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Principle of radial transport in low temperature annular plasmas

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Radial transport in low temperature annular plasmas is investigated theoretically in this paper. The electrons are assumed to be in quasi-equilibrium due to their high temperature and light inertial mass. The ions are not in equilibrium and their transport is analyzed in three different situations: a low electric field (LEF) model, an intermediate electric field (IEF) model, and a high electric field (HEF) model. The universal IEF model smoothly connects the LEF and HEF models at their respective electric field strength limits and gives more accurate results of the ion mobility coefficient and effective ion temperature over the entire electric field strength range. Annular modelling is applied to an argon plasma and numerical results of the density peak position, the annular boundary loss coefficient and the electron temperature are given as functions of the annular geometry ratio and Paschen number. © 2015 AIP Publishing LLC.

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I. INTRODUCTION

A cylindrical plasma is made annular when an inner object is inserted and this implementation is widely seen in low temperature plasma applications ranging from the probe within a plasma column1,2 to the inner quartz tube in a plasma source,3–5 or the central electrode in a plasma jet.6,7 Two dimensional fluid simulations (no azimuthal flow) for a cylindrical plasma8,9 have shown that, by using the variable separation technique, the partial differential equation (PDE) for the momentum conservation can be separated into the forms of two ordinary differential equations (ODEs) in the axial and radial directions. The two ODEs are weakly coupled through the drag term caused by the ionization frequency from the particle conservation. When either (axial or radial) dimension is the main interest for a specific plasma system, the other dimension may be assumed to be “frozen,” i.e., a quiescent or constant plasma flux such that no plasma loss occurs in this dimension. Then, the primarily concerned dimension can be separately solved at the cost of some loss in accuracy which depends on the specific flow in the assumed frozen dimension. This methodology has been widely applied in a number of modelling studies10–12 and its validity has also been verified by experiments.13–15 Axial transport in an annular plasma is similar to that in a cylindrical plasma. The implementation of an inner cylinder in a cylindrical plasma does not change the axial boundary condition and has a small effect on the axial flow except for the value of the coupled drag term. However, radial transport in an annular plasma is quite different from that in a cylindrical plasma. The annular geometry greatly affects the radial boundary conditions: a cylindrical plasma has a central point of maximum ion density and zero electric field, but the central point disappears in an annular plasma and is replaced by an inner wall boundary. In this case, the density peak position becomes a variable in the annulus and the radial transport changes from one direction (outward) in a cylindrical plasma to two directions (outward and inward) in an annular plasma. Hence, the major transport properties of annular plasmas are characterized in the radial dimension and the radial dimension could be reasonably assumed to be frozen as discussed above. The radial transport of collisionless annular plasmas has been previously investigated by using a free-fall model1 and a fluid model.16 Our work focuses on the radial transport of annular collisional plasmas with the interest of single-component, electron-positive-ion, and low temperature discharge. The transport of charged particles is described in terms of “diffusion” and “mobility” due to the advantages of unifying the unmagnetized and magnetized plasmas into a simple algebraic form.11,17 The diffusion represents the momentum balance between the active density gradient and passive collisions, and the mobility represents the balance between the electric field and collisions.18 By using this representation method, the modelling results for unmagnetized annular plasmas, which is the case of the present study, form the basis of the future study for magnetized plasmas.

The radial transport of an unmagnetized plasma is governed by local ambipolarity for which the ions and electrons have the same drifting flux (non-ambipolarity may arise in the magnetized plasma).17,19 The light electrons are in quasi-equilibrium in a low temperature plasma (or a weakly ionized plasma) and a one-temperature Maxwellian distribution is assumed for the electrons. The heavy ions are not in equilibrium and their transport behaves following a more complicated process due to the ion-neutral collisions, i.e., elastic ion-neutral collision and resonant charge transfer collision. The effect of long-range Coulomb collisions, i.e., ion-ion and ion-electron collisions, is negligible in a low temperature plasma, and the ion momentum transfer is determined by the ion-neutral collisions of which description depends on the thermalization treatment for ions and neutrals. The combination of cold neutrals and warm ions is
normally applied, and the opposite case of cold ion beam and warm neutrals has been reported by Fruchtman. Both ions and neutrals remain thermalized in this study and they are connected by using the effective ion temperature. The neutral depletion effect is neglected and the neutrals are assumed to be homogeneously distributed. The radial transport of ions is described by three ion mobility based models: a low electric field (LEF) model, an intermediate electric field (IEF) model, and a high electric field (HEF) model. Since the ion diffusion coefficient becomes a complicated electric-field-dependent parameter, when the ion drift velocity is large compared to the ion thermal velocity (which is the case for IEF and HEF models), its effect was normally neglected in previous HEF studies. The present study follows a consistent path and neglects the diffusion effect for ion transport, i.e., the ions behaving an ion-mobility-governed manner.

In the LEF model, the ion-neutral collisions are dominated by the neutral thermal effect and the electric field effect is negligible. The ion mobility coefficient is independent of the electric field and given by the first Chapman-Enskog approximation. The dependency of the ion mobility coefficient on the electric field appears as the electric field strength is increased. At the upper limit of the electric field strength described by the HEF model, the ion-neutral collisions are dominated by the strong electric field and the neutral thermal effect is negligible (cold gas limit). The ion mobility coefficient is inversely proportional to the square root of the electric field strength. Both the neutral thermal effect and electric field effect are considered in the IEF model, and the ion-mobility coefficient is expressed in terms of the effective ion temperature. To the best of our knowledge, the present paper is the first to use the IEF model to study the ion transport in low temperature plasmas. The total momentum-transfer cross section for ion-neutral collisions (the ion-neutral collision and resonant charge transfer collision) is approximated to be a constant for the low temperature plasma of interest. In a bounded plasma, the LEF dominates the central region of maximum density and the HEF dominates the near-wall presheath region, which accelerates the ions to the Bohm velocity. There are two asymmetric presheath regions near the inner and outer walls across an annular plasma. The presheath width extends as the gas pressure decreases, hence the LEF and HEF regimes dominate the high pressure and low pressure plasmas, respectively. This study shows that the IEF model approaches the LEF and HEF models at their respective electric field strength limits and smoothly connects the center region (dominated by the LEF regime) and the near-wall region (dominated by the HEF regime) within an annular plasma.

II. EQUILIBRIUM OF ELECTRONS

The electron flux ) = and ion flux ) = , representing the mean drift motion of electrons and ions, are given as the sum of “diffusion” due to the density gradient ) (subscripts “i” and “e” denoting ions and electrons, respectively) and “mobility” due to the electric field ) (Ref. 17)

\[
\hat{\Gamma}_e = -\vec{D}_e \nabla n_e - n_e \vec{K}_e \vec{E}, \quad (1a)
\]

\[
\hat{\Gamma}_i = -\vec{\tilde{D}}_i \nabla n_i + n_i \vec{\tilde{K}}_i \vec{E}, \quad (1b)
\]

where ) and ) are the diffusion tensor and mobility tensor, respectively. The non-diagonal terms are zero in the two tensors due to the absence of magnetic field. The swarm motion of electrons and ions is governed by local ambipolarity of ) and the quasi-neutrality ) is held for the bulk plasma. Combining Eqs. (1b) and (1a) yields

\[
(\vec{D}_e - \vec{\tilde{D}}_i) \nabla n + n \vec{\tilde{K}}_e \vec{E} = 0, \quad (2)
\]

An electron loses an energy fraction of \( \sim m_e/m_i \ll 1 \) during an electron-neutral collision, while an ion loses about half of its collisional energy during an ion-neutral collision. As a result, the electron temperature is much higher than the ion temperature \( T_e \gg T_i \). The electrons diffuse quickly along the high pressure \( (n_e \times eT_e) \) gradient and respond easily to the electric field due to the light inertial mass. In a low temperature plasma, the diffusion and mobility are much faster for electrons than for ions, and the diagonal elements of the diffusion and mobility tensors satisfy \( D_{ij}^{(e)} > D_{ij}^{(i)} \) and \( K_{ij}^{(e)} > K_{ij}^{(i)} \) (\( i, j = 1, 2, 3 \) is the index for tensor elements). Equation (2) can be further simplified to give

\[
\vec{\tilde{D}}_e \nabla n + n \vec{\tilde{K}}_e \vec{E} = 0, \quad (3)
\]

which is equivalent to neglecting \( \hat{\Gamma}_e \) in the left hand side (LHS) of Eq. (1a). The above equation shows that the mean drifting motion of electrons is small compared to the motion caused by either the mobility or diffusion. The electrons are in quasi-equilibrium under the forces of electric field and pressure gradient. In this case, the electrons can be described by the Boltzmann relation with the assumption of one-temperature Maxwellian distribution

\[
n = n_0 \exp \left( \frac{\phi}{T_e} \right), \quad (4)
\]

where \( n_0 \) and \( \phi \) are the maximum electron (ion) density and the plasma potential, respectively. Applying the gradient operator to both sides of the Boltzmann relation yields

\[
\vec{E} = -\nabla \phi = -T_e \frac{\nabla n}{n}. \quad (5)
\]

Substituting this formula into Eq. (3) and considering that the density gradient \( \nabla n \) should be nontrivial yields

\[
\vec{\tilde{D}}_e = T_e \vec{\tilde{K}}_e, \quad (6)
\]

which is the famous “Einstein relation” connecting the mobility and diffusion coefficients in an equilibrium state.

III. RADIAL TRANSPORT OF IONS

The ion flux \( \hat{\Gamma}_i \) cannot be neglected for the heavy and non-equilibrium ions in Eq. (1b), and the mean ion drifting
velocity \( \vec{u}_i = \frac{\vec{r}}{r} \) is used to define the plasma boundary where it reaches the Bohm velocity. The ions are bounded by an inner wall at radius \( r_j \) and an outer wall at radius \( r_p \) in an annular plasma. The wall sheath width is normally much smaller than the scale of a plasma and the plasma boundaries are approximately located at the inner and outer walls. The annular plasma can be treated as axially symmetric and no azimuthal flow in the absence of a magnetic field. Hence, on first approximation, a constant axial plasma flux is assumed (frozen axial dimension) and the radial dimension separately solved at the cost of some loss in accuracy (which depends on the specific flow in the assumed frozen dimension) as mentioned in Section I. The radial transport (denoted by the subscript “r”) is of primary interest and given by

\[
\Gamma_r = -D_{fr} \frac{dn}{dr} + nK_{fr}E_r.
\]  

(7)

Combining the Boltzmann relation (5) and the above formula yields

\[
\Gamma_r = -(D_{fr} + T_eK_{fr}) \frac{dn}{dr} = -D_{fr} \frac{dn}{dr},
\]

(8a)

\[
u = \Gamma_r = -D_{fr} \frac{dn}{dr} = D_{fr} \frac{dn}{dr}.
\]

(8b)

where \( D_{fr} = D_{fr} + T_eK_{fr} \) is defined as the effective radial diffusion coefficient. In the LEF regime, the ion diffusion and mobility coefficients \( D_{fr} \) and \( K_{fr} \) are connected by the linear Einstein relation \( D_{fr} = T_eK_{fr} \), and \( D_{fr} \) is rewritten as \( D_{fr} = (T_e + T_{if})K_{fr} \). However, in the IEF and HEF regimes, \( D_{fr} \) and \( K_{fr} \) violate the linear Einstein relation and a nonlinear generalized Einstein relation (GER) should be used.21 The diffusion effect was neglected for the HEM model in previous studies10 for simplicity. The present study follows the same assumption of no diffusion effect for the IEF and HEM models, resulting in the relation

\[ D_{fr} = T_eK_{fr}, \]

(9)

which is satisfied for all the three models now, and the expression for the radial ion flux \( \Gamma_r \) (8a) is determined by \( K_{fr} \).

A. Ion mobility coefficient

The ion mobility coefficient \( K_{fr} \) exhibits different properties in regard to the dominance between the neutral thermal effect and the electric field effect. The formula deduction of \( K_{fr} \) is very complicated18 and not the purpose of this paper. Here, we give a summary of the important results.

In the LEF model for which the electric field effect is small compared to the neutral thermal effect, \( K_{fr} \) can be solved by the first Chapman-Enskog approximation

\[ K_{fr} = \frac{3(\pi)^{\frac{1}{2}}}{8} \frac{e}{n_e\sigma_m} \left( \frac{1}{m_eT_e} \right)^{\frac{1}{2}}, \]

(10)

where \( T_e \) is the neutral gas temperature and \( \sigma_m \) is a cross section averaged over a distribution of ion-neutral collisional energy \( \epsilon_e \).

As \( \sigma_m \) is weakly dependent on the collisional energy of interest in the low temperature plasma, it is approximated to be a constant (hard sphere collision).8,11 In this case, the distinction between \( \sigma_m^c \) and \( \sigma_m \) disappears, \( \sigma_m^c = \sigma_m \).

In the HEM model for which the neutral thermal effect is small compared to the electric field effect (equivalent to the cold gas limit), \( K_{fr} \) is inversely proportional to the square root of the electric field strength

\[
K_{fr} = \frac{\xi_H}{\left( \frac{e}{m_i n_e \sigma_m |E_r|} \right)^{\frac{1}{2}}},
\]

(12)

where the constant \( \xi_H \) slightly varies depending on the chosen mobility model and we use \( \xi_H = (\frac{4}{3})^{\frac{1}{2}} \) from the Smirnov model25,29 It should be noted that the absolute value of \( E_r \) is used in the above formula as \( E_r \) can be either positive or negative within an annulus (always positive in a cylinder).

In the IEF model for which neither the neutral thermal effect or the electric field effect is neglected, an effective ion temperature \( T_{if} \) is defined to include both effects21,22

\[
\frac{3}{2}eT_{if} = \frac{3}{2}eT_e + \frac{1}{2}m_i \nu^2.
\]

(13)

The first term and second term in the right hand side (RHS) represent the contribution of the neutral thermal effect and electric field effect during an ion-neutral collision, respectively. \( \nu = K_{fr}E_r \) is a drift velocity purely caused by the electric field (called the electric drift velocity), equivalent to the ion mean drift velocity \( \nu \) in a homogeneous plasma. In a non-homogeneous plasma \( \nu \) should be less than \( \nu \) which includes the influence of both electric field and density gradient. As the diffusion effect (due to the density gradient) is neglected in the present study, \( \nu \) is satisfied by substituting \( D_{fr} = T_eK_{fr} \) into Eq. (8b). \( K_{fr} \) is given in terms of \( T_{if} \) for the IEF model

\[
\frac{3}{2}eT_{if} = \frac{3}{2}eT_e + \frac{1}{2}m_i \nu^2.
\]

(14)

which is an implicit equation for \( K_{fr} \) and the constant \( \frac{3}{2}eT_{if} \) is taken from the Mason model.18,22 In order to make it explicit, two dimensionless parameters are introduced: a dimensionless electric drift velocity \( \tilde{\nu}_{dr} = \frac{(\frac{3}{2}eT_{if})^{\frac{1}{2}}}{m_i \nu} \) and a dimensionless electric field parameter \( \tilde{E}_r = \frac{3\nu}{4T_e}, \)

where \( x_l = \frac{(\frac{3}{2}eT_{if})^{\frac{1}{2}}}{m_i \nu} \) and \( \sigma_m \). Formula (14) is rewritten as

\[
\tilde{\nu}_{dr} \left( 1 + \frac{2}{3} \tilde{E}_r^2 \right)^{\frac{1}{2}} = \tilde{E}_r,
\]

(15)

which is a quadratic equation of \( \tilde{\nu}_{dr} \) and its real solution is given by

\[
|\tilde{\nu}_{dr}| = \left( \frac{3}{2} \right)^{\frac{1}{2}} \left[ \left( 1 + 8 \tilde{E}_r^2 \right)^{\frac{1}{2}} - 1 \right].
\]

(16)
A dimensionless ion mobility coefficient is defined as $K_{ir} = \frac{u_B}{v_{ci}}$ and it is proportional to $K_{ir}$ of which the final expression is given by

$$K_{ir} = \left(\frac{6\pi e^2}{8} \right)^{1/2} \frac{u_{ih} r_h}{T_e} \left(1 + \frac{1}{2} \frac{v_{ci}^2}{E_r} - 1 \right)^{1/2}. \tag{17}$$

B. Governing equation for radial transport

The radial continuity equation for particle balance is given by

$$\frac{d\Gamma_{ir}}{dr} + \frac{\Gamma_{ir}}{r} - \nu_{iz} n = 0, \tag{18}$$

where $\nu_{iz} = n_i u_{iz}$ is the ionization rate, and the rate constant $\mu_{ic}$ for one-temperature Maxwellian electrons is given by

$$\mu_{ic} = \left(\frac{8e}{\pi m T_e^2}\right)^{1/2} \int_0^\infty \sigma_{ie}(\epsilon) e^{-\epsilon} \, d\epsilon,$$ \tag{19}

where $\epsilon$ and $\sigma_{ie}$ are the electron energy and ionization cross section, respectively.

Substituting the ion flux (8a) into Eq. (18) yields

$$\frac{d^2 \dot{n}}{dr^2} + \frac{1}{r} \frac{d\dot{n}}{dr} + \frac{v_{ci}^2}{E_r} \dot{n} + \frac{1}{D_{fr}} \frac{dD_{fr}}{dr} \frac{d\dot{n}}{dr} = 0, \tag{20}$$

where $\dot{n} = \frac{\dot{n}}{n_0}$ and $\frac{d\Gamma_{ir}}{dr}$ is the normalized ion density and radial position, respectively. The above equation is normally used in the LEF model for which $\frac{D_{fr}}{dr} = \frac{dD_{fr}}{dr} = 0$ as shown in formula (10). In the IEF and HEF models, $D_{fr}$ is a function of the electric field with $\frac{dD_{fr}}{dr} \neq 0$, and it is more convenient to express Eq. (20) in terms of the dimensionless electric field $\tilde{E}_r = \frac{\dot{n}}{n_0}$ as below, where $\eta = \frac{\phi}{T_r}$ is the dimensionless plasma potential and $\tilde{E}_r$ also satisfies $\tilde{E}_r = \frac{\tilde{E}_r}{E_r}r = -\frac{d\tilde{E}_r}{dr}$.

$$\left(1 + \tilde{E}_r \frac{dD_{fr}}{d\tilde{E}_r} \right) \frac{d\tilde{E}_r}{dr} + \tilde{E}_r \frac{d\tilde{E}_r}{dr} - \tilde{E}_r^2 - \frac{v_{ci}^2}{E_r} = 0. \tag{21}$$

Now the radial transport of ions in the annulus can be fully described by adding the boundary conditions, which are given by making the ion mean drift velocity $u_{ir}$ equal to the Bohm velocity $u_B = \left(\frac{6\pi e^2}{m} \right)^{1/2}$ at the inner wall $r_a$ and outer wall $r_b$,

$$(u_{ir})_{r=r_a} = -u_B, \quad (u_{ir})_{r=r_b} = u_B. \tag{22}$$

Replacing $u_{ir}$ by the equation variables $\dot{n}$ (for Eq. (20)) and $\tilde{E}_r$ (for Eq. (21)) using formula (8b) and dimensionless relations defined above yields

$$-\left(\frac{d\dot{n}}{d\tilde{E}_r}\right)_{\tilde{E}_r=\tilde{E}_r} = \left(\tilde{E}_r\right)_{\tilde{E}_r=\tilde{E}_r} = -\frac{u_B r_b}{D_{fr}}, \tag{23}$$

IV. ELECTRIC FIELD BASED MODELS

A. LEF model

$$D_{fr} = K_{ir} T_e$$ is independent of the radial position $\frac{dD_{fr}}{dr} = 0$ for the LEF model as stated above, hence Eq. (20) is reduced to

$$\frac{d^2 \tilde{n}}{dr^2} + \frac{1}{r} \frac{d\tilde{n}}{dr} + \beta_L^2 \tilde{n} = 0, \tag{24}$$

where $\beta_L^2$ satisfies $\beta_L^2 = \frac{v_{ci}^2}{E_r} = \frac{2\sigma_{ie} u_{ih} p_{as}}{3mu_b} \left(\frac{p_g}{T_g^2}\right)^{1/2}$. $\sigma_{ie}$ are the Paschen number for neutral gas and the mean thermal velocity for ion-neutral collisions, respectively. The boundary condition for the LEF model is given by substituting formulas (9) and (10) into (23)

$$-\left(\frac{d\tilde{n}}{d\tilde{E}_r}\right)_{\tilde{E}_r=\tilde{E}_r} = -\frac{u_B r_b}{D_{fr}} = -\frac{2\sigma_{ie} u_{ih} p_{as}}{3mu_b} \frac{T_g}{eT_g} = -\Psi_L, \tag{25a}$$

$$-\left(\frac{d\tilde{n}}{d\tilde{E}_r}\right)_{\tilde{E}_r=\tilde{E}_r} = -\frac{u_B r_b}{D_{fr}} = -\frac{2\sigma_{ie} u_{ih} p_{as}}{3mu_b} \frac{T_g}{eT_g} = \Psi_L. \tag{25b}$$

Equation (24) is a Bessel-type equation and its general solution is given by

$$\tilde{n} = C_1 J_0(\beta_L \tilde{r}) + C_2 Y_0(\beta_L \tilde{r}), \tag{26}$$

where $J_0$ and $Y_0$ are zero order Bessel functions of the first kind and second kind, and $C_1$ and $C_2$ are coefficients to be determined. Substituting the above solution into the boundary condition (25) yields

$$\begin{bmatrix} a_{11} & a_{12} \ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0. \tag{27}$$

The entries in the coefficient matrix are given by

$$a_{11} = \beta_L J_1(\beta_L \frac{r_a}{r_b}) + \Psi_L Y_0(\beta_L \frac{r_a}{r_b}),$$

$$a_{12} = \beta_L J_0(\beta_L \frac{r_a}{r_b}),$$

$$a_{21} = \beta_L J_1(\beta_L \frac{r_a}{r_b}) - \Psi_L J_0(\beta_L \frac{r_a}{r_b}),$$

$$a_{22} = \beta_L J_1(\beta_L \frac{r_a}{r_b}) - \Psi_L Y_0(\beta_L \frac{r_a}{r_b}),$$

where $J_1$ and $Y_1$ are first order Bessel functions of the first kind and second kind, respectively. The determinant of the coefficient matrix in Eq. (27) must vanish for a nontrivial solution $det[a] = 0$, which determines the electron temperature $T_e$. The ratio of $C_2$ to $C_1$ is given by $\frac{C_2}{C_1} = -\frac{a_{11}}{a_{12}} = \kappa$, and solution (26) is rewritten as

$$\tilde{n} = C_1 \left[ J_0(\beta_L \tilde{r}) + \kappa Y_0(\beta_L \tilde{r}) \right], \tag{28}$$

where $C_1$ is determined at the peak position of the normalized radial density profile $\tilde{r}_{fr}$ satisfying $\frac{d\tilde{n}}{d\tilde{r}} = 0$ and $\tilde{n} = 1$. One argument about the annular solution (28) is its convergence for an infinitesimal inner radius $r_a \to 0$, as the Bessel
function of the second kind diverges near zero. Appendix A proves that the solution is convergent for an infinitesimal $r_\alpha$ due to the adjustable constant $\kappa$.

**B. HEF model**

$D_{fr}$ for the HEF model is obtained using formulas (9) and (12) and its derivative with respect to the variable $E_r$ is given by

$$\frac{dD_{fr}}{dE_r} = -\frac{1}{2} \frac{D_{fr}}{E_r}. \quad (29)$$

Substituting formulas (9), (12), and (29) into Eq. (21) yields

$$\frac{dE_r}{dr} + 2 \frac{\bar{E}_r}{r} - 2 \beta_H |\bar{E}_r|^2 = 0, \quad (30)$$

where $\beta_H = \frac{u_i (\frac{e n_i}{m_i})^{\frac{1}{2}}}{n_i e r_i^{\frac{1}{2}}}$. The boundary condition for the HEF model is given by substituting formulas (9) and (12) into (23)

$$\bar{E}_{r=\infty} = \frac{u_B r_b}{D_{fr}} = \left( \frac{\pi \sigma_m P_{ax}}{4 e T_g} \right)^\frac{1}{2} |\bar{E}_r|^2 \Rightarrow (\bar{E}_r)_{r=\infty} \quad (31a)$$

$$\bar{E}_{r=1} = \frac{u_B r_b}{D_{fr}} = \left( \frac{\pi \sigma_m P_{ax}}{4 e T_g} \right)^\frac{1}{2} |\bar{E}_r|^2 \Rightarrow (\bar{E}_r)_{r=1} \quad (31b)$$

Equation (30) is an Abel-type equation with no analytical solution and the boundary value problem (BVP) for the HEF model is numerically solved using the MATLAB solver, “BVP4C,” for which the Hermite-Simpson method was used to solve the ordinary differential equation; an initial solution guess was also evaluated as usually done for most boundary value problems. The validity of this solver has been verified as follow: it was used to numerically solve the LEF model which has an analytical solution as demonstrated in Sec. IV A, and the numerical results calculated from the solver were equal to the results from the analytical solution.

**C. IEF model**

$D_{fr}$ for the IEF model is obtained using formulas (9) and (17) and its derivative with respect to the variable $E_r$ is given by

$$\frac{dD_{fr}}{dE_r} = -\frac{1}{2} \frac{D_{fr}}{E_r} \left[ 1 - \frac{1}{\left( 1 + \frac{8}{3} \bar{E}_r^2 \right)^\frac{1}{2}} \right]. \quad (32)$$

Substituting formulas (9), (17), and (32) into Eq. (21) yields

$$\left[ 1 + \frac{1}{\left( 1 + \frac{8}{3} \bar{E}_r^2 \right)^\frac{1}{2}} \right] \frac{d\bar{E}_r}{dr} + 2 \frac{\bar{E}_r}{r} - 2 \beta H |\bar{E}_r|^2$$

$$- 2 \beta H \left( 1 + \frac{8}{3} \bar{E}_r^2 \right)^\frac{1}{2} = 0, \quad (33)$$

where $\beta_H = \left( \frac{2 e n_i}{m_i} \right)^\frac{1}{2} \frac{P_{ax}}{e r_i^{\frac{1}{2}}}$. The boundary condition for the IEF model is given by substituting formulas (9) and (17) into (23)

$$\bar{E}_{r=\infty} = \frac{u_B r_b}{D_{fr}} = \left( \frac{\pi \sigma_m P_{ax}}{3 T_g} \right)^\frac{1}{2} |\bar{E}_r|^2 \Rightarrow (\bar{E}_r)_{r=\infty} \quad (34a)$$

$$\bar{E}_{r=1} = \frac{u_B r_b}{D_{fr}} = \left( \frac{\pi \sigma_m P_{ax}}{3 T_g} \right)^\frac{1}{2} |\bar{E}_r|^2 \Rightarrow (\bar{E}_r)_{r=1} \quad (34b)$$

Equation (33) is a nonlinear ordinary differential equation (ODE) with no analytical solution and the BVP for the IEF model is numerically solved using the same method as that used in the HEF model.

**V. MODELLING RESULTS**

**A. Unification of IEF model**

The ion mobility coefficient (17) and the effective ion temperature (13) in the IEF model are unified parameters for the LEF and HEF models. In order to check their universal property, a dimensionless ion mobility coefficient $K_{ir}$ and a dimensionless effective ion temperature $\bar{T}_{ir}$ are first defined. The former parameter follows the same definition of $K_{ir} = \frac{u_i}{v_i}$ in the IEF model for all the three models and the latter parameter is given as follow. Substituting $u_B = \left( \frac{2 e n_i}{m_i} \right)^\frac{1}{2} u_B r_b$ into formula (13) gives $\bar{T}_{ir} = \frac{T_e}{T_e} = 1 + \frac{2}{3} \bar{u}_e^2$ for the IEF model. Since the first term and second term in the RHS represent the contribution of the neutral thermal effect and electric field effect, respectively, the definition of $\bar{T}_{ir}$ can be generalized to the LEF and HEM models: In the LEF model, the electric field effect is negligible and $\bar{T}_{ir} = 1$; in the HEF model, the neutral thermal effect is negligible and $\bar{T}_{ir} = 2 \bar{u}_e^2$. The expressions for $K_{ir}$ and $\bar{T}_{ir}$ are summarized as below

$$\text{IEF model : } \bar{T}_{ir} = 1, \quad \bar{T}_{ir} = 1, \quad (35a)$$

$$\text{HEF model : } \bar{T}_{ir} = \frac{4}{3} \left( \frac{2 n_i}{9} \right)^\frac{1}{2} \frac{1}{v_i^2}, \quad \bar{T}_{ir} = \frac{2}{3} \left( K_{ir} \bar{u}_e^2 \right)^2. \quad (35b)$$
IEF model: \[ \hat{K}_\nu = \frac{3}{2} \left( \frac{1 + \frac{8}{3} e^{2 \nu}}{|\nu|} - 1 \right)^{\frac{1}{2}} \]

\[ \hat{T}_f = 1 + \frac{2}{3} (\hat{K}_\nu \nu)^2. \] (35c)

Figures 1(a) and 1(b) present the \( \hat{K}_\nu(\nu) \) curve and \( \hat{T}_f(\nu) \) curve for the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line). Figure 1(a) shows that \( \hat{K}_\nu \) is a decreasing function of \( |\nu| \) for the three models, and Figure 1(b) shows that \( \hat{T}_f \) is an increasing function of \( |\nu| \). In both Figures 1(a) and 1(b), the IEF curve is consistent with the LER curve for the range of \( |\nu| < 0.1 \), and it is consistent with the HER curve for the range of \( |\nu| > 10 \). Hence, \( \hat{K}_\nu \) and \( \hat{T}_f \) for the IEF model are unified parameters for the other two models at their respective electric field strength limits. The IEF transport equation (33) also has a universal property: it is reduced to the LEF transport equation (24) when \( \hat{E}_r \) approaches zero, and it is reduced the HEF transport equation (30) when \( \hat{E}_r \) approaches infinity. A detailed proof is given in Appendix B.

B. Numerical results for argon plasma

We illustrate the annular modelling results for a low-temperature argon plasma. The input parameters are the Paschen number \( P_s \), and the annular geometry ratio \( \frac{a}{b} \); the output parameters are the normalized ion density \( \hat{n} \), the boundary loss coefficient \( L_R \) (defined later in formula (37)), and the electron temperature \( T_e \). The argon has an atomic mass of \( m_\text{g} = 39.95 \) u (u = 1.6605 \times 10^{-27} \text{ kg} \) is the atomic mass unit) and a neutral gas temperature of \( T_g = 0.026 \) V (room temperature). The ionization cross section \( \sigma_\text{ie} \) formulated by Phelps\(^{31} \) is used for the argon ionization calculation (19)

\[ \sigma_\text{ie} = \frac{970}{(\epsilon + 70)^2} (\epsilon - 15.8) + 0.06 \epsilon (\epsilon - 15.8)^2, \] (36)

FIG. 1. (a) \( \hat{K}_\nu(\nu) \) curve obtained by the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line). (b) \( \hat{T}_f(\nu) \) curve obtained by the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line).

FIG. 2. (a) Radial profile of \( \hat{K}_\nu \) for an annular geometry ratio of \( \frac{a}{b} = 0.4 \) with a Paschen number of \( P_s = 0.01 \) Torr cm obtained by the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line). (b) Radial profile of \( \hat{n} \) for \( \frac{a}{b} = 0.4 \) with \( P_s = 0.01 \) Torr cm obtained by the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line).
where $\sigma_{\text{c}}$ is in the unit of $10^{-20} \text{m}^2$. A total momentum-transfer cross section (including the elastic ion-neutral collision and resonant charge transfer collision) of $\sigma_m = 10^{-18} \text{m}^2$ is used for ion-neutral collisions in the low temperature argon plasma. As $\sigma_m$ is a constant, $P_{\text{as}}$ is proportional to the Knudsen number $K_n = \frac{\lambda_i}{r}$, where $\lambda_i$ is the ion mean free path. It should be noted that the collisional annular model has a validity limit of $K_n \leq \frac{2\pi \sigma_m}{\rho_0}$. The annular plasma becomes collisionless when $K_n \gg \frac{2\pi \sigma_m}{\rho_0}$ and the present modelling will be invalid for the collisionless scenario.

Figures 2(a) and 2(b) present the radial profile of $\tilde{K}_r$ and $\hat{n}$ for a typical annular geometry ratio of $\frac{\rho_0}{r_b} = 0.4$ with a low $P_{\text{as}}$ number of $P_{\text{as}} = 0.01 \text{Torr} \cdot \text{cm}$ (for which $\lambda_i$ is comparable to the annular width $(r_b - r_a)$ with $K_n \sim \frac{1}{r_b - r_a}$ = 0.6) obtained by the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line). Figure 2(a) shows that the IEF profile reaches its maximum at $\tilde{r} \sim 0.64$ and the HEF profile approaches infinity at $\tilde{r} \sim 0.64$, indicating a small electric field nearby. The LEF profile remains constant across the radial dimension due to the neglect of electric field effect (similar phenomenon as shown in Figure 1). Figure 2(b) shows that the $\hat{n}$ profiles obtained by the three models are asymmetric around a similar peak position of $\tilde{r}_p \sim 0.64$, which is related to the peak position of IEF and HEF profiles in Figure 2(a). The boundary-to-maximum density ratio is about $\frac{\hat{n}_{a,b}}{\hat{n}} \sim 0.95$ for the LEF model and $\sim 0.8$ for the IEF and HEF models. Figures 3(a) and 3(b) present the radial profile of $\tilde{K}_r$ and $\hat{n}$ for $\frac{\rho_0}{r_b} = 0.4$ with a high $P_{\text{as}}$ number of $P_{\text{as}} = 1.0 \text{Torr} \cdot \text{cm}$ (for which $\lambda_i$ is small compared to $(r_b - r_a)$ with $K_n \sim \frac{1}{r_b - r_a}$) obtained by the LEF model (dashed-dotted line), the IEF model (solid line), and the HEF model (dashed line). Figure 3(a) shows that the IEF profile reaches its maximum at $\tilde{r} \sim 0.67$ and the HEF profile approaches infinity at $\tilde{r} \sim 0.66$ (not completely shown in Figure 3(a) to maintain visual clarity). Figure 3(b) shows that the $\hat{n}$ profiles are asymmetric around a similar peak position of $\tilde{r}_p \sim 0.67$. The boundary density ratio is about $\frac{\hat{n}_{a,b}}{\hat{n}} \sim 0.1$ for the three models.

Figures 2(a) and 3(a) show that the $\tilde{K}_r$ profile of the IEF model is consistent with that of the LEF model in the central peak region (or LEF regime) and consistent with that of the HEF model in the boundary region (or HEF regime). The IEF $\tilde{K}_r$ profile’s central consistency with the LEF profile dominates for a high $P_{\text{as}}$ number (Figure 3(a)) and its boundary consistency with the HEF profile dominates for a low $P_{\text{as}}$ number (Figure 2(a)). The universal property of radial $\tilde{K}_r$ profile obtained by the IEF model in Figures 2 and 3 is consistent with the $\tilde{K}_r(\tilde{r}_p)$ curve in Figure 1(a). The IEF model gives more accurate results of $\tilde{K}_r$, $\tilde{r}_p$, and $\hat{n}$ (determined by the expression of $\tilde{K}_r$) than the LEF and HEF models over the entire electric field strength range and it will be used to further investigate the radial transport properties of the annular plasma.

The shape of radial $\hat{n}$ profile across the annulus is characterized by the density peak position $\tilde{r}_p$ and boundary density ratios. An annular boundary loss coefficient $L_R$, similar to the boundary density parameters $h_R$ and $h_I$ for the cylindrical and plane-parallel plasmas, is defined for an annular plasma as

$$L_R = \frac{r_a}{r_b} \tilde{n}_{a} + \tilde{n}_{a}. \quad (37)$$

It is a generalized boundary density ratio and can be used to estimate the value of the maximum density $n_b$ by considering power balance. Figures 4(a) and 4(b) present the $\tilde{r}_p(\frac{\rho_0}{r_b})$ curve and $L_R(\frac{\rho_0}{r_b})$ curve obtained by the IEF model for different $P_{\text{as}}$ numbers of $P_{\text{as}} = 0.01 \text{Torr} \cdot \text{cm}$ (dashed-dotted line), 0.1 Torr cm (solid line), and 1.0 Torr cm (dashed line). Figure 4(a) shows that $\tilde{r}_p$ is a monotonically increasing function of $\frac{\rho_0}{r_b}$ and the variation is stronger in the low $\frac{\rho_0}{r_b}$ range. $\tilde{r}_p$ approaches to the middle of the annulus (shown as the dotted line) when $\frac{\rho_0}{r_b} > 0.8$. $\tilde{r}_p$ is greater for a higher $P_{\text{as}}$ number compared to a lower $P_{\text{as}}$ number. Figure 4(b) shows that $L_R$ is also a monotonically increasing function of $\frac{\rho_0}{r_b}$ and the...
variation is greater in the high \( \frac{\alpha}{r_a} \) range. In the low \( \frac{\alpha}{r_a} \) range \( L_R \) increases faster for a higher \( P_{as} \) number, while in the high \( \frac{\alpha}{r_a} \) range \( L_R \) increases faster for a lower \( P_{as} \) number. Contrarily to \( \tilde{r}_p \) in Figure 4(a), \( L_R \) is greater for a lower \( P_{as} \) number than for a higher \( P_{as} \) number.

Figures 5(a) and 5(b) present the \( \tilde{r}_p(P_{as}) \) curve and \( L_R(P_{as}) \) curve obtained by the IEF model for different annular geometry ratios of \( \frac{r_a}{r_b} = 0.2 \) (dashed-dotted line), 0.4 (solid line), 0.6 (dashed line), and 0.8 (dotted line). Figure 5(a) shows that \( \tilde{r}_p \) is an increasing function of \( P_{as} \) (consistent with the results in Figure 4(a)) and the total increment \( \Delta \tilde{r}_p \) over the variable range of \( 10^{-3} \text{Torr cm} < P_{as} < 10 \text{Torr cm} \) is reduced for higher \( \frac{r_a}{r_b} \), with an increment of \( \Delta \tilde{r}_p \sim 0.08 \) for \( \frac{r_a}{r_b} = 0.2 \) and \( \Delta \tilde{r}_p \sim 0.003 \) for \( \frac{r_a}{r_b} = 0.8 \). Figure 5(b) shows that \( L_R \) is a reversed-“S”-shape decreasing function of \( P_{as} \) (consistent with the results in Figure 4(b)) and the variation is stronger in the middle range between \( \sim 0.01 \text{Torr cm} \) and \( \sim 1 \text{Torr cm} \) than in the edge ranges. \( L_R \) approaches to zero when \( P_{as} \) reaches a high value of \( P_{as} \sim 10 \text{Torr cm} \). Figures 5(a) and 5(b) show that \( \tilde{r}_p \) is a more robust parameter than \( L_R \) as a function of the \( P_{as} \) number.

Figure 6(a) presents the \( T_e \left( \frac{\alpha}{r_a} \right) \) curve obtained by the IEF model for different \( P_{as} \) numbers of \( P_{as} = 0.01 \text{Torr cm} \) (dashed-dotted line), 0.1 \text{Torr cm} (solid line), and 1.0 \text{Torr cm} (dashed line). \( T_e \) is an increasing function of \( \frac{\alpha}{r_a} \) and the variation is stronger in the high \( \frac{\alpha}{r_a} \) range. The \( T_e \left( \frac{\alpha}{r_a} \right) \) curve exhibits higher values and increases faster for a lower \( P_{as} \) number, with an increment of \( \Delta T_e \sim 12.0 \text{eV} \) for \( P_{as} = 0.01 \text{Torr cm} \), \( \Delta T_e \sim 2.3 \text{eV} \) for \( P_{as} = 0.1 \text{Torr cm} \), and \( \Delta T_e \sim 0.9 \text{eV} \) for \( P_{as} = 1.0 \text{Torr cm} \) over the range of \( 0.01 < \frac{r_a}{r_b} < 0.9 \). Figure 6(b) presents the \( T_e(P_{as}) \) curve obtained by the IEF model for different annular geometry ratios of \( \frac{r_a}{r_b} = 0.2 \) (dashed-dotted line), 0.4 (solid line), 0.6 (dashed line), and 0.8 (dotted line). \( T_e \) is a decreasing function of \( P_{as} \) and the variation is stronger in the high \( \frac{\alpha}{r_a} \) range.
The annular modelling given in this work is ion-mobility-based (the diffusion effect was neglected as mentioned above), and its applicable range is determined by the accuracy of the ion mobility coefficient. The mobility coefficient calculated by the LEF or HEF model has a good accuracy at the low or high electric field limit for which the validity has been verified in previous studies for cylindrical and plane parallel plasmas. The IEF model presents a more unified mobility coefficient over the entire electric field strength range and approaches the results of LEF and HEF models at the respective electric field limits as shown in Section V. Hence, the IEF model gives more accurate results compared to the LEF and HEF models over the entire electric field range. Future studies will aim at comparing the IEF results to other experimental or computational studies, which, to the best of our knowledge, are not yet readily available in the literature.

The annular modelling was applied to a low temperature argon plasma. The normalized radial density profile obtained by the IEF model joins the LEF model in the central peak region and joins the HEF model in the boundary region. The radial profile is asymmetric around the peak position with the latter found closer to the inner wall. The density peak position is an increasing function of the annular geometry ratio and Paschen number. An annular boundary loss coefficient is defined to characterize the boundary density ratios, and it is an increasing function of the annular geometry ratio and a decreasing function of the Paschen number. More ions are lost to the walls for a higher annular geometry ratio, and the radial transport of ions is more effective for a low Paschen number (high Knudsen number) due to less ion-neutral collisions. The electron temperature has a positive correlation with the annular boundary loss coefficient with respect to the variable annular geometry ratio and Paschen number. When the boundary loss of ions is enhanced, the electron temperature increases to provide more ionization and satisfies particle balance.

**APPENDIX A: INFINITESIMAL INNER RADIUS LIMIT OF IEF MODEL**

The convergence of LEF solution (28) at an infinitesimal inner boundary $\tilde{r} = \frac{r_{a}}{r_{b}} \to 0$ is determined by the second term $\kappa Y_{0}(\beta L \tilde{r})$. Define a function $F(\tilde{r})$ as

$$F(\tilde{r}) = \kappa Y_{0}(\beta L \tilde{r}) = - \beta_{L} J_{1} \left( \frac{r_{a}}{r_{b}} \right) + \psi_{L} J_{0} \left( \frac{r_{a}}{r_{b}} \right) Y_{0}(\beta L \tilde{r}).$$  \hspace{1cm} (A1)

When $\tilde{r} = \frac{r_{a}}{r_{b}}$ approaches zero, (1) $J_{0}$ and $J_{1}$ approach unity and zero, respectively, and (2) $Y_{1}$ diverges faster than $Y_{0}$ with $\frac{r_{a}}{r_{b}} \to +\infty$. Then, $F(\tilde{r})$ can be rewritten as

$$F(\tilde{r}) \bigg|_{\tilde{r}=0} = \frac{1}{\beta_{L} J_{1} \left( \frac{r_{a}}{r_{b}} \right) + \psi_{L} J_{0} \left( \frac{r_{a}}{r_{b}} \right) Y_{0}(\beta L \tilde{r})} \sim 0. \hspace{1cm} (A2)$$

**VI. SUMMARY**

The radial transport properties of low temperature annular plasmas were investigated. The electrons were in quasi-equilibrium and governed by the Boltzmann relation.
Hence, the annular solution (28) is convergent and reduced to the classic cylindrical solution of $\hat{n} = I_0(\beta_L \hat{r})$ for an infinitesimal inner boundary.

**APPENDIX B: ELECTRIC FIELD LIMITS OF IEF MODEL**

At the limit of infinitesimal electric field, $\hat{E}_r \to 0$, $(1 + \frac{8}{3} \pi^2 \hat{E}_r^2)^{1/2}$ can be approximated by its first order Taylor expansion $(1 + \frac{8}{3} \pi^2 \hat{E}_r^2)$. Substituting this approximation into Eq. (33) and considering $z_l[\hat{E}_r] \ll 1$ yields

$$\frac{d\hat{E}_r}{\hat{r}} + \frac{\hat{E}_r}{\hat{r}} - \beta_1^* = 0,$$  \hspace{1cm} (B1)

where $\beta_1^* = \frac{(3/2)^2 \beta_1}{z_l} = \frac{9}{5} \frac{e}{\pi^2} \frac{n_e}{m} \frac{B_m}{T_e} \left( \frac{T_e}{T_i} \right)$ is equal to $\beta_L^*$. Substituting $\hat{E}_r = -\frac{d\hat{n}}{dr} \hat{r}$ into the above equation gives

$$\frac{d^2 \hat{n}}{dr^2} + \frac{1}{r} \frac{d\hat{n}}{dr} + \beta_1^* \hat{n} = 0,$$  \hspace{1cm} (B2)

which is exactly the LEF transport equation (24) at the low electric field strength limit.

At the other limit of infinite electric field, $\hat{E}_r \to \infty$, $(1 + \frac{8}{3} \pi^2 \hat{E}_r^2)^{1/2}$ is approximated by $(\frac{3}{2})^{1/2} z_l[\hat{E}_r]$. Substituting this approximation into Eq. (33) and considering $z_l[\hat{E}_r] \gg 1$ yields

$$\frac{d\hat{E}_r}{dr} + 2 \frac{\hat{E}_r}{r} - 2 \beta_1^{**} \hat{E}_r^2 = 0,$$  \hspace{1cm} (B3)

where $\beta_1^{**} = \frac{(3/2)^2 \beta_1}{z_l} = 4 \frac{e}{\pi^2} \frac{n_e}{m} \frac{B_m}{T_e} \left( \frac{T_e}{T_i} \right)$ is 5% higher than $\beta_L$, hence the IEF transport equation (B3) is a good approximation to the HEF transport equation (30) at the high electric field strength limit.