



Q -operators in the six-vertex model

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Abstract

In this paper we continue the study of Q -operators in the six-vertex model and its higher spin generalizations. In [1] we derived a new expression for the higher spin R -matrix associated with the affine quantum algebra $U_q(\widehat{sl}(2))$. Taking a special limit in this R -matrix we obtained new formulas for the Q -operators acting in the tensor product of representation spaces with arbitrary complex spin.

Here we use a different strategy and construct Q -operators as integral operators with factorized kernels based on the original Baxter's method used in the solution of the eight-vertex model. We compare this approach with the method developed in [1] and find the explicit connection between two constructions. We also discuss a reduction to the case of finite-dimensional representations with (half-)integer spins.

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1. Introduction

In our previous paper [1] we derived a new expression for the $U_q(\widehat{sl}(2))$ R -matrix $R_{I,J}(\lambda)$ with a spectral parameter acting in the tensor product of two highest weight representations with arbitrary complex spins I and J . The method we used was based on a 3D approach developed in [2–4]. The result was surprisingly simple and represented in terms of the basic hypergeometric function ${}_4\phi_3$.

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As an application of this new formula for the R -matrix we constructed the Q -operators related to the $U_q(\widehat{sl(2)})$ algebra as special transfer matrices acting in the tensor product of arbitrary highest weight representations. The idea of the construction of the Q -operator in terms of some special transfer matrices belongs to Baxter [5]. It is a key element of his original solution of the 8-vertex model. For the simplest case of the six-vertex model the quantum space is built from 2-dimensional highest weight representations of the $U_q(\widehat{sl(2)})$ algebra at every site of the lattice.

In 1997 Bazhanov, Lukyanov and Zamolodchikov suggested a new method to derive the Q -operators related to the affine algebra $U_q(\widehat{sl(2)})$ [6,7]. Based on the universal R -matrix theory [8] they showed that the Q -operators can be constructed as special monodromy operators with the auxiliary space being an infinite-dimensional representation of the q -oscillator algebra. This method of construction of Q -operators has been actively used and extended in [9–13]. However, the derivation of the local Q -operators from the universal R -matrix [14] quickly becomes unbearable for higher spins.

Taking a special limit $I \rightarrow \infty$ [15–17] in the R -matrix from [1] we generalized the construction of [6,7] to the case of arbitrary complex spin s . In the limit $2s = I \rightarrow \mathbb{Z}_+$ the Verma module becomes reducible and contains an $(I + 1)$ -dimensional invariant subspace. Our Q -operators allow a natural non-singular reduction to this subspace.

Interestingly, there is a different approach to the construction of the Q -operators based on the famous Baxter’s “propagation through the vertex” method [18]. Baxter’s ideas were developed by Bazhanov and Stroganov [19] in their study of the six-vertex model at roots of unity. They considered fundamental L -operators [20] intertwined by the R -matrix of the six-vertex model at the roots of unity $q^N = 1$. In this case, the highest weight representation of the $U_q(\widehat{sl(2)})$ algebra is replaced with a cyclic representation. Based on factorization properties of the $U_q(\widehat{sl(2)})$ L -operator all matrix elements of the Q -operator can be explicitly calculated as simple products involving only a two-spin interaction.

The idea of “factorized” Q -operators was further developed by Pasquier and Gaudin [21] where they constructed the Q -operator for the Toda lattice in the form of an integral operator. The integral Q -operator for the case of the XXX chain was first calculated in [22]. Taking the limit $N \rightarrow \infty$ [23] in the Bazhanov and Stroganov construction [19] one can recover the results of [21] and [22].

The main difference of the above approach from the original Baxter’s method is that the “quantum” representation space is infinite-dimensional. It has the structure of a tensor product of Verma modules with the basis chosen as multi-variable polynomials $p(x_1, \dots, x_M)$, where M is the size of the system.

Initially the Q -operator for the XXX spin chain for complex spins was constructed as the integral operator [22]. Later on a new operator approach was developed where the Q -operator becomes a differential operator of infinite order [24]. This was generalized to the XXZ case when the Q -operator is represented by a q -difference operator of the infinite order [25,26].

It is well known that in the XXZ case the solutions of q -difference equations can be fixed only up to certain periodic functions. This can lead to potential problems when the Yang–Baxter equation is satisfied only up to such periodic functions [27]. A possible resolution in the XXZ case is to use a modular double [28]. The non-compact case and applications of the Q -operators to Liouville theory are discussed in [29–31].

Recently there was a substantial progress in understanding of the structure of a general R -matrix for the XYZ spin chain using elliptic modular double [32] where the elliptic beta integral plays an important role [33]. However, the integral form of the Q -operator for the XXZ spin chain was still missing. In this paper we intend to fill this gap and present the XXZ Q -operator

as an integral operator acting in the space of polynomials. In fact, we shall construct two such operators, one is based on the Askey–Roy extension of the beta integral [34] and another is based on the q -analogue of Barnes’ first lemma [35].

The next question is how these integral Q -operators are connected with the Q -operators derived in [1]. We find the explicit relation between them extending ideas of [36] for the XXX case, where a connection between the Q -operators of [24] and the Q -operators of [12] was found.

The paper is organized as follows. In Section 2 we remind some basic facts about the XXZ chain at arbitrary spin and explain the construction of the Q -operators derived in [1]. We also calculate the Q -operators at some special value of the spectral parameter where they have a very simple form. In Section 3 we introduce a polynomial representation of the $U_q(sl(2))$ algebra and rewrite the transfer-matrix of the six-vertex model in the form of a finite order q -difference operator in the space of M -variable polynomials. In Section 4 we construct the XXZ Q -operator as the integral operator acting in the space of polynomials and calculate its explicit action. In Section 5 we prove a commutativity of the integral Q -operators and the transfer-matrix at different values of the spectral parameters. In Section 6 we find the explicit connection between the Q -operators from Section 3 and the Q -operators constructed in [1]. Finally, in Conclusion we summarize all results and outline further directions of investigation.

2. The XXZ model at arbitrary spin

In this section we remind some results from [1]. We start with the following $U_q(sl(2))$ L -operator

$$L(\lambda; \phi) = \begin{pmatrix} \phi^{-1}[\lambda q^{H/2}] & \phi^{-1}[q]F \\ \phi[q]E & \phi[\lambda q^{-H/2}] \end{pmatrix} \tag{2.1}$$

where

$$[x] = x - x^{-1}, \tag{2.2}$$

E, F and H are the generators of the quantum algebra $U_q(sl(2))$ and ϕ is the horizontal field.

For any $I \in \mathbb{C}$ one can introduce an infinite-dimensional Verma module V_I^+ with a basis $v_i, i \in \mathbb{Z}_+$. We define the infinite-dimensional representation π_I^+ of $U_q(sl(2))$ by the following action on the module V_I^+

$$Hv_i = (I - 2i)v_i, \quad Ev_i = \frac{[q^i]}{[q]}v_{i-1}, \quad Fv_i = \frac{[q^{I-i}]}{[q]}v_{i+1}. \tag{2.3}$$

We notice that the L -operator (2.1) differs from the L -operator used in [1] by a change $\lambda \rightarrow \lambda q^{-1/2}$ and a simple similarity transformation in quantum space. Namely, basis vectors v_i used in this paper are related to basis vectors \tilde{v}_i in [1] as

$$v_i = q^{-\frac{1}{2}i^2} \tilde{v}_i, \quad i = 0, 1, \dots \tag{2.4}$$

When $I \in \mathbb{Z}_+$, the representation π_I^+ becomes reducible. The vectors $v_i, i > I$ span an irreducible submodule of V_I^+ isomorphic to V_{-I-2}^+ and one can introduce a finite-dimensional module V_I with the basis $\{v_0, \dots, v_I\}$ isomorphic to the quotient module V_I^+ / V_{-I-2}^+ . We denote the corresponding finite-dimensional representation as π_I .

For $I \in \mathbb{Z}_+$ the homogeneous transfer matrix $\mathbf{T}_I(\lambda; \phi)$ with periodic boundary conditions acting in the $(I + 1)^M$ -dimensional quantum space $W = \bigotimes_{i=1}^M V_I$ is defined as

$$\mathbf{T}_I(\lambda; \phi) = \text{Tr}[L_1(\lambda; \phi) \otimes \dots \otimes L_M(\lambda; \phi)] \tag{2.5}$$

where the trace is taken in the auxiliary space \mathbb{C}^2 .

Due to a conservation law in the quantum space W

$$\sum_{k=1}^M i_k = \sum_{k=1}^M i'_k = l, \tag{2.6}$$

the transfer matrix (2.5) has a block-diagonal form

$$\mathbf{T}_I(\lambda; \phi) = \bigoplus_{l=0}^{IM} \mathbf{T}_I^{(l)}(\lambda; \phi). \tag{2.7}$$

We call the subspace in the quantum space W with a fixed l as the l -th sector and denote it W_l .

Due to (2.6) one can introduce a simple equivalence transformation in the quantum space and move all local fields ϕ to the left where they combine into one boundary field $\phi^{-M\sigma_3}$. This is how twist is normally introduced into the XXZ model. Here we prefer to use local fields ϕ because in this case we preserve translational invariance of the spin chain which is obviously broken in the case of a boundary field.

Let us notice that the direct sum expansion (2.6) is also true for $I \in \mathbb{C}$, when the quantum space becomes infinite-dimensional, $W = \bigotimes_{i=1}^M V_I^+$. In this case the sum in (2.6) runs from zero to infinity, but all blocks with a fixed l are still finite-dimensional.

There are two Q -operators $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ which commute with the transfer matrix $\mathbf{T}_I(\lambda; \phi)$ and among themselves for different values of spectral parameters

$$[\mathbf{Q}_{\pm}^{(I)}(\lambda), \mathbf{T}_I(\mu; \phi)] = [\mathbf{Q}_{+}^{(I)}(\lambda), \mathbf{Q}_{-}^{(I)}(\mu)] = [\mathbf{Q}_{\pm}^{(I)}(\lambda), \mathbf{Q}_{\pm}^{(I)}(\mu)] = 0. \tag{2.8}$$

They satisfy the famous Baxter TQ -relation

$$\mathbf{T}_I(\lambda; \phi) \mathbf{Q}_{\pm}^{(I)}(\lambda) = [\lambda/\zeta]^M \mathbf{Q}_{\pm}^{(I)}(q\lambda) + [\lambda\zeta]^M \mathbf{Q}_{\pm}^{(I)}(q^{-1}\lambda), \tag{2.9}$$

where we introduced a variable ζ for later convenience

$$\zeta = q^{I/2}. \tag{2.10}$$

Let us first assume that $I \in \mathbb{Z}_+$. In the case $\phi \neq 1$ the eigenvalues of $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ are polynomials in λ and λ^{-1} up to a simple phase factor. Namely, if we define two operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$

$$\mathbf{Q}_{\pm}^{(I)}(\lambda) = e^{\pm iuhM} \mathbf{A}_{\pm}^{(I)}(\lambda) = \lambda^{\pm hM} \mathbf{A}_{\pm}^{(I)}(\lambda), \tag{2.11}$$

where

$$\lambda = e^{iu}, \quad \phi = q^h, \tag{2.12}$$

then the eigenvalues $\mathcal{A}_{\pm}^{(I)}(\lambda)$ of the operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ in the subspace W_l have the following form

$$\mathcal{A}_{+}^{(I)}(\lambda) = \rho_{+} \prod_{k=1}^l [\lambda/\lambda_k^{+}], \quad \mathcal{A}_{-}^{(I)}(\lambda) = \rho_{-} \prod_{k=1}^{IM-l} [\lambda/\lambda_k^{-}]. \tag{2.13}$$

where λ_k^{\pm} are the solutions of the Bethe Ansatz equations.

The TQ -relation for the operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ takes the form

$$\mathbf{T}_I(\lambda; \phi)\mathbf{A}_{\pm}^{(I)}(\lambda) = \phi^{\pm M}[\lambda/\zeta]^M \mathbf{A}_{\pm}^{(I)}(q\lambda) + \phi^{\mp M}[\lambda\zeta]^M \mathbf{A}_{\pm}^{(I)}(q^{-1}\lambda). \tag{2.14}$$

Let us notice that we could change a λ -dependent normalization of the operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ in such a way that their eigenvalues would become polynomials in λ^2 . Such a change of normalization will result in additional factors $q^{\pm I}$ in the RHS of (2.14).

Operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ satisfy the Wronskian relation

$$\phi^M \mathbf{A}_+^{(I)}(q\lambda)\mathbf{A}_-^{(I)}(\lambda) - \phi^{-M} \mathbf{A}_-^{(I)}(q\lambda)\mathbf{A}_+^{(I)}(\lambda) = \mathbf{Wr}(\phi)\lambda^{IM}(\lambda^{-2}q^{-I}; q^2)_I^M, \tag{2.15}$$

where we defined the q -Pochhammer symbol

$$(x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k). \tag{2.16}$$

The Wronskian $\mathbf{Wr}(\phi)$ was calculated in [1]

$$\mathbf{Wr}(\phi) = (-1)^{IM} \phi^M q^{I-IM} (1 - \phi^{2M} q^{2I-IM}) \mathbf{I}. \tag{2.17}$$

Operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ were constructed in [1] as special transfer matrices acting in the subspace W_I

$$\mathbf{A}_{\pm}^{(I)}(\lambda) = (1 - \phi^{2M} q^{2I-IM}) \times \text{Tr}_{\mathcal{F}_q} \left\{ \underbrace{A_{\pm}^{(I)}(\lambda) \otimes \dots \otimes A_{\pm}^{(I)}(\lambda)}_{M \text{ times}} \right\}, \tag{2.18}$$

where the trace is calculated over the infinite-dimensional Fock space \mathcal{F}_q , spanned by a set of vectors $|n\rangle$, $n = 0, 1, 2, \dots, \infty$. We always choose the field $\phi \in \mathbb{C}$ in (2.18) to ensure a convergence of the geometric series in (2.18) and then analytically continue to all values of ϕ .

Local L -operators acting in the tensor product $\mathcal{F}_q \otimes V_I$ have the form

$$\begin{aligned} [A_+^{(I)}(\lambda)]_{n,i}^{n',i'} &= \delta_{i+n',i'+n} \phi^{2n} (-1)^{i+i'} \lambda^{-i} q^{\frac{1}{2}i(i+1) - \frac{1}{2}i'(i'+1) + i(I+i') + n(I-i-i')} \\ &\times \frac{(q^2; q^2)_{n'}}{(q^2; q^2)_n (q^2; q^2)_i} {}_3\bar{\phi}_2 \left(\begin{matrix} q^{-2i}; q^{-2i'}, \lambda^2 q^{-I} \\ q^{-2I}, q^{2(1+n-i)} \end{matrix} \middle| q^2, q^2 \right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} [A_-^{(I)}(\lambda)]_{n,i}^{n',i'} &= \delta_{i+n,i'+n'} \phi^{2n} \lambda^{i-I} q^{-\frac{1}{2}i(i-1) + \frac{1}{2}i'(i'-1) + i(I+i') + n(I-i-i')} \\ &\times \frac{(\lambda^2 q^{-I+2(i'-n)}; q^2)_{I-i-i'}}{(q^2; q^2)_i} {}_3\bar{\phi}_2 \left(\begin{matrix} q^{-2i}; q^{-2i'}, \lambda^2 q^{-I} \\ q^{-2I}, q^{2(1+n-i')} \end{matrix} \middle| q^2, q^2 \right), \end{aligned} \tag{2.20}$$

where we defined a *regularized* terminating basic hypergeometric series ${}_{r+1}\bar{\phi}_r$ as

$${}_{r+1}\bar{\phi}_r \left(\begin{matrix} lq^{-n}; a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} \middle| q, z \right) = \sum_{k=0}^n z^k \frac{(q^{-n}; q)_k}{(q; q)_k} \prod_{s=1}^r (a_s; q)_k (b_s q^k; q)_{n-k}. \tag{2.21}$$

Such a regularization is necessary, since the parameter $q^{2(1+n-i)}$ in (2.19) may become equal to q^{-2m} , $m = 0, 1, \dots$ and the standard basic hypergeometric series is not defined at these points.

It was shown in [1] that two L -operators (2.19)–(2.20) are related by the following transformation for integer $I \in \mathbb{Z}_+$, $0 \leq i, i' \leq I$

$$[A_{-}^{(I)}(\lambda)]_{n,i}^{n',i'} = [A_{+}^{(I)}(\lambda)]_{n,I-i}^{n',I-i'} \Big|_{\phi \rightarrow \phi^{-1}}. \tag{2.22}$$

In fact, we could use (2.22) to define the second Q -operator $\mathbf{A}_{-}^{(I)}(\lambda)$ in terms of the first one, $\mathbf{A}_{+}^{(I)}(\lambda)$.

Due to the presence of δ -functions in (2.19)–(2.20) both Q -operators act invariantly in subspaces W_l similar to the transfer-matrix $\mathbf{T}_l(\lambda; \phi)$.

The choice of a normalization factor in (2.18) leads to simple asymptotics of both operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ at $\lambda \rightarrow \infty$

$$\begin{aligned} \mathbf{A}_{+}^{(I)}(\lambda) \Big|_{\lambda \rightarrow \infty} &= -(-\lambda)^l \phi^{2M} q^{l-IM} (\mathbf{I} + O(\lambda^{-2})), \\ \mathbf{A}_{-}^{(I)}(\lambda) \Big|_{\lambda \rightarrow \infty} &= (-\lambda)^{IM-l} q^{l-IM} (\mathbf{I} + O(\lambda^{-2})). \end{aligned} \tag{2.23}$$

The L -operator (2.19) can be continued to arbitrary $I \in \mathbb{C}$, since its matrix elements are polynomials in ζ^2 where ζ was defined in (2.10). This defines the Q -operator $\mathbf{A}_{+}^{(I)}(\lambda)$ for any $I \in \mathbb{C}$. The continuation of the second Q -operator to non-integer values of I is more problematic. The L -operator (2.20) has a pre-factor which contains infinitely many zeros and poles in ζ^2 for non-integer I . If we multiply (2.20) by a simple meromorphic function independent of indices

$$(\lambda^{-2} q^{-I}; q^2)_I^{-1} = \frac{(\lambda^{-2} q^I; q^2)_{\infty}}{(\lambda^{-2} q^{-I}; q^2)_{\infty}}, \tag{2.24}$$

then the matrix elements of the L -operator (2.20) contain a ratio of two q -Pochhammer symbols. Since such a ratio is no longer a polynomial in ζ^n and q^n , a calculation of the trace over the auxiliary Fock space becomes a nontrivial problem. Let us also notice that a restriction of both L -operators from generic to positive integer values of I is non-singular and the corresponding transfer-matrices act invariantly in the finite-dimensional quantum space V_I .

Both L -operators (2.19)–(2.20) simplify significantly for two special values of the spectral parameter, $\lambda = \zeta^{\pm 1}$. At $\lambda = \zeta$ (2.19) reduces to

$$\begin{aligned} [A_{+}^{(I)}(\zeta)]_{n,i}^{n',i'} &= \delta_{i+n',i'+n} (-1)^{i'} q^{ii'+\frac{1}{2}i(I+3-i)-\frac{1}{2}i'(i'+1)} \frac{(q^{-2I}; q^2)_i}{(q^2; q^2)_i} \\ &\times \frac{(q^2; q^2)_{n'}}{(q^2; q^2)_n} \phi^{-2n} q^{n(I+i-i')} (q^{-2n}; q^2)_i. \end{aligned} \tag{2.25}$$

Now we can expand the last q -Pochhammer symbol in (2.25) in series in q^{-2n} , take the tensor product of M copies of the L -operator and calculate the trace over the Fock space in (2.18). The result can be written as an M -fold sum. One can remove one summation using the following simple identity

$$\sum_{k=0}^i \frac{(q^{-2i}; q^2)_k}{(q^2; q^2)_k} \frac{q^{2ik}}{1-xq^{-2k}} = -x^{-1} \frac{(q^2; q^2)_i}{(x^{-1}; q^2)_{i+1}}, \quad i \in \mathbb{Z}_+, \quad x \in \mathbb{C}. \tag{2.26}$$

The final result for the action of the operator $\mathbf{A}_{+}^{(I)}(\zeta)$ in the quantum subspace W_l can be written in the following neat form

$$\begin{aligned}
 [\mathbf{A}_+^{(I)}(\zeta)]_{i_1, \dots, i_M}^{i'_1, \dots, i'_M} &= (-1)^{l+1} (1 - \phi^{2M} q^{2l-1M}) q^l \zeta^l \prod_{k=1}^M \frac{(\zeta^{-4}; q^2)_{i_k}}{(q^2; q^2)_{i_k}} (\phi/\zeta)^{2+2k(i'_k-i_k)} \\
 &\times \sum_{s=0}^{l-i_1} \frac{(q^2; q^2)_{i_1}}{(\phi^{2M} \zeta^{-2M} q^{2s}; q^2)_{i_1+1}} \frac{1}{s!} \frac{d^s}{dz^s} \prod_{m=2}^M (z q^{2+2 \sum_{l=1}^{m-1} (i_l - i'_l)}; q^2)_{i_m} \Big|_{z=0}.
 \end{aligned} \tag{2.27}$$

In (2.27) we introduced a dummy variable z to uncouple remaining $M - 1$ summations and converted back to the product of $M - 1$ q -Pochhammer symbols. It is quite surprising that there exist such a simple expression for the transfer-matrix (2.18) at the particular value of $\lambda = \zeta$. We also notice that a derivation of the result (2.27) remains valid for any $I \in \mathbb{C}$. In Section 6 we will show that this formula allows us to explicitly compute matrix elements of the transfer-matrix (2.18) for arbitrary values of λ .

At the point $\lambda = \zeta^{-1}$ the hypergeometric function ${}_3\bar{\phi}_2$ reduces to ${}_2\bar{\phi}_1$ of the argument q^2 and can be calculated using the q -Vandermonde sum. As a result we obtain the expression for the L -operator (2.19) similar to (2.25) and again can calculate the transfer-matrix in a closed form. We shall not use this expression and leave it as an exercise for the reader.

In the case $I \in \mathbb{Z}_+$ the second transfer-matrix $\mathbf{A}_-^{(I)}(\zeta)$ can be obtained from (2.27) using a symmetry (2.22). The difficult case $I \in \mathbb{C}$ will be discussed elsewhere.

3. Polynomial representation

There is another representation of the quantum algebra $U_q(sl(2))$ on the space of polynomials which we also use in this paper.

Let us introduce the polynomial ring $K[x]$ in variable x over the field \mathbb{C} and its multi-variable generalization $K_M[X]$, $X \equiv \{x_1, \dots, x_M\}$ and identify basis vectors in W with monomials in $K_M[X]$

$$v_{i_1, \dots, i_M} \equiv v_{i_1} \otimes \dots \otimes v_{i_M} \sim x_1^{i_1} \dots x_M^{i_M}. \tag{3.1}$$

The ring $K_M[X]$ has a gradation

$$K_M[X] = \bigoplus_{l \in \mathbb{Z}_+} K_M^{(l)}[X], \tag{3.2}$$

where $K_M^{(l)}[X]$ is generated by monomials in M variables of the total degree l . There is an obvious isomorphism between W_l and $K_M^{(l)}[X]$.

In this paper we consider only periodic boundary conditions and always imply periodicity $M + 1 \equiv 1$ and all indices run from 1 to $M \pmod{M}$, i.e. $x_0 = x_M$, etc.

Let us introduce a set of operators \mathcal{X}_i and \mathcal{D}_i , $i = 1, \dots, M$ acting in $K_M[X]$ as

$$\begin{aligned}
 \mathcal{X}_i p(x_1, \dots, x_M) &= x_i p(x_1, \dots, x_M), \\
 \mathcal{D}_i p(x_1, \dots, x_M) &= p(x_1, \dots, qx_i, \dots, x_M).
 \end{aligned} \tag{3.3}$$

The $U_q(sl(2))$ generators H, E, F in $K[x]$ can be realized in terms of one pair of operators \mathcal{X}, \mathcal{D}

$$q^H = \zeta^2 \mathcal{D}^{-2}, \quad E = \mathcal{X}^{-1} \frac{[\mathcal{D}]}{[q]}, \quad F = \mathcal{X} \frac{[\zeta^2 \mathcal{D}^{-1}]}{[q]}, \tag{3.4}$$

where ζ was defined in (2.10).

The L -operator takes the form

$$L(\lambda; \phi) = \begin{pmatrix} \phi^{-1}[\lambda z \mathcal{D}^{-1}] & \phi^{-1} \mathcal{X}[z^2 \mathcal{D}^{-1}] \\ \phi \mathcal{X}^{-1}[\mathcal{D}] & \phi[\lambda z^{-1} \mathcal{D}] \end{pmatrix}. \tag{3.5}$$

Again we can define the transfer-matrix with periodic boundary conditions

$$\mathbf{T}_I(\lambda; \phi) = \text{Tr}[L_1(\lambda; \phi) \otimes \dots \otimes L_M(\lambda; \phi)] \tag{3.6}$$

which is a q -difference operator in M variables acting on polynomials $p(x_1, \dots, x_M) \in K_M[X]$.

The TQ -relation becomes the operator relation acting in $K_M[X]$

$$\mathbf{T}_I(\lambda; \phi) \mathbf{A}_{\pm}(\lambda) = \phi^{\pm M} [\lambda/\zeta]^M \mathbf{A}_{\pm}(q\lambda) + \phi^{\mp M} [\lambda\zeta]^M \mathbf{A}_{\pm}(q^{-1}\lambda). \tag{3.7}$$

Although the transfer-matrix (3.6) is a finite order q -difference operator, Q -operators will be, in general, integral operators. Our goal is to construct Q -operators as special integral operators with factorized kernels acting in $K_M[X]$.

4. Integral Q -operator

In this section we construct the XXZ Q -operator as an integral operator with a factorized kernel. First we consider the case $I \in \mathbb{C}$ and discuss a restriction to the finite-dimensional case later.

We will denote the Q -operator in this section as $\mathbf{Q}_f(\lambda)$. It has polynomial eigenfunctions and should not be confused with the Q -operators $\mathbf{Q}_{\pm}(\lambda)$ from Section 2.

First we assume that the Q -operator can be represented as an integral operator with the kernel $Q_{\lambda}(\mathbf{x}|\mathbf{y})$

$$[\mathbf{Q}_f(\lambda)p](\mathbf{x}) = \oint_{\mathcal{C}} \frac{dy_1}{y_1} \dots \frac{dy_M}{y_M} Q_{\lambda}(\mathbf{x}|\mathbf{y}) p(\mathbf{y}), \tag{4.1}$$

where $\mathbf{x} \equiv \{x_1, \dots, x_M\}$, $\mathbf{y} \equiv \{y_1, \dots, y_M\}$ and the integration contour \mathcal{C} is a proper deformation of the unit circle.

In general, one can expect that the kernel $Q_{\lambda}(\mathbf{x}|\mathbf{y})$ will have a non-local structure, i.e. involve functions depending simultaneously on all variables \mathbf{x} and \mathbf{y} . Our goal is to construct a factorized kernel $Q_{\lambda}(\mathbf{x}|\mathbf{y})$ which can be represented as a product of factors depending on pairs of local variables.

It is well known that this can be achieved by using Baxter’s method of “pair-propagation through a vertex”. As a result the TQ -relation reduces to a set of difference equations in variables x_i for the kernel $Q_{\lambda}(\mathbf{x}|\mathbf{y})$. However, in this approach we cannot fix a dependence of the kernel on the integration variables y_i . This difficulty can be resolved by applying Baxter’s method to a “conjugated” L -operator. Such approach will allow us first to determine a dependence of the kernel on y -variables in (4.1). Then a dependence on x -variables can be fixed by a proper normalization of the Q -operator. In original Baxter’s notations it is equivalent to a construction of the operator $Q_L(\lambda)$.

Let us consider a finite-order q -difference operator M and define its conjugate M^* by

$$\oint_{\mathcal{C}} \frac{dy}{y} f(y)[Mg](y) = \oint_{\mathcal{C}} \frac{dy}{y} [M^*f](y)g(y), \tag{4.2}$$

where functions $f(x)$ and $g(x)$ are properly defined to ensure a convergence of the integral. Here we also assume that the integration contour is invariant under the change of variables $y \rightarrow q^{\pm 1}y$.

In particular, we have

$$\mathcal{X}^* = \mathcal{X}, \quad \mathcal{D}^* = \mathcal{D}^{-1}. \tag{4.3}$$

Using (4.3) one can construct the L -operator conjugated to (3.5) in the sense (4.2)

$$L^*(\lambda; \phi) = \begin{pmatrix} \phi^{-1}[\lambda z \mathcal{D}] & \phi^{-1} \mathcal{X}[qz^2 \mathcal{D}] \\ \phi \mathcal{X}^{-1}[q \mathcal{D}^{-1}] & \phi[\lambda z^{-1} \mathcal{D}^{-1}] \end{pmatrix} \tag{4.4}$$

and define the associated transfer-matrix

$$\mathbf{T}_J^*(\lambda; \phi) = \text{Tr}[L_1^*(\lambda; \phi) \otimes \dots \otimes L_M^*(\lambda; \phi)] \tag{4.5}$$

Now let us introduce a set of local transformations U_i in the auxiliary space \mathbb{C}^2

$$\tilde{L}_i^*(\lambda) = U_{i-1} L_i^*(\lambda) U_i^{-1}, \quad U_i = \begin{pmatrix} 1 & \alpha_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, \dots, M, \tag{4.6}$$

where we defined a set of arbitrary complex parameters $\alpha_i, i = 1, \dots, M$. Obviously the transfer matrix (4.5) is not affected by this transformation of the L -operators.

According to Baxter’s method we now construct a set of functions $f(y_i)$ satisfying

$$[\tilde{L}_i^*(\lambda) f](y_i) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}. \tag{4.7}$$

From (4.7) it is easy to obtain a recurrence relation for the function $f(y_i)$:

$$\frac{f(q^2 y_i)}{f(y_i)} = q^2 \frac{(1 - \frac{y_i}{\lambda \zeta \alpha_{i-1} \phi^2})(1 - \frac{\lambda y_i}{\zeta \alpha_i})}{(1 - \frac{q^2 \lambda \zeta y_i}{\alpha_{i-1} \phi^2})(1 - \frac{q^2 \zeta y_i}{\lambda \alpha_i})}. \tag{4.8}$$

Let us notice that the RHS of (4.8) factorizes into the product of four factors even in the case of arbitrary horizontal field ϕ . As a result the horizontal field can be nontrivially introduced into the periodic “saw”-like structure of the kernel of the Q -operator. To our knowledge this was not known before.

There are infinitely many solutions to the difference equation (4.8). They all differ by functions which are periodic in q^2 , i.e. for any two solutions $f_1(y), f_2(y)$ of (4.8) we have

$$f_1(y) = g(y) f_2(y), \quad g(q^2 y) = g(y). \tag{4.9}$$

The choice of the function $g(y)$ is determined by the condition that Q -operators should commute at different values of spectral parameters as *integral* operators. This can be also reformulated as two statements which are more appropriate in our context. First, we show that the Q -operator maps polynomials to polynomials, i.e. $K_M[X]$ to $K_M[X]$. Second, we require that Q -operators commute on polynomials. Since polynomial functions are dense in the space of continuous complex-valued functions on a finite interval, it would imply a commutativity of Q -operators as integral operators.

In this paper we are going to introduce two types of integral operators. The first one is of the type (4.1) and based on Askey and Roy extension of the beta integral [34] (see also (4.11.2) in [37])

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(\frac{\rho y}{d}, \frac{q d}{\rho y}, \frac{\rho c}{y}, \frac{q y}{\rho c}; q)_{\infty}}{(a y, b y, \frac{c}{y}, \frac{d}{y}; q)_{\infty}} \frac{d y}{y} = \frac{(abcd, \rho, \frac{q}{\rho}, \frac{\rho c}{d}, \frac{q d}{\rho c}; q)_{\infty}}{(ac, ad, bc, bd, q; q)_{\infty}}, \tag{4.10}$$

where $ac, ad, bc, bd \neq q^{-n}$, $n = 0, 1, 2, \dots$, $\rho c d \neq 0$ and the contour \mathcal{C} is a deformation of the unit circle such that the zeros of $(a y, b y; q)_{\infty}$ lie outside the contour and the origin and zeros of $(c/y, d/y; q)_{\infty}$ lie inside the contour.

Motivated by (4.10) we choose a solution of (4.8) as

$$f(y_i) = \mu(\alpha_{i-1}, \alpha_i) \frac{1}{2\pi i} \frac{(\frac{\lambda \zeta \rho_i y_i}{\alpha_{i-1} \phi^2}, \frac{q^2 \alpha_{i-1} \phi^2}{\lambda \zeta \rho_i y_i}, \frac{\lambda \rho_i \alpha_i}{\zeta y_i}, \frac{q^2 \zeta y_i}{\lambda \rho_i \alpha_i}; q^2)_{\infty}}{(\frac{y_i}{\lambda \zeta \alpha_{i-1} \phi^2}, \frac{\alpha_{i-1} \phi^2}{\lambda \zeta y_i}, \frac{\lambda y_i}{\zeta \alpha_i}, \frac{\lambda \alpha_i}{\zeta y_i}; q^2)_{\infty}}, \tag{4.11}$$

where $\mu(\alpha_{i-1}, \alpha_i)$ are some normalization factors and ρ_i are the parameters at our disposal. It is easy to check that (4.11) satisfies (4.8).

Now we are ready to define our Q -operator. The product of M functions (4.11) contains M arbitrary parameters α_i and we identify them with x_i in (4.1). Then we fix the factors $\mu(x_{i-1}, x_i)$ in such a way that the Q -operator has the following normalization

$$\mathbf{Q}_f(\lambda) \cdot \mathbf{1} = \mathbf{1}. \tag{4.12}$$

Choosing $\rho_i = x_{i-1} \lambda^{-1} \phi^2$ we obtain the action of the Q -operator

$$\begin{aligned} [\mathbf{Q}_f(\lambda)g](x_1, \dots, x_M) &= \frac{1}{(2\pi i)^M} \prod_{k=1}^M \frac{(\frac{\lambda^2}{\zeta^2}, \frac{1}{\lambda^2 \zeta^2}, \frac{x_k}{\zeta^2 x_{k-1} \phi^2}, \frac{x_{k-1} \phi^2}{\zeta^2 x_k}, q^2; q^2)_{\infty}}{(\zeta^{-4}, \frac{x_{k-1} \phi^2}{\lambda}, \frac{q^2 \lambda}{x_{k-1} \phi^2}, \lambda x_k, \frac{q^2}{\lambda x_k}; q^2)_{\infty}} \\ &\times \oint_{\mathcal{C}_1} \frac{d y_1}{y_1} \dots \oint_{\mathcal{C}_M} \frac{d y_M}{y_M} g(y_1, \dots, y_M) \\ &\times \prod_{k=1}^M \frac{(\zeta y_k, \frac{q^2}{\zeta y_k}, \frac{x_{k-1} x_k \phi^2}{\zeta y_k}, \frac{q^2 \zeta y_k}{x_{k-1} x_k \phi^2}; q^2)_{\infty}}{(\frac{y_k}{\lambda \zeta x_{k-1} \phi^2}, \frac{x_{k-1} \phi^2}{\lambda \zeta y_k}, \frac{\lambda y_k}{\zeta x_k}, \frac{\lambda x_k}{\zeta y_k}; q^2)_{\infty}}, \end{aligned} \tag{4.13}$$

where contours $\mathcal{C}_1, \dots, \mathcal{C}_M$ are defined similar to the contour in (4.10). Using the Askey–Roy integral (4.10) it is easy to check that the Q -operator defined by (4.13) satisfies the normalization condition (4.12).

Having the explicit form of the kernel $Q_{\lambda}(\mathbf{x}|\mathbf{y})$ we can calculate the diagonal matrix elements in (4.7). Using standard arguments we come to the “left” TQ -relation acting in $K_M[X]$

$$\mathbf{Q}_f(\lambda) \mathbf{T}_f(\lambda; \phi) = \phi^M [\lambda/\zeta]^M \mathbf{Q}_f(q\lambda) + \phi^{-M} [\lambda\zeta]^M \mathbf{Q}_f(q^{-1}\lambda). \tag{4.14}$$

Let us notice that if we only use the relation (4.8), then the diagonal matrix elements in (4.7) will contain “wrong” factors $[\lambda/q\zeta]$ and $[q\lambda\zeta]$ instead of the “right” ones $[\lambda/\zeta]$ and $[\lambda\zeta]$. However, the kernel of the Q -operator in (4.13) contains a pre-factor which we obtained from the normalization condition on the Q -operator. This pre-factor will generate the missing factors $[\lambda/\zeta]$ and $[\lambda\zeta]$ and cancel the “wrong” ones.

The next step is to show that the integral operator defined by (4.13) maps polynomials in M variables into polynomials in M variables. This is achieved by choosing a proper basis. Let us fix real parameters x_i , $i = 1, \dots, M$ and consider a family of polynomials in y_i , $i = 1, \dots, M$

$$P_{\{n\},\{m\}}(y_1, \dots, y_M) = \prod_{i=1}^M p_{n_i, m_i}(y_i), \tag{4.15}$$

where

$$p_{n_i, m_i}(y_i) = \left(\frac{y_i}{\lambda \zeta x_{i-1} \phi^2}; q^2 \right)_{n_i} \left(\frac{\lambda y_i}{\zeta x_i}; q^2 \right)_{m_i} \tag{4.16}$$

and $\{n\} \equiv \{n_1, \dots, n_M\}$, $\{m\} \equiv \{m_1, \dots, m_M\}$ are two arbitrary sequences of positive integers. Obviously, polynomials (4.15) form a basis in $K_M[Y]$. In fact, it is sufficient to choose all $n_i = 0$ and consider only nonzero m_i 's.

Using a simple formula

$$(x; q^2)_n = \frac{(x; q^2)_\infty}{(xq^{2n}; q^2)_\infty} \tag{4.17}$$

and (4.10) one can easily check that the Q -operator (4.13) maps polynomials $p_{n_i, m_i}(y_i)$ into polynomials in x_{i-1}/x_i and x_i/x_{i-1}

$$Q_f(\lambda) : p_{n_i, m_i}(y_i) \rightarrow \frac{\left(\frac{x_i}{\zeta^2 x_{i-1} \phi^2}, \frac{1}{\lambda^2 \zeta^2}; q^2 \right)_{n_i} \left(\frac{x_{i-1} \phi^2}{\zeta^2 x_i}, \frac{\lambda^2}{\zeta^2}; q^2 \right)_{m_i}}{(\zeta^{-4}; q^2)_{n_i + m_i}}. \tag{4.18}$$

Since the kernel of the Q -operator is factorized, it maps a product of polynomials $p_{n_i, m_i}(y_i)$ for different i into the product of RHS's in (4.18). This completely describes the action of the Q -operator (4.13) on the basis of polynomials in y_i , $i = 1, \dots, M$.

To make this action explicit (the LHS of (4.18) involves a dependence on the variables x_i 's) let us use the following simple formula

$$x^m = \sum_{k=0}^m \frac{(q^{-m}; q)_k}{(q; q)_k} q^k (x; q)_k. \tag{4.19}$$

It allows us to expand the powers x^m in terms of q -binomials. Combining it with the action (4.18) and setting $n_i = 0$ we obtain

$$Q_f(\lambda) : y_i^m \rightarrow \left(\frac{\zeta x_i}{\lambda} \right)^m \sum_{k=0}^m \frac{(q^{-2m}, \frac{\lambda^2}{\zeta^2}, \frac{x_{i-1} \phi^2}{x_i \zeta^2}; q^2)_k}{(q^2, \zeta^{-4}; q^2)_k} q^{2k} \tag{4.20}$$

We can simplify it even further by expanding the last q -Pochhammer symbol in the numerator in series and performing a summation in k . The result reads

$$Q_f(\lambda) : y_i^m \rightarrow \left(\frac{\lambda}{\zeta} \right)^m \frac{(q^2; q^2)_m}{(\zeta^{-4}; q^2)_m} \sum_{k=0}^m x_{i-1}^k x_i^{m-k} \left(\frac{\phi^2}{\lambda^2} \right)^k \times \frac{(\lambda^2 \zeta^{-2}; q^2)_k (\lambda^{-2} \zeta^{-2}; q^2)_{m-k}}{(q^2; q^2)_k (q^2; q^2)_{m-k}}. \tag{4.21}$$

This formula clearly demonstrates that the integral operator defined by the action (4.13) maps polynomials in M variables into polynomials in M variables. Moreover, the total degree of the polynomials is conserved, so it acts invariantly in the subspace $K_M^{(l)}[X]$ of homogeneous polynomials of the degree l .

Let us notice that if we set $\lambda = \zeta$, then the action (4.21) is trivial and the Q -operator $\mathbf{Q}_f(\lambda)$ simply becomes the identity operator

$$\mathbf{Q}_f(\lambda) = \mathbf{I}. \tag{4.22}$$

Multiplying (4.21) by

$$\zeta^{2m} \phi^{-m} \mu_i^m \frac{(\zeta^{-4}; q^2)_m}{(q^2; q^2)_m} \tag{4.23}$$

and summing over m from 0 to $+\infty$ we derive the action of the Q -operator on a generating function of polynomials in y_i . Taking the product over $i = 1, \dots, M$ we obtain

$$\mathbf{Q}_f(\lambda) \cdot \prod_{i=1}^M \frac{(\zeta^{-2} \phi^{-1} \mu_i x_i; q^2)_\infty}{(\zeta^2 \phi^{-1} \mu_i x_i; q^2)_\infty} = \prod_{i=1}^M \frac{(\lambda \zeta^{-1} \phi \mu_i x_{i-1}; q^2)_\infty}{(\zeta \lambda^{-1} \phi \mu_i x_{i-1}; q^2)_\infty} \frac{((\lambda \zeta \phi)^{-1} \mu_i x_i; q^2)_\infty}{(\lambda \zeta \phi^{-1} \mu_i x_i; q^2)_\infty}, \tag{4.24}$$

where μ_i are arbitrary complex parameters and we imply a periodicity in M , i.e. $x_0 \equiv x_M$. This formula describes the action of the Q -operator on arbitrary polynomials in M variables and coincides with the formula (3.25) of [26] at $\phi = 1$. However, the method we used is completely different from the method of [26]. Our Q -operator is a well defined integral operator and the Q -operator constructed in [26] is the difference operator of infinite order.

One can construct another integral operator which has the same polynomial action (4.21). The kernel of this integral operator comes from the q -analogue of Barnes’ first lemma discovered by Watson [35]

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(q^{1-c+s}, q^{1-d+s}; q)_\infty}{(q^{a+s}, q^{b+s}; q)_\infty} \frac{\pi q^s ds}{\sin \pi(c-s) \sin \pi(d-s)} \\ &= \frac{q^c}{\sin \pi(c-d)} \frac{(q, q^{1+c-d}, q^{d-c}, q^{a+b+c+d}; q)_\infty}{(q^{a+c}, q^{a+d}, q^{b+c}, q^{b+d}; q)_\infty}, \end{aligned} \tag{4.25}$$

where the contour of integration runs from $-i\infty$ to $i\infty$ and separates increasing and decreasing sequences of poles. To use (4.25) for the construction of the Q -operator we need to make a change of variables. Let us define two new sets of variables t_i, s_i as

$$x_i = q^{2t_i}, \quad y_i = q^{2s_i}, \quad i = 1, \dots, M. \tag{4.26}$$

Then we can write another solution of (4.8)

$$f(y_i) = g(\lambda, y_i) y_i \frac{(q^{2\lambda y_i \zeta}, q^{2y_i \zeta}; q^2)_\infty}{(x_{i-1} \phi^2, \frac{q^2 y_i \zeta}{\lambda x_i}; q^2)_\infty} \frac{(y_i, \frac{\lambda y_i}{\lambda \zeta x_{i-1} \phi^2}, \frac{\lambda y_i}{x_i \zeta}; q^2)_\infty}{(x_i \zeta, \frac{\lambda y_i}{x_i \zeta}; q^2)_\infty} \tag{4.27}$$

where the function $g(\lambda, y)$ satisfies

$$g(q\lambda, qy) = g(\lambda, q^2 y) = g(q^2 \lambda, y) = g(\lambda, y). \tag{4.28}$$

It is easy to see that such function does not affect the TQ -relation because it is periodic in q^2 and shifts in λ and y always enter in pairs. Barnes’ first lemma (4.25) suggests to choose the function $g(\lambda, y)$ as

$$g(\lambda, y_i) = \frac{\pi q^{2s_i}}{\sin \pi(t_i - s_i + v - I/4) \sin \pi(t_{i-1} - s_i + h - v - I/4)}, \tag{4.29}$$

where

$$\lambda = q^{2v}, \quad \phi = q^h. \tag{4.30}$$

It is easy to check that (4.29) satisfies (4.28).

Using (4.25) we can again normalize the Q -operator such that it satisfies (4.12). Then we come to a different integral representation for the Q -operator

$$\begin{aligned} & [\mathbf{Q}_f(\lambda)g](x_1, \dots, x_M) \\ &= \prod_{i=1}^M \frac{(\lambda^{-2}\zeta^{-2}, \lambda^2\zeta^{-2}, x_i/\zeta^2 x_{i-1}\phi^2, x_{i-1}\phi^2/x_i\zeta^2; q^2)_\infty}{(q^2, \zeta^{-4}, q^2\lambda^2 x_i/x_{i-1}\phi^2, x_{i-1}\phi^2/\lambda^2 x_i; q^2)_\infty} \\ &\times \prod_{i=1}^M \frac{\zeta \sin \pi(2v + t_i - t_{i-1} - h)}{2\pi i \lambda x_i} \int_{-i\infty}^{i\infty} ds_1 \dots \int_{-i\infty}^{i\infty} ds_M \prod_{i=1}^M \frac{(q^{2+2s_i}\lambda\zeta, q^{2+2s_i}\zeta; q^2)_\infty}{(\frac{q^{2s_i}}{\lambda\zeta x_{i-1}\phi^2}, \frac{\lambda q^{2s_i}}{\zeta x_i}; q^2)_\infty} \\ &\times \prod_{i=1}^M \frac{\pi q^{2s_i}}{\sin \pi(t_i - s_i + v - I/4) \sin \pi(t_{i-1} - s_i + h - v - I/4)} g(q^{2s_1}, \dots, q^{2s_n}). \end{aligned} \tag{4.31}$$

Repeating the same arguments one can easily show that the integral operator (4.31) has the same polynomial action (4.21) as the Q -operator given by (4.13). A representation by the Askey–Roy integral does not require a change of variables (4.26, 4.30) and probably is more convenient.

We also notice that both integrals appeared in the study of orthogonality relations for q -Hahn polynomials in [38]. It is known that Hahn polynomials appear in the XXX model as the eigenvalues of the Q -operator at $M = 2$ [39,40,22]. A better understanding of this connection with the approach of [38] deserves a further study.

5. TQ -relation and commutativity of the Q -operators

In the previous section we constructed the Q -operator on the space of polynomials and proved that it solves the “left” TQ -relation

$$\mathbf{Q}_f(\lambda)\mathbf{T}_I(\lambda; \phi) = \phi^M[\lambda/\zeta]^M \mathbf{Q}_f(q\lambda) + \phi^{-M}[\lambda\zeta]^M \mathbf{Q}_f(q^{-1}\lambda) \tag{5.1}$$

on the space of polynomials $K_M[X]$.

The next step is to prove a commutativity of the Q -operator with the transfer-matrix. It immediately follows from a commutativity of Q -operators at different values of the spectral parameters

$$[\mathbf{Q}_f(\lambda), \mathbf{Q}_f(\mu)] = 0. \tag{5.2}$$

So let us prove the relation

$$\mathbf{Q}_f(\mu)\mathbf{T}_I(\lambda; \phi) = \mathbf{T}_I(\lambda; \phi)\mathbf{Q}_f(\mu) \tag{5.3}$$

as a consequence of (5.1)–(5.2). Multiplying both parts of (5.3) by $\mathbf{Q}_f(\lambda)$ from the left and using a commutativity (5.2) we get

$$\mathbf{Q}_f(\mu)\mathbf{Q}_f(\lambda)\mathbf{T}_I(\lambda; \phi) = \mathbf{Q}_f(\lambda)\mathbf{T}_I(\lambda; \phi)\mathbf{Q}_f(\mu). \tag{5.4}$$

Substituting the TQ -relation (5.1) into (5.4) we obtain

$$\begin{aligned} &\mathbf{Q}_f(\mu)(\phi^M[\lambda/\zeta]^M\mathbf{Q}_f(q\lambda) + \phi^{-M}[\lambda\zeta]^M\mathbf{Q}_f(q^{-1}\lambda)) \\ &= (\phi^M[\lambda/\zeta]^M\mathbf{Q}_f(q\lambda) + \phi^{-M}[\lambda\zeta]^M\mathbf{Q}_f(q^{-1}\lambda))\mathbf{Q}_f(\mu) \end{aligned} \tag{5.5}$$

and this again follows from (5.2).

Now let us prove the relation (5.2). As mentioned before it is sufficient to prove the commutativity of the Q -operators on polynomials. So let calculate the action of the product of two Q -operators on a monomial x_{i+1}^m

$$\mathbf{Q}_f(\lambda)\mathbf{Q}_f(\mu) \cdot x_{i+1}^m. \tag{5.6}$$

Using the action (4.21) we can represent the result in the following form

$$\mathbf{Q}_f(\lambda)\mathbf{Q}_f(\mu) \cdot x_{i+1}^m = \sum_{0 \leq n+k \leq m} c(\lambda, \mu)_{m,n,k} x_{i-1}^n x_i^{m-n-k} x_{i+1}^k. \tag{5.7}$$

After straightforward calculations one can derive the following explicit expression for $c(\lambda, \mu)_{m,n,k}$

$$\begin{aligned} c(\lambda, \mu)_{m,n,k} &= \phi^{2(m+n-k)} \lambda^{2k-m} \mu^{m-2n} \\ &\times \frac{(\lambda^{-2}\zeta^{-2}; q^2)_k (\mu^{-2}\zeta^{-2}; q^2)_{m-n} (\lambda^2/\zeta^2; q^2)_{m-n-k}}{(\zeta^{-4}; q^2)_m (\zeta^{-4}; q^2)_{m-n} (q^2, \zeta^{-4}; q^2)_n} \\ &\times \frac{(q^2; q^2)_m (\lambda^2/\zeta^2, \mu^2/\zeta^2; q^2)_n}{(q^2; q^2)_k (q^2; q^2)_{m-n-k}} \\ &\times 4\phi_3 \left(q^{-2(m-k-n)}, q^{2+2n-2m}\zeta^4, \frac{1}{\lambda^2\zeta^2}, \frac{q^{2n}\mu^2}{\zeta^2} \middle| q^2, q^2 \right), \end{aligned} \tag{5.8}$$

where

$${}_{r+1}\phi_r \left(q^{-n}; a_1, \dots, a_r \middle| b_1, \dots, b_r \middle| q, z \right) = \sum_{k=0}^n z^k \frac{(q^{-n}; q)_k}{(q; q)_k} \prod_{s=1}^r \frac{(a_s; q)_k}{(b_s; q)_k} \tag{5.9}$$

is the standard terminating basic hypergeometric series. The series in (5.8) also satisfies the balancing condition $q^{1-n}a_1a_2a_3 = b_1b_2b_3$ and $z = q$ (with q replaced by q^2).

The commutativity of the Q -operators (5.2) is equivalent to the symmetry of coefficients $c(\lambda, \mu)_{m,n,k}$

$$c(\lambda, \mu)_{m,n,k} = c(\mu, \lambda)_{m,n,k}. \tag{5.10}$$

This symmetry immediately follows from Sears' transformation (III.15) of terminating balanced series ${}_4\phi_3$ in [37]. Extending (5.7) to the action on $x_1^{k_1} \dots x_M^{k_M}$ we prove (5.2) on the space of polynomials $K_M[X]$.

6. Connection between Q -operators

In Sections 2 and 4 we presented two different approaches for construction of the Q -operators in the XXZ spin chain with spin $I/2$ and arbitrary horizontal field. The approach of Section 4 gives much simpler formulas for the action of one Q -operator (4.21). It is easy to see that this operator maps beyond the finite-dimensional space V_I and the action (4.21) becomes singular for $m > I$, $I \in \mathbb{Z}_+$ due to a presence of the factor $(q^{-2I}; q^2)_m$ in the denominator of the RHS in (4.21). The Q -operators constructed in Section 2 are free from this difficulty.

One can carefully expand near the singular point $\zeta = q^{I/2} + \epsilon$, $\epsilon \rightarrow 0$ and extract the invariant block corresponding to the finite-dimensional subspace V_I . This procedure is technically challenging and has been completed in [24] for the case of the XXX chain. So we first consider a generic case $I \in \mathbb{C}$.

Let us remind that the Q -operators $\mathbf{A}_+^{(I)}(\lambda)$ and $\mathbf{Q}_f(\lambda)$ from Sections 2 and 4 solve the same TQ -relation

$$\mathbf{T}_I(\lambda; \phi)\mathbf{Q}(\lambda) = \phi^M [\lambda/\zeta]^M \mathbf{Q}(q\lambda) + \phi^{-M} [\lambda\zeta]^M \mathbf{Q}(q^{-1}\lambda). \tag{6.1}$$

This TQ -relation was obtained from (2.9) by removing the phase factor corresponding to different periodicity conditions with respect to the spectral parameter for two solutions of (2.9). It implies that two Q -operators $\mathbf{A}_\pm^{(I)}(\lambda)$ cannot mix (at $\phi \neq 1$) and specifying periodicity conditions of the solution we fix it up to a constant matrix multiplier. Therefore, we must have the relation

$$\mathbf{A}_+^{(I)}(\lambda) = \mathbf{A}_0 \mathbf{Q}_f(\lambda), \tag{6.2}$$

where \mathbf{A}_0 is some matrix independent of the spectral parameter.

Now let us remind that at $\lambda = \zeta$, the Q -operator $\mathbf{Q}_f(\lambda)$ becomes the identity operator

$$\mathbf{Q}_f(\lambda) = \mathbf{I}. \tag{6.3}$$

It immediately follows from (6.2) that

$$\mathbf{A}_0 = \mathbf{A}_+^{(I)}(\zeta), \tag{6.4}$$

but we already explicitly calculated $\mathbf{A}_+^{(I)}(\zeta)$ in (2.27). Hence, we get the following general expression for the Q -operator $\mathbf{A}_+^{(I)}(\lambda)$

$$\mathbf{A}_+^{(I)}(\lambda) = \mathbf{A}_+^{(I)}(\zeta) \mathbf{Q}_f(\lambda) \tag{6.5}$$

at arbitrary values of λ . Since $\mathbf{A}_+^{(I)}(\zeta)$ commutes with $\mathbf{A}_+^{(I)}(\lambda)$, the order of the matrix multiplication in the RHS of (6.5) is irrelevant.

Let us derive the explicit form of the matrix elements of the operator $\mathbf{Q}_f(\lambda)$ in $K_M[X]$. We obtain from (4.21)

$$\begin{aligned} & \mathbf{Q}_f(\lambda) \cdot x_1^{j_1} \dots x_M^{j_M} \\ &= \prod_{k=1}^M \left(\frac{\lambda}{\zeta}\right)^{j_k} \frac{(q^2; q^2)_{j_k}}{(\zeta^{-4}; q^2)_{j_k}} \\ & \times \sum_{l_1=0}^{j_1} \dots \sum_{l_M=0}^{j_M} \prod_{k=1}^M \left(\frac{\phi^2}{\lambda^2}\right)^{l_k} \frac{(\lambda^2 \zeta^{-2}; q^2)_{l_k} (\lambda^{-2} \zeta^{-2}; q^2)_{j_k - l_k}}{(q^2; q^2)_{l_k} (q^2; q^2)_{j_k - l_k}} x_k^{j_k - l_k + l_{k+1}} \end{aligned} \tag{6.6}$$

Therefore, for the matrix elements of $\mathbf{Q}_f(\lambda)$ we get

$$\mathbf{Q}_f(\lambda)_{i_1, \dots, i_M}^{j_1, \dots, j_M} = \prod_{k=1}^M \left(\frac{\lambda}{\zeta}\right)^{j_k} \frac{(q^2; q^2)_{j_k}}{(\zeta^{-4}; q^2)_{j_k}} \times \sum_{l_1=0}^{j_1} \prod_{k=1}^M \left(\frac{\phi^2}{\lambda^2}\right)^{l_1+\delta_k} \frac{(\lambda^2 \zeta^{-2}; q^2)_{l_1+\delta_k} (\lambda^{-2} \zeta^{-2}; q^2)_{j_k-l_1-\delta_k}}{(q^2; q^2)_{l_1+\delta_k} (q^2; q^2)_{j_k-l_1-\delta_k}}, \tag{6.7}$$

where

$$\delta_k = \sum_{s=1}^{k-1} (i_s - j_s). \tag{6.8}$$

Comparing (6.7) with (2.27) we see that the divergent factors $(\zeta^{-4}; q^2)_{j_k}$ cancel in the matrix product in the RHS of (6.5) and we get the answer for $\mathbf{A}_+^{(I)}(\lambda)$ which is suitable for a reduction to the finite-dimensional case $I \in \mathbb{Z}_+$.

Let us emphasize that we derived a general expression for the Q -operator $\mathbf{A}_+^{(I)}(\lambda)$ with periodic boundary conditions based on a highly nontrivial L -operator (2.19) which contain the hypergeometric function ${}_3\bar{\phi}_2$. To derive the matrix elements of the Q -operator $\mathbf{A}_+^{(I)}(\zeta)$ we need to multiply M copies of such L -operators and to calculate the trace over the Fock space which seems to be a hopeless problem.

Nevertheless, we now have the explicit expression for $\mathbf{A}_+^{(I)}(\zeta)$ in terms of the product of two much simpler matrices $\mathbf{A}_+^{(I)}(\zeta)$ and $\mathbf{Q}_f(\lambda)$ which are both finite-dimensional in any subspace W_l . The second Q -operator $\mathbf{A}_-^{(I)}(\lambda)$ for $I \in \mathbb{Z}_+$ can be easily constructed from $\mathbf{A}_+^{(I)}(\lambda)$ using the symmetry (2.22).

It is also interesting to look at the relation (6.5) in the limit $\lambda \rightarrow \infty$ where $\mathbf{A}_+^{(I)}(\lambda)$ is proportional to the identity operator (see (2.23)). In the subspace W_l we obtain

$$\mathbf{Q}_f(\lambda)|_{\lambda \rightarrow \infty} = \lambda^l \mathbf{Q}_\infty (1 + O(\lambda^{-2})), \tag{6.9}$$

where

$$[\mathbf{Q}_\infty]_{i_1, \dots, i_M}^{j_1, \dots, j_M} = \zeta^{-l} \prod_{k=1}^M \frac{(q^2; q^2)_{j_k}}{(\zeta^{-4}; q^2)_{j_k}} \sum_{l_1=0}^{i_1} \prod_{k=1}^M \frac{(q^{-2j_k}; q^2)_{l_1+\delta_k}}{(q^2; q^2)_{l_1+\delta_k}} (q^{j_k} \phi \zeta^{-1})^{2(l_1+\delta_k)}. \tag{6.10}$$

Substituting this into (6.5) at $\lambda \rightarrow \infty$ and using (2.23) we have

$$\mathbf{A}_+^{(I)}(\zeta) \mathbf{Q}_\infty = (-1)^{l+1} \phi^{2M} q^{l-IM} \mathbf{I}, \tag{6.11}$$

i.e. the matrices $\mathbf{A}_+^{(I)}(\zeta)$ and \mathbf{Q}_∞ are the inverses of each other up to a simple constant. In fact, the inversion of the matrix \mathbf{Q}_∞ in W_l without knowing the explicit answer (2.27) is a difficult problem.

Finally, the relations similar to (6.5), (6.11) were obtained in [36] for the XXX chain using the operator approach. So we can look at them as a generalization of results [36] to the case of the XXZ chain.

7. Conclusion

In this paper we constructed a new integral representation for the XXZ Q -operator at arbitrary complex spin and horizontal field. In the polynomial basis the action of the Q -operator looks surprisingly simple and the matrix elements are expressed in terms of a single sum. The generating function of this action coincides with the generating function for the Q -operator [26] constructed as a q -difference operator of infinite order. Our integral Q -operator is directly related to the Q -operator $\mathbf{A}_+(\lambda)$ constructed in [1] based on the approach developed in [6,7]. Using this connection we calculated explicitly the action of the Q -operator $\mathbf{A}_+(\lambda)$ for arbitrary complex spins. A reduction to (half-) integer spins is non-singular and results in the explicit formulas for both Q -operators $\mathbf{A}_\pm(\lambda)$.

However, when spin is complex a natural duality (2.22) between $\mathbf{A}_+(\lambda)$ and $\mathbf{A}_-(\lambda)$ disappears and construction of the second Q -operator $\mathbf{A}_-(\lambda)$ becomes problematic. It is easy to understand because the eigenvalues of $\mathbf{A}_-(\lambda)$ are no longer polynomials. At this stage we don't know how to construct the integral analog of $\mathbf{A}_-(\lambda)$ for complex spins.

Let us notice that in the elliptic case the Q -operators for the periodic XYZ spin chain constructed in [41] have a natural symmetry between $\mathbf{Q}_+(\lambda)$ and $\mathbf{Q}_-(\lambda)$. However, they were represented as formal infinite series of difference operators with elliptic coefficients and it is difficult to compare their action in any appropriate basis. It looks like in the trigonometric limit such simple duality disappears and $\mathbf{A}_-(\lambda)$ is a more complicated operator comparing to $\mathbf{A}_+(\lambda)$ in the case of arbitrary spins. This is justified by calculations in [24] for the XXX chain where the action of the second Q -operator on polynomials has a more complicated structure.

Another direction of generalization is to extend our results to the inhomogeneous case. Since the kernel of the integral operator $\mathbf{Q}_f(\lambda)$ is factorized, its generalization to the inhomogeneous case is straightforward. Small lattice calculations (up to $M = 3$, $l = 3$) show that the matrix \mathbf{A}_0 in (6.2) remains essentially the same as in the homogeneous case but no longer commutes with $\mathbf{Q}_f(\lambda)$. We plan to return to these questions elsewhere.

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