Approximate amenability for Banach sequence algebras

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Abstract. We consider when certain Banach sequence algebras A on the set $\mathbb{N}$ are approximately amenable. Some general results are obtained, and we resolve the special cases where $A = \ell^p$ for $1 \leq p < \infty$, showing that these algebras are not approximately amenable. The same result holds for the weighted algebras $\ell^p(\omega)$.

1. Introduction

The concept of amenability for a Banach algebra A, introduced by Johnson in 1972 [**7**], has proved to be of enormous importance in Banach algebra theory (see [**1**], for example). In [**3**] several modifications of this notion were introduced; in this paper we shall focus on one of these, that of *approximate amenability*. We recall the definition in Definition 1.1, below.

Let A be an algebra, and let X be an A-bimodule. A *derivation* is a linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A)\,.$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a, \;\; A \to X$. Then ad_x is a derivation; these are the *inner* derivations.

Let A be a Banach algebra, and let X be a Banach A-bimodule. A continuous derivation $D : A \to X$ is *approximately inner* if there is a

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net (x_α) in X such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A)\,,$$

so that $D = \lim_\alpha ad_{x_\alpha}$ in the strong-operator topology of $\mathcal{B}(A)$.

The dual of a Banach space X is denoted by X'; in the case where X is a Banach A-bimodule, X' is also a Banach A-bimodule. For the standard dual module definitions, see [**1**].

DEFINITION 1.1. [**3**] *Let A be a Banach algebra. Then A is* approximately amenable *if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner.*

The qualifier *sequential* prefixed to the above definition specifies that there is a sequence of inner derivations approximating the given continuous derivation.

We remark that, in [**3**], the notion of uniform approximate amenability was also introduced: a Banach algebra A is *uniformly approximately amenable* if, for each Banach A-bimodule X, each continuous derivation $D : A \to X'$ is the limit of a sequence of inner derivations in the norm topology of $\mathcal{B}(A, X')$. In fact, it has recently been shown independently by Pirkovskii [**10**] and Ghahramani [**4**] that a uniformly approximately amenable Banach algebra is already amenable.

Of course, each amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras which are not amenable are constructed in [**3**]. For example, let (A_n) be a sequence of unital, amenable Banach algebras. Then the sum $c_0(A_n)$ is always approximately amenable, but is not necessarily amenable [**3**, Example 6.1]. Further, it has been shown by Ghahramani and Stokke [**5**] that the Fourier algebra $A(G)$ is approximately amenable for each amenable, discrete group G, but it is known that $A(G)$ is not always amenable for an amenable group G [**9**]. Examples of semigroup algebras of the form $\ell^1(S)$ that are approximately amenable but not amenable are given in [**2**]. Nevertheless there is something of a shortage of 'natural' examples of approximately amenable Banach algebras which are not amenable. In this paper, we shall consider when certain Banach sequence algebras on $\mathbb{N}$ are approximately amenable, a question left open in [**3**]. In particular, we shall consider the standard Banach sequence algebras $\ell^p = \ell^p(\omega)$, where $1 \leq p < \infty$ and ω is a weight on $\mathbb{N}$.

2. Basic constructions

When determining whether or not our Banach algebras are approximately amenable, we shall work from a characterization of approximately amenable Banach algebras which is a modification of that given in [**3**].

Let A be Banach algebra. The projective tensor product $A\widehat{\otimes}A$ is a Banach A-bimodule, under the operations defined by

$$c\cdot a\otimes b = ca\otimes b,\quad a\otimes b\cdot c = a\otimes bc\quad (a,b,c\in A)\,,$$

and there is a continuous linear A-bimodule homomorphism $\pi : A\widehat{\otimes}A\to A$ such that $\pi(a\otimes b)=ab\ (a,b\in A)$ [**1**].

Proposition 2.1. *Let A be a Banach algebra. Then A is approximately amenable if and only if, for each $\varepsilon>0$ and each finite subset S of A, there exist $F\in A\otimes A$ and $u,v\in A$ such that $\pi(F)=u+v$ and, for each $a\in S$:*

(i) $\|a\cdot F - F\cdot a + u\otimes a - a\otimes v\| < \varepsilon$ *;*

(ii) $\|a-au\|<\varepsilon$ *and* $\|a-va\|<\varepsilon$ *.*

Proof. Suppose that A is approximately amenable. Then by [**3**, Corollary 2.2] there are nets (M_α) in $(A\widehat{\otimes}A)''$, and (U_α) and (V_α) in A'' such that, for each $a\in A$:

(i) $a\cdot M_\alpha - M_\alpha\cdot a + U_\alpha\otimes a - a\otimes V_\alpha \to 0$;

(ii) $a - a\cdot U_\alpha\to 0$ and $a - V_\alpha\cdot a\to 0$;

(iii) $\pi''(M_\alpha) - U_\alpha - V_\alpha\to 0$.

(This corrects a typographical error in [**3**].) In each case convergence is in the $\|\cdot\|$-topology.

Let Y denote the Banach space $(A\widehat{\otimes}A)\oplus A\oplus A\oplus A$. For each $a\in A$, define a convex set in Y by setting

$$\begin{aligned}K_a := \{(a\cdot m\ &- m\cdot a + u\otimes a - a\otimes v,\\ &a-au,\ a-va,\ \pi(m)-u-v) : m\in A\widehat{\otimes}A,\ u,v\in A\}\,.\end{aligned}$$

For the specified finite subset S of A,

$$K := \prod\{K_a : a\in S\}$$

is a convex set in the Banach space Y^S. The conditions above show that the weak closure of K in Y^S contains the zero element 0 of Y^S. By Mazur's theorem, it follows that 0 belongs to the $\|\cdot\|$-closure of K in Y^S. Thus, with $\varepsilon>0$ as specified, there exist $F\in A\widehat{\otimes}A$ and $u,v\in A$ such that clauses (i) and (ii) of the proposition are satisfied and, further, such that $\|\pi(F) - u - v\| < \varepsilon$. By modifying F and u slightly, we may suppose, further, that $F \in A \otimes A$ and that $\pi(F) = u + v$.

Conversely, suppose that the condition in the proposition is satisfied. Consider the set $D := (0,1) \times \mathcal{F}(A)$, where $\mathcal{F}(A)$ is the family of finite subsets of A, and order D by setting

$$(\varepsilon_1, S_1) \preccurlyeq (\varepsilon_2, S_2) \quad \text{whenever} \quad \varepsilon_1 \geq \varepsilon_2 \quad \text{and} \quad S_1 \subseteq S_2\,.$$

Then $(D, \preccurlyeq)$ is a directed set. The conditions show that there exist nets (F_α) in $A\widehat{\otimes}A$, and $(u_\alpha), (v_\alpha)$ in A such that $\pi(F_\alpha) = u_\alpha + v_\alpha$ and such that, for each $a \in A$, we have:

$$a \cdot F_\alpha - F_\alpha \cdot a + u_\alpha \otimes a - a \otimes v_\alpha \to 0\,;$$
$$a - a u_\alpha \to 0\,, \quad a - v_\alpha a \to 0\,.$$

Thus we have satisfied the conditions of [**3**, Corollary 2.2], and so A is approximately amenable. □

Corollary 2.2. *Let A be a Banach algebra with identity e. Then A is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset S of A, there exists $G \in A \otimes A$ with $\pi(G) = e$ and such that*

$$\|a \cdot G - G \cdot a\| < \varepsilon \quad (a \in S)\,.$$

Proof. Suppose that such a G exists, and set $u = v = e$ and $F = G + e \otimes e$. Then $\pi(F) = u + v$, and F, u, v satisfy the conditions of Proposition 2.1.

Conversely, suppose that F, u, v satisfy the above condition for a finite subset S and with $\varepsilon/3\|e\|$ replacing ε, and set

$$G = F - u \otimes e - e \otimes v + e \otimes e\,.$$

Then $\pi(G) = e$, and

$$\|a \cdot G - G \cdot a\| \leq \|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| + \|a - au\| + \|a - va\| < \varepsilon\,,$$

and so A is approximately amenable by Proposition 2.1. □

For comparison, we recall [**1**], [**8**] that a Banach algebra A is amenable if and only if there is a constant $C > 0$ such that, for each $\varepsilon > 0$ and each finite subset S of A, there exists $F \in A \otimes A$ with $\|F\| \leq C$ such that, for each $a \in S$, we have:

(i) $\|a \cdot F - F \cdot a\| < \varepsilon\,;$

(ii) $\|a - a\pi(F)\| < \varepsilon\,.$

We remark that (ii) of Proposition 2.1 is exactly the condition that A has both left and right approximate units [**1**, Definition 2.9.10]. We do not know whether or not an approximately amenable Banach algebra necessarily has (two-sided) approximate units.

We now give a variation of Proposition 2.1 in the case where A is commutative. For each Banach algebra A, there is an isometry $\iota : A\widehat{\otimes}A \to A\widehat{\otimes}A$ such that $\iota(a \otimes b) = b \otimes a \ (a, b \in A)$.

PROPOSITION 2.3. *Let A be a commutative Banach algebra. Then A is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset S of A, there exist $F \in A \otimes A$ with $\iota(F) = F$ and $u \in A$ such that $\pi(F) = 2u$, and, for each $a \in S$:*

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes u\| < \varepsilon$;

(ii) $\|a - au\| < \varepsilon$.

PROOF. Since A is commutative,

$$\iota(a \cdot F) = \iota(F) \cdot a \quad (a \in A, F \in A\widehat{\otimes}A)\,.$$

Suppose that A is approximately amenable, and take $\varepsilon > 0$ and a finite subset S of A. By Proposition 2.1, there exist F, u, and v satisfying conditions (i) and (ii) of that result. For each $a \in S$, we have

$$\|\iota(F) \cdot a - a \cdot \iota(F) + a \otimes u - v \otimes a\| < \varepsilon\,.$$

Set $G = (F + \iota(F))/2$ and $w = (u + v)/2$. Then $\iota(G) = G$ and $\pi(G) = 2w$. Further,

$$\|a \cdot G - G \cdot a + w \otimes a - a \otimes w\| < \varepsilon \quad \text{and} \quad \|a - aw\| < \varepsilon\,.$$

Thus the specified conditions are satisfied (with w for u).

The converse is immediate. □

3. Banach sequence algebras

We now introduce the specific algebras that will be considered in this paper. As usual c_{00} will be the subalgebra of $\mathbb{C}^{\mathbb{N}}$ consisting of sequences having finite support.

DEFINITION 3.1. *A* Banach sequence algebra on $\mathbb{N}$ *is a Banach algebra A which is a subalgebra of $\mathbb{C}^{\mathbb{N}}$ such that $c_{00} \subset A$.*

For example, $c_0 = c_0(\mathbb{N})$ and $\ell^p = \ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$ are Banach sequence algebras on $\mathbb{N}$.

Let $(A, \|\cdot\|)$ be a Banach sequence algebra on $\mathbb{N}$. Then

$$\|a\| \geq |a|_{\mathbb{N}} \quad (a \in A)\,,$$

where $|\cdot|_{\mathbb{N}}$ denotes the uniform norm on $\mathbb{N}$. In the case where c_{00} is dense in A, the algebra A is natural on $\mathbb{N}$ [**1**, Proposition 4.1.35].

Throughout we write δ_i for the characteristic function of $\{i\}$ for $i \in \mathbb{N}$, and set

$$e_n = \sum_{i=1}^{n} \delta_i \quad (n \in \mathbb{N}),$$

so that $(e_n) \subset c_{00} \subset A$. When convenient we identify $a \in A$ both as the sequence (a_i) and as the formal sum $\sum_i a_i\delta_i$. We shall also identify $A \otimes A$ with a space of functions on $\mathbb{N} \times \mathbb{N}$ by setting

$$(a \otimes b)(i,j) = a_i b_j \quad (a, b \in A,\, i,j \in \mathbb{N});$$

in particular, $\delta_i \otimes \delta_j = \delta_{(i,j)}$, the characteristic function of $\{(i,j)\}$, for $i,j \in \mathbb{N}$. We shall also sometimes write $F = \sum_{i,j} F(i,j)\delta_{(i,j)}$ for $F \in c_{00} \otimes c_{00}$. Note that

$$(a \cdot F)(i,j) = a_i F(i,j)\,, \quad (F \cdot a)(i,j) = a_j F(i,j) \quad (i,j \in \mathbb{N})\,,$$

and that $\pi(F) = \sum_i F(i,i)\delta_i$.

Definition 3.2. *Let A be a Banach sequence algebra on $\mathbb{N}$, and let $a \in A$. For $F \in c_{00} \otimes c_{00}$, set*

$$\Delta_a(F) = a \cdot F - F \cdot a + \pi(F) \otimes a - a \otimes \pi(F)\,.$$

Clearly $\Delta_a(F) \in c_{00} \otimes c_{00}$ whenever $a \in c_{00}$.

Proposition 3.3. *Let A be a Banach sequence algebra with c_{00} dense in A. Then A is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset S of A, there exists $F \in c_{00} \otimes c_{00}$ with $\iota(F) = F$ such that, for each $a \in S$:*

(i) $\|\Delta_a(F)\| < \varepsilon$;

(ii) $\|a - a\pi(F)\| < \varepsilon$.

Proof. Suppose that A is approximately amenable, and take $\varepsilon > 0$ and a finite subset S of A. Let F and u be given by Proposition 2.3. Since c_{00} is dense in A, the space $c_{00} \otimes c_{00}$ is dense in $A \otimes A$, and so we can replace F by an element $G \in c_{00} \otimes c_{00}$ such that (i) and (ii) of that proposition remain true, with $v = \pi(G)/2$ replacing u. Now replace G by

$$H = G + \textstyle\sum_i (v_i - \pi(G)_i)\delta_i \otimes \delta_i\,,$$

noting that the number of non-zero summands in the above sum is finite. This does not affect clauses (i) or (ii) of Proposition 2.3, and now $\pi(H) = v$. Thus conditions (i) and (ii) of the current proposition are satisfied.

The converse is similar. $\square$

We shall later consider only Banach sequence algebras A which are self-adjoint. In such a situation the map $a \mapsto \overline{a}$ is necessarily continuous on A. It follows that we may replace F by $F + \overline{F}$, and so take F to be real-valued. Similarly, we may also suppose that the elements of the 'test sets' S are real-valued.

PROPOSITION 3.4. *Let A be a Banach sequence algebra. Suppose that there is $\eta > 0$ such that, for each $\varepsilon > 0$ and each finite subset S of A, there exists $u \in c_{00}$ with*

$$\|u\| \geq \eta \quad \text{and} \quad \|a - au\| \cdot \|u\| < \varepsilon \,. \tag{3.1}$$

Then A is approximately amenable.

PROOF. Take u as given by (3.1), with ε replaced by $\varepsilon\eta/2$. Set

$$F = u \otimes u + \textstyle\sum_i (u_i - u_i^2)\delta_i \otimes \delta_i \,.$$

Then $\pi(F) = u$ and, for each $a \in S$, we have

$$\|a \cdot F - F \cdot a - a \otimes u + u \otimes a\| = \|au \otimes u - a \otimes u + u \otimes a - u \otimes au\| < \varepsilon$$

and $\|a - au\| < \varepsilon$. By Proposition 3.3, A is approximately amenable.

The converse is immediate. □

More general forms of this result for Banach function algebras on discrete spaces can be shown by the same sort of argument; see, for example, [**5**, Proposition 3.16].

We make the *conjecture* that the sufficient condition in Proposition 3.4 is in fact also necessary for A to be approximately amenable. Indeed, we do not know an example of a Banach sequence algebra which is approximately amenable, but which does not have a bounded approximate identity. It is also conceivable that each Banach sequence algebra A such that c_{00} is dense in A and $A = A^2$ is approximately amenable.

COROLLARY 3.5. *Let A be a Banach sequence algebra such that A has a bounded approximate identity contained in c_{00}. Then A is sequentially approximately amenable.*

PROOF. It is standard that A has a sequential bounded approximate identity, say (u_n), in c_{00} [**1**, Corollary 2.9.18], and satisfying $\inf_n \|u_n\| \geq 1$. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of A. Then, for each $n \in \mathbb{N}$, there exists $i = i(n)$ such that $\|x_j - x_j u_{i(n)}\| < 1/n$ for $1 \leq j \leq n$. Following Proposition 3.4, we set

$$F_n = u_{i(n)} \otimes u_{i(n)} + \textstyle\sum_{j \in \mathbb{N}} (u_{i(n),j} - u_{i(n),j}^2)\delta_j \otimes \delta_j \,.$$

Then, for each $a \in A$ and $\varepsilon > 0$, we have

$$\|a \cdot F_n - F_n \cdot a - a \otimes u_{i(n)} + u_{i(n)} \otimes a\| = 2\|au_{i(n)} - a\| \cdot \|u_{i(n)}\| < \varepsilon$$

for n sufficiently large. Thus $(F_n, u_{i(n)})$ gives a sequence with the required properties of [**3**, Corollary 2.2]. The sequential variant of [**3**, Theorem 2.1] holds (with the same argument), and so A is sequentially approximately amenable. □

Special cases of the above corollary have been shown in [**4**], where it is also shown that the converse holds for certain Banach sequence algebras.

We wish to stress that the function F specified in Proposition 3.3 must satisfy conditions (i) and (ii) for each *finite* collection S of elements. The following shows that, for many Banach sequence algebras A, we can find F to satisfy these conditions for each single element $a \in A$. Indeed the Banach sequence algebra ℓ^1 satisfies the conditions of Proposition 3.6 below, but we shall see that it is not approximately amenable. To determine whether or not such an algebra A is approximately amenable, we must look at sets S with at least 2 elements.

We introduce the following notation. Let A be a Banach sequence algebra on $\mathbb{N}$. For each $a \in A$ and each finite or cofinite subset T of $\mathbb{N}$, set

$$P_T : a \mapsto \textstyle\sum_i \{a_i\delta_i : i \in T\}\,, \quad A \to A\,.$$

We also write $P_n = P_{\{1,\dots,n\}}$ and $Q_n = I - P_n$ for $n \in \mathbb{N}$. The family $\mathcal{C}$ of cofinite subsets of $\mathbb{N}$ will be directed by reverse set inclusion.

Proposition 3.6. *Let A be a Banach sequence algebra, and let $a \in A$. Suppose that*

$$\lim\{\|P_C a\| : C \in \mathcal{C}\} = 0\,. \tag{3.2}$$

Then, for each $\varepsilon > 0$, there exists $F \in c_{00} \otimes c_{00}$ such that

$$\|\Delta_a(F)\| < \varepsilon \quad \text{and} \quad \|a - a\pi(F)\| < \varepsilon\,. \tag{3.3}$$

Proof. Let $\{B_i : i \in \mathbb{Z}^+\}$ be the partition of $\mathbb{N}$ such that a takes the constant value a_i on B_i for $i \in \mathbb{N}$, such that a takes the value 0 on B_0, and such that $a_i \neq a_j$ whenever $i, j \in \mathbb{Z}^+$ and $i \neq j$. Note that, by (3.2), each B_i for $i \in \mathbb{N}$ is finite. For $n \in \mathbb{N}$, set

$$D_n = \bigcup_{i=1}^{n} B_i \quad \text{and} \quad E_n = \bigcup_{i=n+1}^{\infty} B_i\,,$$

and set $\mu(n) = \min E_n$, so that $\mu(n) \to \infty$ as $n \to \infty$.

Fix $\varepsilon > 0$, and take $n_0 \in \mathbb{N}$ such that $\|P_C a\| < \varepsilon$ for each cofinite subset C of $\mathbb{N}$ with $\min C > n_0$. Next choose $n_1 \in N$ such that $n_1 > n_0$

and $\mu(n_1) > n_0$. Set $C = E_{n_1} \cup (B_0 \cap [n_1, \infty))$, so that D_{n_1} is finite and C is cofinite with $\min C > n_0$. Take u to be the characteristic function of D_{n_1}, so that

$$a - au = a\chi_{\mathbb{N}\setminus D_{n_1}} = P_C a\,,$$

and hence

$$\|a - au\| = \|P_C a\| < \varepsilon\,.$$

By (3.2), we may choose $m_0 \in \mathbb{N}$ with $m_0 > n_1$ and such that

$$|D_{n_1}| \cdot \|Q_{m_0} a\| < \varepsilon/2\,. \tag{3.4}$$

Now define F as follows.

(a) For $j, k \leq n_1$, set

$$F = 1 \quad \text{on } B_j \times B_k\,;$$

(b) for $j \leq n_1$ and $n_1 < k \leq m_0$, set

$$F = \frac{-a_k}{a_j - a_k} \quad \text{on } B_j \times B_k\,;$$

(c) by symmetry for $k \leq n_1$ and $n_1 < j \leq m_0$; and

(d) at remaining points, $F = 0$.

Note that $u \in c_{00}$, $F \in c_{00} \otimes c_{00}$, and $\pi(F) = u$. Set $\Delta_a = \Delta_a(F)$.

Clearly Δ_a is zero except on the sets $(B_j \times B_k) \cup (B_k \times B_j)$ where $j \leq n_1$ and $k > m_0$. On the set

$$\left(\bigcup_{j \leq n_0} B_j\right) \times \left(\bigcup_{k > m_0} B_k\right)\,,$$

we see that $a \cdot F - F \cdot a$ and $a \otimes u$ are zero, and that $u \otimes a = u \otimes Q_{m_0} a$. A similar formula holds when j and k are interchanged. Note that $Q_{m_0} a \otimes u$ and $u \otimes Q_{m_0} a$ have disjoint supports in $\mathbb{N} \times \mathbb{N}$. Thus

$$\begin{aligned} \|\Delta_a\| &= 2\|Q_{m_0} a \otimes u\| = 2\left\|\sum\{Q_{m_0} a \otimes \delta_r : r \in D_{n_1}\}\right\| \\ &\leq 2|D_{n_1}| \cdot \|Q_{m_0} a\| < \varepsilon \end{aligned}$$

by (3.4). This establishes (3.3). □

Note the explicit dependence of F on the element a in clause (b), above. One is tempted to try the 'more obvious' definition

$$F_{i,j} = \begin{cases} 1 & (i, j \leq n)\,, \\ 0 & (\text{otherwise})\,, \end{cases}$$

for suitably large $n \in \mathbb{N}$, so that $\pi(F) = e_n$. In this case, F is independent of a. Suppose that S is a finite subset of c_{00} (rather than A). Then our function F satisfies (i) and (ii) of Proposition 3.3 for each $a \in S$

(for sufficiently large $n \in \mathbb{N}$). However this choice of F does not work for all $a \in A$. For example, take $A = \ell^1$, and set $a = \sum_j j^{-3/2}\delta_j \in A$. Then

$$\|\Delta_a\| = \sum_{j=n+1}^{\infty} j^{-3/2}\|\delta_j \otimes e_n - e_n \otimes \delta_j\| = 2n \sum_{j=n+1}^{\infty} j^{-3/2} \geq 4$$

for each $n \in \mathbb{N}$.

In fact, let $A = \ell^1$, and let S be a finite subset of A^2. Then we *claim* that for each $\varepsilon > 0$, there exists $F \in A \otimes A$ such that (3.3) holds for each $a \in S$. This may add some credence to our conjecture that $A^2 = A$ for an approximately amenable Banach sequence algebra.

To prove this claim, we first recall Pringsheim's theorem: for a monotonic decreasing sequence $(a_i) \in A$, one has $\lim_i ia_i = 0$.

Now take $a = (a_i) \in A$ with $0 \leq a_i \leq 1 \;\; (i \in \mathbb{N})$. Certainly $a_i \to 0$, and so there is a permutation σ of $\mathbb{N}$ such that $a_{\sigma(j)} \leq a_{\sigma(i)}$ for $j \geq i$ in $\mathbb{N}$. Thus $ia_{\sigma(i)} \to 0$. Fix $\varepsilon \in (0,1)$, and take $n \in \mathbb{N}$ such that $ja_{\sigma(j)} < \varepsilon/2$ for $j \geq n$ and also $\sum_{j=n+1}^{\infty} a_j < \varepsilon$. Set $B = \sigma^{-1}(\mathbb{N}_n) \cup \mathbb{N}_n$, where $\mathbb{N}_n = \{1, \dots, n\}$. Then set $u = \chi_B$, the characteristic function of B, and

$$F_{i,j} = \begin{cases} 1 & (i,j \in B)\,, \\ 0 & (\text{otherwise})\,. \end{cases}$$

We see that

$$s := |B| \sum \{a_i^2 : i \in \mathbb{N} \setminus B\} \leq 2n \sum_{j=n+1}^{\infty} a_{\sigma(j)}^2 \leq \frac{n\varepsilon^2}{2} \sum_{j=n+1}^{\infty} j^{-2} < \varepsilon\,.$$

Thus

$$\|a^2 \cdot F - F \cdot a^2 + u \otimes a^2 - a^2 \otimes u\| = 2\|Q_B a^2\|\,\|u\| = s < \varepsilon\,,$$

and we have built in the fact that $\|a^2 - ua^2\| < \varepsilon$. It follows that the conditions of (3.3) are satisfied for a^2.

For finitely many elements in A^2, it suffices to consider the case where each of them is real-valued, and hence we need only consider differences of finitely many squares of non-negative elements of A, say the elements are $a^{(1)}, \dots, a^{(k)}$. We then have finitely many permutations $\sigma_1, \dots, \sigma_k$ of $\mathbb{N}$ that respectively render each of these latter sequences decreasing. We argue as above, with $n \in \mathbb{N}$ chosen so that, for each $1 \leq i \leq k$, we have $ja^{(i)}_{\sigma_i(j)} < \varepsilon/2k$ for $j \geq n$ and also $\sum_{j=n+1}^{\infty} a_j^{(i)} < \varepsilon$. Finally, we set

$$B = \mathbb{N}_n \cup \bigcup_{i=1}^{k} \sigma_i^{-1}(\mathbb{N}_n)\,.$$

The above claim now follows.

4. Approximate amenability for ℓ^p

Take $1 \le p < \infty$. Then ℓ^p is a Banach sequence algebra, and c_{00} is dense in ℓ^p. These algebras are discussed in [**1**, Example 4.1.42].

It is well known that ℓ^p is weakly amenable, but not amenable. Clearly the sequence (e_n) is an approximate identity for ℓ^p such that $\|e_n\|_p = n^{1/p}$ $(n \in \mathbb{N})$. Certainly each $a \in \ell^p$ satisfies equation (3.2) above.

It is shown in [**3**, Example 6.3] that ℓ^p is not sequentially approximately amenable. In this section we show that ℓ^p is not approximately amenable.

To this end, some preliminaries and further notations are needed.

First, note that the map

$$T : \ell^p \times \ell^p \to \ell^p(\mathbb{N} \times \mathbb{N}) : T(x,y)(i,j) = x_i y_j \,,$$

is bilinear with $\|T\| = 1$, and so there is a map

$$\widetilde{T} : \ell^p \,\widehat{\otimes}\, \ell^p \to \ell^p(\mathbb{N} \times \mathbb{N})$$

with $\widetilde{T}(x \otimes y) = T(x,y)$ $(x, y \in \ell^p)$ and $\|\,\widetilde{T}\| = 1$. Let $H \in c_{00} \otimes c_{00}$. Then

$$\sum_{i,j} |H(i,j)|^p \le \|H\|^p \,, \tag{4.1}$$

where $\|H\|$ denotes the norm of H in $\ell^p \,\widehat{\otimes}\, \ell^p$. (Of course, equality holds in the case where $p = 1$.)

Fix throughout $\gamma_j = \dfrac{1}{j(j+1)}$ and set $\gamma = (\gamma_j)$. Note that γ is positive, decreasing, and satisfies

$$k\gamma_k \le \sum_{j=k+1}^{\infty} \gamma_j \,. \tag{4.2}$$

Now let $\eta = (\eta_j) \in \ell^1$ be positive and decreasing, and define elements a, b in ℓ^p by

$$a = \sum_{j=1}^{\infty} \eta_j^{1/p} \delta_{2j-1} \,, \quad b = \sum_{j=1}^{\infty} \eta_j^{1/p} \delta_{2j} \,.$$

We show that, for a suitable choice of η and for a certain $\varepsilon > 0$, there is no element $F \in c_{00} \otimes c_{00}$ such that both the following inequalities are true:

$$\|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon \,; \tag{4.3}$$

$$\|a - \pi(F)a\| + \|b - \pi(F)b\| < \varepsilon \,. \tag{4.4}$$

It would then follow from Proposition 3.3 that ℓ^p is not approximately amenable.

Throughout, we set $u = \pi(F)$. As we remarked earlier, we may suppose that F (and u) are real-valued.

We first make a small reduction. We may suppose that $\varepsilon < \eta_1^{1/p}$. Now assume that F satisfies (4.4), with ε replaced by $\varepsilon/2$. Then $\eta_1^{1/p}(1-u_1) < \eta_1^{1/p}/2$, and so $u_1 > 1/2$. By replacing u and F by u/u_1 and F/u_1, respectively, we find new elements $F \in c_{00} \otimes c_{00}$ and $u \in c_{00}$ such that $\pi(F) = u$ and

$$u_1 = 1 \quad \text{and} \quad \|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon\,.$$

Thus we may always suppose that $u_1 = 1$.

We shall need to estimate $\|\Delta_x\| = \|\Delta_x(F)\|$ for $x = a, b$, and for this we shall use (4.1). Thus we require lower bounds for $|\Delta_x(m,n)|$ for $m, n \in \mathbb{N}$.

First consider the points $(2i-1, 2j)$, where $i, j \in \mathbb{N}$. For convenience, define $s = F_{2i-1,2j}$. We calculate the values

$$\begin{aligned}\Delta_a(2i-1,2j) &= \eta_i^{1/p}(s - u_{2j})\,,\\ \Delta_b(2i-1,2j) &= \eta_j^{1/p}(u_{2i-1} - s)\,.\end{aligned}$$

In the case where $i \le j$, so that $\eta_i \ge \eta_j$, geometrical considerations show that

$$|s-u_{2j}|^p\eta_i + |u_{2i-1}-s|^p\eta_j \ge \eta_j\left(|u_{2i-1}-u_{2j}|/2\right)^p\,.$$

In a similar manner, the points $(2i, 2j-1)$ taken with $i \le j-1$ and $j \ge 2$, so that $\eta_i \ge \eta_j$, lead to the estimate

$$|t-u_{2j-1}|^p\eta_i + |u_{2i}-t|^p\eta_j \ge \eta_j\left(|u_{2i}-u_{2j-1}|/2\right)^p\,,$$

where $t = F_{2i,2j-1}$.

[At the points $(2i-1, 2j-1)$ and $(2i, 2j)$, where $i, j \in \mathbb{N}$, we have

$$\begin{aligned}\Delta_a(2i-1,2j-1) &= (\eta_i^{1/p} - \eta_j^{1/p})F_{2i-1,2j-1} - \eta_i^{1/p}u_{2j-1} + \eta_j^{1/p}u_{2i-1}\,,\\ \Delta_b(2i,2j) &= (\eta_i^{1/p} - \eta_j^{1/p})F_{2i,2j} - \eta_i^{1/p}u_{2j} + \eta_j^{1/p}u_{2i}\,,\\ \Delta_a(2i,2j) &= \Delta_b(2i-1,2j-1) = 0\,.\end{aligned}$$

Since $\eta_i \ne \eta_j$ for $i \ne j$, there are choices of the values of F at the points $(2i-1, 2j-1)$ and $(2i, 2j)$ giving zero values to both Δ_a and Δ_b at all these points. We shall not use this fact.]

For $u = (u_i) \in c_{00}$, set

$$\Phi_p(\eta, u) = \sum_{j=1}^{\infty}\eta_j\sum_{i=1}^{j}|u_{2i-1}-u_{2j}|^p + \sum_{j=2}^{\infty}\eta_j\sum_{i=1}^{j-1}|u_{2i}-u_{2j-1}|^p\,. \quad (4.5)$$

It follows from (4.1), the above estimates, and the simple inequality

$$(\|\alpha\| + \|\beta\|)^p \geq \|\alpha\|^p + \|\beta\|^p \quad (\alpha, \beta \in \mathbb{C}),$$

that

$$2^p(\|\Delta_a(F)\| + \|\Delta_b(F)\|)^p \geq \Phi_p(\eta, u).$$

Set

$$\theta_p(\eta) = \inf\{\Phi_p(\eta, u) : u \in c_{00},\, u_1 = 1\}.$$

We seek to show that, for suitable choice of η, we have $\theta_p(\eta) > 0$, for then (4.3) fails for any ε with $0 < \varepsilon < \min\{\theta_p(\eta)^{1/p}, \eta_1^{1/p}\}/2$, and so ℓ^p is not approximately amenable.

We note that $\Phi_p(\eta, u)$ is reduced if every value of u_i outside $[0,1]$ is replaced by its nearest neighbour in $[0,1]$. Thus we may suppose throughout that

$$0 \leq u_i \leq 1 \quad (i \in \mathbb{N}).$$

For $d \geq 2$, consider the set

$$S_d = \{u \in c_{00} : u_1 = 1,\, u_i \in [0,1] \ (i = 1, \dots, d),\, u_i = 0 \ (i > d)\}.$$

Certainly

$$\alpha_d = \min\{\Phi_p(\eta, u) : u \in S_d\} > 0,$$

and this minimum is attained. The question is whether or not

$$\lim_{d \to \infty} \alpha_d > 0.$$

Suppose for the moment that $p = 1$, and take $\eta = \gamma$. Thus, in this case, $\Phi_1(\eta, u)$ from (4.5) becomes

$$\Phi_1(\gamma, u) = \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|.$$

Consider the values of $\Phi_1(\gamma, u)$ for sequences $u \in S_d$, where $d \geq 2$. Indeed, take such a point u with $u_d > 0$. We *claim* that, by setting $u_d = 0$, the value of $\Phi_1(\gamma, u)$ is reduced.

To establish this claim, first suppose that $d = 2k+1$ for some $k \in \mathbb{N}$. By the change specified, we first increase each term in the summand

$$\gamma_{k+1} \sum_{i=1}^{k} |u_{2i} - u_{2k+1}|$$

by at most $u_{2k+1}\gamma_{k+1}$, and so the sum itself increases by at most $ku_{2k+1}\gamma_{k+1}$. On the other hand, we decrease the term

$$\sum_{j=k+1}^{\infty} \gamma_j |u_{2k+1} - u_{2j}| = \left(\sum_{j=k+1}^{\infty} \gamma_j\right) u_{2k+1}$$

by u_{2k+1} times the sum $\sum_{j=k+1}^{\infty}\gamma_j$ of the tail. Other terms are not affected. However, for each $k \in \mathbb{N}$, we have

$$k\gamma_{k+1} \leq k\gamma_k \leq \sum_{j=k+1}^{\infty}\gamma_j$$

by (4.2), and so, in total, the value of $\Phi_1(\gamma, u)$ has been decreased.

Now suppose that $d = 2k$ for some $k \in \mathbb{N}$. By the change specified, we firstly increase each term in the summand

$$\gamma_k \sum_{i=1}^{k} |u_{2i-1} - u_{2k}|$$

by at most $u_{2k}\gamma_k$, and so the sum itself increases by at most $ku_{2k}\gamma_k$. On the other hand, we decrease the term

$$\sum_{j=k+1}^{\infty}\gamma_j|u_{2k} - u_{2j-1}|$$

by u_{2k} times the sum $\sum_{j=k+1}^{\infty}\gamma_j$ of the tail. Other terms are not affected. Once again, (4.2) ensures that the value of $\Phi_1(\gamma, u)$ has been decreased.

By continuing, we see that, subject to the constraints we have imposed, and in particular that $u \in c_{00}$ and $u_1 = 1$, the minimum value of $\Phi_1(\gamma, u)$ is attained at the point $v = (1, 0, 0, \ldots)$, and so

$$\theta_1 = \Phi_1(\gamma, v) = \sum_{j=1}^{\infty}\gamma_j = 1\,.$$

Hence we obtain the required contradiction, at least in the case where $p = 1$.

Now consider the case where $p > 1$. Again we should like to show that $\theta_p(\eta) > 0$ for suitable η. The above method for the case that $p = 1$ does not now work; indeed the minimum value $\min\{\Phi_p(\eta, u) : u \in S_d\}$ need not occur at the point $u = (1, 0, 0, \ldots)$, and in fact, perhaps surprisingly, it does not necessarily occur at a decreasing sequence u of S_d. In fact we cannot explicitly calculate $\theta_p(\eta)$, but we obtain a lower bound by the use of Hölder's inequality.

With $1/p + 1/q = 1$, choose $\alpha > 0$ so small that $1 - p\alpha/q > 1/2$. Then we have

$$\delta = \sum_{j=1}^{\infty} j\gamma_j^{1+\alpha} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty}\gamma_j^{1-p\alpha/q} < \infty\,,$$

and so, in particular, the formula $\eta_j = \gamma_j^{1-p\alpha/q}\,(j \in \mathbb{N})$ defines a sequence $\eta \in \ell^1$ which is positive and decreasing.

Note that

$$\frac{1+\alpha}{q} + \left(1 - \frac{p}{q}\alpha\right)\frac{1}{p} = \frac{p+q}{pq} = 1$$

and that $\gamma_j = \eta_j^{1/p} \cdot \gamma_j^{(1+\alpha)/q}$. For each $u \in c_{00}$ with $u_1 = 1$ we apply Hölder's inequality to the sequence $(x_r y_r)$, where (x_r) has generic term $\eta_j^{1/p}|u_{2i-1} - u_{2j}|$ or $\eta_j^{1/p}|u_{2i} - u_{2j-1}|$, and (y_r) has the corresponding generic term $\gamma_j^{(1+\alpha)/q}$. Thus we obtain

$$\begin{aligned} 1 \le \Phi_1(\gamma, u) = & \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}| \\ \le & \left(\sum_{j=1}^{\infty} \sum_{i=1}^{j} \gamma_j^{1+\alpha} + \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} \gamma_j^{1+\alpha} \right)^{1/q} \Phi_p(\eta, u)^{1/p} \\ \le & \ (2\delta)^{1/q} \Phi_p(\eta, u)^{1/p} . \end{aligned}$$

It follows that $\theta_p(\eta) \ge (2\delta)^{-p/q} > 0$, as required.

Thus we have the following result.

THEOREM 4.1. *The Banach sequence algebras $\ell^p(\mathbb{N})$, $1 \le p < \infty$, are not approximately amenable.*

It is immediate that $\ell^p(S)$ is not approximately amenable for any infinite set S, since there is a continuous epimorphism $\ell^p(S) \to \ell^p(\mathbb{N})$.

Take $1 \le p < \infty$. In [**3**, Corollary 7.1] it was shown that the Banach algebras ℓ^p are essentially amenable, that is, any derivation into the dual of a neo-unital bimodule is inner. From Theorem 4.1 we conclude that essential amenability does not imply approximate amenability. It also follows by the Plancherel theorem that $L^2(\mathbb{T})$ fails to be approximately amenable, though by [**6**, Theorem 4.5] it is pseudo-contractible, that is, it admits a central (unbounded) approximate diagonal.

We finally consider a weighted variant of the ℓ^p algebras.

Let $\omega \in [1, \infty)^{\mathbb{N}}$. For $p \ge 1$, we consider

$$\ell^p(\omega) = \{f \in \mathbb{C}^{\mathbb{N}} : f \cdot \omega \in \ell^p\},$$

where $f \cdot \omega$ denotes the sequence with the ith coordinate $(f \cdot \omega)(i) = f_i\,\omega_i$ $(i \in \mathbb{N})$. With the norm

$$\|f\|_{p,\omega} = \|f \cdot \omega\|_p \quad (f \in \ell^p(\omega))\,,$$

$\ell^p(\omega)$ is a Banach algebra under pointwise operations. As previously, the map $T\colon \ell^p(\omega) \times \ell^p(\omega) \to \ell^p(\omega \otimes \omega) = \ell^p(\omega \otimes \omega, \mathbb{N} \times \mathbb{N})$ given by

$$T(x, y)(i, j) = x_i y_j \quad (x, y \in \ell^p(\omega),\ i, j \in \mathbb{N})$$

defines a contractive operator $\widetilde{T}\colon \ \ell^p(\omega)\widehat{\otimes}\ell^p(\omega) \to \ell^p(\omega \otimes \omega)$, where $\omega \otimes \omega$ denotes the weight on $\mathbb{N} \times \mathbb{N}$ such that $\omega \otimes \omega(i, j) = \omega_i \omega_j$ $(i, j \in \mathbb{N})$. As for the case of ℓ^p, we aim to show that for some $\varepsilon > 0$ and elements $a, b \in \ell^p(\omega)$, there is no $F \in c_{00} \otimes c_{00}$ such that both the following inequalities are true:

$$\|\Delta_a(F)\|_{p,\omega\otimes\omega} + \|\Delta_b(F)\|_{p,\omega\otimes\omega} < \varepsilon\,;$$
$$\|a - \pi(F)a\|_{p,\omega} + \|b - \pi(F)b\|_{p,\omega} < \varepsilon\,.$$

We take $\gamma = (\gamma_i)$ and $\eta = (\eta_i)$ the same as in the proof of Theorem 4.1. Set

$$\eta'_j = \frac{\eta_j}{\omega^p_{2j-1}}\,, \quad \eta''_j = \frac{\eta_j}{\omega^p_{2j}} \quad (j \in \mathbb{N})\,,$$

and define

$$a = \sum_{j=1}^{\infty} (\eta'_j)^{1/p} \delta_{2j-1}\,, \quad b = \sum_{j=1}^{\infty} (\eta''_j)^{1/p} \delta_{2j}\,,$$

so that $a, b \in \ell^p(\omega)$. Now for $F \in c_{00} \otimes c_{00}$ and $u = \pi(F)$, then following the same argument as in the proof of Theorem 4.1, we find that

$$2^p(\|\Delta_a(F)\|_{p,\omega\otimes\omega} + \|\Delta_b(F)\|_{p,\omega\otimes\omega})^p \geq \Phi_p(\eta, u)\,,$$

where $\Phi_p(\eta, u)$ is given by Equation (4.5). This finally shows that the value of $\|\Delta_a\|_{p,\omega\otimes\omega} + \|\Delta_b\|_{p,\omega\otimes\omega}$ is bounded away from 0 as a function of $F \in c_{00} \otimes c_{00}$. We therefore conclude with the following theorem.

THEOREM 4.2. *The Banach sequence algebras $\ell^p(\omega)$, $1 \leq p < \infty$, are not approximately amenable for any weight ω.* □

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