Self-similarity in the conformal framework of quiescent cosmology and the Weyl curvature hypothesis

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Declaration

This thesis is an account of research undertaken between February 2012 and October 2012 at Research School of Physics and Engineering, College of Physical and Mathematical Sciences, Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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This thesis is dedicated to my loved ones,
with whom most good things are made possible.

Abstract

The prevalent theory of cosmological inflation, which attempts to explain the high degree of isotropy observed from the Earth's location in the Universe, has been criticised for being ad hoc and thermodynamically unsound. A viable alternative to inflation is the combined theory of quiescent cosmology and the Weyl curvature hypothesis, in which cosmological models are studied within a mathematical framework that features conformal transformations between physical and unphysical spacetimes. The focus of this thesis is to augment the conformal framework by incorporating a symmetry-related spacetime property known as self-similarity, or scale invariance.

An initial obstacle to this purpose is the lack of a satisfactory definition in the literature for asymptotic self-similarity, i.e. approximate self-similarity at early or late times in a cosmological model's evolution. In this thesis, we conduct an example-driven development of a working definition that is both suitable for use in the conformal framework and sufficiently concordant with existing notions of asymptotic self-similarity. The definition is an asymptotic generalisation of the homothetic equation (which formalises the property of exact self-similarity), and is modified appropriately to generate better agreement with various results in the dynamical systems approach to cosmology.

One unavoidable difficulty with our working definition is that the asymptotic self-similarity of a specified cosmological model is generally not trivial to determine: the existence of a vector field satisfying given conditions is required under the definition, but no universal method of constructing said vector field is provided. We derive several propositions and theorems that seek to address this problem, although such results are limited in their applicability.

After settling on an adequate working definition of asymptotic self-similarity, we employ it in the conformal framework of quiescent cosmology and the Weyl curvature hypothesis. Example spacetimes that have been studied within the framework are examined for self-similarity in this thesis; most significantly, we are able to demonstrate asymptotic self-similarity for the Friedmann–Lemaître–Robertson–Walker models, i.e. the class of all isotropic and homogeneous cosmological models (with some exceptions).

To better understand the characterisation of self-similarity in the conformal framework, we detail the conditions under which it is preserved by conformal transformations. We also investigate the relationships between self-similarity and other symmetry-related spacetime properties in the framework: many of these properties are shown to be pairwise independent via relevant counterexamples, but whether self-similarity stands completely apart remains an open question.

It is hoped that the definition and analysis of asymptotic self-similarity in this thesis will contribute an additional facet to the conformal framework, thereby facilitating further research on quiescent cosmology and the Weyl curvature hypothesis.

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Chapter 1

Introduction

A key concept in cosmology — the study of the Universe's structure and evolution — describes the fact that on large enough scales, space "looks the same" in every direction from the Earth. More precisely: large-scale observations of the Universe at any time, as well as the physical laws implicit in such observations, are independent of direction. This concept is known formally as *spatial isotropy* (henceforth isotropy) at a point.

Isotropy at the Earth's location in the Universe is evident in observations such as the spatial distribution of galaxies on scales of at least 300 million light-years [1], or the uniformity of the cosmic microwave background of early-Universe radiation to one part in 100,000 [2]. It is worth emphasising that while exact isotropy and its negation, *anisotropy*, exist as well-defined mathematical concepts (see Definition 2.4), being able to talk about high or low isotropy is more useful for physical purposes. As it turns out, the degree of isotropy may be defined and quantified in several ways (see Section 2.2.1).

Closely related to isotropy is the concept of *spatial homogeneity* (henceforth homogeneity) and its negation, *inhomogeneity* (see Definition 2.5). In the cosmological context, homogeneity is essentially the statement that the Earth occupies a "typical" location in the Universe, i.e. large-scale observations of the Universe at any time are independent of position. There is no direct evidence for a homogeneous Universe. However, homogeneity follows from the observed isotropy by assuming the Earth is not in a central, specially favoured position (named the Copernican principle by Bondi [3]), since an inhomogeneous universe can appear isotropic only to an observer in a special position [4].

The relationship between isotropy and homogeneity can be muddled through usage of the terms in a non-cosmological sense. For example, the cosmic microwave background itself is often said to be isotropic and homogeneous, but what its uniformity indicates about the Universe is isotropy at the Earth's location and homogeneity confined to the edge of the observable universe. In general, isotropy at a point (or *local isotropy*) and homogeneity are independent properties of a universe. However, isotropy everywhere (or *global isotropy*) implies homogeneity, while local isotropy plus homogeneity implies global isotropy [2]. Extending the argument in the previous paragraph, it follows that the Universe is also globally isotropic; this collective assumption of global isotropy and homogeneity in the Universe is known as the *cosmological principle*.

Another accepted fact about the Universe is that space itself is expanding, in the sense that any two fixed points are growing apart with time. This idea was explored independently by Friedmann and Lemaître [5, 6] in the 1920s, and corroborated by Hubble's [7] observation that galaxies recede from the Earth at speeds increasing with distance. The Universe's expansion strongly indicates the initial existence of an origin for space and time, i.e. a past *cosmological singularity* (henceforth singularity) known as the Big Bang.

As gravity is the dominant interaction on cosmological scales, much of modern cosmology seeks to model the Universe's expansion within the framework of Einstein's [8] general theory of relativity — the most consistent theory of gravity available. In general relativity, space and time are combined into a single continuum known as a *spacetime*, which may be used as a *cosmological model* of a universe (see Definitions 2.1 and 2.3). The Universe's evolution after the Big Bang is assumed to be governed by the *Einstein field equations* (EFE): a system of 10 nonlinear partial differential equations, whose solution with specified boundary conditions gives a *metric tensor* (henceforth metric) describing the geometry and causal structure of a spacetime.

Cosmological models of the Universe

The metric describing the class of all globally isotropic, homogeneous and expanding cosmological models was investigated independently by Robertson and Walker [9, 10] in the 1930s; as such, both metric and class bear the name *Friedmann–Lemaître–Robertson–Walker* (FLRW). It is perhaps unsurprising that the FLRW models yield many predictions in good agreement with observation, since the cosmological principle is not a particularly far-fetched assumption. While there are several problems with the standard near-FLRW description of the Universe, these models still serve as adequate approximations in different epochs of the Universe's evolution, and are worth studying in detail if only for their tractability (see Section 2.1.3).

One serious flaw in the standard near-FLRW picture is its failure to address the *horizon problem*, which essentially asks how the Universe can obey the cosmological principle in the first place. Under typical assumptions on the nature of matter in the early Universe, a high degree of isotropy and homogeneity near the Big Bang is improbable. The resultant anisotropy or inhomogeneity cannot be smoothed out through any form of physical interaction as the Universe expands, since sufficiently separated regions of space grow apart faster than the speed of light [11].

A notable attempt at tackling the horizon problem was introduced by Misner [12] in 1968. The problematic assumption of highly isotropic and homogeneous initial conditions is not made, and the Universe near the Big Bang is modelled by a homogeneous but anisotropic model instead. In this non-FLRW model, the high degree of anisotropy can be smoothed out through particle collisions or other dissipative processes, such that the Universe obeys the cosmological principle after its early evolution. However, the observed photon–baryon ratio in the Universe has been used to rule out the theory in its full generality [13]. Misner's approach is known commonly as *chaotic cosmology*, although the term may also serve as a broader label for any approach that assumes highly anisotropic or inhomogeneous initial conditions.

The widely accepted theory of *cosmological inflation*, introduced by Guth [14] in 1981, is one such approach. As in Misner's theory, any anisotropy or inhomogeneity near the Big Bang is smoothed out through dissipative processes. This takes place rapidly, however, since the early Universe is posited to expand slowly at first. A period of exponential expansion follows shortly; it "freezes in" the isotropy and homogeneity, and brings the early Universe up to speed for the onset of normal expansion as per the near-FLRW picture. While inflation is lauded for its ability to resolve the horizon problem and explain a wealth of other observations, it is not without criticism. For example, all proposed mechanisms for the exponential phase are based on speculative ideas in particle physics; also, the incorporation of dissipative processes in inflation — and chaotic cosmology — arguably violates the second law of thermodynamics [15].

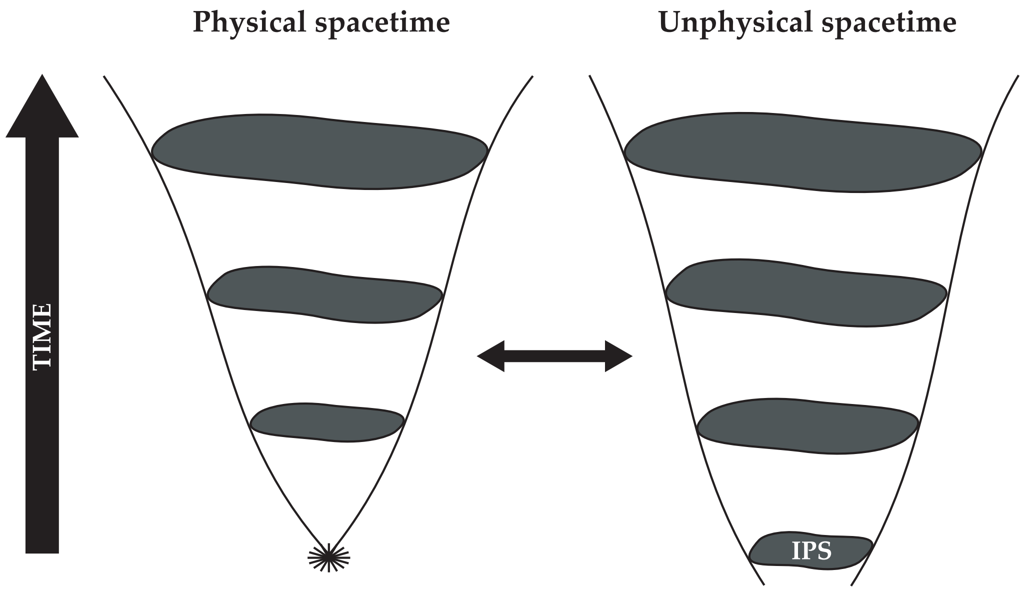

Figure 1.1: Schematic representation of a conformal transformation between spacetimes, with one spatial dimension suppressed. An IPS is a regular spacelike hypersurface in the unphysical spacetime that corresponds to the singularity in the physical spacetime.

Quiescent cosmology and the Weyl curvature hypothesis

Chaotic cosmology and the inflationary paradigm seem to have been motivated by the assumed improbability of near-FLRW models being valid near the Big Bang. This improbability is disputable, as demonstrated by Barrow's [16] introduction of *quiescent cosmology* in 1978. Indeed, near-FLRW behaviour in the early Universe is probable and thermodynamically stable if the nature of matter is taken to be "stiff" (see Table 2.1) at high densities — a speculative but by no means implausible assumption. With highly isotropic and homogeneous initial conditions, the horizon problem ceases to exist.

Just as there are thermodynamic arguments against chaotic cosmology and inflation, the thermodynamically stable initial conditions in quiescent cosmology also appear to violate the second law's assertion that the *entropy* (loosely, a system's proximity to equilibrium) of the Universe should increase with time from a minimal initial level. A possible solution to this issue emerged in 1979, with Penrose's [17] introduction of *gravitational entropy* and the *Weyl curvature hypothesis*. Gravitational entropy is the entropy associated with the gravitational "clumping" of matter. The Weyl curvature hypothesis states that gravitational entropy is related to the Weyl tensor, a measure of a spacetime's intrinsic curvature, and that this tensor vanishes near the Big Bang. Hence the maximal initial level of matter entropy in quiescent cosmology poses no problem to the second law, since the Weyl curvature hypothesis implies that the total entropy can still increase with time from a minimal initial level.

For a cosmological model to be compatible with quiescent cosmology and the Weyl curvature hypothesis (QC–WCH), it must exhibit isotropy, homogeneity and minimal gravitational entropy near its past singularity — at least in an approximate sense. The isotropic and entropic constraints are formalised in the definition of an *isotropic past singularity* (IPS), introduced by Goode and Wainwright [18] in 1985. Briefly, a spacetime admits an IPS if there exists a *conformal* (i.e. angle-preserving) transformation relating it to an unphysical counterpart that is regular at the time of the singularity (see Figure 1.1).

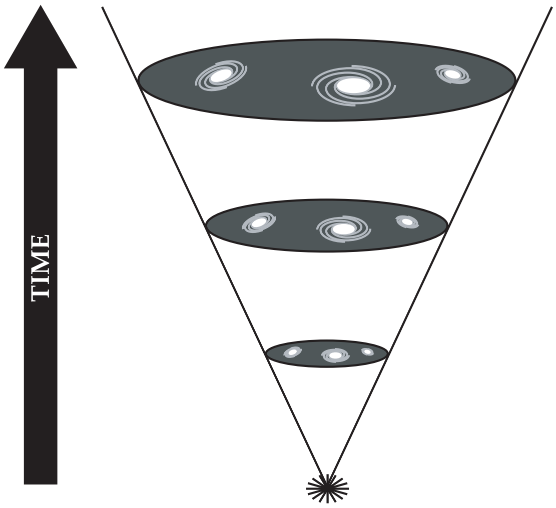

Figure 1.2: Schematic representation of an exactly self-similar cosmological model, with one spatial dimension suppressed. The spacelike hypersurfaces are similar to one another at all times.

The feasibility of the IPS as a preliminary framework for QC–WCH has been investigated with promising results, e.g. the possibility of explaining and modelling the formation of galaxies in the Universe [19].

A further implication of the Weyl curvature hypothesis is that the Universe evolves towards anisotropy and inhomogeneity, since the clumpiness of matter increases with time. Hence a full framework for QC–WCH must also be able to describe cosmological models that exhibit anisotropy, inhomogeneity and maximal gravitational entropy near their future states. In 2009, Höhn and Scott [20] extended the conformal framework of the IPS to introduce definitions for an *anisotropic future endless universe* (AFEU) and an *anisotropic future singularity* (AFS), along with their isotropic counterparts. A spacetime that admits an AFEU or AFS is thought to be compatible with the isotropic and entropic constraints of QC–WCH (further research in this area is ongoing). These new definitions, taken together with the IPS, are the leading candidate for a framework to put the ideas of Barrow and Penrose on firm mathematical footing.

Self-similarity in the conformal framework

To facilitate the study of cosmological models in the context of QC–WCH, it is desirable to fit other common properties of spacetimes into the conformal framework. Incorporating the property known as *self-similarity* is the focus of this thesis. In classical hydrodynamics, self-similarity is precisely the notion of scale invariance with time, and often allows simplification of the governing partial differential equations to ordinary ones [21]. Since the Universe is modelled as an expanding fluid in relativistic cosmology, it might conceivably be self-similar during some phase of its evolution (see Figure 1.2); as such, standard techniques for deriving and analysing self-similar solutions are both applicable and relevant to the EFE. This fact was first realised in 1971 by Cahill and Taub [22], who used it to characterise spherically symmetric self-similar spacetimes.

Self-similarity in cosmological models has since been investigated in considerable breadth and depth. Many exactly self-similar solutions have been found or identified (e.g. a restricted subclass of the flat FLRW models). These solutions play a central role in the *dynamical systems* approach to cosmology, which primarily uses the assumption of exact homogeneity to reduce the EFE to a system of ordinary differential equations in time. The dynamical systems approach is able to qualitatively describe the evolution of homogeneous cosmological models, a large class of which turn out to be approximated by exactly self-similar solutions at early, intermediate or late times [21]. Solutions that are nearly self-similar at early or late times are said to exhibit *asymptotic self-similarity*; they are of particular interest for the purposes of this thesis, since the conformal framework of QC–WCH deals with cosmological models in those regimes.

The most prominent obstacle to fitting self-similarity into the conformal framework is definitional in nature. While exact self-similarity has several adaptable and equivalent definitions in the literature, the existing definition for asymptotic self-similarity only makes sense in the dynamical systems approach. It is not immediately generalisable to the conformal framework, due to the restrictive assumption of homogeneity and the disparate nature of the two approaches. Formulating a definition that is compatible with the conformal framework and a majority — if not all — of the dynamical systems results is a difficult but not prohibitive problem. A number of candidate definitions are proposed and tested in this thesis.

With an adequate working definition of asymptotic self-similarity, the relationships between isotropy, homogeneity and self-similarity may be thoroughly explored in the early and late regimes. In particular, it is important to ascertain if and how these properties are connected, or alternatively to prove their mutual independence. Another interesting line of enquiry is whether the conformal transformations in the framework preserve exact and/or asymptotic self-similarity. Such questions are addressed in this thesis; it is hoped that their resolution will strengthen the existing conformal framework, paving the way for further research on the combined theory of QC–WCH.

1.1 Chapter outline

The central concepts in this chapter are formally expanded upon in Chapter 2, which lends a degree of self-containment to this thesis by providing a short summary of the technical background requisite for its purposes. Specifically, we give a more detailed introduction to the conformal framework of QC–WCH and the spacetime property of exact self-similarity, and highlight the need for a refined definition of asymptotic self-similarity by examining existing interpretations in the literature.

A heuristic approach to the development of said definition is adopted in Chapter 3. We shortlist and discuss three candidate definitions of asymptotic self-similarity, before settling on one that promises compatibility with both the conformal framework and the dynamical systems approach. Several example cosmological models are analysed under the preliminary definition, and the findings from this investigation are used to guide our formulation of the eventual working definition.

With a suitable definition of asymptotic self-similarity in hand, Chapter 4 moves on to the primary task of integrating self-similarity into the conformal framework of QC–WCH. The translation of self-similarity under conformal transformations is explored, along with its relationship to other symmetry-related spacetime properties. We also consider additional example models that have been studied in the conformal framework.

Finally, Chapter 5 recapitulates the key results obtained in this thesis, and presents possible avenues of future study that arise from various issues and open problems identified during the course of our research.

1.2 Preliminaries

Readers are assumed to have at least a rudimentary knowledge of differential geometry, general relativity and relativistic cosmology. Influential treatments of these subjects have been given by Ellis and Hawking [1, 23], and serve as the backdrop to this thesis. The material presented here builds on ideas and results introduced in an earlier thesis by Cain [24], but otherwise stands on its own.

1.2.1 Conventions

- Latin indices run from 0 to 3, while Greek indices run from 1 to 3.
- The Einstein summation convention applies only to lower–upper pairs of indices. Hence $X_\mu Y^\mu = X_1Y^1 + X_2Y^2 + X_3Y^3$, while $X_\mu Y_\mu = (X_1Y_1, X_2Y_2, X_3Y_3)$.
- The metric signature is $(-,+,+,+)$.
- The sign convention for the Riemann and Ricci tensors is $R_{ab} = R^c{}_{acb}$.
- Partial derivatives are denoted by commas, covariant derivatives with respect to a physical metric are denoted by semicolons, and covariant derivatives with respect to an unphysical metric are denoted by colons.
- Symmetrisation is denoted by round brackets, e.g.

$$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right). \tag{1.1}$$

  Skew-symmetrisation is denoted by square brackets, e.g.

$$T_{[abc]} = \frac{1}{6}\left(T_{abc} + T_{cab} + T_{bca} + T_{acb} - T_{bac} - T_{cba}\right). \tag{1.2}$$

- Geometrised units such that $c = 8\pi G = 1$ are used, where c is the speed of light in vacuum and G is the gravitational constant.
- We write $f(t) = O(g(t))$ as $t \to t_0$ if there exists $K > 0$ such that

$$\lim_{t\to t_0}\left|\frac{f(t)}{g(t)}\right| \le K. \tag{1.3}$$

  We write $f(t) = o(g(t))$ as $t \to t_0$ if

$$\lim_{t\to t_0}\frac{f(t)}{g(t)} = 0. \tag{1.4}$$

  We write $f(t) \sim g(t)$ as $t \to t_0$ if

$$\lim_{t\to t_0}\frac{f(t)}{g(t)} = 1. \tag{1.5}$$

1.2.2 Abbreviations

AFEU	Anisotropic future endless universe (Def. 2.9a)
AFS	Anisotropic future singularity (Def. 2.10a)
AHVF	Asymptotically homothetic vector field (Def. 3.4)
CKVF	Conformal Killing vector field (Eq. (2.18))
EFE	Einstein field equations (Eq. (2.2))
FIU	Future isotropic universe (Def. 2.8a)
FLRW	Friedmann–Lemaître–Robertson–Walker
HVF	Homothetic vector field (Eq. (2.18))
IFS	Isotropic future singularity (Def. 2.7a)
IPS	Isotropic past singularity (Def. 2.6a)
KVF	Killing vector field (Eq. (2.8))
QC–WCH	Quiescent cosmology and the Weyl curvature hypothesis

Table 1.1: Glossary of abbreviations.

Chapter 2

Background

In this chapter, we give an overview of the background required to integrate the concept of self-similarity into the conformal framework of quiescent cosmology and the Weyl curvature hypothesis. Section 2.1 is a brief introduction to spacetime and its symmetry properties, as well as the fundamentally important Friedmann–Lemaître–Robertson–Walker models. Past and future states in the conformal framework are defined and discussed in Section 2.2, while Section 2.3 introduces self-similarity and identifies differing notions of asymptotic self-similarity in the literature.

2.1 Introduction to relativistic cosmology

With its roots in Einstein's general theory of relativity, the mathematical framework of modern cosmology has been extensively developed over the past century. As such, a short primer on the subject is necessary for this thesis, but can by no means be exhaustive. Unless otherwise cited, the material in this section is summarised and adapted from expositions by Ellis et al. [1, 23, 25, 26, 27]. Formulae for various canonical curvature and kinematic quantities are relegated to Appendix A.

2.1.1 Formalism

In general relativity, the collection of all events in space and time is known as spacetime. It can be modelled mathematically as a topological space, equipped with additional structure that allows the definition of geometric notions and causal relationships on the space.

Definition 2.1 (Spacetime): A spacetime $(\mathcal{M}, \mathrm{g})$ is a four-dimensional smooth manifold $\mathcal{M}$ that is connected and Hausdorff, along with a nondegenerate C^2 Lorentzian metric tensor g on $\mathcal{M}$.

The manifold $\mathcal{M}$ essentially comprises one temporal and three spatial dimensions, and locally resembles Euclidean space $\mathbb{R}^4$ in that it can be covered by coordinate patches. More precisely, there exists a collection of injective maps $\varphi_i : \mathcal{U}_i \to \mathbb{R}^4$ where $\bigcup_i \mathcal{U}_i = \mathcal{M}$. Calculus may be performed since $\mathcal{M}$ is smooth, i.e. the maps $\varphi_i \circ \varphi_j^{-1}$ (when they exist) are of class C^∞, or infinitely differentiable, in the Euclidean sense. The connected and Hausdorff conditions are more technical: the former is imposed since we would have no knowledge of any disconnected component of the Universe, while the latter rules out certain pathological behaviour.

The metric $\mathbf{g}$ is a symmetric tensor field of type $(0, 2)$ on $\mathcal{M}$; it generalises the notion of an inner product on the tangent space at each point $p \in \mathcal{M}$, and may be described in local coordinates (x^a) by $\mathbf{g} = g_{ab} dx^a \otimes dx^b$ (or, for brevity, the tensor components g_{ab}). Since $\mathbf{g}$ is nondegenerate, continuous and Lorentzian, the matrix $[g_{ab}]$ of components is invertible, while the signs of its eigenvalues at any $p \in \mathcal{M}$ are given by $(-, +, +, +)$.

A considerable amount of structure on $\mathcal{M}$ is added by the specification of $\mathbf{g}$. For example, $\mathbf{g}$ defines a unique covariant derivative (a generalisation of the directional derivative) ∇ on $\mathcal{M}$, by requiring that $\nabla\mathbf{g}$ in the direction of any vector field on $\mathcal{M}$ is everywhere zero. We use said covariant derivative in this thesis (see (A.3)), and write

$$g_{ab;c} = 0 \tag{2.1}$$

in local coordinates (where the semicolon denotes covariant differentiation). Also, since $\mathbf{g}$ is the analogue of an inner product on $\mathcal{M}$, it defines angles and lengths in similar fashion. Hence the metric $\mathbf{g}$ is often synonymous with its associated line element $ds^2 = g_{ab} dx^a dx^b$, which represents the infinitesimal length determined by the coordinate displacement $x^a \to x^a + dx^a$. Finally, the Lorentzian requirement on $\mathbf{g}$ defines a causal structure on $\mathcal{M}$, with a vector field $\mathbf{X}$ on $\mathcal{M}$ classified respectively as timelike, null (lightlike) or spacelike if $\mathbf{g}(\mathbf{X}, \mathbf{X})$ is everywhere negative, zero or positive. This allows, among other things, the definition of a cosmic time function.

Definition 2.2 (Cosmic time function): A cosmic time function T on a spacetime $(\mathcal{M}, \mathbf{g})$ is a smooth function on $\mathcal{M}$ whose gradient ∇T is everywhere timelike and future-directed.

In other words, T increases into the spacetime's designated future with $\mathbf{g}(\nabla T, \nabla T) < 0$; we also impose smoothness on T here, in accordance with several definitions in the literature that use cosmic time functions. A spacetime admits a cosmic time function if and only if it is stably causal (admits no closed timelike curves) [28], and so we restrict our study in this thesis to such spacetimes. The existence of T on $(\mathcal{M}, \mathbf{g})$ allows $\mathcal{M}$ to be foliated into three-dimensional spacelike "slices" of constant T, such that local coordinates (T, x^μ) may be chosen with T as coordinate time.

A spacetime may be extended to a cosmological model of a universe, by specifying a collection of world lines for fundamental observers in the spacetime. This is made precise in the following definition.

Definition 2.3 (Cosmological model): A cosmological model $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ is a spacetime $(\mathcal{M}, \mathbf{g})$, along with a timelike C^2 unit vector field $\mathbf{u}$ on $\mathcal{M}$ that generates a congruence with somewhere positive expansion.

The fundamental four-velocity field $\mathbf{u}$ satisfies $\mathbf{g}(\mathbf{u}, \mathbf{u}) = -1$, and is nowhere zero on $\mathcal{M}$ since it generates a congruence, i.e. a collection of integral curves. Kinematic quantities such as expansion θ, shear σ and vorticity ω may be defined for a spacetime via the specification of $\mathbf{u}$ (see (A.11)–(A.17)). As the Universe is expanding, the definition of a cosmological model excludes spacetimes where θ is everywhere zero/negative (e.g. Minkowski space, the spacetime of special relativity). We note, however, that the distinction between Definitions 2.1 and 2.3 might be blurred in this thesis: most of the spacetimes studied are cosmological models, while $\mathbf{u}$ is not always required for the

Matter content	Equation of state	γ
Vacuum energy	$p = -\mu$	0
Dust	$p = 0$	1
Radiation	$p = (1/3)\mu$	$4/3$
Stiff fluid	$p = \mu$	2

Table 2.1: Equations of state for common types of perfect fluids.

analysis. This is also the case in the cosmological literature, where the terms space(time), (cosmological) model, cosmology and universe are often used interchangeably.

General relativity's central assumption is that the geometry of a spacetime interacts with its matter content, which is represented by the stress–energy tensors $\mathbf{T}_i$ of the matter components M_i, and that this interaction is governed by the Einstein field equations (EFE). In geometrised units such that $c = 8\pi G = 1$, and with the cosmological constant term treated as a matter component M_Λ, the EFE are written in local coordinates as

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}, \tag{2.2}$$

where R_{ab} and R are derived from the Riemann tensor describing spacetime curvature (see (A.5)–(A.7)), and $\mathbf{T} = \sum_i \mathbf{T}_i$.

In practice, the form of $\mathbf{T}$ (or each $\mathbf{T}_i$) may be restricted by making assumptions on the matter content. The simplest one is that there is no matter at all, in which case $T_{ab} = 0$, and the EFE reduce via contraction to the vacuum field equations

$$R_{ab} = 0. \tag{2.3}$$

Another possible assumption is that the matter is a non-tilted γ-law perfect fluid (henceforth perfect fluid), whose four-velocity field is the fundamental four-velocity field $\mathbf{u}$. Its stress–energy tensor is given by

$$T_{ab} = \mu u_a u_b + p\left(g_{ab} + u_a u_b\right), \tag{2.4}$$

where the energy density μ and pressure p are related by a γ-law equation of state

$$p = (\gamma - 1)\,\mu. \tag{2.5}$$

Table 2.1 gives the values of γ for common types of perfect fluids: vacuum energy, which corresponds to a nonzero cosmological constant $\Lambda = \mu$; dust (or nonrelativistic matter); radiation (or relativistic matter); and stiff fluids, in which the speed of sound is the speed of light [16]. We note that all the examples studied in this thesis are either vacuum or perfect fluid spacetimes.

As the Riemann tensor $R^a_{\ bcd}$ and its various traces can be expressed in terms of the metric g_{ab} and its derivatives, it is seen from (2.2)–(2.4) that the EFE is a system of 16 partial differential equations for the metric components (in the case of perfect fluids, $\mathbf{u}$ must be specified first). Ten of these are distinct, since the tensors in (2.2) are symmetric. Furthermore, contracting the Bianchi identities

$$R_{ab[cd;e]} = 0 \tag{2.6}$$

twice and substituting (2.2) into the result yields the four conservation equations

$$T^{ab}{}_{;b} = 0, \tag{2.7}$$

which express the local conservation of energy and momentum — and reduce the EFE to six independent equations. In relativistic cosmology, all valid models are exact solutions to these six equations with appropriate boundary conditions; regardless of whether the metric is known explicitly or studied implicitly, the dynamical evolution of such models is fully compatible with the EFE.

2.1.2 Spacetime symmetries

As mentioned in the discussion following Definition 2.2, a spacetime $(\mathcal{M}, \mathbf{g})$ may be foliated into spacelike hypersurfaces $\mathcal{S}_T \subset \mathcal{M}$ of constant T. In this thesis, the cosmic time function is constructed such that $T = 0$ corresponds to a cosmological past/future state (which is either a singularity or an endless universe). We note here that $\mathcal{S}_0$ is contained not in $\mathcal{M}$ but in its boundary $\partial\mathcal{M}$, and may be defined as a spacelike slice of some unphysical extended manifold $\widetilde{\mathcal{M}} \supset \mathcal{M}$ (equipped with some metric $\widetilde{\mathbf{g}}$ that is conformally related to $\mathbf{g}$ on $\mathcal{M}$). The concepts of isotropy and homogeneity introduced in Chapter 1 are now made precise [2].

Definition 2.4 (Spatial isotropy): A spacetime $(\mathcal{M}, \mathbf{g})$ is spatially isotropic at a point $p \in \mathcal{S}_T \subset \mathcal{M}$ if, for any two unit tangent vectors $v, w \in \mathcal{T}_p\mathcal{S}_T$, there exists an isometry ϕ on the spacelike hypersurface $\mathcal{S}_T$ such that $\phi_{*p}(v) = w$. If the spacetime is spatially isotropic at each $p \in \mathcal{S}_T$, it is spatially isotropic on $\mathcal{S}_T$. If the spacetime is spatially isotropic on each $\mathcal{S}_T \subset \mathcal{M}$, it is spatially isotropic.

Definition 2.5 (Spatial homogeneity): A spacetime $(\mathcal{M}, \mathbf{g})$ is spatially homogeneous on a spacelike hypersurface $\mathcal{S}_T \subset \mathcal{M}$ if, for any two points $p, q \in \mathcal{S}_T$, there exists an isometry ϕ on $\mathcal{S}_T$ such that $\phi(p) = q$. If the spacetime is spatially homogeneous on each $\mathcal{S}_T \subset \mathcal{M}$, it is spatially homogeneous.

An isometry is a smooth, invertible map of a manifold into itself that leaves the equipped metric invariant. In Definitions 2.4 and 2.5, ϕ is defined on the submanifold $\mathcal{S}_T$ with the restricted metric $\mathbf{g}|_{\mathcal{S}_T}$ such that $\phi_*(\mathbf{g}|_{\mathcal{S}_T}) = \mathbf{g}|_{\mathcal{S}_T}$, where $\phi_* : \mathcal{TS}_T \to \mathcal{TS}_T$ is the differential (or pushforward) of ϕ. An isotropic spacetime is necessarily homogeneous since, for each spacelike slice, isotropy everywhere implies homogeneity.

While useful for intuitive purposes, Definitions 2.4 and 2.5 are not particularly tractable. Equivalent formulations of isotropy and homogeneity for a spacetime $(\mathcal{M}, \mathbf{g})$ may be obtained via its r-dimensional isometry group G_r, which is the Lie group formed from the set of all isometries on $(\mathcal{M}, \mathbf{g})$. Each one-dimensional subgroup of G_r determines a collection of curves whose tangent field is known as a Killing vector field (KVF); conversely, each KVF generates a one-parameter group of isometries. The set of all KVFs $\mathbf{K}$ on $\mathcal{M}$ is the solution space to $\mathcal{L}_\mathbf{K}\mathbf{g} = 0$ (where $\mathcal{L}$ denotes the Lie derivative, another generalisation of the directional derivative on a manifold). In local coordinates (see (A.4)), this gives the Killing equation

$$K_{(a;b)} = 0, \tag{2.8}$$

Spacetime(s)	d	s	r
Minkowski	6	4	10
Einstein	3	4	7
FLRW	3	3	6
Kantowski–Sachs	1	3	4
Bianchi	$0, 1, 3$	3	$3, 4, 6$
G_2	0	2	2
Szekeres	0	0	0

Table 2.2: Isometry-based classification for common types of spacetimes.

which may be solved completely to recover information on the isometry group — specifically, the orbits of points in $\mathcal{M}$ under the action of G_r. The s-dimensional ($s < r$) orbit of a point $p \in \mathcal{M}$ is the set of points into which p is mapped when all elements of G_r act on it; this also determines the d-dimensional ($d = r - s$) isotropy subgroup of p. Since $\mathcal{M}$ is four-dimensional, we have $0 \leq r \leq 10$ and $0 \leq s \leq 4$.

Now, a spacetime $(\mathcal{M}, \mathbf{g})$ is homogeneous (in the sense of Definition 2.5) if $s \geq 3$, where the orbits are the spacelike hypersurfaces $\mathcal{S}_T$ for $s = 3$. It is also isotropic (in the sense of Definition 2.4) if $d \geq 3$. Various other types of spacetimes may be located within an isometry-based classification (see Table 2.2). One such class comprises the Bianchi cosmologies, which are homogeneous models that admit an isometry (sub)group $G_3 \subseteq G_r$ acting transitively on the spacelike hypersurfaces. The three-dimensional Lie algebra associated with G_3 has a KVF basis $\{\mathbf{K}_\mu\}$, relative to which the Lie bracket may be expanded such that $[\mathbf{K}_\mu, \mathbf{K}_\nu] = C^\rho_{\mu\nu}\mathbf{K}_\rho$. These structure constants $C^\rho_{\mu\nu}$ allow further classification of the Bianchi models. Spacetimes with Bianchi representations include a subclass of the Friedmann–Lemaître–Robertson–Walker (FLRW) models, which are introduced in Chapter 1 and elaborated upon in the following section.

2.1.3 FLRW models

The FLRW models are characterised by the most general metric describing an isotropic and homogeneous spacetime. In curvature-normalised coordinates, the line element associated with this metric is given by

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + f_k^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right),$$
$$f_k(\chi) = \frac{\sin\sqrt{k}\,\chi}{\sqrt{k}} = \begin{cases} \sin\chi, & k = 1, \\ \chi, & k = 0, \\ \sinh\chi, & k = -1, \end{cases} \tag{2.9}$$

where the scale factor a is arbitrary up to consistency with the EFE. The fundamental four-velocity field is then given by $\mathbf{u} = \partial/\partial t$.

Following the sign of k in (2.9), the spacelike hypersurfaces $\mathcal{S}_t$ of an FLRW model have uniform positive/zero/negative curvature. The first case corresponds to closed elliptic spaces $\mathbb{S}^3$, the second to flat Euclidean spaces $\mathbb{R}^3$, and the third to open hyperbolic spaces $\mathbb{H}^3$. Coordinate domains for $k = 0, -1$ are $t, \chi \in (0, \infty)$, $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$, while the only difference for $k = 1$ is $\chi \in (0, 2\pi)$. Other common incarnations of the FLRW line element are the reduced-circumference form

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right), \tag{2.10}$$

which is obtained from (2.9) via the coordinate transformation $r = f_k(\chi)$, and the isotropic form [29]

$$ds^2 = -dt^2 + \frac{a^2(t)}{\left(1+\frac{1}{4}k\rho^2\right)^2}\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right), \tag{2.11}$$

which is obtained from (2.9) via the coordinate transformation $\rho = 2\tan\left((1/2)\sqrt{k}\,\chi\right)/\sqrt{k}$.

We note here that (2.9) encompasses line elements for some special spacetimes that are technically FLRW, e.g. Minkowski space ($a(t) = 1$ and $k = 0$), the Milne universe ($a(t) = t$ and $k = -1$), the Einstein static universe ($a(t) = 1$ and $k = 1$), and the de Sitter universe ($a(t) = \exp\left(\sqrt{(1/3)\Lambda}\,t\right)$ and $k = 0$). For the purposes of this thesis, however, we exclude such spacetimes from the FLRW label, and focus on models with a single perfect fluid component where $\gamma \in (0, 2]$. Our FLRW models are also required to be expanding in some epoch, which rules out certain time-reversed variants.

For a single-component perfect fluid FLRW model, the EFE reduce to a system of two ordinary differential equations in time; these are the Friedmann equations

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(3\gamma - 2)\,\mu, \tag{2.12}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\mu - \frac{k}{a^2}, \tag{2.13}$$

which may be solved exactly (for a and/or μ) by specifying γ, k and appropriate initial conditions. It is convenient to define here the Hubble parameter

$$H = \frac{\dot{a}}{a}, \tag{2.14}$$

which measures the expansion rate of the universe, and the deceleration parameter

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right), \tag{2.15}$$

which measures whether the expansion is speeding up or slowing down. Furthermore, we note that

$$H = \frac{1}{3}\theta, \tag{2.16}$$

where the expansion θ is determined by $\mathbf{u}$. Equation (2.16) defines a generalised Hubble parameter for non-FLRW models, which in turn defines a generalised scale factor and deceleration parameter via (2.14) and (2.15) respectively.

The high degree of symmetry in the FLRW models leads to strong restrictions on their kinematic quantities; in particular, these models are shear-free ($\sigma = 0$) and irrotational ($\omega = 0$). They also have an everywhere zero Weyl tensor, which indicates their incompatibility with the Weyl curvature hypothesis (see Section 2.2). Finally, as mentioned in Section 2.1.2, the single-component perfect fluid FLRW models admit Bianchi representations: the closed models are Bianchi IX, the flat models are Bianchi I/VII$_0$, and the open models are Bianchi V/VII$_h$. It is useful to locate the FLRW models within the Bianchi classification, since Bianchi cosmologies play a central role in the dynamical systems approach to cosmology (see Section 2.3.2.2), which in turn provides many examples and results that guide the search for a definition of asymptotic self-similarity in Chapter 3.

2.2 Conformal framework of QC–WCH

Quiescent cosmology is the idea that the Universe is approximately FLRW (i.e. highly isotropic and homogeneous) near the Big Bang; the Weyl curvature hypothesis offers thermodynamic justification for this via the concept of gravitational entropy. Hence a cosmological model that is compatible with quiescent cosmology and the Weyl curvature hypothesis (QC–WCH) must exhibit isotropy and minimal gravitational entropy in the past (which is taken to be a singularity), and tend towards anisotropy and maximal gravitational entropy in the future. The past constraints are captured in the definition of an isotropic singularity, while a set of future state definitions has been formulated in an attempt to accommodate the future constraints.

2.2.1 Isotropic singularities

The following definition of an isotropic (past) singularity is due to Goode and Wainwright [18, 19], with some technical amendments by Scott [30, 31].

Definition 2.6a (Isotropic past singularity): A spacetime $(\mathcal{M}, \mathbf{g})$ admits an isotropic past singularity (IPS) if there exists a spacetime $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$, a cosmic time function T on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$, and a conformal factor $\Omega(T)$ such that:

1. $\mathcal{M}$ is the open submanifold $T > 0$;
2. $\mathbf{g} = \Omega^2 \widetilde{\mathbf{g}}$ on $\mathcal{M}$, with $\widetilde{\mathbf{g}}$ regular (at least C^2 and nondegenerate) on an open neighbourhood of $T = 0$;
3. $\Omega(0) = 0$, and $\exists\, c > 0$ such that $\Omega \in C^0[0, c] \cap C^2(0, c]$ and $\Omega > 0$;
4. $L_0 := \lim_{T \to 0^+} L(T)$ exists and $L_0 \neq 1$, where $L := \Omega'' \Omega / \Omega'^2$.

Essentially, the physical spacetime $(\mathcal{M}, \mathbf{g})$ is conformally related to an unphysical counterpart $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$, in which $\mathcal{S}_0$ is a regular spacelike hypersurface known as an IPS. The singularity in $(\mathcal{M}, \mathbf{g})$ that corresponds to $\mathcal{S}_0$ arises solely due to the vanishing of the conformal factor Ω for $T = 0$. When applying Definition 2.6a in the cosmological context, however, the behaviour of the fundamental four-velocity field $\mathbf{u}$ (equivalently, the timelike congruence it generates) near the singularity must also be considered [18].

Definition 2.6b (IPS fluid congruence): With any timelike unit vector field $\mathbf{u}$ on $(\mathcal{M}, \mathbf{g})$, we may associate a timelike unit vector field $\widetilde{\mathbf{u}}$ on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ such that $\widetilde{\mathbf{u}} = \Omega \mathbf{u}$ on $\mathcal{M}$. The vector field $\mathbf{u}$ is regular at the IPS if $\widetilde{\mathbf{u}}$ is regular (at least C^2) on an open neighbourhood of $T = 0$. It is also orthogonal to the IPS if $\widetilde{\mathbf{u}}$ is orthogonal to $T = 0$.

Now, it is not obvious from Definitions 2.6 how the admission of an IPS characterises an (asymptotically) isotropic cosmological model, in the sense of Definition 2.4. To show this, it is necessary to consider various isotropy-related curvature and kinematic quantities that provide different measures of a model's isotropy. Specifically, we have Weyl isotropy (where the Weyl tensor C_{abcd} is everywhere zero), Ricci isotropy relative to $\mathbf{u}$ (where the anisotropic parts Σ_a and $\Sigma_a^{\ b}$ of the Ricci tensor relative to $\mathbf{u}$ are everywhere zero), and kinematic isotropy relative to $\mathbf{u}$ (where the shear σ, vorticity ω and acceleration vector $\dot{u}^a$ are everywhere zero) [18]. As mentioned in Chapter 1, the Weyl tensor is also directly associated with gravitational entropy. Formulae for these quantities are given in Appendix A.

In general, all curvature and kinematic quantities might diverge at the singularity; for an asymptotic notion of isotropy (and low gravitational entropy), it is sufficient to require that C_{abcd} and the other quantities are dominated respectively by the Ricci tensor R_{ab} and the expansion θ. Hence a cosmological model that admits an IPS also exhibits isotropy and minimal gravitational entropy near its singularity, in the sense that

$$\lim_{T\to 0^+} \frac{C_{abcd}C^{abcd}}{R_{ab}R^{ab}} = \lim_{T\to 0^+} \frac{\Sigma_a \Sigma^a}{\theta^4} = \lim_{T\to 0^+} \frac{\Sigma_{ab}\Sigma^{ab}}{\theta^4}$$
$$= \lim_{T\to 0^+} \frac{\sigma^2}{\theta^2} = \lim_{T\to 0^+} \frac{\omega^2}{\theta^2} = \lim_{T\to 0^+} \frac{\dot{u}_a \dot{u}^a}{\theta^2} = 0 \quad (2.17)$$

for a fundamental four-velocity field $\mathbf{u}$ that is orthogonal to the IPS [18]. In this thesis, we study models that admit an IPS at which $\mathbf{u}$ is regular — in which case $\mathbf{u}$ may be chosen as orthogonal to the IPS, since a hypersurface-orthogonal congruence always exists.

Many general and model-specific results regarding isotropic singularities have been derived by Scott and Ericksson [32, 33, 34, 35]. For example, vacuum spacetimes do not admit an IPS; neither do shear-free, perfect fluid models that are not FLRW. Interestingly, while FLRW models might be expected to admit an IPS on account of their isotropy, only those that are initially decelerating ($\lim_{T\to 0^+} q \in (0, \infty)$) actually do so. Another example that admits an IPS is a radiation-filled universe in the Kantowski–Sachs class of homogeneous but anisotropic non-Bianchi models (see Section 4.2.2.2). Finally, we note that Definitions 2.6 are generalisable to a time-symmetric form: this defines an isotropic singularity in the future, and is detailed in the following section.

2.2.2 Future states

The conformal framework of the IPS offers a powerful method of modelling the Universe at early times, as the singular behaviour at $T = 0$ may be "removed" via the conformal transformation. An attempt to extend this method to late times has been made by Höhn and Scott [20, 36, 37, 38], who have defined and studied four possible future states for the Universe. Two of these are isotropic in nature and incompatible with the Weyl curvature hypothesis, but have been included for completeness. The first is simply a time-reversed analogue of the IPS, and describes a spacetime that collapses to a singularity in a highly isotropic manner.

Definition 2.7a (Isotropic future singularity): A spacetime $(\mathcal{M}, \mathbf{g})$ admits an isotropic future singularity (IFS) if there exists a spacetime $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, a cosmic time function T on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, and a conformal factor $\Omega(T)$ such that:

1. $\mathcal{M}$ is the open submanifold $T < 0$;
2. $\mathbf{g} = \Omega^2 \overline{\mathbf{g}}$ on $\mathcal{M}$, with $\overline{\mathbf{g}}$ regular (at least C^2 and nondegenerate) on an open neighbourhood of $T = 0$;
3. $\Omega(0) = 0$, and $\exists\, c > 0$ such that $\Omega \in C^0[-c, 0] \cap C^2[-c, 0)$ and $\Omega > 0$;
4. $L_0 := \lim_{T\to 0^-} L(T)$ exists and $L_0 \neq 1$, where $L := \Omega''\Omega/\Omega'^2$.

The main difference from Definition 2.6a is that $T < 0$ and approaches zero (the future singularity) from below. As an added distinction, the unphysical manifold and metric are denoted with bars instead of tildes; this convention may be extended to all quantities defined on $\widetilde{\mathcal{M}}$ and $\overline{\mathcal{M}}$ if clarity is necessary, although past and future states are analysed separately in this thesis (as they are, in principle, completely independent). The second isotropic future state is defined in similar fashion to the IFS, and corresponds to a spacetime that expands forever in a highly isotropic manner.

Definition 2.8a (Future isotropic universe): A spacetime $(\mathcal{M}, \mathbf{g})$ admits a future isotropic universe (FIU) if there exists a spacetime $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, a cosmic time function T on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, and a conformal factor $\Omega(T)$ such that:

1. $\mathcal{M}$ is the open submanifold $T < 0$;
2. $\mathbf{g} = \Omega^2 \overline{\mathbf{g}}$ on $\mathcal{M}$, with $\overline{\mathbf{g}}$ regular (at least C^2 and nondegenerate) on an open neighbourhood of $T = 0$;
3. $\lim_{T\to 0^-} \Omega = \infty$, and $\exists\, c > 0$ such that $\Omega \in C^2[-c, 0)$ and $\Omega > 0$ is strictly increasing on $[-c, 0)$;
4. $L_0 := \lim_{T\to 0^-} L(T)$ exists and $L_0 \neq 1, 2$, where $L := \Omega''\Omega/\Omega'^2$ and $L \in C^0[-c, 0)$.

As intuition might suggest, an FIU essentially differs from an IFS in that the conformal factor Ω increases strictly and without bound as $T \to 0^-$ (since the physical metric $\mathbf{g}$ is expected to blow up in an ever-expanding spacetime, instead of becoming degenerate as at a singularity). For cosmological models, the regularity/orthogonality of a timelike congruence at these isotropic future states must also be defined. This is done exactly as in Definition 2.6b [20].

Definition 2.7b/2.8b (IFS/FIU fluid congruence): With any timelike unit vector field $\mathbf{u}$ on $(\mathcal{M}, \mathbf{g})$, we may associate a timelike unit vector field $\overline{\mathbf{u}}$ on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ such that $\overline{\mathbf{u}} = \Omega\mathbf{u}$ on $\mathcal{M}$. The vector field $\mathbf{u}$ is regular at the IFS/FIU if $\overline{\mathbf{u}}$ is regular (at least C^2) on an open neighbourhood of $T = 0$. It is also orthogonal to the IFS/FIU if $\overline{\mathbf{u}}$ is orthogonal to $T = 0$.

As with the IPS, we focus on models that admit an IFS/FIU at which $\mathbf{u}$ is regular. Isotropic future states are appropriately named, as cosmological models that admit them exhibit Weyl isotropy and kinematic isotropy relative to $\mathbf{u}$ in the asymptotic sense of (2.17) [20]. Possible future states for the FLRW models have been studied in detail by Threlfall [39]: only models that are eventually decelerating ($\lim_{T\to 0^-} q \in (0, \infty)$) admit an IFS, while only those that are eventually accelerating with a divergent scale factor ($\lim_{T\to 0^-} q \in (-\infty, 0)$ and $\lim_{T\to 0^-} a = \infty$) admit an FIU. Hence there is an entire subclass of FLRW models that admit neither an IFS nor an FIU, even though they are isotropic in the usual sense. We return to these in the context of anisotropic future states (and the corresponding timelike congruences), whose definitions follow.

Definition 2.9a (Anisotropic future endless universe): A spacetime $(\mathcal{M}, \mathbf{g})$ admits an anisotropic future endless universe (AFEU) if there exists an extended manifold $\overline{\mathcal{M}} \supset \mathcal{M}$, a symmetric C^0 tensor field $\overline{\mathbf{g}}$ of type $(0,2)$ on $\overline{\mathcal{M}}$, a cosmic time function T on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, and a conformal factor $\Omega(T)$ such that:

1. $\mathcal{M}$ is the open submanifold $T < 0$;
2. $\mathbf{g} = \Omega^2\overline{\mathbf{g}}$ on $\mathcal{M}$, with $\overline{\mathbf{g}}$ at least C^0 on an open neighbourhood of $T = 0$ and degenerate, but not causally degenerate, on $T = 0$;
3. $\lim_{T\to 0^-} \Omega = \infty$, and $\exists\, c > 0$ such that $\Omega \in C^2[-c, 0)$ and $\Omega > 0$ is strictly increasing on $[-c, 0)$;
4. $L_0 := \lim_{T\to 0^-} L(T)$ exists and $L_0 \neq 1$, where $L := \Omega''\Omega/\Omega'^2$ and $L \in C^0[-c, 0)$;
5. $\lim_{T\to 0^-} \Omega^6 |\det \overline{\mathbf{g}}\,| = \infty$.

Definition 2.10a (Anisotropic future singularity): A spacetime $(\mathcal{M}, \mathbf{g})$ admits an anisotropic future singularity (AFS) if there exists an extended manifold $\overline{\mathcal{M}} \supset \mathcal{M}$, a symmetric C^0 tensor field $\overline{\mathbf{g}}$ of type $(0,2)$ on $\overline{\mathcal{M}}$, a cosmic time function T on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$, and a conformal factor $\Omega(T)$ such that:

1. $\mathcal{M}$ is the open submanifold $T < 0$;
2. $\mathbf{g} = \Omega^2\overline{\mathbf{g}}$ on $\mathcal{M}$, with $\overline{\mathbf{g}}$ at least C^0 on an open neighbourhood of $T = 0$ and degenerate, but not causally degenerate, on $T = 0$;
3. $\lim_{T\to 0^-} \Omega = \infty$, and $\exists\, c > 0$ such that $\Omega \in C^2[-c, 0)$ and $\Omega > 0$ is strictly increasing on $[-c, 0)$;
4. $L_0 := \lim_{T\to 0^-} L(T)$ exists and $L_0 \neq 1$, where $L := \Omega''\Omega/\Omega'^2$ and $L \in C^0[-c, 0)$;
5. $\lim_{T\to 0^-} \Omega^8 |\det \overline{\mathbf{g}}\,| = 0$.

Definition 2.9b/2.10b (AFEU/AFS fluid congruence): With any timelike unit vector field $\mathbf{u}$ on $(\mathcal{M}, \mathbf{g})$, we may associate a timelike unit vector field $\overline{\mathbf{u}}$ on $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ such that $\overline{\mathbf{u}} = \Omega\mathbf{u}$ on $\mathcal{M}$. The vector field $\mathbf{u}$ is regular at the AFEU/AFS if $\overline{\mathbf{u}}$ is regular (at least C^2) on an open neighbourhood of $T = 0$. It is also orthogonal to the AFEU/AFS if $\overline{\mathbf{u}}$ is orthogonal to $T = 0$.

Definition 2.9b/2.10b is analogous to Definition 2.7b/2.8b; again, we focus on models that admit an AFEU/AFS at which $\mathbf{u}$ is regular. A distinguishing feature of the anisotropic future states is that the unphysical "metric" $\overline{\mathbf{g}}$ must be degenerate on the spacelike hypersurface $\mathcal{S}_0$ (but non-causally such that $\overline{\mathbf{g}}(\overline{\mathbf{u}}, \overline{\mathbf{u}}) \neq 0$ on $\mathcal{S}_0$), since any regular choice of $\overline{\mathbf{g}}$ leads inevitably to Weyl isotropy and kinematic isotropy relative to $\mathbf{u}$ in the asymptotic sense of (2.17) [20]. The pair $(\overline{\mathcal{M}}, \overline{\mathbf{g}})$ is defined separately, as it does not strictly constitute a spacetime under the nondegeneracy requirement of Definition 2.1. While the conformal factor Ω blows up for both anisotropic future states, the difference between an AFEU and an AFS still lies in the unboundedness or degeneracy of the physical metric $\mathbf{g}$ (which is reflected by the limiting behaviour of $\det \mathbf{g}$ as $T \to 0^-$).

The anisotropic future states, with their degenerate conformal structures, are considerably harder to study than their isotropic counterparts. Still, the continuity of $\overline{\mathbf{g}}$ and assumed regularity of $\overline{\mathbf{u}}$ at $\mathcal{S}_0$ makes the unphysical spacetime in an AFEU-/AFS-admitting model far more tractable than the physical one.

It is of concern, however, that some isotropic spacetimes (e.g. Minkowski space, or the subclass of FLRW models that do not admit an IFS/FIU) have been shown to admit an anisotropic future state [39]; this might necessitate a slight revision of the anisotropic definitions to exclude exact isotropy and the admission of an isotropic future state. Furthermore, models that admit an AFEU/AFS are not immediately compatible with the Weyl curvature hypothesis, in that gravitational entropy (which may be measured by the quotient $C_{abcd}C^{abcd}/R_{ab}R^{ab}$ in (2.17)) might not be maximal at the future state [36]. Further research on these and other issues is ongoing.

2.3 Self-similarity in cosmology

Self-similar solutions in general relativity were first studied by Cahill and Taub [22]; these correspond to spacetimes that are scale-invariant (usually with a time-dependent scale), and are often straightforward to derive or analyse as the additional symmetry allows simplification of the EFE. A cosmological model might be exactly self-similar, or only approximately so in different epochs of its evolution. Following Eardley [40], we loosely define asymptotic self-similarity here as the notion of approximate self-similarity at early or late times, and seek a more precise working definition in Chapter 3.

Three other dichotomies for self-similarity in the context of general relativity have been identified by Carr and Coley [21]. One of these distinguishes between continuous self-similarity (in which all dimensionless quantities are preserved) and discrete self-similarity (in which all dimensionless quantities repeat themselves on some spacetime scale). Another makes the distinction between geometric self-similarity (of the spacetime metric) and physical self-similarity (of the matter content), although the former implies the latter for perfect fluid spacetimes. Finally, there are self-similar solutions that possess exact self-similarity or pass into the self-similar regime in a regular manner (self-similarity of the first kind), and more general ones that do not (self-similarity of the second kind). We restrict our study in this thesis to continuous, geometric self-similarity of the first kind for vacuum and perfect fluid spacetimes.

2.3.1 Exact self-similarity

The isometry framework introduced in Section 2.1.2 may be expanded to include more general symmetries of a spacetime $(\mathcal{M}, \mathbf{g})$, i.e. homothetic symmetries (also known as homotheties or similarities) and conformal symmetries. A conformal symmetry preserves the metric up to a general point-dependent scaling factor λ, which is constant in the case of a similarity [41]. One-parameter groups of such symmetries are generated by C^2 vector fields $\mathbf{X}$ on $\mathcal{M}$ that satisfy $\mathcal{L}_{\mathbf{X}}\mathbf{g} = 2\lambda\mathbf{g}$. In local coordinates, this gives the conformal Killing equation

$$X_{(a;b)} = \lambda g_{ab}, \tag{2.18}$$

where the scaling factor $\lambda \neq 0$ is a C^2 function on $\mathcal{M}$ and $\mathbf{X}$ is known as a conformal Killing vector field (CKVF). If $\lambda \neq 0$ is a constant, (2.18) may be termed the homothetic equation and $\mathbf{X}$ is known as a homothetic vector field (HVF). If $\lambda = 0$, the Killing equation (2.8) is recovered and $\mathbf{X}$ is a KVF. Our focus in this section is on the homothetic equation, since exactly self-similar spacetimes are characterised by the existence of an HVF.

While an isometry on a spacetime $(\mathcal{M}, \mathbf{g})$ leaves $\mathbf{g}$ invariant, a similarity on $(\mathcal{M}, \mathbf{g})$ induces a constant scale transformation $\mathbf{g} \to e^{2\lambda}\mathbf{g}$. We are concerned primarily with nontrivial ($\lambda \neq 0$) similarities whose associated HVF has a nonzero timelike component. Hence an exactly self-similar spacetime typically exhibits scale invariance with time, i.e. the spacelike hypersurfaces $\mathcal{S}_T$ are similar to one another at all times. For the exactly self-similar spacetimes in this thesis, an important result due to Eardley [40] states that the r-dimensional isometry group G_r of $(\mathcal{M}, \mathbf{g})$ is a subgroup of the $(r+1)$-dimensional similarity group H_{r+1} (the set of all similarities on $(\mathcal{M}, \mathbf{g})$). In other words, there exists precisely one (up to a constant factor) HVF for an exactly self-similar spacetime.

Many specific examples of exact self-similarity are known, in that the spacetime metric is given explicitly and admits an HVF. Two simple ones are presented here for future reference. First we have the Milne universe, an expanding vacuum spacetime whose line element is given by

$$ds^2 = -dt^2 + t^2 \left(d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right), \tag{2.19}$$

i.e. (2.9) with $a(t) = t$ and $k = -1$. This line element also describes a future-directed light cone in Minkowski space, and may accordingly be cast as the Minkowski line element (in spherical coordinates)

$$ds^2 = -d\tilde{t}^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{2.20}$$

via the coordinate transformation

$$\tilde{t} = t\cosh\chi, \qquad r = t\sinh\chi. \tag{2.21}$$

Another common form for the Milne line element is given by [42]

$$ds^2 = -d\bar{t}^2 + \bar{t}^2 \left(dx^2 + e^{2x}\left(dy^2 + dz^2\right)\right), \tag{2.22}$$

which may be obtained from the Minkowski line element (in Cartesian coordinates)

$$ds^2 = -d\tilde{t}^2 + du^2 + dv^2 + dw^2 \tag{2.23}$$

via the coordinate transformation

$$\bar{t} = \sigma, \qquad x = \ln\left|\frac{\tilde{t}+u}{\sigma}\right|, \qquad y = \frac{v}{\tilde{t}+u}, \qquad z = \frac{w}{\tilde{t}+u}, \tag{2.24}$$

where $\sigma = (\tilde{t}^2 - u^2 - v^2 - w^2)^{1/2}$. It is straightforward to verify from (2.18) and (2.19) that the HVF for the Milne universe is given (in the coordinates of (2.19)) by

$$X^a = (t, 0, 0, 0). \tag{2.25}$$

The other example is the class of exactly self-similar Bianchi I cosmologies, whose line element is given by [42]

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2. \tag{2.26}$$

These include as special cases the “circle” of Kasner vacuum solutions, where

$$p_1 = \frac{1}{3}\left(1 - 2\cos\psi\right), \qquad p_{2,3} = \frac{1}{3}\left(1 + \cos\psi \pm \sqrt{3}\sin\psi\right) \tag{2.27}$$

for $\psi \in [0, 2\pi)$, and the flat FLRW models with $\gamma \in (0, 2]$, where

$$p_1 = p_2 = p_3 = \frac{2}{3\gamma}. \tag{2.28}$$

In accordance with (2.9), the line element for the latter may also be written as

$$ds^2 = -dt^2 + t^{\frac{4}{3\gamma}}\left(d\chi^2 + \chi^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right), \tag{2.29}$$

i.e. $a(t) = t^{2/(3\gamma)}$ and $k = 0$. The HVF for these Bianchi I cosmologies is given by

$$X^a = \left(t, (1 - p_1)\,x, (1 - p_2)\,y, (1 - p_3)\,z\right). \tag{2.30}$$

The tractability of exactly self-similar solutions is such that all vacuum/perfect fluid Bianchi cosmologies admitting an HVF (with nonzero timelike component) have already been found, as proven by Hsu and Wainwright [43]. Furthermore, a Hamiltonian formulation of the EFE has been used by Uggla [44] to systematically derive several classes of inhomogeneous, exactly self-similar spacetimes. Research on exactly self-similar spacetimes is actively ongoing, since such models are amenable to in-depth investigation and also approximate a large variety of spacetimes, either at intermediate times or in the asymptotic regime [45].

2.3.2 Asymptotic self-similarity

Three distinct notions of the term “asymptotic self-similarity” are identifiable in the literature, and it is not immediately clear if or how they are equivalent. We discuss these approaches in general and with respect to the conformal framework of QC–WCH.

2.3.2.1 Spherically symmetric approach

Exact self-similarity in spherically symmetric spacetimes has been studied extensively, due to the applicability and relative simplicity of such models. We note here that spherical symmetry in the cosmological context is a property of the spacelike hypersurfaces, and does not necessarily imply that the spacetime is isotropic. Analysis of spherically symmetric, exactly self-similar spacetimes may be carried out in various coordinate systems; we focus on the comoving approach used by Carr and Coley [46, 47] to provide a thorough classification of such spacetimes. The full method is fairly involved, and only its main features are presented.

Following Cahill and Taub [22], the general line element for a spherically symmetric, exactly self-similar spacetime may be written (in spherical coordinates) as

$$ds^2 = -e^{\alpha(z)}dt^2 + e^{\beta(z)}dr^2 + S^2(z)\,r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{2.31}$$

where $z = r/t$ is a dimensionless "similarity variable", and the functions α, β and S are arbitrary up to consistency with the EFE. For a perfect fluid spacetime, α and β may be eliminated by introducing the function $x(z) = ((1/2)\mu r^2)^{-(\gamma-1)/\gamma}$ and integrating the conservation equations (2.7). This yields the line element

$$ds^2 = -C_\alpha x^2 z^{\frac{4(\gamma-1)}{\gamma}} dt^2 + C_\beta S^{-4} x^{\frac{2}{\gamma-1}} dr^2 + S^2 r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{2.32}$$

where C_α and C_β are constants of integration.

The remaining EFE may be cast as a system of four ordinary differential equations in $\ln|z|$ for S and x (i.e. $\dot{S} = z dS/dz$), and analysed qualitatively using standard methods. Three exactly self-similar solutions (which might be unphysical) are explicitly obtainable for each value of γ; these are given by

$$S_1(z) = z^{-\frac{2}{3\gamma}}, \qquad x_1(z) = z^{-\frac{2(\gamma-1)}{\gamma}}, \tag{2.33}$$

$$S_2(z) = S_{2,0} z^{-1}, \qquad x_2(z) = x_{2,0} z^{-\frac{2(\gamma-1)}{\gamma}}, \tag{2.34}$$

$$S_3(z) = S_{3,0}, \qquad x_3(z) = x_{3,0}, \tag{2.35}$$

where the constants $S_{i,0}$ and $x_{i,0}$ depend on γ. Equations (2.33), (2.34) and (2.35) correspond respectively to FLRW, Kantowski–Sachs and static spacetimes, although suitable coordinate transformations are required in order to demonstrate this explicitly.

We now consider asymptotic solutions of the form

$$S(z) = S_i(z)\, e^{A(z)}, \qquad x(z) = x_i(z)\, e^{B(z)}, \tag{2.36}$$

where S_i and x_i are given by (2.33)–(2.35), such that the governing equations for S and x may be cast as ordinary differential equations in $\ln|z|$ for A and B. These solutions are said to be asymptotically FLRW/Kantowski–Sachs/static, in the sense that they asymptote to (2.33)–(2.35) as $A, B \to 0$. Having been constructed from (2.31), however, they are exactly self-similar as well. Hence "asymptotic self-similarity" in this context refers to the existence of asymptotic relationships among spherically symmetric, exactly self-similar spacetimes — and differs from the notion of asymptotic self-similarity that we are after.

2.3.2.2 Dynamical systems approach

As implied in Chapter 1, usage of the term "asymptotic self-similarity" is most prevalent in a subfield of cosmology that employs general methods in dynamical systems theory to deduce the qualitative behaviour of cosmological models. Application of such methods to homogeneous models was initiated by Collins [48, 49], then developed extensively by Wainwright, Hsu and Hewitt [50, 51, 52] in their comprehensive analysis of Bianchi cosmologies. The dynamical systems approach has been generalised (with limited success) to inhomogeneous and anisotropic models with a G_2 isometry group [53, 54, 55], and more recently to fully inhomogeneous models that admit no isometries [56].

Central to the dynamical systems approach is the orthonormal frame formalism pioneered by Ellis and MacCallum [57, 58]. Instead of choosing the usual coordinate basis $\{\partial/\partial x^a\}$ for a spacetime $(\mathcal{M}, \mathbf{g})$ such that

$$\mathbf{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = g_{ab}, \qquad ds^2 = g_{ab} dx^a dx^b, \tag{2.37}$$

we switch to an orthonormal basis $\{\mathbf{e}_a\}$ that satisfies

$$\mathbf{g}\left(\mathbf{e}_a, \mathbf{e}_b\right) = \eta_{ab}, \qquad ds^2 = \eta_{ab}\boldsymbol{\omega}^a\boldsymbol{\omega}^b, \tag{2.38}$$

where $[\eta_{ab}] = \mathrm{diag}\left(-1,1,1,1\right)$ and $\{\boldsymbol{\omega}^a\}$ is the corresponding dual basis of one-forms. Expanding the Lie bracket relative to $\{\mathbf{e}_a\}$ such that $[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c_{\ ab}\mathbf{e}_c$ yields the 24 independent commutation functions $\gamma^c_{\ ab}$, and the Jacobi identities

$$\left[\left[\mathbf{e}_a, \mathbf{e}_b\right], \mathbf{e}_c\right] + \left[\left[\mathbf{e}_c, \mathbf{e}_a\right], \mathbf{e}_b\right] + \left[\left[\mathbf{e}_b, \mathbf{e}_c\right], \mathbf{e}_a\right] = 0 \tag{2.39}$$

may be written as

$$\mathbf{e}_{[d}\gamma^c_{\ ab]} - \gamma^c_{\ e[d}\gamma^e_{\ ab]} = 0. \tag{2.40}$$

Equations (2.2), (2.7) and (2.40) essentially allow the EFE to be cast as a system of first-order evolution equations for the commutation functions, as opposed to second-order partial differential equations for the metric components. If a fundamental four-velocity field $\mathbf{u}$ is specified, the commutation functions may be decomposed into various geometric and kinematic variables (including the generalised Hubble parameter H), and the EFE recast in terms of these variables.

For a homogeneous cosmological model $(\mathcal{M}, \mathbf{g}, \mathbf{u})$, the vector field $\mathbf{e}_0$ is typically chosen to be $\mathbf{u} = \partial/\partial T$, where T is a cosmic time function determined by $\mathbf{u}$. The EFE simplify to a system of first-order ordinary differential equations in T, which may be analysed qualitatively using the methods of dynamical systems theory. If $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ is further assumed to be exactly self-similar, the EFE become purely algebraic; this implies that exactly self-similar models correspond to equilibrium points in the state space of the system. The number of commutation function variables describing the state of a cosmological model is also reduced by the assumption of homogeneity. In the case of Bianchi cosmologies, the state vector $(\mathbf{x}, H)$ is six-dimensional, i.e. it comprises three curvature variables, two shear variables, and the expansion variable H.

Now, defining dimensionless time $\tau = \ln|a/a_0|$ (where a is the generalised scale factor) and five dimensionless, expansion-normalised variables $\mathbf{y} = \mathbf{x}/H$, it follows from (2.14) and (2.15) (with $\dot{a} = da/dT$) that

$$\frac{dT}{d\tau} = \frac{1}{H}, \tag{2.41}$$

$$\frac{dH}{d\tau} = -\left(1+q\right)H. \tag{2.42}$$

Equation (2.42) gives the evolution of H, which is decoupled in that the evolution equations for $\mathbf{y}$ are an autonomous system of ordinary differential equations $d\mathbf{y}/d\tau = \mathbf{f}(\mathbf{y})$. Hence the effects of expansion are essentially scaled away by this transformation to dimensionless variables, and the dynamical evolution of a Bianchi model may be analysed in a five-dimensional state space.

The notion of asymptotic self-similarity in the dynamical systems approach follows naturally from this framework: a cosmological model is asymptotically self-similar in the past (future) if there exists a past (future) attractor for its dynamical evolution in state space. Such attractors (i.e. equilibrium points where $d\mathbf{y}/d\tau = 0$) represent exactly self-similar models in state space. For example, the open and closed FLRW models with $\gamma \in (2/3, 2]$ are asymptotically self-similar in the past to the corresponding flat FLRW models; in the future, however, the open models approach the Milne universe while the

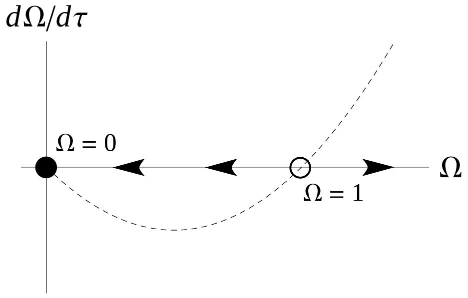

Figure 2.1: Graph of $d\Omega/d\tau$ against Ω for (2.48), with $\gamma \in (2/3, 2]$. All stable (solid) and unstable (open) equilibrium points where $d\Omega/d\tau = 0$ have also been plotted.

closed models asymptote to the time-reversed, contracting flat "models" [59, 60].

These are the simplest results in the dynamical systems approach, as the state space for the FLRW models is reduced to just one dimension by the high degree of symmetry. The first three may be derived in a few short steps; a different analysis is required for closed models in the future, since they recollapse and the expansion-normalised variables diverge at the point of maximal expansion ($H = 0$). First we define the density parameter

$$\Omega = \frac{\mu}{3H^2}, \tag{2.43}$$

which allows (2.12) and (2.15) to be written jointly as

$$q = \frac{1}{2}(3\gamma - 2)\,\Omega. \tag{2.44}$$

From (2.13), (2.14) and (2.43), the value of Ω is related to the geometry of the model by

$$\Omega > 1 \Leftrightarrow k = 1, \qquad \Omega = 1 \Leftrightarrow k = 0, \qquad \Omega < 1 \Leftrightarrow k = -1. \tag{2.45}$$

Next, we eliminate a from the Friedmann equations (2.12) and (2.13) such that

$$\dot{\mu} = -3H\gamma\mu, \tag{2.46}$$

which yields (via (2.41) and (2.42))

$$\frac{d\Omega}{d\tau} = (2q - (3\gamma - 2))\,\Omega. \tag{2.47}$$

Substituting (2.44) into (2.47), we arrive at

$$\frac{d\Omega}{d\tau} = (3\gamma - 2)(\Omega - 1)\,\Omega, \tag{2.48}$$

which describes the (expansion-normalised) dynamical evolution of a single-component perfect fluid FLRW model in terms of its density. Equation (2.48) has an unstable equilibrium point at $\Omega = 1$ and a stable one at $\Omega = 0$ (see Figure 2.1). Hence the open and closed models are asymptotically self-similar in the past to the flat models, and the open models are asymptotically self-similar in the future to the Milne universe (which has $\Omega = 0$ as it is empty). We also note that all FLRW models with $\gamma = 2/3$ are exactly self-similar, since they correspond to equilibrium points of (2.48) as well.

Asymptotic self-similarity in the dynamical systems approach agrees with the notion of approximate self-similarity at early and late times, but only up to the effects of expansion. This is not necessarily undesirable, as a similar caveat exists in the conformal framework of QC–WCH: spacetimes that admit an isotropic past/future state are kinematically isotropic up to expansion effects as well (see (2.17)). There are some problems with fitting the dynamical systems definition into the conformal framework, however.

Firstly, due to the difficulty of extending dynamical systems theory to systems of partial differential equations such as the unsimplified EFE, the dynamical systems approach is restricted mainly to homogeneous models for now. On the other hand, the conformal framework is far more general as it focuses on the geometric properties of a given spacetime instead of its dynamical behaviour. Another obstacle is that the dynamical systems definition for an asymptotically self-similar model depends on the existence of an exactly self-similar counterpart for it to asymptote to; it is not immediately clear how this is reflected in the properties of the asymptotic model itself. The definition of asymptotic self-similarity that we are after is ideally an intrinsic one, since spacetimes are dealt with individually in the conformal framework.

A successful generalisation of the dynamical systems definition to the conformal framework is likely to use the fact that all dimensionless variables are preserved by the flow of an HVF (see Section 3.1.3), and/or incorporate some form of normalisation with respect to expansion (see Section 5.1). Even if such ideas are not adopted, it is still desirable for the eventual working definition to demonstrate good agreement with the numerous asymptotic self-similarity results that exist within the dynamical systems approach.

2.3.2.3 Homothetic equation approach

Although exact self-similarity in a spacetime corresponds precisely to the existence of an HVF, there have been no significant attempts in the literature to formulate a definition of asymptotic self-similarity based on the homothetic equation, i.e. (2.18) with constant $\lambda \neq 0$. However, Cain [24] explores a couple of ways in which this might be done.

The first approach is arguably the most natural, and involves searching for a solution to the homothetic equation with an approximation to the metric at early or late times. One example suited to this method is a Heckmann–Schücking dust universe, whose line element is given by [40]

$$ds^2 = -dt^2 + t^{2p_1} (t + t_0)^{\frac{4}{3}-2p_1} dx^2 + t^{2p_2} (t + t_0)^{\frac{4}{3}-2p_2} dy^2 + t^{2p_3} (t + t_0)^{\frac{4}{3}-2p_3} dz^2, \quad (2.49)$$

where the exponents p_i satisfy (2.27) and $t_0 > 0$ is a constant. At early times, $t \to 0^+$ and (2.49) is approximated by the Kasner line element (2.26) (after rescaling each spatial coordinate); at late times, $t \to \infty$ and (2.49) is approximated by the flat FLRW line element (2.29) with $\gamma = 1$ (in Cartesian coordinates). These line elements admit the HVF (2.30) with p_i given respectively by (2.27) and (2.28), which agrees with the fact that the Heckmann–Schücking solutions are asymptotically self-similar in the past and future within the dynamical systems approach [42].

On the other hand, this method contradicts the result in Section 2.3.2.2 that the open and closed FLRW models with $\gamma \in (2/3, 2]$ are asymptotically self-similar in the past. As a specific example, the line element for the open radiation ($\gamma = 4/3$) model is given by

$$ds^2 = -dt^2 + (2C + t)\, t \left(d\chi^2 + \sinh^2 \chi \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right)\right), \quad (2.50)$$

i.e. (2.9) with $a(t) = \sqrt{(2C+t)t}$ and $k = -1$, where $C = \sqrt{(1/3)\mu a^4}$ is a constant [2]. At early times, $t \to 0^+$ and we have

$$ds^2 \sim -dt^2 + 2Ct\left(d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right), \tag{2.51}$$

which fails to admit a solution to the homothetic equation.

The second approach appears slightly more general, in that an approximate solution to the homothetic equation (with the exact metric) is sought instead. This is formalised by Cain [24] as requiring the existence of a vector field $\mathbf{H}$ on a spacetime $(\mathcal{M}, \mathbf{g})$ such that

$$\lim_{T\to 0^{+(-)}} \left(H_{(a;b)} - \lambda g_{ab}\right) = 0, \tag{2.52}$$

where the cosmic time function $T \to 0^{+(-)}$ as the spacetime's past (future) is approached, and $\lambda \neq 0$ is a constant. Taking $\mathbf{H} = \lambda\mathbf{X}$ for the open radiation FLRW model (2.50), where

$$X^a = \left(t, \frac{1}{2}\chi, 0, 0\right) \tag{2.53}$$

is also the HVF (2.30) for the flat radiation model in spherical coordinates, the nonzero components of the difference tensor $D_{ab} := H_{(a;b)} - \lambda g_{ab}$ are given by

$$D_{11} = \frac{1}{2}\lambda t^2, \tag{2.54}$$

$$D_{22} = \lambda\left(\frac{1}{2}t^2\chi\cosh\chi + Ct\left(\chi\cosh\chi - \sinh\chi\right)\right)\sinh\chi, \tag{2.55}$$

$$D_{33} = \lambda\left(\frac{1}{2}t^2\chi\cosh\chi + Ct\left(\chi\cosh\chi - \sinh\chi\right)\right)\sinh\chi\sin^2\theta. \tag{2.56}$$

The terms in (2.54)–(2.56) clearly vanish in the limit as $t \to 0^+$. Hence the open radiation FLRW model is asymptotically self-similar in the past according to (2.52) with $T = t$, which agrees with the dynamical systems result. Contrary to expectation, however, it is not asymptotically self-similar in the future with the Milne HVF (2.25) in place of (2.53). Furthermore, the Heckmann–Schücking solutions studied in the first approach are no longer asymptotically self-similar in the future with the flat dust FLRW HVF (2.30).

Several problems exist in Cain's [24] investigation of these candidate definitions for asymptotic self-similarity. The most serious ones pertain to the second approach, i.e. the attempt to formalise the notion of approximate self-similarity via the limit of a difference of terms in (2.52). This is misguided on two counts.

Firstly, the time-dependent metric components g_{ab} typically vanish as a spacetime's past (singularity) is approached, and so the corresponding components of $H_{(a;b)}$ are also required to vanish if (2.52) is to be satisfied. However, simply taking the limit of two vanishing quantities is the wrong way of comparing their asymptotic behaviour, as it disregards the rates at which they approach zero and permits trivial/degenerate equivalence. For example, the tensor components (2.55) and (2.56) vanish as $t \to 0^+$ if and only if we neglect terms that are $O(t)$, which essentially implies that the open radiation FLRW model (2.50) is being considered in the degenerate regime ($ds^2 \sim -dt^2$).

Secondly, the time-dependent metric components g_{ab} typically blow up as an ever-expanding spacetime's future is approached, and so the corresponding components of $H_{(a;b)}$ are also required to blow up if (2.52) is to be satisfied. Hence the difference tensor

D_{ab} tends to blow up as well (e.g. the terms in (2.54)–(2.56) are unbounded as $t \to \infty$), and (2.52) will not be satisfied unless $H_{(a;b)} = \lambda g_{ab}$ to begin with, i.e. $\mathbf{H}$ is an HVF and the spacetime is exactly self-similar. This explains the negative results for the open radiation FLRW and Heckmann–Schücking models in the future.

A homothetic equation-based approach to a definition of asymptotic self-similarity is worth developing, but some care is required since the difference between the two candidate approaches is more subtle than expected. The second type of definition is particularly promising, as it lends itself to generalisation by shifting the asymptotic process into the homothetic equation. While these approaches might appear simplistic (much of the spacetime's dynamical information is not taken into account), they are far more tractable than the dynamical systems approach; also, they narrow the notion of asymptotic self-similarity down to an intrinsic property of individual spacetimes, which is more compatible with the conformal framework of QC–WCH.

Chapter 3

Defining asymptotic self-similarity

In this chapter, we execute our strategy for formulating a definition of asymptotic self-similarity that is compatible with the conformal framework and the set of dynamical systems results. Section 3.1 introduces three candidate approaches that correspond to distinct notions of asymptotic self-similarity, while building a case for developing the homothetic equation approach in particular. Various examples from the set of dynamical systems results are examined in Section 3.2 under this preliminary definition, with a view to improving the eventual working definition that is formalised and discussed in Section 3.3. The chapter closes with a brief commentary in Section 3.4 on the difficulties that arise with the definition, and the possible role of conformal Killing vector fields.

3.1 Candidate definitions

Motivated by the existing notions of asymptotic self-similarity in the literature, as well as the discussion in Section 2.3.2 of their relevance to the conformal framework, three preliminary definitions are formulated and presented here for further consideration.

3.1.1 Exact mapping approach

The first approach outlined in Section 2.3.2.3 has heuristic and practical merit due to its intuitiveness and simplicity. Loosely, a spacetime $(\mathcal{M}, \mathbf{g})$ is said to be asymptotically self-similar if it is exactly self-similar in an asymptotic regime. If a cosmic time function T is defined such that $T = 0$ corresponds to the spacetime's past/future state, the asymptotic regime may be taken as an arbitrarily small open submanifold $\mathcal{U} \subset \mathcal{M}$ whose boundary $\partial\mathcal{U}$ contains the spacelike hypersurface $\mathcal{S}_0$. This notion of asymptotic self-similarity is then equivalent to the existence of an exact mapping between $(\mathcal{U}, \mathbf{g}|_{\mathcal{U}})$ and its counterpart on some exactly self-similar spacetime, and is made precise in the following definition.

Definition 3.1 (Exact mapping approach): A spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar to an exactly self-similar spacetime $(\mathcal{M}', \mathbf{g}')$ in the past (future) if there exist cosmic time functions T on $(\mathcal{M}, \mathbf{g})$ and T' on $(\mathcal{M}', \mathbf{g}')$ such that $T, T' \to 0^{+(-)}$ respectively as each spacetime's past (future) is approached, open submanifolds $\mathcal{U} \subset \mathcal{M}$ and $\mathcal{U}' \subset \mathcal{M}'$ such that $T, T' = 0$ are contained respectively in $\partial\mathcal{U}$ and $\partial\mathcal{U}'$, and a diffeomorphism $\theta : \mathcal{U} \to \mathcal{U}'$ such that

$$\theta_* \left(\mathbf{g}|_{\mathcal{U}} \right) = \mathbf{g}'|_{\mathcal{U}'} \,. \tag{3.1}$$

In other words, the pair $(\mathcal{U}, \mathbf{g}|_{\mathcal{U}})$ is isometric to the pair $(\mathcal{U}', \mathbf{g}'|_{\mathcal{U}'})$ (since θ is a smooth, invertible map whose differential carries $\mathbf{g}|_{\mathcal{U}}$ into $\mathbf{g}'|_{\mathcal{U}'}$). Following Hawking and Ellis [23], spacetimes that are isometric to one another are taken to be in the same equivalence class, which justifies the statement that $(\mathcal{M}, \mathbf{g}) \equiv (\mathcal{M}', \mathbf{g}')$ in the asymptotic regime.

As illustrated in Section 2.3.2.3, the exact mapping approach has an equivalent form in local coordinates (T, x^μ) on $\mathcal{M}$: a spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar if its associated line element ds^2 reduces as $T \to 0^{+(-)}$ to the line element ds'^2 for an exactly self-similar spacetime $(\mathcal{M}', \mathbf{g}')$ (via any required coordinate transformation). This line element formulation is valid as all the geometric information of a spacetime is captured implicitly in the metric (since it is defined on the manifold), which in turn is described locally by the associated line element. It is also clearly more convenient to work with than the original form of Definition 3.1.

The lack of ambiguity and ease of use provided by the line element formulation stems from the purely geometric nature of the exact mapping approach. Another key advantage offered by Definition 3.1 is versatility. The idea of a spacetime being asymptotically self-similar to an exactly self-similar counterpart is well-defined as in the dynamical systems approach, but compatibility with the conformal framework is maintained since prior knowledge of the exactly self-similar spacetime is not required in principle. Indeed, a spacetime is asymptotically self-similar under Definition 3.1 as long as the asymptotic form of its line element yields a solution to the homothetic equation.

There are several drawbacks with the exact mapping approach that rule it out as a working definition. Crucially, it fails to demonstrate agreement with numerous results in the dynamical systems approach, e.g. the closed FLRW models being asymptotically self-similar in the past to the flat models. The discrepancy boils down to the fact that Definition 3.1 actually describes the much stronger notion of a spacetime being asymptotically identical to another; the closed and flat FLRW models do not satisfy this, as their spacelike hypersurfaces are diffeomorphic to $\mathbb{S}^3$ and $\mathbb{R}^3$ respectively. Furthermore, not every spacetime has a line element that reduces to an asymptotic form (see Section 3.2.1.5), while the exactness of the approach itself resists any attempt at modification.

3.1.2 Homothetic equation approach (revised)

Another candidate approach to a definition of asymptotic self-similarity is more directly based on the homothetic equation than the exact mapping approach, and essentially involves casting (2.18) itself into a suitable asymptotic form. Equation (2.52) has been ruled out as such a form in Section 2.3.2.3, due to inherent problems with taking the limit of a difference of terms. It is also unclear if a homothetic equation approach is necessarily weaker (in the definitional sense) than the exact mapping approach, but this is required in order to generate better agreement with the dynamical systems results.

Our idea, then, is that an asymptotically self-similar spacetime $(\mathcal{M}, \mathbf{g})$ should admit a vector field $\mathbf{X}$ such that both sides of (2.18) are equivalent up to some order of cosmic time T as the past/future state is approached, i.e. (2.18) is satisfied asymptotically. Furthermore, $\mathbf{X}$ is permitted to be "approximately homothetic" in an even broader sense than the asymptotic one: the scaling factor $\lambda \neq 0$ may be a function on the spacelike hypersurface $\mathcal{S}_0$ instead of a constant, and (2.18) is considered as the conformal Killing equation rather than the homothetic equation. A more rigorous statement of this approach follows.

Definition 3.2 (Homothetic equation approach): A spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar in the past (future) if there exists a cosmic time function T such that $T \to 0^{+(-)}$ as the spacetime's past (future) is approached, and a vector field $\mathbf{X}$ on $\mathcal{M}$ such that

$$\mathcal{L}_{\mathbf{X}}\mathbf{g} \sim 2\lambda\mathbf{g} \tag{3.2}$$

as $T \to 0^{+(-)}$ for some function $\lambda \neq 0$ on $T = 0$.

The asymptotic regime may be formalised as the open submanifold $\mathcal{U}$ introduced in Definition 3.1, but the looser expression "as $T \to 0^{+(-)}$" is chosen in this case to emphasise the asymptotic process. Since (3.2) is a tensor relation, it is not strictly well-defined with respect to the asymptotic notation of (1.5) (as a quotient of tensors is generally meaningless). In local coordinates (T, x^μ), however, we obtain the 10 distinct relations $X_{(a;b)} \sim \lambda g_{ab}$ as $T \to 0^{+(-)}$; these generalise (2.18) to asymptotic form, and are well-defined (for nonzero terms) since each tensor component is a function of (T, x^μ). We naturally require correspondence between the everywhere zero components of $X_{(a;b)}$ and g_{ab} as well.

With the option of point-dependence in the scaling factor λ, Definition 3.2 is weaker than the exact mapping approach by construction. As it turns out, the homothetic equation approach is sufficiently weak to ensure the asymptotic self-similarity of some FLRW models (see Sections 3.2.1.2 and 3.2.1.3). It also allows further classification of asymptotically self-similar spacetimes according to whether they satisfy (3.2) with constant λ or not, and lends itself to modification if an improved working definition is desired.

However, it remains to be seen if the full set of dynamical systems results can be reproduced by the homothetic equation approach, in which the spacetime's dynamical information is largely unused. Definition 3.2 also contributes nothing to the notion that a spacetime might be asymptotically self-similar to another (although this is not a problem in the context of the conformal framework), since the vector field $\mathbf{X}$ is defined only on one spacetime and generally has no canonical counterpart on the other. Finally, the amount of freedom in (3.2) (as opposed to, say, the exact homothetic equation) may make it difficult to find a vector field that satisfies Definition 3.2 — or to show that none exists.

3.1.3 Dimensionless variables approach

In accordance with Section 2.3.2.2, the dynamical systems definition of asymptotic self-similarity for a Bianchi cosmology may be stated in terms of the dimensionless variables that describe its evolution in state space: the limits of all such variables exist as the model's past/future state is approached (and equal the constant values of the same variables on some exactly self-similar model). This statement may be applicable to a general cosmological model in principle, if not in practice, and motivates a related definition that is compatible with both the dynamical systems approach and the conformal framework.

An HVF $\mathbf{X}$ on a spacetime $(\mathcal{M}, \mathbf{g})$ induces a scale transformation $\mathbf{T} \to e^{D\lambda}\mathbf{T}$ for every tensorial object $\mathbf{T}$ on $(\mathcal{M}, \mathbf{g})$, where D is the physical dimension of $\mathbf{T}$. Following Eardley [40], the metric $\mathbf{g}$ is assigned $D = 2$ such that $\mathbf{g} \to e^{D\lambda}\mathbf{g}$. This determines the physical dimension of other objects on $(\mathcal{M}, \mathbf{g})$, e.g. vector fields and various kinematic scalars have $D = -1$, while local coordinates (x^a) and commutation functions $\gamma^c{}_{ab}$ are dimensionless ($D = 0$). We then have $\mathcal{L}_{\mathbf{X}}\mathbf{T} = D\lambda\mathbf{T}$ by analogy with the homothetic equation; in particular, the Lie derivative of any dimensionless scalar along $\mathbf{X}$ is everywhere zero.

Hence all dimensionless variables on an exactly self-similar spacetime are constant along the integral curves of its HVF [61].

To fulfil the dynamical systems notion of asymptotic self-similarity, the limits of the dimensionless variables that describe a spacetime's evolution need only exist (and need not be evaluated as per the dynamical systems approach). In other words, it may suffice to require that the change of these variables along the integral curves of some vector field vanishes in the limit as the spacetime's past/future state is approached.

Definition 3.3 (Dimensionless variables approach): A spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar in the past (future) if there exists a cosmic time function T such that $T \to 0^{+(-)}$ as the spacetime's past (future) is approached, and a vector field $\mathbf{X}$ on $\mathcal{M}$ such that

$$\lim_{T\to 0^{+(-)}} \mathcal{L}_{\mathbf{X}}\delta_i = 0 \tag{3.3}$$

for each $\delta_i \in \Delta$, where Δ is a finite set of dimensionless variables on $\mathcal{M}$ that fully describes the dynamical evolution of $(\mathcal{M}, \mathbf{g})$.

The set Δ is finite, since it may always be taken as the set of 24 independent commutation functions $\gamma^c_{\ ab}$ for a general spacetime. As mentioned in Section 2.3.2.2, the cardinality of Δ is reduced to five for Bianchi cosmologies and just one (the density parameter Ω) for FLRW models. Definition 3.3 is likely to reproduce most results in the dynamical systems approach, and may potentially be applied to spacetimes that lie beyond its scope. The definition also suits the conformal framework, as it makes no mention of the exactly self-similar spacetime to which $(\mathcal{M}, \mathbf{g})$ asymptotes.

However, there are serious issues that hinder development of the dimensionless variables approach. For one, deriving explicit expressions for elements of the set Δ is usually no less involved than solving the EFE in full generality, and indirect methods of verifying (3.3) are required. Even if Δ is obtainable, it will in general be extremely unwieldy due to its size. Without reducing the cardinality of Δ (i.e. the number of independent constraints in (3.3)), the practicality of Definition 3.3 is severely limited and all but rules it out as a working definition.

3.2 Fine-tuning the homothetic equation approach

In this section, we focus our attention on moulding the homothetic equation approach into a suitable working definition. The exact mapping approach (with its tractability and possible relationship to Definition 3.2) is also investigated in parallel, while the dimensionless variables approach is given no further consideration in this thesis.

According to Definition 3.2 in local coordinates (T, x^μ), a spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar if there exists a vector field $\mathbf{X}$ such that

$$X_{(a;b)} \sim \lambda g_{ab} \tag{3.4}$$

as $T \to 0^{+(-)}$. Implicit in (3.4) is the requirement that each component of $X_{(a;b)}$ corresponding to a zero (nonzero) component of g_{ab} must itself be zero (nonzero). Depending on the coordinate dependence of the vector components X^a, this may lead to additional constraints on the choice of $\mathbf{X}$. For example, if g_{ab} is diagonal and X^a has full coordinate dependence such that

$$X^a = \left(X^0\left(T, x^\mu\right), X^1\left(T, x^\mu\right), X^2\left(T, x^\mu\right), X^3\left(T, x^\mu\right)\right), \tag{3.5}$$

the zero components of g_{ab} yield six constraint equations

$$X_{(0;1)} = X_{(0;2)} = X_{(0;3)} = X_{(1;2)} = X_{(1;3)} = X_{(2;3)} = 0, \tag{3.6}$$

while the nonzero relations in (3.4) yield three limit equations

$$\lim_{T\to 0^{+(-)}} \frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^{+(-)}} \frac{X_{(1;1)}}{g_{11}} = \lim_{T\to 0^{+(-)}} \frac{X_{(2;2)}}{g_{22}} = \lim_{T\to 0^{+(-)}} \frac{X_{(3;3)}}{g_{33}} \tag{3.7}$$

that determine the scaling factor $\lambda \neq 0$ on the spacelike hypersurface $\mathcal{S}_0$.

It becomes clear that demonstrating asymptotic self-similarity (i.e. solving (3.4)) for a general spacetime is not trivial, considering the number of constraint equations that may arise. Some or all of these equations may be eliminated by making assumptions on the coordinate dependence of X^a, e.g. "simple" coordinate dependence where each vector component depends only on the coordinate it corresponds to, or "faithful" coordinate dependence where, say, X^2 and X^3 are both functions of (x^2, x^3) if $g_{23} \neq 0$. The following propositions show that such restrictions on coordinate dependence are sufficient (but not necessary) for correspondence between the zero components of $X_{(a;b)}$ and g_{ab}.

Proposition 3.1: If a vector field $\mathbf{X}$ on a spacetime $(\mathcal{M}, \mathbf{g})$ has simple coordinate dependence in local coordinates (x^a), each zero component of g_{ab} corresponds to a zero component of $X_{(a;b)}$.

Proof: First we have

$$\begin{aligned} X_{(a;b)} &= \frac{1}{2}\left(X_{a;b} + X_{b;a}\right) \\ &= \frac{1}{2}\left(g_{ac}X^c{}_{;b} + g_{bc}X^c{}_{;a}\right) \\ &= \frac{1}{2}\left(g_{ac}X^c{}_{,b} + g_{bc}X^c{}_{,a} + \left(\Gamma_{acb} + \Gamma_{bca}\right)X^c\right) \\ &= \frac{1}{2}\left(g_{ac}X^c{}_{,b} + g_{bc}X^c{}_{,a} + g_{ab,c}X^c\right). \end{aligned} \tag{3.8}$$

Since X^a has simple coordinate dependence, we have

$$X^a = \left(X^0\left(x^0\right), X^1\left(x^1\right), X^2\left(x^2\right), X^3\left(x^3\right)\right), \tag{3.9}$$

which reduces (3.8) to

$$X_{(a;b)} = \frac{1}{2}\left(g_{ab}\left(X^a{}_{,a} + X^b{}_{,b}\right) + g_{ab,c}X^c\right). \tag{3.10}$$

Hence each zero component of g_{ab} corresponds to a zero component of $X_{(a;b)}$. ■

Proposition 3.2: If a vector field $\mathbf{X}$ on a spacetime $(\mathcal{M}, \mathbf{g})$ has faithful coordinate dependence in local coordinates (x^a), and g_{ab} has exactly six (four diagonal and two off-diagonal) nonzero components, each zero component of g_{ab} corresponds to a zero component of $X_{(a;b)}$.

Proof: Without loss of generality, suppose $g_{23} \neq 0$. Since X^a has faithful coordinate dependence, we have

$$X^a = \left(X^0\left(x^0\right), X^1\left(x^1\right), X^2\left(x^2, x^3\right), X^3\left(x^2, x^3\right)\right). \tag{3.11}$$

Considering all cases in (3.8), we have

$$X_{(0;1)} = X_{(0;2)} = X_{(0;3)} = X_{(1;2)} = X_{(1;3)} = 0, \tag{3.12}$$

while $X_{(0;0)}$, $X_{(1;1)}$, $X_{(2;2)}$, $X_{(2;3)}$ and $X_{(3;3)}$ are in general nonzero. Hence each zero component of g_{ab} corresponds to a zero component of $X_{(a;b)}$. ■

Proposition 3.1 is particularly useful, as it holds for all spacetimes and implies that vector fields with simple coordinate dependence may be taken as a starting point when applying Definition 3.2. If no such solution to (3.4) can be found, coordinate dependence may be added via Proposition 3.2 without inducing further constraints. The conditions of Proposition 3.2 are not overly restrictive, as many line elements in the literature are either diagonal or have one additional off-diagonal term. Proposition 3.2 does not extend easily to more general cases, however (e.g. $X_{(1;2)} \neq 0$ if $g_{12} = 0$ and $g_{13}, g_{23} \neq 0$, or $X_{(2;2)} \neq 0$ if $g_{22} = 0$ and $g_{23} \neq 0$). Finally, we note here that all known exactly self-similar spacetimes admit an HVF with simple coordinate dependence (in some set of local coordinates) [42].

3.2.1 Guiding examples

We now apply Definitions 3.1 and 3.2 to various vacuum/perfect fluid cosmologies that are known to be asymptotically self-similar in the dynamical systems approach. For each example in Chapter 3, the fundamental four-velocity field is hypersurface-orthogonal and given by $\mathbf{u} = (-g_{00})^{-1/2}\partial/\partial t$.

3.2.1.1 Example: Heckmann–Schücking

The class of all Bianchi I dust solutions was discovered independently by Robinson [62] and Heckmann and Schücking [63] in the 1960s. These homogeneous and anisotropic models have the line element (2.49) with p_i given by (2.27), and are said to asymptote in the past and future to the Kasner vacuum solutions and the flat dust FLRW models respectively [42]. As demonstrated in Section 2.3.2.3, the Heckmann–Schücking solutions satisfy Definition 3.1 since (2.49) reduces to the Kasner/FLRW line elements in the past/future asymptotic regimes. Hence it is reasonable to expect that Definition 3.2 should also be satisfied via the Kasner/FLRW HVFs.

At early times, we take $T = t$ and choose $\mathbf{X}$ as the Kasner HVF (2.30) with p_i given by (2.27); this vector field is well-defined due to the exact mapping between the Heckmann–Schücking and Kasner models in the past asymptotic regime. The simple coordinate dependence of (2.30) ensures correspondence between the zero components of $X_{(a;b)}$ and g_{ab}, while the nonzero components yield

$$\frac{X_{(0;0)}}{g_{00}} = 1, \qquad \frac{X_{(i;i)}}{g_{ii}} = 1 + \frac{(2 - 3p_i)\,T}{3\,(T + t_0)}. \tag{3.13}$$

Since the limits as $T \to 0^+$ of all terms in (3.13) exist and are equal (to 1), the Heckmann–Schücking solutions are asymptotically self-similar in the past under Definition 3.2.

At late times, we take $T = -1/t$ such that $T \to 0^-$ as $t \to \infty$; the line element and coordinate basis transform respectively as

$$dt^2 = \frac{1}{T^4}dT^2, \qquad \frac{\partial}{\partial t} = T^2\frac{\partial}{\partial T}, \tag{3.14}$$

such that the flat dust FLRW HVF (2.30) with p_i given by (2.28) becomes

$$X^a = \left(-T, \frac{1}{3}x, \frac{1}{3}y, \frac{1}{3}z\right). \tag{3.15}$$

The vector field (3.15) has simple coordinate dependence, and yields

$$\frac{X_{(0;0)}}{g_{00}} = 1, \qquad \frac{X_{(i;i)}}{g_{ii}} = \frac{(1+3p_i)\,t_0T - 3}{3\,(t_0T - 1)}. \tag{3.16}$$

Again, the limits as $T \to 0^-$ of all terms in (3.16) exist and are equal (to 1); hence the Heckmann–Schücking solutions are also asymptotically self-similar in the future under Definition 3.2. This is a promising start, and indicates that an exact mapping between spacetimes in the asymptotic regime is preserved by the asymptotic process of the homothetic equation approach. As an artefact of this asymptotic process, however, the choice of $\mathbf{X}$ in Definition 3.2 is not restricted to an exactly self-similar spacetime's HVF should Definition 3.1 be satisfied. For example, any vector field of the form $X^a = (-T + o(T), (1/3)x, (1/3)y, (1/3)z)$ as $T \to 0^-$ may be used in place of (3.15).

3.2.1.2 Example: FLRW (open, radiation)

An example cosmology that satisfies the exact mapping approach in the future, but not in the past, is the open radiation FLRW model discussed in Section 2.3.2.3. This Bianchi V/VII$_h$ solution has the line element (2.50), which reduces as $t \to \infty$ to the Milne line element (2.19). At late times, we take $T = -1/t$ and choose $\mathbf{X}$ as the Milne HVF (2.25) (transformed accordingly via (3.14)) such that

$$\frac{X_{(0;0)}}{g_{00}} = 1, \qquad \frac{X_{(1;1)}}{g_{11}} = \frac{X_{(2;2)}}{g_{22}} = \frac{X_{(3;3)}}{g_{33}} = \frac{CT-1}{2CT-1}. \tag{3.17}$$

Since the limits as $T \to 0^-$ of all terms in (3.17) exist and are equal (to 1), the open radiation FLRW model is asymptotically self-similar in the future under Definition 3.2.

At early times, the open radiation FLRW model asymptotes to the flat radiation model in the dynamical systems approach. The exact mapping approach is not satisfied, however: although the open manifold $\mathbb{R}\times\mathbb{H}^3$ is diffeomorphic to the flat manifold $\mathbb{R}^4$, the two models with their equipped metrics are not isometric in the past. Indeed, taking $T = t$ and simply choosing $\mathbf{X}$ as the analogue of the flat radiation FLRW HVF (2.53) yields

$$\lim_{T\to0^+}\frac{X_{(0;0)}}{g_{00}} = \lim_{T\to0^+}\frac{X_{(1;1)}}{g_{11}} = 1,$$
$$\lim_{T\to0^+}\frac{X_{(2;2)}}{g_{22}} = \lim_{T\to0^+}\frac{X_{(3;3)}}{g_{33}} = \frac{1}{2}\left(1+\chi\coth\chi\right), \tag{3.18}$$

where the last two limits vary with χ across the spacelike hypersurface $\mathcal{S}_0$.

The lack of an exact mapping to guide the choice of $\mathbf{X}$ calls for more general methods of solving (3.4). An obvious one is to assume simple coordinate dependence (i.e. to choose $\mathbf{X}$ as the general vector field (3.9)), in which case the three limit equations (3.7) evaluate to

$$\begin{aligned}\lim_{T\to 0^+}\frac{d}{dT}X^0(T) &= \lim_{T\to 0^+}\left(X^0(T)\frac{C+T}{(2C+T)T}+\frac{d}{d\chi}X^1(\chi)\right)\\ &= \lim_{T\to 0^+}\left(X^0(T)\frac{C+T}{(2C+T)T}+X^1(\chi)\coth\chi+\frac{d}{d\theta}X^2(\theta)\right)\\ &= \lim_{T\to 0^+}\left(X^0(T)\frac{C+T}{(2C+T)T}+X^1(\chi)\coth\chi+X^2(\theta)\cot\theta+\frac{d}{d\phi}X^3(\phi)\right). \end{aligned} \tag{3.19}$$

Existence of the first two limits yields $X^0(T)\sim KT$ as $T\to 0^+$ for some constant $K\neq 0$, while equality yields $X^1(\chi)=(1/2)K\chi$. Hence the first two limits are constant while the last two limits depend at least on χ, and Definition 3.2 cannot be satisfied with the assumption of simple coordinate dependence.

Since coordinate dependence must be added and Proposition 3.2 may not be applied (the line element (2.50) is diagonal), additional constraints on $\mathbf{X}$ inevitably come into play. The spherical symmetry of the model motivates the choice

$$X^a=\left(X^0(T,\chi),X^1(T,\chi),0,0\right), \tag{3.20}$$

and the resultant constraint equation $X_{(0;1)}=0$ evaluates to

$$\frac{\partial}{\partial\chi}X^0(T,\chi)=(2C+T)T\frac{\partial}{\partial T}X^1(T,\chi). \tag{3.21}$$

Equations (3.19) are essentially unchanged, but the last equation is redundant as $X^2=X^3=0$. We further assume the vector components are separable functions and choose $X^1(T,\chi)=f(T)\sinh\chi$, such that (3.19) reduces to

$$\lim_{T\to 0^+}\frac{\partial}{\partial T}X^0(T,\chi)=\lim_{T\to 0^+}\left(X^0(T,\chi)\frac{C+T}{(2C+T)T}+f(T)\cosh\chi\right). \tag{3.22}$$

Integrating (3.21) and substituting the result into (3.22), we obtain a limit equation in f; one possible solution to this equation is given by $f(T)=e^{T/C}$, which yields

$$X^a=\left(\frac{1}{C}(2C+T)Te^{\frac{T}{C}}\cosh\chi,e^{\frac{T}{C}}\sinh\chi,0,0\right), \tag{3.23}$$

$$\begin{aligned}\frac{X_{(0;0)}}{g_{00}} &= \frac{1}{C^2}\left(2C^2+4CT+T^2\right)e^{\frac{T}{C}}\cosh\chi,\\ \frac{X_{(1;1)}}{g_{11}} &= \frac{X_{(2;2)}}{g_{22}}=\frac{X_{(3;3)}}{g_{33}}=\frac{1}{C}(2C+T)e^{\frac{T}{C}}\cosh\chi. \end{aligned} \tag{3.24}$$

Hence the limits as $T\to 0^+$ of all terms in (3.24) exist and are equal (to a point-dependent scaling factor $\lambda(\chi)=2\cosh\chi$), and the open radiation FLRW model is asymptotically self-similar in the past under Definition 3.2.

The vector field (3.23) is smooth, since each component is separable into smooth functions of a single variable. No pathological behaviour exists in λ either: the scaling factor is smooth and nowhere zero on the spacelike hypersurface $\mathcal{S}_0$, although it does blow up as $\chi \to \infty$. Interestingly enough, the asymptotic form of (3.23) as $T, \chi \to 0^+$ is precisely the analogue of the flat radiation FLRW HVF (2.53) (up to a factor of two); this might reflect the fact that the open line element (2.50) reduces as $T, \chi \to 0^+$ to the flat line element (2.29) with $\gamma = 4/3$ (after rescaling the spatial coordinates).

3.2.1.3 Example: FLRW (closed, radiation)

At late times, the closed radiation FLRW model behaves very differently from the open model as it recollapses to a singularity. According to the dynamical systems approach, however, it is still asymptotically self-similar in the future (to the time-reversed, contracting flat "model") [60]. The closed model is Bianchi IX, and its line element is given by

$$ds^2 = -dt^2 + (2C - t)\, t \left(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta\; d\phi^2\right)\right), \tag{3.25}$$

i.e. (2.9) with $a(t) = \sqrt{(2C - t)t}$ and $k = 1$, where $C = \sqrt{(1/3)\mu a^4}$ is a constant [2]. Although (3.25) is similar to (2.50) in form, we have $t \in (0, 2C)$ in this case since the future state is the singularity corresponding to $t = 2C$; furthermore, $\chi \in (0, 2\pi)$ is essentially a third angle instead of a radial coordinate. For our analysis, we cast (3.25) as [20]

$$ds^2 = C^2T^2 \left(-4\; dT^2 + \left(2 - T^2\right)\left(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta\; d\phi^2\right)\right)\right) \tag{3.26}$$

via the coordinate transformation

$$T = -\sqrt{\frac{1}{C}\,(2C - t)}, \tag{3.27}$$

such that $T \in (-\sqrt{2}, 0)$ and $T \to 0^-$ as $t \to 2C^-$.

The exact mapping approach is clearly not satisfied by the closed radiation FLRW model, which is not isometric to the flat model in any regime (since the closed manifold $\mathbb{R} \times \mathbb{S}^3$ is not even diffeomorphic to the flat manifold $\mathbb{R}^4$). As in the case of the open model, the assumption of simple coordinate dependence also fails to yield a solution to (3.4). We then consider the separable and spherically symmetric ansatz (3.20) with $X^1(T, \chi) = f(T)\sin\chi$, such that the constraint and limit equations reduce to

$$X_{(0;1)} = 0, \qquad \lim_{T\to 0^-} \frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^-} \frac{X_{(1;1)}}{g_{11}}. \tag{3.28}$$

One possible solution to (3.28) is given by

$$X^a = \left(\frac{1}{2\sqrt{2}}\left(2 - T^2\right)\sin\sqrt{2}\,T\cos\chi, \cos\sqrt{2}\,T\sin\chi, 0, 0\right), \tag{3.29}$$

which yields

$$\frac{X_{(0;0)}}{g_{00}} = \left(\frac{1}{2}\left(2 - T^2\right)\cos\sqrt{2}\,T + \frac{1}{2}\left(2 - 3T^2\right)\frac{\sin\sqrt{2}\,T}{\sqrt{2}\,T}\right)\cos\chi,$$
$$\frac{X_{(1;1)}}{g_{11}} = \frac{X_{(2;2)}}{g_{22}} = \frac{X_{(3;3)}}{g_{33}} = \left(\cos\sqrt{2}\,T + \left(1 - T^2\right)\frac{\sin\sqrt{2}\,T}{\sqrt{2}\,T}\right)\cos\chi. \tag{3.30}$$

Since the limits as $T \to 0^-$ of all terms in (3.30) exist and are equal (to a point-dependent scaling factor $\lambda(\chi) = 2\cos\chi$), the closed radiation FLRW model is asymptotically self-similar in the future under Definition 3.2. Again, the vector field (3.29) and λ are smooth. The scaling factor is now bounded on the spacelike hypersurface $\mathcal{S}_0$, but vanishes at $\chi = \pi/2, 3\pi/2$; this is not a major concern, as such values of χ correspond to distinguished points on the manifold $\mathcal{M}$ where the timelike component of (3.29) vanishes as well.

At early times, we take $T = t$ and choose (using the same procedure)

$$X^a = \left(\frac{1}{C}\left(2C - T\right) T e^{-\frac{T}{C}} \cos\chi, e^{-\frac{T}{C}} \sin\chi, 0, 0\right), \tag{3.31}$$

which yields

$$\frac{X_{(0;0)}}{g_{00}} = \frac{1}{C^2}\left(2C^2 - 4CT + T^2\right) e^{-\frac{T}{C}} \cos\chi,$$
$$\frac{X_{(1;1)}}{g_{11}} = \frac{X_{(2;2)}}{g_{22}} = \frac{X_{(3;3)}}{g_{33}} = \frac{1}{C}\left(2C - T\right) e^{-\frac{T}{C}} \cos\chi \tag{3.32}$$

and the same scaling factor $\lambda(\chi) = 2\cos\chi$. Hence the closed radiation FLRW model is also asymptotically self-similar in the past under Definition 3.2.

3.2.1.4 Example: Joseph

This Bianchi V vacuum solution was discovered by Joseph [64] in 1966; it is past asymptotic in the dynamical systems approach to the Kasner vacuum solution (2.26) with $p_1 = 1/3$ and $p_{2,3} = (1 \pm \sqrt{3})/3$, and future asymptotic to the Milne universe (2.19) [42]. The line element is given by

$$ds^2 = \sinh 2t \left(-dt^2 + dx^2 + e^{2x}\left(\left(\tanh t\right)^{\sqrt{3}} dy^2 + \left(\tanh t\right)^{-\sqrt{3}} dz^2\right)\right). \tag{3.33}$$

At late times, (3.33) reduces as $t \to \infty$ to

$$ds^2 = \frac{1}{2} e^{2t}\left(-dt^2 + dx^2 + e^{2x}\left(dy^2 + dz^2\right)\right), \tag{3.34}$$

which transforms to the alternate Milne line element (2.22) via

$$\bar{t} = \frac{1}{\sqrt{2}} e^t. \tag{3.35}$$

Hence the exact mapping approach is satisfied, and we expect Definition 3.2 to be satisfied as well with the Milne HVF $X^a = (\bar{t}, 0, 0, 0)$ (i.e. (2.25) in the coordinates of (2.22)). Taking $T = -1/t$, the Milne HVF transforms via (3.35) and (3.14) to

$$X^a = \left(T^2, 0, 0, 0\right), \tag{3.36}$$

which yields

$$\frac{X_{(0;0)}}{g_{00}} = \frac{X_{(1;1)}}{g_{11}} = -\coth\frac{2}{T}, \qquad \frac{X_{(2;2)}}{g_{22}}, \frac{X_{(3;3)}}{g_{33}} = -\coth\frac{2}{T} \mp \sqrt{3}\,\mathrm{csch}\,\frac{2}{T}. \tag{3.37}$$

Since the limits as $T \to 0^-$ of all terms in (3.37) exist and are equal (to a constant scaling factor $\lambda = 1$), the Joseph solution is indeed asymptotically self-similar in the future under Definition 3.2.

At early times, it is unclear if an exact mapping exists between the Joseph and Kasner solutions. Transforming (3.33) via

$$\widetilde{t} = \frac{1}{3}\left(2t\right)^{\frac{3}{2}} \tag{3.38}$$

and taking the asymptotic form as $\widetilde{t} \to 0^+$ yields (after rescaling each spatial coordinate)

$$ds^2 \sim -d\widetilde{t}^2 + \widetilde{t}^{\frac{2}{3}} dx^2 + e^{Cx}\left(\widetilde{t}^{\frac{2}{3}(1+\sqrt{3})} dy^2 + \widetilde{t}^{\frac{2}{3}(1-\sqrt{3})} dz^2\right), \tag{3.39}$$

where $C = 2/3^{1/3}$. Each term in (3.39) has the same dependence on coordinate time as in the Kasner line element (2.26) with $p_1 = 1/3$ and $p_{2,3} = (1 \pm \sqrt{3})/3$, although the similarity ends with the nontrivial presence of the spatial factor e^{Cx}. It may be possible to map (3.39) to (2.26) via a coordinate transformation of comparable complexity to (2.24), but it is just as likely that no such transformation exists.

Furthermore, a vector field that satisfies Definition 3.2 has yet to be found: assuming simple coordinate dependence fails to yield a solution to (3.4), and even considering the full set of constraint equations (3.6) with a separable ansatz does not appear to work. There is no requirement for the vector components to be separable, however, and we are unable to rule out the existence of a more general vector field that fulfils the conditions of the homothetic equation approach. Hence it is not possible at this stage to comment on the past asymptotic self-similarity of the Joseph solution under Definition 3.2.

3.2.1.5 Example: Ellis–MacCallum

This Bianchi VI_0 vacuum solution was discovered by Ellis and MacCallum [58] in 1969, and has the line element

$$ds^2 = t^{-\frac{1}{2}} e^{t^2}\left(-dt^2 + dx^2\right) + t\left(e^{2x} dy^2 + e^{-2x} dz^2\right). \tag{3.40}$$

According to the dynamical systems approach, (3.40) is past asymptotic to the Kasner vacuum solution (2.26) with $p_1 = -1/3$ and $p_2 = p_3 = 2/3$; it is also asymptotically self-similar in the future to the Taub form of flat spacetime, whose line element and HVF are given respectively by [42]

$$ds^2 = -d\bar{t}^2 + \bar{t}^2 d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2, \tag{3.41}$$

$$X^a = \left(\bar{t}, 0, \bar{y}, \bar{z}\right). \tag{3.42}$$

At early times, no exact mapping has been found between the Ellis–MacCallum and Kasner solutions. Transforming (3.40) via

$$\widetilde{t} = \frac{4}{3} t^{\frac{3}{4}} \tag{3.43}$$

and taking the asymptotic form as $\widetilde{t} \to 0^+$ yields (after rescaling each spatial coordinate)

$$ds^2 \sim -d\widetilde{t}^2 + \widetilde{t}^{-\frac{2}{3}} dx^2 + \widetilde{t}^{\frac{4}{3}}\left(e^{Cx} dy^2 + e^{-Cx} dz^2\right), \tag{3.44}$$

where $C = 2(3/4)^{1/3}$. As in the case of the Joseph solution, (3.44) is similar in form to the

Kasner line element (2.26) with $p_1 = -1/3$ and $p_2 = p_3 = 2/3$, but only up to the spatial factors $e^{\pm Cx}$. Since no vector field has been found to satisfy the homothetic equation approach either, it is not possible at this stage to comment on the past asymptotic self-similarity of the Ellis–MacCallum solution under Definitions 3.1 and 3.2.

At late times, however, this example cosmology offers a new twist on our characterisations of the exact mapping and homothetic equation approaches. Attempts to find an exact mapping between the Ellis–MacCallum and Taub solutions have been unsuccessful, and are hampered by the fact that the line element (3.40) is non-analytic in the future asymptotic regime. This may be seen by, say, taking $T = -1/t$ and noting that the inverse metric g^{ab} contains terms with the factor e^{-1/T^2} (which is non-analytic in the limit as $T \to 0^-$ since its Taylor series vanishes). In other words, (3.40) has no power series representation as $t \to \infty$, which hinders the search for a coordinate transformation between its asymptotic form and (3.41).

On the other hand, a vector field that satisfies Definition 3.2 is easily found. We take $T = -1/t$ and assume simple coordinate dependence in solving the limit equations (3.7), which yields

$$X^a = \left(-T^3, 0, y, z\right), \tag{3.45}$$

$$\frac{X_{(0;0)}}{g_{00}} = 1 - \frac{5}{4}T^2, \qquad \frac{X_{(1;1)}}{g_{11}} = 1 - \frac{1}{4}T^2, \qquad \frac{X_{(2;2)}}{g_{22}} = \frac{X_{(3;3)}}{g_{33}} = 1 + \frac{1}{2}T^2. \tag{3.46}$$

Since the limits as $T \to 0^-$ of all terms in (3.46) exist and are equal (to 1), the Ellis–MacCallum solution is asymptotically self-similar in the future under Definition 3.2.

The similarity of (3.45) to the Taub HVF (3.42) seems to indicate the existence of an exact mapping between the Ellis–MacCallum and Taub solutions as $t \to \infty$. Indeed, an approximate mapping has been found via the (non-analytic) coordinate transformation

$$\bar{t} = t^{-\frac{5}{4}} e^{\frac{1}{2}t^2}, \qquad \bar{x} = tx, \qquad \bar{y} = t^{\frac{1}{2}} e^{x} y, \qquad \bar{z} = t^{\frac{1}{2}} e^{-x} z, \tag{3.47}$$

where the spatial Taub coordinates $(\bar{x}, \bar{y}, \bar{z})$ are assumed to be bounded. Then we have $x, y, z = o(1)$ as $t \to \infty$, and the Taub line element (3.41) becomes

$$\begin{aligned} ds^2 &= t^{-\frac{1}{2}} e^{t^2} \left(-\left(1 + o\left(1\right)\right) dt^2 + \left(1 + o\left(1\right)\right) dx^2 + o\left(1\right) dt\; dx\right) \\ &\quad + t\left(e^{2x} dy^2 + e^{-2x} dz^2 + o\left(1\right) dx\; dy + o\left(1\right) dx\; dz\right) + o\left(1\right) dt\; dy + o\left(1\right) dt\; dz, \end{aligned} \tag{3.48}$$

i.e. it is approximately equivalent to the Ellis–MacCallum line element (3.40) as $t \to \infty$. This mapping also explains the cubic dependence on cosmic time in (3.45), as the vector field transforms via (3.14) and (3.47) to

$$X^a = \left(\bar{t}\left(1 - \frac{5}{4t^2}\right), \frac{x}{t}, \bar{y}\left(1 + \frac{1}{2t^2}\right), \bar{z}\left(1 + \frac{1}{2t^2}\right)\right), \tag{3.49}$$

which asymptotes to the Taub HVF (3.42) as $t \to \infty$. Nevertheless, since (3.47) does not constitute an exact mapping due to the off-diagonal terms in (3.48) (the $dt\; dx$ term even blows up faster than the dy^2 and dz^2 terms), we temporarily withhold comment on the future asymptotic self-similarity of the Ellis–MacCallum solution under Definition 3.1 (see Section 3.3.1).

3.2.2 Asymptotic self-similarity breaking

The Joseph and Ellis–MacCallum examples in Section 3.2.1 illustrate that the asymptotic self-similarity of a spacetime under the homothetic equation approach might in general be rather difficult — if not impossible — to determine. For Definition 3.2 to be of any credible use, however, we ideally require it to exclude spacetimes that exhibit asymptotic self-similarity breaking (i.e. the negation of asymptotic self-similarity) in the dynamical systems approach.

Asymptotic self-similarity breaking occurs in the dynamical systems approach when a cosmological model fails to asymptote to an exactly self-similar counterpart in state space. This typically manifests itself in the unbounded and/or oscillatory behaviour of various dimensionless variables in the asymptotic regime, such that their limits do not exist [65]. While asymptotic self-similarity is known to break in this way for several classes of models, explicit solutions in the literature that possess such behaviour are comparatively rare. We now consider some of these examples, explicit or otherwise, in the context of Definition 3.2; the question of how asymptotic self-similarity breaking (in the dynamical systems approach) relates to Definition 3.1 is addressed in Section 3.3.

3.2.2.1 Example: Szekeres (decaying)

An explicit and tractable example of asymptotic self-similarity breaking is found in the class of irrotational dust models discovered by Szekeres [66] in 1975. These solutions are fully inhomogeneous, i.e. they admit no KVFs. The Szekeres solutions may be interpreted as exact linear perturbations of dust FLRW models via a reformulation due to Goode and Wainwright [67, 68], in which the line element is given by

$$ds^2 = -dt^2 + t^{\frac{4}{3}} \left(\left(A\left(x,y,z\right) + k_+\left(x\right) t^{\frac{2}{3}} + k_-\left(x\right) t^{-1} \right)^2 dx^2 + dy^2 + dz^2 \right),$$
$$A\left(x,y,z\right) = a\left(x\right) y + b\left(x\right) z + c\left(x\right) + \frac{5}{9} k_+\left(x\right)\left(y^2+z^2\right), \quad (3.50)$$

where the functions a, b, c and $k_\pm$ are arbitrary but sufficiently differentiable. This line element has a growing mode corresponding to the presence of the term $k_+ t^{2/3}$, as well as a decaying mode corresponding to the presence of the term $k_- t^{-1}$.

Now, the decaying Szekeres solution with $a = b = k_+ = 0$, $c(x) = C$ and $k_-(x) = Kx$ (where $C, K > 0$ are constants) is known to exhibit asymptotic self-similarity breaking at early times, in that a given dimensionless variable blows up as $t \to 0^+$ [69]. We apply the homothetic equation approach to this solution; the line element (3.50) reduces to

$$ds^2 = -dt^2 + t^{\frac{4}{3}} \left(\left(C + Kxt^{-1} \right)^2 dx^2 + dy^2 + dz^2 \right), \quad (3.51)$$

while taking $T = t$ and assuming $\mathbf{X}$ has simple coordinate dependence yields the three limit equations

$$\lim_{T\to 0^+} \frac{d}{dT} X^0\left(T\right) = \lim_{T\to 0^+} \left(X^0\left(T\right) \frac{2CT - Kx}{3T\left(CT+Kx\right)} + X^1\left(x\right) \frac{K}{CT+Kx} + \frac{d}{dx} X^1\left(x\right) \right)$$
$$= \lim_{T\to 0^+} \left(X^0\left(T\right) \frac{2}{3T} + \frac{d}{dy} X^2\left(y\right) \right) = \lim_{T\to 0^+} \left(X^0\left(T\right) \frac{2}{3T} + \frac{d}{dz} X^3\left(z\right) \right). \quad (3.52)$$

If the (T,x)-dependent terms in (3.52) are considered as functions of T with constant x, a "solution" is found without much difficulty. For $x \neq 0$, we have

$$X^a = \left(t, \frac{2}{3}x, \frac{1}{3}y, \frac{1}{3}z\right), \tag{3.53}$$

$$\frac{X_{(0;0)}}{g_{00}} = \frac{X_{(2;2)}}{g_{22}} = \frac{X_{(3;3)}}{g_{33}} = 1, \qquad \frac{X_{(1;1)}}{g_{11}} = 1 + \frac{CT}{3\,(CT+Kx)}, \tag{3.54}$$

where the limits as $T \to 0^+$ of all terms in (3.54) exist and are equal (to 1). This is not the case for $x = 0$, however, since the last equation in (3.54) evaluates to $X_{(1;1)}/g_{11} = 4/3$ instead. Such pathological behaviour is undesirable in the ideal notion of an asymptotically self-similar spacetime, and should be disallowed even if the limits are equal but to some discontinuous scaling factor λ on the spacelike hypersurface $\mathcal{S}_0$. In any case, by properly considering the (T,x)-dependent terms in (3.52) and (3.54) as functions on the manifold $\mathcal{M}$, it is clear that their limits as $(T,x) \to (0^+, x)$ do not even exist. Hence the vector field (3.53) fails to satisfy Definition 3.2.

It is possible that every vector field on $\mathcal{M}$ fails to satisfy Definition 3.2 in similar fashion to (3.53) (i.e. via the non-existence of limits), which would show that the decaying Szekeres solution (3.51) is not asymptotically self-similar under the homothetic equation approach. Such a claim cannot be substantiated at this stage, however. Nevertheless, this example cosmology has highlighted the need to exclude spacetimes with any form of discontinuous behaviour from the eventual working definition of asymptotic self-similarity.

3.2.2.2 Example: Wainwright–Marshman

Another explicit solution that exhibits asymptotic self-similarity breaking at early times is found in the class of irrotational stiff fluid models discovered by Wainwright and Marshman [70] in 1979. These solutions are inhomogeneous but admit a G_2 isometry group, and may be used to model the presence of gravitational waves in otherwise homogeneous cosmologies.

Asymptotic self-similarity is known to break in an oscillating subclass of the Wainwright–Marshman solutions, in that given dimensionless variables are unbounded and/or oscillatory as the past singularity is approached [71]. The line element for this subclass is given by

$$\begin{aligned} ds^2 &= e^{n(t-x)}\left(-dt^2 + dx^2\right) + t^{\frac{1}{2}}\left(dy + m\,(t-x)\,dz\right)^2 + t^{\frac{3}{2}} dz^2, \\ m\,(t-x) &= -\alpha\,\mathrm{Si}\left(\frac{\beta}{t-x}\right), \qquad n\,(t-x) = -\frac{\alpha^2}{2\,(t-x)} + \frac{\alpha^2}{4\beta}\sin\left(\frac{2\beta}{t-x}\right), \end{aligned} \tag{3.55}$$

where $\alpha, \beta > 0$ are constants and Si denotes the sine integral (i.e. $\mathrm{Si}\,(z) = \int_0^z \sin(s)/s\; ds$). Coordinate domains for (3.55) are $t \in (x, \infty)$, $x \in (0, \infty)$ and $y, z \in (-\infty, \infty)$, such that $t \to x^+$ as the singularity is approached. Transforming to cosmic time $T = t - x$ and choosing the most general vector field (3.5), we find that the five limit equations

$$\begin{aligned} \lim_{T\to 0^+} \frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^+} \frac{X_{(0;1)}}{g_{01}} &= \lim_{T\to 0^+} \frac{X_{(1;1)}}{g_{11}} \\ &= \lim_{T\to 0^+} \frac{X_{(2;2)}}{g_{22}} = \lim_{T\to 0^+} \frac{X_{(2;3)}}{g_{23}} = \lim_{T\to 0^+} \frac{X_{(3;3)}}{g_{33}} \end{aligned} \tag{3.56}$$

necessarily contain terms with the factors $\sin(\beta/T)$ and $\mathrm{Si}(\beta/T)$. These terms do not approach $T = 0$ in a smooth manner, since their limits and/or the limits of their derivatives as $T \to 0^+$ do not exist.

Encouragingly, such behaviour is consistent with the oscillatory nature of asymptotic self-similarity breaking in the dynamical systems approach, where the past/future state is either undefined or approached via increasingly rapid oscillations. Hence imposing a sufficient degree of differentiability on the asymptotic process in Definition 3.2 appears to be justifiable, and furthermore ensures that the oscillating Wainwright–Marshman solutions are not asymptotically self-similar under the homothetic equation approach.

3.2.2.3 Example: Davidson

This inhomogeneous radiation solution was discovered by Davidson [72] in 1991; it admits a G_2 isometry group and describes a cylindrically symmetric universe that expands irrotationally. In cylindrical coordinates, the line element is given by

$$ds^2 = -\left(1+\rho^2\right)^{\frac{6}{5}} dt^2 + t^{\frac{4}{3}}\left(1+\rho^2\right)^{\frac{2}{5}} d\rho^2 + t^{\frac{4}{3}}\left(1+\rho^2\right)^{-\frac{2}{5}} \rho^2 d\phi^2 + t^{-\frac{2}{3}}\left(1+\rho^2\right)^{-\frac{2}{5}} dz^2. \quad (3.57)$$

The Davidson solution is an example of a G_2 cosmology with a diagonal and separable line element. Such models have been classified and studied generally in the dynamical systems approach; this particular solution belongs to a subclass of diagonal, separable G_2 cosmologies that are not known to be future asymptotic to an exactly self-similar model [73]. It is unclear if said subclass constitutes a concrete example of asymptotic self-similarity breaking, since the analysis of G_2 cosmologies in the dynamical systems approach is largely incomplete. The late-time evolution of the Davidson solution itself has not been examined in detail either.

In any case, the existence of an explicit member of the subclass allows the homothetic equation approach to be applied. Taking $T = -1/t$ and assuming $\mathbf{X}$ has simple coordinate dependence fails to yield a solution to (3.4), while choosing the cylindrically symmetric (and separable) ansatz

$$X^a = \left(X^0(T,\rho), X^1(T,\rho), 0, 0\right) \quad (3.58)$$

does not appear to work either. These non-results are inconclusive either way, however, and it is not possible at this stage to comment on the future asymptotic self-similarity of the Davidson solution under Definition 3.2.

3.2.2.4 Example: Wainwright–Hancock–Uggla

Asymptotic self-similarity is known to break at late times for perfect fluid Bianchi VII_0/VIII cosmologies of sufficient generality, in that a given dimensionless variable is typically (depending on the matter content) unbounded and oscillatory as the future state is approached [65, 74, 75].

Although the late-time evolution of Bianchi VII_0/VIII cosmologies is more conveniently studied in the dynamical systems (orthonormal frame) formalism, a future asymptotic form for the general Bianchi VII_0 line element relative to a coordinate basis has been derived by Wainwright, Hancock and Uggla [74]. For a Bianchi VII_0 dust model at late times, this line element is given by

$$ds^2 \sim -\frac{1}{H_0^2}e^{3\tau}d\tau^2 + l_0^2 e^{2(\tau-2\beta_+(\tau))}\left(\boldsymbol{\omega}^1\right)^2$$
$$+ l_0^2 e^{2\left(\tau+\beta_+(\tau)+\sqrt{3}\,\beta_-(\tau)\right)}\left(\boldsymbol{\omega}^2\right)^2 + l_0^2 e^{2\left(\tau+\beta_+(\tau)-\sqrt{3}\,\beta_-(\tau)\right)}\left(\boldsymbol{\omega}^3\right)^2,$$
$$\beta_+(\tau) = 2C^2 e^{-\tau}, \qquad \beta_-(\tau) = \frac{1}{2}H_0 l_0 C e^{-\tau}\sin\left(\frac{4}{H_0 l_0}e^{\frac{1}{2}\tau} + \psi_0\right), \quad (3.59)$$

where $H_0, l_0, C, \psi_0 > 0$ are constants, $\tau \in (-\infty, \infty)$, and the one-forms $\boldsymbol{\omega}^\mu$ can be chosen canonically as [76]

$$\boldsymbol{\omega}^1 = dx, \qquad \boldsymbol{\omega}^2 = \cos x\; dy - \sin x\; dz, \qquad \boldsymbol{\omega}^3 = \sin x\; dy + \cos x\; dz. \quad (3.60)$$

We note that taking $C = 0$ in (3.59) (and transforming coordinates appropriately) yields the flat dust FLRW line element (2.29), which is unsurprising as the flat FLRW models admit a Bianchi VII_0 representation.

Numerous Bianchi VII_0 results exist in the literature, but an exact line element has yet to be found. For the purposes of this thesis, then, we apply the homothetic equation approach to the asymptotic form (3.59). Taking $T = -e^{-\tau}$ (such that $T \to 0^-$ as $\tau \to \infty$) and choosing the most general vector field (3.5), we find that the four limit equations

$$\lim_{T\to 0^-}\frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^-}\frac{X_{(1;1)}}{g_{11}} = \lim_{T\to 0^-}\frac{X_{(2;2)}}{g_{22}} = \lim_{T\to 0^-}\frac{X_{(2;3)}}{g_{23}} = \lim_{T\to 0^-}\frac{X_{(3;3)}}{g_{33}} \quad (3.61)$$

necessarily contain terms with factors of similar form to $\sin(1/\sqrt{-T})$ and $\exp(T\sin(1/\sqrt{-T}))$. These terms do not approach $T = 0$ in a smooth manner, since their limits and/or the limits of their derivatives as $T \to 0^-$ do not exist. As in the case of the Wainwright–Marshman solutions, imposing a sufficient degree of differentiability on Definition 3.2 ensures that the Bianchi VII_0 dust models are not asymptotically self-similar under the homothetic equation approach (up to the fact that the asymptotic form (3.59) has been used in place of an exact line element).

3.2.2.5 Example: Mixmaster

Vacuum Bianchi IX cosmologies are well known to exhibit stochastic oscillatory behaviour as the past singularity is approached; they were named Mixmaster universes by Misner [77], who studied them as possible models for the dissipative processes posited in chaotic cosmology (introduced in Chapter 1). Such stochastic oscillations are not exclusive to the Mixmaster universes, however, and have been described in more general models by Belinskii, Lifshitz and Khalatnikov [78] via a chaotic map (i.e. a discrete evolution function that is exponentially sensitive to initial conditions). In this picture, the early-time evolution of a Mixmaster-like model is closely approximated by an infinite (and hence increasingly rapid) sequence of generalised Kasner solutions that are linked by generalised vacuum Bianchi II solutions [79].

Mixmaster-like behaviour occurs in vacuum/perfect fluid Bianchi VIII/IX cosmologies of sufficient generality, and leads to asymptotic self-similarity breaking in the dynamical systems approach since the singularity is approached via increasingly rapid oscillations [65]. Although the homothetic equation approach is not strictly applicable without explicit knowledge of the metric $\mathbf{g}$, it is clear that the stochastic oscillations between

Kasner epochs in a Mixmaster-like model prevent the scaling factor λ from being well-defined in the past asymptotic regime. Hence we may also conclude that such models are not asymptotically self-similar in the past under Definition 3.2.

3.3 Working definition

From our analysis of example cosmologies in Section 3.2, several improvements to the homothetic equation approach are required. The main imperative is to introduce some notion of differentiability into Definition 3.2, for added compatibility with the nature of asymptotic self-similarity breaking in the dynamical systems approach. More precisely, the quotient terms in the limit equations should be sufficiently differentiable functions on the manifold $\mathcal{M}$, while their limits as $T \to 0^{+(-)}$ should exist and equal some sufficiently differentiable function on the spacelike hypersurface $\mathcal{S}_0$.

It is also desirable to remove any ambiguity caused by the asymptotic notation of (1.5) in Definition 3.2, which necessitates a proper reformulation in local coordinates. With these objectives in mind, we amend the homothetic equation approach accordingly and arrive at the following working definition of asymptotic self-similarity.

Definition 3.4 (Asymptotic self-similarity): A spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar in the past (future) if there exists a cosmic time function T such that $T \to 0^{+(-)}$ as the spacetime's past (future) is approached, and a C^2 vector field $\mathbf{X}$ on $\mathcal{M}$ such that

$$X_{(a;b)} = f_{ab} g_{ab} \tag{3.62}$$

in some set of local coordinates (T, x^μ), where the functions $f_{ab} \neq 0$ on $\mathcal{M}$ satisfy

$$f_{ab} \in C^1(\mathcal{M}), \qquad \lim_{T \to 0^{+(-)}} f_{ab}(T, x^\mu) = \lambda(x^\mu) \tag{3.63}$$

for some C^1 function $\lambda \neq 0$ on $T = 0$. If λ is constant on $T = 0$, the spacetime is also uniformly self-similar in the past (future).

By analogy with the homothetic equation (2.18) and the definition of an HVF, the vector field $\mathbf{X}$ is termed a past/future asymptotically homothetic vector field (AHVF). We are concerned primarily with AHVFs that have a nonzero timelike component (i.e. the notion of asymptotic scale invariance with time), but note that Definition 3.4 has been formulated more generally in order not to exclude spacetimes that asymptotically exhibit spatial scale invariance.

Definition 3.4 is an improvement over its predecessor in several ways. Most prominently, the asymptotic notation in (3.2) and (3.4) has been recast as a limiting process for the 10 distinct functions f_{ab} (i.e. the quotient terms $X_{(a;b)}/g_{ab}$) in (3.62) and (3.63). This lends precision and practicality to Definition 3.4, while preserving the central requirement in Definition 3.2 that $X_{(a;b)}$ and g_{ab} are equivalent up to some order of cosmic time T as the past/future state is approached. The new definition also deals with the everywhere zero components of $X_{(a;b)}$ and g_{ab} rather neatly, since the existence of $f_{ab} \neq 0$ immediately implies correspondence between said components.

With the asymptotic process shifted into the functions f_{ab}, imposing differentiability on Definition 3.4 becomes straightforward. The AHVF $\mathbf{X}$ is required to be at least C^2 on $\mathcal{M}$, in accordance with the C^2 conditions of Definition 2.1 and the conformal framework.

A C^1 degree of differentiability is imposed on each f_{ab} and the scaling factor λ, where the latter condition rules out the possibility of λ behaving discontinuously on $\mathcal{S}_0$ (as raised in Section 3.2.2.1). However, as seen in the FLRW examples, λ is permitted to vanish at distinguished points on $\mathcal{M}$ (more precisely, on a subset of $\mathcal{S}_0$ with measure zero).

Finally, spacetimes that admit a constant scaling factor may be thought of as being uniformly self-similar, i.e. asymptotically self-similar in a uniform (across $\mathcal{S}_0$) sense; they come closest to exact self-similarity in the asymptotic regime, and are identified under Definition 3.4 as a special subclass of asymptotically self-similar spacetimes.

Although Definition 3.4 is derived from the coordinate-invariant tensor relation (3.2), its formulation in local coordinates might appear problematic at first glance. Crucially, (3.62) is not a tensor equation as the object f_{ab} is not tensorial; in general, then, the existence of the functions f_{ab} will depend on the set of coordinates they are considered in (since correspondence between the zero/nonzero components of $X_{(a;b)}$ and g_{ab} in other coordinates might be broken). The definition only requires (3.62) to hold in some coordinate frame, however, and allows the choice of a different AHVF (with the same asymptotic behaviour) in another frame if necessary. We revisit this point in Section 5.1.

Armed with a formal working definition, we derive a couple of preliminary results that characterise asymptotic self-similarity in the context of other similarity-related properties. Firstly, a simple proposition linking the existence of HVFs, CKVFs and AHVFs follows immediately from Definition 3.4.

Proposition 3.3: A spacetime $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar in the past (future) if it admits an HVF, or a CKVF whose associated scaling factor is C^1 and nonzero in the limit as the spacetime's past (future) is approached.

Proof: If $\mathbf{X}$ is an HVF on $(\mathcal{M}, \mathbf{g})$, its associated scaling factor λ is a nonzero constant. Hence $\mathbf{X}$ is a uniform AHVF with $f_{ab} = \lambda$, and $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar in the past (future). If $\mathbf{X}'$ is a CKVF on $(\mathcal{M}, \mathbf{g})$ that satisfies the conditions of Proposition 3.3, its associated scaling factor $\lambda' \neq 0$ is a C^2 function on $\mathcal{M}$ that is C^1 and nonzero in the limit as the spacetime's past (future) is approached. Hence $\mathbf{X}'$ is an AHVF with $f'_{ab} = \lambda'$, and $(\mathcal{M}, \mathbf{g})$ is asymptotically self-similar in the past (future). ■

An exactly self-similar spacetime is asymptotically self-similar by Proposition 3.3, along with spacetimes that admit the specified class of CKVFs (see Section 3.4).

We also return to the notion of an exact mapping (between a spacetime and an exactly self-similar counterpart) in the asymptotic regime, which is related to Definition 3.4 and the dynamical systems definition of asymptotic self-similarity by the following theorem.

Theorem 3.1: If a spacetime $(\mathcal{M}, \mathbf{g})$ admits an exact mapping to an exactly self-similar spacetime $(\mathcal{M}', \mathbf{g}')$ in the past (future) asymptotic regime, $(\mathcal{M}, \mathbf{g})$ is uniformly self-similar in the past (future). Furthermore, $(\mathcal{M}, \mathbf{g})$ is past (future) asymptotic to $(\mathcal{M}', \mathbf{g}')$ in the dynamical systems approach.

Proof: For $(\mathcal{M}, \mathbf{g})$ and $(\mathcal{M}', \mathbf{g}')$ to be isometric in the past (future), $\mathcal{M}$ and $\mathcal{M}'$ must be diffeomorphic; we then use the same set of local coordinates (T, x^μ) for both spacetimes, where $T \to 0^{+(-)}$ as the spacetimes' past (future) is approached. From the line element formulation of Definition 3.1, we see that $g_{ab} \sim g'_{ab}$ as $T \to 0^{+(-)}$. Now, we choose $\mathbf{X}$ on $\mathcal{M}$ to be the analogue of the C^2 HVF $\mathbf{H}$ on $\mathcal{M}'$ such that $X^a = H^a$. Since (T, x^μ) may always be chosen such that H^a has simple coordinate dependence, each zero component of g_{ab} corresponds to a zero component of $X_{(a;b)}$ by Proposition 3.1. We take $f_{ab} = \lambda$ for the zero components of g_{ab}, where $\lambda \neq 0$ is the constant scaling factor associated with $\mathbf{H}$. For the nonzero components, we have

$$f_{ab} = \frac{X_{(a;b)}}{g_{ab}} = \frac{\frac{1}{2}\left(g_{ab}\left(X^a_{\ ,a} + X^b_{\ ,b}\right) + g_{ab,c}X^c\right)}{g_{ab}} \tag{3.64}$$

from (3.10). Then $f_{ab} \in C^1(\mathcal{M})$ since $g_{ab}, X^a \in C^2(\mathcal{M})$, and we also have

$$\lim_{T\to 0^{+(-)}} f_{ab} = \frac{H_{(a;b)}}{g'_{ab}} = \lambda. \tag{3.65}$$

Hence $\mathbf{X}$ is a uniform AHVF, and $(\mathcal{M}, \mathbf{g})$ is uniformly self-similar in the past (future).

Furthermore, since $\mathbf{g} \to \mathbf{g}'$ as $T \to 0^{+(-)}$, any orthonormal frame (2.38) for $(\mathcal{M}, \mathbf{g})$ approaches a corresponding orthonormal frame for $(\mathcal{M}', \mathbf{g}')$ in the past (future) asymptotic regime. In other words, all dimensionless variables constructed from the commutation functions $\gamma^c_{\ ab}$ on $\mathcal{M}$ asymptote to their counterparts on $\mathcal{M}'$. Hence $(\mathcal{M}, \mathbf{g})$ is past (future) asymptotic to $(\mathcal{M}', \mathbf{g}')$ in the dynamical systems approach. ■

As alluded to in the discussion following Proposition 3.2, it is reasonable to assume the existence of local coordinates (T, x^μ) in which the HVF $\mathbf{H}$ has simple coordinate dependence [42]. We also note that $f_{ab} \neq 0$ for the nonzero components of g_{ab}, since $\lim_{T\to 0^{+(-)}} f_{ab} \neq 0$.

Theorem 3.1 is useful on several counts. For one, it shows that an isometry between spacetimes in the asymptotic regime is preserved by the asymptotic process of Definition 3.4, such that the search for an exact mapping to an exactly self-similar spacetime becomes an alternative method of finding a uniform AHVF. Conversely, any form of asymptotic self-similarity breaking (under Definition 3.4 or in the dynamical systems approach) implies via the contrapositive of Theorem 3.1 that no such mapping exists. Hence all the example cosmologies in Section 3.2.2 do not satisfy Definition 3.1; we summarise these and other results in the following section.

3.3.1 Summary of examples

In formulating Definition 3.4, we have considered the asymptotic self-similarity of various example cosmologies under a homothetic equation-based approach, as well as an exact mapping definition for the purpose of comparison. Guided by existing results in the dynamical systems approach, 10 (classes of) models have been examined in detail: five of these are both past and future asymptotic to known exactly self-similar models, while the other five exhibit asymptotic self-similarity breaking at early or late times (although, as mentioned in Section 3.2.2.3, the Davidson result is not completely established).

The asymptotic self-similarity of these cosmologies under the dynamical systems, homothetic equation and exact mapping definitions is presented in Table 3.1. While the analysis of several examples under Definition 3.4 remains inconclusive, the homothetic equation results that have been obtained show perfect agreement with those in the dynamical systems approach. As seen from the FLRW models, however, the exact mapping results do not. Theorem 3.1 verifies that Definition 3.1 is the strongest of the three definitions being considered; this allows the existence of an exact mapping to be ruled out in

Model(s)	P/F	DS	Def. 3.4 (uniform)	Def. 3.1
Heckmann–Schücking	P	Y	Y (Y)	Y
	F	Y	Y (Y)	Y
FLRW (open, radiation)	P	Y	Y (N⋆)	N
	F	Y	Y (Y)	Y
FLRW (closed, radiation)	P	Y	Y (N⋆)	N
	F	Y	Y (N⋆)	N
Joseph	P	Y	? (?)	?
	F	Y	Y (Y)	Y
Ellis–MacCallum	P	Y	? (?)	?
	F	Y	Y (Y)	Y⋆
Szekeres (decaying)	P	N	? (N⋆)	N†
Wainwright–Marshman	P	N	N (N)	N†
Davidson	F	N	? (N⋆)	N†
Wainwright–Hancock–Uggla	F	N	N (N)	N†
Mixmaster	P	N	N (N)	N†

†: Theorem 3.1 ⋆: Conjecture 3.1

Table 3.1: Past and/or future asymptotic self-similarity of the example cosmologies in Chapter 3, according to the dynamical systems, homothetic equation and exact mapping definitions.

the five examples that exhibit asymptotic self-similarity breaking.

There appears to be a correlation between uniform self-similarity and Definition 3.1, in that an exact mapping always yields a uniform AHVF (i.e. the exactly self-similar model's HVF) in accordance with Theorem 3.1, while no uniform AHVFs have been found for the examples that do not satisfy Definition 3.1. However, it is not trivial to prove that an example is asymptotically but not uniformly self-similar (e.g. the FLRW models); AHVFs with simple coordinate dependence have generally been ruled out in such cases, but it is possible to conceive of more general AHVFs admitting spatially dependent functions $f_{ab}(T, x^\mu)$ that become constant in the limit as $T \to 0^{+(-)}$.

On the other hand, a strong argument has been made in Section 3.2.1.5 for the existence of an exact mapping in the Ellis–MacCallum (future) example, even though uniform self-similarity has been demonstrated without the HVF from such a mapping. With the apparent correlation between uniform self-similarity and Definition 3.1 in mind, we offer the following conjecture.

Conjecture 3.1: If a spacetime $(\mathcal{M}, \mathbf{g})$ is uniformly self-similar in the past (future), it admits an exact mapping to an exactly self-similar spacetime $(\mathcal{M}', \mathbf{g}')$ in the past (future) asymptotic regime.

Taken together with Theorem 3.1, Conjecture 3.1 essentially posits that uniform self-similarity and Definition 3.1 are equivalent notions of self-similarity. It supports the existence of an exact mapping in the Ellis–MacCallum example, while allowing us to make speculative claims regarding the breaking of uniform self-similarity in the FLRW, Szekeres and Davidson examples.

3.4 Further discussion

Several difficulties arise when using Definition 3.4 to determine the asymptotic self-similarity of a general spacetime (where the line element might be extremely complex). As evident from the Joseph and Ellis–MacCallum solutions at early times, attempting to find an AHVF via the constraint and limit equations is daunting if a non-separable ansatz with full coordinate dependence is required. Furthermore, showing that the functions f_{ab} are C^1 is not trivial if they are multivariate and non-separable.

Proving that a spacetime is not asymptotically self-similar might be even harder than proving it is, although any solution with inherent oscillatory behaviour similar to that in the Wainwright–Marshman or Mixmaster models is unlikely to satisfy the differentiability conditions of Definition 3.4. By such reasoning, we have drawn our conclusions on the general Bianchi VII_0 dust models from the Wainwright–Hancock–Uggla asymptotic form with confidence.

It is clearly desirable to devise additional methods of finding AHVFs. One hitherto unexplored possibility is to search for the specified class of CKVFs that serve as AHVFs by Proposition 3.3. For example, the highly symmetric FLRW models admit nine independent CKVFs (one of these is homothetic if the model is exactly self-similar) [80]; the simplest is the hypersurface-orthogonal vector field $\mathbf{X} = a(t)\partial/\partial t$, where a is the FLRW scale factor and t is coordinate time. In the case of the open radiation model with $T = t$, this CKVF is given by

$$X^a = \left(\sqrt{(2C+T)\,T}, 0, 0, 0\right), \tag{3.66}$$

which yields the divergent (as $T \to 0^+$) scaling factor

$$\lambda\,(T) = \frac{C+T}{\sqrt{(2C+T)\,T}}. \tag{3.67}$$

Another CKVF is given by

$$X^a = \left(\frac{1}{C}\,(2C+T)\,T\cosh\chi, \frac{1}{C}\,(C+T)\sinh\chi, 0, 0\right), \tag{3.68}$$

which yields the scaling factor

$$\lambda\,(T,\chi) = \frac{2}{C}\,(C+T)\cosh\chi, \qquad \lim_{T\to 0^+}\lambda = 2\cosh\chi. \tag{3.69}$$

Since (3.68) satisfies the conditions of Proposition 3.3, it is a past AHVF. We note that (3.68) is asymptotically equivalent to the AHVF (3.23), and accordingly admits the same function $\lambda(\chi) = 2\cosh\chi$ on the spacelike hypersurface $\mathcal{S}_0$.

While CKVFs are (in principle) a convenient source of AHVFs, they rarely exist and are usually homothetic when they do [41]. In fact, most spacetimes that admit a CKVF are very simple and highly symmetric to begin with [81]. Nevertheless, since CKVFs are relatively well-documented in the literature, we may employ them whenever they are available and concordant with the conditions of Proposition 3.3 (see Section 4.2.1).

Chapter 4

Self-similarity in the conformal framework of QC–WCH

In this chapter, we seek to integrate exact self-similarity and the working definition of asymptotic self-similarity into the conformal framework of quiescent cosmology and the Weyl curvature hypothesis. Section 4.1 details the conditions under which these properties are preserved by the conformal transformations in the framework, while additional example cosmologies with known conformal structures are examined for exact and asymptotic self-similarity in Section 4.2. The results in this thesis are consolidated in Section 4.3 to provide new insights into the relationships between isotropy, homogeneity and self-similarity in the asymptotic regime.

4.1 Self-similarity under conformal transformations

A key aspect of fitting self-similarity into the conformal framework is determining if and how the notions of exact self-similarity (i.e. the homothetic equation (2.18) with constant $\lambda \neq 0$) and asymptotic self-similarity (i.e. Definition 3.4) translate under the conformal transformations in the framework. Specifically, we are looking to ascertain whether self-similarity (breaking) in a physical spacetime necessarily implies similar behaviour in its unphysical counterpart, and vice versa.

It is clear that such an investigation cannot be conducted using the dynamical systems definition of asymptotic self-similarity: the unphysical spacetimes in the conformal framework are generally not exact solutions to the EFE (with physically viable matter content), much less solutions that are Bianchi. On the other hand, it is perfectly meaningful for an unphysical spacetime $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ to be asymptotically self-similar under Definition 3.4. Indeed, while the functions $\widetilde{f}_{ab}$ might even be well-defined on all of $\widetilde{\mathcal{M}}$ (and in particular on the spacelike hypersurface $\mathcal{S}_0$), they need only be C^1 on the physical submanifold $\widetilde{\mathcal{M}} \cap \mathcal{M}$ and approach some C^1 function $\widetilde{\lambda} \neq 0$ on the past/future state $T = 0$.

As it turns out, asymptotic self-similarity is largely preserved by the conformal transformations in the framework — up to the fulfilment of certain technical requirements on the conformal factor and relevant vector field.

Theorem 4.1: Suppose a physical spacetime $(\mathcal{M}, \mathbf{g})$ admits an unphysical past (future) counterpart $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ in the conformal framework. If a vector field $\mathbf{X}$ on $(\mathcal{M}, \mathbf{g})$ is an AHVF with associated function λ, and the function $\Lambda := \mathbf{X}(\ln \Omega)$ is C^1 and differs from λ in the limit as the spacetime's past (future) is approached, $\mathbf{X}$ is an AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$.

Proof: First we have $\mathcal{M} \subset \widetilde{\mathcal{M}}$ and $\mathbf{g} = \Omega^2(T)\widetilde{\mathbf{g}}$ from the conformal framework, where the cosmic time function $T \to 0^{+(-)}$ as either spacetime's past (future) is approached. The Leibniz rule for the Lie derivative yields

$$\mathcal{L}_{\mathbf{X}}\widetilde{\mathbf{g}} = \mathcal{L}_{\mathbf{X}}\left(\Omega^{-2}\mathbf{g}\right) = \left(-2\Omega^{-3}\mathcal{L}_{\mathbf{X}}\Omega\right)\mathbf{g} + \Omega^{-2}\left(\mathcal{L}_{\mathbf{X}}\mathbf{g}\right). \tag{4.1}$$

In local coordinates (T, x^μ) such that $\mathbf{X}$ is an AHVF on $(\mathcal{M}, \mathbf{g})$, it follows from (4.1) that

$$\begin{aligned} X_{(a:b)} &= -\Omega^{-3}X^c\Omega_{,c}g_{ab} + \Omega^{-2}X_{(a;b)} \\ &= -\Omega^{-3}X^0\Omega_{,0}g_{ab} + \Omega^{-2}f_{ab}g_{ab} \\ &= \left(f_{ab} - X^0\frac{\Omega'}{\Omega}\right)\widetilde{g}_{ab} \\ &= \left(f_{ab} - X^0\left(\ln\Omega\right)_{,0}\right)\widetilde{g}_{ab} \\ &= \left(f_{ab} - \Lambda\right)\widetilde{g}_{ab}, \end{aligned} \tag{4.2}$$

where the colon denotes covariant differentiation with respect to $\widetilde{\mathbf{g}}$, and $\Lambda \in C^1(\mathcal{M})$ since $\mathbf{X}, \Omega \in C^2(\mathcal{M})$. Then each $(f_{ab} - \Lambda)$ is C^1 on $\widetilde{\mathcal{M}} \cap \mathcal{M}$, and $(\lambda - \lim_{T\to 0^{+(-)}} \Lambda) \neq 0$ is C^1 on $T = 0$. Hence $\mathbf{X}$ is an AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$. ■

The added conditions on the function Λ in Theorem 4.1 (i.e. $\lim_{T\to 0^{+(-)}} \Lambda$ is C^1 and differs from λ) are crucial but not overly restrictive. Furthermore, X^0 is typically $O(T)$ as $T \to 0^{+(-)}$ while $\Omega'/\Omega = O(1/T)$ for an analytic conformal factor, such that Λ is typically $O(1)$ (and hence bounded in the limit as $T \to 0^{+(-)}$). We also note that Λ is permitted to vanish on $T = 0$.

Now, exact self-similarity is essentially a stronger notion of uniform self-similarity, which is in turn a special case of asymptotic self-similarity. It is unsurprising, then, that a couple of corollaries regarding uniform and exact self-similarity may be derived by imposing stronger constraints on Λ in Theorem 4.1.

Corollary 4.1: In addition to the conditions of Theorem 4.1, if $\mathbf{X}$ is a uniform AHVF on $(\mathcal{M}, \mathbf{g})$ and Λ is constant in the limit as the spacetime's past (future) is approached, $\mathbf{X}$ is a uniform AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$.

Proof: As given in the proof of Theorem 4.1, except $(\lambda - \lim_{T\to 0^{+(-)}} \Lambda) \neq 0$ is constant on $T = 0$. Hence $\mathbf{X}$ is a uniform AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$. ■

Corollary 4.2: In addition to the conditions of Corollary 4.1, if $\mathbf{X}$ is an HVF on $(\mathcal{M}, \mathbf{g})$ and Λ is constant, $\mathbf{X}$ is an HVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$.

Proof: As given in the proof of Corollary 4.1, except each $(f_{ab} - \Lambda)$ equals a constant scaling factor $(\lambda - \Lambda) \neq 0$. Hence $\mathbf{X}$ is an HVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$. ■

From Theorem 4.1 and its corollaries, any given degree of self-similarity in the physical spacetime may be used to demonstrate an equal or lower degree of self-similarity in the unphysical spacetime. We note, however, that the latter spacetime is not necessarily restricted to said degree of self-similarity; for example, a physical spacetime that is only asymptotically self-similar might be conformally related to an exactly self-similar counterpart. This is alluded to in the following theorem (and its corollaries), which is essentially the converse of Theorem 4.1 with a similar proof.

Theorem 4.2: Suppose a physical spacetime $(\mathcal{M}, \mathbf{g})$ admits an unphysical past (future) counterpart $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ in the conformal framework. If a vector field $\widetilde{\mathbf{X}}$ on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ is an AHVF with associated function $\widetilde{\lambda}$, and the function $\widetilde{\Lambda} := \widetilde{\mathbf{X}}(\ln\Omega)$ is C^1 and differs from $-\widetilde{\lambda}$ in the limit as the spacetime's past (future) is approached, $\widetilde{\mathbf{X}}$ is an AHVF on $(\mathcal{M}, \mathbf{g})$.

Proof: In local coordinates (T, x^μ) such that $\widetilde{\mathbf{X}}$ is an AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$, we have

$$\begin{aligned}\widetilde{X}_{(a;b)} &= \Omega\widetilde{X}^c\Omega_{,c}\widetilde{g}_{ab} + \Omega^2\widetilde{X}_{(a:b)} \\ &= \Omega\widetilde{X}^0\Omega_{,0}\widetilde{g}_{ab} + \Omega^2\widetilde{f}_{ab}\widetilde{g}_{ab} \\ &= \left(\widetilde{f}_{ab} + \widetilde{X}^0\frac{\Omega'}{\Omega}\right) g_{ab} \\ &= \left(\widetilde{f}_{ab} + \widetilde{X}^0 \left(\ln\Omega\right)_{,0}\right) g_{ab} \\ &= \left(\widetilde{f}_{ab} + \widetilde{\Lambda}\right) g_{ab}. \end{aligned} \tag{4.3}$$

Then each $(\widetilde{f}_{ab} + \widetilde{\Lambda})$ is C^1 on $\mathcal{M} \subset \widetilde{\mathcal{M}}$, and $(\widetilde{\lambda} + \lim_{T\to 0^{+(-)}}\widetilde{\Lambda}) \neq 0$ is C^1 on $T = 0$. Hence $\widetilde{\mathbf{X}}$ is an AHVF on $(\mathcal{M}, \mathbf{g})$. ■

Corollary 4.3: In addition to the conditions of Theorem 4.2, if $\widetilde{\mathbf{X}}$ is a uniform AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ and $\widetilde{\Lambda}$ is constant in the limit as the spacetime's past (future) is approached, $\widetilde{\mathbf{X}}$ is a uniform AHVF on $(\mathcal{M}, \mathbf{g})$.

Proof: As given in the proof of Theorem 4.2, except $(\widetilde{\lambda} + \lim_{T\to 0^{+(-)}}\widetilde{\Lambda}) \neq 0$ is constant on $T = 0$. Hence $\widetilde{\mathbf{X}}$ is a uniform AHVF on $(\mathcal{M}, \mathbf{g})$. ■

Corollary 4.4: In addition to the conditions of Corollary 4.3, if $\widetilde{\mathbf{X}}$ is an HVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ and $\widetilde{\Lambda}$ is constant, $\widetilde{\mathbf{X}}$ is an HVF on $(\mathcal{M}, \mathbf{g})$.

Proof: As given in the proof of Corollary 4.3, except each $(\widetilde{f}_{ab} + \widetilde{\Lambda})$ equals a constant scaling factor $(\widetilde{\lambda} + \widetilde{\Lambda}) \neq 0$. Hence $\widetilde{\mathbf{X}}$ is an HVF on $(\mathcal{M}, \mathbf{g})$. ■

The theorems and corollaries in this section are constructive existence results for (A)HVFs; they describe, but do not fully characterise, how self-similarity translates under the conformal transformations in the framework. On the other hand, there is no analogous result for the breaking of self-similarity. This may be seen by, say, considering the contrapositive of Theorem 4.1: if $\mathbf{X}$ is not an AHVF on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$, we cannot conclude that it is not one on $(\mathcal{M}, \mathbf{g})$ unless Λ satisfies the specified conditions. Hence the non-existence of AHVFs on $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ necessarily carries over to $(\mathcal{M}, \mathbf{g})$ only if every vector field on $\mathcal{M}$ fulfils the requirements of Theorem 4.1 — which is clearly untrue.

Conveniently, Theorem 4.2 provides another method of finding AHVFs for models that admit an isotropic past/future state in the conformal framework. As the unphysical metric in this case is regular on an open neighbourhood of the spacelike hypersurface $\mathcal{S}_0$, the search for an AHVF is potentially simpler on the unphysical spacetime. One example is the closed radiation FLRW model studied in Section 3.2.1.3, whose line element (3.26) has been cast into a form compatible with the admission of an IFS (where $\Omega(T) = -CT$). The line element of the unphysical spacetime follows as

$$d\widetilde{s}^2 = -4\, dT^2 + \left(2 - T^2\right)\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\; d\phi^2\right)\right), \tag{4.4}$$

which is regular near and on $T = 0$. Although the constraint equations resulting from (4.4) are identical to those from (3.26), the limit equations are slightly less complex and may be solved to yield the unphysical AHVF (3.29) with associated function $\widetilde{\lambda}(\chi) = 2\cos\chi$. Applying Theorem 4.2, we then have

$$\widetilde{\Lambda}\left(T,\chi\right) = \frac{1}{2}\left(2 - T^2\right)\frac{\sin\sqrt{2}\,T}{\sqrt{2}\,T}\cos\chi, \qquad \lim_{T\to 0^-}\widetilde{\Lambda} = \cos\chi \neq -\widetilde{\lambda}, \tag{4.5}$$

i.e. (3.29) is a future AHVF for the closed radiation FLRW model as well.

4.2 Examples in the conformal framework

In this section, we discuss the self-similarity of several additional cosmologies whose conformal structures have been studied in the literature, and identify the examples in Chapter 3 that are known to admit or preclude a conformal past/future state.

4.2.1 Self-similarity of FLRW models

As mentioned in Section 2.2, the conformal structures of FLRW models have been thoroughly investigated and classified via the asymptotic properties of the scale factor a and deceleration parameter q [35, 39]. The exact self-similarity of perfect fluid FLRW models is also well known: any flat model with $\gamma \in (0, 2]$ admits an HVF (see (2.28)–(2.30)), as does any model with $\gamma = 2/3$. In the latter case, the line element is given by (2.9) with $a(t) = t/C$, where $C > 0$ is a constant and $C < 1$ for $k = -1$ [42]. The empty Milne universe (2.19) is recovered by taking $C = 1$ for $k = -1$; accordingly, all FLRW models with $\gamma = 2/3$ share the Milne HVF (2.25).

While some open and closed FLRW models are known to be asymptotically self-similar in the dynamical systems approach, we may demonstrate under Definition 3.4 the asymptotic self-similarity of all FLRW models at early and late times. This result is not restricted to perfect fluid models, and exploits the existence of CKVFs in its proof.

Theorem 4.3: All FLRW models are asymptotically self-similar in the past (future).

Proof: First we have the general FLRW line element (2.11) in isotropic form, which may be cast as

$$ds^2 = a^2\left(\tau\right)\left(-d\tau^2 + \left(1 + \frac{1}{4}k\rho^2\right)^{-2}\left(d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta\; d\phi^2\right)\right)\right) \tag{4.6}$$

via the coordinate transformation $\tau = \int 1/a(t)\; dt$. Assuming the scale factor a has a (one-sided) generalised power series representation in the asymptotic regime, we write

$$a\left(\tau\right) = \sum_{i=1}^{\infty} C_i\left|\tau - \tau_0\right|^{r_i}, \tag{4.7}$$

where $C_1 > 0$, $r_i < r_{i+1} \in \mathbb{R}$ and $\tau \to \tau_0^{+(-)}$ as the spacetime's past (future) is approached. We note that $r_1 = -1$ corresponds to the special case of de Sitter-like spacetimes, which we have excluded from the FLRW label in Section 2.1.3. Transforming to cosmic time $T = \tau - \tau_0$ yields

$$a(T) = \sum_{i=1}^{\infty} C_i |T|^{r_i}, \qquad a'(T) = \sum_{i=1}^{\infty} +(-)\, r_i C_i |T|^{r_i - 1}, \tag{4.8}$$

while the line element (4.6) is essentially unchanged. As $T \to 0^{+(-)}$, we see that

$$\frac{a'}{a} \sim \begin{cases} +(-)\, r_1 |T|^{-1}, & r_1 \neq 0, \\ +(-)\, r_2 C_2 / C_1 |T|^{r_2 - 1}, & r_1 = 0, \end{cases} \tag{4.9}$$

i.e. $a'/a = O(1/|T|)$ in all cases.

For $k = 1$, we consider the CKVF given by

$$X^a = \left(\frac{4-\rho^2}{4+\rho^2} \sin T, \rho \cos T, 0, 0 \right), \tag{4.10}$$

which yields the scaling factor

$$\lambda(T,\rho) = \frac{4-\rho^2}{4+\rho^2} \left(\cos T + (-) \frac{a'}{a} \sin |T| \right), \qquad \lim_{T \to 0^{+(-)}} \lambda = \frac{4-\rho^2}{4+\rho^2} (1 + C), \tag{4.11}$$

where $C \in \{r_1, 0\}$ is a constant. Since (4.10) satisfies the conditions of Proposition 3.3 for $r_1 \neq -1$, it is an AHVF for all closed FLRW models in the past (future).

For $k = 0$, we consider the CKVF given by

$$X^a = (T, \rho, 0, 0), \tag{4.12}$$

which yields the scaling factor

$$\lambda(T) = 1 + (-) \frac{a'}{a} |T|, \qquad \lim_{T \to 0^{+(-)}} \lambda = 1 + C, \tag{4.13}$$

where $C \in \{r_1, 0\}$ is a constant. Since (4.12) satisfies the conditions of Proposition 3.3 for $r_1 \neq -1$, it is a uniform AHVF for all flat FLRW models in the past (future).

For $k = -1$, we consider the CKVF given by

$$X^a = \left(\frac{4+\rho^2}{4-\rho^2} \sinh T, \rho \cosh T, 0, 0 \right), \tag{4.14}$$

which yields the scaling factor

$$\lambda(T,\rho) = \frac{4+\rho^2}{4-\rho^2} \left(\cosh T + (-) \frac{a'}{a} \sinh |T| \right), \qquad \lim_{T \to 0^{+(-)}} \lambda = \frac{4+\rho^2}{4-\rho^2} (1 + C), \tag{4.15}$$

where $C \in \{r_1, 0\}$ is a constant. Since (4.14) satisfies the conditions of Proposition 3.3 for $r_1 \neq -1$, it is an AHVF for all open FLRW models in the past (future). Hence all FLRW models are asymptotically self-similar in the past (future). ■

The proof of Theorem 4.3 hinges on assuming the generalised power series in (4.7) converges locally to a in the asymptotic regime, which is certainly the case for all known FLRW models in the literature [82]. We note that $a'/a = \pm\dot{a}$ (where $\dot{a} = da/dt$), and should not be confused with the generalised Hubble parameter $H = \dot{a}/a$. Also, (4.10)/(4.14) and (4.12) correspond respectively to the CKVFs $\mathbf{H}^*$ and $\mathbf{H}$ given by Maartens and Maharaj [80]. Finally, the χ-coordinate domain for $k = -1$ transforms to $\rho \in (-2, 2)$, i.e. the scaling factor on $T = 0$ is well-defined for the open models.

Theorem 4.3 applies to any class of spacetimes described by the FLRW line element, with two exceptions. Firstly, while AHVFs may be found on non-evolving spacetimes such as the Einstein static universe, it is meaningless for these to be asymptotically self-similar at "early" or "late" times (although they may be interpreted as being asymptotically self-similar at all times). Secondly, when $r_1 = -1$ in (4.7), a is approximately exponential in coordinate time t; such scale factors generally describe de Sitter-like spacetimes with a positive cosmological constant Λ. These spacetimes of constant (four-dimensional) curvature include the de Sitter universe as a specific example, which we attend to in the following section.

4.2.2 Further examples

At present, only about 15 distinct (classes of) cosmologies have been studied within the conformal framework. We have examined a handful of these for exact and asymptotic self-similarity; null results for the Carneiro–Marugan [35, 36] and Senin [39] models are neither informative nor presented in this thesis. As in Chapter 3, the fundamental four-velocity field is hypersurface-orthogonal and given by $\mathbf{u} = (-g_{00})^{-1/2}\partial/\partial t$.

4.2.2.1 Example: de Sitter

The de Sitter universe is a vacuum energy spacetime with exponential expansion; it possesses distinctive mathematical properties while serving as a physical model for the exponential phase of cosmological inflation (introduced in Chapter 1) [2]. In spherical coordinates, the line element is given by

$$ds^2 = -dt^2 + e^{2Ct}\left(d\chi^2 + \chi^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right), \tag{4.16}$$

i.e. (2.9) with $a(t) = e^{Ct}$ and $k = 0$, where $C = \sqrt{(1/3)\Lambda}$ is a constant. We note that asymptotic self-similarity is not well-defined for the de Sitter universe in the dynamical systems approach, since the model corresponds to an exceptional equilibrium point that is not exactly self-similar [51].

Although there is no singularity in the de Sitter universe, we may consider $t = 0$ as corresponding to an unphysical past state. At early times, we take $T = t$ and choose the hypersurface-orthogonal CKVF

$$X^a = \left(e^{CT}, 0, 0, 0\right), \tag{4.17}$$

which yields the scaling factor

$$\lambda(T) = Ce^{CT}, \qquad \lim_{T\to 0^+}\lambda = C. \tag{4.18}$$

Since (4.17) satisfies the conditions of Proposition 3.3, it is a uniform AHVF for the de

Sitter universe. However, the model is uniformly accelerating with a finite scale factor as its past state is approached ($q = -1$ and $\lim_{T\to 0^+} a = 1$); from results by Ericksson [35] and Threlfall [39], we may conclude that the de Sitter universe does not admit a past conformal state at which $\mathbf{u} = \partial/\partial T$ is regular.

At late times, we take $T = -e^{-Ct}$ such that (4.16) transforms to

$$ds^2 = \frac{1}{T^2}\left(-\frac{1}{C^2}dT^2 + d\chi^2 + \chi^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right). \tag{4.19}$$

This line element happens to be in conformal form $\mathbf{g} = \Omega^2\widetilde{\mathbf{g}}$ with $\Omega(T) = -1/T$ and $\widetilde{\mathbf{g}}$ regular, but is incompatible with the admission of an FIU since $L_0 = L = \Omega''\Omega/\Omega^2 = 2$ (see Definition 2.8a). The hypersurface-orthogonal CKVF becomes

$$X^a = (C, 0, 0, 0), \tag{4.20}$$

which yields the divergent (as $T \to 0^-$) scaling factor

$$\lambda(T) = -\frac{C}{T}. \tag{4.21}$$

We consider in similar fashion the eight remaining CKVFs $\mathbf{H}$, $\mathbf{M}_{\mu 0}$ and $\mathbf{K}_a$ given by Maartens and Maharaj [80]; interestingly, each yields a scaling factor of the form $\lambda(T, x^\mu) = f(x^\mu)/T$ as well.

It is mentioned in Section 3.2.1.1 that AHVFs appear to fall into asymptotic equivalence classes, such that spacetimes admit only a finite number of AHVF classes; furthermore, as seen in Section 3.4, AHVFs are known to be asymptotically equivalent to (some) CKVFs when the latter exist. Since every CKVF for the de Sitter universe yields a scale factor that blows up as $T \to 0^-$, there is a strong indication of asymptotic self-similarity breaking at late times. Hence we are reasonably justified in offering the following conjecture, whose significance is made clear in Section 4.3.

Conjecture 4.1: The de Sitter universe is not asymptotically self-similar in the future.

4.2.2.2 Example: Kantowski–Sachs

One example cosmology that has been given much attention within the conformal framework is found in the class of homogeneous and anisotropic models discovered by Kantowski and Sachs [83, 84] in 1966. These irrotational perfect fluid solutions are non-Bianchi, in that their G_4 isometry group fails to admit a G_3 subgroup acting transitively on the spacelike hypersurfaces (which have the topology of $\mathbb{S}^2 \times \mathbb{R}$ [49]). The recollapsing radiation model studied in the framework has a form due to Wainwright [85]; its line element is given by

$$ds^2 = -A(t)\, dt^2 + t\left(\frac{1}{A(t)}dx^2 + \frac{A^2(t)}{b^2}\left(dy^2 + f^2(y)\, dz^2\right)\right) \tag{4.22}$$

with $A(t) = 1 - (4/9)b^2 t$ and $f(y) = \sin y$. Asymptotic self-similarity is not well-defined for the Kantowski–Sachs models in the dynamical systems approach, although those that admit a G_2 isometry subgroup are in general asymptotically self-similar [73]. The asymptotic self-similarity of this particular model is not given in the literature.

At early times, we take $T = t$ and choose the ansatz

$$X^a = \left(X^0\left(T,x,y\right), X^1\left(T,x,y\right), X^2\left(T,x,y\right), 0\right) \tag{4.23}$$

with $X^2 = f(T,x)\sin y$, such that the constraint and limit equations reduce to

$$X_{(0;1)} = X_{(0;2)} = X_{(1;2)} = 0, \qquad \lim_{T\to 0^+}\frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^+}\frac{X_{(1;1)}}{g_{11}} = \lim_{T\to 0^+}\frac{X_{(2;2)}}{g_{22}}. \tag{4.24}$$

No solution to (4.24) exists if f is further assumed to be separable, although the choice

$$X^a = \left(-\frac{2}{3}A^{-\frac{3}{2}}T\cosh bx\cos y, \frac{1}{b}A^{\frac{3}{2}}\sinh bx\cos y, A^{-\frac{3}{2}}\cosh bx\sin y, 0\right) \tag{4.25}$$

is a "near" solution in that

$$\lim_{T\to 0^+}\frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^+}\frac{X_{(1;1)}}{g_{11}} = \lim_{T\to 0^+}\frac{X_{(2;2)}}{g_{22}} = \lim_{T\to 0^+}\frac{X_{(3;3)}}{g_{33}} = \pm\frac{2}{3}\cosh bx\cos y, \tag{4.26}$$

where only the first limit is negative. It is unclear if such sign discrepancies are inherent (such that the radiation Kantowski–Sachs model cannot be asymptotically self-similar), or whether a solution to (4.24) might exist for non-separable f. On the other hand, it is straightforward to show that the model admits an IPS by taking $T = \sqrt{2t}$ in (4.22) [18]. We note that attempts to find an AHVF via the resultant unphysical spacetime and Theorem 4.2 have also been unsuccessful.

At late times, we take $T = -A^2$ such that $T \in (-1, 0)$ and $T \to 0^-$ as the future singularity corresponding to $t = 9/(4b^2)$ is approached; accordingly, (4.22) transforms to

$$ds^2 = \Omega^2\left(T\right)\left(-\frac{81}{64b^4}dT^2 + \frac{9}{4b^2}\left(1-\left(-T\right)^{\frac{1}{2}}\right)\left(dx^2 + \frac{1}{b^2}\left(-T\right)^{\frac{3}{2}}\left(dy^2 + \sin^2 y\, dz^2\right)\right)\right),$$
$$\Omega\left(T\right) = \left(-T\right)^{-\frac{1}{4}}, \tag{4.27}$$

which is in a form compatible with the admission of an AFS [20]. However, no AHVF has been found (on either the physical or unphysical spacetime) via this coordinate transformation. Hence it is not possible at this stage to comment on the future asymptotic self-similarity of the radiation Kantowski–Sachs model.

4.2.2.3 Example: Kantowski

The radiation Kantowski–Sachs model has a related counterpart in the class of all Bianchi III radiation solutions, discovered by Kantowski [83] in 1966; the line element of this ever-expanding model is given by (4.22) with $A(t) = 1 + (4/9)b^2 t$ and $f(y) = \sinh y$. In the dynamical systems approach, Bianchi III solutions are past asymptotic to Bianchi I solutions and future asymptotic to the Bianchi III form of flat spacetime, whose line element and HVF are given respectively by [42]

$$ds^2 = -d\bar{t}^2 + d\bar{x}^2 + \bar{t}^2\left(d\bar{y}^2 + e^{2\bar{y}}d\bar{z}^2\right), \tag{4.28}$$

$$X^a = \left(\bar{t}, \bar{x}, 0, 0\right). \tag{4.29}$$

At early times, the radiation Kantowski model admits an IPS (as in the Kantowski–

Sachs case, this is seen by taking $T = \sqrt{2t}$ in (4.22)) [18]. Its line element reduces to

$$ds^2 \sim -dt^2 + t\left(dx^2 + \frac{1}{b^2}\left(dy^2 + \sinh^2 y\, dz^2\right)\right) \tag{4.30}$$

as $t \to 0^+$, but it is unclear if an exact mapping of (4.30) to the Bianchi I line element (2.26) exists. No AHVF has been found via the usual procedure either, although taking $T = t$ and choosing the ansatz (4.23) with $X^2 = f(T, x)\sinh y$ yields another "near" solution

$$X^a = \left(-\frac{2}{3}A^{-\frac{3}{2}}T\cos bx\cosh y, \frac{1}{b}A^{\frac{3}{2}}\sin bx\cosh y, A^{-\frac{3}{2}}\cos bx\sinh y, 0\right), \tag{4.31}$$

$$\lim_{T\to 0^+}\frac{X_{(0;0)}}{g_{00}} = \lim_{T\to 0^+}\frac{X_{(1;1)}}{g_{11}} = \lim_{T\to 0^+}\frac{X_{(2;2)}}{g_{22}} = \lim_{T\to 0^+}\frac{X_{(3;3)}}{g_{33}} = \pm\frac{2}{3}\cos bx\cosh y, \tag{4.32}$$

where only the first limit is negative. Hence it is not possible at this stage to comment on the past asymptotic self-similarity of the radiation Kantowski model.

At late times, we take $T = -A^{-1}$ such that $T \in (-1, 0)$ and $T \to 0^-$ as $t \to \infty$; accordingly, (4.22) transforms to

$$ds^2 = \Omega^2(T)\left(-\frac{81}{16b^4}dT^2 + \frac{9}{4b^2}(1+T)T^2\left(-T^3dx^2 + \frac{1}{b^2}\left(dy^2 + \sinh^2 y\, dz^2\right)\right)\right),$$
$$\Omega(T) = (-T)^{-\frac{5}{2}}, \tag{4.33}$$

which is in a form compatible with the admission of an AFEU [20]. Assuming simple coordinate dependence and solving the limit equations (3.7) yields

$$X^a = \left(-\frac{2}{3}T, x, 0, 0\right), \tag{4.34}$$

$$\frac{X_{(0;0)}}{g_{00}} = 1, \qquad \frac{X_{(1;1)}}{g_{11}} = \frac{X_{(2;2)}}{g_{22}} = \frac{X_{(3;3)}}{g_{33}} = \frac{2T+3}{3T+3}. \tag{4.35}$$

Since the limits as $T \to 0^-$ of all terms in (4.35) exist and equal a constant scaling factor $\lambda = 1$, the radiation Kantowski model is uniformly self-similar in the future. It is unsurprising that the AHVF (4.34) resembles the Bianchi III HVF (4.29), in light of Conjecture 3.1. Finally, we note that this result continues a trend of uniform self-similarity for cosmologies that are future asymptotic to flat (Minkowski) spacetime in the dynamical systems approach, i.e. the open radiation FLRW, Joseph and Ellis–MacCallum models (see Table 3.1).

4.2.2.4 Example: Szekeres (growing)

The inhomogeneous Szekeres solutions introduced in Section 3.2.2.1 have also been studied within the conformal framework, but in the absence of the decaying mode (i.e. $k_- = 0$ in (3.50)). Transforming to cosmic time $T = 3t^{1/3}$ casts the line element (3.50) into a form compatible with the admission of an IPS [19, 35], while the admission of an AFEU is demonstrated by taking $T = -1/t$ [36]. Within the dynamical systems approach, the Szekeres solutions are in general asymptotically self-similar at early times (to Bianchi I solutions), but not necessarily at late times [86, 87, 88]. The growing Szekeres solutions in particular are known to admit FLRW-like singularities [67, 68].

Model(s)	P/F	(A)HVF	Conformal state(s)
Heckmann–Schücking	P	Uniform	Not found
	F	Uniform	AFEU
Wainwright–Marshman	P	None	None
Davidson	F	Not uniform*	AFS
FLRW (flat, $\gamma \in (0, 2/3)$)	P	Exact	None
	F	Exact	FIU
FLRW (flat, $\gamma \in (2/3, 2]$)	P	Exact	IPS
	F	Exact	Not isotropic
FLRW ($\gamma = 2/3$)	P	Exact	None
	F	Exact	None
FLRW (general)	P	Asymptotic‡	IPS
	F	Asymptotic‡	All
de Sitter	P	Uniform	None
	F	None**	Not found
Kantowski–Sachs	P	Not found	IPS
	F	Not found	AFS
Kantowski	P	Not found	IPS
	F	Uniform	AFEU
Szekeres (growing)	P	Exact	IPS
	F	Exact	AFEU

$\ddagger$: Theorem 4.3 $\star$: Conjecture 3.1 $\star\star$: Conjecture 4.1

Table 4.1: Past and/or future self-similarity of the example perfect fluid cosmologies in Chapters 3 and 4 that admit or preclude conformal states.

For our analysis, we focus on the growing Szekeres solution with $a = b = c = k_- = 0$ and $k_+(x) = Kx^r$ (where $K, r > 0$ are constants), which has the line element

$$ds^2 = -dt^2 + t^{\frac{4}{3}} \left(K^2 x^{2r} \left(\frac{5}{9} \left(y^2 + z^2\right) + t^{\frac{2}{3}} \right)^2 dx^2 + dy^2 + dz^2 \right). \tag{4.36}$$

As it turns out, the vector field

$$X^a = \left(t, -\frac{1}{3\,(r+1)} x, \frac{1}{3} y, \frac{1}{3} z \right) \tag{4.37}$$

solves the homothetic equation (2.18) with $\lambda = 1$. Hence (4.37) is an HVF, and the growing Szekeres solution (4.36) serves as a handy example of exact self-similarity in an inhomogeneous cosmological model.

4.2.3 Summary of examples

Most of the example cosmologies studied in the conformal framework are perfect fluid spacetimes, due to their physical significance over vacuum spacetimes (which do not admit an IPS [35]). Table 4.1 lists all such models that have also been examined for exact or asymptotic self-similarity in Chapters 3 and 4.

The Heckmann–Schücking and Davidson examples from Chapter 3 are known to admit anisotropic conformal states [39]. While analysis of the Davidson solution under

Definition 3.4 remains inconclusive, we claim that no uniform AHVF exists by Conjecture 3.1. Furthermore, spacetimes in which the past singularity is not point-like do not admit an IPS at which $\mathbf{u}$ is regular [35]; this allows us to rule out the existence of a past conformal state for the Wainwright–Marshman solutions (where the singularity corresponding to $t = x$ oscillates between the "barrel" and "pancake" types [25, 71]).

Our conclusions on the conformal states admitted (or precluded) by the FLRW models in Table 4.1 are based on general results by Ericksson [35] and Threlfall [39]. For the flat FLRW models, the scale factor is given by $a(t) = t^{2/(3\gamma)}$ as per (2.29), such that $\lim_{T\to 0^{+(-)}} a = 0(\infty)$. The deceleration parameter (2.15) evaluates to

$$q = \frac{1}{2}\left(3\gamma - 2\right), \tag{4.38}$$

such that the models with $\gamma \in (0, 2/3)$ or $\gamma \in (2/3, 2]$ are, respectively, uniformly accelerating or decelerating. Hence the former admit an FIU but not an IPS, while the latter admit an IPS but not an IFS/FIU. For the FLRW models with $\gamma = 2/3$ (and any spatial curvature), no conformal states are admitted as we necessarily have $L_0 = 1$. Finally, it is possible for a general FLRW model (which is at least asymptotically self-similar by Theorem 4.3) to admit any conformal state, isotropic or otherwise; even the past isotropic universe, an unphysical time-reversed analogue of the FIU, is allowed.

At first glance, Table 4.1 indicates no obvious relationship between self-similarity and the admission of (isotropic) conformal states. This is largely borne out by closer scrutiny of the logical connectives among various symmetry-related spacetime properties, which is conducted in the following section.

4.3 Isotropy, homogeneity and self-similarity

Definition 3.4 provides a formal, intrinsic notion of approximate self-similarity in the asymptotic regime, just as the admission of an isotropic conformal state in a spacetime represents the property of asymptotic isotropy (in the Ricci-/expansion-dominated sense of (2.17)). Noticeably, there is no asymptotic counterpart for homogeneity; since isotropy implies homogeneity, a spacetime that admits an isotropic conformal state may loosely be interpreted as being asymptotically homogeneous as well. We note that a provisional definition of asymptotic spatial homogeneity within an isotropic conformal structure has been offered by Ericksson [35], and essentially requires homogeneity of the regular spacelike hypersurface $\mathcal{S}_0$ in the unphysical spacetime. This definition is not analogous to an isotropic conformal state, however, and has not been further developed in the literature.

For the purposes of this thesis, then, there are five symmetry-related spacetime properties in the conformal framework: isotropy, homogeneity, exact self-similarity, asymptotic self-similarity, and the admission of an isotropic conformal state at which the fundamental four-velocity field $\mathbf{u}$ is regular. These properties are abbreviated as logical statements in Table 4.2, with 10 pairwise relationships among them. A couple of implications are immediately identified, i.e. isotropy implies homogeneity and exact self-similarity implies asymptotic self-similarity. From various counterexample cosmologies in Table 4.3, pairwise independence is demonstrated in seven other cases (one is based on Conjecture 4.1). All known relationships are then presented schematically in Figure 4.1.

Even if Conjecture 4.1 is shown to be untrue and all de Sitter-like spacetimes are asymptotically self-similar, a definitive result is still obtained. Since every isotropic (and homogeneous) cosmological model is described by the FLRW line element (2.9), we may

Statement	Spacetime is/admits:
ISO	Spatially isotropic (Def. 2.4)
HOM	Spatially homogeneous (Def. 2.5)
ESIM	Exactly self-similar (Eq. (2.18) with constant $\lambda \neq 0$)
ASIM	Asymptotically self-similar (Def. 3.4)
ICS	Isotropic conformal state at which **u** is regular (Defs. 2.6–2.8)

Table 4.2: Symmetry-related spacetime properties in the conformal framework.

conclude that isotropy implies asymptotic self-similarity by Theorem 4.3.

The remaining pairwise relationship, namely that between asymptotic self-similarity and the admission of an isotropic conformal state, is partially established. Spacetimes that are asymptotically self-similar in the past/future do not necessarily admit a corresponding isotropic past/future state, e.g. the FLRW models with $\gamma = 2/3$. A counterexample has yet to be found for the converse case, as explicit instances of asymptotic self-similarity breaking are comparatively rare and difficult to verify in the first place. One possible counterexample is the de Sitter universe (again, assuming Conjecture 4.1), where the admission of an FIU is enough to establish the pairwise independence of asymptotic self-similarity and the admission of an isotropic conformal state. This is an unlikely scenario, however, as Definition 2.8a appears to have been formulated with the exclusion of de Sitter-like behaviour in mind [36].

On the other hand, it seems considerably harder to prove that the admission of an isotropic conformal state implies asymptotic self-similarity. The challenge is essentially to construct an AHVF on a spacetime via its isotropic conformal structure, where the only significant property to work with is regularity of the unphysical spacetime near and on $T = 0$. It might be simpler to find an AHVF on such an unphysical spacetime (and apply Theorem 4.2), since each nonzero metric component g_{ab} typically does not vanish or blow up as $T \to 0^{+(-)}$, i.e. the homothetic equation may be directly verified on $T = 0$. However, this possibility is not trivial to show in the general case and has not been successfully exploited at this stage.

For a general spacetime that admits an isotropic conformal state, even imposing additional structure such as homogeneity (on either the spacetime or the regular spacelike hypersurface $\mathcal{S}_0$) does not yield any insight into its asymptotic self-similarity. Once again, we are limited by the difficulty of finding AHVFs on spacetimes of greater complexity than the FLRW models. Hence the relationship between asymptotic self-similarity and the admission of an isotropic conformal state remains an open problem, albeit one that might be resolved with further study or amendments to the existing framework.

Connective	Counterexample
ISO $\not\Rightarrow$ ESIM	FLRW (open, radiation)
ESIM $\not\Rightarrow$ ISO	Kasner
ISO $\not\Rightarrow$ ASIM	de Sitter**
ASIM $\not\Rightarrow$ ISO	Heckmann–Schücking
ISO $\not\Rightarrow$ ICS	FLRW ($\gamma = 2/3$)
ICS $\not\Rightarrow$ ISO	Kantowski–Sachs
HOM $\not\Rightarrow$ ESIM	Heckmann–Schücking
ESIM $\not\Rightarrow$ HOM	Szekeres (growing)
HOM $\not\Rightarrow$ ASIM	Mixmaster
ASIM $\not\Rightarrow$ HOM	Szekeres (growing)
HOM $\not\Rightarrow$ ICS	FLRW ($\gamma = 2/3$)
ICS $\not\Rightarrow$ HOM	Szekeres (growing)
ESIM $\not\Rightarrow$ ICS	FLRW ($\gamma = 2/3$)
ICS $\not\Rightarrow$ ESIM	FLRW (open, radiation)

$\star\star$: Conjecture 4.1

Table 4.3: Counterexample cosmologies that illustrate the pairwise independence of various spacetime properties in Table 4.2.

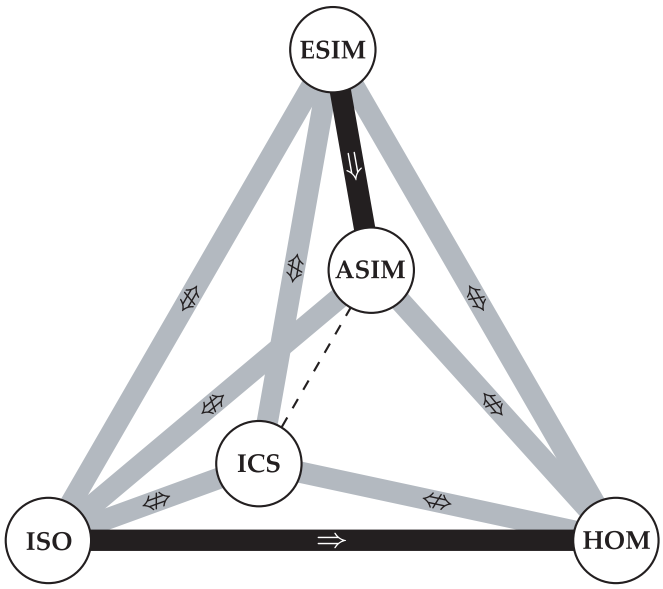

Figure 4.1: Schematic representation of relationships among the spacetime properties in Table 4.2. While isotropy implies homogeneity and exact self-similarity implies asymptotic self-similarity (black lines), the properties are otherwise independent (grey lines). It is not known at this stage if the admission of an isotropic conformal state implies asymptotic self-similarity (dashed line).

Chapter 5

Conclusion

The research in this thesis has been directed solely by the purpose of integrating self-similarity (as a spacetime property) into the conformal framework of quiescent cosmology and the Weyl curvature hypothesis; these cosmological constructs are introduced conceptually in Chapter 1, and formalised from a technical perspective in Chapter 2.

As the conformal framework deals with spacetimes in the asymptotic regime, it is crucial to address the lack of a satisfactory definition for asymptotic self-similarity in the literature. Three existing usages of the term are identified in Section 2.3.2: it refers to asymptotic relationships among exactly self-similar spacetimes in the spherically symmetric approach, describes spacetimes that evolve towards self-similarity in the powerful but specialised dynamical systems approach, and is formally defined but developed improperly in the homothetic equation approach. None of these are suitable working definitions as they stand, and so an improved alternative is required.

To this end, much emphasis has been placed on formulating a robust definition of asymptotic self-similarity in Chapter 3. Two geometric approaches and one that is closer in nature to the dynamical systems definition are shortlisted in Section 3.1. The exact mapping approach is overly rigid but retained for comparative purposes, while the dimensionless variables approach is discarded for its lack of practicality. Our focus is on a homothetic equation-based definition, under which an asymptotically self-similar spacetime admits a vector field satisfying (2.18) in a given asymptotic sense as the spacetime's past/future state is approached.

It is immediately evident that such a definition needs to be weakened sufficiently for agreement with the dynamical systems results on the asymptotic self-similarity of the open and closed FLRW models (this is done by permitting point-dependent scaling factors on the spacelike hypersurface $\mathcal{S}_0$). Also, the definition is more naturally formulated in local coordinates, which leads to a couple of propositions on coordinate dependence that guide the choice of ansatz in the search for AHVFs.

Various example cosmologies that exhibit asymptotic self-similarity (breaking) in the dynamical systems approach are investigated under a preliminary definition in Section 3.2, where two issues promptly surface. Firstly, a common cause of asymptotic self-similarity breaking in spacetimes is attributable to oscillatory behaviour that becomes increasingly rapid at early or late times; this is implemented in the working definition by imposing a C^1 degree of differentiability on the asymptotic process.

Of more concern, it is generally difficult to find AHVFs (or rule out their existence) on spacetimes that are less symmetric than the FLRW models. Several results that facilitate the search for AHVFs are obtained in Section 3.3: a specified class of CKVFs may serve as AHVFs (by Proposition 3.3), while the HVF from any exact mapping between the spacetime and an exactly self-similar counterpart in the asymptotic regime may also be used (by Theorem 3.1). However, such results are limited in their applicability.

The eventual working definition is deemed adequate despite this difficulty, and is considered within the conformal framework in Chapter 4. Section 4.1 analyses how exact and asymptotic self-similarity translate under the conformal transformations in the framework; as it turns out, these properties are preserved with the fulfilment of certain conditions on the conformal factor and relevant vector field. Furthermore, by Theorem 4.2, the possibility of finding AHVFs on a spacetime via its unphysical counterpart provides another method of applying Definition 3.4 within the conformal framework.

Additional example cosmologies whose conformal structures have been studied in the literature are examined for exact and asymptotic self-similarity in Section 4.2. The most significant result is Theorem 4.3, whose proof exploits the existence of CKVFs. This theorem asserts asymptotic self-similarity for the class of all isotropic and homogeneous cosmological models (i.e. spacetimes described by the FLRW line element), with the possible exception of de Sitter-like spacetimes. We also demonstrate past asymptotic self-similarity for the de Sitter universe, and conjecture that it is not asymptotically self-similar at late times.

Armed with a substantial number of example cosmologies (from Chapters 3 and 4), we are able to draw conclusions on the relationships among the five symmetry-related spacetime properties in the conformal framework. It is shown in Section 4.3 via various counterexamples that most of these properties are pairwise independent, apart from isotropy implying homogeneity and exact self-similarity implying asymptotic self-similarity. However, analysis of the relationship between asymptotic self-similarity and the admission of an isotropic conformal state remains inconclusive.

5.1 Future research directions

There is considerable room for improvement in the working definition of asymptotic self-similarity developed in this thesis. As raised in Section 3.3, the definition is not coordinate-invariant; we illustrate this here by first introducing the standard coordinate transformation matrices

$$\Phi_{a'}{}^{a} := \frac{\partial x^{a}}{\partial x'^{a'}}, \qquad \Phi^{a'}{}_{a} := \frac{\partial x'^{a'}}{\partial x^{a}}. \tag{5.1}$$

Now, suppose $\mathbf{X}$ is an AHVF on a spacetime $(\mathcal{M}, \mathbf{g})$ in some set of local coordinates (T, x^{μ}), such that the functions $f_{ab} = X_{(a;b)}/g_{ab}$ satisfy (3.63). Then the corresponding functions in another set of coordinates $(T', x'^{\mu'})$ are given by

$$f'_{a'b'} = \frac{X_{(a';b')}}{g_{a'b'}} = \frac{\Phi_{a'}{}^{a}\Phi_{b'}{}^{b}X_{(a;b)}}{\Phi_{a'}{}^{a}\Phi_{b'}{}^{b}g_{ab}}. \tag{5.2}$$

Each function $f'_{a'b'}$ generally still equals the scaling factor λ in the limit as $T \to 0^{+(-)}$. However, (5.2) might not be well-defined as both the dividend and divisor are now sums of terms, and it is conceivable for one to sum to zero but not the other (since f_{ab} does not factor out of the dividend). In other words, correspondence between the zero/nonzero components of $X_{(a';b')}$ and $g_{a'b'}$ might be broken — which is a minor hassle, as we would prefer the same AHVF to hold in all coordinate frames. It is not immediately clear how or whether to resolve this issue, since a formulation in local coordinates is necessary to make the definition precise (and to distinguish it from the conformal Killing equation).

If greater compatibility with the dynamical systems approach is desired, the working definition may be amended or even reworked to incorporate approaches that have not been explored in this thesis. For example, the definition may be reformulated relative to an expansion-normalised basis $\{\mathbf{v}_{a'}\} = \{H^{-1}\delta_{a'}{}^{a}\partial/\partial x^{a}\}$ such that

$$g_{a'b'} = \mathbf{g}\left(\mathbf{v}_{a'}, \mathbf{v}_{b'}\right) = H^{-2} g_{ab}, \tag{5.3}$$

where H is the generalised Hubble parameter. Another possibility is to employ the (unphysical) dimensionless metric $\widehat{\mathbf{g}} = H^2\mathbf{g}$ in some way; relative to the usual coordinate basis $\{\partial/\partial x^a\}$, the metric components are given by

$$\widehat{g}_{ab} = \widehat{\mathbf{g}}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = H^2 g_{ab}. \tag{5.4}$$

The general idea with these approaches is to somehow take into account the effects of a spacetime's expansion, in accordance with the dynamical systems definition.

Several avenues of future research are available if Definition 3.4 is accepted as a suitable working definition of asymptotic self-similarity. New example cosmologies must be studied and — despite the difficulty of finding or ruling out AHVFs — more conclusive results are required in order to augment the definition. At present, there appears to be good agreement in Table 3.1 between Definition 3.4 and the dynamical systems definition, but this is not expected to persist as further results are obtained. It is also desirable to prove (or disprove) Conjectures 3.1 and 4.1, such that we may better understand the role of an exact mapping in the asymptotic regime and the anomalous behaviour of de Sitter-like spacetimes.

The analysis in Chapter 4 of exact and asymptotic self-similarity within the conformal framework may be expanded upon as well. A complete characterisation of how self-similarity translates under the conformal transformations in the framework would be welcome; specifically, we are looking to get a firm handle on the origin and implication of the technical conditions in Theorems 4.1 and 4.2 (plus their corollaries). Uncovering the relationship between asymptotic self-similarity and the admission of an isotropic conformal state is also important — especially if the latter turns out to imply the former, which would indicate that the spacetime property of self-similarity is not stand-alone and has a significant part to play in the conformal framework.

Appendix A

Formulae in relativistic cosmology

Various curvature and kinematic quantities in relativistic cosmology are required for or relevant to the purposes of this thesis. Formulae for these quantities in local coordinates (x^a) on a cosmological model $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ are sourced from reference material by Ellis et al. [1, 18, 23], and are listed in this appendix for the reader's convenience.

A.1 Curvature quantities

- The inverse metric g^{ab} is given implicitly by

$$g^{ab}g_{bc} = \delta^a_{\ c}, \tag{A.1}$$

  i.e. $[g^{ab}] = [g_{ab}]^{-1}$.

- The Christoffel symbols (of the second kind) are given by

$$\Gamma^a_{\ bc} = \frac{1}{2}g^{ad}\left(g_{db,c} + g_{dc,b} - g_{bc,d}\right). \tag{A.2}$$

- The covariant derivative (with respect to the Levi-Civita connection) of a type-(r, s) tensor $\mathbf{T}$ along the vector field $\partial/\partial x^c$ is given by

$$\begin{aligned} T^{a_1\ldots a_r}{}_{b_1\ldots b_s;c} = T^{a_1\ldots a_r}{}_{b_1\ldots b_s,c} + \Gamma^{a_1}_{\ cd}T^{d\ldots a_r}{}_{b_1\ldots b_s} + \ldots + \Gamma^{a_r}_{\ cd}T^{a_1\ldots d}{}_{b_1\ldots b_s} \\ - \Gamma^d_{\ b_1c}T^{a_1\ldots a_r}{}_{d\ldots b_s} - \ldots - \Gamma^d_{\ b_sc}T^{a_1\ldots a_r}{}_{b_1\ldots d}. \end{aligned} \tag{A.3}$$

- The Lie derivative of a type-(r, s) tensor $\mathbf{T}$ along the vector field $\mathbf{X}$ is given by

$$\begin{aligned} (\mathcal{L}_XT)^{a_1\ldots a_r}{}_{b_1\ldots b_s} = X^cT^{a_1\ldots a_r}{}_{b_1\ldots b_s;c} - X^{a_1}{}_{;c}T^{c\ldots a_r}{}_{b_1\ldots b_s} - \ldots - X^{a_r}{}_{;c}T^{a_1\ldots c}{}_{b_1\ldots b_s} \\ + X^c{}_{;b_1}T^{a_1\ldots a_r}{}_{c\ldots b_s} + \ldots + X^c{}_{;b_s}T^{a_1\ldots a_r}{}_{b_1\ldots c}. \end{aligned} \tag{A.4}$$

- The Riemann tensor is given by

$$R^a_{\ bcd} = \Gamma^a_{\ bd,c} - \Gamma^a_{\ bc,d} + \Gamma^a_{\ ce}\Gamma^e_{\ bd} - \Gamma^a_{\ de}\Gamma^e_{\ bc}. \tag{A.5}$$

- The Ricci tensor is given by

$$R_{ab} = R^c_{\ acb}. \tag{A.6}$$

- The scalar curvature is given by

$$R = R^a_{\ a}. \tag{A.7}$$

- The Weyl tensor (on a four-dimensional manifold) is given by

$$C_{abcd} = R_{abcd} + g_{a[d}R_{c]b} + g_{b[c}R_{d]a} + \frac{1}{3}Rg_{a[c}g_{d]b}. \tag{A.8}$$

A.2 Kinematic quantities

- The projection tensor into the rest space of an observer moving with four-velocity $\mathbf{u}$ is given by

$$h_{ab} = g_{ab} + u_a u_b. \tag{A.9}$$

- The acceleration vector is given by

$$\dot{u}^a = u^a_{\ ;b}u^b. \tag{A.10}$$

- The expansion tensor and scalar are given respectively by

$$\theta_{ab} = h_{(a}^{\ \ c}h_{b)}^{\ \ d}u_{c;d}, \tag{A.11}$$

$$\theta = \theta^a_{\ a} = u^a_{\ ;a}. \tag{A.12}$$

- The shear tensor and scalar are given respectively by

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab}, \tag{A.13}$$

$$\sigma = \sqrt{\frac{1}{2}\sigma_{ab}\sigma^{ab}}. \tag{A.14}$$

- The vorticity tensor, vector and scalar are given respectively by

$$\omega_{ab} = h_{[a}^{\ \ c}h_{b]}^{\ \ d}u_{c;d}, \tag{A.15}$$

$$\omega^a = \frac{1}{2}V\epsilon^{abcd}u_b\omega_{cd}, \tag{A.16}$$

$$\omega = \sqrt{\omega_a\omega^a} = \sqrt{\frac{1}{2}\omega_{ab}\omega^{ab}}, \tag{A.17}$$

where $V = |\det \mathbf{g}|^{-1/2}$ and ϵ^{abcd} is the Levi-Civita symbol.

- The anisotropic parts of the Ricci tensor relative to $\mathbf{u}$ are given by

$$\Sigma_a = -h_a^{\ b}R_b^{\ c}u_c, \tag{A.18}$$

$$\Sigma_a^{\ b} = h_a^{\ c}h_d^{\ b}R_c^{\ d} - \frac{1}{3}h_a^{\ b}h_d^{\ c}R_c^{\ d}. \tag{A.19}$$

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