Proceedings of the
American Control Conference
Baltimore, Maryland • June 1994

Time-varying problems of H^∞ control and absolute stabilizability of uncertain systems

Andrey V. Savkin Ian R. Petersen
Department of Electrical Engineering

Australian Defence Force Academy
Campbell, 2600, Australia
61-6-268-8443 `irp@ee.adfa.oz.au.`

Abstract

This paper considers an absolute stabilization problem for a class of time-varying uncertain systems in which the uncertainty satisfies an integral quadratic constraint. The main result of the paper shows that a state feedback absolute stabilization problem has a solution if and only if there exists a solution to a corresponding infinite horizon, time-varying, state feedback H^∞ control problem. The paper also shows that if either of these problems can be solved via the use of nonlinear state feedback control then it can also be solved via the use of linear state feedback control. Furthermore, the required controller can be constructed by solving a Riccati differential equation. The paper also considers the special case of periodic time-varying systems.

1. Introduction

An important idea to emerge in recent years is the connection between the Riccati equation approach to H^∞ control and the problem of stabilizing an uncertain system; e.g., see [1–4]. In each of these papers, the uncertain system is time-invariant. That is, the linear part of the system is a linear time-invariant system. However, recent advances in H^∞ control theory have extended the H^∞ control problem to the case of time-varying linear systems; e.g., see [5–9]. This motivates us to extend the results of [4] to the case of time-varying uncertain systems. That is, uncertain systems in which the linear part of the system is time-varying.

As in [4], we consider a class of uncertain systems in which the uncertainty is required to satisfy a certain "Integral Quadratic Constraint" which originated in the work of Yakubovich; e.g., see [4,10–15]. Also, the notion of stability considered is that of absolute stability; see also [4, 10–15]. One reason for the use of this framework in [4] was that it enabled the following result to be established: If an uncertain system can be absolutely stabilized via nonlinear control, then it can be absolutely stabilized via linear control. In this paper, we establish a corresponding result for time-varying uncertain systems (for the state feedback case).

The proof of the result given in [4] is based on a corresponding result from H^∞ control theory. This result states for a linear time-invariant H^∞ control problem, if the problem can be solved via a nonlinear controller, then there will also exist a linear controller which solves the same H^∞ control problem; e.g., see [5]. However as yet, no such result has been established for time-varying H^∞ control problems although some closely related results do exist. For example, in [7] the use of nonlinear controllers is considered. However, this paper only considers time-varying H^∞ control problems over a finite time interval. Such H^∞ control problems are not applicable to the problem of absolutely stabilizing an uncertain system. Also, references [6, 8, 9] consider time-varying, infinite time H^∞ control problems. However in these papers, attention is limited to linear controllers. One of the contributions of this paper is fill this gap in the existing time-varying H^∞ control theory. That is, we show that for a time-varying, infinite-time, state feedback, H^∞ control problem, if the problem can be solved using a nonlinear controller, then the problem can also be solved via the use of a linear state feedback controller. We also consider the special case in which the time-varying system is periodic.

2. Definitions

Consider the time-varying uncertain system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + C(t)w(t); \\ z(t) &= K(t)x(t) + G(t)u(t) \end{aligned} \qquad (2.1)$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^p$ is the *disturbance input*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^q$ is the *error output*, $A(\cdot), B(\cdot), C(\cdot), K(\cdot)$ and $G(\cdot)$ are real, piecewise-continuous, matrix functions bounded in the Euclidean norm on the interval $[0, \infty)$.

We will consider the standard H^∞ control problem for the system (2.1) with a nonlinear full information controller of the form

$$u(t) = U[t, x(\cdot)|_0^t, w(\cdot)|_0^t]. \qquad (2.2)$$

For the the class of controllers under consideration, it is assumed that the following causality type assumption is satisfied:

Assumption 2.1 The controller $U(\cdot)$ is such that given any time $t^* > 0$, then $x(0) = 0$ and $w(t) = 0$ for $t \in [0, t^*]$ implies $u(t) = U[t, x(\cdot)|_0^t, w(\cdot)|_0^t]$ satisfies $u(t) = 0$ for $t \in [0, t^*]$.

Note that in contrast to existing results on the time-varying, infinite horizon, H^∞ control problem (see [6, 8, 9]), we allow for the possibility of nonlinear controllers.

H^∞ Norm Bound Condition In the H^∞ control problem under consideration, the H^∞ norm bound condition is as follows: There exists a constant $c_0 > 0$ such that

$$J \triangleq \sup \frac{\int_{t_0}^{\infty} \|z(t)\|^2 dt}{c_0\|x(t_0)\|^2 + \int_{t_0}^{\infty} \|w(t)\|^2 dt} < 1 \qquad (2.3)$$

where the supremum is taken over all $t_0 \geq 0$ and all $w(\cdot) \in \mathbf{L}_2[t_0, \infty)$. Here $\|\cdot\|$ denotes the standard Euclidean norm and $\mathbf{L}_2[t_0, \infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[t_0, \infty)$.

Definition 2.1 A controller (2.2) is said to solve the H^∞ *control problem* defined by the system (2.1) and norm bound condition (2.3) if the following conditions hold:

(i) For any $t_0 \geq 0$, any initial condition $x(t_0) = x_0$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0, \infty)$, the closed loop system (2.1), (2.2) has a unique solution $[x(\cdot), u(\cdot)]$ which is defined on $[t_0, \infty)$ and $[x(\cdot), u(\cdot)] \in \mathbf{L}_2[t_0, \infty)$.

(ii) Condition (2.3) is satisfied for the closed loop system (2.1), (2.2).

As well as considering the above H^∞ control problem, we will also consider a related problem of absolute stabilizability for a class of uncertain systems. This class of uncertain systems is also defined by the state equations (2.1) but in this case $w(t)$ is treated as an *uncertainty input* and $z(t)$ is treated as an *uncertainty output*; e.g., see [4].

System Uncertainty The uncertainty in the above uncertain system is described by an equation of the form:

$$w(t) = \phi(t, z(\cdot)|_0^t) \qquad (2.4)$$

where the following Integral Quadratic Constraint is satisfied.

Definition 2.2 (Integral Quadratic Constraint; see also [4, 10–17].) An uncertainty function of the form (2.4) is an *admissible uncertainty* for the system (2.1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and any corresponding solution to equations (2.1), (2.4) defined on an existence interval $(0, t_*)$ (that is t_* is the upper time limit for which the solution exists), then for any $t_0 \geq 0$ there exists a sequence $\{t_i\}_{i=1}^{\infty}$ and a constant $d \geq 0$ such that $t_i \to t_*$, $t_i \geq t_0$ and

$$\int_{t_0}^{t_i} (\|w(t)\|^2 - \|z(t)\|^2) dt \leq d \quad \forall i. \qquad (2.5)$$

Note that t_* and t_i may be equal to infinity.

Remarks Given an admissible uncertainty of the form (2.4), then for every solution to the equations (2.1), (2.4), there will be a corresponding parameter $d \geq 0$ such that condition (2.5) is satisfied. The value of the parameter d is a "measure of mismatch" from the following $\mathbf{L}_2[t_0, \infty]$ induced norm bound condition on the uncertainty:

$$\int_{t_0}^{t_i} (\|w(t)\|^2 - \|z(t)\|^2)dt \leq 0 \quad \forall i.$$

In references [16] and [17], a number of examples are given of physical systems in which the uncertainty naturally fits into the above framework. In particular, note that the uncertainty description given above allows for $w(t)$ to depend dynamically on $z(t)$). Indeed in this case, one could interpret the "measure of mismatch" d as being due to a non-zero initial condition on the uncertainty dynamics. Thus, we bound the size of this initial condition in terms of the parameter d.

Also, it is clear that the uncertain system (2.1), (2.4) allows for uncertainty satisfying a *pointwise* norm bound condition. In this case, the uncertain system would be described by the state equations

$$\dot{x}(t) = [A(t) + C(t)\Delta(t)K(t)]x(t) + [B(t) + C(t)\Delta(t)G(t)]u(t); \quad \|\Delta(t)\| \leq 1 \qquad (2.6)$$

where $\Delta(t)$ is the uncertainty matrix and $\|\cdot\|$ denotes the standard induced matrix norm; e.g., see [3, 18, 19]. To verify that such uncertainty is admissible for the uncertain system (2.1), (2.2), let $w(t) = \Delta(t)[K(t)x(t)+G(t)u(t)]$ where $\|\Delta(t)\| \leq 1$ for all $t \geq 0$. Then $w(\cdot)$ satisfies condition (2.5) with any t_i and with $d = 0$.

If in the system (2.6), we replace the condition $\|\Delta(t)\| \leq 1$ for all $t \geq 0$ by a requirement that $\Delta(\cdot)$ is any bounded matrix function on an interval $[0, T_0]$ and $\|\Delta(t)\| \leq 1$ for all $t > T_0$, then $w(\cdot)$ satisfies condition (2.5) with any t_i. However in this case, the constant d will non-zero in general. Such uncertainty would not be allowed in the uncertain systems considered in [3, 18, 19].

We now introduce a corresponding notion of stabilizability for the uncertain system (2.1), (2.4).

Definition 2.3 (See also [4, 14].) The uncertain system (2.1), (2.4) is said to be ***absolutely stabilizable via nonlinear control*** if there exists a controller of the form (2.2) and a constant $c > 0$ such that the following conditions hold:

(i) For any $t_0 \geq 0$, any initial condition $x(t_0) = x_0$ and any uncertainty input $w(\cdot) \in \mathbf{L}_2[t_0, \infty)$, the closed loop system (2.1), (2.2) has a unique solution $x(\cdot)$ which is defined on $[t_0, \infty)$.

(ii) Given an admissible uncertainty for the uncertain system (2.1), (2.4), then any solution to the closed loop uncertain system (2.1), (2.4), (2.2) satisfies $[x(\cdot), u(\cdot), w(\cdot)] \in \mathbf{L}_2[0, \infty)$ (hence, $t_* = \infty$) and for any $t_0 \geq 0$ the following inequality is satisfied

$$\int_{t_0}^{\infty} (\|x(t)\|^2 + \|u(t)\|^2 + \|w(t)\|^2)dt \leq c[\|x(t_0)\|^2 + d].$$

Observation 2.1 It follows that if the uncertain system (2.1), (2.4) is absolutely stabilizable, then the corresponding closed loop uncertain system (2.1), (2.4), (2.2) will have the property that $x(t) \to 0$ as $t \to \infty$ for any admissible uncertainty $w(\cdot)$. Indeed, since $[x(\cdot), u(\cdot), w(\cdot)] \in \mathbf{L}_2[0, \infty)$, we can conclude from equation (2.1) and the boundedness of its coefficients that $\dot{x}(\cdot) \in \mathbf{L}_2[0, \infty)$. However, using the fact that $x(\cdot) \in \mathbf{L}_2[0, \infty)$ and $\dot{x}(\cdot) \in \mathbf{L}_2[0, \infty)$, it now follows that $x(t) \to 0$ as $t \to \infty$.

To conclude this section, we present two definitions regarding the exponential stability and detectability of a linear time-varying system.

Definition 2.4 The linear time-varying system $\dot{x}(t) = A(t)x(t)$ is said to be ***exponentially stable*** if there exist constants $c > 0$ and $\nu > 0$ such that

$$\|x(t)\| \leq c \exp[-\nu(t - t_0)]\|x(t_0)\|$$

for all $t \geq t_0 \geq 0$ and all its solutions.

Definition 2.5 2.3 The pair $(A(\cdot), K(\cdot))$ is said to be ***detectable*** if there exists a bounded matrix function $L(t)$ such that the system $\dot{x}(t) = (A(t) - L(t)K(t))x(t)$ is exponentially stable.

3. The Main Result

In this section, we present the main result of this paper which establishes necessary and sufficient conditions for the solvability of the standard H^∞ control problem (2.1), (2.3) and for the absolute stabilizability of the time-varying uncertain system (2.1), (2.4). The conditions are given in terms of the existence of a bounded, non-negative definite, exponentially stabilizing solution to a game type Riccati differential equation.

Our main result requires that the coefficients of the system (2.1) satisfy a number of additional assumptions.

Assumptions Let the matrix function $E(\cdot)$ be defined by $E(t) := G'(t)G(t)$. Then the coefficients of the system (2.1) are required to satisfy the following additional assumptions:

3.1 There exists a constant $\delta > 0$ such that $E(t) \geq \delta I$ for all $t \geq 0$.

3.2 The matrix functions $K(\cdot)$ and $G(\cdot)$ satisfy the condition $K(\cdot)'G(\cdot) \equiv 0$.

3.3 The pair $(A(\cdot), K(\cdot))$ is detectable.

Remarks Assumptions 3.1-3.3 are technical assumptions required to ensure that the underlying H^∞ problem is "non-singular"; e.g., see [20].

The Riccati differential equation under consideration is defined as follows:

$$\begin{aligned}-\dot{P}(t) &= A(t)'P(t) + P(t)A(t) + K(t)'K(t)\\ &\quad + P(t)[C(t)C(t)' - B(t)E(t)^{-1}B(t)']P(t). \qquad (3.1)\end{aligned}$$

We are now in a position to present our main result.

Theorem 3.1 *Consider the uncertain system (2.1), (2.4) and suppose that Assumptions 3.1-3.3 are satisfied. Then the following statements are equivalent:*

(i) The uncertain system (2.1), (2.4) is absolutely stabilizable via a nonlinear controller of the form (2.2).

(ii) The controller (2.2) solves the H^∞ control problem (2.1), (2.3).

(iii) The Riccati differential equation (3.1) has a bounded, non-negative definite solution $P(\cdot)$ on the interval $[0, \infty)$ such that the linear system

$$\begin{aligned}\dot{x}(t) = (A(t) - [&B(t)E(t)^{-1}B(t)'\\ &- C(t)C(t)']P(t))x(t) \qquad (3.2)\end{aligned}$$

is exponentially stable.

If condition (iii) holds then, the uncertain system (2.1), (2.4) is absolutely stabilizable via a linear time-varying state feedback controller of the form

$$u(t) = -E^{-1}(t)B'(t)P(t)x(t). \qquad (3.3)$$

The same controller also solves to H^∞ control problem (2.1), (2.3).

The following corollary is an immediate consequence of the above theorem and the remarks following the definition of the uncertain system (2.1), (2.4).

Corollary 3.1 *The uncertain system (2.6) with norm bounded uncertainty will be absolutely stabilizable via the linear controller (3.3) if condition (iii) of Theorem 3.1 is satisfied.*

Observation 3.1 In the above theorem, condition (iii) is given in terms of the existence of a bounded, non-negative definite, exponentially stabilizing solution to the Riccati equation (3.1) defined on the interval $[0, \infty)$. It follows from the proof of this theorem that if such a solution exists, it can be obtained as the limit $P(t) = \lim_{T\to\infty} P_T(t)$ where $P_T(\cdot)$ is the solution to the Riccati equation (3.1) defined over the finite time interval $[0, T]$ and with boundary condition $P_T(T) = 0$.

4. Systems with Periodic Coefficients

In this section, we will consider systems of the form (2.1) in which the coefficient matrices are periodic with period $\Omega > 0$. That is $A(t + \Omega) = A(t)$; $B(t + \Omega) = B(t)$; $C(t + \Omega) = C(t)$; $K(t + \Omega) = K(t)$; $G(t + \Omega) = G(t)$ for all $t \geq 0$.

We are now in a position to present our result for systems with periodic coefficients.

Theorem 4.1 *Consider the uncertain system (2.1), (2.4) with $\Omega-$periodic coefficients and suppose that Assumptions 3.1-3.3 are satisfied. Then the following statements are equivalent:*

(i) The uncertain system (2.1), (2.4) is absolutely stabilizable via a nonlinear controller of the form (2.2).

- *(ii) The controller (2.2) solves the* H^∞ *control problem (2.1), (2.3).*
- *(iii) The Riccati differential equation (3.1) has a non-negative definite, periodic solution* $P(t) = P(t+\Omega)$ *on the interval* $[0, \infty)$ *such that the system (3.2) is exponentially stable.*

If condition (iii) holds then, the uncertain system (2.1), (2.4) is absolutely stabilizable via a linear periodic state feedback controller of the form (3.3). The same controller also solves to H^∞ *control problem (2.1), (2.3).*

Acknowledgement: This work was supported by the Australian Research Council.

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