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Dissipative Systems Theory: Analysis and Synthesis

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A thesis submitted for the degree of Doctor of Philosophy
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Department of Systems Engineering
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To my parents
Harpini and Soedomo,
and my family
Titiek and Fadhil.

In the name of God the Merciful, the Compassionate,
Praise and Glory belong to God, the Lord of the Universe,
May God pour His bless onto the Prophet and his purified Family and his chosen friends.
Declaration

These doctoral studies were conducted with supervision from Dr. Matthew James, and advisory from Dr. Michael Green and Dr. Iven Mareels, all of the Department of Engineering, Faculty of Engineering and Information Technology, The Australian National University.

The work contained in this thesis, except where explicitly stated, is original research performed by the author under the guidance of Matt James. This work has not been submitted for a degree at any other university or institution.

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Abstract

Finite $L_2$ gain (or $H_\infty$) and passivity (or positive real) methods have recently played an important role in a large number of robust, high performance engineering designs for both nonlinear and linear systems. This has renewed interest in the classical concept of dissipative systems. In particular, in various finite gain or passivity system synthesis methods in the literature, one studies a relevant dissipation inequality and looks for an appropriate solution to it. When such a solution exists, one then constructs the desired system by using this solution.

The main theme of the thesis is the development of a framework for general dissipative systems analysis and synthesis. We firstly present a numerical method for testing dissipativity of a given system. We characterize a dissipative system in terms of a weak (viscosity) solution to a partial differential inequality (PDI) which is the relevant dissipation inequality for the system being considered and develop a finite-difference based discretization method that results in a partial difference inequality approximating the PDI. We then propose two iterative methods to solve the partial difference inequality. We report a number of computational experiment results to demonstrate the utility of the method.

Under certain circumstances, strict dissipativity is of the main concern. We provide characterization of a strongly stable, strictly quadratic dissipative nonlinear system in terms of a solution to a PDI or a solution to a partial differential equation (PDE), in the viscosity sense. When the solution to the PDE is smooth, then it also has a stabilizing (in some sense) property. These results generalize the strict bounded real lemma in the linear $H_\infty$ control literature. We also provide characterization of a stable, strictly quadratic dissipative linear system in terms of a stabilizing solution to an algebraic Riccati equation (ARE). Connections between quadratic dissipative systems and finite gain related systems are given.

In the thesis, we propose a synthesis method for a general dissipative control problem for nonlinear and linear systems with state feedback. We express the solution to the
problem in terms of a solution to a Hamilton-Jacobi-Isaacs (HJI) PDI/PDE in the nonlinear systems case (algebraic Riccati equation/inequality in the linear systems case). In particular, in the case of nonlinear systems with a general quadratic supply rate, we show that whenever there exists a static state feedback control that renders the closed loop system dissipative, then there exists a solution to the Hamilton-Jacobi-Isaacs PDI/PDE in the viscosity solution. This extends and generalizes a number of synthesis results in the nonlinear $H_\infty$ control literature.

We then consider a general dissipative output feedback control problem and propose a solution by employing the recently developed information state method. We formulate an information state and then convert the original output feedback problem into a new full state one in which the information state provides the appropriate state. The dynamics of the information state takes the form of a controlled PDE. We then solve the new problem by using game theoretic methods leading to a (infinite dimensional) HJI PDI. This is the relevant (controlled) dissipation inequality for the output feedback problem at hand. The solution is then specialized to bilinear and linear systems yielding finite dimensional solutions.

As a by product, we formulate and solve a general dissipativity filtering problem for nonlinear and linear systems. The problem takes the nonlinear $H_\infty$ filtering as a special case. As in the control case, the solution to the filtering problem is expressed in terms of a controlled PDE describing the dynamics of the corresponding information state and a (infinite dimensional) HJI PDI. When specialized to linear systems with a general quadratic supply rate, the solution reduces to new finite dimensional linear filters with the (central) linear $H_\infty$ filter appearing as a special one.

Finally, we propose application of general dissipativity control methods to two stabilization problems. In the first problem we look for a controller that stabilizes linear systems possessing sector bounded nonlinearities at their inputs and outputs. In the second one, we look for a controller that stabilizes an uncertain nonlinear system consisting of a nonlinear nominal model and an unknown nonlinear model belonging to a class of general dissipative systems described in terms of a specific supply rate function. In either case, we pose the stabilization problem as a dissipativity control synthesis one for a related system.
# Contents

Declaration

Acknowledgements

Abstract

Notation and Abbreviations

1 Introduction

1.1 Finite Gain and Passivity Performance Measures

1.1.1 Finite Gain

1.1.2 Passivity

1.2 Dissipative Systems

1.3 Existing Synthesis Theory

1.3.1 Finite Gain Synthesis

1.3.2 Passivity Synthesis

1.4 Thesis Outline

1.4.1 Computation of the $H_\infty$ Norm of Nonlinear Systems

1.4.2 State Feedback Analysis and Synthesis

1.4.3 Output Feedback Synthesis

1.4.4 Filter Synthesis

1.4.5 Applications
1.4.6 Results for Discrete Time Systems ........................................... 15
1.4.7 Summary of Contributions ...................................................... 16

2 Computational Methods for Dissipative Systems .......................... 17

2.1 Introduction ............................................................................... 17
2.2 Computation of the $H_\infty$ Norm ............................................. 18
2.3 The $H_\infty$ Norm .................................................................... 18
2.4 The Numerical Schemes .............................................................. 22
  2.4.1 Approximation in Value Space ............................................... 24
  2.4.2 Approximation in Policy Space .............................................. 26
  2.4.3 Remarks on Feasibility ............................................................. 27
2.5 Examples .................................................................................... 27
  2.5.1 Example 1 ........................................................................... 28
  2.5.2 Example 2 ........................................................................... 29
  2.5.3 Example 3 ........................................................................... 30
  2.5.4 Example 4 ........................................................................... 30
  2.5.5 Example 5 ........................................................................... 31
2.6 Extension to General Dissipativity Cases ................................. 32
2.7 Examples: general dissipativity cases ........................................ 34
  2.7.1 Example 6 ........................................................................... 34
  2.7.2 Example 7 ........................................................................... 34
2.8 Conclusions ............................................................................... 35
2.9 Figures ....................................................................................... 36

3 Strictly Dissipative Systems ......................................................... 51

3.1 Introduction ............................................................................... 51
3.2 General Dissipative Systems ...................................................... 52
3.3 Strictly Dissipative Nonlinear Systems ......................... 54
3.4 Strictly Dissipative Linear Systems .......................... 63
3.5 Conclusions ............................................ 71

4 State Feedback Synthesis .............................. 73
  4.1 Introduction .................................. 73
  4.2 Problem Formulation .............................. 74
  4.3 General Dissipativity Control Synthesis ................. 75
  4.4 Nonlinear Strict Dissipativity Control Synthesis ...... 78
  4.5 Linear Strict Dissipativity Control Synthesis .......... 82
  4.6 Locally Smooth Solutions .......................... 87
  4.7 Conclusions ....................................... 92

5 Output Feedback Synthesis: General Case .................. 93
  5.1 Introduction ....................................... 93
  5.2 Finite Time Problem ................................ 94
    5.2.1 Information State Solution ..................... 96
    5.2.2 Verification .................................... 100
    5.2.3 A Certainty Equivalence Principle ................ 101
  5.3 Infinite Time Problem ................................ 104
    5.3.1 Information State Solution ..................... 104
    5.3.2 A Storage Function ............................ 108
  5.4 Conclusions ....................................... 109

6 Output Feedback Synthesis: Finite Dimensional Cases .... 111
  6.1 Introduction ....................................... 111
  6.2 Finite Time Problem ................................ 112
    6.2.1 Information State Solution ..................... 112
6.2.2 Verification .................................................. 114
6.2.3 Linear Systems Case ........................................... 116

6.3 Infinite Time Problem ........................................... 118
6.3.1 Linear Systems Case ........................................... 119

6.4 Examples ......................................................... 125
6.4.1 Example 1: $H_\infty$ case .................................. 126
6.4.2 Example 2: Mixed performance case ..................... 127

6.5 Conclusions ..................................................... 127

6.6 Figures .......................................................... 128

7 Filter Synthesis .................................................. 131

7.1 Introduction .................................................... 131

7.2 Finite Time Problem ............................................ 133
7.2.1 Information State Filter .................................. 134
7.2.2 Certainty Equivalence Filter ............................... 138
7.2.3 Linear $H_\infty$ Filtering .................................. 139

7.3 Infinite Time Problem ........................................... 141

7.4 General Dissipative Filters ................................... 142
7.4.1 Linear Systems Case ....................................... 143

7.5 Conclusions ..................................................... 145

8 Applications ....................................................... 147

8.1 Introduction .................................................... 147

8.2 Linear Systems with Sector Bounded Nonlinearities ............ 148
8.2.1 Nonlinearity at the Input .................................. 149
8.2.2 Nonlinearities at the Input and Output .................. 155
8.2.3 Examples ..................................................... 163
List of Figures

1.1 A typical robust stabilization problem. .................................................. 2
1.2 Linear system with nonlinearity at the input and output. ...................... 15

2.1 Example 1. The value space iteration converges for $\gamma \geq 2.1$ and diverges for $\gamma \leq 1.9$. Therefore $\| G \|_{H_\infty} \in (1.9, 2.1]$. ........................................ 37
2.2 Example 1. Numerical solution of the Riccati equation converges for $\gamma \geq 2.1$ and diverges for $\gamma \leq 1.9$. Therefore $\| G \|_{H_\infty} \in (1.9, 2.1]$. ........................................ 38
2.3 Example 2. The value space iteration converges for $\gamma \geq 0.67$ and diverges for $\gamma \leq 0.65$. Therefore $\| G \|_{H_\infty} \in (0.65, 0.67]$. ........................................ 39
2.4 Example 2. The policy space iteration converges for $\gamma \geq 0.67$ and diverges for $\gamma \leq 0.65$. ......................................................... 40
2.5 Example 2. Comparison of methods for $\gamma = 0.67$. ......................... 41
2.6 Example 3. Comparison of methods for $\gamma = 0.21$. ......................... 42
2.7 Example 4. The value space iteration converges for $\gamma \geq 0.21$ and diverges for $\gamma \leq 0.19$. Therefore $\| G \|_{H_\infty} \in (0.19, 0.21]$. ........................................ 43
2.8 Example 4. The policy space iteration converges for $\gamma \geq 0.21$ and diverges for $\gamma \leq 0.19$. ......................................................... 44
2.9 Example 4. Comparison of methods for $\gamma = 0.21$. ......................... 45
2.10 Example 5. The value space iteration with local velocity converges for $\gamma \geq 0.19$ and diverges for $\gamma \leq 0.15$. ................................. 46
2.11 Example 5. The policy space iteration with local velocity converges for $\gamma \geq 0.19$ and diverges for $\gamma \leq 0.15$. ................................. 47
2.12 Example 6. The value space iteration converges for all $\theta \in [0,1]$. Thus the system is both finite gain and passive. ........................................ 48

2.13 Example 7. The value space iteration converges. Therefore, the system is passive (or positive real). ........................................ 49

6.1 Example 1. ($H_\infty$ case ) Open loop unstable bilinear system; the value function $W^*(\hat{x},Y)$ (top); size of domains (middle); trajectories of disturbance $w(t)$, $v(t)$ and state $x(t)$ (bottom). The controller is stabilizing. . . . 128

6.2 Example 2. (Mixed performance case) (a) $\theta = 0.0$ (positive real), (b) $\theta = 0.1429$, (c) $\theta = 0.2857$, (d) $\theta = 0.4286$. ........................................ 129

6.3 Example 2. (Mixed performance case) (a) $\theta = 0.5714$, (b) $\theta = 0.7143$, (c) $\theta = 0.8571$, (d) $\theta = 1.0$ ($H_\infty$ with $\gamma = 0.9$). .................. 130

7.1 Information State Filter. ........................................ 138

7.2 Certainty Equivalence Filter. ........................................ 139

8.1 Linear system with nonlinearity at the input. ........................................ 151

8.2 New system configuration. ........................................ 151

8.3 Linear system with nonlinearity at the input and output. ........................................ 156

8.4 New system configuration. ........................................ 156

8.5 Example 1. State trajectories. ........................................ 167

8.6 Example 1. Input trajectories. ........................................ 167

8.7 Example 2. State trajectories. ........................................ 168

8.8 Example 2. Input and output trajectories. ........................................ 168

8.9 Example 3. Uncontrolled and controlled output trajectories. ................ 169

8.10 Example 3. Input and output trajectories. ........................................ 169

8.11 A robust stabilization problem. ........................................ 170

A.1 Information state filter. ........................................ 206
Notation and Abbreviations

Notation

In this section we shall describe the notation that will be used throughout the thesis.

\( \mathbb{R} \) denotes the real line and \( \mathbb{R}^n \) denotes \( n \)-dimensional euclidean space. If \( A \) is a \( m \times n \) matrix, \( A' \) denotes its transpose. For \( z \in \mathbb{R}^n \) we write \( |z| = (z'z)^{\frac{1}{2}} \) for the euclidean norm. \( \sigma_{\text{max}}(A)(\sigma_{\text{min}}(A)) \) denotes the maximum (minimum) singular value of \( A \).

We write \( C^1(\mathbb{R}^n) \) for the class of all differentiable functions in \( \mathbb{R}^n \). If \( z \) is in \( \mathbb{R}^n \) and the function \( V: \mathbb{R}^n \to \mathbb{R} \) is in \( C^1(\mathbb{R}^n) \), we write \( \nabla_z V = [\frac{\partial V}{\partial z_1}, \ldots, \frac{\partial V}{\partial z_n}] \). If \( Y \) is a \( n \times n \) real matrix and the function \( W: \mathbb{R}^{n \times n} \to \mathbb{R} \) is in \( C^1(\mathbb{R}^{n \times n}) \), we write \( \nabla_Y W \) as the \( n \times n \) matrix with \( \frac{\partial W}{\partial Y_{ij}} \) as its \( ij \)-th entry.

We write \( L^2([0,T], \mathbb{R}^n) \) \( (L^2([0,M], \mathbb{R}^n)) \) for the space of all square integrable functions on the continuous interval \([0,T], T \geq 0 \) \( (\text{on the discrete interval } [0,M], M \geq 0) \) taking values in \( \mathbb{R}^n \). We write \( L^2([0,\infty), \mathbb{R}^n) \) \( (L^2([0,\infty), \mathbb{R}^n)) \) for the space of all functions satisfying

\[ \int_0^\infty z(t)'z(t) dt < +\infty \quad (\Sigma_{i=0}^\infty z_i'^2 z_i < +\infty). \]

If \( z \in L^2([0,T], \mathbb{R}^n) \), \( |z|_{2,T} \) denotes the 2-norm \( \|z\|_{2,T} = (\int_0^T z(t)'z(t) dt)^{\frac{1}{2}} \). A similar definition applies to \( \|z\|_{2,M} \) for \( z \in L^2([0,\infty), \mathbb{R}^n) \).

If \( F \) is a real-valued function on a set \( U \) which has a minimum on \( U \), then we write

\[ \text{argmin}_{v \in U} F(v) = \{ v^* \in U : F(v^*) \leq F(v), \forall v \in U \}. \]

The notation \( \text{argmax} \) is defined in a similar way.

xv
Abbreviations

The following abbreviations will be adopted in the thesis.

ARE  Algebraic Riccati equation
ARI  Algebraic Riccati inequality
DPE  Dynamic programming equation
ODE  Ordinary differential equation
PDE  Partial differential equation
PDI  Partial differential inequality
RDE  Riccati differential equation
HJB  Hamilton-Jacobi-Bellman
HJI  Hamilton-Jacobi-Isaacs
Chapter 1

Introduction

1.1 Finite Gain and Passivity Performance Measures

In recent years there has been an increasing number of applications of the finite $L_2$ gain (or $H_\infty$) and passivity (or positive real) performance measures in control designs for both linear and nonlinear systems. In this section we shall briefly discuss some of these applications.

1.1.1 Finite Gain

In the linear systems case, it is well known that finite gain performance measure is an effective tool for addressing practical control problems (see, for example, the recent textbook [36] for a thorough discussion). The use of the notion of $H_\infty$ norm for robust control designs is introduced in the seminal paper [115]. The design problems in practice may be classified, in a broad sense, into: (i) robust stabilization with respect to unstructured uncertainties and (ii) performance shaping for nominal systems. By employing small-gain type techniques, the stability of an uncertain system consisting of a nominal model and a description of unstructured uncertainty can be examined (see [36], [78] for a complete discussion on this matter). The uncertainty that is present in the system may take the form of additive or multiplicative perturbation and may arise from input channels or output channels. Typically one assumes that the uncertainties are of bounded $H_\infty$ norm. The small-gain result guarantees stability of the uncertain system if the product of the $H_\infty$ norm of the uncertainty model and the $H_\infty$ norm of the nominal model is strictly
less than one [36], [78]. In the case of performance shaping, there are two approaches for achieving desirable performance. In the first approach, which is commonly called the *loop shaping design*, one shapes a number of closed loop quantities that reflect major closed loop behaviour in an iterative scheme ([78], [36]). The singular values of the sensitivity and the complementary sensitivity functions are quantities that are commonly used in this approach. The second approach is usually carried out by formulating the performance criterion to be achieved as a disturbance attenuation criterion.

In the nonlinear systems case, robust stability results can be obtained, in principle, by applying small-gain type theorems [3], [48], [44]. Figure 1.1 illustrates a typical robust stabilization problem.

![Figure 1.1: A typical robust stabilization problem.](image)

As pointed out in [5], in practice knowledge concerning the size and location of uncertainties may not be easy to obtain. Moreover, in the nonlinear systems case small-gain type results are typically only sufficient. Thus, the resulting controller may be conservative. These may limit the utility of a nonlinear finite gain design method for robust stabilizations. In the case of performance shaping, the first approach is not easily extendable to the nonlinear case because of the lack of understanding about quantities that
1.1 Finite Gain and Passivity Performance Measures

characterize major properties of closed loop systems [5]. However, the second approach
can be extended directly to the nonlinear case. At present there have been many applica­
tions of the use of finite gain performance criteria for solving various types of disturbance
attenuation problems. These applications are reported in [4] and [95] for control tasks
for robots, in [71] and [28] for spacecraft control applications and in [5] for reactor tank
control, to name a few. Another important application of the second approach is in
model matching problems [3], and in approximate input-output linearization problems
[5]. The latter problems are crucial in cases where exact feedback linearization methods
do not apply [5].

1.1.2 Passivity

The application of passivity, or positive real, performance measure has so far been in
system stabilization problems [39], [91], [77], [14], [31], [88], [81], [15], [62]. The paper
[39] considers a stabilization problem for uncertain systems in which the uncertainty is
described as specific positive-real parametric perturbation. It is shown in this paper
that the special form of the uncertainties arises naturally in the modeling of (natural)
frequency uncertainties of a lightly damped flexible structure. Thus, the uncertainty
description takes into account phase information rather than the common gain informa­
tion. A robust stabilization result with respect to unstructured uncertainties can also be
obtained based on a positive real performance criterion [40]. In this case, the uncertain
part of the model (see Figure 1.1) is assumed to be positive real. In particular, in [40]
it is shown using a Lyapunov function argument that if the controlled nominal model
$\Sigma^K$ is positive real then the uncertain system is stable. This stability is asymptotic if
either the nominal system or the uncertain system is strictly positive real. Such a result
is termed a positivity theorem [40].

The positivity theorem can be extended to solve a robust stabilization problem for
nonlinear systems, under certain conditions (see for example [45], [101]). Such a result
can be particularly useful in situations where the uncertain systems are inherently
passive such as Euler-Lagrange mechanical systems (see [81] for description of this class
of systems). Another application of a passive performance measure is in stabilization of
systems with known model. In [14], it is shown that under certain conditions passive
systems are stabilizable by a simple static output feedback controller. Thus, a possible
design procedure is firstly to passify the system with respect to certain inputs and out-
puts and then to construct a stabilizing static output feedback controller. Geometric conditions under which a nonlinear system is passifiable by a smooth state feedback control are given in [14]. Such a design method is commonly termed a passivity based design one (see also [81], [15], [62]). On the other hand, in the paper [88], the authors consider a passivity aimed design method in an adaptive fashion. Thus, the goal of the design is to render the controlled system passive. The usefulness of this method stems from the fact that a strictly passive system can be asymptotically stable under certain conditions [44], [45].

1.2 Dissipative Systems

The previous discussion which is, by no means, complete, is presented to point out the versatility of the finite gain and passivity performance measures in control designs for both linear and nonlinear systems. Finite gain and passivity are special concepts of the more general one of dissipativity. This concept is formulated in [105], [106] and is utilized for stability analysis of dynamical systems, and is further developed in [44], [45], [46] and [47]. Central to the concept of dissipative system is the so-called energy supply rate function \( r(z, w) \), in which \( z \) and \( w \) are the output and input of the system respectively. This function can be thought of as the rate of (generalized) energy that is injected into the system from some external source. A system is called dissipative with respect to the supply rate function \( r \) if

\[
\begin{align*}
-\int_0^T r(z(t), w(t))dt & \leq \beta(x_0), \\
\forall w, \forall T & \geq 0,
\end{align*}
\]

for some finite function \( \beta \), with \( \beta(0) = 0 \), where \( x_0 \) denotes the initial state of the system. This is an external description of dissipativity [44]. When \( x_0 = 0 \), i.e., the system is initially unexcited, this inequality says that an initially unexcited dissipative system can only absorb energy [44]. In [105], [44], it is shown that dissipative systems can be characterized in terms of a set of energy-like functions \( V \) (called storage functions). These function are non-negative and satisfy the following (integral) dissipation inequality

\[
V(x(t_0)) + \int_{t_0}^{t_1} r(z(t), w(t))dt \geq V(x(t_1)),
\]

for all time variables \( t_0, t_1 \), with \( t_1 \geq t_0 \), for all admissible inputs \( w \). The function \( V(x) \) can be interpreted as the internal (generalized) energy stored in the system when it stays
1.3 Existing Synthesis Theory

at the state \( z \). Thus, the increment of energy within a dissipative system is always less than or equal to the amount of the total energy injected into it (i.e., the system actually dissipates energy). The infinitesimal form of this inequality is given by (provided \( V \) is sufficiently smooth)

\[
\nabla_z V(z)f(z,w) - r(h(z,w),w) \leq 0,
\]

for all \( w \), in which \( f(z,w) \) denotes the vector field describing the dynamics of the system and \( h(z,w) \) is the output function, i.e., \( z = h(z,w) \). Under appropriate observability assumptions it is shown in [44], [45], [14] that the storage function \( V \) serves as a Lyapunov function and, therefore, its existence ensures the system stability. Under additional conditions on the supply rate functions, asymptotic stability can be obtained by utilizing the La Salle invariance principle [44], [45], [46]. An interesting feature here is that inequality (1.2) does not uniquely define storage functions. However, these functions satisfy an inequality

\[
V_a(z) \leq V(z) \leq V_r(z),
\]

where the bounds \( V_a \) and \( V_r \) are called the available and required storage functions respectively [105]. The available storage function \( V_a \) is defined by

\[
V_a(z) = \sup_{T \geq 0, w \in L_2([0,T], \mathbb{R}^d)} \{ -\int_0^T r(z(t),w(t)) dt : X(0) = x \}.
\]

This function describes the maximum amount of energy that can be retrieved from the system initialized at \( z \). As we shall see later in the thesis, it is the available storage function that plays the fundamental role in the synthesis of dissipative systems.

Finite gain and passive systems are dissipative systems with respect to the specific supply rate functions \( \frac{1}{2}(\gamma w'w - z'z) \) and \( z'w \) respectively [106], [44], [14]. These special forms of supply rate functions, together with inequality (1.1) or (1.3) are key elements in the achievement of desirable performance in the various applications discussed previously in Section 1.1.

1.3 Existing Synthesis Theory

In the literature, synthesis theories for finite gain and passive systems have been addressed independently, either in the state feedback or output feedback case. Finite gain control synthesis problem is addressed and solved using various techniques in the papers [99], [6], [11], [56], [66] for nonlinear systems and [23], [82], [83] and [73] for linear systems, to
name but a few, whereas the positive real control synthesis theory is presented in [14], [31], [88] for nonlinear systems and [39], [91] and [85] for linear systems.

1.3.1 Finite Gain Synthesis

In the linear systems case, a complete finite $L_2$ gain (or $H_\infty$) synthesis theory by using either state or output feedback is presented in [23], [82], [83] and [73]. Key in the techniques employed in [23], [83] and [73] are game theoretic methods. The paper [82] proposes a rather elementary approach which is based on a bounded real lemma (BRL) expressed in terms of an ARE. This Riccati equation can be interpreted as the dissipation inequality for finite gain dissipative linear systems. In this literature, the solution to the output feedback control synthesis problem is presented in terms of solutions of two uncoupled algebraic Riccati equations (ARE's) plus a coupling condition.

In the nonlinear systems case, finite gain state feedback synthesis problems are fairly well understood (see for example [6], [99], [10], [56], [66] and the references therein). In this case, one considers a HJI PDI/PDE and finds appropriate solutions to it [6], [99], [10], [56], [66]. When such solutions exist, an optimal control law solving the problem can be constructed using these solutions. The HJI PDI is the relevant (controlled) dissipation inequality for the state feedback problem and is the dynamic programming equation for an underlying dynamic game problem. An important issue here is that in general the (available) storage function need not be smooth (first order differentiable) everywhere. This is a common feature in the traditional optimal control or game theory (see [34], [35], [26]). In such a situation the weak (viscosity) notion of solutions plays the key role [35], [26], [10], [29], [90], [51]. Locally smooth solutions may exist, however, when the control problem for the corresponding linearized system is solvable (by using a linear controller). This result is obtained using differential geometry methods in [97], [98], [99].

Finite gain control synthesis problems using output feedback are addressed in [99], [6], [50], [11], [66], [56], [57]. The principle difficulty in the output feedback case arises from the lack of complete information concerning the state of the system. In [6], [99], [66] the authors employ a state estimate, called the minimum stress estimate in [104] (in a stochastic control problem), that maximizes a related worst-case cost function which is consistent with past measurements. The output feedback control law is constructed via replacement of the state variable in the expression of a related state feedback law by
this state estimate. This state feedback control law is obtained by solving the control synthesis problem assuming that the state variable is available for control. This control scheme is termed the *certainty equivalence control* (CEC). Under the assumptions that (i) relevant value functions are smooth and (ii) the minimum stress estimate is uniquely determined, this control scheme is proven to be optimal (i.e., it achieves minimax) in [6], [25], [66]. A different approach is taken in [11] in which a special structure of controller is postulated. The parameters or function defining the controller is then determined by solving related HJI inequalities. These approaches in general lead to sufficient conditions for the solvability of the output feedback control problem. That is, if the corresponding HJI inequalities proposed in the approaches admit solutions, then the control problem is solved. However, conditions under which solutions to these inequalities exist are not obtained.

In the papers [56], [57] the authors propose an approach to solve the finite gain output feedback control problem by introducing the notion of state called *information state*. This notion is commonly used to solve stochastic control problems [67], [28]. In [54], [55], an information state is used to solve a risk-sensitive stochastic control problem. The information state is a causal function of the measurement and evolves forward in time according to a (controlled) partial differential equation (PDE) and, hence, is infinite dimensional, in general. The key step in this approach is a reformulation the original partial state information (or output feedback) control problem as a new, but equivalent, problem with full state information, in which the information state plays the role as the state. In the new problem, the original state space model is replaced by the information state dynamics and the original cost function by a new expression purely in terms of the information state. The new problem is then solved using dynamic programming methods leading to an infinite dimensional HJI PDI, which is the relevant (controlled) dissipation inequality for the output feedback control problem. In the discrete time systems case, in [56] it is shown that the information state approach leads to sufficient as well as necessary conditions for the solvability of the output feedback control problem. In particular, it is proven that when the control problem is solved by any output feedback controller, then solutions to related inequalities exist. The underlying system theoretic idea of the information state approach applies well to the continuous time systems case. The related inequalities are presented in [92], [57], [42]. The mathematical difficulties in the continuous time case arise in the rigorous interpretation of the infinite dimensional HJI
Preliminary results regarding this matter are reported in [57] in which a definition of viscosity solutions to the infinite dimensional PDI is given. Further study on the stationary behaviour of the inequalities arising in the information state approach, i.e., the equation describing the information state dynamics and the infinite dimensional PDI, are presented in [42], resulting in deeper insights into the information state theory and connections with the J-inner/outer factorization theory.

1.3.2 Passivity Synthesis

Passivity or positive real synthesis methods for linear systems are presented in [85], [39] and [91]. In [85] and [39], the authors utilize the Cayley transformation to convert the positive real problem to a finite gain problem for a new system constructed from the original system. This is an indirect technique and in general leads to non-unique solutions [91]. On the other hand, the paper [91] develops a direct synthesis theory based on the positive real lemma, paralleling the technique in [82]. Interestingly, the solution to the output feedback positive real control problem is expressed in terms of solutions to two uncoupled Riccati equations plus a coupling condition, mimicking the solution for finite gain problem.

Passivity synthesis theory for nonlinear systems is presented in the papers [14], [31] and [88]. In [14] the authors present geometric conditions (namely relative degree 1, minimum phaseness) under which a system is passifiable by using a smooth state feedback. When the conditions hold this approach leads to the existence of a smooth storage function. This result is extended to the case of static output feedback control in [31]. In [88], an adaptive scheme to construct a controller that renders the closed loop system passive is presented.

1.4 Thesis Outline

To date, there has been little literature on synthesis theories for general dissipative systems, even for linear systems. In the linear systems case, the Cayley transformation may be employed to convert passivity control problem to a finite gain problem for a related system [85], [39]. This is an indirect approach and in general does not lead to unique solutions [91]. Moreover, in the nonlinear systems case, it may be difficult to obtain a state space representation of the corresponding transformed system. This prevents
us from employing state space based dynamic optimization techniques such as dynamic programming as they require an expression of the state space model. In addition, a transformation based technique does not apply in the case of nonquadratic dissipativity, in general.

The existence of a general synthesis theory could offer theoretical as well as practical advantages. From a theoretical point of view, such a theory could allow us to focus our study on a single type of equation. One could then concentrate on the development of a numerical method to solve the equation to obtain design results with desirable dissipativity properties.

Once a general dissipative synthesis theory is established, then we are in a good position to explore the various dissipativity performance criteria (the theory could offer) to achieve performance enhancement in systems designs or, perhaps, to solve previously unsolved design problems. The previous overview on the applications of finite gain and passivity performance measures confirms the versatility of these specific measures. Thus, it would not be too much to expect that a general dissipativity performance measure could find its place in even wider range of applications.

This thesis contributes to the development of a general dissipative systems synthesis theory. In particular, it studies a number of issues related to the computation for dissipative systems, control synthesis methods with either state or output feedback and filter synthesis. Continuous time and discrete time dynamical systems are considered. In addition, it studies some applications of dissipative performance measures for some robust stabilization problems. An outline of the thesis is as follows.

### 1.4.1 Computation of the $H_\infty$ Norm of Nonlinear Systems

Computing the $H_\infty$ norm of a nonlinear system in a state space model can be carried out by testing whether the system has $L_2$ gain bounded (from above) by a prescribed value, $\gamma$. The value of $\gamma$ is then reduced or increased accordingly. The boundedness test can be cast as one of determining the solvability of a related dissipation inequality. This inequality takes the form as a partial differential inequality (PDI) of Hamilton-Jacobi-Bellman type given by

$$\sup_{w \in W} \{ \nabla_x V(x)f(x,w) - (\gamma^2|w|^2 - |h(x,w)|^2) \} \leq 0 \text{ in } \mathbb{R}^n.$$  \hspace{1cm} (1.5)
In this dissipation inequality the corresponding supply rate function is given by

\[ r(z, w) = \gamma^2 w'w - z'z. \]

In the linear systems case, this inequality reduces to an algebraic Riccati inequality (ARI) and an efficient numerical method is available for solving it [12]. In this thesis, a numerical method for solving the PDI (1.5) is proposed. The method employs a finite difference scheme similar to that employed in [69] to discretize the PDI (1.5). This yields the following partial difference inequality

\[ V^\delta(x) \geq \sup_{w \in W^\delta} \left\{ \sum_{x \in M^\delta(x)} P^\delta(x, z; w) V^\delta(z) - \frac{\delta}{\lambda^\delta} \left( \gamma^2 |w|^2 - |h(x, w)|^2 \right) \right\}. \quad (1.6) \]

The partial difference inequality (1.6) may be interpreted as a dynamic programming equation for an optimal control problem of a related Markov chain (see [69]). In [51], [35] it is shown that the inequality (1.6) converges to the original PDI (1.5) in the viscosity sense. The contribution of the thesis lies in the development of two iterative schemes for solving the partial difference inequality (1.6), namely the value space and the policy space iteration schemes (see Chapter 6 of [69] for a treatment of such schemes for stochastic control problems). In the first scheme, a related finite horizon value function is computed. The length of the horizon is iteratively increased until the value function converges (within some numerical tolerance). Convergence of this iteration is proven. Since the value space scheme is based on a finite horizon approximation, it typically requires a large number of iterations to converge. The second scheme is based on iteration in the policy or control space introduced by Bellman (see [69]). Given an old value function and an old policy, a new policy that optimizes a related functional is sought at each iteration. A new value function is then computed using the new policy by solving a linear equation. The iteration is carried out until the resulting value function converges. When it converges, the policy scheme typically takes a smaller number of iterations. In addition to the basic value space and policy space schemes, some acceleration techniques based on the results in [17] and [69] are presented. Numerical experiment was conducted for a number of low dimension linear and nonlinear systems. The numerical method is implemented on Sun4 Workstation and on Connection Machine (CM 5) supercomputer employing the FORTRAN 90 language. The computation results are described in Chapter 2.

In Chapter 2, we also present an extension of the numerical method for testing more
1.4 Thesis Outline

general dissipative systems. In the general case, the PDI (1.5) is replaced by
\[ \sup_{w \in W} \{ \nabla_x V(x) f(x, w) - r(h(x, w), w) \} \leq 0 \text{ in } \mathbb{R}^n, \]  
(1.7)
where \( r(x, w) \) is the general (possibly nonquadratic) supply rate function being considered. The discretized version of the PDI is given by
\[ V^\delta(x) \sup_{w \in W^\delta} \{ \sum_{z \in N_d(x)} p^\delta(z, z; w) V^\delta(z) - \frac{\delta}{\lambda_d^\delta} r(h(x, w), w) \}. \]  
(1.8)
The corresponding value space and policy space iterations for solving (1.8) can then be developed in a similar manner.

1.4.2 State Feedback Analysis and Synthesis

In this thesis we propose a general dissipative control synthesis method using full state feedback. The general dissipative performance measure is expressed in terms of a supply rate function which is allowed to be nonquadratic. When the supply rate is set to be quadratic, it includes finite gain, passivity, a mixture between finite gain and passivity, and sector bounded performance measures as special cases.

In Chapter 3, known results related to general dissipative systems are reviewed, and, in the quadratic supply rate case, i.e., when
\[ r(z, w) = \frac{1}{2} (w'Qw + 2w'Sz + z'Rz), \]
the results are extended to a strict dissipativity case. In particular, we characterize a strict (quadratic) dissipative nonlinear system in terms of the PDI
\[ \sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x)(A(x) + B_1(x)w) - r(h(x, w), w) + \frac{\delta}{2} x'x \} \leq 0, \]  
(1.9)
with \( \delta > 0 \), or in terms of the partial differential equation (PDE)
\[ \sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x)(A(x) + B_1(x)w) - r(h(x, w), w) \} = 0, \]  
(1.10)
such that, when \( V \) is smooth, the vector field
\[ A^*(x) = A(x) + B_1(x) \tilde{Q}^{-1}(x)[B_1(x)^' \nabla_x V(x)' - \tilde{S}(x)C_1(x)] \]
enjoys a stability property. These results are obtained without any assumptions regarding controllability or observability of the system and, thus, extend the strict bounded real lemma (SBRL) for linear systems in [82].
In the general dissipative control synthesis case, game theoretic methods are undertaken and solutions to the control problem are expressed in terms of solutions to the PDI (of Hamilton-Jacobi-Isaacs type)

$$\inf_{u \in \mathbb{R}^n} \sup_{w \in \mathbb{R}^m} \{ \nabla_x V(x)(A(x) + B_1(x)w + B_2(x)u) - r(h_1(x, w, u), w) \} \leq 0. \quad (1.11)$$

We show that whenever there exists a static state feedback control that control problem then there exists a solution $V$ to the HJI PDI in the viscosity sense. We deduce stability of dissipative systems by assuming a detectability property. This result is described in Chapter 4. In the quadratic dissipativity case, we express the solution to the control problem in terms of a solution to the PDI (1.11) or a solution to a HJI PDE, in the viscosity sense. This extends the results in [51], [10], [90]. A detailed proof of this result is presented in Appendix C.

While in general, the PDI (1.11) need not have globally smooth (say, first order differentiable) solutions, under certain geometric conditions there may exist locally smooth solutions. In [97], [98], [99] it is shown in the case of $H_\infty$ control problem for nonlinear system that if the control problem for the corresponding linearized system has a particular solution, which is expressed in terms of a stabilizing solution of an algebraic Riccati equation, then the related PDI has a smooth solution locally and the problem for nonlinear systems is also solvable locally by a smooth state feedback control law. In Chapter 4, this result is extended to a general quadratic dissipativity control case.

1.4.3 Output Feedback Synthesis

In Chapter 5 we study a general dissipative control synthesis problem for nonlinear systems using output feedback. The approach undertaken follows the information state method recently developed for $H_\infty$ control synthesis in the papers [54], [55], [56], [57] and [42]. The solution is obtained by first reformulating the original partial state optimization problem as a new, but equivalent, full state problem in which the information state replaces the original state. The new problem is then solved by employing the dynamic programming method, leading to an infinite dimensional dissipation inequality. The controller state, that is the information state $p$, lives in an infinite dimensional space, and its evolution is governed by a controlled PDE. The contribution of the work in the thesis lies in the extension of the information state method to solve a general dissipative
1.4 Thesis Outline

control synthesis problem. This lifts the results in Chapter 4 for state feedback case up to the output feedback case. In the general dissipativity case, the infinite dimensional (controlled) dissipation inequality is given by

\[ \sup_{y \in \mathbb{R}^{y}} \inf_{u \in U} \{ \langle \nabla_p W, F(p, u, y) \rangle \} \leq 0, \]  

(1.12)
in which, \( F(p, u, y) \) denotes the information state dynamics. A new feature in the general dissipativity case is that the sup and inf operations in (1.12) need not commute in general, i.e., the Isaacs condition does not always hold. As a result, a saddle point strategy need not exist, in general. In Chapter 5, we also evaluate the validity of a certainty equivalence controller (CEC) for the general dissipativity control synthesis problem. This certainty equivalence controller is previously proposed in [104], [6], [25], [66] in the case of \( H_{\infty} \) control.

In Chapter 6 we consider a quadratic dissipative output feedback control problem for a class of bilinear systems, including linear systems, for which the information state method leads to a finite dimensional solution, under certain condition concerning the initial information state. In particular, if we choose a quadratic initial information state, then it remains quadratic in the future. The quadratic form is completely determined by some finite dimensional quantities, denoted by \( \rho \), which is governed by a set of ordinary differential equations (ODE's). As a result, we may regard the finite dimensional quantities \( \rho \) as our finite dimensional information state having the ODE's as its dynamics, i.e.,

\[ \dot{\rho} = \tilde{F}(\rho, u, y). \]  

(1.13)
The relevant dissipation inequality now is finite dimensional given by

\[ \sup_{y \in \mathbb{R}^{y}} \inf_{u \in U} \{ \nabla_{\rho} W(\rho) \tilde{F}(\rho, u, y) \} \leq 0. \]  

(1.14)
Moreover, in the linear systems case the solution of this inequality can be expressed in terms of two ARE's plus a coupling condition. The coupling condition characterizes the domain, which is a subset of the information state space, on which the solution \( W \) to (1.14) is finite. This provides a new insight into the coupling condition familiar in the linear \( H_{\infty} \) control theory [23], [82], [36] or in the linear positive real control theory [91].

1.4.4 Filter Synthesis

In this thesis, we study a general dissipative filtering problem for nonlinear systems. A special case of this problem is the nonlinear \( H_{\infty} \) filtering which currently attracts
considerable attention [25], [63], [66], [60] and [89]. In Chapter 7, we first present a
solution to the nonlinear $H_{\infty}$ filtering problem by applying the information state method. The solution to the filtering problem is expressed in terms of two PDE's: the first PDE governs the evolution of the relevant information state and the second one is a (infinite dimensional) HJI PDE providing a means for constructing the optimal filtering strategy. This result presents a new feature in nonlinear filtering theory since in the traditional stochastic or deterministic filtering problems, the solution involves one equation, namely the Zakai equation for describing an unnormalized conditional density in the stochastic filtering case [116], [22], or the Mortensen equation in the deterministic estimation case [43]. We compare our information state solution with the certainty equivalence filter proposed in [25], [60], [66] in the $H_{\infty}$ filtering case, in which the solution is expressed in terms of one equation. The synthesis results are then extended to a general dissipativity filtering case. The information solution is then specialized to linear systems recovering the linear (central) $H_{\infty}$ filter and producing new results for linear filtering.

1.4.5 Applications

In Chapter 8, we propose two applications of dissipativity performance criteria for some robust stabilization problems. In the first application, we consider linear systems possessing sector bounded nonlinearities at their input and output. We look for a controller that stabilizes the system for all admissible input and output nonlinearities. This problem is cast as one of finding a controller for a new, related linear system that renders the new closed loop dissipative with respect to a certain quadratic supply rate function. We provide a method for synthesizing such a controller and prove stability under a closed loop observability assumption. This result can be used to achieve some pre-specified gain and phase stability margins at input and output simultaneously.
In the second application, we consider a class of uncertain nonlinear systems consisting of nonlinear nominal models and general dissipative unknown parts (see Figure 1.1). We propose a synthesis method (based on the results in Chapter 4) which yields a controller that stabilizes all the uncertain systems in the class being considered. This is an extension of the results in [48] and complements the analysis results in [45], [3].

1.4.6 Results for Discrete Time Systems

A large portion of the synthesis results for continuous time systems are also valid in the discrete time systems case. In the thesis, we provide a solution to discrete time dissipative control system synthesis problems using state or output feedback and to a solution to a filtering problem. In the state feedback control case, the relevant dissipation inequality is described by

$$ V(x) \geq \inf_{u \in \mathbb{R}^m} \sup_{w \in \mathbb{R}^d} \{ V(f(x, u) + w) - r(h_1(x, u, w), w) \}, \quad (1.15) $$

and in the output feedback control case, the (infinite dimensional) dissipation inequality is given by

$$ W(p) \geq \sup_{y \in \mathbb{R}^r} \inf_{u \in \mathbb{R}^m} \{ W(F(p, u, y)) \}, \quad (1.16) $$

in which $F(p, u, y)$ denotes the dynamics of the discrete time information state.

In the case of a general dissipative filtering, the (infinite dimensional) dissipation inequality is given by

$$ W(p) \geq \sup_{y \in \mathbb{R}^r} \inf_{\hat{z} \in \mathbb{R}^s} \{ W(\tilde{F}(p, \hat{z}, y)) \}, \quad (1.17) $$

Figure 1.2: Linear system with nonlinearity at the input and output.
where $\bar{F}(p, \hat{z}, y)$ denotes the information state dynamics for the filtering problem.

1.4.7 Summary of Contributions

The principal contributions of the thesis are listed below.

- Computation for dissipative nonlinear systems.
- Characterization of strictly quadratic dissipative nonlinear or linear systems.
- Synthesis of general dissipative nonlinear/linear systems using static state feedback.
- Synthesis of general dissipative nonlinear systems using output feedback.
- Synthesis of quadratic dissipative bilinear/linear systems using output feedback.
- Synthesis of general dissipative filters for nonlinear/linear systems.
- General dissipative state and output feedback control synthesis and filter synthesis for discrete time systems.
- Application of general dissipativity control methods in a number of robust stabilization problems.
Chapter 2

Computational Methods for Dissipative Systems

2.1 Introduction

In this chapter we study the problem of computing the $H_\infty$ norm of input-output maps arising from nonlinear state space models and propose an approach to the numerical solution of this problem based on the finite difference method.

For linear systems, a number of techniques are available. For instance, the Bounded Real Lemma can be used ([36], [23], [12]), and the numerical problem becomes one of solving a Riccati-type matrix inequality (or equation) ([1]). One solution to this numerical problem involves numerically integrating a matrix differential equation. An interesting difficulty is that the matrix inequality may have infinitely many solutions.

Our proposed approach for nonlinear systems is similar in spirit. We make use of a version of the Bounded Real Lemma, in which a partial differential inequality (PDI) replaces the matrix inequality ([44], [51]). This PDI also may have many solutions, one of which is the so called available storage function ([105]). We then show that this function is the limit of a corresponding finite horizon storage function. The problem of solving this PDI is addressed using the finite difference method ([17], [69]) and convergence of the numerical scheme is proven using weak (or viscosity) solution methods ([16], [7], [17], [20]); see Theorem 2.3 below. We show that the discrete version of the PDI corresponds to an infinite horizon problem for a controlled Markov chain. The problem of solving the
discrete PDI is then addressed using value space and the policy space approximations, and various acceleration methods ([17], [69]). Some numerical examples of computing the $H_\infty$ norm for one and two dimensional systems are provided. Preliminary results were reported in [53].

2.2 Computation of the $H_\infty$ Norm

For linear systems, there are a number of methods available for computing the $H_\infty$ norm ([23], [36], [12]). One approach mentioned above is to combine the iterative search over $\gamma$ with a technique for solving matrix inequalities (or equations). This procedure is analogous to the method we present in Section 2.4. An alternative well-known and simple method involves the calculation of the eigenvalues of the Hamiltonian matrix $H^\gamma$ parameterized by $\gamma$: $\|G\|_{H_\infty} < \gamma$ if and only if $H^\gamma$ has no imaginary eigenvalues. Of course, one could also appeal directly to the frequency domain definition.

In the case of nonlinear systems, it has been proven in [97], [98] and [99] that if the linearization has $H_\infty$ norm $< \gamma$, then locally the nonlinear system has $H_\infty$ norm $< \gamma$, and conversely. This is a very useful result, and can be readily implemented. It can serve as a starting point for the global methods we propose in this chapter.

The PDI gives a global characterization of the $H_\infty$ norm, and can be used as a means of determining the $H_\infty$ norm globally. In Section 2.4 we present a numerical method for achieving this.

2.3 The $H_\infty$ Norm

The systems we consider are described by state space models of the form

$$\begin{align*}
\dot{x}(t) &= f(x(t), w(t)), \quad t > 0, \quad x(0) = x_0, \\
z(t) &= h(x(t), w(t)), \quad t \geq 0,
\end{align*}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in W$, a closed subset of $\mathbb{R}^m$ containing the origin, and $z(t) \in \mathbb{R}^p$. We assume $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $h \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^p)$, and that the first order derivatives are bounded. In addition, we suppose $f(0,0) = 0$ and $h(0,0) = 0$, so that $x = 0$ is an equilibrium for the uncontrolled system. Given a control $w : [0, \infty) \to W$, the solution at time $t \geq 0$ with initial condition $x_0$ is denoted $x(t) = \psi_w(t)x_0$; the
2.3 The $H_\infty$ Norm

The corresponding output is $z(t) = h(\psi_w(t)x_0, w(t))$. We regard an input $w \in L_2([0, T], W)$ as an element of $L_2([0, \infty), W)$ by setting $w(t) = 0$ for $t > T$.

In applications, $w$ is a disturbance input whose effect on a given output quantity $z$ is to be measured, for example using the $H_\infty$ norm of the input–output map $G$ relating $w$ and $z$, given $z(0) = 0$, defined by

$$(Gw)(t) = h(\psi_w(t)0, w(t)) \quad \text{for all } t \geq 0.$$ If $G$ maps $L_2([0, \infty), W) \to L_2([0, \infty), \mathbb{R}^p)$, then the "$H_\infty$ norm of $G$" can be defined by

$$\|G\|_{H_\infty} = \sup_{w \in L_2([0, \infty), W), w \neq 0} \frac{\|Gw\|_2}{\|w\|_2},$$

where $\| \cdot \|_2$ is the usual $L_2$ norm. This definition is a direct time domain analogue of the frequency domain definition for linear systems.

The number $\|G\|_{H_\infty}$ may be computed iteratively as follows ([23], [36], [12]). Select $\gamma > 0$ and test if $\|G\|_{H_\infty} \leq \gamma$. Then adjust $\gamma$ accordingly and repeat until a desired accuracy is achieved. The problem then is to determine whether or not $\|G\|_{H_\infty} \leq \gamma$.

By definition, we will say that "$\|G\|_{H_\infty} \leq \gamma$" (i.e. has $L_2$ gain $\leq \gamma$) if and only if

$$\int_0^T (\gamma^2|w(t)|^2 - |z(t)|^2) \, dt \geq 0, \quad x(0) = 0,$$

for all $T \geq 0$ and all $w \in L_2([0, T], W)$. Inequality (2.3) holds if the system (2.1) is dissipative ([105], [44]) with respect to the supply rate $\gamma^2|w|^2 - |z|^2$, i.e. if there exists a non-negative function $S$ such that $S(0) = 0$ satisfying the dissipation inequality

$$S(x) \geq \sup_{T \geq 0, w \in L_2([0, T], W)} \left\{ S(x(T)) - \int_0^T (\gamma^2|w(r)|^2 - |z(r)|^2) \, dr : x(0) = x \right\}.$$ Such functions $S$ are called storage functions. Conversely, if the system (2.1) is reachable from 0 and (2.3) holds, then (2.1) is dissipative. One particular storage function is the available storage, defined by

$$S_a(x) = \sup_{T \geq 0, w \in L_2([0, T], W)} \left\{ -\int_0^T (\gamma^2|w(r)|^2 - |z(r)|^2) \, dr : x(0) = x \right\}.$$ Note that $S_a$ is a solution of (2.5), see [44], and [51]. Thus determining whether or not $\|G\|_{H_\infty} \leq \gamma$ is equivalent to the solvability of (2.4). This in turn is equivalent to the
solvability of a PDI. This leads to the following theorem, which is essentially a version of the Bounded Real Lemma ([44], [51], [105]).

**Theorem 2.1** Assume that the system (2.1) is reachable from 0. Then \( \| G \|_{H_\infty} \leq \gamma \) if and only if there exists a non-negative function \( S \) satisfying \( S(0) = 0 \) and the PDI

\[
\sup_{w \in W} \left\{ \nabla_x S(x) f(x, w) - \left( \gamma^2 |w|^2 - |h(x, w)|^2 \right) \right\} \leq 0 \text{ in } \mathbb{R}^n,
\]

(2.5)
in the viscosity sense.

**Proof.** Assume there exists a non-negative solution of the PDI (2.5) such that \( S(0) = 0 \). Then Theorem 3.1 of [51] implies that \( S \) is a storage function, i.e \( S \) satisfies (2.4). This implies \( \| G \|_{H_\infty} \leq \gamma \). Conversely, assume (2.1) is reachable from 0 and \( S_a(x) \) is a storage function. By Theorem 3.1 of [51] \( S_a(x) \) satisfies (2.5).

The PDI (2.5) is understood in the weak (or viscosity) sense (see Appendix C, or [51], [8]), and is a nonlinear analogue of a Riccati-type matrix inequality (see [1], equation (7.3.10)). One difficulty in using Theorem 2.1 is the lack of uniqueness of storage functions, or equivalently, solutions of the PDI (2.5). This issue is well known for linear systems, where the matrix inequality has infinitely many solutions, and is discussed at length in [1] together with techniques for finding solutions.

One solution of the PDI is the available storage \( S_a \). To approximate this, consider the finite horizon problem

\[
S_a(x, t) = \sup_{w \in L^2([0,t],W)} \left\{ -\int_0^t \left( \gamma^2 |w(r)|^2 - |z(r)|^2 \right) dr : x(0) = x \right\}.
\]

This function is a solution, in the viscosity sense, of the PDE

\[
\begin{cases}
-\frac{\partial S}{\partial t} + \sup_{w \in W} \left\{ \nabla_x S(x)' f(x, w) - \left( \gamma^2 |w|^2 - |h(x, w)|^2 \right) \right\} = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\
S(x, 0) = 0 \text{ in } \mathbb{R}^n.
\end{cases}
\]

(2.6)
The following theorem shows that \( S_a(x, t) \) serves as an approximation to \( S_a \), a solution of (2.5), for sufficiently large \( t \) ([105], [106]).

**Theorem 2.2** Suppose that the available storage \( S_a \) exists and is finite. Then

\[
\lim_{t \to \infty} S_a(x, t) = S_a(x).
\]
Proof. The key idea in the proof is the non-decreasing property of $S_a(x,t)$. Given $t^* \geq t$, we will show that $S_a(x,t) \leq S_a(x,t^*)$.

Define
\[
J(x,t;w(\cdot)) = -\int_0^t (\gamma^2|w(r)|^2 - |z(r)|^2) \, dr, \quad x(0) = x, \quad w(\cdot) \in L_2([0,t],W).
\]

Given $w(\cdot) \in L_2([0,t],W)$, define $w^*(\cdot) \in L_2([0,t^*],W)$ by
\[
w^*(r) = \begin{cases} 
w(r) & \text{if } 0 \leq r \leq t, \\
0 & \text{if } t < r \leq t^*,
\end{cases}
\]
and denote by $W^*(0,t^*) \subset L_2([0,t^*],W)$ the class of all such inputs $w^*(\cdot)$. Then
\[
J(x,t^*;w^*(\cdot)) = -\int_0^{t^*} (\gamma^2|w^*(r)|^2 - |z(r)|^2) \, dr, x(0) = x, w^*(\cdot) \in W^*(0,t^*)
\]
\[
= -\int_0^t (\gamma^2|w(r)|^2 - |z(r)|^2) \, dr - \int_t^{t^*} (-|z(r)|^2) \, dr 
\geq J(x,t;w(\cdot)).
\]

Since $W^*(0,t^*) \subset L_2([0,t^*],W)$, we have
\[
S_a(x,t^*) \geq \sup_{w^* \in W^*(0,t^*)} J(x,t^*;w^*(\cdot))
\geq \sup_{w \in L_2([0,t],W)} J(x,t;w(\cdot))
= S_a(x,t).
\]

Next, observe that, for all $t \geq 0$,
\[
0 \leq S_a(x,t) \leq S_a(x).
\]

Therefore, $\lim_{t \to \infty} S_a(x,t)$ exists and is finite. Then following [105],
\[
S_a(x) = \sup_{t \geq 0, w \in L_2([0,t],W)} \left\{ -\int_0^t (\gamma^2|w(r)|^2 - |z(r)|^2) \, dr : x(0) = x \right\}
= \sup_{t \geq 0} \sup_{w \in L_2([0,t],W)} \left\{ -\int_0^t (\gamma^2|w(r)|^2 - |z(r)|^2) \, dr : x(0) = x \right\}
= \sup_{t \geq 0} S_a(x,t)
= \lim_{t \to \infty} S_a(x,t).
\]

The last equation follows since $S_a(x,t)$ is non-decreasing in $t$. \qed

This theorem suggests that computing $S_a(x,t)$ for sufficiently large $t$ will provide an approximation to $S_a$. The discrete analogue of this result is the basis for the value iteration method in Section 2.4.1.
2.4 The Numerical Schemes

We use a finite difference scheme similar to those presented in [69] for approximating solutions to dynamic programming equations arising in stochastic optimal control. For the deterministic problem considered here, the scheme is very similar to those presented in [7], [17], and [30].

Let \((\mathbb{R}^n)^\delta\) denote a coordinate grid of size \(\delta > 0\), centred at the origin. Define a system of neighborhoods \(N_\delta(x)\) for \(x \in (\mathbb{R}^n)^\delta\) by

\[
N_\delta(x) = \{ z \in (\mathbb{R}^n)^\delta : z = x \text{ or } z = x \pm \delta e_i, \text{ for some } i = 1, \ldots, n \}.
\]

Here, \(e_i \in \mathbb{R}^n\) denotes the \(i\)-th unit vector, \(i = 1, \ldots, n\). Write \(W^\delta = W \cap (\mathbb{R}^n)^\delta\). Define

\[
\lambda^\delta = \sup_{x \in (\mathbb{R}^n)^\delta, w \in W^\delta} |f(x, w)|_1,
\]

where \(|v|_1 \triangleq |v_1| + \cdots + |v_n|\) for \(v \in \mathbb{R}^n\), and

\[
p^\delta(x, z; w) = \begin{cases} 
1 - |f(x, w)|_1/\lambda^\delta & \text{if } z = x, \\
f_i^\pm(x, w)/\lambda^\delta & \text{if } z = x \pm \delta e_i, \ i = 1, \ldots, n, \\
0 & \text{if } z \notin N_\delta(x).
\end{cases}
\]

A finite difference analogue of the PDI (2.5) is the discrete inequality

\[
S^\delta(x) \geq \sup_{w \in W^\delta} \left\{ \sum_{z \in N_\delta(x)} p^\delta(x, z; w) S^\delta(z) - \frac{\delta}{\lambda^\delta} \left( \gamma^2 |w|^2 - |h(x, w)|^2 \right) \right\}
\]

for \(x \in (\mathbb{R}^n)^\delta\). For details on deriving such discretizations, see [69]. The discretization can be interpreted in terms of a controlled Markov chain \(\xi_k\), with transition probabilities \(p^\delta(x, z; w)\) and finite state space \((\mathbb{R}^n)^\delta\) ([69]). Indeed,

\[
P(\xi_{k+1}^k = z | \xi_l^l, w_l, 0 \leq l \leq k) = p^\delta(\xi_k^k, z; w)
\]

for all admissible control policies \(w \in W^\delta\). Iteration of (2.10) yields a dissipation inequality for the discrete stochastic system. This interpretation, while not used explicitly in this paper, can often be very useful.

The number \(\|G\|_{H_\infty}\) can be approximated with the aid of the discrete inequality (2.10).
2.4 The Numerical Schemes

**Theorem 2.3** Let \( \gamma > 0 \). If there exists \( \delta_0 > 0 \) such that for all \( 0 < \delta \leq \delta_0 \) the inequality (2.10) has a non-negative solution \( S^\delta \) satisfying \( \lim_{\delta \downarrow 0} S^\delta(0) = 0 \) and
\[
\sup_{0 < \delta \leq \delta_0} \sup_{x \in \mathbb{R}^n} |S^\delta(x)| < \infty \quad \text{for all } R > 0,
\]
then there exists a storage function \( S \) satisfying the PDI (2.5) and hence \( \| G \|_{\infty} \leq \gamma \).

**Proof.** We follow the general convergence technique described in [16]. Define
\[
S(x) = \liminf_{\delta \downarrow 0, x^\delta \to x} S^\delta(x^\delta).
\]
Then \( S \) is non-negative, l.s.c., and \( S(0) = 0 \). We now show that \( S \) satisfies the PDI (2.5).

Let \( \phi \in C^1(\mathbb{R}^n) \) and assume, without loss of generality, that \( S - \phi \) attains a strict local minimum at \( x_0 \). There is a subsequence \( x^\delta \), which we again index by \( \delta \), such that
\[
S^\delta(x^\delta) \to S(x_0), \quad x^\delta \to x_0
\]
as \( \delta \to 0 \), and \( S^\delta - \phi \) has a local minimum at \( x^\delta \in (\mathbb{R}^n)^\delta \). Then (2.10) and
\[
S^\delta(x) - S^\delta(x^\delta) \geq \phi(x) - \phi(x^\delta), \quad x \in N^\delta(x^\delta),
\]
for \( \delta \) small imply
\[
\sup_{w \in W^\delta} \left\{ \frac{\phi(x^\delta \pm \delta e_i) - \phi(x^\delta)}{\delta} f^\delta_i(x^\delta, w) - \left( \gamma^2 |w|^2 - |h(x^\delta, w)|^2 \right) \right\} \leq 0.
\]
Send \( \delta \downarrow 0 \) to obtain
\[
\sup_{w \in W} \left\{ \nabla_x \phi(x) f(x_0, w) - \left( \gamma^2 |w|^2 - |h(x_0, w)|^2 \right) \right\} \leq 0.
\]
This proves that \( S \) satisfies (2.5). Hence by Theorem 2.1, we conclude that \( \| G \|_{\infty} \leq \gamma \).
\( \square \)

In order to use Theorem 2.3, an effective numerical procedure is needed for solving the discrete inequality (2.10). This is non-trivial, since, as one might expect, there may be many solutions of (2.10). Actually, we will compute an approximation to a particular solution of (2.10) namely, the discrete analogue of the available storage \( S_0(x) \), defined by
\[
S^\delta_0(x) = \sup_{k \geq 0, w \in W_{0,k-1}} E^w_x \left\{ -\sum_{l=0}^{k-1} \left( \gamma^2 |w_l|^2 - |h(x_l, w_l)|^2 \right) \frac{\delta}{\lambda^2} \right\},
\]
where \( E^w_x \) denotes expectation with respect to the chain \( \xi^\delta_t \) with \( \xi^\delta_0 = x \).

The two main methods for solving infinite horizon problems for Markov Chain are called approximation in value space, and approximation in policy space (see Chapter 6 of [69]). Also, combinations and variants of those methods are commonly employed.
2.4.1 Approximation in Value Space

This method calculates $S^d_a(x)$ as a limit of the corresponding finite horizon storage function $S^d_a(x,k)$ as $k \uparrow \infty$.

Consider an explicit finite difference approximation to (2.6), with a time partition

$$t_k = k\delta/\lambda^d, \quad k = 0, 1, \ldots,$$

defined by

$$S^d_a(x,k) = \sup_{w \in W^d} \left\{ \sum_{z \in N(z)} p^d(z,z;w) V^d_a(z,k-1) - \frac{\delta}{\lambda^d} (\gamma^2 |w|^2 - |h(z,w)|^2) \right\}$$

$$S^d_a(x,0) = 0$$

(2.11)

for $x \in (\mathbb{R}^n)^d$, $k = 1, \ldots$. The function $S^d_a(x,k)$ defined by (2.11) has the representation

$$S^d_a(x,k) = \sup_{w \in W_{0,k-1}} E_x \left\{ -\sum_{l=0}^{k-1} (\gamma^2 |w_l|^2 - |h(x_l,w_l)|^2) \frac{\delta}{\lambda^d} \right\}$$

(2.12)

**Theorem 2.4** Assume that the discrete available storage, $S^d_a$, exists and is finite. Then,

$$\lim_{k \to \infty} S^d_a(x,k) = S^d_a(x).$$

**Proof.** The proof employs the Markov chain interpretation of the discretization and the probabilistic version of the technique used in the previous section to obtain the monotonicity. For any $k \geq 1$ define $J^d_a(x,k;w)$ as follows

$$J^d_a(x,k;w) = E^w_x \left\{ -\sum_{l=0}^{k-1} (\gamma^2 |w_l|^2 - |h(x_l,w_l)|^2) \frac{\delta}{\lambda^d} \right\},$$

where $w \in W_{0,k-1}$. Let $k^* \geq k$. We show that

$$S^d_a(x,k) \leq S^d_a(x,k^*).$$

Given $w \in W_{0,k-1}$, define $w^* \in W_{0,k^*-1}$ by

$$w^*_l = \begin{cases} w_l & \text{if } 0 \leq l \leq k - 1, \\ 0 & \text{if } k \leq l \leq k^* - 1, \end{cases}$$
and denote by $W^*_{0,k-1} \subset W_{0,k-1}$ the class of such policies $w^*$. Then, for each $w^* \in W^*_{0,k-1}$,

$$J^\delta(x, k^*; w^*) = E^w_k \left\{ - \sum_{i=0}^{k-1} \left( \gamma^2 |w_i|^2 - |h(x_i, w_i)|^2 \right) \frac{\delta}{\lambda^i} \right\}$$

$$= E^w_k \left\{ - \sum_{i=0}^{k-1} \left( \gamma^2 |w_i|^2 - |h(x_i, w_i)|^2 \right) \frac{\delta}{\lambda^i} - \sum_{i=k}^{k^*-1} \left( -|h(x_i, w_i)|^2 \right) \frac{\delta}{\lambda^i} \right\}$$

$$\geq E^w_k \left\{ - \sum_{i=0}^{k-1} \left( \gamma^2 |w_i|^2 - |h(x_i, w_i)|^2 \right) \frac{\delta}{\lambda^i} \right\}$$

$$= E^w_k \left\{ - \sum_{i=0}^{k-1} \left( \gamma^2 |w_i|^2 - |h(x_i, w_i)|^2 \right) \frac{\delta}{\lambda^i} \right\}$$

$$= J^\delta(x, k; w). \quad (2.13)$$

Therefore,

$$S^\delta_a(x, k^*) = \sup_{w^* \in W^*_{0,k^*-1}} J^\delta(x, k^*, w^*)$$

$$\geq \sup_{w \in W_{0,k-1}} J^\delta(x, k; w)$$

$$= S^\delta_a(x, k). \quad (2.14)$$

This proves the monotonicity property for $S^\delta_a(x, k)$. Finally, we note that

$$0 \leq S^\delta_a(x, k) \leq S^\delta_a(x)$$

for all $k \geq 0$. Therefore, the limit exists and is finite.

This theorem guarantees that an approximate solution to the discrete inequality (2.10) can be obtained by fixing $\delta > 0$ sufficiently small and iterating (2.11) forward in time until a stationary solution is obtained.

Our value space approximation scheme is summarised as follows.

**Step 1.** Select the discretization size $\delta > 0$.

**Step 2.** Set $p = 0$ and choose $\gamma_0 > 0$.

**Step 3.** Set $\gamma = \gamma_p$ and iterate (2.11) forward in time.

**Step 4.** If a stationary solution is obtained, then $\| G \|_{H^\infty} \leq \gamma$, and choose $\gamma_{p+1} < \gamma$.

 Otherwise, choose $\gamma_{p+1} > \gamma$.

**Step 5.** Repeat Steps 2—4, until a desired accuracy is achieved.
Step 6. If necessary, reduce the discretization size $\delta > 0$ and repeat.

Remark 2.1 To implement this scheme, the grid $(\mathbb{R}^n)^\delta$ must be truncated to a finite grid, say $D^\delta = D \cap (\mathbb{R}^n)^\delta$, where $D$ is a bounded domain in $\mathbb{R}^n$. Appropriate boundary conditions must be imposed, such as Neumann type: $\partial S/\partial \nu = 0$ on $\partial D$. This modification can be done by projecting the vector fields at the boundary points of $D^\delta$ along the boundary. To do this, let $f_i(x^i, w), x^i \in \partial D_i, i = 1, 2, ..., n$ be the $i^{th}$ component of the vector field $f(\cdot, \cdot)$ on the $x_i$-axis boundary. If the component points outwards of the boundary $\partial D_i$ then we set $f_i(x^i, w) = 0$.

2.4.2 Approximation in Policy Space

This second method for approximating $S^\delta(x)$, based on a classical method introduced by Bellman, involves two steps ([69]). For $k = 0, 1, 2, ...$

Stage 1. Given $w_k^\delta(x)$ and $S_k^\delta(x; w_k^\delta(x))$, compute the new policy $w_{k+1}^\delta(x)$ by carrying out the following optimization over the policy space

$$w_{k+1}^\delta(x) = \arg\max_{\alpha \in W^\delta} \left\{ \sum_{z \in N^\delta(x)} p(x, z; \alpha) S_k^\delta(x; w_k^\delta(z)) - \frac{\delta}{\lambda} (\gamma^2 |\alpha|^2 - |h(x, w_k^\delta(x))|^2) \right\}.$$

Stage 2. Given the updated $w_{k+1}^\delta(x)$, compute the new $S_{k+1}^\delta(x; w_{k+1}^\delta(x))$ by solving the following simultaneous linear equations

$$S_{k+1}^\delta(x; w_{k+1}^\delta(x)) = \left\{ \sum_{z \in N^\delta(x)} p(x, z; w_{k+1}^\delta(x)) S_k^\delta(x; w_{k+1}^\delta(z)) \right\} - \frac{\delta}{\lambda} (\gamma^2 |w_{k+1}^\delta(x)|^2 - |h(x, w_{k+1}^\delta(x))|^2)\right\}.$$

To implement this method, the linear equations arising in the second step can be solved iteratively by using methods such as Jacobi or Gauss-Seidel iterations. The procedure of using this method to compute the $H_\infty$ norm is similar to that of using the value space method and is summarised as follows.

Step 1. Select the discretization size $\delta > 0$.

Step 2. Set $p = 0$ and choose $\gamma_0 > 0$.

Step 3. Set $\gamma = \gamma_p$ and carry out the two stages described above with $w_0^\delta(x) = 0$ and $S^\delta(x; w_0^\delta(x)) = 0$. 
2.5 Examples

Step 4. If the value of $|S^f_0(z; w_{k+1}(x)) - S^f_0(z; w_k(x))|$ is small enough, then $\| G \|_{H_{\infty}} \leq \gamma$, and choose $\gamma_{p+1} < \gamma$. Otherwise, choose $\gamma_{p+1} > \gamma$.

Step 5. Repeat Steps 2—4, until a desired accuracy is achieved.

Step 6. If necessary, reduce the discretization size $\delta > 0$ and repeat.

2.4.3 Remarks on Feasibility

- An inherent limitation of any finite difference or finite element scheme is the well-known "curse of dimensionality". The examples below show that our schemes are feasible for low dimensional systems.

- An important issue is the selection of a range of values of $\gamma$ to search through. An ad hoc approach is to try various values and see what happens. A more systematic approach would be to obtain first some a priori estimates, or perhaps to use the local results of [97] [98], [99] to obtain starting values.

2.5 Examples

In this section we describe the use of the value space and the policy space methods presented in the previous section to compute the $H_{\infty}$ norm of a number of low dimensional systems. We also illustrate the use of two techniques: (1) local velocity, and (2) acceleration method of [17] to speed up the basic methods. The local velocity technique replaces $\lambda^f$ in (2.9) by $\lambda^f(w) = \sup_{x \in \mathbb{R}^n} |f(x,w)|_1$, or by $\lambda^f(x,w) = |f(x,w)|_1$. This technique increases the size of the time increment. Since $\lambda^f \geq \lambda^f(w) \geq \lambda^f(x,w)$, the use of $\lambda^f(x,w)$ gives the largest time increment and may result in the fastest algorithm, see [69]. We apply this technique to both the value space and the policy space methods. The second technique can be described briefly as follows (see [17] for details). Denote $T^f(\cdot)$ as the dynamic programming operator defined on the RHS of (2.11), i.e.,

$$T^f \left( S^f_0(\cdot, \cdot) \right)(x) = \sup_{w \in W} \left\{ \sum_{z \in N_4(x)} p^f(x,z;w) S^f_0(z, \cdot) - \delta \frac{\lambda^f}{\lambda^2} \gamma^2 |w|^2 - |h(x,w)|^2 \right\}.$$  

Given $S^f_0(x, k-1)$, $x \in \mathbb{D}^f$, $k = 1, 2, \ldots$, 

- compute $S^{f,0}_0(x, k)$ using

$$S^{f,0}_0(x, k) = T^f \left( S^f_0(\cdot, k-1) \right)(x),$$  

• compute \( S_a^k(x,k) \) using

\[
S_a^k(x,k) = S_a^k(x,k-1) + \alpha^*(S_a^{k,0}(x,k) - S_a^k(x,k-1)),
\]

in which \( \alpha^* \) is defined by

\[
\alpha^* = \max\{\alpha \in \mathbb{R}^+: T^\delta\left(S_a^{k,\alpha}(\cdot,k-1)\right)(x) \geq S_a^{k,\alpha}(x,k-1), \forall x \in D^\delta\},
\]

where \( S_a^{k,\alpha}(x,k-1) = S_a^k(x,k-1) + \alpha(S_a^{k,0}(x,k) - S_a^k(x,k-1)) \).

We apply this technique to speed up the value space method. A significant acceleration will be achieved if \( \alpha^* \gg 1 \).

### 2.5.1 Example 1

In this example, we consider a one-dimensional linear system for which the \( H_\infty \) norm can be calculated explicitly ([36]). The system, given by \( f(x,w) = -0.5x + w, h(x) = x, W = \mathbb{R} \), defines a linear state space system corresponding to the transfer function \( G(s) = 1/(s + 0.5) \). The \( H_\infty \) norm of \( G(s) \) can be evaluated by any of the standard methods, giving \( \| G \|_\infty = 2 \).

For linear systems the PDI (2.5) can be solved explicitly, providing a useful check for any iterative methods. Solutions of (2.5), if they exist, are of the form

\[
S(x) = Px^2,
\]

where \( P \) satisfies the Riccati inequality

\[
-P + P^2/\gamma^2 + 1 \leq 0. \tag{2.15}
\]

If \( \gamma > 2 \), all solutions of (2.15) are real and positive. If \( \gamma < 2 \), no real solutions exist. This implies \( \| G \|_{H_\infty} = 2 \).

Figure 1 illustrates the use of the value space algorithm of §4.1. In the simulations,

- \( D = [-1, 1] \) and the condition \( \partial S/\partial v = 0 \) is imposed on the boundary \( \partial D = \{-1, 1\} \) (c.f. Remark 2.1).

- \( \delta = 0.05 \), and \( D^\delta \) consists of 41 equally spaced points in the interval \([-1, 1]\).
2.5 Examples

- $W^\delta$ consists of 201 equally spaced points in the interval $[-50, 50]$.

- $\lambda^\delta = 51$, and the time step size is $\Delta = \delta / \lambda^\delta = 0.00098$.

- The basic value space algorithm was run for 25,000 iterations.

In the figure, the vertical axis denotes the value function $S_k(x, k)$. If $\gamma \geq 2.1$, the value space algorithm converges and gives a solution of the discrete inequality (2.10), and so we conclude from Theorem 2.3 that $\| G \|_{H_\infty} \leq 2.1$. On the other hand, if $\gamma \leq 1.9$ the algorithm diverges. Thus, iterating further, one concludes $\| G \|_{H_\infty} \in (1.9, 2.1]$.

By way of comparison, Figure 2 shows simulation results obtained by numerically solving the Riccati inequality (2.15) using the scheme

$$
\begin{cases}
    P_k &= P_{k-1} + \Delta \left( -P_{k-1} + P_{k-1}^2 / \gamma^2 + 1 \right) \\
    P_0 &= 0.
\end{cases}
$$

(2.16)

Here,

- the time step size is $\Delta = 0.05$.

- The algorithm was run for 500 (300) iterations for $\gamma = 2.1$ ($\gamma = 1.9$ resp.).

If $\gamma \geq 2.1$, this algorithm converges to a solution of the Riccati inequality (2.15), and if $\gamma \leq 1.9$ the algorithm diverges. Thus $\| G \|_{H_\infty} \in (1.9, 2.1]$.

2.5.2 Example 2

In the following example, we consider a one-dimensional nonlinear system on $\mathbb{R}$ given by

$$
f(x, w) = -x \sqrt{2} x^4 + 4x^2 + 1 - \frac{x}{\sqrt{2} x^4 + 4x^2 + 1} + (1 + x^2)w
$$

The $H_{\infty}$ norm of this system is shown theoretically to be less than $\frac{1}{2} \sqrt{2} \approx 0.707$, see [97]. We employ both the value space and the policy space methods to solve the inequality (2.10). The linear equations arising in the second stage of the policy space is solved by using a Jacobi type iteration.

To do the simulations we set

- $D = [-1, 1]$ and the condition $\partial S / \partial \nu = 0$ on $\partial D = \{-1, 1\}$. 
Computational Methods for Dissipative Systems

- $\delta = 0.04$, and $D^\delta$ consists of 51 equally spaced points in the interval $[-1, 1]$.
- $W^\delta$ consists of 401 equally spaced points in the interval $[-80, 80]$.
- The basic value space method was run for 60,000 iterations and its modified versions were run for 3,000 iterations.
- The basic policy space method was run for 300 iterations.

The value space and the policy space methods always provide convergent solutions of the discrete inequality (2.10) for $\gamma \geq 0.67$ while they result in divergent solutions for $\gamma \leq 0.65$. Thus, one concludes that the $H_\infty$ norm of the system lies in $(0.65, 0.67]$. This conclusion is consistent with the prediction that $\| G \|_{H_\infty} \leq \frac{1}{3}\sqrt{2}$ in [97]. Figure 2.3 and Figure 2.4 depict the simulation results of using the basic value space and the basic policy space methods respectively, for $\gamma = 0.67$ and $\gamma = 0.65$. As shown in the figures, the policy space method converges much faster than the value space method.

The relative performances of the methods speeded up using local velocity technique and acceleration technique of [17] are depicted in Fig. 2.5. In this figure, the vertical axis denotes the value of $S_a(x, k)$ at $x = 1$.

2.5.3 Example 3

Next, we consider a nonlinear one-dimensional system $f(x, w) = -5x(1 + \sin^2 x) + w$ defined on $\mathbb{R}$. We worked on the same $D^\delta$ and $W^\delta$ spaces as those used in the previous example. The basic value space and policy space methods were run for 10,000 and 300 iterations respectively, while the modified value space methods were run for 3,000 iterations. Both methods converge to approximately the same solutions for $\gamma \geq 0.21$ and diverge for $\gamma \leq 0.19$. Thus, $\| G \|_{H_\infty}$ lies in $(1.9, 2.1]$. Figure 2.6 shows the comparison of methods for $\gamma = 0.21$. The vertical axis shown in the figure denotes the value of $S_a(x, k)$ at $x = 1$.

2.5.4 Example 4

Now we consider a two-dimensional linear system given by $\dot{x} = Ax + w, h(x) = x$, in which

$$A = \begin{bmatrix} -5.0 & -0.1 \\ -0.1 & -1.0 \end{bmatrix},$$
and $W = \mathbb{R}^2$ defines a linear state space system corresponding to the transfer function $G(s) = (s + 1)/(s^2 + 6s + 4.99)$. By employing the Bounded Real Lemma we know that the $H_\infty$ norm of this system is 0.2.

To compute the $H_\infty$ norm using our method, we use both the value space and the policy space iterations. We implement these methods on Connection Machine computers and employ the FORTRAN 90 language. The linear equations arising in the policy space is solved using a Jacobi type iteration. The following setting is used.

- $D = [-1.55, 1.55] \times [-1.55, 1.55]$ and the condition $\partial S/\partial \nu = 0$ is imposed on the boundary.
- $\delta = 0.05$, and $D^\delta$ consists of $63 \times 63$ points in $[-1.55, 1.55] \times [-1.55, 1.55]$.
- $W^\delta$ consists of 201 equally spaced points in the interval $[-50, 50]$.
- The basic value space method was run for 6,000 iterations and its modified versions were run for 500 iterations.
- The basic policy method was run for 100 iterations.

The basic value space and policy space methods provide convergent solutions of the discrete inequality (2.10) for $\gamma \geq 0.21$ and result in divergent solutions for $\gamma \leq 0.19$. Thus, one concludes that the $H_\infty$ norm of the system lies in $(0.19, 0.21)$. This conclusion is approximately the same as that obtained by employing the Bounded Real Lemma. Figure 2.7 and Figure 2.8 illustrate the use of the basic value space and the basic policy space methods respectively, for $\gamma = 0.21$ and $\gamma = 0.19$. In each of these figures, the vertical axis denotes the maximum value of $S_\alpha(x, k)$ over $x$. Figure 2.9 illustrates the relative performances of the modified methods.

### 2.5.5 Example 5

In the last example, we adopt from [102] a simple two-dimensional nonlinear system described by $\dot{z} = f(x_1, x_2, w), h(x_1, x_2) = x_1$, in which

$$f(x_1, x_2, w) = \left[ \begin{array}{c} x_2 \\ -5.5886x_1 - 7.5923x_2 - \sqrt{x_1^2 + x_2^2 \sin x_1 + w} \end{array} \right].$$
where \( W = \mathbb{R}^2 \). To compute the \( H_\infty \) norm of this system, we use the value space and the policy space methods and implement these algorithms on Connection Machine computers. The setting for carrying out simulation was the same as that in the previous example.

The value space and policy space methods provide convergent solutions for \( \gamma \geq 0.19 \) and result in divergent solutions for \( \gamma \leq 0.15 \). Thus, one concludes that the \( H_\infty \) norm of the system lies in \((0.15, 0.19]\). The basic value space requires more than 18,000 iterations to converge, while the value space modified using acceleration technique does not speed up the method significantly. On the other hand, the value space with local velocity converges in 6,000 iterations. The performance of the policy space is also improved by using the local velocity technique. Figure 2.10 and Figure 2.11 show the simulation results of using the value space and the policy space methods respectively, for \( \gamma = 0.19 \) and \( \gamma = 0.15 \), in which both methods are modified using the local velocity technique.

### 2.6 Extension to General Dissipativity Cases

The numerical method developed in this chapter can be extended to compute storage functions for more general dissipativity performance measures. Detailed treatment concerning general dissipative systems is presented in Chapter 3. In this section we write down the relevant PDI's for the purpose of computation. We consider the general supply rate \( r(z,w) \)

\[
r : \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R},
\]

which is assumed to be \( C^1(\mathbb{R}^q,\mathbb{R}^d) \), with at most quadratic growth, i.e. \(|r(z,w)| \leq \alpha (1 + |z|^2 + |w|^2)\) for some \( \alpha > 0 \), for all \( z \in \mathbb{R}^q \), \( w \in \mathbb{R}^d \). Moreover, we assume that \( r(z,0) \leq 0 \) for all \( z \in \mathbb{R}^q \), with \( r(0,0) = 0 \). We say that the system \( \Sigma \) is dissipative with respect to the supply rate \( r(z,w) \) if the following inequality holds

\[
- \int_0^T r(z(t),w(t))dt \leq \beta(x_0), \tag{2.18}
\]

for all \( x_0 \in \mathbb{R}^n \), \( w \in L_2([0,T],\mathbb{R}^d) \), for all \( T \geq 0 \), for some \( \beta \geq 0 \), with \( \beta(0) = 0 \). This inequality is a generalization of the norm inequality in (2.3) which is defined for \( x_0 = 0 \). The finite function \( \beta \) takes into account the effect of nonzero initial condition. This replaces the reachability assumption concerning the system \( \Sigma \). The supply rate may take various forms including those of finite \( L_2 \) gain and passivity performance measures (see Chapter 3).
2.6 Extension to General Dissipativity Cases

The corresponding dissipation inequality generalizing (2.5) is given by (see also [105], [44], [51])

\[
\sup_{w \in W} \{\nabla_z S(z) f(x, w) - r(h(z, w), w)\} \leq 0 \text{ in } \mathbb{R}^n.
\]  

(2.19)

To test whether a given system is dissipative with respect to a specified supply rate function \( r \) one needs to solve this PDI (c.f. Theorem 2.1). Employing the finite difference scheme developed in the previous section yields the following discrete analogs of (2.19) given by

\[
S^d(z) \geq \sup_{w \in W^d} \left\{ \sum_{z \in N_d(z)} p^d(z, z; w) S^d(z) - \frac{\delta}{\lambda^d} r(h(z, w), w) \right\}
\]

(2.20)

for \( z \in (\mathbb{R}^n)^{\delta} \). A solution to this discrete inequality can be obtained using value space and policy space iterations developed in Section 2.4.

In the value space iteration case one computes the function

\[
S_{a}^d(x, k) = \sup_{w \in W_{0, k-1}} E_x \left\{ - \sum_{l=0}^{k-1} r(h(x_l, u_l), u_l) \frac{\delta}{\lambda^d} \right\},
\]

with initial \( S_{a}^d(x, 0) = 0 \), which is a generalization of (2.12). Following the proof of Theorem 2.4, the function \( S_{a}^d(x, k) \) is monotonic nondecreasing in \( k \). Its limit, when exists, is the available storage function given by

\[
S_{a}^d(x) = \sup_{k \geq 0, w \in W_{k-1}} E_x \left\{ - \sum_{l=0}^{k-1} r(h(x_l, u_l), u_l) \frac{\delta}{\lambda^d} \right\}.
\]

This serves as a solution to the discrete dissipation inequality (2.20).

The policy space iteration is carried out by solving the following equations. For \( k = 0, 1, 2, \ldots \):

**Stage 1.** given \( w_{k}^d(x) \) and \( S_{a}^d(x; w_{k}^d(x)) \), compute the new policy \( w_{k+1}^d(x) \) given by

\[
w_{k+1}^d(x) = \arg \max_{\alpha \in W^d} \{ \sum_{z \in N^d(x)} p(x, z; \alpha) S_{a}^d(x; w_{k}^d(x)) - \frac{\delta}{\lambda^d} r(h(x, w_{k}^d(x)), w_{k}^d(x)) \},
\]

**Stage 2.** given the updated \( w_{k+1}^d(x) \), compute the new \( S_{a}^d(x, w_{k+1}^d(x)) \) by solving the linear equations

\[
S_{a}^d(x, w_{k+1}^d(x)) = \{ \sum_{z \in N^d(x)} p(x, z; w_{k+1}^d(x)) S_{a}^d(x, w_{k+1}^d(x)) - \frac{\delta}{\lambda^d} r(h(x, w_{k+1}^d(x)), w_{k+1}^d(x)) \}.
\]
2.7 Examples: general dissipativity cases

In this section we present two examples illustrating the use of our numerical method for testing the dissipativity of a given system.

2.7.1 Example 6

We consider a one-dimensional nonlinear system described by

\[ \dot{x} = -5x(1 + \sin^2 x) + w, \quad z = x + w. \]

We test the dissipativity of the system with respect to the supply rate function given by

\[ r(z, w) = \theta(w'w - \gamma^2) + (1 - \theta)(2w'z), \]

in which \( \theta \) takes value in \([0, 1]\) and \( \gamma = 1.5 \). To carry out the simulations we set

- \( D = [-1.2, 1.2] \) and the condition \( \partial S/\partial \nu = 0 \) on \( \partial D = \{-1.2, 1.2\} \).
- \( \delta = 0.04 \), and \( D^\delta \) consists of 61 equally spaced points in the interval \([-1.2, 1.2] \).
- \( W^\delta \) consists of 1201 equally spaced points in the interval \([-48, 48] \).
- The basic value space method was run for 3,000 iterations.

The value space iteration converges to a solution to the inequality (2.20) for all \( \theta \) in \([0, 1]\). We conclude that the system being tested has finite \( L_2 \) gain less than 1.5 and is passive, at least locally around \( x = 0 \). The simulation results are depicted in Figure 8.8. In the figure, we plot the values of \( S_a(x, \cdot) \) at \( x = 1 \) for \( \theta = 0.0, 0.4, 0.6, 1.0 \).

2.7.2 Example 7

Next, we consider a two-dimensional linear system given by

\[ \dot{x} = \begin{bmatrix} -5.0 & 1.0 \\ -1.0 & 0.0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \]

\[ z = [0 \ 0.5]x + 0.1w. \]

By employing the Positive Real Lemma we know that this system is positive real. The following setting was used.
2.8 Conclusions

- \( D = [-1.24, 1.24] \times [-1.24, 1.24] \) and the condition \( \partial S/\partial \nu = 0 \) is imposed on the boundary.
- \( \delta = 0.04 \), and \( D^\delta \) consists of 63 \times 63 points in \([-1.24, 1.24] \times [-1.24, 1.24] \).
- \( W^\delta \) consists of 496 equally spaced points in the interval \([-198.4, 198.4] \).
- the value space method with local velocity was run for 10,000 iterations.

The value space iteration converges to a solution of the discrete inequality (2.20). Thus, we conclude that the system being tested is positive real. The simulation result is depicted in Figure 8.9, in which the plot of \( \max_{x \in D^\delta} S_n(x, k) \) is shown.

2.8 Conclusions

In this chapter we have studied the computational problem for testing dissipativity of nonlinear systems. We firstly develop in detail a numerical method for computing the \( H_\infty \) norm for nonlinear systems and then indicate how the method extend to more general dissipativity cases. We have showed that the problem of testing dissipativity of a given system can be cast as the one of solving a first-order nonlinear partial differential inequality (PDI), which is interpreted in the viscosity sense. Solutions of the PDI are storage functions of the system being considered. The discrete time version of this PDI has been formulated based on the finite difference scheme, resulting in a partial difference inequality which converges to the corresponding continuous time PDI in the viscosity sense. We have proposed the use of value space and policy space iterations to obtain a solution of the partial difference inequality and have discussed a number of acceleration techniques.

In the examples presented, we have showed that the proposed numerical methods provide a good result for testing a number of dissipativity properties, namely, the \( H_\infty \) norm (or \( L_2 \) gain), passivity, and a mixture between finite gain and passivity. The use of the value space method requires stationary solutions to be obtained before a conclusion can be drawn. This typically requires a large number of iterations. On the other hand, the result of using the policy space method depends on the success in solving the linear algebraic equations arising in the second stage of this method. In the examples we considered, the linear equations were easily solved in a few number of Jacobi type
we considered, the linear equations were easily solved in a few number of Jacobi type iterations. For systems with higher dimension, the linear equations become even larger in size and become more difficult to solve. Furthermore, the linear equation does not enjoy contraction property, since this equation corresponds to an infinite horizon optimal control problem without discounting factor. Effort is still needed to investigate the convergence properties of the policy space method.

We have illustrated the use the local velocity technique and the acceleration technique of [17] to speed up the basic value space and policy space methods. These techniques can improve the performance of the basic methods significantly. The use of the local velocity leads to a larger time step. This may reduce the accuracy of the algorithms. In this case, trade-off between speed of convergence and accuracy may be required.

2.9 Figures

We shall now present the figures showing the simulation results for Example 1 to Example 7 described in Section 2.5 and Section 2.7.
Figure 2.1: Example 1. The value space iteration converges for \( \gamma \geq 2.1 \) and diverges for \( \gamma \leq 1.9 \). Therefore \( \| G \|_{H_\infty} \in (1.9, 2.1] \).
Figure 2.2: Example 1. Numerical solution of the Riccati equation converges for $\gamma \geq 2.1$ and diverges for $\gamma \leq 1.9$. Therefore $\|G\|_{H_{\infty}} \in (1.9, 2.1]$. 
Figure 2.3: Example 2. The value space iteration converges for $\gamma \geq 0.67$ and diverges for $\gamma \leq 0.65$. Therefore $\|G\|_{\infty} \in (0.65, 0.67]$. 
Figure 2.4: Example 2. The policy space iteration converges for $\gamma \geq 0.67$ and diverges for $\gamma \leq 0.65$. 
Figure 2.5: Example 2. Comparison of methods for $\gamma = 0.67$. 
Figure 2.6: Example 3. Comparison of methods for $\gamma = 0.21$. 

---:policy
- - - - :value + local velocity
- - - - - - - - :value + acceleration
- - - - - - - - - - - - - - - - - :value
Figure 2.7: Example 4. The value space iteration converges for $\gamma \geq 0.21$ and diverges for $\gamma \leq 0.19$. Therefore $\|G\|_{H_\infty} \in (0.19, 0.21]$. 
Figure 2.8: Example 4. The policy space iteration converges for $\gamma \geq 0.21$ and diverges for $\gamma \leq 0.19$. 
Figure 2.9: Example 4. Comparison of methods for $\gamma = 0.21$. 
Figure 2.10: Example 5. The value space iteration with local velocity converges for $\gamma \geq 0.19$ and diverges for $\gamma \leq 0.15$. 
Figure 2.11: Example 5. The policy space iteration with local velocity converges for $\gamma \geq 0.19$ and diverges for $\gamma \leq 0.15$. 
Figure 2.12: Example 6. The value space iteration converges for all \( \theta \in [0,1] \). Thus the system is both finite gain and passive.
Figure 2.13: Example 7. The value space iteration converges. Therefore, the system is passive (or positive real).
Chapter 3

Strictly Dissipative Systems

3.1 Introduction

Characterization of general (non strict) dissipative systems are studied in [105], [106], [107], [44], [45], [46], [99], [51] and [8] for linear and nonlinear systems. In particular, it is shown in this literature that a dissipative system possesses a storage function that satisfies a dissipation inequality. This result is obtained without any a priori assumption regarding the stability of the systems. When the systems are zero state detectable, then dissipativity implies stability [45], [46], [99], [51]. The dissipation inequality does not uniquely determined a storage function. The existence of any solution of this inequality implies that the system is dissipative.

In [1], [82], [36], [106], [107] the authors present characterization of asymptotically stable linear systems which have strictly bounded $L_2$ gain in terms of a particular solution of an algebraic Riccati equation termed the stabilizing solution. While there are many solutions to the ARE, the stabilizing one is unique [13] and it corresponds to a strict finite gain property [82], [36].

In this chapter, we study asymptotically stable nonlinear and linear systems which are strictly dissipative with respect to a general quadratic supply rate function. In the nonlinear systems case we shall show that strict dissipativity is characterized by the existence of: (i) a solution to a PDI and (ii) a solution to a PDE, in the viscosity sense. Moreover, if the solution to the PDE is smooth, it is in fact stabilizing (in some sense). These results generalize those in [82] to the nonlinear systems case with a general
quadratic dissipativity. In the linear systems case we show that strict dissipativity is characterized by the existence of (i) a solution to an ARI in the strict sense and (ii) a stabilizing solution to an ARE. This is an extension of the results in [82], [36] to a general quadratic dissipativity. In [97], [98], [99] characterization of locally strict finite gain nonlinear systems are given in terms of a smooth PDE. The result is obtained by using geometric methods.

3.2 General Dissipative Systems

We consider the class of nonlinear systems described by

\[
\Sigma : \begin{cases} 
\dot{x}(t) = A(x(t)) + B_1(x(t))w(t), \ x(0) = x_0, \\
z(t) = C_1(x(t)) + D_{11}(x(t))w(t).
\end{cases}
\]  

(3.1)

In this description, \(x \in \mathbb{R}^n\) denotes the state vector, \(z \in \mathbb{R}^q\) the output vector and \(w \in \mathbb{R}^d\) the input vector. In the control synthesis case, usually one regards \(z\) as the output to be controlled and \(w\) as the disturbance. We assume that \(A(\cdot), B_1(\cdot), C_1(\cdot), D_{11}(\cdot)\) are smooth functions, \(A(\cdot)\) and \(C_1(\cdot)\) are globally Lipschitz continuous and \(B_1(\cdot), D_{11}(\cdot)\) are bounded, and that \(A(0) = 0, C_1(0) = 0\). We consider the general supply rate \(r(z, w)\)

\[
r : \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R},
\]  

(3.2)

which is assumed to be \(C^1(\mathbb{R}^q, \mathbb{R}^d)\), with at most quadratic growth, i.e. \(|r(z, w)| \leq \alpha (1 + |z|^2 + |w|^2)\) for some \(\alpha > 0\), for all \(z \in \mathbb{R}^q, w \in \mathbb{R}^d\). Throughout the chapter, we consider supply rate functions that satisfy

\[
r(z, 0) \leq 0,
\]  

(3.3)

for all \(z \in \mathbb{R}^q\), with \(r(0, 0) = 0\).

**Definition 3.1** We say that the system \(\Sigma\) is dissipative with respect to the supply rate \(r(z, w)\) (3.2) if for each initial condition \(x_0 \in \mathbb{R}^n\) the map \(\Sigma_{x_0}\) is dissipative with respect to the supply rate \(r(z, w)\) (3.2), which means

\[
-\int_0^T r(z(t), w(t))dt \leq \beta(x_0),
\]  

(3.4)

for all \(w \in L_2([0, T], \mathbb{R}^d)\), for all \(T \geq 0\), for some \(\beta \geq 0\), with \(\beta(0) = 0\). We say that \(\Sigma\) is strictly dissipative with respect to the supply rate \(r(z, w)\) (3.2) if the inequality in (3.4)
is replaced by

\[-\int_0^T r(z(t), w(t)) dt \leq -\frac{1}{2} \epsilon \int_0^T w'(t) w(t) dt + \beta(x_0),\]  

(3.5)

for some \( \epsilon > 0 \).

It is known that dissipative systems can be characterized in terms of a set of energy-like functions called storage functions, which satisfy dissipation type inequalities. For nonlinear systems with general (non quadratic) supply rate these inequalities take the form of the following partial differential inequality (PDI) (see [105], [44], [45], [98], [51])

\[\sup_{w \in \mathbb{R}^d} \{\nabla_x V(x) (A(x) + B_1(x) w) - r(h(x, w), w)\} \leq 0,\]  

(3.6)

where \( h(z, w) = C_1(z) + D_{11}(z) w \), and for linear systems with quadratic supply rate, they reduce to algebraic Riccati inequalities (ARI's) (see [106], [107] and (3.8) below). Furthermore, under appropriate detectability assumptions, dissipative systems are asymptotically stable.

**Definition 3.2** We say that \( \Sigma \) is (zero state) detectable if \( w = 0 \) and \( \lim_{t \to \infty} z(t) = 0 \) implies \( \lim_{t \to \infty} x(t) = 0 \).

The above-mentioned known results are summarized in the next theorem and the remarks following it.

**Theorem 3.1** (See [105], [44], [99], [51], [10].) Assume that \( \Sigma \) is dissipative with respect to the supply rate \( r(z, w) \) in (3.2). Then there exists a non-negative solution \( V \), with \( V(0) = 0 \) to the PDI (3.6) in the viscosity sense (see the Appendix C for a definition). Conversely, if there exists a non-negative function \( V \), with \( V(0) = 0 \) that satisfies the PDI (3.6) in the viscosity sense, then \( \Sigma \) is dissipative with respect to the supply rate \( r(z, w) \) in (3.2), and the function \( V_* \) defined by

\[ V_*(x) = \liminf_{y \to x} V(y),\]  

is a lower semicontinuous storage function.

**Remark 3.1** Suppose that the supply rate \( r(z, w) \) in (3.2) satisfies a stronger condition

\[ r(z, 0) \leq -c_1 |z|^2, \]  

(3.7)
for all \( z \), for some constant \( c_1 > 0 \). Assume that \( \Sigma \) is dissipative with respect to \( r(z,w) \).

Then, by using inequality (3.4) with \( w = 0 \), and using (3.7) we get the inequality

\[
0 \leq c_1 \int_0^T |h(x(t),0)|^2 dt \leq - \int_0^T r(h(x(t),0),0) dt \leq \beta(x_0) < +\infty,
\]

for all \( T \geq 0 \), for all initial conditions \( x_0 \) in \( \mathbb{R}^n \). This implies that \( \lim_{t \to \infty} |h(x(t),0)|^2 = 0 \) and, therefore, \( h(x(t),0) \to 0 \) as \( t \to \infty \). If the system \( \Sigma \) is detectable then \( x(t) \to 0 \) as \( t \to \infty \). Thus, the equilibrium \( x = 0 \) is asymptotically stable. This provides a way of showing the asymptotic stability of dissipative systems under (zero-state) detectability assumption. Related results are presented in [45] (based on a Lyapunov function argument), [99] and in [56].

**Remark 3.2** Suppose that \( \Sigma \) is linear, i.e., \( A(x) = Ax, \; B_1(x) = B_1, \; C_1(x) = C_1x, \) and \( D_{11}(x) = D_{11} \), where \( A, B_1, C_1, D_{11} \) are constant matrices with appropriate sizes, and the supply rate has a quadratic form, i.e. \( r(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz) \), for some constant matrices \( Q, S, R \), with \( Q = Q' \) and \( R = R' \leq 0 \), such that \( Q = Q + S D_{11} + D_{11}'S' + D_{11}'RD_{11} > 0 \). Then the function \( V \) in Theorem 3.1 is quadratic given by \( V(x) = \frac{1}{2}x'Xx \), where \( X \) is a non negative matrix satisfying the following ARI (see a related result in Theorem 3 of [106])

\[
A'X + XA + [XB_1 - C_1'S']\bar{Q}^{-1}[B_1'X - \bar{S}C_1] - C_1'RC_1 \leq 0,
\]

(3.8)

where \( \bar{S} = S + D_{11}R \). Arguing as in the previous remark, if \( R < 0 \) (which implies the condition (3.7) holds) and the matrices pair \((A, C_1)\) is completely observable, then \( A \) is an asymptotically stable matrix, i.e., each eigenvalue of \( A \) has a negative real part.

**Remark 3.3** In this section and the rest of this chapter, we use the notation \( V \) the denote storage functions, instead of \( S \) used in Chapter 2. This is done to maintain the consistency with the notation in Chapter 4, in the case of control synthesis.

### 3.3 Strictly Dissipative Nonlinear Systems

In the paper [82], it is shown that if a linear system has a strict finite gain property and is asymptotically stable, then it possesses a storage function with a stabilizing property. This result, which is expressed in terms of an algebraic Riccati equation, is obtained
without assuming controllability or observability of the system. We shall now present an extension of these results to a more general type of dissipativity for nonlinear systems.

We consider the nonlinear systems described in (3.1) and a general quadratic supply rate of the form

\[ r_q(z, w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz), \]

where \( Q, R \) are symmetric matrices. We write \( \bar{Q}(x) = Q + SD_{11}(x) + D_{11}'(x)S' + D_{11}'(x)RD_{11}(x) \) and \( \bar{S}(x) = S + D_{11}(x)'R \). We assume that

\[ R \leq 0, \quad \text{and} \quad \inf_{z \in \mathbb{R}^n} \{\sigma_{\min}(\bar{Q}(x))\} = k > 0. \]

This assumption implies \( \bar{Q}(x) > 0 \) for all \( x \in \mathbb{R}^n \). This quadratic supply rate includes the following performance measures as a few special cases (see [44], [45]):

1. \( H_{\infty} \) (or finite gain) performance; when \( Q = \gamma^2 I, S = 0, R = -I \),
2. positive real (or passivity) performance; when \( Q = R = 0, S = I \),
3. mixed performance; when \( Q = \theta \gamma^2 I, R = -\theta I, S = (1 - \theta)I, \theta \in [0, 1] \),
4. sector bounded performance; when \( Q = -\frac{1}{2}(K_1'K_2 + K_2'K_1), S = \frac{1}{2}(K_1 + K_2)', \quad R = -I, \) for some constant matrices \( K_1, K_2 \).

Remark 3.4 The multiplication with the factor \( \frac{1}{2} \) in the expression (3.9) for a quadratic supply rate function will result in simpler formulas. We will follow this convention for the rest of this thesis whenever we deal with quadratic supply rate functions.

Before presenting our results, we shall make the following definitions.

Definition 3.3 A vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) is called strongly stable if there exists \( c > 0 \) such that

\[ (f(x) - f(y))(x - y) \leq -c|x - y|^2, \]

for all \( x, y \) in \( \mathbb{R}^n \). If \( f \) is globally asymptotically stable, it is called weakly stable.

Condition (3.11) implies global exponential stability. The following theorem provides characterization of a stable, strictly (quadratic) dissipative system, generalizing the strict bounded real lemma (SBRL) of [82].
Theorem 3.2  Consider the open loop system $\Sigma$ (3.1) and the supply rate $r_q(x,w)$ in (3.9) such that the assumption (3.10) holds. Suppose that the vector field $A(x)$ is strongly stable and $\Sigma$ is strictly dissipative with respect to the supply rate $r_q(x,w)$. Then:

i. there exists a finite function $\bar{V} > 0$, with $\bar{V}(0) = 0$, satisfying the PDI

$$\sup_{w \in \mathbb{R}^d} \{ \nabla_x \bar{V}(x)(A(x) + B_1(x)w) - r(h(x,w),w) + \frac{\delta}{2} x'x \} \leq 0, \quad (3.12)$$

for some $\delta > 0$, where $h(x,w) = C_1(x) + D_{11}(x)w$, in the viscosity sense, and

ii. there exists a finite function $V \geq 0$ with $V(0) = 0$, satisfying the PDE

$$\sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x)(A(x) + B_1(x)w) - r(h(x,w),w) \} = 0, \quad (3.13)$$

in the viscosity sense. Moreover, if $V$ is smooth, then the vector field

$$A^*(x) = A(x) + B_1(x)\bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)]$$

is weakly stable.

Proof. For any $w \in L_2([0,T],\mathbb{R}^d)$ and any $T \geq 0$, the strict dissipativity of $\Sigma$ implies

$$-\int_0^T r_q(z(t),w(t))dt \leq -\frac{1}{2} \int_0^T w'(t)w(t)dt + \beta(x_0), \text{ for some } \epsilon > 0,$$

and the strong stability of $A$ implies (see [33])

$$\frac{1}{2} \int_0^T \dot{z}'(t)z(t)dt \leq \frac{1}{2} \tilde{\gamma} \int_0^T w'(t)w(t)dt + \tilde{\beta}(x_0), \text{ for some } \tilde{\gamma} > 0,$$

for some $\tilde{\beta} \geq 0$ with $\tilde{\beta}(0) = 0$, where $\beta$, $\tilde{\beta}$, $\epsilon$, $\tilde{\gamma}$ are independent of $T,w$. Multiplying both sides of the last inequality by $\epsilon/\tilde{\gamma}$, and adding the result to the first one yields, for any $w \in L_2([0,T],\mathbb{R}^d)$, $T \geq 0$,

$$-\int_0^T (r_q(z(t),w(t)) - \frac{1}{2} \delta \dot{z}'(t)z(t))dt \leq \beta(x_0) + \delta \tilde{\beta}(x_0) = \tilde{\beta}(x_0),$$

where $\delta = \epsilon/\tilde{\gamma} > 0$. This implies that the system $\hat{\Sigma}$ defined by

$$\hat{\Sigma} : \left\{ \begin{array}{l}
\dot{x}(t) = A(x(t)) + B_1(x(t))w(t), \quad x(0) = x_0, \\
\dot{z}(t) = \begin{bmatrix} C_1(x(t)) \\ \delta \frac{1}{2} x(t) \end{bmatrix} + \begin{bmatrix} D_{11}(x(t)) \\ 0 \end{bmatrix} w(t),
\end{array} \right.$$
3.3 Strictly Dissipative Nonlinear Systems

is dissipative with respect to the supply rate

\[ \dot{r}_q(\dot{z},w) = \frac{1}{2}(w'Qw + 2w'\dot{S}\dot{z} + \dot{z}'\dot{R}\dot{z}), \]

where \( \dot{S} = [S\ 0] \), \( \dot{R} = \text{diag}(R,-I) \). Write \( \dot{h}(x,w) = \left[ C_1(x(t)) \right] + \left[ D_{11}(x(t)) \right] w(t). \)

Since

\[ \dot{r}_q(\dot{h}(x,w),w) = \frac{1}{2}(w'Qw + 2w'Sh(x,w) + h(x,w)'Rh(x,w) - \delta x'x), \]

Theorem 3.1 of [51] implies there exists a finite function \( \hat{V} \geq 0 \), with \( \hat{V}(0) = 0 \), solving the PDI

\[ \sup_{w \in \mathbb{R}^d} \{ \nabla_x \hat{V}(x)'(A(x)+B_1(x)w) - \frac{1}{2}(w'Qw + 2w'Sh(x,w) + h(x,w)'Rh(x,w) - \delta x'x) \}\]

\[ = \sup_{w \in \mathbb{R}^d} \{ \nabla_x \hat{V}(x)'(A(x)+B_1(x)w) - \dot{r}_q(\dot{h}(x,w),w) \}\]

\[ \leq 0, \]

in the viscosity sense. This proves the existence of the PDI (3.12). The function \( \hat{V} \) has the integral representation

\[ \hat{V}(x) \geq \sup_{T \geq 0, w \in L^2([0,T],\mathbb{R}^d)} \{ \hat{V}(x(T)) - \frac{1}{2} \int_0^T (w(t)'Qw(t) + 2w(t)'Sh(x(t),w(t))) dt : x(0) = x \}. \]

Suppose \( \hat{V}(x) = 0 \), for some \( x \). Then, (3.15) implies, with \( T \geq 0 \), \( w = 0 \)

\[ 0 = \hat{V}(x) \geq \delta \frac{1}{2} \int_0^T x'(t)x(t)dt \geq 0. \]

Thus \( x(t) = 0 \) for all \( t \in [0, T] \). Uniqueness of solutions to ODE then implies that \( x = x(0) = 0 \). This proves assertion (i).

Next, define the function \( V \) by

\[ V(x) = \sup_{T \geq 0, w \in L^2([0,T],\mathbb{R}^d)} \{ -\frac{1}{2} \int_0^T (w(t)'Qw(t) + 2w(t)'Sh(x(t),w(t))) dt : x(0) = x \}. \]

Clearly we have \( 0 \leq V \leq \hat{V} \). We claim that \( V \) solves the PDE (3.13) in the viscosity sense. The proof, which adopts the techniques employed in [90], is given in Appendix C.

Next, suppose \( V \) is smooth, and set

\[ w = w^*(x) = \tilde{Q}^{-1}(x)[B'_1(x)\nabla_x V(x)' - \tilde{S}(x)C_1(x)] \]
Strictly Dissipative Systems

(which achieves the maximum in the PDE (3.13)). Then we get

\[ V(x(0)) - V(x(T)) = -\frac{1}{2} \int_0^T (w^*(t)'Qw^*(t) + 2w^*(t)'Sh(x(t), w^*(t))) + h(x(t), w^*(t))'Rh(x(t), w^*(t)))dt. \]

(3.16)

The representation (3.15) implies that for \( w = w^* \)

\[ \dot{V}(x(0)) - \dot{V}(x(T)) \geq -\frac{1}{2} \int_0^T (w^*(t)'Qw^*(t) + 2w^*(t)'Sh(x(t), w^*(t))) + h(x(t), w^*(t))'Rh(x(t), w^*(t)) - \delta x(t)'x(t))dt. \]

(3.17)

Subtracting (3.16) from (3.17) we get

\[ S(x(0)) - S(x(T)) \geq \frac{1}{2} \delta \int_0^T x(t)'x(t)dt, \]

where \( S = \dot{V} - V \geq 0, \) with \( S(0) = 0. \) Since the last inequality implies

\[ S(x(0)) \geq \frac{1}{2} \delta \int_0^T x(t)'x(t)dt, \]

for all \( T \geq 0 \) we conclude that any initial \( x(0) \) produces a \( L_2([0, \infty), \mathbb{R}^n) \) trajectory \( x(\cdot); \) therefore \( x(t) \to 0. \) This shows the weak stability of the vector field \( A^*. \)

We have the following converse results.

**Theorem 3.3** Consider the open loop system \( \Sigma (3.1) \) and the supply rate \( r_q(z, w) \)

in (3.9) such that the assumption (3.10) holds. Suppose there exists a smooth solution \( V \geq 0, \) with \( V(0) = 0, \) to the PDE (3.13), in the classical sense such that the vector field \( A^* \) is strongly stable. Then the vector field \( A \) is weakly stable and \( \Sigma \) is dissipative with respect to the supply rate \( r_q(z, w). \) Moreover, if the vector field \( w^*(x) = \tilde{Q}^{-1}(x)[B_1(x)'\nabla_x V(x) - \tilde{S}(x)C_1(x)] \) satisfies \( |w^*(x)| \leq \alpha|x|, \) for all \( x, \) for some \( \alpha > 0, \) then the dissipativity is strict.

**Proof.** We first write

\[ \dot{z} = A(z), \ z(0) = x_0, \]

\[ \dot{\xi} = A^*(\xi), \ \xi(0) = x_0. \]

Under the hypothesis, the PDE (3.13) can be rewritten as (after evaluating the max operation)

\[ \nabla_x V(x)A(x) + \frac{1}{2}[\nabla_x V(x)B_1(x) - C_1(x)'\tilde{S}(x)']\tilde{Q}^{-1}(x) \]

\[ \times [B_1(x)'\nabla_x V(x)' - \tilde{S}(x)C_1(x)] - C_1(x)'RC_1(x) = 0, \]
3.3 Strictly Dissipative Nonlinear Systems

which implies (since \(-R \geq 0\))

\[
\nabla_x V(x)A(x) + \frac{1}{2}[\nabla_x V(x)B_1(x) - C_1(x)'\bar{S}(x)']\bar{Q}^{-1}(x)\bar{Q}(x) \\
\times \bar{Q}^{-1}[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C_1(x)] \leq 0.
\]

Therefore, we have

\[
V(x(0)) \geq \frac{1}{2}k \int_0^T \left([\nabla_x V(x(t))B_1(x(t)) - C_1(x(t))'\bar{S}(x(t)')]\bar{Q}^{-1}(x(t))
\right)(\bar{Q}^{-1}(x(t))[B_1(x(t))'\nabla_x V(x(t))' - \bar{S}(x(t))C_1(x(t))])dt,
\]

with \(k = \inf_{x \in \mathbb{R}^n} \{\sigma_{\min}(\bar{Q}(x))\}\), which shows that the trajectory of \(w^*(t) = w^*(x(t)) = \bar{Q}^{-1}(x(t))[B_1(x(t))'\nabla_x V(x(t))' - \bar{S}(x(t))C_1(x(t))]\) is in \(L_2([0, \infty), \mathbb{R}^d)\). Now, define \(e = \xi - x\). Then, by the hypothesis that \(A^*(x)\) is strongly stable, and using a technique similar to that in [33], one can obtain the following estimate for \(e(t)\)

\[
\frac{1}{2}e'(t)e(t) \leq \frac{1}{2} \exp(-\bar{c}t)e'(0)e(0) + \frac{1}{2}\bar{k} \int_0^t \exp(-\bar{c}(t-s))
\]

\[
\times|\bar{Q}^{-1}(x(s))[B_1(x(s))'\nabla_x V(x(s))' - \bar{S}(x(s))C_1(x(s))]|^2 ds,
\]

for some constants \(\bar{c} > 0, \bar{k} > 0\). Therefore \(e(\cdot)\) is in \(L_2([0, \infty), \mathbb{R}^n)\). In particular, \(e(t) \to 0\). Since \(\xi(t) \to 0\), we conclude that \(z(\cdot)\) is asymptotically stable, and thus \(A\) is weakly stable. Next, by direct calculation we get

\[
-\int_0^T r_q(z(s), w(s))ds
\]

\[
= -\frac{1}{2} \int_0^T ([w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)] + 2dV/ds)ds
\]

\[
= -\frac{1}{2} \int_0^T [w(s) - w^*(s)]'\bar{Q}[w(s) - w^*(s)]ds + V(x_0) - V(x(T)),
\]

\[
\leq -\frac{1}{2}k \int_0^T [w(s) - w^*(s)]'[w(s) - w^*(s)]ds + V(x_0),
\]

since \(V \geq 0\), where

\[
w^*(s) = w^*(x(s)) = \bar{Q}^{-1}(x(s))[B_1(x(s))'\nabla_x V(x(s))' - \bar{S}(x(s))C_1(x(s))].
\]

The last inequality shows that \(\Sigma\) is dissipative with respect to the supply rate \(r_q(z, w)\) (3.9), with \(\beta = V\). To show the strict dissipativity, consider the following system

\[
\Sigma^{-1} : \begin{cases} 
\dot{x}(t) = A^*(x(t)) + B_1(x(t))(w(t) - w^*(t)), \quad x(0) = x_0, \\
w(t) = w^*(t) + (w(t) - w^*(t)).
\end{cases}
\]
Since $A^*$ is strongly stable, we have (see [33])

$$\frac{1}{2} \int_0^T x'(t)x(t)dt \leq \frac{1}{2} \gamma^* \int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt + \beta^*(x_0),$$

(3.20)

for all $T \geq 0$, for all $w - w^* \in L_2([0,T],\mathbb{R}^d)$, for some $\beta^* \geq 0$ with $\beta^*(0) = 0$, for some $\gamma^* > 0$. Furthermore, by using $w = w^* + (w - w^*)$, the inequality (3.20) and the hypothesis $|w^*(x)| \leq \alpha|x|$, for all $x$, we have

$$\frac{1}{2} \int_0^T w'(t)w(t)dt \leq \frac{1}{2} \int_0^T w^*(t)'w^*(t)dt + \frac{1}{2} \int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt$$

$$\leq \frac{1}{2}k_1 \int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt + \beta_1(x_0),$$

(3.21)

where $1 = \alpha \gamma^* + 1$, $\beta_1 = \alpha \beta^*$. By substituting (3.21) in (3.18) we finally get

$$- \int_0^T r_q(z(s),w(s))ds$$

$$\leq -\frac{1}{2}(k/k_1) \int_0^T w(s)'w(s)ds + V(x_0) + \beta_1(x_0)/k_1,$$

for all $w \in L_2([0,T],\mathbb{R}^d)$, for all $T \geq 0$, which shows the required strict dissipativity with $\epsilon = k/k_1 > 0$, and $\beta(x) = V(x) + \beta_1(x)/k_1$. This completes the proof.

\[\square\]

**Remark 3.5** Under the hypothesis of Theorem 3.3, the function $V$ stated in the hypothesis serves as the available storage function. To see this, we integrate the PDE (3.13) for fixed $T \geq 0, w \in L_2([0,T],\mathbb{R}^d)$ yielding

$$V(x) \geq -\frac{1}{2} \int_0^T r_q(z(s),w(s))ds + V(x(T)).$$

Since $V \geq 0$, we have

$$V(x) \geq -\frac{1}{2} \int_0^T r_q(z(s),w(s))ds.$$

Since this inequality holds for all $T \geq 0, w \in L_2([0,T],\mathbb{R}^d)$, we get

$$V(x) \geq \sup_{T \geq 0, w \in L_2([0,T],\mathbb{R}^d)} \left\{ -\frac{1}{2} \int_0^T r_q(z(s),w(s))ds \right\}$$

$$= V_a(x),$$

(3.22)

in which $V_a$ is the available storage function. It remains to show that $V \leq V_a$. We write the integral expression for $V$, with $w = w^*$, $w^*$ is defined in (3.19), as

$$V(x) = -\frac{1}{2} \int_0^T r_q(z(s),w^*(s))ds + V(x(T)),$$

(3.23)
for all $T \geq 0$, in which $\xi(\cdot)$ is the trajectory produced by

$$\dot{\xi} = A^*(\xi), \; \xi(0) = z.$$ 

Since $A^*$ is strongly stable, by the hypothesis, we have $\lim_{T\to\infty} \xi(T) = 0$. Moreover, the continuity of $V(\cdot)$ implies $\lim_{T\to\infty} V(\xi(T)) = V(\lim_{T\to\infty} \xi(T)) = V(0) = 0$. Therefore, we have the expression

$$V(x) = \lim_{T\to\infty} \{ -\frac{1}{2} \int_0^T r_q(z(s), w^*(s))ds + V(\xi(T)) \}$$

$$= \lim_{T\to\infty} \{ -\frac{1}{2} \int_0^T r_q(z(s), w^*(s))ds \} + \lim_{T\to\infty} V(\xi(T))$$

$$= \lim_{T\to\infty} \{ -\frac{1}{2} \int_0^T r_q(z(s), w^*(s))ds \}$$

Next, fixed $T \geq 0$ and let $w = w^*$. Clearly we have

$$V_a(x) \geq -\frac{1}{2} \int_0^T r_q(z(s), w^*(s))ds.$$ 

Sending $T \to \infty$ yields

$$V_a(x) \geq V(x). \quad (3.24)$$

Combining (3.22) and (3.24) shows that $V(x) = V_a(x)$. 

The following corollary shows the connection between the dissipativity of the system $\Sigma$ (3.1) and the finite gain property of a related system $\tilde{\Sigma}$ given by

$$\tilde{\Sigma}: \begin{cases} \dot{z}(t) = \tilde{A}(x(t)) + \tilde{B}_1(x(t))w(t), \; z(0) = z_0, \\ z(t) = \tilde{C}_1(x(t)), \end{cases} \quad (3.25)$$

where $\tilde{A}(x) = A(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)C_1(x)$, $\tilde{B}_1(x) = B_1(x)\bar{Q}^{-\frac{1}{2}}(x)$, and $\tilde{C}_1(x) = (\tilde{S}(x)\bar{Q}^{-1}(x)\tilde{S}(x) - R)\frac{1}{2}C_1(x)$.

**Corollary 3.1** Consider the system $\Sigma$ and the quadratic supply rate $r_q(z, w)$ in (3.9). Assume that the vector field $A(x)$ is strongly stable and the system $\Sigma$ is strictly dissipative with respect to the supply rate $r_q(z, w)$ in (3.9). Then, there exists a solution $V \geq 0$, with $V(0) = 0$, to the PDI

$$\nabla_x V(x)\tilde{A}(x) + \frac{1}{2} \nabla_x V(x)\tilde{B}_1(x)\tilde{B}_1(x)'\nabla_x V(x)' + \frac{1}{2} \tilde{C}_1(x)\tilde{C}_1(x)' \leq 0 \quad (3.26)$$

in the viscosity sense. In particular, the system $\tilde{\Sigma}$ has finite gain less than 1. If, moreover $V$ is smooth, then the vectorfield $\tilde{A}(x) + \tilde{B}_1(x)\tilde{B}_1(x)\nabla_x V(x)'$ is weakly stable.
Conversely, assume that the vector field \( \tilde{A}(x) = A(x) - B_1(x) \bar{Q}^{-1}(x) \tilde{S}(x) C_1(x) \) is strongly stable and the system \( \tilde{\Sigma} \) has finite gain strictly less than 1. Then there exists a solution \( V \geq 0 \), with \( V(0) = 0 \), to the PDI (3.13) in the viscosity sense. In particular, the system \( \Sigma \) is dissipative with respect to the supply rate \( \tau_q(z, w) \) in (3.9). If \( V \) is smooth, then \( A^*(x) = A(x) + B_1(x) \bar{Q}^{-1}(x) [B_1(x)' \nabla_z V(x)' - \tilde{S}(x) C_1(x)] \) is weakly stable.

\[ \text{Proof.} \] Under the hypothesis, Theorem 3.2 implies there exists a solution \( V \geq 0 \) to the PDI (3.13) in the viscosity sense. This PDI can be rewritten as in (3.26). Therefore, it admits the following integral representation (see Theorem 3.1 of [51])

\[
V(x) \geq \sup_{T \geq 0, w \in L^2([0,T], \mathbb{R}^d)} \{ V(x(T)) - \frac{1}{2} \int_0^T (w(t)'w(t) + C_1(x(t))C_1(x(t)))dt : x(0) = x \},
\]

where \( x(\cdot) \) is the trajectory generated by the system \( \tilde{\Sigma} \) in (3.25). This implies that \( \tilde{\Sigma} \) has finite gain less than 1. If \( V \) is smooth, then results in part (ii) of Theorem 3.2 implies that the vector field

\[
\tilde{A}(x) + B_1(x)B_1(x)'V_xV(x)' = A(x) + B_1(x) \bar{Q}^{-1}(x) [B_1(x)' \nabla_z V(x)' - \tilde{S}(x) C_1(x)]
\]

is weakly stable.

Conversely, if the system \( \tilde{\Sigma} \) has finite gain strictly less than 1 with \( \tilde{A} \) strongly stable, then Theorem 3.2 implies there exists a solution to the PDI (3.26) in the viscosity sense. Rewriting this PDI as in (3.13) and applying Theorem 3.1 of [51] results in the representation

\[
V(x) \geq \sup_{T \geq 0, w \in L^2([0,T], \mathbb{R}^d)} \{ V(x(T)) - \frac{1}{2} \int_0^T (w(t)'Qw(t) + 2w(t)'Sh(x(t), w(t)) + h(x(t), w(t))'Rh(x(t), w(t)))dt : x(0) = x \},
\]

where \( x(\cdot) \) is the trajectory produced by \( \Sigma \) in (3.1). Thus the system \( \Sigma \) is dissipative with respect to the supply rate \( \tau_q(z, w) \) (3.9). Moreover, if \( V \) is smooth, then by Theorem 3.2 the vector field

\[
A^*(x) = A(x) + B_1(x) \bar{Q}^{-1}(x) [B_1(x)' \nabla_z V(x)' - \bar{S}(x) C_1(x)]
\]

is weakly stable. This completes the proof. \( \Box \)
3.4 Strictly Dissipative Linear Systems

In this section we consider linear systems described by

\[ \dot{z}(t) = Ax(t) + B_1w(t), \quad z(0) = z_0, \]
\[ z(t) = C_1z(t) + D_{11}w(t), \quad (3.27) \]

where \( A, B_1, C_1, D_{11} \) are appropriately sized constant matrices. The supply rate we consider has a quadratic form

\[ r_q(z, w) = \frac{1}{2}(w'^TQw + 2w'Sz + z'Rz), \quad (3.28) \]

where \( Q, R \) are symmetric matrices. We assume

\[ R \preceq 0 \text{ and } \bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0, \quad (3.29) \]

and write \( \bar{S} = S + D_{11}'R \). The strong and weak stability concepts can both be replaced by the Hurwitz stability criterion. We shall now present the analog of Theorem 3.2 for linear systems (see also the bounded real lemmas in [82], [36] and a more general dissipativity case in [106]).

**Theorem 3.4**  Consider the open loop linear system \( \Sigma_1 \) (3.27) and the supply rate \( r_q(z, w) \) in (3.28) such that the assumption (3.29) holds and assume that the matrix \( A \) is asymptotically stable. Suppose that the system \( \Sigma_1 \) is strictly dissipative with respect to the supply rate \( r_q(z, w) \). Then:

i. there exists a solution \( X \succeq 0 \) to the following ARE

\[ A'X + AX + [XB_1 - C_1'S'][\bar{Q}^{-1}[B_1'X - \bar{S}C_1] - C_1'R\bar{C}_1 = 0, \quad (3.30) \]

such that the matrix \( A^* = A + B\bar{Q}^{-1}[B_1'X - \bar{S}C_1] \) is asymptotically stable, and

ii. there exists a solution \( \bar{X} \succ 0 \) to the following ARI

\[ A'\bar{X} + \bar{X}A + [\bar{X}B_1 - C_1'S'][\bar{Q}^{-1}[B_1'\bar{X} - \bar{S}C_1] - C_1'R\bar{C}_1 < 0. \quad (3.31) \]

**Proof.** We follow closely the technique used in the proof of Theorem 3.7.1 in [36] (see also [73], [1]).
First note that the strict dissipativity implies that with \( x(0) = 0 \) we have for any \( T \geq 0 \)

\[
- \int_0^T r_q(z(t), w(t)) dt \leq - \frac{1}{2} \epsilon \int_0^T w'(t)w(t) dt,
\]

(3.32)
for all \( w \in L_2([0, T], \mathbb{R}^d) \). We shall show that this implies that for any \( T \geq 0 \), the following Riccati differential equation (RDE)

\[
- \dot{X} = A'\dot{X}(t) + \ddot{X}(t)A + [\ddot{X}(t)B_1 - C_1'\bar{S}]\bar{Q}^{-1}[B_1'\dot{X}(t) - \bar{S}C_1] - C_1'R C_1,
\]

(3.33)
with \( \dot{X}(T, T) = 0 \), has no finite escape time in \([0, T] \), and moreover, \( \dot{X}(t, T) \geq 0 \). Having done this, we shall show that the limit \( 0 \leq X = \lim_{T \rightarrow \infty} \ddot{X}(t, T) \) is the required stabilizing solution to the ARE (3.30).

Consider the following two point boundary value problem (TPBVP)

\[
\begin{bmatrix}
  \dot{x}(t) \\
  \dot{p}(t)
\end{bmatrix} =
\begin{bmatrix}
  A - B_1\bar{Q}^{-1}\bar{S}C_1 & B_1\bar{Q}^{-1}B_1' \\
  -C_1'(\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1 & -(A - B_1\bar{Q}^{-1}\bar{S})'
\end{bmatrix}
\begin{bmatrix}
  x(t) \\
  p(t)
\end{bmatrix},
\]

(3.34)

where \( 0 \leq t_0 \leq T \). We shall show that this TPBVP has no nontrivial solution on \([t_0, T] \), for all \( 0 \leq t_0 \leq T \). The result follows trivially if \( t_0 = T \). Thus, we assume \( t_0 < T \).

Consider any solution \( x(\cdot), p(\cdot) \) to the TPBVP and set \( w = \ddot{w} \), where

\[
\ddot{w}(t) = \begin{cases} 
\bar{Q}^{-1}(B_1'p(t) - \bar{S}C_1z(t)), & t_0 < t \leq T, \\
0, & 0 \leq t \leq t_0.
\end{cases}
\]

Note that applying \( \ddot{w} \) on \( \Sigma_i \) in (3.27) with \( x(0) = 0 \) yields \( x(t) = 0, z(t) = 0, t \in [0, t_0] \).

By direct calculation, we get

\[
- \int_0^T r_q(z(t), \ddot{w}(t)) dt = - \int_0^T r_q(z(t), \ddot{w}(t)) dt
\]

\[
= - \frac{1}{2} \int_{t_0}^T (p(t)'B_1\bar{Q}^{-1}B_1'p(t) - x'(t)C_1'\bar{S}'\bar{Q}^{-1}\bar{S} - R)C_1x(t)) dt
\]

\[
= - \frac{1}{2} \int_{t_0}^T d(\dot{x}(t)'p(t))/dt dt
\]

\[
= - \frac{1}{2} (x(T)'p(T) - x(t_0)'p(t_0))
\]

\[
= 0.
\]
Thus, we have (using (3.32)) \( 0 = -\int_{t_0}^{T} r_q(z(t), \dot{w}(t)) dt \leq -\frac{1}{2}\epsilon \int_{t_0}^{T} \dot{w}(t)' \dot{w}(t) dt \leq 0 \), from which we conclude that \( \dot{w}(\cdot) = 0 \) on \([t_0, T]\). Since \( \bar{Q} > 0 \) we must have \( B_1 \bar{Q}^{-1} \bar{S} C_1 x(\cdot) \) on \([t_0, T]\). The TPBVP now becomes

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{p}(t)
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
C_1 R C_1 & -A'
\end{bmatrix}
\begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix},
\begin{bmatrix}
x(t_0) \\
p(t_0)
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix},
\tag{3.34}
\]

which implies \( x(\cdot) = 0, \ p(\cdot) = 0 \) on \([t_0, T]\). Thus, we conclude that the TPBVP has only trivial solutions on \([t_0, T]\) for all \( t_0 \in [0, T] \). Next, let \( \Phi(t, T) \) denote the transition matrix associated with the TPBVP (3.34), i.e.

\[
\begin{bmatrix}
\Phi_{11}(t, T) & \Phi_{12}(t, T) \\
\Phi_{21}(t, T) & \Phi_{22}(t, T)
\end{bmatrix} =
\begin{bmatrix}
A - B_1 \bar{Q}^{-1} \bar{S} C_1 & B_1 \bar{Q}^{-1} B_1' \\
-C_1' (\bar{S}' \bar{Q}^{-1} \bar{S} - R) C_1 & -(A - B_1 \bar{Q}^{-1} \bar{S})'
\end{bmatrix}
\times
\begin{bmatrix}
\Phi_{11}(t, T) & \Phi_{12}(t, T) \\
\Phi_{21}(t, T) & \Phi_{22}(t, T)
\end{bmatrix},
\]

\( \Phi(T, T) = I \). The fact that the TPBVP has only trivial solution implies that \( \Phi_{11}(t, T) \) is non-singular for all \( 0 \leq t \leq T \). To see this, pick any vector \( v \) such that \( \Phi_{11}(t_0, T)v = 0 \) for some \( t_0 \in [0, T] \). Setting \( x(T) = v, \ p(T) = 0 \) we get

\[
\begin{bmatrix}
x(t_0) \\
p(t_0)
\end{bmatrix} =
\begin{bmatrix}
\Phi_{11}(t_0, T) \\
\Phi_{21}(t_0, T)
\end{bmatrix} v =
\begin{bmatrix}
0 \\
\Phi_{21}(t_0, T)
\end{bmatrix} v.
\]

By the previous results, we must have \( x(\cdot) = 0 \) and \( p(\cdot) = 0 \) for all \( t_0 \leq t \leq T \). In particular \( p(t_0) = 0 \). Thus, we have

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
\Phi_{11}(t_0, T) \\
\Phi_{21}(t_0, T)
\end{bmatrix} v = \Phi(t, T) \begin{bmatrix}
v \\
0
\end{bmatrix}.
\]

Since the transition matrix \( \Phi(t, T) \) is nonsingular for all \( t \in [0, T] \), we must have \( v = 0 \). This shows that \( \Phi_{11}(t_0, T) \) is nonsingular for all \( t_0 \in [0, T] \). Next, it is straightforward to see that the matrix \( \bar{X}(t, T) \) defined by

\[
\bar{X}(t, T) = \Phi_{21}(t, T) \Phi_{11}(t, T)^{-1}
\]

solves the following RDE

\[
-\dot{X} = A' \bar{X} + \bar{X} A + [\bar{X} B_1 - C_1' \bar{S}] \bar{Q}^{-1} [B_1' \bar{X} - \bar{S} C_1] - C'_1 R C_1,
\tag{3.35}
\]

with terminal condition \( \bar{X}(T, T) = 0 \). To show that \( \bar{X}(t, T) \geq 0 \) for all \( t \in [0, T] \), define the quadratic function \( \bar{V}(x, t) = \frac{1}{2} x' \bar{X}(t, T) x \). This function has the integral
representation

\[ \tilde{V}(x, t) = \sup_{w \in L_2([0,T], \mathbb{R}^d)} \{ -\frac{1}{2} \int_t^T (w(s)'Qw(s) + 2w(s)'Sh(x(s), w(s)) + h(x(s), w(s))'Rh(x(s), w(s)))ds : x(t) = x \} . \]

Setting \( w = 0 \) we get

\[ \tilde{V}(x, t) \geq \{ -\frac{1}{2} \int_t^T h(x(s), 0)'Rh(x(s), 0)ds : x(t) = x \} \geq 0, \]

for all \( x \in \mathbb{R}^n \), for all \( t \in [0,T] \), since \( R \leq 0 \). Thus, we conclude that \( \tilde{X}(t, T) \geq 0 \) for all \( t \in [0,T] \).

Next, we shall show that the limit

\[ X = \lim_{T \to \infty} \tilde{X}(t, T) \quad (3.36) \]

exists. Differentiating both sides of the RDE (3.35) yields the matrix differential equation

\[ -\dot{X} = (A + B_1Q^{-1}[B_1'X - SC_1])'\dot{X} + \dot{X}(A + B_1Q^{-1}[B_1'X - SC_1]), \]

with \( -\dot{X}(T, T) = C_1'(\bar{S}'Q^{-1}\bar{S} - R)C_1 \geq 0 \). The solution to this equation is given by

\[ \dot{X}(t, T) = -\hat{\Phi}(t, T)C_1'(\bar{S}'Q^{-1}\bar{S} - R)C_1T(t, T), \]

where \( \dot{\Phi}(t, T) \) is the transition matrix associated with \( (A + B_1Q^{-1}[B_1'X - SC_1])' \). This shows that \( \dot{X}(t, T) \) is monotonically nonincreasing in \( t \). Furthermore, since the matrix \( A \) is asymptotically stable, the zero-input response (i.e. the response obtained by setting \( w = 0 \)) \( x_{z_0} \) of \( \Sigma^0 \) satisfies

\[ \int_0^T z_{x_0}(t)'z_{x_0}(t)dt \leq \alpha x_0'x_0, \]

for all \( T \geq 0 \), for all \( x_0 \), for some \( \alpha > 0 \). By using this inequality and (3.32), and employing the Cauchy-Schwarz inequality one can get

\[ -\int_0^\infty \tau_g(z(t), w(t))dt \leq c x_0'x_0, \quad (3.37) \]

for all \( x_0 \in \mathbb{R}^n \) for some \( c > 0 \). Set the particular \( w \) as follows

\[ \bar{w}(t) = \begin{cases} Q^{-1}(B_1'\tilde{X}(t; T) - \bar{S}C_1)x(t), & 0 \leq t \leq T, \\ 0, & T < t. \end{cases} \]
3.4 Strictly Dissipative Linear Systems

Note that \( \dot{Q}^{-1}(B_1 \dot{X}(t; T) - \dot{S}C_1)x(t) \) maximizes

\[
- \int_0^T r_q(z(t), w(t))dt, \quad x(0) = x_0,
\]

and we have

\[
- \int_0^T r_q(z(t), w^*(t))dt = \frac{1}{2} x_0' \dot{X}(0; T)x_0.
\]

Then,

\[
- \int_0^\infty r_q(z(t), w(t))dt \geq - \int_0^T r_q(z(t), w^*(t))dt = \frac{1}{2} x_0' \dot{X}(0; T)x_0.
\]

(3.38)

Combining (3.37) and (3.38) yields the following upper bound for \( \dot{X}(0, T) \)

\[
\dot{X}(0, T) \leq c_1 I,
\]

for some constant \( c_1 > 0 \) which is independent of \( t, T \) and \( x_0 \). This calculation can be repeated for all \( t \leq T \) yielding

\[
\dot{X}(t, T) \leq c_1 I,
\]

for all \( t \leq T \). Now, since \( 0 \leq \dot{X}(t, T) \leq c_1 I \) and \( \dot{X}(t, T) \) is monotonic nonincreasing in \( t \), the limit \( X \) in (3.36) exists and is nonnegative. Moreover, because the solution of the RDE (3.35) depends continuously upon its terminal condition, one can conclude that the limit \( X = \lim_{T \to \infty} \dot{X}(t, T) \) solves the ARE in (3.30).

We shall now show that the matrix \( A^* = A + B_1Q^{-1}(B_1'X - \dot{S}C_1) \) is asymptotically stable. First, we note that by the Plancherel’s theorem the strict dissipation inequality (3.32) implies

\[
- \frac{1}{2} \int_{-\infty}^\infty \dot{w}(j\omega)^*H(j\omega)\dot{w}(j\omega) d\omega \leq - \frac{\epsilon}{2} \int_{-\infty}^\infty \dot{w}(j\omega)^*\dot{w}(j\omega) d\omega
\]

for all \( \dot{w} \in L_2((-\infty, \infty), \mathbb{R}^d) \), where \( \dot{w} \) denotes the Fourier transform of \( w \) and \( H(j\omega) \) is given by

\[
H(j\omega) = Q + SG(j\omega) + G(-j\omega)'S' + G(-j\omega)'RG(j\omega), \quad (3.39)
\]

where \( G(j\omega) = D_{11} + C_1(j\omega I - A)^{-1}B_1 \). This inequality implies (as can be shown by using a simple contradiction argument (see [105])) that \( H(j\omega) \) satisfies the positive definiteness condition

\[
H(j\omega) > \eta I, \quad (3.40)
\]
for all \( \omega \in \mathbb{R} \), for some constant \( \eta > 0 \). Next, define the transfer matrix \( W(j\omega) = W + W L(j\omega I - A)^{-1} B \), where \( W \) is the square root of \( Q \), i.e., \( W'W = \bar{Q} > 0 \) (hence, \( W \) is invertible), and \( L = -\bar{Q}^{-1}[B'_1 X - \bar{S}C_1] \). Then, we obtain the following factorization

\[
H(j\omega) = W(-j\omega)'W(j\omega).
\]

The condition (3.40) implies that \( W(j\omega) \) has no purely imaginary zeros. Moreover, the stability of the matrix \( A \) guarantees there is no unobservable/uncontrollable modes on the \( j\omega \) axis. Noting that \( W^{-1}(j\omega) = W^{-1} - L(j\omega I - (A - B_1 L))^{-1}B_1 W^{-1} \), with \( A - B_1 L = A + B_1 \bar{Q}^{-1}[B'_1 X - \bar{S}C_1] = A^* \), we conclude that the matrix \( A^* \) has no purely imaginary eigenvalues. Finally, using a technique similar to that in the proof of Lemma 3.7.7 of [36], one can show that the existence of the limit \( X \geq 0 \) given in (3.36), satisfying the ARE (3.30) implies that the matrix \( A^* = A + B_1 \bar{Q}^{-1}[B'_1 X - \bar{S}C_1] \) has no eigenvalues with positive real parts. Combining these results, we conclude that the matrix \( A^* \) is asymptotically stable. This completes the proof of assertion (i).

To show assertion (ii), rewrite the ARE (3.30) as

\[
\tilde{A}'X + X\tilde{A} + X\tilde{B}_1\tilde{B}'_1 X + \tilde{C}'\tilde{C}_1 = 0,
\]

where \( \tilde{A} = A - B_1 \bar{Q}^{-1}\bar{S}C_1, \tilde{B}_1 = B_1 \bar{Q}^{-\frac{1}{2}}, \) and \( \tilde{C}_1 = (\bar{S}'\bar{Q}^{-1}\bar{S} - R)^{\frac{1}{2}}C_1 \). Note that from the identity

\[
\tilde{A} + \tilde{B}_1\tilde{B}'_1 X = A - B_1 \bar{Q}^{-1}\bar{S}C_1 + B_1 \bar{Q}^{-1}B'_1 X = A^*,
\]

\( \tilde{A} + \tilde{B}_1\tilde{B}'_1 X \) is an asymptotically stable matrix. This implies, by employing Theorem 2.1 of [82], there exists a positive definite matrix \( \tilde{X} \) solving the ARI (3.31). The proof is completed.

The converse results are given below.

**Theorem 3.5** Consider the open loop linear system \( \Sigma_l \) (3.27) and the supply rate \( r_q(x, w) \) in (3.28) such that the assumption (3.29) holds. Suppose that there exists a solution \( X \geq 0 \) to the ARE (3.30) such that the matrix \( A^* = A + B_1 \bar{Q}^{-1}[B'_1 X - \bar{S}C_1] \) is asymptotically stable. Then, the matrix \( A \) is asymptotically stable and the linear system \( \Sigma_l \) in (3.27) is strictly dissipative with respect to the supply rate \( r_q(x, w) \) in (3.28).

**Proof.** The ARE (3.30) may be rewritten as

\[
A'X + XA + \tilde{C}'\tilde{C} = 0,
\]
where \( \tilde{C} = \begin{bmatrix} (\tilde{Q}^{-\frac{1}{2}}[B_1'X - \tilde{S}C_1]) \\ (-R)^{\frac{1}{2}}C_1 \end{bmatrix} \). Since \( A^* = A + [B_1\tilde{Q}^{-\frac{1}{2}}0]\tilde{C} \) is asymptotically stable, the pair \((A, \tilde{C})\) is detectable, and since \( X \geq 0 \) solves the above (linear) Lyapunov equation, we conclude that \( A \) is asymptotically stable.

Next, for any output trajectory \( z(\cdot) \) produced by \( \Sigma_t \) in (3.27) with \( z(0) = x_0 \) we have

\[
-\int_0^T r_q(z(s), w(s))ds = -\frac{1}{2}\int_0^T [w(s) - w^*(s)]'[\tilde{Q}[w(s) - w^*(s)]ds + \frac{1}{2}(x_0'Xx_0 - z'(T)Xz(T))
\]

\[
\leq -\frac{1}{2}\sigma_{\text{min}}(\tilde{Q})\int_0^T [w(s) - w^*(s)]'[w(s) - w^*(s)]ds + \frac{1}{2}x_0'Xx_0,
\]

where \( w^* = \tilde{Q}^{-1}[B_1'X' - \tilde{S}C_1]x \), for all \( w \in L_2([0,T], R^d) \) for all \( T \geq 0 \). This shows the dissipativity of \( \Sigma_t \) with respect to \( r_q(z, w) \), with \( \beta(x) = \frac{1}{2}x'Xx \). To get the strict dissipativity, consider the following system

\[
\Sigma_t^{-1}: \begin{cases} \dot{x}(t) = A^*x(t) + B_1\dot{w}(t), \ x(0) = x_0, \\ w(t) = w^*(t) + \dot{w}(t), \end{cases}
\]

where \( \dot{w}(t) = w(t) - w^*(t) \). The output \( w \) can be written as \( w = w_{x_0} + w_\dot{w} \), where \( w_{x_0} \) and \( w_\dot{w} \) denote the zero-input and the zero-initial responses respectively. Since, by the hypothesis, \( A^* \) is asymptotically stable, the map \( \Sigma_t^{-1} : \dot{w} \to w \), with initial condition \( x_0 = 0 \), has finite \( L_2 \) gain, i.e.,

\[
\frac{1}{2}\int_0^T w_{\dot{w}}(t)w_{\dot{w}}(t)dt \leq \frac{1}{2}\gamma^*\int_0^T |\dot{w}(t)|^2dt,
\]

for all \( T \geq 0 \), for all \( \dot{w} \in L_2([0,T], R^d) \), for some \( \gamma^* > 0 \). Moreover, by setting \( \dot{w} = 0 \) in (3.42), the stability of \( A^* \) implies

\[
\frac{1}{2}\int_0^T w_{x_0}(t)'w_{x_0}(t)*dt \leq \frac{1}{2}\alpha^*x_0'x_0,
\]

for all \( T \geq 0 \), for all \( x_0 \in R^n \), for some \( \alpha^* > 0 \). Combining (3.43) and (3.44) and using the triangular inequality we get,

\[
\frac{1}{2}\int_0^T w'(t)w(t)dt \leq \frac{1}{2}\int_0^T w_{x_0}(t)'w_{x_0}(t)dt + \frac{1}{2}\int_0^T w_{\dot{w}}w_{\dot{w}}(t)dt \\
\leq \frac{1}{2}\alpha^*x_0'x_0 + \frac{1}{2}\gamma^*\int_0^T (w(t) - w^*(t))'(w(t) - w^*(t))dt.
\]
By substituting (3.45) in (3.41) we finally get

\[- \int_0^T r_q(z(s), w(s)) ds \leq - \frac{1}{2} \sigma_{\min}(\bar{Q}) \bar{\gamma}^* \int_0^T w(s)' w(s) ds + \frac{1}{2} \gamma_0 (\sigma_{\min}(\bar{Q}) \alpha^* I + X) x_0,\]

for all \( w \in L_2([0, T], \mathbb{R}^d), \) for all \( T \geq 0, \) which shows the required strict dissipativity with \( \epsilon = \sigma_{\min}(\bar{Q}) / \gamma^* > 0, \) and \( \beta(z) = \frac{1}{2} z' (\sigma_{\min}(\bar{Q}) \alpha^* I + X) z. \) This completes the proof.

\[\Box\]

**Remark 3.6**

In view of the discussion in Remark 3.5, we see that the function \( V(z) = \frac{1}{2} z' X z, \) where \( X \) satisfies the hypothesis of Theorem 3.5 is the available storage function for the system (3.27) with the supply rate given in (3.28).

We shall now present the analogy of Corollary 3.1 for linear systems. Consider the following system \( \tilde{\Sigma}_l \) which is related to the system \( \Sigma_l \) in (3.27)

\[
\tilde{\Sigma}_l: \left\{ \begin{array}{l}
\dot{x}(t) = \tilde{A} x(t) + \tilde{B}_1 w(t), \quad x(0) = x_0, \\
\tilde{z}(t) = \tilde{C}_1 x(t),
\end{array} \right.
\]

(3.46)

where \( \tilde{A} = A - B_1 \bar{Q}^{-1} S C_1, \) \( \tilde{B}_1 = B_1 \bar{Q}^{-\frac{1}{2}}, \) and \( \tilde{C}_1 = (S' \bar{Q}^{-1} S - R)^{-\frac{1}{2}} C_1. \)

**Corollary 3.2** Consider the open loop linear system \( \Sigma_l \) (3.27) and the supply rate \( r_q(z, w) \) in (3.28) such that the assumption (3.29) holds and assume that the matrix \( A \) is asymptotically stable. Suppose that the linear system \( \Sigma_l \) is strictly dissipative with respect to the supply rate \( r_q(z, w). \) Then, there exists a solution \( X \geq 0 \) to the ARE

\[
\tilde{A}' X + X \tilde{A} + X \tilde{B}_1 \tilde{B}_1' X + \tilde{C}_1 \tilde{C}_1 = 0,
\]

(3.47)

such that the matrix \( \tilde{A} + \tilde{B}_1 \tilde{B}_1' X \) is asymptotically stable. In particular, the matrix \( \tilde{A} \) is asymptotically stable and the system \( \tilde{\Sigma}_l \) has \( H_\infty \) norm strictly less than 1.

Conversely, assume that the matrix \( \tilde{A} = A - B_1 \bar{Q}^{-1} S C_1 \) is asymptotically stable and the system \( \tilde{\Sigma}_l \) has \( H_\infty \) norm strictly less than 1. Then, there exists a solution \( X \geq 0 \) to the ARE in (3.30) such that the matrix \( A^* = A + B_1 \bar{Q}^{-1} [B_1' X - \bar{S} C_1] \) is asymptotically stable. In particular, the matrix \( A \) is asymptotically stable and the system \( \Sigma_l \) (3.27) is strictly dissipative with respect to the supply rate \( r_q(z, w) \) in (3.28).
3.5 Conclusions

Proof. Under the hypothesis, Theorem 3.4 implies there exists a solution \( X \geq 0 \) to the ARE (3.30) such that \( A^* \) is asymptotically stable. Note that the ARE in (3.30) can be rewritten as in (3.47) and that \( A^* = \tilde{A} + \tilde{B}_1 \tilde{B}_1^T X \). This implies, by Theorem 3.4, that the related system \( \tilde{\Sigma}_t \) has \( H_\infty \) norm strictly less than 1. To show that the matrix \( \tilde{A} \) is asymptotically stable. Rewrite the ARE (3.47) as

\[
\tilde{A}'X + X\tilde{A} + [X\tilde{B}_1 \tilde{C}_i']^T [X\tilde{B}_1 \tilde{C}_i'] = 0.
\]

Since the matrix \( A^* = \tilde{A} + [\tilde{B}_1 \ 0][X\tilde{B}_1 \tilde{C}_i'] \) is stable, the pair \((\tilde{A}, [X\tilde{B}_1 \tilde{C}_i'])\) is detectable. Since \( X \geq 0 \) solves the above (linear) Lyapunov equation, we conclude that \( \tilde{A} \) is an asymptotically stable matrix.

Conversely, under the hypothesis Theorem 3.4 implies there exists \( X \geq 0 \) solving the ARE (3.47) such that \( \tilde{A} + \tilde{B}_1 \tilde{B}_1^T X \) is asymptotically stable. Since \( X \) also solves the ARE (3.30) such that \( A^* \) is asymptotically stable, Theorem 3.4 implies that the system \( \Sigma_t \) is strictly dissipative with respect to the supply rate \( \tau_q(z, w) \) in (3.28). To show that \( A \) is asymptotically stable, rewrite the ARE (3.30) as

\[
A'X + XA + ([XB_1 - C'_1 \tilde{S}_1']\tilde{Q}^{-1} W' - C'_1 V') \left[ \frac{W\tilde{Q}^{-1} [B'_1 X - \tilde{S}_C_1]}{VC_1} \right],
\]

where \( W'W = \tilde{Q} > 0, V'V = -R \geq 0 \). Since

\[
A^* = A + [B_1 W^{-1} \ 0] \left[ \frac{W\tilde{Q}^{-1} [B'_1 X - \tilde{S}_C_1]}{VC_1} \right]
\]

is asymptotically stable, the pair \((A, \left[ \frac{W\tilde{Q}^{-1} [B'_1 X - \tilde{S}_C_1]}{VC_1} \right])\) is detectable. Since \( X \geq 0 \) solves the above Lyapunov equation, we conclude that \( A \) is asymptotically stable. \( \square \)

3.5 Conclusions

In this chapter we have studied nonlinear and linear systems which are strictly dissipative with respect to a general quadratic supply rate function. In the nonlinear systems case, we have characterized a strongly-stable strictly dissipative system in terms of the viscosity solutions of a strict PDE or a PDE without any assumptions regarding system's detectability/stabilizability. Moreover, when the solution to the PDE is, in fact, smooth, it also possesses a kind of stabilizing property. The results can be regarded as a generalization of the (linear) strictly bounded real lemma in [82] to the nonlinear systems case with a general quadratic dissipativity. In the linear systems case, we have showed that
an asymptotically-stable strictly (quadratic) dissipative linear system can be characterized in terms of a strict ARI or an ARE with a stabilizing property. The techniques we employ in the linear case are rather elementary involving an analysis of a two-point boundary-value problem as described in [36], [73].

In addition, we have established a connection between a quadratic dissipative system and a finite gain related system in both nonlinear and linear cases. The results are obtained by studying the corresponding PDI's/ARI's instead of employing the Cayley transformation.
Chapter 4

State Feedback Synthesis

4.1 Introduction

In this chapter we study a general dissipative control synthesis problem using a static state feedback control law. We first consider a general (non-strict) dissipativity case in which the supply rate may be non quadratic, and show that the existence of a state feedback control law solving the problem is equivalent to the existence of viscosity solutions to a related controlled PDI of Hamilton-Jacobi-Isaacs (HJI) type (see also [10], [90] in the case of $H_\infty$ control).

In the strict quadratic dissipativity nonlinear systems case we show that the existence of a state feedback control solving the dissipativity control problem is equivalent to the existence of: (i) a solution to a HJI PDI or (ii) a solution to a max-min type HJI PDE, both are in the viscosity sense. Moreover, if a smooth solution to a related min-max HJI PDE exists, then under additional assumptions regarding the system's structure and regarding the solution of the PDE, this solution is stabilizing. These min-max PDE and max-min PDE coincide when the Isaacs condition holds as in the $H_\infty$ control case. In the linear systems case we express the solution to the strict dissipativity control problem in terms of a stabilizing solution to an ARE. Our linear synthesis results are based on the analysis parts in Section 3.4 and extend those in [82] and [91]. We finally show that a locally smooth solution to the PDE exists whenever the control problem for the linearized system admits a solution (by a linear control law). This extends the results in [97], [98], [99].
4.2 Problem Formulation

We consider a general class of nonlinear systems described by

\[
\Sigma: \begin{cases} 
\dot{x}(t) = A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad x(0) = x_0, \\
x(t) = C_1(x(t)) + D_{11}(x(t))w(t) + D_{12}(x(t))u(t).
\end{cases}
\] (4.1)

In this description, \( x \in \mathbb{R}^n \) denotes the state vector, which is available for control. The vector \( z \in \mathbb{R}^q \) represents the quantity to be controlled. The disturbance \( w \in \mathbb{R}^d \) corrupts the state and the vector \( u \in \mathbb{R}^m \) denotes the control. We assume that \( A(\cdot), B_1(\cdot), B_2(\cdot), C_1(\cdot), D_{11}(\cdot), D_{12}(\cdot) \) are smooth functions, \( A(\cdot) \) and \( C_1(\cdot) \) are globally Lipschitz continuous, \( B_1(\cdot), B_2(\cdot), D_{11}(\cdot) \) and \( D_{12}(\cdot) \) are bounded, and that \( A(0) = 0, C_1(0) = 0 \). We further assume \( D_{12}(x)'D_{12}(x) = E_2(x) > 0 \), for all \( x \).

The admissible class of control laws is the set of static functions of the state \( u = K(x) \) with

\[
K: \mathbb{R}^n \to \mathbb{R}^m,
\]

such that, when applied to the system (4.1) they result in a unique solution for all \( t \geq 0 \).

We denote the class by \( S \). Admissible disturbances are all signals \( w \in L_2([0,\infty), \mathbb{R}^d) \).

We consider the general supply rate

\[
r : \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R},
\] (4.2)

which is assumed to be \( C^1(\mathbb{R}^q, \mathbb{R}^d) \), with at most quadratic growth, i.e. \( |r(z,w)| \leq \alpha(1 + |z|^2 + |w|^2) \) for some \( \alpha > 0 \), for all \( z \in \mathbb{R}^q \), \( w \in \mathbb{R}^d \). Throughout the chapter, we consider supply rate functions that satisfy

\[
r(z,0) \leq 0,
\] (4.3)

for all \( z \in \mathbb{R}^q \), with \( r(0,0) = 0 \).

Let \( \Sigma^u_{x_0} \) denote the map from \( w \) to \( z \), under the control law \( u \), with initial condition \( x_0 \).

**Definition 4.1** The dissipative control problem is to find \( u \in S \) such that:

1. the closed loop system \( \Sigma^u \) is asymptotically stable when \( w = 0 \), and
4.3 General Dissipativity Control Synthesis

2. $\Sigma^u$ is dissipative (see Definition 3.1) with respect to the supply rate $r(z, w)$ in (4.2).

We make the following definition regarding closed loop (zero state) detectability.

Definition 4.2 We say that the closed loop system $\Sigma^u$ is (zero state) detectable if, under the control law $u$, the conditions $w = 0$ and $\lim_{t \to \infty} z(t) = 0$ implies $\lim_{t \to \infty} z(t) = 0$.

4.3 General Dissipativity Control Synthesis

The next results are concerned with a general (non strict) dissipative control synthesis problem.

Theorem 4.1 Consider the system $\Sigma$ in (4.1) and the supply rate $r(z, w)$ in (4.2) satisfying (4.3). Assume that there exists $u^*(z) = K^*(z) \in S$ such that the closed loop system $\Sigma^{u^*}$ (4.1) is dissipative with respect to the supply rate $r(z, w)$ (4.2). Then there exists a solution $V$ to the following PDI

$$\inf_{u \in \mathbb{R}^n} \sup_{w \in \mathbb{R}^d} \{\nabla_z V(x) (A(x) + B_1(x) w + B_2(x) u) \}
- r(h_1(x, w, u), w) \leq 0,$$

(4.4)

where $h_1(x, w, u) = C_1(x) + D_{11}(x) w + D_{12}(x) u$, such that $V \geq 0$ and $V(0) = 0$, in the viscosity sense.

Conversely, assume that there exists $V \in C^1$ solving the PDI (4.4) in the classical sense such that $V \geq 0$ and $V(0) = 0$. Suppose that the control law $u^* = u^*(z)$ attains minimum on the left hand side of the PDI (4.4) i.e.

$$u^* = u^*(x) \in \arg \min_{u \in \mathbb{R}^n} \{\sup_{w \in \mathbb{R}^d} \{\nabla_z V(x) (A(x) + B_1(x) u + B_2(x) u) \}
- r(h_1(x, w, u), w)\}.$$ 

(4.5)

Then the closed loop $\Sigma^{u^*}$ is dissipative with respect to the supply rate $r(z, w)$ (4.2). Moreover if: (i) the supply rate $r(z, w)$ satisfies

$$r(z, 0) \leq -c_1|z|^2,$$

(4.6)
for all \( z \neq 0 \), for some constant \( c_1 > 0 \), with \( r(0,0) = 0 \), and (ii) the closed loop system \( \Sigma u^* \) is zero state detectable, then \( \Sigma u^* \) is asymptotically stable.

**Proof.** Under the hypothesis, Theorem 3.1 implies there exists a solution \( V \geq 0 \), with \( V(0) = 0 \), to the PDI

\[
\sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x)(A(x) + B_1(x)w + B_2(x)K^*(x)) - r(h_1(x, w, K^*(x)), w) \} \leq 0,
\]

in the viscosity sense. Taking the minimum over \( u(x) = K(x) \in \mathcal{S} \) yields

\[
\inf_{u \in \mathbb{R}^m} \sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x)(A(x) + B_1(x)w + B_2(x)u) - r(h_1(x, w, u), w) \} \leq 0.
\]

This proves the existence of a solution to the PDI (4.4).

Conversely, if a smooth solution \( V \in C^1 \) to the PDI (4.4) exists, such that \( V \geq 0 \), and \( V(0) = 0 \), then by using the control law \( u^*(x) = K^*(x) \) defined in (4.5), we get

\[
0 \geq \int_0^T \sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x(t))(A(x(t)) + B_1(x(t))w + B_2(x(t))u^*(x(t)))
\]

\[
- r(h_1(x(t), w, u^*(x(t))), w) \} dt
\]

\[
\geq \int_0^T (\nabla_x V(A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u^*(x(t))))
\]

\[
- r(h_1(x(t), w(t), u^*(x(t))), w(t)) \} dt
\]

\[= V(x(T)) - V(x(0)) - \int_0^T r(h_1(x(t), w(t), u^*(x(t))), w(t)) dt.\]

Since \( V \geq 0 \), the last inequality implies,

\[
V(x(0)) \geq - \int_0^T r(h_1(x(t), w(t), u^*(x(t))), w(t)) dt,
\]

which shows that \( \Sigma u^* \) is dissipative with respect to \( r(x, w) \), with \( \beta u^*(x) = V(x) \). Furthermore, by setting \( w = 0 \) in the last inequality we get

\[
V(x(0)) \geq - \int_0^T r(h_1(x(t), 0, u^*(x(t))), 0) dt.
\]

By (4.6), the last inequality implies (see Remark 3.3) \( h_1(x(t), 0, u^*(t)) \to 0 \) as \( t \uparrow \infty \). By the zero state detectability assumption of \( \Sigma u^* \) we conclude that \( x(t) \to 0 \) asymptotically.

\[\Box\]

**Remark 4.1** Suppose that the supply rate is quadratic as in (3.9) such that \( Q(x) > 0 \),

\[
\bar{R}(x) = \bar{S}(x)'\bar{Q}^{-1}(x)\bar{S}(x) - R > 0 \quad \text{for all } x,
\]

where \( \bar{S}(x) = S + D_{11}(x)'R \). Then the PDI
(4.4) can be written as (by direct evaluation)
\[ \nabla_x V(x)(A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x)) \]
\[ -\frac{1}{2} \nabla_x V(x)(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')\nabla_x V(x)'
\]
\[ +\frac{1}{2} C_1(x)'\bar{D}(x)'\bar{R}(x)\bar{D}(x)C_1(x) \leq 0, \tag{4.7} \]

where \( \bar{B}(x) = B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x), \bar{E}(x) = D_{12}(x)'\bar{R}(x)D_{12}(x) > 0 \) and \( \bar{D}(x) = I - D_{12}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x), \) and the optimal control in (4.5) is given by
\[ u^*(x) = -\bar{E}^{-1}(x)(\bar{B}(x)'\nabla_x V(x) + D_{12}(x)'\bar{R}(x)C_1(x)). \tag{4.8} \]

**Remark 4.2** The condition (4.6) does not hold in the positive real (or passivity) case (here \( r(z,0) = 0 \) for all \( w) \). Without the condition (4.6) holding, closed loop stability can be pursued in the following manner (see [44], [45]), which works in the quadratic supply rate case, including the positive real case. Assume that \( V \) is smooth and positive definite, i.e., \( V(x) > 0 \) for all \( x \neq 0 \), with \( V(0) = 0 \). The PDI (4.7) may be rewritten as
\[ \nabla_x V(x)A^{u^*}(x) + \frac{1}{2}[\nabla_x V(x)B_1(x) - C^{u^*}(x)\bar{S}(x)']\bar{Q}^{-1}(x) \]
\[ [B_1(x)'\nabla_x V(x)' - \bar{S}(x)C^{u^*}(x)] - C^{u^*}(x)'RC^{u^*}(x) = 0, \]
where \( A^{u^*}(x) = A(x) + B_2(x)u^*(x) \) and \( C^{u^*}(x) = C_1(x) + D_{12}u^*(x) \). Then, along any trajectory \( x(t) \) with \( u(t) = u^*(x(t)) \) and \( w = 0 \), we have (see also [44], [45])
\[ \frac{dV}{dt} = -L(x(t))'L(x(t)) + h_1'(x(t),0,u^*(x(t)))Rh_1(x(t),0,u^*(x(t))), \]
\[ \leq h_1'(x(t),0,u^*(x(t)))Rh_1(x(t),0,u^*(x(t))), \]
for some bounded function \( L(x) \). If \( R \leq 0 \), then the closed loop system is (Lyapunov) stable since \( V \) serves as a Lyapunov function. Thus, we conclude that a positive real system possessing a smooth and positive definite storage function is Lyapunov stable (see also [45], [14]). When \( R < 0 \), the La Salle invariance principle implies that \( x(\cdot) \) converges to the set \( \{x : h_1(x,0,u^*(x)) = 0\} \). Thus, if \( \Sigma^{u^*} \) is zero state detectable, we have \( \lim_{t \to \infty} x(t) = 0 \), i.e. \( \Sigma^{u^*} \) is asymptotically stable. **Remark 4.3** If the system in (4.1) is linear, i.e. \( A(x) = Ax, B_1(x) = B_1, B_2(x) = B_2, C_1(x) = C_1x, D_{11}(x) = D_{11}, D_{12}(x) = D_{12} \) where \( A, B_1, B_2, C_1, D_{11} \) and \( D_{12} \) are
constant matrices, and the supply rate is quadratic as above, then the PDI (4.4) has an explicit expression given by 

\[ V(x) = \frac{1}{2}x'Xx, \]  

where \( X = X' \geq 0 \) solves the following ARI

\[ X(A - \bar{B}\bar{E}^{-1}D_2\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) + \left(A - \bar{B}\bar{E}^{-1}D_2\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1\right)'X 
- X(\bar{B}\bar{E}^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1')X + C_1'\bar{D}'\bar{R}\bar{D}C_1 \leq 0, \]  

(4.9)

where \( \bar{B} = B_2 - B_1\bar{Q}^{-1}\bar{S}D_2, \ \bar{E} = D_2'\bar{R}D_2 \) and \( \bar{D} = I - D_2\bar{E}^{-1}D_2'\bar{R} \) and the optimal control law is linear in \( x \) given by 

\[ u^*(x) = K^*x, \]  

where

\[ K^* = -\bar{E}^{-1}(\bar{B}'X + D_2'\bar{R}C_1). \]  

(4.10)

When \( R < 0 \), closed loop asymptotic stability will be obtained if the matrices pair \((A + B_2K^*, C_1 + D_2K^*)\) is completely observable. The ARI in (4.9) has infinitely many solutions and each of them will result in dissipative closed loop system. However, only one of them will result in a strict dissipativity property. This particular case will be discussed in more details in Section 4.5

\[ \bullet \]

4.4 Nonlinear Strict Dissipativity Control Synthesis

In this section we present control synthesis results for strict dissipativity, which are partly based on the analysis results in Section 3.3. We consider the systems \( \Sigma (4.1) \) with quadratic supply rate \( r_q(z,w) \) given by

\[ r_q(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz), \]  

(4.11)

where \( Q, R \) are symmetric matrices. We write \( \bar{Q}(x) = Q + SD_{11}(x) + D_{11}'(x)S' + 
D_{11}(x)RD_{11}(x), \bar{S}(x) = S + D_{11}(x)'R \) and \( \bar{R}(x) = \bar{S}(x)'\bar{Q}^{-1}(x)\bar{S}(x) - R \). We assume that

\[ R \leq 0, \ \inf_{x \in \mathbb{R}^n} \{\sigma_{\min}(\bar{Q}(x))\} = k > 0, \ \text{and} \ \bar{R}(x) > 0, \]  

(4.12)

for all \( x \).

Theorem 4.2 Consider the system in (4.1) and the quadratic supply rate \( r_q(z,w) \) in (4.11) satisfying (4.12). Assume that there exists a controller \( u^* = K^*(x) \in S \) such that the closed loop system \( \Sigma u^* (4.1) \) is strictly dissipative with respect to the supply rate \( r_q(z,w) \) and the vector field \( A^{u^*}(x) = A(x) + B_2(x)u^*(x) \) is strongly stable. Then the following statements hold true.
4.4 Nonlinear Strict Dissipativity Control Synthesis

1. There exists a function $\hat{V} > 0$, with $\hat{V}(0) = 0$, satisfying the PDI

$$\inf_{u \in \mathbb{R}^n} \sup_{w \in \mathbb{R}^d} \{ \nabla_x \hat{V}(x)(A(x) + B_1(x)w + B_2(x)u)$$

$$- r_q(h_1(x, w, u), w) + \delta \frac{1}{2} x' x \} \leq 0,$$

(4.13)

for some $\delta > 0$, where $h_1(x, w, u) = C_1(x) + D_{11}(x)w + D_{12}(x)u$, in the viscosity sense.

2. There exists a function $V \geq 0$ with $V(0) = 0$, satisfying the PDE

$$\sup_{w \in \mathbb{R}^d} \inf_{u \in \mathbb{R}^n} \{ \nabla_x V(x)(A(x) + B_1(x)w + B_2(x)u)$$

$$- r_q(h_1(x, u, w), w) \} = 0,$$

(4.14)

in the viscosity sense.

3. Assume that $D_{12}(x) = I$. Assume there exists a positive semidefinite solution $V \in C^1(\mathbb{R}^n)$ to the PDE

$$\inf_{u \in \mathbb{R}^n} \sup_{w \in \mathbb{R}^d} \{ \nabla_x V(x)(A(x) + B_1(x)w + B_2(x)u)$$

$$- r_q(h_1(x, u, w), w) \} = 0.$$

(4.15)

Then we have the following results concerning stabilizing properties of $V$.

a. Assume that $-A^{xx}(x) \triangleq -(A(x) - B_2(x)C_1(x))$ is Lyapunov stable. If the function $V(x) > 0$ for all $x \neq 0$, with $V(0) = 0$, and $V$ is a Lyapunov function for $-A^{xx}(x)$, then the vector field

$$A^*(x) \triangleq A(x) - B_2(x)C_1(x) - ([B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)]\bar{R}^{-1}(x)$$

$$[B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)]' - B_1(x)\bar{Q}^{-1}(x)B_1(x)'\nabla_x V(x)'$$

(4.16)

is stable in the sense that $V(x)$ is a Lyapunov function for $A^*(x)$. Thus, $V$ is a stabilizing (in the sense of Lyapunov stability) solution to the PDE (4.15). In particular, if $-A^{xx}(x)$ is exponentially stable (i.e., we have $c_1|x|^2 \leq V(x) \leq c_2|x|^2$, $\nabla_x V(x)(-A^{xx}(x)) \leq -c_3|x|^2$, for some $c_1 > 0, c_2 > 0, c_3 > 0$), then so is $A^*(x)$.

b. Assume that $A^{xx}(x) \triangleq (A(x) - B_2(x)C_1(x))$ is asymptotically stable. Then $V = 0$ is a stabilizing solution to the PDE (4.15).
Proof. Under the hypothesis, Theorem 3.2 implies there exists a solution $\dot{V} > 0$, with $\dot{V}(0) = 0$, to the PDI

$$\sup_{w \in \mathbb{R}^d} \{ \nabla_x \dot{V}(x)(A(x) + B_1(x)w + B_2(x)K^*(x)) - r_q(h_1(x, w, K^*(x)), w) + \frac{1}{2}x'x \} \leq 0,$$

in the viscosity sense. Taking the infimum over $u = K(x) \in S$ yields

$$\inf_{u \in \mathbb{R}^m} \sup_{w \in \mathbb{R}^d} \{ \nabla_x \dot{V}(x)(A(x) + B_1(x)w + B_2(x)u) - r_q(h_1(x, w, u), u) + \frac{1}{2}x'x \} \leq 0.$$

This proves the existence of a viscosity solution to the PDI (4.13).

Next, define the function $V$ by

$$V(x) \triangleq \inf_{K \in \mathcal{U}_c} \sup_{T \geq 0, w \in L^2(\mathbb{R}^d)} \{-\int_0^T r_q(h_1(x(t), w(t), K(x(t))), w(t)) dt : x(0) = x\},$$

in which $\mathcal{U}_c$ denotes the class of causal maps $K : \mathcal{W} \to \mathcal{U}$, in which $\mathcal{W}, \mathcal{U}$ denote the spaces of admissible disturbance and controls respectively. Clearly $V \geq 0$. Since $K^* \in \mathcal{S} \subset \mathcal{U}_c$, we have

$$V(x) \leq \sup_{T \geq 0, w \in L^2(\mathbb{R}^d)} \{-\int_0^T r_q(h_1(x(t), w(t), K^*(x(t))), w(t)) dt : x(0) = x\} \leq \beta K^*(x).$$

Thus $V$ is finite. We claim that $V$ is a viscosity solution of the PDE (4.14). The proof follows closely the techniques employed in [90] and is given in Appendix C. This completes the proof of assertion (ii).

To prove assertion (iii), suppose there exists a smooth solution $V$ to the PDE (4.15). Evaluation of the inf and sup operations in the PDE (4.15), yields, with $D_{12}(x) = I$,

$$\nabla_x V(x)(A(x) - B_2(x)C_1(x)) - \frac{1}{2} \nabla_x V(x)((B_2(x) - B_1(x)\tilde{Q}^{-1}(x)S(x))^R^{-1}(x)$$

$$((B_2(x) - B_1(x)\tilde{Q}^{-1}(x)S(x))' - B_1(x)\tilde{Q}^{-1}(x)B_1(x)')\nabla_x V(x)' = 0.$$ 

This PDE can be re-expressed in terms of the vector fields $A^*(x)$ and $A^{\times x}(x)$ as follows

$$\nabla_x V(x) A^*(x) - \nabla_x V(x)(-A^{\times x}(x)) = 0.$$

If $V > 0$ is a Lyapunov function for $-A^{\times x}(x)$ (i.e. $\nabla_x V(x)(-A^{\times x}(x)) \leq 0$ for all $x$), then, along any trajectory produced by $A^*(x)$ we have

$$\frac{dV}{dt}(x(t)) = \nabla_x V(x(t))A^*(x(t)) = \nabla_x V(x(t))(-A^{\times x}(x(t))) \leq 0.$$
Assertion concerning exponential stability can be verified in the same manner. This proves assertion (iiia).

If $A^{xx}(x) \hat{=} (A(x) - B_2(x)C_1(x))$ is asymptotically stable, then with $V = 0$ (which solves the PDE (4.15)) we have

$$A^*(x) = A^{xx}(x),$$

and hence, $A^*(x)$ is an asymptotically stable vector field. This completes the proof. □

Conversely, we have the following results.

**Theorem 4.3** Consider the system in (4.1) and the quadratic supply rate $r_q(z, w)$ in (4.11) satisfying (4.12). Assume that there exists $V \in C^1$ solving the PDE (4.17)

$$\nabla_x V(x)(A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x))$$

$$-\frac{1}{2}\nabla_x V(x)(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')\nabla_x V(x)'$$

\hspace{1cm} (4.17)

(this is the PDE (4.15) with $D_{12} = I$) where $\bar{B}(x) = B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x)$, $\bar{E}(x) = D_{12}(x)'\bar{R}(x)D_{12}(x) > 0$ and $\bar{D}(x) = I - D_{12}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)$, in the classical sense with $V \sim 0$ and $V(0) = 0$, such that the vector field

$$A^*(x) = A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x)$$

$$-\frac{1}{2}\nabla_x V(x)(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')$$

\hspace{1cm} (4.18)

is strongly stable. Then, by employing the control law $u^*(x)$ defined by

$$u^*(x) = -\bar{E}^{-1}(x)(\bar{B}(x)\nabla_x V(x)' + D_{12}(x)'\bar{R}(x)C_1(x)).$$

\hspace{1cm} (4.19)

(which attains the minimum on the left hand side of the PDE (4.15)) the closed loop vector field $A^{u*}(x) = A(x) + B_2(x)u^*(x)$ is weakly stable and the closed loop system $\Sigma^{u*}$ (4.1) is dissipative with respect to $r_q(z, w)$. Moreover, if the vector field $w^{u*}(x) = \bar{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C^{u*}(x)]$, with $C^{u*}(x) = C_1(x) + D_{12}(x)u^*(x)$, satisfies $|w^{u*}(x)| \leq \alpha|x|$, for all $x$, for some $\alpha > 0$, then the dissipativity is strict.

**Proof.** First, rewrite the PDE (4.17) as follows

$$\nabla_x V(x)A^{u*}(x) + \frac{1}{2}[\nabla_x V(x)B_1(x) - C^{u*}(x)'\bar{S}(x)']\bar{Q}^{-1}(x)$$

$$[B_1(x)'\nabla_x V(x)' - \bar{S}(x)C^{u*}(x)] - C^{u*}(x)'RC^{u*}(x) = 0,$$
with \(A^u(x) = A(x) + B_2(x)u^*(x)\), \(C^u(x) = C_1(x) + D_{12}(x)u^*(x)\). The results then simply follow by performing calculations similar to those in the proof of Theorem 3.3. □

**Remark 4.4** Following the discussion in Remark 3.5, we see that the value function \(V\) stated in Theorem 4.3 serves as the available storage function for the closed loop system

\[
\Sigma^{u^*} : \begin{cases} 
\dot{z} = (A(x) + B_2(x)u^*(x)) + B_1(x)w, \ x(0) = x_0, \\
\dot{z} = (C_1(x) + D_{12}(x)u^*(x)) + D_{11}(x(t))w(t),
\end{cases}
\]

in which \(u^*(x)\) is given in (4.19).

**Remark 4.5** In the PDE (4.15), the Isaacs condition does not hold, i.e., the order in which the inf and sup operations is carried out is not interchangeable, in general. Thus, a saddle point strategy does not exist in general. As a consequence, the completion of square technique (see for example [97] in the finite gain case) cannot be used in the verification of optimality of the control law (4.19).

### 4.5 Linear Strict Dissipativity Control Synthesis

We present results analogous to those in the previous section for linear systems with quadratic supply rate. In particular, we characterize the strict dissipativity property in terms of the stabilizing solution to an ARE. These results extend known ones in the literature on \(H_\infty\) control or positive real control (see [36], [82], [91]).

We consider linear models given by

\[
\Sigma_1 : \begin{cases} 
\dot{z}(t) = Az(t) + B_1w(t) + B_2u(t), \ z(0) = z_0, \\
z(t) = C_1z(t) + D_{11}w(t) + D_{12}u(t),
\end{cases}
\]

in which \(A \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^{n \times d}, B_2 \in \mathbb{R}^{n \times m}, C_1 \in \mathbb{R}^{q \times n}, D_{11} \in \mathbb{R}^{q \times d}, D_{12} \in \mathbb{R}^{q \times m}\) are constant matrices, with \(D_{12}\) satisfying \(\bar{D}_{12}D_{12} = E_1 > 0\).

We consider a quadratic supply rate \(r_q(z,w)\) as in (4.11) satisfying

\[
R \leq 0, \ \bar{Q} = Q + SD_{11} + D_{11}'S' + D_{11}'RD_{11} > 0,
\]

\[
\bar{R} = S\bar{Q}^{-1}S - R > 0,
\]

where \(\bar{S} = S + D_{11}'R\). We assume that \(\text{rank}\{M(j\omega)\} = n + m\), where

\[
M(j\omega) = \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}.
\]
4.5 Linear Strict Dissipativity Control Synthesis

**Theorem 4.4** Consider the system in (4.20) and the quadratic supply rate \( r_q(z, w) \) in (4.11) satisfying (4.21). Assume that there exists a linear feedback law \( u^*(z) = K^*x \), such that \( \Sigma u^* \) is strictly dissipative with respect to the quadratic supply rate \( r_q(z, w) \) (4.11). Then, there exists a solution \( X \geq 0 \) to the following ARE

\[
X(A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) + (A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)'X \\
-X(\bar{B}E^{-1}B' - B_1\bar{Q}^{-1}B_1')X + C_1D'\bar{R}DC_1 = 0,
\]

(4.23)

such that the matrix

\[
A^* = (A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) - (\bar{B}E^{-1}B' - B_1\bar{Q}^{-1}B_1')X
\]

(4.24)

is asymptotically stable.

**Proof.** We follow the proofs in [82] and [91]. Under the hypothesis, the closed loop system

\[
\dot{x}(t) = A''x(t) + B_1w(t), \quad x(0) = x_0,
\]

\[
z(t) = C''x(t) + D_{11}w(t),
\]

where \( A'' = A + B_2K^* \), \( C'' = C_1 + D_{12}K^* \) is strictly dissipative with respect to the supply rate (3.28), such that the matrix \( A'' \) is asymptotically stable. Theorem (3.4) implies there exists a matrix \( \bar{X} > 0 \) solving the following ARI

\[
(\bar{A}'')'\bar{X} + \bar{X}\bar{A}'' + \bar{X}\bar{B}_1\bar{B}_1'\bar{X} + (\bar{C}_1')'\bar{C}_1'' < 0,
\]

where \( \bar{A}'' = A'' - B_1\bar{Q}^{-1}\bar{S}C'' \), \( \bar{B}_1 = B_1\bar{Q}^{-\frac{1}{2}} \), and \( \bar{C}_1'' = (\bar{S}'\bar{Q}^{-1}\bar{S} - R)^\frac{1}{2}C'' \). This ARI can be rewritten in terms of the parameters of the original system as follows

\[
(A''')'\bar{X} + \bar{X}A''' + [\bar{X}B_1 - (C''')'\bar{S}'\bar{Q}^{-1}[B_1'\bar{X} - \bar{S}C'''] - (C''')'RC''' < 0.
\]

Write \( W = \bar{X}^{-1} > 0 \). By multiplying both sides of the above inequality from left and right by \( W \), and rearranging the resulting inequality we get

\[
W(A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1) + (A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)'W \\
-(\bar{B}E^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1') + WC_1D'\bar{R}DC_1W + Z\bar{E}^{-1}Z' < 0,
\]

where \( \bar{B} = B_2 - B_1\bar{Q}^{-1}\bar{S}D_{12}, \quad \bar{E} = D_1\bar{R}D_{12} > 0, \quad \bar{D} = I - D_{12}\bar{E}^{-1}D_1\bar{R} \) and \( Z = W(K'' + C_1'\bar{R}D_2\bar{E}^{-1})\bar{E} + \bar{B} \). Since \( \bar{E} > 0 \), the above inequality implies

\[
W(A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)' + (A - \bar{B}E^{-1}D_1\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1)'W \\
-(\bar{B}E^{-1}\bar{B}' - B_1\bar{Q}^{-1}B_1') + WC_1D'\bar{R}DC_1W < 0.
\]

(4.25)
Under the assumption $\text{rank}\{M(j\omega)\} = n+m$, where $M(j\omega)$ is given in (4.22), the relation

$$
\begin{bmatrix}
\dot{A} - j\omega I & B\bar{E}^{-\frac{1}{2}} \\
\bar{C} & D_{12}\bar{E}^{-\frac{1}{2}}
\end{bmatrix} = \begin{bmatrix}
I & -B_1\bar{Q}^{-1}\bar{S} \\
0 & I
\end{bmatrix} \begin{bmatrix}
A - j\omega & B_2 \\
C_1 & D_{12}
\end{bmatrix}
$$

where $\dot{A} = A - B\bar{E}^{-1}D_{12}\bar{R}C_1 - B_1\bar{Q}^{-1}\bar{S}C_1$, $\bar{C} = \bar{D}C_1$ implies that the pair $(\dot{A}, \bar{C})$ has no unobservable modes on the imaginary axis. This allows us to perform the following decomposition (using appropriate coordinate transformation)

$$
\dot{A} = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}, \quad \bar{C} = [C_{11} \ 0],
$$

where $A_{22}$ is stable and the pair $(-A_{11}, C_{11})$ is detectable (since $A_{11}$ contains either modes that are observable through $C_{11}$ or modes that are anti stable). In the new coordinates, write $\bar{B} = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$, $B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$ and $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{bmatrix}$. Substituting these matrices into the ARI (4.25), and taking the $(1,1)$ block inequality yields

$$
W_{11}A_{11}' + A_{11}W_{11} - B_{21}\bar{E}^{-1}B_{21}' + B_{11}\bar{Q}^{-1}B_{11}' + W_{11}C_{11}'\bar{R}C_{11}W_{11} < 0,
$$

that is there exists a positive definite matrix $P$ such that

$$
W_{11}A_{11}' + A_{11}W_{11} - B_{21}\bar{E}^{-1}B_{21}' + B_{11}\bar{Q}^{-1}B_{11}' + W_{11}C_{11}'\bar{R}C_{11}W_{11} + P = 0. \quad (4.26)
$$

Since $(-A_{11}, C_{11})$ is detectable, the comparison theorem of Riccati equations (Theorem 2.2 of [84]) implies there exists a matrix $W_{11E} \geq W_{11} > 0$ solving the ARE

$$
W_{11E}A_{11}' + A_{11}W_{11E} - B_{21}\bar{E}^{-1}B_{21}' + B_{11}\bar{Q}^{-1}B_{11}' + W_{11E}C_{11}'\bar{R}C_{11}W_{11E} = 0, \quad (4.27)
$$

with $W_{11E} \geq W_{11}$ such that all eigenvalues of $-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11})$ are in the closed LHP of the complex plane. We shall show that $-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11})$ is, in fact, asymptotically stable. Substracting (4.27) from (4.26) yields

$$
Z(-A_{11} + W_{11E}C_{11}'\bar{R}C_{11})' + (-A_{11} + W_{11E}C_{11}'\bar{R}C_{11})Z + P + ZC_{11}'\bar{R}C_{11}Z = 0,
$$

with $Z = W_{11E} - W_{11} \geq 0$, $P + ZC_{11}'\bar{R}C_{11}Z > 0$. By applying the standard (linear) Lyapunov theorem, we conclude that $-(A_{11} + W_{11E}C_{11}'\bar{R}C_{11})$ is asymptotically stable.
and \( W_{11E} > W_{11} \). Now, straightforward calculation shows that the matrix \( X_{11E} \geq 0 \) defined by

\[
X_E = \begin{bmatrix}
W_{11E}^{-1} & 0 \\
0 & 0
\end{bmatrix},
\]

satisfies

\[
X_E (A - \tilde{B} \tilde{E}^{-1} D_{12} \tilde{R} C_1 - B_1 \tilde{Q}^{-1} \tilde{S} C_1) + (A - \tilde{B} \tilde{E}^{-1} D_{12} \tilde{R} C_1 - B_1 \tilde{Q}^{-1} \tilde{S} C_1)' X_E \\
- X_E (\tilde{B} \tilde{E}^{-1} \tilde{B}' - B_1 \tilde{Q}^{-1} B_1') X_E + C_1' \tilde{R} \tilde{D} C_1
\]

\[
= A_{11}' W_{11E}^{-1} + W_{11E}^{-1} A_{11} - W_{11E}^{-1} (B_{21} \tilde{R}^{-1} B_{21}' - B_{11} \tilde{Q}^{-1} B_{11}') W_{11E}^{-1} + C_1' \tilde{R} C_1
\]

\[
= 0.
\]

Moreover, we have

\[
A^* = (A - \tilde{B} \tilde{E}^{-1} D_{12} \tilde{R} C_1 - B_1 \tilde{Q}^{-1} \tilde{S} C_1) - (\tilde{B} \tilde{E}^{-1} \tilde{B}' - B_1 \tilde{Q}^{-1} B_1') X_E
\]

\[
= \begin{bmatrix}
A_{11} - (B_{21} \tilde{E}^{-1} B_{21}' - B_{11} \tilde{Q}^{-1} B_{11}') W_{11E}^{-1} & 0 \\
A_{21} - (B_{22} \tilde{E}^{-1} B_{21}' - B_{12} \tilde{Q}^{-1} B_{11}') W_{11E}^{-1} & A_{22}
\end{bmatrix}.
\]

Since \( A_{22} \) is asymptotically stable and the matrix

\[
A_{11} - (B_{21} \tilde{E}^{-1} B_{21}' - B_{11} \tilde{Q}^{-1} B_{11}') W_{11E}^{-1} = W_{11E} (- (A_{11} + W_{11E} C_1' \tilde{R} C_1)) W_{11E}^{-1}
\]

is also asymptotically stable, the above expression shows that the matrix \( A^* \) is asymptotically stable. This proves that \( X_E \) is the required stabilizing solution to the ARE (4.23).

The converse results are as follows.

**Theorem 4.5** Consider the system in (4.20) and the quadratic supply rate \( r_q(z, w) \) in (4.11) satisfying (4.21). Suppose there exists a solution \( X \) to the ARE (4.23) such that \( X \geq 0 \) and the matrix \( A^* \) in (4.24) is asymptotically stable. Then, under the control law \( u^*(x) = K^* x \), where

\[
K^* = -\tilde{E}^{-1} ((B_2 - B_1 \tilde{Q}^{-1} \tilde{S} D_{12})' X + D_{12} \tilde{R} C_1),
\]

the closed loop matrix \( A u^* = A + B_2 K^* \) is asymptotically stable, and the closed loop system \( \Sigma_1 u^* \) in (4.20) is strictly dissipative with respect to the supply rate \( r_q(z, w) \) (4.11).
Proof. Write $A^{u^*} = A + B_2K^*$, $C^{u^*} = C_1 + D_{12}K^*$, where

$$K^* = -E^{-1}((B_2 - B_1\bar{Q}^{-1}\bar{S}D_{12})'X + D_{12}'\bar{R}C_1).$$

Then the ARE in (4.23) can be rewritten as

$$(A^{u^*})'X + XA^{u^*} + [XB_1 - (C^{u^*})'\bar{S}'\bar{Q}^{-1}[B_1'X - \bar{S}C^{u^*}]) - (C^{u^*})'RC^{u^*} = 0.$$}

Moreover, the matrix $A^{u^*} + B_1\bar{Q}^{-1}[B_1'X - \bar{S}C^{u^*}] = A^*$ is asymptotically stable. Therefore, by Theorem 3.5 the closed loop system $\Sigma^{u^*}$ is strictly dissipative with respect to $r_q(z, w)$ in (4.11) and the closed loop matrix $A^{u^*}$ is asymptotically stable. □

Remark 4.6 From the discussion in Remark 3.5 we conclude that the function $V(z) = \frac{1}{2}z'Xz$, where $X$ is stated in Theorem 4.5, is the available storage function for the closed loop system

$$\dot{x}(t) = (A + BK^*)x(t) + B_1w(t), \quad x(0) = x_0,$$

$$z(t) = (C_1 + D_{12}K^*)x(t) + D_{11}w(t),$$

in which $K^*$ is given in (4.28).

Remark 4.7 When the expression on the LHS of the PDI (4.4) is not strictly concave in $w$, then the extremum $w^*$, when they exist, may not be unique. This is an example of a singular control problem. The PDI may then break up into a set of equations. We illustrate this situation for linear systems case. Suppose that $A$, $B_1$, $B_2$, $C_1$ are constant matrices, and $D_{11} = 0$. The supply rate is quadratic as in (3.28). Assume that $S = I$, and $\bar{Q} = Q = 0$. Then the PDI (4.4) becomes

$$\inf_{u \in \mathbb{R}^m} \sup_{w \in \mathbb{R}^d} \{\nabla_z V(z)'(A + B_1w + B_2u) - \frac{1}{2}(2w'C_1z + z'C_1'R C_1x)\} \leq 0,$$

which is linear in $w$. Now assuming that $V(z) = \frac{1}{2}z'Xz$, and $u = Kz$, for some constant matrices $X \geq 0$, $K$, this PDI breaks down into two equations

$$B_1X = (C_1 + K), \quad X \geq 0,$$

$$\min_K \{X(A + B_2K) + (A + B_2K)'X - (C_1 + K)'R(C_1 + K)\} \leq 0.$$

As an example, suppose that $A$, $B_1$, $B_2$, $C_1$ are scalars, and assume that $A - B_2C_1 < 0$, $B_1(B_2 + 2B_1) > 0$. Then we have, after substituting $K = B_1X - C_1$ into the last inequality, the following optimization problem

$$\min_X \{2(A - B_2C_1)X + B_1(B_2 + 2B_1)X^2\} \leq 0, \quad X \geq 0,$$
which has the unique minimizer given by $X^* = -(A - B_2C_1)/(B_1(B_2 + 2B_1)) > 0$. The optimal feedback policy is then given by $u^* = K^*z$, where $K^* = B_1X^* - C_1$.

4.6 Locally Smooth Solutions

In the $H_\infty$ control case, it is known that the control problem for the nonlinear systems (4.1) is locally solvable if the problem for its linearized model is solvable [97], [98], [99]. In this section we present an extension of this result to the more general quadratic supply rate case. We consider the nonlinear systems $\Sigma$ in (4.1) and its linearization given by

$$\Sigma_{lin} : \begin{cases} 
\dot{x}(t) = A_1x(t) + B_{11}w(t) + B_{21}u(t), \quad x(0) = x_0, \\
\dot{z}(t) = C_{11}x(t) + D_{111}w(t) + D_{121}u(t), 
\end{cases}$$

where $A_1 = \frac{\partial A}{\partial x}(0)$, $C_{11} = \frac{\partial C_1}{\partial x}(0)$, $B_{11} = B_1(0)$, $B_{21} = B_2(0)$, $D_{111} = D_{11}(0)$ and $D_{121} = D_{12}(0)$. We denote $D'_{121}D_{121} = E_{11} > 0$. As in the previous section, we consider a quadratic supply rate

$$r_q(z, w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz),$$

where $Q$, $S$, $R$ are constant matrices, with $Q$, $R$ symmetric, satisfying $R \leq 0$. We assume that

$$\begin{align*}
\bar{Q}_1 &= Q + SD_{111} + D'_{111}S' + D'_{111}RD_{111} > 0, \\
\bar{R}_1 &= S_1\bar{Q}_1^{-1}S_1 - R > 0,
\end{align*}$$

where $S_1 = S + D'_{111}R$, and that

$$\begin{align*}
\bar{Q}(x) &= Q + SD_{11}(x) + D_{111}(x)'S' + D_{111}(x)'RD_{111}(x) > 0, \\
\bar{R}(x) &= S(x)'\bar{Q}^{-1}(x)S(x) - R > 0,
\end{align*}$$

for all $x$, where $S(x) = S + D_{11}(x)'R$.

We consider the manifold $T^*\mathbb{R}^n$ (dimension($T^*\mathbb{R}^n$) = $2n$), with coordinates $z_1$, $\ldots$, $z_n$, $p_1$, $\ldots$, $p_n$, endowed with a natural symplectic form $\tilde{\omega}$. In the next theorem we shall need the following Hamiltonian function $H(x, p)$ defined on $T^*\mathbb{R}^n$

$$H(x, p) = p'(A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)S_1C_1(x))$$

$$-\frac{1}{2}p'(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')p$$

$$+\frac{1}{2}C_1(x)'\bar{D}(x)'\bar{R}(x)\bar{D}(x)C_1(x)$$

(4.34)
where \( \bar{B}(x) = B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x) \), \( \bar{E}(x) = D_{12}(x)'\bar{R}(x)D_{12}(x) \) and \( \bar{D}(x) = I - D_{12}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x) \).

**Theorem 4.6** Consider the nonlinear system \( \Sigma \) (4.1), its linearization \( \Sigma_{\text{lin}} \) (4.29) and the supply rate (4.30) and assume that conditions (4.31) to (4.33) hold. Assume that
\[
\text{rank} \begin{bmatrix} A_1 - j\omega I & B_{21} \\ C_{11} & D_{121} \end{bmatrix} = n + m,
\]
for all real \( \omega \). Suppose there exists a linear feedback control law \( u^*(x) = K^*x \), such that the linearized system \( \Sigma_{\text{lin}}^{u^*} \) (4.29) is strictly dissipative with respect to the quadratic supply rate \( r_q(z, w) \) in (4.30), and the closed loop matrix \( A^{u^*} = A_1 + B_{21}K^* \) is asymptotically stable. Then there exists a smooth solution \( V \) to the PDE
\[
\nabla_x V(x)(A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x))
\]
\[
-\frac{1}{2} \nabla_x V(x)(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')\nabla_x V(x)'
\]
\[
\frac{1}{2} C_1(x)'\bar{D}(x)'\bar{R}(x)\bar{D}(x)C_1(x) = 0
\]
with \( V(x) \geq 0, \ V(0) = 0 \), on a neighbourhood \( W \) of \( x_0 = 0 \), such that the vector fields
\[
A^*(x) = A(x) - \bar{B}(x)\bar{E}^{-1}(x)D_{12}(x)'\bar{R}(x)C_1(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}C_1(x)
\]
\[
-\frac{1}{2}(\bar{B}(x)\bar{E}^{-1}(x)\bar{B}(x)' - B_1(x)\bar{Q}^{-1}(x)B_1(x)')\nabla_x V(x)'
\]
and
\[
A^{u^*}(x) = A(x) + B_2(x)u^*(x),
\]
where
\[
u^*(x) = -\bar{E}^{-1}(x)((B_2(x) - B_1(x)\bar{Q}^{-1}(x)\bar{S}(x)D_{12}(x))'\nabla_x V(x)' + D_{12}(x)\bar{R}(x)C_1(x)),
\]
are locally exponentially stable. Furthermore, the closed loop systems \( \Sigma^{u^*} \) (4.1) is locally dissipative with respect to the supply rate \( r_q(z, w) \) in (4.30) (i.e., the integral dissipation inequality holds for all \( T \geq 0, \) for all \( w \in L_2([0,T], \mathbb{R}^d) \) such that the corresponding trajectory \( x(\cdot) \) does not leave \( W \).

**Proof.** Under the hypothesis, Theorem 4.4 implies there exists a nonnegative matrix \( X \) solving the ARE
\[
X(A_1 - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{11l}\bar{Q}_l^{-1}\bar{S}_lC_{1l}) + (A_1 - \bar{B}_l\bar{E}_l^{-1}D_{12l}'\bar{R}_lC_{1l} - B_{11l}\bar{Q}_l^{-1}\bar{S}_lC_{1l})'X
\]
\[
-\bar{X}(\bar{B}_l\bar{E}_l^{-1}\bar{B}_l' - B_{11l}\bar{Q}_l^{-1}B_{11l})X + C_{1l}'\bar{D}_l'\bar{R}_l\bar{D}_lC_{1l} = 0.
\]
such that the matrix

\[ A^* = (A_l - \tilde{B}_l \tilde{E}_l^{-1} D_{12l} \tilde{R}_l C_{1l} - B_{1l} Q_l^{-1} S_l C_{1l}) - X(\tilde{B}_l \tilde{E}_l^{-1} \tilde{B}_l - B_{1l} Q_l^{-1} B_{1l}' ) \]  

(4.41)

and

\[ A^* = (A_l - B_{2l} D_{12l} \tilde{R}_l C_{1l}) - B_{2l} \tilde{E}_l^{-1} [B_{2l} - B_{1l} Q_l^{-1} S_l D_{12l}]' \]  

(4.42)

are asymptotically stable, in which expressions

\[
\begin{align*}
\tilde{B}_l = B_{2l} - B_{1l} Q_l^{-1} S_l D_{12l}, & \quad \tilde{E}_l = D_{12l} \tilde{R}_l D_{12l} \quad \text{and} \quad \tilde{D}_l = I - D_{12l} \tilde{E}_l^{-1} D_{12l} \tilde{R}_l.
\end{align*}
\]

Define the Hamiltonian matrix

\[ \text{Ham} = \left[ \begin{array}{c}
(A_l - \tilde{B}_l \tilde{E}_l^{-1} D_{12l} \tilde{R}_l C_{1l} - B_{1l} Q_l^{-1} S_l C_{1l}) \tilde{B}_l \tilde{E}_l^{-1} \tilde{B}_l'

C_{1l} \tilde{D}_l \tilde{R}_l \tilde{D}_l C_{1l} - (A_l - \tilde{B}_l \tilde{E}_l^{-1} D_{12l} \tilde{R}_l C_{1l} - B_{1l} Q_l^{-1} S_l C_{1l})'
\end{array} \right].
\]

Direct calculation yields

\[
\begin{bmatrix}
I & 0 \\
X & I
\end{bmatrix}
\text{Ham}
\begin{bmatrix}
I & 0 \\
-X & I
\end{bmatrix}
= \begin{bmatrix}
A^* & \tilde{B}_l \tilde{E}_l^{-1} \tilde{B}_l' \\
0 & -(A^*)'
\end{bmatrix}.
\]

Therefore, the eigenvalues of \( \text{Ham} \) are symmetric with respect to the imaginary axis. Since the matrix \( A^* \), with dimension \( n \), is asymptotically stable, we conclude \( \text{Ham} \) has no purely imaginary eigenvalues. Moreover, the relation

\[ \text{Ham} \begin{bmatrix}
I \\
X
\end{bmatrix} = \begin{bmatrix}
I \\
X
\end{bmatrix} A^* \]

implies that

\[ \text{span}(\begin{bmatrix}
I \\
X
\end{bmatrix}) \text{ is stable eigenspace of } \text{Ham}. \]

(4.43)

Next, consider the Hamiltonian vector field \( X_H \) on \( T^* \mathbb{R}^n \), associated with the Hamiltonian function \( H(x,p) \) in (4.34), defined by

\[ X_H(x,p) = \begin{bmatrix}
\frac{\partial H'}{\partial p} \\
-\frac{\partial H'}{\partial x}
\end{bmatrix}(x,p). \]

Note that \( (0,0) \) is an equilibrium point of \( X_H \). By direct calculation, we get the linearization matrix of \( X_H \) around \((0,0)\) given by

\[
DX_H(0,0) = \begin{bmatrix}
\frac{\partial^2 H}{\partial x \partial p} & \frac{\partial^2 H}{\partial p^2} \\
-\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p \partial z}
\end{bmatrix}
= \text{Ham}.
\]
Thus, the linearization of $X_H$ around $(0,0)$ equals $Ham$. Since $Ham$ has no pure imaginary eigenvalues, $X_H$ is hyperbolic in $(0,0)$. Proposition A.4 of [98] says there exists a stable (invariant) manifold $N^-$ of $X_H$, with dimension $n$, passing through $(0,0)$, such that the restriction of $X_H$ on $N^-$ is asymptotically stable (with regard to $(0,0)$). Moreover, $N^-$ is tangent at $(0,0)$ to the stable eigen space of $DX_H(0,0)$, i.e. (using (4.43))

$$T_{(0,0)}N^- = \text{stable eigenspace of } Ham$$

(4.44)

By Proposition A.5 of [98], $N^-$ is also a Lagrangian submanifold of $T^*R^n$ (i.e., $N^-$ has dimension equals $n$ and the restriction of the symplectic form $\omega$ on $N^-$ equals 0). From (4.44), we see that $N^-$ may be parametrized by the $z$ coordinate locally around $(0,0)$. By Proposition A.6 of [98], there exists a smooth function $V$, with $V(0) = 0$, $\nabla_z V(0) = 0$, defined on a neighborhood $W^0$ of $z = 0$ such that for all $z \in W^0$, $N^-$ is given by

$$N^- = \{(z,p) : p = \partial V \over \partial z, \ z \in W^0\}.$$ 

Now we shall show that $V$ solves the PDE (4.36) locally. Note that since the Lie derivative $L_{X_H} H = {\partial H \over \partial z} \cdot {\partial H \over \partial p} + {\partial H \over \partial p} \cdot (-{\partial H \over \partial z}) = 0$, $H(z,p)$ is constant along any integral trajectory of $X_H$. Since the restriction of $X_H$ on $N^-$ is asymptotically stable, $X_H$ produces a trajectory that goes to $(0,0)$ as $t \to \infty$ for any initial condition $(x_0,p_0)$ in $N^-$. In particular, along any trajectory which starts from $(x_0, \nabla_z V(x_0))$, $x_0 \in W^0$, we have

$$H(x_0, \nabla_z V(x_0)) = H(z(t), p(t))$$

$$= \lim_{t \to \infty} H(z(t), p(t))$$

$$= H(\lim_{t \to \infty} z(t), \lim_{t \to \infty} p(t)) \text{ (by continuity of } H)$$

(4.45)

$$= H(0, 0)$$

$$= 0.$$ 

Thus $H(z, \nabla_z V(z)) = 0$ on $W^0$. This shows that $V$ solves the PDE (4.36) on $W^0$. Next, note that the equation (4.45) together with (4.44) implies that

$$\frac{\partial^2 V}{\partial z^2} (0) = X,$$

where the matrix $X$ is the solution to the ARE (4.40) such that the matrices $A^r_\tau$ and $A^r_\tau^*$ in (4.41) and (4.42) are asymptotically stable. However, these matrices are the linearization
of the vector fields $A^*(x)$ and $A^{u*}(x)$ in (4.37) and (4.38) respectively, around $x = 0$, i.e., $A_i^* = \frac{\partial A^*_i}{\partial x}(0)$, $A_{i^{u*}} = \frac{\partial A^{u*}}{\partial x}(0)$. Therefore, we conclude that $A^*(x)$ and $A^{u*}(x)$ are locally exponentially stable. To see that $V \geq 0$ locally, rewrite the PDE (4.36) as

$$
\begin{align*}
\nabla_x V(x)A^{u*}(x) + \frac{1}{2}[\nabla_x V(x)B_1(x) - C_u(x)\tilde{S}(x)]Q^{-1}(x) & \\
[B_1(x)\nabla_x V(x)' - \tilde{S}(x)Cu^*(x)] - Cu^*(x)'RCu^*(x) = 0,
\end{align*}
$$

with $Cu^*(x) = C_1(x) + D_{12}(x)u^*(x)$. Since $\tilde{Q}(x) > 0$ for all $x$, and $R \leq 0$, we have $\nabla_x V(x)A^{u*}(x) \leq 0$. Integrating this inequality along the trajectory produced by $A^{u*}$, with $z(0) = z_0$, yields

$$
V(x(t)) - V(z_0) \leq 0,
$$

provided $z(\cdot)$ stays in $W^0$. Since $A^{u*}(x)$ is locally exponentially stable, choosing $z_0$ in the domain of attraction for $A^{u*}$ results in a trajectory $x(t)$ that goes to 0 as $t \to \infty$.

Thus

$$
0 = V(0) = V(\lim_{t \to \infty} x(t)) = \lim_{t \to \infty} V(x(t)) \leq V(z_0).
$$

Finally, we shall show that under the policy $u^*(x)$ in (4.39), the closed loop system $\Sigma^{u*}$ (4.1) is locally dissipative with respect to the supply rate $r_q(z, w)$. Let $W$ denote the domain on which the function $V$ is defined and is nonnegative. By rewriting the PDE (4.36) as in (4.46) we see that under the policy $u^*(x)$ the following inequality holds

$$
-\int_0^T r(z(s), w(s))ds
$$

$$
= -\frac{1}{2}\int_0^T [(w(s) - w^*(s))']\tilde{Q}[w(s) - w^*(s)] + 2dV/ds)ds
$$

$$
= -\frac{1}{2}\int_0^T [w(s) - w^*(s)]'\tilde{Q}[w(s) - w^*(s)]ds + V(x(0)) - V(x(T))
$$

$$
\leq -\frac{1}{2}\int_0^T [w(s) - w^*(s)]'\tilde{Q}[w(s) - w^*(s)]ds + V(x(0)),
$$

where $w^*(s) = w^*(x(s)) = \tilde{Q}^{-1}(x)[B_1(x)'\nabla_x V(x)' - \tilde{S}(x)Cu^*(x)]$, for all $x(0) \in W$, and for all $T \geq 0$, $w \in L_2([0, T])$ such that $x(\cdot)$ does not leave $W$. This shows the (local) dissipativity of $\Sigma^{u*}$ with respect to the supply rate $r_q(z, w)$ in (4.30), and completes the proof. □
4.7 Conclusions

In this chapter we have formulated and solved a general dissipativity control synthesis problem for nonlinear and linear systems. In either nonlinear case or linear case, the dissipativity performance measure includes the finite gain (or $H_\infty$) measure, passivity (or positive real) measure and a mixture of them as special cases.

We have expressed the solution to the problem in terms of a PDI (ARI) in the case of nonlinear (linear) systems. The PDI (ARI) is the relevant (controlled) dissipation inequality for the state feedback control at hand. We interpret the PDI using the viscosity solution notion. We have also illustrated an extension of our approach to solve a singular control problem.

Finally, we have also shown that a local smooth solution to the PDI exists whenever the control problem for the linearized system admits a solution via a linear (state feedback) controller. This generalizes those results in [97], [98], [99].
Chapter 5

Output Feedback Synthesis: General Case

5.1 Introduction

In this chapter we study a general dissipativity control synthesis problem for nonlinear systems with output feedback. We consider a general dissipative performance measure which includes finite gain, passivity, a mixture between finite gain and passivity performance measures as a few special cases. We formulate the problem as an output feedback dynamic game in which a cost function corresponding to the dissipativity performance measure is considered. In this game the controller seeks to minimize the cost whereas disturbance signals act as the opponents which attempt to maximize it. We employ the information method which is developed in the papers [54], [55], [56], [57] and is studied further in [42]. Key to this method is the notion of state called the information state. This quantity is a causal function of the output and its evolution is governed by a nonlinear first order partial differential equation, which takes the form of an extension of the Mortensen equation arising in the deterministic estimation [43] or of the Zakai equation arising in the stochastic filtering [116], [22]. It turns out that the cost function can be re-expressed purely in terms of the information state and this allows us to view the original partial state dynamic game problem as a new, but equivalent full state one in which the information state replaces the original state. We solve the new full state problem using dynamic programming methods which leads to an infinite dimensional HJI PDE. Thus, the resulting controller dynamic has the information state as its state governed
by a PDE. The optimal control law is constructed by solving the HJI equation for each point in the information state space yielding an information state feedback control.

In the finite gain synthesis case, the authors in [6], [25], [66] propose a method for solving output feedback control called the certainty equivalence principle (CEP). Under certain assumptions, this method results in an optimal minimax performance. In this chapter we examine the extension of this principle to a general dissipativity case.

In the infinite time case, the solution to the control problem is expressed in terms of a (infinite dimensional) HJI PDI which is the appropriate dissipation inequality and the control law is time invariant. We deduce some sort of internal stability under closed loop detectability and reachability assumptions following the technique in [56]. We also present a formula for a closed loop storage function, extending the result in [42].

We maintain the technicality involved in this chapter at a minimum. Specifically, we shall assume that solutions to the relevant PDE's are smooth. A complete theory concerning regularity of the first PDE, which is the dynamics of the information state, is available in the literature (see for example [20], [35]). On the other hand, the second (infinite dimensional) PDE is new and rigorous treatment concerning it is still under development. Preliminary mathematical results are obtained in [57] in which a definition of viscosity solutions to the PDE is provided. The results in [57] are presented for the finite gain control synthesis case. However, they apply to more general dissipativity cases. In the Appendix B we shall consider a model problem which bridges the results in this chapter to the mathematical results in [57].

5.2 Finite Time Problem

We consider a class of nonlinear systems described by the equation

\[
\begin{align*}
\dot{x}(t) &= A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad x(0) = x_0, \\
z(t) &= C_1(x(t)) + D_{11}(x(t))w(t) + D_{12}(x(t))u(t), \\
y(t) &= C_2(x(t)) + D_{21}(x(t))w(t) + D_{22}(x(t))u(t).
\end{align*}
\] (5.1)

In this description, \( x \in \mathbb{R}^n \) denotes the state vector, which is partially observed through a measured output quantity \( y \in \mathbb{R}^p \). The initial condition \( x_0 \) is assumed to be unknown. The vector \( z \in \mathbb{R}^q \) represents the quantity to be controlled. The disturbance \( w \in \mathbb{R}^d \)
5.2 Finite Time Problem

corrupts the state and the output quantities and the vector \( u \in U \subset \mathbb{R}^m \) denotes the control. We assume that the maps \( A : \mathbb{R}^n \to \mathbb{R}^n, B_1 : \mathbb{R}^n \to \mathbb{R}^{n \times d}, B_2 : \mathbb{R}^n \to \mathbb{R}^{n \times m}, C_1 : \mathbb{R}^n \to \mathbb{R}^q, C_2 : \mathbb{R}^n \to \mathbb{R}^p \) are smooth and globally Lipschitz continuous and \( D_{11} : \mathbb{R}^n \to \mathbb{R}^{q \times d}, D_{12} : \mathbb{R}^n \to \mathbb{R}^{q \times m}, D_{21} : \mathbb{R}^n \to \mathbb{R}^{p \times d} \) and \( D_{22} : \mathbb{R}^n \to \mathbb{R}^{p \times m} \) are smooth and bounded with bounded first derivatives, and that \( A(0) = 0, C_1(0) = 0, C_2(0) = 0 \). For simplicity we assume \( D_{12} = I, D_{21} = I \). This assumption corresponds to the one-block problem in the linear \( H_\infty \) control case.

We consider the admissible control strategies as the set of causal maps of the observation

\[
u : L_2([0,T], \mathbb{R}^p) \to L_2([0,T], \mathbb{R}^m)
\]
such that, when applied to (5.1) they result in unique solutions to (5.1) for \( t \geq 0 \). The causality means that for any \( 0 \leq t_0 \leq s \leq T \), if \( y_1, y_2 \in L_2([t_0,T], \mathbb{R}^p) \) and \( y_1(r) = y_2(r) \) a.e. \( r \in [t_0,s] \), then \( u[y_1](r) = u[y_2](r) \) a.e. \( r \in [t_0,s] \). The control signal is obtained via \( u(r) = u[y](r), r \in [t_0,T] \). We denote the class of such strategies by \( U \). In the sequel, we use the notation \( f[y] \) to mean that \( f \) is a causal function of \( y \). Admissible disturbances are all signals \( w \in L_2([0,T], \mathbb{R}^d) \).

We consider the general supply rate \( r(z,w) \)

\[
r : \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R}, \tag{5.2}
\]

which is assumed to be \( C^1(\mathbb{R}^q, \mathbb{R}^d) \), with at most quadratic growth, i.e. \( |r(z,w)| \leq \alpha(1 + |z|^2 + |w|^2) \) for some \( \alpha > 0 \), for all \( z \in \mathbb{R}^q, w \in \mathbb{R}^d \). Throughout the chapter, we assume that

\[
r(z,0) \leq 0, \tag{5.3}
\]

for all \( z \in \mathbb{R}^q \), with \( r(0,0) = 0 \). Let \( \Sigma_{x_0}^u \) denote the map from \( w \) to \( z \) under the control policy \( u \) and initial condition \( x_0 \). Given a fixed finite time interval \([0,T]\), the finite time dissipative output feedback control problem is to find \( u \in U \) such that for any initial condition \( x_0 \in \mathbb{R}^n \) the map \( \Sigma_{x_0}^u \) is dissipative with respect to the supply rate \( r \) in (5.2), which means there exists a finite quantity \( \beta_M^u(x) \geq 0 \) such that

\[
\left\{ \begin{array}{l}
-\int_0^T r(z(t),w(t))dt \leq \beta_M^u(x_0),

\forall w \in L_2([0,T], \mathbb{R}^d).
\end{array} \right. \tag{5.4}
\]

We assume that \( \beta_M^u(0) = 0 \).
We use game-theoretic methods to solve this problem. For \( u \in U \) define the cost functional related to the dissipative control problem as follows

\[
J_{p,T}(u) \triangleq \sup_{w \in L^2([0,T],\mathbb{R}^d), x_0 \in \mathbb{R}^n} \{ p(x_0) - \int_0^T r(z(t), w(t)) \, dt \},
\]

where \( p \) lives in a function space \( \mathcal{X} \). Clearly, the map \( \Sigma^u \) is dissipative if and only if

\[
J_{\alpha,T}(u) \leq 0,
\]

for some \( \alpha(z) = -\beta(z) \leq 0 \), with \( \alpha(0) = 0 \).

In what follows we make use of the "sup-pairing" ([54], [56], [57]):

\[
(p, q) = \sup_{x \in \mathbb{R}^n} \{ p(x) + q(x) \}
\]

for \( p, q \in \mathcal{X} \). If \( \Sigma^u \) is dissipative, then for \( p \in \mathcal{X} \), we have

\[
p(x_0) - \int_0^T r(z(t), w(t)) \, dt \leq p(x_0) + \beta^u_T(x_0) \leq (p, \beta^u_T),
\]

for all \( w \in L^2([0,T],\mathbb{R}^d) \). This implies

\[
J_{p,T}(u) \leq (p, \beta^u_T).
\]

Also, it is immediate that, using (5.3), we have, for any \( u \in U \),

\[
J_{p,T}(u) \geq \sup_{x_0 \in \mathbb{R}^n} \{ p(x_0) - \int_0^T r(z(t), 0) \, dt \} \geq (p, 0).
\]

Thus, we have the relation

\[
\{ p \in \mathcal{X} : (p, 0), (p, \beta^u_T) \ \text{finite} \} \subset \text{dom} J_{p,T}(u),
\]

in which \( \text{dom} J_{p,T}(u) \) is the set of \( p \in \mathcal{X} \) on which \( J_{p,T}(u) \) is finite. The idea is that a solution to the dissipative control problem will be obtained by minimizing \( J_{p,T}(u) \) over \( U \). This is a zero-sum dynamic game problem with partial observation. In this game, the initial condition is assumed unknown and is considered as a part of the disturbance.

### 5.2.1 Information State Solution

In this section we solve the partially observed game problem associated with the dissipative control problem by following the information state method developed in [56], [57].

For fixed output \( y \in L^2([0,T],\mathbb{R}^p) \) and control signal \( u \in L^2([0,T],\mathbb{R}^m) \), we define the information state \( p_t(x) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R} \) by

\[
p_t(x) \triangleq \alpha(x_0) - \int_0^t r(z(s), y(s) - C_2(\xi(s)) - D_{22}(\xi(s))u(s)) \, ds,
\]

(5.8)
5.2 Finite Time Problem

in which \( \xi(\cdot) \) is the solution of

\[
\dot{\xi}(s) = A(\xi(s)) + B_1(\xi(s))(y(s) - C_2(\xi(s)) - D_{22}(\xi(s))u(s)) + B_2(\xi(s))u(s), \quad 0 \leq s \leq t,
\]

(5.9)

with \( \xi(t) = z \). This quantity describes the maximum or worst possible control cost (over the disturbance space) which is conditioned on the past measurement \( y(s) \), \( 0 \leq s \leq t \) and the constraint \( z(t) = z \). By the definition in (5.8), \( p_t \) is a causal function of \( y \).

As we shall see the information state actually summarizes information available in the measurement in a manner which is suitable for the control task (see also [56], [57]).

We observe that any \( w \in L_2([0, T), \mathbb{R}^d) \) generates an output \( y \in L_2([0, T], \mathbb{R}^p) \) through the dynamics (5.1) and that, conversely, any given \( y \in L_2([0, T], \mathbb{R}^p) \) generates a disturbance \( w \in L_2([0, T], \mathbb{R}^d) \) through the dynamics (5.9) by setting \( w(t) = y(t) - D_{22}(z(t))u(t) - C_2(z(t)), 0 \leq t \leq T \). Thus, there exists a natural bijection between \( L_2([0, T], \mathbb{R}^d) \) and \( L_2([0, T], \mathbb{R}^p) \).

By applying dynamic programming methods, we see that the dynamics of \( p_t \) is given by

\[
\begin{cases}
\dot{p}_t = F(p_t, u(t), y(t)), \\
p_0 = \alpha,
\end{cases}
\]

(5.10)
in which

\[
F(p, u, y) = -\nabla_x p(A + B_1(y - D_{22}u - C_2) + B_2u) - r((C_1 + D_{11}(y - C_2 - D_{22}u) + u), (y - C_2 - D_{22}u)).
\]

(5.11)
The dynamics (5.10) takes an extended form of the Mortensen equation arising in the deterministic estimation case [43]. In the control problem at hand, we shall regard the DPE (5.10) as a new (infinite dimensional) system to be controlled, with the state variable \( p \), control input \( u \) and disturbance \( y \).

The sense in which equation (5.10) is to be understood depends on the smoothness of \( \alpha \) and on the regularity of \( u(\cdot) \) and \( y(\cdot) \). Detailed discussion regarding this matter is provided in Appendix B.

The following representation result is an extension of the one obtained in [56], [57] in the case of finite gain control.
Theorem 5.1  For any $u \in U$ such that $J_{\alpha,T}(u)$ is finite, we have

$$J_{\alpha,T}(u) = \sup_{y \in L_2([0,T],\mathbb{R}^p)} \{ (p_T,0) : p_0 = \alpha \}$$

(5.12)

Proof. By using equation (5.5) and recalling the bijection from $L_2([0,T],\mathbb{R}^d)$ to $L_2([0,T],\mathbb{R}^p)$ induced by equations (5.1) and (5.9), it then follows that

$$J_{\alpha,T}(u) = \sup_{w \in L_2([0,T],\mathbb{R}^d), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) - \int_0^T r(z(t), w(t)) dt \}
= \sup_{y \in L_2([0,T],\mathbb{R}^p), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) - \int_0^T r(z_y(t), w_y(t)) dt \}
= \sup_{y \in L_2([0,T],\mathbb{R}^p), x \in \mathbb{R}^n} \{ \alpha(z_0) - \int_0^T r(z_y(t), w_y(t)) dt : x(T) = x \}
= \sup_{y \in L_2([0,T],\mathbb{R}^p), x \in \mathbb{R}^n} \{ p_T(x) \}
= \sup_{y \in L_2([0,T],\mathbb{R}^p)} \{ (p_T,0) : p_0 = \alpha \},$$

in which expression $z_y = C_1(x) + D_{11}(x)(y - C_2(x) - D_{22}(x)u) + u, \ w_y = y - C_2(x) - D_{22}(x)u$. This completes the proof. \(\Box\)

The expression in the left hand side of (5.12) involves the original state $x$ with dynamics given by (5.1) which is only partially observed (through $y$). On the other hand, the right hand side expression involves the new state $p$ with dynamics given in (5.10). The information state $p_t$ is completely known (by solving equation (5.10)). By Theorem 5.1 we may regard (5.10) as our new dynamical system with inputs $u$ and $y$ and seek a control law $u = K(p[y], y)$ that minimizes the right hand side of (5.12). This control law will then solve our original partial observation dynamic game problem. With regard to the dynamics (5.11), the control law is a full information one since it observes both the state $p$ and the input $y$. We note that the new cost function takes the Mayer-form [34].

Note that if the map $\Sigma^u$ is dissipative, then for $p \in \mathcal{X}$, we have

$$p_t(x) \leq (p_t, 0) \leq \sup_{y \in L_2([0,t],\mathbb{R}^p)} \{ (p_t,0) : p_0 = p \}
\leq J_{p,T} \leq (p, \beta^u_T),$$

for all $0 \leq t \leq T$. Thus, $p_t$ is bounded from above.
5.2 Finite Time Problem

We employ dynamic programming methods to solve the complete observation problem. To this end, define the value function $W$ for $(p,t) \in \mathcal{X} \times [0,T]$ by

$$W(p,t;T) \triangleq \inf_{u \in U} \sup_{y \in L^2([0,T], \mathbb{R}^p)} \{(p_T,0) : p_t = p\}.$$  \hfill (5.13)

Suppose that under the control $u^o$ the map $\Sigma^{u^o}$ is dissipative. Then,

$$W(p,0;T) \leq J_{p,T}(u^o) \leq (p, \beta^o_T).$$

Moreover, since $J_{p,T}(u) \geq (p,0)$, for all $u \in U$, we have

$$W(p,0;T) \geq (p,0).$$

Using these estimates, we conclude that

$$\text{dom} J_{.,T}(u^o) \subset \text{dom} W(.,0;T),$$

where $\text{dom} W(.,0;T)$ is a set in $\mathcal{X}$ on which $W(.,0;T)$ is finite. For $p = \beta_T$, we have

$$0 = \langle - \beta_T, 0 \rangle = W(- \beta_T, 0;T) \leq (p, \beta_T) = 0,$$

which implies $W(- \beta_T, 0;T) = 0$. Thus, $- \beta_T \in \text{dom} W(.,0;T)$. By a standard calculation employing dynamic programming methods, the value function $W$ satisfies the following equation (see [57], [26], [29])

$$W(p,t;T) = \inf_{u \in U} \sup_{y \in L^2([t,r], \mathbb{R}^p)} \{(p_T,0) : p_t = p\},$$

for all $t \leq r \leq T$. Passing $r$ to $t$ leads us, formally, to the following infinite dimensional HJI dynamic programming equation ([56], [57]).

$$\frac{\partial W}{\partial t} + \sup_{y \in \mathbb{R}^p} \inf_{u \in U} \{(\nabla_p W, F(p,u,y))\} = 0 \text{ in } \mathcal{X} \times [0,T], \hfill (5.14)$$

$$W(p,T;T) = (p,0) \text{ in } \mathcal{X}.$$

In this equation, $\nabla_p W \in \mathcal{X}^*$ ($\mathcal{X}^*$ is the dual space of $\mathcal{X}$) denotes the Frechet gradient of the value function $W$ with respect to $p$. The expression $\langle \nabla_p W, F(p,u,y) \rangle$ denotes the Frechet directional derivative of $W$ in the direction $F(p,\cdot,\cdot)$ evaluated at $p \in \mathcal{X}$.

Since, in the expression (5.11) for $F(p,u,y)$, the variables $u$ and $y$ are coupled, in general the inf and sup operations in the left hand side of (5.14) do not commute, i.e., the Isaacs condition does not hold. This is a new feature of the general dissipative control
problem. The DPE (5.14) is the Hamilton-Jacobi-Isaacs (HJI) equation for the partially observed game associated with the problem.

**Remark 5.1** In [56] in the case of a finite gain \((H_\infty)\) control problem for discrete time nonlinear systems, it is shown that necessary as well as sufficient conditions for the solvability of the problem can be expressed in terms a discrete time, infinite dimensional dynamic programming equation which is the discrete time analog of (5.14). In the continuous time case, the difficulty lies in the rigorous interpretation of equation (5.14) (but see Appendix B, or [57]).

### 5.2.2 Verification

The optimal control law can be determined by employing the DPE (5.14). Suppose there exists smooth (i.e., Frechet differentiable) solutions \(p\) and \(W\) to the DPE's (5.10) and (5.14) such that \(p_0 = -\beta \in \text{dom}W(\cdot, 0; T), W(-\beta, 0; T) = 0\) for some \(\beta \geq 0\) with \(\beta(0) = 0\) and the trajectory \(p_t\) corresponding to \(p_0 = -\beta\) satisfies \(p_t \in \text{dom}W(\cdot, t; T)\) with \(W(p_t, t; T) \geq (p_t, 0)\) for all \(0 \leq t \leq T\). Suppose the control law

\[
u^*[y](t) = u^*[p[y], y(t), t], \quad 0 \leq t \leq T
\]

attains the minimum in the left hand side of the HJI equation (5.14), i.e.,

\[
u^*[p[y], y(t), t] \in \operatorname{argmin}_{u \in U} \{\langle \nabla_p W(p, t), F(p, u, y(t))\rangle\}
\]

for each \(p_t \in \text{dom}W(\cdot, t; T)\). Then, by applying \(u^*[p[y], y(t), t]\) on (5.10) and integrating equation (5.14) along the trajectory produced by (5.10) which starts from \(p_0 = -\beta\), we have, for any \(y \in L_2([0, T], \mathbb{R}^p)\),

\[
(p_T, 0) - W(-\beta, 0; T)
\]

\[
= \int_0^T \frac{\partial W}{\partial s}(p_s, s) + \{\langle \nabla_p W(p_s, s; T), F(p_s, u^*(p[y], y(s), s), y(s))\rangle\} ds
\]

\[
\leq \int_0^T \frac{\partial W}{\partial s}(p_s, s) + \sup_{y \in \mathbb{R}^p} \{\langle \nabla_p W(p_s, s; T), F(p_s, u^*(p[y], y, s), y)\rangle\} ds
\]

\[
\leq 0.
\]

Since \(W(-\beta, 0; T) = 0\), we have, for any \(y \in L_2([0, T], \mathbb{R}^p)\),

\[
(p_T, 0) \leq 0.
\]
Due to the infinite nature of the equation (5.14), the computational requirement is high. In [6], [25] and [66], a certainty equivalence principle (CEP) has been proposed to solve an output feedback dynamic game problem. This method, when it is valid, is
computationally less intensive because it replaces the infinite dimensional (5.14) with a related state feedback DPE (which is finite dimensional). In the bilinear systems case [94], and in the general nonlinear systems case [57], both are for the finite gain or $H_{\infty}$ case, the conditions for the validity of the CEP were examined using the information state solution framework. The technique employed in in [57] for general nonlinear systems involves the Gateaux derivative notion. We shall now examine the certainty equivalence for general dissipativity for systems in (5.1) using more elementary techniques employed in [6], [25].

We consider the following value function corresponding to the dissipative control problem with state feedback, i.e. the problem in which the controller has the form $u(t) = K_t(x(t)) \in S$, where $K_t(x)$ is a static (or memoryless) function of $x$ (see Chapter 4),

$$V(x,t) = \inf_{K \in S} \sup_{w \in L_2([t,T],\mathbb{R}^d)} \left\{ -\frac{1}{2} \int_t^T r(h_1(x(s),K(x(s)),w(s)),w(s))ds + \hat{V}(z(T)) : z(t) = x \right\}$$

(5.17)

The corresponding DPE is given by

$$\frac{\partial V}{\partial t} + \inf_{u \in \mathbb{R}^n} \sup_{w \in \mathbb{R}^d} \left\{ \nabla_z V(z)(A(z) + B_1(z)w + B_2(z)u) - r(h_1(x,u,w),w) \right\} = 0,$$

(5.18)

with $V(x,T) = \hat{V}(x)$, where $h_1(x,u,w) = C_1(x) + D_{11}(x)w + u$. In equation (5.18), we use the order of operations inf sup, instead of sup inf, because we deal with static state feedback control functions which are independent of $w$ (see Chapter 4). We assume that $V$ is smooth and there exists a policy $K^*(x)$ which achieves the minimum in the left hand side of the DPE (5.18), i.e. with $K = K^*$ we have

$$\frac{\partial V}{\partial t} + \left\{ \nabla_z V(z)(A(z) + B_1(z)w + B_2(z)K^*(z)) - r(h_1(x,K^*(x),w),w) \right\}$$

$$\leq \frac{\partial V}{\partial t} + \sup_{w \in \mathbb{R}^d} \left\{ \nabla_z V(z) \cdot (A(z) + B_1(z)w + B_2(z)K^*(z)) - r(h_1(x,K^*(x),w),w) \right\}$$

$$= 0.$$

For fixed observation path $y_{0,t}$, we define the maximum stress function $G_t(y)$ by

$$G_t(y) \overset{\Delta}{=} \max_{z \in \mathbb{R}^n} \{ p_t(z) + V(z,t) \},$$

where $p_t(z)$ is the information state given in (5.8), and the maximum stress estimate $\tilde{x}_t$ by

$$\tilde{x}_t \in \text{argmax}_{z \in \mathbb{R}^n} \{ p_t(z) + V(z,t) \}.$$
5.2 Finite Time Problem

Both $G_t$ and $\tilde{e}_t$ depend on the observation $y$ through $p_t$. The certainty equivalence policy is defined by

$$u^{CE}(t) \triangleq u^{CE}[y](t) = K^*(\tilde{e}_t).$$

(5.19)

**Lemma 5.1** Assume the following conditions:

(i) there exists a smooth function $V(x, t)$ solving the DPE (5.18) with the policy $K^*_I(x)$ attaining the minimum in the DPE,

(ii) for all $y$ and $t \in [0, T]$, the maximum stress estimate $\tilde{e}_t$ is unique (i.e. the sets $\arg\max_{z \in \mathbb{R}^n} \{p_t(z) + p_t(x(t))\}$ are singleton).

Then

$$J_{\alpha, T}(u^{CE}) \leq \inf_{u \in \mathcal{U}} \sup_{w \in L^2([0, T], \mathbb{R}^n)} \{\alpha(x_0) - \int_0^T \{h_1(x(t), u(t), w(t), w(t))\}dt\},$$

(5.20)

where the cost $J_{\alpha, T}(u)$ is defined in (5.5).

**Proof.** Under the hypothesis we have the expression, for fixed $y_0, t$,

$$G_t(y) = \alpha(\xi_0) - \int_0^t \{h_1(x(s), y(s) - C_2(\xi(s)) - D_{22}(\xi(s))u(s))ds + V(\tilde{e}_t, t),$$

in which $\xi(\cdot)$ is the solution of

$$\dot{\xi}(s) = A(\xi(s)) + B_1(\xi(s))(y(s) - C_2(\xi(s)) - D_{22}(\xi(s))u(s)) + B_2(\xi(s))u(s), \quad 0 \leq s \leq t,$

with $\xi(t) = \tilde{e}_t$. By Danskin's theorem (see the Appendix in [6]), the directional derivative of $G_t$ is given by

$$dG_t/dt = \frac{\partial V}{\partial t}(\tilde{e}_t, t) - \nabla_xp(A(\tilde{e}_t) + B_1(\tilde{e}_t)(y - D_{22}(\tilde{e}_t)u - C_2(\tilde{e}_t)) + B_2(\tilde{e}_t)u$$

$$- \{h_1(\tilde{e}_t, u, y - C_2(\tilde{e}_t) - D_{22}(\tilde{e}_t)u), (y - C_2(\tilde{e}_t) - D_{22}(\tilde{e}_t)u)\}.$$

Also, since $\tilde{e}_t$ is the unique maximizer, we have

$$\nabla_x V(\tilde{e}_t, t) + \nabla_x p_t(\tilde{e}_t) = 0.$$
Using this relation, and setting \( u(t) = u^C(t) = K_1^*(\tilde{z}_t) \) we get

\[
dG_t/dt = \frac{\partial V}{\partial t}(\tilde{z}_t, t)
\]

\[ + \nabla_z V(\tilde{z}_t, t)(A(\tilde{z}_t) + B_1(\tilde{z}_t)(y - D_{22}(\tilde{z}_t)K_1^*(\tilde{z}_t) - C_2(\tilde{z}_t)) + B_2(\tilde{z}_t)u)
\]

\[ - \tau(h_1(\tilde{z}_t, K_1^*(\tilde{z}_t), y - C_2(\tilde{z}_t) - D_{22}(\tilde{z}_t)K_1^*(\tilde{z}_t)), y - C_2(\tilde{z}_t) - D_{22}(\tilde{z}_t)K_1^*(\tilde{z}_t))
\]

\[ \leq \sup_{y \in \mathbb{R}^d} \left\{ \frac{\partial V}{\partial t}(\tilde{z}_t, t) ight\}
\]

\[ + \nabla_z V(\tilde{z}_t, t)(A(\tilde{z}_t) + B_1(\tilde{z}_t)(y - D_{22}(\tilde{z}_t)K_1^*(\tilde{z}_t) - C_2(\tilde{z}_t)) + B_2(\tilde{z}_t)K_1^*(\tilde{z}_t))
\]

\[ - \tau(h_1(\tilde{z}_t, K_1^*(\tilde{z}_t), y - C_2(\tilde{z}_t) - D_{22}(\tilde{z}_t)K_1^*(\tilde{z}_t)), y - C_2(\tilde{z}_t) - D_{22}(\tilde{z}_t)K_1^*(\tilde{z}_t)))
\]

\[ = 0. \]

Therefore we have,

\[ G_T(y) \leq G_0 \]

\[ = \inf_{u \in \mathcal{S}} \sup_{w \in L_2([0,T], \mathbb{R}^d), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) - \int_0^T \tau(h_1(x(t), u(t), w(t)), w(t)) dt \}. \]

Since this inequality holds for all \( y(\cdot) \in L_2([0,T], \mathbb{R}^p) \), we have

\[ \sup_{x_0 \in \mathbb{R}^n, w \in L_2([0,T], \mathbb{R}^d)} \{ \alpha(x_0) - \int_0^T \tau(z(s), w(s)) ds \}
\]

\[ = \sup_{y \in L_2([0,T], \mathbb{R}^p)} \{ G_T(y) \}
\]

\[ \leq \inf_{u \in \mathcal{S}} \sup_{w \in L_2([0,T], \mathbb{R}^d), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) - \int_0^T \tau(h_1(x(t), u(t), w(t)), w(t)) dt \}, \]

and the result follows. \( \square \)

### 5.3 Infinite Time Problem

In this section, we write down the relevant equations for general dissipative control on infinite time horizon. We also obtain some internal stability results under appropriate closed loop detectability/reachability assumptions.

#### 5.3.1 Information State Solution

Let \( \Sigma_{x_0}^u \) denote the map from \( w \) to \( z \) under the control policy \( u \) and initial condition \( x_0 \). The \textit{dissipative control problem} is to find \( u \in \mathcal{U} \) such that:
5.3 Infinite Time Problem

(i) the close loop system $\Sigma^u$ is asymptotically stable when no disturbances are present (i.e. $w = 0$), and

(ii) the close loop system $\Sigma^u$ is dissipative with respect to the supply rate $r(z, w)$ in (5.2), that is, for any initial condition $z_0 \in \mathbb{R}^n$ the map $\Sigma^u_{z_0}$ is dissipative with respect to the supply rate $r(z, w)$, which means there exists a finite quantity $\beta^u(x) \geq 0$ such that

$$-\int_0^T r(z(t), w(t)) dt \leq \beta^u(x_0),$$

(5.21)

for all $w \in L_2([0, T], \mathbb{R}^d)$, for all $T \geq 0$. We assume that $\beta^u(0) = 0$.

In the rest of this section we assume that the measurement equation in (5.1) takes the form

$$y(t) = C_2(x(t)) + w(t) + D_{22}(x(t), u(t)),$$

(5.22)

where $D_{22}(\cdot, \cdot)$ is a smooth and bounded function (that is, the expression $D_{22}(x)u$ is replaced by $D_{22}(x, u)$). This implies that

$$|C_2(x) + w + D_{22}(x, u)| \leq C(|x| + |w|),$$

(5.23)

for some constant $C > 0$.

To solve this problem we consider the cost functional

$$J_p(u) = \sup_{T \geq 0} J_{p,T}(u),$$

in which $J_{p,T}(u)$ is defined in (5.5), and minimize it over $u \in U$. We see that $\Sigma^u$ is dissipative with respect to $r(z, w)$ if and only if $J_{-\beta} \leq 0$, or, by Theorem 5.1, if and only if

$$\sup_{T \geq 0, y \in L_2([0, T], \mathbb{R}^p)} \{(p_T, 0) : p_0 = -\beta\} \leq 0,$$

for some $\beta \geq 0$ with $\beta(0) = 0$. Minimization of $J_p(u)$ over $u \in U$ leads to the following (stationary) value function $W(p)$

$$W(p) = \inf_{u \in U} \sup_{T \geq 0, y \in L_2([0, T], \mathbb{R}^p)} \{(p_T, 0) : p_t = p\}.$$

(5.24)

Suppose that for some $u^o$, the system $\Sigma^{u^o}$ is dissipative. This implies

$$(p, 0) \leq W(p) \leq (p, \beta^{u^o}).$$

(5.25)
In particular, $W(p)$ is finite on the set $\{ p \in X : (p, 0), (p, \beta u^*) \text{ finite} \}$. Moreover, we have

$$0 = (-\beta u^*, 0) \leq W(-\beta u^*) \leq (-\beta u^*, \beta u^*) = 0,$$

which implies $W(-\beta u^*) = 0$. By the dynamic programming principle, the value $W$ satisfies the following equation

$$W(p) \geq \inf_{u \in U} \sup_{y \in L^2([t, \tau], \mathbb{R}^p)} \{ W(p_t, \tau) : p_t = p \}$$

for all $\tau \geq t$ which formally leads to the following (infinite dimensional) stationary HJI PDI ([56], [57], [42]).

$$\sup_{y \in \mathbb{R}^p} \inf_{u \in U} \{ \langle \nabla_p W, F(p, u, y) \rangle \} \leq 0. \quad (5.27)$$

Conversely, suppose there exists smooth (i.e., Frechet differentiable) solutions $p$ and $W$ to the DPE's (5.10) and (5.27) such that $p_0 = -\beta \in \text{dom} W(\cdot), W(-\beta) = 0$ for some $\beta \geq 0$ with $\beta(0) = 0$ and $p_t \in \text{dom} W(\cdot)$ with $W(p_t) \geq (p_t, 0)$ for all $0 \leq t$. Suppose the control law

$$u^*[y] = u^*(p[y], y) \quad (5.28)$$

attains the minimum in the left hand side of the HJI equation (5.27) for each $p \in \text{dom} W(\cdot)$. Then, following the calculation in Section 5.2.2 the control policy $u^*[y](t) = u^*(p[y], y(t)), \ t \geq 0,$ will result in the dissipativity of $\Sigma u^*$, with respect to the supply rate $r$ in (5.2).

The asymptotic stability of the system $\Sigma u^*$ can be deduced from its dissipativity provided the supply rate $r(z, w)$ in (5.2) satisfies

$$r(z, w) \leq -c|z|^2, \quad (5.29)$$

for all $z$, for some constant $c > 0$, and the closed loop system $\Sigma u^*$ is zero-state detectable (see Definition 4.2). To see this, setting $w = 0$ and using (5.29) we get

$$0 \leq c \int_0^T |z(t)|^2 dt \leq -\int_0^T r(z(t), 0) dt \leq \beta u^*(x_0),$$

$$\forall T \geq 0.$$ This implies that $z(t) \to 0$ as $t \to \infty$. By the closed loop detectability assumption we have $x(t)$ goes to 0 asymptotically.

To achieve internal stability the information state dynamics is required to have some sort of stability. Given input $u \in U$ and output $y \in L^2([0, \infty), \mathbb{R}^p)$, we say that the
5.3 Infinite Time Problem

information state dynamics (5.11) is stable if for each \( x \in \mathbb{R}^n \) there exist \( T_x \geq 0 \), \( C_x \geq 0 \) such that

\[
|p_t(x)| \leq C_x \quad \text{for all } t \geq T_x,
\]

(5.30)

provided the initial information state \( p_0 \) satisfies the growth condition

\[-\bar{a}_1|x|^2 - \bar{a}_2 \leq p_0(x) \leq -a_1|x|^2 - a_2,\]

(5.31)

for some non-negative constants \( \bar{a}_1, \bar{a}_2, a_1, a_2 \). We say that \( \Sigma^u \) is \( L_2 - z \) detectable if \( z \in L_2([0, \infty), \mathbb{R}^q) \) implies \( x \in L_2([0, \infty), \mathbb{R}^n) \). For \( u \in U \) and \( y \in L_2([0, \infty), \mathbb{R}^p) \), we say that the closed loop \( \Sigma^u \) is \( (w, y) \)-uniformly reachable if for all \( x \in \mathbb{R}^n \) there exists a function \( 0 \leq \alpha(x) < +\infty \) such that for all \( t \geq 0 \) sufficiently large there exists some \( x_0 \in \mathbb{R}^n \), \( w \in L_2([0, t], \mathbb{R}^d) \) such that \( x(0) = x_0, x(t) = x, y(s) = C_2(x(s)) + w(s) + D_{22}(x(s))u(s), 0 \leq s \leq t \) and

\[
|\alpha(x)|^2 + \int_0^t |w(s)|^2 ds \leq \alpha(x).
\]

(5.32)

Now assume that: (i) the supply rate \( r(z, w) \) satisfies (in addition to satisfying (5.29))

\[-r(z, w) \geq -b|w|^2,\]

(5.33)

for some constants \( b > 0 \) (which holds in the finite gain control case) and (ii) the closed loop \( \Sigma^u \) is \( L_2 - z \) detectable and \( (w, y) \) uniformly reachable. Then, if the initial \( p_0 \) satisfies the growth condition (5.31) the information dynamics (5.11) is stable. To see this, first note that the PDI (5.27) and the inequality (5.25) imply

\[
p_t(x) \leq (p_t, 0)
\]

\[
\leq W(p_t) \leq W(p) \leq (p, \beta),
\]

for all \( t \geq 0 \). Thus, \( p_t \) is bounded from above for all \( t \geq 0 \) provided \( (p, \beta) \) is finite. The condition on the supply rate (5.29) implies \( z \in L_2([0, \infty), \mathbb{R}^q) \) and under the \( L_2 - z \) detectability and condition (5.23) this implies \( y \in L_2([0, \infty), \mathbb{R}^p) \). By the uniform reachability assumption we also have, for all \( t \geq 0 \) sufficiently large,

\[
p_t(x) \geq p_0(x_0) - \int_0^t r(z(s), w(s)) ds
\]

\[
\geq p_0(x_0) - b \int_0^t |w(s)|^2 ds
\]

\[
\geq (b - \bar{a}_1)|x_0|^2 - \bar{a}_2 - b\alpha(x)
\]

Thus, \( p_t \) is eventually bounded. Therefore, the information state dynamics is stable, in this sense.
5.3.2 A Storage Function

In this section we construct a storage function for the output feedback control problem at hand, following the result in [42]. For fixed \( y, u \), the state equation in (5.1) can be re-written as

\[
\dot{x} = A(x) + B_1(x)(y - C_1(x) - D_{22}(x)u) + B_2(x)u,
\]

with \( x(0) = x_0 \), and the (information state) controller is given by

\[
\begin{cases}
\dot{p} = F(p, u, y), p_0 = \alpha, \\
u = u^*(p[y], y),
\end{cases}
\]

in which, \( F \) and \( u^* \) are given in (5.11) and (5.28) respectively. These systems define a closed loop system with the state \((x, p)\). Next, define the function \( e(x, p) \) by

\[
e(x, p) \triangleq -p(x) + W(p).
\] (5.34)

Since \( W(p) \geq (p, 0) \), \( e \) is nonnegative. By the definition, we have the derivatives

\[
\nabla_x e(x, p) f(x) = -\nabla_x p(x) f(x)
\]

\[
\langle \nabla_p e(x, p), F(p) \rangle = -\text{Eval}^p(F) + \langle \nabla_p W(p), F(p) \rangle.
\]

The first term in the second expression follows by evaluating the Gateaux derivative of the function \( G(p) = -p \). \( \text{Eval}^p \) is the evaluation map

\[
\text{Eval}^p(F) = F(p(x)).
\]

Using these expressions, we have, with \( u = u^* \), for any \( y \)

\[
\begin{align*}
\nabla_x e(A(x) + B_1(x)(y - C_1(x) - D_{22}(x)u^*) + B_2(x)u^*) \\
+ \langle \nabla_p e, F(p, u^*, y) \rangle - r(\bar{h}_1(x, y, u^*, (y - C_2(x) - D_{22}(x)u^*))
\end{align*}
\]

\[
= -\nabla_x p(x)(A(x) + B_1(x)(y - C_2(x) - D_{22}(x)u^*) + B_2(x)u^*)
\]

\[
-(-\nabla_x p(x)(A(x) + B_1(x)(y - D_{22}(x)u^* - C_2(x)) + B_2(x)u^*)
\]

\[
- r(\bar{h}_1(x, y, u^*), (y - C_2(x) - D_{22}(x)u^*))
\]

\[
+ \langle \nabla_p W(p), F(p, u^*, y) \rangle - r(\bar{h}_1(x, y, u^*), (y - C_2(x) - D_{22}(x)u^*))
\]

\[
= \langle \nabla_p W(p), F(p, u^*, y) \rangle
\]

\[
\leq 0,
\]

in which \( \bar{h}_1(x, u, y) = C_1(x) + D_{11}(x)(y - C_2(x) - D_{22}u) + u \). This is the dissipation inequality satisfied by \( e(x, p) \), showing that \( e \) is a storage function. In general, \( e \) need not the available storage function.
5.4 Conclusions

A general dissipative output feedback control for a wide class of nonlinear systems has been studied in this chapter. Utilizing the information state method developed in [54], [56], [57], we have expressed the solution to the problem in terms of an infinite dimensional PDE/PDI, which is the relevant dissipation inequality for the control problem being considered. We then deduce in Section 5.3 some sort of internal stability under a closed loop detectability/reachability assumption. We have also obtained an expression for a storage function, generalizing that in [42].

While leading to an optimal controller, in general, solving the PDE/PDI numerically is a very difficult task (due to its infinite dimensional nature). However, our solution provides a theoretical guideline for developing approximate, and hence, suboptimal solutions. We have provided some conditions under which the certainty equivalence principle developed in [6], [104] results in an optimal controller. These conditions are similar to those employed in [6] for the case of finite gain control.

While, in general, the solution to the dissipative output feedback control problem is infinite dimensional, we shall consider in the next chapter a specific quadratic dissipative control problem for a class of systems that leads to a finite dimensional solution.
Chapter 6

Output Feedback Synthesis:
Finite Dimensional Cases

6.1 Introduction

In this chapter we study a dissipative control synthesis problem for a special class of systems consisting of bilinear and linear systems with a quadratic supply rate function. For this special case, we shall see that the corresponding information state is completely determined by a set of finite dimensional quantities governed by ODE's. One of these ODE's takes the form of a filtering Riccati differential equation. We regard these ODE's as the (finite dimensional) dynamics of the information state. This situation resembles that of the Kalman filtering in which the conditional density of the process is Gaussian and, therefore, it is completely determined by the conditional mean and covariance, which are finite dimensional quantities [67]. We then express the solution to the problem in terms of a (finite dimensional) Hamilton-Jacobi-Isaacs equation. An example illustrating the computation of a solution to this equation is provided.

In the linear systems case, we show that for a number of types of quadratic supply rate functions, the solution to the HJI equation can be expressed in terms of a solution of a control Riccati differential equation, which is coupled to the solution of the above-mentioned filtering RDE. The types of the supply rate functions include the $H_\infty$, positive real and a mixture between $H_\infty$ and positive real performance measures. The coupling condition characterizes a set contained in the domain of the solution to the HJI equation. We provide a numerical example illustrating the performance of the controller
corresponding to the mixed performance measure case.

6.2 Finite Time Problem

We consider the class of systems described by
\[
\begin{align*}
\dot{x}(t) &= A_u(t)x(t) + B_1w(t) + B_2u(t), \quad x(0) = x_0, \\
z(t) &= C_1x(t) + D_{11}w(t) + u(t), \\
y(t) &= C_2z(t) + w(t) + D_{22}u(t),
\end{align*}
\] (6.1)
in which \(A_u \triangleq A + \sum_{i=1}^{m} A_i u_i\) with \(A_i \in \mathbb{R}^{n \times n}, i = 0, 1, \ldots, m,\) and \(u_i, i = 1, 2, \ldots, m\) are scalars, \(B_1 \in \mathbb{R}^{n \times d}, B_2 \in \mathbb{R}^{n \times m}, C_1 \in \mathbb{R}^{q \times n}, C_2 \in \mathbb{R}^{p \times n}, D_{11} \in \mathbb{R}^{q \times d}, D_{22} \in \mathbb{R}^{p \times m}\) are constant matrices. We also assume \(U = \mathbb{R}^m\). Linear systems are obtained by setting \(A_i = 0, i = 1, 2, \ldots, m\).

We consider quadratic supply rates of the form [44]
\[
r(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz),
\] (6.2)
in which \(Q \succeq 0, R \preceq 0, S\) are constant matrices of appropriate sizes. This supply rate corresponds to: (i) \(H_\infty\) control problem when \(Q = \gamma^2 I, R = -I,\) and \(S = 0,\) (ii) positive real control problem when \(Q = R = 0, S = \frac{1}{2} I,\) and (iii) mixed problem (see also Chapter 3).

6.2.1 Information State Solution

In the case at hand, the information state evolves according to the DPE
\[
\begin{align*}
-\frac{\partial p_t}{\partial t} &= \nabla_z p(A_u x + B_1(y - D_{22} u - C_2 z) + B_2 u) \\
&= -r((C_1 x + D_{11}(y - C_2 z - D_{22} u) + u), (y - C_2 z - D_{22} u)),
\end{align*}
\] (6.3)
with initial condition \(p_0.\) The special form of the dynamics (6.1) allows us to solve (6.3) for \(p_t\) explicitly in terms of finite dimensional quantities, as shown in the following lemma.

**Lemma 6.1** Assume \(p_0(x) = -\frac{1}{2}(x - \hat{x})'P(x - \hat{x}) + \varphi,\) for some \(\hat{x} \in \mathbb{R}^n,\) \(0 < P \in \mathbb{R}^{n \times n},\) \(\varphi \in \mathbb{R}.\) Then we have
\[
p_t(x) = -\frac{1}{2}(x - \hat{x}(t))'Y(t)^{-1}(x - \hat{x}(t)) + \varphi(t),
\] (6.4)
6.2 Finite Time Problem

where \( \dot{x} \in \mathbb{R}^n, Y \in \mathbb{R}^{n \times n}, \) and \( \varphi \in \mathbb{R} \) satisfy the ODEs

\[
\begin{align*}
\dot{x}(t) &= A\dot{x}(t) + B_1 y(t) + B_2 u(t), \\
\dot{Y}(t) &= (A_u(t) - B_1 C_2)Y(t) + Y(t)(A_u(t) - B_1 C_2)' - Y(t)(C_2^T Q C_2 \\
&\quad + (C_1 - D_{11} C_2)' R (C_1 - D_{11} C_2) + (C_2^T S D_{11} C_1 + C_1^T D_{11}^T S' C_2)) \\
&\quad - (C_2^T S C_1 + C_1^T S' C_2) Y(t), \\
\dot{\varphi}(t) &= -\frac{1}{2} (\dot{w}'(t)Q\dot{w}(t) + 2\dot{w}'(t)S\dot{z}(t) + \ddot{z}(t)R\ddot{z}(t)),
\end{align*}
\]

with initial conditions \( \dot{x}(0) = \dot{x}, Y(0) = P^{-1} > 0, \varphi(0) = \varphi, \) and in which expression

\[
\begin{align*}
\dot{A} &= (A_u - B_1 C_2) - Y(t)(C_2^T Q C_2 + (C_1 - D_{11} C_2)' R (C_1 - D_{11} C_2) \\
&\quad + (C_2^T S D_{11} C_1 + C_1^T S' C_2)), \\
\dot{B}_1 &= B_1 + Y(C_2^T Q - C_1^T S' + C_2^T D_{11}^T S' + S D_{11} - (C_1 - D_{11} C_2)' R D_{11}, \\
\dot{B}_2 &= (B_2 - B_1 D_{22}) - Y(C_2^T Q D_{22} + (C_1 - D_{11} C_2)' S' D_{22} \\
&\quad + C_2^T S (I - D_{11} D_{22}) - (C_1 - D_{11} C_2)' R (I - D_{11} D_{22}), \\
\dot{w} &= y - C_2 \dot{x} - D_{22} u, \quad \dot{z} = C_1 \dot{x} + D_{11} \dot{w} + u.
\end{align*}
\]

\textbf{Proof.} The result follows by substituting \( p_t(x) \) in (6.4) together with the ODE's in (6.5) into equation (6.3). \( \square \)

If we define the finite dimensional quantity by \( \rho(t) \overset{\Delta}{=} (\dot{x}(t), Y(t), \varphi(t)) \), equation (6.5) can be rewritten as

\[
\dot{\rho}(t) = \dot{F}(\rho(t), u(t), y(t)),
\]

where \( \dot{F}(\rho, u, y) \) denotes the dynamics of \( \rho \) in the right hand side of (6.5) with initial condition \( \rho(0) = \rho \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R} \). Thus the finite dimensional quantity \( \rho \) is identified with the quadratic information state \( p \) in (6.4). We denote this information state \( p^p \).

From the expression in (6.4) we see that \( (p^p_0, 0) = \varphi(t) \). Following the proof of Theorem 5.1 we have the representation for \( J \) in terms of \( \rho \) as follows

\[
J_{\rho,T}(u) = J_{p^p,T}(u)
= \sup_{y \in L^2([0,T], \mathbb{R}^n)} \{ \varphi(T) : \rho(0) = \rho \}.
\]

Thus, our new problem is to minimize the right hand side of (6.7) over \( u \in U \) constrained on the dynamics given in (6.6). We employ dynamic programming methods to solve this
problem. Define the value function $W(p, t)$ defined by
\[
W(p, t; T) \triangleq \inf_{u \in U} J_{p, T}(u) = \inf_{u \in U} \sup_{y \in L^2([t, T], \mathbb{R})} \{ \varphi(T) : \rho(t) = \rho \}.
\] (6.7)

This function solves the dynamic programming equation (see [58], [94])
\[
\frac{\partial W}{\partial t} + \sup_{y \in \mathbb{R}^n} \inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial W}{\partial y} (\bar{A} \dot{x} + \bar{B}_1 y + \bar{B}_2 u) \right\}
+ \left\langle \frac{\partial W}{\partial y}, ((A_u - B_1 C_2) Y + Y (A_u - B_1 C_2)' - Y (C_2' Q C_2 + (C_1 - D_{11} C_2)' R \right.
\left. \times (C_1 - D_{11} C_2) + (C_2 S D_{11} C_1 + C_1' D_{11} S' C_2) - (C_2 S C_1 + C_1' S' C_2)) Y) \right\rangle
\right.
\left. - \frac{\partial W}{\partial \rho} \frac{1}{2} (\dot{\omega}' Q \dot{\omega} + 2 \dot{\omega}' S \dot{z} + \dot{z}' R \dot{z}) \right\} = 0,
\]
\[
W(p, T; T) = \varphi.
\]

In the above equation, we use $\langle \cdot, \cdot \rangle$ to denote the matrix dot product. In particular, if $A, B$ are $n \times n$ matrices with entries $a_{ij}, b_{ij}$, we have
\[
\langle (A, B) \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}.
\]

In general, the inf and sup operations in the left hand side of equation (6.8) do not commute, i.e. Isaacs condition does not hold. Also, in general, equation of this type need not have smooth solutions in the classical sense. In such cases, we could interpret this equation in the weak (viscosity) sense [90], [8].

**Remark 6.1** From the definition of $W(p, t)$ in (6.16) and the expressions for $p^\rho$ in (6.4) and in (6.5), we may write
\[
W(p, t; T) = W^*(\dot{x}, Y, t; T) + \varphi.
\]

In particular, we have $\frac{\partial W}{\partial \rho} = 1$.

**Remark 6.2** In general, the value function $W(p, T)$ need not finite for all points $p \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R}$ (i.e., $W$ has a nontrivial domain). This issue will be illustrated by presenting an example in Section 6.4.

### 6.2.2 Verification

Optimal controls can be determined by employing the DPE (6.8). The following result says that if there exists a smooth solution to equation (6.8), then the control law that
achieves the minimum in the left hand side of equation (6.8) will render the closed loop system dissipative.

**Theorem 6.1** Assume that \( p_0' (x) \) be described by equations (6.4) and (6.5) and that the matrix
\[
\bar{R} = D_{22}'SD + D'S'D_{22} - D_{22}'QD_{22} - D'R\bar{D},
\]
where \( D = I - D_{11}D_{22} \), is positive definite. Assume there exists smooth solution \( \tilde{W}(\rho(t), t; T) \in C^1 \) to the DPE (6.8) such that \( \bar{W}(\rho_0, 0; T) = 0 \) for some \( \rho_0 \) in the set
\[
\mathcal{R} = \{(\hat{x}, Y, \varphi) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R} : \hat{x} = 0, \ \varphi = 0\}
\]
and define the control law \( u^*(\rho(t), y(t)) \) by
\[
u^*(\rho, y, t) = -\bar{R}^{-1}((A_{n} + \bar{B}_2)\hat{z}W' + K'(Y, \nabla Y) - \Gamma_1(y - C_{2}\hat{x}) - \Gamma_2C_{1}\hat{x}), \quad (6.9)
\]
in which
\[
\begin{align*}
A_n &= [A_1 \hat{x} | A_2 \hat{x} | \ldots | A_m \hat{x}], \\
K(Y, \nabla Y) &= [(\nabla Y W, (A_1Y + Y A_1')] | \ldots | (\nabla Y W, (A_mY + Y A_m'))], \\
\Gamma_1 &= D'RD_{11} + S'D_{22}(SD_{11} + Q), \quad \Gamma_2 = D'R - D_{22} S, \\
\Gamma_3 &= \Gamma_1'\bar{R}^{-1}\Gamma_2 + D_{11}'RS.
\end{align*}
\]
Then we have
\[
J_{\rho, T}(u^*) = \sup_{y \in L^2([0, T], \mathbb{R}^n)} \{ \varphi(T) : \rho(0) = \rho \} \leq \tilde{W}(\rho, 0; T).
\]
In particular, setting \( \rho(0) = \rho_0 \) in (6.5) yields an optimal controller \( u^*(\rho, y, t) \) that solves the dissipative control problem.

**Proof.** First note that the controller \( u^* \) in (6.9) achieves the minimum in the right hand side of (6.8). For simplicity, we shall rewrite the DPE (6.8) as
\[
\frac{\partial W}{\partial t} + \sup_{y \in \mathbb{R}^n} \inf_{u \in \mathbb{R}^m} \{ \nabla_{\rho} \hat{F}(\rho(t), u(t), y(t)) \} = 0, \quad (6.10)
\]
with \( W(\rho, T; T) = \varphi \), where \( \hat{F} \) is given in (6.6). Applying \( u^* \) on the dynamics (6.6) and integrating equation (6.10) along the trajectory produced by (6.6) we get, for any
\[ y \in L_2([0, t], \mathbb{R}^p), \]
\[ \tilde{W}(\rho(T), T; T) - \tilde{W}(\rho(0), 0; T) \]
\[ = \int_0^T \left( \frac{\partial \tilde{W}}{\partial s} + \nabla_\rho \tilde{W} \nabla^*(\rho(s), y(s), s, y(s)) \right) ds \]
\[ \leq \int_0^T \left( \frac{\partial \tilde{W}}{\partial s} + \sup_{y \in \mathbb{R}^p} \nabla_\rho \tilde{W} \nabla^*(\rho(s), y(s), y(s)) \right) ds \]
\[ \leq 0. \]

Since \( \tilde{W}(\rho, T; T) = \varphi \), we have, for any \( y \in L_2([0, T], \mathbb{R}^p) \),
\[ \varphi(T) \leq \tilde{W}(\rho(0), 0; T). \]

Since this inequality holds for all \( y \in L_2([0, T], \mathbb{R}^p) \), we get, by using the representation in (6.7)
\[ J_{p^*, T}(u^*) = \sup_{y \in L_2([0, T], \mathbb{R}^p)} \{ \varphi(T) : \rho(0) = \rho \} \leq \tilde{W}(\rho, 0; T). \]

In particular, setting \( \rho(0) = \rho_0^0 \) yields
\[ J_{p^*, T}(u^*) \leq \tilde{W}(\rho, 0; T) = 0, \]
which shows that the closed loop system is dissipative with respect to the quadratic supply rate \( r(z, w) \) (6.2) with \( \beta = -\rho_0^0 \).

6.2.3 Linear Systems Case

In the case of linear systems, i.e. when \( A_i = 0, \ i = 1, 2, \ldots, m \), it is possible to express the value function \( W(\hat{x}, Y, \varphi, t; T) \) explicitly in terms of \( \hat{x}, Y, \varphi \) for several cases. Specifically, it may have a quadratic form
\[ W(\hat{x}, Y, \varphi, t; T) = \frac{1}{2} \hat{x}'X(t)[I - YY(t)]^{-1}\hat{x} + \varphi, \quad (6.11) \]
in which \( X(t) \) is a symmetric matrix solving a particular RDE.

Theorem 6.2 Assume that \( p_t(x) \) be given by the expression (6.4) and (6.5). We have the following:
(i) if $Q = R = 0$, $S = I$, then $W(\hat{x}, Y, \varphi, t)$ has an explicit form as in (6.11) where $X(t)$ solves the RDE

$$-\dot{X} = X(A - B_2C_1) + (A - B_2C_1)'X - X(B_1B_2' + B_2B_1' - B_2(D_{11} + D_{11}')B_2')X,$$

(6.12)

with $X(T) = 0$,

(ii) if $Q = \gamma^2 I$, $R = -I$, $S = 0$, and $D_{11} = 0$, then $W(\hat{x}, Y, \varphi, t)$ has an explicit form as in (6.11) where $X(t)$ solves the RDE

$$-\dot{X} = X(A - B_2C_1) + (A - B_2C_1)'X - X(\gamma^2 B_1B_1' - B_2B_2')X,$$

(6.13)

with $X(T) = 0$,

(iii) if $Q = \gamma^2 \theta_1 I$, $R = -\theta_1 I$, $S = \theta_2 I$, with $\theta_1 \geq 0, \theta_2 \geq 0$ (but they are not simultaneously equal to 0), and $D_{11} = 0$, then $W(\hat{x}, Y, \varphi, t)$ has an explicit form as in (6.11) where $X(t)$ solves the RDE

$$-\dot{X} = X(A - B_2C_1) + (A - B_2C_1)'X - \theta X(\theta_1 B_1B_1' - \gamma^2 \theta_1 B_2B_2' - \theta_2 (B_1B_1' + B_2B_2'))X,$$

(6.14)

where $\theta = \frac{1}{(\gamma^2 \theta_1^2 + \theta_2^2)}$, with $X(T) = 0$.

\textbf{Proof.} The results follow by substituting the derivatives

$$\nabla_x W(\hat{x}, Y, \varphi, t; T) = \hat{x}'(I - YX(t))^{-1},$$

$$\nabla_Y W(\hat{x}, Y, \varphi, t; T) = \frac{1}{2}(I - YX(t))^{-1}X(t)\hat{x}'X(t)(I - YX(t))^{-1},$$

$$\nabla_{\varphi} W(\hat{x}, Y, \varphi, t; T) = 1,$$

into the DPE (6.8), and from the fact that $W(\hat{x}, Y, \varphi, t; T) = \varphi$.

\textbf{Remark 6.3} The first part of this theorem corresponds to the positive real control problem recently presented in [91] and the second one corresponds to the $H_\infty$ control...
problem. The third part corresponds to a mixture between the $H_\infty$ control (when $\theta_1 = 1, \theta_2 = 0$) and the positive real control (when $\theta_1 = 0, \theta_2 = 1$). It is possible to obtain a similar expression for $D_{11} \neq 0$, with the price of lengthy and tedious calculation. The HJI equation (6.8) serves as the basic equation for obtaining explicit solutions for various combinations of $Q, S$ and $R$.

\begin{remark}
The values of $Y$ such that $YX(t) > I$ do not belong to the domain of $W$. This is because when exists, $W$ satisfies the inequality $W(\hat{x}, Y, \phi, t; T) \geq (p^\rho, 0) = \phi$ which necessitates that $(I - YX(t))^{-1}X(t) \geq 0$. From the expression (6.11) we conclude that the set
\begin{equation*}
\{(\hat{x}, Y, \phi) : YX(t) < I\}
\end{equation*}
is contained in the domain of $W$.
\end{remark}

## 6.3 Infinite Time Problem

In the infinite time case, we seek to minimize the cost function
\begin{equation*}
J_\rho(u) = J_\rho^*(u) = \sup_{T \geq 0, y \in L_2([0,T], \mathbb{R}^p)} \{\varphi(T) : \rho(0) = \rho\},
\end{equation*}
where $\rho$ satisfies the dynamics (6.6). The relevant value function $W(\rho)$ is given by
\begin{equation*}
W(\rho) = \inf_{u \in U} J_\rho(u) = \inf_{u \in U} \sup_{T \geq 0, y \in L_2([0,T], \mathbb{R}^p)} \{\varphi(T) : \rho(t) = \rho\}. \tag{6.16}
\end{equation*}
satisfies the dissipation inequality (see [58], [94])
\begin{equation*}
\sup_{p \in \mathbb{R}^p} \inf_{u \in \mathbb{R}^m} \left\{\frac{\partial W}{\partial \rho} \left(\dot{A}\hat{x} + B_1 y + B_2 u\right) + \langle \frac{\partial W}{\partial \rho}, ((A - B_1 C_2)Y + Y(A - B_1 C_2)' - Y(C_2^T Q C_2 + (C_1 - D_{11} C_2)' R \right. \right.
\end{equation*}
\begin{equation*}
\times (C_1 - D_{11} C_2) + (C_2^T S D_{11} C_1 + C_1^T D_{11} S' C_2') - (C_2^T S C_1 + C_1^T S' C_2) Y)) \left. \right\} - \frac{\partial W}{\partial \rho} \left(\frac{1}{2} \left(\dot{W}^T Q \dot{W} + 2 \dot{W}^T S \hat{x} + \hat{x}' R \hat{x}\right)\right) \leq 0. \tag{6.17}
\end{equation*}
In general, this inequality could be interpreted in the viscosity sense [90], [8]. When a smooth solution $\tilde{W}$ exists such that $\tilde{W}(\rho^0, 0) = 0$ for some $\rho^0$ in the set $\mathcal{R} = \{(\hat{x}, Y, \phi) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R} : \hat{x} = 0, \ \phi = 0\}$ the optimal control law
\begin{equation*}
u^*(\hat{x}, Y, \phi, y) = -\tilde{R}^{-1}((A_{\hat{x}} + B_2)' \nabla_{\hat{x}} W'(\hat{x}, Y, \phi) + K'(Y, \nabla_Y W) - \Gamma_1(y - C_2 \hat{x}) - \Gamma_2 C_1 \hat{x}). \tag{6.18}
\end{equation*}
results in the dissipative closed loop $\Sigma u^*$. Moreover, the asymptotic stability of $\Sigma u^*$ and the ultimate boundedness of the information state can be deduced by assuming closed loop detectability and reachability (see Section 5.3).

**Remark 6.5** Define the function $e(x, \rho)$ by

$$e(x, \rho) = -p^\rho(x) + W(\rho).$$

Since $W(\rho) \geq (p^\rho, 0)$, $e$ is nonnegative. It is a simple matter to show that $e$ satisfies the dissipation inequality

$$\nabla_x e(Au^*x + B_1(y - C_2x - D_{22}u^*) + B_2u^*) + \nabla_\rho W(\rho)f(\rho, u^*, y)$$

$$-r(h_1(x, y, u^*), (y - C_2x - D_{22}u^*)) \leq 0,$$

for all $y$, with the equality achieved at $y = y^*$. Thus, $e$ serves as a storage function.

### 6.3.1 Linear Systems Case

In the case of linear systems, the stationary value function $W(\hat{x}, Y, \varphi)$ may have a quadratic form given by

$$W(\hat{x}, Y, \varphi) = \frac{1}{2} \hat{x}'X[I - YY^{-1}x + \varphi,$$

in which $X$ is a symmetric matrix solving an ARE. In particular:

(i) if $Q = R = 0$, $S = I$, then $W(\hat{x}, Y, \varphi)$ is quadratic as in (6.20) in which $X$ is determined by the ARE

$$X(A - B_2C_1) + (A - B_2C_1)'X - X(B_1B_2' + B_2B_1' - B_2(D_{11} + D_{11}'B_2')X = 0,$$

(ii) if $Q = \gamma^2 I$, $R = -I$, $S = 0$, and $D_{11} = 0$, then $W(\hat{x}, Y, \varphi)$ is quadratic as in (6.20) in which $X$ is determined by the ARE

$$X(A - B_2C_1) + (A - B_2C_1)'X - X(\gamma^2 B_1B_2' - B_2B_2')X = 0,$$

(iii) if $Q = \gamma^2 \theta_1 I$, $R = -\theta_1 I$, $S = \theta_2 I$, with $\theta_1 \geq 0, \theta_2 \geq 0$ (but they are not simultaneously equal to 0), $D_{11} = 0$ and $D_{22} = I$, then $W(\hat{x}, Y, \varphi)$ is quadratic as in (6.20) in which $X$ is determined by the ARE

$$X(A - B_2C_1) + (A - B_2C_1)'X - \theta X(\theta_1 B_1B_2' - \gamma^2 \theta_1 B_2B_2' - \theta_2 (B_2B_2' + B_2B_2'))X = 0,$$

where $\theta = \frac{1}{(\gamma^2 \theta_1 + \theta_2)}$. 


In any of the three cases, the optimal control policy is given by

$$u^*(\hat{x}, y, \varphi, y) = -\bar{R}^{-1}(\bar{B}'_2 X [I - YX]^{-1} \hat{x} - \Gamma_1 (y - C_2 \hat{x}) - \Gamma_2 C_1 \hat{x}).$$

(6.24)

Thus, the controller is, in general, time varying due to the variation of the $Y(t)$ component of the information state $\rho(t)$. This resembles the results in [83], [65] for $H_\infty$ control with unknown initial state. However we may pick up a stationary solution $Y$ to the second RDE in (6.5), i.e., a solution to the ARE

$$\begin{align*}
(A - B_1 C_2)Y + Y(A - B_1 C_2)' - Y(C_2' Q C_2 + (C_1 - D_{11} C_2)') \\
\times R(C_1 - D_{11} C_2) + (C_2'SD_{11} C_1 + C_1'D_{11}'S'C_2) - (C_2'SC_1 + C_1'S'C_2))Y = 0,
\end{align*}$$

(6.25)

and then obtain a time invariant controller that yields the desired dissipativity. Again, asymptotic stability will be obtained by assuming closed loop detectability.

In the next results, we consider the mixed performance measure case and show that if we choose the stabilizing solutions $X$ and $Y$ to the corresponding AREs, then we obtain closed loop internal stability without assuming closed loop detectability. Moreover, the resulting dissipativity is in the strict sense. We say that the linear closed loop systems $\Sigma_t^K$ is \textit{internally asymptotically stable} if both $x(t)$ and $\hat{x}(t)$ go to 0 as $t \to \infty$. In the mixed performance case we assume that $Q = \gamma^2 \theta_1 I$, $R = -\theta_1 I$, $S = \theta_2 I$, with $\theta_1 \geq 0, \theta_2 \geq 0$ (but they are not simultaneously equal to 0). Moreover, for simplicity we assume $D_{11} = 0$ and $D_{22} = I$.

We first write down the state space model of the closed loop system for the mixed performance case. The system is described by

$$\begin{align*}
\Sigma_t : \\
\begin{aligned}
\dot{x}(t) &= A x(t) + B_1 w(t) + B_2 u(t), \\
z(t) &= C_1 x(t) + u(t), \\
y(t) &= C_2 x(t) + w(t) + u(t), 
\end{aligned}
\end{align*}$$

(6.26)

and the controller by

$$\begin{align*}
K : \\
\begin{aligned}
\dot{\hat{x}}(t) &= \hat{A} \hat{x}(t) + \hat{B}_1 y(t) + \hat{B}_2 u(t), \\
u(t) &= K_1 \hat{x}(t) + K_2 y(t), 
\end{aligned}
\end{align*}$$

(6.27)

where

$$K_1 = -\bar{R}^{-1}((B_2 - B_1)'X + \lambda_1 C_2 - \lambda_2 C_1),$$

$$K_2 = \bar{R}^{-1}(\theta_2 - \gamma^2 \theta_1),$$
6.3 Infinite Time Problem

In which \( \check{R} = (2\theta_2 - \gamma^2 \theta_1 + \theta_1)I, \) \( \lambda_1 = \theta_2 - \gamma^2 \theta_1, \) \( \lambda_2 = -(\theta_1 + \theta_2). \) Defining the augmented state

\[
x_a = \begin{bmatrix} x \\
e \end{bmatrix},
\]

where \( e = x - \bar{x}, \bar{x} = [I - XY]^{-1}\bar{x}, \) we obtain the following state space representation of the closed loop map from \( w \rightarrow z \)

\[
\dot{x}_a(t) = \begin{bmatrix} A_{xz} & A_{xe} \\ A_{ez} & A_{ee} \end{bmatrix} x_a(t) + \begin{bmatrix} B_x \\ B_e \end{bmatrix} w
\]

\[
= A_d x_a(t) + B_d w(t), \tag{6.28}
\]

\[
z(t) = \begin{bmatrix} C_x & C_e \end{bmatrix} x_a(t) + D_w w(t)
\]

with initial condition \( x_a(0) = \begin{bmatrix} x_0 \\ e_0 \end{bmatrix}, \) in which

\[
A_{xz} = A - B_2(\check{R} - \lambda_1)^{-1}(B_2 - B_1)'X - B_2 C_1,
\]

\[
A_{xe} = B_2(\check{R} - \lambda_1)^{-1}((B_2 - B_1)'X + \lambda_1 C_2 - \lambda_2 C_1),
\]

\[
A_{ez} = [I - XY]^{-1}((X B_2 + (\theta_2 C_2 + \theta_1 C_1) \psi^{-1}(B_2 - B_1)'X
\]

\[
-((B_2 C_1 - B_1 C_2)'X - X \Omega_{x} X)),
\]

\[
A_{ee} = \check{A} - [I - XY]^{-1}((\theta_1 C_1 + \theta_2 C_2)' + X B_2)(\check{R} - \lambda_1)^{-1}
\]

\[
\times((B_2 - B_1)'X + \lambda_1 C_2 - \lambda_2 C_1),
\]

\[
B_x = B_1 + B_2 \check{R}^{-1} \lambda_1 (I - \check{R}^{-1} \lambda_1)^{-1},
\]

\[
B_e = -Y[I - XY]^{-1}((B_2 X + C_2)' \check{R}^{-1} \lambda_1 (I - \check{R}^{-1} \lambda_1)^{-1} + (B_1 X - \theta_2 C_1)'
\]

\[
+\theta_1 (\check{R}^{-1} \lambda_1 (I - \check{R}^{-1} \lambda_1)^{-1} C_1 + \gamma^2 C_2)'),
\]

\[
C_x = -(\check{R} - \lambda_1)^{-1}(B_2 - B_1)'X,
\]

\[
C_e = (\check{R} - \lambda_1)^{-1}(B_2 - B_1)'X + \check{R}^{-1} \lambda_1 (I - \check{R}^{-1} \lambda_1)^{-1} C_2 + C_1,
\]

\[
D_w = \check{R}^{-1} \lambda_1 (I - \check{R}^{-1} \lambda_1)^{-1},
\]
where
\[
\begin{align*}
\Omega_X &= -\theta(\theta_1 B_1 B_1' - \gamma^2 \theta_1 B_2 B_2' - \theta_2 (B_1 B_2' + B_2 B_1')) ,
\tilde{A} &= A - B_1 C_2 + Y[I - XY]^{-1} \Omega Y[I - XY]^{-1} ,
\Omega_Z &= X(B_2 C_1 - B_1 C_2) + (B_2 C_1 - B_1 C_2)' X - X \Omega_Z X - \Omega_Y ,
\Omega_Y &= -\theta_1 (\gamma^2 C_2' C_2 - C_1' C_1) - \theta_2 (C_2' C_1 + C_1' C_2) .
\end{align*}
\] (6.29)

**Theorem 6.3**  Assume $Q = \gamma^2 \theta_1 I$, $R = -\theta_1 I$, $S = \theta_2 I$, with $\theta_1 \geq 0, \theta_2 \geq 0$ (but they are not simultaneously equal to 0), $D_{11} = 0$, and $D_{22} = I$. Assume there exist nonnegative solutions $Y, X$ to the algebraic Riccati equations
\[
(A - B_1 C_2) Y Y' (A - B_1 C_2)' Y - Y (\theta_1 (\gamma^2 C_2' C_2 - C_1' C_1) - \theta_2 (C_2' C_1 + C_1' C_2)) Y = 0 ,
\] (6.30)
\[
X(A - B_2 C_1) + (A - B_2 C_1)' X - \theta X (\theta_1 B_1 B_1' - \gamma^2 \theta_1 B_2 B_2' - \theta_2 (B_1 B_2' + B_2 B_1')) X = 0 ,
\] (6.31)
such that the matrices $A_Y, A_X$ defined by
\[
A_Y = (A - B_1 C_2) - Y (\theta_1 (\gamma^2 C_2' C_2 - C_1' C_1) - \theta_2 (C_2' C_1 + C_1' C_2)) ,
A_X = (A - B_2 C_1) - \theta (\theta_1 B_1 B_1' - \gamma^2 \theta_1 B_2 B_2' - \theta_2 (B_1 B_2' + B_2 B_1')) X
\]
are asymptotically stable, and the coupling condition
\[
XY < I
\] (6.32)
is satisfied. Then the closed loop $\Sigma_{\xi}^C$ is internally asymptotically stable and is strictly dissipative with respect to the supply rate
\[
r_{\xi}(w, z) = \frac{1}{2} (\theta_1 (\gamma^2 w^t w - z^t z) + 2 \theta_2 w^t z).
\]

**Proof.** We shall prove the result by showing that under the hypothesis, we can construct a nonnegative solution $X_d$ to the following ARE for the closed loop system (6.28)
\[
A_d' X_d + X_d A_d + [X_d B_d - C_d S'] Q^{-1} [B_d' X_d - \tilde{S} C_d] - C_d' R C_d = 0 ,
\] (6.33)
such that the matrix $A_d' = A_d + B_d \tilde{Q}^{-1} [B_d' X_d - \tilde{S} C_d]$ is asymptotically stable. Then, by Theorem 3.5 in Section 3.4, the results follow.
Define the nonnegative matrix $Z$ by

$$Z = Y[I - XY]^{-1}.$$  

By the hypothesis, $Z \geq 0$. By a simple algebraic calculation, we see that $Z$ solves the ARE

$$(A - B_1C_2)Z + Z(A - B_1C_2)^T + Z\Omega_Z Z = 0,$$  

(6.34)

where $\Omega_Z$ is given in (6.29). Moreover, from the identity

$$[I - YX]^{-1}((A - B_1C_2) - Y\Omega_Y)[I - YX]$$  

$$= (A - B_1C_1)Z + Z\Omega_Z,$$

we conclude that $(A - B_1C_1)Z + Z\Omega_Z$ is an asymptotically stable matrix. By some tedious, but otherwise straightforward, algebraic calculation, the ARE (6.34) can be rewritten as

$$A_{ee}Z + ZA_{ee}' + [B_e - ZC_e'S][\bar{Q}]^{-1}[B_e' - \bar{S}_{ee}Z] - ZC_e'RC_eZ = 0,$$  

(6.35)

such that the matrix $A_{ee} - [B_e - ZC_e'S][\bar{Q}]^{-1}\bar{S}_{ee} - ZC_e'RC_e$ is asymptotically stable. This implies that there exists a nonnegative matrix $W$ such that

$$WA_{ee} + A_{ee}'W + [WB_e - C_e'S][\bar{Q}]^{-1}[B_e'W - \bar{S}_{ee}] - C_e'RC_e = 0,$$  

(6.36)

with the matrix $A_{ee} + B_e\bar{Q}^{-1}[B_e'W - \bar{S}_{ee}]$ being asymptotically stable. We shall construct a solution to the closed loop ARE (8.38) using $X$ and $W$.

Define the closed loop matrix $\hat{X}_d$ by

$$\hat{X}_d = \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix},$$  

(6.37)

where $W \geq 0$ is obtained from (6.34). Substituting this into the closed loop ARE (8.38) results in the following algebraic equations

$$A_{d}X_d + X_dA_d + [X_dB_d - C_d'S][\bar{Q}]^{-1}[B_d'X_d - \bar{S}_{d}C_d] - C_d'RC_d$$  

$$= \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}.$$
where
\[
\Phi_{11} = A'_{x}X + XA_{xx} + [XB_{x} - C'_{x}S]\bar{Q}^{-1}[B_{x}X - \bar{S}C_{x}] - C_{x}RC_{x}
\]
\[
= X(A - B_{2}C_{1}) + (A - B_{2}C_{1})'X
- \theta X(\theta_{1}B_{1}'X - \gamma_{1}^{2}\theta_{1}B_{2}'X - \theta_{2}(B_{1}B_{2}' + B_{2}B_{1}'))X
\]
= 0,
\]
\[
\Phi_{12} = (A'_{ex} + (XB_{x} - C'_{x}S)\bar{Q}^{-1}B_{x}')W
+ X(A_{ex} - B_{e}\bar{Q}^{-1}SC_{e}) + C_{x}RC_{e}
= 0 \cdot W + X \cdot 0 = 0
\]
= \Phi_{21},
\]
\[
\Phi_{22} = WA_{ee} + A'_{ee}W + [WB_{e} - C'_{e}S]\bar{Q}^{-1}[B'_{e}W - \bar{S}C_{e}] - C_{e}RC_{e}
\]
= 0.

Thus, \(\bar{X}_{d}\) solves the closed loop ARE (8.38). Moreover, we have
\[
A_{x}^* = A_{d} + B_{d}\bar{Q}^{-1}[B_{d}'\bar{X}_{d} - \bar{S}C_{d}]
= \begin{bmatrix}
A_{xx} + B_{x}\bar{Q}^{-1}(B_{x}'X - \bar{S}C_{x}) & A_{xe} + B_{x}\bar{Q}^{-1}(B_{e}'W - \bar{S}C_{e}) \\
A_{ee} + B_{e}\bar{Q}^{-1}(B_{e}'X - \bar{S}C_{e}) & A_{ee} + B_{e}\bar{Q}^{-1}(B_{e}'W - \bar{S}C_{e})
\end{bmatrix}
= \begin{bmatrix}
A_{x} & A_{xe} + B_{x}\bar{Q}^{-1}(B_{e}'W - \bar{S}C_{e}) \\
0 & A_{ee} + B_{e}\bar{Q}^{-1}(B_{e}'W - \bar{S}C_{e})
\end{bmatrix}
\]
which is asymptotically stable since \(A_{x}\) and \(A_{ee} + B_{e}\bar{Q}^{-1}(B_{e}'W - \bar{S}C_{e})\) are asymptotically stable. From the proof of Theorem 3.5 in Section 3.4, this implies that the closed loop matrix \(A_{d}\) is asymptotically stable. In particular, when \(w = 0\), \((x(t), e(t))\) go to \((0, 0)\) as \(t \to \infty\). Therefore \(\bar{x}(t) = [I - XY](x(t) - e(t))\) also goes to 0 asymptotically. This proves internal stability. Furthermore, the closed loop map \(\Sigma_{d}\) from \(w\) to \(z\) satisfies
\[
\left\{ \begin{align*}
-\int_{0}^{T} r_{q}(w(t), z(t))dt & \leq -\frac{1}{2} \epsilon \int_{0}^{T} |w(t)|^{2}dt + \frac{1}{2} z_{0}'(X + W)x_{0}, \\
\forall w \in L_{2}([0, T], \mathbb{R}^{p}), \forall T \geq 0,
\end{align*} \right.
\]
where
\[
r_{q}(w, z) = \frac{1}{2}(\theta_{1}(\gamma^{2}w'w - z'z) + 2\theta_{2}w'z).
\]
This completes the proof. \(\square\)
6.4 Examples

**Remark 6.6** Following the discussion in Remark 3.5, we see that under the hypothesis of Theorem 6.3, the function $V_d(x, e) = \frac{1}{2}[x' e'] \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} e \\ z \end{bmatrix}$ serves as the available storage function for the closed loop system (6.28). We also note the identity (by a simple calculation)

$$V_d(x, x - \bar{x}) = -p'(x) + W([I - YX]^{-1} \bar{x}, Y, \varphi)$$

$$= -p'(x) + W(\rho)$$

$$= e(x, \rho),$$

(see (6.19)). Thus, under the choice of $X$ and $Y$ in Theorem 6.3, the function $e(x, \rho)$ serves as the available storage function.

**Remark 6.7** In the frequency domain, the mixed performance case corresponds to the following inequality

$$\theta_1 \int_{-\infty}^{\infty} (\bar{z}^*(j\omega) \bar{z}(j\omega) - \gamma^2 \bar{w}^*(j\omega) \bar{w}(j\omega))d\omega$$

$$- \theta_2 \int_{-\infty}^{\infty} (\bar{w}^*(j\omega)(\bar{G}^*(j\omega) + \bar{G}(j\omega)) \bar{w}(j\omega))d\omega \leq 0,$$

for all $\bar{w}(j\omega) \in L_2(-\infty, \infty)$, in which $j = (-1)^{\frac{1}{2}}, \omega \in \mathbb{R}$, $\bar{w}$ and $\bar{z}$ denote the Fourier transforms of $w$ and $z$ respectively, and $\bar{G}(j\omega)$ denotes the transfer matrix from $\bar{w}(j\omega)$ to $\bar{z}(j\omega)$. If we set $0 \leq \theta_1 \leq 1$ and $\theta_2 = 1 - \theta_1$, then by varying $\theta_1$ we obtain a smooth transition between $H_\infty$ performance and positive real performance measures. We would expect that this could capture the advantages of both performance measures.

6.4 Examples

In this section we present two numerical examples to illustrate the optimal information state solution developed in this chapter. In the first example we consider a $H_\infty$ control problem for a bilinear system and compute the value function and the optimal control for the problem. The numerical method we employ is an extension of that described in Chapter 2. We illustrate the domain issue of the value function $W$ indicated in Section 6.2.1. In the second example, we consider a mixed performance control for a linear model. In particular we shall depict the frequency domain behaviour of the closed loop system when we transit from positive real to $H_\infty$ performance measures.
6.4.1 Example 1: $H_\infty$ case

We consider an open loop ($u = 0$) unstable bilinear system with the state space model given by

$$
\begin{align*}
\dot{x}(t) &= (0.5 + u(t))x(t) + w(t) + u(t), \quad x(0) = x_0, \\
2z(t) &= \begin{bmatrix} 2x(t) \\ u(t) \end{bmatrix}, \\
y(t) &= x(t) + v(t).
\end{align*}
$$

The control problem is to find an output feedback controller such that

$$
\frac{1}{2} \int_0^T |x(t)|^2 dt \leq \frac{1}{2} \int_0^T (|w(t)|^2 + |v(t)|^2) dt + \beta(x_0),
$$

for all $x_0 \in \mathbb{R}^n, w \in L^2([0,T], \mathbb{R}^d), v \in L^2([0,T], \mathbb{R}^p)$, for all $T \geq 0$, where $\beta \geq 0$ with $\beta(0) = 0$. In the linear $H_\infty$ literature, this problem corresponds to a special case of the four block one. We solve the problem using the information state technique developed in this chapter. A detailed expression for the resulting equations is presented in [94].

To obtain the desired controller, we solve equation (6.17) numerically by employing an extension of the computational method described in Chapter 2. The numerical experiment shows that we can obtain a solution $W(\rho)$ to the DPE (6.17) for $\gamma = 6.0$ or larger. Recall that we may write $W(\rho) = W^*(\hat{x}, Y) + \varphi$ (see Remark 6.1). The plot of $W^*(\hat{x}, Y)$ for $\gamma = 6.0$ is shown in Figure 6.1 (top). As mentioned in Remark 6.2, the function $W$ has a nontrivial domain, in general. The numerical experiment shows that the size of the domain is determined by the $Y$ component of $\rho$. At the points $(\hat{x}, Y)$ near the boundary of the domain, the function $W^*$ blows up. As a measure of the size of the domain we take the point $(0, Y)$ in the $(\hat{x}, Y)$ coordinate system at which the value function $W^*$ starts to blow up. Figure 6.1 (middle) shows that the size of the domain of $W^*(\hat{x}, Y)$ reduces as the number of iterations increases.

Figure 6.1 (bottom) shows the plot of a state response under some disturbances. At $t = 22.5$ seconds, the disturbance is pulled out from the system and, as one can see, the state approaches zero rapidly, showing that the controller is stabilizing. The offset from the origin ($z = 0$) happens because of the truncation and discretization of the controller space $U$. 
6.4.2 Example 2: Mixed performance case

The next example illustrate the mixed performance case presented in the previous sections (see Remark 6.7). We consider an unstable model given by

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 8 & -4 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 5 \\ 0 \end{bmatrix} u(t), \quad x(0) = x_0, \\
z(t) &= [1 \ 0]z(t) + u(t), \\
y(t) &= [-5 \ 0]z(t) + w(t) + u(t).
\end{align*}
\]

The dissipative performance index is given by

\[-\frac{1}{2} \int_{0}^{T} (\theta w(t)'w(t) - \gamma^2 z(t)'z(t)) + 2(1 - \theta)w(t)'z(t))dt \leq \beta(x_0),\]

for all \( w \in L_2([0,T],\mathbb{R}^d) \), and all \( T \geq 0 \), with \( \beta \geq 0, \beta(0) = 0 \). In the simulation, we solve the related AREs for some values of \( \theta \in [0,1] \). The simulation results are illustrated in Figure 6.2 and 6.3, in which the Nyquist plots of the resulting closed loop systems are depicted. In Figure 6.2 (a), \( \theta = 0.0 \) This corresponds to the positive real case. As can be seen from the figure, the closed loop system is, indeed, positive real and has a relatively large \( H_\infty \) norm (the norm is the largest distance from the points in the plot to the \((0,0)\) point). As the value of \( \theta \) increases, the Nyquist plot shifts to the left and the \( H_\infty \) norm reduces slightly (see Figure 6.2 (b) to Figure 6.3 (d)). In Figure 6.3 (d), \( \theta = 1.0 \) corresponding to the \( H_\infty \) case. The controller results in the smallest \( H_\infty \) norm, but non-positive real closed loop. It is interesting to note that when \( \theta = 0.8571 \) (i.e., Figure 6.3 (c)), the closed loop system is positive real and has a small \( H_\infty \) norm. Thus, one could say that, as far as both the gain and the phase are concerned, the controller corresponding to \( \theta = 0.8571 \) is better than the one that corresponds to \( \theta = 1.0 \), which is the \( H_\infty \) controller or the one that corresponds to \( \theta = 0.0 \), the positive real controller.

6.5 Conclusions

In this chapter we have studied a quadratic dissipative control problem for bilinear (and linear) systems in which the solution can be expressed in terms of finite dimensional equations. We have presented some numerical examples illustrating the computational issues for solving the equations. In the linear case, the solution can be simply expressed in terms of two ARE plus a coupling condition. When the stabilizing solution to the ARE's
are chosen, we obtain an expression for the available storage function for the closed loop system.

6.6 Figures

We shall now present the relevant figures for Example 1 and 2 in Section 6.4.
Figure 6.1: Example 1. ($H_{\infty}$ case) Open loop unstable bilinear system; the value function $W^* (\hat{x}, Y)$ (top); size of domains (middle); trajectories of disturbance $w(t)$, $v(t)$ and state $x(t)$ (bottom). The controller is stabilizing.
Figure 6.2: Example 2. (Mixed performance case) (a) $\theta = 0.0$ (positive real), (b) $\theta = 0.1429$, (c) $\theta = 0.2857$, (d) $\theta = 0.4286$. 
Figure 6.3: Example 2. (Mixed performance case) (a) \( \theta = 0.5714 \), (b) \( \theta = 0.7143 \), (c) \( \theta = 0.8571 \), (d) \( \theta = 1.0 \) \((H_\infty \text{ with } \gamma = 0.9)\).
Chapter 7

Filter Synthesis

7.1 Introduction

In recent years, a number of researchers have addressed the problem of robust or $H_\infty$ filtering (see [80], [87] in the linear systems case, and [25], [63], [66], [60] in the nonlinear systems case). The motivation for this activity appears to be twofold: (i) to provide an interpretation of the filter that occurs in the output feedback $H_\infty$ control problem, and (ii) to obtain filters which are robust with respect to uncertainties or disturbances.

In the underlying filter problem, there is an unobserved state $x$, a measured observation quantity $y$, and a quantity $z$, which is a function of the state $x$, for which an estimate $\hat{z}$ is desired. In the standard stochastic filtering set up ($H_2$ filtering), the objective is to compute least square estimates of $z(t) = h(x(t))$ given the past observations $y(s), 0 \leq s \leq t$, which turn out to be conditional expectations $E[h(x(t))|y(s), 0 \leq s \leq t]$. In the linear systems case, this leads to the Kalman Filter [67], whereas in the nonlinear case, it leads to the Kushner-Stratonovich and Duncan-Mortensen-Zakai equations for describing the evolution of conditional distributions [116], [22], [43].

In the $H_\infty$ filtering case, which is deterministic, one seeks to minimize the influence of the disturbances on the estimation error such that

$$\frac{1}{2} \int_0^T (z(t) - \hat{z}(t))^2 dt \leq \frac{1}{2} \int_0^T \gamma^2 |w(t)|^2 dt,$$

(7.1)

for all disturbances $w$. In the standard finite time horizon set up, the time horizon length $T$ and the so-called performance level $\gamma > 0$ are fixed. In practice, it is desirable to find
a filter such that the above inequality holds for as small $\gamma > 0$ as possible. This will result in filters which are robust with respect to disturbances and model uncertainty [87]. Nonlinear robust $H_\infty$ filtering problem has been addressed by a number of authors in [25], [63], [66] and [60] using different approaches. A common feature in this literature is that the filtering policy is determined by a single DPE.

In this chapter we study the nonlinear robust $H_\infty$ filtering problem by employing the information state approach. We shall show that a solution to the $H_\infty$ filtering problem for a general class of nonlinear systems can be expressed in terms of solutions of two dynamic programming equations (DPEs): the first equation describes the dynamics of the information state and is driven by measurements and filtering signals, and the second one provides a means for constructing the optimal filtering policy. These equations are of similar forms to those appearing in Chapter 5 (see also [57], [42]). In the finite time case the resulting filter depends on the horizon length, in general, and this dependency vanishes in the infinite time case.

In this chapter we also study the relation between the information state filter and the certainty equivalence (CE) filter obtained in [25] and [66]. The CE filtering involves only one DPE (which is our first DPE). We shall explain the reason underlying this result by using the information state solution framework. Interestingly, when the CE principle holds, the optimal filtering policy is independent of the horizon length, even for the finite time case. We then show that in the linear systems case the information state filter is finite dimensional and recovers the familiar linear $H_\infty$ filter. Finally, we present an information state solution to more general dissipative filter synthesis problems paralleling the results in Chapter 5 and Chapter 6 in the case of control synthesis.

As the case in Chapter 5, we maintain the technicality involved in this chapter at a minimum. In particular, we assume that the relevant DPEs possess smooth solutions. In the Appendix B we shall consider a model filtering problem which bridges to the mathematical results in [57].
7.2 Finite Time Problem

We consider the class of nonlinear systems modeled by

\[
\begin{align*}
\dot{x}(t) &= A(x(t)) + B(x(t))w(t), \quad x(0) = x_0, \\
z(t) &= C_1(x(t)), \\
y(t) &= C_2(x(t)) + D_{21}(x(t))w(t).
\end{align*}
\]

(7.2)

In this expression, \(x \in \mathbb{R}^n\) denotes the state vector, which is partially observed through a measured output quantity \(y \in \mathbb{R}^p\). The initial condition \(x_0\) is assumed to be unknown. The disturbance \(w \in \mathbb{R}^d\) corrupts the state and the output of the system. The output \(z \in \mathbb{R}^q\), which is a function of the state \(x\), represents the quantity to be estimated. We assume that the maps \(A : \mathbb{R}^n \to \mathbb{R}^n, B : \mathbb{R}^n \to \mathbb{R}^{nxd}, C_1 : \mathbb{R}^n \to \mathbb{R}^q, C_2 : \mathbb{R}^n \to \mathbb{R}^p\) and \(D_{21} : \mathbb{R}^n \to \mathbb{R}^{pxd}\) are smooth functions with \(A, B, C_1, C_2\) are globally Lipschitz continuous, \(D_{21}\) is bounded, and \(A(0) = 0, B(0) = B, C_1(0) = 0, C_2(0) = 0,\) and \(D_{21}(0) = D_{21}\), where \(B, D_{21}\) are constant matrices. We further assume that \(D_{21}(x)D_{21}(x)' = E_2(x) > 0\) for all \(x\).

We consider the admissible filtering strategies as the set of causal maps of the observation

\(\hat{z} : L_2([0,T], \mathbb{R}^p) \to L_2([0,T], \mathbb{R}^q)\).

The causality means that for any \(0 \leq t_0 \leq s \leq T\), if \(y_1, y_2 \in L_2([t_0,T], \mathbb{R}^p)\) and \(y_1(r) = y_2(r)\) a.e. \(r \in [t_0,s]\), then \(\hat{z}[y_1](r) = \hat{z}[y_2](r)\) a.e. \(r \in [t_0,s]\). The filtering signal is obtained via \(\hat{z}(r) = \hat{z}[y](r), r \in [t_0,T]\). We denote the class of such strategies by \(\mathcal{Z}\). In the sequel, we use the notation \(\hat{f}[y]\) to mean that \(\hat{f}\) is a causal function of \(y\). Admissible disturbances are all signals \(w \in L_2([0,T], \mathbb{R}^d)\).

We denote \(e\) the difference between the to-be estimated quantity \(z\) and its estimate \(\hat{z}\), i.e. \(e = z - \hat{z}\). Let us denote the map from \(w\) to \(e\), with initial condition \(x_0\), under the filtering policy \(\hat{z}\) by \(\Sigma_{x_0}^\hat{z}\). Given \(\gamma > 0\) and time interval \([0,T]\), the finite time horizon robust \(H_\infty\) filtering problem is to find \(\hat{z} \in \mathcal{Z}\) such that for any initial condition \(x_0 \in \mathbb{R}^n\) the map \(\Sigma_{x_0}^\hat{z}\) is finite gain, which means there exists a finite quantity \(\beta^\hat{z}_\Sigma(x) \geq 0\) such that

\[
\frac{1}{2} \int_0^T |e(t)|^2 dt \leq \frac{1}{2} \gamma^2 \int_0^T |w(t)|^2 dt + \beta^\hat{z}_\Sigma(x_0),
\]

(7.3)

for all \(w \in L_2([0,T], \mathbb{R}^d)\). We assume that \(\beta^\hat{z}_\Sigma(0) = 0\). Inequality (7.3) is our replacement of the previous one in (7.1).
We use game-theoretic methods to solve this problem. For \( z \in \mathcal{Z} \) define the cost functional related to the filtering problem as follows

\[
J_{\alpha,T}(z) \triangleq \sup_{w \in L_2([0,T], \mathbb{R}^d), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) + \frac{1}{2} \int_0^T [e(t)]^2 - \gamma^2 |w(t)|^2 \} dt,
\]

where \( \alpha \) belongs to a function space \( \mathcal{X} \). From the definition, we immediately see that if the map \( \Sigma^\mathcal{Z} \) is finite gain, then

\[
(\alpha, 0) \leq J_{\alpha,T}(z) \leq (\alpha, \beta^\mathcal{Z}_T),
\]

where \( (\cdot, \cdot) \) denotes the sup pairing \( (p, q) = \sup_{z} \{ p(z) + q(z) \} \), for functions \( p, q \in \mathcal{X} \) (see Chapter 5, see also [56], [57]). Thus, \( J_{\alpha,T}(z) \) is finite on the set \( \{ \alpha \in \mathcal{X} : (\alpha, 0), (\alpha, \beta^\mathcal{Z}_T) \ \text{finite} \} \subset \text{dom} J_{\alpha,T}(z) \). The idea is that a solution to the robust filtering problem will be obtained by minimizing \( J_{\alpha,T}(z) \) over \( \mathcal{Z} \). This is a zero-sum dynamic game problem with partial observation. In this game, the initial condition is assumed unknown and is considered as a part of the disturbances.

### 7.2.1 Information State Filter

In this section we formally solve the partially observed game problem associated with the filtering problem by following the information state method developed in [56] and [57]. For fixed output \( y \in L_2([0,T], \mathbb{R}^p) \) and filtering signal \( \hat{z} \in L_2([0,T], \mathbb{R}^q) \), we define the information state by

\[
p_t(z) \triangleq \sup_{x_0 \in \mathbb{R}^n, w \in L_2([0,t], \mathbb{R}^d)} \{ \alpha(x_0) + \frac{1}{2} \int_0^t [e(s)]^2 - \gamma^2 |w(s)|^2 \} ds
\]

\[
: y(s) = C_2(z(s)) + D_{21}(z(s))w(s), \ 0 \leq s \leq t, \ z(t) = z.
\]

This quantity gives the worst-case filtering cost up to time \( t \) which is consistent with the output record \( y \) and the constraint \( z(t) = x \). As we shall see later in this section, this quantity extracts information available in the observation which is suitable for the filtering task. The information state \( p_t \) has dynamics given by

\[
p_t = F(p_t, \hat{z}(t), y(t)),
\]

with initial condition \( p_0 = \alpha \) in \( \mathcal{X} \), in which

\[
F(p, \hat{z}, y) = \sup_{w \in \mathbb{R}^d} \{ -\nabla_p (A(z) + B(z)w) + \frac{1}{2} |\hat{z} - C_1(z)|^2 - \frac{1}{2} \gamma^2 |w|^2 \}
\]

\[
: y = C_2(z) + D_{21}(z)w.
\]
7.2 Finite Time Problem

The right hand side of the above expression is a constrained maximization and can be evaluated using the standard Lagrange multiplier technique to yield

\[
F(p, \bar{z}, y) = -\nabla_x p(A(x) + B(x)D_{21}(x)'E_2(x)^{-1}(y - C_2(x))) \\
+ \frac{1}{2} \gamma^{-2} \nabla_x pB(x)(I - D_{21}(x)'E_2(x)^{-1}D_{21}(x))B(x)'\nabla_x p' \\
+ \frac{1}{2}(|\bar{z} - C_1(x)|^2 - \gamma^2(y - C_2(x))'E_2(x)^{-1}(y - C_2(x))).
\]

(7.9)

When \( y = 0, \bar{z} = 0 \) this expression gives a nonlinear generalization of the filtering Riccati differential equation in the linear systems case (see [80], [87]).

Remark 7.1

If in (7.2) we have \( B(x)D_{21}(x)' = 0 \), the information state dynamics becomes

\[
F(p, \bar{z}, y) = -\nabla_x p(A(x) + \frac{1}{2} \gamma^{-2} \nabla_x pB(x)B(x)'\nabla_x p' \\
+ \frac{1}{2}(|\bar{z} - C_1(x)|^2 - \gamma^2(y - C_2(x))'E_2(x)^{-1}(y - C_2(x))).
\]

If in (7.2), \( D_{21}(x) = I \), the information state dynamics reduces to

\[
F(p, \bar{z}, y) = -\nabla_x p(A(x) + B(x)(y - C_2(x))) \\
+ \frac{1}{2}(|\bar{z} - C_1(x)|^2 - \gamma^2(y - C_2(x))(y - C_2(x))).
\]

This case corresponds to the 1-block problem in the linear \( H_\infty \) filter literature.

The cost function (7.4) can be represented in terms of \( p_t(x) \) (see also [57] Theorem 3.6).

Theorem 7.1 For any \( \bar{z} \in \mathcal{Z} \) we have

\[
J_{\alpha,T}(\bar{z}) = \sup_{y \in L_2([0,T],\mathbb{R}^r)} \{(p_T,0) : p_0 = \alpha \}.
\]

(7.10)
Proof. For all \( z \in Z \) we have,

\[
\sup_{y \in L^2([0,T], R^r)} \{ (p_T, 0) : p_0 = p \} = \sup_{y \in L^2([0,T], R^r)} \sup_{x \in \mathbb{R}^n} \{ \sup_{w \in L^2([0,T], R^d)} \{ p(x_0) \\
+ \frac{1}{2} \int_0^T [\|e(s)\|^2 - \gamma^2 \|w(s)\|^2] ds : \ y(s) = C_2(x(s)) + D_{21}(x(s))w(s) \}, \ 0 \leq s \leq T : x(T) = x \} \},
\]

\[
= \sup_{y \in L^2([0,T], R^r)} \sup_{w \in L^2([0,T], R^d)} \{ p(x_0) \\
+ \frac{1}{2} \int_0^T [\|e(s)\|^2 - \gamma^2 \|w(s)\|^2] ds \}
\]

\[
= J_{p_T, T}(z).
\]

This completes the proof.

In (7.10), the left hand side expression is the original filtering cost function expressed in terms of the state \( z \) (see (7.4)), which is only partially observed, while the right hand side one is an equivalent representation of \( J \) in terms of the information state \( p \), which is a completely known quantity obtained through (7.7) and (7.8). This shows that the information state \( p \) extracts information relevant for the optimization problem at hand which is available in the measurement. Thus, as in the case of output feedback control in Chapter 5, we now have a completely observed dynamic game problem in which the information state \( p \) replaces the original state \( z \).

We employ dynamic programming methods to solve the complete observation problem. To this end, define the value function \( W(p, t; T) \) for \( (p, t) \in \mathcal{X} \times [0, T] \) by

\[
W(p, t; T) \triangleq \inf_{\hat{z} \in Z} \sup_{y \in L^2([0,T], R^r)} \{ (p_T, 0) : p_t = p \}.
\]

(7.11)

Suppose that under the filtering policy \( \hat{z}^o \), the map \( \Sigma^{\hat{z}^o} \) is finite gain. Then using (7.5) and (7.10) we have

\[
0 = (-\beta_T^{\hat{z}^o}, 0) \leq W(-\beta_T^{\hat{z}^o}, 0; T) \leq J_{-\beta_T^{\hat{z}^o}, T}(\hat{z}^o) \leq (-\beta_T^{\hat{z}^o}, \beta_T^{\hat{z}^o}) = 0.
\]

(7.12)

Thus, \( W(-\beta_T^{\hat{z}^o}, 0; T) = 0 \). The dynamic programming principle then leads us to the following infinite dimensional dynamic programming equation for \( W \) ([57], [42]).

\[
\frac{\partial W}{\partial t} + \inf_{\hat{z} \in Z} \sup_{y \in R^r} \{ (\nabla_p W, F(p, \hat{z}, y)) \} = 0 \text{ in } \mathcal{X} \times [0, T],
\]

(7.13)
7.2 Finite Time Problem

with terminal condition \( W(p,T;T) = (p,0) \) in \( \mathcal{X} \). In this equation, we use the notation \( \nabla_p W \) to denote, say, the Frechet derivative of \( W \) with respect to \( p \) which lives in the dual space \( \mathcal{X}^* \). Thus, \( (\nabla_p W,F(p)) \) is the directional derivative of \( W \) at \( p \) in the direction \( F(p) \). Since in the expression (7.8) for \( F(p,z,y) \), the variables \( z \) and \( y \) are not coupled, the order in which the operations of inf and sup being carried out is interchangeable, i.e., the Issacs condition holds. The DPE (7.13) is the Hamilton-Jacobi-Isaacs (HJI) equation for the partially observed game associated with the filtering problem.

Conversely, suppose that smooth solutions to DPEs (7.13) and (7.8) exist, with \( p_0 = -\beta \), \( W(-\beta,0;T) = 0 \) for some function \( \beta \geq 0 \) with \( \beta(0) = 0 \). Suppose also that the filtering policy \( \hat{z}^*(p,t;T) \) achieves minimum in the left hand side of (7.13), i.e.,

\[
\hat{z}^*(p,t;T) \in \text{argmin}_{\hat{z} \in \mathbb{R}^*} \{ \sup_{y \in \mathbb{R}^p} \{(\nabla_p W,F(p,\hat{z},y))}\}.
\] (7.14)

Then, we obtain by integrating the left hand side of (7.13) along the trajectory produced by (7.8) (see also Section 5.2.2)

\[
J_{-\beta,T}(\hat{z}^*) = \sup_{y \in \mathcal{L}_2([0,T],\mathbb{R}^p)} \{(p_T,0) : p_0 = -\beta\} = W(-\beta,0;T) = 0.
\]

(In this case the equality is achieved because the Isaacs condition holds in (7.13).) This implies that the map \( \Sigma \hat{z}^* \) is finite gain. Moreover, since \( (\nabla_p W(p,t;T),\cdot) \) is a linear map from \( \mathcal{X} \to \mathbb{R} \), and that for all variables \( v \) independent of \( x \), \( (\nabla_p W(p,t;T),v) = v \), we get the following expression

\[
(\nabla_p W(p,t;T),F(p,\hat{z},y)) = (\nabla_p W(p,t;T),-\nabla_x p(A + BD_2 E_2^{-1}(y - C_2))
+ \frac{1}{2}\gamma^2 \nabla_x B(I - D_1 D_2^{-1} D_{21}) B' \nabla_x p' + \frac{1}{2}(|C_1|^2 - \gamma^2 C_2^2 E_2^{-1} C_2)
- \hat{z}'(\nabla_p W(p,t;T),C_1) + \gamma^2 y'(\nabla_p W(p,t;T),C_2) + \frac{1}{2}|\hat{z}|^2 - \gamma^2 |y|^2).
\]

In particular, the optimal filtering strategy can be obtained by solving the equation

\[
\frac{\partial}{\partial \hat{z}} \{(\nabla_p W(p,t;T),F(p,\hat{z},y))\} = 0.
\]

This yields

\[
\hat{z}^*(p,t;T) = (\nabla_p W(p,t;T),C_1).
\]

Thus, the filtering policy depends on \( T \). Similarly, the maximizing (or the worst) output is given by

\[
y^*(p,t;T) = (\nabla_p W(p,t;T),-E_2^{-1} D_{21} B' \nabla_x p' - C_2).
\]

The structure of the \( H_{\infty} \) filter is depicted in the following figure.
7.2.2 Certainty Equivalence Filter

In [25], [66], a certainty equivalence principle (CEP) has been proposed to solve the $H_{\infty}$ filtering problem. This method, when valid, is computationally less intensive. In the bilinear systems case [94], and in the general nonlinear systems case [57], both are for the finite gain or $H_{\infty}$ case, it is shown how the CEP fits within the information state framework, under appropriate assumptions. We shall now discuss the CE principle for $H_{\infty}$ filtering problem.

We consider the state feedback value function $V(x, t; T)$, related to the filtering problem in §2.1, whose evolution is described by the DPE

$$
\begin{align*}
\frac{\partial V}{\partial t} + \inf_{z \in \mathbb{R}^n} \sup_{u \in \mathbb{R}^d} \{ \nabla_z V(z)(A(z) + B(z)w) + \frac{1}{2}(|\dot{z} - C_1(z)|^2 - \gamma^2|w|^2) \} = 0,
\end{align*}
$$

with terminal condition $V(x, T; T) = 0$. Evaluation of the DPE (7.15) yields $V = 0$. We denote the state feedback filter that achieves minimum in the left hand side of the DPE by $\hat{z}^*(x) = C_1(x)$. The certainty equivalence filter is defined by

$$
\begin{align*}
\hat{z}^{CE}(t) = \hat{z}^{CE}[y](t) \overset{\Delta}{=} \hat{z}^*(\mathcal{R}(p_t[y])) = C_1(\mathcal{R}(p_t[y])) = C_1(\hat{x}(t)).
\end{align*}
$$

where $\hat{x}(t) = \mathcal{R}(p_t[y]) \in \arg\max_{x \in \mathbb{R}^n} \{ p_t(x) \}$. In [25], it is shown that under the assumptions: (i) $p$ is smooth and (ii) the set $\arg\max_{x \in \mathbb{R}^n} \{ p_t(x) \}$ is singleton for all $t$, implementation of the CE filtering policy results in the following performance

$$
J_{\alpha, T}(z^{CE}) \leq \inf_{\hat{z} \in \mathcal{Z}} \sup_{\omega \in L_2([0, T], \mathbb{R}^d), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) + \frac{1}{2} \int_0^T [e(t)]^2 - \gamma^2|w(t)|^2 dt \},
$$

i.e., the CE filter is minimax. We shall now put the CE filter within the information state framework. Fix a point $p^1$ in $\mathcal{X}$ and assume that $\hat{x}^1 = \mathcal{R}(p^1)$ is unique. Define
the function $\tilde{W}(p, t; T) = (p, 0)$. By definition we have $\frac{\partial W}{\partial t}(p^1, \cdot) = 0$ and $\nabla_x p_t(z^1) = 0$. Moreover, as shown in Theorem 6.1 of [57], $\tilde{W}(p, t; T)$ is Gateaux differentiable at $p^1$ and $\langle \nabla_p \tilde{W}(p^1, t; T), q \rangle = q(z^1)$ for $q \in \mathcal{X}$, i.e., $\partial_p \tilde{W}(p^1, \cdot)$ is the evaluation map. Substituting these derivatives into the left hand side of the DPE (7.13) yields

$$
\frac{\partial W}{\partial t}(p^1, \cdot) + \inf_{z \in \mathbb{R}^*} \sup_{y \in \mathbb{R}^*} \{\langle \nabla_p \tilde{W}(p^1, \cdot), F(p^1, z, y) \rangle \}
= 0 + \inf_{z \in \mathbb{R}^*} \sup_{y \in \mathbb{R}^*} \sup_{w \in \mathbb{R}^*} \{-\nabla_x p(A(z^1) + B(z^1)w)
+ \frac{1}{2}|z - C_1(z^1)|^2 - \frac{1}{2} \gamma^2 |w|^2 : y = C_2(z^1) + D_{21}(z^1)w \}
= 0.
$$

This shows that $\tilde{W}(p, t; T)$ satisfies the DPE (7.13) (in the Gateaux sense) at $p^1$.

**Remark 7.2** When the CE principle holds, the resulting optimal filter is completely determined by the DPE for $p$, and therefore, is independent of $T$. However, in general the CEP is only suboptimal (see [57]). In this case, the optimal filter is determined by two DPEs, namely the equations (7.8) and (7.13).

![Figure 7.2: Certainty Equivalence Filter.](image)

### 7.2.3 Linear $H_\infty$ Filtering

In this section, we consider the $H_\infty$ filtering problem for linear systems. As we shall see, this leads to finite dimensional solutions and recovers the linear (central) $H_\infty$ filter [80], [87]. The systems are described as in (7.2) with $A(x) = Ax, B(x) = B, C_1(x) = C_1x, C_2(x) = C_2$ and $D_{21}(x) = D_{21}$, with $D_{21}D_{21} = E_2 > 0$, in which $A, B, C_1, C_2, D_{21}$
are constant matrices with appropriate sizes.

In this case, the information state (7.6) evolves according to the DPE

\[
\dot{p}_t = -\nabla_x p(A + BD_{21}E_2^{-1}(y - C_2x) + \frac{1}{2}\gamma^{-2}pB(I - D_{21}E_2^{-1}D_{21})B'\nabla_x p' + \frac{1}{2}(|\hat{x} - C_1x|^2 - \gamma^2(y - C_2x)'E_2^{-1}(y - C_2x)),
\]

(7.17)

with initial condition \(p_0\). If we choose \(p_0\) to be quadratic, i.e. \(p_0(x) = -\frac{1}{2}x'Px\) for some positive definite matrix \(P\) (this choice corresponds to the assumption of quadratic functions \(\beta_2(x)\) in (7.3)), the information state in (7.6) is finite dimensional.

**Lemma 7.1** Suppose that \(p_0(x) = -\frac{1}{2}x'Px\) for some positive definite matrix \(P\). Then \(p_t(x)\) has the explicit solution given by

\[
p_t(x) = -\frac{1}{2}\gamma^2(z - \hat{x}(t))'Y(t)^{-1}(z - \hat{x}(t)) + \varphi(t),
\]

(7.18)

where \(\hat{x} \in \mathbb{R}^n, Y \in \mathbb{R}^{nxn}, \) and \(\varphi \in \mathbb{R}\) satisfy the ODEs

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + (BD_{21} + Y(t)C_2')E_2^{-1}(y(t) - C_2\hat{x}(t))
- \gamma^{-2}Y(t)C_1'(\hat{x}(t) - C_1\hat{x}(t)),

\dot{Y}(t) &= (A - BD_{21}E_2^{-1}C_2)Y(t) + Y(t)(A - BD_{21}E_2^{-1}C_2)'Y(t)
- Y(t)(C_2E_2^{-1}C_2 - \gamma^{-2}C_1'C_1)Y(t) + B(I - D_{21}E_2^{-1}D_{21})B',

\dot{\varphi}(t) &= \frac{1}{2}((\hat{x}(t) - C_1\hat{x}(t))'(\hat{x}(t) - C_1\hat{x}(t))
- \gamma^2(y(t) - C_2\hat{x}(t))'E_2^{-1}(y(t) - C_2\hat{x}(t))).
\end{align*}
\]

(7.19)

with initial conditions \(\hat{x}(0) = 0, Y(0) = \gamma^2P^{-1} > 0, \varphi(0) = 0\).

**Proof.** The result follows by substituting \(p_t(x)\) in (7.18) together with the ODEs in (7.19) into the DPE (7.17). \(\square\)

Defining the finite dimensional quantity by \(\rho(t) \triangleq (\hat{x}(t), Y(t), \varphi(t))\), equation (7.19) can be rewritten as

\[
\dot{\rho}(t) = \dot{F}(\rho(t), \hat{x}(t), y(t)),
\]

with initial condition \(\rho(0) = \rho \in \mathbb{R}^n \times \mathbb{R}^{nxn} \times \mathbb{R}\), in which \(\dot{F}\) denotes the right hand side of (7.19). Thus the finite dimension quantity \(\rho\) is identified with the quadratic information state \(p\) in (7.18). We denote the quadratic information state by \(p^\rho\).
7.3 Infinite Time Problem

Since, \((p^\rho_t, 0) = \varphi(t)\), the equation (7.10) yields the representation for \(J\) given by

\[
J_{\rho, T}(\hat{z}) = J_{\rho^*, T}(\hat{z}) = \sup_{\hat{z} \in \mathbb{Z}} \{ \varphi(T) : \rho(0) = \rho \}.
\]

The value function \(W(\rho, t; T)\) defined by

\[
W(\rho, t; T) = \inf_{\hat{z} \in \mathbb{Z}} J_{\rho, T}(\hat{z}) = \inf_{\hat{z} \in \mathbb{Z}} \sup_{\hat{z} \in \mathbb{Z}} \{ \varphi(T) : \rho(t) = \rho \}.
\]

solves the dynamic programming equation (see [94])

\[
\begin{align*}
\frac{\partial W}{\partial t} + \inf_{\hat{z} \in \mathbb{Z}} \sup_{\varphi \in \mathbb{Z}} & \{ \frac{\partial W}{\partial \hat{z}} (A\hat{\hat{z}} + (BD_{21} + YC_2)E_2^{-1}(y - C_2\hat{\hat{z}}) - \gamma^{-2}YC_1'(\hat{\hat{z}} - C_1\hat{\hat{z}}) ) \\
+ (\frac{\partial W}{\partial Y}, ((A - BD_{21}E_2^{-1}C_2)Y + Y(A - BD_{21}E_2^{-1}C_2)' - Y(C_2'E_2^{-1}C_2 - \gamma^{-2}C_1'C_1)Y \\
+ B(I - D_{21}E_2^{-1}D_{21})B')) & + \frac{\partial W}{\partial \rho} \frac{1}{2} (\hat{z} - C_1\hat{\hat{z}})^2 - \gamma^2(y - C_2\hat{\hat{z}})'E_2^{-1}(y - C_2\hat{\hat{z}})) \} = 0,
\end{align*}
\]

(7.21)

with terminal condition \(W(\rho, T; T) = (p^\rho, 0) = \varphi\). This DPE is the appropriate one for the linear \(H_\infty\) filtering problem.

**Lemma 7.2** The value function \(W(\hat{z}, Y, \varphi, t)\) has the explicit solution given by

\[
W(\rho, t; T) = W(\hat{z}, Y, \varphi, t; T) = \varphi.
\]

**Proof.** This result can be verified by directly substituting \(W\) into the LHS of (7.21) (since \(\frac{\partial W}{\partial \hat{z}} = 0, \frac{\partial W}{\partial Y} = 0, \frac{\partial W}{\partial \rho} = 1\) and \(W(\rho, T; T) = \varphi\)).

The optimal filtering policy is given by

\[
\begin{align*}
\hat{z}^*(t) &= \hat{z}^*(\rho(t), t; T) = C_1\hat{\hat{z}}(t) + \gamma^{-2}(\frac{\partial W}{\partial \rho})^{-1}Y(t)C_1'\frac{\partial W}{\partial \hat{z}} = C_1\hat{\hat{z}}(t),
\end{align*}
\]

which is independent of \(T\) and has the structure of the (central) linear \(H_\infty\) filter. We note that in the linear systems case the optimal information state filter coincides with the certainty equivalence filter.

### 7.3 Infinite Time Problem

In this section we write down the appropriate equations for infinite time horizon \(H_\infty\) filtering. We consider the class of admissible filtering strategies \(Z\) as the set of causal
maps of the observation $z : L_2([0,\infty), \mathbb{R}^p) \to \mathbb{Z}$. For given $\gamma > 0$, the infinite time horizon robust $H_\infty$ filtering problem is to find $\hat{z} \in \mathbb{Z}$ such that for any initial condition $z_0$ the following inequality holds

$$\frac{1}{2} \int_0^T |e(t)|^2 dt \leq \frac{1}{2} \gamma^2 \int_0^T |w(t)|^2 dt + \beta^2(z_0),$$

for all $w \in L_2([0,T], \mathbb{R}^d)$, and all $T \geq 0$, for some finite $\beta^2(x) \geq 0$, which is independent of $T$, with $\beta^2(0) = 0$. To solve this problem we consider the cost functional

$$J_p(\hat{z}) = \sup_{T \geq 0} J_{p,T}(\hat{z}),$$

in which $J_{p,T}(\hat{z})$ is defined in (7.4). We see that $\Sigma^\hat{z}$ is finite gain if and only if $J_{-\beta}(\hat{z}) \leq 0$, or, equivalently, by the representation (7.10), $\Sigma^\hat{z}$ is finite gain if and only if

$$\sup_{T \geq 0, y \in L_2([0,\infty), \mathbb{R}^p)} \{(p_T, 0) : p_0 = -\beta \} \leq 0,$$

for some $\beta \geq 0$ with $\beta(0) = 0$. This leads us to the following (stationary) value function

$$W(p) = \inf_{\hat{z}} \sup_{T \geq 0, y \in L_2([0,\infty), \mathbb{R}^p)} \{(p_T, 0) : p_t = p \}.$$  \tag{7.23}

By the dynamic programming principle, the value function $W$ satisfies the following stationary (infinite dimensional) partial differential inequality (PDI)

$$\inf_{\hat{z}} \sup_{y \in \mathbb{R}^p} \{(\nabla_p W(p, \hat{z}, y)) \leq 0.$$ \tag{7.24}

The optimal filtering strategy is then given by

$$\hat{z}^*(p) = (\nabla_p W(p), C_1)$$

which is independent of $T$, and the worst output by

$$y^*(p) = (\nabla_p W(p), -E_2^{-1}D_{21}B' \cdot \nabla_{zz} - C_2).$$

### 7.4 General Dissipative Filters

The information state method employed in the previous section provides a framework for achieving filters with more general dissipative performance measures. Dissipative performances are desirable when we consider uncertain systems (i.e., nominal plants plus uncertainties) with dissipative uncertain components. If the dissipativity of the nominal systems and the uncertain systems are matched, in some sense, then the uncertain systems
enjoy stability properties (see [44]). In the filtering context, we say that the map \( \Sigma^\delta_w \) is **dissipative** with respect to the supply rate function \( r(e,w) \) if there exists a finite quantity \( \beta^\delta_T(x) \geq 0 \) such that

\[
-\int_0^T r(e(t),w(t))dt \leq \beta^\delta_T(x_0),
\]

(7.25)

for all \( w \in L_2([0,T],\mathbb{R}^d) \) (see [110] for discrete time formulation). We assume that \( \beta^\delta_T(0) = 0 \). The corresponding information state for this problem is given by

\[
\dot{p}_t = \sup_{w \in \mathbb{R}^d} \{-\nabla_x p(A(x) + B(x)w) - r(\tilde{z} - C_1(x), w) : y = C_2(x) + D_21(x)w\},
\]

(7.26)

with initial \( p_0 = \alpha \), and the DPE for constructing the optimal filter is given by

\[
\frac{\partial W}{\partial t} + \sup_{p \in \mathbb{R}^d} \inf_{\hat{z} \in \mathbb{R}^d} \{ (\nabla_p W, F(p, \hat{z}, y) ) \} = 0 \text{ in } \mathcal{X} \times [0,T],
\]

(7.27)

with \( W(p,T;T) = (p,0) \) in \( \mathcal{X} \), where \( F(\cdot) \) denotes the dynamics of \( p \) in the RHS of (7.26). In general, the Isaacs condition may not hold in (7.27). If the supply rate is quadratic, i.e.,

\[
r(e, w) = \frac{1}{2}(w'Qw + 2w'Se + e'RE),
\]

where \( Q, S, R \) are constant matrices with \( Q > 0, R \leq 0 \), then evaluation of the right hand side of (7.26) yields

\[
F(p, \hat{z}, y) = -\nabla_x p(A(x) - B(x)Q^{-1}\tilde{D}(x)Se + B(x)Q^{-1}D_21(x)'E_2(x)^{-1}v)
\]

\[
+ \frac{1}{2}\nabla_x pB(x)Q^{-1}\tilde{D}(x)B(x)'\nabla_x p' + \frac{1}{2}e'(S'Q^{-1}\tilde{D}(x)S - R)e
\]

\[
-\frac{1}{2}v'\tilde{E}_2(x)^{-1}v - v'E_2(x)^{-1}D_21(x)Q^{-1}Se,
\]

with \( e = \hat{z} - C_1(x), \) \( v = y - C_2(x), \) where

\[
\tilde{D}(x) = (I - D_21(x)'E_2(x)^{-1}D_21(x)Q^{-1}), \quad E_2(x) = D_21(x)Q^{-1}D_21(x)'.
\]

When \( Q = \gamma I, S = 0, R = -I, \) the \( H_\infty \) filter results. In the quadratic case, the optimal filter is given by (assuming that \((\nabla_p W(p,t;T), S'Q^{-1}\tilde{D}S - R) \) is invertible)

\[
2^*(p,t;T) = -((\nabla_p W(p,t;T), S'Q^{-1}\tilde{D}S - R))^{-1}
\]

\[
\times (\nabla_p W(p,t;T), S\tilde{D}'Q^{-1}B'\nabla_x p' - SQ^{-1}D_21E_2^{-1}(y - C_2)).
\]

### 7.4.1 Linear Systems Case

In the linear systems case we obtain finite dimensional dissipative filters. In particular, the information state is quadratic as given below.
Lemma 7.3  Suppose that \( p_0(x) = -\frac{1}{2}x^TPx \) for some positive definite matrix \( P \). Then \( p_t(x) \) has the explicit solution given by

\[
p_t(x) = -\frac{1}{2}(x - \hat{x}(t))^\prime Y(t)^{-1}(x - \hat{x}(t)) + \varphi(t),
\]

where \( \hat{x} \in \mathbb{R}^n, Y \in \mathbb{R}^{nxn} \), and \( \varphi \in \mathbb{R} \) satisfy the ODEs

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + BQ^{-1}(D_2E_2^{-1}(y(t) - C_2\hat{x}(t)) - \bar{D}S(\hat{x}(t) - C_1\hat{x}(t))) \\
&\quad + Y(t)((C_2'E_2^{-1} + C_1'SQ^{-1}D_2E_2^{-1})(y(t) - C_2\hat{x}(t)) \\
&\quad + (C_2'E_2^{-1}D_2Q^{-1}S - C_1'\bar{S})(\hat{x}(t) - C_1\hat{x}(t))), \\
\dot{Y}(t) &= \bar{A}Y(t) + Y(t)\bar{A}' + Y(t)\Omega_Y Y(t) + BQ^{-1}\bar{D}B', \\
\dot{\varphi}(t) &= \frac{1}{2}((\hat{x}(t) - C_1\hat{x}(t))^\prime \bar{S}(\hat{x}(t) - C_1\hat{x}(t)) - (y(t) - C_2\hat{x}(t))^\prime E_2^{-1}(y(t) - C_2\hat{x}(t)) \\
&\quad - 2(\hat{x}(t) - C_1\hat{x}(t))^\prime S'Q^{-1}D'E_2^{-1}(y(t) - C_2\hat{x}(t)))
\end{align*}
\]

with initials \( \hat{x}(0) = \hat{x}_0, Y(0) = P^{-1} > 0, \varphi(0) = \varphi \), in which \( \bar{A} = A + BQ^{-1}(DSC_1 - D_2E_2^{-1}C_2) \), \( \Omega_Y = C_1\bar{S}C_1 - C_2'E_2^{-1}C_2 - (C_1'SQ^{-1}D_2E_2^{-1}C_2 + C_2'E_2^{-1}D_2Q^{-1}SC_1) \) and \( \bar{S} = S'Q^{-1}\bar{D}S - R > 0. \)

Proof. The result follows by substituting \( p_t(x) \) in (7.28) together with the ODEs in (7.29) into the DPE (7.26).

The dissipative generalization of the DPE (7.21) is given by

\[
\begin{align*}
\frac{\partial W}{\partial t} + \sup_{p \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} \{ \frac{\partial W}{\partial x}(\hat{x}(t)) \\
&\quad + BQ^{-1}(D_2E_2^{-1}(y(t) - C_2\hat{x}(t)) - \bar{D}S(\hat{x}(t) - C_1\hat{x}(t))) \\
&\quad + Y(t)((C_2'E_2^{-1} + C_1'SQ^{-1}D_2E_2^{-1})(y(t) - C_2\hat{x}(t)) \\
&\quad + (C_2'E_2^{-1}D_2Q^{-1}S - C_1'\bar{S})(\hat{x}(t) - C_1\hat{x}(t))) \\
&\quad - 2(\hat{x}(t) - C_1\hat{x}(t))^\prime S'Q^{-1}D'E_2^{-1}(y(t) - C_2\hat{x}(t))) \\
&\quad + \frac{\partial W}{\partial \varphi} \frac{1}{2}((\hat{x}(t) - C_1\hat{x}(t))^\prime \bar{S}(\hat{x}(t) - C_1\hat{x}(t)) - (y(t) - C_2\hat{x}(t))^\prime E_2^{-1}(y(t) - C_2\hat{x}(t)) \\
&\quad - 2(\hat{x}(t) - C_1\hat{x}(t))^\prime S'Q^{-1}D'E_2^{-1}(y(t) - C_2\hat{x}(t))) \} = 0,
\end{align*}
\]

with terminal condition \( W(\rho, T; T) = (p^\rho, 0) = \varphi \). In this DPE, the Isaacs condition does not hold in general.
Lemma 7.4  The value function \( W(\hat{x}, Y, \varphi, t; T) \) has the explicit solution given by
\[
W(\rho, t; T) = W(\hat{x}, Y, \varphi, t; T) = \rho.
\]

Proof. This result follows easily by directly substituting \( W \) into the LHS of (7.30) by using the derivatives \( \frac{\partial W}{\partial \hat{x}} = 0, \frac{\partial W}{\partial Y} = 0, \frac{\partial W}{\partial \varphi} = 1 \) and noting that \( W(\rho, T; T) = \rho \).

The optimal filtering policy is given by
\[
\hat{x}^*(\hat{x}, Y, y) = \tilde{S}^{-1}S\hat{Q}^{-1}D_{21}'E_2^{-1}(y - C_2\hat{x}) + C_1\hat{x}.
\]

Thus, as in the \( H_\infty \) filtering case, the quadratic dissipative filters are determined by one Riccati differential equation and is independent of the horizon length \( T \). A new feature in the general case is that the filtering policy depends explicitly on \( y \).

7.5 Conclusions

In this chapter we have formulated a general dissipative filtering problem for nonlinear systems and presented a solution by applying the information state method. A new feature in our solution is that it is expressed in terms of two dynamic programming equations. We have clarified the connection with existing solution in [25], [66], in the case of \( H_\infty \) filtering, which is expressed in terms of one equation describing the information state dynamics. In the case of linear systems, our solution yields a finite dimensional equation, which, for the \( H_\infty \) filtering problem, recovers the results in the literature [80], [87], [36].
Chapter 8

Applications

8.1 Introduction

In the previous chapters we have studied control/filtering synthesis problems for general dissipative performance measures that are beyond the familiar finite gain and passivity measures. In this chapter we consider two applications of the general dissipativity measures to robust stabilization problems for linear and nonlinear systems.

In the first application, we consider a class of nonlinear systems consisting of linear systems possessing sector bounded nonlinearities at their input and output [79], [77], [61], [37], [40]. Such problems are commonly encountered in practice in which the actuators or sensors fail to operate in their linear regions or are defective [61]. The problem can also be interpreted as the one of achieving specified gain and phase stability margins. In [79], [77] the authors consider a similar problem where nonlinearities occur only at input. In [79], a state feedback controller is proposed while in [77], the author proposes an output feedback controller. In [61], the author considers a stabilization for flexible structures with input and output nonlinearities and present a synthesis method by exploiting the model structure. In a recent paper [117], the authors consider a $H_\infty$ control problem for nonlinear systems with sector bounded nonlinearities.

In the literature, it is known that a LQ regulator can deliver $\frac{1}{2}$ to $\infty$ gain and $-60$ to $+60$ degree phase stability margins (see for example [2]). However, when an observer is used to estimate the state, the stability margins simply disappear [2]. In this case, one typically attempts to recover the margins in an iterative manner.
We propose a control method for stabilizing linear systems with specified sector bounded nonlinearities at their input and output. The method is obtained by firstly posing the stabilization problem as one of achieving specific quadratic dissipativity performance measure for a new, related, linear systems, and then synthesizing a controller that achieve the required dissipativity for the new systems. In the state feedback case, in which nonlinearities occur only at input, we express the desired controller in terms of a positive definite solution to an algebraic Riccati equation. We show that the resulting controller is stabilizing the linear systems for all nonlinearities within the specified sector bounds. The stability proof employs the Lyapunov methods. This result is extended to the case where the nominal model is nonlinear. We then consider an output feedback case in which nonlinearities occur at input and output simultaneously. We employ the information method for output feedback controller synthesis (see Chapter 6). We express the desired controller in terms of positive definite solutions of two ARE's satisfying a coupling condition. Stability results are obtained in a similar way as in the state feedback case. We demonstrate the utility of our control method by presenting a number of examples.

The second application concerns a robust stabilization problem for nonlinear systems, in which the uncertain parts are described as dissipative dynamical systems. Special cases of this problem are uncertain systems with finite $L_2$ gain or passive uncertain components. The importance of this problem setting is to allow for flexibility in characterization of uncertainties. This may lead to less conservative control design (see for example [37], [40] for some discussions on this matter for linear systems). We present a controller that stabilizes all the uncertain systems in an admissible class based on the results in Chapter 4. Our result complements those in [45] and extends those in [3], [48].

8.2 Linear Systems with Sector Bounded Nonlinearities

We shall review briefly the concept of sector bounded nonlinear functions. For a given input vector $u = [u_1 \ldots u_m]' \in \mathbb{R}^m$, we consider the class of nonlinear memoryless functions of $u$, $\phi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ of the form

$$\phi(u, t) = \begin{bmatrix} \phi_1(u_1, t) \\ \vdots \\ \phi_m(u_m, t) \end{bmatrix},$$

(8.1)
8.2 Linear Systems with Sector Bounded Nonlinearities

in which expression \( \phi_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \ i = 1, 2, \ldots, m \) are nonlinear memoryless functions. We assume that \( \phi(u, \cdot) \) is locally Lipschitz in \( u \) satisfying

\[
\phi(0, \cdot) = 0. \tag{8.2}
\]

**Definition 8.1** Given diagonal matrices \( L_1 \) and \( L_2 \), we say that \( \phi(u, t) \) is inside the sector \((L_1, L_2)\) if

\[
(\phi_i(u, t) - L_{1i}u_i)(\phi_i(u, t) - L_{2i}u_i) \leq 0, \quad i = 1, 2, \ldots, m, \tag{8.3}
\]

for all \( u \in \mathbb{R}^m, \ \ t \geq 0, \) where \( L_{1i}, L_{2i} \) are the \( i \)-th entries of \( L_1, L_2 \) respectively (see also [37], [77], [79], [40]). We say that \( \phi \) is strictly inside sector \((L_1, L_2)\) if

\[
(\phi_i(u, t) - L_{1i}u_i)(\phi_i(u, t) - L_{2i}u_i) < 0, \tag{8.4}
\]

for all \( u \in \mathbb{R}^m - \{0\}, \ t \geq 0. \)

For such a sector bounded function, we write \( \phi \) is (or is strictly) \( \in \) sector\((L_1, L_2)\). In cases where \( \phi \) is time invariant, we write \( \phi = \phi(u) \). If \( u \in \mathbb{R} \) and \( \phi(\cdot, t) : \mathbb{R} \rightarrow \mathbb{R} \), the above inequality can be re-expressed as \( L_1 \phi(u, t)/u \leq L_2 \), i.e., the graph of \( \phi(\cdot, t) \) lies in a region within two lines of slopes \( L_1 \) and \( L_2 \). Saturation, dead-zone and hysteresis are sector bounded nonlinearities that are commonly found in practice [61].

**Remark 8.3** The inequality (8.3) in the above definition is stronger than

\[
(\phi(u, t) - L_1u)(\phi(u, t) - L_2u) \leq 0 \tag{8.5}
\]

used in [40]. Our definition requires that each component \( \phi_i \) is sector bounded. Clearly, (8.3) implies (8.5).

8.2.1 Nonlinearity at the Input

We now consider a state feedback stabilization problem, in which nonlinearities occur at input. The control systems are described by (see Figure 8.1)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu_n(t), \quad x(0) = x_0, \\
\Sigma_{\phi}^K : \quad u_n(t) &= \phi(u(t), t), \\
u(t) &= Kx(t),
\end{align*} \tag{8.6}
\]
where $x \in \mathbb{R}^n$ and $u, u_n \in \mathbb{R}^m$ and $\phi$ is a (possibly unknown) nonlinear function inside the sector $\text{sector}(L_1, L_2)$ for some specified diagonal matrices $L_1, L_2$. We assume that $L_1, L_2$ satisfy
\[ L_{1i} < 1, \ L_{2i} > 1, \ i = 1, 2, \ldots, m. \quad (8.7) \]

The control task is to find a linear state feedback law $u = Kz$ such that $\Sigma^K$ is asymptotically stable for all nonlinear functions $\phi \in \text{sector}(L_1, L_2)$.

We reconfigure the system (8.6) as an interconnected system (see Figure 8.2) consisting of a new controlled linear system
\[
\dot{\Sigma}^K : \begin{cases}
\dot{z}(t) = Az(t) + Bu(t) + Bw(t), \ z(0) = z_0, \\
z(t) = u(t), \\
u(t) = Kz(t),
\end{cases}
\]
and a memoryless nonlinearity $\Phi$ given by
\[
\Phi : \Phi(z, t) = z - \phi(z, t) = \begin{bmatrix} z_1 - \phi_1(z_1, t) \\ \vdots \\ z_m - \phi_m(z_m, t) \end{bmatrix}.
\]

The interconnection is obtained by setting $w = -\Phi(z, t)$. Clearly, the original nonlinear system $\Sigma^\phi$ in (8.6) is equivalent with the interconnection of the new linear system $\dot{\Sigma}^K$ (8.8) and the nonlinearity $\Phi$ (8.9). Note that since $\phi$ satisfies (8.3), we have
\[
(\Phi_i(z_i, t) - (1 - L_{2i})z_i)(\Phi_i(z_i, t) - (1 - L_{1i})z_i) \leq 0, \quad (8.10)
\]
for $i = 1, 2, \ldots, m$, which implies
\[
(\Phi(z, t) - (I - L_2)z)'(\Phi(z, t) - (I - L_1)z) \leq 0,
\]
that is $\Phi$ is inside $\text{sector}((I - L_2), (I - L_1))$. 
If \( \phi \) is strictly inside sector \((L_1, L_2)\) (i.e., inequality (8.4) holds), then
\[
(\Phi(z, t) - (I - L_2)z)'(\Phi(z, t) - (I - L_1)z) < 0,
\]
for all \( z \neq 0 \), with \((\Phi(0, t) - (I - L_2)0)'(\Phi(0, t) - (I - L_1)0) = 0\) (since \(\Phi(0, t) = 0 - \phi(0, t) = 0\)).
As we shall see, a solution to the problem can be obtained by finding a linear feedback law \( u = Kx \) that renders the new closed loop \( \hat{S}^K \) dissipative with respect to the quadratic supply rate

\[
\tau_q(x, w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz),
\]

with appropriately chosen \( Q, S \) and \( R \). The following results provide a way for constructing such a controller (see Chapter 4).

**Theorem 8.1** Consider the nonlinear system \( \Sigma^K_{\phi} \) in (8.6) in which the nonlinearity \( \phi \) is inside sector\((L_1, L_2)\) with \( L_1, L_2 \) satisfying (8.7). Assume there exists a positive definite solution \( X \) to the ARE

\[
XA + A'X - X(B(I - Q^{-1}S)\bar{R}^{-1}(I - Q^{-1}S)'BB' - BQ^{-1}B')X = 0.
\]

where \( Q = -\frac{1}{2}(\bar{L}_1\bar{L}_2 + \bar{L}_1') > 0, \ S = \frac{1}{2}(\bar{L}_1 + \bar{L}_2), \ R = -I, \ \bar{R} = S'Q^{-1}S + I, \) with \( \bar{L}_1 = -(I - L_2)^{-1}, \ \bar{L}_2 = -(I - L_1)^{-1} \). Then, under the optimal control law \( u^* = K^*x \), where

\[
K^* = -\bar{R}^{-1}(I - Q^{-1}S)'B'X,
\]

the closed loop system \( \Sigma^K_{\phi} \) is Lyapunov stable. If: (i) \( \phi \) is time invariant (i.e., \( \phi = \phi(u) \)) and is strictly inside sector\((L_1, L_2)\) and (ii) the matrices pair \( (A+BK^*, K^*) \) is completely observable, the stability is asymptotic.

**Proof.** First, rewrite the ARE (8.13) as

\[
XA^{K^*} + (A^{K^*})'X + (XB - (C^{K^*})'S')Q^{-1}(B'X - SC^{K^*}) + (C^{K^*})'C^{K^*} = 0,
\]

where \( A^{K^*} = A + BK^* , \ C^{K^*} = K^* \). Define the function \( V(x) = \frac{1}{2}x'Xx \) (thus, \( V \) is positive definite and proper). Evaluation of \( dV/dt \) along the trajectory of the systems (8.8), under the control law \( u^* = K^*x \), yields

\[
dV/dt(x(t)) = -\frac{1}{2}(w(t) - w^*(t))'(w(t) - w^*(t)) + \frac{1}{2}(w(t)'Qw(t) + 2w(t)'S(z(t) - z(t)'z(t)) \\
\leq + \frac{1}{2}(w(t)'Qw(t) + 2w(t)'S(z(t) - z(t)'z(t)) \\
= -\frac{1}{2}(z(t) - \bar{L}_1w(t))'(z(t) - \bar{L}_2w(t)),
\]

in which \( w^* = Q^{-1}(B'X - SC^{K^*})x \). The right hand side expression of the last inequality
8.2 Linear Systems with Sector Bounded Nonlinearities

This expression can be rewritten as

\[-(z(t) - \bar{L}_1w(t))'(z(t) - \bar{L}_2w(t))\]

\[= -\sum_{i=1}^{m} L_{1i}\bar{L}_{2i}(-(1 - L_{1i})z_i - w_i)(-(1 - L_{2i})z_i - w_i)\]

\[= \sum_{i=1}^{m} (-\bar{L}_{1i}\bar{L}_{2i})(-(1 - L_{1i})z_i + \Phi_i(z_i))(-(1 - L_{2i})z_i + \Phi_i(z_i)).\]

From (8.10), and noting that $-\bar{L}_{1i}\bar{L}_{2i} > 0, i = 1, 2, \ldots m$, we conclude that

\[-(z(t) - \bar{L}_1w(t))'(z(t) - \bar{L}_2w(t)) \leq 0.\]

Thus,

\[dV(z(t))/dt \leq -\frac{1}{2}(z(t) - \bar{L}_1w(t))'(z(t) - \bar{L}_2w(t))\]

\[(8.15)\]

\[\leq 0,\]

which shows that $V$ is a Lyapunov function and, therefore, the interconnection of $\bar{\Sigma}^{K*}$ and $\Phi$ or, equivalently, the nonlinear system $\Sigma^{K*}_\phi$ in (8.6) is Lyapunov stable.

Next, suppose that $\phi$ is time invariant and is strictly inside sector($L_1, L_2$) (this implies that $\Phi$ is time invariant and inequality (8.11) holds with $(\Phi(0) - (I - L_2)0)'(\Phi(0) - (I - L_1)0) = 0$). Let $z^1(\cdot)$ denote the trajectory starting from $z^1 \in \mathbb{R}^n$ and $L^+$ denote its $\omega$-limit set, which is nonempty, compact and invariant (see Lemma 3.1 of [64]). Since $V$ is nonnegative and is nondecreasing along the trajectory, we have $\lim_{t \to \infty} V(z^1(t)) = a \geq 0$. By continuity of $V$, $V(\bar{z}) = a$ at each point $\bar{z} \in L^+$. Let $\bar{z}$ be in $L^+$ and $\bar{z}(\cdot)$ denote the corresponding trajectory. By invariance of $L^+$ and using (8.15), we have

\[0 = V(\bar{z}(T)) - V(\bar{z}) \leq -\frac{1}{2} \int_0^T (z(t) - \bar{L}_1w(t))'(z(t) - \bar{L}_2w(t))dt \leq 0,
\]

for all $T \geq 0$ implying (by the strict sector boundedness of $\Phi$) that $z(t) = K^*\bar{z}(t) = 0$ and also $w(t) = -\Phi(z(t)) = 0$, for all $t \geq 0$. By detectability assumption, $\lim_{t \to \infty} \bar{z}(t) = 0$ and, therefore, $a = 0$. Thus, $\lim_{t \to \infty} V(z^1(t)) = 0$, i.e., $\lim_{t \to \infty} z^1(t) = 0$ (by continuity and positive definiteness of $V$). This shows that the interconnection of $\bar{\Sigma}^{K*}$ and $\Phi$ or, equivalently, the nonlinear system $\Sigma^{K*}_\phi$ in (8.6) is asymptotically stable.

The design procedure is summarized as follows.

- Step 1: For given diagonal matrices $L_1$ and $L_2$ specifying the nonlinearities, with $L_{1i} < 1, L_{2i} > 1, i = 1, 2, \ldots, m$, set

\[S = \frac{1}{2}(\bar{L}_1 + \bar{L}_2), \quad Q = -\bar{L}_1\bar{L}_2,
\]

where $\bar{L}_1 = -(I - L_2)^{-1}$, $\bar{L}_2 = -(I - L_1)^{-1}$.
Step 2: Find a positive definite solution $X$ to the ARE (8.13). If no such solution exists increase the entries of $L_1$ and/or reduce the entries of $L_2$ and go to Step 1; otherwise, construct the control law using

$$u^* = K^* x,$$

where $K^* = -\tilde{R}^{-1}(I - Q^{-1}S)'B'X$. This control law solves the problem with asymptotic stability provided $(A + BK^*, K)$ is completely observable. Otherwise, only Lyapunov stability is guaranteed.

Remark 8.4 The result in this section can be extended to a nonlinear case, in principal. In this case, the model (8.6) is replaced by a nonlinear model

\[
\begin{align*}
\dot{x}(t) &= A(x(t)) + B(x(t))u_n(t), \quad x(0) = x_0, \\
u_n(t) &= \phi(u(t)), \\
u(t) &= K(x(t)),
\end{align*}
\]

where $\phi$ is a nonlinear function inside the sector $\text{sector}(L_1, L_2)$. A static state feedback stabilizing controller can be obtained if there exists a smooth positive definite solution $V$ to the PDE

\[
\nabla_x V(x)A(x) - \frac{1}{2} \nabla_x V(x)(B(x)(I - Q^{-1}S)\tilde{R}^{-1}(I - Q^{-1}S)'B(x)'

- B(x)Q^{-1}B(x)')\nabla_x V(x)' = 0.
\]

where $Q = -\frac{1}{2}(\tilde{L}_1 \tilde{L}_2 + \tilde{L}_2 \tilde{L}_1) > 0$, $S = \frac{1}{2}(\tilde{L}_1 + \tilde{L}_2)$, $R = -I$, $\tilde{R} = S'Q^{-1}S + I$, with $\tilde{L}_1 = -(I - L_2)^{-1}$, $\tilde{L}_2 = -(I - L_1)^{-1}$. If such a solution exists, then following the proof of Theorem (8.1), the controller

$$K^*(x) = -\tilde{R}^{-1}(I - Q^{-1}S)'B(x)\nabla_x V(x)' ,$$

will result in a stable closed loop system, in the sense of Lyapunov.

Asymptotic stability can also be obtained provided: (i) $\phi$ is time invariant (i.e., $\phi = \phi(u)$) and is strictly inside sector $\text{sector}(L_1, L_2)$ and (ii) the vector field $(A(x) + B(x)K^*(x))$ is detectable from $z = K^*(x)$. Here, detectability means that $w = 0$ and $\lim_{t\to\infty} z(t) = 0$ imply $\lim_{t\to\infty} z^*(t) = 0$, where $z^*(\cdot)$ is the trajectory produced by $A + BK^*$.
8.2 Linear Systems with Sector Bounded Nonlinearities

8.2.2 Nonlinearities at the Input and Output

In this section we consider linear systems with sector bounded nonlinearities at input and output. The systems are described by (see Figure 8.3)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu_n(t), \quad x(0) = x_0, \\
y(t) &= Cx(t), \\
\Sigma^K_\phi : & \begin{cases} 
  u(t) = K(y_n(t)), \\
y_n(t) = \phi_y(y(t)), \\
u_n(t) = \phi_u(u(t)),
\end{cases}
\end{align*}
\]

where \( y, y_n \in \mathbb{R}^p, \phi_u, \phi_y \) are inside the sectors \( \text{sector}(L_{1u}, L_{2u}) \) and \( \text{sector}(L_{1y}, L_{2y}) \), respectively. We assume that \( L_{1u}, L_{2u}, L_{1y} \) and \( L_{2y} \) satisfy

\[
\begin{align*}
L_{1ui} &< 1, \quad L_{2ui} > 1, \quad i = 1, 2, \ldots, m, \\
L_{1yj} &< 1, \quad L_{2yj} > 1, \quad j = 1, 2, \ldots, p.
\end{align*}
\]

The control problem is to find a linear dynamical output feedback controller \( u = K(y) \), such that \( \Sigma^K_\phi \) is stable for all \( \phi_u \in \text{sector}(L_{1u}, L_{2u}), \phi_y \in \text{sector}(L_{1y}, L_{2y}) \).

We shall first reconfigure the systems \( \Sigma^K_\phi \) as an interconnected system consisting of a new, controlled, linear system (see also Figure 8.4)

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) + Bw(t), \quad x(0) = x_0, \\
z(t) &= \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ Cz(t) \end{bmatrix}, \\
y(t) &= Cz(t) + v(t),
\end{align*}
\]

where \( z = [z_1', z_2']' \in \mathbb{R}^{m+p}, w \in \mathbb{R}^d, v \in \mathbb{R}^p \), and the nonlinearities

\[
\Phi : \begin{align*}
\Phi_u(z_1, t) &= z_1 - \phi_u(z_1, t), \\
\Phi_y(z_2, t) &= z_2 - \phi_y(z_2, t).
\end{align*}
\]

The interconnection is given by \( w(t) = -\Phi_u(z_1(t), t), \quad v(t) = -\Phi_y(z_2(t), t) \). Since \( \phi_u \in \text{sector}(L_{1u}, L_{2u}), \phi_y \in \text{sector}(L_{1y}, L_{2y}) \), we have

\[
\begin{align*}
(\Phi_u(z_{1i}, t) - (1 - L_{2ui})z_{1i})(\Phi_u(z_{1i}, t) - (1 - L_{1ui})z_{1i}) &\leq 0, \\
(\Phi_y(z_{2j}, t) - (1 - L_{2yj})z_{2j})(\Phi_y(z_{2j}, t) - (1 - L_{1yj})z_{2j}) &\leq 0,
\end{align*}
\]

for \( i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, p \). This implies that \( \Phi_u \) and \( \Phi_y \) are inside the sectors \( \text{sector}((I - L_{1u}), (I - L_{2u})) \) and \( \text{sector}((I - L_{1y}), (I - L_{2y})) \) respectively.
Motivated by the discussions in the previous section, we shall solve the problem by finding a linear dynamical output feedback controller $u = K(y)$, such that the closed
8.2 Linear Systems with Sector Bounded Nonlinearities

Loop \( \Sigma^K \) dissipative with respect to the following supply rate

\[
\tau(z, w, v) = \frac{1}{2}([w' v']Q \begin{bmatrix} w \\ v \end{bmatrix} + 2[w' v']Sz + z'Rz),
\]

(8.21)
in which \( Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \), with \( Q_1 > 0 \), \( Q_2 > 0 \), \( S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \), \( R = -I \), for some appropriately chosen matrices \( Q_1, Q_2, S_1 \) and \( S_2 \).

To obtain the desired controller, we pose the problem as one of finding an output feedback controller \( u = K(y) \) which minimizes the cost functional

\[
J_p(K) = \sup_{T \geq 0, w, v \in L_2([0, T], R^{d_f})} \sup_{z_0 \in R^n} \{p(z_0)
- \int_0^T ([w(t)' v(t)']Q \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} + 2[w(t)' v(t)']Sz(t) + z(t)'Rz(t)) dt \}.
\]

(8.22)

We solve the optimization problem by employing the information state technique developed in Chapter 5 and Chapter 6. The controller formulae in the linear systems case in Chapter 6 do not apply here, in view of the state space model in (8.18). In the case at hand, the information state is given by the dynamic programming equation (DPE)

\[
\begin{cases}
-\frac{\partial p}{\partial t} = \sup_{w \in R^{d_f}} \{-\nabla_x p \cdot (Ax + B_1w + B_2u) \\
-\frac{1}{2}([w' (y - Cz)']Q \begin{bmatrix} w \\ (y - Cz) \end{bmatrix} + 2[w' (y - Cz)']Sz + z'Rz),
\end{cases}
\]

(8.22)

with initial condition \( p_0 \). If we choose \( p_0 \) to be quadratic, i.e., \( p_0(x) = \frac{1}{2}x'Px \), with \( P > 0 \), the special form of the dynamics (8.18) allows us to solve (8.22) for \( p_t \) explicitly in terms of finite dimensional quantities, as given below (see also Chapter 6).

\[
p_t(z) = \frac{1}{2} (z - \hat{z}(t))'Y(t)^{-1}(z - \hat{z}(t)) + \varphi(t),
\]

(8.23)

where \( \hat{z} \in R^n, Y \in R^{n \times n} \), and \( \varphi \in R \) satisfy the ODEs

\[
\begin{cases}
\dot{z}(t) = (A + Y(Q + C'(S_2 + S_2' - Q_2)C))\hat{z}(t) + B(I - Q_1^{-1}S_1)u \\
-YC'(S_2' - Q_2)Y(t),
\end{cases}
\]

(8.24)

\[
\dot{Y}(t) = AY(t) + Y(t)A' - Y(t)(C'(Q_2 - (S_2 + S_2'))C - \tilde{Q})Y(t) + BQ_1^{-1}B',
\]

\[
\dot{\varphi}(t) = \frac{1}{2}(u'\tilde{R}u + \hat{z}'\tilde{Q}\hat{z} - (y - C\hat{z})'Q_2(y - C\hat{z}) - 2(y - C\hat{z})'S_2\hat{z}),
\]

where \( \tilde{Q} = C'C, \tilde{R} = I + S_1'Q_1^{-1}S_1 \), with initial conditions \( \hat{z}(0) = 0 \), \( Y(0) = P^{-1} > 0 \), \( \varphi(0) = 0 \).
Denoting the finite dimensional quantity \( \rho \) by \( \rho(t) = (\dot{\varphi}(t), Y(t), \varphi(t)) \), equation (8.24) defines the dynamics for \( \rho \) as follows

\[
\dot{\rho}(t) = \dot{F}(\rho(t), u(t), y(t)),
\]

where \( \dot{F}(\rho, u, y) \) denotes the right hand side of (8.24), with initial condition \( \rho(0) = \rho \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R} \). We shall denote the quadratic information state in (8.23) by \( \rho^\phi \). Our control problem can be recast as one of controlling (8.25) such that the cost functional

\[
J_\rho(u) = \sup_{T \geq 0, y \in L_2([0, T], \mathbb{R}^m)} \{ \varphi(T) : \rho(0) = \rho \}.
\]

is minimized. This is a full observation optimization problem with the state \( \rho \). To solve this problem define the value function \( W(\rho) \) by

\[
W(\rho) \triangleq \inf_{u \in L_2([0, \infty), \mathbb{R}^m)} \sup_{T \geq 0, y \in L_2([0, T], \mathbb{R}^m)} \{ \varphi(T) : \rho(t) = \rho \}.
\]

This function solves the dynamic programming equation (see Chapter 6)

\[
\inf_{u \in \mathbb{R}^m} \sup_{y \in \mathbb{R}^n} \left\{ \frac{\partial W}{\partial \rho} (A + Y(\dot{Q} + C'(S_2 + S_2' - Q_2)C)) \dot{\varphi} + B(I - Q_1^{-1}S_1)u - YC'(S_2' - Q_2)y \right. \\
+ \langle (\frac{\partial W}{\partial \rho}, (AY + YA' - Y(C'(Q_2 - (S_2 + S_2'))C - \dot{Q})Y + BQ_1^{-1}B')) \right. \\
\left. + \frac{\partial W}{\partial \rho} \frac{1}{2}(u'\tilde{R}u + \dot{y}'\tilde{Q}\dot{y} - 2(y - C\dot{y})'S_2C\dot{y}) \right. \\
\left. - (y - C\dot{y})'Q_2(y - C\dot{y}) \right\} = 0,
\]

The optimal controller that achieves the minimum in the left hand side of the DPE (8.27) is given by

\[
u^*(\dot{\varphi}) = -\tilde{R}^{-1}(I - Q_1^{-1}S_1)'B'\nabla_{\dot{\varphi}}W'.
\]

Furthermore, the DPE has the explicit solution given by

\[
W(\dot{\varphi}, Y, \varphi) = \frac{1}{2} \dot{\varphi}'X[I - YX]^{-1}\dot{\varphi} + \varphi,
\]

in which \( X \) a symmetric positive semidefinite solution of the ARE

\[
XA + A'X - X(B(I - Q_1^{-1})\tilde{R}^{-1}(I - Q_1^{-1}S_1)'B' - BQ_1^{-1}B')X + C'(I + S_2'Q_2^{-1}S_2)C = 0,
\]

satisfying the coupling condition

\[
XY(t) < I,
\]

for all \( t \geq 0 \).
In summary, employing the information method to minimize the cost function (8.26) leads to the following expression for the controller

\[ \Sigma_c : \begin{align*}
\dot{z}(t) &= (A + Y(t)(\bar{Q} + C'(S_2 + S_2' - Q_2)C))\dot{z}(t) + B(I - Q_1^{-1}S_1)u \\
- Y(t)C'(S_2' - Q_2)y(t), \\
u(t) &= -\tilde{R}^{-1}\tilde{B}'X(I - Y(t)X)^{-1}\dot{z}(t),
\end{align*} \tag{8.32}
\]

with initial conditions \( \dot{z}(0) = 0 \), in which \( X \) and \( Y(t) \) are positive definite solutions to the ARE \((8.30)\) and the RDE in \((8.24)\). This controller is time varying due to the variation of \( Y(t) \).

In the next results, we shall use a stationary value of \( Y(t) \), i.e., a solution to the ARE

\[ AY + YA' - (C'(Q_2 - (S_2 + S_2'))C - \bar{Q})Y + BQ_1^{-1}B' = 0. \tag{8.33} \]

This results in a time invariant controller. Let us denote the closed loop state by \( z_c = [z' \ e']' \), in which we take \( e = x - (I - YX)^{-1}\dot{x} \). We then obtain the following dynamics of the closed loop system

\[ \begin{align*}
\begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A - B\tilde{R}^{-1}\tilde{B}'X & B\tilde{R}^{-1}\tilde{B}'X \\ -\tilde{A} - B\tilde{R}^{-1}\tilde{B}'X - KyC & \tilde{A} + B\tilde{R}^{-1}\tilde{B}'X \end{bmatrix} \begin{bmatrix} z \\ e \end{bmatrix} \\
&\quad + \begin{bmatrix} B & 0 \\ B & -Ky \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix},
\end{align*} \tag{8.34}\]

\[ z = \begin{bmatrix} -\tilde{R}^{-1}\tilde{B}'X & \tilde{R}^{-1}\tilde{B}'X \\ C & 0 \end{bmatrix} z_c,
\]

where \( \tilde{A} = (BQ_1^{-1}B' - \tilde{B}\tilde{R}^{-1}\tilde{B}')X + (I - YX)^{-1}YC'((S_2 + S_2') - Q_2 - S_2'Q_2^{-1}S_2)C \), \( \tilde{B} = B(I - Q_1^{-1}S_1) \), \( K_y = -(I - YX)^{-1}YC'(S_2' - Q_2) \), and \( \tilde{A} = \tilde{A} - A \). The following results say that if positive definite solutions \( X, Y \) to the AREs \((8.30)\) and \((8.33)\), with appropriately chosen matrices \( Q_1, Q_2, S_1 \) and \( S_2 \), such that the coupling condition \((8.31)\) holds, then a stabilizing controller for the systems \( \Sigma^K_\phi \) \((8.16)\) can be obtained.

**Theorem 8.2** Consider the nonlinear systems \( \Sigma^K_\phi \) in \((8.16)\) with the sector...
bounded nonlinearities

$$\phi_u \in \text{sector}(L_{1u}, L_{2u}), \phi_y \in \text{sector}(L_{1y}, L_{2y}),$$

in which \(L_{1u}, L_{2u}, L_{1y}, L_{2y}\) satisfy (8.17). Assume there exist positive definite solutions to the AREs (8.30) and (8.33) with \(Q_1 = -L_{1u}L_{2u} > 0, Q_2 = -L_{1y}L_{2y} > 0, S_1 = -\frac{1}{2}(L_{1u} + L_{2u}), S_2 = -\frac{1}{2}(L_{1y} + L_{2y}), \tilde{R} = S_1^TQ_1^{-1}S + I, \) in which \(L_{1u} = -(I - L_{2u})^{-1}, L_{2u} = -(I - L_{1u})^{-1}\) and \(L_{1y} = -(I - L_{2y})^{-1}, L_{2y} = -(I - L_{1y})^{-1}\), such that the coupling condition \(\rho(XY) < 1\) is satisfied. Then the interconnection of the linear system \(\Sigma^K\) with \(\Phi\) or equivalently, the nonlinear systems \(\Sigma_{\Phi}^K\) (8.16), is Lyapunov stable. If: (i) \(\phi_u, \phi_y\) are time invariant and are strictly inside \((L_{1u}, L_{2u})\) and \((L_{1y}, L_{2y})\) respectively and (ii) the pair of closed loop matrices \((A_d, C_d)\) is observable, then the stability is asymptotic.

**Proof.** First define the matrix \(Z = Y(I - XY)^{-1}\) which is positive definite, by the hypothesis. A simple algebraic calculation shows that the matrix \(W = Z^{-1} > 0\) solves the ARE

$$W(\tilde{A} + \tilde{B}\tilde{R}^{-1}\tilde{B}'X) + (\tilde{A} + \tilde{B}\tilde{R}^{-1}\tilde{B}'X)'W$$

$$+ W(ZC'(S_2 - Q_2)Q_2^{-1}(S_2 - Q_2) + BQ_1^{-1}B')W + X\tilde{B}\tilde{R}^{-1}\tilde{B}'X = 0. \tag{8.35}$$

Next, define the matrix \(X_d\) by

$$X_d = \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix}.$$  

Then, by some algebraic calculation we see that \(X_d\) solves the closed loop ARE

$$A'_dX_d + X_dA_d + (X_dB_d - C_dS_1)Q^{-1}(B'_dX_d - SC_d) + C_dC_d = 0. \tag{8.36}$$

Define the positive definite function \(V_d(x_d) = \frac{1}{2}x_d'X_dx_d\). Evaluating \(dV_d/dt\) along the trajectory \(x_d(\cdot)\) produced by the closed loop system (8.34) yields

$$dV_d/dt(x_d(t))$$

$$= -((w(t) - w_d^*)'Q(w(t) - w_d^*)$$

$$+ \frac{1}{2}([w(t)'v(t)']*Q\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} + 2[w(t)'v(t)']*Sz(t) + z(t)'Rz(t)),$$

$$\leq \frac{1}{2}([w(t)'v(t)']*Q\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} + 2[w(t)'v(t)']*Sz(t) + z(t)'Rz(t),$$

$$= -\frac{1}{2}((z_1(t) - L_{1u}w(t))'(z_1(t) - L_{2u}w(t)) + ((z_2(t) - L_{1y}v(t))'(z_2(t) - L_{2y}v(t),$$
where \( w^*_d = Q^{-1}(B'_d X_d - SC_d)x_d \). Since \( \phi_u \in \text{sector}(L_{1u}, L_{2u}), \phi_y \in \text{sector}(L_{1y}, L_{2y}) \), we have, by using (8.20),

\[
-((z_1(t) - \bar{L}_{1u}w(t))'(z_1(t) - \bar{L}_{2u}w(t)) + ((z_2(t) - \bar{L}_{1y}v(t))'(z_2(t) - \bar{L}_{2y}v(t))
\]

\[
= -\sum_{i=1}^{m} \bar{L}_{1u} \bar{L}_{2u}((-1 - L_{1ui})z_{1i} - w_i)((-1 - L_{2ui})z_{1i} - w_i)
\]

\[
-\sum_{j=1}^{p} L_{1yj} \bar{L}_{2yj}((-1 - L_{1yj})z_{2j} - v_j)((-1 - L_{2yj})z_{2j} - v_j)
\]

\[
= \sum_{i=1}^{m} ((-\bar{L}_{1ui} \bar{L}_{2ui})((-1 - L_{1ui})z_{1i} + \Phi_{ui}(z_{1i}))((-1 - L_{2ui})z_{1i} + \Phi_{ui}(z_{1i}))
\]

\[
+\sum_{j=1}^{p} ((-\bar{L}_{1yj} \bar{L}_{2yj})((-1 - L_{1yj})z_{2j} + \Phi_{yj}(z_{2j}))+((-1 - L_{2yj})z_{2j} + \Phi_{yj}(z_{2j}))
\]

\[\leq 0.\]

Thus,

\[
dV_d/dt(x_d(t)) \leq -\frac{1}{2}((z_1(t) - \bar{L}_{1u}w(t))'(z_1(t) - \bar{L}_{2u}w(t))
\]

\[+((z_2(t) - \bar{L}_{1y}v(t))'(z_2(t) - \bar{L}_{2y}v(t))
\]

\[\leq 0,\]

which shows that \( V_d \) is a Lyapunov function and, therefore, the interconnection of \( \Sigma^K \) and \( \Phi \) or, equivalently, the nonlinear system \( \Sigma^K \) is Lyapunov stable. If, moreover, \( \phi_u \) and \( \phi_y \) are strictly inside \( (L_{1u}, L_{2u}) \) and \( (L_{1y}, L_{2y}) \) respectively, then following the asymptotic stability proof in Theorem 8.1, if the pair \( (A_d, C_d) \) is observable, we have \( x_d(t) \to 0 \) as \( t \to \infty \). This proves the asymptotic stability. □

The design procedure is summarized as follows.

- **Step 1:** For given diagonal matrices \( L_{1u}, L_{2u}, L_{1y} \) and \( L_{2y} \), with \( L_{1ui} < 1, L_{2ui} > 1, i = 1, 2, \ldots, m, \) and \( L_{1yj} < 1, L_{2yj} > 1, j = 1, 2, \ldots, p, \) set

\[
S_1 = \frac{1}{2}(\bar{L}_{1u} + \bar{L}_{2u}), \quad Q_1 = -\bar{L}_{1u} \bar{L}_{2u},
\]

\[
S_2 = \frac{1}{2}(\bar{L}_{1y} + \bar{L}_{2y}), \quad Q_2 = -\bar{L}_{1y} \bar{L}_{2y},
\]

where

\[
\bar{L}_{1u} = -(I - L_{2u})^{-1}, \quad \bar{L}_{2u} = -(I - L_{1u})^{-1},
\]

\[
\bar{L}_{1y} = -(I - L_{2y})^{-1}, \quad \bar{L}_{2y} = -(I - L_{1y})^{-1}.
\]

- **Step 2:** Find positive definite solutions \( X \) and \( Y \) to the AREs (8.30) and (8.33). If no such solutions exist increase the entries of \( L_{1u}, L_{1y} \) and/or reduce the entries of \( L_{2u}, L_{2y} \) and go to Step 1; otherwise, construct the control law using (8.32). This controller solves the problem with asymptotic stability provided \( (A_d, C_d) \) is completely observable. Otherwise, only Lyapunov stability is guaranteed.
Remark 8.5  The $H_\infty$ technique can be used directly to achieve closed loop system with some sector bounded performance measure. In this technique, one uses the supply rate given by

$$r(z,w,v) = \frac{1}{2}(w'v')Q_{H_\infty} \begin{bmatrix} w \\ v \end{bmatrix} + 2[w'v']S_{H_\infty}z + z'R_{H_\infty}z,$$

in which $Q_{H_\infty} = \gamma^2 I$, $S_{H_\infty} = 0$, $R_{H_\infty} = -I$. However, the requirement that $S_{H_\infty} = 0$ restricts us to use sector bounds that satisfy

$$(I - L_{2w})^{-1} + (I - L_{1w})^{-1} = 0, (I - L_{2y})^{-1} + (I - L_{1y})^{-1} = 0,$$

i.e., the upper and lower sectors have to be symmetric with respect to the line with slope 1. This restriction may lead to a conservative control design.

The controller synthesis method we employ here can be regarded as a direct one for achieving a general quadratic dissipativity performance measure for the system (8.18), with the supply rate $r$ given in (8.21). Indeed, we have the following result which is interesting in its own right.

Corollary 8.1  Assume there exist nonnegative solutions $X,Y$ to the AREs (8.33) and (8.30) for some $Q > 0, S$ such that the matrices

$$A - Y(C'(Q_2 - (S_2 + S_2'))C - \bar{Q})$$

$$A - (B(I - Q_1^{-1})\bar{R}^{-1}(I - Q_1^{-1}S_1)'B' - BQ_1^{-1}B')X$$

are asymptotically stable and the coupling condition $\rho(XY) < 1$ is satisfied. Then, the new controlled system $\tilde{\Sigma}^K$ in (8.18) is strictly dissipative with respect to the supply rate $r$ in (8.21), that is we have, with $x_d(0) = 0$,

$$-\frac{1}{2}\int_0^T ([w(t)'v(t)']Q \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} + 2[w(t)'v(t)']Sz(t) + z(t)'Rz(t))dt$$

$$\leq -\frac{\epsilon}{2}\int_0^T [w(t)'v(t)'] \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} dt,$$

for all $w,v \in L_2([0,T], \mathbb{R}^{d+p})$, for all $T \geq 0$, for some constant $\epsilon > 0$. Moreover, the equilibrium state $x_d = 0$ is asymptotically stable.

Proof. The proof follows those in [82] abd [91]. First define the matrix $Z = Y(I - XY)^{-1}$ which is nonnegative definite, by the hypothesis. A simple algebraic calculation
shows that $Z$ solves the ARE
\[
(\bar{A} + \bar{B}\bar{R}^{-1}\bar{B}'X)Z + Z(\bar{A} + \bar{B}\bar{R}^{-1}\bar{B}'X)' = 0,
\]
(8.37)
such that the matrix
\[
\bar{A} + \bar{B}\bar{R}^{-1}\bar{B}'X + ZX\bar{B}\bar{R}^{-1}\bar{B}'X
\]
is asymptotically stable. This implies there exists $W \geq 0$ satisfying the ARE (8.35) such that the matrix
\[
\bar{A} + \bar{B}\bar{R}^{-1}\bar{B}'X + (ZC'(S'_2 - Q_2)Q_2^{-1}(S_2 - Q_2)CZ + BQ_1^{-1}B')W
\]
is asymptotically stable. Next, following the calculation in the proof of Theorem 8.2, we see that the matrix $X_d$ defined by
\[
X_d = \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix}
\]
solves the closed loop ARE
\[
A'_dX_d + X_dA_d + (X_dB_d - C'_dS')Q^{-1}(B'_dX_d - SC_d) + C'_dC_d = 0.
\]
(8.38)
Moreover, we have the expression
\[
(A_d - B_dQ^{-1}SC_d) + B_dQ^{-1}B'_dX_d)
\]
in which $A_{11} = A - \bar{B}\bar{R}^{-1}\bar{B}'X + BQ_1^{-1}B'W$, $A_{12} = \bar{B}\bar{R}^{-1}\bar{B}'X + BQ_1^{-1}B'W$ and $A_{22} = \bar{A} + \bar{B}\bar{R}^{-1}\bar{B}'X + (ZC'(S'_2 - Q_2)Q_2^{-1}(S_2 - Q_2)CZ + BQ_1^{-1}B')W$. Since $A_{11}$ and $A_{22}$ are asymptotically stable matrices, we conclude that $A_d - Q^{-1}SC_d) + B_dQ^{-1}B'_dX_d$ is asymptotically stable. We then conclude the results by applying Theorem 3.5 in Chapter 3.

8.2.3 Examples

We now present a number examples illustrating utility of the control design methods proposed in the previous sections.

Example 1
We consider a state feedback stabilization problem described as follows

\[
\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n(t),
\]

\[u(t) = K z(t),\]

\[u_n(t) = \phi(u(t)),\]

in which \(\phi\) is strictly inside sector(0.1, 1.1). Note that here the LQR method cannot be applied to solve the problem since it only tolerates up to 50 percent gain reduction tolerance. Setting \(L_1 = 0.1\) and \(L_2 = 1.1\) yields the design parameters \(Q_1 = 11.11, S_1 = 4.44\). Substituting these parameters to the ARE (8.30) results in a positive definite solution \(X\). Applying Theorem 8.1 we obtain the required stabilizing controller as given in (8.14). Simulation results are shown in the Figure 8.5 and Figure 8.6.

In the simulation we use the nonlinear function \(\phi\) as follows

\[\phi(u) = \begin{cases} 
    u, & \text{if } |u| \leq 0.5, \\
    0.5 + 0.1(u - 0.5), & \text{if } u > 0.5, \\
    -0.5 - 0.1(-0.5 - u), & \text{if } u < -0.5.
\end{cases}\]

Figure 8.5 shows the state trajectories \(x(t) = [x_1(t) \ x_2(t)]'\) while Figure 8.6 shows the trajectories of \(u(t) = K^*x(t)\) (unperturbed control) and \(u_n(t) = \phi(u(t))\) (perturbed control). These figures demonstrate that the resulting controller is stabilizing under the nonlinearity within the specified sector.

**Example 2**

In the next example, we consider a stabilization problem, in which nonlinearities occur at input and output. The control system is described as follows

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n(t), \quad y(t) = [1 \ 0] x(t),
\]

\[u(t) = K(y_n(t)),\]

\[y_n(t) = \phi_y(y(t)), u_n(t) = \phi_u(u(t)),\]

in which \(\phi_u\) and \(\phi_y\) are strictly inside sector(0.35, 1.05) and sector(0.65, 1.05) respectively. Setting \(L_{1u} = 0.35, L_{2u} = 1.05, L_{1y} = 0.65\) and \(L_{2y} = 1.05\) yields the design parameters \(Q_1 = 30.77, S_1 = 9.23, Q_2 = 57.14\) and \(S_2 = 8.57\). Applying these parameters to the AREs (8.30) and (8.33) results in positive definite solutions \(X\) and \(Y\) which satisfy \(\rho(XY) < 1\) (in fact \(X\) and \(Y\) are also stabilizing solutions). Applying Theorem 8.2 we
obtain the required stabilizing controller as given in (8.32). Simulation results are shown in the Figure 8.7 and Figure 8.8.

In the simulation we use the nonlinear function $\phi_u$ given by

$$
\phi_u(u) = \begin{cases} 
u, & \text{if } |\nu| \leq 1.0, \\ 1.0 + 0.35(u - 1.0), & \text{if } u > 1.0, \\ -1.0 - 0.35(-1.0 - u), & \text{if } u < -1.0, \\ \end{cases}
$$

and $\phi_y$ given by

$$
\phi_y(y) = \begin{cases} 
u, & \text{if } |\nu| \leq 1.0, \\ 1.0 + 0.65(y - 1.0), & \text{if } y > 1.0, \\ -1.0 - 0.65(-1.0 - y), & \text{if } y < -1.0. \\ \end{cases}
$$

Figure 8.7 shows the state trajectories $x(t) = [x_1(t) \ x_2(t)]'$ while Figure 8.8 shows the trajectories of the unperturbed signal $u(t), y(t)$ and the perturbed signals $u_n(t) = \phi_u(u(t))$ and $y_n(t) = \phi_y(y(t))$. These figures demonstrate that the resulting controller is stabilizing under the input and output nonlinearities within the specified sectors.

**Example 3**

In the next example, we tune the performance of a stable system having a saturated input and a sector bounded nonlinear output. The model is given by

$$
\dot{x}(t) = \begin{bmatrix} -0.3 & 1.0 & 0.0 \\ 0.0 & -0.3 & 1.0 \\ 0.0 & 0.0 & -0.3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_n(t), \ y(t) = [1 \ 0 \ 0] x(t),
$$

$u(t) = K(y_n(t)),$

$y_n(t) = \phi_y(y(t)), u_n(t) = \phi_u(u(t)),$

where $\phi_u$ is a saturation nonlinearity given by

$$
\phi(u) = \begin{cases} 
u, & \text{if } |\nu| \leq 1, \\ \frac{\nu}{|\nu|}, & \text{otherwise}. \\ \end{cases}
$$

(The state space matrices $A, B, C$ corresponding to this model is taken from [96]. However, here we multiply the original entries of $A$ in [96] by 10 to allow for positive definite solutions of the AREs (8.30) and (8.33).) In the case at hand, $\phi_u$ is inside sector(0.00, 1.01). We suppose that the output nonlinearity $\phi_y$ is strictly inside sector(0.98, 1.02). Setting $L_{1u} = 0.0, L_{2u} = 1.01, L_{1y} = 0.5$ and $L_{2y} = 1.05$ yields the design parameters $Q_1 = 99.90, S_1 = 49.50, Q_2 = 2,500$ and $S_2 = 0$. Applying these parameters to
the AREs (8.30) and (8.33) results in positive definite solutions $X$ and $Y$, which satisfy $\rho(XY) < 1$. Applying Theorem 8.2 we obtain the required stabilizing controller as described in (8.32). Simulation results are shown in the Figure 8.9 and Figure 8.10.

Figure 8.9 shows the trajectories of $y(t)$ when: (i) $u = 0$ (i.e., uncontrolled case); shown by the dashed line, and (ii) the controller (8.32) is connected and is subjected to nonlinearities $\phi_u, \phi_y$; shown by the solid line. As can be seen, the controller is able to tune the transient response by reducing the maximum overshoot and the settling time of the system. An explanation for this could be that under the constructed controller, the closed loop map from $[w' v']'$ to $z$ in configuration (8.18) is, indeed, sector bounded dissipative (see Corollary 8.1). Figure 8.10 shows the trajectories of $u(t)$, $u_n(t) = \phi_u(u(t))$, $y(t)$ and $y_n(t) = \phi_y(y(t))$.

8.3 Figures

We shall now present the relevant figures.
Figure 8.5: Example 1. State trajectories.

Figure 8.6: Example 1. Input trajectories.
Figure 8.7: Example 2. State trajectories.

Figure 8.8: Example 2. Input and output trajectories.
Figure 8.9: Example 3. Uncontrolled and controlled output trajectories.

Figure 8.10: Example 3. Input and output trajectories.
8.4 Nonlinear Systems with Dissipative Uncertainties

In this section we consider an application of the synthesis theory developed in Chapter 4 to a stabilization problem for a class of uncertain nonlinear systems. This class is described by an interconnection of a fixed (known) nominal system and an unknown system possessing a certain dissipativity property (see Figure 8.11 below). This property may be boundedness of the $H_\infty$ norm [36], [3], [48], positive realness [39], or sector boundedness [37], [91].

![Figure 8.11: A robust stabilization problem.](image)

The uncertain system, denoted by $\Sigma_{G,\Delta}$, consists of a (known) nominal system $G$, where

$$G : \begin{cases} \dot{z}(t) = A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad z(0) = x_0, \\ z(t) = C_1(x(t)) + D_{11}(x(t))w(t) + D_{12}(x)u(t), \end{cases}$$

(8.39)

with $x \in \mathbb{R}^n$, $w \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^q$, and an unknown system $\Delta \in \mathcal{D}$

$$\Delta : \begin{cases} \dot{\xi}(t) = F_\Delta(\xi(t)) + G_\Delta(\xi(t))\eta(t), \quad \xi(0) = \xi_0, \\ \zeta(t) = H_\Delta(\xi(t)) + J_\Delta(\xi(t))\eta(t). \end{cases}$$

(8.40)
with \( \xi \in \mathbb{R}^{n_{\Delta}}, \eta \in \mathbb{R}^{q}, \zeta \in \mathbb{R}^{d} \), in which \( \mathcal{D} \) denotes the class of uncertainties. The dimension of the state space realization of \( \Delta, n_{\Delta} \) may be arbitrary, but finite. Thus, the input-output map of each system \( \Delta \in \mathcal{D} \) is completely described by the maps \( F_{\Delta} : \mathbb{R}^{n_{\Delta}} \to \mathbb{R}^{n_{\Delta}}, G_{\Delta} : \mathbb{R}^{n_{\Delta}} \to \mathbb{R}^{n_{\Delta} \times q}, H_{\Delta} : \mathbb{R}^{n_{\Delta}} \to \mathbb{R}^{d} \) and \( J_{\Delta} : \mathbb{R}^{n_{\Delta}} \to \mathbb{R}^{d \times q} \). We assume that for each \( \Delta \in \mathcal{D} \), the functions \( F_{\Delta}, G_{\Delta}, H_{\Delta}, J_{\Delta} \) satisfy the regularity assumptions described in Section 4.2, with \( F_{\Delta}(0) = 0, H_{\Delta}(0) = 0 \). To ensure the wellposedness of the interconnected systems \( \Sigma_{G, \Delta} \), we require that the matrices

\[
I + J_{\Delta}(\xi)D_{11}(x), \quad I + D_{11}(x)J_{\Delta}(\xi)
\]  

be nonsingular for all \( x, \xi \).

We consider the following class of uncertainties.

**Definition 8.2** The class of admissible uncertainties \( \mathcal{D} \) is the set of all systems described in (8.40), which are dissipative with respect to the supply rate \( r_{\mathcal{D}}(\zeta, \eta) \) possessing positive definite and smooth (i.e. \( C^{1} \)) storage functions \( V_{\mathcal{D}} \).

To stabilize the uncertain system \( \Sigma_{G, \Delta} \) for all \( \Delta \in \mathcal{D} \), we select an appropriate supply rate \( r_{G}(z, w) \), and attempt to find a controller \( u = K(x) \in S \) to attain dissipativity of the closed loop \( G^{u} \) with respect to \( r_{G}(z, w) \).

**Theorem 8.3** Suppose there exist a supply rate \( r_{G}(z, w) \), and a constant \( \alpha > 0 \) such that the following condition holds

\[
r_{G}(z, w) + \alpha r_{\mathcal{D}}(-w, z) \leq 0,
\]  

for all \( z \) and \( w \). Assume there exists a positive definite function \( V_{G} \in C^{1} \) solving the following PDI

\[
\inf_{u \in \mathbb{R}^{m}} \sup_{w \in \mathbb{R}^{q}} \{ \nabla_{w} V_{G}(z) \cdot (A(z) + B_{1}(z)w + B_{2}(z)u) \}
\]  

\[
- r_{G}(h_{1}(z, w, u)) \leq 0,
\]

where \( h_{1}(z, w, u) = C_{1}(z) + D_{11}(z)w + D_{12}(z)u \). Then the control \( u^{*}(z) \) which attains minimum on the LHS of the PDI (8.43) will stabilize (in the sense of Lyapunov) the uncertain system \( \Sigma_{G, \Delta} \) for all \( \Delta \in \mathcal{D} \). If, furthermore, we have: (i) the condition (8.42) holds in the strict sense, i.e.

\[
r_{G}(z, w) + \alpha r_{\mathcal{D}}(-w, z) < 0,
\]
for all \((z,w) \neq (0,0)\) with \(r_G(0,0) + \alpha r_D(0,0) = 0\), and (ii) the systems \(G\mathbf{u}^*\) and each of \(\Delta \in \mathcal{D}\) are zero-state detectable, then the control \(\mathbf{u}^*(x)\) will asymptotically stabilize the uncertain system \(\Sigma_{G,\Delta}\) for all \(\Delta \in \mathcal{D}\).

**Proof.** Suppose \(\mathbf{u}^*(x)\) attains minimum on the LHS of PDI (8.43). Then with \(u = u^*\), the PDI (8.43) implies, for any \(w\),

\[
\frac{dV_{\Delta}}{dt} = \nabla_x V_g(x) \cdot (A_{u^*}(x) + B_1(x)w)
\leq r_g(h_1(x,w,u^*(x)),w),
\]

where \(A_{u^*}(x) = A(x) + B_2(x)u^*(x)\). Since each \(\Delta \in \mathcal{D}\) is dissipative with respect to \(r_D(\zeta,\eta)\), Theorem 3.1 implies, for any \(\eta\),

\[
\alpha \frac{dV_{\Delta}}{dt} = \alpha \nabla_x V_{\Delta}(\xi) \cdot (F_\Delta(\xi) + G_\Delta(\xi)\eta)
\leq \alpha r_\Delta(H_1(\xi,\eta),\eta),
\]

where \(H_1(\xi,\eta) = H_\Delta(\xi) + J_\Delta(\xi)\eta\). Now, define the function

\[
V_{\Delta}(x,\xi) = V_g(x) + \alpha V_{\Delta}(\xi).
\]

Clearly \(V_{\Delta}(x,\xi)\) is positive definite, i.e., \(V_{\Delta}(x,\xi) > 0\) for all \((x,\xi) \neq (0,0)\) and satisfies \(V_{\Delta}(0,0) = 0\). By evaluating \(\frac{dV_{\Delta}}{dt}\) along the trajectory of \((x(\cdot),\xi(\cdot))\) we get

\[
\frac{dV_{\Delta}}{dt} = \frac{dV_g}{dt} + \alpha \frac{dV_{\Delta}}{dt}
= \nabla_x V_g(x) \cdot (A_{u^*}(x) + B_1(x)w) + \alpha \nabla_x V_{\Delta}(\xi) \cdot (F_\Delta(\xi) + G_\Delta(\xi)\eta)
\leq r_g(h_1(x,w,u^*(x)),w) + \alpha r_\Delta(\xi,\eta)
\leq 0.
\]

Therefore \(V_{\Delta}(x,\xi)\) is a Lyapunov function for the uncertain system \(\Sigma_{G,\Delta}\), and hence, \(\Sigma_{G,\Delta}\) is Lyapunov stable for all \(\Delta \in \mathcal{D}\).

Suppose that, instead of the condition (8.42), we have

\[
r_g(z,w) + \alpha r_D(-w,z) < 0,
\]

for all \((z,w) \neq (0,0)\), with \(r_g(0,0) + \alpha r_D(0,0) = 0\). Then, by the La Salle invariance principle [64], \((x(\cdot),\xi(\cdot))\) approaches the set

\[
\mathcal{N} = \{(x,\xi) : h_1(x,0,u^*(x)) = 0, \text{ and } H_1(\xi,0) = 0\}
\]
asymptotically. Therefore, by the zero-state detectability of $G_u^*$ and each of $\Delta \in \mathcal{D}$, $(\mathbf{x}(t), \xi(t))$ goes to $(0, 0)$ as $t \to \infty$ (see the proof of Theorem 8.1), and thus the uncertain system $\Sigma_{G, \Delta}$ is asymptotically stable for all $\Delta \in \mathcal{D}$ (see also related results in [48], [44], [45], ).

8.5 Conclusions

In this chapter we have addressed the problem of stabilizing linear systems possessing sector bounded nonlinearities at their input and output. We have proposed a dissipativity control design method to solve the problem. In this method, the original stabilization problem is recast as a dissipativity control synthesis problem for a new, related, linear system. We have expressed the desired controller in terms of a solution to an ARE in the state feedback case, and in terms of solutions to ARE's plus a coupling condition, in the output feedback case. We obtain stability results by using the Lyapunov methods and the La Salle invariance principle. In the state feedback case, we have provided an extension of the results to a nonlinear model case. We have also shown that in the output feedback case, our synthesis method can be regarded as a direct one for obtaining an output feedback controller that renders the closed loop system strictly quadratic dissipative. This extends partially the results in [82], [91]. Finally, a number of examples have been presented to demonstrate the utility of the proposed control method.
Chapter 9

Conclusions

9.1 Overview of The Thesis

The main theme of the thesis has been a general dissipative control and filtering synthesis for nonlinear and linear systems in either continuous or discrete time (see Appendix A). The dissipativity performance measure is expressed in terms of a general form of supply rate function which includes finite gain and passivity measures and a mixture of them as specific forms.

In Chapter 2 and Chapter 3 of the thesis, we have provided some analysis and computational results for dissipative systems. We make use of a generalization of the bounded real lemma to characterize a dissipative system. This characterization is expressed in terms of a (Hamilton-Jacobi-Bellman) PDI which is interpreted in the viscosity sense. We have developed a numerical method that solves this inequality by utilizing a finite difference discretization scheme. We have provided a number of numerical examples demonstrating the utility of the method, at least for low order systems. In the case of strictly dissipative systems possessing a strong stability, we characterize them in terms of a solution of a strict PDI or a stabilizing solution of a PDE. These results, besides being interesting in their own right, serve as a partial and intuitive basis for the synthesis results in the following chapters.

In Chapter 4 we have presented a general dissipative control synthesis method with full state feedback. We have expressed the solution to the control synthesis problem in terms of a solution of a PDI/PDE of Hamilton-Jacobi-Isaacs type, which is the relevant
controlled dissipation inequality for the problem. In particular, we have shown that whenever there exists any static state feedback controller rendering the closed loop system dissipative, then there exists a solution to the PDI/PDE in the weak (viscosity) sense. Stability of the closed loop system is deduced from the dissipativity property under a closed loop detectability assumption. We have also presented some synthesis results for linear systems in which the solution is expressed in terms of a solution of an ARI/ARE.

When full information regarding the state is not available, we have solved the general dissipative control problem by employing the information state method [56], [57], in which the notion information state is used. We have shown that the original partial information optimization problem can be cast as a new full information one in which the information state plays the role as the required state. The dynamics of the information state takes the form of a controlled PDE. Thus, the partial information control problem is solved by controlling the dynamics of the information state, leading to an infinite dimensional (Hamilton-Jacobi-Isaacs) PDE/PDI. Relation between the information state controller with the certainty equivalence (CE) controller has been examined, providing a deeper insight into the CE solution. These results are presented in Chapter 5. In Chapter 6 we have specialized these results to a class of bilinear systems (which includes linear systems), in which the corresponding information state can be expressed in terms of finite dimensional quantities whose dynamics is given by a set of ODE's. We have solved the control problem for this special class of systems by controlling the ODE's leading to a finite dimensional (HJI) PDE/PDI. Moreover, in the linear systems case we have been able to obtain an explicit solution to this PDI in terms of two ARE's plus a coupling condition, mimicking the results in linear $H_{\infty}$ control (see [23], [82]) and linear positive real control [91]. Stability results for linear systems have also been presented utilizing the stabilizing solutions to the ARE's.

In Chapter 7 we have formulated and solved a general dissipative filtering problem, which includes $H_{\infty}$ filtering as a special case, by applying the information state method. We have shown that a solution to the problem can be obtained by employing the notion of information state, and then controlling its dynamics. This approach leads to two equations: a PDE describing the dynamics of the information state and a (HJI) PDE/PDI for controlling the dynamics of the information state. We have also specialized the results to linear systems recovering the familiar linear $H_{\infty}$ filter.

Discrete time results analogous to the control and filtering synthesis results in Chapter
9.2 Issues for Future Research

4, 5, 6 and 7 have been provided in the thesis (see Appendix A). However, the results for discrete time systems are not as complete as their counterparts in continuous time systems are (see the next section).

Finally, in Chapter 8 we have considered two applications of the dissipativity performance measure for robust stabilization problems. In the first application we have considered a robust stabilization problem for linear systems possessing sector bounded nonlinearity at the input and output. In particular, we have chosen a particular quadratic dissipativity performance measure which is appropriate to obtain solutions to the problem and provided the required controller synthesis method. A number of numerical examples have also been provided. In the second application, we consider a stabilization problem for uncertain nonlinear systems in which the uncertain parts are characterized by a general dissipative measure. Stability results are obtained using the Lyapunov methods and the La Salle principle. The potential advantage of the general dissipativity design method could be that it allows for a tighter characterization of uncertainties, and, as a consequence, leads to a less conservative design.

We shall now discuss several issues for further development of the results in the thesis.

9.2 Issues for Future Research

1. General dissipative robust control design. To date, finite $L_2$ gain or $H_\infty$ is one of the most popular control design methods for achieving high performance, robust control designs (see, for example, a survey in [5]). This method concentrates on information regarding the gain of the uncertainties [5], [36]. However, as pointed out in the papers [40] and [37], when other information regarding sector bounds or phase of the uncertainties is available, then a sector bounded or a positive real control design method would be more appropriate and leads to a less conservative design. A further issue would then be to characterize a wide class of engineering design problems in which information regarding sector bounds or phase of the uncertainties is, in fact, available in practice. In the thesis, we have also proposed a performance measure which is a mixture between $H_\infty$ and positive real measures. The performance measure is expressed as a weighted (with the weighting parameter takes value in $[0, 1]$) summation between them. It is interesting to investigate the effect of changing this parameter on the stability/performance robustness of the
closed loop system.

In the nonlinear case, it is natural to formulate a cost function which reflect the desired performance in terms of nonquadratic functions [41]. In fact, robust control design results for nonlinear systems by utilizing non quadratic performance criteria have been obtained in [41] and the references therein. In the future, it would be interesting to explore nonquadratic forms of supply rate functions to achieve a robust design for nonlinear systems. A preliminary result on a stability robustness analysis of nonlinear systems with a general dissipative uncertainty has been presented in Chapter 8 of this thesis.

2. Computational and Approximation issues. The discretization employed in the thesis has been the explicit finite difference method. This results in a simpler expression for iteration schemes. However, implicit schemes could offer faster rate of convergence [69]. Exploring various implicit discretization schemes to obtain a faster numerical method could be an important research in the future. Convergence proofs of these schemes could be obtained by using viscosity solutions methods [69], [35].

It could also be interesting to develop a numerical method to solve the dynamics of the information state. This work will be needed if one wishes to obtain nonlinear filters. When the filtering is carried out on-line, then the speed of the numerical method will be an important issue.

In the output feedback control and filtering cases, we have expressed the optimal solutions in terms of an infinite dimensional equation. Implementation of such equation in practice is not desirable, especially when real time computation is involved. To overcome this barrier, one needs to develop finite dimensional approximations. The key step would be to obtain a finite dimensional approximate of the dynamics of the information state. In [92], an extended-Kalman-filter based approximation scheme is developed. Extension of this result to more general dissipative cases could be an interesting work. An alternative to this approach could be the use of orthogonal polynomials such as Legendre and Chebychev polynomials. These polynomials have been successfully used to obtain approximate solutions to a class of PDE's (see for example [93] and the references therein).

3. Further discrete time results. In this thesis (see Appendix A), we have pre-
9.2 Issues for Future Research

Presented some partial synthesis results for discrete time systems. Work remains to be done to characterize strictly dissipative nonlinear discrete time systems. In the linear systems case, it would be interesting to look for Riccati equations based expression for the controller or filter in the case of a general quadratic dissipativity. The importance of discrete time synthesis results stem from the fact that many engineering designs rely on digital computers as their signal processors.

4. Stability analysis. In the output feedback control problem for nonlinear systems (see Chapter 5), we have presented some stability results under the assumptions of closed loop detectability/reachability. In practice, these conditions are difficult to check. In the case of control problem for linear systems, one expresses the solution in terms of stabilizing solutions to AREs [23], [82], [91]. This leads to a closed loop stability without any assumptions regarding closed loop detectability/reachability. We would expect that these results could be generalized, in some manner, to the nonlinear systems case. In the state feedback control, we have obtained some results in this direction. In the output feedback control case, some progress has been made in [42].

5. Some variations. In the situations where systems parameters are not known, one may like to seek for an adaptive implementation of controllers. Results in this direction has been obtained in [24], [18] for a finite gain performance measure and in [88] for passive performance measures. It is interesting to develop an adaptive scheme that achieve a general dissipative performance measure.

Another variation which could be of importance is to develop a multiobjective dissipativity/$H_2$ control scheme. In [72], a mixed $H_{\infty}/H_2$ control scheme is obtained based on a Nash equilibrium approach leading to cross-coupled AREs. These results are extended to nonlinear systems in [74] employing cross-coupled HJI equations. It is interesting to extend the results even further to a more general dissipativity case.
Bibliography


Appendix A

Discrete Time Results

In this appendix we present some results on dissipative control/filtering synthesis problems for discrete time systems. We employ similar methods as those in the continuous time cases presented in Chapter 4, Chapter 5 and Chapter 7. As we shall see, discrete time versions of dissipation inequalities play the key roles for obtaining solutions. In the control synthesis case the results presented here extend those in [56]. Different approaches to discrete time control/filtering synthesis problems are given in [24], [75], [76], [89] for nonlinear systems and in [6], [36], [108], [109], [87] for linear systems.

A.1 Control Synthesis

We consider discrete time systems described by

\[
\begin{align*}
\Sigma_d: & & x_{k+1} &= f(x_k, u_k) + w_k, & x_0 = x, \\
& & z_{k+1} &= h_1(x_k, u_k, w_k), \\
& & y_{k+1} &= h_2(x_k, u_k) + v_k, & k = 0, 1, \ldots .
\end{align*}
\]

(A.1)

In this expression, \( x_k \in \mathbb{R}^n \) denotes the state variable, \( z_k \in \mathbb{R}^q \) and \( y_k \in \mathbb{R}^p \) denote the to-be-controlled output and the measurement vectors respectively. The vectors \( w_k \in \mathbb{R}^n, v_k \in \mathbb{R}^p \) denote system and measurement disturbances. The initial condition \( x \) is assumed to be unknown and is considered as a part of the disturbances. We assume that \( x = 0 \) is an equilibrium. In particular, we assume \( f(0, 0) = 0, h_1(0, 0, 0) = 0, h_2(0, 0) = 0 \). The admissible disturbances are all signals \( w \) and \( v \) in \( l_2([0, \infty), \mathbb{R}^n) \) and \( l_2([0, \infty), \mathbb{R}^p) \) respectively.
We consider a general supply rate function $r(z, w, v)$ as the function

$$r : \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}.$$  \hspace{1cm} (A.2)

As in the continuous time case, we assume that

$$r(z, 0, 0) \leq 0,$$  \hspace{1cm} (A.3)

and $r(0, 0, 0) = 0$. The dissipative control problem is to find a control law $u$ in a specified admissible class such that:

1. the closed loop map $\Sigma_d^u$ is asymptotically stable when $w = 0$, $v = 0$,

2. the closed loop map $\Sigma_d^u$ is dissipative with respect to the supply rate $r$ in (A.2), that is for any initial $z_0$ we have

$$-\sum_{i=0}^{M-1} r(z_{i+1}, w_i, v_i) \leq \beta^u(z),$$  \hspace{1cm} (A.4)

for all $(w, v)$ in $l_2([0, M - 1], \mathbb{R}^{n+p})$ for all $M \geq 0$, for some function $\beta^u$ satisfying $\beta^u \geq 0$ with $\beta^u(0) = 0$. This function depends on the control law being employed.

### A.2 State Feedback Synthesis

In this section we consider a specific synthesis problem in which the full state vector $z$ is available for control and the measurement disturbance is zero. In particular, we have the measurement equation $y_{k+1} = z_k$. Thus, our model reduces to

$$\Sigma_d : \left\{ \begin{array}{l} x_{k+1} = f(x_k, u_k) + w_k, \quad x_0 = z, \\ z_{k+1} = h_1(x_k, u_k, w_k), \quad k = 0, 1, \ldots \end{array} \right.$$  \hspace{1cm} (A.5)

We consider supply rate functions of the form $r(z, w)$ satisfying

$$r(z, 0) \leq 0,$$  \hspace{1cm} (A.6)

for all $z \in \mathbb{R}^q$ with $r(0, 0) = 0$.

Furthermore, we consider admissible control laws as the class of static functions of the state $u = K(z)$, in which $K(\cdot)$ is a memoryless function. We denote this class $\mathcal{S}$.

We shall first make the following definition.
Definition A.1 We say that the closed loop system \( \Sigma_d^u \) is (zero state) detectable if, under the control law \( u \), the conditions \( w = 0 \) and \( \lim_{k \to \infty} x_k = 0 \) implies \( \lim_{k \to \infty} z_k = 0 \).

The following result is analogous to that in Theorem 4.1 in Chapter 4.

Theorem A.1 Consider the system \( \Sigma_d \) in (A.5) and the supply rate \( r(z, w) \) satisfying (A.6) with \( r(0,0) = 0 \). Assume there exists a control \( u^* = K^*(x) \in S \) such that the system \( \Sigma_d \) is dissipative with respect to the supply rate \( r \). Then, there exists a solution \( V \) to the discrete time dissipation inequality

\[
V(x) \geq \inf_{u \in \mathbb{R}^m} \sup_{w \in \mathbb{R}^n} \{ V(f(x,u)+w) - r(h_1(x,u,w),w) \}, \tag{A.7}
\]

in \( \mathbb{R}^n \), such that \( V \geq 0 \) with \( V(0) = 0 \). Conversely, assume there exists a solution \( V \) to the discrete time dissipation inequality (A.7) such that \( V \geq 0 \) with \( V(0) = 0 \). Assume that the control law \( u^* \) achieves minimum in the right hand side of (A.7), i.e.,

\[
u^* \in \arg\min_{u \in \mathbb{R}^m} \{ \sup_{w \in \mathbb{R}^n} \{ V(f(x,u)+w) - r(h_1(x,u,w),w) \} \}.
\]

Then, applying \( u^* \) to the system (A.5) results in a closed loop dissipativity. Moreover, if: (i) the supply rate \( r \) satisfies

\[
r(z,0) \leq -c_1|z|^2, \text{ for some } c_1 > 0, \tag{A.8}
\]

and (ii) \( \Sigma_d^u \) is detectable, then the closed loop system is asymptotically stable.

Proof. Under the hypothesis, the function \( V \) defined by

\[
V(x) \triangleq \sup_{M \geq 0, w \in h_1(0,M-1)R^n} \{ -\sum_{i=0}^{M-1} r(h_1(x_i,K^*(x_i),w_i),w_i) : x_0 = x \}
\]

is bounded from above by \( \beta^u \). Moreover, by definition \( V \geq 0 \) since it is defined as the supremum over a set that contains 0. Thus, \( V \) is finite. By dynamic programming methods \( V \) satisfies the discrete time dissipation inequality (A.7). This proves the existence of a solution to (A.7).

Conversely, the hypothesis implies that on applying \( u^* \) on \( \Sigma_d \) we have for all \( x_0 \in \mathbb{R}^n \),
for all \( w \in l_2([0, M-1], \mathbb{R}^n) \) for all \( M \geq 0 \),
\[
(V(x_M) - V(x_0)) - \sum_{i=0}^{M-1} r(h_1(x_i, u^*(x_i), w_i), w_i)
\]
\[
= \sum_{i=0}^{M-1} \{ V(f(x_i, u^*(x_i)) + w_i) - V(x_i) - r(h_1(x_i, u^*(x_i), w_i), w_i) \}
\]
\[
\leq \sum_{i=0}^{M-1} \sup_{w_i \in \mathbb{R}^n} \{ V(f(x_i, u^*(x_i)) + w_i) - V(x_i) - r(h_1(x_i, u^*(x_i), w_i), w_i) \}
\]
\[
\leq 0.
\]
Since \( V \geq 0 \), this inequality implies
\[
-\sum_{i=0}^{M-1} r(z_i+1, w_i) \leq V(x_0),
\]
for all \( x_0 \in \mathbb{R}^n \), for all \( w \in l_2([0, M-1], \mathbb{R}^n) \) for all \( M \geq 0 \). This proves the dissipativity of the closed loop system \( \Sigma_d^u \) with \( \beta^u = V \). In particular, setting \( w = 0 \) and using (A.8) we get
\[
0 \leq c_1 \sum_{i=0}^{M-1} |z_i+1|^2 \leq -\sum_{i=0}^{M-1} r(z_i+1, 0) \leq \beta^u(x_0),
\]
for all \( M \geq 0 \). This implies that \( \lim_{k \to \infty} z_k = 0 \). By the closed loop detectability assumption, this implies that \( x_k \) goes to 0 asymptotically. This proves the asymptotic stability. \( \square \)

### A.3 Output Feedback Synthesis

In this section we turn to the general synthesis problem in which the measurement equation takes a general form as described in (A.1). We shall first present some results for finite time case and then extend them to the infinite time case.

#### A.3.1 Finite Time Case

We consider the admissible control laws as the set of causal maps of the measurement
\[
u : l_2([0, M-1], \mathbb{R}^p) \to l_2([0, M-1], \mathbb{R}^m).
\]
We denote the class by \( U \). In the sequel, we use the notation \( f[y] \) to mean that \( f \) is a causal function of \( y \). Admissible disturbances are all signals \( w, v \in l_2([0, M-1], \mathbb{R}^{n+p}) \).

Given a fixed finite time interval \([0, M-1] \), the finite time dissipative output feedback control problem is to find \( u \in U \) such that for any initial condition \( x_0 \in \mathbb{R}^n \) the map \( \Sigma^u \) is dissipative with respect to the supply rate \( r \) in (A.2), which means
\[
-\sum_{i=0}^{M-1} r(z_i+1, w_i, v_i) \leq \beta^u_M(x),
\]
(A.9)
for all \(w, v\) in \(l_2([0, M - 1], \mathbb{R}^{n+p})\), for some function \(\beta_M^u\) satisfying \(\beta_M^u \geq 0\) with \(\beta_M^u(0) = 0\).

As in the continuous time case in Chapter 5, we shall rephrase the control problem in terms of a dynamic game. For \(u \in U\) we define the cost functional \(J_{p,M}\) as follows

\[ J_{p,M}(u) \triangleq \sup_{(w,v) \in l_2([0, M-1], \mathbb{R}^{n+p})} \sup_{x_0 \in \mathbb{R}^n} \{ p(x_0) - \Sigma_{i=0}^{M-1} r(z_{i+1}, w_i, v_i) \}. \] (A.10)

Obviously, the map \(\Sigma^u\) is dissipative with respect to \(r(z, w, v)\) if and only if

\[ J_{\alpha,M}(u) \leq 0 \]

for some \(\alpha(x) = -\beta_M^u(x) \leq 0\), with \(\beta_M^u(0) = 0\). Moreover, if \(\Sigma^u\) is dissipative the following inequalities hold (see Chapter 5)

\[ (p, 0) \leq J_{p,M}(u) \leq (p, \beta_M^u). \]

Thus, we have the relation

\[ \{ p : (p, 0), (p, \beta) \text{ finite} \} \subset \text{dom} J_{p,M}. \]

The output feedback control problem can be solved by minimizing \(J_{p,M}(u)\) over \(U\). This is a zero-sum dynamic game problem with a partial observation. Note that \(p = -\beta_M^u\) belongs to \(\text{dom} J_{p,M}\).

We solve the problem by employing the information state technique. For a fixed output path \(y \in l_2([1, k], \mathbb{R}^p)\) the information state \(p_k(\cdot)\) is given by

\[ p_k(x) \triangleq \sup_{w \in l_2([0, k-1], \mathbb{R}^*), x_0 \in \mathbb{R}^*} \{ p(x_0) - \Sigma_{i=0}^{k-1} r(z_{i+1}, w_i, y_{i+1} - h_2(x_i, u_i)) : x_k = x \}. \] (A.11)

Applying dynamic programming methods, we see that \(p_k(x)\) satisfies the recursion

\[ p_{k+1}(x) = F(p_k, u_k, y_{k+1})(x), \]

\[ p_0 = p, \quad k = 0, 1, \ldots, M - 1, \]

where

\[ F(p, u, y)(x) = \sup_{\xi} \{ p(\xi) - r(h_1(\xi, u, x - f(\xi, u)), x - f(\xi, u), y - h_2(\xi, u)) \}. \]

In terms of \(p_k\) we have the following representation of \(J_{p,M}\) (see also [56]).

\textbf{Theorem A.2} For any \(u \in U\), we have

\[ J_{p,M}(u) = \sup_{y \in l_2([0, M-1], \mathbb{R}^p)} \{ (p_M, 0) : p_0 = p \}. \] (A.13)
Proof. The right hand side of (A.13) can be rewritten as

\[
\sup_{y \in \ell_2([1, M], \mathbb{R}^p)} \{ (p_M, 0) : p_0 = p \} 
\]

\[
= \sup_{y \in \ell_2([1, M], \mathbb{R}^p)} \{ \sup_{x \in \mathbb{R}^n} \{ \sup_{w \in \ell_2([0, M - 1], \mathbb{R}^n), x_0 \in \mathbb{R}^n} \{ p(x_0) 
\]

\[-\sum_{i=0}^{M-1} r(z_{i+1}, w_i, y_{i+1} - h_2(z_i, u_i)) : x_k = x \} : p_0 = p \} ,
\]

\[
= \sup_{y \in \ell_2([1, M], \mathbb{R}^p)} \sup_{w \in \ell_2([0, M - 1], \mathbb{R}^n), x_0 \in \mathbb{R}^n} \{ p(x_0) 
\]

\[-\sum_{i=0}^{M-1} r(z_{i+1}, w_i, y_{i+1} - h_2(z_i, u_i)) \}
\]

\[
= \sup_{y \in \ell_2([0, M - 1], \mathbb{R}^p), w \in \ell_2([0, M - 1], \mathbb{R}^n), x_0 \in \mathbb{R}^n} \{ p(x_0) - \sum_{i=0}^{M-1} r(z_{i+1}, w_i, u_i) \}
\]

\[
= J_{p, M}(u).
\]

This completes the proof. 

Thus, minimizing \( J_{p, M} \) can equivalently be carried out by minimizing the right hand side of equation (A.13). This is a completely observed dynamic game problem in which the information state \( p_k \) provides the appropriate state and has the dynamics given in (A.12). In the new game problem, the measurement variable acts as a malicious agent that seeks to maximize the cost function

\[
\{(p_M, 0) : p_0 = p \},
\]

while the controller seeks to minimize it. We employ dynamic programming methods to solve the problem. To this end, define the value function \( W(p, k) \) by

\[
W(p, k; M) \triangleq \inf_{u \in U} \sup_{y \in \ell_2([k+1, M], \mathbb{R}^p)} \{(p_M, 0) : p_k = p \} \quad (A.14)
\]

which satisfies the following dynamic programming equation

\[
W(p, k; M) = \sup_{y \in \mathbb{R}^p} \inf_{u \in \mathbb{R}^m} \{ W(F(p, u, y), k + 1) \},
\]

\[
W(p, M; M) = (p, 0),
\]

\( k = 0, 1, \ldots, M \). This equation is the appropriate dynamic programming equation for the partially observed dynamic game associated with the output feedback control problem. In this equation the Isaacs condition need not hold in general. We express the solution to the control problem in terms of the equations (A.12) and (A.15).
Theorem A.3 Suppose there exists a control \( u^o \in U \) such that the closed loop system \( \Sigma_d^{u^o} \) is dissipative with respect to \( r \) in (A.2). Then there exist solutions \( p, W \) to the DPE's (A.12) and (A.15), such that \( p_0 = -\beta \) and \( W(-\beta,0;M) = 0 \), for some \( \beta(x) \geq 0 \), with \( \beta(0) = 0 \), and \( W(p,0;M) \geq (p,0) \) for all \( p \in \text{dom}W(\cdot,0;M) \).

Conversely, suppose there exist solutions to the DPE's (A.12) and (A.15) such that \( p_0 = -\beta \) and \( W(-\beta,0;M) = 0 \), for some \( \beta(x) \geq 0 \) with \( \beta(0) = 0 \), and such that \( p_k \in \text{dom}W(\cdot,k;M) \), \( k = 0,1,\ldots,M-1 \). Let \( u^* \) be the control law that achieves minimum in the right hand side of (A.15), i.e.,

\[
\begin{align*}
u^*_k &= \arg\min_{u \in \mathbb{R}^m} \{W(F(p_k, u, y), k + 1)\} \\
&= u^*(p[y], y, k),
\end{align*}
\]

\( k = 0,1,\ldots,M-1 \), for each \( p_k \in \text{dom}W(\cdot,k;M) \). Then \( u^* \) solves the finite time dissipative output feedback control problem.

Proof. If there exists a filter \( u^o \in U \) solving the dissipative control problem, then with \( p_0 = -\beta u^o \)

\[
J_{p_0,M}(u^o) \leq 0,
\]

and moreover

\[
0 = (-\beta u^o, 0) \leq W(-\beta u^o, 0;M) \leq J_{p_0,M}(u^o) \leq 0.
\]

Therefore \( W(-\beta u^o, 0;M) = 0 \). By dynamic programming methods, \( W \) solves the DPE (A.15). This proves the existence of a solution to (A.15).

Conversely, suppose that equations (A.12) and (A.15) have solutions \( p, W \) such that \( p_0 = -\beta \) and \( W(-\beta,0;M) = 0 \), with \( \beta(x) \geq 0 \), and \( \beta(0) = 0 \). Then, employing the control law \( u^* \) in (A.16), we have for any \( y \in l_2([1,M], \mathbb{R}^p), \)

\[
(W(p_M,M;M) - W(p_0,0;M))
\]

\[
= \sum_{i=0}^{M-1} \{W(F(p_i, u^*(p_i,y_i,i),y_i), i+1, M) - W(p_i,i;M)\}
\]

\[
\leq \sum_{i=0}^{M-1} \sup_{y \in \mathbb{R}^p} \{W(F(p_i, u^*(p_i,y,i), y), i+1;M) - W(p_i,i;M)\}
\]

\[
= 0.
\]

Since \( W(p_M,M;M) = (p_M,0) \), this inequality implies

\[
(p_M,M) \leq W(p_0,0),
\]
for all $y \in l_2([1, M], \mathbb{R}^p)$. Therefore we have

$$\sup_{y \in l_2([1, M], \mathbb{R}^p)} \{(p_M, M) : p_0 = p \} \leq W(p_0, 0; M).$$

In particular, choosing $p_0 = -\beta$, we get, using Theorem A.2,

$$J_{-\beta, M}(u^*) = \sup_{y \in l_2([0, M-1], \mathbb{R}^p)} \{(p_M, 0) : p_0 = -\beta \} \leq W(-\beta, 0; M) = 0.$$

The last inequality implies that $\Sigma_{\beta}^{y^*}$ is dissipative with respect to $r$. This completes the proof. \hfill \Box

**Remark A.1** To summarize, we express the solution to the output feedback dissipative control problem in terms of: (i) a solution $p_k$ to equation (A.12), (ii) a solution $W(p, k)$ to equation (A.15) and (iii) a coupling condition between $p$ and $W$ given by $p_k \in \text{dom} W_k$, $k = 0, 1, \ldots, M - 1$.

### A.3.2 Example: A Linear $H_\infty$ Control Problem

We shall now look at a special case, namely the linear $H_\infty$ control problem, in which equations (A.12) and (A.15) admit finite dimensional solutions. In this problem, the model is given by

$$\begin{cases}
    x_{k+1} = A x_k + w_k + B_2 u_k, & x_0 = z, \\
    z_{k+1} = \begin{bmatrix} C_1 z_k \\ D_{12} u_k \end{bmatrix}, \\
    y_{k+1} = C_2 z_k + u_k, & k = 0, 1, \ldots,
\end{cases} \tag{A.17}$$

where $A, B_2, C_1, C_2$ and $D_{12}$ are constant matrices with appropriate dimension. We assume $D_{12} D_{12}^T = E_1 > 0$. The supply rate function takes a quadratic form given by

$$r(x, w, v) = \frac{1}{2}(-|x|^2 + \gamma^2 (|w|^2 + |v|^2)),$$

where $\gamma > 0$. The information state is given by the recursion

$$p_{k+1}(x) = \sup_{\xi} \{ p_k(\xi) + \frac{1}{2} (x_k' C_1' C_1 x_k + u_k' B_1 u_k) \\
- \gamma^2 (|x - (A \xi + B_2 u_k)|^2 + |y_{k+1} - C_2 \xi|^2) \}$$

$k = 0, 1, \ldots, M - 1$, with $p_0 = p$. As in the nonlinear systems case, in general, $p$ lives in a (infinite dimensional) function space. However if we choose $p_0$ to be quadratic, i.e.,
A.3 Output Feedback Synthesis

\[ p_0(x) = -\frac{1}{2}\gamma^2(x - \hat{x})'Y^{-1}(x - \hat{x}) + \varphi \] with positive definite matrix \( Y > 0 \), then \( p_k \) takes the quadratic form

\[ p_k(x) = -\frac{1}{2}\gamma^2(x - \hat{x}_k)'Y_k^{-1}(x - \hat{x}_k) + \varphi_k, \tag{A.18} \]

where \( \hat{x}_k, Y_k, \) and \( \varphi_k \) satisfy the recursions

\[
\begin{cases}
\hat{x}_{k+1} = \bar{A}(Y_k) + Bu_k + \bar{B}_1(Y_k)\bar{v}_k, & \hat{x}_0 = \hat{x}, \\
Y_{k+1} = AY_k(I + (C_2'C_2 - \gamma^{-2}C_1'C_1)Y_k)^{-1}A' + I, & Y_0 = Y > 0, \\
\varphi_{k+1} = \varphi_k + \frac{1}{2}(\bar{v}_k'\bar{Q}(Y_k)\bar{v}_k + u_k'R_{uk} - \gamma^2\bar{v}_k'\bar{v}_k), & \varphi(0) = \varphi,
\end{cases} \tag{A.19} \]

for \( k \in [0, M - 1] \), with

\[
\begin{align*}
\bar{v}_k &= \Phi_0^{\frac{1}{2}}(Y)(y_{k+1} - (C_2 + \Phi_0^{-1}(Y)\Phi_k(Y))\hat{x}), \\
\bar{A}(Y) &= A(I - \gamma^{-2}YQ)^{-1}, \\
\bar{B}_1(Y) &= AY(I + (C_2'C_2 - \gamma^{-2}Q)Y)^{-1}C_2\Phi_0^{-\frac{1}{2}}(Y), \\
\bar{Q}(Y) &= Q(I - \gamma^{-2}YQ) \geq 0, \\
\Phi_0(Y) &= I - C_2Y(I + (C_2'C_2 - \gamma^{-2}Q)Y)^{-1}C_2 > 0, \text{ and} \\
\Phi_k(Y) &= \gamma^{-2}C_2Y(I + (C_2'C_2 - \gamma^{-2}Q)Y)^{-1}Q, \quad Q = C_1'C_1.
\end{align*} \tag{A.20} \]

Now, if we define the finite dimensional quantity by \( \rho \triangleq (\hat{x}, Y, \varphi) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R} \), equation (A.19) can be rewritten as

\[ \rho_{k+1} = \hat{F}(\rho_k, u_k, y_{k+1}), \tag{A.21} \]

with initial condition \( \rho_0 = \rho \). Thus the finite dimension quantity \( \rho \) is identified with the quadratic information state \( p \). We denote this quadratic information state \( p^\rho \). Next, since, \( (p^\rho, 0) = \varphi \), Theorem A.2 yields the representation for \( J \) as follows

\[
J_{\rho,M}(u) = J_{p^\rho,M}(u) = \sup_{p \in \mathcal{L}_2([1,M],[\mathbb{R}^\rho])} \{ \varphi_M : \rho_0 = \rho \}.
\]

The value function \( W(\rho, k; M) = W(p^\rho, k; M) \) defined by

\[
W(\rho, k; M) \triangleq \inf_{u \in U} \sup_{y \in \mathcal{L}_2([1,M],[\mathbb{R}^\rho])} \{ \varphi_M : \rho_0 = \rho \}
\]

satisfies the following (infinite dimensional) DPE

\[
W(\rho, k; M) = \sup_{\bar{u} \in \mathbb{R}^\rho} \inf_{u \in \mathbb{R}^m} \{ W(\hat{F}(\rho, u, \bar{v}), k + 1) \}, \tag{A.22} \]

\[
W(\rho, M; M) = \varphi, \quad k = 0, 1, \ldots, M - 1,
\]
where \( \hat{P}(\cdot) \) is defined in the right hand side of (A.21) and (A.19). The reversion of order of the inf and sup operations is possible in this equation since the Isaacs condition holds. This DPE has the explicit solution given by

\[
W(\rho, k; M) = \frac{1}{2} \hat{P} X_k [I - \gamma^{-2} Y X_k]^{-1} \hat{P} + \varphi, \quad W(\rho, M; M) = \varphi,
\]

where \( X_k \) obeys the recursion

\[
X_k = A_k' X_{k+1} (I + (B_2 E_1^{-1} B_2' - \gamma^{-2} I) X_{k+1})^{-1} A + C_1 C_1, \quad X_M = 0,
\]

with \( I - \gamma^2 X_{k+1} > 0, \) \( k = 0, 1, \ldots, M - 1 \) such that the coupling condition

\[
(A Y_k \hat{A}(Y_k)' + I) X_{k+1} < \gamma^2 I
\]

holds for \( k = 0, 1, \ldots, M - 1 \). The optimal control is given by

\[
u^*_k = u^*(_k Y, Y, k) = -E_2^{-1} B_2' X_{k+1} (I + (B_2 E_1^{-1} B_2' - \gamma^{-2} I) X_{k+1})^{-1} A[I - \gamma^{-2} Y X_k]^{-1} \hat{P},
\]

for all \( 0 \leq k \leq M - 1 \).

### A.3.3 Infinite Time Case

In this section, we present the relevant equations for general dissipative control on infinite time horizon. We also obtain asymptotical stability under appropriate closed loop detectability assumptions. We consider the admissible control laws as the set of causal maps of the measurement

\[
\hat{u} : l_2([0, \infty), R^p) \to U.
\]

We denote the class by \( U \). Admissible disturbances are all signals \( w, v \in l_2([0, \infty), R^{n+p}) \).

To solve the infinite horizon problem we consider the cost functional

\[
J_p(u) = \sup_{M \geq 0} J_{p,M}(u),
\]

in which \( J_{p,M}(u) \) is defined in (A.10), and minimize it over \( u \in U \). We see that \( \Sigma u \) is dissipative with respect to \( \tau(z, w, v) \) if and only if \( J_{-\beta} \leq 0 \), or, by Theorem A.2, if and only if

\[
\sup_{M \geq 0, \nu \in L_2([1,M], R^p)} \{(p_M, 0) : p_0 = -\beta \} \leq 0,
\]

for some \( \beta \geq 0 \) with \( \beta(0) = 0 \). Minimization of \( J_p(u) \) over \( u \in U \) leads to the following (stationary) value function \( W(p) \)

\[
W(p) = \inf_{u \in U} \sup_{M \geq 0, \nu \in L_2([1,M], R^p)} \{(p_M, 0) : p_0 = p \}.
\] (A.23)
A.3 Output Feedback Synthesis

Suppose that for some $u^0$, the systems $\Sigma^{u^*}$ is dissipative. This implies

$$(p, 0) \leq W(p) \leq (p, \beta^{u^*}).$$  \hfill (A.24)

In particular, $W(p)$ is finite on the set

$$\text{dom} W = \{p : (p, 0), (p, \beta^{u^0}) \text{ finite}\}.$$

Moreover, we have

$$0 = (-\beta^{u^0}, 0) \leq W(-\beta^{u^0}) \leq (-\beta^{u^0}, \beta^{u^0}) = 0,$$

which implies $W(-\beta^{u^*}) = 0$. By dynamic programming principle, the value $W$ satisfies the following discrete time (infinite dimensional) dissipation inequality

$$W(p) \geq \sup_{y \in \mathbb{R}^p} \inf_{u \in \mathbb{R}^m} \{W(F(p,u,y))\}. \hfill (A.25)$$

This is the appropriate dissipation inequality for the output feedback dissipative control problem.

Conversely, assume that equations (A.12) and (A.25) have solutions $p, W$ such that with $p_0 = -\beta$, for some function $\beta(x) \geq 0$ satisfying $\beta(0) = 0$, $p_k \in \text{dom} W$ for all $k \geq 0$ for all sequence $y$ and $W(-\beta) = 0$. Then, following the proof of the converse part of Theorem A.3 the control law $u^*$ defined by

$$u^* = \arg \min_{u \in \mathbb{R}^m} \{W(F(p_k,u,y))\}$$

results in the dissipativity of $\Sigma^{u^*}$ with respect to the supply rate $r$ in (A.2).

The asymptotic stability of the system $\Sigma^{u^*}$ can be deduced from the dissipativity property provided the following conditions hold: (i) the supply rate $r(w, z, v)$ satisfies

$$r(z, 0, 0) \leq -c|z|^2,$$  \hfill (A.26)

for all $z$, for some constant $c > 0$, and (ii) the closed loop system $\Sigma^{u^*}_d$ is zero-state detectable. This can be established using a similar calculation as in the state feedback case in Section A.2. To achieve internal stability the information state system is required to have some kind of stability. Indeed the DPE (A.25) implies that the information state $p_t$ is bounded from above, provided $p_0$ belongs to $\text{dom} W$. This is because, the DPE (A.25) and inequality (A.24) imply

$$p_k(x) \leq (p_k, 0) \leq W(p_k) \leq W(p) \leq (p, \beta),$$

for all $k \geq 0$. A lower bound can be obtained by assuming uniform reachability assumption as described in Chapter 4.
A.4 Filter Synthesis

The purpose of this section is to present discrete time results for filter synthesis analogous to those in Chapter 7. We shall begin by considering a general dissipative filtering problem. We express the solution to the problem in terms of two recursions. The first recursion moving forward in time describes the dynamics of the corresponding information state while the second one is a backward recursion determining the optimal filtering policy. We shall then specialize the general theory to a linear $H_\infty$ filtering problem.

We consider the class of nonlinear systems modelled by

$$\begin{align*}
  x_{k+1} &= f(x_k) + w_k, \quad x_0 = x, \\
  \dot{z}_d &= h_1(x_k), \\
  y_{k+1} &= h_2(z_k) + u_k, \quad k = 0, 1, \ldots, M - 1.
\end{align*}$$

(A.27)

In this expression, $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, and $z_k \in \mathbb{R}^q$ denote state, output and to-be estimated vectors respectively. The vectors $w_k \in \mathbb{R}^n, u_k \in \mathbb{R}^p$ denote system and measurement disturbances. The initial condition $x$ is assumed to be unknown and is considered as a part of the disturbances.

A.4.1 Finite Time Case

We consider the admissible filtering strategies as the set of causal maps of the observation $\hat{z} : l_2([1, M], \mathbb{R}^p) \rightarrow l_2([1, M], \mathbb{R}^q)$. We denote the class of such strategies by $Z$. Admissible disturbances are all $l_2$ bounded signals $w, u \in l_2([0, M - 1], \mathbb{R}^{n+p})$.

Let $e = \hat{z} - z$ denote the estimation error and $\Sigma^\delta_d$ denote the map from $w_k$ to $e$ under the filtering policy $\hat{z}$. In the filtering case, the supply rate takes the form

$$r(e, w, v).$$

(A.28)

We assume that $r(e, 0, 0) \leq 0$ for all $w, v$ and $r(0, 0, 0) = 0$. In the $H_\infty$ filtering problem, one has $r(e, w, v) = \frac{1}{2}(-|e|^2 + \gamma^2(|w|^2 + |v|^2))$. Given a fixed finite time interval $[1, M]$, and a supply rate function $r(e, w, v)$ we say that the map $\Sigma^\delta$ is dissipative with respect to the supply rate $r(e, w, v)$ (A.28) if for each initial condition $x_0$ the map $\Sigma^\delta_{x_0}$ is dissipative.
with respect to $r(e, w, v)$, which means there exists a finite quantity $\beta^2_M(x) \geq 0$, with $\beta^2_M(0) = 0$, such that

$$-\Sigma_{i=0}^{M-1} r(e_i, w_i, v_i) \leq \beta^2_M(x),$$

(A.29)

for all $(w, v) \in l_2([0, M - 1], R^{n+p})$. Thus, the finite time horizon dissipative filtering problem is to find $\hat{z} \in Z$ such that the map $\Sigma^2$ is dissipative with respect to the supply rate $r(e, w, v)$ (A.28).

We use game-theoretic methods to solve this problem (see Chapter 7). For $\hat{z} \in Z$ define the cost functional related to the filtering problem as follows

$$J_{p, M}(\hat{z}) \triangleq \sup_{(w, v) \in l_2([0, M - 1], R^{n+p})} \sup_{x_0} \{p(x_0) - \Sigma_{i=0}^{M-1} r(e_{i+1}, w_{i+1}, v_{i+1})\}.$$  

(A.30)

We see that the map $\Sigma^2$ is dissipative if and only if $J_{\alpha, M}(\hat{z}) \leq 0$, for some $\alpha(x) = -\beta(x) \leq 0$, with $\beta(0) = 0$. Moreover, if $\Sigma^2$ is dissipative the following inequalities hold

$$(\beta, 0) \leq J_{p, M}(\hat{z}) \leq (\beta, \beta).$$

Thus, for dissipative map $\Sigma^2$, we have the relation

$$\{p : (p, 0), (p, \beta^2) \text{ finite}\} \subset \text{dom} J_{p, M}$$

We seek to minimize $J_{\alpha, M}(\hat{z})$ over $Z$. This is a zero-sum dynamic game problem with partial observation.

For a fixed output path $y \in l_2([1, k], R^p)$ and filtering signal $\hat{z} \in l_2([0, k - 1], R^q)$, we define the information state $p_k(\cdot)$ by

$$p_k(x) \triangleq \sup_{(w, v) \in l_2([0, k - 1], R^{n+p})} \sup_{x_0 \in R^n} \{p(x_0) - \frac{1}{2} \Sigma_{i=0}^{k-1} r(e_{i+1}, w_{i+1}, v_{i+1} - h_2(x_i) : x_k = x)\}.$$  

(A.31)

Applying dynamic programming methods, we see that $p_k(\cdot)$ satisfies the recursion

$$p_{k+1}(x) = F(p_k(\hat{z}, y_{k+1}))(x),$$

$$p_0 = p, \quad k = 0, 1, \ldots, M - 1,$$

where

$$F(p, \hat{z}, y)(x) = \sup_\xi \{p(\hat{z})$$

$$+ r(\xi - h_1(\xi), x - f(\xi), y_{k+1} - h_2(\xi))\}.$$  

In terms of $p_k$ we have the following representation result.

**Theorem A.4** If $J_{p, M}(\hat{z})$ is finite, then

$$J_{p, M}(\hat{z}) = \sup_{y \in l_2([0, M - 1], R^p)} \{(p_M, 0) : p_0 = p\}.$$
Discrete Time Results

Proof. The proof is similar to that of Theorem A.2.

Thus, we now have a completely observed dynamic game corresponding to the filtering problem. Employing dynamic programming methods leads us to the value function $W(p, k; M)$ defined by

$$W(p, k; M) \triangleq \inf_{\tilde{z} \in \mathcal{Z}([k, M-1], \mathbb{R}^s)} \sup_{y \in \mathcal{Y}([k+1, M], \mathbb{R}^p)} \{(p_M, 0) : p_k = p\}.$$

Using dynamic programming principle we see that $W$ satisfies the following dynamic programming equation

$$W(p, k; M) = \sup_{y \in \mathcal{Y}^+} \inf_{\tilde{z} \in \mathcal{Z}^+} \{W(F(p, \tilde{z}, y), k + 1; M)\},$$

$$W(p, M; M) = (p, 0),$$

$k = 0, 1, \ldots, M$. This equation is the appropriate dynamic programming equation for the partially observed game associated with the dissipative filtering problem. The next theorem says that equations (A.32) and (A.34) serve as the key recursions to obtain the optimal estimate $\hat{z}^*$ of $z = h_1(x)$. The optimal estimate is obtained by finding the policy $\hat{z}^*(p, k)$ which attains minimum in the right hand side of (A.34)

$$\hat{z}^*[y]_k = \arg\min_{\tilde{z} \in \mathcal{Z}^+} \left\{ \sup_{y \in \mathcal{Y}^+} W(F(p_k, \tilde{z}, y), k + 1; M) \right\}$$

$$= \hat{z}^*[y]_k, y, k),$$

$k = 0, 1, \ldots, M - 1$. We note that the estimate $\hat{z}^*$ depends on $p$ and on $y$. We now present the solution to the robust $H_\infty$ filtering problem.

**Theorem A.5** Suppose there exists a solution to the finite time dissipative filtering problem. Then there exist solutions to the DPE's (A.32) and (A.34), such that $p_0 = -\beta$ and $W(-\beta, 0; M) = 0$, for some $\beta(x) \geq 0$, with $\beta(0) = 0$, and $W(p, 0; M) \geq (p, 0)$ for all $p \in \text{dom}W(\cdot, 0; M)$.

Conversely, suppose there exist solutions to the DPE's (A.32) and (A.34) such that $p_0 = -\beta$ and $W(-\beta, 0; M) = 0$, for some $\beta(x) \geq 0$ with $\beta(0) = 0$, and such that $p_k \in \text{dom}W(\cdot, k; M), k = 0, 1, \ldots, M - 1$. Let the filter $\hat{z}^*$ be defined by (A.35) for each $p_k \in \text{dom}W(\cdot, k; M)$. Then $\hat{z}^*$ solves the finite time dissipative filtering problem.

Proof. If there exists a filter $\hat{z}^* \in \mathbb{Z}$ solving the filtering problem, then with $p_0 =$
\( -\beta z^* \)

\[ J_{p_0,M}(z^*) \leq 0, \]

and moreover

\[ 0 = (-\beta z^*, 0) \leq W(-\beta z^*, 0; M) \leq 0. \]

Therefore \( W(-\beta z^*, 0; M) = 0. \) By dynamic programming methods, \( W \) solves the DPE (A.34)

Conversely, suppose that equations (A.32) and (A.34) have solutions such that \( p_0 = -\beta \) and \( W(-\beta, 0; M) = 0, \) with \( \beta(x) \geq 0, \) and \( \beta(0) = 0. \) Then it follows from the proof of the converse part of Theorem A.3 that the filtering policy defined in (A.35) solves the dissipative filtering. In particular, we have, with \( p_0 = -\beta, \)

\[ J_{p_0,M}(z^*) \leq W(p_0, 0; M) = 0, \]

which implies that the map \( \Sigma z^* \) is dissipative on \([0, M]. \) This completes the proof. \( \square \)

**Remark A.2** Similar to the control synthesis case, in general we express the solution to dissipative dissipative problem in terms of: (i) solution to equation (A.32), (ii) solution to equation (A.34) and (iii) a coupling condition \( p_k \in \text{dom}W_k, \) \( k = 0, 1, \ldots, M - 1. \) We shall see in the next section the case in which the function \( W \) has a trivial solution and therefore the coupling condition holds trivially.

**Remark A.3** Since, in general, the value function \( W \) depends on the time horizon length \( M, \) i.e. \( W(p, k; M) = W(p, k; M), \) the resulting optimal filtering policy also depends on the time horizon length \( M, \) i.e. \( \hat{z}_k = \hat{z}(p, k; M). \) As \( M \) tends to infinity, the value function (A.33) becomes stationary, i.e. \( W(p, k; M) \to W(p) \) and therefore the filtering policy is independent of the time horizon length. This case is discussed in more detail in the later section.
A.4.2 Example: A Linear $H_\infty$ Filtering Problem

In this section, we specialize the results in the previous section to a linear $H_\infty$ filtering problem. The model is given by

$$
\begin{aligned}
    x_{k+1} &= Ax_k + w_k, \quad x_0 = x, \\
    z_k &= C_1 x_k, \\
    y_{k+1} &= C_2 z_k + v_k, \quad k = 0, 1, \ldots, M - 1,
\end{aligned}
$$

where $A, C_1,$ and $C_2$ are appropriately sized constant matrices. The recursion of the information state is given by

$$
p_{k+1}(z) = \sup_{\xi} \{p_k(\xi) + \frac{1}{2}(|\hat{x}_k - C_1 \xi|^2) \\
- \gamma^2 (|z - Ax_k|^2 + |y_{k+1} - C_2 z_k|^2)\},
$$

$k = 0, 1, \ldots, M - 1$, with $p_0 = p$. As in the nonlinear systems case, in general, $p$ lives in a (infinite dimensional) function space. However if we choose $p_0$ to be quadratic, i.e., $p_0(x) = -\frac{1}{2}\gamma^2(x - \hat{x})^T Y^{-1}(x - \hat{x}) + \varphi$ with positive definite matrix $Y > 0$, then $p_k$ takes the quadratic form

$$
p_k(x) = -\frac{1}{2}\gamma^2 (x - \hat{x}_k)^T Y_k^{-1} (x - \hat{x}_k) + \varphi_k,
$$
in which the information state components \( \rho = (\hat{z}, Y, \varphi) \) are finite dimensional quantities satisfying the following recursions

\[
\begin{align*}
\hat{x}_{k+1} &= A\hat{x}_k + A(Y_k^{-1} - (\gamma^{-2}C_1'C_1 - C_2'C_2))^{-1} \\
& \quad (C_2'\hat{v}_k - \gamma^{-2}C_1'\hat{e}_k), \\
\hat{x}_0 &= 0, \\
Y_{k+1} &= A(Y_k^{-1} - (\gamma^{-2}C_1'C_1 - C_2'C_2))^{-1}A' + I, \\
Y_0 &= Y > 0, \\
\varphi_{k+1} &= \varphi_k + \frac{1}{2}(\hat{e}_k'\Phi_k^{-1}\hat{e}_k - \hat{v}_k'\hat{v}_k), \\
\varphi_0 &= 0, \text{ with } \Phi_k = (I - \gamma^{-2}C_1Y_kC_1') > 0,
\end{align*}
\]

(A.38)

for all \( k = 0, 1, \ldots, M - 1 \), where \( \hat{u}_k = y_{k+1} - C_2\hat{x}_k \), \( \hat{e}_k = \hat{x}_k - C_1\hat{x}_k \), and where

\[
\begin{align*}
\hat{v}_k &= \Phi_k^{1/2}(Y)(y_{k+1} - (C_2 + \Phi_k^{-1}(Y)\Phi_b(Y))\hat{x}), \\
\Phi_a(Y) &= I - C_2Y(I + (C_2'C_2 - \gamma^{-2}C_1'C_1)Y)^{-1}C_2 > 0, \\
\Phi_b(Y) &= \gamma^{-2}C_2Y(I + (C_2'C_2 - \gamma^{-2}C_1'C_1)Y)^{-1}C_1'C_1.
\end{align*}
\]

The choice of quadratic \( p_0(x) \) corresponds to the assumption that the bounding function \( \beta(x) \) in the previous section is quadratic, i.e. \( \beta(x) = \gamma^2x'Y^{-1}x \). This choice is analogous to assuming Gaussian initial probability density function in the linear stochastic filtering case.

Now, the information state is completely determined by the finite dimensional quantity \( \rho \) by \( \rho_k = (\hat{z}_k, Y_k, \varphi_k) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R} \) satisfying the dynamic

\[
\rho_{k+1} = \hat{F}(\rho_k, \hat{x}_k, y_{k+1}),
\]

with initial condition \( \rho_0 = \rho \), where \( \hat{F} \) is described in equation (A.38). Next, since, \( (p^0, 0) = \varphi \), Theorem A.4 yields the representation for \( J \) as follows

\[
J_{p,M}(\hat{z}) = J_{p^0,M}(\hat{z}) = \sup_{y \in [1, M]} \{ \varphi_M : \rho_0 = \rho \},
\]

(A.39)

The value function \( W(\rho, k; M) = W(p^0, k; M) \) defined by

\[
W(\rho, k; M) = \inf_{\hat{z}, \varphi} \sup_{y \in [1, M, R^p]} \{ \varphi_M : \rho_0 = \rho \}
\]

satisfies the recursion

\[
\begin{align*}
W(\rho, k; M) &= \inf_{\hat{z}} \sup_{\varphi} \{ W(\hat{F}(\rho, \hat{z}, \hat{v}), k + 1) \}, \\
W(\rho, M; M) &= \varphi, \quad k = 0, 1, \ldots, M - 1,
\end{align*}
\]

(A.40)
where \( \hat{F}(\cdot) \) is defined in the right hand side of (A.38). This DPE has the explicit solution given by

\[
W(\rho, k; M) = W(\hat{x}, Y, \varphi, k)
\]

as can be verified by directly substituting \( W \) into the left hand side of (A.40), and the optimal filtering policy is then given by

\[
\bar{z}^*_k = \bar{z}^*(\rho_k, k) = C_1 \hat{x}_k, \quad k = 0, 1, \ldots, M - 1,
\]

which is independent of \( M \). This leads to the following structure of the (central) linear \( H_\infty \) filter (see Shaked and Theodor [87])

\[
\begin{aligned}
\dot{x}_{k+1} &= A \dot{x}_k + A(Y_k^{-1} - (\gamma^{-2} C'_1 C_1 - C' C))^{-1} \\
&\quad \times C'(y_{k+1} - C \bar{z}_k), \quad \dot{x}_0 = 0, \\
\bar{z}^*_k &= C_1 \dot{x}_k, \\
Y_{k+1} &= A(Y_k^{-1} - (\gamma^{-2} C'_1 C_1 - C' C))^{-1} A' + I, \\
Y_0 &= Y > 0,
\end{aligned}
\]

\[k = 0, 1, \ldots, M - 1.\]

**Remark A.4** In the case at hand, the value function has the trivial solution as given in (A.41). It is finite whereever \( \varphi \) is. Therefore the domain of \( W \) is the whole information state space.

### A.4.3 Infinite Time Case

In this section we write down the appropriate equations for infinite time horizon dissipative filtering. We consider the class of admissible filtering strategies \( Z \) as the set of causal maps of the observation \( \hat{z} : l_2([0, \infty), \mathbb{R}^p) \to Z \). The infinite time horizon dissipative filtering problem is to find \( \hat{z} \in Z \) such that for any initial condition \( z_0 \) the following inequality holds

\[
\Sigma_{i=0}^{M-1} r(e_i, w_i, u_i) \leq \beta^{\hat{z}}(x),
\]

for all \((w, v) \in \mathcal{W}_0\), for all \( M \geq 0 \), for some function \( \beta^{\hat{z}} \geq 0 \) independent of \( M \) with \( \beta^{\hat{z}}(0) = 0 \).
To solve this problem we consider the cost functional

$$J_p(z) = \sup_{M \geq 0} J_{p,M}(z),$$

in which $J_{p,M}(z)$ is defined in (A.30). We see that $\Sigma^\beta$ is dissipative if and only if $J_{p}(-\beta) \leq 0$, or, equivalently, by the representation (A.4), $\Sigma^\beta$ is dissipative if and only if

$$\sup_{M \geq 0, y \in l^2([0,\infty), \mathbb{R}^p)} \{(p_M, 0) : p_0 = -\beta\} \leq 0,$$

for some $\beta \geq 0$ with $\beta(0) = 0$. This leads us to the following (stationary) value function

$$W(p) = \inf_{\hat{z} \in \mathbb{Z}} \sup_{M \geq 0, y \in l^2([0,\infty), \mathbb{R}^p)} \{(p_M, 0) : p_0 = p\}. \quad (A.43)$$

By the dynamic programming principle, the value function $W$ satisfies the following discrete time dissipation inequality

$$W(p) \geq \sup_{y \in \mathbb{R}^p} \inf_{\hat{z} \in \mathbb{R}^n} \{W(F(p, \hat{z}, y))\}. \quad (A.44)$$

The optimal filtering strategy is obtained by finding

$$\hat{z}^* = \hat{z}^*(p, y) \in \arg\min_{\hat{z} \in \mathbb{R}^n} \{W(F(p, \hat{z}, y))\}.$$

Thus, the optimal estimate is independent of $k$ and, in general, depends on $y$ directly as well as indirectly through $p$. 
Appendix B

Model Problems

In Appendix A we develop a method, based on the notion of information state, for solving general dissipative output feedback control and filtering problems for discrete time nonlinear systems. In general, the solution involves a finite dimensional dynamic programming equation describing the dynamic of the information state and an infinite dimensional one which determines the optimal controls or filters. The similar information state idea is also employed in Chapter 5 and Chapter 7 of the thesis in the study of output feedback control and filtering problems respectively for continuous time nonlinear systems. In these chapters we assume that the corresponding dynamic programming equations possess smooth (i.e., Frechet differentiable) solutions.

In general these equations do not have smooth solutions, however. This is a common situation even in very simple optimal control or game problems [20], [35], [26]. Mathematical results concerning the second (infinite dimensional) dynamic programming equation is still at an infancy stage. In [57] the authors study a specific finite gain output feedback control problem for a class of nonlinear systems under certain smoothness and boundedness assumptions. The authors then provide a definition of viscosity solutions of the dynamic programming equation corresponding to the control problem and show that the value function arising from the problem indeed satisfies the dynamic programming equation in the viscosity sense.

The purpose of this appendix is to study a particular dissipative control and filtering synthesis for a class of nonlinear systems in which the mathematical results developed in [56] apply directly. This study serves as a bridge connecting the results in Chapter 5
and Chapter 7, which are obtained under the assumption that certain value function are sufficiently smooth, to the rigorous treatment in [57].

B.1 Control Synthesis

In this section we consider a class of nonlinear systems described by the equation

\[
\begin{align*}
\dot{x}(t) &= A(x(t)) + B_1(x(t))w(t) + B_2(x(t))u(t), \quad x(0) = x_0, \\
 z(t) &= C_1(x(t)) + D_{11}(x(t))w(t) + u(t), \\
y(t) &= C_2(x(t)) + w(t) + D_{22}(x(t))u(t).
\end{align*}
\]

(B.1)

In this description, \( x \in \mathbb{R}^n \) denotes the state vector, which is partially observed through a measured output quantity \( y \in \mathbb{R}^p \). The initial condition \( x_0 \) is assumed to be unknown. The vector \( z \in \mathbb{R}^q \) represents the quantity to be controlled. The disturbance \( w \in \mathbb{R}^d \) corrupts the state and the output quantities and the vector \( u \in U \subset \mathbb{R}^m \) denotes the control. We assume that the maps \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n, B_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}, B_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}, C_1 : \mathbb{R}^n \rightarrow \mathbb{R}^q, C_2 : \mathbb{R}^n \rightarrow \mathbb{R}^p, D_{11} : \mathbb{R}^n \rightarrow \mathbb{R}^{q \times d} \) and \( D_{22} : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times m} \) are smooth and bounded with bounded first derivatives, and that \( A(0) = 0, C_1(0) = 0, C_2(0) = 0 \).

The admissible control strategies are the set of \( U \)-valued causal maps of the measurement

\[
u : \mathcal{Y}(t) \rightarrow U(t),
\]

where \( \mathcal{Y}(t) = L_2([t,T], \mathbb{R}^p) \), \( U(t) = L_2([t,T], U) \). Thus, the control signal is obtained via \( u(r) = u[y(r)], r \in [t,T] \). The causality means that if \( y_1, y_2 \in \mathcal{Y}(t) \) and \( y_1(r) = y_2(r) \) a.e. \( r \in [t,T] \) then \( u[y_1](r) = u[y_2](r) \) a.e. \( r \in [t,T] \). We denote the class of such strategies by \( U(t) \). Admissible disturbances are all \( L_2 \) signals \( w \in \mathcal{W}(t) = L_2([t,T], \mathbb{R}^d) \).

We consider the general supply rate \( r(z, w) \)

\[
r : \mathbb{R}^q \times \mathbb{R}^d \rightarrow \mathbb{R},
\]

(B.2)

which is assumed to be bounded and has bounded derivatives with respect to \( w \) and \( z \). We assume that \( r(z, 0) \leq 0, \forall z \in \mathbb{R}^q \). Let \( \Sigma_u \) denote the map from \( w \) to \( z \) under the control policy \( u \) and initial condition \( x_0 \). Given a fixed finite time interval \([0,T]\), the finite time dissipative control problem is to find \( u \in U(0) \) such that for any initial
B.1 Control Synthesis

condition \( x_0 \in \mathbb{R}^n \) the map \( \Sigma^u_{x_0} \) is dissipative with respect to the supply rate \( r \) in (B.2), which means there exists a finite quantity \( \beta^u_T(x) \geq 0 \) such that

\[
\begin{align*}
- \int_0^T r(w(t), z(t)) dt & \leq \beta^u_T(x_0), \\
\forall w \in \mathcal{W}(0).
\end{align*}
\]  

We assume that \( \beta^u_T(0) = 0 \).

In the discussion to follow we use two function spaces which were introduced in [57] for technical purposes. We denote \( \mathcal{X} \) the Banach space of continuous functions with at most linear growth, i.e.

\[
\mathcal{X} = \{ p \in C(\mathbb{R}^n) : \| p \| < \infty \},
\]

where the norm \( \| \cdot \| \) is defined by \( \| p \| = \sup_{z \in \mathbb{R}^n} \{ p(z) \} \), and denote \( \mathcal{X}^1 \) the Banach space of continuously differentiable functions with bounded derivatives

\[
\mathcal{X}^1 = \{ p \in C^1(\mathbb{R}^n) : \| p \|_1 < \infty \},
\]

equipped with the norm \( \| \cdot \|_1 \) defined by \( \| p \|_1 = \sup_{z \in \mathbb{R}^n} \{ \frac{p(z)}{1+|z|} \} + \sup_{z \in \mathbb{R}^n} \{ |\nabla_z p(z)| \} \), in which \( \nabla_z p \) is the gradient of \( p \). Finally we define the function space

\[
\mathcal{D} = \{ p \in C(\mathbb{R}^n) : p(z) \leq -c_1|z| + c_2, \ \forall z \in \mathbb{R}^n, \text{ for some } c_1 > 0, c_2 \in \mathbb{R} \}.
\]

We assume that the function \( \beta^u_T(x) \) in (B.3) satisfies \( -\beta^u_T(x) \in \mathcal{D} \cap \mathcal{X} \).

We use game-theoretic methods to solve this problem. For \( u \in U(0) \) define the cost functional related to the dissipative control problem as follows

\[
J_{a,T}(u) \overset{\Delta}{=} \sup_{w \in \mathcal{W}(0), x_0 \in \mathbb{R}^n} \{ \alpha(x_0) - \int_0^T r(w(t), z(t)) dt \},
\]  

where \( \alpha \in \mathcal{D} \cap \mathcal{X} \). Under our assumptions, \( J_{a,T}(u) \) is finite. Obviously the map \( \Sigma^u \) is dissipative if and only if \( J_{a,T}(u) \leq 0 \), for some \( \alpha(x) = -\beta(x) \leq 0 \), with \( \alpha(0) = 0 \). The idea is that a solution to the dissipative control problem will be obtained by minimizing \( J_{a,T}(u) \) over \( U(0) \). This is a zero-sum dynamic game problem with partial observation.

B.1.1 Information State Solution

In this section we solve the partially observed game problem associated with the dissipative control problem by following the information state method developed in [56], [57].
For fixed output \( y \in \mathcal{Y}(0) \) and control signal \( u \in \mathcal{U}(0) \), we define the information state by

\[
p_t(x) = \alpha(\xi_0) - \int_0^t r(y(s) - C_2(\xi(s)) - D_{22}(\xi(s))u(s), z(s))ds,
\]

(B.5)
in which \( \xi(\cdot) \) is the solution of

\[
\dot{\xi}(s) = A(\xi(s)) + B_1(\xi(s))(y(s) - C_2(\xi(s)) - D_{22}(\xi(s))u(s)) + B_2(\xi(s))u(s), \quad 0 \leq s \leq t,
\]

(B.6)
with \( \xi(t) = z \).

The information state \( p_t \) has dynamics given by

\[
\begin{cases}
\dot{p}_t = F(p_t, u(t), y(t)), \\
p_0 = \alpha,
\end{cases}
\]

(B.7)
in which

\[
F(p, u, y)
= -\nabla_x p(A(x) + B_1(x)(y - D_{22}(x)u - C_2(x)) + B_2(x)u)
- r((C_1(x)x + D_{11}(x)(y - C_2(x) - D_{22}(x)u) + u), (y - C_2(x) - D_{22}(x)u)).
\]

(B.8)
The sense in which this equation is to be understood depends on the smoothness of \( \alpha \) and on the regularity of \( u(\cdot) \) and \( y(\cdot) \). Indeed, if \( \alpha \in \mathcal{D} \cap \mathcal{X}^1 \), then \( p_t(x) \) is the unique solution (in the classical sense) of (B.7), and if \( \alpha \) is not differentiable, i.e. \( \alpha \in \mathcal{D} \cap \mathcal{X} \), then \( p_t \) solves the dynamics (B.7) in the viscosity sense (see [57] Lemma 3.2, and 3.4).

The cost function in (B.4) can be expressed in terms of \( p_t(x) \), as in [57].

**Theorem B.1** For any \( u \in \mathcal{U}(0) \) we have

\[
J_{\alpha,T}(u) = \sup_{y \in \mathcal{Y}(0)} \{ (p_T, 0) : p_0 = \alpha \}.
\]

(B.9)

Thus, we now have a completely observed dynamic game problem in which the information state \( p_t \) provides the appropriate state.

We employ dynamic programming methods to solve the complete observation problem. To this end, define the value function \( W(p, t; T) \) for \( (p, t) \in \mathcal{D} \cap \mathcal{X} \times [0, T] \) by

\[
W(p, t; T) \triangleq \inf_{u \in \mathcal{U}(t)} \sup_{y \in \mathcal{Y}(t)} \{ (p_T, 0) : p_t = p \}.
\]

(B.10)
B.1 Control Synthesis

The boundedness assumptions of the functions in (B.1) guarantee the finiteness of this function (see Lemma 4.1 of [57]). The dynamic programming principle then leads us to the following infinite dimensional dynamic programming equation ([56], [57], [42]).

\[
\frac{\partial W}{\partial t} + \sup_{u \in \mathbb{R}^p} \inf_{u \in U} \{ (\nabla_p W, F(p, u, y)) \} = 0 \quad \text{in} \quad D \cap X^1 \times [0, T],
\]

\[
W(p, T; T) = (p, 0) \quad \text{in} \quad D \cap \mathcal{X}.
\]

(B.11)

In this equation, \( \nabla_p W \in X^* \) (\( X^* \) is the dual space of \( X \)) denotes the gradient of the value function \( W \) with respect to \( p \). This quantity lives in the dual space \( X^* \) of the Banach space \( \chi; \) for \( \lambda \in \chi^* \), \( (\lambda, p) \) denotes the value of \( \lambda \) evaluated at \( p \in \chi \). Since, in the expression (B.8) for \( F(p, u, y) \), the variables \( u \) and \( y \) are coupled, in general the inf and sup operations on the left hand side of (B.11) do not commute, i.e. the Isaacs condition does not hold. The DPE (B.11) is the Hamilton-Jacobi-Isaacs (HJI) equation for the partially observed game associated with the dissipative control problem.

We make the following definition concerning solutions of this equation.

**Definition B.1** We say \( W : D \cap \mathcal{X} \times [0, T] \to \mathbb{R} \) is a smooth solution of the DPE (B.11) if: (i) \( W \) is \( \mathcal{X} \)-Frechet differentiable, with continuous derivatives \( (\nabla_p W, \frac{\partial W}{\partial t}) \) on \( D \cap X^1 \times [0, T] \), and (ii) \( W \) satisfies the equation (B.11) in \( D \cap X^1 \times (0, T) \).

In general the DPE (B.11) may not have smooth solutions and one must appeal to a weaker concept of solution. We take the following definition of viscosity solution to the DPE (B.11) introduced in [57].

**Definition B.2** A function \( W \in C(D \cap \mathcal{X} \times [0, T]) \) is called: a viscosity subsolution (supersolution) of the DPE (B.11) if for all \( \phi \in \mathcal{C} \), whenever there exists \( (p', t') \in D \cap X^1 \times (0, T) \), such that \( W(p', t'; T) - \phi(p', t') \) achieves maximum (minimum) over \( D \cap X^1 \times [0, T] \) with \( \{(p', t'; T) - \phi(p', t')\} = 0 \), then

\[
\frac{\partial \phi}{\partial t}(p', t') + \inf_{\mathbb{Z} \in \mathbb{Z}} \sup_{v \in \mathbb{R}^p} \{ (\nabla_v \phi(p', t'), F(p', \hat{z}, y)) \} \geq (\leq)0;
\]

a viscosity solution if \( W \) is both a subsolution and a supersolution.

We now present the solution to the dissipative control problem.

**Theorem B.2** Suppose there exists a solution to the finite time dissipative control problem. Then there exist solutions to the PDEs (B.8) and (B.11), in the viscosity sense,
such that \( p_0 = -\beta \) and \( W(-\beta, 0; T) = 0 \), for some \( \beta(x) \geq 0 \), with \( \beta(0) = 0 \). Conversely, suppose there exist smooth solutions of the PDEs (B.8) and (B.11) such that \( p_0 = -\beta \) and \( W(-\beta, 0; T) = 0 \), for some \( \beta(z) \geq 0 \), with \( \beta(0) = 0 \). Assume that the control \( u^* \) achieves minimum in the right hand side of (B.11), i.e.,

\[
\begin{align*}
\mathbf{u}^*(p, t; T) &\in \arg\min_{\mathbf{u} \in \mathcal{U}} \{ \langle \nabla_p W(p, t; T), F(p, u, y) \rangle \}.
\end{align*}
\]

Then \( u^* \) solves the finite time dissipative control problem.

**Proof.** If there exists a control \( u^0 \in \mathcal{U}(0) \) solving the dissipative control problem, then with \( p_0 = -\beta u^0 \)

\[
J_{p_0, T}(u^0) \leq 0,
\]

and moreover

\[
0 = (-\beta u^0, 0) \leq W(-\beta u^0, 0; T) \leq 0.
\]

Therefore \( W(-\beta u^0, 0; T) = 0 \). From Lemma 3.4 and Theorem 4.9 of [57], solutions to equations (B.8) and (B.11) exist at least in the preliminary viscosity sense.

Conversely, suppose that equations (B.8) and (B.11) have smooth solutions such that \( p_0 = -\beta \) and \( W(-\beta, 0; T) = 0 \), with \( \beta(z) \geq 0 \), and \( \beta(0) = 0 \). Then Theorem 5.1 of [57] implies that the control policy defined in (B.12) is optimal. Therefore, with \( p_0 = -\beta \),

\[
J_{p_0, T}(u^*) \leq W(-\beta, 0; T) = 0,
\]

which implies that the system \( \Sigma u^* \) is dissipative on \([0, T]\). \( \square \)

### B.2 Filter Synthesis

In this section we consider a particular filtering problem for a class of nonlinear system under the same assumptions as in (B.1). The systems are described by

\[
\begin{align*}
\dot{z}(t) &= f(z(t)) + g(x(t), w(t)), \quad x(0) = x_0, \\
z(t) &= h_1(x(t)), \\
y(t) &= h_2(x(t)) + w(t).
\end{align*}
\]

We assume that the maps \( f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, h : \mathbb{R}^n \to \mathbb{R}^p \) and \( h_1 : \mathbb{R}^n \to Z \subseteq \mathbb{R}^g \) are smooth and bounded with bounded first derivatives, satisfying
f(0) = 0, g(0) = 0, h(0) = 0, h_1(0) = 0. We assume that Z is compact. Let e denote the estimation error, i.e., e = \hat{z} - z. We consider the general supply rate \( r(e, w) \)

\[
r : \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R},
\]

which is assumed to be bounded and has bounded derivatives with respect to \( w \) and \( e \). We assume that \( r(e, 0) \leq 0, \forall e \in \mathbb{R}^q \). Given a finite time interval \([0, T]\), the finite time dissipative filtering problem is to find \( \hat{z} \in Z \) such that the following inequality holds

\[
- \int_0^T r(e(t), w(t)) dt \leq \beta_\omega^\delta(x_0),
\]

for all \( w \in L_2([0, T], \mathbb{R}^p) \), for some finite function \( \beta_\omega^\delta \geq 0 \) satisfying \( \beta_\omega^\delta(0) = 0 \). For technical convenience, we assume that the function \( \beta_\omega^\delta(x) \) in (B.15) satisfies \(-\beta_\omega^\delta(x) \in \mathcal{D} \cap \mathcal{X}\).

### B.2.1 Information State Solution

For the particular problem at hand the corresponding information state evolves according to the dynamics

\[
\dot{p}_t = F(p_t, \hat{z}(t), y(t)),
\]

with initial \( p_0 = \alpha \), in which

\[
F(p, \hat{z}, y) = -\nabla_x p(f(x) + g(x, y - h(x))) - r(\hat{z} - h_1(x), y - h_2(x)).
\]

The regularity of (B.16) depends on that of \( \hat{z}(\cdot) \) and \( y(\cdot) \) (see Lemma 3.2 and Lemma 3.4 of [57]). The corresponding value function \( W(p, t; T) \) satisfies the following infinite dimensional dynamic programming equation ([57], [42]).

\[
\frac{\partial W}{\partial t} + \inf_{\hat{z} \in Z} \sup_{y \in \mathbb{R}^p} \{ (\nabla_p W(p, \hat{z}, y), F(p, \hat{z}, y)) \} = 0 \text{ in } \mathcal{D} \cap \mathcal{X}^1 \times [0, T],
\]

with \( W(p, T; T) = (p, 0) \) in \( \mathcal{D} \cap \mathcal{X} \). We now present the solution to the dissipative filtering problem at hand. The proof is similar to that of Theorem B.2.

**Theorem B.3** Suppose there exists a solution to the finite time dissipative filtering problem. Then there exist solutions to the DPEs (B.17) and (B.18), in the viscosity sense, such that \( p_0 = -\beta \) and \( W(-\beta, 0; T) = 0 \), for some \( \beta(x) \geq 0 \), with \( \beta(0) = 0 \). Conversely, suppose there exist smooth solutions to the DPEs (B.17) and (B.18) such that \( p_0 = -\beta \) and \( W(-\beta, 0; T) = 0 \), for some \( \beta(x) \geq 0 \). Then the filter \( \hat{z}^* \) defined by

\[
\hat{z}^*(p, t; T) \in \arg\min_{\hat{z} \in Z} \left\{ \sup_{y \in \mathbb{R}^p} \{ (\nabla_p W(p, \hat{z}, y), F(p, \hat{z}, y)) \} \right\}
\]

(B.19)
solves the finite time dissipative filtering problem.
Appendix C

Viscosity Solutions Proof

C.1 Definition of viscosity solutions

We recall the definition of weak (viscosity) solutions for nonlinear first order partial differential inequalities ([20],[35], [90]). In what follows, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is locally bounded above and lower semicontinuous.

A locally bounded function $V$ is said to satisfy the PDI

$$H(x, \nabla_x V) \leq 0 \text{ in } \mathbb{R}^n$$

in the weak (viscosity) sense if for every $\phi \in C^1(\mathbb{R}^n)$ and any local minimum $x_0 \in \mathbb{R}^n$ of $V_\ast - \phi$ one has

$$H(x_0, \nabla_x \phi(x_0)) \leq 0,$$

where $V_\ast(x) \triangleq \liminf_{y \to x} V(y)$ is the lower semi continuous envelope of $V$.

A locally bounded function $V$ is called a viscosity subsolution (respectively supersolution) of the PDE

$$H(x_0, \nabla_x \phi(x_0)) = 0.$$

if for every $\phi \in C^1(\mathbb{R}^n)$ and any local maximum (respectively minimum) $x_0 \in \mathbb{R}^n$ of $V^\ast - \phi$ (respectively $V_\ast - \phi$) one has

$$H(x_0, \nabla_x \phi(x_0)) \geq 0 \text{ (respectively } \leq 0).$$

Here, $V^\ast(x) \triangleq \limsup_{y \to x} V(y)$ is the upper semicontinuous envelope of $V$. 

221
C.2 Viscosity Solutions Proof

We shall now show that the value function $V$ defined by

$$V(x) = \inf_{K \in U_c} \sup_{T\geq 0, w \in L^2([0,T],\mathbb{R}^d)} \left\{ - \int_0^T \tau_q(z(t), w(t)) dt \right\}$$

(C.1)

where $\tau_q(z,w) = \frac{1}{2}(w'Qw + 2w'Sz + z'Rz)$, and $h_1(x,w,u) = C_1(x) + D_{11}(x)w + D_{12}(x)u$ is a viscosity solution of the PDE

$$H(x, \nabla_x V) = 0,$$

(C.2)

where

$$H(x, \nabla_x V) = \sup_{w \in \mathbb{R}^d} \inf_{u \in \mathbb{R}^m} \{ \nabla_x V(x)(A(x) + B_1(x)w + B_2(x)u)$$

$$- \frac{1}{2}(w'Qw + 2w'Sh_1(x,w,u) + h_1(x,w,u)'Rh_1(x,w,u)) \}.$$ 

(C.3)

We follow the techniques employed in [90]. The result also holds when the set $U_c$ is a singleton such as in the open loop problem in Theorem 3.2 in Chapter 3.

Consider the following monotonically increasing change of parameter

$$\tau_{u,w}(t) = \int_0^t (1 + |u(s)|^2 + |w(s)|^2) ds.$$ 

Suppose that for all $u_1, u_2 \in L^2([0,\infty),\mathbb{R}^m)$, $w_1, w_2 \in L^2([0,\infty),\mathbb{R}^d)$, we define $u \in L^2([0,\infty),\mathbb{R}^m)$ as

$$u(t) = \begin{cases} u_1(t), & [0, t_{u_1,w_1}(\tau_1)), \\ u_2(t), & [t_{u_1,w_1}(\tau_1), \infty), \end{cases}$$

and similarly for $w \in L^2([0,\infty),\mathbb{R}^d)$. Then we have [90]

$$u(t_{u,w}(\tau)) = \begin{cases} u_1(t_{u_1,w_1}(\tau)), & [0, \tau_1), \\ u_2(t_{u_2,w_2}(\tau - \tau_1)), & [\tau_1, \infty). \end{cases}$$

The same property holds similarly for $w$ ([90]).

Consider now the reparametrized state $y(\tau) = x(t(\tau))$. The dynamics of $y$ is given by

$$dy/d\tau = (A(y(\tau)) + B_1(y(\tau))w(\tau) + B_2(y(\tau))u(\tau)) \cdot dt/d\tau$$

$$= (A(y(\tau)) + B_1(y(\tau))w(\tau) + B_2(y(\tau))u(\tau)) \cdot (1 + |u(\tau)|^2 + |w(\tau)|^2)^{-1}$$

(C.4)

$$= f(y(\tau), u(\tau), w(\tau)).$$
with initial condition $y(0) = x(t(0)) = x(0) = x$, where

$$f(y, u, w) = (A(y) + B_1(y)w + B_2(y)u) \cdot (1 + |u|^2 + |w|^2)^{-1}.$$ (C.5)

Moreover, in terms of $\tau$ we have

$$V(x) = \inf_{K \in \mathcal{U}_c} \sup_{\tau \geq 0, w \in L^2([0, \tau], \mathbb{R}^d)} \{ -\int_0^\tau \tilde{r}(h_1(y(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma : y(0) = x \},$$ (C.6)

where

$$\tilde{r}(h_1(z, w, u), w) = \frac{1}{2}(w'Qw + 2w'Sh_1(z, w, u) + h_1(z, w, u)'Rh_1(z, w, u)) \cdot (1 + |u|^2 + |w|^2)^{-1}.$$ (C.7)

The corresponding PDE is given by

$$\mathcal{H}(x, \nabla_x V) = 0$$ (C.8)

where

$$\mathcal{H}(x, \nabla_x V) = \sup_{w \in \mathbb{R}^d} \inf_{w \in \mathbb{R}^m} \{ \nabla_x V(x)f(x, u, w) - \tilde{r}(h_1(x, w, u), w) \}.$$ Clearly, the functions $f$ and $\tilde{r}$ enjoy the following regularity

$$|f(x, u, w) - f(x_0, u, w)| \leq L_\rho |x - x_0|,$$

$$|\tilde{r}(h_1(x, w, u), w) - \tilde{r}(h_1(x_0, w, u), w)| \leq L_\rho |x - x_0|,$$ (C.9)

for all $x, x_0 \in B(0, \rho), \rho > 0$, for all $u \in \mathbb{R}^m$, for all $w \in \mathbb{R}^d$, for some constant $L_\rho$ depending only on $\rho$. Furthermore, the trajectory $y(\cdot)$ enjoys the following regularity

(see Remark 3.2 in [90])

$$|y(s) - x| \leq \gamma(\tau),$$ (C.10)

for all $x \in B(0, \rho), \in [0, \tau]$, where $\gamma > 0$ is nondecreasing function which is continuous at $0$, and $\gamma(0) = 0$. In particular, $|y(\tau) - x| \leq o(1)$ as $\tau \to 0$, uniformly in $u, w$ for all $x \in B(0, \rho)$.

Note that since $-r_q(z, 0) \geq 0$ for all $z$, we have

$$-\tilde{r}(z, 0) \geq 0, \ \forall z \in \mathbb{R}^q.$$ This implies that, for each $K \in \mathcal{U}_c$, the finite time horizon cost function $J^K(x, \tau)$ defined by

$$J^K(x, \tau) = \sup_{w \in L^2([0, \tau], \mathbb{R}^d)} \{-\int_0^\tau \tilde{r}(h_1(y(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma \}$$
is monotonically increasing in $\tau$. In particular, we have

$$J^K(x) = \sup_{\tau \geq 0, w \in L_2([0,\tau], \mathbb{R}^d)} \{ -\int_{\tau}^{\tau} \tilde{r}(\sigma) d\sigma \}$$

$$= \sup_{\tau \geq 0} J^K(x, \tau)$$

$$= \lim_{\tau \to \infty} J^K(x, \tau). \tag{C.11}$$

We will need the following dynamic programming principle.

**Lemma C.1** The value function $V$ in (C.6) satisfies the dynamic programming principle

$$V(x) = \inf_{K \in \mathcal{U}} \sup_{w \in L_2([0,\tau], \mathbb{R}^d)} \{ -\int_{\tau}^{\tau} \tilde{r}(h_1(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma + V(y(\tau)) \}$$

for all $\tau \geq 0$. \tag{C.12}

**Proof.** From Lemma 3.4 in [90], we have

$$V(x) \geq \inf_{K \in \mathcal{U}} \sup_{w \in L_2([0,\tau], \mathbb{R}^d)} \{ -\int_{\tau}^{\tau} \tilde{r}(h_1(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma + V(y(\tau)) \}$$

for all $\tau \geq 0$. It remains to show that the reverse inequality holds. For fixed $\tau \geq 0$, let $W(x)$ denote the right hand side of (C.12), i.e.

$$W(x) = \inf_{K \in \mathcal{U}} \sup_{w \in L_2([0,\tau], \mathbb{R}^d)} \{ -\int_{\tau}^{\tau} \tilde{r}(h_1(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma + V(y(\tau)) \}.$$

By the definition of $W$, there exists $K^* \in \mathcal{U}$ such that

$$W(x) \geq \sup_{w \in L_2([0,\tau], \mathbb{R}^d)} \{ -\int_{\tau}^{\tau} \tilde{r}(h_1(\sigma), w(\sigma), K^*[w](\sigma)), w(\sigma)) d\sigma + V(y(\tau)) \} - \frac{\varepsilon}{2},$$

for all $w$. Moreover, the definition of $V$ implies there exists $K^{**} \in \mathcal{U}$ such that

$$V(y(\tau)) \geq \sup_{\tau \geq \tau, w \in L_2([\tau,\tau], \mathbb{R}^d)} \{ -\int_{\tau}^{\tau} \tilde{r}(h_1(\sigma), w(\sigma), K^{**}[w](\sigma)), w(\sigma)) d\sigma \} - \frac{\varepsilon}{2}.$$

Combining the last two inequalities, and defining the control law

$$\bar{K}[w](\tau) = \begin{cases} K^*[w](\tau), & [0, \bar{\tau}] \\ K^{**}[w](\tau), & [\bar{\tau}, \infty) \end{cases}$$

we get

$$W(x) \geq -\int_{\tau}^{\tau} \tilde{r}(h_1(\sigma), w(\sigma), \bar{K}[w](\sigma)), w(\sigma)) d\sigma - \varepsilon,$$
C.2 Viscosity Solutions Proof

for all \( w \in L_2([0, \tau], \mathbb{R}^d) \), for all \( \tau \geq \bar{\tau} \). Thus, using \( \bar{K} \), and the monotonicity in (C.11) we have

\[
W(x) \geq \sup_{\tau \geq \bar{\tau}, w \in L_2([0, \tau], \mathbb{R}^d)} \left\{ -\frac{1}{2} \int_0^\tau \bar{r}(h_1(y(\sigma), w(\sigma), \bar{K}[w](\sigma)), w(\sigma)) d\sigma \right\} - \epsilon
\]

\[
= \sup_{\tau \geq 0, w \in L_2([0, \tau], \mathbb{R}^d)} \left\{ -\frac{1}{2} \int_0^\tau \bar{r}(h_1(y(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma \right\} - \epsilon.
\]

Therefore,

\[
W(x) \geq \inf_{K \in \mathcal{U}_c} \sup_{\tau \geq 0, w \in L_2([0, \tau], \mathbb{R}^d)} \left\{ -\frac{1}{2} \int_0^\tau \bar{r}(h_1(y(\sigma), w(\sigma), K[w](\sigma)), w(\sigma)) d\sigma \right\} - \epsilon,
\]

\[
= V(x) - \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we get the reverse inequality. This completes the proof. \( \square \)

Now we show that \( V \) is a viscosity solution to the PDE (C.2). First, it follows from second part of the proof of Proposition 3.5 of [90] that \( V \) is a viscosity supersolution of the PDE

\[
\mathcal{H}(x, p) = 0,
\]

where \( \mathcal{H} \) is given in (C.8).

Next, we show that \( V \) is also a viscosity subsolution of the PDE (C.8). Let \( x_0 \in \text{argmax}\{V^* - \phi\} \), where \( \phi \in C^1 \), and \( V^*(x_0) = \phi(x_0) \). Assume that \( V \) is not a viscosity subsolution of the PDE (C.8), i.e.

\[
\mathcal{H}(x_0, \nabla_x \phi(x_0)) < 0. \quad (C.13)
\]

Following the proof of Proposition 3.5 of [90], this inequality implies that there exists a control law \( \bar{K}[w] \in \mathcal{U}_c \) such that the following inequality (involving \( \phi \)) holds

\[
\phi(x) - \phi(y(\tau)) + \int_0^\tau \bar{r}(h_1(y(s), w(s), \bar{K}[w](s)), w(s)) ds \geq \epsilon \tau, \quad (C.14)
\]

for all \( w \in L_2([0, \tau], \mathbb{R}^d) \), for all \( x \in B(x_0, \frac{\epsilon}{2}) \), for sufficiently small quantities \( \tau > 0, \epsilon > 0 \).

Now, consider the sequence \( \{x_n\} \) such that \( \lim_{n \to \infty} x_n = x_0 \), \( \lim_{n \to \infty} V(x_n) = V^*(x_0) \).

Then, there exists large enough \( N \) such that \( x_n \in B(x_0, \epsilon/2) \) for all \( n \geq N \). Setting \( x = x(0) = x_n, n \geq N \), the inequality (C.14) becomes

\[
\phi(x_n) - \phi(y(\tau)) + \int_0^\tau \bar{r}(h_1(y(s), w(s), \bar{K}[w](s)), w(s)) ds \geq \epsilon \tau. \quad (C.15)
\]

Adding \( 0 = V^*(x_0) - \phi(x_0) \) to both sides of this inequality and using \( V^*(y(\tau)) \leq \phi(y(\tau)) \) (this is because \( V^*(y(\tau)) - \phi(y(\tau)) \leq V^*(x_0) - \phi(x_0) = 0 \), for sufficiently small \( \tau > 0 \),
yields
\[ V^*(x_0) - \phi(x_0) + \phi(x_n) - V(y(\tau)) + \int_0^\tau \tilde{r}(h_1(y(s), w(s), \bar{K}[w](s)), w(s))ds \geq \epsilon\tau \] (C.16)
and then
\[ V^*(x_0) \geq \epsilon\tau - \phi(x_n) + \phi(x_0) - \int_0^\tau \tilde{r}(h_1(y(s), w(s), \bar{K}[w](s)), w(s))ds + V(y(\tau)), \] (C.17)
for all \( w \in L_2([0, \tau], \mathbb{R}^d) \). By employing Lemma C.1, we get
\[
V^*(x_0) \geq \epsilon\tau - \phi(x_n) + \phi(x_0)
+ \sup_{w \in L_2([0, \tau], \mathbb{R}^d)} \{- \int_0^\tau \tilde{r}(h_1(y(s), w(s), \bar{K}[w](s)), w(s))ds + V(y(\tau)) \}
\geq \epsilon\tau - \phi(x_n) + \phi(x_0)
+ \inf_{K \in \mathcal{U}e} \sup_{w \in L_2([0, \tau], \mathbb{R}^d)} \{- \int_0^\tau \tilde{r}(h_1(y(s), w(s), \bar{K}[w](s)), w(s))ds + V(y(\tau)) \}
= \epsilon\tau - \phi(x_n) + \phi(x_0) + V(x_n).
\] (C.18)
Sending \( n \to \infty \) yields
\[ V^*(x_0) \geq \epsilon\tau + V^*(x_0), \]
which is a contradiction. Therefore, we must have
\[ \mathcal{H}(x_0, \nabla_x \phi(x_0)) \geq 0, \] (C.19)
i.e., \( V \) is a viscosity subsolution of the PDE (C.8). Since \( V \) is also a viscosity supersolution, we conclude that \( V \) is a viscosity solution to the PDE (C.8). Finally, it follows from the proof of Theorem 4.9 of [90] that
\[ \mathcal{H}(x_0, \nabla_x \phi(x_0)) \geq 0 \] (respectively \( \leq 0 \))
implies
\[ H(x_0, \nabla_x \phi(x_0)) \geq 0 \] (respectively \( \leq 0 \),
in which \( H \) is given in (C.3). This completes the proof that \( V \) is a viscosity solution of the PDE (C.2).