Small and Large Time Stability of the Time taken for a Lévy Process to Cross Curved Boundaries

Philip S. Griffin and Ross A. Maller*

Syracuse University and Australian National University

Abstract

This paper is concerned with the small time behaviour of a Lévy process X. In particular, we investigate the *stabilities* of the times, $\overline{T}_b(r)$ and $T_b^*(r)$, at which X, started with $X_0 = 0$, first leaves the space-time regions $\{(t,y) \in \mathbb{R}^2 : y \le rt^b, t \ge 0\}$ (one-sided exit), or $\{(t,y) \in \mathbb{R}^2 : |y| \le rt^b, t \ge 0\}$ (two-sided exit), $0 \le b < 1$, as $r \downarrow 0$. Thus essentially we determine whether or not these passage times behave like deterministic functions in the sense of different modes of convergence; specifically convergence in probability, almost surely and in L^p. In many instances these are seen to be equivalent to relative stability of the process X itself. The analogous large time problem is also discussed.

Keywords: Lévy process; passage times across power law boundaries; relative stability; overshoot; random walks.

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1 Introduction

There is a strand of research, going back to [4], and continuing most recently in [2], in which the local behaviour of a Lévy process X_t is compared with that of power law functions, t^b, $b \ge 0$. Here we address this question, but take a different line, by asking for properties of the first exit time

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of the process out of space-time regions bounded, either on one side or both sides, by power law functions. Our aim is to give a very general study of the small time stability, as the boundary level $r \to 0$, of the one-sided exit time

$$\overline{T}_b(r) = \inf\{t \geq 0 : X_t > rt^b\},\ r \geq 0, \tag{1.1}$$

and the 2-sided exit time,

$$T_b^*(r) = \inf\{t \geq 0 : |X_t| > rt^b\},\ r \geq 0, \tag{1.2}$$

when $0 \leq b < 1$. (We adopt the convention that the inf of the empty set is $+\infty$.) While not the primary motivation for this paper, in Section 5 we also include results on stability for large times as the boundary level $r \to \infty$. When $b = 0$ such results form part of classical renewal theory for Lévy processes.

The restriction of b to the interval $[0, 1)$ in (1.1) and (1.2) involves no loss of generality, since as we show below in Proposition 3.1, neither passage time can be relatively stable when $b \geq 1$. Thus unless otherwise mentioned we keep $0 \leq b < 1$ in what follows. Our study will draw out similarities as well as differences between the behaviours of $\overline{T}_b(r)$ and $T_b^*(r)$ with respect to differing modes of stability. By *relative stability at 0* of $\overline{T}_b(r)$, we will mean that $\overline{T}_b(r)/C(r)$ converges in probability to a finite nonzero constant (which by rescaling we can take as 1), as $r \to 0$ for a finite function $C(r) > 0$. We will show that this is precisely equivalent to the *positive relative stability at 0* of the process X, i.e., to the property that

$$\frac{X_t}{B(t)} \xrightarrow{\mathrm{P}} +1,\ \text{as } t \to 0, \tag{1.3}$$

for some norming function $B(t) > 0$. The corresponding result for the two-sided exit is that $T_b^*(r)$ is relatively stable at 0 iff X_t is relatively stable at 0 in the two-sided sense, i.e., if

$$\frac{|X_t|}{B(t)} \xrightarrow{\mathrm{P}} 1,\ \text{as } t \to 0, \tag{1.4}$$

for some function $B(t) > 0$. The statements of these results are similar, and this is exploited in one direction of the proof, but the proofs in the opposite direction are completely different.

We also consider relative stability in the a.s. sense and in L^p. In the former case the results for the one-sided and two-sided exit times are again similar, see Theorem 3.2, and we are again able to exploit this in one direction. In the case of stability in L^p, the behaviour of the two exit times is significantly different, see Theorem 3.4. Section 3 contains a complete discussion of these results.

Given the equivalences between the relative stability of $\overline{T}_b(r)$ and $T^*_b(r)$, and the relative stability of the original process X, we begin Section 2 by reprising, and where necessary extending, the properties of a relatively stable X. Our main results, related to the stability of $\overline{T}_b(r)$ and $T^*_b(r)$, are then given in Section 3. Proofs of these results can be found in Section 4, together with some preliminary results which may be of independent interest. Finally Section 5 contains results in the large time setting. We strive for, and mostly achieve, definitive (necessary and sufficient) conditions.

We conclude this section by introducing some of the notation that will be needed in the remainder of the paper. The setting is as follows. Suppose that $X = \{X_t : t \geq 0\}$, $X_0 = 0$, is a Lévy process defined on $(\Omega, \mathcal{F}, P)$, with triplet (γ, σ^2, Π), Π being the Lévy measure of X, $\gamma \in \mathbb{R}$ and $\sigma \geq 0$. Thus the characteristic function of X is given by the Lévy-Khintchine representation, $E(e^{i\theta X_t}) = e^{t\Psi(\theta)}$, where

$$\Psi(\theta) = \mathrm{i}\theta\gamma - \sigma^2\theta^2/2 + \int_{\mathbb{R}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}})\Pi(\mathrm{d}x), \text{ for } \theta \in \mathbb{R}. \tag{1.5}$$

If X is of bounded variation, then $\sigma = 0$ and the Lévy-Khintchine exponent may be expressed in the form

$$\Psi(\theta) = \mathrm{i}\theta\mathrm{d} + \int_{\mathbb{R}} (e^{\mathrm{i}\theta x} - 1)\Pi(\mathrm{d}x), \text{ for } \theta \in \mathbb{R}, \tag{1.6}$$

where $\mathrm{d} = \gamma - \int x\mathbf{1}_{\{|x|\leq 1\}})\Pi(\mathrm{d}x)$ is called the drift of X. We will sometimes include a subscript, as in for example d_X, to make clear the process we are referring to. X is a compound Poisson process if $\sigma_X = 0$, $\Pi_X(\mathbb{R}) < \infty$ and $\mathrm{d}_X = 0$.

Let $\overline{\Pi}$ and $\overline{\Pi}^{\pm}$ denote the tails of Π, thus

$$\overline{\Pi}^{+}(x) = \Pi\{(x,\infty)\},\ \overline{\Pi}^{-}(x) = \Pi\{(-\infty,-x)\},\ \text{and}\ \overline{\Pi}(x) = \overline{\Pi}^{+}(x) + \overline{\Pi}^{-}(x),$$

for $x > 0$, and define kinds of Winsorised and truncated means $A(x)$ and $\nu(x)$ by

$$\begin{aligned} A(x) &:= \gamma + \overline{\Pi}^{+}(1) - \overline{\Pi}^{-}(1) + \int_1^x \left(\overline{\Pi}^{+}(y) - \overline{\Pi}^{-}(y)\right) \mathrm{d}y \\ &= \gamma + x\left(\overline{\Pi}^{+}(x) - \overline{\Pi}^{-}(x)\right) + \int_{1<|y|\le x} y\Pi(\mathrm{d}y) \\ &=: \nu(x) + x\left(\overline{\Pi}^{+}(x) - \overline{\Pi}^{-}(x)\right),\ x > 0, \end{aligned} \tag{1.7}$$

where $\int_{1<|y|\le x} = -\int_{x<|y|\le 1}$ if $x < 1$. Similarly, for variances, we set

$$\begin{aligned} U(x) &:= \sigma^2 + 2\int_0^x y\overline{\Pi}(y)\mathrm{d}y \\ &= \sigma^2 + x^2\overline{\Pi}(x) + \int_{0<|y|\le x} y^2\Pi(\mathrm{d}y) \\ &=: V(x) + x^2\overline{\Pi}(x),\ x > 0. \end{aligned} \tag{1.8}$$

Note that, because $\int_{\{|x|\le 1\}} x^2\Pi(\mathrm{d}x) < \infty$, we have

$$\lim_{x\to 0} xA(x) = \lim_{x\to 0} x\nu(x) = 0. \tag{1.9}$$

2 Small Time Relative Stability of X

Recall that X is *relatively stable* (in probability, as $t \to 0$), denoted $X \in RS$ at 0, if there is a nonstochastic function $B(t) > 0$ such that

$$\frac{X_t}{B(t)} \xrightarrow{\mathrm{P}} +1,\ \text{or}\ \frac{X_t}{B(t)} \xrightarrow{\mathrm{P}} -1,\ \text{as}\ t \to 0. \tag{2.1}$$

(We abbreviate this to $X_t/B(t) \xrightarrow{\mathrm{P}} \pm 1$.) If (2.1) holds with a "+" sign we say that X_t is *positively relatively stable* as $t \to 0$, denoted $X \in PRS$; a minus sign gives negative relative stability, NRS.

Various properties of relative stability at 0 are developed in [9]. We need only assume $B(t) > 0$ for $t > 0$: $B(t)$ is not assumed *a priori* to be nondecreasing, but can always be taken as such. Further properties of relative stability in probability at 0, and of the norming function $B(t)$, are summarized in the next proposition.

Proposition 2.1 *There is a non-stochastic function $B(t) > 0$ such that*

$$\frac{X_t}{B(t)} \xrightarrow{\mathrm{P}} \pm 1, \text{ as } t \to 0, \tag{2.2}$$

if and only if

$$\sigma^2 = 0 \quad \text{and} \quad \frac{A(x)}{x\overline{\Pi}(x)} \to \pm\infty, \text{ as } x \to 0. \tag{2.3}$$

(The + or − signs should be taken together in (2.2) and (2.3).) If these hold, then $|A(x)|$ is slowly varying as $x \to 0$, and $B(t)$ is regularly varying of index 1 and $B(t) \sim t|A(B(t))|$ as $t \to 0$. Further, $B(t)$ may be chosen to be continuous and such that $t^{-b}B(t)$ is strictly increasing for all $0 \le b < 1$.

In addition, we have

$$\frac{|X_t|}{B(t)} \xrightarrow{\mathrm{P}} 1, \text{ as } t \to 0, \tag{2.4}$$

for a non-stochastic function $B(t) > 0$, if and only if

$$\sigma^2 = 0 \quad \text{and} \quad \frac{|A(x)|}{x\overline{\Pi}(x)} \to \infty, \text{ as } x \to 0, \tag{2.5}$$

and this is equivalent to (2.2) and (2.3) (with either the + or − sign). Thus (2.4) implies that $\lim_{t\to 0} P(X_t > 0) = 1$ or $\lim_{t\to 0} P(X_t < 0) = 1$, just as (2.5) implies that $A(x)$ is of constant sign for all small x, that is, $A(x) > 0$ for all small $x > 0$ or $A(x) < 0$ for all small $x > 0$.

Further, the following conditions are also each equivalent to (2.4):
there exist constants $0 < c_1 < c_2 < \infty$ and a non-stochastic function $\widetilde{B}(t) > 0$ such that

$$\lim_{t\to 0} P\left(c_1 < \frac{|X_t|}{\widetilde{B}(t)} < c_2\right) \to 1; \tag{2.6}$$

there is a nonstochastic function $\widehat{B}(t) > 0$ such that every sequence $t_k \to 0$ contains a subsequence

$t_{k'} \to 0$ *with*

$$\frac{X_{t_{k'}}}{\widehat{B}(t_{k'})} \xrightarrow{\mathrm{P}} c', \tag{2.7}$$

where c' *is a constant with* $0 < |c'| < \infty$ *which may depend on the choice of subsequence.*

Note: If $\Pi(\mathbb{R}) = 0$ then $A(x) = \gamma$ for all $x > 0$, and the meaning of the limit in (2.3) is that $\gamma > 0$ when the limit is ∞ and $\gamma < 0$ when the limit is $-\infty$. This corresponds to the case that $X_t = \gamma t + \sigma W_t$ is Brownian motion with drift, and it's clear that $X \in PRS$ ($X \in NRS$) at 0 iff $\sigma^2 = 0$ and $\gamma > 0$ ($\gamma < 0$). In this case $B(t) = |\gamma| t$. Similarly the meaning of the limit in (2.5) when $\Pi(\mathbb{R}) = 0$ is that $\gamma \neq 0$.

Proof of Proposition 2.1. See Theorem 2.2 of [9] for the equivalence of (2.2) and (2.3), and the properties of $B(\cdot)$ and $A(\cdot)$. (A blanket assumption of $\Pi(\mathbb{R}) > 0$ is made in [9], but it is unnecessary in any of the instances where references are made to [9] in this paper. One way to see this is to add an independent rate 1 Poisson process to X and use that the resulting process agrees with X at sufficiently small times.) The strict monotonicity of $t^{-b}B(t)$ for $0 \le b < 1$ follows easily from the regular variation of B; see for example, Section 1.5.2 of [3].

Clearly (2.3) implies (2.5) and the converse holds by continuity of A. Further, it is trivial that (2.2) implies (2.4) and (2.4) implies (2.6). Also (2.6) implies (2.7) because, under (2.6), every sequence $t_k \to 0$ contains a subsequence $t_{k'} \to 0$ such that $X_{t_{k'}}/\widetilde{B}(t_{k'}) \to Z'$, where Z' is an infinitely divisible random variable with $P(c_1 \le |Z'| \le c_2) = 1$. Thus, Z' is bounded a.s., hence is a constant, c', say, with $|c'| \in [c_1, c_2]$. Hence we may take $\widehat{B} = \widetilde{B}$ in (2.7). Thus to complete the proof of Proposition 2.1, it suffices to show (2.7) implies (2.5).

Assume (2.7) holds. Then every sequence $t_k \to 0$ contains a subsequence $t_{k'} \to 0$ with

$$\frac{X_{t_{k'}}}{\widehat{B}(t_{k'})} \xrightarrow{\mathrm{P}} c', \tag{2.8}$$

for some $c' \neq 0$. We first show that this condition holds if $\widehat{B}$ is replaced by any function D with $D(t) \in \mathcal{L}_t$ for all $t > 0$, where

$$\mathcal{L}_t = \{\text{limit points of } \widehat{B} \text{ at } t\} \cup \{\widehat{B}(t)\}.$$

Since $P(X_t = 0) > 0$ for some $t > 0$ precisely when X is compound Poisson, it follows from (2.8) that $P(X_t \neq 0) = 1$ for all $t > 0$. Thus if $0 \in \mathcal{L}_t$, then along some sequence $s \to t$, we have $|X_s|/\widehat{B}(s) \xrightarrow{\mathrm{P}} \infty$. From this it follows that $0 \notin \mathcal{L}_t$ if t is sufficiently small. Now take any sequence $t_k \to 0$. Choose s_k so that

$$\frac{\widehat{B}(s_k)}{D(t_k)} \to 1 \quad \text{and} \quad \frac{X_{|t_k - s_k|}}{D(t_k)} \xrightarrow{\mathrm{P}} 0.$$

The former is possible since $D(t_k) \in \mathcal{L}_{t_k}$, and the latter since $X_t \xrightarrow{\mathrm{P}} 0$ as $t \to 0$. Now choose a subsequence $s_{k'}$ of s_k so that $X_{s_{k'}}/\widehat{B}(s_{k'}) \xrightarrow{\mathrm{P}} c'$ where $c' \neq 0$. Then

$$\frac{X_{t_{k'}}}{D(t_{k'})} = \frac{X_{s_{k'}}}{\widehat{B}(s_{k'})} \frac{\widehat{B}(s_{k'})}{D(t_{k'})} + \frac{X_{t_{k'}} - X_{s_{k'}}}{D(t_{k'})} \xrightarrow{\mathrm{P}} c'.$$

Thus

$$\text{every sequence } t_k \to 0 \text{ contains a subsequence } t_{k'} \to 0 \text{ with } X_{t_{k'}}/D(t_{k'}) \xrightarrow{\mathrm{P}} c' \neq 0. \tag{2.9}$$

From the convergence criteria for infinitely divisible laws, e.g. Theorem 15.14 of Kallenberg [15], this is equivalent to every sequence $t_k \to 0$ containing a subsequence $t_{k'} \to 0$ such that for every $\varepsilon > 0$,

$$\lim_{t_{k'} \to 0} t_{k'} \overline{\Pi}(\varepsilon D(t_{k'})) = 0, \quad \lim_{t_{k'} \to 0} \frac{t_{k'} V(D(t_{k'}))}{D^2(t_{k'})} = 0, \quad \text{and} \quad \lim_{t_{k'} \to 0} \frac{t_{k'} A(D(t_{k'}))}{D(t_{k'})} = c' \neq 0. \tag{2.10}$$

From this we may conclude that,

$$\lim_{t \to 0} \frac{D(t)|A(D(t))|}{V(D(t))} = \infty, \quad \text{and} \quad \lim_{t \to 0} \frac{|A(D(t))|}{D(t)\overline{\Pi}(D(t))} = \infty. \tag{2.11}$$

Observe that $D(t)|A(D(t))| \to 0$ as $t \to 0$ by (1.9), since necessarily $D(t) \to 0$. Hence, $\sigma^2 \le V(D(t)) \to 0$, proving the first condition in (2.5). Next, let

$$D_1(t) = \liminf_{s \to t} \widehat{B}(s), \quad D_2(t) = \limsup_{s \to t} \widehat{B}(s) \vee \widehat{B}(t).$$

Since D_1 and D_2 satisfy (2.9), it follows easily that

$$\limsup_{t \to 0} \frac{D_2(t)}{D_1(t)} < \infty. \tag{2.12}$$

Now given $x > 0$, let

$$t_x = \inf\{s : \widehat{B}(s) \ge x\}.$$

Then $t_x < \infty$ for sufficiently small x, $D_1(t_x) \le x \le D_2(t_x)$, and $t_x \to 0$ as $x \to 0$. Further

$$\begin{aligned} |A(D_1(t_x)) - A(x)| &\le |x\overline{\Pi}(x) - D_1(t_x)\overline{\Pi}(D_1(t_x))| + \int_{D_1(t_x)<|y|\le x} |y|\Pi(\mathrm{d}y) \\ &\le 2D_2(t_x)\overline{\Pi}(D_1(t_x)), \end{aligned}$$

hence by (2.11) and (2.12),

$$\frac{A(x)}{A(D_1(t_x))} \to 1 \quad \text{as } x \to 0. \tag{2.13}$$

Thus by (2.11), (2.12) and (2.13),

$$\frac{|A(x)|}{x\overline{\Pi}(x)} \ge \frac{|A(D_1(t_x))|}{D_1(t_x)\overline{\Pi}(D_1(t_x))} \frac{|A(x)|}{|A(D_1(t_x))|} \frac{D_1(t_x)}{D_2(t_x)} \to \infty,$$

which proves the second condition in (2.5). □

Remarks: (i) An equivalence for PRS, i.e., (1.3), is (2.3) holding with a "+" sign. Then $A(x) > 0$ for all small x. Symmetrically, for NRS.

(ii) For any family of events A_t, we say that A_t occur with probability approaching 1 (WPA1) as $t \to L$ if $\lim_{t \to L} P(A_t) = 1$. ($L$ may be 0 or ∞.) We sometimes describe a situation like (2.6), i.e, for a stochastic function Y_t, $t \ge 0$, there exist constants $0 < c_1 < c_2 < \infty$ such that

$$\lim_{t\to L} P\left(c_1 < Y_t < c_2\right) = 1, \tag{2.14}$$

by writing $Y_t \asymp 1$ WPA1 as $t \to L$. The strict inequalities may be replaced by non-strict ones.

(iii) It is possible to have $A(x) \to 0$ as $x \to 0$, and also $X \in RS$ as $t \to 0$. For example, take $\sigma^2 = 0$, and define

$$\overline{\Pi}^{+}(x) = \frac{1}{x(\log x)^2},\ 0 < x < e^{-1}, \quad \overline{\Pi}^{+}(x) = 0,\ x \ge e^{-1},$$

and $\overline{\Pi}^{-}(x) \equiv 0$. Then $A(x) = \gamma - 1 - 1/\log x$, for $x \le e^{-1}$. Thus if $\gamma = 1$ then $A(x) \to 0$ as $x \to 0$. Further $A(x)/x\overline{\Pi}(x) \to \infty$ as $x \to 0$, so that $X \in PRS$ as $t \to 0$. In this case, $X(t)/B(t) \xrightarrow{\text{P}} 1$ where $B(t) = t/|\log t|$.

In studying $\overline{T}_b$ and T_b^* we will need the following corresponding maximal processes;

$$\overline{X}_t := \sup_{0\le s\le t} X_s, \quad \text{and} \quad X_t^* := \sup_{0\le s\le t} |X_s|.$$

Lemma 2.1 *Let t_k be any sequence with $t_k \to 0$ as $k \to \infty$. Assume $X_{t_k}/B(t_k) \xrightarrow{\text{P}} a \in \mathbb{R}$, where $B(t_k) > 0$, when $k \to \infty$. Then*

$$(i)\ \frac{X_{t_k}^*}{B(t_k)} \xrightarrow{\text{P}} |a|, \text{ and } (ii)\ \frac{\overline{X}_{t_k}}{B(t_k)} \xrightarrow{\text{P}} \max(a,0), \text{ as } k \to \infty. \tag{2.15}$$

Conversely, (i) with $a \in \mathbb{R}$ implies $|X_{t_k}|/B(t_k) \xrightarrow{\text{P}} |a|$ as $k \to \infty$. Finally, as a partial converse to (ii), if $a > 0$ and $\overline{X}_t/B(t) \xrightarrow{\text{P}} a$ as $t \to 0$, then $X_t/B(t) \xrightarrow{\text{P}} a$ as $t \to 0$.

Proof of Lemma 2.1. Assume $X_{t_k}/B(t_k) \xrightarrow{\text{P}} a$ as $t_k \to 0$. Use the decomposition in [9], Lemma 6.1, to write

$$X_t = t\nu(h) + X_t^{(S,h)} + X_t^{(B,h)}, \quad t > 0, h > 0, \tag{2.16}$$

where $X_t^{(B,h)}$ is the component of X containing the jumps larger than h in modulus, and $X^{(S,h)}$ is then defined through (2.16). We will use (2.16) with $t = t_k$ and $h = B(t_k)$. As in (2.10), $t_k\overline{\Pi}(B(t_k)) \to 0$ as $t_k \to 0$, so

$$P(\sup_{0\le s\le t_k} |X_s^{(B,h)}| = 0) = e^{-t_k\overline{\Pi}(B(t_k))} \to 1. \tag{2.17}$$

Next, $X_t^{(S,h)}$ is a mean 0 martingale with variance $tV(h)$, so by applying Doob's inequality to the submartingale $(X_t^{(S,h)})^2$, for any $\varepsilon > 0$,

$$P\left(\sup_{0\le s\le t_k} |X_s^{(S,h)}| > \varepsilon B(t_k)\right) \le \frac{t_k V(B(t_k))}{\varepsilon^2 B^2(t_k)} \to 0,$$

using (2.10) again. A third use of (2.10) gives $t_k\nu(B(t_k)) \sim aB(t_k)$, so from (2.16),

$$\frac{X^*_{t_k}}{B(t_k)} = \sup_{0\le s\le t_k} \frac{|X_s|}{B(t_k)} = \sup_{0\le s\le t_k} \frac{s|\nu(B(t_k)|}{B(t_k)} + o_p(1) \xrightarrow{\text{P}} |a|.$$

Thus (i) is proved and (ii) follows similarly.

Conversely, let (i) hold. Then $X^*_{t_k}/B(t_k)$ is stochastically bounded and the inequality $P(|X_{t_k}| > xB(t_k)) \le P(X^*_{t_k} > xB(t_k))$ for $x > 0$ shows that $X_{t_k}/B(t_k)$ is also stochastically bounded. Thus every subsequence of $\{t_k\}$ contains a further subsequence $t_{k'} \to 0$ along which $X_{t_{k'}}/B(t_{k'}) \xrightarrow{\text{D}} Z'$, for some infinitely divisible random variable Z' with $|Z'| \le |a|$. As a bounded infinitely divisible random variable, Z' is degenerate at z', say. But then $X^*_{t_{k'}}/B(t_{k'}) \xrightarrow{\text{P}} |z'|$ by the converse direction already proved. Hence by (i), $|z'| = |a|$ and since this is true for all subsequences we get $|X_{t_k}|/B(t_k) \xrightarrow{\text{P}} |a|$.

Finally assume $\overline{X}_t/B(t) \xrightarrow{\text{P}} a$ as $t \to 0$ where $a > 0$. We may assume $B(t)$ is nondecreasing; see for example Lemma 4.1. We first show that

$$\limsup_{t\to 0} \frac{B(t)}{B(2t)} < 1. \tag{2.18}$$

If not, there exists a sequence $t_k \to 0$ so that $B(t_k)/B(2t_k) \to 1$ as $k \to \infty$. Thus

$$\frac{\overline{X}_{t_k}}{B(t_k)} \xrightarrow{\text{P}} a \quad \text{and} \quad \frac{\overline{X}_{2t_k}}{B(t_k)} \xrightarrow{\text{P}} a, \quad \text{as } k \to \infty. \tag{2.19}$$

Let $\tau_k = \inf\{t : X_t > aB(t_k)/2\}$ and set $Y_t = X_{\tau_k+t} - X_{\tau_k}$. Then $P(\tau_k \le t_k) \ge P(\overline{X}_{t_k} > aB(t_k)/2) \to 1$, and on $\{\tau_k \le t_k\}$,

$$\overline{Y}_{t_k} \le \overline{X}_{2t_k} - X_{\tau_k} \le (\overline{X}_{2t_k} - \overline{X}_{t_k}) + (\overline{X}_{t_k} - aB(t_k)/2).$$

By (2.19),

$$\frac{\overline{X}_{2t_k} - \overline{X}_{t_k}}{B(t_k)} + \frac{\overline{X}_{t_k} - aB(t_k)/2}{B(t_k)} \xrightarrow{\text{P}} \frac{a}{2},$$

and hence

$$P(\overline{X}_{t_k} > 3aB(t_k)/4) = P(\overline{Y}_{t_k} > 3aB(t_k)/4) \to 0,$$

which is a contradiction. Thus (2.18) holds. Now write

$$\overline{X}_{2t} = \overline{X}_t \vee (X_t + \overline{X}'_t), \tag{2.20}$$

where $\overline{X}'_t = \sup_{t \le s \le 2t}(X_s - X_t)$ is an independent copy of $\overline{X}_t$. Given any sequence $t_k \to 0$ we may choose a further subsequence $t_{k'} \to 0$ so that

$$\frac{B(t_{k'})}{B(2t_{k'})} \to \lambda'$$

where necessarily $\lambda' \in [0, 1)$ by (2.18). Setting $t = t_{k'}$ in (2.20), dividing throughout by $B(2t_{k'})$ and taking limits, we see that

$$\frac{X_{t_{k'}}}{B(2t_{k'})} \xrightarrow{\text{P}} a(1 - \lambda') > 0. \tag{2.21}$$

Thus with $\widehat{B}(t) = B(2t)$ in (2.7), it follows that $X \in RS$ and since the subsequential limits in (2.21) are positive, $X \in PRS$. Thus for some function $D(t) > 0$, $X_t/D(t) \xrightarrow{\text{P}} 1$. Then from part (ii), $\overline{X}_t/D(t) \xrightarrow{\text{P}} 1$. Hence $D(t) \sim aB(t)$ and the proof is complete. $\square$

Remark: We are unsure whether a subsequential version of the converse to (ii), with $a > 0$, holds. Since it will not be needed in this paper we do not pursue it further.

One final result which will prove useful below is the following;

Proposition 2.2 *Suppose* $X_t/B(t) \xrightarrow{\mathrm{P}} 1$*, where* $B(t) > 0$*, when* $t \to 0$*, and let* $Y_t(\lambda) := X_{\lambda t}/B(t)$ *for* $\lambda \geq 0$*. Then for every* $\delta > 0$*,* $0 < \eta \leq T < \infty$ *and* $0 \leq b < 1$*,*

$$P(\sup_{\eta \leq \lambda \leq T} |\lambda^{-b} Y_t(\lambda) - \lambda^{1-b}| > \delta) \to 0 \quad \text{as } t \to 0.$$

Proof of Proposition 2.2. By a result of Skorohod, see for example Theorem 15.17 of [15], for every $\delta > 0$

$$P(\sup_{0 \leq \lambda \leq T} |Y_t(\lambda) - \lambda| > \delta) \to 0 \quad \text{as } t \to 0. \tag{2.22}$$

Thus

$$P(\sup_{\eta \leq \lambda \leq T} |\lambda^{-b} Y_t(\lambda) - \lambda^{1-b}| > \delta) \leq P(\sup_{\eta \leq \lambda \leq T} |Y_t(\lambda) - \lambda| > \delta \eta^b) \to 0$$

by (2.22). □

3 Relative Stability of $\overline{T}_b(r)$ and $T^*_b(r)$ for Small Times

Recall we always assume, unless explicitly stated otherwise, that $0 \leq b < 1$. The first two theorems concern the relative stability in probability or almost surely of $\overline{T}_b(r)$ and $T^*_b(r)$, as $r \to 0$. These are shown to be equivalent to positive relative stability at 0 of X and relative stability at 0 of X, in the relevant mode of convergence, respectively. Proposition 3.1 demonstrates that there is no loss of generality in assuming $0 \leq b < 1$, since $\overline{T}_b(r)$ and $T^*_b(r)$ cannot be relatively stable, in probability (and hence also a.s.), as $r \to 0$, when $b \geq 1$.

Theorem 3.1 *(a) Assume* $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ *as* $t \to 0$*, where* $B(t) > 0$*. Then* $B(t)/t^b$ *may be chosen to be continuous and strictly increasing, in which case* $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ *as* $r \to 0$*, where* $C(r)$ *is the inverse to* $B(t)/t^b$*.*

Conversely, assume $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ *as* $r \to 0$*, where* $C(r) > 0$*. Then C(r) may be taken to be continuous and strictly increasing with inverse* C^{-1}*, in which case* $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ *as* $t \to 0$*, where* $B(t) = t^b C^{-1}(t)$*.*

(b) The same result holds if X *and* $\overline{T}_b(r)$ *are replaced by* $|X|$ *and* $T_b^*(r)$ *respectively in (a).*

In either case, (a) or (b), the function $C(r)$ *is regularly varying with index* $1/(1-b)$ *as* $r \to 0$*.*

In the example given prior to Lemma 2.1, $X(t)/B(t) \xrightarrow{\mathrm{P}} 1$ where $B(t) = t/|\log t|$. Thus $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ and $T_b^*(r)/C(r) \xrightarrow{\mathrm{P}} 1$ where $C(r) = \big((1-b)^{-1} r |\log r|\big)^{1/(1-b)}$.

Remark: Implicit in Theorem 3.1 we understand, is that $\overline{T}_b(r)$ and $T_b^*(r)$ are finite WPA1 as $r \to 0$, as a consequence of their relative stability when $X \in PRS$ or $X \in RS$.

Corollary 3.1 *Assume* $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ *as* $t \to 0$*, where* $B(t) > 0$*. Then* $P(\overline{T}_b(r) = T_b^*(r)) \to 1$ *as* $r \to 0$*.*

Proposition 3.1 *Suppose* $b \geq 1$*. Then neither* $\overline{T}_b(r)$ *nor* $T_b^*(r)$ *can be relatively stable, in probability, as* $r \to 0$*.*

Theorem 3.2 *(a)* $T_b^*(r)$ *is almost surely (a.s.) relatively stable, i.e.,* $T_b^*(r)/C(r) \to 1$*, a.s., as* $r \to 0$*, for a finite function* $C(r) > 0$*, iff* X *has bounded variation with drift* $\mathrm{d_X} \neq 0$*.*

(b) $\overline{T}_b(r)$ *is almost surely relatively stable, i.e.,* $\overline{T}_b(r)/C(r) \to 1$*, a.s., as* $r \to 0$*, for a finite function* $C(r) > 0$*, iff* X *has bounded variation with drift* $\mathrm{d_X} > 0$*.*

In either case, (a) or (b), the function $C(r)$ *may be chosen as* $C(r) = (r/|\mathrm{d}_X|)^{1/(1-b)}$*.*

The next theorem deals with the position of the process after exiting, in the setting of Theorem 3.1.

Theorem 3.3 *(a) Suppose* $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ *as* $t \to 0$*, where* $B(t) > 0$ *satisfies the regularity conditions of Theorem 3.1. Let* C *be the inverse of* $B(t)/t^b$*. Then, as* $r \to 0$*,*

$$\frac{X_{\overline{T}_b(r)}}{r(\overline{T}_b(r))^b} \xrightarrow{\mathrm{P}} 1, \quad \frac{X_{\overline{T}_b(r)}}{B(\overline{T}_b(r))} \xrightarrow{\mathrm{P}} 1, \quad \frac{X_{\overline{T}_b(r)}}{B(C(r))} \xrightarrow{\mathrm{P}} 1, \text{ and } \frac{X_{\overline{T}_b(r)}}{r(C(r))^b} \xrightarrow{\mathrm{P}} 1. \tag{3.1}$$

(b) Suppose $|X_t|/B(t) \stackrel{\mathrm{P}}{\longrightarrow} 1$ *as* $t \to 0$. *Then* (3.1) *holds with* $|X|$ *and* $T_b^*(r)$ *in place of* X *and* $\overline{T}_b(r)$ *respectively.*

Remark: By Theorem 4.2 of [9], $X_t/B(t) \to 1$ a.s. as $t \to 0$ for some $B(t) > 0$, is equivalent to X having bounded variation with drift $\mathrm{d}_X > 0$, and in that case $B(t) \sim \mathrm{d}_X t$. Using this, it is then easy to see that the analogous result to Theorem 3.3 holds when $\stackrel{\mathrm{P}}{\longrightarrow}$ is replaced throughout by a.s. convergence.

In the results so far, $\overline{T}_b(r)$ and $T_b^*(r)$ have behaved very similarly. This is not the case when it comes to stability in L^p as our final result shows. When considering $E\overline{T}_b(r)$, the immediate problem arises as to whether or not the expectation is finite. As shown in Theorem 1 of [11], finiteness of $E\overline{T}_b(r)$ for some (all) $r > 0$ is equivalent to $X_t \to \infty$ a.s. as $t \to \infty$. Since our aim is to study the local behaviour of X for small times, imposing a large time condition is most unnatural. Thus, we remove the issue of finiteness of $E\overline{T}_b(r)$, by studying instead, $E(\overline{T}_b(r) \wedge \varepsilon)$ as $r \to 0$ for small ε. Similarly for $E(T_b^*(r) \wedge \varepsilon)$.

Theorem 3.4 *(a) Assume* X *has bounded variation with drift* $\mathrm{d_X} > 0$, *then*

$$\lim_{\varepsilon \to 0} \lim_{r \to 0} \frac{E(\overline{T}_b(r) \wedge \varepsilon)}{C(r)} = 1, \tag{3.2}$$

where $C(r) = (r/\mathrm{d}_X)^{1/(1-b)}$.

(b) Fix a function $C(r) > 0$; *then* $T_b^*(r)/C(r) \stackrel{\mathrm{P}}{\longrightarrow} 1$ *iff* $(T_b^*(r) \wedge \varepsilon)/C(r) \to 1$ *in* L^p *for some (all)* $p > 0$ *and some (all)* $\varepsilon > 0$. *In particular, if* $T_b^*(r)/C(r) \stackrel{\mathrm{P}}{\longrightarrow} 1$, *then for every* $p > 0$ *and* $\varepsilon > 0$

$$\lim_{r \to 0} \frac{E(T_b^*(r) \wedge \varepsilon)^p}{C(r)^p} = 1. \tag{3.3}$$

By standard uniform integrability arguments, see for example Theorem 4.5.2 of [12], (3.2) implies that $(\overline{T}_b(r) \wedge \varepsilon)/(r/\mathrm{d}_X)^{1/(1-b)} \to 1$ in L^p as $r \to 0$ then $\varepsilon \to 0$, if $p \le 1$. However, unlike (3.3), (3.2) does not extend to convergence of the p-th moment for $p > 1$. Nor does (3.2) hold without taking the additional limit as $\varepsilon \to 0$. To illustrate this, let $X_t = at - N_t$ where N_t is a rate one Poisson process and $a > 0$. Then X has bounded variation with $\mathrm{d}_X = a$. Clearly $\overline{T}_b(r) = (r/a)^{1/(1-b)}$ if $N_{(r/a)^{1/(1-b)}} = 0$, while $\overline{T}_b(r) \geq a^{-1}$ if $N_{(r/a)^{1/(1-b)}} \geq 1$. Thus for any $p > 0$, if $\varepsilon < a^{-1}$ and r is sufficiently small that $(r/a)^{1/(1-b)} < \varepsilon$, then

$$E(\overline{T}_b(r) \wedge \varepsilon)^p = (r/a)^{p/(1-b)} e^{-(r/a)^{1/(1-b)}} + \varepsilon^p (1 - e^{-(r/a)^{1/(1-b)}}).$$

Hence, with $p = 1$, we obtain

$$\lim_{r \to 0} \frac{E(\overline{T}_b(r) \wedge \varepsilon)}{(r/\mathrm{d}_X)^{1/(1-b)}} = 1 + \varepsilon,$$

showing that the limit on $\varepsilon \to 0$ is needed in (3.2). If $p > 1$, then

$$\lim_{r \to 0} \frac{E(\overline{T}_b(r) \wedge \varepsilon)^p}{(r/\mathrm{d}_X)^{p/(1-b)}} = \infty,$$

for every $\varepsilon > 0$, so the first moment convergence in (3.2) does not extend to p-th moment convergence for any $p > 1$.

Remarks: (i) Although parts (a) and (b) of Theorem 3.1 are similar in content, the proof for T_b^* is quite different to that for $\overline{T}_b$. To prove (a) we need to establish *a priori* certain regularity properites of $C(r)$, whereas the proof of (b) relies heavily on the fact that bounded infinitely divisible distributions must be degenerate.

(ii) In Theorem 3.2 we deduce results for $\overline{T}_b$ from those for T_b^*, whereas in Theorem 3.3, we do the opposite.

(iii) We restricted ourselves to the boundary functions $t \mapsto t^b$ in this paper for clarity of exposition, though it's clear that many of our arguments will go through for more general regularly varying or even dominated varying functions.

4 Proofs

We set out some preliminary results. Throughout, take $0 \leq b < 1$. A key to proving Theorem 3.1 for $\overline{T}_b(r)$ is to obtain the *a priori* regularity of $C(r)$ contained in the following Proposition;

Proposition 4.1 *Suppose* $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ *as* $r \to 0$ *for a finite function* $C(r) > 0$*. Then* $C(r)$ *may be chosen to be continuous and strictly increasing.*

We emphasize that no assumptions are being made on C beyond positivity. This creates several difficulties which could be avoided if we were to assume, for example, that C is regularly varying. Such an assumption, however, would clearly be unsatisfactory, and, as we show, unnecessary. The main purpose of Proposition 4.1, which is somewhat hidden in the proof of Theorem 3.1, is that from $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ we can conclude that $C(r) - C(r-) = o(C(r-))$ as $r \to 0$. This latter condition is actually all that is needed, but proving the stronger continuity simplifies matters at several points.

The proposition will be proved by a series of lemmas. Recalling (2.14) we begin with the following elementary result which we will apply below to the processes $\overline{X}, X^*, \overline{T}_b$ and T_b^*:

Lemma 4.1 *Let* W_t *be any nonnegative, nondecreasing stochastic process with* $W_t \to 0$ *a.s. as* $t \to 0$*. If* $W_t/D(t) \asymp 1$ *WPA1 as* $t \to 0$ *for some non-stochastic function* $D(t) > 0$*, then* $D(t) \to 0$ *and may be chosen to be nondecreasing. If* $W_t/D(t) \xrightarrow{\mathrm{P}} 1$ *as* $t \to 0$ *for some function* D*, then again* $D(t) \to 0$ *and may be chosen to be nondecreasing.*

Proof of Lemma 4.1. Suppose that $P(c_1 < W_t/D(t) < c_2) \to 1$ for some $0 < c_1 < c_2 < \infty$, as $t \to 0$. This trivially implies $D(t) \to 0$ as $t \to 0$. To avoid pathological cases where $D(t) \to 0$ as $t \to 1$ for example, choose t_0 small enough that $P(c_1 < W_t/D(t) < c_2) \geq 1/2$ for all $0 < t \leq t_0$. Then $0 < \inf_{t \leq s \leq t_0} D(s) \leq \sup_{t \leq s \leq t_0} D(s) < \infty$ for all $0 < t \leq t_0$. Let $D^*(t) = \inf_{t \leq s \leq t_0} D(s)$ for $0 < t \leq t_0$. It then suffices to show

$$1 \leq \liminf_{t \to 0} \frac{D(t)}{D^*(t)} \leq \limsup_{t \to 0} \frac{D(t)}{D^*(t)} \leq \frac{c_2}{c_1}.$$

Only the final inequality requires proof. If this did not hold there would be a sequence $t_k \to 0$ with $D(t_k)/D^*(t_k) \to a > c_2/c_1$. Thus for some sequence $s_k \to 0$, $s_k \geq t_k$, we have $D(t_k)/D(s_k) \to a$.

But this leads to a contradiction since

$$\frac{W_{t_k}}{D(t_k)} \le \frac{W_{s_k}}{D(s_k)} \frac{D(s_k)}{D(t_k)},$$

and the LHS is $\ge c_1$ WPA1, whereas the RHS is $\le c_2/a < c_1$ WPA1. $\square$

Lemma 4.1 clearly applies to $\overline{X}$ and X^*. For application of Lemma 4.1 to $\overline{T}_b$ and T_b^*, note that in general $\overline{T}_b(r)$ (respectively $T_b^*(r)$) need not converge to 0 a.s. as $r \to 0$. However, when $\overline{T}_b(r)/C(r) \asymp 1$ (respectively $T_b^*(r)/C(r) \asymp 1$) WPA1 as $r \to 0$, almost sure convergence of $\overline{T}_b(r)$ and $T_b^*(r)$ to 0 does occur. This is because $\overline{T}_b(r) \downarrow \overline{T}_0(0)$ a.s. as $r \downarrow 0$, and if $P(\overline{T}_0(0) = 0) < 1$, then, combined with $\overline{T}_b(r)/C(r) \asymp 1$, we would have $P(\overline{T}_0(0) > c) = 1$ for some $c > 0$. Hence $X_t \le 0$ for all t and so $\overline{T}_b(r) = \infty$ for all $r > 0$; but this contradicts $\overline{T}_b(r)/C(r) \asymp 1$ WPA1. Similarly for T_b^*. For later reference, we note that the same argument holds if $\overline{T}_b$ is replaced by $\overline{T}_f$, where

$$\overline{T}_f(r) = \inf\{t \ge 0 : X_t > rf(t)\},\ r \ge 0, \tag{4.1}$$

and f is any function for which $f(t) > 0$ for $t > 0$ and $f(t) \to 0$ as $t \to 0$. Similarly for T_f^*.

Lemma 4.2 *Suppose there is a (nondecreasing, without loss of generality) function $C(r) > 0$ such that*
(a) $T_b^*(r)/C(r) \xrightarrow{\mathrm{P}} 1$ or (b) $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to 0$.
Then for every $\beta > 1$, in either case,

$$\limsup_{r\to 0} \frac{C(\beta r)}{C(r)} < \infty. \tag{4.2}$$

Proof of Lemma 4.2. (a) Let

$$Y_s := X_{T_b^*(r)+s} - X_{T_b^*(r)},\ s \ge 0, \tag{4.3}$$

and $T^*_{b,Y}(r)$ be the corresponding two-sided passage time, viz,

$$T^*_{b,Y}(r) := \inf\{s \geq 0 : |Y_s| > rs^b\},\ r \geq 0. \tag{4.4}$$

Fix $\varepsilon \in (0,1)$ so that $\xi := 2^{1-b}((1-\varepsilon)/(1+\varepsilon))^b > 1$ and set

$$\begin{aligned}
A_r^+ :&= \{X_{T_b^*(r)} > 0,\ Y_{T^*_{b,Y}(r)} > 0,\ \frac{T_b^*(r)}{C(r)} \in (1-\varepsilon, 1+\varepsilon),\\
&\qquad \frac{T^*_{b,Y}(r)}{C(r)} \in (1-\varepsilon, 1+\varepsilon),\ T_b^*(\xi r) \geq (1-\varepsilon)C(\xi r)\},\\
A_r^- :&= \{X_{T_b^*(r)} < 0,\ Y_{T^*_{b,Y}(r)} < 0,\ \frac{T_b^*(r)}{C(r)} \in (1-\varepsilon, 1+\varepsilon),\\
&\qquad \frac{T^*_{b,Y}(r)}{C(r)} \in (1-\varepsilon, 1+\varepsilon),\ T_b^*(\xi r) \geq (1-\varepsilon)C(\xi r)\}.
\end{aligned}$$

Then on A_r^+ we have

$$\begin{aligned}
X_{T_b^*(r)+T^*_{b,Y}(r)} &= X_{T_b^*(r)} + Y_{T^*_{b,Y}(r)}\\
&\geq r(T_b^*(r))^b + r(T^*_{b,Y}(r))^b\\
&> 2r((1-\varepsilon)C(r))^b\\
&= \xi r\,(2(1+\varepsilon)C(r))^b\\
&\geq \xi r\left(T_b^*(r) + T^*_{b,Y}(r)\right)^b.
\end{aligned}$$

Hence, still on A_r^+, we have

$$(1-\varepsilon)C(\xi r) \leq T_b^*(\xi r) \leq T_b^*(r) + T^*_{b,Y}(r) \leq 2(1+\varepsilon)C(r). \tag{4.5}$$

Replacing X by $-X$ (which does not change T_b^* or $T^*_{b,Y}$) in this argument shows that (4.5) also holds on A_r^-.

Since $P(A_r^+ \cup A_r^-) > 0$ for small r (in fact, $\liminf_{r\to 0} P(A_r^+ \cup A_r^-) \geq 1/2$), we have for small r

$$\frac{C(\xi r)}{C(r)} \leq \frac{2(1+\varepsilon)}{1-\varepsilon},$$

proving that

$$\limsup_{r\to 0} \frac{C(\xi r)}{C(r)} \le 2.$$

Thus (4.2) holds for $\beta = \xi$, and the general result holds, for the two-sided case, by iteration and monotonicity.

(b) Exactly the same argument works for the one-sided case if T_b^* is replaced by $\overline{T}_b$ throughout, including in the definition of Y in (4.3), and $T_{b,Y}^*(r)$ is replaced by the corresponding one-sided exit time in (4.4). In fact the one-sided case is slightly simpler in that there is no need to consider the events A_r^- since $P(A_r^+) \to 1$ as $r \to 0$. □

Lemma 4.3 *Suppose there is a function $C(r) > 0$ such that $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to 0$. Then, with $\lambda := 2^{1-b}$, we have*

$$\liminf_{r\to 0} \frac{C(\lambda r)}{C(r)} \ge 2, \tag{4.6}$$

and, with $\beta = \lambda^n$, $n = 1, 2, \ldots$, we have

$$\liminf_{r\to 0} \frac{C(\beta r)}{C(r)} \ge \beta^{1/(1-b)}, \tag{4.7}$$

or, equivalently, with $\alpha = \lambda^{-n}$, $n = 1, 2, \ldots$,

$$\liminf_{r\to 0} \frac{C(r)}{C(\alpha r)} \ge \frac{1}{\alpha^{1/(1-b)}}. \tag{4.8}$$

Proof of Lemma 4.3. Fix $\varepsilon \in (0, 1)$ and let

$$Z_s := X_{(1-\varepsilon)C(r)+s} - X_{(1-\varepsilon)C(r)}, \ s \ge 0,$$

and $\overline{T}_{b,Z} := \inf\{t \ge 0 : Z_t > rt^b\}$, $r \ge 0$. Then on

$$A_r := \left\{(1-\varepsilon)C(r) < \overline{T}_b(r), (1-\varepsilon)C(r) < \overline{T}_{b,Z}(r), \overline{T}_b(\lambda r) \le (1+\varepsilon)C(\lambda r)\right\}$$

we claim

$$X_t \leq 2^{1-b} r t^b, \text{ for all } 0 \leq t \leq 2(1-\varepsilon)C(r). \tag{4.9}$$

This is trivial for $0 \leq t \leq (1-\varepsilon)C(r)$, while for $0 < s \leq (1-\varepsilon)C(r)$, on A_r,

$$\begin{aligned} X_{(1-\varepsilon)C(r)+s} &= X_{(1-\varepsilon)C(r)} + Z_s \\ &\leq r\left((1-\varepsilon)C(r)\right)^b + rs^b \\ &\leq 2^{1-b} r \left((1-\varepsilon)C(r) + s\right)^b, \end{aligned}$$

where the last inequality follows from convexity of $x \mapsto x^b$, $0 \leq b < 1$, which implies $x^b + y^b \leq 2^{1-b}(x+y)^b$, for $x, y > 0$. Thus we get (4.9). So, on A_r, we have $\overline{T}_b(\lambda r) \geq 2(1-\varepsilon)C(r)$, where, recall, $\lambda = 2^{1-b}$. Since $P(A_r) > 0$ for small r (in fact $P(A_r) \to 1$ as $r \to 0$) this gives

$$2(1-\varepsilon)C(r) \leq (1+\varepsilon)C(\lambda r).$$

Letting $r \to 0$ then $\varepsilon \to 0$ yields (4.6).

For (4.7), let $\beta = \lambda^n$ and write

$$\frac{C(\beta r)}{C(r)} = \prod_{k=1}^{n} \frac{C(\lambda^k r)}{C(\lambda^{k-1} r)},$$

from which we get $\liminf_{r \to 0} C(\beta r)/C(r) \geq 2^n$. But $2^n = (\lambda^{1/(1-b)})^n = \beta^{1/(1-b)}$.

Finally, (4.8) follows immediately from (4.7). □

Now we need a little analysis. Fix $n \geq 1$, $a = a_n > 0$ and $\alpha \in (0,1)$, and consider the following curves for $t \geq a$:

$$y_1 = r_n t^b \qquad \text{and} \qquad y_2 = \alpha r_{n+1}(t-a)^b + r_{n+1} a^b,$$

where for notational convenience we let $r_n = 1/n$. (A picture which also includes the curve $y = r_{n+1} t^b$ is helpful). We wish to estimate where these curves intersect. For this it is more convenient to consider them in the new coordinate system:

$$s = t - a, \qquad Y = y - r_{n+1}a^b,$$

in which they become

$$Y_1 = r_n(s+a)^b - r_{n+1}a^b := f_1(s) \qquad \text{and} \qquad Y_2 = \alpha r_{n+1}s^b := f_2(s). \tag{4.10}$$

Elementary calculus shows that $f_1' = f_2'$ iff

$$s = \frac{(\alpha n)^{1/(1-b)}a}{(n+1)^{1/(1-b)} - (\alpha n)^{1/(1-b)}}.$$

Thus the curves cross at most twice. We will show that they cross at exactly 2 points and estimate the positions of these points.

To do this, first note that the function

$$g(x) := \frac{(1+x)^b - 1}{x^b}, \ x > 0, \tag{4.11}$$

is strictly increasing on $(0,\infty)$, with $g(x) \downarrow 0$ as $x \downarrow 0$, and $g(x) \uparrow 1$ as $x \uparrow \infty$. For $\alpha \in (0,1)$ define $c(\alpha) = g^{-1}(\alpha)$, and, for $c > 0$,

$$R_n(c) := \frac{ca}{(\alpha n)^{1/b}} \quad \text{and} \quad \widehat{R}_n(c) := ca. \tag{4.12}$$

Now we need:

Lemma 4.4 *Define f_1, f_2, $R_n(c)$, and $\widehat{R}_n(c)$ as in* (4.10) *and* (4.12)*. Fix $\alpha \in (0,1)$, but allow $a = a_n > 0$ to vary with n. Then for large n,*

$$f_1(R_n(c)) \gtrless f_2(R_n(c)) \text{ if } c \lessgtr 1; \tag{4.13}$$

and

$$f_1(\widehat{R}_n(c)) \overset{<}{\underset{>}{}} f_2(\widehat{R}_n(c)) \text{ if } c \overset{<}{\underset{>}{}} c(\alpha). \tag{4.14}$$

Consequently, for any $\varepsilon \in (0,1)$, if n is sufficiently large, then

$$f_2(s) > f_1(s) \quad \textit{for all} \quad s \in \big(R_n(1+\varepsilon), \widehat{R}_n((1-\varepsilon)c(\alpha))\big). \tag{4.15}$$

Proof of Lemma 4.4. First, as $n \to \infty$,

$$\begin{aligned} n(n+1)f_1(R_n(c)) &= (n+1)\left(1 + c/(\alpha n)^{1/b}\right)^b a^b - na^b \\ &= (n+1)\left(1 + bc/(\alpha n)^{1/b} + O(1/n^{2/b})\right) a^b - na^b \\ &\to a^b, \text{ since } b < 1, \end{aligned}$$

while

$$n(n+1)f_2(R_n(c)) = \alpha n (ca/(\alpha n)^{1/b})^b = c^b a^b,$$

which proves the first statement.

For the second, we have that

$$\begin{aligned} \frac{f_1(\widehat{R}_n(c))}{f_2(\widehat{R}_n(c))} &= \frac{(n+1)(1+c)^b a^b - na^b}{\alpha n (ca)^b} \\ &= \frac{(1+c)^b - 1}{\alpha c^b} + \frac{(1+c)^b}{\alpha n c^b}. \end{aligned}$$

Since the second term on the RHS tends to 0, the result then follows from the definition of $c(\alpha)$ and the monotonicity of g.

Finally, (4.15) follows immediately from (4.13) and (4.14). □

Lemma 4.5 *Suppose $\overline{T}_b(r)/C(r) \xrightarrow{\text{P}} 1$, as $r \to 0$, where $C(r) > 0$. Then with $r_n = 1/n$, we have*

$$\limsup_{n\to\infty} \frac{C(r_n)}{C(r_{n+1})} \leq 1.$$

Proof of Lemma 4.5. Fix $\varepsilon \in (0,1)$ small enough that

$$\frac{b^{1/(1-b)}(1+\varepsilon)^3}{(1-\varepsilon)^2} < 1. \tag{4.16}$$

Recall $c(\alpha) = g^{-1}(\alpha)$, where g is defined in (4.11). As $x \downarrow 0$, we have

$$\frac{(1+x)^b - 1}{x^b} \sim \frac{bx}{x^b} = bx^{1-b},$$

thus $bc(\alpha)^{1-b} \sim \alpha$, as $\alpha \downarrow 0$, or, equivalently,

$$\lim_{\alpha \downarrow 0} \frac{c(\alpha)}{\alpha^{1/(1-b)}} = \frac{1}{b^{1/(1-b)}}.$$

Hence we can choose $\alpha > 0$ of the form $\alpha = \lambda^{-k}$ small enough that

$$\frac{\alpha^{1/(1-b)}}{b^{1/(1-b)}} < (1+\varepsilon)c(\alpha). \tag{4.17}$$

For $n \geq 1$ let

$$A_n = \left\{ \frac{\overline{T}_b(r_n)}{C(r_n)} \in (1-\varepsilon, 1+\varepsilon), \frac{\overline{T}_b(r_{n+1})}{C(r_{n+1})} \in (1-\varepsilon, 1+\varepsilon), \frac{\overline{T}_{b,Y}(\alpha r_{n+1})}{C(\alpha r_{n+1})} \in (1-\varepsilon, 1+\varepsilon) \right\}$$

where $Y_s = X_{\overline{T}_b(r_{n+1})+s} - X_{\overline{T}_b(r_{n+1})},\ s \ge 0$, and $\overline{T}_{b,Y}(r) := \inf\{s \ge 0 : Y_s > rs^b\},\ r \ge 0$. Then on A_n we have, for n sufficiently large, depending on α,

$$\begin{aligned}
\frac{(1+\varepsilon)\overline{T}_b(r_{n+1})}{(\alpha(n+1))^{1/b}} &\le \frac{(1+\varepsilon)^2 C(r_{n+1})}{(\alpha(n+1))^{1/b}} \\
&\le (1-\varepsilon)C(\alpha r_{n+1}) \quad \text{(by Lemma 4.2, and taking } n \text{ large enough)} \\
&< \overline{T}_{b,Y}(\alpha r_{n+1}) \\
&< (1+\varepsilon)C(\alpha r_{n+1}) \\
&\le (1+\varepsilon)^2 \alpha^{1/(1-b)} C(r_{n+1}) \quad \text{(by (4.8))} \\
&\le \frac{(1+\varepsilon)^2}{(1-\varepsilon)} \alpha^{1/(1-b)} \overline{T}_b(r_{n+1}) \\
&\le \frac{(1+\varepsilon)^3}{(1-\varepsilon)} b^{1/(1-b)} c(\alpha) \overline{T}_b(r_{n+1}) \quad \text{(by (4.17))} \\
&\le (1-\varepsilon)c(\alpha)\overline{T}_b(r_{n+1}) \quad \text{(by (4.16))}.
\end{aligned}$$

Thus with $a = a_{n+1} = \overline{T}_b(r_{n+1})$ in (4.10) and (4.12), we have shown that on A_n, for large n,

$$R_{n+1}(1+\varepsilon) < \overline{T}_{b,Y}(\alpha r_{n+1}) < \widehat{R}_{n+1}((1-\varepsilon)c(\alpha)). \tag{4.18}$$

This means that, on A_n,

$$\begin{aligned}
X_{\overline{T}_b(r_{n+1})+\overline{T}_{b,Y}(\alpha r_{n+1})} &= X_{\overline{T}_b(r_{n+1})} + Y_{\overline{T}_{b,Y}(\alpha r_{n+1})} \\
&\ge r_{n+1}(\overline{T}_b(r_{n+1}))^b + f_2(\overline{T}_{b,Y}(\alpha r_{n+1})) \\
&> r_{n+1}(\overline{T}_b(r_{n+1}))^b + f_1(\overline{T}_{b,Y}(\alpha r_{n+1})) \quad \text{(by (4.15) and (4.18))} \\
&= r_n(\overline{T}_b(r_{n+1}) + \overline{T}_{b,Y}(\alpha r_{n+1}))^b
\end{aligned}$$

by (4.10), and so

$$\overline{T}_b(r_n) \le \overline{T}_b(r_{n+1}) + \overline{T}_{b,Y}(\alpha r_{n+1}).$$

Since $P(A_n) > 0$ for large n this implies

$$(1-\varepsilon)C(r_n) \le (1+\varepsilon)C(r_{n+1}) + (1+\varepsilon)C(\alpha r_{n+1}).$$

Hence

$$\limsup_{n\to\infty} \frac{C(r_n)}{C(r_{n+1})} \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\left(1 + \limsup_{n\to\infty} \frac{C(\alpha r_{n+1})}{C(r_{n+1})}\right) \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\left(1+\alpha^{1/(1-b)}\right),$$

by (4.8). Now let $\alpha \downarrow 0$ then $\varepsilon \downarrow 0$ to complete the proof. □

Proof of Proposition 4.1. By Lemma 4.1 and the paragraph following, we may assume C is nondecreasing and $C(r) \to 0$ as $r \to 0$. With $r_n = 1/n$ as above, define

$$D(r_n) = C(r_n)(1 + r_n), \quad n \ge 1,$$

and interpolate $D(r_n)$ linearly for $0 < r \le 1$. Then D is continuous and strictly increasing on $(0, 1]$. Thus to complete the proof of Proposition 4.1, it suffices to show that

$$\frac{D(r)}{C(r)} \to 1 \text{ as } r \to 0.$$

This follows easily from Lemma 4.5, because, given $r \in (0, 1]$, by letting n satisfy $r_{n+1} < r \le r_n$, we obtain

$$D(r) \le D(r_n) = C(r_n)(1 + r_n) \le [C(r) + (C(r_n) - C(r_{n+1}))]\,(1 + r_n),$$

hence

$$\frac{D(r)}{C(r)} \le \left\{1 + \frac{C(r_n) - C(r_{n+1})}{C(r_{n+1})}\right\}(1 + r_n) \to 1 \text{ as } n \to \infty,$$

while

$$D(r) \ge D(r_{n+1}) \ge C(r_{n+1}) \ge C(r) - (C(r_n) - C(r_{n+1})),$$

and so, also,

$$\frac{D(r)}{C(r)} \ge 1 - \frac{C(r_n) - C(r_{n+1})}{C(r_{n+1})} \to 1 \text{ as } n \to \infty.$$

□

Proof of Theorem 3.1. We first show that if X is positively relatively stable, then so are $\overline{T}_b$ and T_b^*. This will prove one direction of (a), and also one direction of (b) because if $|X_t|/B(t) \stackrel{\mathrm{P}}{\longrightarrow} 1$, then by Proposition 2.1, either $X \in PRS$ or $X \in NRS$, and in the latter case we simply apply the result to $-X$ rather than X, which does not change T_b^*. Thus assume there is a non-stochastic function $B(t) > 0$ such that $X_t/B(t) \stackrel{\mathrm{P}}{\longrightarrow} 1$, as $t \to 0$, assumed to have the properties listed in Proposition 2.1. Let $C(r)$ be the inverse function to $B(t)/t^b$, uniquely defined because $B(t)/t^b$ is chosen continuous and strictly increasing. Thus we have $B(C(r)) = rC(r)^b$. Since $T_b^*(r) \le \overline{T}_b(r)$, it suffices to show

$$P\left(\overline{T}_b(r) > (1+\varepsilon)C(r)\right) \to 0 \ \text{ and } \ P\left(T_b^*(r) < (1-\varepsilon)C(r)\right) \to 0 \ \text{ as } r \to 0 \tag{4.19}$$

for every $\varepsilon > 0$. Now for any $\eta \in (0, 1+\varepsilon)$,

$$\begin{aligned} P\left(\overline{T}_b(r) > (1+\varepsilon)C(r)\right) &\le P\left(\sup_{0<t\le(1+\varepsilon)C(r)} \frac{X_t}{t^b} \le r\right) \\ &\le P\left(\sup_{\eta\le\lambda\le1+\varepsilon} \frac{\lambda^{-b}X_{\lambda C(r)}}{B(C(r))} \le 1\right) \to 0 \end{aligned}$$

by Proposition 2.2. A similar argument shows that for any $\varepsilon \in (0,1)$ and $\eta \in (0, 1-\varepsilon)$

$$P\left(T_b^*(r) < (1-\varepsilon)C(r)\right) \le P\left(T_b^* < \eta C(r)\right) + P\left(\sup_{\eta\le\lambda\le1-\varepsilon} \frac{\lambda^{-b}|X_{\lambda C(r)}|}{B(C(r))} \ge 1\right).$$

As above, the second term converges to 0 as $r \to 0$ by Proposition 2.2. For the first we use the Lévy process version of Remark 2.1 and Proposition 2.1 in [7], which translated into our notation, and using that A is slowly varying, gives, for some universal constant c,

$$\begin{aligned} P\left(T_b^*(r) < \eta C(r)\right) &\le \frac{c\eta C(r)A(r(\eta C(r))^b)}{r(\eta C(r))^b} \\ &\sim \frac{c\eta^{1-b}C(r)A(rC(r)^b)}{rC(r)^b} \\ &\to c\eta^{1-b} \end{aligned}$$

as $r \to 0$, since $B(C(r)) = rC(r)^b$, $A(\cdot)$ is slowly varying at 0, and $tA(B(t))/B(t) \to 1$ as $t \to 0$, by Proposition 2.1. Letting $\eta \to 0$ completes the proof of (4.19).

We now come to the converse direction. We first consider (a). Thus assume there exists a finite function $C(r) > 0$ such that $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to 0$. Then $C(r) \to 0$ as $r \to 0$, and we may assume that $C(\cdot)$ is continuous and strictly increasing. Thus $B(t) := t^b C^{-1}(t)$ is uniquely defined and $t^{-b}B(t) \downarrow 0$ as $t \downarrow 0$. We first show that, for each $\delta > 0$,

$$\lim_{t\to 0} P\left(\frac{X_t}{B(t)} > 1+\delta\right) = 0. \tag{4.20}$$

To see this, take $t > 0$ and $\lambda > 0$, and define $r = C^{-1}(t/\lambda)$, so that $\lambda C(r) = t$. On the event $\{\overline{T}_b(r)/C(r) > \lambda\}$ we have $X_s \le rs^b$ for all $0 \le s \le \lambda C(r) = t$, and hence

$$\overline{X}_t \le t^b C^{-1}(t/\lambda) = \lambda^b B(t/\lambda)$$

on that event. Thus for every $0 < \lambda < 1$,

$$\liminf_{t\to 0} P\left(\frac{\overline{X}_t}{B(t/\lambda)} \le \lambda^b\right) \ge \liminf_{r\to 0} P\left(\frac{\overline{T}_b(r)}{C(r)} > \lambda\right) = 1. \tag{4.21}$$

Now given $\delta > 0$, choose $0 < \varepsilon < 1$ so that $(1-\varepsilon)^b + \varepsilon^b < 1+\delta$. This is possible since $(1-\varepsilon)^b + \varepsilon^b \downarrow 1$ as $\varepsilon \downarrow 0$. Hence

$$P\left(\frac{X_t}{B(t)} > 1+\delta\right) \le P\left(\frac{\overline{X}_t}{B(t)} > 1+\delta\right) \le P\left(\frac{\overline{X}_{(1-\varepsilon)t}}{B(t)} > (1-\varepsilon)^b\right) + P\left(\frac{\overline{X}_{\varepsilon t}}{B(t)} > \varepsilon^b\right) \to 0$$

as $r \to 0$, by (4.21).

Next, still assuming that $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to 0$, we show that, for each $\delta > 0$,

$$\lim_{t\to 0} P\left(\frac{X_t}{B(t)} > 1-\delta\right) = 1. \tag{4.22}$$

To see this, take $\delta > 0$ and choose $\varepsilon \in (0, 1)$ small enough that

$$(2\varepsilon)^b \le \delta \quad \text{and} \quad (1 + 2\varepsilon)(1 - \varepsilon) > 1. \tag{4.23}$$

Given $t > 0$, set $r = C^{-1}(t/(1-\varepsilon))$; thus $t = (1-\varepsilon)C(r)$. Now since $\overline{T}_b(r)/C(r) \xrightarrow{\text{P}} 1$, we have

$$1 - \varepsilon < \frac{\overline{T}_b(r)}{C(r)} < (1 + 2\varepsilon)(1 - \varepsilon)$$

WPA1 as $r \to 0$. On this event

$$X_s > rs^b \text{ for some } s \in [(1-\varepsilon)C(r), (1+2\varepsilon)(1-\varepsilon)C(r)] = [t, (1+2\varepsilon)t].$$

Thus for this s,

$$X_s > rs^b \ge C^{-1}\left(\frac{t}{1-\varepsilon}\right) t^b \ge C^{-1}(t)\, t^b = B(t). \tag{4.24}$$

Now suppose (4.22) fails. Then along a subsequence $t_k \to 0$, with probability bounded away from 0, we have

$$X_s - X_{t_k} > B(t_k) - (1-\delta)B(t_k) = \delta B(t_k),$$

for some $s \in [t_k, (1+2\varepsilon)t_k]$, by (4.24). Thus with probability bounded away from 0

$$\overline{X}_{2\varepsilon t_k} > \delta B(t_k) = p_k(2\varepsilon t_k)^b, \tag{4.25}$$

where

$$p_k := \frac{\delta B(t_k)}{(2\varepsilon t_k)^b} \to 0. \tag{4.26}$$

But

$$C^{-1}(t_k) = \frac{B(t_k)}{t_k^b} = \frac{(2\varepsilon)^b p_k}{\delta} \le p_k$$

by (4.23), and so, $t_k \le C(p_k)$. Hence by (4.25) and (4.26), with probability bounded away from 0, $\overline{T}_b(p_k) \le 2\varepsilon t_k \le 2\varepsilon C(p_k)$ where $p_k \to 0$. This contradicts the relative stability of $\overline{T}_b(\cdot)$. Thus (4.22) holds and together with (4.20) this proves $X \in PRS$.

We now consider (b). Thus suppose $T_b^*(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to 0$ for a function $C(r) > 0$. Then $C(r) \to 0$ as $r \to 0$, and we may assume, by Lemma 4.1, that $C(\cdot)$ is nondecreasing. (Note that we are not assuming *a priori* that $C(\cdot)$ is continuous and strictly increasing, in this proof.) For any $t > 0$, define $C^{-1}(t) = \inf\{r > 0 : C(r) \geq t\}$. Then

$$C(C^{-1}(t)-) \leq t \leq C(C^{-1}(t)+),$$

and so by Lemma 4.2, for some constants $0 < c_1 \leq c_2 < \infty$

$$c_1 \; C(C^{-1}(t)) \leq t \leq c_2 \; C(C^{-1}(t)), \tag{4.27}$$

if t is sufficiently small.

Observe that for any $\lambda > 0$,

$$P\left(T_b^*(r) \leq \lambda C(r)\right) \geq P\left(\sup_{0<t\leq \lambda C(r)} \frac{|X_t|}{t^b} > r\right) \geq P\left(\frac{X^*_{\lambda C(r)}}{(\lambda C(r))^b} > r\right).$$

The LHS tends to 0 as $r \to 0$ for $\lambda \in (0,1)$, so we have, for such λ,

$$\lim_{r \to 0} P\left(X^*_{\lambda C(r)} \leq \lambda^b D(r)\right) = 1 \tag{4.28}$$

where $D(r) = rC(r)^b$. Now take any sequence $t_k \to 0$ and define $r_k = C^{-1}(t_k)$. Note that by (4.27), for large k

$$c_1 \; C(r_k) \leq t_k \leq c_2 \; C(r_k). \tag{4.29}$$

Setting $\lambda = 1/2$, it follows from (4.28) that along a further subsequence $t_{k'}$,

$$\frac{X_{C(r_{k'})/2}}{D(r_{k'})} \xrightarrow{\mathrm{D}} Z'(1/2),$$

where $|Z'(1/2)| \le (1/2)^b$ a.s. Since $Z'(1/2)$ is infinitely divisible, this means $Z'(1/2)$ is degenerate at a constant, $c'(1/2)$, say. By considering characteristic functions for example, this then implies

$$\frac{X_{\lambda C(r_{k'})}}{D(r_{k'})} \xrightarrow{\mathrm{P}} c'(\lambda), \tag{4.30}$$

for all $\lambda > 0$, where the constants $c'(\lambda)$ satisfy $c'(\lambda) = \lambda c'(1)$.

We next show $c'(1) \neq 0$. If not, (4.30) gives $X_{\lambda C(r_{k'})}/D(r_{k'}) \xrightarrow{\mathrm{P}} 0$ for every $\lambda > 0$, and so by Lemma 2.1 $X^*_{\lambda C(r_{k'})}/D(r_{k'}) \xrightarrow{\mathrm{P}} 0$ for every $\lambda > 0$. Fix $\varepsilon \in (0,1)$. Then for any $\delta \in (0,1)$, $s, r > 0$,

$$\begin{aligned} P\left(T_b^*(r) \ge s\right) &\ge P\left(\sup_{0<t\le s} \frac{|X_t|}{t^b} \le r\right) \\ &\ge P\left(\sup_{\delta s<t\le s} |X_t| \le r(\delta s)^b, \sup_{0<t\le \delta s} \frac{|X_t|}{t^b} \le r\right) \\ &\ge P\left(\sup_{\delta s<t\le s} |X_t - X_{\delta s}| \le \varepsilon r(\delta s)^b, \sup_{0<t\le \delta s} \frac{|X_t|}{t^b} \le (1-\varepsilon) r\right) \\ &\ge P\left(X^*_{(1-\delta)s} \le \varepsilon r(\delta s)^b\right) P\left(T_b^*((1-\varepsilon)r) > \delta s\right). \end{aligned} \tag{4.31}$$

Now by Lemma 4.2, for r small enough

$$1 \le \frac{C(r)}{C((1-\varepsilon)r)} \le c_\varepsilon,$$

for some $c_\varepsilon < \infty$. Thus if we let

$$\delta = \frac{1-\varepsilon}{(1+\varepsilon)c_\varepsilon}, \quad \lambda = (1-\delta)(1+\varepsilon)$$

and substitute $r = r_k$, $s = (1+\varepsilon)C(r_k)$ into (4.31), we obtain for large k

$$P\left(T_b^*(r_k) \ge (1+\varepsilon)C(r_k)\right) \ge P\left(X^*_{\lambda C(r_k)} \le \varepsilon[\delta(1+\varepsilon)]^b D(r_k)\right) P\left(\frac{T_b^*((1-\varepsilon)r_k)}{C((1-\varepsilon)r_k)} > 1-\varepsilon\right).$$

But along the sequence k', the LHS converges to 0 while both terms on the RHS converge to 1. Thus it must be the case that $c'(1) \neq 0$.

By again considering characteristic functions, (4.30) easily extends to

$$\frac{X_{\lambda_{k'}C(r_{k'})}}{D(r_{k'})} \xrightarrow{\text{P}} c'(\lambda),$$

if $\lambda_{k'} \to \lambda \in [0, \infty)$. By choosing a further subsequence k'' of k' if necessary, it follows from (4.29) that for some $\widehat{\lambda} \in (0, \infty)$,

$$\frac{t_{k''}}{C(r_{k''})} \to \widehat{\lambda}.$$

Consequently

$$\frac{X_{t_{k''}}}{D(r_{k''})} \xrightarrow{\text{P}} c'(\widehat{\lambda}),$$

where $c'(\widehat{\lambda}) = \widehat{\lambda}c'(1) \neq 0$. Hence every sequence $t_k \to 0$ contains a subsequence $t_{k''} \to 0$ with $X_{t_{k''}}/D(r_{k''})$ converging to a finite nonzero constant. Thus, by Proposition 2.1, we have $X \in RS$.

Finally under (a) or (b), the proofs show that $C(r)$ may be taken as the inverse of the continuous and strictly increasing function $B(t)/t^b$ where $B(t)$ is regularly varying with index 1. Hence $C(r)$ is regularly varying with index $1/(1-b)$ and may be taken to be continuous and strictly increasing. □

Proof of Corollary 3.1. Assume $X_t/B(t) \xrightarrow{\text{P}} 1$, where $B(t)/t^b$ is continuous and strictly increasing and $B(t)$ is regularly varying with index 1. Then by Theorem 3.1

$$\frac{T_b^*(r)}{C(r)} \xrightarrow{\text{P}} 1 \quad \text{and} \quad \frac{\overline{T}_b(r)}{C(r)} \xrightarrow{\text{P}} 1,$$

where C is the inverse of $B(t)/t^b$. In particular $B(C(r)) = rC(r)^b$. Fix $\varepsilon \in (0, 1)$. On

$$A_r = \left\{ \frac{T_b^*(r)}{C(r)} \in (1-\varepsilon, 1+\varepsilon), \frac{\overline{T}_b(r)}{C(r)} \in (1-\varepsilon, 1+\varepsilon), T_b^*(r) \neq \overline{T}_b(r) \right\},$$

it must be the case that $X_{T_b^*(r)} < 0$ and so

$$X_{T_b^*(r)+s} - X_{T_b^*(r)} > 2r((1-\varepsilon)C(r))^b = 2(1-\varepsilon)^b B(C(r))$$

for some $0 \le s \le 2\varepsilon C(r)$. Hence if $\liminf_{r\to 0} P(\overline{T}_b(r) = T_b^*(r)) < 1$, then $\limsup_{r\to 0} P(A_r) > 0$ and so

$$\limsup_{r\to 0} P(\overline{X}_{2\varepsilon C(r)} > 2(1-\varepsilon)^b B(C(r))) > 0.$$

Using the regular variation of B, this contradicts $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ if ε is sufficiently small, by Lemma 2.1. □

Proof of Proposition 3.1. We will prove this in a little more generality than stated. Assume $f : (0,\infty) \mapsto (0,\infty)$ is such that $f(x) \to 0$ as $x \to 0$, and there exists $\varepsilon > 0$ for which

$$f(x+y) \ge f(x) + f(y), \quad \text{for all } \ 0 < x \le \varepsilon y \tag{4.32}$$

if y is sufficiently small.

For the one-sided exit, recall the definition of $\overline{T}_f(r)$ in (4.1), and assume there is a $C(r) > 0$ such that $\overline{T}_f(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to 0$. By Lemma 4.1 and the paragraph following it, $C(r) \to 0$ and we may assume $C(r)$ is nondecreasing. Fix $\varepsilon \in (0, 1/2)$ so that (4.32) holds. Observe that if r is sufficiently small,

$$rf(s) + rf((1-\varepsilon/2)C(r)) \le rf((1-\varepsilon/2)C(r) + s) \tag{4.33}$$

for all $0 \le s \le \varepsilon(1-\varepsilon/2)C(r)$. Since $\varepsilon(1-\varepsilon/2) \ge 3\varepsilon/4$, (4.33) holds in particular for all $0 \le s \le 3\varepsilon C(r)/4$. Now let

$$Y_s := X_{(1-\varepsilon/2)C(r)+s} - X_{(1-\varepsilon/2)C(r)}, \ s \ge 0$$

and

$$\overline{T}_f^Y(r) := \inf\{s \ge 0 : Y_s > rf(s)\}, \ r \ge 0.$$

Then by (4.33)

$$P\left(\overline{T}_f(r) \le (1+\varepsilon/4)C(r)\right) \le P\left(\overline{T}_f(r) \le (1-\varepsilon/2)C(r)\right) + P\left(\overline{T}_f^Y(r) \le 3\varepsilon C(r)/4\right). \tag{4.34}$$

Since the LHS of (4.34) tends to 1, while $P\left(\overline{T}_f(r) \le (1-\varepsilon/2)C(r)\right)$ tends to 0, we may conclude that

$$\lim_{r\to 0} P\left(\overline{T}_f(r) \le 3\varepsilon C(r)/4\right) = \lim_{r\to 0} P\left(\overline{T}_f^Y(r) \le 3\varepsilon C(r)/4\right) = 1,$$

which is a contradiction, since $3\varepsilon/4 < 1$.

The proof for the 2-sided exit is virtually the same; simply replace $\overline{T}_f^Y(r)$ by $T_f^{*,Y}(r) := \inf\{s > 0 : |Y_s| > rf(s)\}$, $r \ge 0$. (4.34) holds with this replacement. □

If $f(x) = x^b$ with $b \ge 1$ it's easy to check that f satisfies (4.32). More generally, if, for small y, f' is increasing and

$$f'(y) \ge \frac{f(x)}{x} \text{ for } 0 < x \le \varepsilon y,$$

then (4.32) holds. For example, $f(x) = x/|\log x|$ satisfies this condition. If (4.32) holds for all x, y and $\varepsilon = 1$, then f is superadditive, so the proposition holds for this class of functions also.

Before proceeding to the proof of Theorem 3.2, we need some preliminary results which may be of independent interest. We begin with a corollary to a result of Erickson [13]. In it we allow for the possibility of a killed subordinator Y, that is, a process obtained from a proper subordinator $\mathcal{Y}$ by killing at an independent exponential time $e(q)$ with mean q^{-1}; thus

$$Y_t = \begin{cases} \mathcal{Y}_t, & \text{if } t < e(q), \\ \partial, & \text{if } t \ge e(q), \end{cases}$$

where ∂ is a cemetery state. The extension from proper to killed subordinators is trivial, but is needed below.

Proposition 4.2 *Let Y be a (possibly killed) subordinator, then*

$$\lim_{t\to 0} \frac{Y_{t-}}{Y_t} = 1 \text{ a.s. } \quad \textit{iff} \quad \mathrm{d}_Y > 0.$$

Proof of Proposition 4.2. Since killing does not affect the drift, it suffices to prove the result for proper subordinators. If $\mathrm{d}_Y > 0$ then $Y_t/t \to \mathrm{d}_Y$ a.s. by Proposition 3.8 of [1]. This is easily seen to imply $Y_{t-}/t \to \mathrm{d}_Y$ a.s., and so

$$\lim_{t\to 0} \frac{Y_{t-}}{Y_t} = 1 \text{ a.s.} \tag{4.35}$$

Conversely, assume (4.35) holds and by way of contradiction assume $\mathrm{d}_Y = 0$. Clearly (4.35) implies that Y can not be compound Poisson. Since $\mathrm{d}_Y = 0$, and $\sigma_Y = 0$ since Y is a subordinator, it must be the case that $\Pi(\mathbb{R}) = \infty$. We may now apply Theorem 2 of [13]. In the terminology of [13], under (4.35), the function $h(x) = x$ is not a small gap function, and hence

$$\int_0^1 \frac{x\Pi(\mathrm{d}x)}{\int_0^x \overline{\Pi}(y)\mathrm{d}y} < \infty; \tag{4.36}$$

see Theorem 2 and the first paragraph of page 459 in [13]. Thus

$$\frac{\int_0^z x\Pi(\mathrm{d}x)}{\int_0^z \overline{\Pi}(y)\mathrm{d}y} \le \int_0^z \frac{x\Pi(\mathrm{d}x)}{\int_0^x \overline{\Pi}(y)\mathrm{d}y} \to 0 \text{ as } z \to 0.$$

Since

$$\int_0^x \overline{\Pi}(y)\mathrm{d}y = x\overline{\Pi}(x) + \int_0^x y\Pi(\mathrm{d}y),$$

(4.36) then implies that

$$\int_0^1 \frac{\Pi(\mathrm{d}x)}{\overline{\Pi}(x)} < \infty. \tag{4.37}$$

But this is a contradiction since $\Pi(\mathbb{R}) = \infty$. □

Remark: Let $\Delta Y_t = Y_t - Y_{t-}$. Then Proposition 4.2 can be rephrased as

$$\lim_{t\to 0} \frac{\Delta Y_t}{Y_{t-}} = 0 \text{ a.s.} \quad \textit{iff} \quad \mathrm{d}_Y > 0.$$

By a similar argument, one can check that if Y is not a compound Poisson subordinator, then

$$\limsup_{t\to 0} \frac{\Delta Y_t}{Y_{t-}} = \infty \text{ a.s.} \quad \textit{iff} \quad \mathrm{d}_Y = 0.$$

At this point we need to introduce a little fluctuation theory for which we refer to [1], [5] or [17]. Let L_t denote the local time of X at its maximum and $(L_t^{-1}, H_t)_{t\geq 0}$ the bivariate ascending ladder process of X. If $X_t \to -\infty$ a.s then (L^{-1}, H) may be obtained from a proper bivariate subordinator by exponential killing. Let $\kappa(\cdot,\cdot)$ denote the Laplace exponent of (L^{-1}, H). Then

$$\kappa(\alpha,\beta) = k + \mathrm{d}_{L^{-1}}\alpha + \mathrm{d}_H\beta + \int_{t\geq 0}\int_{h\geq 0}\left(1 - e^{-\alpha t-\beta h}\right)\Pi_{L^{-1},H}(\mathrm{d}t,\mathrm{d}h), \quad \alpha,\beta \geq 0, \tag{4.38}$$

where $\mathrm{d}_{L^{-1}} \geq 0$ and $\mathrm{d}_H \geq 0$ are drift constants, $\Pi_{L^{-1},H}$ is the Lévy measure of (L^{-1}, H) and $k \geq 0$ is the killing rate.

Lemma 4.6 *For any Lévy process X,*

$$X \textit{ is of bounded variation and } \mathrm{d}_X > 0 \textit{ iff } \mathrm{d}_{L^{-1}} > 0 \textit{ and } \mathrm{d}_H > 0.$$

In that case $\mathrm{d}_X = \mathrm{d}_H/\mathrm{d}_{L^{-1}}$.

Proof of Lemma 4.6. By Theorem 2.2*(b)(ii)* of [14], X is of bounded variation and $\mathrm{d}_X > 0$ iff $\sigma = 0$, $\mathrm{d}_{L^{-1}} > 0$ and $\mathrm{d}_H > 0$, in which case $\mathrm{d}_X = \mathrm{d}_H/\mathrm{d}_{L^{-1}}$. Thus it suffices to show that if $\mathrm{d}_{L^{-1}} > 0$ and $\mathrm{d}_H > 0$, then $\sigma = 0$. If X is compound Poisson then so is H, and consequently $\mathrm{d}_H = 0$. But $\mathrm{d}_H > 0$, and so X can not be compound Poisson. Thus by Corollary 4.4(v) of [5], since $\mathrm{d}_{L^{-1}} > 0$, the downward ladder height process $\widehat{H}$ is compound Poisson. Hence $\mathrm{d}_{\widehat{H}} = 0$, and so by Corollary 4.4(i) of [5], $\sigma = 0$. □

The following result is a companion to Theorem 4.2 in [9].

Proposition 4.3 *Assume that*

$$\lim_{t\to 0}\frac{\overline{X}_t}{B(t)} = 1 \ a.s. \tag{4.39}$$

for some function $B(t) > 0$, then X is of bounded variation with $\mathrm{d}_X > 0$.

Proof of Proposition 4.3. Since $\overline{X}_t/B(t) \xrightarrow{\mathrm{P}} 1$ as $t \to 0$, it follows from Lemma 2.1 that $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ as $t \to 0$. Hence by Proposition 2.1 we may assume that B is increasing, continuous and regularly varying with index 1. By continuity of B it follows easily from (4.39) that

$$\lim_{t\to 0}\frac{\overline{X}_{t-}}{B(t)} = 1 \text{ a.s.} \tag{4.40}$$

Fix $t < L_\infty$. If $L^{-1}_{t-} < L^{-1}_t$, then $\overline{X}_{L^{-1}_{t-}} = \overline{X}_{(L^{-1}_t)-}$ since $\overline{X}$ does not increase on the interval (L^{-1}_{t-}, L^{-1}_t). Hence by (4.39) and (4.40),

$$\frac{B(L^{-1}_{t-})}{B(L^{-1}_t)} = I(L^{-1}_{t-} = L^{-1}_t) + I(L^{-1}_{t-} < L^{-1}_t)\frac{B(L^{-1}_{t-})}{\overline{X}_{L^{-1}_{t-}}}\,\frac{\overline{X}_{(L^{-1}_t)-}}{B(L^{-1}_t)} \to 1 \text{ a.s.} \tag{4.41}$$

Consequently, by regular variation of B,

$$\lim_{t\to 0}\frac{L^{-1}_{t-}}{L^{-1}_t} = 1 \text{ a.s.}$$

Thus $\mathrm{d}_{L^{-1}} > 0$ by Proposition 4.2. Further

$$\lim_{t\to 0}\frac{H_{t-}}{H_t} = \lim_{t\to 0}\frac{\overline{X}_{(L^{-1}_{t-})-}}{\overline{X}_{L^{-1}_t}} = 1 \text{ a.s.}$$

by (4.39)-(4.41). Hence $\mathrm{d}_H > 0$, again by Proposition 4.2. The result now follows from Lemma 4.6.
□

Proof of Theorem 3.2. (a) Assume that X has bounded variation with drift $\mathrm{d}_X \neq 0$. Then by Theorem 4.2 of [9]

$$\lim_{t\to 0}\frac{X_t}{t} = \mathrm{d}_X \ \text{ a.s.} \tag{4.42}$$

This is easily seen to imply $t^{-1}X_{t-} \to \mathrm{d}_X$ a.s., and so

$$\lim_{t\to 0}\frac{\Delta X_t}{t} = 0 \ \text{ a.s.} \tag{4.43}$$

where $\Delta X_t = X_t - X_{t-}$. Now

$$r(T^*_b(r))^b \le |X_{T^*_b(r)}| \le r(T^*_b(r))^b + |\Delta X_{T^*_b(r)}|,$$

so dividing through by $T_b^*(r)$ and using (4.42) and (4.43) we obtain

$$\lim_{r\to 0} r(T_b^*(r))^{b-1} = |\mathrm{d}_X| \text{ a.s.,} \tag{4.44}$$

or equivalently

$$\lim_{r\to 0} \frac{T_b^*(r)}{r^{1/(1-b)}} = \frac{1}{|\mathrm{d}_X|^{1/(1-b)}} \text{ a.s.} \tag{4.45}$$

Conversely, assume $T_b^*(r)/C(r) \to 1$ a.s. as $r \to 0$. Then by Theorem 3.1, we may assume C is regularly varying with index $1/(1-b)$, continuous, strictly increasing and $C(r) \to 0$ as $r \to 0$. Let $B(t) = t^b C^{-1}(t)$ and fix $\eta < 1 < \lambda$. Then a.s. for small t, we have $\eta t < T_b^*(C^{-1}(t)) < \lambda t$. For such t

$$X_{\lambda t}^* \ge X_{T_b^*(C^{-1}(t))}^* \ge C^{-1}(t)(T_b^*(C^{-1}(t)))^b > C^{-1}(t)(\eta t)^b = \eta^b B(t) \tag{4.46}$$

and

$$X_{\eta t}^* \le X_{T_b^*(C^{-1}(t))-}^* \le C^{-1}(t)(T_b^*(C^{-1}(t)))^b < C^{-1}(t)(\lambda t)^b = \lambda^b B(t). \tag{4.47}$$

Since $B(t)$ is regularly varying with index 1 as $t \to 0$, it then easily follows from (4.46) and (4.47) that $\lim_{t\to 0} X_t^*/B(t) = 1$ a.s. An examination of the proof of Theorem 4.2 in [9], shows that from this we may conclude that X has bounded variation with $\mathrm{d}_X \ne 0$.

(b) Now assume that X has bounded variation with drift $\mathrm{d}_X > 0$. Then $\lim_{t\to 0} t^{-1} X_t = \mathrm{d}_X > 0$ a.s., and so $P(\overline{T}_b(r) = T_b^*(r)$ for all small $r) = 1$. Consequently, from part (a),

$$\lim_{r\to 0} \frac{\overline{T}_b(r)}{r^{1/(1-b)}} = \frac{1}{\mathrm{d}_X^{1/(1-b)}} \text{ a.s.} \tag{4.48}$$

The proof of the converse for $\overline{T}_b(r)$ is virtually the same as for $T_b^*(r)$. Arguing as above, first show $\lim_{t\to 0} \overline{X}_t/B(t) = 1$ a.s., and then use Proposition 4.3 to complete the proof. □

Proof of Theorem 3.3.

(a) Assume the first condition fails. Fix $\xi > 1$ so that, with

$$A_r = \left\{ X_{\overline{T}_b(r)} > \xi r(\overline{T}_b(r))^b \right\},$$

we have $\limsup_{r\to 0} P(A_r) > 0$. Now $\overline{T}_b(r) = \overline{T}_b(\xi r)$ on A_r, and by Theorem 3.1, for arbitrary $\delta \in (0,1)$, $\overline{T}_b(r) \le (1+\delta)C(r)$ and $\overline{T}_b(\xi r) \ge (1-\delta)C(\xi r)$ hold WPA1 as $r \to 0$, hence

$$\limsup_{r\to 0} P\left(A_r \cap \{\overline{T}_b(r) \le (1+\delta)C(r), \overline{T}_b(\xi r) \ge (1-\delta)C(\xi r)\}\right) > 0. \tag{4.49}$$

Thus

$$\liminf_{r\to 0} \frac{C(\xi r)}{C(r)} \le \frac{1+\delta}{1-\delta}.$$

Letting $\delta \downarrow 0$ we get a contradiction, since by Theorem 3.1, $C(r)$ is regularly varying with index $1/(1-b)$. Hence

$$\frac{X_{\overline{T}_b(r)}}{r(\overline{T}_b(r))^b} \xrightarrow{\mathrm{P}} 1, \text{ as } r \to 0.$$

Then, since B is regularly varying with index 1, $B(C(r)) = rC(r)^b$ and $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$, the remaining relationships in Theorem 3.3 follow.

(b) Assume $|X_t|/B(t) \xrightarrow{\mathrm{P}} 1$ as $t \to 0$. By Proposition 2.1, there is no loss of generality in assuming $X_t/B(t) \xrightarrow{\mathrm{P}} 1$. The result then follows from Corollary 3.1 and part (a). □

Proof of Theorem 3.4. (a) Assume that X has bounded variation with drift $\mathrm{d}_X > 0$. Then by Theorem 3.2, for every $\varepsilon > 0$,

$$\frac{\overline{T}_b(r) \wedge \varepsilon}{(r/\mathrm{d}_X)^{1/(1-b)}} \to 1 \text{ a.s. as } r \to 0.$$

Hence by Fatou's Lemma,

$$\liminf_{r\to 0} \frac{E(\overline{T}_b(r) \wedge \varepsilon)}{(r/\mathrm{d}_X)^{1/(1-b)}} \ge 1. \tag{4.50}$$

Letting $\varepsilon \to 0$ proves a lower bound for (3.2).

For the upper bound, we first prove the result for $b = 0$. Recall the bivariate ascending ladder process (L^{-1}, H) of X, and its Laplace exponent $\kappa(\cdot,\cdot)$ given by (4.38). Clearly

$$\lim_{q\to\infty} \frac{\kappa(q,0)}{q} = \mathrm{d}_{L^{-1}},$$

and by Lemma 4.6, we have $\mathrm{d}_{L^{-1}} > 0$, $\mathrm{d}_H > 0$ and $\mathrm{d}_X = \mathrm{d}_H/\mathrm{d}_{L^{-1}}$. For $q > 0$, let $e(q)$ be independent of X and have exponential distribution with mean q^{-1}. Then a straightforward calculation shows that for any $\varepsilon \in (0, \infty]$ (with the obvious interpretation when $\varepsilon = \infty$),

$$E(\overline{T}_0(r) \wedge \varepsilon \wedge e(q)) = \int_0^\infty e^{-qs} P(\overline{T}_0(r) \wedge \varepsilon > s)\mathrm{d}s = q^{-1} P(\overline{T}_0(r) \wedge \varepsilon > e(q)). \tag{4.51}$$

Hence by (8) on p.174 of [1],

$$E(\overline{T}_0(r) \wedge e(q)) = \frac{\kappa(q,0)}{q} V^q(r),$$

where

$$V^q(r) = \int_0^\infty E(e^{-qL_t^{-1}}; H_t \le r)\mathrm{d}t.$$

Now the Laplace transform of V^q is given by

$$\widehat{V}^q(\lambda) := \lambda \int_0^\infty e^{-\lambda r} V^q(r)\mathrm{d}r = \frac{1}{\kappa(q,\lambda)}, \quad \lambda > 0,$$

and so

$$\lambda \widehat{V}^q(\lambda) = \frac{\lambda}{\kappa(q,\lambda)} \to \frac{1}{\mathrm{d}_H} \quad \text{as } \lambda \to \infty.$$

Thus

$$\frac{V^q(r)}{r} \to \frac{1}{\mathrm{d}_H} \quad \text{as } r \to 0,$$

by Karamata's Tauberian Theorem; see Theorem 1.7.1′ in [3]. Hence

$$\lim_{q\to\infty} \lim_{r\to 0} \frac{E(\overline{T}_0(r) \wedge e(q))}{r} = \lim_{q\to\infty} \lim_{r\to 0} \frac{\kappa(q,0)V^q(r)}{qr} = \frac{\mathrm{d}_{L^{-1}}}{\mathrm{d}_H} = \frac{1}{\mathrm{d}_X}. \tag{4.52}$$

Next fix $\varepsilon > 0$ and $q \in (0,\infty)$. Then

$$\begin{aligned}
E(\overline{T}_0(r)\wedge\varepsilon) &= E(\overline{T}_0(r)\wedge\varepsilon; \overline{T}_0(r)\wedge\varepsilon \le e(q)) + E(\overline{T}_0(r)\wedge\varepsilon; \overline{T}_0(r)\wedge\varepsilon > e(q)) \\
&\le E(\overline{T}_0(r)\wedge e(q)) + \varepsilon P(\overline{T}_0(r)\wedge\varepsilon > e(q)) \\
&= E(\overline{T}_0(r)\wedge e(q)) + \varepsilon q E(\overline{T}_0(r)\wedge\varepsilon\wedge e(q)) \quad \text{(by (4.51))} \\
&\le (1+\varepsilon q) E(\overline{T}_0(r)\wedge e(q)).
\end{aligned}$$

Thus for every $q \in (0,\infty)$,

$$\lim_{\varepsilon\to 0}\lim_{r\to 0} \frac{E(\overline{T}_0(r)\wedge\varepsilon)}{r} \le \lim_{r\to 0} \frac{E(\overline{T}_0(r)\wedge e(q))}{r}.$$

Letting $q\to\infty$ and using (4.52), proves the upper bound in (3.2) for $b=0$.

To deal with the upper bound when $0<b<1$, introduce

$$Y_t = \frac{X_t}{\mathrm{d}_X} - bt,\ t \ge 0.$$

Then Y has bounded variation with drift $\mathrm{d}_Y = 1-b > 0$. Take $r>0$ and let $\lambda_r = r/\mathrm{d}_X$. Consider the function

$$f(t) := \lambda_r t^b - bt,\ t\ge 0.$$

This increases from 0 at $t=0$ to a maximum of $\lambda_r^{1/(1-b)}(1-b)$ at $t=\lambda_r^{1/(1-b)}$, then decreases to 0 at $t=(\lambda_r/b)^{1/(1-b)}$. Hence it is non-negative for $t\in[0,(\lambda_r/b)^{1/(1-b)}]$ and lies entirely below the horizontal line of height $\lambda_r^{1/(1-b)}(1-b)$ for $t\ge 0$. Thus with $\overline{T}_0^Y(\lambda) = \inf\{t\ge 0: Y_t > \lambda\}$, we have

$$\overline{T}_b(r) \le \overline{T}_0^Y(\lambda_r^{1/(1-b)}(1-b)).$$

Thus invoking the $b=0$ result just proved, for Y, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{r\to 0} \frac{E(\overline{T}_0(r)\wedge\varepsilon)}{(r/\mathrm{d}_X)^{1/(1-b)}} &\le (1-b)\lim_{\varepsilon\to 0}\lim_{r\to 0} \frac{E(\overline{T}_0^Y(\lambda_r^{1/(1-b)}(1-b))\wedge\varepsilon)}{\lambda_r^{1/(1-b)}(1-b)} \\
&= \frac{1-b}{\mathrm{d}_Y} = 1.
\end{aligned}$$

(b) Assume $T_b^*(r)/C(r) \xrightarrow{\text{P}} 1$, and fix $p > 0$ and $\varepsilon > 0$. By Proposition 2.1 and Theorem 3.1, $A(r)$ is slowly varying at 0, and we may assume without loss of generality that $A(r) > 0$ for small r. It then follows from [19] (see also (4.3) of [7]), when translated to the current notation, that for every $r > 0, t > 0$ and $m \geq 1$,

$$P(T_b^*(r) \geq t) \leq P(T_0^*(rt^b) \geq t) \leq \left(\frac{cmr}{t^{1-b}A(rt^b)}\right)^m, \tag{4.53}$$

where $c \in (0, \infty)$ denotes an unimportant constant that may change from one usage to the next. Furthermore, again by Proposition 2.1 and Theorem 3.1, we may assume that $C^{-1}(t) = t^{-b}B(t)$ where $tA(B(t))/B(t) \to 1$ as $t \to 0$. Setting $t = C(r)$, and using $B(C(r)) = rC(r)^b$, this gives

$$\lim_{r \to 0} \frac{C(r)A(rC(r)^b)}{rC(r)^b} = 1. \tag{4.54}$$

Choose $\xi > 0$ sufficiently small that $1 - b - b\xi > 0$. Since A is slowly varying at 0, there is a function $\widehat{A}$ such that $A(r) \sim \widehat{A}(r)$ as $r \to 0$, and $r^\xi \widehat{A}(r)$ is increasing on $(0, a]$ for some $a \in (0, \varepsilon]$. For any $p > 0$, write

$$\begin{aligned}
E(T_b^*(r) \wedge \varepsilon)^p &= \int_0^\varepsilon pt^{p-1} P(T_b^*(r) \geq t)\mathrm{d}t \\
&\leq \int_0^{C(r)} pt^{p-1}\mathrm{d}t + \int_{C(r)}^a pt^{p-1} \left(\frac{cmr}{t^{1-b}A(rt^b)}\right)^m \mathrm{d}t + \int_a^\varepsilon pt^{p-1} P(T_b^*(r) \geq a)\mathrm{d}t \\
&= I + II + III.
\end{aligned} \tag{4.55}$$

Clearly $I = C(r)^p$, while by (4.53)

$$III \leq \varepsilon^p \left(\frac{cmr}{a^{1-b}A(ra^b)}\right)^m \sim \varepsilon^p \left(\frac{cmr}{a^{1-b}A(r)}\right)^m = o(r^{(1-\eta)m}), \quad \text{as } r \to 0,$$

for any $\eta > 0$, since A is slowly varying. Now $C(r)$ is regularly varying with exponent $1/(1-b)$, thus by choosing m sufficiently large we see that $III = o(C(r)^p)$. Finally for II we observe that if $C(r) \leq t \leq a$, then

$$t^{b\xi}\widehat{A}(rt^b) \geq C(r)^{b\xi}\widehat{A}(rC(r)^b).$$

Hence

$$t^{p-1}\left(\frac{r}{t^{1-b}\widehat{A}(rt^b)}\right)^m \le t^{p-1-m(1-b-b\xi)}C(r)^{m(1-b-b\xi)}\left(\frac{rC(r)^b}{C(r)\widehat{A}(rC(r)^b)}\right)^m.$$

Recalling that $1-b-b\xi>0$, we see that if m is sufficiently large that $m(1-b-b\xi)>p$, then

$$\int_{C(r)}^a t^{p-1}\left(\frac{r}{t^{1-b}\widehat{A}(rt^b)}\right)^m \mathrm{d}t \le cC(r)^p$$

for small r by (4.54). Hence by (4.55)

$$\limsup_{r\to 0}\frac{E(T_b^*(r)\wedge\varepsilon)^p}{C(r)^p}<\infty.$$

Thus $(T_b^*(r)\wedge\varepsilon)/C(r)$ is bounded in L^p for every $p>0$, which together with $T_b^*(r)/C(r)\xrightarrow{\mathrm{P}}1$ proves convergence in L^p. The converse direction is trivial. □

Remark: If $X_t\to\infty$ a.s. as $t\to\infty$, then $E\overline{T}_0(r)<\infty$ for some (every) $r>0$. In that case, if in addition $EL_1^{-1}<\infty$ and $d_H>0$, then by Theorem 2.3 of [14]

$$\lim_{r\to 0}\frac{E\overline{T}_0(r)}{r}=\frac{EL_1^{-1}}{d_H}.$$

If $0\le b<1/2$ then $ET_b^*(r)^p<\infty$ for every $r>0$, $p>0$, and the proof of (3.3) can be modified to show

$$\lim_{r\to 0}\frac{ET_b^*(r)^p}{C(r)^p}=1.$$

5 Relative Stability of $\overline{T}_b(r)$ and $T_b^*(r)$ for Large Times

In this section we briefly summarise relative stability of $\overline{T}_b(r)$ and $T_b^*(r)$ for large times. All proofs are omitted. In many cases they parallel the proofs given for small times although in some cases there are nontrivial differences. We must first discuss the definitions of $\overline{T}_b(r)$ and $T_b^*(r)$. It is possible for X to cross the t^b boundary for small t, but not for large t. For example, when $\sigma^2>0$

we have by [18] that

$$\limsup_{t\downarrow 0} \frac{X_t}{\sqrt{2\sigma^2 t \log|\log t|}} = 1 \text{ a.s.},$$

thus $\limsup_{t\downarrow 0} X_t/\sqrt{t} = +\infty$ a.s., while we can have in addition that X_t drifts to $-\infty$ a.s. as $t \to \infty$. If we took the infimum in (1.1) over all $t > 0$, we may have that $\overline{T}_{1/2}(r)$ is finite, in fact, takes value 0, for all $r > 0$, even though $\limsup_{t\to\infty} X_t/\sqrt{t} < \infty$ a.s. Since we are interested in the behaviour of X for large t, we prevent this kind of behaviour by taking the inf in (1.1) over $t \geq 1$. Thus we define

$$\overline{T}_b(r) = \inf\{t \geq 1 : X_t > rt^b\},\ r \geq 0, \tag{5.1}$$

and

$$T_b^*(r) = \inf\{t \geq 1 : |X_t| > rt^b\},\ r \geq 0. \tag{5.2}$$

We always assume, unless explicitly stated otherwise, that $0 \leq b < 1$. The results up to Theorem 5.4 below, parallel those for small times if $r \to 0$ is replaced by $r \to \infty$ and the drift d_X is replaced by the mean EX_1 at the appropriate points.

Theorem 5.1 *(a) Assume $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ as $t \to \infty$, where $B(t) > 0$. Then $B(t)/t^b$ may be chosen to be continuous and strictly increasing, in which case $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to \infty$, where $C(r)$ is the inverse to $B(t)/t^b$.*

Conversely, assume $\overline{T}_b(r)/C(r) \xrightarrow{\mathrm{P}} 1$ as $r \to \infty$, where $C(r) > 0$. Then $C(r)$ may be taken to be continuous and strictly increasing with inverse C^{-1}, in which case $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ as $t \to \infty$, where $B(t) = t^b C^{-1}(t)$.

(b) The same result holds if X and $\overline{T}_b(r)$ are replaced by $|X|$ and $T_b^*(r)$ respectively in (a).

In either case, (a) or (b), the function $C(r)$ is regularly varying with index $1/(1-b)$ as $r \to \infty$.

Corollary 5.1 *Assume $X_t/B(t) \xrightarrow{\mathrm{P}} 1$ as $t \to \infty$, where $B(t) > 0$. Then $P(\overline{T}_b(r) = T_b^*(r)) \to 1$ as $r \to \infty$.*

Proposition 5.1 *Suppose $b \geq 1$. Then neither $\overline{T}_b(r)$ nor $T_b^*(r)$ can be relatively stable, in probability, as $r \to \infty$.*

Theorem 5.2 *(a) $T_b^*(r)$ is almost surely (a.s.) relatively stable, i.e., $T_b^*(r)/C(r) \to 1$, a.s., as $r \to \infty$, for a finite function $C(r) > 0$, iff $E|X_1| < \infty$ and $\mu := EX_1 \neq 0$.*
(b) $\overline{T}_b(r)$ is almost surely relatively stable, i.e., $\overline{T}_b(r)/C(r) \to 1$, a.s., as $r \to \infty$, for a finite function $C(r) > 0$, iff $E|X_1| < \infty$ and $\mu = EX_1 > 0$.
In either case, (a) or (b), the function $C(r)$ may be chosen as $C(r) = (r/|\mu|)^{1/(1-b)}$.

Theorem 5.3 *(a) Suppose $X_t/B(t) \xrightarrow{\text{P}} 1$ as $t \to \infty$, where $B(t) > 0$ satisfies the regularity conditions of Theorem 5.1. Let C be the inverse of $B(t)/t^b$. Then, as $r \to \infty$,*

$$\frac{X_{\overline{T}_b(r)}}{r(\overline{T}_b(r))^b} \xrightarrow{\text{P}} 1, \quad \frac{X_{\overline{T}_b(r)}}{B(\overline{T}_b(r))} \xrightarrow{\text{P}} 1, \quad \frac{X_{\overline{T}_b(r)}}{B(C(r))} \xrightarrow{\text{P}} 1, \text{ and } \frac{X_{\overline{T}_b(r)}}{r(C(r))^b} \xrightarrow{\text{P}} 1. \tag{5.3}$$

(b) Suppose $|X_t|/B(t) \xrightarrow{\text{P}} 1$ as $t \to \infty$. Then (5.3) holds with $|X|$ and $T_b^*(r)$ in place of X and $\overline{T}_b(r)$ respectively.

Remark: The analogous result to Theorem 5.3 holds when $\xrightarrow{\text{P}}$ is replaced by a.s. convergence throughout.

For the large time version of Theorem 3.4, it is no longer appropriate to truncate the passage times since $C(r) \to \infty$ as $r \to \infty$.

Theorem 5.4 *(a) Suppose $E|X_1| < \infty$ and $\mu = EX_1 > 0$. Then*

$$\lim_{r \to \infty} \frac{E\overline{T}_b(r)}{C(r)} = 1. \tag{5.4}$$

where $C(r) = (r/\mu)^{1/(1-b)}$.
(b) Fix a function $C(r)$; then $T_b^*(r)/C(r) \xrightarrow{\text{P}} 1$ iff $T_b^*(r)/C(r) \to 1$ in L^p for some (all) $p > 0$. In particular, if $T_b^*(r)/C(r) \xrightarrow{\text{P}} 1$, then for every $p > 0$

$$\lim_{r \to \infty} \frac{ET_b^*(r)^p}{C(r)^p} = 1. \tag{5.5}$$

Remarks: (i) Suppose $EX_1^2 < \infty$ and $EX_1 = 0$. Then X cannot be relatively stable, so neither can $\overline{T}_b(r)$ or $T_b^*(r)$. If $\overline{\Pi}(x_0) = 0$ for some $x_0 > 0$, then $EX_1^2 < \infty$, and so relative stability of $\overline{T}_b(r)$ or $T_b^*(r)$ hinges on whether $EX_1 = 0$ or not.

(ii) In addition to covering one-sided passage times and also dealing with the important case $b = 1/2$, omitted in [6], [7], [8], Theorem 5.1 and associated results provide a more general approach than that of [8], where the norming functions are assumed *a priori* to have strong regularity properties, such as regular variation, whereas we make no such assumptions.

(iii) Siegmund [20] contains the random walk version of (5.4). He mentions extensions of his result and possible applications to sequential confidence intervals and hypothesis tests. We expect that similar extensions can be carried out in the general Lévy case.

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