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Planar Multimode Waveguides and Devices

A thesis submitted for the degree of Doctor of Philosophy of the Australian National University

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Canberra,
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Declaration

This thesis is an account of research undertaken in the Optical Sciences Centre and the Plasma Research Laboratory within the Research School of Physical Sciences and Engineering at the Australian National University between March 1994 and October 1997 while I was enrolled for the Doctor of Philosophy degree.

The research has been conducted under the supervision of Prof. J. D. Love, Dr. F. Ladouceur, Dr. A. Durandet, Dr A. Ankiewicz and Dr. R. W. Boswell. However, unless specifically stated otherwise, the material presented within this thesis is my own.

None of the work presented here has been submitted for any degree at this or any other institution of learning.

Daniel R. Beltrami
July, 1998
Publications


Abstract

A range of planar multimode waveguides and devices are considered in this thesis, including both step- and graded-index profiles. The first half of this thesis is concerned with the design analysis, fabrication and characterisation of a planar graded-index (GRIN) lens. The remaining half deals with modelling the power transmission of light through several step-profile multimode devices and bent waveguides, using ray theory and finite-difference beam propagation method (FD-BPM) simulations. The devices examined include acute-angled X-junctions, symmetric Y-junctions and modified multimode interference (MMI) couplers.
Acknowledgements

Wow, its over! I must say that I owe a debt of gratitude to many people for making this long held ambition of mine a reality. The support I received ranged from the highly technical to the deeply personal, but it all helped me hang in there in the end.

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CHAPTER 1

Introduction

Only a little more
I have to write,
Then I'll give o'er,
And bid the world good-night.

Robert Herrick 1591-1674,
*His Poetry his Pillar.*

The work contained in this thesis is concerned with multimoded planar waveguides and devices. This arose out of the intention to further develop and extend the expertise in single-mode planar waveguides and devices into the multimode arena, here at the Australian National University (ANU). This move to diversify into multimoded planar technologies was spurred by J.D. Love and R.W. Boswell, who realised that the plasma deposition reactor that was already used for the fabrication of single-moded devices within the Plasma Research Laboratory, could also be used for the in-house development of novel multimoded devices.

My involvement in this process began with the planar graded-index (GRIN) lens, which has the appeal of simplicity as only one fabrication step is required; that being the deposition of silica to form the waveguide. This proved to be a very important first step before trying to embark upon the more ambitious task of fabricating multimoded planar devices. Approximately half of the material presented in this thesis is a detailed account of the design analysis, fabrication and characterisation of the planar GRIN lens. The remainder of this thesis is concerned with modelling a range of simple multimoded devices that are likely to become essential elements in the production of multimoded integrated optical circuits. However, fabricating these devices is a more complicated process which requires the design of a photolithographic mask. With such a mask, it is foreseen that the expertise and machinery is already in place at the ANU to facilitate the fabrication of these devices.
Planar optical waveguide technology has historically been dominated by the interests of telecommunications carriers which presently represents the largest market for integrated optical devices. As a consequence of the powerful trend towards single-mode technologies, multimode devices have until recently been overlooked. Increasing interest in multimode optical technologies has arisen from possible applications to fields that were not foreseen just a few years ago. For instance, the development of planar optical sensors has replaced issues surrounding fibre compatibility by far more elementary requirements related to the routing of light through optical circuits. Another example is the planar graded-index lens, which is a multimode planar waveguide with imaging properties that lends itself to applications as diverse as confocal microscopy [1], to newly developed optical storage devices [2], and optical networks [3]. So despite the comparatively "limited" functionality of multimoded planar waveguides and devices, the simple optical functions they are capable of performing is generating an increasing amount of interest.

Currently, rapid improvements to fabrication technologies are also making multimoded technology look more attractive. Polymeric waveguides are now capable of being fabricated with very high accuracy at potentially low-cost, where it takes no more time to make a multi-, as opposed to, a single-mode waveguide. Moulding and casting techniques have already been developed for fabricating passive polymer multimode waveguiding structures [4, 5] and there is currently a lot of interest in multimode Y-junctions for the interconnection of optical networks.

The content of this thesis is obviously going to entail a mixture of experiment and theory, and the work is divided in two parts. The first half of this thesis provides the details relating to the design, fabrication and characterisation of the planar GRIN lens, with a heavy emphasis on experiment. The remaining half, models the behaviour of a range of simple passive step-profile multimoded devices on a purely theoretical basis.

The field of graded-index optics is introduced to the reader in chapter 2, where the concept and uses of axial, radial and spherical gradients are established. Properties of the planar GRIN lens with a parabolic refractive index profile are investigated using a variety of techniques. These techniques include ray and modal analysis as well as the beam propagation method (BPM). The imaging properties of the planar GRIN lens are also compared with those of step-index multimode waveguides, arising from self-imaging effects within the waveguide. This comparison is made in order to justify the effort of fabricating a graded index profile. The fabrication of
such a planar device is not a task to be undertaken lightly.

The concepts of rays and modes are also introduced to the reader in this chapter. All of the waveguides considered throughout this thesis are assumed to satisfy the weak guidance approximation [6, 7]. This approximation applies equally well to multimode waveguides as it does to single mode waveguides, and is an essential ingredient for any modal or BPM analysis.

A numerical tool that is relied upon quite heavily throughout the thesis, is the Finite-Difference BPM (FD-BPM). This scheme is explained at some length in chapter 2. However, modifications are made to this technique in the analysis of bent multimode waveguides which are covered in chapter 9. None of the computer code has been included either in the text nor in appendices in order to avoid both boring the reader and obscuring the results.

The fabrication process is covered in chapter 3 where the lenses are formed using the Helicon Plasma Enhanced Chemical Vapour Deposition (PECVD) reactor to deposit the graded index silica. The influence of the dopant gas CF$_4$ during deposition on both the deposition rate and the refractive index is investigated. Other issues covered in this chapter relate to the stoichiometry and scattering loss of the deposited films and the influence of altering the ion bombardment energy to improved film quality.

Chapter 4 deals with the characterisation of the GRIN lenses. A prism coupling technique is presented which enables the recovery of the refractive index profile of the lenses. A step-profile planar waveguide is also included as a control feature to see how well this technique applies to different cases. A thin layer of specially dye-doped polymer spin coated onto the top of the GRIN lens, was used to measure the refocussing period of the lens at two different wavelengths. This was then compared with FD-BPM and ray tracing results, obtained from the recovered refractive index profile. This is the first time such a technique has been applied to the characterisation of optical waveguides. Closing this chapter is an examination of the output characteristics of the fabricated lenses.

Simple step-profile planar waveguide devices are modelled in subsequent chapters using ray tracing to derive expressions for the power transmission characteristics for each device. These results were compared with those obtained from the FD-BPM. The excitation of rays and modes required to adequately describe a diffuse light
source, such as a light emitting diode (LED) is established in chapter 5, providing the foundation for the following analysis.

The palate of passive waveguide devices considered are acute- and right-angled X-junctions, Y-junctions and a modified Multimode Interference (MMI) coupler. Expressions for the excess loss are derived for both the acute- and right-angled junctions in chapter 6, and expressions for the power transmission through a symmetric Y-junction, and a modified Multimode Interference (MMI) coupler are provided in chapters 7 and 8, respectively.

Finally in chapter 9, bends in multimode waveguide are investigated using ray and FD-BPM techniques to determine how light behaves in such structures and to quantify the bend loss. Ray invariants are introduced to classify the behaviour of rays in the bent waveguide in order to calculate the losses.

Chapter 10 concludes this thesis with a brief overview of the results, and includes suggestions for future work.
CHAPTER 2

The planar graded-index (GRIN) lens

But optics sharp it needs, I ween,
To see what is not to be seen.

John Trumbull 1750-1831,
McFingal.

2.1 What are graded-index devices?

Most of us are probably familiar with the idea that light travels through most homogeneous media in a rectilinear fashion. This is in contrast to materials that do not possess a homogeneous refractive index distribution, where light propagates along a curved path. The field of optics describing the passage of light through these materials is known as gradient index (GRIN) optics, which possesses a long and rich history. The most familiar example of a gradient index phenomenon is a mirage [8, Sectn. 9-1]. This typically occurs on planar surfaces such as roads or tarmacs, where the heat absorbed by these surfaces causes the air temperature to rise close to the surface, creating a temperature gradient in the air. According to the gas law, the density decreases close to the surface in question, and so too does the refractive index. The result is that the mirrored appearance of a mirage is not a result of reflection off a hot road surface, but is caused by a gradual refraction of the light, giving it a curved trajectory. Another type of mirage, which is sometimes seen at sea, is created by a stratum of hot air well above the earth’s surface and behaves in the same manner as hot air near the ground. The result is that an object below the horizon is brought into view by the abnormal change in the refractive index of the air. It may interest the reader to know that ships moored off the French coast across the English channel have been seen from Dover some 35 kilometres away [9].

Optical materials with a distributed refractive index profile are referred to as gradient-index materials. Most graded-index profiles can be accurately represented by a polynomial expansion [8, 10]. One such example is a polynomial expansion in both the radial coordinate $r$ and the optical axis coordinate $z$. In this instance, the

\footnote{This ignores materials with any kind of intensity-dependent photo-refractive effect.}
refractive index distribution is given by

\[ n(r, z) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} n_{ij} r^{2i} z^j \]  

(2.1)

where the coefficients \( n_{ij} \) are the coefficients of the index of refraction polynomial. The origins of this notation for the index of refraction polynomial lie in the development of aberration theory of gradient index materials [11]. Two special cases arise from equation (2.1), one of which is the case where the coefficients are only in the form of \( n_{0j} \), representing a pure axial gradient in the \( z \) direction and the other takes the form of \( n_{i0} \), representing a pure radial gradient. These two cases are illustrated in figure 2.1. Note that it is possible for a pure radial gradient to consist of all powers of \( r \) and not just the even ones as represented in equation (2.1). The inclusion of odd powers in \( r \) in polynomial expansions of the refractive index profile can have a significant effect on the aberrations present in any imaging system relying on a radial GRIN profile [12].

Needless to say, in addition to axial and radial gradient index profiles, there are spherical gradient index profiles. However, spherical GRIN profiles still remain more of a theoretical curiosity than a practical reality, as will be discussed later (see section 2.1.3).

2.1.1 Axial Gradients

The usefulness of axial gradient index profiles has been limited to the correction of spherical aberration in lenses, as an axial gradient cannot add power to a lens. However, developments of UV photosensitive materials could lead to axially graded planar waveguide devices, such as dispersion compensators [13] and mode transformers. This is due to the fact that such a graded index profile is only capable of bending the path of light to a very small extent. Figure 2.2 illustrates how an axial gradient can be used to correct for spherical aberration. In fact the effect of an axial gradient on monochromatic aberrations is exactly equivalent to that of an aspheric surface [11]. The way in which an axial gradient is incorporated into lens design is not to correct the spherical aberration of individual lens elements to zero, but to correct the total amount of spherical aberration of all the lens elements to zero [8]. To date, axially graded profiles have been incorporated into collimators [14], eyepieces [15] and photographic objectives [16].
2.1.2 Radial Gradients

Radial gradient index profiles have an additional utility compared to an axially graded profile, in that not only is it possible to use the gradient for aberration correction, but it can also be used to modify the focal length of a lens. Recent interest in radial gradients has been derived mainly from multimode optical fibres for telecommunications applications [17]. Two refractive index profiles are commonly fabricated, the step-index profile and the graded-index profile, the latter being approximately parabolic. In the graded-index profile, the rays which travel the longest distance traverse the regions of lowest index and hence go faster than on-axis rays. This tends to equalise the transit times (relative to a step index fibre) and thus minimise dispersion [18, 19]. Surprisingly, early interest in the subject sprung from entomology, where results published in 1889 concluded that the eyes of some insects were comprised of radially graded index components [20].
Figure 2.2: An example of how a spherically aberrated lens (a) can be corrected (b) using an axial gradient index profile (c).

The Wood Lens

The first device with a radial GRIN profile known to have been made is the Wood lens which was developed in 1905 [9]. The lens was made from gelatine using a test tube as the mold, and then cut into thin sections which were sandwiched between two plates of glass. By either soaking the gelatine in water or allowing the gelatine to dry out, Wood was able to create an internal refractive index profile that either increased or decreased, respectively, as a function of the radial distance.
What he demonstrated experimentally was that each radial GRIN profiles he created performed as either a converging or diverging lens, depending on whether the index was a decreasing or increasing function of the radial distance, respectively. Furthermore, the converging lens produces ray paths inside the material which are sinusoidal along the meridional plane. Similarly, in fibres with a parabolic profile, the ray trajectories are sinusoidal or helical-like along the fibre such that the rays do not impinge upon the boundaries of the fibre.

Cylindrical and Planar GRIN lenses

Before continuing this discussion, it is important to note that there are two types of radial GRIN profiles. The radial gradients mentioned thus far possess a cylindrical symmetry and lenses made from materials with such a refractive index profile are commonly referred to as gradient-index rod lenses. Lenses with radial gradients in only one dimension are termed gradient-index slab lenses or planar gradient-index lenses. Beyond telecommunication, GRIN rod lenses have important commercial applications, such as rod lens arrays for photocopying and facsimile machines [22–24] and as relay rods for endoscopes [25]. Recently, it has been pointed out that an elliptical graded index cladding can be used to give efficient excitation of a rare-earth doped fibre core [26].

Unlike a rod GRIN lens, with its cylindrical geometry, the slab geometry presents a much greater technical challenge in fabricating the required parabolic refractive index profile. Despite these difficulties, the planar GRIN lenses with a parabolic refractive index profile has a number of potential applications, including for example, its use as a 1 × 7 optical star coupler [3, 27] or for maximising the coupling efficiency between a laser diode and a fibre [28]. Furthermore, planar GRIN lenses fabricated with a narrow cross-section can be used as micro-cylindrical lenses. There is already a need for such a device in the current development of 3D optical memory devices [2], as a thin cylindrical lens is required to focus the addressing beam down to a thin beam waist. One advantage a planar GRIN lens has over a conventional cylindrical lens is its size, where a planar GRIN lens with a smaller cross-section could conceivably improve the above-mentioned memory device by increasing the resolution and hence its storage density. Furthermore, a planar GRIN lens has the potential to deliver significant improvements to confocal microscopy [1]. So hopefully it is clear that interest in the planar GRIN lens is not a purely theoretical

†These planes contain rays which cross the optical axis [21, Sectn. 2-2].
interest and that such a device has important applications.

2.1.3 Spherical Gradients

In addition to axial and radial graded-index profiles, spherical GRIN profiles are also possible. As the name suggests, spherical gradient-index profiles possess a refractive index distribution which is symmetric about a point, so that surfaces of constant index are spheres. The Maxwell fish eye lens was the first spherical gradient lens considered and was conceived by Maxwell in 1854 [29].

This is a non-trivial example of a device which has the ability to sharply image every point in a region of space. In this case, only points on the surface of the lens and within the lens are sharply imaged, as is illustrated in figure 2.3. The are several problems with such a device in that firstly, the images of extended objects suffer from severe aberration, and secondly, it is highly doubtful that it is possible to fabricate such a lens with existing technology and as yet no possible applications have not yet been identified. Another type of spherical lens with perhaps greater utility is the Luneburg lens, which focuses parallel rays to a point on the surface of the sphere (see figure 2.3).

![Figure 2.3: A schematic example of how the Maxwell fish eye lens (a) and the Luneburg lens (b) focus rays. Each lens operates in a very different manner, as every point on the surface or within the fish eye lens is sharply imaged whereas a Luneburg lens focuses parallel rays entering the lens to a point on the opposite surface of the lens. The circles represent the outline of the spherical lenses.](image-url)
2.2 Analysis of the planar GRIN Lens

There is more than one way to skin a cat and the same is true for analysing a planar GRIN lens. Three methodologies will be employed to gain insight into the workings of this device - ray optics, electromagnetic analysis and the use of the finite-difference beam propagation method (FD-BPM) to simulate the field propagating through the device. The purpose of the analysis is to better understand how the device works, as this will ultimately assist in the design, fabrication and characterisation of the GRIN lens.

2.2.1 Ray Analysis

Ray tracing is based on the premise that when light propagates through a device whose characteristic dimensions are much greater than the wavelength, then its behaviour can be accurately described by rays obeying a set of geometrical rules. These rules are referred to as the postulates of ray optics. Before stating these postulates it is important to realise that the optical path length, $L$, along a given path between two points $a$ and $b$ in an inhomogeneous medium is defined as

$$L \equiv \int_a^b n(r) \, ds,$$

(2.2)

where $ds$ is the differential element of length along the path and $r$ is the position vector; thus it is proportional to the transit time.

The postulates of ray optics state that:

- Light travels in the form of rays, which are emitted by light sources and can be observed when they reach an optical detector.

- An optical medium is characterised by its refractive index, $n$, which is defined as the ratio of the speed of light in free space to that in the medium.

- Fermat's principle is obeyed. This states that the optical path length between two points $a$ and $b$ is an extremum relative to neighbouring paths, i.e. it is a local minimum (or maximum):

$$\delta \int_a^b n(r) \, ds = 0.$$

(2.3)

The extremum is usually a minimum and therefore

light rays travel along the path of least time.
Ray equation

If the ray trajectory is described by the functions $x(s)$, $y(s)$ and $z(s)$ where $s$ is the physical path length, then the position vector $\mathbf{r}(s)$ is expressed as

$$\mathbf{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$$  \hspace{1cm} (2.4)

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are orthogonal cartesian unit vectors parallel to the $x$, $y$ and $z$ axes, respectively, and

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$  \hspace{1cm} (2.5)

Now the optical path length (eq. (2.2)) can be recast, introducing the dependence of the integrand on the curve as

$$L = \int_a^b n(x,y,z) \left( \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 \right)^{1/2} \, ds,$$  \hspace{1cm} (2.6)

resulting in the Euler-Lagrange equations [30] which can be expressed as the following three partial differential equations:

$$\frac{d}{ds} \left( n \frac{dx}{ds} \right) = \frac{\partial n}{\partial x}; \quad \frac{d}{ds} \left( n \frac{dy}{ds} \right) = \frac{\partial n}{\partial y}; \quad \frac{d}{ds} \left( n \frac{dz}{ds} \right) = \frac{\partial n}{\partial z}. $$  \hspace{1cm} (2.7)

The more compact vector form of equation (2.7) is

$$\frac{d}{ds} \left( n(\mathbf{r}) \frac{d\mathbf{r}}{ds} \right) = \nabla n(\mathbf{r}),$$  \hspace{1cm} (2.8)

where $\nabla n$ is the vector gradient of $n$. Equation (2.8) is commonly referred to as either the ray equation or the eikonal equation, and is the basic formula for determining the possible ray paths that can pass through a medium whose refractive index profile is specified by $n$. Note that for the case when $n = 1$ the optical path length coincides with the product of the geometrical length between points $a$ and $b$, and $n$.

Paraxial approximation

One of the techniques for solving the ray equation is to employ an approximation known as the paraxial approximation [31, Sectn. 1.3], which assumes that the ray trajectory is almost parallel to the optical axis, i.e. the $z$ axis, so that $ds \approx dz$. 

12
Upon applying this approximation and assuming that the refractive-index profile is translationally invariant, i.e. \( n = n(x, y) \), the ray equation (2.8) simplifies to
\[
\frac{d}{dz} \left( n \frac{dx}{dz} \right) \approx \frac{\partial n}{\partial x}, \quad \frac{d}{dz} \left( n \frac{dy}{dz} \right) \approx \frac{\partial n}{\partial y}.
\] (2.9)

As we shall see in section 4.1.3, there is only a very slight variation in index across the planar GRIN, thereby ensuring this condition is met in practice. These two partial differential equations are appropriately termed the paraxial ray equations. Once the refractive index profile is specified, equation (2.9) may be solved to provide the ray trajectories \( x(z) \) and \( y(z) \).

For a graded-index slab whose refractive index \( n = n(x) \) is uniform along \( y \) and \( z \), whilst varying continuously along \( x \), the refractive-index profile \( n = n(x) \) and the ray paths of paraxial rays in the \( x-z \) plane satisfy the ordinary differential equation
\[
\frac{d}{dz} \left( n \frac{dx}{dz} \right) = \frac{dn}{dx}.
\] (2.10)

which can be rearranged as
\[
\frac{d^2 x}{dz^2} = \frac{1}{n} \frac{dn}{dx} = \frac{d}{dx} \left( \log n \right).
\] (2.11)

Solution of this equation gives the ray path for a particular profile. For a general refractive index profile, there is no explicit analytical solution, but in the case of the parabolic profile, a simple solution is available for a planar geometry.

Parabolic refractive index profile

Consider the one-dimensional refractive index profile which is parabolic with a profile expressible as
\[
n^2(x) = \begin{cases} 
n_{co}^2 \left( 1 - 2\Delta x^2 / \rho^2 \right), & \text{for } |x| \leq \rho \\
n_{cl}^2, & \text{for } |x| > \rho \end{cases}
\] (2.12)

where \( x \) is the coordinate in the cross-section, \( \Delta \) is the relative index difference, \( n_{co} \) is the maximum core index on axis, \( n_{cl} \) is the uniform cladding index and \( \rho \) is the half-width. The relative index difference is given by
\[
\Delta = \frac{n_{co}^2 - n_{cl}^2}{2n_{co}^2}
\] (2.13)
and it is assumed that $\Delta \ll 1$. Utilising this condition, the refractive index profile within the graded index region can be approximated by

$$n(x) \approx n_{co} \left(1 - \frac{\Delta x^2}{\rho^2}\right), \quad (2.14)$$

so that once this is substituted into equation (2.11) we obtain a simple differential equation which describes the ray paths for this particular profile

$$\frac{d^2x}{dz^2} = -\left(\frac{n_{co}}{n(x)}\right) \frac{2\Delta}{\rho^2} x \approx \frac{2\Delta}{\rho^2} x, \quad (2.15)$$

since $n(x) \approx n$. Given an initial position of, $x_0$, and initial slope, $\tan \theta_0$, at $z = 0$, the ray trajectory is expressible as

$$x(z) = x_0 \cos \left(\frac{z\sqrt{2\Delta}}{\rho}\right) + \frac{\rho \tan \theta_0}{\sqrt{2\Delta}} \sin \left(\frac{z\sqrt{2\Delta}}{\rho}\right), \quad (2.16)$$

so that the ray path oscillates about the optical axis of the slab with a maximum deviation from the optical axis of

$$x_{max} = \left(x_0^2 + \frac{\rho^2 \tan^2 \theta_0}{2\Delta}\right)^{1/2}. \quad (2.17)$$

Now provided $x_{max} \leq \rho$, the ray remains confined to the slab with the ray period expressed as

$$z_p = \frac{2\pi \rho}{\sqrt{2\Delta}}. \quad (2.18)$$

This means that irrespective of the launch angle or position, provided the rays are paraxial, the ray period is the same for all the rays. Furthermore, this graded-index slab can act as a cylindrical lens, whose focussing power in the $x$-$z$ plane depends on both the launch conditions and the length of the lens. These features of the planar GRIN lens are illustrated in figure 2.4.
Figure 2.4: A schematic illustration of how a GRIN lens of length $L_1$ focusses rays, as in (a). The ray paths emanating from the GRIN lens can be altered by either changing the launch conditions for the same length of lens, as in (b), or by changing the length of the lens from $L_1$ to $L_2$ for the same input, as in (c).
2.2.2 Modal Analysis

Another means of describing the propagation of light through non-absorbing optical waveguides is to examine the bound modes of the waveguide. The bound modes themselves are solutions of the source-free Maxwell equations, which can be cast into the following vector wave equations [21, Sectn. 30-10]

\begin{align}
\nabla^2 \mathbf{E} + k^2 n^2 \mathbf{E} &= \nabla (\nabla \cdot \mathbf{E}); \\
\nabla \times \{\nabla \times \mathbf{H} n^{-2} \} &= k^2 \mathbf{H},
\end{align}

(2.19)

which govern the behaviour of the components of the electric field \( \mathbf{E}(x, y, z) \) and the magnetic field \( \mathbf{H}(x, y, z) \) in an anisotropic medium, where \( k = 2\pi / \lambda \) is the wavenumber and \( \lambda \) is the wavelength of a monochromatic source.

Despite the elegant appearance of the vector wave equations (2.19), they are unfortunately rather cumbersome to use in practice. However, if we assign cartesian coordinate with axes orientated such that the \( z \)-axis is parallel with the optical axis, and the \( x-y \) plane is in the waveguide cross-section, and the refractive index profile is assumed to be translationally invariant, i.e. \( n = n(x, y) \), then the electric and magnetic fields can be expressed in the separable forms

\begin{align}
\mathbf{E}(x, y, z) &= \mathbf{e}(x, y) e^{i(\beta z - \omega t)}; \\
\mathbf{H}(x, y, z) &= \mathbf{h}(x, y) e^{i(\beta z - \omega t)},
\end{align}

(2.20)

where \( \mathbf{e} \) and \( \mathbf{h} \) describe the transverse dependence of the electric and magnetic fields, respectively. The field’s phase is described in terms of the accumulated phase change at position \( z \) along the waveguide at time \( t \), for a field with propagation constant \( \beta \) and frequency \( \omega \). Note that the time dependence will be implicitly assumed from now on. Furthermore, the transverse dependence of the electric and magnetic fields are comprised of transverse components, \( \mathbf{e}_t(x, y) \) and \( \mathbf{h}_t(x, y) \), and a longitudinal components, \( e_z(x, y) \) and \( h_z(x, y) \), respectively. So the expressions for the reduced fields are expressible as

\begin{align}
\mathbf{e}(x, y) &= \mathbf{e}_t(x, y) + e_z \mathbf{z}; \\
\mathbf{h}(x, y) &= \mathbf{h}_t(x, y) + h_z \mathbf{z},
\end{align}

(2.21)

where \( \mathbf{z} \) is the cartesian unit vector parallel to the \( z \) axis.

Now provided that there is only a slight variation in the refractive index profile of the waveguide, i.e. \( \Delta \ll 1 \), then the vector equations (2.19) can be replaced by a scalar wave equation involving only a single component of the electric field, with
negligible loss of accuracy [32, Sectn. 3]:

\[
\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 n^2(x, y) - \beta^2 \right\} e_t(x, y) \approx 0.
\]

(2.22)

This is referred to as the weak-guidance approximation [6, 7].

The fields which satisfy the scalar wave equation (2.22) and are bound at infinity are referred to as the bound modes of the waveguide, each with a discrete value for the propagation constant \( \beta \). Fields which are bound modes of a waveguide have the property that their field distribution remains invariant while propagating along the length of the waveguide and that the flow of power is always parallel to the waveguide axis. Furthermore, any guided field of the waveguide can be deconvolved into a linear superposition of all the bound modes of the guide. The range of bound mode propagation constants is given by [21, Sectn. 13-2]

\[
k n_{cl} \leq \beta \leq k n_{co},
\]

(2.23)

where the largest value for \( \beta \) corresponds to the fundamental mode of the guide and the lowest value for \( \beta \) corresponds to the highest-order mode. A mode is at cutoff when \( \beta = kn_{cl} \), as this is the smallest permissible value that the propagation constant for a bound mode may have.

Modal Parameters

The physical quantities of a waveguide determine the modal fields, including the number of modes that a guide will support. A useful parameter which embodies these physical quantities is referred to as the waveguide parameter [21, Sectn. 11-17] and is defined as

\[
V \equiv k \rho n_{co} \sqrt{2\Delta}.
\]

(2.24)

A waveguide is multimoded provided \( V \gg 1 \). In a multimode waveguide, the number of modes \( N \), which can propagate can be large. For instance the number of TE-polarised bound modes in a step-index planar waveguide is

\[
N = \text{Int}\{2V/\pi\}.
\]

In addition, dimensionless modal parameters \( U_j \) and \( W_j \) are defined to facilitate the description of the modal fields in the core and cladding, respectively; those corresponding to the \( j \)-th order mode with propagation constants \( \beta_j \) are

\[
U_j \equiv \rho \sqrt{k^2 n_{co}^2 - \beta_j^2}; \quad W_j \equiv \rho \sqrt{\beta_j^2 - k^2 n_{cl}^2},
\]

(2.25)
where the waveguide and modal parameters are related by
\[ V^2 = U_j^2 + W_j^2. \]  
(2.26)

It is important to note that ray optics generally gives results very close to wave theory provided that \( V \) is large enough [33]. In fact the agreement is good when the fields are tightly confined to the core, and this occurs for guides when \( V \gg 1 \) [34].

**Parabolic Profile**

Propagation through the planar GRIN lens can be analysed in terms of the bound modes of the parabolic profile. Since the variation in this profile between the maximum on-axis index and the uniform index of the buffer and cladding layers is small, the modal analysis can be conducted in the weak-guidance approximation, rather than the full set of Maxwell’s equations. In this approximation, the transverse modal electric field has the single transverse electric-field component in the form
\[ E(x, z) = \psi(x)e^{i\beta z} \]  
(2.27)
where \( z \) is the distance along the axis. The transverse field variation \( \psi(x) \) is a solution of the one-dimensional scalar wave equation
\[ \left\{ \frac{d^2}{dx^2} + k^2n^2(x) - \beta^2 \right\} \psi(x) = 0, \]  
(2.28)
where \( n(x) \) denotes the complete refractive index profile given by eq. (2.12). This equation can be solved exactly for the clad parabolic profile in terms of Whittaker functions [21, Ch.14], but if we limit our attention to just the core parabolic variation, then these solutions simplify to Gauss-Laguerre polynomials [21, Ch.12]. This approximation makes little difference to the propagation constant values of lower-order modes only a small number of highest-order modes are affected, but within the accuracy of the fabrication and characterisation procedures of sections 3.4 and 4.1.3, respectively, we can safely use the infinite parabolic profile.

**Infinite Parabolic Profile**

The values of the propagation constant follow from the eigenvalue equation, which for the infinite parabolic profile is derived from the solutions of the scalar wave equation by imposing continuity on the solutions and their first derivatives at the interfaces between the parabolic core region and the surrounding cladding.
regions of uniform index. The eigenvalue equation for a planar waveguide with the $j$-th mode of a parabolic refractive index profile is [21, Sectn. 14-4]

$$U_j^2 = (2j + 1)V,$$  \hspace{1cm} (2.29)

where $j = 0, 1, 2, \ldots$. Substituting the expression for $V$, eq. (2.24), into the above eigenvalue equation and re-arranging yields

$$\beta_j = kn_{co} \left( 1 - \frac{(2j + 1)\sqrt{2\Delta}}{k\rho n_{co}} \right)^{1/2} \approx kn_{co} - \frac{(2j + 1)\sqrt{2\Delta}}{2\rho}$$  \hspace{1cm} (2.30)

since $\Delta \ll 1$. Hence the beat length between any pair of adjacent modes is given by

$$z_b = \left| \frac{2\pi}{\beta_j - \beta_{j+1}} \right| = \frac{2\pi\rho}{\sqrt{2\Delta}},$$  \hspace{1cm} (2.31)

which is the same as the ray period, eq. (2.18). For parabolic guides the temporal band-width is higher than that for a step index waveguide [35] because the optical path lengths are partly equalised. The index which minimises dispersion in a fibre is close to $n^2(r) = n_{co}^2(1 - 2\Delta\{r^q - \frac{2}{3}\Delta r^4\})$ where $r$ is the radial variable with $q = 2 - \frac{4}{3}\Delta$ [36], which is not parabolic. In a slab guide (and for meridional rays in a fibre), all rays have the same transit time for the "sech" profile. However, unlike dispersion, the focussing properties of the refractive index profile are optimised when it is parabolic.

### 2.2.3 Propagation Methodologies

Suppose a monochromatic beam of light illuminates the input face of a planar GRIN lens. What happens to the beam and how does it evolve as it propagates through the lens? We know that rays travel along sinusoidal trajectories, so how do rays compare with fields? For the sake of simplicity, we shall restrict ourselves to considering the parabolic core index variation and assume that the input beam is Gaussian and that is propagating through a guide that satisfies the weak-guidance approximation. The beam's field in this instance can be approximated by

$$E(x, z) = \psi(x, z)e^{ikn_{co}z}, \quad \psi(x, z) = e^{-\frac{1}{2}(a(z)x^2 - 2b(z)x + c(z))},$$  \hspace{1cm} (2.32)

in terms of functions $a(z)$, $b(z)$ and $c(z)$, where the beam is initially paraxial with respect to the optical axis and the components of the electric field satisfy the scalar Helmholtz equation, which is expressed as

$$(\nabla^2 + k^2n^2)E = 0.$$  \hspace{1cm} (2.33)
Plugging eq. (2.32) into the Helmholtz equation and assuming that the field envelope is slowly varying, i.e. \(|\partial^2 \psi / \partial z^2| \ll 1\), then the field is approximately a solution to:

\[
2ikn_{co} \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + k^2 (n^2(x) - n_{co}^2) \psi = 0. \tag{2.34}
\]

With the parabolic refractive index profile expressed in this instance as \(n^2(x) = n_{co}^2(1 - \alpha^2 x^2)\), where \(\alpha^2 = 2\Delta/\rho^2\) in eq. (2.12), the above equation becomes

\[
2ikn_{co} \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} - k^2 \gamma^2 x^2 \psi = 0, \tag{2.35}
\]

where \(\gamma = an_{co}\). Substituting \(\psi(x, z)\) of eq. (2.32) into eq. (2.35) and comparing like terms we find that the functions \(a\), \(b\) and \(c\) must satisfy the following conditions

\[
\begin{align*}
ink_{co} \frac{da}{dz} &= a^2 - k^2 \gamma^2; & ikn_{co} \frac{db}{dz} &= ab; & ikn_{co} \frac{dc}{dz} &= b^2 - a. \tag{2.36}
\end{align*}
\]

These differential equations are readily solved, yielding

\[
\begin{align*}
a(z) &= \frac{k\gamma (1 + \zeta e^{-i2\alpha z})}{1 - \zeta e^{-i2\alpha z}}; & b(z) &= \frac{b_0 (1 - \zeta) e^{-i\alpha z}}{1 - \zeta e^{-i2\alpha z}}; \\
c(z) &= c_0 + i\alpha z - \frac{b_0^2 (1 - \zeta) (1 - e^{-i2\alpha z})}{2k\gamma (1 - \zeta e^{-i2\alpha z})} + \ln \left| \frac{1 - \zeta e^{-i2\alpha z}}{1 - \zeta} \right|; \hspace{1cm} (2.37)
\end{align*}
\]

with \(\zeta = \frac{a_0 - k\gamma}{a_0 + k\gamma}\), where \(a_0 = a(0)\); \(b_0 = b(0)\) and \(c_0 = c(0)\).

These are the equations of motion that determine approximately how a Gaussian beam evolves through a medium with an infinite parabolic refractive index profile.

An example of how the Gaussian beam with an initially narrow beam waist propagates when it is initially incident either on and off centre can be viewed in figures 2.5 (a) and (b), respectively. Figure 2.5 (a) resembles the propagation of a Gaussian beam through free-space with periodic focussing from lenses.

### 2.2.4 Finite-difference method

Unfortunately the method presented above is too restrictive despite the appeal of having an approximate solution in analytic form. What is required is a propagator that is capable of propagating the arbitrary input field with more complicated and realistic refractive index profiles. The finite-difference (FD) method was used
Figure 2.5: A monochromatic ($\lambda = 632.8\text{nm}$) Gaussian beam propagating through an infinite parabolic refractive index distribution whose core index maximum is located at $x = 0$. View (a) shows the beam propagating when initially located at the centre of GRIN profile ($x = 0$), while view (b) illustrates the characteristic manner in which an uncentred beam oscillates in a sinusoidal fashion about the optical axis. The parabolic index in this instance has a core maximum index of 1.46 with $\alpha = 1.17 \times 10^{-2}\text{µm}^{-2}$. Lighter shades indicate higher field intensities.
for this purpose and has been implemented for all the simulations presented in this thesis from herein.

The method involves propagating wavefronts which are paraxial in either one or two dimensions (even though all the simulations presented here only propagate in one dimension) in a medium with a refractive index profile that does not possess a rapidly varying gradient in the longitudinal direction. The choice of the FD method in preference to other beam propagation methods (BPMs), most notably the fast Fourier transform (FFT) BPM, is primarily because it is not as severely limited by the gradient of the refractive index profile. Furthermore, the numerical efficiency is particularly good and is technically superior to that for the FFT-BPM, and the FD-method is much more stable with respect to grid size [37]. Note that, in their basic forms, neither method accounts for back reflected waves, although they could be modified to take these into account [38].

The FD-method relies on propagating a monochromatic wave, $E(x, z) = u(x, z) \exp(ik\bar{n})$ using the scalar Helmholtz equation (2.33), where $\bar{n}$ denotes the background refractive index. The background refractive index is most often assigned to the value of $n_{cl}$ or to the average taken from the lowest and highest values of the refractive index profile in the computational domain. When this expression is inserted into eq. (2.33) and the slowly-varying term involving $\partial^2 \psi / \partial z^2$ is omitted, we then obtain

$$2ik\bar{n} \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + k^2 \left( n^2(x, z) - \bar{n}^2 \right) u = 0. \tag{2.38}$$

The conditions for neglecting $\partial^2 \psi / \partial z^2$ are that the gradient of $n(x, z)$ along $z$ must be small, that the field amplitude be slowly-varying and, more importantly, that the field must be paraxial. The FD method of solution of eq. (2.38) involves re-writing it as [38]

$$\frac{\partial u}{\partial z} = A(x, z) \frac{\partial^2 u}{\partial x^2} + B(x, z) u, \tag{2.39}$$

where $A(x, z) = i/2k\bar{n}$ and $B(x, z) = ik(n^2(x, z) - \bar{n}^2)/2\bar{n}$.

Discretisation

Using the notation $u_s^r = u(x_s, z_r)$, the field $u(x, z)$ is discretised so that the transverse grid extends from the left boundary at $x_0 = p$ to the right boundary at $x_N = q$, with $N+1$ grid points and $s = 0, 1, 2, \ldots, N$. The location of each grid point
is given by \( x_s = p + s \delta x \) where \( \delta x = (p - q)/N \). Similarly the longitudinal grid is \( [z_r = r \delta z; r = 0, 1, \ldots] \). The discretisation of eq.(2.39) follows the Crank-Nicholson scheme which estimates the finite differences at a fictitious half-step \((s, r + 1/2)\), so that

\[
\frac{\partial u}{\partial z} = \frac{u_{r+1}^s - u_r^s}{\delta z},
\]

\[
A \frac{\partial^2 u}{\partial x^2} \to \frac{A_{s}^{r+\frac{1}{2}}}{2} \left\{ \frac{u_{s-1}^r - 2u_s^r + u_{s+1}^r}{(\delta x)^2} + \frac{u_{s-1}^{r+1} - 2u_s^{r+1} + u_{s+1}^{r+1}}{(\delta x)^2} \right\}; \quad (2.40)
\]

\[
Bu \to \frac{B_{s}^{r+\frac{1}{2}}}{2} (u_r^s + u_s^r).
\]

Once these expressions are substituted into eq.(2.39), the field must satisfy the following system of equations

\[
a_s u_{s-1}^{r+1} + b_s u_s^{r+1} + a_s u_{s+1}^{r+1} = d_s, \quad \text{for } s = 1, \ldots, N - 1; \quad (2.41)
\]

and the expressions for \( a_s, b_s \) and \( d_s \) are:

\[
a_s = -\frac{\delta z}{(\delta x)^2} A_{s}^{r+\frac{1}{2}};
\]

\[
b_s = 2 \left( 1 + \frac{\delta z}{(\delta x)^2} A_{s}^{r+\frac{1}{2}} \right) + \delta z B_{s}^{r+\frac{1}{2}};
\]

\[
d_s = \left\{ 2 \left( 1 - \frac{\delta z}{(\delta x)^2} A_{s}^{r+\frac{1}{2}} \right) + \delta z B_{s}^{r+\frac{1}{2}} \right\} u_r^s + \frac{\delta z}{(\delta x)^2} A_{s}^{r+\frac{1}{2}} (u_{s-1}^r + u_{s+1}^r).
\]

Thus, given the field \( u_r^s \) (for \( s = 0, \ldots, N \)) at \( z = r \delta z \), the next step involves solving the tridiagonal system of linear equations in eq.(2.41) with the appropriate boundary conditions, to obtain the field, \( u_{r+1}^s \) at \( z = (r + 1) \delta z \). The solution to this system of equations can also be shown to be stable \([39]\). Also the FD-method is a more stable propagator than the FFT-BPM \([37]\). The accuracy of the FD-method primarily relies upon the step size of \( \delta x \) and \( \delta z \). However, as this is purely a numerical solution, the accuracy of this technique will deteriorate as the computational domain becomes larger.

2.2.5 Boundary Conditions

Transparent boundary conditions \([40, 41]\) are implemented for all FD-method simulations presented in this thesis. The transparent boundary condition allows
outgoing radiation to propagate freely while suppressing the flux of incoming radiation. Reflections from the boundaries are usually very small, however conditions can arise where the reflections can be substantial. In order to overcome this impediment, an absorbing strip is placed adjacent to both the left and right boundaries to eliminate any reflected radiation from re-entering the computational domain. Implementing transparent boundary conditions involves the adjustment of \( u_0^r \) and \( u_{N-1}^r \) prior to evaluating the field for the \( r+1 \) propagation step. The method assumes that the radiation flux leaving the computational domain is given by plane-waves at the left and right boundaries which are represented by \( u_0^r \exp(ik_1x) \) and \( u_0^r \exp(ik_r x) \), respectively. The modified values for the field elements \( u_0^r \) and \( u_{N-1}^r \) and the constants \( k_1 \) and \( k_r \) are determined by:

\[
\frac{u_0^r}{u_1^r} = \frac{u_1^r}{u_2^r} = \exp(ik_1 \delta x) \quad \text{and} \quad \frac{u_{N-1}^r}{u_{N-2}^r} = \frac{u_{N-1}^r}{u_{N-2}^r} = \exp(ik_r \delta x). \tag{2.43}
\]

In order to ensure the outflow of radiation, the real part of \( k_1 \) and \( k_r \) are required to be positive and the boundary conditions for the next propagation step are given by

\[
u_0^{r+1} = u_1^{r+1} \exp(ik_1 \delta x); \quad u_{N-1}^{r+1} = u_{N-1}^{r+1} \exp(ik_r \delta x). \tag{2.44}\]

### 2.3 Multi-mode interference and Self-imaging

So far we have only discussed the self-imaging properties of graded-index waveguides with a parabolic refractive index profile. However, under certain conditions optical images can also be formed in waveguides with a uniform refractive index and uniform thickness [42]. To understand this situation, assume a thick dielectric slab of uniform refractive index, acting as a multimode waveguide, is illuminated by a monochromatic coherent object of wavelength \( \lambda \), located at \( z = 0 \). This has the effect of exciting a large number, \( M \), of propagating modes and a continuum of unguided (radiation) modes, all initially with the same phase. In an ideal guide, each guided mode propagates independently of all others with a propagation constant \( \beta_j \), while the unguided modes will attenuate quickly, so that beyond a certain distance along the guide, only the guided modes will remain. For those lower-order modes far from cut off propagating along the step-profile planar waveguide [21, Sectn. 12-3]

\[
U_j \cong (j + 1) \frac{\pi}{2} \frac{V}{V + 1}, \tag{2.45}
\]

where \( V \) is the waveguide parameter defined by eq. (2.24). Utilising the definition for \( U_j \), eq. (2.25), an expression for the propagation constants of each mode can be
obtained:
\[
\beta_j \cong k n_{co} \left( 1 - \frac{(j + 1)^2 \pi^2 \Delta}{4(V + 1)^2} \right). \tag{2.46}
\]

By defining \( L_\pi \) as
\[
L_\pi = \frac{\pi}{\beta_0 - \beta_1} \cong \frac{4(V + 1)^2}{3\pi k n_{co} \Delta}, \tag{2.47}
\]
then propagation constants spacing can be written as
\[
\beta_0 - \beta_j = \frac{j(j + 2)\pi}{3L_\pi}. \tag{2.48}
\]

As these modes propagate along the guide, each with a different phase velocity, they rapidly move out of phase and interfere in some complicated way. The field profile at a distance \( z \) along the guide can be written as a superposition of all the guided mode field distributions as
\[
\Psi(z) = \sum_{j=0}^{M-1} c_j \psi_j \exp[-i\beta_j z]. \tag{2.49}
\]
where the coefficients \( c_j \) are calculated from the overlap integrals of the input field profile \( \Psi(0) \) with the modes of the waveguide, \( \psi_j \). Taking the phase of the fundamental mode as a common factor out of the sum and redefining \( \Psi(z) \), the field profile becomes
\[
\Psi(z) = \sum_{j=0}^{M-1} c_j \psi_j \exp[i(\beta_0 - \beta_j)z] = e^{i\beta_0 z} \Psi(z). \tag{2.50}
\]

At some distant plane at \( z = L \), the phases of all the guided modes roughly coincide and a one-dimensional image of the object field is formed in the waveguide’s transverse cross-section. This phenomenon is referred to as self-imaging, and it occurs when the phase factor in eq. (2.50) satisfies the condition
\[
\exp \left[ i \frac{j(j + 2)\pi}{3L_\pi} \right] = (-1)^j. \tag{2.51}
\]

When \( j \) is even, the phases of all the modes at \( z = L \) must differ by integer multiples of \( 2\pi \), with the result that all the guided modes will be in phase, as they were at \( z = 0 \); thus the image is a direct replica of the object field. Otherwise, phase changes corresponding to integer multiples of \( \pi \) result in reproductions of the object field which are alternatingly erect \((j \text{ even})\) and inverted \((j \text{ odd})\). Therefore, erect and inverted single images of the object field at unit magnification occur when
\[
L = p(3L_\pi) \quad \text{with} \quad p = 1, 2, 3 \ldots, \tag{2.52}
\]
for \( p \) even (erect) and \( p \) odd (inverted), respectively.
Symmetric Object field

If the object field only excites the even symmetric modes, i.e.,
\[ c_j = 0 \text{ for } j = 1, 3, 5, \ldots, \]  
then, as we shall see, the length \( L \) required to obtain single images becomes four times shorter. This is because \( j(j + 2) \) is a multiple of 4 when \( j \) is even. Therefore, single images of a symmetric object field at unit magnification will occur for
\[ L_p = p \left( \frac{3L_\pi}{4} \right) \text{ with } p = 1, 2, 3 \ldots, \]
with the first inverted image of the object field occurring when
\[ L_1 = \frac{(V + 1)^2}{\pi kn_{ce} \Delta}. \]
This mechanism is referred to in the literature as *symmetric* interference [43].

Even though a step-index waveguide is capable of creating images of the object field, the length required for the periodic refocussing of the object field to occur strongly depends on the object field itself. If a Gaussian beam is used as the object field, tilting it with respect to the optical axis of the waveguide will result in exciting the odd modes of the guide and hence changing the refocussing period by a factor of four. The GRIN lens however possesses a refocussing period that is much shorter, and that is independent of the modes excited by the object field illuminating it. Furthermore, these phase coincidences in the step-index waveguide are only approximate and they deteriorate with each increasing image order, \( p \). Hence the reproduction of the object field also deteriorates, with an increase in the number of contributing modes that form the image [42].

Quantitative Illustrations

An example of self-imaging in a step-index waveguide is presented in figure 2.6; utilising a monochromatic Gaussian beam at a wavelength of 632.8\( \text{nm} \), as the object field. In this example, the core index of the guide is 1.462, \( \Delta = 1.359 \times 10^{-2} \) and the half-width is 10\( \mu \text{m} \) (\( V = 24.0 \)). These simulations required a step-size of \( \delta x = 2\text{nm} \) and \( \delta z = 0.1\mu \text{m} \), to ensure that either individual modes or a superposition of modes of the waveguide do not radiate power from the waveguide as a result of there being insufficient points to adequately discretise the refractive index profile.
According to eq. (2.55), the image of the object field should occur at $z = 1003\mu m$ along the guide for a centred beam, which, from visual inspection is approximately where the image occurs ($z \approx 1000\mu m$) in figure 2.6 (a). However, in figure 2.6 (b) the process is more complicated as the beam is off-centre and a combination of odd and even modes are excited. This results in a partial refocussing of the object field occurs at $z = 1000\mu m$. This is because the phases of the even modes coincide at this point but the phases of the odd modes do not. Hence, the inverted image at $z \approx 1000\mu m$ is not as close a resemblance of the object field as compared with the image formed four times further away at $z \approx 4000\mu m$ where the phases of all the modes coincide. Compared with waveguides possessing a parabolic refractive index profile, the images of the object field are much further apart in step index waveguides with the same $V$ value.

Another example involves using the same Gaussian beam as the object field, rather in this instance it is tilted at a slight angle to the input face of the waveguide. In this instance the inverted image of the object field appears four times further away as compared with the examples illustrated in figure 2.6. A demonstration of this can be seen in figure 2.7 where it Gaussian beam is tilted five degrees to the left, away from the optical axis, in the same step-index waveguide depicted in figure 2.6. The results are that the inverted image does not appear at $z \approx 1000\mu m$ but at $z \approx 4000\mu m$, as shown in figures 2.7(a) and (b), respectively. Although, careful visual inspection of figure 2.7 reveals that the object field is weakly refo-cussed every $1000\mu m$ due to the coincident phases of the even modes.

Despite the simplicity of fabricating a step index waveguide, its imaging properties as a lens are far poorer than a parabolic GRIN waveguide. A GRIN lens is a far more flexible instrument where the focussing power of the lens can be altered by either changing the launch conditions or the length of the guide. Furthermore, as the ray-period and beat-length of the light propagating through the GRIN lens are independent of wavelength, the imaging properties are also dependent of the source wavelength. Hence, the GRIN lens serves a far greater utility as a lens than compared with a step-index waveguide.
Figure 2.6: Image formation in a step-index waveguide using a monochromatic Gaussian beam (λ = 632.8nm) as the object field. The core and cladding indices of this device are 1.462 and 1.442, respectively, with half-width ρ = 10µm. The object field is in the centre of the guide in (a) and is off-centre in (b). In (a), only the even modes are excited whereas in (b) both even and odd modes are excited. However in (b), a partially refocussed image of the object field appears at z ≈ 1000µm as in (a) due to the phases of all the contributing even modes coinciding there. This simulation utilised the finite-difference method with step sizes of δx = 2nm and δz = 0.1µm.
Figure 2.7: A Gaussian beam tilted with respect to the optical axis of the step-index waveguide, by 5 degrees to the left at the same wavelength in the same step-index waveguide in figure 2.6. This results in the excitation of all the guided modes and consequently the self-image of the object field does not appear at \( z \approx 1000\mu m \) (a), but forms 4 times further away at \( z \approx 4000\mu m \) (b). View (a) shows the first quarter of view (b). Again the step sizes for these simulation are \( \delta x = 2nm \) and \( \delta z = 0.1 \mu m \).
CHAPTER 3

Fabrication

In Italy for thirty years under the Borgias they had warfare, terror, murder, bloodshed - they produced Michelangelo, Leonardo da Vinci and the Renaissance. In Switzerland they had brotherly love, five hundred years of democracy and peace, and what did they produce ... ? The cuckoo clock.

Orson Welles 1915-1985, From the film The Third Man.

3.1 Introduction

Fabricating a planar GRIN lens with a parabolic profile is technically difficult. This contrasts sharply with GRIN rod lenses, which are simpler to produce and have been commercially available for some time. As discussed previously (see section 2.1.2), planar GRIN lenses have potential applications where a rod GRIN lens would be unsuitable. The intended purpose of our planar GRIN lenses is to function as a cylindrical lens, some tens of microns wide, that produces a focussed narrow beam waist that extends out from the end of the guide. Recently, two techniques for producing planar GRIN have been presented: one using a molecular stuffing process [27] and the other an indiffusion ion exchange and cementing technique [44]. Unfortunately, neither of these two methods is capable of producing a sufficiently thin GRIN lens. Our fabricated GRIN lenses rely on plasma-enhanced chemical vapour deposition (PECVD) in a Helicon plasma reactor [45]. It proved to have two novel properties, the planar GRIN lenses being the smallest produced to date and demonstrating a new application for an existing fabrication technology. However, it is insightful to understand how the other two earlier processes work in order to address the relative merits of all three fabrication techniques.
3.2 Molecular Stuffing Process

In principle, the molecular stuffing process [27] consists of the deposition of an index-modifying dopant from solution into a porous glass preform until the concentration is uniform, and then removing the dopant in such a way to achieve the required concentration profile. The porous glass preform is produced by heat-treating a borosilicate glass below its miscibility temperature, so that the glass separates into a relatively insoluble silica-rich interconnective phase and a soluble low-silica phase. The soluble phase is dissolved in acid and the remaining porous skeleton is then immersed into an aqueous solution of the dopant. The solution is allowed to diffuse into the pores until the concentration of the dopant is constant – this constitutes the stuffing. To create the desired profile, the glass is then placed in a solvent for a prescribed period of time until the required concentration profile of the dopant has been achieved. The index gradient is correlated with the concentration profile and it is the unstuffing process that creates the refractive index profile. The remaining two steps involve precipitating the remaining dopant and heating the glass to collapse the pore structure.

Two planar devices have been fabricated using this technique. One used CsNO₃ as a dopant, and was 3.4 mm thick with a numerical aperture \( NA = 0.205 \). The second used TlNO₃ as the dopant, and had a cross-section of 2.5 mm with a \( NA = 0.318 \). A major difficulty with this technique is that commercially available glasses are manufactured in such a manner that they do not phase separate. Hence this technique is only suited to specially manufactured glasses that phase separate. Furthermore, if the phase separation is not uniform, then the resulting gradient is not uniform [35], and the method is only capable of producing profiles that are parabolic out to about 60% of the lens' radius. In fact, the overall shape of the recovered refractive index distribution is closer to Gaussian in shape. Because this technique relies essentially on ion diffusion, the best profiles are obtained by trial-and-error so that the reproducibility of this technique relies heavily on accumulated practical experience.

3.3 Ion Exchange and Cementing

Using an alternative approach, a gradient-index slab lens with a large NA was produced [44]. In this fabrication process, the index profile is generated using an indiffusion ion exchange and cementing technique by placing a glass plate into a
molten salt bath of Ag$^+$ ions. Some of the alkali ions of the glass are replaced with the Ag$^+$ ions of the bath to form a graded-index layer close to the surface of the plate. The treated plate is then cut longitudinally through its centre into two thinner plates, and the graded-index faces of these plates are brought into contact with each other and cemented together in such a way as to form the near parabolic profile. Finally the two ends of the guide are polished.

The planar GRIN lens produced by this technique had a thickness of 2.4 mm and an NA=0.7. Although, the resulting profile was close to parabolic but cementing the plates together probably results in a lens with a significant scattering loss. The main drawback of this method is that it is only capable of producing profiles that are limited to being Gaussian, Lorentzian, linear or close to one of these [35].

Unfortunately, neither the molecular stuffing process nor the ion exchange and cementing processes produce lenses with a small enough cross-section for our needs. Furthermore, neither technique can accurately control the refractive index profile as they both rely upon the diffusion of ions in the material to effect the change in the refractive index. These approaches are difficult to control in practice and they produce refractive index profiles with significant variability. However, a distinct advantage of the ion exchange and cementing technique is that it is capable of producing planar lenses with a large numerical aperture.

3.4 PECVD Fabrication

Rather than use either of the above-mentioned techniques, the planar GRIN lenses were fabricated using PECVD (plasma enhanced chemical vapour deposition) to deposit graded index silica onto a silicon wafer, as illustrated in figure 3.1. There are several advantages in using this fabrication technique. This deposition technique does not rely upon ion diffusion to create the change in the refractive index, and the refractive index is directly controlled during deposition by adjusting the flow rate of the dopant gas. Another advantage is that only one fabrication step is required to complete the lens with a parabolic profile of arbitrary thickness and NA, unlike the molecular stuffing and ion exchange techniques. However, the deposition rate places a practical limitation on the maximum thickness that is achievable, due to the time required to deposit the material. Thus, it is unlikely that a film thicker than 40 $\sim$ 50$\mu$m could be produced using this technique.
3.4.1 The Helicon plasma reactor

In the present case, the Helicon plasma reactor developed at the Australian National University [46] was used for the depositions. The reactor is illustrated schematically in figure 3.2, and was used to deposit graded-index, fluorine-doped silicon dioxide films (silica) onto p-type, (100) orientated silicon substrates, 100 mm in diameter.

The deposition of silica was achieved by using a mixture of silane (SiH$_4$) and oxygen, at flow rates of 12 sccm $^1$ and 96 sccm, respectively, where the flow rates are regulated by mass flow controllers. For these depositions, O$_2$ is chosen in place of N$_2$O in order to avoid the absorption peak at 1.5µm, owing to the presence of N-H bonds occurring in PECVD-deposited silica [47]. The oxygen is introduced at the top of the diffusion chamber and the silane is introduced via four tubes located above and directed towards the wafer. With these gases, a plasma is generated in an upper cylindrical glass tube, 30 cm long and 15cm in diameter, and is surrounded by the helicon antenna (see fig.3.2). Once formed, the plasma diffuses downwards

---

$^1$The units sccm refer to standard cubic centimetres per minute at STP.
Figure 3.2: Cross-section of the Helicon plasma reactor used to deposit the PECVD films onto 100mm diameter silicon wafers. The wafers are loaded into the loadlock which is then evacuated down to a base pressure of approximately $4 \times 10^{-4}$ Torr and then transferred to the reactor chamber for processing. The loadlock chamber enables the reactor chamber to be kept under constant vacuum, and also minimises the influx of dust and other contaminants into the reactor chamber.
into the lower cylindrical aluminium chamber, 30 cm deep and 35 cm in diameter, which houses the substrate table. The molecules of SiH₄ and O₂ dissociate in the radio frequency (rf) plasma at 13.56 MHz, forming the active species (precursors) that react on the surface and on the walls of the reactor, resulting in the formation of a thin film of SiOₓ.

The rf power at 13.56 MHz is coupled to the plasma via the helicon antenna [48] surrounding the glass tube, and is matched by a π-tuning network, in the presence of an axial magnetic field. The excitation frequency of 13.56 MHz generates a Helicon wave with a wavelength of ~ 22 cm, hence the choice of the excitation frequency was an important consideration in building a reactor chamber of a convenient size [49]. The source solenoid surrounding the glass tube is used to create an axial magnetic field in order to generate the Helicon plasma, but also to obtain the maximum energy transfer into the plasma. Furthermore, the diffusion of the high-density plasma into the reactor chamber is controlled by a chamber solenoid, and has been optimised to improve the confinement and uniformity of the plasma [46, 49]. Typically, the reactor is evacuated down to a base pressure of approximately 4 x 10⁻⁶ Torr using a turbomolecular pump (at 1000 l/s) placed on top of the source tube. Once the gases are introduced into the chamber, the total pressure rises to approximately 3 mTorr and all the depositions are carried out with a source power of 800 W. These deposition conditions have previously been used for the fabrication of buried channel waveguide (BCW) devices [50].

3.4.2 Ion bombardment

The Helicon diffusion plasma reactor generates high plasma densities without high plasma potentials. One of the more interesting features of the Helicon reactor is that it allows for independent control of both the plasma generation and the interaction between the plasma and the surface by independently biasing the substrate. Utilising these degrees of freedom, the film is subjected to an ion bombardment during deposition by independently biasing the substrate.

Films deposited at floating potential (i.e. without ion bombardment) were found to possess a columnar structure throughout the bulk of the material that leads to significant scattering loss of light in buried channel waveguides. This loss can be significantly reduced by annealing the deposited material at high temperature. These films are also birefringent, due to the effect of the anisotropy in the material. How-
ever, by correctly selecting the ion bombardment energy, these unwanted features can be eliminated. This in turn further improves the film quality without having to anneal the film after the deposition has been completed [51].

The method for accelerating ions from the plasma onto an insulating surface, such as SiO$_x$, consists of applying a separate rf voltage (at 13.56MHz) to the substrate through a blocking capacitor. In the steady state, a negative voltage, known as the self bias, develops on this capacitor. The energy of the ions in this case is introduced through the difference between the plasma potential and the self bias. It has been established [51, 52] that, to a good approximation, the ion bombardment energy $W^+$, in practice, is given by

$$W^+ = \frac{1}{2} e V_{pp},$$  \hspace{1cm} (3.1)

where $e$ is the electronic charge and $V_{pp}$ is the peak-to-peak voltage. In our case, an applied peak-to-peak voltage is maintained at 600V, resulting in an ion bombardment energy of 300eV. The justification for this ion bombardment energy is that it is the lowest energy required to produce films with the best optical properties. The films possess very little columnar structure, with minimal birefringence, low surface roughness and high density. These features are related to the stress in the deposited films, which is minimal when the ion bombardment energy is in the range of $300 \sim 400eV$ [51].

It has been previously established that when using a high ratio of oxygen to silane (8 in this case), there is enough oxygen in the gas phase for the stoichiometry of the deposited film to be close to 2, i.e. SiO$_2$ [46]. The choice for this particular ratio and flow rates for oxygen and silane were determined experimentally, in order to produce the best deposited film quality, combined with the fastest deposition rates. Higher flow rates of silane increased the deposition rate and the optimal film quality occurred when the ratio of oxygen to silane was approximately equal to 8. It was also found that the poorer quality films usually peeled from the silicon wafer substrate during and/or after deposition, as well as suffering from higher scattering losses.

Throughout the deposition process, the substrate holder is water cooled and the wafer is clamped onto the substrate holder with helium circulating in the gap between the wafer and the holder, at a pressure of about 2.5 Torr, to ensure good thermal contact. Under these conditions the wafer temperature remains below 100°C during deposition. Finally, the deposited layers were not annealed after deposi-
tion, and the propagation losses were reasonably low, estimated to be the order of 1.0 dB/cm [50].

3.4.3 Controlling the refractive index

The control of the refractive index of the deposited material in the Helicon is a key issue in the fabrication of the graded-index layer. We chose to control the refractive index by doping the layer with fluorine [53, 54] by introducing a small amount of tetrafluoro-methane (CF₄), thus avoiding the use of toxic gases such as germane or phosphine as used in other conventional PECVD techniques. CF₄ is a useful source of fluorine as it readily dissociates in the plasma.

It should be noted that it is possible to alter the refractive index profile by adjusting the ratio of the flow rates of silane to oxygen entering the reactor chamber. The advantage of this set up is that it obviates the need to introduce a dopant gas such as CF₄. However, the process is much harder to control due to the fact that changing the gas mixture alters the growth rate and the quality of the deposited material. Material deposited in this manner is of much poorer quality and has a large number of scattering centres that greatly enhance its loss. This is primarily why an alternative approach was sought to alter the refractive index of the deposited films. Furthermore, fluorine doping was the only available option on the Helicon for varying the index.

The high concentration of oxygen in the gas phase ensures that the carbon is removed by the formation of CO and CO₂, and thus does not contaminate the films. Confirmation of this mechanism is provided by the fourier infrared transmission spectrum (FTIR) of a 1.0 μm thick layer of silica deposited with CF₄ present; it is displayed in figure 3.3. Due to the fact that there is very little CF₄ present during the depositions, so that the transmission spectrum cannot be used to confirm the presence of fluorine in the deposited films. Consequently, the spectrum is the same regardless of whether or not CF₄ is introduced into the plasma during the film’s deposition. However, the presence of fluorine in the silica films has been verified using X-ray analysis with an energy dispersive spectrometer [50]. With these deposition parameters, a typical infrared transmission spectrum shows the presence of hydrogen-containing bonds such as Si-OH (∼ 3620 cm⁻¹) and Si-H stretching and bending bond modes (∼ 2300 cm⁻¹, ∼ 880 cm⁻¹, respectively) and including Si-O-Si stretching and rocking bond modes (∼ 1090 cm⁻¹, ∼ 820 cm⁻¹ respectively) and
SiO double bonds (~ 600 cm\(^{-1}\)). These results are consistent with other findings relating to the chemical bonds that typically appear in silicon dioxide films, using infrared spectroscopy [55–57].

![Figure 3.3: Infrared transmission spectrum of a typical SiO\(_2\) film fabricated in the Helicon plasma reactor. This spectrum shows the presence of Si-OH (~ 3620 cm\(^{-1}\)), Si-H stretching and bending bond modes (~ 2300 cm\(^{-1}\), ~ 880 cm\(^{-1}\), respectively), Si-O-Si stretching and rocking bond modes (~ 1090 cm\(^{-1}\), ~ 820 cm\(^{-1}\) respectively) and SiO double bonds (~ 600 cm\(^{-1}\)).](image)

### 3.4.4 Ellipsometry & Prism Coupling

A single wavelength ellipsometer, at 632.8 nm, is mounted on the diffusion chamber of the Helicon and is used as an in situ diagnostic tool, to monitor the thickness of the growing films during the deposition process. This tool is particularly useful for ensuring that test wafers deposited with different flow rates of dopant gas can be grown to approximately the same thickness. It also helps ensure that the deposition is working normally. However, the main drawback of this single-wavelength ellipsometer is that it is not capable of accurately measuring the refractive index to the required precision of four significant figures. Furthermore, the nonreproducibility of the wafers alters the precision of the measurement for each deposition.

For these reasons, the thickness and refractive index of the films were obtained
after they were deposited using a prism coupler, at 632.8 nm, because this technique is known to be more precise. The results, illustrating the variation in the deposition rate and the refractive index with respect to the flow rate of CF$_4$, are displayed in figures 3.4 (a) and (b), respectively. It should be noted that two batches of GRIN lenses were produced during May 1995 and February 1997. During both these periods the deposition process was characterised for the effects of the variation of CF$_4$ on the refractive index and deposition rates of the grown films. As shown in figure 3.4, there is a considerable variation in the effect the dopant gas had on the deposited films between these two results. However, the significant result is that the variation in the refractive index of the films depends linearly on the flow rate of the dopant gas for both cases and the maximum attainable refractive index in both sets of results is the same, within experimental error. The deposition rates are still constant, to a good approximation, and are within 5 nm/min of each other.

The difference between the results taken in May 1995 and February 1997 is attributable to the removal, from the walls of the reactor chamber, of the silica deposits that had built up over time. The reactor was cleaned, because once too much silica deposits on the walls of the chamber, it begins to flake off, with the risk that a piece (or pieces) of silica can fall off and contaminate the wafer during the deposition. A detailed explanation of the mechanisms that create these effects is beyond the scope of this thesis. However, it is likely that there is more fluorine being consumed on the walls of the chamber, and that removing the silica from the reactor chamber can perhaps lead to a change in the plasma potentials inside the chamber, altering the deposition conditions. All the GRIN lenses were produced within two weeks of completing the characterisation for the effects of CF$_4$. Finally, the results obtained in 1997 show that the refractive index of the deposited films extend in a linear fashion down to as low 1.428 with a CF$_4$ flow rate of 15 sccm.

3.4.5 Deposition Analysis

During the deposition process, the flow rate of CF$_4$ is the only variable that is be altered in time in order to create the desired refractive index profile of the deposited films, and is expressed as $R(t)$. It is clear from the experimental results illustrated in figure (3.4), that the flow rate, $R$, influences both the refractive index and the growth rate of the deposited films. Furthermore, these experimental results also demonstrate that the change imposed on both the refractive index and the growth

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Footnote: A thorough explanation of the workings of prism couplers will be provided in Chapter 4.
Figure 3.4: Variation in (a) refractive index and in (b) deposition rate of silica grown as functions of the flow rate of CF₄ dopant in sccm. The difference between the results taken in May 1995 and February 1997 is attributable to the removal of silica from the walls of the reactor chamber. This was done to reduce the amount of silica flaking off the walls and contaminating the wafer surface.
rate, depends linearly on the flow rate of the dopant gas during the deposition process. Hence the deposition rate can be expressed approximately as

\[
\frac{dx}{dt} = \sigma R(t) + \gamma,
\]

(3.2)

where \( \sigma \ll \gamma \). In fact, the experimental results demonstrate that, to an excellent approximation, the growth rate can be assumed to be constant and the term \( \sigma R \) can be discarded altogether. This means that the thickness of the material deposited after time \( t \) is simply \( x = \gamma t \).

The functional dependence of the parabolic refractive index of the film on the flow rate \( R \) is given by

\[
n^2(R) = (mR + n_{\text{max}})^2 = n_{\text{co}}^2 \left(1 - 2 \Delta x^2 / \rho^2\right),
\]

(3.3)

where \( m \) is the absolute value of the slope of either line in fig. 3.4 (a), \( n_{\text{max}} \) is the maximum attainable refractive index (i.e. when \( R = 0 \)) and the relative index difference \( \Delta = (n_{\text{co}}^2 - n_{\text{cl}}^2) / (2n_{\text{co}}^2) \). Rearranging equation (3.3), so that \( R \) is the subject of the equation, and assigning \( n_{\text{co}} \) the same value as \( n_{\text{max}} \), results in an expression which determines how the flow rate must evolve in time, in order to produce a parabolic refractive index profile; thus

\[
R(t) = \frac{n_{\text{co}} \left\{ 1 - [1 - 2\Delta(\rho - \gamma t)^2 / \rho^2]^{1/2} \right\}}{m} \text{ (sccm)}.
\]

(3.4)

Initially, the parabolic refractive-index profile for the first GRIN lens was sandwiched between uniform layers of lower-index material, and thus the complete device comprised the three regions deposited on the silicon substrate, as shown schematically in figure (3.1) (c).

As the upper cladding layer made characterising the core refractive index profile a much more difficult task, it was omitted from subsequent lenses. However, the lower buffer layer serves a more important task, as its purpose is to isolate the evanescent field of the light propagating in the graded guiding region from the higher-index silicon substrate, thereby minimising the tunnelling loss to it [32]. In electromagnetic terms, the light is very tightly confined to the guiding region as \( V \gg 1 \), with minimal evanescent field outside of this region, so that a relatively thin buffer layer is sufficient.
3.4.6 Deposition

The whole deposition process is carried out in one step. Initially the buffer layer is deposited, with a constant refractive index \( n_c \) to a thickness of 2\( \mu m \). Once this is completed, the flow rate of CF\(_4\) is controlled manually, stepped 0.1 sccm at a time, and at predetermined time intervals governed by eq. (3.4). By doing so, we obtain a reasonably smooth increase of index above the buffer layer to the centre of the lens and a corresponding decrease beyond the centre of the graded-index region, to produce the symmetric profile. The overall time for depositing the complete core layer is \( 2\rho/\gamma \). The maximum attainable core index was 1.462 ± 0.001 and the specifications of the fabricated GRIN lenses are provided in table 3.1.

<table>
<thead>
<tr>
<th>Core width 2( \rho ) (( \mu m ))</th>
<th>Relative Index Difference ( \Delta )</th>
<th>Numerical aperture</th>
<th>Growth rate ( \gamma ) (nm/min)</th>
<th>Slope m eq. (3.4)</th>
<th>Buffer layer</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.0</td>
<td>1.3586 %</td>
<td>0.24</td>
<td>101</td>
<td>0.006</td>
<td>1.437</td>
</tr>
<tr>
<td>10.0</td>
<td>2.0323 %</td>
<td>0.30</td>
<td>96</td>
<td>0.002</td>
<td>1.428</td>
</tr>
<tr>
<td>10.0</td>
<td>1.3596 %</td>
<td>0.24</td>
<td>96</td>
<td>0.002</td>
<td>1.441</td>
</tr>
<tr>
<td>10.0</td>
<td>2.0323 %</td>
<td>0.30</td>
<td>96</td>
<td>0.002</td>
<td>1.428</td>
</tr>
<tr>
<td>5.0</td>
<td>2.0323 %</td>
<td>0.30</td>
<td>96</td>
<td>0.002</td>
<td>1.428</td>
</tr>
</tbody>
</table>

Table 3.1: A summary of the specifications of the fabricated GRIN lenses. A total of five lenses were successfully fabricated using the Helicon plasma reactor. The core index of the first GRIN lens in the table is 1.462 ± 0.001 while that of the others is 1.461 ± 0.001.

For comparison, a step index waveguide was also fabricated with the same 20\( \mu m \) thickness of the central guiding region as the GRIN lens, but with uniform refractive index 1.462, between the buffer and cladding layers each 2.0\( \mu m \) thick with a refractive index of 1.437.
CHAPTER 4

Waveguide Characterisation

If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties.

Francis Bacon 1561-1626,
The Advancement of Learning.

4.1 Recovering the refractive index profile

Francis Bacon's words have a striking poignancy, in that once the silica-based PECVD material has been deposited, how certain can you be that it is what you believe it to be? This question arises whether the deposited material is believed to be a step-index waveguide, a GRIN lens or something else. It is important to investigate the various properties of the waveguides in order to determine the reproducibility of the fabrication process and the specifications of the resulting planar waveguides. The main parameters that characterise such films are the refractive index profile and the thickness. There are several methods that can possibly determine these parameters. However, one method that is particularly well adapted to this problem is the prism coupling technique [58, 59]; this can quickly provide accurate results.

4.1.1 Prism Coupler

This method involves coupling a laser beam, with the use of a prism, into a planar dielectric slab [58–60], as is illustrated schematically in figure 4.1. The slab in this instance is a layer of silica deposited onto a silicon wafer where the refractive index, $n_p$, of the prism is greater than the refractive index of the silica. In this arrangement, the prism is pushed into contact with the wafer by means of a pneumatically-operated coupling head. This minimises the air gap between the dielectric slab and the prism for the purposes of improving the coupling efficiency. The laser beam is incident onto the base of the prism at an angle, $\theta_p$, such that it undergoes total internal reflection. The light reflected off the base of the prism then
leaves the prism and is measured by an adjacent photodetector in figure 4.1. The incident and reflected beams at the base of the prism produce a travelling wave which is parallel to the base of the prism, with a propagation constant \( \beta_p = k n_p \sin \theta_p \). The evanescent field produced by the travelling wave extends out of the prism into the space separating the slab waveguide and the prism. As the transverse field distribution of the evanescent field decays exponentially, the waveguide layer must be as close as possible to the prism. In practice the waveguiding film is not perfectly flat, so only part of the base of the prism is physically touching the waveguiding film. So when the prism and the film are brought into contact with each other, a small dark spot appears on the base of the prism which corresponds to the region where the two are in physical contact. Directing the laser beam at this small dark spot provides the best conditions for coupling light into the film, and hence this region is referred to as the "coupling spot". Strong coupling of light into the waveguide layer occurs for angles where \( \beta_p \) closely matches the propagation constants of the modes of the waveguide layer underneath the prism. This results in a sharp drop in the intensity of light striking the photodetector, thus enabling the calculation of the propagation constants of the modes of the waveguiding film. All the measurements presented here relied upon a Metricon model 2010 prism coupler, equipped with a TE-polarised Helium-Neon (HeNe) laser operating at 632.8 nm. Finally, the measurements were obtained from the centre of the wafer in each case and as the variation in refractive index and thickness was only slight across the wafer, the variation in the refractive-index profile will be correspondingly small and within the tolerances of the experimental measurements.

### Effective Index Measurement

This machine automatically measures the effective indices of the modes of the guide, rather than the propagation constants (i.e. \( n_{eff} = \beta/k \)), and lists them. For a given waveguiding film, the value(s) of the propagation constants depends only on the film thickness and refractive index. For deposited films with a uniform refractive index profile, the thickness and refractive index can be determined as soon as the first two mode angles are measured, by using a computer algorithm [58, 60]. In this way, the prism coupler was used to measure the films produced under various conditions in order to characterise the effect of the dopant gas on the deposition rate and refractive index of the films (see fig 3.4). Later, the same machine was used to measure the effective indices of the GRIN lenses and the step index waveguide in order to reconstruct the refractive index profiles of the fabricated devices.
Figure 4.1: Optical power can be coupled into (or out of) a slab waveguide using a prism. A TE-polarised Helium-Neon (HeNe) laser at a wavelength of 633.8 nm is used as the source. A photo-detector receives the light emanating from the prism, and detects the intensity of the light as the incident angle of the laser light is changed continuously. The propagation constants of the modes of the slab guide are calculated from the angles corresponding to sharp dips in the intensity of the light striking the photo-detector. These measurements were taken using a Metricon model 2010 prism coupler.
Effect of the Silicon Substrate

An important point to note is that all of the thin silica films produced are deposited on a silicon substrate with an index of $3.877 + 0.016i$ at a wavelength of 630nm at 25°C [61], so its real part is much higher than the silica film. This immediately begs the question - how can the silica layer behave as a waveguide if it is deposited onto a silicon substrate? The answer lies in the power reflectance, $R$, of the transverse electric (TE) and transverse magnetic (TM) polarised waves, incident on the silica/silicon (SiO$_2$/Si) interface. Note that here, the TE polarisation has its electric field, $E$, normal to the plane of incidence, and its magnetic field, $B$, parallel to it. Similarly, the TM polarisation has $E$ parallel to the plane of incidence and $B$ normal to it. The prism coupler can select either polarisation by using a half-wave plate to rotate the polarisation state of the incoming laser beam from the TE to the TM state. The complex amplitude reflectance for the TE and TM polarisations are

$$r_{TE} = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

and

$$r_{TM} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)},$$

respectively, where $\theta_i$ and $\theta_t$ are defined in figure 4.2 (a). The power reflectance, $R$, defined to be the ratio of the power flow, along a direction normal to the boundary, of the reflected wave to that of the incident wave and is simply expressed as

$$R = |r|^2. \quad (4.2)$$

A plot of the power reflectance, $R$, as a function of the angle of the incident wave is shown in figure 4.2 (b) for both the TE and TM polarisations. What it demonstrates is that the first few bound modes of the guide, which correspond to shallow angles of incidence at the silica/silicon boundary, possess power reflectance of $\sim 80\%$ for the highest-order mode of a TE polarised wave propagating in the deposited film, compared with $\sim 15\%$ for the same order TM polarised mode. Typically, the thin films being characterised for the effect of the CF$_4$ dopant were about 1$\mu$m thick, usually with three modes. The power reflectance for these three modes lies in the shaded region of figure 4.2 (b), highlighting the fact that the power reflectances for TE-polarised waves are significantly higher than those for TM polarised waves.

The power reflectances for the next two TM modes are of the order of 1%. As a result of this, the TM modes are very difficult to observe, and, although a TM mode
may exist, it may be difficult to see. Hence, for TM polarised waves, the prism coupling technique is only capable of measuring the effective index of the fundamental TM mode but cannot be relied upon for measuring the next two TM modes. In practice, what occurs is that the first TM mode is usually observed, but the next two modes are missing. This is in agreement with other findings which showed that TM modes for low index films on high index substrates (such as silicon or metals) are very difficult to observe [63]. Hence, all the measurements involving the prism coupler utilised only TE-polarised beams of coherent light for characterising these thin films. Finally, results from the prism coupler corroborated the data obtained from the in situ ellipsometer, mounted on the deposition chamber, indicating that the deposited material has the same index of refraction through its cross-section, provided the flow rate of the dopant gas is the same throughout the deposition.

4.1.2 Recovery of the Effective Indices

Using a prism coupler, the refractive-index profile for the fabricated graded-index and the step-index slab waveguides were determined by the measurement of the effective index values of the discrete bound modes. To access the modal fields, which are tightly confined to the central regions of the GRIN and step waveguides, it was necessary to remove the cladding layer from those samples which had been fabricated with a cladding layer (refer to sec. 3.4.5), using a plasma etch. Whilst the first fabricated GRIN lens had a top cladding layer to protect the core region this was omitted from subsequent waveguide fabrication to facilitate the characterisation. The plasma etch was performed using a reactive ion etch. This enabled the effective index values of the highest-order modes (smallest values of effective index) to be measured. Access to the lower-order modes (higher values of effective index) for those samples already etched, was possible only by etching further to remove part of the core. Unfortunately, each etch back of the surface damages the exposed top surface of the waveguide, reducing the coupling efficiency. Etching is necessary so that the evanescent field from the prism coupler can couple into these modes, because the transverse evanescent field of each mode attenuates rapidly in the cladding. One consequence of removing a small part of the core is that although it may modify the effective indices of the lowest-order modes slightly, the effective indices of the highest-order modes are changed significantly. This was not a problem because the effective indices of the highest-order modes had been measured before the core was etched. The result of this procedure was that the effective indices of almost all the guided modes of the planar guides, apart from a small number of the
Figure 4.2: The power reflectance, $R$, of light waves incident at the silica/silicon boundary (a) depends strongly on the polarisation of the incident light (b). The shaded region in (b) shows the power reflectance range for the the first three bound modes, ignoring the imaginary part of $n_t$. It is evident that the TE polarised modes have power reflectances which are significantly higher than the corresponding TM polarised modes. For the TM polarisation, $R$ reaches zero at the Brewster angle, $\theta_B = \tan^{-1}(n_t/n_i)$.
lower-order modes, were measured. Alternatively, waveguides deposited without a top cladding layer only required one measurement and no etching in order to obtain most of the effective indices.

4.1.3 Profile reconstruction

Armed with a set of experimentally measured effective indices, the refractive index profile of the waveguide can be obtained from the distribution of the effective indices [64]. Firstly, a monotonically-decreasing continuous function is calculated as the best fit to the distribution of effective indices; this is referred to as the effective-index function \( N(m) \), where \( m \) denotes the mode number. Thus \( N(m) = \beta_m/k \) is the effective index of the \( m \)th-order mode and \( \beta_m \) is the corresponding propagation constant. If \( x(m) \) corresponds to the position within the profile where the index is equal to \( N(m) \), then a WKB analysis (see appendix A) leads to the following eigenvalue equation for the symmetric graded and step profiles:

\[
I = k \int_{-x(m)}^{x(m)} [n^2(x) - N^2(m)]^{1/2} dx = \left( m + \frac{1}{2} \right) \pi, \quad (4.3)
\]

where \( \lambda \) is the source wavelength, \( n[x(m)] = N(m) \), and the peak index of the guide occurs when \( m = -\frac{1}{2} \). This relationship is based on the Wentzel-Kramers-Brillouin (WKB) approximation to the Helmholtz equation. The WKB integral, equation (4.3), can be discretised and expressed as

\[
I \approx 2k \left\{ (x_1 - x_0)(\bar{N}_1^2 - N_i^2)^{1/2} + (x_2 - x_1)(\bar{N}_2^2 - N_i^2)^{1/2} + \cdots + (x_i - x_{i-1})(\bar{N}_i^2 - N_i^2)^{1/2} \right\}, \quad (4.4)
\]

where \( x_0 = 0 \). Substituting eq. (4.4) into eq. (4.3), the resulting expression can be rearranged thus

\[
x_i = \frac{\alpha_i - 2k \sum_{j=1}^{i-1} \{x_j[(\bar{N}_j^2 - N_i^2)^{1/2} - (\bar{N}_{j+1}^2 - N_i^2)^{1/2}]\}}{2k(\bar{N}_i^2 - N_i^2)^{1/2}}, \text{ for } i = 1, 2, 3, \ldots \quad (4.5)
\]

where \( \alpha_i = (m_i + 0.5)\pi \) and \( \bar{N}_i = \frac{(N_i + N_{i-1})}{2} \), then the refractive index profile can be computed recursively from the effective index function \( N(m) \). In fact, for parabolic profiles the effective index function \( N(m) \) is a straight line since the effective indices are equally spaced. Hence, the effective-index function is expressed as

\[
N = N_0 - bm \quad (4.6)
\]
The parabolic refractive-index profile corresponding to the effective index function described by equation (4.6), is given by

$$n^2(x) = n_0^2 - c^2x^2, \quad (4.7)$$

where $n_0$ is the peak index and $c$ is a constant. Using these expressions for $n(x)$ and $N(m)$, equation (4.3) can be re-cast as

$$I = k \int_{-x(m)}^{x(m)} [n^2(x) - N^2(m)]^{1/2} dx = ck \int_{-x(m)}^{x(m)} [a^2(m) - x^2]^{1/2} dx$$

$$= ck \left[ \frac{1}{2} x \sqrt{a^2(m) - x^2} + \frac{a^2(m)}{2} \sin^{-1} \left( \frac{x}{a(m)} \right) \right]_{-x(m)}, \quad (4.8)$$

where

$$a^2(m) = \frac{n_0^2 - N_0^2 - b^2m^2 + 2N_0 bm}{c^2}. \quad (4.9)$$

As $n[x(m)] = N(m)$, equating equations (4.6) and (4.7) provides the result that $x(m) = a(m)$, simplifying equation (4.8) and providing the result that

$$a^2(m) = \frac{2m + 1}{ck}. \quad (4.10)$$

Substituting the expression for $a(m)$ into equation (4.10), we finally obtain two simple expressions

$$n_0 = N_0 + \frac{b}{2} \quad \text{and} \quad c = k(n_0^2 - N_0^2), \quad (4.11)$$

relating the unknown quantities $n_0$ and $c$, which describe the parabolic refractive-index profile, to the known quantities $N_0$ and $b$, which are obtained from the experimentally measured set of effective-indices.

The refractive-index profiles for the five graded-index samples were found to have profiles that were very close to the designed parabolic profile for each lens. A comparison of the parameter values of the designed and recovered profiles are shown in table 4.1. The effective index function and the recovered profiles for the GRIN slab – with design parameters NA = 0.24 and width 20 \(\mu\)m – and the step-index slab waveguides are shown in figures 4.3 and 4.4, respectively. The discrepancy between the design and the fabricated step profiles in figure 4.4 is a short-coming associated with the WKB approach of this inversion technique [65].
and this limitation is discussed fully in Appendix A. Finally, a comparison of the designed and recovered profiles for the other fabricated GRIN lenses is presented in figure 4.5.

<table>
<thead>
<tr>
<th>Design Parameters</th>
<th>Recovered Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ (µm)</td>
<td>$\rho$ (µm)</td>
</tr>
<tr>
<td>Maximum core index $n_{co}$</td>
<td>Maximum core index $n_{co}$</td>
</tr>
<tr>
<td>Relative index $\Delta$</td>
<td>Relative index $\Delta$</td>
</tr>
<tr>
<td>10.0</td>
<td>1.4620</td>
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<tr>
<td>10.0</td>
<td>1.4610</td>
</tr>
</tbody>
</table>

Table 4.1: A comparison of the design and recovered parameters for the fabricated GRIN lenses. All of the graded-index samples were found to have profiles very close to the designed parabolic profiles.

4.2 Beat Length and Poly(N-vinylpyrrolidone)

A means of independently verifying that the GRIN slab has a parabolic refractive-index profile is to measure the beat length between the modes that propagate within the waveguide. It has previously been shown, in sec. 2.2.2, that the beat length between consecutive modes, along the $z$ axis of the GRIN, is given by

$$z_b = \left| \frac{2\pi}{\beta_j - \beta_{j+1}} \right| = \frac{2\pi \rho}{\sqrt{2\Delta}},$$

which is the same as the ray period, eq. (2.18).

A recently developed fluorescence technique [66] was used to demonstrate visually periodic refocussing in the parabolic profile. A sample of the GRIN lens was taken from the only wafer that had a cladding layer deposited over the graded-index layer (NA=0.24, $\rho = 10.0\mu m$). This sample was obtained by cleaving a square portion from the centre of the wafer, approximately 2cm wide, and the remaining portion of the wafer was used for the purposes of recovering the refractive index profile using the prism coupler. Once cleaved, this sample was etched to remove
Figure 4.3: Effective index function $N(m)$ is determined from (a) for the graded-index lens, and the recovered refractive-index profile is shown as the solid curve in (b). The dashed curve in (b) represents the design profile for a GRIN lens with a $NA = 0.24$ and a core width, $2p = 20.0\mu m$. The fact that the modes are equally spaced in (a) implies that the profile is close to parabolic.
Figure 4.4: Effective index function $N(m)$ is determined from (a) for the step-index waveguide, and the recovered refractive-index profile is shown as the solid curve in (b). The dashed curve, in view (b), represents the design profile for the step-index waveguide with a $NA = 0.27$ and a core width, $2\rho = 20.0\mu m$. See appendix A for a discussion on the discrepancy between the recovered and design profiles.
Figure 4.5: A comparison of the recovered refractive-index profiles through the waveguide cross-section, depicted as solid curves for each of the remaining GRIN lenses that were fabricated. The dashed curves represent the design profile in each case. The numerical aperture and the core width chosen for these GRIN lenses are given underneath each figure.
the top cladding layer. It was then spin-coated with a 2.4µm thick layer of the polymer poly(N-vinylpyrrolidone) doped with 4.81% of a xanthene dye, Phloxine B. The chemical structure of poly(N-vinylpyrrolidone) and Phloxine B are illustrated in figure 4.6 (a) and (b), respectively.

![Chemical structure of Phloxine B and PVP](image)

Figure 4.6: *Chemical structure of Phloxine B (a) and the polymer poly(N-vinylpyrrolidone) or PVP (b)* [67].

It is believed that the mechanism responsible for the upconversion involves the first triplet state of the dye resulting in the generation of delayed fluorescence by these states [66]. What is unique about xanthene dyes is that the phenomenon of upconversion, which is usually associated with the use of high light intensities from pulsed lasers, is achieved by using the output from a low intensity HeNe laser. Consequently, laser light was launched into the (polished) end of the guiding region using the focussed beam waist created by a low-NA microscope objective, which resulted in the formation of the characteristic beat node and antinode pattern, shown schematically in figure 4.7. At each antinode, the evanescent field extended into the polymer layer, exciting the dye and producing a fluorescent band across the top surface; this was observed with a fluorescence microscope.
The presence of Phloxine B dye in the polymer coating allowed the measurement of the beat spacing by two different (exciter) wavelengths. This was due to the fact that the dye exhibits both normal fluorescence behaviour (it fluoresces at a maximum intensity around 576 nm) when excited by a frequency-doubled Nd:YAG laser (532 nm) and also exhibits an upconversion (delayed fluorescence) behaviour that gives the same 576 nm output wavelength when excited by a He-Ne laser (632.8 nm). Figures 4.8(a) and (b) show colour fluorescence photographs of the resulting beat patterns on the top surface of the GRIN waveguide when 532-[Fig. 4.8(a)] or 632.8-nm [Fig. 4.8(b)] laser light was launched onto its end face. Note that the colour in the photographs does not accurately depict the colours seen by the naked eye. This is due to the reciprocity failure of colour film associated with long exposure times (greater than a few minutes).

\(^{1}\text{Nd:YAG denotes Neodymium: Yttrium-Aluminium Garnet (Nd}_{x}\text{Y}_{3-x}\text{Al}_{5}\text{O}_{12}) which serves as an efficient compact source of 1.064}\mu\text{m laser radiation. When passed through a second-harmonic-generating crystal, the frequency of the laser radiation doubles, thereby providing a strong source of radiation at 532nm in the green [68].}
Prism Excitation

Another means for generating the beat spacing is to use a prism coupler. As has been stated earlier, a prism coupler is ideal for coupling light into an optical waveguide. In this case the light is coupled through the polymer into the waveguide. As the polymer is quite soft it allows for strong coupling of light into the waveguide, thereby making it possible to couple into more than one mode simultaneously. The modes which are excited propagate through the GRIN lens, creating the same pattern of nodes and antinodes that were observed using the low NA microscope objective. A spectacular image of this effect, with the same beat spacing as figures 4.8(a) and (b), can be seen in figure 4.9 where the light is launched – at the same exciter wavelength – using a prism coupler instead of a low NA lens.

One of the concerns with this technique of depositing the dyed-doped polymer onto of the GRIN lens relates to the fact that near-cutoff modes strongly depend on the outer layer with the propagation characteristics differing from those in an ideal guide where the outer medium is removed [69]. In using a prism coupler, it is a simple matter to avoid exciting modes that are near-cutoff. However, when a Gaussian beam is used to illuminate the end-face of the guide, it is less certain if the polymer is likely to change the beat-length of the guide in any way. Two approaches were taken to overcome this uncertainty. The first involved comparing the beat spacings from coated and uncoated portions of the GRIN lens. Due to the etching and all the handling the cleaved sample had been subjected to, the surface had become rough enough to see a very faint scattered beat pattern without the use of the polymer coating. The green Nd:YAG laser, the brighter of the two, was used to observe these scattered fringes in order to make a direct comparison between the beat spacing with and without the polymer coating. The image in figure 4.10, is in fact a composite of two images taken at the same magnification with the polymer coated guide on the left and the uncoated guide on the right, showing that the beat-length of the two are indeed the same. This image provides experimental evidence that the polymer coating does not alter the beat spacing, regardless of whether or not the field distribution of the modes close to cut-off have been changed. The other technique for confirming this experimental finding is to use the finite-difference BPM outlined in section 2.2.4.
Figure 4.8: Beat pattern in polymer-coated top surface of GRIN slab as imaged with a fluorescence microscope. (a) Yellow (576nm peak) fluorescent bands, spaced at 330µm±10%, are produced wherever evanescent fields of beat antinodes penetrate the Phloxine B dye-impregnated polymer coating when excited by green laser light (532nm), launched into waveguide at right. (b) Yellow (also 576nm peak) upconversion fluorescent bands are excited at the same wavelength by a HeNe laser (632.8nm) and show a similar spacing at 360µm±7% (wavelength independence). Red blotches are due to scattering of laser light from tiny imperfections which have bloomed in size due to the long exposure required to capture the faint upconversion fluorescent bands. (These photographs were taken by Dr. Carol Cogswell of the School of Physics, University of Sydney.) Scale bar = 1mm.
Figure 4.9: Demonstration of the appearance of the beat pattern, where a prism coupler has been used to couple light into the waveguide. The prism is located to the right of the image and the coupling spot corresponds to a bright disc underneath the prism. The soft polymer enables very strong coupling to take place, where several modes are excited simultaneously, thereby producing the characteristic beat pattern. The light coupled into the polymer propagates to the left, and the edge of the waveguide can be seen at the far left. Scale bar = 2mm.

Figure 4.10: A composite image showing an uncoated (right) and coated (left) GRIN lens at the same magnification. The light in both instances is propagating from right to left and the beat spacing in both images is the same, confirming experimentally that the presence of the polymer coating does not affect the spatial period of the observable beat-pattern structure. The green fringes are very faint and are the result of the scattering from surface imperfections when the field is located close to the surface. (The two images in this composite were kindly taken by Dr. Carol Cogswell.) Scale bar = 1mm.
4.3 Finite Difference BPM Analysis

The FD-BPM was used to simulate the propagation along the GRIN waveguide with the experimentally-recovered refractive-index profile (fig. 4.3(b)), both with and without the polymer coating. The thickness and refractive index of the polymer layer were obtained using the prism coupler (for the TE-polarisation) and were found to be 2.4µm and 1.5332, respectively. A narrow Gaussian beam, described by

\[ \exp \left( -\frac{(x - x_0)^2}{\sigma^2} \right), \]

(4.13)

centred at \( x_0 \) with width \( \sigma \), was used to mimic the focussed laser light, which used a low NA microscope objective to illuminate the end-face of the GRIN lens. The BPM simulations provided results that were consistent with those produced by both ray tracing and modal analyses, for both the 532- and 632.8-nm excitation wavelengths. A benefit of this technique is that the insight gained into the way in which the field propagating through the core couples into the polymer layer is used as a means of validating the experimental results obtained.

Figure 4.11 shows the results of several FD-BPM simulations when a Gaussian beam, whose position \( x_0 \) and width \( \sigma \) are defined in eq. (4.13), is launched into a GRIN lens both without, figs. 4.11(a) and (b), and with the polymer layer, figs. 4.11(c) and (d), for the purposes of comparison. The simulations presented in figure 4.11 have step sizes of \( \Delta x = 2.0\text{nm} \) and \( \Delta z = 0.1\mu\text{m} \) at a wavelength of 632.8 nm. The fringes observed in figs. 4.11(c) and (d) are due to the evanescent fields of the modes in the core region of the GRIN lens that extend into the polymer layer. As the refractive-index of the polymer is much higher than the GRIN core, the air-polymer-silica profile supports a number of modes, the result of which is the creation of standing waves within the polymer, between the air/polymer and the polymer/silica boundaries. It is the field associated with these standing waves that induces either the fluorescence or the delayed fluorescence (upconversion), depending on the excitation wavelength that produces the coloured fringes shown in figures 4.8(a) and 4.8(b). Furthermore, the simulations show how it is possible for the fringes to be seen without the use of the polymer layer. Under these conditions, the surface roughness results in scattering light towards the observer from regions where the field in the core is close to the surface. However, these fringes are extremely faint and are easily obscured from scattered light from defects within the deposited film.

Many simulations were conducted and the position and width of the input Gaus-
Figure 4.11: Finite difference BPM simulations using the recovered GRIN lens profile (see fig. 4.3(b)) with a narrow Gaussian input at different positions, \(x_0\), and widths, \(\sigma\), with [(c) and (d)], and without it [(a) and (b)] the polymer layer. As the field propagates through the core region, it follows a sinusoidal-like trajectory that brings it close to the polymer layer. When the field in the core is close to the polymer layer, the evanescent field extends from the core into the polymer layer, forming standing waves that appear as fringes on the left-hand-side of the core. These fringes correspond to those observed in figs. 4.8, 4.9, and 4.10.
Figure 4.12: The above plots give the ratios of the power in the polymer layer relative to that in the graded-index core of the GRIN lens, expressed as a percentage. The plots (a) and (b) above correspond to the simulations depicted in figures 4.11(c) and 4.11(d), respectively.
sian beam were altered to investigate how this affects the intensity of the fringes. In fact, the choice of \( x_0 \) and \( \sigma \) strongly influence the ratio of the power in the polymer to that in the graded-index core. The values cited in figure 4.11 provided the strongest coupling into the polymer layer, while beams that were centred on-axis, \( x_0 = 0 \), resulted in very weak coupling. A plot of the ratio of the power in the polymer to that in the core is shown in figure 4.12 for the fields propagating in figs. 4.11(c) and 4.11(d). These results demonstrate that, under the best coupling conditions, the ratio of power in the polymer to that in the core is at most \( \sim 0.6\% \) when the core’s field is concentrated near the surface. As a means of comparison, consecutive pairs of modes of the GRIN lens were excited to see how the light coupled into the guide using a prism coupler would behave. The modes in a planar GRIN lens with a parabolic profile are accurately described by Hermite-Gauss polynomials, and the field resulting from the superposition of two consecutive modes propagate in a sinusoidal manner with the same period as a Gaussian beam input field.

### 4.3.1 Comparison of Beat lengths

The results from the simulations suggest that the polymer layer alters the field inside the GRIN lens by creating ripples in the field (higher spatial frequencies) that would not normally be there. However, the simulations and the experimental observations suggest that there is no noticeable change in the spacing of the fringes arising from the parabolic index profile of these GRIN lenses. A comparison between the measured beat spacing resulting from the different laser wavelengths and the design and recovered profiles is presented in table 4.2. The results for the two different excitation wavelengths are consistent (to within experimental error) with the value of 361 \( \mu m \) predicted from the reconstructed index profile. This is an important finding, in that it suggests that these lenses do not suffer from chromatic aberration, and, like commercially available GRIN rod lenses, are ideally suited to focus white light.

### 4.4 Output characteristics

The results presented thus far provide very strong evidence that the graded index profiles are indeed parabolic. Now that this has been established, the output
A piece from each GRIN lens was cleaved, and the cleaved faces were later polished, courtesy of Sydney University's Electron Microscope Unit. The source consisted of a 5 mW 632.8 nm HeNe laser beam which passed through a microscope objective that focussed onto a pinhole. The diverging light emanating from the pinhole was then collimated with a lens and the wide collimated beam was used to fill the back aperture of a 10x/0.3 microscope objective. The light from the microscope objective was focussed onto the polished face of a selected piece of a GRIN lens. The light emanating from the GRIN lens was imaged by another microscope objective forming a collimated beam that was focussed onto a CCD (Charge Coupled Device) camera for the purposes of capturing the images onto a computer. By moving the focal plane of the microscope objective imaging the GRIN lens output, it is possible to build up a processed image, showing how the light propagates out of the GRIN waveguide. Of the images captured, a sample of four of these images are shown in figure 4.13.

These images illustrate that, depending upon the position and focal plane of the incident light on the GRIN lens, the GRIN lenses are capable of focussing and defocussing the incident light. The light in these images is propagating from right to left, where the terminating face of the GRIN lens is denoted by the white vertical bar. The scale bar in each image is 10µm long and 1µm thick. Unfortunately, the focal plane of these GRIN lenses could not be extended beyond a few tens of microns, however the beamwaist of the output can be quite narrow (~ 5µm in width).

<table>
<thead>
<tr>
<th>Excitation wavelength</th>
<th>Measured beat spacing</th>
<th>Calculated beat spacing</th>
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</thead>
<tbody>
<tr>
<td>533 nm</td>
<td>330µm ± 10%</td>
<td>Design profile 381µm</td>
</tr>
<tr>
<td>632.8 nm</td>
<td>360µm ± 7%</td>
<td>Recovered profile 361µm</td>
</tr>
</tbody>
</table>

Table 4.2: A comparison of the beat spacings measured at two different wavelengths with the designed and recovered recovered profiles (fig. 4.3(b)) of a fabricated GRIN lens. Note that the measured results are consistent with the result predicted from the reconstructed profile.
Figure 4.13: The output light intensity distribution from several different GRIN lenses captured with a CCD camera. The light is propagating from left to right and the end-face of the GRIN lens is denoted by the vertical white bar. The light source wavelength is 632.8 nm. The lenses have the ability to produce an output beam waist of \( \sim 5\mu m \). (These images were obtained with the generous assistance of Mr. Paul Cronin of the School of Physics, University of Sydney.)
4.5 Discussion

This work has demonstrated that several planar, graded-index lenses have been successfully fabricated with a parabolic index profile with a PECVD Helicon reactor. The parabolic index profile has been recovered with a prism coupler, and was later confirmed with the measurement of the beat length by means of a polymer coating. As the recovered profiles so closely match the design parameters that were initially proposed, the fabrication process can be viewed as being highly reliable. This is definitely a consideration in favour of this fabrication technology. However, due to the small cross-section of the fabricated GRIN lenses, they are not particularly easy to use and may not be suitable for any practical purpose. This suggests that thicker lenses are needed, and that the deposition time required using the PECVD process might rule it out as being a commercially viable fabrication technique. Nonetheless, it has shown its worth as a process that can be used to successfully create one-dimensional parabolic refractive-index profiles, or indeed any profile shape that is required, and so provides the designers of silica waveguide devices a previously unrealised degree of freedom. Finally, the Phloxine B dye-doped PVP polymer could prove itself to be extremely useful as a diagnostic tool for fabricated multimode waveguide circuits, through its ability to allow easy observation of the evanescent fields close to the waveguide's boundary. The propagation characteristics of multimode devices are much harder to characterise than single-moded devices, and this dye-doped polymer could be used to determine, for example, the power splitting ratio of a multimode Y-junction, the power loss associated with a multimode X-junction or the transmission losses through a bent multimode guide. These are just a few possibilities that lend themselves to future investigation.
CHAPTER 5

Planar Multimoded Devices

The rule of the road is a paradox quite,
Both in riding and driving along;
If you keep to the left, you are sure to be right,
If you keep to the right you are wrong.

Henry Erskine 1746-1817,
The Rule of the Road.

5.1 Introduction

Continuing the theme of multimoded guided wave optics, the remainder of this thesis will examine various other multimoded waveguides and devices. However, unlike the previously examined GRIN lens – with its parabolic refractive-index profile – the multimoded optical waveguides and devices that will be considered here will be restricted to those with step-index profiles, due to the anticipation of fabricating planar devices using PECVD processing. The following chapters will examine the power transmission characteristics of simple passive multimoded devices, including bent multimode waveguides, using both ray optical and numerical techniques for their analysis. In this way we can obtain a quantitative comparison between the two different methodologies, which will be shown to confirm the accuracy of the ray tracing technique.

Back in the early 1970’s, fabrication of low-loss single-mode communications fibres had not been perfected and the first optical transmission experiments and subsequent systems had to rely on propagation along multimode fibres. What further enhanced the appeal of multimode fibres was the relative ease of launching light into these fibres and splicing them, as compared with single mode fibres. These multimode fibres operated in the first telecommunications window, with excitation by laser diodes or light emitting diodes, using wavelengths around 850nm. With relatively large core diameters, of the order of 50 – 100µm, these fibres guided a large number of modes with correspondingly large \( V \) values, typically in the range of 50-100. This early transmission work stimulated a significant body of theoretical
research work on the analysis of the excitation and the propagation of light along passive loss-less multimode waveguides and fibres. A large part of this work relied upon simple classical ray-tracing techniques that were able to determine, in particular, pulse dispersion effects, which were vital for designing the first transmission systems. However, as events unfolded, multimode fibres were superseded in favour of single-mode fibre waveguides, with a much larger bandwidth, and devices that were subsequently developed. This development occurred because inter-modal dispersion could thus be eliminated, and the problems associated with launching into and splicing small fibre core diameters, necessary for single mode transmission, were overcome. During the last two decades or so, long-distance optical communications has evolved towards the almost exclusive use of single-mode transmission. This has seen single-mode fibre not only dominate though its use in trunk lines, but also in inter-exchange networks. As the optical communications sector is by far the greatest market for integrated optical circuits and devices, it is then understandable that interest in single-moded technologies totally outweighed the activities of the earlier work developed for multimoded devices. It is not surprising then that work in developing passive multimoded components was put aside and not pursued further.

Despite this impasse, multimode integrated optical technologies were seen by some [70] to possess a possible application in fields that were not foreseen in the early stages of its development. Multimode integrated devices lend themselves to performing simple sets of elementary optical functions monolithically at a cost which falls dramatically as the number of identical components increases, in contrast to all-fibre and micro-optic components [70]. Furthermore, the issue of fibre compatibility is not such as significant an issue as an increasing number of optical applications are becoming fibreless, such as the development of planar optical sensors [71, 72]. Hence, the issues surrounding fibre compatibility have been replaced by a far more elementary set of requirements, namely, that the planar optical waveguide trap most of the power emitted from an LED, and propagate this light with minimum loss along the waveguide circuit, comprising straight segments, junctions and bends. Using multimode optical waveguides to route the light emitted from LEDs does not allow for a high level of complexity in the performed functions, but will confine the light along long paths, redirect beams using bends, and split or recombine light paths using Y-junctions and X-junctions to keep optical circuits compact and carry more light power than their single moded counterparts. These are just a few examples of the possible uses of multimode waveguides and devices, replacing the need for discrete lenses, mirrors and beam splitters if the same functions are to be performed by free-space components.
Furthermore, recent interest in multimode propagation has been revived by the application of multimoded fibres in short-distance networks, where dispersion and attenuation effects are not a significant issue. There is also an increasing interest in the use of polymers as the transmission medium for multimode waveguides and devices, due to the very high accuracy and potentially low-cost fabrication of polymeric waveguides and that it takes no more time to make a multi-, as opposed to, a single-mode waveguide. In fact, this has already developed to the stage where the moulding and casting techniques for fabricating passive polymer waveguiding structures have already been demonstrated [4, 5]. Accordingly, it is timely to investigate propagation through simple multimode devices, such as X- and Y-junctions and couplers which will be required in such compact planar networks, using both ray-tracing and numerical techniques for comparative purposes, including the eventual fabrication of these devices using PECVD and reactive ion etching technologies [50].

5.2 Multimode Waveguide Excitation

The question of the excitation of a multimode waveguide is an important one, as multimode devices, unlike single-moded devices, are highly sensitive to the launch conditions. For instance, the branching ratio of a Y-junction is not always unity, as it depends on the distribution of the modes entering the junction. Fortunately, this problem can be avoided by ensuring that the source of illumination for these devices is a diffuse (or Lambertian) source [21, Sectn. 4-1], which is the most typical source in practice, and approximates the output from an incoherent light-emitting diode (LED). It has an intensity distribution given by

\[ I(\theta_0) = I_0 \cos \theta_0, \quad -\pi/2 \leq \theta_0 \leq \pi/2, \]

(5.1)

for light emitted at angle \( \theta_0 \) to the normal of each source element, whose area emits light in all forward directions. When using such a source, we assume that all the bound ray directions carry equal power for the small bound-ray angles involved in the analysis.

In integrated optical waveguide circuits, the LED surface will most likely be positioned against the input waveguide port where the excitation efficiency \( I_e \) has previously been determined to be [73]

\[ I_e = \frac{A_e}{A_s} \left( \frac{NA}{n_{co}} \right)^2, \]

(5.2)
where $A_c$ and $A_s$ are the contacting areas of the waveguide core and of the LED source, respectively, and $n_{co}$ and NA are respectively the core-refractive index and numerical aperture of the waveguide.

In such an arrangement, overfilling the numerical aperture and the cross-section is preferable to underfilling. This is an important consideration, as it affects the stability of the optical functions performed by the waveguide elements further down, such as branching elements and bends. Overfilling the NA and core cross-section ensures that all rays in the waveguide cross-section are equally excited, avoiding the occurrence of a bundle (or bundles) of rays in the core cross-section from being preferentially excited [70].

5.3 Ray Tracing

Previously, ray tracing has been used to examine a variety of multimode devices. Ray tracing techniques have established the optimum conditions for achieving a 1:1 branching ratio for multimode Y-junctions [74] and the modal distribution of the output [75]. It has also been shown that the choice of the branching angle can be used to selectively transmit the waveguide modes into the output arms [76]. Furthermore, approximate asymptotic expressions for the power transmission through multimoded X-junctions have also been derived [77]. However, our aim is to obtain accurate expressions for the power transmission characteristics of a variety of multimoded optical devices, which as yet has not been done. As modal analysis becomes increasing cumbersome as $V$ increases, ray tracing proves to be the most convenient and straightforward means of obtaining those results.

5.3.1 Source of Illumination

The source of illumination is assumed to be a diffuse light source, which is well approximated by a light emitting diode. Of the light launched into the guide, all the unbound refracting rays rapidly leak out of the guide, leaving only the bound rays in the guide. The steady state input into the guide is therefore taken to consist of all possible ray paths at each point in the core cross-section, occupying a wedge of half-angle equal to the complementary critical angle $\theta_c$, which is defined by:

$$\theta_c = \sin^{-1}\left(\left(\frac{n_{cl}^2 - n_{co}^2}{n_{co}}\right)^{1/2}\right) \approx \left(\frac{n_{cl}^2 - n_{co}^2}{n_{co}}\right)^{1/2} = \sqrt{2\Delta},$$

(5.3)
where $\Delta \ll 1$ is the relative index difference, which is assumed to be small (i.e. weak-guidance).

5.3.2 Total Power

One consequence of weak-guidance is that all the bound rays propagate in directions close to the waveguide axis. For a Lambertian source, each ray increment, $d\theta$, carries approximately the same power for $-\theta_c \leq \theta \leq \theta_c$, which is evident from equation (5.1), when $\theta_c$ is small. Hence the total power carried in the guide is given by only considering the bound rays. By referring to the geometry in figure 5.1, the total power, $I_T$, is calculated as:

$$I_T = \int_{-\rho}^{\rho} dx \int_{-\theta_c}^{\theta_c} d\theta = 4\rho\theta_c. \quad (5.4)$$

The only rays which enter the device and contribute to the power output are those which make it into an output port and then remain guided by it. These considerations will determine expressions for the limits of integration for the angles and positions of the rays that are appropriate for each device. Once the integral has been evaluated, we obtain the remaining power output from the device, $I_R$, or the corresponding normalised quantity $I_R/I_T$. This calculation also determines the excess loss of the device.

5.4 Finite-difference BPM Analysis

Until recently, the only useful alternative approach to ray-tracing for quantifying the propagation in multimode waveguides relied on evaluating a superposition of all the propagating electromagnetic bound modes. For a square-core waveguide the number of such modes increases as $V$, as this method becomes increasingly cumbersome as the $V$ value becomes larger. However, with the advent of beam propagation methods (BPM), in particular the finite-difference BPM, which solve the governing scalar wave equation directly as a function of distance along the waveguide, it is thereby possible to avoid modal considerations altogether. Hence, in this way, a direct comparison can be made between ray-tracing and FD-BPM solutions. The geometry and coordinates for the step-profile planar waveguide can be seen in figure 5.2.
Figure 5.1: A diffuse source which is exciting all the bound rays of the guide equally within the range of angles $-\theta_c \leq \theta \leq \theta_c$.

Figure 5.2: Geometry, axes and parameters for the step-profile slab waveguide.
The finite-difference approach, outlined in some detail in section 2.2.4, will be used to examine the multimoded waveguides and devices in the following chapters. Special care was taken to account for radiation leakage, by using transparent boundary conditions (section 2.2.5) together with absorbing regions, in order to minimise the reflections at the domain boundaries. This eliminates uncertainty about whether the power within the core consists of any previously rejected component of the field that has subsequently reflected off the domain boundary and passed back into the waveguide core. Furthermore, care has been taken in ensure that the high spatial frequencies present in the high-order modes of the waveguides are accurately represented.

5.4.1 Input Field

It is not immediately obvious what input field conditions should be chosen for the BPM simulations to ensure that they are equivalent to the steady state input from an Lambertian light source. It should be noted that a totally incoherent source and a Lambertian source are strictly not the same [78]. However for a waveguide that obeys weak guidance, the efficiency with which these two sources excite the modes of the waveguide are to a good approximation the same.

Our approach is to first consider an extended coherent source, that is assumed to consist of the superposition of equal amplitude plane waves incident upon the end-face of the waveguide.

Suppose one of these normalised plane waves, \( \phi_p \), is incident in a uniform medium of index \( n_{cl} \) upon the end-face of the slab waveguide of half-width \( \rho \) at an angle \( \theta \) to the z-axis, and is expressed as

\[
\phi_p = \frac{1}{\sqrt{2\rho}} e^{i k n_{cl} e' x'} \cap \left( \frac{x'}{\rho} \right) + \text{c.c.,} \tag{5.5}
\]

where \( \cap (x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.6} \)

and c.c. represents the complex conjugate. This is shown schematically in figure 5.3, where the primed and unprimed coordinates are related by the following coordinate transformation:

\[
z' = x \sin \theta + z \cos \theta;
x' = x \cos \theta - z \sin \theta. \tag{5.7}
\]
Figure 5.3: A schematic representation of a plane wave incident on the end-face of a waveguide.

Substituting these expressions into equation (5.5) gives rise to the following

$$\phi_p = \sqrt{\frac{2}{\rho}} \cos \left\{ kn_{cl}(x \sin \theta + z \cos \theta) \right\} \cap \left( \frac{x \cos \theta - z \sin \theta}{\rho} \right)$$

(5.8)

where $-\theta_c \leq \theta \leq \theta_c$ for bound mode excitation, and, since $\theta_c$ is assumed small, we can apply the small angle approximation to equation (5.8) to yield:

$$\phi_p = \sqrt{\frac{2}{\rho}} \cos \left\{ kn_{cl}(x \theta + z) \right\} \cap \left( \frac{x}{\rho} - \frac{z}{\rho} \right).$$

(5.9)

The normalised input field $\phi_p$ excites the $j$th mode with an amplitude given by the overlap integral $\langle \phi_p | \psi_j \rangle$ for any modal field $\psi_j$. Furthermore, if the modes of the waveguide $\psi_j$ are assumed to be far from cut off, i.e. $V \equiv W \gg 1$, the asymptotic form for $U_j$ is given by [21, Sectn. 12-3]:

$$U_j \equiv \rho(k^2 n_{co}^2 - \beta_j^2)^{1/2} \approx (j + 1) \frac{\pi}{2} \left\{ \frac{V}{V + 1} \right\}$$

(5.10)

where $j = 0, 1, 2, \ldots$, and the modal fields have been normalised to unit normalisation. Provided the multimode waveguide has a relatively large $V$-value, we can make the approximation of neglecting the field in the cladding, i.e.

$$\int_{-\infty}^{\infty} |\psi_j^2| \, dx \approx \int_{-\rho}^{\rho} |\psi_j^2| \, dx = 1.$$  

(5.11)
The expressions for the modal fields in the core ($x \in [-\rho, \rho]$) are:

$$
\psi_j = \begin{cases} 
\sqrt{\frac{2U_j}{\rho(2U_j + \sin(2U_j))}} \cos(U_j x / \rho) \approx \frac{1}{\sqrt{\rho}} \cos(U_j x / \rho) & \text{Even modes,} \\
\sqrt{\frac{2U_j}{\rho(2U_j - \sin(2U_j))}} \sin(U_j x / \rho) \approx \frac{1}{\sqrt{\rho}} \sin(U_j x / \rho) & \text{Odd modes.}
\end{cases}
$$

(5.12)

Using equations (5.9) and (5.12) to evaluate the overlap integral

$$
\langle \phi_p | \psi_j \rangle = \int_{-\infty}^{\infty} \phi_p^* \psi_j \, dx,
$$

(5.13)

where $\phi_p^*$ denotes the conjugate wavefunction, and results in the following expression

$$
\langle \phi_p | \psi_j \rangle = \begin{cases} 
2\sqrt{2} \left( \frac{\sin(\rho kn_1 \theta + U_j)}{\rho kn_1 \theta + U_j} + \frac{\sin(\rho kn_1 \theta - U_j)}{\rho kn_1 \theta - U_j} \right) & \text{Even modes,} \\
0 & \text{Odd modes,}
\end{cases}
$$

(5.14)

demonstrating the point that a coherent diffuse light source only excites the even modes of the waveguide. This must be the case as the excitation is totally symmetric. The accuracy of this expression decreases for higher order modes, as $U_j$ becomes larger, and an increasing proportion of field for these modes extends outside the core region.

### 5.4.2 Modal Excitation

We obtain the result for the diffuse light source by adding coherent excitation at different angles as shown above. Hence, the amplitude coefficient for each excited even mode is found by integrating $\langle \phi_p | \psi_j \rangle$ over $[-\theta_c, \theta_c]$. This ensures that the phase factor $e^{i\phi_j} \neq e^{i\phi_{j+1}}$ at different points $j$ along the input face of the waveguide. In doing so, the results are, to a good approximation, that all the even modes are excited equally, which corresponds to equal excitation of all the bound rays of the planar waveguide.

However, how does this compare with a totally or partially incoherent diffuse source? Thus we allow for phase differences between different plane wave directions at the input. From a ray optical perspective it makes no difference, and all the bound rays of the planar waveguide are equally excited. A totally incoherent diffuse source will result in the excitation of all the bound modes of the waveguide with approximately equal efficiency [79]. But it should be noted, that it says nothing about the relative phases of each mode if they are all excited. A combination of modes with
random relative phases at the input (i.e. $\psi(0) = \sum_j \psi_j \exp(i\eta_j)$) still corresponds to the excitation of the whole range of $\beta$ values in ray terms (i.e. $kn_{cl} \leq \beta \leq kn_{co}$) but can result in an asymmetry in the core of the waveguide. Whereas, equal excitation of all the even modes of the planar waveguide tells us that the light intensity along the cross-section normal to the waveguide axis must be symmetric. So for example, in the case of a symmetric Y-junction splitter, equal excitation of all the modes can result in an unequal split of power between the output arms. However, for $V \gg 1$ there are sufficient modes ($M = 2V/\pi$) to ensure that the split is near 50:50, and, in any case, different combinations of initial random phases would produce slightly different output condition, as is the case with an LED where the output at each position has a phase which varies randomly with time. It turns out in practice that, with enough runs with different phases for each of the modes the overall result tends to average out to the results obtained using either the excitation of all the even or all the odd modes.

It should be stressed that various electromagnetic situations are accurately modelled with the same ray input conditions. This occurs because the phase relationships between modes and their discrete nature are ignored in the ray result. Hence, the ray results always become exact as $V \to \infty$.

In order to test the validity of this scheme, we present results involving equal excitation of all even modes and all odd modes independently; the results clearly show that these conditions lead essentially to the same qualitative behaviour, and, consequently, they show that the most important aspect is to have a sufficient sampling of modes to mimic the continuous behaviour of ray analysis. In these cases, the resulting field possesses either odd or even symmetry and mimics a Lambertian source. However, as the LED is not coherent, the chosen phase for each of the modes that form the input field could be purely arbitrary or random. As explained above, this situation is still accurately modelled by using the whole range of rays.

Rather than use the approximate expressions for the odd- and even-TE modal field components of equation (5.12), only the exact expressions for TE modal field components will be used to calculate the input conditions for the FD-BPM simulations. The expressions for the exact even TE modal fields are given by [21, Sectn.
The expressions for the exact odd TE modal fields are

\[
\psi_j = \begin{cases} 
\frac{\cos(U_j X)}{\cos U_j} & \text{if } |X| \equiv |x/\rho| \leq 1, \\
\frac{\exp(-W_j |X|)}{\exp(-W_j)} & \text{if } |X| > 1,
\end{cases}
\] (5.15)

and the expressions for the exact odd TE modal fields are

\[
\psi_j = \begin{cases} 
\frac{\sin(U_j X)}{\sin U_j} & \text{if } |X| \equiv |x/\rho| \leq 1, \\
\frac{X \exp(-W_j |X|)}{|X| \exp(-W_j)} & \text{if } |X| > 1.
\end{cases}
\] (5.16)

Furthermore, the values for \( U_j \) and \( W_j \) are obtained by numerically solving the eigenvalue equations

\[
W = U \tan U; \text{ and } W = -U \cot U,
\] (5.17)

for either the even or odd modes, respectively, with \( V^2 = U^2 + W^2 \).
6.1 Excess Loss

An important issue arising from the design of integrated optical circuits relates to the losses incurred by waveguide junctions. As the propagation losses in buried channel waveguides have been reported to be as low as 0.026 dB/cm [80], the excess loss incurred using junctions should be minimal and not contribute significantly to the overall loss. From a designer's perspective, X-junctions are necessary to help overcome the topological constraint that currently limits optical circuitry to a single plane [32, Sectn. 13]. Furthermore, as optical networks are currently becoming increasingly sophisticated, possessing many channels and devices, there is obviously a requirement for an X-junction with very low loss both, to keep these networks compact, and to ensure that propagation through a series of X-junctions results in minimal overall loss.

Restrictions placed on the size of optical circuits mean that not all X-junctions are likely to be at right-angles. With this in mind, it is important to be able to quantify the excess loss the X-junction is likely to introduce, as a function of the intersecting angle of the junction. In this way, we can place limits on the range of angles of intersecting junctions that possess reasonably low excess losses.

Previously, expressions for the excess loss have been calculated for both single mode right-angle X-junctions [81, 82], and single mode acute-angle X-junctions [83]. Multimode X-junctions easily lend themselves to a ray-optical analysis of the excess-
losses associated with the junction. An approximate closed-form expression for the excess losses of an acute-angle multimode X-junction has already been obtained [77]. As we shall see later, this approximation relies on neglecting a proportion of the rays entering the junction region, and this approximation eventually breaks down.

In this chapter, we will calculate closed-form expressions for the excess loss for both right- and acute-angled multimode X-junctions using ray-analysis, accounting for all guided rays entering the junction. The predicted excess loss for both the right- and acute-angled junctions will be compared with finite-difference Beam Propagation Method (FD-BPM) calculations, including a comparison with the approximate closed form solutions mentioned above. We also consider design strategies for reducing excess loss by raising the value of the refractive index in the intersecting junction region.

6.2 Physical Model

A plane view of both the right- and acute-angle X-junctions are shown in figure 6.1(a) and (b), respectively. Both of these X-junctions consist of intersecting identical multimode waveguides of core width $2\rho$, with a step-profile with uniform core index $n_c$ surrounded by an infinite cladding of uniform index $n_{cl}$. The region where the two waveguides intersect also consists of a uniform core index $n_c$.

The arrangement of the acute-angle junction in figure 6.1(b) is similar to the right-angle junction of figure 6.1(a), and the angle $\gamma$ gives the departure from the right angle intersection; furthermore, the angle $\gamma$ is in radians. In either case, power is input from the left, enters the device through port 1, propagates across the intersection between $AO$ and $AO'$ and spreads into the side arms of the junction due to the lack of guidance in this region. A portion of the power propagating through the intersecting junction region is then recaptured by the outgoing port 2.

As the light source is assumed to be diffuse (section 5.3.1), the steady-state input into port 1 consists of the equal excitation of all the rays in the core cross-section within the cone of angles $-\theta_c \leq \theta \leq \theta_c$, where $\theta_c$ is defined by equation (5.3). On the other hand, the input conditions for the FD-BPM simulations will consist of equal excitation of all the even modes in port 1, and also runs with all the odd modes for comparative purposes.
The simulated runs were performed using a wavelength of 632.8nm with step sizes of $\delta x = 2nm$ and $\delta z = 0.1\mu m$ in the computational domain. These step-sizes were chosen on a trial-and-error basis, so that the results from the FD-BPM converge to three significant figures.

The core width, $2\rho = 20.0\mu m$, with core and cladding refractive indices of 1.46 and 1.44, respectively, in each simulated run (i.e. $V \cong 24$) with a computational domain that is $80\mu m$ wide and $1mm$ in length. The junction region is placed $100\mu m$ away from the start of the computational domain and the excess loss is calculated from the power captured by port 2. The power is calculated by integrating the field over waveguide core cross-section, i.e. $-\rho \leq x \leq \rho$.

6.2.1 Right-Angle X-junctions

Before tackling the acute-angle X-junction, we first consider the behaviour of the right-angle X-junction. The result obtained from this analysis can be used to check the expressions obtained for the acute-angle X-junction in the limit as $\gamma \to 0$.

In order to determine the excess loss from the rays propagating across the intersection of the two waveguide cores, from port 1 to port 2, it is sufficient to trace the ray paths emanating from each position along the line segment $\overline{AO}$ in figure 6.1(a). Of these ray paths, some will pass into the core of the intersecting waveguide either above $\overline{AA'}$ or below $\overline{OO'}$ and be lost. The summation of power in all these "lost" rays determines the excess loss due to the junction itself, as the rays that reach the core of port 2 must necessarily remain bound.

By referring to figure 6.2, we can see that, over the cross-section $\overline{AO}$, the lossy rays at A subtend angles to the $z$ axis in the range $0 \leq \theta \leq \theta_c$. However, at a position L only the ray with angle $\theta_c$ is lost. Hence, between L and O, all the upward pointing rays are captured by port 2. By symmetry, the same number of downward propagating rays are lost.

The $x$ axis in figure 6.2 is parallel to the cross-section $\overline{AO}$, where $x = 0$ at O and $x = 2\rho$ at A. Therefore, the position of L has coordinate

$$x = 2\rho(1 - \tan \theta_c),$$

(6.1)
Figure 6.1: Schematic diagrams of (a) the right-angle X-junction and (b) the acute-angle X-junction.
and the range of lossy rays at $x$ between $A$ and $L$ is given by

$$\tan^{-1}\left(\frac{2\rho - x}{2\rho}\right) \leq \theta \leq \theta_c. \quad (6.2)$$

Therefore, the ratio of excess loss to the total incident power in port $1$, for all the rays propagating across the junction, is given by

$$P_{\text{loss}} = \frac{2}{\tan^{-1}(2\rho_2 - x)} \frac{d\theta}{dx} = -\frac{\ln|\cos \theta_c|}{\theta_c} \approx \frac{\theta_c}{2} \quad \text{for} \quad \theta_c \ll 1. \quad (6.3)$$

Interestingly this result is independent of the guide width and the excess loss and varies linearly with the critical angle $\theta_c$.

The corresponding loss in dB is given by

$$-10 \log_{10}\left\{1 - \frac{\theta_c}{2}\right\} \approx 2.17\theta_c \quad (6.4)$$

on changing base and expanding the logarithm. To quantify the result, table 6.1 evaluates the loss for various values of the relative index difference $\Delta$. In a concatenation of X-junctions, the accumulated loss could become quite significant if the relative index difference is not kept very small. However, there are strategies for further reducing excess loss, which are outlined in section 6.5.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$\theta_c$</th>
<th>Excess loss dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.10</td>
<td>0.22</td>
</tr>
<tr>
<td>0.004</td>
<td>0.09</td>
<td>0.19</td>
</tr>
<tr>
<td>0.003</td>
<td>0.08</td>
<td>0.17</td>
</tr>
<tr>
<td>0.002</td>
<td>0.06</td>
<td>0.14</td>
</tr>
<tr>
<td>0.001</td>
<td>0.04</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 6.1: Excess loss (dB) calculated by equation 6.4 for a multimoded right-angle X-junction for a range of $\Delta$. 

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6.2.2 Acute-Angle X-junctions

The geometry of the acute-angle X-junction being considered is shown schematically in figure 6.1(b), which illustrates the intersection of two identical waveguides at an arbitrary angle \( \gamma \), which is a measure of the departure from the right angle situation \( (\gamma = 0) \). Furthermore, we would expect intuitively that the loss from this junction would gradually increase as the angle \( \gamma \) increases from 0 because the length of the unguided region \( \overline{AA'} \) increases with increasing \( \gamma \).

In a same fashion as before, the excess loss caused by propagating across the intersection is determined by tracing the ray paths emanating from each position along the line segment \( \overline{AO} \). Similarly, a proportion of these rays will be captured by the outgoing port 2, but some will pass through \( \overline{AA'} \) or \( \overline{OO'} \) and are lost. The expression for the excess loss will be the summation of the power in all the lost rays.

Propagation in port 1 of the X-junction is described by all possible ray paths in the core cross-section \( \overline{AO} \) occupying a cone for which half its angle is equal to \( \theta_c \). The length of the line segment \( \overline{AO} \) is given by \( 2\rho \sec \gamma \), so that the length of \( \overline{AO} \) depends on the angle \( \gamma \). The rays emanating from port 1 into the junction can be

![Figure 6.2: The ray geometry and coordinates used for determining the excess loss for rays propagating through the right-angle X-junction.](image-url)
split into upward and downward pointing rays, as illustrated in figures 6.3(a) and (b), respectively.

For the upward pointing rays at A, all the rays with angles in the range $0 \leq \theta \leq \theta_c$ are lost. At some position, L, below A (see figure 6.3(a)) only the ray with angle $\theta_c$ is lost. In a normalised coordinate system where O is at the origin and A is at 1, the position of L is given by:

$$
L = \begin{cases} 
1 - \sin \theta_c \sec(\theta_c + \gamma) & \text{if } 0 \leq \gamma \leq \pi/2 - 2\theta_c, \\
0 & \text{if } \pi/2 - 2\theta_c < \gamma \leq \pi/2.
\end{cases} \quad (6.5)
$$

Hence rays below L are not lost and some proportion of the rays entering the junction above L are lost. Now in between A and L at X, the range of angles which are lost are bounded by $\vartheta(X) \leq \theta \leq \theta_c$ where

$$
\vartheta(X) = \frac{(1 - X) \cos \gamma}{(1 + \sin \gamma) - X \sin \gamma}. \quad (6.6)
$$

Similarly for the downward pointing rays at O, all the rays with angles in the range $0 \leq \theta \leq \theta_c$ are lost. Furthermore, at some position $\tilde{L}$, below A (see figure 6.3(a)), only the ray subtending the angle $\theta_c$ below the horizontal is lost. The position of $\tilde{L}$ is given by

$$
\tilde{L} = \sin \theta_c \sec(\theta_c - \gamma). \quad (6.7)
$$

Below $\tilde{L}$, only rays with angles between $\tilde{\vartheta}(X)$ and $\theta_c$ are lost, where

$$
\tilde{\vartheta}(X) = \frac{X \cos \gamma}{1 - X \sin \gamma}. \quad (6.8)
$$

Hence the excess power loss for the X-junction is given by

$$
P_{\text{loss}}(\gamma, \theta_c) = \frac{1}{2\theta_c} \left( \int_{\tilde{L}}^{1} \int_{\vartheta(X)}^{\theta_c} \ d\theta dX + \int_{0}^{\tilde{L}} \int_{\vartheta(X)}^{\theta_c} \ d\theta dX \right), \quad (6.9)
$$
which, when evaluated, yields

\[
P_{\text{loss}}(\gamma, \theta_c) = \begin{cases} 
\frac{\cos \gamma}{2\theta_c \sin^2 \gamma} \ln \left| \frac{1}{1 - \tan^2 \theta_c \tan^2 \gamma} \right| + \frac{\sin \theta_c}{2} (\sec(\theta_c - \gamma) + \sec(\theta_c + \gamma)) \\
\quad + \cot \gamma \sin \frac{\theta_c}{2} (\sec(\theta_c - \gamma) - \sec(\theta_c + \gamma)), \\
0 < \gamma \leq \pi/2 - 2\theta_c
\end{cases}
\]

\[
P_{\text{loss}}(\gamma, \theta_c) = \begin{cases} 
\frac{\cos \gamma}{2\theta_c \sin^2 \gamma} \ln \left| \frac{(1 + \sin \gamma) \cos \theta_c \cos \gamma}{\cos(\theta_c - \gamma)} \right| + \frac{1 + \sec(\theta_c - \gamma) \sin \theta_c}{2} \\
\quad + \cot \gamma \sin \frac{\theta_c}{2} (\sec(\theta_c - \gamma) \sin \theta_c - 1), \\
\pi/2 - 2\theta_c < \gamma \leq \pi/2
\end{cases}
\]  

(6.10)

In order to check the validity of this result, it is important to examine the values for the excess loss when \( \gamma \) equals 0 and \( \pi/2 \). According to equation (6.10), the excess loss for a right angled X-junction is given by \( \lim_{\gamma \to 0} P_{\text{loss}}(\gamma, \theta_c) = \theta_c/2 \), which is the same as the result provided by equation (6.3). Finally, equation (6.10) predicts that the excess loss accounts for all the power incident into the junction, as \( \lim_{\gamma \to \pi/2} P_{\text{loss}}(\gamma, \theta_c) = 0 \).

6.3 Numerical Results

The two-dimensional FD-BPM described earlier (section 2.2.4) was used to determine the excess loss of the multimoded right- and acute-angle X-junctions. The input field consists of the summation of either normalised even or odd modal fields, with the resulting field being normalised as well. The power was calculated only over the core region of the guide at regular intervals throughout the simulation. A comparison of the excess loss calculated from both ray tracing and FD-BPM simulated runs can be seen in figure 6.4, and the agreement is very good. The greatest deviation between the FD-BPM and the ray results occurs for \( \gamma = 4\pi/9 \) (80 degrees), whereas all the other points lie quite close to the curve. Two simulations of the TE-polarised electric field propagating through an acute-angle X-junction are
6.2.2 Acute-Angle X-junctions

The geometry of the acute-angle X-junction being considered is shown schematically in figure 6.1(b), which illustrates the intersection of two identical waveguides at an arbitrary angle \( \gamma \), which is a measure of the departure from the right angle situation (\( \gamma = 0 \)). Furthermore, we would expect intuitively that the loss from this junction would gradually increase as the angle \( \gamma \) increases from 0 because the length of the unguided region \( \overline{AO} \) increases with increasing \( \gamma \).

In a same fashion as before, the excess loss caused by propagating across the intersection is determined by tracing the ray paths emanating from each position along the line segment \( \overline{AO} \). Similarly, a proportion of these rays will be captured by the outgoing port 2, but some will pass through \( \overline{AA'} \) or \( \overline{OO'} \) and are lost. The expression for the excess loss will be the summation of the power in all the lost rays.

Propagation in port 1 of the X-junction is described by all possible ray paths in the core cross-section \( \overline{AO} \) occupying a cone for which half its angle is equal to \( \theta_c \).

The length of the line segment \( \overline{AO} \) is given by \( 2\rho \sec \gamma \), so that the length of \( \overline{AO} \) depends on the angle \( \gamma \). The rays emanating from port 1 into the junction can be
Another finding from the BPM simulations was that the input field consisting of even modal fields produces a consistently lower excess loss for each of the simulated runs presented in figure 6.4. We believe this is because the simulated waveguide possesses 16 modes, and therefore the highest-order mode was an odd mode. Even though the field is tightly confined to the core, the input field, consisting of odd symmetric modes would have a slightly higher proportion of its field remain outside the waveguide's core compared with an input field consisting of even symmetric modes. We believe that for guides with higher V values and hence more modes this effect will become less noticeable.

Another finding from the simulated runs was that randomising the phase of the individual modes comprising the input field did not alter the simulation outcome. This is due to the different phase velocities of each of the modes in the input field, so that when the field arrives at the junction, the phase of each mode is different, whether or not they were all the same to start with at \( z = 0 \). The input field for the simulations, depicted in figure 6.5, comprises the modes which all have the same phase initially.

### 6.4 Comparison with an approximate solution

Previously, an expression for the excess loss of an acute-angle X-junction was obtained [77]. The analysis relied upon neglecting a proportion of rays entering the junction in order to simplify the analysis. The excess loss is calculated from the rays passing through \( \overline{A_0O} \) and neglects rays reflecting off the \( \overline{A_0A} \) plane and passing out through \( \overline{O_0} \) in figure 6.3.

The excess loss found by making this approximation is given by

\[
P_{\text{loss}}(\gamma, \theta_c) = -\frac{\ln |\cos \theta_c|}{\theta_c \cos \gamma} \approx \frac{\theta_c}{2 \cos \gamma}, \text{ for } \theta_c \ll 1,
\]

which is only valid provided \( \gamma < \pi/2 - \theta_c \). A comparison of the predicted excess loss from both equations, (6.10) and (6.11) is shown in figure 6.6. As the numerical aperture of the waveguides being considered increases, so too does \( \theta_c \), decreasing the range of angles for which equation. (6.11) holds true.
Figure 6.4: A comparison of the percentage excess loss of $P_{\text{loss}}(\gamma, \theta_k)$, for a waveguide with $\theta_k = 0.166$, between results obtained using ray tracing (solid line) and FD-BPM simulations (dots and triangles) as a function of angle $\gamma$ (in radians). Input fields, consisting of even or odd symmetric modes, are depicted by solid dots and triangles, respectively.
Figure 6.5: Finite difference-BPM simulations of an input field consisting of the even modes of the guide passing through an acute-angle X-junction with (a) $\gamma = 40$ and (b) $\gamma = 85$ degrees. The excess loss in (a) is approximately 10% of the incident power whereas in (b) it is as high as 60% of the incident power. Even at a junction angle of 40 degrees (a) the fields of the guide suffer little disruption, whereas the 85 degree junction causes a significant distortion of the fields propagating through the waveguide. Note that the z axis is compressed by approximately a factor of 10 compared with the x axis.
The two solutions are in excellent agreement and only begin to deviate at $\gamma \approx \pi/3$, with the approximate solution rapidly increasing and approaching infinity in the limit of $\gamma$ approaching $\pi/2$. In stark contrast to this, our expression predicts an excess loss of 100% in the limit when $\gamma \to \pi/2$.

![Figure 6.6: A comparison of the results obtained from our result, equation (6.10), represented by the solid curve and the approximate result, equation (6.11), represented by the dashed curve, as a function of angle $\gamma$ for a waveguide with $\theta_c = 0.166 (\approx 9.5^\circ)$.](image)

6.5 Strategy for Reducing Excess Loss

As we have shown, the excess loss across a right-angle X-junction with the same step profile as the arms can only be reduced by decreasing the relative index difference $\Delta$. However, to maintain the same multimodedness, a commensurate increase in the waveguide cross-section would be required, which may not always be appropriate from a practical viewpoint. Fortunately, there is a strategy for reducing excess loss without having to drastically reduce $\Delta$. 

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6.5.1 Superposed Profiles

Consider the step-profile, right-angle X-junction depicted in figure 6.7. The only difference between this and figure 6.1(a), is that the index in the intersecting region of the two waveguides has a refractive index which has been raised by the difference between the core and cladding indices, i.e. \( n_{co} - n_{cl} \), to produce an index of \( 2n_{co} - n_{cl} \), so that the relative index difference in this region is twice that in the adjacent four cores.

The effect of the increased index in this square region is to retain guidance of the rays across the intersection. If we examine refraction and reflection at the cross-section \( \overline{AA'} \) in figure 6.7, then Snell’s law is applied to determine the relationship between the incident ray angle \( \theta_i \) and the refracted ray angle \( \theta_t \), as shown in figure 6.8, hence

\[
n_{co} \sin \theta_i = (2n_{co} - n_{cl}) \sin \theta_t. \tag{6.12}
\]

Since both \( \theta_i \) and \( \theta_t \) are small, and \( \Delta \approx (n_{co} - n_{cl})/n_{co} \), this expression reduces to

\[
\theta_t = \frac{\theta_i}{1 + \Delta}. \tag{6.13}
\]

Consequently, \( \theta_t < \theta_i \) and since \( 0 \leq \theta_i \leq \theta_c \) then all rays ray which are bound in the input port, remain bound in the intersection and are bound on transmission across the opposite interface of the intersection into output port 2.

There will be some loss of power from these rays due to reflection from the core-intersection interface, as indicated in figure 6.8. The loss due to rays reflecting at the two boundaries has been previously calculated to be [77]

\[
P_{loss} = \frac{(n_{co} - n_{cl})^2}{2n_{cl}^2} \tag{6.14}
\]

for a right-angled multimoded X-junction. With this modification to the junction region, the loss from a multimode X-junction is now of the same order as the loss from a single mode X-junction.
Figure 6.7: Right-angle X-junction with the raised index in the intersection.

Figure 6.8: Reflection and refraction at the interface between a core and the raised-index intersection.
6.5.2 Practical Realisation

The use of a raised intersection clearly reduces excess loss to negligibly small values, but introduces complexity into the fabrication procedure. In terms of PECVD fabrication, existing technology is based on the deposition or etching of uniform layers of material. To produce the raised index intersections adds complexity to the fabrication process, whereby additional deposition and etching would be required including the design of an extra mask. Plus there is also the consideration of having to align each mask.

However, there is an alternative approach which would avoid these additional fabrication steps altogether. If the X-junction were to be fabricated using a photosensitive core material, such as germanosilicate, the increased index in the intersection could be introduced by exposing just the junction to intense UV laser light, e.g. from an Eximer laser. This would have the effect of raising the index in the intersection and the abrupt change in figure 6.7 could be approximated by an adiabatic increase in index. Furthermore, this scheme for reducing the excess loss from waveguide junctions has been shown to work in practice [77].

6.6 Discussion

The results provided by our ray analysis are in good agreement with the FD-BPM simulations of the X-junctions. The close agreement strongly suggests that the choice for the input field serves as an appropriate modal representation of a diffuse light source. The results also show that the excess loss, to a good approximation, is constant for intersecting junction angles, $\gamma$, up to 40 degrees. This eliminates the restriction of ensuring that all multimode X-junctions should be at right-angles in order to minimise the excess loss from them.
Y-junctions

It is because nations tend towards stupidity and baseness that mankind moves so slowly; it is because individuals have a capacity for better things that it moves at all.

George Gissing 1857-1903,
*The Private Papers of Henry Ryecroft*, ‘Spring’, XVI.

7.1 Introduction

The symmetric Y-junction is a basic device in any lightwave circuit for splitting optical signals. In the case of the single-mode Y-junction [32, Chapt. 14], the signal is guaranteed to be split equally between the two output arms and also independently of wavelength as long as the device remains single-moded. Furthermore, the excess loss, or difference between the input power and the total power in the output ports is relatively small, typically around 0.3dB for splitting angles of less than one degree. Intuitively, we would expect the corresponding symmetric multimode Y-splitter to behave similarly, i.e. split light equally and independently of the wavelength, but the excess loss should be much smaller because the propagating light is much more tightly bound to the core.

If the single-mode Y-junction becomes asymmetric, with one arm differing from the other in, say, width, the fundamental mode propagating along the stem will be deflected entirely into the arm with the higher effective index, and no power will exit the smaller arm. It requires only a very small asymmetry, of the order of a few percent, for this to occur. By comparison, an asymmetric multimode Y-junction can split light into virtually any desired ratio between 0:1 and 1:0 through appropriate choice of arm widths. This aspect of splitting has been investigated elsewhere in the case of a low-power tap using both ray-tracing and BPM analysis [84].

As discussed in earlier chapters, we must take into account of the nature of the
source exciting the multimode Y-junction. In the symmetric case, the splitting ratio may not be unity because of its dependence on the mode distribution of the incoming light. Similarly, the excess loss from recombining light signals is also dependent upon the modal distribution.

The unequal splitting ratio can arise as a result of the waveguide excitation under-filling the numerical aperture, or by connecting Y-junctions in series [75]. Overcoming the first obstacle merely involves ensuring that the light source overfills the numerical aperture of the waveguide. The second can be overcome by choosing the optimum length between junctions connected in series to achieve a stable splitting ratio, and a mode scrambler has been investigated to minimise the modal dependence of the splitting ratio in this case [74]. These effects were investigated using a ray tracing technique to simulate the multimode branching waveguides and a phase space expression with the ray position and slope to describe the modal distribution of the output light [74].

Other investigations, utilising FFT-BPM simulations, have reported that the radiation losses are mode-dependent, and that they are higher for the even modes than for the odd modes in a branching waveguide with only a small number of TE-polarised modes [85]. Furthermore, it was found that the difference increases with the separation angle and with the guidance strength $V$. More recently, theoretical work has established that it is feasible to transmit a useful fraction of the incident optical power into the output arms of a symmetric Y-junction at large branching angles [76], and, further, that an appropriate choice for the branching angle of the Y-junction can be used to selectively transmit particular waveguide modes into the output arms.

Currently, interest in producing multimode Y-junctions is developing rapidly, and already polymeric multimode Y-junctions have been fabricated with 50:50 splitting ratios [5]. Multimode Y-junctions allow for more complex topologies than point-to-point links, and those fabricated from polymers are ideally suited for mass production. With these current developments in mind, it is therefore timely to develop models using ray tracing techniques to establish, more formally, the power transmission characteristics of simple symmetric multimode Y-junction splitters.

In this chapter, we establish closed-form expressions for symmetric multimode Y-Junctions, and with the additional flexibility of output arms having adjustable widths. These results are compared with those obtained from the aforementioned
FD-BPM (section 2.2.4) simulations.

7.2 Physical Model

A schematic plane view of the symmetric multimode Y-junction considered in this chapter is illustrated in figure 7.1. The slab waveguides forming the junction are comprised of a step-profile guide, with uniform core index \( n_{co} \) surrounded by an infinite cladding of uniform index \( n_{cl} \). The light enters the Y-junction from the left, through the waveguide labeled "input port". The width of the input port is \( 2\rho \), with the junction region commencing at the cross-section \( AB \). The output arms connected to the junction region have an angular separation of \( 2\Omega \) (in radians) and the width of each arm is \( w \).

A novel feature of this Y-junction is that the width \( w \) of both output arms is allowed to vary by simply adjusting the distance \( d \) in figure 7.1 between the start of the "Y" and the intersection of the two arms. This added flexibility should provide optical circuit designers with an added degree of freedom, in addition to the angular separation of the output arms. By geometry

\[
w = d \sin \Omega + \rho \cos \Omega.
\]  

(7.1)

For convenience, we define \( d_\rho \) to be the value of \( d \) when \( w = 2\rho \). Hence

\[
d_\rho = \frac{2 - \cos \Omega}{\sin \Omega} \rho
\]  

(7.2)

and we define a normalised value of \( d \) by setting

\[
D \equiv \frac{d}{d_\rho} = \frac{d \sin \Omega}{\rho (2 - \cos \Omega)}.
\]  

(7.3)

We can then express \( w \) in terms of \( D \) by

\[
w = \rho (2D + \{1 - D\} \cos \Omega)
\]  

(7.4)

so that \( w = 2\rho \) when \( D = 1 \) and the stem and arms have a common width. If \( D < 1 \), the arms will be narrower than the stem and approach \( \rho \cos \Omega \) as \( D \to 0 \), and, conversely for \( D > 1 \), the arm width increases monotonically beyond \( 2\rho \).

Once again, the source of illumination is taken to be a diffuse light source with equal excitation of all the rays in the core cross-section of the input port. Furthermore, the input condition for the FD-BPM simulations consists of equal excitation
of all the even modes, or all the odd modes or all the guided modes for reasons explained earlier. The case consisting of all the guided modes has been introduced to determine the effect this has on the power transmission through the Y-junction when the splitting ratio of the power emanating from the two output arms is not 1:1.

The core width of the input waveguide was fixed at 20.0\( \mu m \) for all the simulated runs, and the cases D = 1 and 3 were investigated. The core and cladding refractive indices were chosen to be the same as those used for the X-junction results, i.e. 1.46 and 1.44, respectively. The cross-section \( \overline{AB} \) was positioned 300\( \mu m \) from the start of all of the simulated runs to allow any radiated power to exit the domain of the Y-junction, and the power was calculated from the average power in the core of the output arms over the last 200\( \mu m \) of the computational domain. These simulations were performed at a wavelength of 632.8nm with step sizes of \( \delta x = 2\, nm \) and \( \delta z = 0.1\, \mu m \) in the computational domain. The size of the computational domains were varied in width in order to accommodate the junction, although the length in each case was kept fixed at 1\( mm \). Finally, with these values \( V = 24 \) for the input port, which consequently supports a total of 16 even and odd modes.

7.3 Power Transmission

7.3.1 Ray Analysis

Due to the symmetry of the device and the distribution of rays, it is sufficient to calculate the power emerging from one of the output ports and then double the result to obtain the total output power for the device. In this instance the output port under consideration is taken to be the top output port. The strategy here is to categorise the rays entering the junction region into three classes and calculate the power, \( I_i \), that each contributes to the output of the device. These three categories are:

- \( I_1 \rightarrow \) The upward pointing rays between O and B;
- \( I_2 \rightarrow \) The upward pointing rays between A and O;
- \( I_3 \rightarrow \) The downward pointing rays between A and O.

Each of these cases is illustrated in figure 7.2 with the geometry and coordinates used for the subsequent analysis. When calculating the power, it necessary to identify those rays entering the output ports that remain guided, as the optical axis of the output arm has been tilted by angle \( \Omega \). Rays that enter an output port will only remain guided if they are incident on the walls of the output port at angles
Figure 7.1: Schematic illustration of the geometry of the symmetric Y-junction considered for the analysis. Both $d$ and $\Omega$ are adjustable parameters which alter the width, $w$, of the output arms (equation (7.1)).
Figure 7.2: The ray geometry and coordinates used for determining the power transmission for the rays comprising (a) $I_1$, (b) $I_2$ and (c) $I_3$ propagating through the symmetric Y-junction in figure 7.1.
less than $\theta_c$ relative to the port’s axis.

### 7.3.2 Lossless Y-junctions

The rays passing through the cross-section $\overline{AB}$ which enter into the top output port are split up into the three parts defined above. The $x$ axis is parallel to the line segment $\overline{AB}$ and the normalised variable $X = x/\rho = 0$ at $O$ and $X = 1$ at position $A$. Similarly, $\tilde{X}$ is parallel to $\overline{AB}$, $\tilde{X} = 0$ at position $B$ and $\tilde{X} = 1$ at position $O$.

In the analysis, all the refracting rays entering the output arms are neglected as their power is rapidly lost from the core at successive reflections/refractions along the output ports. For the case when $\Omega < \theta_c$, the rays included in $I_1$ or $I_2$ remain bound, and hence all the rays in $I_1$ and $I_2$ contribute to the power in the output. Of the rays in $I_3$, the ray subtending the largest angle that enters the output arm occurs at $X = 1$ (figure 7.2(c)), and is given by

$$\theta_{max} = \tan^{-1}\left(\frac{\sin \Omega}{(2 - \cos \Omega)D}\right) \leq \theta_c \quad \text{(7.5)}$$

since $\theta_c$ is assumed small and in this instance $\Omega < \theta_c$. Applying the small angle approximation to equation (7.5) results in

$$\theta_{max} \approx \frac{\Omega}{D} H\left(\frac{\theta_c - \Omega}{D}\right), \quad \text{(7.6)}$$

where $H(x)$ is the Heaviside step function:

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases} \quad \text{(7.7)}$$

None of the rays in $I_3$ are lost from the output port provided that $\theta_{max} + \Omega \leq \theta_c$. Under these conditions none of the rays emanating from the input port into the junction arms are lost due to refraction, so this symmetric Y-junction is lossless provided that

$$\Omega \leq \frac{\theta_c D}{D + 1}. \quad \text{(7.8)}$$

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This is an important result from a design perspective, as the excess loss from the junction should be kept to a minimum. A plot of this result is given in figure 7.3, with the shaded region denoting the choices of $D$ and $\Omega/\theta_c$ giving rise to a lossless Y-junction. Note that when $D = 1$, this requires $\Omega \leq \theta_c/2$, i.e. the angle separating the arms is $\theta_c$.

Figure 7.3: A plot of the curve $\Omega/\theta_c = D/(D + 1)$, where the shaded region corresponds to the regime where the symmetric Y-junction in figure 7.1 is lossless. The dashed line shows the asymptote for $\Omega \to \theta_c$ as $D \to \infty$. 

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7.3.3 Lossy Y-junctions

Case 1: Power in $I_1$

When $\Omega < \theta_c$, the rays considered in $I_1$ are bound between $\bar{X} = 1$ and $\bar{X} = L$, as shown in figure 7.2(a), where $L$ is the normalised position where only the ray at angle $\theta_c$ enters the output arm. In this case, the expression for $L$ is derived from

$$\tan \theta_c = \frac{(1 - L)\rho}{d}, \quad (7.9)$$

where $0 \leq L \leq 1$. By directly substituting in the expression for $d$, equation (7.1), and making the appropriate small angle approximations, the position for $L$ is given by

$$L = \left(\frac{\Omega - \theta_c D}{\Omega}\right) H(\Omega - \theta_c D). \quad (7.10)$$

In fact this expression for $L$ holds for $0 \leq \Omega \leq 2\theta_c$, although the small angle approximation deteriorates as $\Omega$ increases. Of the rays between $A$ and $L$, the range of ray angles at position $\bar{X}$ entering the output arm is $(1 - \bar{X})\Omega/D \leq \theta \leq \theta_c$.

For $\theta_c < \Omega \leq 2\theta_c$, all rays with angles less than $\Omega - \theta_c$ are lost due to these rays refracting out of the junction arms. As a consequence of this, the minimum angle, $\theta_{\text{min}}(\bar{X})$, that a ray may have at position $\bar{X}$ and contribute to $I_1$ is

$$\theta_{\text{min}}(\bar{X}) = \frac{(1 - \bar{X})\Omega}{D} H\left(\frac{(1 - \bar{X})\Omega}{D} - (\Omega - \theta_c)\right) + (\Omega - \theta_c) H\left((\Omega - \theta_c) - \frac{(1 - \bar{X})\Omega}{D}\right). \quad (7.11)$$

Therefore the power contributed by the rays comprising $I_1$ within the range of branching angles $0 \leq \Omega \leq 2\theta_c$ is simply

$$P_1 = \int_L^1 \int_{\theta_{\text{min}}(\bar{X})}^{\theta_c} d\theta d\bar{X}. \quad (7.12)$$

Case 2: Power in $I_2$

Evaluating $I_2$ is far more straightforward, as all the rays contribute to the power output for $\Omega \leq \theta_c$, and only rays with angles between $\theta_c$ and $\Omega - \theta_c$ contribute to the power output once $\Omega > \theta_c$. Thus the power contributed by the rays in $I_2$ is

$$P_2 = \int_0^{\theta_c} \int_{(\Omega - \theta_c) H(\Omega - \theta_c)}^{\theta_c} d\theta dX. \quad (7.13)$$
Case 3: Power in $I_3$

The rays in $I_3$ that are captured by the output port are those between $0 \leq X \leq 1$ whose range of angles at position $X$ captured by the output port is

$$0 \leq \theta \leq \frac{X\Omega}{D} H\left(\theta_c - \frac{\Omega}{D}\right). \quad (7.14)$$

However, this holds only for $\Omega \leq \theta_c D/(D + 1)$. Beyond this limit, it is possible for a proportion of rays at $X$ to subtend angles larger than $\theta_c - \Omega$. In this case, those rays will refract out of the output port at the core cladding interface. Furthermore, it is obvious that once $\Omega > \theta_c$, there will be no contribution from the rays in $I_3$ to the output power. Hence those rays which form part of the transmitted power of the Y-junction are within the range of angles $0 \leq \theta \leq \xi(X) H(\theta_c - \Omega)$, where

$$\xi(X) = \begin{cases} \frac{X\Omega}{D} H\left(\left(\theta_c - \Omega\right) - \frac{\Omega}{D}\right) & \text{if } 0 \leq \Omega \leq \frac{\theta_c D}{D + 1}, \\ + \left(\theta_c - \Omega\right) H\left(\frac{\Omega}{D} - \left(\theta_c - \Omega\right)\right) & \text{if } \Omega > \frac{\theta_c D}{D + 1}. \end{cases} \quad (7.15)$$

Hence, the component of the transmitted power in $I_3$ is given by

$$P_3 = \int_0^1 \int_0^{\xi(X) H(\theta_c - \Omega)} d\theta dX. \quad (7.16)$$

7.3.4 Ray result

Obtaining the expressions for the power transmission is now simply a matter of evaluating equations (7.12), (7.13) and (7.16). The total power transmission for the symmetric Y-junction can be expressed as a fraction of the input power as

$$P(\theta_c, \Omega, D) = \frac{1}{2\theta_c} \sum_{i=1}^{3} P_i, \quad (7.17)$$

where the expressions for the $P_i$'s are provided below in equations (7.18) and (7.19), for $0 \leq \Omega \leq \theta_c$ and $\theta_c < \Omega \leq 2\theta_c$, respectively. A feature of this Y-junction is that no bound power enters the output ports once $\Omega \geq 2\theta_c$, and, conversely, the junction is lossless provided that $0 < \Omega \leq \theta_c D/(D + 1)$. 

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For $0 \leq \Omega \leq \theta_c$ we have

$$I_1 = \begin{cases} \frac{\theta_c - \frac{\Omega}{2D}}{\frac{\theta_c^2 D}{2\Theta}} & 0 \leq \Omega \leq \theta_c D, \\ \theta_c D < \Omega \leq \theta_c, \end{cases}$$

$$I_2 = \theta_c,$$

$$I_3 = \begin{cases} \frac{\Omega}{2D} \frac{(\theta - \Omega)(\Omega(D + 2) - \theta D)}{2\Theta} & 0 \leq \Omega \leq \frac{\theta D}{D+1}, \\ \frac{\theta D}{D+1} < \Omega \leq \theta_c, \end{cases}$$

while for $\theta_c < \Omega \leq 2\theta_c$ we find

$$I_1 = (2\theta_c - \Omega)D/2$$

if $0 \leq D \leq 1$,

$$I_1 = \begin{cases} \frac{\theta_c (D + 1) - \frac{\Omega^2 (D^2 + 1) + \theta_c^2 D^2}{2Nd}}{2\theta_c - \Omega} & \theta_c < \Omega \leq \theta_c D, \\ \theta_c D < \Omega \leq 2\theta_c, \end{cases}$$

$$I_2 = 2\theta_c - \Omega,$$

$$I_3 = 0.$$

### 7.4 Numerical Results

A comparison between the ray and FD-BPM results for a Y-junction with $D = 1$ and the one with $D = 3$ is given in figures 7.4(a) and (b), respectively. These cases correspond to Y-junctions with equal width stem and arms, and to arms which are significantly wider than the stem. The agreement between these two techniques is good, with the FD-BPM results demonstrating that using either the even or odd modes confirms the predicted ray result, viz that the Y-junction is lossless provided $0 < \Omega \leq \theta_c D/(D + 1)$.

Figure 7.4(a) also compares the transmission of an input field that excites all the modes of the guide equally. Even though this is not strictly equivalent to a diffuse light source because of the limited number of modes, it is interesting to note that
the summed output from the two output arms is in excellent agreement with the ray result. However, the branching ratio was found to depend heavily on the phase of the modes when they enter the junction region, as would be expected.

In practice, the phase on the field entering the junction can be altered by either changing the length of the input waveguide or by adjusting the phases of each of the modes that comprise the input field at the start of the simulation. It was found that, under the right conditions, it is possible to achieve equal powers in the two output arms, but in the simulations presented here, the split in power was found to be approximately in the ratio of 42:58, where the left output arm has the smaller output power.

The FD-BPM simulations of the fields propagating through the symmetric Y-junction for a variety of angles can be seen in figures 7.5, 7.6 and 7.7 for an input field comprised of even, odd and all bound modes (of which there are 16), respectively. In each case, no power was captured by the output arms when $\Omega = 2\theta_e$.

7.5 Discussion

The ray results given by equations (7.18) and (7.19) are in excellent agreement with the FD-BPM simulations conducted for input fields described by equal excitation of all the even, all the odd and all modes. The results show that for reasonably large branching angles, the symmetric Y-junction is still lossless and that the total output power of the device decreases linearly with increasing branch angle $\Omega$, and that the output power of the device is zero for $\Omega = 2\theta_e$. This can be contrasted with the single-mode Y-junction, where branching angles of more than 1 degree introduce a large excess loss.
Figure 7.4: Comparison of the results obtained using ray analysis (solid line) and the FD-BPM simulations for symmetric Y-junctions with (a) $D = 1$ and (b) $D = 3$, for the total power remaining in both the output arms as a function of branch angle $\Omega$. The FD-BPM simulations confirm the ray result that the junction is in both cases lossless for $\Omega < \theta_c/2$ for $D = 1$ and $\Omega < 3\theta_c/4$ for $D = 3$ and that no power is transmitted through the output arms for $\Omega \geq 2\theta_c$. 
Figure 7.5: FD-BPM simulations of the evolution of the input field described by equal excitation of all the even modes of the input port at a wavelength of 632.8nm. The input arm in these simulations is 300µm in length and 20µm in width, and the refractive indices of the core and cladding are 1.46 and 1.44, respectively. For these simulations the parameter \( D = 1 \), so that the width of each output arm is the same as the input arm. For \( \Omega < \theta_c/2 \) [(a) and (b)], the junction is lossless, and for \( \Omega = \theta_c \) [(c)] the excess loss is 25%. When \( \Omega = 2\theta_c \) [(d)] no power is captured by the output arms and fans out into the cladding between the two arms. Note the power is split equally between the two output arms.
Figure 7.6: *FD-BPM* simulations of the evolution of the input field described by equal excitation of all the odd modes of the input port, using the same waveguiding parameters in figure 7.5. Again the waveguide is lossless for $\Omega < \theta_c/2$ [(a) and (b)], with 25% excess loss when $\Omega = \theta_c$ (c), and there is no transmitted power in the output arms when $\Omega = 2\theta_c$ (d). The power in each output arm is equal, but the field distributions are different from those in figure 7.5.
Figure 7.7: FD-BPM simulations involving the equal excitation of all the guided modes of the input port, where $V = 24$ so there are 16 modes. These simulations are the same as those presented in figures 7.5 and 7.6, apart from the fact that the input field is different. The behaviour of the total output from the junction is the same as before, but the power is not split equally between the two output arms. For these simulations, the ratio of power in the output arms is approximately 42:58, where the left output arm captures less power than the right.
CHAPTER 8

Modified MMI Coupler

Quelqu’un disait d’un homme très personnel: il brûlerait votre maison pour se faire cuire deux œufs.

- Someone said of a great egotist: ‘He would burn your house down to cook himself a couple of eggs.’

Nicolas Chamfort 1741-1794, Charactères et anecdotes.

8.1 Introduction

Beyond using symmetric Y-junctions to split light signals in multimoded planar waveguide circuits, it is worth considering what other devices could conceivably perform the same function. A novel device that achieves this for single-moded waveguides is the Multimode Interference (MMI) coupler. This is a simple device that relies on self-imaging effects in a multimode waveguide to split or combine light signals used for single moded transmission [43, 86]. The principles of multimode self-imaging, which were previously discussed in section 2.3, rely on the formation of multiple copies of the single-mode waveguide’s field after entering a multimoded waveguide. At the cross-section along the multimoded section where these multiple images occur, single-moded waveguides are positioned to capture the output of each image. This behaviour is illustrated in figure 8.1 using FD-BPM simulations to propagate light through two different 3dB MMI couplers. This raises the question: are MMI couplers capable of performing useful tasks using multimode waveguides as opposed to single moded ones? Furthermore, can it function as a 3dB splitter and is it possible to use it to tap power off from the input light beam?

As yet there are no published works that have considered this approach of applying the MMI device to multimoded light pipes. In this chapter, ray tracing will be applied to obtain expressions governing the power transmission characteristics of the modified device, and these results will be compared with those from
Figure 8.1: FD-BPM simulations of a single input being split into two using a MMI coupler at a wavelength of 632.8 nm. With core and cladding refractive indices of 1.46 and 1.44, respectively, the input and output waveguides are 1.05 µm wide (V=2.5) with a multimode coupling region that is 20 µm wide (so V=24). The simulations illustrate how a waveguide (a) centred with respect to the multimode region and (b) off-centre, acts as a 3dB splitter. The multimode region is 500.55 µm long and commences at $z = 250$ µm in (a) and is 667.42 µm long and commences at $z = 100$ µm in (b).
finite-difference BPM simulations.

8.2 Physical Model

The multimode interference coupler examined here is a variation on "conventional" MMI couplers that have raised so much interest in recent years, and are currently being used in interferometric switches [87, 88], modulators [89] and ring lasers [90, 91]. Here, the single-mode input and output ports have been replaced with multimode input and output ports, as shown in figure 8.2.

The input and output ports attached to the coupler are all $2\rho$ wide with a gap, $2\varepsilon$, separating the two output ports. Hence, the width of the coupler region is $2(\rho + \varepsilon)$, and its length is $L$. The input/output ports and the coupling region are all comprised of step-profile guides of core index $n_{co}$ and cladding index $n_{cl}$. Light enters the coupling region from the left via the input port and spreads out into the wider waveguiding region. Depending on the length of the device, a proportion of the incident light will be captured by ports 1 and 2 where there is likely to be only a small amount of leakage, provided that the gap separating the output waveguides is small.

A diffuse light source is assumed to be the source of illumination, comprising the equal excitation of all the rays in the core cross-section or the input port. For the FD-BPM simulations, only equal excitation of all the even or all the odd modes will be considered here for reasons discussed earlier. For the simulated runs, the input and output waveguides are $20\mu m$ wide with a gap of $5\mu m$ separating the output ports. For these runs, the core and cladding refractive indices were 1.46 and 1.44, respectively, and the source of illumination was assumed to be a monochromatic source at 632.8nm. With this choice of parameter values, the input and output waveguides have 16 modes ($V = 24$) and the coupler region has 35 modes ($V = 54$). The step sizes used for the simulated runs were $\delta x = 2nm$ in the transverse direction and $\delta z = 0.1\mu m$ longitudinally, as shown in figure 8.6[(a) and (c)], with a computational domain that is $60\mu m$ wide.

8.3 Ray Analysis

Most of the insight so far gained into conventional devices – those with single-moded input and output ports – has been obtained using modal propagation analysis
Figure 8.2: Schematic of the “modified” MMI coupler with multimode input and output ports all of which are $2p$ wide. The length of the device is given by $L$ and the output ports are separated by $2\varepsilon$. The analysis here considers light entering the coupler region from the left through the input port and considers the output powers in both ports 1 and 2.

This scheme essentially deconvolves the input field into the modes of the coupler region. These modes are then propagated to the output waveguides where the resulting field is coupled with the modes of the output waveguides. However, as the input and output ports are now multimoded, modal propagation analysis becomes significantly more cumbersome, and ray tracing is therefore much better suited to analyse our “modified” MMI coupler.

The rays entering the modified MMI coupler in figure 8.2 are capable of reflecting off the top and bottom core/cladding boundaries of the coupler so that upward pointing rays at the input can conceivably leave via output port 2 rather than output port 1. The same argument applies to the downward pointing rays. In fact, depending on the length $L$ of the device, a proportion of the input rays will undergo several reflections before reaching the output end of the device. Based on these considerations, the rays entering the device split into two categories: upward pointing rays and downward pointing rays, as illustrated in figures 8.3 (a) and (b), respectively. The contribution of each to the output power in either port 1 or port 2 is considered separately in the following analysis.

The notation convention used here, denotes the power comprised of all of the upward pointing rays entering the primary image of output port 1 as $I_0$; i.e. those that have not undergone any reflections within the core/cladding boundaries, as shown in figure 8.3(a). The power in the subsequent images of output port 1 are
Figure 8.3: The ray geometry and coordinates used for determining the proportion of (a) upward and (b) downward rays entering either port 1 or port 2.
denoted $I_1, I_2, \ldots, I_i$ and so forth. The subscript referring to the number of reflections the rays have undergone in order to enter port 1. It follows then, that the total contribution of the rays initially directed upwards through the cross-section $\overline{AB}$, to the output power in port 1 is simply $I_0 + I_1 + \cdots + I_i$, such that $I_{i+1} = 0$. Similarly, the power in the images of port 2 are denoted $\Pi_1, \Pi_2, \ldots, \Pi_i$, and the total power comprised of upward pointing rays in port 2 is the sum of these terms. As the upward rays must reflect at least once in order to enter port 2, the term $\Pi_0$ is neglected as its value is obviously zero.

For the downward pointing rays the same applies; however, the notation is slightly different. The powers in the images of either output port (1 or 2) are now denoted by $I_0, I_1, \ldots, I_i$ and $\Pi_0, \Pi_1, \ldots, \Pi_i$ respectively, as shown in figure 8.3(b).

The power in any of the images of either output port (1 or 2) is calculated by the following integral

$$\int_0^{2\rho} \int_{\theta_{\text{min}}(x)}^{\theta_{\text{max}}(x)} d\theta \, dx,$$  \hspace{1cm} (8.1)

where the position of $x$ differs in figures 8.3 (a) and (b). The power calculated consists of the segment of ray angles at each position $x$, over the cross-section $\overline{AB}$, for rays entering the image of the output port being considered.

If the image of the output port is bounded by $x \in [P - 2\rho, P]$ at $z = L$, then the expressions for $\theta_{\text{max}}(x)$ and $\theta_{\text{min}}(x)$ are given by:

$$\theta_{\text{max}}(x) = \theta(P, x) H(\theta_c - \theta(P, x)) + \theta_c H(\theta(P, x) - \theta_c); \hspace{1cm} (8.2)$$

$$\theta_{\text{min}}(x) = \theta(P - 2\rho, x) H(\theta_c - \theta(P - 2\rho, x)) + \theta_c H(\theta(P - 2\rho, x) - \theta_c); \hspace{1cm} (8.3)$$

$$\theta(P, x) = \tan^{-1} \left( \frac{P - x}{L} \right), \hspace{1cm} (8.4)$$

and $H(x)$ is the Heaviside function defined by equation (7.7). By directly substituting equations (8.2) - (8.4) into equation (8.1), while also applying the small angle
approximation, the expression for equation (8.1) becomes:

\[ \int_0^{2p} \theta_c \, dx - \int_0^{2p} \theta_c \, dx = 0, \]

if \( L\theta_c \leq P - 4\rho, \)

\[ \int_0^{2p} \theta_c \, dx - \int_{P-2p-L\theta_c}^{2p} \left( \frac{P - 2\rho - x}{L} \right) \, dx - \int_0^{P-2p-L\theta_c} \theta_c \, dx \]

\[ = \frac{(P - 4\rho - L\theta_c)^2}{2L}, \]

if \( P - 4\rho < L\theta_c \leq P - 2\rho, \)

\[ \int_{P-L\theta_c}^{2p} \frac{P - x}{L} \, dx - \int_0^{P-L\theta_c} \theta_c \, dx - \int_0^{2p} \frac{P - 2\rho - x}{L} \, dx \]

\[ = \frac{8\rho^2 - (P - L\theta_c)^2}{2L}, \]

if \( P - 2\rho < L\theta_c \leq P, \)

\[ \int_0^{2p} \left( \frac{P - x}{L} \right) \, dx - \int_0^{2p} \left( \frac{P - 2\rho - x}{L} \right) \, dx = \frac{4\rho^2}{L}, \]

if \( L\theta_c > P. \)

The expression for \( P \) depends on which image of port 1 or port 2 the rays are entering, including whether the rays entering the output port are pointing upwards or downwards. The symbol \( P \) is then replaced with either \( P_{i}^{(1)} \) or \( P_{i}^{(2)} \) for the upper bound for the upward pointing rays entering the image of port 1 or port 2, respectively. Otherwise, \( P \) is replaced by \( \tilde{P}_{i}^{(1)} \) or \( \tilde{P}_{i}^{(2)} \), for the downward rays entering either port 1 or port 2, respectively. This is further illustrated in figures 8.3 (a) and (b).
Hence, the expressions for $P_i^{(1)}$, $P_i^{(2)}$, $\tilde{P}_i^{(1)}$ and $\tilde{P}_i^{(2)}$ are given by:

\begin{align*}
P_i^{(1)} &= 2\rho (i + 1 - 2[i/2]) + 2\eta[i/2], \text{ for } i = 0, 1, 2, \ldots; \\
P_i^{(2)} &= 2\rho \left(i - 2 \left[\frac{i - 1}{2}\right]\right) + \eta \left[\frac{i - 1}{2}\right], \text{ for } i = 1, 2, 3, \ldots; \\
\tilde{P}_i^{(1)} &= 2\rho (i + 1) + 2\eta \left[\frac{i + 1}{2}\right], \text{ for } i = 0, 1, 2, \ldots; \\
\tilde{P}_i^{(2)} &= 2\rho (i + 1) + 2\eta \left[\frac{i + 2}{2}\right], \text{ for } i = 0, 1, 2, \ldots,
\end{align*}

where $\eta = 2(\rho + \varepsilon)$, and $[x]$ is the largest integer $\leq x$.

Hence the ratio of the output power in port 1 to the power in the input port ($4\rho \theta_c$) is expressed as

\begin{equation}
\frac{I_{\text{Port}1}}{I_T} = \frac{1}{4\rho \theta_c} \left( \sum_{i=0}^{\infty} I_i \bigg|_{P=P_i^{(1)}} + \sum_{i=1}^{\infty} \tilde{I}_i \bigg|_{P=\tilde{P}_i^{(1)}} \right) 
\end{equation}

Similarly, the ratio of the output power in port 2 to the power in the input port is

\begin{equation}
\frac{I_{\text{Port}2}}{I_T} = \frac{1}{4\rho \theta_c} \left( \sum_{i=1}^{\infty} \Pi_i \bigg|_{P=P_i^{(2)}} + \sum_{i=0}^{\infty} \tilde{\Pi}_i \bigg|_{P=\tilde{P}_i^{(2)}} \right) 
\end{equation}

Using these results, the output power predicted using ray theory for a modified MMI coupler with the output ports separated by 5 $\mu$m and 10 $\mu$m is shown in figures 8.4 (a) and (b), respectively. These results confirm that, if the coupler to function as a 3dB splitter then, the excess loss is higher when the output ports are more widely separated, and that the length, $L$, of the coupler increases as the separation of the output ports increases.

### 8.4 Numerical Results

The FD-BPM simulations compare well with the ray tracing results in figure 8.5. However, in contrast to the results obtained for the X- and Y- junctions, best results were obtained with a random variation in phase for each of the modes forming the input fields used in these simulations. This can be done by either scrambling the phase of each mode before starting the simulation, or by increasing the length of the input port, as this effectively scrambles the phases of the modes entering the coupler region. The option chosen for the simulations was to scramble the phase
Figure 8.4: Power transmission through port 1 (solid line) and port 2 (dashed line) as predicted by ray theory for the modified MMI coupler shown in figure 8.2. The core and cladding indices are 1.46 and 1.44, respectively, with $\rho = 10.0\mu m$ and $\varepsilon = 2.5\mu m$ in view (a) and $\varepsilon = 5.0\mu m$ in view (b). The excess losses are 11% in (a) and 20% in (b).
before commencing the simulated runs, in order to keep the computation domain as small as possible.

An example of how the input field propagates through the modified coupler is shown in figure 8.6 (a) and (b) for different coupler lengths. The simulations also confirm that, for the case where \( c = 2.5\mu m \), the excess loss is of the order of a few percent of the incident power and that this provides a sufficient gap to prevent the fields in the output ports from coupling together.

### 8.5 Discussion

The ray tracing analysis has provided a simple method by which the power transmission can be calculated for the modified MMI coupler considered in this chapter. The results compare well with FD-BPM simulations performed on the same waveguiding structures, and this helps validate the results derived using ray analysis.

The modified MMI coupler could prove to be quite useful in multimoded waveguide circuits, as either a space-saving alternative to a symmetric Y-junction or as a power tap. The appeal of the device is that the excess loss is small and that it occupies little space. The amount of power tapped off depends only on the length of the device and a 3dB split only requires a device a few hundred microns in length. A drawback of the device, however, is that it cannot channel most of (or all) the power into port 2. So, unlike conventional MMI couplers, this one serves a much simpler utility.
Figure 8.5: Comparison between ray (full and dashed lines) and BPM (dots and triangles) results for the modified MMI coupler depicted in figure 8.2. For these simulations $\rho = 10.0 \mu m$, $\varepsilon = 2.5 \mu m$, $n_{co} = 1.46$ and $n_{cl} = 1.44$. 
Figure 8.6: FD-BPM simulations of a multimode input passing through the modified MMI coupler configuration in figure 8.2 with the refractive index profile for each displayed in (a) and (c). For these simulations $\rho = 10\mu m$, $\varepsilon = 2.5\mu m$, $n_{co} = 1.46$ and $n_{cl} = 1.44$ at a wavelength of 632.8 nm. The length of the coupling region for these simulations is $160\mu m$ [(a) and (b)] and $280\mu m$ [(c) and (d)], achieving a power split between port 1 and port 2 of approximately 75:25 and 50:50, respectively.
Bend loss in Multimode Waveguides

Discovery consists of seeing what everybody has seen and thinking what nobody has thought.

Albert Szent-Györgyi 1893-1986, Quoted in The Scientist Speculates.

Optical waveguide bends are a vital component in optical waveguide circuits and are incorporated to connect the various devices and elements, as well as keeping the overall optical circuit design compact. But, even in the simplest circuits, small radius bends are crucial to this goal. However, small radius bends are responsible for significant radiation losses which can impair the performance of a circuit design, as tight bend radii are often required in order to maximise device yield and lower production costs [32, Chapter 10]. Accordingly, there has been a great deal of interest in modelling the radiation loss associated with waveguide bends.

Continuing the theme of comparing FD-BPM solutions with ray-tracing results, radiation loss calculations for multimoded step-profile planar waveguide bends have already been investigated using ray-tracing [92], but these results have not yet been compared with other techniques. It is certainly useful to compare these results with the FD-BPM approach, in order to identify any potential shortcomings with the former.

In order to undertake this comparison, the FD-BPM for straight waveguides needs modification by recasting this scheme into cylindrical coordinates. It then becomes well adapted to the analysis of multimode step-profile waveguide bends without any loss of accuracy [93], as no additional approximations have been incorporated into this propagation method. This technique is not the only approach, as it is possible to calculate the equivalent straight waveguide to a curved one by making a conformal transformation of the refractive index profile to cylindrical coordinates [94, 95]. Both of these methods were found to provide the same results, however, recasting the propagator into cylindrical coordinates proved to be the eas-
iest scheme to implement in practice, and therefore became the method of choice for the analysis contained in this chapter.

Most of the studies of radiative losses of bent planar dielectric waveguides have been conducted on single- or, at most, few-moded guides. These studies have predominately used an electromagnetic approach [96–99] to calculate the losses, which becomes increasingly cumbersome, and less accurate, as the number of modes increases. So for bent dielectric waveguides with large $V$ values, a ray optics approach incorporating loss coefficients is a method that is well suited for the calculation of bending losses in multimode waveguides.

9.1 Ray Analysis

9.1.1 Bent step-index planar waveguides

The step-index planar waveguide considered in the ray analysis and shown in figure 9.1 consists of a straight segment, of width $2\rho$, leading into a curved section of fixed radius $R$. The analysis assumes that a diffuse source illuminates the waveguide, equally exciting all the bound rays of the guide and that the straight segment is sufficiently long so that we can assume that any leaky rays excited by the source have lost their power by the beginning of the bend. With this in mind, only the bound rays remain in the core. Furthermore, any ray which is leaky in the straight section has a much higher attenuation in the bend and is therefore neglected from the following analysis.

The bent waveguide consists of two interfaces at $r = R \pm \rho$, where $r$ is the cylindrical radius from $O$ and the assumption is made that the profile shape is unaffected by the bend. The bend commences at the cross-section $XX'$, where the angle $\phi = 0$ relative to the centre of the bend, and the core and cladding indices of the waveguide are given by $n_{co}$ and $n_{cl}$, respectively. Due to the azimuthal symmetry of the bend, the characteristic shape of each ray trajectory follows a polygonal shape around the bend [21, Sectn. 9-1], as shown in figures 9.1 and 9.2.
Figure 9.1: A straight step-index planar waveguide, illuminated by a diffuse source, leads into a bend of constant radius, $R$, at the cross-section $XX'$. The waveguide is $2\rho$ wide with core and cladding indices $n_{co}$ and $n_{cl}$, respectively. At each reflection at the outer boundary of the bend, the ray loses a fraction, $T$, of its power.

9.1.2 Ray invariants

The ray trajectories within the waveguide, whether it is bent or straight, are found from the eikonal equation [21, Sectn. 1-18]

\[
\frac{d}{ds} \left\{ n(r) \frac{dr}{ds} \right\} = \nabla n(r),
\]

where $s$ is the length along the ray path and $r$ is the position vector of a point on the ray path. For the straight portion of the waveguide, the refractive index profile depends only on $x$, so that at each position, $r(x, z) = x \hat{x} + z \hat{z}$, and the ray path is determined by the two component equations

\[
\frac{d}{ds} \left\{ n(x) \frac{dx}{ds} \right\} = \frac{dn(x)}{dx}; \quad \frac{d}{ds} \left\{ n(x) \frac{dz}{ds} \right\} = 0,
\]
that give rise to the result that
\[ n(x) \frac{dz}{ds} = \text{constant} = \beta, \quad (9.3) \]
where \( \beta \) is the ray invariant. By introducing the angle \( \theta_z(x) \), being defined as angle between the tangent to the ray trajectory at \( x \) and the \( z \) axis, it then follows that within the core of the straight segment of the waveguide
\[ \beta = n_{co} \cos \theta_z(x). \quad (9.4) \]
This is the expression for the ray invariance along a step-profile waveguide [100]. For bound rays we have \( n_{cl} < \beta \leq n_{co} \) and for refracting rays \( 0 \leq \beta < n_{cl} \). This classifies all the rays in the straight waveguide [21, Sectn. 1-3].

Similarly, there is a ray invariant, \( \tilde{\beta} \), associated with each ray trajectory around the bend. In this instance, the ray position is given by \( r(r, \phi) \) and \( dr/ds \) is a unit vector tangent to the ray path, with \( dr = dr\hat{r} + r d\phi \),. As the refractive index depends only on \( r \), the eikonal equation (9.1) becomes
\[ \frac{d}{ds} \left( n(r) \left\{ \frac{dr}{ds} \hat{r} + r \frac{d\phi}{ds} \hat{\phi} \right\} \right) = \frac{dn(r)}{dr} \hat{r}. \quad (9.5) \]
By also noting that
\[ \frac{d}{ds} \hat{r} = \dot{\phi} \hat{\phi}; \quad \frac{d}{ds} \phi = -\dot{\phi} \hat{r}; \quad \dot{\phi} \equiv \frac{d\phi}{ds}, \quad (9.6) \]
the radial and angular components of equation (9.5) reduce to the following two equations
\[ \frac{d}{ds} \left( n(r) \frac{dr}{ds} \right) - n(r) r^2 \dot{\phi}^2 = \frac{dn(r)}{dr}; \quad (9.7) \]
\[ \frac{d}{ds} \left( n(r) r \frac{d\phi}{ds} \right) + n(r) \frac{dr}{ds} \dot{\phi} = 0. \quad (9.8) \]
Equation (9.8) further simplifies to
\[ \frac{d}{ds} \left( n(r) r^2 \dot{\phi} \right) = 0, \quad (9.9) \]
which gives rise to the result that
\[ n(r) r^2 \dot{\phi} = \text{constant}. \quad (9.10) \]
By defining $\theta(r)$ as the angle between the ray path and the azimuthal direction at coordinate $(r, \phi)$, as shown in figure 9.1, it then follows that $\rho = \cos \theta(r)$. Using this, the ray invariant $I_b$, associated with each path on the bend, is expressible in dimensionless form as [101]

$$I_b = \frac{r}{R + \rho} n(r) \cos \theta(r) = n_c \cos \theta \theta \cos \theta' \theta,$$

(9.11)

where $\theta$ and $\theta'$ are angles between the ray path and the tangent to the outer or inner interface, respectively, as illustrated in figure 9.2.

At the beginning of the bend, where $\theta_z(x) = \theta_{00}(r_0)$ and $r = r_0$ at $\phi = 0$, we see that

$$\bar{\beta} = n_c \cos \theta_z(x) = n_c \cos \theta_{00}(r_0)$$

(9.12)

and that by substituting this into equation (9.11) we obtain the relationship between the ray invariant on the straight waveguide and that in the bend:

$$I_b = \frac{r_0}{R + \rho} \bar{\beta}.$$  

(9.13)

This relates ray angle $\theta$ to angle $\theta_z$ at the start of the bend:

$$\cos \theta = \frac{r_0}{R + \rho} \cos \theta_z.$$  

(9.14)

As we have seen previously, on the straight waveguide the bound-ray invariant must satisfy $n_{cl} < \bar{\beta} < n_c$, so the ray invariant, $I_b$, on the waveguide bend must satisfy

$$n_{cl}(R - \rho)/((R + \rho) \leq I_b \leq n_c),$$

(9.15)

since $R - \rho \leq r_0 \leq R + \rho$.

### 9.1.3 Classification of rays

Rays on the straight waveguide are simply categorised as either bound or refracting rays, whereas on the bent waveguide the rays are classed as either tunnelling or refracting rays. The ray invariant for tunnelling rays satisfies equation (9.15), and these rays are so named because they appear to tunnel a finite distance from the core-cladding interface into the cladding before reappearing as propagating
Figure 9.2: Ray paths on a bent step-profile planar waveguide of radius $R$ (i.e. the distance from $O$ to the dashed path). The ray paths around the bend either (i) reflect alternately from the inner and outer interfaces or (ii) reflect from the outer interface only – these rays are also known as whispering-gallery rays.

Qualitatively, the paths of the tunneling rays in the core all have a turning point or caustic, which for the step-profile waveguide is simply the core-cladding interface at $r = R + \rho$. A tunnelling ray in the core reappears at a position $r_{rad}$, known as the radiation caustic, in the cladding so that between $R + \rho$ and $r_{rad}$, the fields of the rays are evanescent and no ray can propagate. This mechanism gives rise to the transfer of power from the core to the cladding in a way which is analogous to frustrated total internal reflection [21, Sectn. 2-7]. The radiation caustic for the step-profile waveguide for a ray labelled by $\bar{l}_b$ is given by

$$r_{rad} = (R + \rho)\bar{l}_b/n_{cl} = \frac{n_{co}}{n_{cl}}(R + \rho)\cos\theta_{\phi}. \quad (9.16)$$

According to equation (9.15), $\bar{l}_b$ is finite, so every ray in the bent waveguide is leaky. Thus, there are no bound rays and, similarly, there are no bound modes either.
Refracting and Tunnelling Rays

The radiation caustic provides a criterion for delineating tunnelling rays from refracting rays. For tunnelling rays $r_{rad} > R + \rho$, whereas refracting rays originate on the surface and $r_{rad} = R + \rho$. Therefore, if $\theta_\phi > \theta_c$ the ray refracts at the outer interface, but tunnels if $\theta_\phi < \theta_c$. Similarly, at the inner interface, rays do not undergo refraction provided $\theta'_\phi < \theta_c$ and no power is lost, as this interface is convex to the incident ray [103]. We can see from equation (9.11) that $\theta'_\phi < \theta_\phi$, so provided only the bound rays of the straight waveguide enter the bend there should be no radiative losses from the inner interface.

Whispering-gallery rays

Whispering-gallery rays are those that are only incident on the outer waveguiding boundary. A whispering-gallery ray trajectory is shown in figure 9.2, labelled as path (ii). According to equation (9.11) whispering-gallery rays satisfy the condition

$$\cos \theta_\phi > \frac{R - \rho}{R + \rho}. \quad (9.17)$$

Only rays with angles $\theta_\phi$ close to 0 satisfy this condition in practice for bends with large bend radii. The angular separation, $\phi_p$, between successive points of reflection for whispering-gallery rays is

$$\phi_p = 2\theta_\phi, \quad (9.18)$$

and for successive reflections of rays following path (i) in figure 9.2 by

$$\phi_p = \theta_\phi - \theta'_\phi. \quad (9.19)$$

Now the rays in the bent waveguide are fully classified, so that, given the ray position, $r_0$, and the angle, $\theta_z$, at the start of the bend, we can determine whether that ray is a refracting or tunnelling ray; and if it is a tunnelling ray, whether that ray is a whispering-gallery ray or not.

9.1.4 Ray power attenuation

We now know that every ray on the bend is lossy and that all the rays on the bend are either refracting or tunneling rays. The ray power, $P(\phi)$, after travelling
the angular displacement, $\phi$, through the bend (assuming that the waveguide is non-absorbing) can be expressed as [92]

$$P(\phi) = P(0)e^{-\gamma \phi},$$  \hspace{1cm} (9.20)

where $\gamma$ represents the dimensionless power attenuation coefficient.

In order to calculate the power attenuation coefficient, consider a ray that enters the bend with initial power $P_0$ at $\phi = 0$. This ray loses a fraction, $T_1$, of its power at the first reflection involving power loss, so that now the ray power, $P_1$, is given by

$$P_1 = P_0(1 - T_1).$$ \hspace{1cm} (9.21)

Similarly, after the second lossy reflection at which the ray loses a fraction, $T_2$, of its power, so the amount of power, $P_2$, remaining is now

$$P_2 = P_0(1 - T_1)(1 - T_2).$$ \hspace{1cm} (9.22)

Thus after making $n$ of these lossy reflections, the remaining power, $P_n$, can be expressed as [104]

$$P_n = P_0 \prod_{i=1}^{n} (1 - T_i).$$ \hspace{1cm} (9.23)

Furthermore, the angular displacement between each reflection is invariant so that after $n$ reflections the angular separation is simply $\sum_{i=1}^{n} \phi_p = n\phi_p$. Now the attenuation coefficient can be determined by recasting equation (9.20) as

$$-\gamma n\phi_p = \ln \left( \frac{P(\phi)}{P_0} \right) = \sum_{i=1}^{n} \ln (1 - T_i),$$ \hspace{1cm} (9.24)

by replacing $P(\phi)$ with $P_n$.

**Power transmission coefficients**

The losses $T_i$ that occur at the boundaries can be calculated from the power transmission coefficients for either tunnelling or refracting rays. The expression for the power transmission for tunnelling rays has the closed form [105]

$$T_i = \frac{4 \sin \theta_p}{\sin \theta_c} \left( 1 - \frac{\sin^2 \theta_p}{\sin^2 \theta_c} \right)^{1/2} \exp \left\{ -\frac{2}{3} n_c k (R + \rho) \left( \cos^2 \theta_c - \cos^2 \theta_p \right)^{3/2} \right\},$$ \hspace{1cm} (9.25)
where \( k = 2\pi/\lambda \). For refracting rays, the transmission coefficient, when \( n_{co} \approx n_{cl} \), is given by [21, Sectn. 7-1]

\[
T_i = \frac{4 \sin \theta_\phi (\sin^2 \theta_\phi - \sin^2 \theta_e)^{1/2}}{\sin \theta_\phi + (\sin^2 \theta_\phi - \sin^2 \theta_e)^{1/2}}.
\] (9.26)

This expression calculates the refracted ray loss from the outer boundary (i.e., at \( r = R + \rho \)). However, to calculate the refracted ray transmission loss at the inner boundary, we only need to replace the angle \( \theta_\phi \) by \( \theta'_e \) in the above expression.

Unfortunately, the Fresnel transmission coefficient, given by equation (9.26), incorrectly implies that \( T_i = 0 \) when \( \theta_\phi = \theta_c \), thus significantly underestimating the loss for refracting rays at, and also close to \( \theta_c \). Despite this, \( T_i \) rapidly increases for angles greater than the critical angle, as illustrated in figure 9.3, and it is only for the refracted ray angles very close to the critical angle that need a closer examination. In practice, for large bend radii, a large proportion of rays refracting out of the bend have ray angles of incidence that are close to \( \theta_c \). It is important, therefore, to find an expression that provides a more accurate expression for the transmission loss for rays close to the critical angle.

For refracting rays where \( \theta_\phi = \theta_c \), the transmission coefficient is approximately [103]

\[
T_i \approx \frac{3.18}{\theta_c} \left( \frac{1}{k n_{co} (R + \rho)} \right)^{1/3}.
\] (9.27)

The approach here uses the above expression when \( \theta_\phi \approx \theta_c \) in place of equation (9.26). Furthermore, equation (9.27) must be treated with care, as very tight bend radii will result in \( T_i > 1 \). In these cases equation (9.27) cannot be used, and equation (9.26) is used for all refracting rays.

It should be noted that in the paper by Winkler et. al. [92], the Fresnel equation for refracted rays is incorrectly quoted as

\[
T_i = \frac{4 \sin \theta_\phi}{\sin \theta_c} \left( \frac{\sin^2 \theta_\phi}{\sin^2 \theta_c} - 1 \right)^{1/2},
\] (9.28)

which very rapidly exceeds 1.0 as \( \theta_\phi \) increases. Furthermore, in the analysis the authors do not take account of the fact that the transmission loss is zero for rays impinging on the core/cladding boundary at the critical angle. However, the results obtained appear to be the same as those in figure 9.4.
9.1.5 Power Loss Calculations

As the angles for each successive reflection for each ray are the same, the power transmission coefficients are therefore invariant as well. Using this, the expression for the ray attenuation coefficient, equation (9.24), for $\gamma$ can be re-expressed as

$$
\gamma = -\frac{\ln (1 - T_i)}{\phi_p}.
$$

(9.29)

The total ray power $P_T(\phi)$ at angle $\phi$ is determined by integrating the expression for the individual ray power, equation (9.20), over all ray directions over the core cross-section. By assuming that the waveguide is illuminated by a diffuse source, the initial power, $P(0)$, is virtually the same for all rays and for the purposes of this analysis, they are treated as all being equal. Hence the total ray power is calculated by the following expression

$$
P_T(\phi) = P(0) \int_{R-\rho}^{R+\rho} dr \int_{-\delta_c}^{\delta_c} \exp(-\gamma \phi) d\theta_z.
$$

(9.30)
9.1.6 Discussion

The normalised power \( P_r(z)/P(0) \) is displayed as a function of the normalised distance \( z/\rho = R\phi/\rho \) along the bent waveguide axis in figure 9.4, for a waveguide with \( V = 50 \), \( \theta_c = 0.1 \) and core refractive index \( n_{co} = 1.52 \). The essential features are that the losses become increasingly significant as the normalised radius of curvature, \( R/\rho \), decreases. There is a transition region, where the losses are dominated by refracted rays corresponding to the initial rapid decrease in each curve, and a 'steady state' region corresponding to the slower decrease where the power decreases approximately exponentially for bends which are not too tight, and where the losses are determined by the tunnelling rays alone. Interestingly enough, the transition region ends at \( \approx 10^2 R/\rho \) for the various bend radii for the particular waveguide parameters considered here. Furthermore, the losses increase as the numerical aperture, or \( \theta_c \), decreases even if \( V \) remains fixed.

9.2 Modified FD-BPM

The finite-difference BPM considered here applies to a bent dielectric slab waveguide with the TE polarisation given in figure 9.5. This configuration has the electric field polarised parallel to the \( y \)-axis, so the scalar wave equation can be expressed in terms of a single component \( E(r, \phi) \) in cylindrical coordinates as

\[
\{ \nabla^2 + k^2 n^2(r) \} E(r, \phi) = 0. \tag{9.31}
\]

The slowly varying envelope form of this equation is obtained by making the substitution \( E(r, \phi) = u(r, \phi) \exp(i\tilde{k}R\phi) \) and neglecting the term \( \partial^2 u/\partial \phi^2 \), yielding [93]

\[
\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 n^2(r) - \frac{\tilde{k}^2 R^2}{r^2} \right\} u(r, \phi) = -\frac{2i\tilde{k}R}{r^2} \frac{\partial u(r, \phi)}{\partial \phi}, \tag{9.32}
\]

with \( \tilde{k} = k\tilde{n} = 2\pi\tilde{n}/\lambda \), where \( k \) is the free space wavenumber, \( \lambda \) is the wavelength and \( \tilde{n} \) is the background refractive index which is taken to be the mean index \( \tilde{n} = (n_{co} + n_{cl})/2 \). By applying the variable transformations

\[
s = R\phi; \quad r = R + x, \tag{9.33}
\]

where \( R \) is radius measured from \( O \) to the centre of the waveguide, and the curvature is expressed as \( \kappa = 1/R \), the final form for the scalar wave equation is obtained;
Figure 9.4: Ray optics calculation of normalised power remaining in a guide for a bent step-profile waveguide with \( V = 50 \), \( \theta_c = 0.1 \) and \( n_{co} = 1.52 \), for a range of bend radii as a function of the normalised distance \( z/\rho \) along the axis of the bent waveguide. View (b) shows the power transmission for a range of bend radii between \( R/\rho = 200 \) and 2000.
thus
\[
\left\{ \frac{\partial^2}{\partial x^2} + \frac{\kappa}{1 + \kappa x} \frac{\partial}{\partial x} + k^2 n^2(x) - \frac{\bar{k}^2}{(1 + \kappa x)^2} \right\} u(x, s) = -\frac{2i\bar{k}}{(1 + \kappa x)^2} \frac{\partial u(x, s)}{\partial s}.
\]

(9.34)

By recasting the scalar wave equation into the form of equation (9.34), we obtain an expression that is robust for the purposes of modelling bent waveguides [93]. Furthermore, equation (9.34) reduces to the correct form for straight waveguides when \( \kappa = 0 \), by setting \( s = z \).

The finite-difference form of equation (9.34) is found by applying the standard Crank-Nicolson scheme [106] to it. We obtain:

\[
\begin{align*}
&\{-[\sigma_q^p + 1]^2 - 2\alpha + \gamma \varepsilon_q^{p+1} - \varrho[\sigma_q^{p+1}]^2\}u_q^{p+1} \\
&\quad + \{\alpha + \tau \mu_q^p\}u_q^{p+1} - \{\alpha - \tau \mu_q^p\}u_q^{p+1} \\
&= \{-[\sigma_q^p]^2 + 2\alpha - \gamma \varepsilon_q^p + \varrho[\sigma_q^p]^2\}u_q^p \\
&\quad - \{\alpha + \tau \mu_q^p\}u_q^{p+1} - \{\alpha - \tau \mu_q^p\}u_q^{p+1},
\end{align*}
\]

(9.35)

where
\[
\begin{align*}
\alpha &= \frac{i \delta s}{4\bar{k}(\delta x)^2}; \quad \gamma = \frac{i\bar{k} \delta s}{4\bar{n}^2}; \quad \varrho = \gamma \bar{n}^2; \quad \varepsilon_q^p = [n^2(x)]_q^p; \\
\tau &= \frac{i \delta s}{8\bar{k} \delta x}; \quad \mu_q^p = \frac{\kappa^p}{1 + \kappa^p x_q}; \quad \sigma_q^p = \frac{1}{1 + \kappa^p x_q}.
\end{align*}
\]

(9.36)

The notation convention used here differs slightly from that used to describe the finite difference scheme in section 2.2.4. The superscripts and subscripts correspond to the longitudinal and transverse directions of the field amplitude, hence \( u_q^p = u(x_q, s_p) \). The field \( u(x, s) \) is discretised so that the transverse grid extends from the left boundary at \( x_0 \) to the right boundary at \( x_N \), with \( N + 1 \) grid points and \( q = 0, 1, 2, \ldots , N \). The location of each grid point is given by \( x_q = x_0 + q \delta x \) where \( \delta x = (x_0 - x_N)/N \). Similarly the longitudinal grid is \( [s_p = p \delta s; p = 0, 1, \ldots ] \).

The main advantages of this method are that the computational domain has been reduced considerably by making the waveguide bend a longitudinally invariant structure and that the refractive index profile does not have to be continually updated for each step as the field propagates along the bend. These considerations prove to be important in reducing the demand for computational resources, thereby
Figure 9.5: A schematic of a bent dielectric slab waveguide with fixed radius $R$ (a) with the TE-polarisation corresponding to the electric field in the $y$ direction, which is orthogonal to the plane of the bend, and the equivalent invariant structure (b).
significantly reducing the time to conduct each simulation.

All of the following simulations use transparent boundary conditions (see section 2.2.5) as well as absorbing boundaries to prevent the power radiating from the bend from being reflected off the computational domain boundaries and back into the waveguide. In calculating the power in the waveguide, the power orthogonality condition is given by [93]

\[
\int_{-R}^{\infty} \frac{u_i u_j}{1 + \kappa x} \, dx = A_i \delta_{ij},
\]

(9.37)

where in this equation \( \delta_{ij} \) is the Kronecker delta and should not be confused with \( \delta x \) and \( \delta s \) used previously. In the above equation \( u_i \) and \( u_j \) corresponds to the field at the same longitudinal position with transverse coordinate \( x_i = x_0 + i \times \delta x \) and \( x_j = x_0 + j \times \delta x \), respectively.

9.2.1 Physical Model

The bent step-profile waveguides considered here all possessed the following parameters: \( V = 50; \theta_s = 0.1 \) and \( n_{co} = 1.52 \), with a source wavelength of \( \lambda = 632.8nm \). It should be noted that these parameters were chosen to be the same as those used by Winkler et. al. [92] in order to compare the ray and finite-difference BPM results obtained here to their ray results. A straight waveguide with these parameters is 66.4\( \mu m \) wide (\( \rho = 33.2\mu m \)) and supports a total of 32 modes. The computational domain for each propagation is 160\( \mu m \) wide and 5\( mm \) in length with steps sizes \( \delta x = 1nm \) and \( \delta s = 0.1\mu m \), representing a grid discretised by 160,000 \times 50,000 points. The simulations were kept as narrow as possible so that the absorbing region did not interfere with the fields in the core. Propagation was not pursued beyond 5\( mm \), mainly due to the reduced accuracy of the results and the time required in running the simulations.

The simulation examined the power loss of waveguides with normalised bend radii of \( R/\rho = 2000, 800, 400, 200 \) and 100, using, respectively, equal excitation of all the even, all the odd and all the guided modes of the straight waveguide entering the bend to compare with the ray tracing results shown in figures 9.4.
9.3 Comparison of Bend Losses

A comparison of the FD-BPM results and the ray tracing results can be found in figures 9.6(a) and 9.6(b) for equal excitation of all even and all odd modes of the waveguide, respectively, for various normalised bend radii. A comparison is also provided for excitation using both the even and the odd input fields against the ray result for \( R/\rho = 100 \) in figure 9.7.

Equal excitation of both all even and all odd modes results in similar behaviour to the all even-mode or all odd-mode cases, whereby no power is lost from the guide until the field propagates beyond a normalised distance of \( z/\rho \lesssim 10 \). Beyond this region, there is a rapid transition that is generally much shorter than that predicted by ray theory, finally reaching a steady state at a normalised propagation length of \( z/\rho \approx 50 \). However, the onset of steady state radiative loss predicted by ray theory generally occurs at \( z/\rho \approx 10^2 \).

Equal excitation of all the even modes, however, provides the closest agreement with the ray results, and the steady state radiative loss closely matches that predicted by ray theory. The results obtained using all the odd modes for the normalised power remaining in the core is approximately 10% less than that for ray theory in the steady state region for normalised bend radii of 800 and 600. For bend radii smaller than this the results more closely coincide with the ray result. Figures 9.8 and 9.9 show how all the even and all the odd modes, respectively, propagate through different waveguide bends.

Finally, there is a limit to how small \( R/\rho \) can get before the FD-BPM becomes inaccurate. The fundamental assumption that is eventually violated is the requirement that the fields are paraxial. The accuracy of the results provided by the modified FD-BPM used here, appears to deteriorate once \( R/\rho \to 100 \) and so it is not worth comparing ray and FD-BPM results for \( R/\rho < 100 \).

9.4 Discussion

The ray and FD-BPM results are in close agreement for the steady state losses, provided the bend radii are not too tight, whereas the predicted outcomes of the two techniques provide more contrasting results when examining the transition region. The ray results show that the refracting rays rapidly leak out of the bent guide,
Figure 9.6: Comparison of the FD-BPM (curves with ripples) and ray-theory (smooth curves) results for the fraction of initial power remaining along a bend when $V = 50$, $\theta_c = 0.1$ and $n_{co} = 1.52$, for a range of normalised bend radii. The input fields consist of equal excitation of (a) all even modes, and (b) all odd modes, with different random relative phases.
Figure 9.7: Comparison of two separate FD-BPM simulations (rippled curves) and ray-theory (smooth curve) for the fraction of initial power remaining along a bend when $V = 50$, $\theta_b = 0.1$ and $n_{co} = 1.52$, for $R/\rho = 100$. The input field for these two simulations consists of equal excitation of all even modes or all the odd modes. The results of these two simulations are the same, however, the deviation from the ray result has increased compared with the plots in figure 9.6 as the paraxial approximation used in the FD-BPM is becoming invalid.
Figure 9.8: FD-BPM simulations depicting the evolution of the input field consisting of equal excitation of all even modes through a multimode bent waveguide when $V = 50$, $\theta_k = 0.1$ and $n_{co} = 1.52$. These simulations are performed at a wavelength of 632.8nm for a range of normalised bend radii.
Figure 9.9: FD-BPM simulations depicting the evolution of the input field consisting of equal excitation of all odd modes through a multimode bent waveguide when $V = 50$, $\theta_e = 0.1$ and $n_{co} = 1.52$. These simulations are performed at a wavelength of 632.8nm for a range of normalised bend radii.
and that the power smoothly decreases until essentially the only power remaining in the waveguide is attributable to the tunnelling rays.

The FD-BPM’s results behave in a noticeably different manner. At first no power appears to radiate from the waveguide and after the field has propagated a short distance the waveguide begins to radiate power, rapidly dropping down to the steady state. It is believed that this discrepancy is attributable to the finite number of modes the waveguide supports, and that as the number increases, the results should more closely match the ray theory behaviour. Unfortunately, the FD-BPM simulations are limited by both the number of modes that can be represented and the paraxial approximation, which requires that $n_{co} \approx n_{cl}$, and places a limit on the number of modes that can be accurately modelled by this means.

Furthermore, if we interpret the propagation constant, $\beta$, for each mode as the $z$-component of the local wave vector, then in a straight core of index $n_{co}$ [32, Sectn. 3.4.4]:

$$\beta = kn_{co} \cos \theta_z,$$

where $k = 2\pi/\lambda$ is the wavenumber and $\theta_z$ is the angle between the local wave vector and the direction of propagation. The ray interpretation assumes that there is a continuum of $\beta$-values between $kn_{cl}$ and $kn_{co}$. However, this is not the case for the FD-BPM simulations as, for example, the highest-order odd in the simulations is an odd mode for which $\theta_z = 0.098$ compared with $\theta_c = 0.1$. Consequently, in ray terms, the directions of the rays constituting this mode, do not initially impinge on the bent core-cladding interface at the critical angle and lose power by tunnelling, whereas if this mode had been propagating at cutoff, i.e. $\theta_z = \theta_c$, then power would be initially lost much more rapidly through refraction. In other words, the initial build up of transition loss is delayed by this discretisation.

This practical limitation placed on the BPM simulations means that ultimately the method is not as well suited to the study of waveguide bends for calculating loss for smaller bend radii as ray tracing is. However, since the two techniques are in good agreement with each other for $R/\rho \geq 200$, which might be expected to be the situation in practical bent waveguides, the FD-BPM technique could prove very useful in this situation.
CHAPTER 10

Concluding remarks

There's a good time coming, it's almost here,
'Twas a long, long time on the way.


Having presented the results of my work for the past three and a half years in the preceding chapters, it is worth summarising the main points of this thesis and reflecting upon future avenues of research. The main features of the work relating to the GRIN lens are:

- The PECVD Helicon plasma reactor has clearly demonstrated its capacity to fabricate both step-index and graded-(parabolic) index multimode profiles both reliably and reproducibly. This was established by using prism coupling techniques to recover the refractive index profile of the deposited films which demonstrated that the fabricated profiles closely matched the design profiles.

- This was further corroborated by measuring the refocussing period, or beat length, of the GRIN lenses using a specially developed polymer coating. The refocussing period obtained was the same, to within experimental error, as that predicted by the recovery of the refractive index profile using the prism coupler. Furthermore, this measurement was obtained at two different wavelengths and the results suggest that the fabricated planar GRIN lenses are not subject to chromatic aberration.

- The use of the Phloxine B dye-doped PVP polymer to recover the refocussing period, could also prove to be a useful diagnostic tool for characterising multimode waveguide devices which are otherwise difficult to characterise. The ability to detect evanescent fields close to the core/cladding interface could prove to be an easy way to measure the excess loss from X- and Y-junctions and multimode waveguide bends.
• It has also been demonstrated that the fabricated GRIN lenses can focus or defocus light by altering the illumination entering the lens, which is a property predicted using ray theory.

The remainder of the thesis used ray theory and FD-BPM simulations to examine several simple step-profile multimode devices and bends. The main findings are that:

• A diffuse source such as an LED, which would typically be used to illuminate multimode waveguides, is known to consist of approximately equal excitation of all the bound ray paths in the core cross-section. It was established from the simulations that this is approximately the same as equally exciting all the even bound modes of the multimode waveguide. The close agreement with the results provided using ray theory strongly suggests that this serves as an appropriate modal representation of a diffuse light source.

• Expressions for the excess loss of multimode X-junctions were presented that show that the excess loss, to a good approximation, is constant for intersecting junction angles, less than 40 degrees. The excess loss of multimode X-junctions is much higher than than of single moded junctions. A simple strategy is proposed that is capable of reducing multimode X-junction losses to the same order as that of single moded X-junctions.

• Multimode Y-junction were found to exhibit the unusual ability to remain lossless for reasonably large branching angles, provided that \( \Omega < \theta_d D/(D + 1) \). Beyond this limit, the total output power of the device decreases linearly with increasing branch angle \( \Omega \), with zero output power once \( \Omega = 2\theta_d \).

• The modified MMI coupler proved it could either serve as a miniature 3dB splitter or as a power tap. This device is not lossless, however, if designed appropriately the losses would be small. The modified MMI coupler could prove to be a useful alternative to the Y-junction as a means of splitting the input signal.

• Finally, for multimode waveguide bends the FD-BPM and ray theory results show that the losses are negligible provided \( R/\rho > 1000 \). For large bend radii the two techniques provide similar results, however, as the bend radii becomes tighter short comings of the FD-BPM simulations become responsible for a departure from the ray results. It was found that the FD-BPM simulations were not accurate for bend radii \( R/\rho < 200 \).
Both techniques identified that initially there is a transitional region where the guide rapidly loses power, and a steady state region after which the power decreases exponentially with $z/p$. The two techniques behave in the same way in the steady state region but differently in the transitional region. The discrepancy is thought to arise from a discretisation problem whereby as greater number of modes will more closely mimic the ray results.

Finally, where does this lead? Naturally, the next step would be to fabricate these devices and characterise them. To do so requires a mask for the fabrication process, and such a mask has been designed and produced. A copy of this mask has been reproduced for the readers interest in figure 10.1. With the expertise and facilities available at the ANU, it is highly likely that these devices will be fabricated in the near future. Who knows, they may all be household items in years to come.
Figure 10.1: Mask design where each rectangular cell contains a series of the same device with differing parameters. Each cell is duplicated twice with dimensions of 3 cm × 1.5 cm on a wafer 3 inches in diameter. This mask has on it X- and Y-junctions, modified MMI couplers and multimode Bends.
The WKB approximation for the scalar Helmholtz equation

The scalar wave equation is an eigenvalue equation giving the eigenmodes of propagation in a waveguide, so it is formally identical with Schrödinger's wave equation. Thus, methods used for the solution of quantum-mechanical problems, such as the Wentzel-Kramers-Brillouin (WKB) approximation, are immediately applicable to less generalised forms of the scalar wave equation, such as the Helmholtz equation. This approximation was first applied to planar geometries by Gordon in 1966 [107] and was later extended to account for cylindrical geometries, useful for describing fibre waveguides, by Gloge and Marcatili [108].

This approximation concerns the a beam of light that is travelling paraxially, where the electromagnetic fields are always nearly perpendicular to the direction of propagation, which is chosen to be the z-axis through a weakly-guiding transparent medium. The refractive-index of the weakly-guiding waveguide through which the beam of light is traveling has the form

\[ n(x) = n_0 \left( 1 - \frac{1}{2} f(x) \right) \]  

(A.1)

where it is assumed that \(|f(x)| \ll 1\) in the range of interest, so that \(n^2 \cong n_0^2(1 - f(x))\), and that

\[ \frac{\partial^2 f}{\partial x^2} \geq 1; \quad f(0) = 0; \quad f(x) = f(-x). \]  

(A.2)

By invoking the paraxial approximation, we may then use the scalar Helmholtz wave equation (2.33) to determine the propagation of harmonic fields \(\exp(-i\omega t)\) through the weakly-guiding medium being considered. As the refractive-index profile is \(x\)-dependent, the harmonic fields are therefore independent of \(y\), thus

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 n_0^2 (1 - f(x)) \psi = 0. \]  

(A.3)
Using equation (A.3), we are then equipped to find the eigenvalues of the propagation constant $\beta$ in the $z$ direction where

$$\psi \propto \exp(i\beta z); \quad \frac{\partial^2 \psi}{\partial z^2} = -\beta^2 \psi$$  \hspace{1cm} (A.4)

which, upon its substitution back into equation (A.3), yields

$$\frac{d^2 \psi}{dx^2} + \left[\beta_0^2(1 - f(x)) - \beta^2\right] \psi = 0,$$  \hspace{1cm} (A.5)

where the substitution $n_0^2 k^2 = \beta_0^2$ has been made. With the further substitution

$$1 - \beta^2/\beta_0^2 = \xi,$$  \hspace{1cm} (A.6)

equation (A.5) becomes

$$\frac{d^2 \psi}{dx^2} + \beta_0^2(\xi - f(x))\psi = 0.$$  \hspace{1cm} (A.7)

Equation (A.7) bears a formal resemblance to the Schrödinger equation for a particle in a one-dimensional potential well, which has been recognised for some time [109]. The solutions to this equation consist of a set of eigenfunctions $\psi_m$, describing the transverse distributions of the propagating modal fields, with corresponding eigenvalues $\xi_m$, that provide the propagation constants. By introducing the notation

$$p(x) = \sqrt{\xi - f(x)}$$  \hspace{1cm} (A.8)

equation (A.7) can be expressed far more succinctly as

$$\frac{d^2 \psi}{dx^2} + \beta_0^2 p^2 \psi = 0.$$  \hspace{1cm} (A.9)

Consider the points $x = \pm x_{tp}$, where $f(x_{tp}) = \xi$. As $f(x)$ has been defined to be a monotonically increasing function of $|x|$, then for $|x| < x_{tp}$, the quantity $p^2$ is positive, and therefore the solution to equation (A.9) dictate that $\psi$ is oscillatory. Whereas for $|x| > x_{tp}$, $p^2$ is negative and therefore $\psi$ has a decreasing exponential behaviour. Hence, the points $|x| = \pm x_{tp}$ are inflection points of $\psi$. From a ray optical perspective, the points $|x| = \pm x_{tp}$ correspond to turning points, where the ray has zero slope and, correspondingly, the turning point is the maximum excursion of the rays from the optical axis [33,107]. Near the turning point, any graded index is locally linear, thus equation (A.9) leads to an equation that can be expressed as

$$\frac{d^2 \psi}{dx^2} = \eta \psi.$$  \hspace{1cm} (A.9)

We require a solution which is finite everywhere, and the function $\psi = Ai(X)$ satisfies this (while $Bi(X)$ does not). Here $Ai$ and $Bi$ are the Airy functions. This solution connects the two regions because $Ai(-\eta)$ is oscillatory, while $Ai(\eta)$ is exponentially decreasing. Furthermore, $Ai''(0) = 0$ so the turning point corresponds to the inflection point of the field, not its maximum.
The WKB Approximation

We now derive the solution to equation (A.9), valid in the range $|x| < x_{tp}$ provided $p$ is slowly-varying [107, 110] whereby we neglect terms including $dp/dx$. The next step involves making the substitution $\eta = \sqrt{p}\psi$, hence

$$\frac{d\psi}{dx} = -\frac{1}{p^{\frac{1}{2}}} \frac{dp}{dx} \eta + p^{-\frac{1}{2}} \frac{d\eta}{dx} \approx p^{-\frac{1}{2}} \frac{d\eta}{dx};$$

(A.10)

Furthermore, by making the assignment that

$$\frac{dz}{dx} = \beta_0 p,$$

(A.11)

then $\eta''(x) + \beta_0^2 p^2 \eta = 0$ becomes $\eta''(z) + \eta = 0$, whose solution is proportional to $\cos(z + \alpha)$, where $\alpha$ is an arbitrary constant. Accordingly, the solution to equation (A.9) is

$$\psi \propto p^{-\frac{1}{2}} \cos \left\{ \beta_0 \int_{0}^{x} p \, dx' + \alpha \right\}.$$  

(A.12)

Since equation (A.2) stipulates that $f(x)$ is symmetric, it can easily be shown from equation (A.5) that the modal fields, $\psi$, must either be symmetric or anti-symmetric in $x$. This therefore means that $\alpha$ must be a multiple of $\pi/2$. The eigenvalues of $\xi$ are then determined by matching $\psi$ through the inflection point at $x = x_{tp}$. This is achieved by linearising $f(x)$ in the vicinity of $x_{tp}$ resulting in the following asymptotic connection formulæ through $x = x_{tp}$:

$$\psi \propto p^{-\frac{1}{2}} \cos \left( \theta - \frac{\pi}{4} \right) \quad x < x_{tp};$$

$$\rightarrow \left( \frac{2\pi \theta}{3p} \right)^{\frac{1}{3}} \text{Ai}(-\theta) \quad x \lesssim x_{tp};$$

(A.13)

$$\rightarrow \left( \frac{2\pi \theta}{3p} \right)^{\frac{1}{3}} \text{Ai}(\theta) \quad x \gtrsim x_{tp};$$

$$\rightarrow \frac{1}{2} p^{-\frac{1}{2}} \exp(-\theta) \quad x > x_{tp},$$

where

$$\theta = \theta(x) = \left| \beta_0 \int_{x_{tp}}^{x} |p(x')| \, dx' \right|. \quad (A.14)$$
The phase of $\psi$ as it approaches the turning point ($x < x_{tp}$) must be equal to $-\pi/4$ in order that the connection through the turning point does not give rise to an increasing exponential in the region $x > x_{tp}$, which of course is not allowable [110]. Now the expression for $\psi$ when $x < x_{tp}$ in equation (A.13) can be equated with equation (A.12), replacing $\alpha$ by $m\pi/2$ where $m$ is an integer. Hence

$$\beta_0 \int_0^{x_{tp}} p \, dx = \left( m + \frac{1}{2} \right) \frac{\pi}{2},$$  \hspace{1cm} (A.15)$$

where even and odd values of $m$ correspond to even and odd symmetrical modal fields, respectively. Furthermore it is quite straightforward to see that extending the WKB method for profiles extending from $-x_{tp}$ to $x_{tp}$ the result is simply

$$\beta_0 \int_{-x_{tp}}^{x_{tp}} p \, dx = 2\beta_0 \int_0^{x_{tp}} p \, dx = \left( m + \frac{1}{2} \right) \pi.$$  \hspace{1cm} (A.16)$$

A final word on this approximation. As $f(x)$ is defined to be strictly monotonically increasing then the refractive index profiles that can be applied to this approximation must thereby by symmetric and strictly monotonically decreasing. Hence, the WKB approximation is obviously well suited to parabolic profiles, but is not ideal for step-index waveguides. As a result the reconstructed step-index profiles will look as they do in figure 4.4(b).
Bibliography


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