity of Brownian motion, $W_{ct} \stackrel{d}{=} c^{1/2} W_t$ and the fact $\partial_1 K_{H'}(cu, cy) = c^{H'-3/2} \partial_1 K_{H'}(u, y)$, we have that $Z_{ct}^{q,H} = c^H Z_t^{q,H}$. This argument can be made rigorous (see Proposition 14.3.5 in [GKS12]).

Next, (iii) follows from the fact that $L_{t+s} - L_s = L_{t-s}$, so that $I_q(L_{t+s}) - I_q(L_s) = I_q(L_{t-s})$, for all $q \geq 1$ and $s, t \in [0, 1]$. Using the translation invariance of Gaussian white noise, it can be shown that $I_q(L_{t-s}) \overset{d}{=} I_q(L_t)$ (see Proposition 14.3.5 in [GKS12]). Therefore, $Z^{q,H}_{t+s} - Z^{q,H}_s \overset{d}{=} Z^{q,H}_t$.

The other results directly follow from the fact that $Z^{q,H}_t$ is a H-self-similar process with stationary increments by using Proposition 4.3.3, Proposition 4.3.5, and Proposition 4.3.8. □

From Proposition 5.1.4, the Hermite process has zero mean and covariance function R_H. In addition, if $q = 1$, then it is in the first Wiener chaos, so it is also Gaussian. Therefore, when $q = 1$, the Hermite process is fractional Brownian motion with $H \in (1/2, 1)$. It can easily be checked that $L_{1,t}(s) = K_H(t, s)$, so we have a integral representation of fractional Brownian motion on a finite interval,

$$B^H_t = \int_0^t K_H(t, s)\, dB_s.$$

This is why we previously called K_H the kernel of fractional Brownian motion. Note this is only valid when $H \in (1/2, 1)$. When $H \in (0, 1/2)$, the kernel is different. However, we will not need make use of that case.

Definition 5.1.5. When $q = 2$, the Hermite process $Z^{2,H}$ is known as the **Rosenblatt process**.

For $q \geq 2$, the Hermite process is an example of an non-Gaussian self-similar process with stationary increments.

5.2 Convergence of the Quadratic Variation of the Hermite Process

In this section we study the the convergence of the discrete quadratic variations of the Hermite process. We will then use the limiting distribution to construct an estimator for the Hurst parameter and derive its limiting distribution.

Suppose that the Hermite process $Z^{q,H}$ is observed at discrete times $t = 0, t_1, \ldots, t_n$, where $t_i = i/n$. Then the discrete centered p-variation statistic of $Z^{q,H}$ reduces to

$$U_n(p) = \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{\left| Z^{q,H}_{\frac{i+1}{n}} - Z^{q,H}_{\frac{i}{n}} \right|^k}{\mathrm{E}\left(\left| Z^{q,H}_{\frac{i+1}{n}} - Z^{q,H}_{\frac{i}{n}} \right|^k \right)} - 1 \right).$$

In the case where $p = 2$, using Proposition 5.1.4 (v), the discrete centered quadratic variation statistic of $Z^{q,H}$ is

$$U_n := U_n(2) = \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{\left(Z^{q,H}_{\frac{i+1}{n}} - Z^{q,H}_{\frac{i}{n}} \right)^2}{\mathrm{E}\left(\left(Z^{q,H}_{\frac{i+1}{n}} - Z^{q,H}_{\frac{i}{n}} \right)^2 \right)} - 1 \right)$$

$$= n^{2H-1} \sum_{i=0}^{n-1} \left(\left(Z^{q,H}_{\frac{i+1}{n}} - Z^{q,H}_{\frac{i}{n}} \right)^2 - n^{-2H} \right).$$

The main result of this section is to prove that U_n converges to normal random variable in $q = 1$ case when $H \in (1/2, 3/4)$, otherwise it coverages to a Rosenblatt random variable.

In the fractional Brownian motion case, when $q = 1$ and $Z^{q,H} = B^H$, U_n may converge to either a normal random variable or a Rosenblatt random variable.

Theorem 5.2.1. *Let $q = 1$, then we have:*

(i) If $H \in (0, \frac{3}{4})$, then $b_{1,H} n^{-\frac{1}{2}} U_n \xrightarrow{d} N(0,1)$ as $n \to \infty$, where

$$b_{1,H} = \left(2 + \sum_{q=1}^{\infty} \left(2q^{2H} - (q-1)^{2H} - (q+1)^{2H} \right)^2 \right)^{-1/2}.$$

(ii) If $H = \frac{3}{4}$, then $b_{2,H} (n \log(n))^{-\frac{1}{2}} U_n \xrightarrow{d} \mathcal{N}(0,1)$ as $n \to \infty$, where

$$b_{2,H} = \frac{1}{2H(2H-1)}.$$

(iii) If $H \in (\frac{3}{4}, 1)$, then $b_{3,H} n^{1-2H} U_n \xrightarrow{d} Z^{2,2H-1}$ as $n \to \infty$, where

$$b_{3,H} = \left(\frac{4H-3}{2H^2(2H-1)} \right)^{1/2}.$$

Proof. These results follow from Corollary 4.4.4 with $t = 1$, so that we obtain limit theorems for random variables. Put $f = H_2$, the second Hermite polynomial, and set the underlying Gaussian process to fractional Gaussian noise, $X = X^H$. □

In the case of a Hermite process of order $q \geq 2$, the convergence of U_n is always a Rosenblatt random variable. Note that we cannot apply Theorem 4.4.3 because the Hermite process is non-Gaussian, but interestingly the result turns out as expected.

Theorem 5.2.2. *Let $q \geq 2$, then $c_{q,H} n^{2-2H'} U_n \to Z_1^{2,2H'-1}$ in $L^2(\Omega)$ as $n \to \infty$, where*

$$c_{q,H} = \left(\frac{(4H'-3)(4H'-2)\left(((H'-1)(q-1)+1)((2H'-2))(q-1)+1\right)^2}{4(H'(2H'-1))^{2q} d_{q,H}^4 (q-1)!^2 q^4} \right)^{1/2}$$

We will use the next section to prove this result. Note that using the fourth moment theorem or other normal approximation methods will fail here as the limit is non-Gaussian. Thus, we will need to work more directly with the properties of the Wiener chaos.

5.3 Proof of Theorem 5.2.2

The following proof is from Chronopoulou, Tudor and Veins [CTV11, TV09].

Lemma 5.3.1. *Let* $f_{i,n} := L_{(i+1)/n} - L_{i/n}$ *and* $I_i := [(i+1)/n, i/n]$. *Then for all* $r = 1, \dots, q-1$, *we have*

$$(f_{i,n} \otimes_r f_{i,n})(t_1, \dots, t_{2q-2r}) = (H'(2H'-1))^q d_{q,H}^2 \int_{I_i} \int_{I_i} \partial K(u, t_1) \dots \partial K(u, t_{q-r}) \partial K(v, t_{q-r+1}) \dots \partial K(v, t_{2q-2r}) |u-v|^{(2H'-2)q} \, dudv. \quad (5.3.1)$$

Proof. Using the definition of a contraction, we have

$$\begin{aligned}(f_{i,n} \otimes_r f_{i,n})(t_1, \dots, t_{2q-2r}) &= \int_{[0,1]^r} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}})(t_1, \dots, t_{q-r}, s_1, \dots, s_r) \\ &\quad (L_{\frac{i+1}{n}} - L_{\frac{i}{n}})(t_{q-r+1}, \dots, t_{2q-2r}, s_1, \dots, s_r) \, ds_1 \dots ds_r \\ &= A_{(i+1)/n,(i+1)/n} - A_{(i+1)/n,i/n} - A_{i/n,(i+1)/n} + A_{i/n,i/n},\end{aligned}$$

where

$$\begin{aligned}A_{a,b} = d_{q,H}^2 &\int_{[0,1]^r} 1_{[0,a]^q}(t_1, \dots, t_{q-r}, s_1, \dots, s_r) 1_{[0,b]^q}(t_{q-r+1}, \dots, t_{2q-2r}, s_1, \dots, s_r) \\ &\left(\int_{J_{1,a}} \partial K_{H'}(u, t_1) \dots \partial K_{H'}(u, t_{q-r}) \partial K_{H'}(u, s_1) \dots \partial K_{H'}(u, s_r) \, du \right) \\ &\left(\int_{J_{2,b}} \partial K_{H'}(v, t_{q-r+1}) \dots \partial K_{H'}(v, t_{2q-2r}) \partial K_{H'}(v, s_1) \dots \partial K_{H'}(v, s_r) \, dv \right) \\ &ds_1 \dots ds_r \\ J_{1,a} &:= [t_1 \vee \dots \vee t_{q-r} \vee s_1 \vee \dots \vee s_r, a], \\ J_{2,b} &:= [t_{q-r+1} \vee \dots \vee t_{2q-2r} \vee s_1 \vee \dots \vee s_r, b].\end{aligned}$$

Now we can apply Fubini's theorem, followed by (5.1.5) to get

$$\begin{aligned}A_{a,b} &= d_{q,H}^2 1_{[0,a]^{q-r} \times [0,b]^{q-r}}(t_1, \dots, t_{2q-2r}) \int_{t_1 \vee \dots \vee t_{q-r}}^a \int_{t_{q-r+1} \vee \dots \vee t_{2q-2r}}^b \\ &\quad \partial K_{H'}(u, t_1) \dots \partial K_{H'}(u, t_{q-r}) \partial K_{H'}(v, t_{q-r+1}) \dots \partial K_{H'}(v, t_{2q-2r}) \\ &\quad \left(\int_0^{u \wedge v} \partial K_{H'}(u, s) \partial K_{H'}(v, s) \, ds \right)^r dudv \\ &= (H'(2H'-1))^r d_{q,H}^2 \int_{t_1 \vee \dots \vee t_{q-r}}^a \int_{t_{q-r+1} \vee \dots \vee t_{2q-2r}}^b \partial K_{H'}(u, t_1) \dots \partial K_{H'}(u, t_{q-r}) \\ &\quad \partial K_{H'}(v, t_{q-r+1}) \dots \partial K_{H'}(v, t_{2q-2r}) |u-v|^{(2H'-2)r} \, dudv.\end{aligned}$$

In the last equality, the indicator function can be dropped because $\partial K(u, t)$ as a function of t is supported on $[0, u]$ which is a subset of $[0, a]$, while $\partial K(v, t)$ is supported on $[0, v]$. Let $B(u, v)$ be the integrand in $A_{a,b}$ and note that the integrand is the same for all these terms. Now we have

$$A_{(i+1)/n,(i+1)/n} - A_{(i+1)/n,i/n} = (H'(2H'-1))^r d_{q,H}^2 \int_{t_1 \vee \dots \vee t_{q-r}}^{\frac{i+1}{n}} \int_{I_i} B(u, v) \, dudv$$

$$= (H'(2H'-1))^r d_{q,H}^2 \int_{I_i} \int_{t_1 \vee \cdots \vee t_{q-r}}^{\frac{i+1}{n}} B(u,v)\, dvdu. \tag{5.3.2}$$

Using a similar argument, we also have

$$A_{i/n,(i+1)/n} - A_{i/n,i/n} = (H'(2H'-1))^r d_{q,H}^2 \int_{I_i} \int_{t_{q-r} \vee \cdots \vee t_{2q-2r}}^{\frac{i}{n}} B(u,v)\, dvdu. \tag{5.3.3}$$

Finally, combining (5.3.2) and (5.3.3) yields.

$$(f_{i,n} \otimes_r f_{i,n})(t_1, \ldots, t_{2q-2r}) = (H'(2H'-1))^r d_{q,H}^2 \int_{I_i} \int_{I_i} \partial K(u,t_1) \ldots \partial K(u,t_{q-r})$$
$$\partial K(v, t_{q-r+1}) \ldots \partial K(v, t_{2q-2r}) |u-v|^{(2H'-2)r}\, dudv. \tag{5.3.4}$$

We can now calculate the inner product using Fubini's theorem

$$\langle f_{i,n} \otimes_r f_{i,n}, f_{j,n} \otimes_r f_{j,n} \rangle_{\mathcal{H}^{\otimes 2q-2r}}$$
$$= (H'(2H'-1))^{2r} d_{q,H}^4 \int_{[0,1]^{2q-2r}} \left(\int_{I_i} \int_{I_i} B(u,v)\, dudv \right) \left(\int_{I_j} \int_{I_j} B(u',v')\, du'dv' \right)$$
$$dt_1 \ldots dt_{2q-2r}$$
$$= (H'(2H'-1))^{2r} d_{q,H}^4 \int_{I_i} \int_{I_i} \int_{I_j} \int_{I_j} \left(\int_0^1 \partial K(u,t) \partial K(u',t)\, dt \right)^{q-r}$$
$$\left(\int_0^1 \partial K(v,t) \partial K(v',t)\, dt \right)^{q-r} |u-v|^{(2H'-2)r} |u'-v'|^{(2H'-2)r}\, dv'du'dvdu$$
$$= (H'(2H'-1))^{2q} d_{q,H}^4 \int_{I_i} \int_{I_i} \int_{I_j} \int_{I_j} |u-v|^{(2H'-2)r} |u'-v'|^{(2H'-2)r}$$
$$|u-u'|^{(2H'-2)(q-r)} |v-v'|^{(2H'-2)(q-r)}\, dv'du'dvdu.$$

Note that $\partial K(u,t)\partial K(u',t)$ as a function of t is nonzero only when $u > t$ and $u' > t$, so that we can replace integration over $[0,1]$ with integration over $[0, u \wedge u']$. This allows (5.1.5) to be used at the last line with a similar result for v. Now make the change of variables

$$x = \left(u - \frac{i}{n}\right) n, \quad y = \left(v - \frac{i+1}{n}\right) n, \quad x' = \left(u' - \frac{j}{n}\right) n, \quad y' = \left(v' - \frac{j+1}{n}\right) n,$$

which gives

$$\langle f_{i,n} \otimes_r f_{i,n}, f_{j,n} \otimes_r f_{j,n} \rangle_{\mathcal{H}^{\otimes 2q-2r}} = (H'(2H'-1))^{2q} d_{q,H}^4 n^{-(2H'-2)2q-4} \int_{[0,1]^4} |x-y|^{(2H'-2)r}$$
$$|x'-y'|^{(2H'-2)r} |x-x'+i-j|^{(2H'-2)(q-r)}$$
$$|y-y'+i-j|^{(2H'-2)(q-r)}\, dxdydx'dy'. \tag{5.3.5}$$

This concludes the proof of the lemma. □

Proof of Theorem 5.2.2. In this proof, we work on the Hilbert space $\mathcal{H} = L^2([0,1])$ and assume that $q \geq 2$. We complete this proof in 4 steps.

Step 1: Compute the Wiener chaos expansion of U_n.

Define the kernel of $Z^{q,H}_{(i+1)/n} - Z^{q,H}_{i/n}$ as $f_{i,n} := L_{(i+1)/n} - L_{i/n}$. Then using Corollary 1.4.8 and Proposition 5.1.4 (ii), we have

$$\left(Z^{q,H}_{(i+1)/n} - Z^{q,H}_{i/n}\right)^2 = q!\,\|f_{i,n}\|^2_{\mathcal{H}^{\otimes q}} + \sum_{r=0}^{q-1} r!\binom{q}{r}^2 I_{2q-2r}\left(f_{i,n}\otimes_r f_{i,n}\right)$$
$$= n^{-2H} + \sum_{r=0}^{q-1} r!\binom{q}{r}^2 I_{2q-2r}\left(f_{i,n}\otimes_r f_{i,n}\right).$$

Therefore,

$$U_n = T_{2q} + c_{2q-2}T_{2q-2} + \cdots + c_2 T_2, \tag{5.3.6}$$

where

$$T_{2q-2r} := n^{2H-1} I_{2q-2r}\left(\sum_{i=0}^{n-1} f_{i,n}\otimes_r f_{i,n}\right) \quad \text{and} \quad c_{2q-2r} := r!\binom{q}{r}^2, \tag{5.3.7}$$

for all $r = 0, 1, \ldots, q-1$.

Step 2: Show that T_2 is the leading term of the Wiener chaos expansion.

From (5.3.7), the orthogonality property and the fact that $\left\|\widetilde{f}_{i,n}\right\|_{\mathcal{H}^{\otimes q}} \le \|f_{i,n}\|_{\mathcal{H}^{\otimes q}}$ we have that

$$\mathrm{E}(T^2_{2q-2r}) \le (2q-2r)!n^{4H-2}\left\|\sum_{i=0}^{n-1} f_{i,n}\otimes_r f_{i,n}\right\|^2_{\mathcal{H}^{\otimes 2q-2r}} \tag{5.3.8}$$
$$= (2q-2r)!n^{4H-2}\sum_{i,j=0}^{n-1}\langle f_{i,n}\otimes_r f_{i,n}, f_{j,n}\otimes_r f_{j,n}\rangle_{\mathcal{H}^{\otimes 2q-2r}}.$$

This sum decomposes into a diagonal term C_n when $i=j$ and a non-diagonal term C'_n when i, so that $\mathrm{E}(T^2_{2q-2r}) \le C_n + C'_n$. Using Lemma 5.3.1 we get

$$C_n = (2q-2r)!(H'(2H'-1))^{2q}d^4_{q,H}n^{-2}\sum_{i=0}^{n-1}\int_{[0,1]^4}|x-y|^{(2H'-2)r}|x'-y'|^{(2H'-2)r}$$
$$|x-x'|^{(2H'-2)(q-r)}|y-y'|^{(2H'-2)(q-r)}\,dxdydx'dy'.$$

Since $H\in(1/2,1)$, the exponents $(2H'-2)r, (2H'-2)(q-r) > -1$, so the integral is bounded. Therefore $C = O(n^{-3})$, which approaches 0 as $n\to\infty$. Next, we turn to the non-diagonal term.

$$C'_n = (2q-2r)!(H'(2H'-1))^{2q}d^4_{q,H}n^{-2}2\sum_{i>j}\int_{[0,1]^4}|x-y|^{(2H'-2)r}|x'-y'|^{(2H'-2)r}$$
$$|x-x'+i-j|^{(2H'-2)(q-r)}|y-y'+i-j|^{(2H'-2)(q-r)}\,dxdydx'dy'$$
$$= (2q-2r)!(H'(2H'-1))^{2q}d^4_{q,H}n^{-2}2\sum_{i=0}^{n-2}\sum_{j=1}^{n-1-i}\int_{[0,1]^4}|x-y|^{(2H'-2)r}|x'-y'|^{(2H'-2)r}$$
$$|x-x'+j|^{(2H'-2)(q-r)}|y-y'+j|^{(2H'-2)(q-r)}\,dxdydx'dy'$$

$$= (2q-2r)!(H'(2H'-1))^{2q} d_{q,H}^4 n^{-2} 2 \sum_{i=0}^{n-2} \sum_{j=1}^{n-1-i} \int_{[0,1]^4} |x-y|^{(2H'-2)r} |x'-y'|^{(2H'-2)r}$$
$$|x-x'+j|^{(2H'-2)(q-r)} |y-y'+j|^{(2H'-2)(q-r)} \, dx dy dx' dy'$$
$$= (2q-2r)!(H'(2H'-1))^{2q} d_{q,H}^4 n^{-2} 2 \sum_{j=1}^{n-1} (n-j) \int_{[0,1]^4} |x-y|^{(2H'-2)r} |x'-y'|^{(2H'-2)r}$$
$$|x-x'+j|^{(2H'-2)(q-r)} |y-y'+j|^{(2H'-2)(q-r)} \, dx dy dx' dy'$$
$$= (2q-2r)!(H'(2H'-1))^{2q} d_{q,H}^4 n^{(2H'-2)(2q-2r)} 2 \int_{[0,1]^4} |x-y|^{(2H'-2)r} |x'-y'|^{(2H'-2)r}$$
$$\left(\frac{1}{n} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \left| \frac{x-x'}{n} + \frac{j}{n} \right|^{(2H'-2)(q-r)} \left| \frac{y-y'}{n} + \frac{j}{n} \right|^{(2H'-2)(q-r)} \right) dx dy dx' dy'.$$

Now the term in the brackets is a Riemann sum, as $n \to \infty$ we have

$$\lim_{n\to\infty} n^{-(2H'-2)(2q-2r)} C_n' = 2(2q-2r)!(H'(2H'-1))^{2q} d_{q,H}^4 \left(\int_0^1 \int_0^1 |x-y|^{(2H'-2)r} \, dx dy \right)^2$$
$$\left(\int_0^1 (1-x) x^{(2H'-2)(2q-2r)} \, dx \right)$$
$$= \frac{2(2q-2r)!(H'(2H'-1))^{2q} d_{q,H}^4}{\left(((H'-1)r+1)((2H'-2)r+1)\right)^2}$$
$$\frac{1}{((2H'-2)(2q-2r)+1)\left((2H'-2)(2q-2r)+2\right)}.$$

where the first integral was evaluated using (5.1.3). Since $H \in (1/2, 1)$ implies that $(2H'-2)(2q-2r) > -1$, the off-diagonal term is dominant and we have $C_n' = O(n^{(2H'-2)(2q-2r)})$ as $n \to \infty$. Therefore, for all $r = 0, \dots, q-1$,

$$\mathrm{E}(T_{2q-2r}^2) = O(n^{(2H'-2)(2q-2r)}) \tag{5.3.9}$$

as $n \to \infty$.

Thus, the dominant term in U_n is T_2 which occurs when $r = q-1$. In this case, the inequality in (5.3.8) can be replaced with equality because $f_{i,n} \otimes_{q-1} f_{i,n}$ are symmetric functions. Therefore, $U_n \sim c_2 T_2$ and have proved that

$$\lim_{n\to\infty} c_{q,H}^2 n^{2(2-2H')} \mathrm{E}(U_n^2) = 1,$$

where

$$c_{q,H}^2 = \frac{(4H'-3)(4H'-2)\left(((H'-1)(q-1)+1)((2H'-2))(q-1)+1)\right)^2}{4(H'(2H'-1))^{2q} d_{q,H}^4 (q-1)!^2 q^4}.$$

In order to show that $c_{q,H} n^{2-2H'} U_n \xrightarrow{L^2(\Omega)} Z_1^{2,2H'-1}$, (5.3.6) and the orthogonality property implies that it suffices to show $c_2 c_{q,H} n^{2-2H'} T_2 \xrightarrow{L^2(\Omega)} Z_1^{2,2H'-1}$ and $n^{2-2H'} T_{2q-2r} \xrightarrow{L^2(\Omega)} 0$ for all $r = 0, \dots, q-2$. Since we have proven to latter, we will now prove the former. Let f_n be the kernel of $c_2 c_{q,H} n^{2-2H'} T_2$. By combining (5.3.7) and (5.3.4) that is

$$f_n(t_1, t_2) = c_{q,H}' (H'(2H'-1))^{q-1} d_{q,H}^2 n^{2-2H'} n^{2H-1} \sum_{i=0}^{n-1} \int_{I_i} \int_{I_i} \partial K(u, t_1) \partial K(v, t_2)$$

$$|u-v|^{(2H'-2)(q-1)}\,dudv,$$

where $c'_{q,H} := c_2 c_{q,H}$. It is enough to show that f_n converges in $L^2([0,1]^2)$ to the kernel of $Z_1^{2,2H'-1}$ which is

$$L_{1,2}(t_1,t_2) = d_{2,2H'-1} 1_{[0,1]^2}(t_1,t_2) \int_{t_1 \vee t_2}^1 \partial K(u,t_1)\partial K(u,t_2)\,du.$$

Step 3: Prove that the kernel of T_2' converges almost everywhere to the kernel of a Rosenblatt random variable using a Riemann sum argument.

For large n, $\partial K(u,t)$ is approximately equal to $\partial K(i/n,t)$ when $u \in I_i$ and $u > t$. Hence,

$$f_n(t_1,t_2) \sim c'_{q,H}(H'(2H'-1))^{q-1} d_{q,H}^2 n^{2-2H'} n^{2H-1} \sum_{i=0}^{n-1} \int_{I_i}\int_{I_i} 1_{\{i/n > t_1 \wedge t_2\}}(t_1,t_2)$$
$$\partial K\left(\frac{i}{n},t_1\right)\partial K\left(\frac{i}{n},t_2\right)|u-v|^{(2H'-2)(q-1)}\,dudv. \tag{5.3.10}$$

Using a change of variables then applying (5.1.3), it can easily be shown that

$$\int_{I_i}\int_{I_i}|u-v|^{(2H'-2)(q-1)}\,dudv = \frac{n^{(2H'-2)(q-1)-2}}{((H'-1)(q-1)+1)((2H'-2)(q-1)+1)}. \tag{5.3.11}$$

It can also be readily verified that $c'_{q,H}(H'(2H'-1))^{q-1}d_{q,H}^2 = d_{2,2H'-1}$. Combining this fact with (5.3.10) and (5.3.11) yields

$$f_n(t_1,t_2) \sim d_{2,2H'-1} n^{-1} \sum_{i=0}^{n-1} 1_{\{i/n > t_1\wedge t_2\}}(t_1,t_2)\partial K\left(\frac{i}{n},t_1\right)\partial K\left(\frac{i}{n},t_2\right).$$

This is a Riemann sum, so taking $n \to \infty$ we have

$$\lim_{n\to\infty} f_n(t_1,t_2) = d_{2,2H'-1} 1_{[0,1]^2}(t_1,t_2)\int_{t_1\vee t_2}^1 \partial K(u,t_1)\partial K(u,t_2)\,du.$$

Thus, f_n converges to $L_{1,2}$ almost everywhere as $n \to \infty$.

Step 4: Finally, prove that the kernel of T_2' converges in $L^2([0,1]^2)$ to the kernel of a Rosenblatt random variable using a Cauchy sequence argument.

To complete the proof we will show that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This implies that f_n converges in $L^2([0,1]^2)$, which must coincide with the pointwise limit proved in the previous step. It suffices to prove that $\langle f_n, f_m\rangle_{\mathcal{H}^{\otimes 2}}$ converges to a constant as $n,m\to\infty$. Notice that

$$f_n = K n^{2-2H'} n^{2H-1} \sum_{i=0}^{n-1} f_{i,n} \otimes_{q-1} f_{i,n},$$

for some K constant with respect to n. Then using (5.3.5), we have

$$\langle f_n, f_m\rangle_{\mathcal{H}^{\otimes 2}} = K^2(nm)^{-1}\int_{[0,1]^4}|x-y|^{(2H'-2)(q-1)}|x'-y'|^{(2H'-2)(q-1)}\left|\frac{x}{n}-\frac{x'}{m}+\frac{i}{n}-\frac{j}{m}\right|^{2H'-2}$$

$$\left|\frac{y}{n} - \frac{y'}{m} + \frac{i}{n} - \frac{j}{m}\right|^{2H'-2} dxdydx'dy'.$$

The terms x/n, x'/m, y/n, and y'/m are negligible for large n and m. Thus, this is a Riemann sum, so taking $n, m \to \infty$ gives

$$\lim_{n,m\to\infty} \langle f_n, f_m \rangle_{\mathcal{H}^{\otimes 2}} = K^2 \left(\int_{[0,1]^2} |x-y|^{2H'-2}\, dxdy \right)^2 \int_{[0,1]^2} |x-y|^{(2H'-2)(q-1)}\, dxdy.$$

Since $H \in (1/2, 1)$, the exponents $2H'-2, (2H'-2)(q-1) > -1$, so the integral is bounded and the limit approaches a constant. Thus, $\|f_n - f_m\|_{\mathcal{H}^{\otimes 2}} \to 0$ as $n, m \to \infty$, so that f_n is Cauchy. Therefore, f_n converges to $L_{1,2}$ in $L^2([0,1]^2)$ as $n \to \infty$. This completes the proof of the theorem. □

5.4 Estimating Hurst Parameter

Let the discrete uncentered quadratic variation of $Z^{q,H}$ be

$$S_N = \frac{1}{n} \sum_{i=0}^{n-1} \left(Z^{q,H}_{\frac{i+1}{n}} - Z^{q,H}_{\frac{i}{n}} \right)^2.$$

Then it immediately follows that $1 + U_n = n^{2H} S_n$, which motivates

$$\hat{H}_n = -\frac{\log S_n}{2 \log n}$$

as a estimator for the Hurst parameter H.

In this section we will show that $\hat{H}_n$ is a consistent estimator for H and derive its limiting distribution.

Proposition 5.4.1. *The estimator $\hat{H}_n \to H$ almost surely as $n \to \infty$.*

Proof. For simplicity, K will denote a constant with respect to n in this proof, but will not necessarily be the same constant. Using the Wiener chaos expansion of U_n, the orthogonality property, and Hölder's inequality, we have

$$\mathrm{E}(|U_n|^p) = \mathrm{E}\left(\left| \sum_{r=0}^{q-1} c_{2q-2r} T_{2q-2r} \right|^p \right) \leq \sum_{r=0}^{q-1} c_{2q-2r}\, \mathrm{E}(|T_{2q-2r}|^p). \tag{5.4.1}$$

Using (5.3.9) for all $r = 0, \ldots, q-1$, there exists a positive integer m_1, such that for all $n > m_1$, we have $\mathrm{E}(T^2_{2q-2r}) \leq K n^{(2H'-2)(2q-2r)}$. So for sufficiently large n, T_2 is the dominant term. Combining this with the hypercontractivity property of Wiener-Itô integrals yields

$$\mathrm{E}(|T_{2q-2r}|^p) \leq \mathrm{E}(T^2_{2q-2r})^{p/2} \leq \mathrm{E}(T_2^2)^{p/2} \leq K n^{(2H'-2)p}, \tag{5.4.2}$$

where $p > 1$. Therefore, (5.4.1) and (5.4.1) gives $\mathrm{E}(|U_n|^p) \leq K n^{(2H'-2)p}$. Combining this result with Markov's inequality, we have

$$\mathrm{P}(|U_n| > \epsilon) = \mathrm{P}(|U_n|^p > \epsilon^p) \leq \epsilon^{-p}\, \mathrm{E}(|U_n|^p) \leq K \epsilon^{-p} n^{(2H'-2)\epsilon} \leq K n^{\left(-\frac{\log(\epsilon)}{\log(n)} - \frac{2H-2}{q}\right)p}.$$

Now all all $\epsilon > 0$,

$$-\frac{\log(\epsilon)}{\log(n)} - \frac{2H-2}{q} < 0, \tag{5.4.3}$$

when $\epsilon > n^{(2H-2)/q}$. Since $H \in (0,1)$, it follows that there exists a positive integer m_2, such that if $n > m_2$, then (5.4.3) holds. Now choose $p > 1$ large enough that $\left(-\frac{\log(\epsilon)}{\log(n)} - \frac{2H-2}{q}\right) p < -1$. Therefore,

$$\sum_{n=0}^{\infty} \mathrm{P}(|U_n| > \epsilon) \leq K + K \sum_{n=m_1 \vee m_2}^{\infty} n^{\left(-\frac{\log(\epsilon)}{\log(n)} - \frac{2H-2}{q}\right)p} < \infty.$$

By Proposition A.3.4, this implies that $U_n \xrightarrow{a.s.} 0$. Next, we can write

$$\log(1+U_n) = -2(\hat{H}_n - H)\log(n).$$

Combining this with the fact that $\log(1+x)/x \to 0$ as $x \to \infty$, we have that

$$U_n = 2(H - \hat{H}_n)\log(n)(1+o(1)). \tag{5.4.4}$$

Thus, as $n \to \infty$, $\hat{H}_n \xrightarrow{a.s.} H$ because $U_n \xrightarrow{a.s.} 0$. □

Theorem 5.4.2. *Let $q = 1$, then we have:*

(i) If $H \in (\frac{1}{2}, \frac{3}{4})$, then $b'_{1,H} n^{-\frac{1}{2}} \log(n)(\hat{H}_n - H) \xrightarrow{d} \mathcal{N}(0,1)$ as $n \to \infty$, where $b'_{1,H} = 2b_{1,H}$.

(ii) If $H = \frac{3}{4}$, then $b'_{2,H}(n\log(n))^{-\frac{1}{2}}(\hat{H}_n - H) \xrightarrow{d} \mathcal{N}(0,1)$ as $n \to \infty$, where $b'_{2,H} = 2b_{2,H}$

(iii) If $H \in (\frac{3}{4}, 1)$, then $b'_{3,H} n^{1-H} \log(n)(\hat{H}_n - H) \xrightarrow{d} Z^{2,H}$ as $n \to \infty$, where $b'_{3,H} = 2b_{3,H}$

In the case of a Hermite process of order $q \geq 2$, the convergence of U_n is always a Rosenblatt random variable.

Theorem 5.4.3. *Let $q \geq 2$, then $c'_{q,H} n^{2-2H'} \log(n)(\hat{H}_n - H) \to Z_1^{2,2H'-1}$ in $L^2(\Omega)$ as $n \to \infty$, where $c'_{q,H} = 2c_{q,H}$*

Proof. In Theorem 5.2.2, we proved that

$$\mathrm{E}\left(\left|c_{q,H} n^{2-2H'} U_n - Z_1^{2,2H'-1}\right|^2\right) \to 0,$$

as $n \to \infty$. Now using (5.4.4), we have

$$\mathrm{E}\left(\left|2c_{q,H} n^{2-2H'} \log(n)(H - \hat{H}_n)(1+o(1)) - Z_1^{2,2H'-1}\right|^2\right) \to 0.$$

Taking $n \to \infty$ gives the required conclusion. □

Proposition 5.4.4. *Let $q \geq 2$ and $\hat{H}'_n = 1 + (\hat{H}_n - 1)/q$, then in $L^1(\Omega)$,*

$$c'_{q,H} n^{2-2\hat{H}'_n} \log(n)(\hat{H}_n - H) \to Z_1^{2,2H'-1}$$

as $n \to \infty$.

Proof. It suffices to show that

$$\lim_{n\to\infty} \mathrm{E}\left(\left|\left(n^{2-2\hat{H}_n'} - n^{2-2H'}\right)\left(H-\hat{H}_n\right)\log(n)\right|\right) = 0. \tag{5.4.5}$$

Choose $\epsilon \in (0, 2-2H')$ and recalling the definition of H',

$$C := \left\{\hat{H}_n' > 2H' - 1 + \epsilon/2\right\} = \left\{\hat{H}_n > 2H - 1 + q\epsilon/2\right\}.$$

Now we can write

$$\begin{aligned}\mathrm{E}\left(\left|\left(n^{2-2\hat{H}_n'} - n^{2-2H'}\right)\left(H-\hat{H}_n\right)\log(n)\right|\right) &= \mathrm{E}\left(\left|\left(n^{2-2\hat{H}_n'} - n^{2-2H'}\right)\left(H-\hat{H}_n\right)\log(n)\right| 1_C\right) \\ &\quad + \mathrm{E}\left(\left|\left(n^{2-2\hat{H}_n'} - n^{2-2H'}\right)\left(H-\hat{H}_n\right)\log(n)\right| 1_{C^c}\right) \\ &=: A + B.\end{aligned}$$

First we estimate A. Let $x = (2-2H') \vee (2-\hat{H}_n')$ and $y = (2-2H') \wedge (2-\hat{H}_n')$. Also, note that $e^z - 1$ is a convex function so $e^z - 1 \le ze^z$. Thus we have

$$\begin{aligned}\left|n^{2-2\hat{H}_n'} - n^{2-2H'}\right| &= e^{y\log(n)}\left(e^{(x-y)\log(n)} - 1\right) \\ &\le n^y \log(n)(x-y)n^{x-y} \\ &= 2\log(n)n^x\left|H' - \hat{H}_n'\right| \\ &= \frac{2}{q}\log(n)n^x\left|H - \hat{H}_n\right|.\end{aligned}$$

Then

$$\begin{aligned}A &\le \frac{2}{q}\,\mathrm{E}\left(1_C n^x \left|H-\hat{H}_n\right|^2 \log(n)^2\right) && (5.4.6)\\ &= \frac{2}{q} n^{x-2(2-2H')}\,\mathrm{E}\left(1_C n^{2(2-2H')}\left|H-\hat{H}_n\right|^2\log(n)^2\right). && (5.4.7)\end{aligned}$$

When $\omega \in C$ and either case $x = 2-2H'$ or $x = 2-2\hat{H}_n'$, we have $x - 2(2-2H') < -\epsilon$, so that

$$A \le \frac{2}{q} n^{-\epsilon}\,\mathrm{E}\left(\left|n^{2(2-2H')}\left(H-\hat{H}_n\right)^2\log(n)^2\right|\right).$$

The expectation converges to a constant due to Theorem 5.4.3, so $A \to 0$ as $n \to \infty$. If $\omega \notin C$, then $A = 0$.

Next, we estimate B. Since $\omega \in C^c$ and $\epsilon < 2-2H'$, this implies that $\left|n^{2-2H'} - n^{2-2\hat{H}_n}\right| \le n^{2-2\hat{H}_n}$ and $\left|H-\hat{H}_n\right| \le H$, and then using Hölder's inequality, we have

$$\begin{aligned}B &\le H\log(n)\,\mathrm{E}\left(1_{C^c} n^{2-2\hat{H}_n}\right) \\ &\le H\log(n)\,\mathrm{P}(C^c)^{1/p}\,\mathrm{E}\left(n^{2-2\hat{H}_n}2q\right)^{1/(2q)},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{2q} = 1$. Using Markov's inequality, followed by Theorem 5.4.3, we have

$$\begin{aligned} \mathrm{P}(C^c)^{1/p} &\leq \frac{\mathrm{E}\left(\left|H - \hat{H}_n\right|^2\right)^{1/p}}{(1 - H - q\epsilon/2)^{2/p}} \\ &\leq K n^{-2(2-2H')/p}, \end{aligned}$$

where K is a constant that does not depend on n. Now, we can write $1 + U_n = n^{2q(2-2\hat{H}_n')} n^{-2q(2-2H')}$ so that

$$\mathrm{E}\left(N^{2-2\hat{H}_n} 2q\right)^{1/(2q)} \leq n^{2-2H'}\,\mathrm{E}(1 + U_n)^{1/(2q)} = N^{2-2H'},$$

since $\mathrm{E}(U_n) = 0$. Putting this together, we have

$$B \leq KH \log(n) n^{-2(2-2H')(2/p-1)}.$$

AS $2q \geq 4$, we have $p \leq 4/3$ and $-2(2 - 2H')(2/p - 1) < 0$. So $B \to 0$ as $n \to \infty$. □

Note that all of limit theorems in the section also hold in distribution, as convergence in distribution is weaker than convergence in $L^q(\Omega)$.

There are other interesting applications of Malliavin calculus to parameter estimation for Hermite processes. For example Breton, Nourdin and Peccati [BNP09] find exact confidence intervals for the Hurst parameter of fractional Brownian motion using Malliavin calculus.

Conclusion

Limit theorems through the use of Malliavin calculus and Stein's method has been a recent development in probability theory and stochastic analysis. In this text, we have aimed to provide a comprehensive overview of this subject.

In particular, we have provided an introduction to the Wiener chaos and Malliavin calculus, which gives the foundation for this theory. Using these tools with Stein's method allows for the derivation of total variation bounds to prove the fourth moment theorem. A corollary to the fourth moment theorem was then used to prove the Breuer-Major theorem, a central limit theorem for partial sum processes under short range dependence. In the case of long range dependence, there is a noncentral limit theorem whereby these partial sum processes converge to Hermite processes, a class of self-similar processes with stationary increments that live on the Wiener chaos that are in general non-Gaussian.

An application of these central and non-central limit theorems was to find the limiting distribution of an estimator of the Hurst parameter of fractional Brownian motion. In the case of a Hermite process, using the properties of the Wiener chaos, it was shown that the estimator of the Hurst parameter always converges to a Rosenblatt distribution.

In this text, we have given only a very small sample applications of this theory. However, there is a large diversity of interesting applications and open questions in the literature. For example, in the multivariate setting it is unknown if a central or non-central limit theorem continues to hold for sequences of random vectors where components have both short and long range dependence when the Hermite rank is greater than 2 [BT13a]. Also, it is surprising that still not much is known about the Hermite process.

Recently, there have also been interesting theoretical developments using the techniques of Malliavin calculus. For example, Nourdin and Peccati [NP13] have computed exact rates of convergence of the fourth moment theorem in the total variation bound. It has also been shown by Arizmendi [Ari13] that for infinitely divisible distributions, the fourth moment condition is sufficient for convergence to a normal distribution. There have also been new applications, for example, Bardet and Tudor [BT13b] use these methods to investigate the distribution for the Whittle estimator of the Hurst parameter of the Rosenblatt distribution. So this is an exciting area of active research.

Appendix A

Analysis and Probability

In this appendix we list some miscellaneous results from measure theory, functional analysis and probability theory that are used in the main text.

A.1 Measure Theory

The following result is a corollary to Lusin's lemma and allows us to approximate indicator functions by a sequence of bounded and continuous functions.

Lemma A.1.1 (Lusin). *Let $a > 0$, $B \subseteq [-a, a]$ be a Borel set on $\mathbb{R}$, and μ be finite measure on $[-a, a]$. Then there exists a sequence of continuous functions $(g_n)_{n\geq 1}$ with support included in $[-a, a]$, such that $g_n(x) \in [0, 1]$ and $g_n(x) \to 1_B(x)$ μ almost everywhere as $n \to \infty$.*

Proof. See corollary to Theorem 2.24 in [Rud87]. □

A.2 Closed and Closable Operators

Many important linear operators are unbounded. However, some of these operators, such as the derivative, turn out to be closed. In this section, we will discuss the construction of closed extensions and see that it is possible to uniquely define the adjoint of unbounded operators that are closed.

This summary on closed and closable operators is from Section 2.7 in [Jac01]. These results apply to Banach spaces, as these results will be required to prove that the Malliavin derivative $D : \mathcal{S} \subseteq L^q(\Omega) \to L^q(\Omega \to \mathbb{H})$ is a closable operator for $q \in [1, \infty)$. When $q = 2$, this reduces to the Hilbert space case, and these results on closed and closable operators can be found in most standard textbooks on functional analysis, such as [Kre78] and [Con90].

Definition A.2.1. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. The **direct sum** of $\mathcal{X}$ and $\mathcal{Y}$ is the Banach space $\mathcal{X} \oplus \mathcal{Y}$, with elements (x, y) where $x \in \mathcal{X}, y \in \mathcal{Y}$. The usual operations on $\mathcal{X} \oplus \mathcal{Y}$ are defined component-wise and the **graph norm** is

$$\|(x, y)\|^2_{\mathcal{X}\oplus\mathcal{Y}} = \|x\|^2_{\mathcal{X}} + \|y\|^2_{\mathcal{Y}}.$$

Definition A.2.2. Let $A : \mathrm{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ be a linear operator with its domain denoted by $\mathrm{dom}(A)$, then the **graph** of A is defined as

$$\mathrm{gra}(A) := \{(x, A(x)) \in \mathcal{X} \oplus \mathcal{Y} \,|\, x \in \mathrm{dom}(A)\}.$$

Definition A.2.3. A linear operator $A : \operatorname{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ is **densely defined** if $\operatorname{dom}(A)$ is dense in $\mathcal{X}$.

Definition A.2.4. An linear operator $A : \operatorname{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ is **closed** if $\operatorname{gra}(A)$ is a closed set in $\mathcal{X} \oplus \mathcal{Y}$.

Proposition A.2.5. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. The following are equivalent:*

(i) The operator A is closed.

(ii) For all sequences $(x_n)_{n\geq 1} \subseteq \operatorname{dom}(A)$, if $x_n \to x \in \mathcal{X}$ and $A(x_n) \to y \in \mathcal{Y}$, then $x \in \operatorname{dom}(A)$ and $A(x) = y$.

(iii) The domain $\operatorname{dom}(A)$ equipped with the graph norm is a Banach space.

Definition A.2.6. Let $A : \operatorname{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ and $B : \operatorname{dom}(B) \subseteq \mathcal{X} \to \mathcal{Y}$ be linear operators, then A is an **extension** of B if $\operatorname{dom}(B) \subseteq \operatorname{dom}(A)$ and $A(y) = B(y)$ for all $y \in \mathcal{Y}$. Further, A is a **closable operator** if it is the closed extension of some operator.

Let $\overline{X}$ denote the closure of $X \subseteq \mathcal{X}$.

Proposition A.2.7. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. The following are equivalent:*

(i) The operator A is closable.

(ii) For all sequences $(x_n)_{n\geq 1} \subseteq \operatorname{dom}(A)$ such that $x_n \to 0 \in \mathcal{X}$ and $A(x_n) \to y \in \mathcal{Y}$, then $y = 0$.

(iii) There exists some operator with graph $\overline{\operatorname{gra}(A)}$.

Usually, we use Proposition A.2.7 (ii) to check that an operator is closable. From Definition A.2.6, it is clear that it is possible to enlarge the domain of a closable operator in a consistent way to construct a closed operator. From Proposition A.2.7 (iii) and Proposition A.2.5 (ii), this construction should be done as follows: consider all sequences $(x_n)_{x=1}^{\infty} \subseteq \operatorname{dom}(A)$ such that $x_n \to x \in \mathcal{X}$ and $A(x_n) = y \in \mathcal{Y}$, then include all such x in $\operatorname{dom}(A)$ to form $\operatorname{dom}(\overline{A})$ and set $\overline{A}(x) = y$. The result is an operator $\overline{A} : \operatorname{dom}(\overline{A}) \subseteq \mathcal{X} \to \mathcal{Y}$, which is a closed extension of A. The closed extension is unique when A is densely defined. This is summarized in the following proposition. Proposition A.2.5 (iii), implies that $\operatorname{dom}(\overline{A})$ is the closure of $\operatorname{dom}(A)$ with respect to the graph norm.

Proposition A.2.8. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Let $A : \operatorname{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ be a densely defined linear operator. Then A has a unique closed extension of $\overline{A} : \operatorname{dom}(\overline{A}) \subseteq \mathcal{X} \to \mathcal{Y}$.*

In the special case of Hilbert spaces, it is possible to define an adjoint for closed operators.

Proposition A.2.9. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces and $A : \operatorname{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ is a densely defined and closed linear operator. Then there exists a unique adjoint $A^* : \operatorname{dom}(A^*) \subseteq \mathcal{Y} \to \mathcal{X}$. Furthermore,*

$$\operatorname{dom}(A^*) = \{y \in \mathcal{Y} \,|\, x \mapsto \langle A(x), y\rangle_{\mathcal{Y}} \textit{ is a bounded for } x \in \mathcal{X}\} \tag{A.2.1}$$

and A^* is closed.

A.3 Probability Theory

The following result gives a criteria for convergence in moments.

Proposition A.3.1. *Let $r \geq 1$ be an integer. If $X_n \xrightarrow{d} X$ and $\sup_{n\geq 0} E(|X_n|^{r+\epsilon}) < \infty$ where $\epsilon > 0$, then $\mathrm{E}(X_n^r) \to \mathrm{E}(X^r)$ and $X \in L^r(\Omega)$.*

Proof. See corollary to Theorem 25.12 in [Bil95]. □

The next two proposition are from Section 6.6 in [Res99].

Proposition A.3.2. *Let $X_n, X \in L^1(\Omega)$ for all $n \geq 1$. If $X_n \to X$ in L^1 as $n \to \infty$, then $\mathrm{E}(X_n) \to \mathrm{E}(X)$ as $n\infty$. In particular, the conclusion holds when $X_n \to X$ in L^2.*

Proposition A.3.3. *Let $p \in [1, \infty]$. If $X_n \to X$ in L^p as $n \to \infty$, then $\|X_n\|_p \to \|X\|_p$ as $n \to \infty$*

The following is a simple criteria for almost sure convergence.

Proposition A.3.4. *Let X_n, X be random variables. If for each $\epsilon > 0$, we have $\sum_{n=1}^{\infty} \P(|X_n - X| > \epsilon) < \infty$, then $X_n \xrightarrow{a.s.} X$ as $n\infty$.*

Proof. See Corollary 5.2.2 of [Ros06]. □

Appendix B

Hilbert Spaces

We list some concepts and results about orthonormal bases and the direct sums and tensor products of Hilbert spaces. The facts about direct sums and tensor products are not always found in the standard textbooks on functional analysis. This will mainly be used in Chapter 1 and 2.

Throughout this appendix we will assume that the Hilbert space $\mathcal{H}$ is real and separable.

B.1 Hilbert Space Valued Functions

Let $(T, \mathcal{B}, \mu)$ be a measure space and $\mathcal{H}$ be a Hilbert space. Then for $p \geq 1$, $L^p(T \to \mathcal{H}) := L^p(T \to \mathcal{H}, \mathcal{B}, \mu)$ is the set of $\mathcal{H}$-valued functions, f, such that f is $\mathcal{B}$-measurable and

$$\int_T \|f(t)\|_{\mathcal{H}}^p \, d\mu(t) < \infty.$$

It turns out that $L^2(T \to \mathcal{H})$ is a Hilbert space equipped with the inner product

$$\langle f, g \rangle_{L^2(T \to \mathcal{H})} = \int_T \langle f(t), g(t) \rangle_{\mathcal{H}} \, d\mu(t)$$

(see Section II.1, Example 6 in [RS80]).

B.2 Orthogonality

The definition and result in this section are from [KR97]. These books are also a useful reference for other basic facts about Hilbert spaces.

Definition B.2.1. Let $\mathcal{H}$ be a Hilbert space. If the span of $(e_i)_{i \in \mathbb{N}}$ is dense in $\mathcal{H}$, then it is a **total set**. If $(e_i)_{i \in \mathbb{N}}$ is both an orthonormal sequence and a total set, then it is called an **orthonormal basis** of $\mathcal{H}$.

Proposition B.2.2. *Let $(e_i)_{i \in \mathbb{N}}$ be a orthonormal sequence in a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) The set $(e_i)_{i \in \mathbb{N}}$ is total.

(ii) If f is orthogonal to e_i for all $i \in \mathbb{N}$, then $f = 0$.

(iii) For all $f \in \mathcal{H}$, we have the orthogonal expansion

$$f = \sum_{i=0}^{\infty} \langle f, e_i \rangle_{\mathcal{H}} e_i.$$

Proof. See Theorem 2.2.9 in [KR97]. □

B.3 Direct Sum of Hilbert Spaces

While finite direct sums are an elementary part of Hilbert space theory, here we look at infinite direct sums. The results in this section can be found in Section 2.3 in [BEH08].

Definition B.3.1. Let $(\mathcal{H}_i)_{i\in\mathbb{N}}$ be a family of mutually orthogonal subspaces of the Hilbert space $\mathcal{H}$. We say that $\mathcal{H}$ is the **orthogonal direct sum** of $(\mathcal{H}_i)_{i\in\mathbb{N}}$,

$$\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i,$$

if $\mathcal{H}$ equals the closure of the span of

$$\bigcup_{q=0}^{\infty} \mathcal{H}_i.$$

The orthogonal direct sum defined above is also known as the internal direct sum, as opposed to the external direct sum which applies when $(\mathcal{H}_i)_{i\in\mathbb{N}}$ is only assumed to be a family of Hilbert spaces, not necessarily subspaces of a larger Hilbert space. More information can be found in [KR97], however, all direct sums that we encounter will be internal direct sums.

More concretely, when a Hilbert space has an orthogonal direct sum decomposition, it means that we can write the elements of the Hilbert space via an orthogonal expansion.

Proposition B.3.2. *Let $(\mathcal{H}_i)_{i\in\mathbb{N}}$ be a family of mutually orthogonal subspaces of the Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) The orthogonal direct sum of $(\mathcal{H}_i)_{i\in\mathbb{N}}$ is $\mathcal{H}$.

(ii) All $f \in \mathcal{H}$ can be written as

$$f = \sum_{q=0}^{\infty} f_q,$$

where $f_i \in \mathcal{H}_i$, f_i is orthogonal to f_j for all $i \neq j$, and $\sum_{i=0}^{\infty} \|f_i\|_{\mathcal{H}} < \infty$.

B.4 Tensor Product of Hilbert Spaces

This definition for the tensor product of two Hilbert spaces is taken from [BEH08] and is standard.

Definition B.4.1. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. The Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the **tensor product** of $\mathcal{H}_1$ and $\mathcal{H}_2$ if there exists a bilinear map $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $(f_1, f_2) \mapsto f_1 \otimes f_2$ which satisfies:

(i) For all $f_1, g_1 \in \mathcal{H}_1$, $f_2, g_2 \in \mathcal{H}_2$, we have $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle f_1, g_1 \rangle_{\mathcal{H}_1} \langle f_2, g_2 \rangle_{\mathcal{H}_2}$.

(ii) The set $\{f_1 \otimes f_2 \,|\, f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2\}$ is total in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

From this definition, the existence and uniqueness of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not immediately clear. See Section 2.4 of [BEH08] for a proof that the $\mathcal{H}_1 \otimes \mathcal{H}_2$ exists and is unique up to isomorphism.

Proposition B.4.2. *If U_1 is a total set in $\mathcal{H}_1$ and U_2 is a total set in $\mathcal{H}_2$, then $\{f_1 \otimes f_2 \,|\, f_1 \in U_1, f_2 \in U_2\}$ is a total set in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, if $(e_{i_1})_{i_1 \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_1$ and $(e_{i_2})_{i_2 \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_2$, then $\{e_{i_1} \otimes e_{i_2} \,|\, i_1, i_2 \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

Proof. See proposition 2.4.4 in [BEH08]. □

The tensor product is associative, meaning that $(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3$ is isomorphic to $\mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)$, so either can be defined as $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ (see Appendix E.1 in [Jan97]). Thus, tensor powers can be defined inductively.

Definition B.4.3. Let $\mathcal{H}$ be a Hilbert space and $q \geq 1$. The tensor product of $\mathcal{H}$ with itself q times is the qth **tensor power** of $\mathcal{H}$, denoted $\mathcal{H}^{\otimes q}$. We set $\mathcal{H}^{\otimes 1} = \mathcal{H}$.

Proposition B.4.4. *Let $q \geq 1$. Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_q} \,|\, i_1, \ldots, i_q \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}^{\otimes q}$.*

Proof. This follows from Proposition B.4.2. □

Due to this proposition, each $f \in \mathcal{H}^{\otimes q}$ has an orthogonal expansion

$$f = \sum_{i_1, \ldots, i_q = 0}^{\infty} a_{i_1, \ldots, i_q} e_{i_1} \otimes \cdots \otimes e_{i_j}.$$

Let $\mathcal{S}_q$ be the set of all permutations of $\{1, \ldots, q\}$. Now we can define the symmetrization operator.

Definition B.4.5. Let $\mathcal{H}$ be a Hilbert space and $q \geq 1$. Let $f \in \mathcal{H}^{\otimes q}$. The **symmetrization** of f is defined as

$$\widetilde{f} := \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_n} \sum_{i_1, \ldots, i_q = 0}^{\infty} a_{i_1, \ldots, i_q} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(j)}}.$$

A function f is said to be **symmetric** if $f = \widetilde{f}$. We may also use the notation $\mathrm{sym}(f)$ instead of $\widetilde{f}$.

Definition B.4.6. Let $\mathcal{H}$ be a Hilbert space and $q \geq 1$. The qth **symmetric tensor power** of $\mathcal{H}$, denoted $\mathcal{H}^{\odot q}$ is the range of the symmetrization operator applied to $\mathcal{H}^{\otimes q}$. We set $\mathcal{H}^{\odot 1} = \mathcal{H}$.

The symmetrization operator is a orthogonal projection so that $\mathcal{H}^{\odot q}$ is a closed subspace of $\mathcal{H}^{\otimes q}$ and a Hilbert space (see Section 2.1 in [Gui72]). The elements of $\mathcal{H}^{\odot q}$ are the symmetric functions in $\mathcal{H}^{\otimes q}$. We will equip $\mathcal{H}^{\odot q}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^{\odot q}} = q! \langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes q}}$ and the norm $\|\cdot\|_{\mathcal{H}^{\odot q}} = \sqrt{q!}\, \|\cdot\|_{\mathcal{H}^{\otimes q}}$. Note that this choice of inner product is not standard in Hilbert space theory, although it is standard in the Malliavin calculus literature. Next, we introduce a few concepts needed to describe the orthonormal basis of $\mathcal{H}^{\odot q}$.

Firstly, we will need the concept of a multiindex which is given in Definition 1.5.1. We will also use the notation given in that definition.

Consider the sequence $f = (f_i)_{i \in \mathbb{N}} \subseteq \mathcal{H}$. We use the notation

$$f^{\otimes a} := \bigotimes_{i=0}^{\infty} f_i^{\otimes a_i},$$

where the term $f_i^{\otimes a_i}$ is omitted whenever $a_i = 0$.

Proposition B.4.7. *Let $q \geq 1$. Let $e = (e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Then*

$$\left\{ \frac{1}{\sqrt{a!}} \operatorname{sym}\left(e^{\otimes a}\right) \,\middle|\, a \in \mathcal{A}_q \right\}$$

is an orthonormal basis for $\mathcal{H}^{\odot q}$.

Proof. See Section 2.1 in [Gui72] and note that an adjustment is made for the fact that we equipped $\mathcal{H}^{\odot q}$ with the inner product $q! \langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes q}}$, instead of $\langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes q}}$. □

In the case where $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B}, \mu)$ is a measure space and μ is a σ-finite and nonatomic measure, the concepts discussed above can be put in a more familiar form.

Proposition B.4.8. *Let $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B}, \mu)$ is a measure space and μ is a σ-finite and nonatomic measure. Then the symmetrization of f is*

$$\widetilde{f}(t_1, \ldots, t_q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_n} f(t_{\sigma(1)}, \ldots, t_{\sigma(q)}). \tag{B.4.1}$$

Proof. See Appendix B in [NP12]. □

Proposition B.4.9. *Let $(T_1, B_1, \mu_1), (T_2, B_2, \mu_2)$ be measure spaces with σ-finite and nonatomic measures. Let $\mathcal{H}$ be a separable Hilbert space Then:*

(i) There exists a unique isomorphism from $L^2(T_1, B_1, \mu_1) \otimes L^2(T_2, B_2, \mu_2) \to L^2(T_1 \times T_2, B_1 \times B_2, \mu_1 \times \mu_2)$ such that $f_1 \otimes f_2 \mapsto f_1 f_2$.

(ii) There exists a unique isomorphism from $L^2(T_1, B_1, \mu_1) \otimes \mathcal{H} \to L^2(T_1 \to \mathcal{H}, B_1, \mu_1)$ such that $(x \mapsto f_1(x)) \otimes f_2 \mapsto (x \mapsto f_1(x) f_2)$.

(iii) There exists a unique isomorphism from $L^2(T_1 \times T_2, B_1 \times B_2, \mu_1 \times \mu_2) \to L^2(T_1 \to L^2(T_2, B_2, \mu_2), B_1, \mu_1)$ *such that* $((x,y) \mapsto f(x,y)) \mapsto (x \mapsto (y \mapsto f(x,y)))$.

Proof. See Theorem II.10 in [RS80]. □

Proposition B.4.10. *Let* $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$*, where* $(T, \mathcal{B}, \mu)$ *is a measure space and* μ *is a* σ*-finite and nonatomic measure. Then for all* $q \geq 1$*,* $\mathcal{H}^{\otimes q}$ *is isomorphic* $L^2(T^q, B^q, \mu^q)$ *to and* $\mathcal{H}^{\odot q}$ *is isomorphic to* $L^2_s(T^q, B^q, \mu^q)$*, where* $L^2_s(T^q, B^q, \mu^q)$ *is the subspace of symmetric functions in* $L^2(T^q, B^q, \mu^q)$*. In both cases the isomorphism sends* $f_1 \otimes \cdots \otimes f_q \mapsto f_1 \ldots f_q$*.*

Proof. The $\mathcal{H}^{\otimes q}$ case follows from Proposition B.4.9 (i). For the $\mathcal{H}^{\odot q}$ case, see Proposition E.16 in [Jan97]. □

B.5 Contractions

The following section on contractions is from Appendix B in [NP12] and also [PT11].

Definition B.5.1. Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Let $p, q \geq 1$, $f \in \mathcal{H}^{\odot p}$, $g \in \mathcal{H}^{\odot p}$. Then for all $r = 0, \ldots, p \wedge q$, the rth **contraction** of f and g is

$$f \otimes_r g = \sum_{i_1, \ldots i_r = 0}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^{\otimes r}} .$$

We denote the symmetrization of $f \otimes_r g$ by $f \widetilde{\otimes}_r g$.

Note that f, g are symmetric functions, $\langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^{\otimes r}} \in \mathcal{H}^{\odot p-r}$ and $\langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^{\otimes r}} \in \mathcal{H}^{\odot q-r}$ (see Section 8.5 in [PT11]). However, $f \otimes_r g \in \mathcal{H}^{\otimes p+q-2r}$ is not necessarily symmetric.

If $p = q = r$, then $f \otimes_r g = \langle f, g \rangle_{\mathcal{H}^{\otimes q}}$. If $r = 0$, then $f \otimes_0 g$ reduce to the tensor product $f \otimes g$.

Proposition B.5.2. *Let* $\mathcal{H} = L^2(T, \mathcal{B}, \mu)$*, where* $(T, \mathcal{B}, \mu)$ *is a measure space and* μ *is a* σ*-finite and nonatomic measure. Let* $p, q \geq 1$*,* $f \in \mathcal{H}^{\otimes p}$*,* $g \in \mathcal{H}^{\otimes p}$*. Then for all* $r = 0, \ldots, p \wedge q$ *we have*

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{T^r} f(t_1, \ldots, t_{p-r}, s) g(t_{p+1}, \ldots, t_{p+q-r}, s) \, d\mu^r(s). \qquad \text{(B.5.1)}$$

Proof. See Appendix B in [NP12]. □

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