Approximate amenability for Banach sequence algebras

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Abstract. We consider when certain Banach sequence algebras $A$ on the set $\mathbb{N}$ are approximately amenable. Some general results are obtained, and we resolve the special cases where $A = \ell^p$ for $1 \leq p < \infty$, showing that these algebras are not approximately amenable. The same result holds for the weighted algebras $\ell^p(\omega)$.

1. Introduction

The concept of amenability for a Banach algebra $A$, introduced by Johnson in 1972 [7], has proved to be of enormous importance in Banach algebra theory (see [1], for example). In [3] several modifications of this notion were introduced; in this paper we shall focus on one of these, that of approximate amenability. We recall the definition in Definition 1.1, below.

Let $A$ be an algebra, and let $X$ be an $A$-bimodule. A derivation is a linear map $D: A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a,b \in A).$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a$, $A \to X$. Then $ad_x$ is a derivation; these are the inner derivations.

Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. A continuous derivation $D: A \to X$ is approximately inner if there is a
net \((x_\alpha)\) in \(X\) such that

\[
D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),
\]

so that \(D = \lim_\alpha ad_{x_\alpha}\) in the strong-operator topology of \(B(A)\).

The dual of a Banach space \(X\) is denoted by \(X'\); in the case where \(X\) is a Banach \(A\)-bimodule, \(X'\) is also a Banach \(A\)-bimodule. For the standard dual module definitions, see \([1]\).

**Definition 1.1.** [3] Let \(A\) be a Banach algebra. Then \(A\) is approximately amenable if, for each Banach \(A\)-bimodule \(X\), every continuous derivation \(D : A \to X'\) is approximately inner.

The qualifier *sequential* prefixed to the above definition specifies that there is a sequence of inner derivations approximating the given continuous derivation.

We remark that, in [3], the notion of uniform approximate amenability was also introduced: a Banach algebra \(A\) is *uniformly approximately amenable* if, for each Banach \(A\)-bimodule \(X\), each continuous derivation \(D : A \to X'\) is the limit of a sequence of inner derivations in the norm topology of \(B(A, X')\). In fact, it has recently been shown independently by Pirkovskii [10] and Ghahramani [4] that a uniformly approximately amenable Banach algebra is already amenable.

Of course, each amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras which are not amenable are constructed in [3]. For example, let \((A_n)\) be a sequence of unital, amenable Banach algebras. Then the sum \(c_0(A_n)\) is always approximately amenable, but is not necessarily amenable [3, Example 6.1]. Further, it has been shown by Ghahramani and Stokke [5] that the Fourier algebra \(A(G)\) is approximately amenable for each amenable, discrete group \(G\), but it is known that \(A(G)\) is not always amenable for an amenable group \(G\) [9]. Examples of semigroup algebras of the form \(\ell^1(S)\) that are approximately amenable but not amenable are given in [2]. Nevertheless there is something of a shortage of ‘natural’ examples of approximately amenable Banach algebras which are not amenable. In this paper, we shall consider when certain Banach sequence algebras on \(\mathbb{N}\) are approximately amenable, a question left open in [3]. In particular, we shall consider the standard Banach sequence algebras \(\ell^p = \ell^p(\omega)\), where \(1 \leq p < \infty\) and \(\omega\) is a weight on \(\mathbb{N}\).
2. Basic constructions

When determining whether or not our Banach algebras are approximately amenable, we shall work from a characterization of approximately amenable Banach algebras which is a modification of that given in [3].

Let $A$ be Banach algebra. The projective tensor product $A \hat{\otimes} A$ is a Banach $A$-bimodule, under the operations defined by

$$c \cdot a \otimes b = ca \otimes b, \quad a \otimes b \cdot c = a \otimes bc \quad (a, b, c \in A),$$

and there is a continuous linear $A$-bimodule homomorphism $\pi: A \hat{\otimes} A \rightarrow A$ such that $\pi(a \otimes b) = ab \quad (a, b \in A)$ [1].

**Proposition 2.1.** Let $A$ be a Banach algebra. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exist $F \in A \hat{\otimes} A$ and $u, v \in A$ such that $\pi(F) = u + v$ and, for each $a \in S$:

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| < \varepsilon$;

(ii) $\|a - au\| < \varepsilon$ and $\|a - va\| < \varepsilon$.

**Proof.** Suppose that $A$ is approximately amenable. Then by [3, Corollary 2.2] there are nets $(M_\alpha)$ in $(A \hat{\otimes} A)''$, and $(U_\alpha)$ and $(V_\alpha)$ in $A''$ such that, for each $a \in A$:

(i) $a \cdot M_\alpha - M_\alpha \cdot a + U_\alpha \otimes a - a \otimes V_\alpha \rightarrow 0$;

(ii) $a - a \cdot U_\alpha \rightarrow 0$ and $a - V_\alpha \cdot a \rightarrow 0$;

(iii) $\pi''(M_\alpha) - U_\alpha - V_\alpha \rightarrow 0$.

(This corrects a typographical error in [3].) In each case convergence is in the $\|\cdot\|$-topology.

Let $Y$ denote the Banach space $(A \hat{\otimes} A) \oplus A \oplus A \oplus A$. For each $a \in A$, define a convex set in $Y$ by setting

$$K_a := \{(a \cdot m - m \cdot a + u \otimes a - a \otimes v, \quad a - au, a - va, \pi(m) - u - v) : m \in A \hat{\otimes} A, u, v \in A\}.$$

For the specified finite subset $S$ of $A$,

$$K := \prod \{K_a : a \in S\}$$

is a convex set in the Banach space $Y^S$. The conditions above show that the weak closure of $K$ in $Y^S$ contains the zero element 0 of $Y^S$. By Mazur’s theorem, it follows that 0 belongs to the $\|\cdot\|$-closure of $K$ in $Y^S$. Thus, with $\varepsilon > 0$ as specified, there exist $F \in A \hat{\otimes} A$ and $u, v \in A$ such that clauses (i) and (ii) of the proposition are satisfied and, further,
such that \( \| \pi(F) - u - v \| < \varepsilon \). By modifying \( F \) and \( u \) slightly, we may suppose, further, that \( F \in A \otimes A \) and that \( \pi(F) = u + v \).

Conversely, suppose that the condition in the proposition is satisfied. Consider the set \( D := (0, 1) \times \mathcal{F}(A) \), where \( \mathcal{F}(A) \) is the family of finite subsets of \( A \), and order \( D \) by setting\( (\varepsilon_1, S_1) \preccurlyeq (\varepsilon_2, S_2) \) whenever \( \varepsilon_1 \geq \varepsilon_2 \) and \( S_1 \subseteq S_2 \).

Then \((D, \preccurlyeq)\) is a directed set. The conditions show that there exist nets \((F_\alpha)\) in \( A \hat{\otimes} A \), and \((u_\alpha)\), \((v_\alpha)\) in \( A \) such that \( \pi(F_\alpha) = u_\alpha + v_\alpha \) and such that, for each \( a \in A \), we have:

\[
a \cdot F_\alpha - F_\alpha \cdot a + u_\alpha \otimes a - a \otimes v_\alpha \to 0;
\]
\[
a - au_\alpha \to 0; \quad a - v_\alpha a \to 0.
\]

Thus we have satisfied the conditions of [3, Corollary 2.2], and so \( A \) is approximately amenable.

**Corollary 2.2.** Let \( A \) be a Banach algebra with identity \( e \). Then \( A \) is approximately amenable if and only if, for each \( \varepsilon > 0 \) and each finite subset \( S \) of \( A \), there exists \( G \in A \otimes A \) with \( \pi(G) = e \) and such that

\[
\|a \cdot G - G \cdot a\| < \varepsilon \quad (a \in S).
\]

**Proof.** Suppose that such a \( G \) exists, and set \( u = v = e \) and \( F = G + e \otimes e \). Then \( \pi(F) = u + v \), and \( F, u, v \) satisfy the conditions of Proposition 2.1.

Conversely, suppose that \( F, u, v \) satisfy the above condition for a finite subset \( S \) and with \( \varepsilon/3\|e\| \) replacing \( \varepsilon \), and set

\[
G = F - u \otimes e - e \otimes v + e \otimes e.
\]

Then \( \pi(G) = e \), and

\[
\|a \cdot G - G \cdot a\| \leq \|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| + \|a - au\| + \|a - va\| < \varepsilon,
\]

and so \( A \) is approximately amenable by Proposition 2.1.

For comparison, we recall [1], [8] that a Banach algebra \( A \) is amenable if and only if there is a constant \( C > 0 \) such that, for each \( \varepsilon > 0 \) and each finite subset \( S \) of \( A \), there exists \( F \in A \otimes A \) with \( \|F\| \leq C \) such that, for each \( a \in S \), we have:

(i) \( \|a \cdot F - F \cdot a\| < \varepsilon \);

(ii) \( \|a - a\pi(F)\| < \varepsilon \).

We remark that (ii) of Proposition 2.1 is exactly the condition that \( A \) has both left and right approximate units [1, Definition 2.9.10]. We
do not know whether or not an approximately amenable Banach algebra necessarily has (two-sided) approximate units.

We now give a variation of Proposition 2.1 in the case where $A$ is commutative. For each Banach algebra $A$, there is an isometry $\iota : A \hat{\otimes} A \to A \hat{\otimes} A$ such that $\iota(a \otimes b) = b \otimes a$ ($a, b \in A$).

**Proposition 2.3.** Let $A$ be a commutative Banach algebra. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exist $F \in A \hat{\otimes} A$ with $\iota(F) = F$ and $u \in A$ such that $\pi(F) = 2u$, and, for each $a \in S$:

1. $\|a \cdot F - F \cdot a + u \otimes a - a \otimes u\| < \varepsilon$;
2. $\|a - au\| < \varepsilon$.

**Proof.** Since $A$ is commutative, $\iota(a \cdot F) = \iota(F) \cdot a$ ($a \in A, F \in A \hat{\otimes} A$).

Suppose that $A$ is approximately amenable, and take $\varepsilon > 0$ and a finite subset $S$ of $A$. By Proposition 2.1, there exist $F, u, v$ satisfying conditions (i) and (ii) of that result. For each $a \in S$, we have

$$\|\iota(F) \cdot a - a \cdot \iota(F) + a \otimes u - v \otimes a\| < \varepsilon.$$ 

Set $G = (F + \iota(F))/2$ and $w = (u + v)/2$. Then $\iota(G) = G$ and $\pi(G) = 2w$. Further,

$$\|a \cdot G - G \cdot a + w \otimes a - a \otimes w\| < \varepsilon \quad \text{and} \quad \|a - aw\| < \varepsilon.$$ 

Thus the specified conditions are satisfied (with $w$ for $u$).

The converse is immediate. \qed

3. **Banach sequence algebras**

We now introduce the specific algebras that will be considered in this paper. As usual $c_{00}$ will be the subalgebra of $\mathbb{C}^\mathbb{N}$ consisting of sequences having finite support.

**Definition 3.1.** A Banach sequence algebra on $\mathbb{N}$ is a Banach algebra $A$ which is a subalgebra of $\mathbb{C}^\mathbb{N}$ such that $c_{00} \subset A$.

For example, $c_0 = c_0(\mathbb{N})$ and $\ell^p = \ell^p(\mathbb{N})$ for $1 \leq p \leq \infty$ are Banach sequence algebras on $\mathbb{N}$.

Let $(A, \| \cdot \|)$ be a Banach sequence algebra on $\mathbb{N}$. Then

$$\|a\| \geq |a|_\mathbb{N} \quad (a \in A),$$

where $| \cdot |_\mathbb{N}$ denotes the uniform norm on $\mathbb{N}$. In the case where $c_{00}$ is dense in $A$, the algebra $A$ is natural on $\mathbb{N}$ [1, Proposition 4.1.35].
Throughout we write $\delta_i$ for the characteristic function of $\{i\}$ for $i \in \mathbb{N}$, and set

$$e_n = \sum_{i=1}^{n} \delta_i \quad (n \in \mathbb{N}),$$

so that $(e_n) \subset c_00 \subset A$. When convenient we identify $a \in A$ both as the sequence $(a_i)$ and as the formal sum $\sum_i a_i \delta_i$. We shall also identify $A \otimes A$ with a space of functions on $\mathbb{N} \times \mathbb{N}$ by setting

$$(a \otimes b)(i, j) = a_i b_j \quad (a, b \in A, i, j \in \mathbb{N});$$

in particular, $\delta_i \otimes \delta_j = \delta_{(i,j)}$, the characteristic function of $\{(i, j)\}$, for $i, j \in \mathbb{N}$. We shall also sometimes write $F = \sum_{i,j} F(i, j) \delta_{(i,j)}$ for $F \in c_00 \otimes c_00$. Note that

$$(a \cdot F)(i, j) = a_i F(i, j), \quad (F \cdot a)(i, j) = a_j F(i, j) \quad (i, j \in \mathbb{N}),$$

and that $\pi(F) = \sum_i F(i, i) \delta_i$.

**Definition 3.2.** Let $A$ be a Banach sequence algebra on $\mathbb{N}$, and let $a \in A$. For $F \in c_00 \otimes c_00$, set

$$\Delta_a(F) = a \cdot F - F \cdot a + \pi(F) \otimes a - a \otimes \pi(F).$$

Clearly $\Delta_a(F) \in c_00 \otimes c_00$ whenever $a \in c_00$.

**Proposition 3.3.** Let $A$ be a Banach sequence algebra with $c_00$ dense in $A$. Then $A$ is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exists $F \in c_00 \otimes c_00$ with $\iota(F) = F$ such that, for each $a \in S$:

(i) $||\Delta_a(F)|| < \varepsilon$;
(ii) $||a - a\pi(F)|| < \varepsilon$.

**Proof.** Suppose that $A$ is approximately amenable, and take $\varepsilon > 0$ and a finite subset $S$ of $A$. Let $F$ and $u$ be given by Proposition 2.3. Since $c_00$ is dense in $A$, the space $c_00 \otimes c_00$ is dense in $A \otimes A$, and so we can replace $F$ by an element $G \in c_00 \otimes c_00$ such that (i) and (ii) of that proposition remain true, with $v = \pi(G)/2$ replacing $u$. Now replace $G$ by

$$H = G + \sum_i (v_i - \pi(G)_i) \delta_i \otimes \delta_i,$$

noting that the number of non-zero summands in the above sum is finite. This does not affect clauses (i) or (ii) of Proposition 2.3, and now $\pi(H) = v$. Thus conditions (i) and (ii) of the current proposition are satisfied.

The converse is similar. \qed
We shall later consider only Banach sequence algebras $A$ which are self-adjoint. In such a situation the map $a \mapsto a$ is necessarily continuous on $A$. It follows that we may replace $F$ by $F + \overline{F}$, and so take $F$ to be real-valued. Similarly, we may also suppose that the elements of the ‘test sets’ $S$ are real-valued.

**Proposition 3.4.** Let $A$ be a Banach sequence algebra. Suppose that there is $\eta > 0$ such that, for each $\varepsilon > 0$ and each finite subset $S$ of $A$, there exists $u \in c_{00}$ with

$$\|u\| \geq \eta \quad \text{and} \quad \|a - au\| \cdot \|u\| < \varepsilon. \quad (3.1)$$

Then $A$ is approximately amenable.

**Proof.** Take $u$ as given by (3.1), with $\varepsilon$ replaced by $\varepsilon \eta/2$. Set

$$F = u \otimes u + \sum_{i}(u_i - u_i^2)\delta_i \otimes \delta_i.$$

Then $\pi(F) = u$ and, for each $a \in S$, we have

$$\|a \cdot F - F \cdot a - a \otimes u + u \otimes a\| = \|au \otimes u - a \otimes u + u \otimes a - u \otimes au\| < \varepsilon$$

and $\|a - au\| < \varepsilon$. By Proposition 3.3, $A$ is approximately amenable.

The converse is immediate. \(\square\)

More general forms of this result for Banach function algebras on discrete spaces can be shown by the same sort of argument; see, for example, [5, Proposition 3.16].

We make the conjecture that the sufficient condition in Proposition 3.4 is in fact also necessary for $A$ to be approximately amenable. Indeed, we do not know an example of a Banach sequence algebra which is approximately amenable, but which does not have a bounded approximate identity. It is also conceivable that each Banach sequence algebra $A$ such that $c_{00}$ is dense in $A$ and $A = A^2$ is approximately amenable.

**Corollary 3.5.** Let $A$ be a Banach sequence algebra such that $A$ has a bounded approximate identity contained in $c_{00}$. Then $A$ is sequentially approximately amenable.

**Proof.** It is standard that $A$ has a sequential bounded approximate identity, say $(u_n)$, in $c_{00}$ [1, Corollary 2.9.18], and satisfying $\inf_n \|u_n\| \geq 1$. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of $A$. Then, for each $n \in \mathbb{N}$, there exists $i = i(n)$ such that $\|x_j - x_ju_i(n)\| < 1/n$ for $1 \leq j \leq n$. Following Proposition 3.4, we set

$$F_n = u_{i(n)} \otimes u_{i(n)} + \sum_{j \in \mathbb{N}}(u_{i(n),j} - u_{i(n),j}^2)\delta_j \otimes \delta_j.$$

Then, for each $a \in A$ and $\varepsilon > 0$, we have
$$
\|a \cdot F_n - F_n \cdot a - a \otimes u_{i(n)} + u_{i(n)} \otimes a\| = 2\|au_{i(n)} - a\| \cdot \|u_{i(n)}\| < \varepsilon
$$
for $n$ sufficiently large. Thus $(F_n, u_{i(n)})$ gives a sequence with the required properties of [3, Corollary 2.2]. The sequential variant of [3, Theorem 2.1] holds (with the same argument), and so $A$ is sequentially approximately amenable.

Special cases of the above corollary have been shown in [4], where it is also shown that the converse holds for certain Banach sequence algebras.

We wish to stress that the function $F$ specified in Proposition 3.3 must satisfy conditions (i) and (ii) for each finite collection $S$ of elements. The following shows that, for many Banach sequence algebras $A$, we can find $F$ to satisfy these conditions for each single element $a \in A$. Indeed the Banach sequence algebra $\ell^1$ satisfies the conditions of Proposition 3.6 below, but we shall see that it is not approximately amenable. To determine whether or not such an algebra $A$ is approximately amenable, we must look at sets $S$ with at least 2 elements.

We introduce the following notation. Let $A$ be a Banach sequence algebra on $\mathbb{N}$. For each $a \in A$ and each finite or cofinite subset $T$ of $\mathbb{N}$, set
$$
P_T : a \mapsto \sum_{i} \{a, \delta_i : i \in T\}, \quad A \to A.
$$
We also write $P_n = P_{\{1, \ldots, n\}}$ and $Q_n = I - P_n$ for $n \in \mathbb{N}$. The family $C$ of cofinite subsets of $\mathbb{N}$ will be directed by reverse set inclusion.

**Proposition 3.6.** Let $A$ be a Banach sequence algebra, and let $a \in A$. Suppose that
$$
\lim\{\|P_C a\| : C \in C\} = 0. \tag{3.2}
$$
Then, for each $\varepsilon > 0$, there exists $F \in c_{00} \otimes c_{00}$ such that
$$
\|\Delta a(F)\| < \varepsilon \quad \text{and} \quad \|a - a\pi(F)\| < \varepsilon. \tag{3.3}
$$

**Proof.** Let $\{B_i : i \in \mathbb{Z}^+\}$ be the partition of $\mathbb{N}$ such that $a$ takes the constant value $a_i$ on $B_i$ for $i \in \mathbb{N}$, such that $a$ takes the value 0 on $B_0$, and such that $a_i \neq a_j$ whenever $i, j \in \mathbb{Z}^+$ and $i \neq j$. Note that, by (3.2), each $B_i$ for $i \in \mathbb{N}$ is finite. For $n \in \mathbb{N}$, set
$$
D_n = \bigcup_{i=1}^{n} B_i \quad \text{and} \quad E_n = \bigcup_{i=n+1}^{\infty} B_i,
$$
and set $\mu(n) = \min E_n$, so that $\mu(n) \to \infty$ as $n \to \infty$.

Fix $\varepsilon > 0$, and take $n_0 \in \mathbb{N}$ such that $\|P_C a\| < \varepsilon$ for each cofinite subset $C$ of $\mathbb{N}$ with $\min C > n_0$. Next choose $n_1 \in \mathbb{N}$ such that $n_1 > n_0$.
and \( \mu(n_1) > n_0 \). Set \( C = E_{n_1} \cup (B_0 \cap [n_1, \infty)) \), so that \( D_{n_1} \) is finite and \( C \) is cofinite with \( \min C > n_0 \). Take \( u \) to be the characteristic function of \( D_{n_1} \), so that
\[
a - au = a\chi_{\mathbb{N}\setminus D_{n_1}} = P_C a,
\]
and hence
\[
\|a - au\| = \|P_C a\| < \varepsilon.
\]
By (3.2), we may choose \( m_0 \in \mathbb{N} \) with \( m_0 > n_1 \) and such that
\[
|D_{n_1}| \cdot \|Q_{m_0} a\| < \varepsilon/2.
\]
(3.4)

Now define \( F \) as follows.

(a) For \( j, k \leq n_1 \), set
\[
F = 1 \quad \text{on } B_j \times B_k;
\]

(b) for \( j \leq n_1 \) and \( n_1 < k \leq m_0 \), set
\[
F = \frac{-a_k}{a_j - a_k} \quad \text{on } B_j \times B_k;
\]

(c) by symmetry for \( k \leq n_1 \) and \( n_1 < j \leq m_0 \); and

(d) at remaining points, \( F = 0 \).

Note that \( u \in c_{00} \), \( F \in c_{00} \otimes c_{00} \), and \( \pi(F) = u \). Set \( \Delta_a = \Delta_a(F) \).

Clearly \( \Delta_a \) is zero except on the sets \((B_j \times B_k) \cup (B_k \times B_j)\) where \( j \leq n_1 \) and \( k > m_0 \). On the set
\[
\left( \bigcup_{j \leq n_0} B_j \right) \times \left( \bigcup_{k > m_0} B_k \right),
\]
we see that \( a \cdot F - F \cdot a \) and \( a \otimes u \) are zero, and that \( u \otimes a = u \otimes Q_{m_0} a \).
A similar formula holds when \( j \) and \( k \) are interchanged. Note that \( Q_{m_0} a \otimes u \) and \( u \otimes Q_{m_0} a \) have disjoint supports in \( \mathbb{N} \times \mathbb{N} \). Thus
\[
\|\Delta_a\| = 2\|Q_{m_0} a \otimes u\| = 2\left\| \sum \{Q_{m_0} a \otimes \delta_r : r \in D_{n_1}\} \right\|
\leq 2|D_{n_1}| \cdot \|Q_{m_0} a\| < \varepsilon
\]
by (3.4). This establishes (3.3).

Note the explicit dependence of \( F \) on the element \( a \) in clause (b), above. One is tempted to try the ‘more obvious’ definition
\[
F_{i,j} = \begin{cases} 
1 & (i, j \leq n), \\
0 & \text{(otherwise),}
\end{cases}
\]
for suitably large \( n \in \mathbb{N} \), so that \( \pi(F) = e_n \). In this case, \( F \) is independent of \( a \). Suppose that \( S \) is a finite subset of \( c_{00} \) (rather than \( A \)). Then our function \( F \) satisfies (i) and (ii) of Proposition 3.3 for each \( a \in S \).
(for sufficiently large $n \in \mathbb{N}$). However this choice of $F$ does not work for all $a \in A$. For example, take $A = \ell^1$, and set $a = \sum_j j^{-3/2} \delta_j \in A$. Then

$$\|\Delta a\| = \sum_{j=n+1}^{\infty} j^{-3/2} \|\delta_j \otimes e_n - e_n \otimes \delta_j\| = 2n \sum_{j=n+1}^{\infty} j^{-3/2} \geq 4$$

for each $n \in \mathbb{N}$.

In fact, let $A = \ell^1$, and let $S$ be a finite subset of $A^2$. Then we claim that for each $\varepsilon > 0$, there exists $F \in A \otimes A$ such that (3.3) holds for each $a \in S$. This may add some credence to our conjecture that $A^2 = A$ for an approximately amenable Banach sequence algebra.

To prove this claim, we first recall Pringsheim’s theorem: for a monotonic decreasing sequence $(a_i) \in \mathbb{A}$, one has $\lim i a_i = 0$.

Now take $a = (a_i) \in A$ with $0 \leq a_i \leq 1$ ($i \in \mathbb{N}$). Certainly $a_i \to 0$, and so there is a permutation $\sigma$ of $\mathbb{N}$ such that $a_{\sigma(j)} \leq a_{\sigma(i)}$ for $j \geq i$ in $\mathbb{N}$. Thus $ia_{\sigma(i)} \to 0$. Fix $\varepsilon \in (0, 1)$, and take $n \in \mathbb{N}$ such that $ja_{\sigma(j)} < \varepsilon/2$ for $j \geq n$ and also $\sum_{j=n+1}^{\infty} a_j < \varepsilon$. Set $B = \sigma^{-1}(N_n) \cup N_n$, where $N_n = \{1, \ldots, n\}$. Then set $u = \chi_B$, the characteristic function of $B$, and

$$F_{i,j} = \begin{cases} 1 & (i, j \in B), \\ 0 & (\text{otherwise}) \end{cases}$$

We see that

$$s := |B| \sum \{a_i^2 : i \in \mathbb{N} \setminus B\} \leq 2n \sum_{j=n+1}^{\infty} a_{\sigma(j)}^2 \leq \frac{n \varepsilon^2}{2} \sum_{j=n+1}^{\infty} j^{-2} < \varepsilon.$$ 

Thus

$$\|a^2 \cdot F - F \cdot a^2 + u \otimes a^2 - a^2 \otimes u\| = 2\|Q_B a^2\| \|u\| = s < \varepsilon,$$

and we have built in the fact that $\|a^2 - u a^2\| < \varepsilon$. It follows that the conditions of (3.3) are satisfied for $a^2$.

For finitely many elements in $A^2$, it suffices to consider the case where each of them is real-valued, and hence we need only consider differences of finitely many squares of non-negative elements of $A$, say the elements are $a^{(i)}, \ldots, a^{(k)}$. We then have finitely many permutations $\sigma_1, \ldots, \sigma_k$ of $\mathbb{N}$ that respectively render each of these latter sequences decreasing. We argue as above, with $n \in \mathbb{N}$ chosen so that, for each $1 \leq i \leq k$, we have $ja^{(i)}_{\sigma(i,j)} < \varepsilon/2k$ for $j \geq n$ and also $\sum_{j=n+1}^{\infty} a^{(i)}_j < \varepsilon$. Finally, we set

$$B = N_n \cup \bigcup_{i=1}^{k} \sigma_i^{-1}(N_n).$$
The above claim now follows.

4. Approximate amenability for $\ell^p$

Take $1 \leq p < \infty$. Then $\ell^p$ is a Banach sequence algebra, and $c_{00}$ is dense in $\ell^p$. These algebras are discussed in [1, Example 4.1.42].

It is well known that $\ell^p$ is weakly amenable, but not amenable. Clearly the sequence $(e_n)$ is an approximate identity for $\ell^p$ such that $\|e_n\|_p = n^{1/p}$ ($n \in \mathbb{N}$). Certainly each $a \in \ell^p$ satisfies equation (3.2) above.

It is shown in [3, Example 6.3] that $\ell^p$ is not sequentially approximately amenable. In this section we show that $\ell^p$ is not approximately amenable.

To this end, some preliminaries and further notations are needed. First, note that the map
\[
T : \ell^p \times \ell^p \to \ell^p(N \times N) : T(x, y)(i, j) = x_i y_j,
\]
is bilinear with $\|T\| = 1$, and so there is a map
\[
\widehat{T} : \ell^p \hat{\otimes} \ell^p \to \ell^p(N \times N)
\]
with $\widehat{T}(x \otimes y) = T(x, y)$ ($x, y \in \ell^p$) and $\|\widehat{T}\| = 1$. Let $H \in c_{00} \otimes c_{00}$.

Then
\[
\sum_{i,j} |H(i, j)|^p \leq \|H\|^p,
\]
(4.1)
where $\|H\|$ denotes the norm of $H$ in $\ell^p \hat{\otimes} \ell^p$. (Of course, equality holds in the case where $p = 1$.)

Fix throughout $\gamma_j = \frac{1}{j(j + 1)}$ and set $\gamma = (\gamma_j)$. Note that $\gamma$ is positive, decreasing, and satisfies
\[
k\gamma_k \leq \sum_{j=k+1}^\infty \gamma_j.
\]
(4.2)

Now let $\eta = (\eta_j) \in \ell^1$ be positive and decreasing, and define elements $a, b$ in $\ell^p$ by
\[
a = \sum_{j=1}^\infty \eta_j^{1/p} \delta_{2j-1}, \quad b = \sum_{j=1}^\infty \eta_j^{1/p} \delta_{2j}.
\]

We show that, for a suitable choice of $\eta$ and for a certain $\varepsilon > 0$, there is no element $F \in c_{00} \otimes c_{00}$ such that both the following inequalities are true:
\[
\|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon; \quad (4.3)
\]
\[
\|a - \pi(F)a\| + \|b - \pi(F)b\| < \varepsilon. \quad (4.4)
\]
It would then follow from Proposition 3.3 that $\ell^p$ is not approximately amenable.

Throughout, we set $u = \pi(F)$. As we remarked earlier, we may suppose that $F$ (and $u$) are real-valued.

We first make a small reduction. We may suppose that $\varepsilon < \eta_i^{1/p}$. Now assume that $F$ satisfies (4.4), with $\varepsilon$ replaced by $\varepsilon/2$. Then $\eta_i^{1/p}(1 - u_1) < \eta_i^{1/p}/2$, and so $u_1 > 1/2$. By replacing $u$ and $F$ by $u/u_1$ and $F/u_1$, respectively, we find new elements $F \in c_{00} \otimes c_{00}$ and $u \in c_{00}$ such that $\pi(F) = u$ and

$$u_1 = 1 \quad \text{and} \quad \|\Delta_a(F)\| + \|\Delta_b(F)\| < \varepsilon.$$  

Thus we may always suppose that $u_1 = 1$.

We shall need to estimate $\|\Delta_x\| = \|\Delta_x(F)\|$ for $x = a, b$, and for this we shall use (4.1). Thus we require lower bounds for $|\Delta_x(m, n)|$ for $m, n \in \mathbb{N}$.

First consider the points $(2i - 1, 2j)$, where $i, j \in \mathbb{N}$. For convenience, define $s = F_{2i-1,2j}$. We calculate the values

$$\Delta_a(2i - 1, 2j) = \eta_i^{1/p}(s - u_{2j}) ,$$

$$\Delta_b(2i - 1, 2j) = \eta_j^{1/p}(u_{2i-1} - s) .$$

In the case where $i \leq j$, so that $\eta_i \geq \eta_j$, geometrical considerations show that

$$|s - u_{2j}|^p \eta_i + |u_{2i-1} - s|^p \eta_j \geq \eta_j \left( |u_{2i-1} - u_{2j}|/2 \right)^p .$$

In a similar manner, the points $(2i, 2j - 1)$ taken with $i \leq j - 1$ and $j \geq 2$, so that $\eta_i \geq \eta_j$, lead to the estimate

$$|t - u_{2j-1}|^p \eta_i + |u_{2i} - t|^p \eta_j \geq \eta_j \left( |u_{2i} - u_{2j-1}|/2 \right)^p ,$$

where $t = F_{2i,2j-1}$.

At the points $(2i - 1, 2j - 1)$ and $(2i, 2j)$, where $i, j \in \mathbb{N}$, we have

$$\Delta_a(2i - 1, 2j - 1) = \left( \eta_i^{1/p} - \eta_j^{1/p} \right) F_{2i-1,2j-1} - \eta_i^{1/p} u_{2j-1} + \eta_j^{1/p} u_{2i-1} ,$$

$$\Delta_b(2i - 1, 2j - 1) = \left( \eta_i^{1/p} - \eta_j^{1/p} \right) F_{2i,2j} - \eta_i^{1/p} u_{2j} + \eta_j^{1/p} u_{2i} ,$$

$$\Delta_a(2i, 2j) = \Delta_b(2i - 1, 2j - 1) = 0 .$$

Since $\eta_i \neq \eta_j$ for $i \neq j$, there are choices of the values of $F$ at the points $(2i - 1, 2j - 1)$ and $(2i, 2j)$ giving zero values to both $\Delta_a$ and $\Delta_b$ at all these points. We shall not use this fact.

For $u = (u_i) \in c_{00}$, set

$$\Phi_p(\eta, u) = \sum_{j=1}^{\infty} \eta_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}|^p + \sum_{j=2}^{\infty} \eta_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|^p . \quad (4.5)$$
It follows from (4.1), the above estimates, and the simple inequality
\[(\|\alpha\| + \|\beta\|)^p \geq \|\alpha\|^p + \|\beta\|^p \quad (\alpha, \beta \in \mathbb{C}),\]
that
\[2^p(\|\Delta_a(F)\| + \|\Delta_b(F)\|)^p \geq \Phi_p(\eta, u).\]
Set
\[\theta_p(\eta) = \inf \{ \Phi_p(\eta, u) : u \in c_0, u_1 = 1 \}.\]
We seek to show that, for suitable choice of \(\eta\), we have \(\theta_p(\eta) > 0\), for then (4.3) fails for any \(\varepsilon\) with \(0 < \varepsilon < \min \{ \theta_p(\eta)^{1/p}, \eta_1^{1/p} \} / 2\), and so \(\ell^p\) is not approximately amenable.

We note that \(\Phi_p(\eta, u)\) is reduced if every value of \(u_i\) outside \([0, 1]\) is replaced by its nearest neighbour in \([0, 1]\). Thus we may suppose throughout that
\[0 \leq u_i \leq 1 \quad (i \in \mathbb{N}).\]

For \(d \geq 2\), consider the set
\[S_d = \{ u \in c_0 : u_1 = 1, u_i \in [0, 1] \quad (i = 1, \ldots, d), u_i = 0 \quad (i > d) \}.\]
Certainly
\[\alpha_d = \min \{ \Phi_p(\eta, u) : u \in S_d \} > 0,\]
and this minimum is attained. The question is whether or not
\[\lim_{d \to \infty} \alpha_d > 0.\]

Suppose for the moment that \(p = 1\), and take \(\eta = \gamma\). Thus, in this case, \(\Phi_1(\eta, u)\) from (4.5) becomes
\[\Phi_1(\gamma, u) = \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|.\]

Consider the values of \(\Phi_1(\gamma, u)\) for sequences \(u \in S_d\), where \(d \geq 2\). Indeed, take such a point \(u\) with \(u_d > 0\). We claim that, by setting \(u_d = 0\), the value of \(\Phi_1(\gamma, u)\) is reduced.

To establish this claim, first suppose that \(d = 2k+1\) for some \(k \in \mathbb{N}\). By the change specified, we first increase each term in the summand
\[\gamma_{k+1} \sum_{i=1}^{k} |u_{2i} - u_{2k+1}|\]
by at most \(u_{2k+1}\gamma_{k+1}\), and so the sum itself increases by at most \(k u_{2k+1}\gamma_{k+1}\). On the other hand, we decrease the term
\[\sum_{j=k+1}^{\infty} \gamma_j |u_{2k+1} - u_{2j}| = \left( \sum_{j=k+1}^{\infty} \gamma_j \right) u_{2k+1}\]
by $u_{2k+1}$ times the sum $\sum_{j=k+1}^{\infty} \gamma_j$ of the tail. Other terms are not affected. However, for each $k \in \mathbb{N}$, we have

$$k\gamma_{k+1} \leq k\gamma_k \leq \sum_{j=k+1}^{\infty} \gamma_j$$

by (4.2), and so, in total, the value of $\Phi_1(\gamma, u)$ has been decreased.

Now suppose that $d = 2k$ for some $k \in \mathbb{N}$. By the change specified, we firstly increase each term in the summand

$$\gamma_k \sum_{i=1}^{k} |u_{2i-1} - u_{2k}|$$

by at most $u_{2k}\gamma_k$, and so the sum itself increases by at most $ku_{2k}\gamma_k$. On the other hand, we decrease the term

$$\sum_{j=k+1}^{\infty} \gamma_j|u_{2k} - u_{2j-1}|$$

by $u_{2k}$ times the sum $\sum_{j=k+1}^{\infty} \gamma_j$ of the tail. Other terms are not affected. Once again, (4.2) ensures that the value of $\Phi_1(\gamma, u)$ has been decreased.

By continuing, we see that, subject to the constraints we have imposed, and in particular that $u \in c_0$ and $u_1 = 1$, the minimum value of $\Phi_1(\gamma, u)$ is attained at the point $v = (1, 0, 0, \ldots)$, and so

$$\theta_1 = \Phi_1(\gamma, v) = \sum_{j=1}^{\infty} \gamma_j = 1.$$

Hence we obtain the required contradiction, at least in the case where $p = 1$.

Now consider the case where $p > 1$. Again we should like to show that $\theta_p(\eta) > 0$ for suitable $\eta$. The above method for the case that $p = 1$ does not now work; indeed the minimum value $\min\{\Phi_p(\eta, u) : u \in S_d\}$ need not occur at the point $u = (1, 0, 0, \ldots)$, and in fact, perhaps surprisingly, it does not necessarily occur at a decreasing sequence $u$ of $S_d$. In fact we cannot explicitly calculate $\theta_p(\eta)$, but we obtain a lower bound by the use of Hölder’s inequality.

With $1/p + 1/q = 1$, choose $\alpha > 0$ so small that $1 - p\alpha/q > 1/2$. Then we have

$$\delta = \sum_{j=1}^{\infty} j^{1+\alpha} \gamma_j^+ < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma_j^{1-p\alpha/q} < \infty.$$
and so, in particular, the formula \( \eta_j = \gamma_j^{1-p\alpha/q} (j \in \mathbb{N}) \) defines a sequence \( \eta \in \ell^1 \) which is positive and decreasing.

Note that
\[
\frac{1 + \alpha}{q} + \left(1 - \frac{p}{q}\alpha\right) \frac{1}{p} = \frac{p + q}{pq} = 1
\]
and that \( \gamma_j = \eta_j^{1/p} \cdot \gamma_j^{(1+\alpha)/q} \). For each \( u \in c_{00} \) with \( u_1 = 1 \) we apply Hölder’s inequality to the sequence \((x_r y_r)\), where \((x_r)\) has generic term \( \eta_j^{1/p}|u_{2i-1} - u_{2j}| \) or \( \eta_j^{1/p}|u_{2i} - u_{2j-1}| \), and \((y_r)\) has the corresponding generic term \( \gamma_j^{(1+\alpha)/q} \). Thus we obtain

\[
1 \leq \Phi_1(\gamma, u) = \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|
\]
\[
\leq \left( \sum_{j=1}^{\infty} \sum_{i=1}^{j} \gamma_j^{1+\alpha} + \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} \gamma_j^{1+\alpha} \right)^{1/q} \Phi_p(\eta, u)^{1/p}
\]
\[
\leq (2\delta)^{1/q} \Phi_p(\eta, u)^{1/p}.
\]
It follows that \( \theta_p(\eta) \geq (2\delta)^{-p/q} > 0 \), as required.

Thus we have the following result.

**Theorem 4.1.** The Banach sequence algebras \( \ell^p(\mathbb{N}) \), \( 1 \leq p < \infty \), are not approximately amenable.

It is immediate that \( \ell^p(S) \) is not approximately amenable for any infinite set \( S \), since there is a continuous epimorphism \( \ell^p(S) \to \ell^p(\mathbb{N}) \).

Take \( 1 \leq p < \infty \). In [3, Corollary 7.1] it was shown that the Banach algebras \( \ell^p \) are essentially amenable, that is, any derivation into the dual of a neo-unital bimodule is inner. From Theorem 4.1 we conclude that essential amenability does not imply approximate amenability. It also follows by the Plancherel theorem that \( L^2(\mathbb{T}) \) fails to be approximately amenable, though by [6, Theorem 4.5] it is pseudo-contractible, that is, it admits a central (unbounded) approximate diagonal.

We finally consider a weighted variant of the \( \ell^p \) algebras.

Let \( \omega \in [1, \infty)^\mathbb{N} \). For \( p \geq 1 \), we consider
\[
\ell^p(\omega) = \{ f \in \mathbb{C}^\mathbb{N} : f \cdot \omega \in \ell^p \},
\]
where \( f \cdot \omega \) denotes the sequence with the \( i \)th coordinate \( (f \cdot \omega)(i) = f_i \omega_i \) \( (i \in \mathbb{N}) \). With the norm
\[
\|f\|_{p,\omega} = \|f \cdot \omega\|_p \quad (f \in \ell^p(\omega)),
\]
$\ell^p(\omega)$ is a Banach algebra under pointwise operations. As previously, the map $T: \ell^p(\omega) \times \ell^p(\omega) \to \ell^p(\omega \otimes \omega) = \ell^p(\omega \otimes \omega, N \times N)$ given by

$$T(x, y)(i, j) = x_i y_j \quad (x, y \in \ell^p(\omega), \ i, j \in \mathbb{N})$$

defines a contractive operator $\tilde{T}: \ell^p(\omega) \hat{\otimes} \ell^p(\omega) \to \ell^p(\omega \otimes \omega)$, where $\omega \otimes \omega$ denotes the weight on $N \times N$ such that $\omega \otimes \omega(i, j) = \omega \omega_j(i, j) (i, j \in \mathbb{N})$. As for the case of $\ell^p$, we aim to show that for some $\varepsilon > 0$ and elements $a, b \in \ell^p(\omega)$, there is no $F \in c_{00} \otimes c_{00}$ such that both the following inequalities are true:

$$\|\Delta_a(F)\|_{p, \omega \otimes \omega} + \|\Delta_b(F)\|_{p, \omega \otimes \omega} < \varepsilon;$$
$$\|a - \pi(F)a\|_{p, \omega} + \|b - \pi(F)b\|_{p, \omega} < \varepsilon.$$

We take $\gamma = (\gamma_i)$ and $\eta = (\eta_i)$ the same as in the proof of Theorem 4.1. Set

$$\eta'_j = \frac{\eta_j}{\omega_{2j-1}}, \quad \eta''_j = \frac{\eta_j}{\omega_{2j}} \quad (j \in \mathbb{N}),$$

and define

$$a = \sum_{j=1}^{\infty} (\eta'_j)^{1/p} \delta_{2j-1}, \quad b = \sum_{j=1}^{\infty} (\eta''_j)^{1/p} \delta_{2j},$$

so that $a, b \in \ell^p(\omega)$. Now for $F \in c_{00} \otimes c_{00}$ and $u = \pi(F)$, then following the same argument as in the proof of Theorem 4.1, we find that

$$2^p(\|\Delta_a(F)\|_{p, \omega \otimes \omega} + \|\Delta_b(F)\|_{p, \omega \otimes \omega})^p \geq \Phi_p(\eta, u),$$

where $\Phi_p(\eta, u)$ is given by Equation (4.5). This finally shows that the value of $\|\Delta_a\|_{p, \omega \otimes \omega} + \|\Delta_b\|_{p, \omega \otimes \omega}$ is bounded away from 0 as a function of $F \in c_{00} \otimes c_{00}$. We therefore conclude with the following theorem.

**Theorem 4.2.** The Banach sequence algebras $\ell^p(\omega)$, $1 \leq p < \infty$, are not approximately amenable for any weight $\omega$. $\square$

**References**


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