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SPECTRAL CURVES AND PARAMETERIZATION OF A DISCRETE INTEGRABLE THREE-DIMENSIONAL MODEL

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We consider a discrete classical integrable model on a three-dimensional cubic lattice. The solutions of this model can be used to parameterize the Boltzmann weights of various three-dimensional spin models. We find the general solution of this model constructed in terms of the theta functions defined on an arbitrary compact algebraic curve. Imposing periodic boundary conditions fixes the algebraic curve. We show that the curve then coincides with the spectral curve of the auxiliary linear problem. For a rational curve, we construct the soliton solution of the model.

Keywords: three-dimensional integrable systems, Bäcklund transformations, spectral curves

1. Introduction

This paper is devoted to describing the periodic and soliton solutions of a general classical three-dimensional discrete integrable model. We show that this description can be given completely analogously to the description of the finite-gap solutions of the hierarchy of the continuous integrable equations, although our way of deriving our discrete systems of equation is nonstandard and is motivated by the approach developed in [1] for discrete integrable spin models. First, we construct the discrete equations of motion from an equivalence condition for the auxiliary linear systems (which replaces the zero-curvature condition for the Lax operators in the standard formulation) and then prove the integrability by counting the number of independent integrals of motion.

Several types of boundary conditions can be considered on the cubic lattice: open boundary, periodic boundary conditions in chosen directions, or completely periodic boundary conditions. Choosing the specific boundary conditions leads to a specific dynamical interpretation of the model: the Cauchy problem, the Bäcklund transformation, or an analogue of a standing wave on the discrete three-dimensional torus.

Starting from the discrete equations of motion for our model, we change variables through a triple of the Legendre variables, which transforms the equations of motion into the trilinear form. These trilinear equations are a generalization of the famous Hirota bilinear difference equation. We then observe that these trilinear equations can be formally solved using Fay's identity for the theta function on an arbitrary algebraic curve.

Some facts observed in this paper are a manifestation of the well-known statement proved more than two decades ago [2] that any discrete integrable system can be solved using algebraic-geometric methods. In this paper, we further develop an alternative approach to three-dimensional discrete integrable systems [1] that does not use the notion of Lax operators and can be applied to both quantum (spin) and classical integrable systems associated with several three-dimensional lattices. Instead of the Lax operators, we

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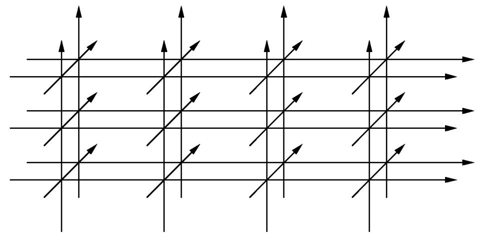

Fig. 1

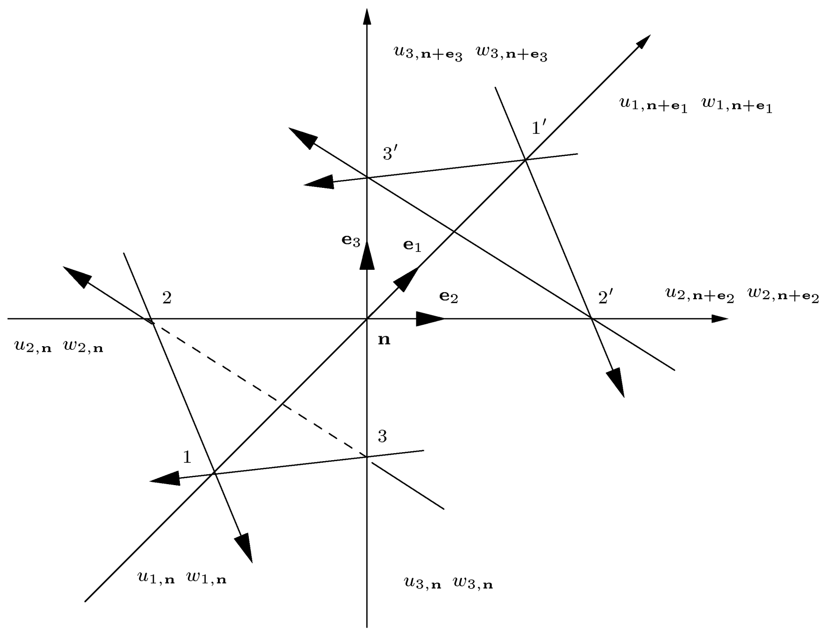

Fig. 2

use the notion of the linear system defined on auxiliary planes. In this paper, we consider only the cubic lattice although these methods may be applied to an arbitrary three-dimensional lattice formed by a set of intersecting planes.

2. Classical discrete integrable system on the cubic lattice

Let the vertices of a cubic lattice spanned by the orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be labeled by the vectors

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3. \tag{1}$$

The cubic lattice is formed by three sets of parallel planes or, equivalently, by three sets of parallel lines (see Fig. 1, where the cubic lattice of the size 3×2×4 is shown as an example). We associate the dynamical variables with each edge of the cubic lattice of a given type α, ($\alpha = 1, 2, 3$ corresponds to the three orthogonal directions of the lattice) as shown in Fig. 2. Namely, the pairs of dynamical variables $u_{\alpha,\mathbf{n}}$, $w_{\alpha,\mathbf{n}}$, $\alpha = 1, 2, 3$, are associated with the edges incoming to the oriented vertex $\mathbf{n}$, while $u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$, $w_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$ are associated with the outgoing edges. In Fig. 2, we also depict two auxiliary planes intersecting the respective incoming and outgoing edges with the number $\mathbf{n}$ (these intersecting planes form the triangles 123 and $1'2'3'$). Each of

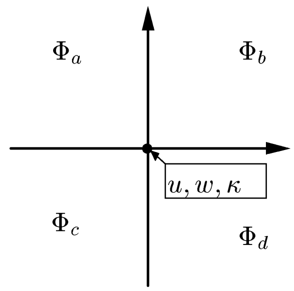

Fig. 3

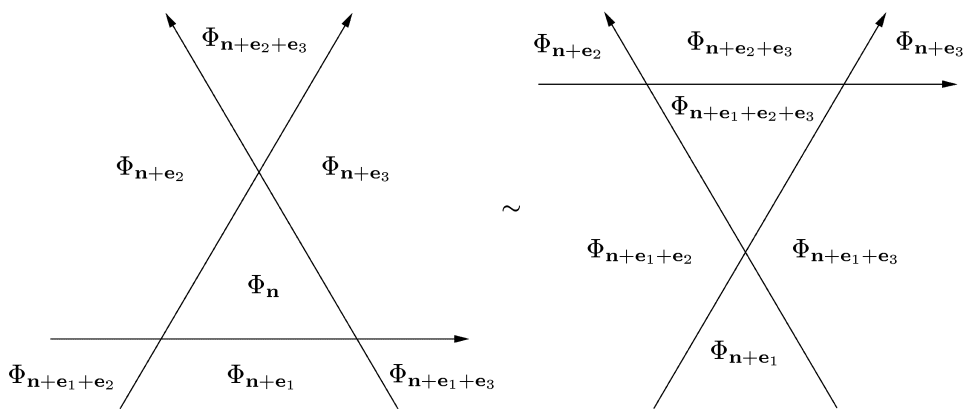

Fig. 4

these planes intersects seven of the eight octants around the vertex of the cubic lattice labeled by the vector $\mathbf{n}$.

We stress that our considerations are strictly local at the moment. Our goal is to obtain the relations between the dynamical variables surrounding a given vertex. We then extend these relations to the whole lattice, thus obtaining the discrete dynamical system.

Each triangle on the auxiliary planes is formed by three lines obtained by the intersection of the corresponding plane with the planes forming the vertex $\mathbf{n}$ of the cubic lattice. The vertices of the auxiliary triangles are thus associated with the edges of the cubic lattice and consequently with a pair of the dynamical variables. We consider two linear problems associated with the auxiliary triangles by the following rules. First, we introduce the linear variables Φ_a, Φ_b, Φ_c, and Φ_d around the vertex on the auxiliary plane according to Fig. 3. The linear problems for each vertex on the auxiliary planes always have the form

$$0 = \Phi_a - \Phi_b \cdot u + \Phi_c \cdot w + \Phi_d \cdot \kappa u w, \tag{2}$$

where $\kappa \in \mathbb{C}$ is an additional parameter associated with the edge of the cubic lattice. We note that the coefficients of linear form (2) are fixed by the orientation of the lines that form the vertex on the auxiliary plane.

The parameterization of the linear variables on the auxiliary planes for the cubic geometry is shown in Fig. 4. It is clear that these linear variables are associated with the internal parts of the cubes in the cubic lattice. According to (2), we can write the system of linear equations

$$
\begin{aligned}
0 &= \Phi_{\mathbf{n}+\mathbf{e}_2} - \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} u_{1,\mathbf{n}} + \Phi_{\mathbf{n}} w_{1,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_3} \kappa_{1,\mathbf{n}} u_{1,\mathbf{n}} w_{1,\mathbf{n}},\\
0 &= \Phi_{\mathbf{n}} - \Phi_{\mathbf{n}+\mathbf{e}_3} u_{2,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1} w_{2,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3} \kappa_{2,\mathbf{n}} u_{2,\mathbf{n}} w_{2,\mathbf{n}},\\
0 &= \Phi_{\mathbf{n}+\mathbf{e}_2} - \Phi_{\mathbf{n}} u_{3,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} w_{3,\mathbf{n}} + \Phi_{\mathbf{n}+\mathbf{e}_1} \kappa_{3,\mathbf{n}} u_{3,\mathbf{n}} w_{3,\mathbf{n}}
\end{aligned}
\tag{3}
$$

for the left triangle shown in Fig. 4 and the linear system

$$
\begin{aligned}
0 = {} & \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} - \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} u_{1,\mathbf{n}+\mathbf{e}_1} + \Phi_{n+\mathbf{e}_1} w_{1,\mathbf{n}+\mathbf{e}_1} + \\
& + \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3} \kappa_{1,\mathbf{n}} u_{1,\mathbf{n}+\mathbf{e}_1} w_{1,\mathbf{n}+\mathbf{e}_1},\\
0 = {} & \Phi_{\mathbf{n}+\mathbf{e}_2} - \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} u_{2,\mathbf{n}+\mathbf{e}_2} + \Phi_{n+\mathbf{e}_1+\mathbf{e}_2} w_{2,\mathbf{n}+\mathbf{e}_2} + \\
& + \Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} \kappa_{2,\mathbf{n}} u_{2,\mathbf{n}+\mathbf{e}_2} w_{2,\mathbf{n}+\mathbf{e}_2},\\
0 = {} & \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} - \Phi_{\mathbf{n}+\mathbf{e}_3} u_{3,\mathbf{n}+\mathbf{e}_3} + \Phi_{n+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} w_{3,\mathbf{n}+\mathbf{e}_3} + \\
& + \Phi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} \kappa_{3,\mathbf{n}} u_{3,\mathbf{n}+\mathbf{e}_3} w_{3,\mathbf{n}+\mathbf{e}_3}
\end{aligned}
\tag{4}
$$

for the right triangle.

Assuming that the dynamical variables do not vanish identically on any edge of the lattice, we can eliminate the linear variables $\Phi_{\mathbf{n}}$ and $\Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}$ from systems (3) and (4). We require that systems (3) and (4) (each now contains only two linear relations for the remaining six linear variables $\Phi_{\mathbf{n}+\mathbf{e}_1}$, $\Phi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}$, etc.) be equivalent. This means that the matrix elements of these linear systems, which are rational functions of all dynamical variables around a given vertex, coincide identically. Moreover, we require that each type of the parameters $\kappa_{\alpha,\mathbf{n}}$ be conserved along the corresponding direction α (which was already taken into account in (4)):

$$
\kappa_{\alpha,\mathbf{n}} = \kappa_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}, \quad \alpha = 1, 2, 3. \tag{5}
$$

As a result, we obtain the recursive relations for the dynamical variables $u_{\alpha,\mathbf{n}}$, $w_{\alpha,\mathbf{n}}$, $u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$, and $w_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}$,

$$
\begin{aligned}
u_{1,\mathbf{n}+\mathbf{e}_1} &= \frac{\kappa_{2,\mathbf{n}} u_{1,\mathbf{n}} u_{2,\mathbf{n}} w_{2,\mathbf{n}}}{\kappa_{1,\mathbf{n}} u_{1,\mathbf{n}} w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}} u_{2,\mathbf{n}} w_{3,\mathbf{n}} + \kappa_{1,\mathbf{n}} \kappa_{3,\mathbf{n}} u_{1,\mathbf{n}} w_{3,\mathbf{n}}},\\
w_{1,\mathbf{n}+\mathbf{e}_1} &= \frac{w_{1,\mathbf{n}} w_{2,\mathbf{n}} + u_{3,\mathbf{n}} w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}} u_{3,\mathbf{n}} w_{3,\mathbf{n}}}{w_{3,\mathbf{n}}},
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
u_{2,\mathbf{n}+\mathbf{e}_2} &= \frac{u_{1,\mathbf{n}} u_{2,\mathbf{n}} u_{3,\mathbf{n}}}{u_{2,\mathbf{n}} u_{3,\mathbf{n}} + u_{2,\mathbf{n}} w_{1,\mathbf{n}} + \kappa_{1,\mathbf{n}} u_{1,\mathbf{n}} w_{1,\mathbf{n}}},\\
w_{2,\mathbf{n}+\mathbf{e}_2} &= \frac{w_{1,\mathbf{n}} w_{2,\mathbf{n}} w_{3,\mathbf{n}}}{w_{1,\mathbf{n}} w_{2,\mathbf{n}} + u_{3,\mathbf{n}} w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}} u_{3,\mathbf{n}} w_{3,\mathbf{n}}},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
u_{3,\mathbf{n}+\mathbf{e}_3} &= \frac{u_{2,\mathbf{n}} u_{3,\mathbf{n}} + u_{2,\mathbf{n}} w_{1,\mathbf{n}} + \kappa_{1,\mathbf{n}} u_{1,\mathbf{n}} w_{1,\mathbf{n}}}{u_{1,\mathbf{n}}},\\
w_{3,\mathbf{n}+\mathbf{e}_3} &= \frac{\kappa_{2,\mathbf{n}} u_{2,\mathbf{n}} w_{2,\mathbf{n}} w_{3,\mathbf{n}}}{\kappa_{1,\mathbf{n}} u_{1,\mathbf{n}} w_{2,\mathbf{n}} + \kappa_{3,\mathbf{n}} u_{2,\mathbf{n}} w_{3,\mathbf{n}} + \kappa_{1,\mathbf{n}} \kappa_{3,\mathbf{n}} u_{1,\mathbf{n}} w_{3,\mathbf{n}}},
\end{aligned}
\tag{8}
$$

where, by virtue of (5), $\kappa_{1,\mathbf{n}}$ is independent of the coordinate n_1 in expansion (1), $\kappa_{2,\mathbf{n}}$ is analogously independent of n_2, and $\kappa_{3,\mathbf{n}}$ is independent of n_3. Such a dependence on not all the coordinates of $\mathbf{n}$ is denoted by

$$
\kappa_{1,\mathbf{n}} = \kappa_{1:n_2,n_3}, \qquad \kappa_{2,\mathbf{n}} = \kappa_{2:n_1,n_3}, \qquad \kappa_{3,\mathbf{n}} = \kappa_{3:n_1,n_2}. \tag{9}
$$

It might be asked why recursive relations (6)–(8) constitute a dynamical system. We introduce the nonzero Poisson bracket for the edges incoming to the vertex with some fixed $\mathbf{n}$,

$$\{u_{\alpha,\mathbf{n}}, w_{\beta,\mathbf{n}}\} = u_{\alpha,\mathbf{n}} w_{\alpha,\mathbf{n}} \delta_{\alpha\beta}. \tag{10}$$

Transformation (6)–(8) is canonical, i.e., it follows from (10) that

$$\{u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha}, w_{\beta,\mathbf{n}+\mathbf{e}_\beta}\} = u_{\alpha,\mathbf{n}+\mathbf{e}_\alpha} w_{\beta,\mathbf{n}+\mathbf{e}_\beta} \delta_{\alpha\beta}. \tag{11}$$

The auxiliary triangles shown in Fig. 2 are fragments of two spacelike surfaces. The set of three-dimensional vertices between these surfaces provides canonical transformation (6)–(8) of the set of dynamical variables from one surface to the other. We call this transformation the local equation of motion in the direction perpendicular to the chosen spacelike surface. In addition to local equations of motion (6)–(8), a complete formulation of the dynamical system requires the specification of boundary conditions.

The type of the boundary conditions, as well as the global characteristics of the model, depends on the choice of the spacelike surface. The described auxiliary lattices are appropriate for the Cauchy problem for the cubic lattice. In Sec. 6, we choose n_1 as the discrete time, while n_2 and n_3 are the discrete space coordinates. System (6) then describes the "time" evolution of the variables $u_{1,\mathbf{n}}$, $w_{1,\mathbf{n}}$, while systems (7) and (8) describe the spatial distribution of the variables $u_{2,\mathbf{n}}$, $w_{2,\mathbf{n}}$, and $u_{3,\mathbf{n}}$, $w_{3,\mathbf{n}}$, which are "auxiliary" for this evolution. All the dynamical quantities, such as integrals of motion, spectral curves, etc., must be calculated for system (6) in this case.

3. The Legendre transform

It follows from (6)–(8) that

$$\begin{aligned} &w_{1,\mathbf{n}} w_{2,\mathbf{n}} = w_{1,\mathbf{n}+\mathbf{e}_1} w_{2,\mathbf{n}+\mathbf{e}_2}, \qquad u_{2,\mathbf{n}} u_{3,\mathbf{n}} = u_{2,\mathbf{n}+\mathbf{e}_2} u_{3,\mathbf{n}+\mathbf{e}_3}, \\ &\frac{u_{1,\mathbf{n}}}{w_{3,\mathbf{n}}} = \frac{u_{1,\mathbf{n}+\mathbf{e}_1}}{w_{3,\mathbf{n}+\mathbf{e}_3}}. \end{aligned} \tag{12}$$

In general, we can use these relations to perform the change of variables

$$\begin{aligned} u_{1,\mathbf{n}} &= u^{(0)}_{1:n_2,n_3} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}, \qquad & w_{1,\mathbf{n}} &= w^{(0)}_{1:n_2,n_3} \frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}, \\ u_{2,\mathbf{n}} &= u^{(0)}_{2:n_1,n_3} \frac{\tau_{1,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}}, \qquad & w_{2,\mathbf{n}} &= w^{(0)}_{2:n_1,n_3} \frac{\tau_{3,\mathbf{n}}}{\tau_{3,\mathbf{n}+\mathbf{e}_1}}, \\ u_{3,\mathbf{n}} &= u^{(0)}_{3:n_1,n_2} \frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}}}, \qquad & w_{3,\mathbf{n}} &= w^{(0)}_{3:n_1,n_2} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_1}}. \end{aligned} \tag{13}$$

After substitution (13), Eqs. (6)–(8) can be rewritten in the trilinear form

$$\begin{aligned} r_{\alpha,\mathbf{n}} \tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta+\mathbf{e}_\gamma} \tau_{\beta,\mathbf{n}} \tau_{\gamma,\mathbf{n}} = \tau_{\alpha,\mathbf{n}} \tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma} \tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta} + \\ + s_{\beta,\mathbf{n}} \tau_{\alpha,\mathbf{n}+\mathbf{e}_\beta} \tau_{\beta,\mathbf{n}+\mathbf{e}_\gamma} \tau_{\gamma,\mathbf{n}} + s^{-1}_{\gamma,\mathbf{n}} \tau_{\alpha,\mathbf{n}+\mathbf{e}_\gamma} \tau_{\beta,\mathbf{n}} \tau_{\gamma,\mathbf{n}+\mathbf{e}_\beta}, \end{aligned} \tag{14}$$

where (α,β,γ) is any *cyclic permutation* of the indices $(1,2,3)$ and the coefficients are

$$
\begin{aligned}
s_{1,\mathbf{n}} &= \frac{\kappa_{3:n_1,n_2} w^{(0)}_{3:n_1,n_2}}{w^{(0)}_{2:n_1,n_3}}, &\qquad r_{1,\mathbf{n}} &= \frac{u^{(0)}_{1:n_2,n_3} u^{(0)}_{3:n_1,n_2}}{w^{(0)}_{1:n_2,n_3} u^{(0)}_{2:n_1,n_3}}, \\
s_{2,\mathbf{n}} &= \frac{u^{(0)}_{3:n_1,n_2}}{w^{(0)}_{1:n_2,n_3}}, &\qquad r_{2,\mathbf{n}} &= \frac{\kappa_{2:n_1,n_3} u^{(0)}_{2:n_1,n_3} w^{(0)}_{2:n_1,n_3}}{\kappa_{1:n_2,n_3}\kappa_{3:n_1,n_2} u^{(0)}_{1:n_2,n_3} w^{(0)}_{3:n_1,n_2}}, \\
s_{3,\mathbf{n}} &= \frac{u^{(0)}_{2:n_1,n_3}}{\kappa_{1:n_2,n_3} u^{(0)}_{1:n_2,n_3}}, &\qquad r_{3,\mathbf{n}} &= \frac{w^{(0)}_{1:n_2,n_3} w^{(0)}_{3:n_1,n_2}}{w^{(0)}_{2:n_1,n_3} u^{(0)}_{3:n_1,n_2}}.
\end{aligned}
\tag{15}
$$

Change of variables (13) admits the following interpretation. Equations (6)–(8) are a kind of Hamiltonian equations of motion for the classical discrete system. Substitution (13) is therefore the Legendre transformation. Finally, Eqs. (14) are the Lagrangian equations of motion. We call $\tau_{\alpha\mathbf{n}}$ the Legendre variables and the coefficients $u^{(0)}_{\alpha:n_\beta,n_\gamma}$ and $w^{(0)}_{\alpha:n_\beta,n_\gamma}$ the preexponentials.

Equations (14) in a sense generalize the famous Hirota bilinear equation [3], which can be obtained from (14) as a special limit where all the parameters $\kappa_{\alpha:n_\beta,n_\gamma}$ vanish in a definite way. Indeed, let the parameters $r_{2\mathbf{n}}$, $s_{3\mathbf{n}}$, and $s^{-1}_{1\mathbf{n}}$, which depend on the parameters κ, tend to infinity as $\varepsilon \to 0$,

$$
r_{2,\mathbf{n}} \sim \varepsilon^{-2} + \frac{1}{2}\varepsilon^{-1}, \qquad s_{3,\mathbf{n}} \sim \varepsilon^{-2} - \frac{1}{2}\varepsilon^{-1}, \qquad s^{-1}_{1,\mathbf{n}} \sim \varepsilon^{-1}. \tag{16}
$$

Choosing $\alpha = 2$, $\beta = 3$, and $\gamma = 1$ and equating the coefficients at ε^{-2} and ε^{-1} in (14), we obtain

$$
\tau_{3,\mathbf{n}} = \tau_{2,\mathbf{n}+\mathbf{e}_3}, \qquad \tau_{1,\mathbf{n}} = \tau_{2,\mathbf{n}+\mathbf{e}_1}. \tag{17}
$$

We note that $\kappa_{1,\mathbf{n}} \sim \varepsilon^2$ and $\kappa_{2,\mathbf{n}} \sim \kappa_{3,\mathbf{n}} \sim \varepsilon$ in limit (16). Imposing one more condition $r_{1,\mathbf{n}} = s_{2,\mathbf{n}} r_{3,\mathbf{n}}$, we find that the remaining two equations in (14) coincide and can be written as a single difference equation for the Legendre variable $\tau_{2,\mathbf{n}}$:

$$
r_{1,\mathbf{n}} \tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3} \tau_{2,\mathbf{n}} = \tau_{2,\mathbf{n}+\mathbf{e}_1} \tau_{2,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} + s_{2,\mathbf{n}} \tau_{2,\mathbf{n}+\mathbf{e}_3} \tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}.
$$

In the homogeneous limit where the parameters $r_{1,\mathbf{n}} = r_1$ and $s_{2\mathbf{n}} = s_2$ are independent of the vertex number $\mathbf{n}$, the latter equation can be rewritten in the canonical Hirota form after an obvious reenumeration of the discrete variables. Because of this analogy, we often call the Legendre variables $\tau_{j\mathbf{n}}$ the triplet of tau functions.

We conclude this section by describing the discrete gauge invariance of linear systems (3) and (4). We require that these systems be invariant under a simultaneous shift of the linear variables

$$
\Phi_{\mathbf{n}} \mapsto \xi_{\mathbf{n}} \Phi_{\mathbf{n}}, \tag{18}
$$

where $\xi_{\mathbf{n}} \in \mathbb{C}$. This invariance requires the corresponding change of the dynamical variables

$$
u_{1,\mathbf{n}} \mapsto \frac{\xi_{\mathbf{n}+\mathbf{e}_2}}{\xi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}} u_{1,\mathbf{n}}, \qquad w_{1,\mathbf{n}} \mapsto \frac{\xi_{\mathbf{n}+\mathbf{e}_2}}{\xi_{\mathbf{n}}} w_{1,\mathbf{n}},
$$

etc. We must then require the coincidence of the gauge transformation of the parameters $\kappa_{\alpha\mathbf{n}}$, $\alpha = 1,2,3$, in both linear systems (3) and (4), i.e.,

$$
\kappa_{1,\mathbf{n}} \mapsto \frac{\xi_{\mathbf{n}} \xi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}}{\xi_{\mathbf{n}+\mathbf{e}_2} \xi_{\mathbf{n}+\mathbf{e}_3}} \kappa_{1,\mathbf{n}} = \frac{\xi_{\mathbf{n}+\mathbf{e}_1} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}}{\xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2} \xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}} \kappa_{1,\mathbf{n}}. \tag{19}
$$

Similar equalities can be obtained for the parameters $\kappa_{2\mathbf{n}}$ and $\kappa_{3\mathbf{n}}$. Any of these requirements leads to the single relation

$$\xi_{\mathbf{n}}\xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}\xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}\xi_{\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3} = \xi_{\mathbf{n}+\mathbf{e}_1}\xi_{\mathbf{n}+\mathbf{e}_2}\xi_{\mathbf{n}+\mathbf{e}_3}\xi_{\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}. \tag{20}$$

The last equation can be easily satisfied by the ansatz

$$\xi_{\mathbf{n}} = \xi_{1:n_2,n_3}\xi_{2:n_1,n_3}\xi_{3:n_1,n_2}. \tag{21}$$

This explains all ξ-dependent factors appearing in the statement of Proposition 1 below.

4. General solution of the classical equations of motion

As was shown, Eqs. (14) generalize the Hirota equations. The structure of (14) repeats the structure of the Hirota equation. In this section, we present a general solution of these classical equations of motion when each trilinear equation (14) is reduced to a pair of bilinear equations. This form is suitable for imposing periodic boundary conditions. The bilinear relations can be identified with Fay's well-known identities for the theta functions associated with an algebraic curve of a finite genus. Periodic boundary conditions fix the form of a general algebraic curve uniquely. The form of this curve depends on the type of boundary conditions. We claim that the obtained solution of trilinear equations (14) is the most general solution for periodic boundary conditions. We start by formulating the necessary algebraic-geometric objects (see, e.g., [4], [5]).

Let Γ be an arbitrary algebraic curve of genus g and $\omega = (\omega_1, \omega_2, \dots, \omega_g)$ be the g-dimensional vector of holomorphic differentials. We use the standard normalization

$$\oint_{a_j} \omega_k = \delta_{k,j}, \qquad \oint_{b_j} \omega_k = \Omega_{k,j}, \tag{22}$$

where a_j and b_j, $j = 1, \dots, g$, are the sets of canonical cycles on Γ.

Let $\mathbf{I}\colon \Gamma^{\otimes 2} \mapsto \mathrm{Jac}(\Gamma)$ be the Jacobi map

$$X, Y \in \Gamma \quad \mapsto \quad \mathbf{I}(X, Y) = \int_X^Y \omega \in \mathrm{Jac}(\Gamma), \tag{23}$$

and let $\Theta_{\boldsymbol{\epsilon}}(\mathbf{v})$, $\mathbf{v} \in \mathbb{C}^g$, $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$, $\boldsymbol{\epsilon}_i \in \mathbb{C}^g$, be the theta function with characteristic $\boldsymbol{\epsilon}$ on the Jacobian $\mathrm{Jac}(\Gamma)$,

$$\Theta_{\boldsymbol{\epsilon}}(\mathbf{v}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp\bigl(i\pi(\mathbf{m}+\boldsymbol{\epsilon}_1, \Omega\mathbf{m}+\boldsymbol{\epsilon}_1) + 2i\pi(\mathbf{m}+\boldsymbol{\epsilon}_1, \mathbf{v}+\boldsymbol{\epsilon}_2)\bigr), \tag{24}$$

and let $E(X, Y) = -E(Y, X)$ be the prime form for $X, Y \in \Gamma$ such that the cross ratio

$$\frac{E(X,Y)E(X',Y')}{E(X,Y')E(X',Y)} = \frac{\Theta_{\boldsymbol{\epsilon}_{\mathrm{odd}}}\bigl(\mathbf{I}(X,Y)\bigr)\Theta_{\boldsymbol{\epsilon}_{\mathrm{odd}}}\bigl(\mathbf{I}(X',Y')\bigr)}{\Theta_{\boldsymbol{\epsilon}_{\mathrm{odd}}}\bigl(\mathbf{I}(X,Y')\bigr)\Theta_{\boldsymbol{\epsilon}_{\mathrm{odd}}}\bigl(\mathbf{I}(X',Y)\bigr)} \tag{25}$$

is a well-defined quasiperiodic function on $\Gamma^{\otimes 4}$. The parameter $\boldsymbol{\epsilon}_{\mathrm{odd}}$ is a nonsingular odd theta characteristic such that $\Theta_{\boldsymbol{\epsilon}_{\mathrm{odd}}}(\mathbf{0}) = 0$. We let $\Theta(\mathbf{v})$ denote the theta function with the zero characteristic.

There is an identity on $\Gamma^{\otimes 4} \otimes \mathrm{Jac}(\Gamma)$, the so-called bilinear Fay identity, which can be written in the form

$$\begin{aligned}\Theta(\mathbf{v})\Theta\bigl(\mathbf{v}+\mathbf{I}(B+D, A+C)\bigr) = {}& \frac{E(A,B)E(D,C)}{E(A,C)E(D,B)}\Theta\bigl(\mathbf{v}+\mathbf{I}(D,A)\bigr)\Theta\bigl(\mathbf{v}+\mathbf{I}(B,C)\bigr) + \\ &+ \frac{E(A,D)E(C,B)}{E(A,C)E(D,B)}\Theta\bigl(\mathbf{v}+\mathbf{I}(B,A)\bigr)\Theta\bigl(\mathbf{v}+\mathbf{I}(D,C)\bigr),\end{aligned} \tag{26}$$

where it is assumed that $\mathbf{I}(B+D, A+C) = \mathbf{I}(B,A) + \mathbf{I}(D,C) = \mathbf{I}(D,A) + \mathbf{I}(B,C)$.

Let $\mathbf{v} \in \mathrm{Jac}(\Gamma)$, X_{n_1}, X'_{n_1}, Y_{n_2}, Y'_{n_2}, Z_{n_3}, Z'_{n_3} ($n_\alpha \in \mathbb{Z}$), and P, Q be arbitrary distinct points on the algebraic curve Γ. We then have the following proposition.

Proposition 1. *Any solution of local equations of motion* (14) *is a particular case of the general solution*

$$\begin{aligned} \tau_{1,\mathbf{n}} &= \xi_{1:n_2,n_3}\Theta\big(\mathbf{I_n} + \mathbf{I}(Q, X_{n_1})\big), \\ \tau_{2,\mathbf{n}} &= \xi_{2:n_1,n_3}\Theta\big(\mathbf{I_n} + \mathbf{I}(Q, Y_{n_2})\big), \\ \tau_{3,\mathbf{n}} &= \xi_{3:n_1,n_2}\Theta\big(\mathbf{I_n} + \mathbf{I}(Q, Z_{n_3})\big), \end{aligned} \tag{27}$$

where

$$\mathbf{I_n} = \mathbf{v} + \sum_{m_1=0}^{n_1-1} \mathbf{I}(X'_{m_1}, X_{m_1}) + \sum_{m_2=0}^{n_2-1} \mathbf{I}(Y'_{m_2}, Y_{m_2}) + \sum_{m_3=0}^{n_3-1} \mathbf{I}(Z'_{m_3}, Z_{m_3}) \tag{28}$$

and the parameters ξ *and points* $X_{n_1}, \ldots, Z'_{n_3}$ *enter the parameterizations of* $\kappa_{\alpha\mathbf{n}}$ *and the preexponentials as*

$$\begin{aligned} \kappa_{1:n_2,n_3} &= -\frac{\xi_{1:n_2,n_3}\xi_{1:n_2+1,n_3+1}}{\xi_{1:n_2+1,n_3}\xi_{1:n_2,n_3+1}} \frac{E(Y'_{n_2}, Z_{n_3})E(Y_{n_2}, Z'_{n_3})}{E(Y'_{n_2}, Z'_{n_3})E(Y_{n_2}, Z_{n_3})}, \\ \kappa_{2:n_1,n_3} &= -\frac{\xi_{2:n_1+1,n_3}\xi_{2:n_1,n_3+1}}{\xi_{2:n_1,n_3}\xi_{2:n_1+1,n_3+1}} \frac{E(X_{n_1}, Z_{n_3})E(X'_{n_1}, Z'_{n_3})}{E(X'_{n_1}, Z_{n_3})E(X_{n_1}, Z'_{n_3})}, \\ \kappa_{3:n_1,n_2} &= -\frac{\xi_{3:n_1,n_2}\xi_{3:n_1+1,n_2+1}}{\xi_{3:n_1+1,n_2}\xi_{3:n_1,n_2+1}} \frac{E(X'_{n_1}, Y_{n_2})E(X_{n_1}, Y'_{n_2})}{E(X_{n_1}, Y_{n_2})E(X'_{n_1}, Y'_{n_2})} \end{aligned} \tag{29}$$

and

$$\begin{aligned} \frac{u^{(0)}_{3:n_1,n_2}}{w^{(0)}_{1:n_2,n_3}} &= -\frac{\xi_{1:n_2,n_3}\xi_{3:n_1,n_2+1}}{\xi_{1:n_2+1,n_3}\xi_{3:n_1,n_2}} \frac{E(Y'_{n_2}, Z_{n_3})E(Y_{n_2}, X_{n_1})}{E(Y'_{n_2}, X_{n_1})E(Y_{n_2}, Z_{n_3})}, \\ \frac{u^{(0)}_{2:n_1,n_3}}{u^{(0)}_{1:n_2,n_3}} &= \frac{\xi_{1:n_2+1,n_3+1}\xi_{2:n_1,n_3}}{\xi_{1:n_2+1,n_3}\xi_{2:n_1,n_3+1}} \frac{E(Y'_{n_2}, Z_{n_3})E(X_{n_1}, Z'_{n_3})}{E(Y'_{n_2}, Z'_{n_3})E(X_{n_1}, Z_{n_3})}, \\ \frac{w^{(0)}_{3:n_1,n_2}}{w^{(0)}_{2:n_1,n_3}} &= \frac{\xi_{2:n_1+1,n_3}\xi_{3:n_1,n_2+1}}{\xi_{2:n_1,n_3}\xi_{3:n_1+1,n_2+1}} \frac{E(Y'_{n_2}, X'_{n_1})E(X_{n_1}, Z_{n_3})}{E(Y'_{n_2}, X_{n_1})E(X'_{n_1}, Z_{n_3})}. \end{aligned} \tag{30}$$

Proof. To prove that substitution (13) with the tau functions given by (27) solves (14), we repeatedly use Fay's identity. For example, Eq. (14) at $\alpha = 1$, $\beta = 2$, and $\gamma = 3$ follows from two Fay identities (26) taken for the respective sets of divisors (A, B, C, D) and (A', B', C', D'), where $A = A' = X_{n_1}$, $B = B' = Y_{n_2}$, $D = D' = Y'_{n_2}$, $C = Z'_{n_3}$, and $C' = Z_{n_3}$. The ratios containing the parameters $\xi_{\alpha:n_\beta,n_\gamma}$ in (29) and (30) appear because of the gauge invariance of linear systems (3) and (4). The gauge parameters $\xi_{\alpha:n_\beta,n_\gamma}$, the divisors $X_{n_1}, X'_{n_1}, \ldots$, and also the period matrix Ω and the point on the Jacobian $\mathbf{v}$ are free parameters of the general solution. Imposing periodic boundary conditions (which, in particular, means fixing the size of the system), we can relate these parameters to the values of the integrals of motion.

We can solve expressions (30) to avoid the ratios of preexponentials. For this, we introduce extra parameters A_{n_1}, B_{n_2}, and C_{n_3}. We then obtain

$$
\begin{aligned}
u_{1,\mathbf{n}} &= \frac{\xi_{1:n_2+1,n_3}}{\xi_{1:n_2+1,n_3+1}} \frac{E(Y'_{n_2}, Z'_{n_3})}{E(Y'_{n_2}, Z_{n_3})} \frac{E(C_{n_3}, Z_{n_3})}{E(C_{n_3}, Z'_{n_3})} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}, \\
w_{1,\mathbf{n}} &= -\frac{\xi_{1:n_2+1,n_3}}{\xi_{1:n_2,n_3}} \frac{E(Z_{n_3}, Y_{n_2})}{E(Z_{n_3}, Y'_{n_2})} \frac{E(B_{n_2}, Y'_{n_2})}{E(B_{n_2}, Y_{n_2})} \frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}, \\
\kappa_{1:n_2,n_3} &= -\frac{\xi_{1:n_2,n_3}\xi_{1:n_2+1,n_3+1}}{\xi_{1:n_2+1,n_3}\xi_{1:n_2,n_3+1}} \frac{E(Y'_{n_2}, Z_{n_3})E(Y_{n_2}, Z'_{n_3})}{E(Y'_{n_2}, Z'_{n_3})E(Y_{n_2}, Z_{n_3})},
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
u_{2,\mathbf{n}} &= \frac{\xi_{2:n_1,n_3}}{\xi_{2:n_1,n_3+1}} \frac{E(X_{n_1}, Z'_{n_3})}{E(X_{n_1}, Z_{n_3})} \frac{E(C_{n_3}, Z_{n_3})}{E(C_{n_3}, Z'_{n_3})} \frac{\tau_{1,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}}, \\
w_{2,\mathbf{n}} &= -\frac{\xi_{2:n_1,n_3}}{\xi_{2:n_1+1,n_3}} \frac{E(Z_{n_3}, X'_{n_1})}{E(Z_{n_3}, X_{n_1})} \frac{E(A_{n_1}, X_{n_1})}{E(A_{n_1}, X'_{n_1})} \frac{\tau_{3,\mathbf{n}}}{\tau_{3,\mathbf{n}+\mathbf{e}_1}}, \\
\kappa_{2:n_1,n_3} &= -\frac{\xi_{2:n_1+1,n_3}\xi_{2:n_1,n_3+1}}{\xi_{2:n_1,n_3}\xi_{2:n_1+1,n_3+1}} \frac{E(X_{n_1}, Z_{n_3})E(X'_{n_1}, Z'_{n_3})}{E(X'_{n_1}, Z_{n_3})E(X_{n_1}, Z'_{n_3})},
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
u_{3,\mathbf{n}} &= \frac{\xi_{3:n_1,n_2+1}}{\xi_{3:n_1,n_2}} \frac{E(X_{n_1}, Y_{n_2})}{E(X_{n_1}, Y'_{n_2})} \frac{E(B_{n_2}, Y'_{n_2})}{E(B_{n_2}, Y_{n_2})} \frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}}}, \\
w_{3,\mathbf{n}} &= -\frac{\xi_{3:n_1,n_2+1}}{\xi_{3:n_1+1,n_2+1}} \frac{E(Y'_{n_2}, X'_{n_1})}{E(Y'_{n_2}, X_{n_1})} \frac{E(A_{n_1}, X_{n_1})}{E(A_{n_1}, X'_{n_1})} \frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_1}}, \\
\kappa_{3:n_1,n_2} &= -\frac{\xi_{3:n_1,n_2}\xi_{3:n_1+1,n_2+1}}{\xi_{3:n_1+1,n_2}\xi_{3:n_1,n_2+1}} \frac{E(X'_{n_1}, Y_{n_2})E(X_{n_1}, Y'_{n_2})}{E(X_{n_1}, Y_{n_2})E(X'_{n_1}, Y'_{n_2})},
\end{aligned}
\tag{33}
$$

where $\tau_{\alpha\mathbf{n}}$ are given by (27). The discrete Baker–Akhiezer function $\Phi_{\mathbf{n}}$, satisfying the whole set of Eqs. (3) (and therefore (4)) and normalized by $\Phi_{\mathbf{0}} = \xi_{\mathbf{0}}$, is given by

$$
\Phi_{\mathbf{n}} = \Phi_{\mathbf{n}}(P) = \xi_{\mathbf{n}} \Phi^{(0)}_{\mathbf{n}}(P) \frac{\Theta\big(\mathbf{v} + \mathbf{I}(Q,P) + \mathbf{I}_{\mathbf{n}}\big)}{\Theta\big(\mathbf{v} + \mathbf{I}(Q,P)\big)},
\tag{34}
$$

where

$$
\begin{aligned}
\Phi^{(0)}_{\mathbf{n}}(P) = {}& \prod_{m_1=0}^{n_1-1} \frac{E(P, X_{m_1})E(A_{m_1}, X'_{m_1})}{E(P, X'_{m_1})E(A_{m_1}, X_{m_1})} \times \\
& \times \prod_{m_2=0}^{n_2-1} \frac{E(P, Y_{m_2})E(B_{m_2}, Y'_{m_2})}{E(P, Y'_{m_2})E(B_{m_2}, Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{E(P, Z_{m_3})E(C_{m_3}, Z'_{m_3})}{E(P, Z'_{m_3})E(C_{m_3}, Z_{m_3})}.
\end{aligned}
\tag{35}
$$

5. Finite lattice with open boundaries

To this point, all the considerations were strictly local and did not take the size and the boundary of the cubic lattice into account. To describe global characteristics, such as integrals of motion, we must nevertheless specify these additional data.

We consider a finite cubic lattice with open boundary conditions and the corresponding Cauchy problem. For a cubic lattice of the size $N_1 \times N_2 \times N_3$, we restrict the coordinates n_1, n_2, n_3 of the three-dimensional vector $\mathbf{n}$ in (1) by

$$
0 \le n_\alpha < N_\alpha, \quad \alpha = 1, 2, 3.
\tag{36}
$$

On the $\Delta = N_1N_2 + N_2N_3 + N_3N_1$ incoming edges of the cubic lattice, 2Δ initial data values can be given,

$$\begin{aligned}
&u_{1:n_2,n_3} \equiv u_{1,0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}, \qquad w_{1:n_2,n_3} \equiv w_{1,0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3},\\
&u_{2:n_1,n_3} \equiv u_{2,n_1\mathbf{e}_1+0\mathbf{e}_2+n_3\mathbf{e}_3}, \qquad w_{2:n_1,n_3} \equiv w_{2,n_1\mathbf{e}_1+0\mathbf{e}_2+n_3\mathbf{e}_3},\\
&u_{3:n_1,n_2} \equiv u_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+0\mathbf{e}_3}, \qquad w_{3:n_1,n_2} \equiv w_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+0\mathbf{e}_3}.
\end{aligned} \tag{37}$$

Recursively applying equations of motion (6)–(8), we define the transformation from initial data (37) to the 2Δ final data:

$$\begin{aligned}
&u'_{1:n_2,n_3} \equiv u_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}, \qquad w'_{1:n_2,n_3} \equiv w_{1,N_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3},\\
&u'_{2:n_1,n_3} \equiv u_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3}, \qquad w'_{2:n_1,n_3} \equiv w_{2,n_1\mathbf{e}_1+N_2\mathbf{e}_2+n_3\mathbf{e}_3},\\
&u'_{3:n_1,n_2} \equiv u_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3}, \qquad w'_{3:n_1,n_2} \equiv w_{3,n_1\mathbf{e}_1+n_2\mathbf{e}_2+N_3\mathbf{e}_3}.
\end{aligned} \tag{38}$$

Assuming that initial data (37) are general, we reach the natural question of how we can invert the parameterizations described by formulas (31)–(33) in order to restore the algebraic-geometric data in terms of (37) and the parameters $\kappa_{\alpha:n_\beta,n_\gamma}$ (the total number of parameters is 3Δ).

In general, the solution of this problem is not unique. Evidently, the same initial data can be parameterized by various sets of algebraic-geometric data related to algebraic curves with a sufficiently high genus. But there exists a unique preferred compact Riemann surface that is minimum (in a certain sense) among all possible curves parameterizing dynamics (6)–(8).

In the Cauchy problem with (37) and (38), this curve appears as follows. Let the initial data be parameterized by a curve Γ. We can write

$$u_{\alpha:n_\beta,n_\gamma} = u_{\alpha:n_\beta,n_\gamma}(\mathbf{v}), \qquad w_{\alpha:n_\beta,n_\gamma} = w_{\alpha:n_\beta,n_\gamma}(\mathbf{v}).$$

Because of (27) and (28), the solution of the Cauchy problem is then

$$u'_{\alpha:n_\beta,n_\gamma} = u_{\alpha:n_\beta,n_\gamma}(\mathbf{v}+\mathbf{T}_\alpha), \qquad w'_{\alpha:n_\beta,n_\gamma} = w_{\alpha:n_\beta,n_\gamma}(\mathbf{v}+\mathbf{T}_\alpha),$$

where

$$\mathbf{T}_1 = \sum_{n_1=0}^{N_1} \mathbf{I}(X'_{n_1}, X_{n_1}), \qquad \mathbf{T}_2 = \sum_{n_2=0}^{N_2} \mathbf{I}(Y'_{n_2}, Y_{n_2}), \qquad \mathbf{T}_3 = \sum_{n_3=0}^{N_3} \mathbf{I}(Z'_{n_3}, Z_{n_3}). \tag{39}$$

The transformation from the initial data to the final data is the *evolution* if

$$\mathbf{T}_1 = \mathbf{T}_2 = \mathbf{T}_3 = \mathbf{T} \mod (\mathbb{Z}^g + \Omega\mathbb{Z}^g). \tag{40}$$

In what follows, we understand all relations of this type modulo $\mathbb{Z}^g + \Omega\mathbb{Z}^g$.

Evolution conditions (40) mean that because of the Abel theorem, there exist two meromorphic functions $\lambda(P)$ and $\mu(P)$, $P \in \Gamma$, with the divisors

$$\begin{aligned}
(\lambda) &= \sum_{n_1\in\mathbb{Z}_{N_1}} X_{n_1} - \sum_{n_1\in\mathbb{Z}_{N_1}} X'_{n_1} - \sum_{n_3\in\mathbb{Z}_{N_3}} Z_{n_3} + \sum_{n_3\in\mathbb{Z}_{N_3}} Z'_{n_3},\\
(\mu) &= \sum_{n_2\in\mathbb{Z}_{N_2}} Y_{n_2} - \sum_{n_2\in\mathbb{Z}_{N_2}} Y'_{n_2} - \sum_{n_3\in\mathbb{Z}_{N_3}} Z_{n_3} + \sum_{n_3\in\mathbb{Z}_{N_3}} Z'_{n_3}.
\end{aligned} \tag{41}$$

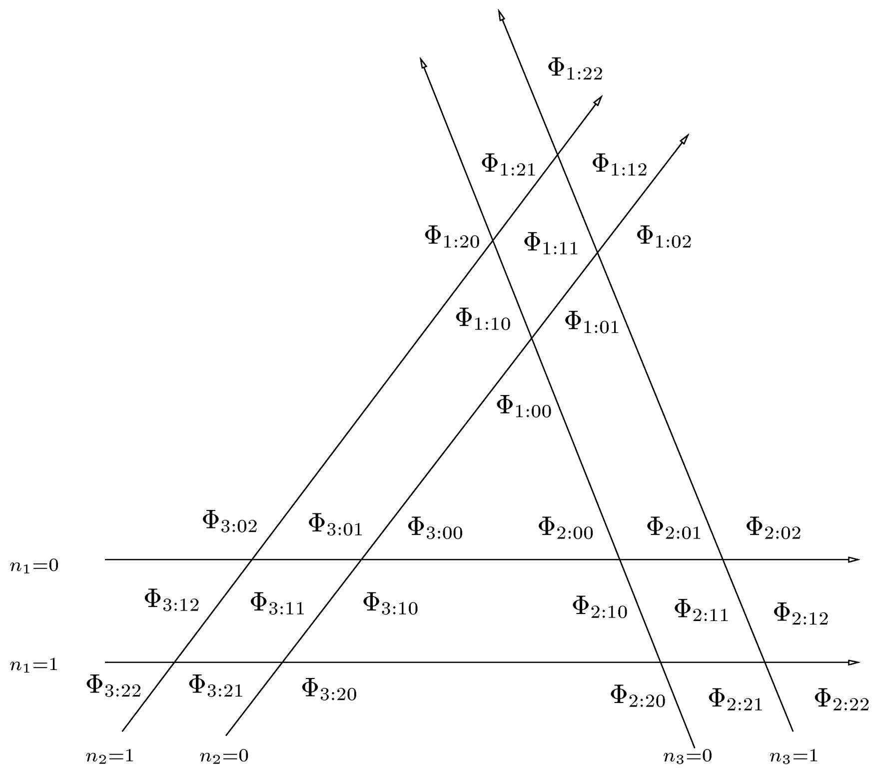

Fig. 5

The curve Γ is then the compact Riemann surface determined by the polynomial equation $J_\triangle(\lambda,\mu)=0$ (see, e.g., Theorem 10-23 in [6]). Moreover, conditions (41) fix the general structure of $J_\triangle(\lambda,\mu)$ (see (48) below). The key observation is that evolution conditions (40) imply that $J_\triangle(\lambda,\mu)$, considered as a functional of the initial data, generates the set of evolution invariants and the curve defined by this polynomial is therefore the spectral curve.

We can derive the polynomial $J_\triangle$ using a linear system of type (3) written for the whole auxiliary plane. We fix the position of the auxiliary planes corresponding to the initial and final data. The auxiliary plane for the initial data crosses all the incoming edges of the cubic lattice, while the plane for the final data intersects all the outgoing edges. These auxiliary planes play the role of the two-dimensional spacelike surfaces, and the discrete evolution process corresponds to translating the auxiliary plane in the direction perpendicular to this spacelike surface.

All the objects, namely, the dynamical and auxiliary linear variables on the spacelike surface, can be labeled by a two-dimensional discrete index. We choose the labeling for the linear variables in a form similar to the labeling of initial data (37) or final data (38):

$$
\begin{aligned}
&\Phi_{0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}=\Phi_{1:n_2,n_3},\\
&\Phi_{n_1\mathbf{e}_1+0\mathbf{e}_2+n_3\mathbf{e}_3}=\Phi_{2:n_1,n_3},\\
&\Phi_{n_1\mathbf{e}_1+n_2\mathbf{e}_2+0\mathbf{e}_3}=\Phi_{3:n_1,n_2},\\
&0\le n_i\le N_i,\quad i=1,2,3.
\end{aligned}
\tag{42}
$$

An example of such an enumeration on the auxiliary plane in the simplest case $N_1=N_2=N_3=2$ is shown in Fig. 5.

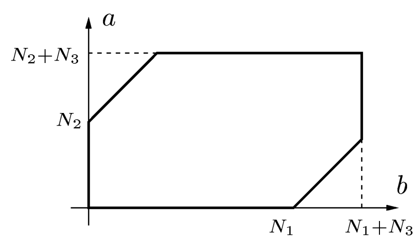

Fig. 6

Let $\mathcal{L}(\lambda,\mu)$ be the matrix of the linear system

$$\begin{aligned}
0 &= \Phi_{1:n_2+1,n_3} - \Phi_{1:n_2+1,n_3+1}u_{1:n_2,n_3} + \Phi_{1:n_2,n_3}w_{1:n_2,n_3} + \\
&\quad + \Phi_{n_2,n_3+1}\kappa_{1:n_2,n_3}u_{1:n_2,n_3}w_{1:n_2,n_3}, \\
0 &= \Phi_{2:n_1,n_3} - \Phi_{2:n_1,n_3+1}u_{2:n_1,n_3} + \Phi_{2:n_1+1,n_3}w_{2:n_1,n_3} + \\
&\quad + \Phi_{2:n_1+1,n_3+1}\kappa_{2:n_1,n_3}u_{2:n_1,n_3}w_{2:n_1,n_3}, \\
0 &= \Phi_{3:n_1,n_2+1} - \Phi_{3:n_1,n_2}u_{3:n_1,n_2} + \Phi_{3:n_1+1,n_2+1}w_{3:n_1,n_2} + \\
&\quad + \Phi_{3:n_1+1,n_2}\kappa_{3:n_1,n_2}u_{3:n_1,n_2}w_{3:n_1,n_2}
\end{aligned} \tag{43}$$

written in the matrix form $0 = \Phi\cdot\mathcal{L}(\lambda,\mu)$, where the linear variables satisfy the identification conditions

$$\Phi_{1:0,n_3} = \Phi_{2:0,n_3}, \qquad \Phi_{1:n_2,0} = \Phi_{3:0,n_2}, \qquad \Phi_{2:n_1,0} = \Phi_{3:n_1,0} \tag{44}$$

and the quasiperiodicity conditions

$$\frac{\Phi_{3:N_1,n_2}}{x} = \frac{\Phi_{1:n_2,N_3}}{z}, \qquad \frac{\Phi_{3:n_1,N_2}}{y} = \frac{\Phi_{2:n_1,N_3}}{z}, \qquad \frac{\Phi_{1:N_2,n_3}}{y} = \frac{\Phi_{2:N_1,n_3}}{x} \tag{45}$$

for the boundary domains on the auxiliary plane. The parameters

$$\lambda = \frac{x}{z}, \qquad \mu = \frac{y}{z} \tag{46}$$

are complex numbers, and we call them the spectral parameters. By virtue of (44) and (45), it is clear that the total number of the independent linear variables in system (43) is Δ and $\mathcal{L}(\lambda,\mu)$ is a $\Delta\times\Delta$ square matrix.

We define

$$J_\triangle(\lambda,\mu) = \det\mathcal{L}(\lambda,\mu)\Bigl(\prod_{n_2,n_3} u_{1:n_2,n_3}\Bigr)^{-1}. \tag{47}$$

In other words, $J_\triangle(\lambda,\mu)$ is a normalized Laurent polynomial ($J_{0,0}=1$) of the spectral parameters λ and μ,

$$\begin{aligned}
J_\triangle &= \sum_{a,b\in\Pi} \lambda^a\mu^{-b}J_{a,b}, \\
\Pi &= \{0\le a\le N_2+N_3,\ 0\le b\le N_1+N_3,\ -N_1\le a-b\le N_2\}.
\end{aligned} \tag{48}$$

The domain Π (the Newton polygon of $J_\triangle$) is shown in Fig. 6. It can be proved (see [1]) that the coefficients of this Laurent polynomial are invariants of the evolution, i.e., they are the same for initial data (37) and final data (38) and can generate the algebraic-geometric data for the solution of these equations with open boundary conditions.

The requirement that linear system (43) have a nontrivial solution is equivalent to the requirement that the spectral parameters lie on the algebraic curve,

$$P = (\lambda, \mu) \in \Gamma_\triangle \quad \Leftrightarrow \quad J_\triangle(\lambda, \mu) = 0. \tag{49}$$

Assuming that all the incoming data together with the parameters $\kappa_{\alpha:n_\beta,n_\gamma}$ are in general position, we can calculate the genus of curve (49) using the Newton polygon. For a cubic lattice of size $N_1 \times N_2 \times N_3$ such that $N_1 \geq N_2 \geq N_3$, the Newton polygon associated with corresponding curve (49) is shown in Fig. 6. The genus of the curve $\Gamma_\triangle$ is equal to the number of points with integer coordinates inside this polygon,

$$g_\triangle = (N_1N_2 + N_1N_3 + N_2N_3) - (N_1 + N_2 + N_3) + 1. \tag{50}$$

Then, $g_\triangle$ coefficients $J_{a,b}$ in the Laurent polynomial $J_\triangle(\lambda, \mu)$ corresponding to the internal points of the Newton polygon with integer coordinates are related to the moduli of the algebraic curve $\Gamma_\triangle$ and consequently to the period matrix $\Omega_\triangle$. The coefficients $J_{a,b}$ corresponding to the perimeter of the polygon are related to the divisors of the functions λ and μ meromorphic on $\Gamma_\triangle$, thus giving the set of parameters $X_{n_1}, \ldots, Z'_{n_3}$. From (41) and (34), we explicitly obtain

$$\begin{aligned}
\lambda(P) &= \prod_{n_1=0}^{N_1-1} \frac{E(P, X_{n_1})E(A_{n_1}, X'_{n_1})}{E(P, X'_{n_1})E(A_{n_1}, X_{n_1})} \prod_{n_2=0}^{N_2-1} \frac{E(P, Y'_{n_2})E(B_{n_2}, Y_{n_2})}{E(P, Y_{n_2})E(B_{n_2}, Y'_{n_2})}, \\
\mu(P) &= \prod_{n_2=0}^{N_2-1} \frac{E(P, Y_{n_2})E(B_{n_2}, Y'_{n_2})}{E(P, Y'_{n_2})E(B_{n_2}, Y_{n_2})} \prod_{n_3=0}^{N_3-1} \frac{E(P, Z'_{n_3})E(C_{n_3}, Z_{n_3})}{E(P, Z_{n_3})E(C_{n_3}, Z'_{n_3})}.
\end{aligned} \tag{51}$$

A different approach was previously used in [7]. Because of Theorem 2 in [7], the solution $\Phi(\lambda, \mu)$ of the linear problem $\Phi \cdot \mathcal{L}(\lambda, \mu) = 0$, as a vector of meromorphic functions on the curve $\Gamma_\triangle$, was given by expression (34) on $\mathrm{Jac}(\Gamma_\triangle)$. Using (34), we can unambiguously restore parameterization (27)–(30).

We conclude the discussion of the Cauchy problem by calculating the degrees of freedom of the algebraic-geometric parameterizations. Having $\Delta = N_1N_2 + N_2M_3 + N_3N_1$, $\Delta' = N_1 + N_2 + N_3$, and $g = \Delta - \Delta' + 1$, we have g moduli of $\Gamma_\triangle$, g complex numbers $\mathbf{v} \in \mathrm{Jac}(\Gamma_\Delta)$, g independent cross ratios of $\xi_\mathbf{n}$, $2\Delta' - 2$ independent divisors $X_{n_1}, \ldots, Z'_{n_3}$, and Δ' arbitrary divisors $A_{n_1}, B_{n_2}, C_{n_3}$; in this, one extraneous degree of freedom corresponding to the arbitrariness of Q is taken into account. We thus obtain 3Δ parameters in total. This proves the completeness of the algebraic-geometric parameterization in terms of the curve $\Gamma_\triangle$.

6. The Bäcklund transformation

The curve $\Gamma_\triangle$ corresponds to initial data in the general position and open boundary conditions. But to apply our results to integrable spin models, it is necessary to specify periodic boundary conditions $\mathbf{T} = 0$, i.e., to set

$$\sum_{n_1 \in \mathbb{Z}_{N_1}} \mathbf{I}(X'_{n_1}, X_{n_1}) = \sum_{n_2 \in \mathbb{Z}_{N_2}} \mathbf{I}(Y'_{n_2}, Y_{n_2}) = \sum_{n_3 \in \mathbb{Z}_{N_3}} \mathbf{I}(Z'_{n_3}, Z_{n_3}) = 0. \tag{52}$$

These boundary conditions reduce the spectral curve $\Gamma_\triangle$. Namely, because of (52), the parameters x, y, and z in (45) become meromorphic functions with the divisors

$$(x) = \sum_{n_1} X_{n_1} - \sum_{n_1} X'_{n_1}$$

and (58) (see below). Because x, y, and z themselves (not only their ratios) are meromorphic functions, the curve can be defined by an algebraic equation for any pair of x, y, or z, i.e.,

$$J_1(y,z) = J_2(x,z) = J_3(x,y) = 0. \tag{53}$$

The Laurent polynomial $J_\triangle(x/z, y/z)$ is then not irreducible, and the condition $J_\triangle = 0$ follows from algebraic relations (53).

In this section, we describe the reduced spectral curve and the meromorphic functions uniformizing it for another choice of the auxiliary plane. The interpretation of the problem now differs from the Cauchy problem for the open cubic lattice. The discrete evolution can now be considered a sequence of Bäcklund transformations in the square auxiliary plane.

We first define the new location of the auxiliary plane. Let it cross the incoming edges for the vertices of the three-dimensional lattice with the coordinates $n_2\mathbf{e}_2 + n_3\mathbf{e}_3$. With this choice, the role of the variables u_α and w_α, $\alpha = 1,2,3$, becomes different: $u_{2\mathbf{n}}$, $w_{2\mathbf{n}}$ and $u_{3\mathbf{n}}$, $w_{3\mathbf{n}}$ are auxiliary, while $u_{1\mathbf{n}}$, $w_{1\mathbf{n}}$ are dynamical. The evolution then means a variation of these dynamical variables with the "time" n_1 with periodic boundary conditions for the second and third directions:

$$u_{1:n_2+N_2,n_3} = u_{1:n_2,n_3+N_3} = u_{1:n_2,n_3}, \qquad w_{1:n_2+N_2,n_3} = w_{1:n_2,n_3+N_3} = w_{1:n_2,n_3}. \tag{54}$$

The auxiliary variables for this evolution also vary from layer to layer according to discrete equations (7) and (8). We preserve the open boundary conditions in the first direction.

We now consider one layer of the cubic lattice corresponding to the initial value $n_1 = 0$. Equations of motion (6) and periodic boundary conditions (54) yield the implicit transformation

$$u_{1:n_2,n_3}, w_{1:n_2,n_3} \mapsto u'_{1:n_2,n_3}, w'_{1:n_2,n_3}. \tag{55}$$

According to (27)–(30), we can define

$$u_{1:n_2,n_3} = u_{1:n_2,n_3}(\mathbf{v}), \qquad w_{1:n_2,n_3} = w_{1:n_2,n_3}(\mathbf{v}), \tag{56}$$

where $\mathbf{v} \in \mathrm{Jac}(\Gamma)$ for some algebraic curve Γ. Then transformation (55) can be written in the form

$$u'_{1:n_2,n_3} = u_{1:n_2,n_3}\big(\mathbf{v} + \mathbf{I}(X'_0, X_0)\big), \qquad w'_{1:n_2,n_3} = w_{1:n_2,n_3}\big(\mathbf{v} + \mathbf{I}(X'_0, X_0)\big). \tag{57}$$

Periodic conditions (54) in parameterization (27)–(30) yield $\sum_{n_2\in\mathbb{Z}_{N_2}} \mathbf{I}(Y'_{n_2} Y_{n_2}) = \sum_{n_3\in\mathbb{Z}_{N_3}} \mathbf{I}(Z'_{n_3}, Z_{n_3}) = 0$ (see (52)). Using the same arguments that led to formula (41), we can conclude that there exist two meromorphic functions y and z on Γ such that

$$(y) = \sum_{n_2} Y_{n_2} - Y'_{n_2}, \qquad (z) = \sum_{n_3} Z_{n_3} - Z'_{n_3}, \tag{58}$$

and a compact Riemann surface Γ is defined by the polynomial equation $J_\square(y,z) = 0$. Moreover, the structure of (y) and (z) fixes the algebraic form of Γ (see (61) below).

We call transformation (55) the *Bäcklund transformation* for the two-dimensional square lattice because it is a canonical transformation preserving the integrals of motion. To specify the integrals of motion, we again consider the linear system for the chosen auxiliary plane. This linear system is a system of N_2N_3 equations

$$\begin{aligned} &j_{n_2,n_3} = 0, && \Phi_{-1,n_3} = y^{-1}\Phi_{N_2-1,n_3}, \\ &\Phi_{n_2,N_3} = z\Phi_{n_2,0}, && \Phi_{-1,N_3} = y^{-1}z\Phi_{N_2-1,0}, \end{aligned} \tag{59}$$

where $0 \le n_2 < N_2$, $0 \le n_3 < N_3$, and the linear form j_{n_2,n_3} is defined by the first equation in system (3) with relabeled auxiliary linear variables Φ_{n_2,n_3}:

$$\begin{aligned} j_{n_2,n_3} = \Phi_{n_2,n_3} - \Phi_{n_2,n_3+1}u_{1:n_2,n_3} + \Phi_{n_2-1,n_3}w_{1:n_2,n_3} + \\ + \Phi_{n_2-1,n_3+1}\kappa_{1:n_2,n_3}u_{1:n_2,n_3}w_{1:n_2,n_3}. \end{aligned} \tag{60}$$

The equations in (59) including the spectral parameters y and z are quasiperiodic boundary conditions for the linear variables Φ_{n_2,n_3}.

Let $L(y,z)$ be the complete matrix of the coefficients of system (59), $j = \Phi \cdot L(y,z)$. We define the Laurent polynomial

$$J_\Box(y,z) = \det L(y,z) = \sum_{a=0}^{N_3}\sum_{b=0}^{N_2} J_{a,b}y^{-a}z^b, \qquad J_{0,0} = 1. \tag{61}$$

The parameters $J_{a,b}$ are then invariants of Bäcklund transformation (55) (see [1]). For the system $\Phi(P) \cdot L(y,z) = 0$ to have a nonzero solution, the spectral parameters y and z must belong to the spectral curve

$$P = (y,z) \in \Gamma_\Box \quad \Leftrightarrow \quad J_\Box(y,z) = 0. \tag{62}$$

Assuming that the dynamical variables $u_{1:n_2,n_3}$, $w_{1:n_2,n_3}$ and the parameters $\kappa_{1:n_2,n_3}$ are general, we find that the genus of spectral curve (62) is $g_\Box = (N_2-1)(N_3-1)$. We recall that form (61) of the polynomial $J_\Box$ follows uniquely from conditions (58).

Conversely, Theorem 2 in [7] claims that the solution of $\Phi(P) \cdot L(y,z) = 0$ as a meromorphic vector-valued function of $\Gamma_\Box$ is given by the formula

$$\begin{aligned} \Phi_{n_2,n_3}(P) = \prod_{m_2=0}^{n_2-1} \frac{E(P,Y_{m_2})}{E(P,Y'_{m_2})}\frac{E(B_{m_2},Y'_{m_2})}{E(B_{m_2},Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{E(P,Z_{m_3})}{E(P,Z'_{m_3})}\frac{E(C_{m_3},Z'_{m_3})}{E(C_{m_3},Z_{m_3})} \times \\ \times \Theta\left(\mathbf{v} + \mathbf{I}(Q,P) + \sum_{m_2=0}^{n_2-1}\mathbf{I}(Y'_{m_2},Y_{m_2}) + \sum_{m_3=0}^{n_3-1}\mathbf{I}(Z'_{m_3},Z_{m_3})\right), \end{aligned} \tag{63}$$

where the theta functions are constructed using the period matrix $\Omega_\Box$ of the algebraic curve $\Gamma_\Box$. Using formula (63), we can find the corresponding expressions for the dynamical variables $u_{1:n_2,n_3}$ and $w_{1:n_2,n_3}$. They coincide with those in the proof of Proposition 1 under the condition that an arbitrary curve Γ is identified with the spectral curve $\Gamma_\Box$. Explicit formulas for the spectral parameters uniformizing the spectral curve $\Gamma_\Box$ are

$$\begin{aligned} z(P) &= \prod_{n_3 \in \mathbb{Z}_{N_3}} \frac{E(P,Z_{n_3})}{E(P,Z'_{n_3})}\frac{E(C_{n_3},Z'_{n_3})}{E(C_{n_3},Z_{n_3})}, \\ y(P) &= \prod_{n_2 \in \mathbb{Z}_{N_2}} \frac{E(P,Y_{n_2})}{E(P,Y'_{n_2})}\frac{E(B_{n_2},Y'_{n_2})}{E(B_{n_2},Y_{n_2})}. \end{aligned} \tag{64}$$

In the case where the algebraic curve used to construct the general solution is rational, one evolution step, $n_1 \mapsto n_1 + 1$, can be identified with creating a soliton. We study this phenomenon in more detail in the next section.

In this section, we have thus established why the curve $\Gamma_\square$ appears when periodic boundary conditions are imposed on the second and third directions. Evidently, the curve $J_\square(y, z)$ must be identified with the curve $J_1(y, z)$ in (53). In the same way, the periodic boundary conditions can be imposed on any other pair of directions, and the corresponding curves J_2 and J_3 are also the spectral determinants. If the periodic boundary conditions are imposed on all three directions, then each of the three relations $J_1 = 0$, $J_2 = 0$, and $J_3 = 0$ must define the same curve, and because the generality of the data is lost in this case, the genus of the curve is restricted to

$$g \le \min\big\{(N_1 - 1)(N_2 - 1),\ (N_2 - 1)(N_3 - 1),\ (N_3 - 1)(N_1 - 1)\big\}. \tag{65}$$

7. The rational limit

In this section, we consider a rational limit of the algebraic-geometric solutions of discrete equation of motion (14). The rational limit for theta function (24) on the Jacobian of an algebraic curve corresponds to the limit (see [8])

$$\mathbf{e}^{i\pi\Omega_{n,n}+2i\pi v_n} = -f_n \tag{66}$$

and

$$\mathbf{e}^{i\pi\Omega_{n,n}} \mapsto 0, \qquad \mathbf{e}^{i\pi\Omega_{k,n}} \mapsto \frac{(q_k - q_n)(p_k - p_n)}{(q_k - p_n)(p_k - q_n)} \stackrel{\text{def}}{=} d_{k,n}. \tag{67}$$

Thus, the set of $d_{k,n}$ arises from the period matrix, as the set of f_n arises from the points on the Jacobian $\mathbf{v}$. The prime forms in the rational limit are the prime forms on the sphere:

$$\frac{E(A, B)E(C, D)}{E(A, D)E(C, B)} = \frac{(A - B)(C - D)}{(A - D)(C - B)}. \tag{68}$$

For $g = 1, 2, \ldots$ and the set of parameters

$$p_0, q_0, f_0; p_2, q_2, f_2; \ldots ; p_{g-1}, q_{g-1}, f_{g-1} = \{p_k, q_k, f_k\}_{k=0}^{g-1}, \tag{69}$$

we define the rational limit of the theta function (see the appendix in [8])

$$H^{(g)}\big(\{p_k, q_k, f_k\}_{k=0}^{g-1}\big) = \frac{\det |q_j^i - f_j p_j^i|_{i,j=0}^{g-1}}{\prod_{i>j}(q_i - q_j)}. \tag{70}$$

We set $H^{(0)} \equiv 1$ by definition. We note that if all the parameters f_k vanish, then $H^{(g)}\big(\{p_k, q_k, 0\}_{k=0}^{g-1}\big) = 1$. Let the function $\sigma_k(z)$ be

$$\sigma_k(z) = \frac{p_k - z}{q_k - z}. \tag{71}$$

For $g = 0$, Eqs. (14) admit the simple solution $\tau_{\alpha\mathbf{n}} = 1$ in the rational limit because of the identity

$$r_\alpha = 1 + s_\beta + s_\gamma^{-1}, \tag{72}$$

where

$$r_\alpha = -\frac{X'_\beta - X'_\gamma}{X_\beta - X_\gamma}\frac{X_\alpha - X_\beta}{X_\alpha - X_\gamma}\frac{X_\alpha - X'_\beta}{X_\alpha - X'_\gamma}, \qquad s_\alpha = -\frac{X_\alpha - X_\gamma}{X_\alpha - X_\beta}\frac{X'_\alpha - X_\beta}{X'_\alpha - X_\gamma}, \tag{73}$$

and (α, β, γ) is any even permutation of the set $(1, 2, 3)$. The notation in (72) and (73) is related to that in (15) and in Proposition 1 as follows: $r_\alpha = r_{\alpha,\mathbf{n}}$, $s_\alpha = s_{\alpha,\mathbf{n}}$, $X_1 = X_{n_1}$, $X'_1 = X'_{n_1}$, $X_2 = Y_{n_2}$, $X'_2 = Y'_{n_2}$, $X_3 = Z_{n_3}$, and $X'_3 = Z'_{n_3}$.

Proposition 2. *The soliton solutions of Eq.* (14) *are*

$$\begin{aligned}\tau_{1,\mathbf{n}} &= H^{(g)}\left(\left\{\frac{I_{\mathbf{n}:k}}{\sigma_k(X_{n_1})}\right\}_{k=0}^{g-1}\right),\\ \tau_{2,\mathbf{n}} &= H^{(g)}\left(\left\{\frac{I_{\mathbf{n}:k}}{\sigma_k(Y_{n_2})}\right\}_{k=0}^{g-1}\right),\\ \tau_{3,\mathbf{n}} &= H^{(g)}\left(\left\{\frac{I_{\mathbf{n}:k}}{\sigma_k(Z_{n_3})}\right\}_{k=0}^{g-1}\right),\end{aligned} \tag{74}$$

where $n_1, n_2, n_3 \in \mathbb{Z}$,

$$I_{\mathbf{n}:k} = f_k\left(\prod_{m_1=0}^{n_1-1}\frac{\sigma_k(X'_{m_1})}{\sigma_k(X_{m_1})}\right)\left(\prod_{m_2=0}^{n_2-1}\frac{\sigma_k(Y'_{m_2})}{\sigma_k(Y_{m_2})}\right)\left(\prod_{m_3=0}^{n_3-1}\frac{\sigma_k(Z'_{m_3})}{\sigma_k(Z_{m_3})}\right), \tag{75}$$

and the parameters $r_{\alpha,\mathbf{n}}$ *and* $s_{\alpha,\mathbf{n}}$ *are defined by formulas* (73).

Proof. The proof of this proposition is based on the rational analogue of Fay's identity

$$\begin{aligned}&(A-D)(C-B)H^{(g)}\left(\left\{f_k\frac{\sigma_k(A)}{\sigma_k(B)}\right\}_{k=1}^{g}\right)H^{(g)}\left(\left\{f_k\frac{\sigma_k(C)}{\sigma_k(D)}\right\}_{k=1}^{g}\right)+\\ &\quad+(A-B)(D-C)H^{(g)}\left(\left\{f_k\frac{\sigma_k(A)}{\sigma_k(D)}\right\}_{k=1}^{g}\right)H^{(g)}\left(\left\{f_k\frac{\sigma_k(C)}{\sigma_k(B)}\right\}_{k=1}^{g}\right)=\\ &\qquad=(A-C)(D-B)H^{(g)}\left(\{f_k\}_{k=1}^{g}\right)H^{(g)}\left(\left\{f_k\frac{\sigma_k(A)\sigma_k(C)}{\sigma_k(B)\sigma_k(D)}\right\}_{k=1}^{g}\right)\end{aligned} \tag{76}$$

described in [8].

So far, parameters (69), p_k, q_k, and f_k, have been considered arbitrary complex parameters, and the solution given by Proposition 2 has been relevant to a lattice infinite in all directions. We now impose periodic boundary conditions (54) in directions 2 and 3 and interpret the evolution in direction 1 as a Bäcklund transformation, which creates solitons. For this, we first note that boundary conditions (54) are equivalent to the algebraic relations for the parameters p and q

$$\prod_{n_2=0}^{N_2-1}\frac{\sigma(Y'_{n_2})}{\sigma(Y_{n_2})} = \prod_{n_3=0}^{N_3-1}\frac{\sigma(Z'_{n_3})}{\sigma(Z_{n_3})} = 1. \tag{77}$$

It can be verified that for the parameters $Y_{n_2}, \ldots, Z'_{n_3}$ in the general position, system of equations (77) has exactly $g = (N_2-1)(N_3-1)$ nonequivalent solutions (equivalence means that if (p,q) is a solution of (77), then (q,p) is also a solution). We choose this set of solutions as sequence (69), leaving the parameters f_k, $k = 1, \ldots, g$, free.

Using this freedom, we can redefine the "amplitudes" f_k as

$$f_k = F_k \cdot \sigma_k(X_k), \tag{78}$$

where, as before, the parameters p_k and q_k of functions (71) are already fixed by system of equations (77) while the parameters X_k are still free. We consider the solutions for the tau functions $\tau_{2\mathbf{n}}$ and $\tau_{3\mathbf{n}}$ given by

Proposition 2 with redefined amplitudes (78) at the initial value of the discrete coordinate $n_1 = 0$. Using the freedom in the parameters X_k, we send

$$X_k \mapsto p_k. \tag{79}$$

In this limit, it is clear that all components of the set $\{I_{\mathbf{n}:k}\}_{k=0}^{g-1}$ vanish and we have $\tau_{2\mathbf{n}} = \tau_{3\mathbf{n}} = 1$ according to definition (70). In other words, we obtain a zero-soliton solution for these tau functions at $n_1 = 0$. We repeat the same procedure at $n_1 = 1$. Namely, we again consider a general solution for these tau functions given by Proposition 2 and then take limit (79). It is clear that only one element that corresponds to $k = 0$ does not vanish in the set $\{I_{\mathbf{n}:k}\}_{k=0}^{g-1}$, and we thus obtain a one-soliton solution. Increasing n_1 results in increasing the number of solitons. It can be seen that the maximum number of solitons that can be created by this procedure is equal to $g = (N_2 - 1)(N_3 - 1)$ (see [8] for a detailed description of this phenomenon in the simplest situation).

Before concluding this section, we give one more explanation of this soliton creation procedure. We consider Eq. (14) at $n_1 = 0$ for $\alpha, \beta, \gamma = 1, 2, 3$ and for the homogeneous, or zero-soliton, tau functions $\tau^{(0)}_{2:n_2,n_3} = \tau^{(0)}_{3:n_2,n_3} = 1$. This is a linear difference equation for the function $\tau_{1:n_2,n_3}$ with respect to the "space" coordinates n_2 and n_3. Using simple algebra, we can verify that in addition to the trivial solution $\tau_{1:n_2,n_3} = 1$, this equation admits a solution of the form

$$\tau_{1:n_2,n_3} = \prod_{m_2=0}^{n_2-1} \frac{\sigma_0(Y'_{m_2})}{\sigma_0(Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{\sigma_0(Z'_{m_3})}{\sigma_0(Z_{m_3})},$$

where the parameter p_0 of the function σ_0 is identified with X_0. The complete solution of the linear difference equation is the linear combination

$$\tau^{(1)}_{1:n_2,n_3} = 1 - F_0 \prod_{m_2=0}^{n_2-1} \frac{\sigma_0(Y'_{m_2})}{\sigma_0(Y_{m_2})} \prod_{m_3=0}^{n_3-1} \frac{\sigma_0(Z'_{m_3})}{\sigma_0(Z_{m_3})} \tag{80}$$

with an arbitrary F_0. Solving Eq. (14) for $\alpha, \beta, \gamma = 2, 3, 1$ and for $\alpha, \beta, \gamma = 3, 1, 2$ with the already found $\tau^{(1)}_{1:n_2,n_3}$, we can now find the values of the tau functions $\tau_{2:n_2,n_3}$ and $\tau_{3:n_2,n_3}$ at the discrete time $n_1 = 1$. They acquire a one-soliton form similar to (80). Using these solutions again in Eq. (14) with $\alpha, \beta, \gamma = 1, 2, 3$, we then find a two-soliton solution for the function $\tau_{1:n_2,n_3}$ and then two-soliton solutions for the functions $\tau_{2:n_2,n_3}$ and $\tau_{3:n_2,n_3}$ at the next value of the discrete time $n_2 = 2$. It is clear that this procedure can be continued, and we can demonstrate the equivalence of the discrete time n_1 to the number of solitons. This simple explanation finally justifies that the discrete dynamics investigated in this paper, which is given by equations of motion (6)–(8), is a set of consecutive Bäcklund transformations.

8. Discussion

It is well known [1] that the dynamics of the parameters of three-dimensional spin models is equivalent to dynamics (6)–(8). Therefore, the classical solutions discussed in this paper can be used to describe spin models of different types. The solutions with periodic boundary conditions can be used to construct general spin models such that their Boltzmann weights are parameterized by theta functions on higher-genus curves. The integrability of such spin models is based on the modified tetrahedron equation [9]. Among these models, there are various generalizations of the chiral Potts model.

On the other hand, the solitonic solutions are also convenient for the completely inhomogeneous Zamolodchikov–Bazhanov–Baxter model and open a way for developing the quantum separation of variables for that model [10]. In particular, these solutions with parameters $f_k \neq 0$ allow constructing the complete family of isospectral deformations of the Zamolodchikov–Bazhanov–Baxter transfer matrix (also see [8] for the realization of a similar program in the case of the relativistic Toda chain model with spin degrees of freedom).

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